Congruence permutable semigroups in special classes of semigroups

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Attila Nagy

Mathematical Institute
Budapest University of Technology and Economics

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Contents

Introduction

An algebraic structure **A** is said to be congruence permutable if $\alpha \circ \beta = \beta \circ \alpha$ is satisfied for arbitrary congruences α and β on A, where $\circ$ is the usual composition of binary relations. The congruence permutable algebraic structures occur in a number of examinations. Here we refer to only papers [HM73], [Idz89], [Kea93], [Nau74] and [VW91] in which congruence permutable varieties of algebraic structures are in the centre of examinations. The congruence permutable algebraic structures are also in the focus of a famous problem (see [Schm69, Problem 3] or [RTW07, Problem CPP]) solved negatively in [RTW07]: Is every distributive algebraic lattice isomorphic to the congruence lattice of some algebraic structure with permuting congruences?

The groups and the rings are well known examples for congruence permutable algebraic structures. Every algebraic structure whose congruence lattice is a chain with respect to inclusion is also congruence permutable. The valuation rings, the Galois rings are well-known examples for algebraic structures whose congruence lattice is a chain with respect to inclusion.

The semigroups are common generalizations of groups and rings. In some respect the theory of semigroups is similar to group theory and ring theory and so the semigroup theoretical investigations are often motivated by comparisons with groups and rings. The semigroups are not congruence permutable, in general. As the groups and the rings are congruence permutable, and the chain rings play an important role in the theory of rings, it is not surprising that a number of papers are published in which the congruence permutable semigroups, especially the Δ-semigroups (semigroups whose lattices of congruences form a chain with respect to inclusion) are investigated in special subclasses of the class of all semigroups.

The aim of this dissertation is to present my results on Δ-semigroups and congruence permutable semigroups. We present our results published in papers [Nag84], [Nag90], [Nag92], [Nag98], [Nag00], [NJ04], [Nag05], [Nag08], [Nag13], [DN10], [JN03], [NZ16].

The dissertation contains an introduction and seven numbered chapters. Chapter 1 contains those basic notions and results which are used in the dissertation. The other chapters are devoted to special subclasses of the class of all semigroups. In Chapter 2, we give a complete description of weakly exponential Δ-semigroups. In Chapter 3, we determine all Δ-semigroups in the class of all RGC_n-commutative semigroups. In Chapter 4, we focus our attention on semigroups which satisfy a non-trivial permutation identity (these semigroups are called permutative semigroups). The main result is that every congruence permutable permutative semigroup is necessarily medial (that is, it satisfies the identity $axyb = ayxb$). In Chapter 5 we deal with the medial semigroups. We determine all medial Δ-semigroups, and characterize a type of medial congruence permutable semigroups. We define the notion of the left and the right reflection on semigroups, and show how we can get a type of congruence permutable medial semigroups from the similar type of commutative congruence permutable semigroups. In Chapter 6, we focus our attention on finite congruence permutable Putcha semigroups. Two types of them are constructed and characterized, using Lemma 3 of the paper [PP80] published by P.P. Pálfy and P. Pudlák. In Chapter 7, we give an application of congruence permutable semigroups.

In the literature of the semigroup theory, the first two papers on the subject were published on Δ-semigroups, in 1969. These two papers are [Sch69] and [Tam69], in which B.M. Schein and T. Tamura, independently, described the commutative Δ-semigroups. By their result, a semigroup is a commutative Δ-semigroup if and only if it is isomorphic to one of the following semigroups: (*i*) G or G^0, where G is a non-trivial subgroup of a quasicyclic p-group (p is a prime); (*ii*) a two-element semilattice; (*iii*) a commutative nil semigroup with chain ordered principal ideals; (*iv*) N^1, where N is a non-trivial commutative nil semigroup with chain ordered principal ideals.

The first paper on congruence permutable semigroups was published in 1975 by H. Hamilton. In his paper [Ham75], the commutative congruence permutable semigroups were described. It is proved that a commutative semigroup is congruence permutable if and only if it is either a commutative group or a commutative nil semigroup with chain ordered principal ideals or an ideal extension of a commutative nil semigroup N by a commutative group G with a zero adjoined such that the orbits of N under the action by G form a commutative nil semigroup with chain ordered principal ideals.

The above mentioned results on commutative semigroups started a process in which many results have been published on Δ-semigroups and congruence permutable semigroups in special subclasses of the class of all semigroups. Here we give a chronological summary of them, focusing on our own results.

1976: In papers [TS72] and [TN72], the authors (T. Tamura, T.E. Nordahl J. Shafer) described the structure of exponential semigroups (a semigroup is called an exponential semigroup if it satisfies the identity $(ab)^n = a^n b^n$ for every positive integer n). Using these result, P.G. Trotter generalized the results of [Sch69] and [Tam69]. He proved in [Tro76] that a semigroup S is an exponential Δ-semigroup if and only if it is isomorphic to one of the following semigroups: (*i*) G or G^0, where G is a non-trivial subgroup of a quasicyclic p-group (p is a prime); (*ii*) a two-element semilattice; (*iii*) B or B^0 or B^1, where B is a two-element rectangular band; (*iv*) an exponential nil semigroup with chain ordered principal ideals; (*v*) an exponential T1 semigroup or an exponential T2R semigroup or an exponential T2L semigroup (see Definition 2.2.1).

1981: The Trotter's result inspired A. Cherubini and C. Bonzini to examine the congruence permutable semigroups in a special subclass of the class of all exponential semigroups. In their paper [BC81], they dealt with the congruence permutable medial semigroups.

1984: In my paper [Nag84], I generalized the results of [Tro76] such that I extended them to a class of semigroups which class is wider than the class of exponential semigroups. I introduced the notion of the weakly exponential semigroup. A semigroup S is said to be weakly exponential if, for every $(a,b) \in S \times S$ and every positive integer m, there is a non-negative integer k such that $(ab)^{m+k} = a^m b^m (ab)^k = (ab)^k a^m b^m$. I proved that every weakly exponential semigroup is a semilattice of weakly exponential archimedean semigroups. Moreover, a semigroup is a weakly exponential archimedean Δ-semigroup if and only if it is isomorphic to either G or B or N, where G is a non-trivial subgroup of a quasicyclic p-group (p is a prime), B is a two-element rectangular band, and N is a nil semigroup with chain ordered principal ideals.

1990: Continuing the above investigation, in my paper [Nag90], I gave a complete description of weakly exponential Δ-semigroups. I proved that a semigroup is a weakly exponential Δ-semigroup if and only if it is isomorphic to one of the following semigroups: (i) G or G^0, where G is a non-trivial subgroup of a quasicyclic p-group (p is a prime); (ii) a two-element semilattice; (iii) B or B^0 or B^1, where B is a two-element rectangular band; (iv) a nil semigroup with chain ordered principal ideals; (v) a T1 semigroup or a T2R semigroup or a T2L semigroup. These results will be presented in Chapter 2 of this dissertation.

1992: In my paper [Nag92], I introduced the notion of the $\mathcal{RC}$-commutative semigroup and determined the $\mathcal{RC}$-commutative Δ-semigroups. I proved that a semigroup is an $\mathcal{RC}$-commutative Δ-semigroup if and only if it is isomorphic to one of the following semigroups: (i) G or G^0, where G is a non-trivial subgroup of a quasicyclic p-group (p is a prime); (ii) a two-element semilattice; (iii) R or R^0, where R is a two-element right zero semigroup; (iv) a commutative nil semigroup with chain ordered principal ideals; (v) N^1, where N is a non-trivial commutative nil semigroup with chain ordered principal ideals. The results of [Nag92] are presented at the end of Chapter 3 of this dissertation.

1995: My above mentioned results on $\mathcal{RC}$-commutative semigroups published in [Nag92] gave an impulse for further examinations of $\mathcal{RC}$-commutative semigroups. In [Jia95], Z. Jiang gave a complete description of congruence permutable $\mathcal{LC}$-commutative semigroups (the $\mathcal{LC}$-commutativity is the dual of the $\mathcal{RC}$-commutativity).

1998-1999: In my paper [Nag98], I introduced the notions of the $\mathcal{GC}_n$-commutativity of semigroups. For a positive integer n, a semigroup is said to be $\mathcal{GC}_n$-commutative if it satisfies the identity $a^n b a^i = a^i b a^n$ for every integer $i \geq 2$. It is clear that the $\mathcal{GC}_n$-commutativity is a generalization of the conditionally commutativity. In [Nag98], I proved some basic results on $\mathcal{GC}_n$-commutative semigroups and such $\mathcal{GC}_n$-commutative semigroups which also has the property $\mathcal{R}$-commutativity. A semigroup satisfying both of the $\mathcal{GC}_n$-commutativity and the $\mathcal{R}$-commutativity is called an $\mathcal{RGC}_n$-commutative semigroup. In [Nag98] and in the collected paper [JN03] (published in 1999 together with J. Ziang) we described the $\mathcal{RGC}_n$-commutative Δ-semigroups. We proved that a semigroup is an $\mathcal{RGC}_n$-commutative Δ-semigroup if and only if it is isomorphic to one of the following semigroups: (*i*) G or G^0, where G is a non-trivial subgroup of a quasicyclic p-group (p is a prime); (*ii*) a two-element semilattice; (*iii*) R or R^0 or R^1, where R is a two-element right zero semigroup; (*iv*) a commutative nil semigroup with chain ordered principal ideals; (*v*) N^1, where N is a non-trivial commutative nil semigroup with chain ordered principal ideals. The results of [Nag98] and [JN03] are presented in Chapter 3 of the dissertation.

2004: The $\mathcal{GC}_n$-commutativity together with the $\mathcal{R}$-commutativity has proven useful in our studies. In their paper [JC04], Z. Jiang and L. Chen associated the notion of the $\mathcal{GC}_n$-commutativity to the right duo property of semigroups (a semigroup is said to be right duo if every right ideal of S is a two sided ideal). A semigroup having both properies is said to be $\mathcal{RDGC}_n$-commutative. The combination of the above mentioned two properties also worked well. Using also the results of my paper [Nag98], Z. Jiang and L. Chen determined all congruence permutable $\mathcal{RDGC}_n$-commutative semigroups.

2004: In his Ph.D. dissertation [Ett70] (supervisor is: T. Tamura), W.A. Etterbeek dealt with the medial Δ-semigroups. The dissertation has often been cited in the literature, but it contains false assertions. The main theorem (Theorem 3.49) of the dissertation states that, apart from the two-element left and right zero semigroups, with or without adjoined zero, all such semigroups are commutative. In the proof of Theorem 3.49 Etterbeek used Theorem 3.45 in which it was asserted that if $S = S_0 \cup \{e\}$ is a right commutative Δ-semigroup such that S_0 is a nil semigroup and e is a right identity element of S, then S is necessarily commutative. The Example of my paper [Nag00] shows that this assertion is false. In our collected paper together with P.R. Jones [NJ04], we gave a review of the Etterbeek's dissertation. We pointed at the incorrect part of the Ph.D. dissertation. We proved that every permutative Δ-semigroup is medial and gave a correct description of the medial Δ-semigroups. We proved that a semigroup S is a medial Δ-semigroup if and only if one of the following conditions holds: (*i*) S is a commutative Δ-semigroup; (*ii*) S is isomorphic to either R or R^0, where R is a two-element right zero semigroup; (*iii*) S is isomorphic to the semigroup $Z = \{0, e, a\}$, obtained by adjoining to a zero semigroup $\{0, a\}$ an idempotent element e that is both a right identity element of Z and a left annihilator of $\{0, a\}$; (*iv*) S is isomorphic to the dual of a semigroup of type (*ii*) or (*iii*). These results are presented in Capter 4 and Chapter 5 of this dissertation.

2005: The fact that every permutative Δ-semigroup is medial inspired me to generalize this result to congruence permutable semigroups. In may paper [Nag05], I begun to deal with the following problem: Is every permutative congruence permutable semigroup medial? I gave a partial answer for this question. I proved that every permutative congruence permutable semigroup is either medial or an ideal extension of a rectangular band by a non-trivial commutative nil semigroup.

2006: P.R. Jones ([Jon06]) and A. Deák ([Dea06]) independently proved that if a permutative congruence permutable semigroup S is an ideal extension of a rectangular band by a non-trivial commutative nil semigroup, then S is medial. This and my results published in [Nag05] together imply that every permutative congruence permutable semigroup is medial. These results are presented in Chapter 4 of this dissertation.

2008: In their paper [BC81] published in 1981, A. Cherubini and C. Bonzini described the congruence permutable medial semigroups. They defined three kinds of semigroups, and showed that every non-archimedean congruence permutable medial semigroup is isomorphic to one of them. In my paper [Nag08], I defined the notion of the left [right] reflection of semigroups, and showed that the congruence permutable medial semigroup of the first kind can be obtained from the non-archimedean commutative congruence permutable semigroups by using the notion of the right and the left reflection. This result is presented in Chapter 5.

2009: In our collected paper [DN10] published together with A. Deák, we investigated the finite congruence permutable Putcha semigroups. We shoved that the finite archimedean congruence permutable semigroups are exactly the finite cyclic nilpotent semigroups and the finite completely simple congruence permutable semigroups. We also shown that if S is a finite non-archimedean congruence permutable Putcha semigroup, then it is a semilattice of a completely simple semigroup $S_1 = M(I, G, J; P)$ with $|I|, |J| \leq 2$ and a semigroup S_0 such that $S_1 S_0 \subseteq S_0$ and S_0 is an ideal extension of a completely simple semigroup by a nilpotent semigroup. Dealing with some special cases, we give a complete characterization of two types of finite congruence permutable non-archimedean Putcha semigroups. In our investigation we used Lemma 3 of the paper [PP80] published by P.P. Pálfy and P. Pudlák several times. The results on finite congruence permutable Putcha semigroups will be presented in Chapter 6 of this dissertation.

2016: In our collected paper [NZ16] published together with M. Zubor, we give an application of congruence permutable semigroups. For an ideal J of a semigroup algebra $\mathbb{F}[S]$, let ϱ_J denote the congruence on the semigroup S which is the restriction of the congruence on $\mathbb{F}[S]$ defined by the ideal J. We show that if S is a semilattice or a rectangular band, then the mapping $\varphi_{\{S;\mathbb{F}\}}: \ J \mapsto \varrho_J$ is a $\circ$-homomorphism if and only if S is congruence permutable. These results of this paper is presented in Chapter 7 of this dissertation.

Chapter 1

Preliminaries

In this chapter we present those basic notions and results on arbitrary semigroups, congruence permutable semigroups and Δ-semigroups which are used in this dissertation. For notations and notions not defined here we refer to the books [CP61], [CP67], [How76] and [Okn91].

1.1 Basic notions and results; general case

A ***semigroup*** is a groupoid in which the operation is associative. A semigroup containing an identity element is called a ***monoid***.

Let S be a semigroup, and 1 be a symbol not representing any element of S. Extend the given binary operation in S to one in $S \cup \{1\}$ by defining $11 = 1$ and $1s = s1 = s$ for every $s \in S$. Then $S \cup \{1\}$ is a monoid (with the identity element 1). We say that this monoid is obtained from S by adjunction an identity element to S.

Similarly, one may adjoin an element 0 to S by defining $00 = 0s = s0 = 0$ for every $s \in S$. Then $S \cup \{0\}$ is a semigroup with the zero 0.

We shall use the following notations. For an arbitrary semigroup S, let

$$S^1 = \begin{cases} S & \text{if } S \text{ has an identity element,} \\ S \cup \{1\} & \textit{otherwise;} \end{cases}$$

and

$$S^0 = \begin{cases} S & \text{if } S \text{ has a zero element, and } |S| > 1, \\ S \cup \{0\} & \textit{otherwise.} \end{cases}$$

Bands

An element e of a semigroup S is called an ***idempotent element*** if $e^2 = e$. An element a of a semigroup S is called a ***regular element*** if there is an element

$x \in S$ such that $axa = a$ is satisfied. It is easy to see that $axa = a$ implies that ax and xa are idempotent elements of S. It is clear that every idempotent element of a semigroup is regular. Thus a semigroup contains an idempotent element if and only if it has a regular element.

A semigroup S is called a ***band*** if every element of S is an idempotent element. A commutative band is called a ***semilattice***.

A semigroup satisfying the identity $ab = a$ [$ab = b$] is called a ***left zero semigroup*** [***right zero semigroup***]. A semigroup satisfying the identity $aba = a$ is called a ***rectangular band***. It is known ([Pet77, II.1.5. Lemma]) that a semigroup is a rectangular band if and only if it is a direct product of a left zero semigroup and a right zero semigroup.

A direct product of a group and a rectangular band is called a ***rectangular group***. If the group is commutative, then we say that the semigroup is a ***rectangular abelian group***. A direct product of a group and a left zero [right zero] semigroup is called a ***left group*** [***right group***].

Congruences on semigroups

Let X be a non-empty set. For arbitrary binary relations α and β on X, $\alpha \circ \beta$ denotes the binary relation on X defined by $(a, b) \in \alpha \circ \beta$ if and only if there is an element $x \in X$ such that $(a, x) \in \alpha$ and $(x, b) \in \beta$. The set $\mathcal{B}_X$ of all binary relations on X is a semigroup with respect to the operation $\circ$.

Definition 1.1.1 *([Lja63]) A non-empty subset H of a semigroup S is called a* ***normal complex*** *of S if $xHy \cap H \neq \emptyset$ implies $xHy \subseteq H$ for every $x, y \in S^1$.*

Theorem 1.1.2 *([Lja63]) If H is a normal complex of a semigroup S, then the relation α_H defined by $a\ \alpha_H\ b$ if and only if $a = b$ or there is a positive integer n and there are elements $x_i, y_i \in S^1$ and $p_i, q_i \in H$ $(i = 1, 2, \ldots, n)$ such that*

$$a = x_1p_1y_1,\ x_1q_1y_1 = x_2p_2y_2, \ldots, x_nq_ny_n = b$$

is the least congruence on S such that H is a congruence class. $\square$

A non-empty subset H of a semigroup S is said to be a ***left [right] unitary subset*** of S if, for every $a, b \in S$, the assumption $ab, a \in H$ [$ba, a \in H$] implies $b \in H$. A left and right unitary subset of a semigroup is said to be a ***unitary subset*** of S.

A non-empty subset H of a semigroup S is called a ***reflexive subset*** of S if, for every $a, b \in S$, $ab \in H$ if and only if $ba \in H$.

For a non-empty subset H of a semigroup S, let

$$\mathcal{R}_H = \{(a, b) \in S \times S : \ (\forall x \in S) \quad ax \in H \quad \text{iff} \quad bx \in H\}.$$

It is easy to see that $\mathcal{R}_H$ is a right congruence on S which is called the ***principal right congruence*** on S. Let $\mathcal{L}_H$ denote the dual of $\mathcal{R}_H$, and let

$$\mathcal{P}_H = \{(a,b) \in S \times S : \ (\forall x, y \in S) \quad xay \in H \quad \text{iff} \quad xby \in H\}.$$

It is easy to see that if H is a reflexive unitary subsemigroup of a semigroup S, then $\mathcal{R}_H = \mathcal{L}_H = \mathcal{P}_H$. Moreover, the next theorem is true.

Theorem 1.1.3 *([CP67]) If H is a reflexive unitary subsemigroup of a semigroup S, then $\mathcal{R}_H$ is a group or a group with zero congruence on S such that H is an identity element of $S/\mathcal{R}_H$.*

Conversely, if α is a group or a group with zero congruence on a semigroup S and H denotes the α-class of S which is the identity of S/α, then H is a reflexive unitary subsemigroup of S and $\alpha = \mathcal{R}_H$.

The right residue $W_H = \{x \in S : \ (\forall a \in S) \ xa \notin H\}$ of H is not empty if and only if S/α has a zero element. In this case the zero of S/α equals W_H. □

Ideals, simple and completely simple semigroups

A non-empty subset A of a semigroup S is called a ***left ideal*** [***right ideal***] of S if $sa \in A$ [$as \in A$] for every $a \in A$ and $s \in S$. A non-empty subset A of a semigroup is called an ***ideal*** of S if it is a left ideal and a right ideal of S, that is, $as, sa \in A$ for every $a \in A$ and $s \in S$.

For an element a of a semigroup S, let $L(a)$ [$R(a)$ $J(a)$] denote the left ideal [right ideal, ideal] of S generated by a. It is clear that $L(a) = S^1a$, $R(a) = aS^1$ and $J(a) = S^1aS^1$.

For an arbitrary semigroup S,

$$\mathcal{L} = \{(a,b) \in S \times S : \ L(a) = L(b)\},$$

$$\mathcal{R} = \{(a,b) \in S \times S : \ R(a) = R(b)\}$$

and

$$\mathcal{J} = \{(a,b) \in S \times S : \ J(a) = J(b)\}$$

are equivalences on S. These equivalences are called the ***Green's equivalences*** on S.

If B is an ideal of an ideal A of a semigroup S, then B is not an ideal of S, in general. But the following theorem is true, which will be used in the dissertation several times.

Theorem 1.1.4 *(Exercises 4. (a) for §2.6 of [CP61]) If A is an ideal of a semigroup S, and if B is an ideal of A such that $B^2 = B$, then B is an ideal of S.* □

If A is an ideal of a semigroup S, then the relation

$$\varrho_A = \{(a, b) \in S \times S : \ a = b \quad \text{or} \quad a, b \in A\}$$

is a congruence on S. This congruence is called the ***Rees congruence*** on S defined by the ideal A. The factor semigroup S/ϱ_A is said to be the ***Rees factor semigroup*** of S defined by the ideal A. This factor semigroup is also denoted by S/A.

If A is an ideal of a semigroup S and Q denotes the Rees factor semigroup S/A, then we also say that S is an ***ideal extension*** (briefly: an extension) of the semigroup A by the semigroup Q.

If A is an ideal of a semigroup S such that there is a homomorphism φ of S onto A which leaves the elements of A fixed, then we say that S is a ***retract extension*** of A (by $Q = S/A$). If this is the case, then the homomorphism φ is called a ***retract homomorphism*** of S onto A, and the ideals A is said to be a ***retract ideal*** of S.

It is easy to see that if a semigroup S is an ideal extension of a subgroup G (with an identity element e) of S, then $s \mapsto es$ is a retract homomorphism of S onto G. Thus an ideal extension of a group by a semigroup with a zero is a retract extension.

An ideal A of a semigroup S is called a ***dense ideal*** of S if, for every congruence α on S, the assumption that the restriction of α to A is the identity relation on A implies that α is the identity relation on S.

An ideal A of a semigroup S is called a ***proper ideal*** of S if $A \neq S$. A semigroup S is called a ***simple semigroup*** if it has no proper ideal.

For arbitrary idempotent elements e and f of a semigroup S, let $e \leq f$ denote the fact that $ef = fe = e$. It is known that $\leq$ is a partial ordering on the set $E(S)$ of all idempotent elements of a semigroup S. If a semigroup contains a zero element 0, then $0 \leq e$ is satisfied for every $e \in E(S)$. An idempotent element e of a semigroup S is said to be a ***primitive idempoten element*** of S if the only idempotents of S under e are itself e and 0 (if S has a zero) and $e \neq 0$.

We say that a semigroup S is a ***completely simple semigroup*** if either $|S| = 1$ or $|S| \geq 2$ and S is a simple semigroup containing a primitive idempotent.

The next theorem characterizes the completely simple semigroups.

Theorem 1.1.5 *([How76, Theorem 2.11. of Chapter III]) Let G be a group, let I, Λ be non-empty sets, and let $P = (p_{\lambda i})$ be a $\Lambda \times I$ matrix with entries in G. Let $S = I \times G \times \Lambda$, and define a binary operation on S by the role that*

$$(i, a, \lambda)(j, b, \mu) = (i, ap_{\lambda j}b, \mu).$$

Then S is a completely simple semigroup, which will be denoted by $\mathcal{M}(G; I, \Lambda; P)$.

Conversely, any completely simple semigroup is isomorphic to one of constructed in this manner. □

The semigroup $\mathcal{M}(G; I, \Lambda; P)$ is called a ***Rees*** $I \times \Lambda$ ***matrix semigroup*** over the group G with sandwich matrix P.

We say that the sandwich matrix P is normalized if all the elements in a given row and in a given column are the identity element of G. By [CP61, Lemma 3.6.], we can suppose that P is normalized.

A monoid S with the identity element e is called a ***bicyclic semigroup*** if it is generated by two elements a, b with the single generating relation $ab = e$.

If a semigroup S is simple but not completely simple, then $|S| \geq 2$ and so it does not contain a zero. By the proof of Theorem 2.54 of [CP61], the following theorem holds.

Theorem 1.1.6 *If e is an idempotent element of a simple semigroup S which is not completely simple, then S contains a bicyclic subsemigroup having e as the identity element.* □

Semilattice decomposition of semigroups

A congruence α of a semigroup S is called a ***semilattice congruence*** if the factor semigroup $I = S/\alpha$ is a semilattice. The α-classes S_i $(i \in I)$ are subsemigroups of S such that $S_iS_j \subseteq S_{ij}$, where ij is the product of i and j in the semilattice I. We also say that the semigroup S is a semilattice I of subsemigroups S_i $(i \in I)$.

A semigroup S is said to be ***semilattice indecomposable*** if the universal relation ω_S is the only semilattice congruence on S.

Let S be a semigroup and σ a relation on S defined by $a\sigma\, b$ if and only if a divides some power of b, that is, $xay = b^m$ for some $x, y \in S^1$ and some positive integer m. Let ϱ be the transitive closure of σ, and let ϱ' defined by a ϱ' b if and only if $a\varrho\, b$ and $b\varrho\, a$.

Theorem 1.1.7 *([Tam68, THEOREM]) ϱ' is a smallest semilattice congruence on a semigroup S, and each ϱ'-class is a semilattice indecomposable semigroup.* □

With other words: every semigroup is decomposable into a semilattice of semilattice indecomposable semigroups. The next result is a consequence of Theorem 1.1.7.

Theorem 1.1.8 *([Tam68, COROLLARY]) A semigroup S is semilattice indecomposable if and only if, for every $a, b \in S$, there is a sequence*

$$a = a_0, a_1, \ldots, a_{k-1}, a_k = b$$

of elements of S such that a_{i-1} divides some power of a_i, $(i = 1, \ldots, k)$. □

Definition 1.1.9 *A semigroup S is called a left [right] archimedean semigroup if, for every $a, b \in S$, there are positive integers m and n such that $a^m \in S^1 b$ and $b^n \in S^1 a$ [$a^m \in bS^1$ and $b^n \in aS^1$]. A semigroup S is said to be an archimedean semigroup if, for every $a, b \in S$, there are positive integers m and n such that $a^m \in S^1 bS^1$ and $b^n \in S^1 aS^1$.*

It is clear that every left archimedean and every right archimedean semigroup is archimedean. By Theorem 1.1.8, the archimedean semigroups (and so the left archimedean semigroups and the right archimedean semigroups) are special semilattice indecomposable semigroups.

Definition 1.1.10 *A semigroup S is called a* ***left [right] Putcha semigroup*** *if, for every $x, y \in S$, the assumption $y \in xS^1$ [$y \in S^1 x$] implies $y^m \in x^2 S^1$ [$y^m \in S^1 x^2$] for some positive integer m.*

A semigroup S is called a ***Putcha semigroup*** *if, for every $x, y \in S$, the assumption $y \in S^1 xS^1$ implies $y^m \in S^1 x^2 S^1$ for some positive integer m.*

The next theorem is about a connection between the archimedean semigroups and the Putcha semigroups.

Theorem 1.1.11 *([Put73]) A semigroup S is a semilattice of archimedean semigroups if and only if S is a Putcha semigroup. In such a case the corresponding semilattice congruence on S equals*

$$\eta = \{(a, b) \in S \times S : \ a^m \in SbS, \ b^n \in SaS \text{ for some positive integers } m, n\}$$

and is the least semilattice congruence on S. □

The next theorem is a characterization of archimedean semigroups containing at least one idempotent element. This result will be used in the dissertation several times.

Theorem 1.1.12 *([Chr69]) A semigroup S is archimedean and contains at least one idempotent element if and only if it is an ideal extension of a simple semigroup containing an idempotent by a nil semigroup.*

A special type of left weakly commutative semigroups will be examined in Chaptert 3.

Definition 1.1.13 *A semigroup S is called a* ***left [right] weakly commutative semigroup*** *if, for every $a, b \in S$, there exist $x \in S$ and a positive integer n such that $(ab)^n = bx$.*

The following theorem shows the connection of the left [right] weakly commutative semigroups and the right [left] archimedean semigroups.

Theorem 1.1.14 *([Nag01, Theorem 4.2]) A semigroup is left [right] weakly commutative if and only if it is a semilattice of right [left] archimedean semigroups.* □

As every right [left] archimedean semigroup is archimedean, the following assertion is true.

Corollary 1.1.15 *Every left [right] weakly commutative semigroup is a semilattice of archimedean semigroups.*

Lemma 1.1.16 *([Mar92]) A left [right] Putcha semigroup is a Putcha semigroup.*

By Theorem 1.1.11 and Lemma 1.1.16, the following assertion is true.

Corollary 1.1.17 *Every left [right] Putcha semigroup is decomposable into a semilattice of archimedean semigroups.*

The following two theorems will be used in the dissertation several times.

Theorem 1.1.18 *([Mar92]) A semigroup is a simple left and right Putcha semigroup if and only if it is completely simple.*

Theorem 1.1.19 *([Mar92]) A semigroup is an archimedean left and right Putcha semigroup containing at least one idempotent element if and only if it is a retract extension of a completely simple semigroup by a nil semigroup.*

Semigroup Algebra

By the ***semigroup algebra*** $\mathbb{F}[S]$ of a semigroup S over a field $\mathbb{F}$, we mean the set of all functions $f: S \mapsto \mathbb{F}$ such that the ***support*** of f (that is the set of all s in S such that $f(s) \neq 0$) is finite or empty, with operation defined for every $f, g \in \mathbb{F}[S]$, $s \in S$, $\alpha \in \mathbb{F}$ as follows:

$$(f+g)(s) = f(s) + g(s)$$

$$(\alpha f)(s) = \alpha f(s)$$

$$(fg)(s) = \begin{cases} \sum_{(t,u)\in A(s)} f(t)g(u) & \text{if } A(s) \neq \emptyset, \\ 0 & \textit{if } A(s) = \emptyset, \end{cases}$$

where $A(s) = \{(t,u) \in S \times S : \ tu = s\}$. $\mathbb{F}[S]$ is an associative $\mathbb{F}$-algebra subject to these operations.

For any $s \in S$, let $f_s : S \mapsto \mathbb{F}$ be the function such that $f_s(s) = 1$, $f_s(t) = 0$ if $t \neq s$. Then $\{f_s : \ s \in S\}$ is a subsemigroup of the multiplicative semigroup of $\mathbb{F}[S]$, which is an $\mathbb{F}$-basis of $\mathbb{F}[S]$. Moreover $s \mapsto f_s$ is a semigroup isomorphism. Thus, as usual, $\mathbb{F}[S]$ will be identified with the set of all finite sums $\sum \alpha_s s$, $\alpha_s \in \mathbb{F}$, $s \in S$, so that it is an $\mathbb{F}$-space with a basis S and the multiplication induced by the multiplication in S.

1.2 Congruence permutable semigroups

Definition 1.2.1 *We say that a semigroup S is a* ***congruence permutable semigroup*** *(or briefly:* ***permutable semigroup****) if $\alpha \circ \beta = \beta \circ \alpha$ is satisfied for every congruences α and β on S.*

In this dissertation we use the expression "congruence permutable".

It is clear that a semigroup S is congruence permutable if and only if the congruences on S form a subsemigroup of the semigroup $\mathcal{B}_S$ of all binary relations on S.

Theorem 1.2.2 *([Ham75]) If S is a congruence permutable semigroup, then the ideals of S form a chain with respect to inclusion.* □

The next result will be used in the dissertation several times.

Theorem 1.2.3 *([Sza70]) The ideals of a semigroup S form a chain with respect to inclusion if and only if the principal ideals of S do it.*

The next two theorem are very useful in our investigation.

Theorem 1.2.4 *([Ham75]) If S is a congruence permutable semigroup and S is homomorphic onto T, then T is a congruence permutable semigroup.* □

Theorem 1.2.5 *([Ham75]) A semilattice Γ is congruence permutable if and only if $|\Gamma| \leq 2$.* □

Remark 1.2.6 *By Theorem 1.1.7, every semigroup is a semilattice of semilattice indecomposable semigroups. Thus Theorem 1.2.4 and Theorem 1.2.5 together imply that every congruence permutable semigroup is either semilattice indecomposable or a semilattice of two semilattice indecomposable semigroups S_0 and S_1 such that $S_0S_1 \subseteq S_0$.*

Theorem 1.2.7 *([Ham75]) If a congruence permutable semigroup S is a semilattice of two semilattice indecomposable subsemigroups S_1 and S_0 such that $S_0S_1 \subseteq S_0$, then S_1 is simple.* □

Theorem 1.2.8 *([Ham75]) If a congruence permutable semigroup S has a proper ideal, then S has no non-trivial group homomorphic image.* □

Lemma 1.2.9 *([Tam67]) Let I be an ideal of a semigroup S. If f is a homomorphism of I onto a non-trivial group G, then there is a homomorphism g of S onto G such that f is the restriction of g to I.* □

Remark 1.2.10 *By Lemma 1.2.9 and Theorem 1.2.8, if a congruence permutable semigroup S has a proper ideal I, then neither S nor I has a non-trivial group homomorphic image.*

The next theorem shows the connection between the congruence classes and the ideals of congruence permutable semigroups.

Theorem 1.2.11 *([Jia95]) If I is an ideal and α is a congruence of a congruence permutable semigroup, then I is a union of α-classes or is contained in an α-class.*

The class of all Δ-semigroups is a subclass of the class of all congruence permutable semigroups. In the next we present those basic results on Δ-semigroups which will be use in the dissertation.

1.3 Δ-semigroups

Definition 1.3.1 *A semigroup S is called a **Δ-semigroup** if the lattice $\mathcal{L}(S)$ of all congruences of S is a chain with respect to inclusion.*

Remark 1.3.2 *If S^1 or S^0 is a Δ-semigroup, then S is also a Δ-semigroup.*

Theorem 1.3.3 *([Tam69]) Every homomorphic image of a Δ-semigroup is also a Δ-semigroup.*

Theorem 1.3.4 *([Sch69, Tam69]) A semigroup S is a commutative Δ-semigroup if and only if it satisfies one of the following conditions:*

(i) *S is isomorphic to G or G^0, where G is a non-trivial subgroup of a quasi-cyclic p-group (p is a prime).*

(ii) *S is isomorphic to a two-element semilattice.*

(iii) *S is isomorphic to a commutative nil semigroup with chain ordered principal ideals.*

(iv) *S is isomorphic to N^1, where N is a non-trivial commutative nil semigroup with chain ordered principal ideals.*

From Theorem 1.3.4, we have the following result which will be used in the dissertation several times.

Theorem 1.3.5 *A semilattice is a Δ-semigroup if and only if it contains at most two elements.*

Remark 1.3.6 *Theorem 1.3.5 and Theorem 1.3.3 together imply that if a semigroup S is a Δ-semigroup, then it is either semilattice indecomposable or a semilattice of two semilattice indecomposable semigroups S_0 and S_1 with $S_0S_1 \subseteq S_0$.*

The next theorem is a consequence of Remark 1.2.10.

Theorem 1.3.7 *([Tam69]) If a Δ-semigroup S contains a proper ideal I, then neither S nor I has a non-trivial group homomorphic image.*

The following theorem is a consequence of Theorem 1.2.2.

Theorem 1.3.8 *If S is a Δ-semigroup, then all the ideals of S form a chain with respect to inclusion.*

The next theorem is about nil Δ-semigroups. A semigroup S with a zero element 0 is called a ***nil semigroup*** if, for every $a \in S$, there is a positive integer n such that $a^n = 0$.

Theorem 1.3.9 *([Nag01, Theorem 1.54 and Theorem 1.56]) Let S be a nil semigroup. The following are equivalent:*

(i) S is a Δ-semigroup;

(ii) the ideals of S form a chain with respect to inclusion;

(iii) the principal ideals of S form a chain with respect to inclusion

(iv) S is a chain with respect to the divisibility ordering.

In that case, each congruence on S is the Rees congruence corresponding to the ideal consisting of the congruence class of 0.

By Theorem 1.2.2 and Theorem 1.3.9, we have the following result.

Theorem 1.3.10 *A nil semigroup is congruence permutable if and only if it is a Δ-semigroup.*

The next theorem is about the non-identity, non Rees congruences on Δ-semigroups.

Theorem 1.3.11 *([Tro76]) Let S be a Δ-semigroup and σ be a non-identity congruence on S which is not a Rees congruence. Then, for some $a \in S$,*

$$[b]_\sigma = I_a, \text{ if } J(b) \subset J(a),$$
$$[b]_\sigma \subseteq J_a, \text{ if } J(b) = J(a),$$
$$[b]_\sigma = \{b\}, \text{ if } J(b) \supset J(a),$$

where J_a denotes the $\mathcal{J}$-class of S containing a and $I_a = J(a) - J_a$.

As a Δ-semigroup is congruence permutable and a non-trivial nil semigroup is not simple, the following theorem is a consequence of Theorem 1.3.12.

Theorem 1.3.12 *([Nag01, Theorem 1.57]) If a Δ-semigroup S is a semilattice of a nil semigroup S_1 and an ideal S_0 of S, then $|S_1| = 1$.*

The next theorem will be used in Chapter 2, when we will characterize the T1 semigroups.

Theorem 1.3.13 *([Nag01, Theorem 1.58]) Let S be a semigroup which is a disjoint union $S = P \cup N$ of a one-element subsemigroup $P = \{e\}$ of S and an ideal N of S such that N is a nil semigroup. Then S is a Δ-semigroup if and only if N is a Δ-semigroup and $S^1eS^1 = S$.* □

Here is a consequence of the previous theorem.

Corollary 1.3.14 *([Nag01, Corollary 1.2]) A nil semigroup with an identity adjoined N^1 is a Δ-semigroup if and only if N is a Δ-semigroup.*

The next theorem will be used sevaral times.

Theorem 1.3.15 *([Tro76], [Nag01, Theorem 1.59]) If a Δ-semigroup S is a semilattice of a subgroup P of a quasicyclic p-group (p is a prime) and a nil semigroup N with $NP \subseteq N$, then either $|N| = 1$ or $|P| = 1$.*

Theorem 1.3.16 *([Nag01, Theorem 1.60]) Let S be a semigroup in which $\alpha \cap \beta = id_S$ implies $\alpha = id_S$ or $\beta = id_S$ for every congruences α and β on S. If S is an ideal extension of a rectangular group K by a semigroup with zero, then K is either a subgroup or a left zero subsemigroup or a right zero subsemigroup of S.*

Corollary 1.3.17 *([Nag01, Corollary 1.3]) If a Δ-semigroup S is an ideal extension of a rectangular group K by a semigroup with zero, then K is either a subgroup or a left zero subsemigroup or a right zero subsemigroup of S. As a special case: if a Δ-semigroup S is a rectangular group, then S is either a group or a left zero semigroup or a right zero semigroup.*

Theorem 1.3.18 *([Tro76]) A non-trivial band is a Δ-semigroup if and only if it is isomorphic to either R or R^1 or R^0, where R is a two-element right zero semigroup, or L or L^1 or L^0, where L is a two-element left zero semigroup, or F, where F is a two-element semilattice.*

The next theorem is a consequence of the previous one.

Theorem 1.3.19 *A left (right) zero semigroup is a Δ-semigroup if and only if it has at most two elements.*

Chapter 2

Weakly exponential semigroups

In [TK54], T. Tamura and N. Kimura proved basic results on the structure of commutative semigroups. They proved that every commutative semigroup is a semilattice of commutative archimedean semigroups. It was also shown that a commutative archimedean semigroup containing an idempotent element is an ideal extension of a commutative group by a commutative nil semigroup. In the literature of the theory of semigroups we can find a number of papers in which the authors extended these results to special classes of semigroups. In [Chr69], J.L. Chrislock defined the notion of the medial semigroup (a semigroup which satisfies the identity $axyb = ayxb$), and generalized the results of T. Tamura and N. Kimura to medial semigroups. He proved that every medial semigroup is a semilattice of medial archimedean semigroups. Moreover, a medial semigroup is archimedean and contains an idempotent element if and only if it is an ideal extension of a rectangular abelian group by a nil semigroup. In [TS72], T. Tamura and J. Shafer introduced the notion of the exponential semigroup (a semigroup which satisfies the identity $(ab)^n = a^n b^n$ for every positive integer n), and generalized the results of J.L. Chrislock to this new kind of semigroups. They proved that every exponential semigroup is a semilattice of exponential archimedean semigroups. Moreover, if an exponential archimedean semigroup contains an idempotent element, then it is an ideal extension of a rectangular abelian group by an exponential nil semigroup. In [TN72], T. Tamura and T.E. Nordahl proved further results on exponential archimedean semigroups. They proved that a semigroup S is an exponential archimedean semigroup containing at least one idempotent element if and only if S is a strict ideal extension of a rectangular abelian group by an exponential nil semigroup. Using these results, P.G. Trotter generalized Schein's results on commutative Δ-semigroups ([Sch69]) to exponential semigroups. In [Tro76], P.G. Trotter determined all possible exponential Δ-semigroups. In order to generalize the results on exponential semigroups, I introduced the notion of the weakly exponential semigroup: a semigroup S with the property that, for every $(a,b) \in S \times S$ and every integer $m \geq 2$, there is a positive integer k such that $(ab)^{m+k} = a^m b^m (ab)^k = (ab)^k a^m b^m$ ([Nag84]). The structure of weakly exponential semigroups and weakly exponential Δ-semigroups are described in my papers [Nag84], [Nag90] and [Nag13]. In this chapter we present the results of them. The chapter contains three sections.

In the first section we deal with the semilattice decomposition of weakly exponential semigroups. We show that every weakly exponential semigroup is a semilattice of weakly exponential archimedean semigroups. We proved that a semigroup is simple and weakly exponential if and only if it is a rectangular abelian group. Using also this result, we show that a semigroup is a weakly exponential archimedean semigroup containing at least one idempotent element if and only if it is a retract extension of a rectangular abelian group by a nil semigroup. We also prove that every weakly exponential archimedean semigroup without idempotent elements has a non-trivial group homomorphic image.

In the second section we characterize all weakly exponential Δ-semigroups. We show that a semigroup is a weakly exponential Δ-semigroup if and only if it is isomorphic one of the following semigroups: (i) G or G^0, where G is a non-trivial subgroup of a quasicyclic p-group (p is a prime); (ii) a two-element semilattice; (iii) R or R^0 or R^1, where R is a two-element right zero semigroup; (iv) L or L^0 or L^1, where L is a two-element left zero semigroup; (v) a nil semigroup with chain ordered principal ideals; (vi) a T1 or a T2R or a T2L semigroup (see Definition 2.2.1).

In the third section we characterize the $T1$ semigroups and the $T2R$ (T2L) semigroups.

2.1 Semilattice decomposition of weakly exponential semigroups

Definition 2.1.1 *([Nag84]) A semigroup S is called a weakly exponential semigroup if, for every $(a,b) \in S \times S$ and every integer $m \geq 2$, there is a positive integer k such that $(ab)^{m+k} = a^m b^m (ab)^k = (ab)^k a^m b^m$.*

We note that, in Definition 2.1.1, the condition that k is a positive integer can be changed over the condition that k is a non-negative integer.

Theorem 2.1.2 *[Nag01, Theorem14.1]) Every weakly exponential semigroup is a left and right Putcha semigroup.*

Proof. Let S be a weakly exponential semigroup. To prove that S is a left Putcha semigroup, assume that $b \in aS^1$ is satisfied for some elements a and b of S. We must to show that there is a positive integer m such that $b^m \in a^2 S^1$. We can suppose $a \neq b$. Then there is an element $y \in S$ such that $b = ay$. As S is weakly exponential, for the integer 2, there is a positive integer k such that

$$b^{2+k} = (ay)^{2+k} = a^2 y^2 (ay)^k \in a^2 S^1.$$

Hence S is a left Putcha semigroup. We can prove, in a similar way, that S is a right Putcha semigroup. □

Theorem 2.1.3 *([Nag84]) Every weakly exponential semigroup is decomposable into a semilattice of weakly exponential archimedean semigroups.*

Proof. Let S be a weakly exponential semigroup. Then S is a left and right Putcha semigroup by Theorem 2.1.2. Then, by Corollary 1.1.17, S is a semilattice Y of archimedean semigroups S_α $(\alpha \in Y)$. It is clear that the semigroup S_α is weakly exponential for every $\alpha \in Y$. □

Theorem 2.1.4 *([Nag84], [Nag85]) A semigroup is simple and weakly exponential if and only if it is a rectangular abelian group.*

Proof. Let S be a simple weakly exponential semigroup. By Theorem 2.1.2, S is a left and right Putcha semigroup and so, by Theorem 1.1.18, it is completely simple. Then, by Theorem 1.1.5, S is isomorphic with a Rees matrix semigroup $\mathcal{M}(G; I, J; P)$ over a group G with a sandwich matrix P. Assume that P is normalized by $p_{j_0,i} = p_{i,j_0} = e$ for all $i \in I,\ j \in J$ and some $i_0 \in I,\ j_0 \in J$, where e is the identity element of G. Then, for an arbitrary integer $m \geq 2$ and every $g \in G,\ i \in I,\ j \in J$, there is a positive integer k such that

$$(i, g(p_{j,i}g)^{m+k-1}, j) = (i, g, j)^{m+k} = ((i, g, j_0)(i_0, e, j))^{m+k}$$

$$= (i, g, j_0)^m (i_0, e, j)^m (i, g, j)^k = (i, g^m, j_0)(i_0, e, j)(i, g, j)^k$$

$$= (i, g^m, j)(i, g, j)^k = (i, g^m, j)(i, g(p_{j,i}g)^{k-1}, j)$$

$$= (i, g^m p_{j,i} g(p_{j,i}g)^{k-1}, j)$$

and so

$$g(p_{j,i}g)^{m+k-1} = g^m p_{j,i} g(p_{j,i}g)^{k-1},$$

that is,

$$(gp_{j,i})^m = g^m p_{j,i}.$$

Then, letting $g = e$, it follows that

$$p_{j,i}^{m-1} = e.$$

Moreover, for a positive integer t and every $g, h \in G$, we get

$$(i_0, (gh)^{m+t}, j_0) = (i_0, gh, j_0)^{m+t} = ((i_0, g, j_0)(i_0, h, j_0))^{m+t}$$

$$= (i_0, g, j_0)^m (i_0, h, j_0)^m ((i_0, g, j_0)(i_0, h, j_0))^t$$

$$(i_0, g^m h^m, j_0)(i_0, (gh)^t, j_0) = (i_0, g^m h^m (gh)^t, j_0)$$

and so

$$(gh)^{m+t} = g^m h^m (gh)^t$$

from which it follows that

$$(gh)^m = g^m h^m.$$

If we apply our above results for $m = 2$, then we get

$$(gh)^2 = g^2h^2, \quad p_{j,i} = e$$

for every $g, h \in G$ and $i \in I$, $j \in J$. Hence G is a commutative group and so $\mathcal{M}(G; I, J; P)$ is isomorphic to a rectangular abelian group.

As the converse statement is obvious, the theorem is proved. □

Theorem 2.1.5 *([Nag84]) A retract extension of a weakly exponential semigroup by a weakly exponential semigroup with a zero is also weakly exponential.*

Proof. Let S be a semigroup which is a retract extension of a weakly exponential semigroup I by a weakly exponential semigroup Q with a zero. Then I is an ideal of S and the Rees factor semigroup S/I is isomorphic to Q. Let p denote a retract homomorphism of S onto I. Let x and y be arbitrary elements of S. Let n be an arbitrary fixed positive integer (with $n \geq 2$). Then there is a positive integer t such that

$$(p(x)p(y))^{n+t} = (p(x))^n(p(y))^n(p(x)p(y))^t = (p(x)p(y))^t(p(x))^n(p(y))^n.$$

If x or y is in I, then

$$(xy)^{n+t} = p((xy)^{n+t}) = (p(x)p(y))^{n+t} = (p(x))^n(p(y))^n(p(x)p(y))^t =$$

$$= p(x^n y^n (xy)^t) = x^n y^n (xy)^t.$$

Similarly,

$$(xy)^{n+t} = (xy)^t x^n y^n.$$

Consider the case when $x, y \notin I$. Then x and y can be considered as the non-zero elements of Q. As Q is a weakly exponential semigroup by the assumption, there is a positive integer k such that (in Q),

$$(xy)^{n+k} = x^n y^n (xy)^k = (xy)^k x^n y^n.$$

Let

$$T = \{t \in N^+ : \ (p(x)p(y))^{n+t} = (p(x))^n(p(y))^n(p(x)p(y))^t =$$

$$= (p(x)p(y))^t(p(x))^n(p(y))^n\}$$

and

$$K = \{k \in N^+ : \ (xy)^{n+k} = x^n y^n (xy)^k = (xy)^k x^n y^n = (xy)^k x^n y^n \text{ in } Q\}.$$

It is clear that there are positive integers t_0 and k_0 such that $T = [t_0, \infty)$ and $K = [k_0, \infty)$.

If there is a positive integer k in K such that $(xy)^{n+k} \neq 0$ in Q, that is, $(xy)^{n+k} \notin I$ in S, then

$$(xy)^{n+k} = x^n y^n (xy)^k = (xy)^k x^n y^n$$

holds in S. Consider the case when $(xy)^{n+k} = 0$ in Q for every $k \in K$ (that is, $(xy)^{n+k} \in I$ for every $k \in K$). Let t be a positive integer which belongs to $K \cap T$. As $t \in K$, we have $(xy)^{n+t} \in I$ and $(xy)^{n+k} = x^n y^n (xy)^k$ in Q. Thus $x^n y^n (xy)^t \in I$ in S. From this and $t \in T$, we get in S:

$$(xy)^{n+t} = p((xy)^{n+t}) = (p(x)p(y))^{n+t} = (p(x))^n (p(y))^n (p(x)p(y))^t =$$

$$p(x^n y^n (xy)^t) = x^n y^n (xy)^t.$$

Similarly,

$$(xy)^{n+t} = (xy)^t x^n y^n.$$

Thus, in all cases, there is a positive integer m such that

$$(xy)^{n+m} = x^n y^n (xy)^m = (xy)^m x^n y^n$$

is satisfied in S. Consequently S is a weakly exponential semigroup. □

Theorem 2.1.6 *([Nag84]) A semigroup is a weakly exponential archimedean semigroup containing at least one idempotent element if and only if it is a retract extension of a rectangular abelian group by a nil semigroup.*

Proof. Let S be a weakly exponential archimedean semigroup containing at least one idempotent element. By Theorem 2.1.2, S is a left and right Putcha semigroup. Then, by Theorem 1.1.19 and Theorem 2.1.4, S is a retract extension of a rectangular abelian group by a nil semigroup.

As the rectangular abelian groups and the nil semigroups are weakly exponential, the converse follows from Theorem 1.1.12 and Theorem 2.1.5. □

Lemma 2.1.7 *([Nag84]) If S is a weakly exponential semigroup then, for every $a \in S$,*

$$S_a = \{x \in S : \ a^i x a^j = a^h \text{ for some positive integers } i, j, k\}$$

is the least reflexive unitary subsemigroup of S containing a.

Proof. Let S be a weakly exponential semigroup and $a \in S$ be arbitrary. To show that S_a is a subsemigroup of S, let $x, y \in S_a$ be arbitrary. Then there are positive integers i, j, k, h, m, n such that

$$a^i x a^j = a^k$$

and

$$a^m y a^n = a^h.$$

As S is a weakly exponential semigroup, (for the integer 2) there is a positive integer t such that

$$(xa^{j+m}ya^{n+i})^{2+t} = (xa^{j+m}y)^2 a^{2(n+i)}(xa^{j+m}ya^{n+i})^t.$$

As S is weakly exponential, (for the integer $2+t$) there is a positive integer s such that

$$(a^i xa^{j+m}ya^{n+i})^{2+t+s} = a^{i(2+t)}(xa^{j+m}ya^{n+i})^{2+t}(a^i xa^{j+m}ya^{n+i})^s.$$

Let $p := k+h$. Then

$$a^{(p+i)(2+t+s)} = (a^i xa^{j+m}ya^{n+i})^{2+t+s}$$

$$= a^{(2+t)i}(xa^{j+m}ya^{n+i})^{2+t}(a^{p+i})^s$$

$$(*) \qquad = a^{(2+t)i}(xa^{j+m}y)^2 a^{2(n+i)}(xa^{j+m}ya^{n+i})^t a^{s(p+i)}$$

$$= a^{(1+t)i}a^i xa^j(a^m yxa^j)(a^m ya^n)a^{n+i}((a^i xa^j)(a^m ya^n))^t a^i a^{s(p+i)}$$

$$= a^{(1+t)i+k+m}yxa^{j+h+n+p(t+s)+i(s+2)}.$$

Hence

$$yx \in S_a,$$

that is, S_a is a subsemigroup of S.

We show that S_a is left unitary. Assume $x, xy \in S_a$ for some $x, y \in S$. Then there are positive integers i, j, k, m, n, h such that

$$a^i xa^j = a^k$$

and

$$a^m xya^n = a^h.$$

Let r denote a positive integer which satisfies $r \geq max\{i-m, j-h\}$. As S is weakly exponential, (for the integer 2) there is a positive integer t such that

$$(a^{r+m}xya^n)^{2+t} = (a^{r+m}x)^2(ya^n)^2(a^{r+m}xya^n)^t.$$

From this we get

$$a^{(2+t)(r+h)} = (a^{r+h})^{2+t} = (a^{r+m}xya^n)^{2+t}$$

$$= (a^{r+m}x)^2(ya^n)^2(a^{r+m}xya^n)^t$$

$$= a^{r+m}xa^{r+m}xya^n ya^n a^{t(r+h)}$$

$$= a^{r+m}xa^{r+h}ya^{t(r+h)+n}$$

$$= a^{m+r-i}a^i xa^j a^{r+h-j}ya^{t(r+k)+n}$$

$$= a^{2r+m+h+k-i-j}ya^{t(r+h)+n}.$$

Hence $y \in S_a$. Consequently, S_a is a left unitary subsemigroup of S. We can prove, in a similar way, that S_a is right unitary in S.

We show that S_a is reflexive in S. Assume $xy \in S_a$ for some $x, y \in S$. As S is weakly exponential, there is a positive integer k such that

$$(xy)^{3+k} = x(yx)^{2+k}y = xy^2x^2(yx)^k y = (xy)(yx)(xy)^{k+1} \in S_a.$$

As S_a is unitary in S, we have

$$yx \in S_a.$$

Hence S_a is reflexive in S. It is clear that $a \in S_a$. We show that S_a is the least reflexive unitary subsemigroup of S which contains a. Assume, in an indirect way, that S has a reflexive unitary subsemigroup V such that $a \in V$ and $V \subset S_a$. Then there is an element $x \in S_a - V$ such that

$$a^i x a^j = a^k \in V$$

for some positive integers i, j, k. As V is unitary in S, we get $x \in V$ which contradict the choosing of x. Thus the lemma is proved. $\square$

Theorem 2.1.8 *([Nag84]) Every weakly exponential archimedean semigroup without idempotent element has a non-trivial group homomorphic image.*

Proof. Let S be a weakly exponential archimedean semigroup without idempotent element. Assume that $S_a \neq S$ for some $a \in S$. By Lemma 2.1.7, S_a is a reflexive unitary subsemigroup of S. Let $x \in S$ be an arbitrary element. As S is archimedean, there are element $t, s \in S$ such that $txs = a^n$ for some positive integer n. As S_a is a reflexive subsemigroup of S containing a, we get $xst \in S_a$. Consequently the right residue of S_a is empty. Then, by Theorem 1.1.3, the principal right congruence $\mathcal{R}_{S_a}$ is a group congruence on S. Hence the factor semigroup $S/\mathcal{R}_{S_a}$ is a non-trivial group homomorphic image of S.

Consider the case when $S_a = S$ for every $a \in S$. Then, for every $a \in S$, we have $a \in S_{a^2}$ and so there are positive integers i, j, k such that $a^{2i}aa^{2j} = a^{2k}$, that is, $a^{2(i+j)+1} = a^{2k}$. One of the exponents is even, the other is odd. From this it follows that the order of a is finite and so S contains an idempotent element. This contradicts the assumption. $\square$

2.2 Weakly exponential Δ-semigroups

Definition 2.2.1 *Let S be a Δ-semigroup which is a semilattice of a semigroup P and a non-trivial nil semigroup N such that $NP \subseteq N$. Then S is called*

(1) a T1 semigroup if P has only one element,

(2) a T2L semigroup if P is a two-element left zero semigroup,

(3) a T2R semigroup if P is a two-element right zero semigroup.

It is easy to check that the T1 semigroups, the T2R semigroups and the T2L semigroups are weakly exponential. In the next we formulate our main theorem on weakly exponential Δ-semigroups.

Theorem 2.2.2 *([Nag90]) A semigroup S is a weakly exponential Δ-semigroup if and only if one of the following satisfied.*

(i) $S \cong G$ *or* G^0*, where* G *is a non-trivial subgroup of a quasicyclic* p*-group (*p *is a prime).*

(ii) $S \cong F$*, where* F *is a two-element semilattice.*

(iii) $S \cong R$ *or* R^0 *or* R^1*, where* R *is a two-element right zero semigroup.*

(iv) $S \cong L$ *or* L^0 *or* L^1*, where* L *is a two-element left zero semigroup.*

(v) S *is a nil semigroup whose principal ideals form a chain with respect to inclusion.*

(vi) S *is a T1 or a T2R or a T2L semigroup (see Definition 2.2.1).*

Proof. Let S be a weakly exponential Δ-semigroup. Then, by Theorem 2.1.3, it is a semilattice of archimedean weakly exponential semigroups. By Remark 1.3.6, S is either archimedean or a disjoint union $S = S_0 \cup S_1$ of an ideal S_0 and a subsemigroup S_1 of S which are archimedean and weakly exponential.

First, assume that S is archimedean. If S has a zero element, then it is a nil semigroup. By Theorem 1.3.9, the principal ideals of S form a chain with respect to inclusion.

In the next, we consider the case when S has no zero element. Then $|S| \geq 2$. If S is simple, then it is a rectangular abelian group by Theorem 2.1.4, that is, S is a direct product of a left zero semigroup L, a right zero semigroup R and an abelian group G. Then, by Corollary 1.3.17, we have either $S = L$ or $S = R$ or $S = G$. In the first case S is a two-element left zero semigroup by Theorem 1.3.18. In the second case (using also Theorem 1.3.18) S is a two-element right zero semigroup. In the third case S is a non-trivial subgroup of a quasicyclic p-group (p is a prime) by Theorem 1.3.4.

Consider the case when S is not simple (and S has no zero element). Then, by Theorem 2.1.8 and Theorem 1.3.7, S has an idempotent element. By Theorem 2.1.6, S is a retract extension of a rectangular abelian group K ($|K| > 1$) by a nil semigroup N. Let δ denote the congruence on S determined by the retract homomorphism. Then

$$\delta \cap \rho_K = id_S,$$

where ρ_K denotes the Rees congruence of S defined by the ideal K of S. As S is a Δ-semigroup and $|K| > 1$, we have

$$\delta = id_S.$$

Then $S = K$ which contradicts the assumption that S is not simple.

Next, consider the case when S is a disjoint union $S = S_0 \cup S_1$ of an ideal S_0 and a subsemigroup S_1 of S, where S_0 and S_1 are archimedean. By Theorem 1.3.3 the Rees factor semigroup $S/S_0 \cong S_1^0$ is a Δ-semigroup. By Remark 1.3.2, S_1 is an archimedean weakly exponential Δ-semigroup. If S_1 is a nil semigroup, then$|S_1| = 1$ by Theorem 1.3.12. Thus S_1 is either a two-element left zero semigroup L or a two-element right zero semigroup R or a subgroup G of a quasicyclic p-group (p is a prime).

If $|S_0| = 1$, then either $S = L^0$ or $S = R^0$ or $S = G^0$ (if $|G| = 1$, then S is a two-element semilattice).

Next, we can suppose that $|S_0| > 1$. Recall that S_0 is a weakly exponential archimedean semigroup. By Theorem 2.1.8 and Theorem 1.3.7, S_0 has an idempotent element. By Theorem 2.1.6, S_0 is a retract extension of a rectangular abelian group $K = L \times R \times G$ (L is a left zero semigroup, R is a right zero semigroup, G is an abelian group) by a nil semigroup. By Theorem 1.3.7, K has no non-trivial group homomorphic images. Hence $K = L \times R$. As $K^2 = K$, Theorem 1.1.4 implies that K is an ideal of S. Consider the case when $|K| > 1$. By Corollary 1.3.17, $K = L$ or $K = R$. Assume that $K = L$. It is easy to see that

$$\alpha = \{(a, b) \in S \times S : \ ax = bx \text{ for all } x \in L\}$$

is a congruence on S such that

$$\alpha|L = id_L.$$

As L is a dense ideal, it follows that

$$\alpha = id_S.$$

Let $x \in L$ and $c \in S$ be arbitrary elements. Then there is a positive integer k such that

$$cx = (cx)^{2+k} = c^2x^2(cx)^k = c^2x$$

which means that

$$(c, c^2) \in \alpha.$$

Then

$$c = c^2.$$

Consequently, S is a band and $S_0 = L$. By Theorem 1.3.18, $S = S_0^1$ and S_0 is a two-element left zero semigroup. We get, in a similar way, that $S_0 = K$ and S is a band in that case when K is a right zero semigroup and so, by Theorem 1.3.18, $S = S_0^1$ and S_0 is a two-element right zero semigroup.

Next, consider the case when $|K| = 1$. Then S_0 is a (non-trivial) nil semigroup.

If $|S_1| = 1$, then S is a T1 semigroup. If S_1 is a two-element right zero semigroup, then S is a T2R semigroup. If S_1 is a two-element left zero semigroup, then S is a T2L semigroup.

If S_1 was a non-trivial subgroup G of a quasicyclic p-group (p is a prime), then S_0 would be trivial by Theorem 1.3.15, which contradicts the assumption that $|S_0| > 1$. Thus the first part of the theorem is proved.

As the semigroups listed in the theorem are weakly exponential Δ-semigroups, the proof is complete. $\square$

2.3 Semigroups T1 and T2R (T2L)

Recall that, if a is an arbitrary element of a semigroup S, then $J(a)$ denotes the ideal of S generated by a. It is known that $J(a) = S^1aS^1$. The set of all elements s of S with $J(s) = J(a)$ is denoted by J_a. The set $J(a) - J_a$ is denoted by I_a. It is known that I_a is either empty or an ideal of S.

Lemma 2.3.1 *If S is a T1 semigroup, then $J_a = \{a\}$ for every $a \in S$.*

Proof. Let S be a T1 semigroup. It is clear that $J_e = \{e\}$. Let $a \in S_0$ be an arbitrary element. Assume

$$J(a) = J(b)$$

for some $b \in S$ with $b \neq a$. Then $b \in S_0$, and

$$ras = b, \quad pbq = a$$

for some $r, s, p, q \in S^1$. Thus

$$prasq = a \quad \text{and} \quad rpbqs = b.$$

As $a \neq b$ and S_0 is a nil semigroup, we get $r, s, p, q \notin S_0$. Then r, s, p, q are in the semilattice $S^1 - S_0$. Thus

$$ras = b = rbs$$

and so

$$b = rpbqs = prbsq = pbq = a$$

which is a contradiction. Thus $J_a = \{a\}$. $\square$

Theorem 2.3.2 *([Nag90]) S is a T1 semigroup if and only if it is a semilattice of a non-trivial nil Δ-semigroup S_0 and a one-element semigroup $S_1 = \{e\}$ such that $S_0S_1 \subseteq S_0$ and $S^1eS^1 = S$.*

Proof. Assume that S is a T1 semigroup. Then it is a Δ-semigroup and a semilattice of a non-trivial nil Δ-semigroup S_0 and a one-element semigroup $S_1 = \{e\}$ such that $S_0S_1 \subseteq S_0$. Let $a \in S_0$ be an arbitrary element. Applying the proof of Lemma 3.3 of [Tro76], the ideal $J(a)$ of S generated by the element a equals the ideal of S_0 generated by a. As the principal ideals of S form a chain with respect to inclusion (by Theorem 1.3.8), the principal ideals of S_0 form a chain with respect to inclusion. By Theorem 1.3.9, S_0 is a Δ-semigroup. As S^1eS^1 and S_0 are ideals of S such that $e \notin S_0$, we get $S^1eS^1 = S$.

Conversely, let S be a semigroup which is a semilattice of a non-trivial nil Δ-semigroup S_0 and a one-element semigroup $S_1 = \{e\}$ such that $S_0S_1 \subseteq S_0$ and $S^1eS^1 = S$. Let α be a non-identity congruence on S. Assume $(e, a) \in \alpha$ for some $a \in S_0$. Then $(e, ae) \in \alpha$ which implies $(e, a^m e) \in \alpha$ for every positive integer m. As S_0 is a nil semigroup, we get $(e, 0) \in \alpha$, where 0 denotes the zero of S_0. Then, for every $x, y \in S^1$, we have $(xey, 0) \in \alpha$. As $S^1eS^1 = S$, we get $(s, 0) \in S$ for every $s \in S$. Consequently, α is the universal relation on S. It means that $\{e\}$ is an α-class for every non-universal congruence on S. Thus the congruences of S form a chain with respect to inclusion. □

Theorem 2.3.3 *([Nag90]) Let S be a T2R semigroup. Then, for every element $a \in S$, we have*

$$J_a = \begin{cases} aS_1, & \text{if } a \in SS_1 \\ \{a\} & \text{otherwise.} \end{cases}$$

Proof. Let u and v denote the elements of S_1. Assume $a \in SS_1$. Then, for example, $a = su$ for some $s \in S$. It is clear that $au = a$. First we show that $av \in J_a$. The inclusion $J(av) \subseteq J(a)$ is obvious. As S_1 is a right zero semigroup, $a = au = avu$ and so $J(a) \subseteq J(av)$. Hence $J(a) = J(av)$ which means that $av \in J_a$. From the previous results, we have

$$aS_1 \subseteq J_a.$$

Next we show that $a \neq b \in J_a$ implies $b = bv$. As $b \in J_a$, we have $J(a) = J(b)$ and so there are elements $x, y, t, s \in S_1$ such that

$$xay = b \quad \text{and} \quad tbs = a.$$

From this it follows that

$$xtbsy = b \quad \text{and} \quad txays = a.$$

Moreover,

$$(xt)^k b(sy)^k = b \quad \text{and} \quad (tx)^k a(ys)^k = a$$

for every positive integer k. As S_0 is a nil semigroup, we get $x, y, t, s \in S_1^1$. In the opposite case $a = b = 0$ which contradicts the assumption $a \neq b$.

If $x = 1$, then $xay = b$ implies $ay = b$. As $b \neq a$, we get $y = v$ (applying also $au = a$). Hence $b = av$.

Assume $x \neq 1$. Then $x \in S_1$ and so $tx = x$. As $txays = a$, we get

$$a = xays = bs.$$

As $a \neq b$, we get $s \in S_1$. Thus $ys = s$ and so

$$a = xays = xas.$$

From this we get

$$a = au = xasu = xau = xa.$$

Thus
$$b = xay = ay.$$
If $y = 1$ or $y = u$, then $a = b$; this is a contradiction. Thus $y = v$ and so $b = av$.

In both cases we get $b = av$. Hence
$$J_a \subseteq aS_1.$$
Consequently $J_a = aS_1$.

In the second part of the proof assume that a is an element of S with $a \notin SS_1$. We show that, for every $b \in S$, the assumption $J(a) = J(b)$ implies $a = b$. Assume
$$J(a) = J(b)$$
for an element $b \in S$. Then there are elements $x, y, t, s \in S^1$ such that
$$xay = b \quad \text{and} \quad tbs = a.$$
Thus
$$xtbsy = b \quad \text{and} \quad txays = a.$$
If one of the elements x, y, t, s is in S_0, then $a = 0 = b$. Thus we can consider the case $x, y, t, s \in S_1^1$. If $ys \in S_1$, then
$$a = txays \in SS_1$$
which is impossible by the assumption $a \notin SS_1$. Hence $y = s = 1$. If $t = 1$, then
$$a = tbs = b.$$
If $x = 1$, then
$$b = xay = a.$$
Consider the case when $x, t \in S_1$. Then $xt = t$ and $tx = x$. As $y = s = 1$ implies $xtb = b$ and $txa = a$, we get $b = xtb = tb$. On the other hand, $tbs = a$ implies $tb = a$ (because $s = 1$). Thus
$$b = tb = a.$$
In all cases we get $a = b$. Consequently
$$J_a = \{a\}.$$
□

Theorem 2.3.4 *([Nag90]) A semigroup S is a T2R semigroup if and only if it satisfies all of the following five conditions.*

(i) S is semilattice of a nil semigroup S_0 and a two-element right zero semigroup S_1 with $S_0S_1 \subseteq S_0$;

(ii) *The principal ideals (equivalently, the ideals) of S form a chain with respect to inclusion;*

(iii) *For every $b \in S_0$, either $b \in bS_1$ or $bS_1 \subseteq S^1bS_0$;*

(iv) *For every $b \in S_0$, either $S_1b = \{b\}$ or $S_1b \cap (S_0bS^1 \cup S^1bS_0 \neq \emptyset$;*

(v) *For every $b \in S$, if $|J_b| = 2$ and $I_b \neq \{0\}$, then, for every $a \in I_b$ there are elements $x, y \in S^1$ such that $xJ_by \cap J_a \neq \emptyset$, but $xJ_by \not\subseteq J_a$.*

Proof. Let S be a T2R semigroup. Then (i) and (ii) are satisfied. We prove that (iii), (iv) and (v) are also satisfied.

The proof of (iii): Let the elements of S_1 denoted by u and v. Assume, in an indirect way, that there is an element $b \in S_0$ such that

$$b \notin bS_1 \quad \text{and} \quad bS_1 \not\subseteq S^1bS_0.$$

From this it follows that $b \neq 0$. If $bu \in S^1bS_0$, then $bv = buv \in S^1bS_0$. Similarly, $bv \in S^1bS_0$ implies $bu = bvu \in S^1bS_0$. Consequently,

$$bS_1 \cap S1bS_0 = \emptyset. \tag{2.1}$$

We show that bS_1^1 is a normal complex of S. Let $x, y \in S^1$ be elements with

$$xbS_1^1y \cap bS_1^1 \neq \emptyset.$$

Then there are elements $t, s \in S_1^1$ such that

$$xbty = bs. \tag{2.2}$$

and so

$$xbtyu = bsu = bu.$$

As $bu \notin S^1bS_0$, we have $y \notin S_0$ and so $y \in S_1^1$. Then

$$xbtyu = xbu$$

and so

$$xbu = bu.$$

Hence

$$x^nbu = bu$$

for every non-negative integer n. From this it follows that $x \notin S_0$, because S_0 is a nil semigroup, $0 \in S^1bS_0$ and $bu \notin S^1bS_0$. Consequently $x \in S_1^1$. Thus

$$x, y \in S_1^1.$$

We have four cases.

Case 1: $s = t = 1$. In this case (2.2) is $b = xby$. If $y \neq 1$, then

$$bS_1^1y = \{xby\} = \{b\} \subseteq bS_1^1.$$

If $y = 1$, then (2.2) is $b = xb$ and so $xbS_1^1 = bS_1^1$, from this it follows that

$$xbS_1^1 y = bS_1^1.$$

Case 2: $s = 1$, $t \neq 1$. In this case $b \in J(bt)$, that is $J(b) \subseteq J(bt)$. As $J(bt) \subseteq J(b)$, we have

$$J(b) = J(bt).$$

As $bt \in SS_1$, Theorem 2.3.3 implies $J_{bt} = bS_1$. Then

$$b \in bS_1,$$

which contradicts the assumption $b \notin bS_1$.

Case 3: $s \neq 1$ and $t = 1$. In this case (2.2) is $xby = bs$. If $y \neq 1$, then

$$xbS_1^1 y = \{xby\} = \{bs\} \subseteq bS_1^1.$$

If $y = 1$, then

$$xbS_1^1 = bsS_1^1 = bS_1.$$

(We note that $1 \neq s \in S_1^1$ and so $sS_1^1 = S_1$, because S_1 is a right zero semigroup.) Thus

$$xbS_1^1 y = xbS_1 \subseteq xbS_1^1 = bS_1 \subseteq bS_1^1.$$

Case 4: $s \neq 1$ and $t \neq 1$. If $y \neq 1$, then

$$xbS_1^1 y = \{xby\} = \{xbty\} = \{bs\} \subseteq bS_1^1.$$

If $y = 1$, then we can suppose that $x \neq 1$. We note that if $bu = ubu$, then

$$bv = buv = ubuv = ubv$$

and so

$$vbv = vubv = ubv = bv.$$

Similarly, $bv = vbv$ implies $bu = ubu$. Thus $bu = ubu$ is satisfied if and only if $bv = vbv$ is satisfied.

Case 4a: $bu \neq ubu$ and $bv \neq vbv$. As $y = 1$, (2.2) has the form $xbt = bs$. Thus, for every $p \in S_1$,

$$pbp = pbsp = pxbtp = xbp = bp$$

which is a contradiction.

Case 4b: $bu = ubu$ and $bv = vbv$. We show that

$$S_1 b = \{b\} \quad \text{or} \quad S_1 b \cap (S^1 bS_0 \cup S_0 bS^1) \neq \emptyset.$$

We shall use that $S_1 bu = \{bu\}$ and $S_1 bv = \{bv\}$, which follows from

$$S_1 bu = S_1 ubu = \{ubu\} = \{bu\}$$

and
$$S_1bv = S_1vbv = \{vbv\} = \{bv\}.$$

Assume, in an indirect way, that
$$S_1b \neq \{b\} \quad \text{and} \quad S_1b \cap (S^1bS_0 \cup S_0bS^1) = \emptyset.$$

First we show that S_1^1b is a normal complex of S. Let $z, w \in S^1$ be arbitrary elements with
$$zS_1^1bw \cap S_1^1b \neq \emptyset.$$

Then there are elements $p, q \in S_1^1$ such that
$$zpbw = qb.$$

It is clear that
$$qb \notin (S^1bS_0 \cup S_0bS^1).$$

(If $q = 1$, then $qb = b \in (S^1bS_0 \cup S_0bS^1)$ would imply $b \in S^1bS_0$ or $b \in S_0bS^1$ from which we would get that $b = 0$; this is a contradiction. If $q \in S_1$, then $qb \in S_1b$ and so $qb \notin (S^1bS_0 \cup S_0bS^1)$ by the indirect assumption $S_1b \cap (S^1bS_0 \cup S_0bS^1)$). As $qb \notin (S^1bS_0 \cup S_0bS^1)$, we have $z, w \in S_1^1$.

If $w = 1$, then
$$zS_1^1bw = sS_1^1b \subseteq S_1^1b.$$

Consider the case when $w \neq 1$. Then $w \in S_1$. If $pz = 1$, then $p = 1$ and $z = 1$ and so
$$qb = zpbw = bw.$$

If $pz \neq 1$, then $pz \in S_1$ and so
$$qb = (pz)bw \in S_1(bw) = \{bw\}$$

(see the above: $S_1bu = \{bu\}$ and $S_1bv = \{bv\}$). In both cases we get $qb = bw$ from which we get
$$zS_1^1bw = S_1^1qb \subseteq S_1^1b.$$

Consequently S_1^1b is a normal complex of S. By Theorem 1.1.2, there is a congruence α on S such that S_1^1b is an α-class of S. The congruence α is not a Rees congruence. If α was a Rees congruence on S, then we would get that $|S_1^1b| = 1$ or S_1^1b is an ideal of S.

In the first case $S_1^1b = \{b\}$ which contradicts $S_1b \neq \{b\}$. Consider the second case. Let s be an arbitrary element of S_0. Then
$$ubs \in S_1^1b.$$

If $ubs = b$, then
$$u^kbs^k = b$$

for every positive integer. As $s \in S_0$ and S_0 is a nil semigroup, we have $b = 0$ which is a contradiction.

If $ubs = ub$ or $ubs = vb$, then

$$S_1b \cap (S^1bS_0 \cup S_0bS^1) \neq \emptyset$$

which is also a contradiction. Consequently α is not a Rees congruence on S, indeed. By Theorem 1.3.11 and the above result, S_1^1b is not an ideal and

$$S_1^1b \neq \{b\} \quad \text{and} \quad S_1^1b \subseteq J_b.$$

By Theorem 2.3.3, $J_b = bS_1$ or $J_b = \{b\}$. Thus

$$S_1^1b \subseteq bS_1 \quad \text{or} \quad S_1^1b = \{b\}.$$

The second case is not true (see the above result). In the first case it follows that

$$b = bu \quad \text{or} \quad b = bv.$$

If $b = bu$, then

$$ub = ubu = bu = b \quad \text{and} \quad vb = vub = ub = b$$

and so $S_1b = \{b\}$ which is a contradiction. The equation $b = bv$ gives a contradiction in a similar way. As we get a contradiction in all cases, we have

$$S_1b = \{b\} \quad \text{or} \quad S_1b \cap (S^1bS_0 \cup S_0bS^1) \neq \emptyset$$

as it was asserted in the beginning of Case 4b. Do not forget that we want to show that bS_1^1 is a normal complex (we can use $s \neq 1$, $t \neq 1$ and $S_1b = \{b\}$ or $S_1b \cap (S^1bS_0 \cup S_0bS^1) \neq \emptyset$).

If $S_1b = \{b\}$, then

$$xbS_1^1y \subseteq xbS_1^1 = bS_1^1.$$

Consider the case when $S_1b \cap (S^1bS_0 \cup S_0bS^1) \neq \emptyset$. Without loss of generality, we can suppose

$$ub \in (S^1bS_0 \cup S_0bS^1).$$

If $ub \in S^1bS_0$, then

$$bu = ubu \in S^1bS_0$$

which is a contradiction.

If $ub \in S_0bS^1$, then $ub = rbz$ for some $r \in S_0$ and $z \in S^1$. As $bu = ubu = rb(zu)$ and $bu \notin S^1bS_0$ (because $bS_1 \cap S^1bS_0 = \emptyset$; see the beginning of the proof), we get $z \in S_1^1$. Thus $bu = rbu$ and so

$$bu = r^kbu$$

for every positive integer k. As $r \in S_0$ and S_0 is a nil semigroup, we get

$$bu = 0 \in S^1bS_0$$

which is a contradiction.

Summarising our results: in all cases we get either a contradiction or the inclusion $xbS_1^1y \subseteq bS_1^1$. Consequently the indirect assumptions $b \notin bS_1$ and $bS_1 \not\subseteq S^1bS_0$ imply that bS_1^1 is a normal complex of S. By Theorem 1.1.2, there is a congruence β on S such hat bS_1^1 is a β-class of S. If β was a Rees congruence, then we would have that $bS_1^1 = \{b\}$ or bS_1^1 is an ideal of S.

The equation $bS_1^1 = \{b\}$ contradicts the assumption $b \notin bS_1$ (see the beginning of the proof of (iii)).

If bS_1^1 was an ideal of S, then the zero 0 of S would be in bS_1^1 and so 0 would be in bS_1, because $b \neq 0$. Then we would have $bS_1 \cap S^1bS_0 \neq \emptyset$ which contradicts (2.1). Thus β is not a Rees congruence on S. It is clear that

$$[b]_\beta = bS_1^1.$$

By Theorem 1.3.11, there is an element $a \in S$ such that

$$[b]_\beta = I_a \quad \text{or} \quad [b]_\beta \subseteq J_a \quad \text{or} \quad [b]_\beta = \{b\}.$$

In the first case bS_1^1 is an ideal; this is a contradiction (see the previus result). In the third case $bS_1^1 = \{b\}$ which is also a contradiction (see the above result). Hence

$$bS_1^1 = [b]_\beta \subseteq J_a = J_b.$$

By Theorem 2.3.3,

$$J_b = bS_1 \quad \text{or} \quad J_b = \{b\}.$$

In the firs case

$$b \in bS_1^1 \subseteq J_b = bS_1$$

which is a contradiction.

In the second case $bS_1^1 = \{b\}$ and so $b \in S_1$. This is a contradiction. As we get a contradiction in all cases, the indirect assumption $b \notin bS_1$ and $bS_1 \not\subseteq S^1bS_0$ is not true. Thus

$$b \in bS_1 \quad \text{or} \quad bS_1 \subseteq S^1bS_0$$

as it is asserted in (iii).

The proof of (iv): Assume, in an indirect way, that there is an element $b \in S_0$ such that

$$\{b\} \neq S_1b \quad \text{and} \quad S_1b \cap (S_0bS^1 \cup S^1bS_0) = \emptyset. \tag{2.3}$$

In this case $b \neq 0$. We show that S_1^1b is a normal complex of S. Assume

$$xS_1^1by \cap S_1^1 \neq \emptyset$$

for some $x, y \in S^1$. We have to show that

$$xS_1^1by \subseteq S_1^1. \tag{2.4}$$

By the assumption $xS_1^1by \cap S_1^1 \neq \emptyset$, there are elements $t, s \in S_1^1$ such that

$$xtby = sb. \tag{2.5}$$

If $x \in S_0$ or $y \in S_0$, then

$$sb = xtby \in S_0bS^1 \cup S^1bS_0.$$

If $s \in S_1$, then

$$S_1b \cap (S_0bS^1 \cup S^1bS_0) \neq \emptyset$$

which contradicts the second assumption in (2.3).

If $s = 1$, then $b = (xt)by$ implies

$$b = (xt)^k by^k$$

for every positive integer k and so $b = 0$, because S_0 is a nil semigroup. This is also a contradiction.

Hence $x, y \in S_1^1$. From this we get

$$sby = (xtby)y = xtby^2 = xtby.$$

We have two cases.

Case 1*: $s = 1$. In this case (2.5) has the following form: $xtby = b$. We have two subcases.

Case 1*a: $t = 1$. In this case

$$xby = b.$$

If $x = 1$, then $S_1^1b = S_1^1by$ and so

$$S_1^1b = xS_1^1b = xS_1^1by;$$

the inclusion (2.4) is satisfied.

If $x \neq 1$, then $x \in S_1$ and so $S_1x = \{x\}$ from which we get

$$S_1b = S_1xby = \{xby\}.$$

Thus

$$S_1by = \{xby^2\} = \{xby\} = S_1b$$

and so

$$xS_1^1by = (\{x\} \cup S_1)by = xby \cup S_1by = \{b\} \cup S_1b = S_1^1b.$$

Thus the inclusion (2.4) is satisfied.

Case 1*b: $t \neq 1$. In this case $b = tby$ and so

$$ub = utby = tby = b$$

and

$$vb = vtby = tby = b,$$

Thus $S_1b = \{b\}$ which is a contradiction (see (2.3)).

Case 2*: $s \neq 1$. Introduce the following notation. If $p \in S_1$, then let p' denote the element of $S_1 - \{p\}$. We show that if $pbq \in S_1 b$ $(p, q \in S_1)$, then

$$p'bq \in S_1 b \tag{2.6}$$

is also satisfied. Assume $pbq \in S_1 b$ for some $p, q \in S_1$. By condition (iii) of our theorem,

$$b \in bS_1 \quad \text{or} \quad bS_1 \subseteq S^1 bS_0.$$

If $bS_1 \subseteq S^1 bS_0$, then

$$pbq \in S^1 bS_0.$$

As $pbq \in S_1 b$ (by the assumption), we get

$$pbq \in S_1 b \cap (S_0 bS^1 \cup S^1 bS_0)$$

which contradict the second assumption in (2.3).

Consider the case when $b \in bS_1$. If $b = bq$, then

$$p'bq = p'b \in S_1 b$$

and so (2.6) is satisfied. Examine the case when $b \neq bq$. Then $b = bq'$ and so

$$S_1 b = S_1 bq'. \tag{2.7}$$

We show that $S_1^1 bS_1^1$ is a normal complex of S. Let $x'', y'' \in S^1$ be elements such that

$$x'' S_1^1 bS_1^1 y'' \cap S_1^1 bS_1^1 \neq \emptyset. \tag{2.8}$$

Then there are elements $t_1, t_2, s_1, s_2 \in S_1^1$ such that

$$x'' t_1 b t_2 y'' = s_1 b s_2. \tag{2.9}$$

Then

$$x'' t_1 b t_2 y'' q' = s_1 b s_2 q' = s_1 b q' = s_1 b$$

applying the above assumption $b = bq'$. As S_0 is a nil semigroup,

$$b \notin S^1 bS_0 \cup S_0 bS^1.$$

This and the second assumption in (2.3) together imply that

$$s_1 b \notin (S^1 bS_0 \cup S_0 bS^1).$$

Thus $x'', y'' \in S_1^1$. Hence

$$x'' S_1^1 bS_1^1 y'' \subseteq S_1^1 bS_1^1$$

and so $S_1^1 bS_1^1$ is a normal complex of S. By Theorem 1.1.2, there is a congruence γ on S such that $S_1^1 bS_1^1$ is a γ-class of S. There are two cases: $S_1^1 bS_1^1 = \{b\}$ or $S_1^1 bS_1^1$ is an ideal of S.

If $S_1^1bS_1^1 = \{b\}$, then $S_1b = \{b\}$ which contradict the first assumption in (2.3). If $S_1^1bS_1^1$ is an ideal of S, then

$$0 \in S_1^1bS_1^1 \cap (S_0bS^1 \cup S_0bS^1).$$

Hence

$$S_1^1bS_1^1 \cap (S_0bS^1 \cup S_0bS^1) \neq \emptyset.$$

Using the above $b = bq'$ and $S_1b = S_1bq'$, we have

$$S_1^1bS_1^1 = S_1^1b \cup S_1^1bq.$$

If $S_1^1b \cap (S_0bS^1 \cup S^1bS_0) \neq \emptyset$, then

$$S_1b \cap (S_0bS^1 \cup S^1bS_0) \neq \emptyset,$$

because $b \neq 0$ and S_0 is a nil semigroup. But this contradict the second assumption in (2.3).

If $S_1^1bq \cap (S_0bS^1 \cup S^1bS_0) \neq \emptyset$, then there is an element $u \in S_1^1$ such that

$$ubq \in (S_0bS^1 \cup S^1bS_0).$$

As $(S_0bS^1 \cup S^1bS_0$ is an ideal of S,

$$ubqq' \in (S_0bS^1 \cup S^1bS_0).$$

On the other hand,

$$ubqq' = ubq' = ub \in S_1^1b.$$

Hence

$$S_1^1b \cap (S_0bS^1 \cup S^1bS_0) \neq \emptyset.$$

This case is the above one. As we get a contradiction in both cases, γ is not a Rees congruence. By Theorem 1.3.11 and the previous results,

$$S_1^1bS_1^1 \subseteq J_b.$$

By Theorem 2.3.3,

$$J_b = \{b\} \quad \text{or} \quad J_b = bS_1.$$

In the first case

$$S_1^1bS_1^1 = \{b\}$$

which is a contradiction (see the previous result).

If $J_b = bS_1$, then the earlier assumptions $b \neq bq$ and $b = bq'$ implies $|bS_1| = 2$. Thus $|S_1^1bS_1^1| = 2$, because $S_1^1bS_1^1 \subseteq J_b = bS_1$ and $S_1^1bS_1^1 = \{b\}$ is impossible. As $b = bq'$ and bq are in $S_1^1bS_1^1$, we get

$$S_1^1bS_1^1 = \{b, bq\}.$$

Hence

$$S_1bq \subseteq S_1^1bS_1^1 = \{b, bq\},$$

that is,

$$p'bq = b \quad \text{or} \quad p'bq = bq.$$

In the case when $p'bp = b$, we get

$$p'bq = p'p'bq = p'b \in S_1b.$$

In the case when $p'bq = bg$, we get

$$p'bq = (pp')bq = p(p'bq) = pbq \in S_1b.$$

In all cases we get either a contradiction or the inclusion $p'bq \in S_1b$. Thus we have proved that $pbq \in S_1b$ implies $p'bq \in S_1b$ for every $p, q \in S_1$. In the next, we consider two subcases.

Case 2*a: $t = 1$ and ($s \in S_1$). In this case (2.5) has the following form: $xby = sb$.

If $y = 1$, then $xb = sb$ and so

$$xS_1^1by = xS_1^1b = \{xb\} \cup xS_1b = \{sb\} \cup S_1b \subseteq S_1^1b.$$

Thus (2.4) is satisfied.

If $y \neq 1$, then $xbx = sb$. Assume $x = 1$. Then $S_1by = S_1sb = \{sb\}$ and so

$$xS_1^1by = \{by\} \cup S_1by = \{sb\} \sup S_1sb = \{sb\} \subseteq S_1^1b.$$

Thus (2.4) is satisfied. In the next consider the case $x \neq 1$. Do not forget that $y \neq 1$ and $s \in S_1$. In this case

$$xS_1^1by = \{xby\} \cup xS_1by = \{xby\} \cup S_1by = \{sb\} \cup \{x, x'\}by = \{sb\} \cup \{x'by\}.$$

As $x, y \in S_1$ and $xby = sb \in S_1b$, we get (see above) $x'by \in S_1b$. Consequently the above $xS_1^1by = \{sb\} \cup \{x'by\}$ equation implies that $xS_1^1by \in S_1^1b$. Thus (2.4) is satisfied.

Case 2*b: $t \neq 1$ (and $s \in S_1$) In this case $tby = sb$ and

$$xS_1^1by = \{\{xby\} \cup xS_1by = \{xby\} \cup S_1by =$$

$$= \{xby\} \cup \{t, t'\}by = \{xby, tby, t'by\} = \{xby, sb, t'by\}.$$

If $y = 1$, then

$$\{xby, sb, t'by\} \subseteq S_1^1b.$$

In this case

$$xS_1^1by \subseteq S_1^1b$$

and so (2.4) is satisfied.

If $y \neq 1$, that is, $y \in S_1$, then $tby = sb \in S_1b$ implies (see above) that $t'by \in S_1b$. As $x = t$ or $x = t'$, the above inclusions imply $xby \in S_1b$. Hence

$$\{xby, sb, t'by\} \subseteq S_1b \subseteq S_1^1b.$$

Consequently

$$xS_1^1by \subseteq S_1^1b.$$

Thus (2.4) is satisfied. From the previous results it follows that S_1^1b is a normal complex of S, indeed. By Theorem 1.1.2, there is a congruence δ on S such that S_1^1b is a δ-class of S.

If $S_1^1b = \{b\}$, then $\{b\} = S_1b$ which contradicts the first assumption in (2.3). If S_1^1b is an ideal of S, then

$$S_1^1b \cap (S_0bS^1 \cup S^1bS_0) \neq \emptyset.$$

As $S_1b \cap (S_0bS^1 \cup S^1bS_0) = \emptyset$ by the second assumption in (2.3), we have

$$b \in S_0bS^1 \cup S^1bS_0$$

which implies $b = 0$ because S_0 is a nil semigroup. This contradicts $b \neq 0$. Thus δ is not a Rees congruence on S. The above results and Theorem 2.3.3 imply that $S_1^1b \subseteq J_b$. From this it follows that b and ub (recall that $S_1 = \{u, v\}$) generate the same two-sided ideal of S. From this it follows that there are elements $x, y \in S^1$ such that

$$xuby = b.$$

Then

$$(xu)^k by^k = b$$

for every positive integer k. As S_0 is a nil semigroup, $x, y \notin S_0$, that is $x, y \in S_1^1$. Thus

$$xuby = uby$$

and so

$$uby = b.$$

Multiply this equation by u (on the left), we get

$$ub = uuby = uby = b.$$

If we apply the above idea for v (instead of u), we get $vb = b$. Thus $S_1b = \{b\}$. This contradict the first assumption in (2.3). As we get a contradiction, the indirect assumption is not true. Hence (iv) must be satisfied.

The proof of (v): Let b be an arbitrary element of S such that $|J_b| = 2$ and $I_b \neq \{0\}$. Let $a \in I_b$ be an arbitrary element. Since $I_b \subseteq J(b)$, then there are elements $x, y \in S^1$ such that $a = xby$. Thus $xJ_by \cap J_a \neq \emptyset$. Assume that, for every $x, y \in S^1$, the assumption $xJ_by \cap J_a \neq \emptyset$ implies $xJ_by \subseteq J_a$. We show that this is impossible which proves (v). Consider the congruence

$$P_{J_a}^* = \{(c, d) \in S \times S : \ (\forall s, t \in S^1) \ sct \in J_a \quad \text{iff} \quad sdt \in J_a\}.$$

It is easy to see that J_a is a union of $P_{J_a}^*$-classes of S, and $J_a \subset I_b$. As S is a Δ-semigroup, $P_{J_a}^* \subseteq \varrho_{I_b}$, where ϱ_{I_b} denotes the Rees congruence on S defined by the ideal I_b of S.

Let b and b' $(b \neq b')$ denote the elements of J_b. If $sbt \in J_a$ for some $s,t \in S^1$, then $sJ_bt \cap J_a \neq \emptyset$ and so, by our previous assumption, $sJ_bt \subseteq J_a$ and so $sb't \in J_a$. Similarly, if $sb't \in J_a$ for some $s,t \in S^1$, then $sbt \in J_a$. Hence $(b,b') \in P^*_{J_a}$. As $P^*_{J_a} \subseteq \varrho_{I_b}$, we get $(b,b') \in \varrho_{I_b}$ which implies $b = b'$, because $b, b' \notin I_b$. This contradict the assumption $b \neq b'$.

Consequently, there are elements $x, y \in S^1$ such that $xJ_bx \cap J_a \neq \emptyset$ and $xJ_by \not\subseteq J_a$, that is, (v) is satisfied.

Conversely, assume that S is a semigroup which satisfies all of the conditions (i) - (v). First we show that S is weakly exponential. Let a and b be arbitrary elements of S. Let n be a positive integer. As S_1 is a right zero semigroup, it is sufficient to deal with the case when one of a and b is in S_0. In this case $ab \in S_0$. As S_0 is a nil semigroup, there is a positive integer k such that $(ab)^k = 0$ and so $(ab)^{n+k} = 0 = a^nb^n(ab)^k = (ab)^ka^nb^n$. Thus S is weakly exponential.

Next we show that S is a Δ-semigroup. Let λ be an arbitrary non-universal congruence on S. Let I_λ denote the λ-class of S containing the zero 0 of S. It is obvious that I_λ is an ideal of S. Let $a, b \in S - I_\lambda$ be arbitrary elements such that $(a,b) \in \lambda$. We show that $J(a) = J(b)$. As the ideals of S form a chain with respect to inclusion, we have

$$J(a) \subseteq J(b) \quad \text{or} \quad J(b) \subseteq J(a).$$

By the symmetry, we can consider, for example, the case $J(a) \subseteq J(b)$.

If $a \in S_1$, then we have $J(a) = S$ by (ii), and so $J(b) = S$. Assume $a \in S_0$. As $J(a) \subseteq J(b)$, there are elements $p, q \in S^1$ such that $a = pbq$. Then $(b, pbq) \in \lambda$ and so

$$(b, p^kbq^k) \in \lambda$$

for every positive integer k. As S_0 is a nil semigroup and $(b, 0) \notin \lambda$, we get $p, q \in S_1^1$. From this it follows that $b \in S_0$ (because $a \in S_0$ and $a = pbq$). We have three cases.

Case 1: $p = 1$. In this case $a = bq$. We can suppose that $q \neq 1$. From condition (iii) it follows that

$$bq \in S^1bS_0 \quad \text{or} \quad b \in bS_1.$$

If bq was in S^1bS_0, then would be elements $x \in S^1$ and $y \in S_0$ such that $bq = xby$. This equation implies $(b, xby) \in \lambda$ from which it follows that

$$(b, x^kby^k) \in \lambda$$

for every positive integer k. As S_0 is a nil semigroup, we would get $(b, 0) \in \lambda$ which contradict $b \in S - I_\lambda$. Hence $b \in bS_1$. (Recall that $q \in S_1$.)

If $b = bq$, then $a = b$ (and so $J(a) = J(b)$). If $b = bq'$ (q' is the other element of S_1), then

$$aq' = bqq' = bq' = b.$$

This equation together with $a = bq$ imply that $J(a) = J(b)$.

Case 2: $q=1$. In this case $a=pb$. We can suppose that $p\neq 1$. By condition (iv),

$$S_1b=\{b\}\quad \text{or}\quad S_1b\cap(S_0bS^1\cup S^1bS_0)\neq\emptyset.$$

If $S_1b=\{b\}$, then $pb=b$ and so $a=b$. Thus we can consider the case $S_1b\cap(S_0bS^1\cup S^1bS_0)\neq\emptyset$. We show that this case is impossible. As $S_1b=\{pb,p'b\}$, we have that one of the elements pb and $p'b$ is in the ideal $S^1bS_0\cup S_0bS^1$.

Assume $pb\in S^1bS_0\cup S_0bS^1$. Then there are elements $x,y\in S^1$ such that $pb=xby$ and one of x and y is in S_0. As $pb=a$ and $(a,b)\in\lambda$, we get $(xby,b)\in\lambda$. From this it follows that

$$(x^kby^k,b)\in\lambda$$

for every positive integer k. As S_0 is a nil semigroup and one of x and y in in S_0, we get $(0,b)\in\lambda$ which contradicts $b\notin I_\lambda$.

Consider the case $p'b\in(S^1bS_0\cup S_0bS^1)$. As $a=pb$, we have

$$p'a=p'(pb)=(p'p)b=pb,$$

because S_1 is a right zero semigroup. As $(a,b)\in\lambda$, we have $(p'a,p'b)\in\lambda$. The equation $p'a=pb$ implies $(pb,p'b)\in\lambda$. As $p'b\in(S^1bS_0\cup S_0bS^1)$, there are elements $x,y\in S^1$ such that x or y is in S_0 and $p'b=xby$. As $a=pb$, we have $(b,pb)\in\lambda$. This and the above $(pb,p'b)\in\lambda$ together imply that $(b,p'b)\in\lambda$ and so $(b,xby)\in\lambda$. From this it follows (see the above) that $(b,0)\in\lambda$ which is a contradiction.

Case 3: $p\neq 1$ and $q\neq 1$. In this case $p,q\in S_1$ and $a=(pb)q$. Recall that $b\in S_0$. By condition (iv),

$$\{b\}=S_1b\quad \text{or}\quad S_1b\cap(S^1bS_0\cup S_0bS^1)\neq\emptyset.$$

If $\{b\}=S_1b$, then $pb=b$ and so $a=bq$. This is the Case 1. Assume $S_1b\cap(S^1bS_0\cup S_0bS^1)\neq\emptyset$. Then pb or $p'b$ in in the ideal $S^1bS_0\cup S_0bS^1$ and so pbq or $p'bq$ is in $S^1bS_0\cup S_0bS^1$. If $pbq\in(S^1bS_0\cup S_0bS^1)$, then $pbq=xby$ for some $x,y\in S^1$ such that x or y is in S_0. Thus $(b,xby)\in\lambda$ from which we get $(b,0)\in\lambda$ (see above). This contradicts $b\notin I_\lambda$. If $p'bq\in(S^1bS_0\cup S_0bS^1)$, then $(a,b)\in\lambda$, $p'ab=p'(pbq)q=pbq=a$ and $(p'bq,p'aq)\in\lambda$ together imply that $(b,p'bq)\in\lambda$. As $p'bq\in(S^1bS_0\cup S_0bS^1)$, there are elements $x,y\in S^1$ such that $p'bq=xby$ and x or y is in S_0. Then $(b,xby)\in\lambda$. From this we get (as above) that $(b,0)\in\lambda$ which is a contradiction.

Summarizing our results, we get $J(a)=J(b)$. Thus $(a,b)\in\lambda$ implies $J(a)=J(b)$ for every $a,b\in S-I_\lambda$. From this it follows that if $b\in S-I_\lambda$ is an arbitrary element, then $[b]_\lambda\subseteq J_b$. It is clear that $J_b\cap=\emptyset$ for every $b\in S-I_\lambda$.

In the next we show that $|J_b|\leq 2$ for every $b\in S-I_\lambda$. Let $b\in S-I_\lambda$ be an arbitrary element. Assume that there are element a and c in $S-I_\lambda$ such that a, b, c are pairwise different, and $J(a)=J(b)=J(c)$. As $J(a)=J(b)$, there are elements $x,y,p,q\in S^1$ such that $b=xcy$ and $c=pbq$. Thus

$$b=xpbqy\quad \text{and}\quad c=pxcyq.$$

Moreover

$$b = (xp)^k b(qy)^k$$

for every positive integer k. As $b \neq 0$ and S_0 is a nil semigroup, we get $x, y, p, q \in S_1^1$. Thus x, y, p, q are idempotent elements and so $by = b$, $cq = c$. Hence

$$by = b = xpbqy = p(xp)b(qy)(qy) = p(xpbqy)qy = (pbq)y = cy$$

and

$$cq = c = pxcyq = x(px)cyqyq = x(pxcyq)yq = xcyq = bq.$$

As $b \neq c$, we have $y, q \in S_1$ and $y \neq q$. If we apply the previous result for b and a, and for c and a, the we get

$$by' = b = ay',$$

$$aq' = a = bq',$$

$$cy'' = c = ay'',$$

$$aq'' = a = cq''$$

for some $y', q', y'', q'' \in S_1$ with $y' \neq q'$ and $y'' \neq q''$. We show that $y'' = y = y'$. Assume $y \neq y'$. As $q' \neq y'$, $q', y' \in S_1$ and $|S_1| = 2$, we get $y = q'$. In this case $b = by = bq' = a$ which is a contradiction. Assume $y \neq y''$. As $q'' \neq y''$, and $q'', y'' \in S_1$, we get $y = q''$. In this case $b = cy = cq'' = a$ which is a contradiction. We get a contradiction in every case. Thus $|J_b| \leq 2$ for every $b \in S - I_\lambda$.

Let $b \in S - I_\lambda$ be an arbitrary element such that $|[b]_\lambda| = 2$. As $[b]_\lambda \subseteq J_b$, $|J_b| \leq 2$ and $|[b]_\lambda| = 2$, we have $J_b = [b]_\lambda$. We show that $J(b) = J_b \cup I_\lambda$. Assume, in an indirect way, that $J(b) \neq J_b \cup I_\lambda$. It is obvious that $J(b) \supset J_b \cup I_\lambda$. Let $a \in J(b) - (J_b \cup I_\lambda)$ be an arbitrary element. Then $0 \neq a \in I_b$ and so $I_b \neq \{0\}$. By condition (v), there are elements $x, y \in S^1$ such that $xJ_by \cap J_a \neq \emptyset$ but $xJ_by \not\subseteq J_a$. As $a \notin (J_b \cup I_\lambda)$, we get $[a]_\lambda \cap (J_b \cup I_\lambda) = \emptyset$. As $xJ_by \cap J_a \neq \emptyset$, there are elements $t \in J_b$ and $p \in J_a$ such that $xty = p$. As $J_b = [b]_\lambda$, we get $(xty, xry) \in \lambda$ for every $r \in J_b$. Thus $xJ_by \in [p]_\lambda \subseteq J_p = J_a$ which contradict the above $xJ_by \not\subseteq J_a$. Consequently $J(b) = J_b \cup I_\lambda$.

Let e and f be arbitrary elements in $S - I_\lambda$ such that $|[e]_\lambda| = 2$ and $|[f]_\lambda| = 2$. Then, by the previous results, $J(e) = J_e \cup I_\lambda$ and $J(f) = J_f \cup I_\lambda$. By condition (ii), we have $J(e) \subseteq J(f)$ or $J(f) \subseteq J(e)$ from which we get $J_e = J_f$ and so $(e, f) \in \lambda$. From this it follows that $S - I_\lambda$ can contain at most one λ-class which contains two elements.

Let λ_1 and λ_2 be arbitrary congruences on S. Assume $I_{\lambda_1} = I_{\lambda_2}$. If λ_1 and λ_2 are Rees congruences, then $\lambda_1 = \lambda_2$. Assume that one of them (for example, λ_1) is not a Rees congruence. Then, by the above results, there is an element $b \in S - I_\lambda$ such that $J(b) = J_b \cup I_{\lambda_1}$, $|J_b| = 2$ and the λ_1-classes of S are I_{λ_1}, J_b and the one-element subsets of $S - J(b)$. If λ_2 is a Rees congruence on S, then $\lambda_2 \subseteq \lambda_1$. If λ_2 is not a Rees congruence on S, then there is an element $c \in S - I_{\lambda_1}$ such that $J(c) = J_c \cup I_{\lambda_1}$, $|J_c| = 2$ and the λ_2-classes of S are

I_{λ_1}, J_c and the one-element subsets of $S - J(c)$. By condition (ii), $J(b) \subseteq J(c)$ or $J(c) \subseteq J(b)$. In both cases we get $J_b = J_c$ and so $\lambda_1 = \Lambda_2$. Consequently, $\lambda_1 \subseteq \lambda_2$ or $\lambda_2 \subseteq \lambda_1$.

Consider the case when $I_{\lambda_1} \neq I_{\lambda_2}$. Then $I_{\lambda_1} \subset I_{\lambda_2}$ or $I_{\lambda_2} \subset I_{\lambda_1}$. By the symmetry , we may assume $I_{\lambda_1} \subset I_{\lambda_2}$. Let $a, b \in S$ be arbitrary elements with $(a, b) \in \lambda_1$. We show that $(a, b) \in \lambda_2$. We may assume $a \neq b$. If $a, b \in I_{\lambda_1}$, then $a, b \in I_{\lambda_2}$ and so $(a, b) \in \lambda_2$. If $a, b \notin I_{\lambda_1}$, then $[a]_{\lambda_1} = \{a, b\}$ and $J(a) = I_{\lambda_1} \cup \{a, b\}$. By condition (ii), $J(a) \subseteq I_{\lambda_2}$ or $I_{\lambda_2} \subseteq J(a)$. If $J(a) \subseteq I_{\lambda_2}$, then $a, b \in I_{\lambda_2}$ and so $(a, b) \in \lambda_2$. The case $I_{\lambda_2} \subset J(a)$ is not possible, because $I_{\lambda_1} \subset I_{\lambda_2}$ and $J(a) = J_a \cup I_{\lambda_1}$. Consequently $\lambda_1 \subseteq \lambda_2$. Hence S is a Δ-semigroup. □

Proposition 2.3.5 *([Nag13]) If b is an element of a T2R semigroup S such that $|J_b| = 2$ and $I_b = \{0\}$, then , for every $x, y \in S^1$, either $0 \notin xJ_by$ or $xJ_by = \{0\}$. Moreover, $J_bS_0 = S_0J_b = \{0\}$ and either $S_1J_b = \{0\}$ or $S_1J_b = J_b$.*

Proof. Let b be an element of a T2R semigroup S such that $|J_b| = 2$ and $I_b = \{0\}$. Then $b \in S_0$. By Theorem 2.3.3, $J_b = bS_1 = \{bu, bv\}$. By [BC80, Lemma 2.7], J_b is a normal complex, that is, $xJ_by \cap J_b \neq \emptyset$ implies $xJ_by \subseteq J_b$ for every $x, y \in S^1$. As $xJ_by \subseteq J(b) = J_b \cup \{0\}$, we get either $0 \notin xJ_by$ or $xJ_by = \{0\}$ for every $x, y \in S^1$.

Next we show that $J_bS_0 = S_0J_b = \{0\}$. If $J_by \neq \{0\}$ for some $y \in S_0$, then $0 \notin J_by$ and so $buy \in J_b$. Thus $buyu = bu$ from which we get $bu(yu)^n = bu$ for every positive integer n. As S_0 is a nil semigroup and $yu \in S_0$, we have $bu = 0$. This is a contradiction. Hence $J_bS_0 = \{0\}$. If $xJ_b \neq \{0\}$ for some $x \in S_0$, then $0 \notin xJ_b$ and so $xbu \in J_b$. Then $xbu = bu$. From this we get $x^nbu = bu$ for every positive integer n. As $x \in S_0$ and S_0 is a nil semigroup, we get $bu = 0$. This is a contradiction. Hence $S_0J_b = \{0\}$.

Next we show that $uJ_b = \{0\}$ if and only if $vJ_b = \{0\}$. Assume $uJ_b = \{0\}$ and $vJ_b \neq \{0\}$. Then $0 \notin vJ_b$ and so $vbu \in J_b$. Then $vbu = bu$ from this we get $bu = vbu = uvbu = ubu = 0$. This is a contradiction. Thus $uJ_b = \{0\}$ implies $vJ_b = \{0\}$. Similarly, $vJ_b = \{0\}$ implies $uJ_b = \{0\}$. Hence $uJ_b = \{0\}$ iff $vJ_b = \{0\}$.

Next we show that either $S_1J_b = \{0\}$ or $S_1J_b = J_b$. First of all, we note that $S_1J_b = J_b$ is satisfied if and only if $ef = f$ is satisfied for every $e \in S_1$ and $f \in J_b$. Assume $S_1J_b \neq \{0\}$. As $uJ_b = \{0\}$ iff $vJ_b = \{0\}$, $uJ_b \neq \{0\}$ and $vJ_b \neq \{0\}$. Thus $0 \notin uJ_b$ and $0 \notin vJ_b$ from which we get that, for every $x \in S_1$, there are elements $y, z \in S_1$ such that $ubx = by$ and $vbx = bz$. Then $uw = w$ and $vw = w$ for every $w \in J_b$, that is, $S_1J_b = J_b$. □

Corollary 2.3.6 *([Nag13]) If S is a T2R semigroup and $b \in S_0$ is arbitrary with $|J_b| = 2$, then $S_0J_b \subseteq I_b$, $J_bS_0 \subseteq I_b$ and either $S_1J_b \subseteq I_b$ or $S_1J_b = J_b$.*

Proof. Let $b \in S_0$ be an arbitrary element of a T2R semigroup S such that $|J_b| = 2$. Using Theorem 1.3.3, it is easy to see that the Rees factor semigroup of S by the ideal I_b is a T2R semigroup, in which $J(b) = J_b \cup \{0\}$. Thus our assertion follows from Proposition 2.3.5. □

Proposition 2.3.7 *([Nag13]) If S is a T2R semigroup, then there is an element $b \in S_0$ such that $|J_b| = 2$.*

Proof. Assume, in an indirect way, that S is a T2R semigroup in which $|J_b| \neq 2$ for every $b \in S_0$. Then, by Theorem 2.3.3, $J_b = \{b\}$ for every $b \in S_0$.

First we show that u and v are left identity elements of S. Let $a \in S_0$ be an arbitrary element. Then $a \in I(u) = S_0 \neq \{0\}$. By (v) of Theorem 2.3.4, there are elements $x, y \in S^1$ such that $xJ_uy \cap J_a \neq \emptyset$ and $xJ_uy \not\subseteq J_a$. As $J_a = \{a\}$, we have $xuy = a$ and $xvy \neq a$ or $xvy = a$ and $xuy \neq a$.

By the symmetry, we can consider only one of the above two cases. Assume, for example, $xuy = a$, $xvy \neq a$. If $x \in S_0$, then $xu \in SS_1$ and so (by Theorem 2.3.3) $J_{xu} = xuS_1 = \{xu, xv\}$. As $xu \in S_0$, we have $|J_{xu}| = 1$ and so $xu = xv$. From this it follows that $xuy = xvy$ which is a contradiction. Thus $x \in S_1^1$ and so $xu = u$. From $uy = xuy = a$ we get $ua = a$ and so we also have $va = a$. Thus u and v are left identity elements of S.

By the previous part of the proof, if a is an arbitrary element of S_0, then there is an element $y \in S_0$ such that $uy = a$ and $vy \neq a$ or $vy = a$ and $uy \neq a$. Both cases are impossible, because $uy = a$ is satisfied if and only if $y = a$ if and only if $vy = a$, because u and v are left identity elements of S. □

Proposition 2.3.8 *([Nag13]) If there exists a T2R semigroup, then there exists a T2R semigroup S which contains an element $b \in S_0$ with $|J_b| = 2$ and $I_b = \{0\}$.*

Proof. Suppose that there exist a T2R semigroup H which is a semilattice of a non-trivial nil semigroup H_0 and a two-element right zero semigroup H_1. By Proposition 2.3.7, there is a element $b \in H_0$ such that $|J_b| = 2$. Denote S the Rees factor semigroup H/I_b defined by the ideal I_b. By Theorem 1.3.3, S is a Δ-semigroup. It is clear that S is a $T2R$-semigroup in which $S_1 = H_1$ and $S_0 = H_0/I_b$. Identifying the elements of $S - \{0\}$ and $H - I_b$, for $b \in S_0$, we have (in S) $|J_b| = 2$ and $I_b = \{0\}$. □

Proposition 2.3.9 *([Nag13]) In every T2R semigroup S there is an element $b \in S_0$ such that $ub \neq b$ and $vb \neq b$.*

Proof. Assume, in an indirect way, that there is a T2R semigroup S in which $ub = vb = b$ is satisfied for every $b \in S_0$. Let $b \in S_0$ be an arbitrary element with $|J_b| = 2$. By Proposition 2.3.7, such element exists. By (v) of Theorem 2.3.4, there are elements $x, y \in S^1$ such that $xJ_uy \cap J_b \neq \emptyset$ and $xJ_uy \not\subseteq J_b$. Let $b^* \in J_b$ denote the element for which $b^* \in xJ_uy$ is satisfied. Then $xuy = b^*$ or $xvy = b^*$. Consider the case $xuy = b^*$ (the proof is similar in the case $xvy = b^*$). By $xJ_uy \not\subseteq J_b$, we have $xvy \notin J_b$. Then $xuy = b^*$ and $xvy \neq b^*$ and so $uy \neq vy$ from which we get $y \notin S$, that is, $y = 1$. Then $xvy = xv = xuv = b^*v \in J_b$ which contradicts $xvy \notin J_b$. □

Proposition 2.3.10 *([Nag13]) In every T2R semigroup S, $S_0^2 = S_0$.*

Proof. It is sufficient to show that, in every T2R semigroup S, $S_0^2 \neq \{0\}$. This implies our assertion, because if $S_0^2 \neq S_0$ was in a T2R semigroup S, then the Rees factor semigroup $H = S/S_0^2$ would be a T2R semigroup in which $H_0 = S_0/S_0^2$ would satisfy $H_0^2 = \{0\}$ contradicting our result.

Assume, in an indirect way, that there is a T2R semigroup S in which $S_0^2 = \{0\}$. By Proposition 2.3.9, $uS_0 \neq S_0$ (and $vS_0 \neq S_0$). Let $a \in S_0 - uS_0$ be an arbitrary element. By (v) of Theorem 2.3.4, there are elements $x, y \in S^1$ such that $xJ_uy \cap J_a \neq \emptyset$ and $xJ_uy \not\subseteq J_a$. Let $a^* \in J_a$ denote the element for which $a^* \in xJ_uy$ is satisfied. Then $xuy = a^*$ or $xvy = a^*$. Consider the case $xuy = a^*$ (the proof is similar in case $xvy = a^*$). Then $xvy \neq a^*$. If $|J_a| = 1$, then $a = a^*$ and so $ua^* \neq a^*$. If $|J_a| = 2$, then $a \in J_a = J_{a^*} = \{a^*u, a^*v\}$ and so there is an element $x \in \{u, v\}$ such that $a = a^*x$. Then $ua^* \neq a^*$, because the opposite case implies

$$a = a^*x = (ua^*)x = u(a^*x) = ua$$

which is a contradiction. Consequently (in both cases) $a^* \notin uS_0$. Thus, from the above equation $xuy = a^*$, it follows that $x \in S_0$. If $y = 1$, then $a^* = xu \in SS_1$ and so, by Theorem 2.3.3, $J_a = J_{a^*} = \{a^*u, a^*v\}$. Then

$$xvy = xv = xuv = a^*v \in J_{a^*} = J_a$$

which is a contradiction. If $y \in S_1$, then $uy = vy$ and so

$$xvy = xuy = a^*$$

which is also a contradiction. If $y \in S_0$, then, using also $x \in S_0$, we have

$$a^* = xuy \in {S_0}^2 = \{0\}$$

from which we get

$$a^* = ua^* \in uS_0.$$

This is a contradiction. As in all cases we get a contradiction, the indirect assumption is not true. □

Corollary 2.3.11 *([Nag13]) There is no finite T2R (T2L) semigroup.*

Proof. Assume that S is a finite T2R semigroup. As S_0 is a finite nil semigroup, it is nilpotent, that is, $S_0^k = \{0\}$ for some positive integer k. Then, by Proposition 2.3.10, $S_0 = \{0\}$ which contradict the assumption that S_0 contains at least two different elements. □

Open problem: Is there an infinite T2R (or a T2L) semigroup?

Chapter 3

$\mathcal{RGC}_n$-commutative semigroups

By Definition 1.1.13, a semigroup S is called a left weakly commutative semigroup if, for every $a, b \in S$, there exist $x \in S$ and a positive integer n such that $(ab)^n = bx$. In this chapter a special type of left weakly commutative semigroups is considered. In [Nag92], I introduced the notion of the $\mathcal{R}$-commutative semigroup. A semigroup S is said to be $\mathcal{R}$-commutative if, for every elements $a, b \in S$, there is an element $x \in S^1$ such that $ab = bax$. It is clear that every $\mathcal{R}$-commutative semigroup is left weakly commutative. In my paper [Nag92], I examined $\mathcal{R}$-commutative semigroups S which have also the property that, for every $a, b \in S$, $ab = ba$ implies $axb = bxa$ for all $x \in S$. A semigroup with this last property is called a conditionally commutative semigroup, and a semigroup which is $\mathcal{R}$-commutative and conditionally commutative is called an $\mathcal{RC}$-commutative semigroup. In [Nag92], I determined all $\mathcal{RC}$-commutative Δ-semigroups. In the examinations the conditionally commutativity of a semigroup S was used only in the following form: S satisfies the identity $aba^2 = a^2ba$. In [Pon94], a semigroup satisfying this identity is called a generalized conditionally commutative (briefly, GC-commutative) semigroup, and it was proved that every GC-commutative semigroup satisfies the identity $axa^i = a^ixa$ for every integer $i \geq 2$. In [Nag98], I generalized the notion of the $\mathcal{GC}$-commutative semigroup. I defined the notion of the $\mathcal{GC}_n$-commutative semigroup (n is a positive integer) as a semigroup which satisfies the identity $a^nba^i = a^iba^n$ for every integer $i \geq 2$. I examined semigroups which are $\mathcal{R}$-commutative and also $\mathcal{GC}_n$-commutative; these semigroups are called $\mathcal{RGC}_n$-commutative semigroups. In [Nag92] and [JN03] we described the $\mathcal{RGC}_n$-commutative Δ-semigroups. In this chapter we present the results of [Nag92], [Nag98] and [JN03]. The chapter contains four sections.

In the first section we present those results on $\mathcal{R}$-commutative semigroups which will be used in the chapter.

In the second section the $\mathcal{GC}_n$-commutative semigroups are investigated. We prove that a semigroup is simple and $\mathcal{GC}_n$-commutative if and only if it is a Rees matrix semigroup over a commutative group.

In the third section the $\mathcal{RGC}_n$-commutative semigroups are considered. We prove that every $\mathcal{RGC}_n$-commutative semigroup is a semilattice of $\mathcal{GC}_n$-commutative archimedean semigroups. Moreover, a semigroup is simple and $\mathcal{RGC}_n$-commutative if and only if it is a right abelian group. Using this result, we prove that every archimedean $\mathcal{RGC}_n$-commutative semigroup with an idempotent element is an ideal extension of a right abelian group by a commutative nil semigroup.

In the fourth section we determine all $\mathcal{RGC}_n$-commutative Δ-semigroups. We show that a semigroup is an $\mathcal{RGC}_n$-commutative Δ-semigroup if and only if it is isomorphic to one of the following semigroups: (i) G or G^0, where G is a non-trivial subgroup of a quasicyclic p-group (p is a prime); (ii) a two-element semilattice; (iii) R or R^0 or R^1, where R is a two-element right zero semigroup; (iv) a commutative nil semigroup with chain ordered principal ideals; (v) N^1, where N is a non-trivial commutative nil semigroup with chain ordered principal ideals. At the end of the section, we present our main result on $\mathcal{RC}$-commutative Δ-semigroups (which was published in [Nag92]).

3.1 $\mathcal{R}$-commutative semigroups

Definition 3.1.1 *A semigroup S is called an $\mathcal{R}$-commutative semigroup if, for every elements $a, b \in S$, there is an element $x \in S^1$ such that $ab = bax$.*

Remark 3.1.2 *Every $\mathcal{R}$-commutative semigroup is left weakly commutative.*

Theorem 3.1.3 *Every $\mathcal{R}$-commutative semigroup is decomposable into a semilattice of archimedean semigroups.*

Proof. Let S be an $\mathcal{R}$-commutative semigroup. Then, by Remark 3.1.2, it is left weakly commutative. Then, by Corollary 1.1.15, S is a semilattice of archimedean semigroups. $\square$

Theorem 3.1.4 *([Nag01, Theorem 5.2]) A semigroup S is $\mathcal{R}$-commutative if and only if the Green's equivalence $\mathcal{R}$ on S is a commutative congruence on S.*

Proof. Let S be an $\mathcal{R}$-commutative semigroup and $a, b, s \in S$ be arbitrary elements with $a \neq b$ and $(a, b) \in \mathcal{R}$. Then

$$aS^1 = bS^1$$

and so

$$a = by,$$

$$b = ax$$

for some $x, y \in S$. As $as = bys = bsyt$ and $bs = axs = asxt'$ for some $t, t' \in S^1$, we get

$$asS^1 = bsS^1,$$

that is,

$$(as, bs) \in \mathcal{R}.$$

Hence $\mathcal{R}$ is right compatible. As $\mathcal{R}$ is a left congruence on an arbitrary semigroup, it is a congruence on S. As $ab = bax$ and $ba = aby$ for some $x, y \in S^1$, we have

$$(ab, ba) \in \mathcal{R}.$$

Hence $\mathcal{R}$ is a commutative congruence on S.

Conversely, assume that S is a semigroup in which the Green's equivalence $\mathcal{R}$ is a congruence. Then, for arbitrary elements $a, b \in S$,

$$(ab, ba) \in \mathcal{R}$$

and so

$$ab = bax$$

for some $x \in S^1$. Hence S is $\mathcal{R}$-commutative. □

Corollary 3.1.5 *([Nag92]) Every $\mathcal{R}$-commutative nil semigroup is commutative.*

Proof. Let S be an $\mathcal{R}$-commutative nil semigroup. It is easy to see that the Green's equivalence $\mathcal{R}$ is the identity relation on a nil semigroup. Thus $S/\mathcal{R} \cong S$. By Theorem 3.1.4, S is commutative. □

We note that the subsemigroups (and so the archimedean components) of an $\mathcal{R}$-commutative semigroup are not necessarily $\mathcal{R}$-commutative.

Lemma 3.1.6 *([Nag92]) Every right ideal of an $\mathcal{R}$-commutative semigroup is a two-sided ideal.*

Proof. Let R be a right ideal of an $\mathcal{R}$-commutative semigroup. Then, for every $r \in R$ and $s \in S$, there is an element x in S^1 such that

$$sr = rsx \in R.$$

So

$$SR \subseteq R,$$

that is, R is also a left ideal of S. □

Lemma 3.1.7 *([Nag92]) If K is an ideal of an $\mathcal{R}$-commutative semigroup such that K is simple, then K is an $\mathcal{R}$-commutative semigroup.*

Proof. Let k_1, k_2 be arbitrary elements of K. It is evident that k_2k_1K is a right ideal of S. By Lemma 3.1.6, k_2k_1K is a two-sided ideal of S and so of K. As K is simple, we get

$$k_2k_1K = K.$$

Then there is an element k in K such that

$$k_1k_2 = k_2k_1k.$$

Hence K is $\mathcal{R}$-commutative. □

3.2 $\mathcal{GC}_n$-commutative semigroups

Definition 3.2.1 *For a positive integer n, a semigroup is called generalized conditionally n-commutative (or $\mathcal{GC}_n$-commutative) if it satisfies the identity $a^n x a^i = a^i x a^n$ for every integer $i \geq 2$.*

We note that if S is a non commutative band with an identity element, then S is $\mathcal{GC}_n$-commutative for every positive integer n, but it is not conditionally commutative. Thus the conditionally commutative semigroups form a proper subclass in the class of $\mathcal{GC}_n$-commutative semigroups for every positive integer n.

Lemma 3.2.2 *([Nag98]) Every $\mathcal{GC}_n$-commutative semigroup satisfies the identity $a^{tn} b a^k = a^k b a^{tn}$ for every positive integer t and every integer $k \geq 2t$.*

Proof. By induction for t. If $t = 1$, then the assertion holds by definition. Assume that the identity holds in a $\mathcal{GC}_n$-commutative semigroup S for some positive integer t and every integer $k \geq 2t$. Let $a, b \in S$ be arbitrary elements and $k \geq 2(t+1)$ an arbitrary integer. Then $k - 2 \geq 2t$ and so

$$a^{(t+1)n} b a^k = a^{tn} a^n b a^{k-2} a^2 = a^{k-2} a^n b a^{tn} a^2 =$$

$$a^{k-2} a^2 b a^{tn} a^n = a^k b a^{(t+1)n}.$$

□

The simple $\mathcal{GC}_n$-commutative and the simple $\mathcal{RGC}_n$-commutative semigroups are very important in our investigation. Before describing them, we prove the following lemma.

Lemma 3.2.3 *([Nag98] If S is a $\mathcal{GC}_n$-commutative semigroup such that $a = x a^{6n} y$ holds for some $a \in S$ and $x, y \in S^1$, then S has an idempotent element.*

Proof. Let S be a semigroup satisfying the condition of the lemma. Then

$$a^{3n} = a^n a a^{n-1} a^n = a^n x a^{6n} y a^{n-1} a^n = [a^n x a^{3n}][a^{3n}(y a^{n-1}) a^n].$$

By Lemma 3.2.2,

$$a^n x a^{3n} = a^{3n} x a^n$$

and

$$a^{3n}(y a^{n-1}) a^n = a^n (y a^{n-1}) a^{3n}.$$

Thus

$$a^{3n} = a^{3n}(x a^{2n} y a^{n-1}) a^{3n}.$$

Hence, a^{3n} is a regular element of S. Consequently S contains an idempotent element. □

Theorem 3.2.4 *([Nag98] A semigroup is simple and $\mathcal{GC}_n$-commutative if and only if it is isomorphic with a Rees matrix semigroup $\mathcal{M}(G;I,J;P)$ over a commutative group G.*

Proof. Let S be a simple $\mathcal{GC}_n$-commutative semigroup. We can suppose that $|S| \geq 2$. Then S has no a zero element. As S is simple, for an arbitrary element $a \in S$, there are elements $x, y \in S^1$ such that $a = xa^{6n}y$. By Lemma 3.2.3, S has an idempotent element f. We show that S is completely simple. Assume, in an indirect way, that S is not completely simple. Then, using Theorem 1.1.6, we can conclude that S contains a bicyclic semigroup $C(p,q)$ such that $pq = f$, $qp \neq f$. It is clear that $C(p,q)$ must be $\mathcal{GC}_n$-commutative and so

$$q^n p^{n+1} = q^n p^{n+1} p^2 q^2 = p^{n+3} q^{n+2} = p$$

which is a contradiction. Consequently S is completely simple and it is isomorphic with a Rees matrix semigroup $\mathcal{M}(G;I,J;P)$ over a group G with a $J \times I$ sandwich matrix P. Suppose that P is normalized, that is, there are elements $i_0 \in I$ and $j_0 \in J$ such that $p_{j_0,i} = p_{j,i_0} = e$ for all $i \in I,\ j \in J$, where e denotes the identity element of G. Let g and h be arbitrary elements in G. Then

$$(i_0, g^n h g^{n+1}, j_0) = (i_0, g, j_0)^n (i_0 h, j_0)(i_0, g, j_0)^{n+1} =$$

$$(i_0, g, j_0)^{n+1} (i_0 h, j_0)(i_0, g, j_0)^n = (i_0, g^{n+1} h g^n, j_0)$$

which implies that

$$g^n h g^{n+1} = g^{n+1} h g^n,$$

that is,

$$hg = gh.$$

Hence G is commutative. Thus the first part of the theorem is proved.

Conversely, assume that a semigroup S is isomorphic with a Rees matrix semigroup over a commutative group. It is a matter of checking to see that S is $\mathcal{GC}_n$-commutative. Thus S is a simple $\mathcal{GC}_n$-commutative semigroup. □

3.3 $\mathcal{RGC}_n$-commutative semigroups

Definition 3.3.1 *([Nag98]) A semigroup which is $\mathcal{R}$-commutative and $\mathcal{GC}_n$-commutative will be called an $\mathcal{RGC}_n$-commutative semigroup .*

We note that if $S = R^1$, where R is a non-trivial right zero semigroup, then S is an $\mathcal{RGC}_n$-commutative semigroup for every positive integer n such that it is not conditionally commutative. Consequently, for every positive integer $n \geq 2$, the class of $\mathcal{RC}$-commutative semigroups is a proper subclass in the class of $\mathcal{RGC}_n$-commutative semigroups.

Theorem 3.3.2 *Every $\mathcal{RGC}_n$-commutative semigroup is a semilattice of archimedean $\mathcal{GC}_n$-commutative semigroups.*

Proof. By Theorem 3.1.3, it is obvious. □

Theorem 3.3.3 *([Nag98]) A semigroup is simple and $\mathcal{RGC}_n$-commutative if and only if it is a right abelian group.*

Proof. Let S be an $\mathcal{RGC}_n$-commutative simple semigroup. By Theorem 3.2.4, S is isomorphic with a Rees matrix semigroup $\mathcal{M}(G; I, J; P)$ over a commutative group G. Assume that P is normalized ($p_{j_0,i} = p_{j,i_0} = e$ for some $i_0 \in I,\ j_0 \in J$ and for all $i \in I,\ j \in J$). Let $a = (i_0, g, j_0)$ and $b = (m, h, j_0)$ be elements of S for some $g, h \in G$ and $m \in I$. As S is simple, $ab = xbay$ for some $x, y \in S$. As S is $\mathcal{R}$-commutative, $xba = baxz$ for some $z \in S^1$. Let $xzy = (k, r, l)$. Then

$$(i_0, gh, j_0) = ab = xbay = ba(xzy) = (m, hgr, l)$$

from which we get $m = i_0$ for all $m \in I$. Thus $|I| = 1$. Consequently, P has only one column and every element of P equals to the identity element e of G. It is clear that J can be considered as a right zero semigroup and $(i_0, g, j) \to (g, j)$ is an isomorphism of S onto the direct product $G \times J$ of the commutative group G and the right zero semigroup J. Thus S is a right abelian group.

As every right abelian group is simple and $\mathcal{RGC}_n$-commutative, the theorem is proved. □

Theorem 3.3.4 *([Nag98]) Every $\mathcal{RGC}_n$-commutative archimedean semigroup containing at least one idempotent element is an ideal extension of a right abelian group by a commutative nil semigroup.*

Proof. Let S be an $\mathcal{RGC}_n$-commutative archimedean semigroup containing at least one idempotent element f. If S has a zero element, then it is a nil semigroup. By Corollary 3.1.5, S is commutative. Next we suppose that S has no zero element. Let f be an arbitrary idempotent of S. As S is archimedean, f is contained by all ideals of S. Hence $K = SfS$ is the kernel of S and so K is simple. It is clear that K is $\mathcal{GC}_n$-commutative. By Lemma 3.1.7, K is also $\mathcal{R}$-commutative. Then, by Theorem 3.3.3, K is a right abelian group. As the Rees factor semigroup S/K is nil and $\mathcal{R}$-commutative, it is also commutative (see Corollary 3.1.5). Thus the theorem is proved. □

Lemma 3.3.5 *([Nag98]) If S is an $\mathcal{R}$-commutative semigroup and I is an ideal of S such that I is $\mathcal{GC}_n$-commutative, then, for aritrary $a \in I$,*

$$\alpha_a = \{(x, y) \in I \times I :\ xa^i = ya^j \textit{ for some positive integers } i \textit{ and } j\}$$

is a congruence on I.

Proof. It is clear that α_a is a left congruence on I. We show that α_a is also right compatible. Let $x, y, s \in I$ be arbitrary elements with $(x, y) \in \alpha_a$. Then, for some positive integers i and j, $xa^i = ya^j$. We can suppose that $i \geq j = tn \geq 2$

for some positive integer t. As S is $\mathcal{R}$-commutative, there is an element $u \in S^1$ such that $sa^i = a^i su$. Thus $xsa^i = xa^i su = ya^j su$ and so

$$xsa^{2i} = ya^j sua^i.$$

If $n = 1$, then

$$a^j sua^i = a^j suaa^{i-1} = asua^j a^{i-1} = asua^i a^{j-1} = a^i sua^j.$$

If $n \geq 2$, then $i \geq j = tn \geq 2t$. Using Lemma 3.2.2 and the fact that I is $\mathcal{GC}_n$-commutative, we get

$$a^j sua^i = a^{tn} sua^i = a^i sua^{tn} = a^i sua^j.$$

Consequently, $a^j sua^i = a^i sua^j$ is satisfied in both cases. Hence

$$xsa^{2i} = ya^j sua^i = ya^i sua^j = ysa^i a^j = ysa^{i+j}.$$

Thus α_a is right compatible and so it is a congruence on I. □

Theorem 3.3.6 *([Nag98]) If S is an $\mathcal{R}$-commutative semigroup and I is an ideal of S such that I is $\mathcal{GC}_n$-commutative and archimedean without idempotents, then I has a non-trivial group homomorphic image.*

Proof. By Lemma 3.3.5, α_a is a congruence on I for arbitrary $a \in I$. As $(sa, s) \in \alpha_a$ for all $s \in I$, we get that the α_a-class of I containing the element a is a right identity element of I/α_a. Let $s \in I$ be arbitrary. Then there are elements $u, v \in I$ and a positive integer m such that $a^m = usv$, because I is archimedean. As S is $\mathcal{R}$-commutative, $a^m = suwv$ for some $w \in S^1$. Thus $a(a^m) = (suwv)a$. As I is an ideal of S and $v \in I$, we have $(a, suwv) \in \alpha_a$ which means that the α_a-class $[uwv]_{\alpha_a}$ of I is a right inverse of the α_a-class $[s]_{\alpha_a}$ of I with respect to the right identity element $[a]_{\alpha_a}$ of I/α_a. Then the factor semigroup I/α_a is a group. Consequently I/α_a is a group for arbitrary $a \in I$. As I does not contain idempotents, $(a, a^2) \notin \alpha_{a^2}$ (a is an arbitrary element of I). Thus I/α_{a^2} is a non-trivial group-homomorphic image of I. □

3.4 $\mathcal{RGC}_n$-commutative Δ-semigroups

Lemma 3.4.1 *([Nag98]) If S is an $\mathcal{R}$-commutative semigroup, then, for arbitrary $a \in S$,*

$$\tau_a = \{(x, y) \in S \times S : \ xa = ya\}$$

is a conguence on S.

Proof. It is clear that τ_a is a left congruence on S. To show that τ_a is also a right congruence, let $x, y, s \in S$ be arbitrary element with $(x, y) \in \tau_a$. Then, for some $u \in S$,

$$xsa = xasu = yasu = ysa,$$

which means that

$$(xs, ys) \in \tau_a.$$

Thus the lemma is proved. $\square$

In this section we will use the following lemma several times.

Lemma 3.4.2 *([Nag98]) If S is an $\mathcal{R}$-commutative Δ-semigroup and I is an ideal of S such that I is $\mathcal{GC}_n$-commutative and a nil extension of a non-trivial right zero semigroup R, then $I = R$.*

Proof. Since $R^2 = R$, then R is an ideal of S by Theorem 1.1.4. It can be easily verified that

$$\eta = \{(a, b) \in S \times S : \ ra = rb \text{ for all } r \in R\}$$

is a congruence on S. It is evident that $\eta | R = id_R$. Thus $\eta \cap \varrho_R = id_S$. As S is a Δ-semigroup,

$$\eta \subseteq \varrho_R \quad \text{or} \quad \varrho_R \subseteq \eta.$$

Then

$$\eta = id_S \quad \text{or} \quad \varrho_R = id_S.$$

As $|R| \geq 2$, we have $\eta = id_S$. As I is $\mathcal{GC}_n$-commutative,

$$ra^n = ra^k ra^n = ra^n ra^k = ra^k$$

for all $a \in I$, $r \in R$ and $k \geq 2$. It means that $(a^n, a^k) \in \eta$ and so $a^n = a^k$ for all $a \in I$ and $k \geq 2$. As I is a nil extension of R, we get $a^2 \in R$ for all $a \in I$. From $a^2 = a^3$, we get $(a, a^2) \in \tau_a$, where τ_a is defined in Lemma 3.4.1. Assume $I - R \neq \emptyset$. Let $a \in I - R$ be an arbitrary element. As $a^2 \in R$, we have $(a, a^2) \notin \varrho_R$. By the above, $(a, a^2) \in \tau_a$. As S is a Δ-semigroup, we get $\varrho_r \subseteq \tau_a$. As $a^2 \in R$, we have $(r, a^2) \in \varrho_R \subseteq \tau_a$ for all $r \in R$. As $(a, a^2) \in \tau_a$ we have $(r, a) \in \tau_a$ for all $r \in R$. Thus

$$ra = a^2 = ra^2$$

for all $r \in R$, because $a^2 \in R$ and R is a right zero semigroup. Then $(a, a^2) \in \eta$. As $\eta = id_S$, we get $a = a^2 \in R$ which is a contradiction. Hence $R = I$. $\square$

Theorem 3.4.3 *([Nag98]) A semigroup is an archimedean $\mathcal{RGC}_n$-commutative Δ-semigroup if and only if it is isomorphic to either G or R or N, where G is a non-trivial subgroup of a quasicyclic p-group (p is a prime), R is a right zero semigroup of order 2 and N is a commutative nil Δ-semigroup.*

Proof. Let S be an archimedean $\mathcal{RGC}_n$-commutative Δ-semigroup. If S has a zero element, then S is an $\mathcal{R}$-commutative nil semigroup and so it is commutative by Corollary 3.1.5.

Next we can suppose that S has no zero element. First suppose that S has no proper ideals. Then S is simple. By Theorem 3.3.3, S is a direct product of a commutative group G and a right zero semigroup R. Consequently S is isomorphic with either G or R (see Corollary 1.3.17). In the first case Theorem 1.3.4 implies that S is a subgroup of a quasicyclic p-group (p is a prime). In the second case Theorem 1.3.19 implies that S is a right zero semigroup of order 2.

Consider the case when S has a proper ideal. Then, By Theorem 1.3.7 and Theorem 3.3.6, S has an idempotent element. Thus, by Theorem 3.3.4, S is a nil extension of a direct product of a commutative group G and a right zero semigroup R. By Theorem 1.3.7, $|G| = 1$. Thus S is a nil extension of the right zero semigroup R. Applying Lemma 3.4.2 for $I = S$, we get $S = R$ which contradicts the assumption that S has a proper ideal.

As the semigroups listed in the theorem are archimedean $\mathcal{RGC}_n$-commutative Δ-semigroups, the theorem is proved. $\square$

Lemma 3.4.4 *([Nag98]) If an $\mathcal{RGC}_n$-commutative semigroup is a semilattice of a commutative group S_1 and a nil semigroup S_0 such that $S_0S_1 \subseteq S_0$, then $Hb \subseteq bS_1$ is satisfied for every subgroup H of S_1 and every element $b \in S^1$.*

Proof. Let S be an $\mathcal{RGC}_n$-commutative semigroup satisfying the conditions of the lemma. Let H be an arbitrary subgroup of S_1 and b be an arbitrary element in S^1. We can suppose that $b \in S$. Let $h \in H$ be arbitrary. As S is $\mathcal{R}$-commutative, there is an element $t \in S^1$ such that $hb = bht$. If $t \in S_1^1$, then $hb \in bS_1$. If $t \in S_0$, then $bh = hbs$ for some $s \in S^1$. As $st \in S_0$, $(st)^n = 0$ for some positive integer n. Thus

$$hb = hb(st) = hb(st)^n = 0$$

and also

$$bh = 0.$$

Let h^* be an arbitrary element in H. Then $h^* = h'h$ for some $h' \in H$. Thus

$$h^*b = h'hb = 0,$$

that is, $Hb = \{0\}$. Let $g \in S_1$ be arbitrary. Then, for some $h'' \in S_1$, $hh'' = g$ which implies that $bg = bhh'' = 0$. Thus $bS_1 = \{0\}$. Consequently, $Hb \subseteq bS_1$. $\square$

Lemma 3.4.5 *If an $\mathcal{RGC}_n$-commutative Δ semigroup is a semilattice of archimedean semigroups S_1 and S_0 such that $S_0S_1 \subseteq S_0$, then S_0 is either a nil semigroup or a non-trivial right zero semigroup.*

Proof. By Theorem 3.3.6 and Theorem 1.3.7, S_0 has an idempotent element. If S_0 contains a zero element, then it is a nil semigroup. Consider the case when S_0 does not contain a zero element. Let f be an idempotent element of S_0. As S_0 is archimedean, $K = S_0fS_0$ is the kernel of S_0 such that $|K| \geq 2$ and S_0 is a nil extension of K. It is clear that K is simple and $\mathcal{GC}_n$-commutative. As $K^2 = K$ and K is an ideal of S_0 (which is an ideal of S), Theorem 1.1.4 implies that K is an ideal of S. As S is $\mathcal{R}$-commutative and K is simple, Lemma 3.1.7

implies that K is $\mathcal{R}$-commutative. Hence K is an $\mathcal{RGC}_n$-commutative simple semigroup. By Theorem 3.3.3, K is a direct product of a commutative group G and a right zero semigroup R. By Theorem 1.3.7, we can suppose that $K = R$. As S_0 is a nil extension of R ($|R| \geq 2$), Lemma 3.4.2 implies $S_0 = R$. $\square$

In the remainder of this section, S_1 and S_0 denote the semilattice components of an $\mathcal{RGC}_n$-commutative Δ-semigroup S.

Lemma 3.4.6 *([Nag98, JN03]) If S is an $\mathcal{RGC}_n$-commutative Δ-semigroup such that S_1 is a commutative group, then either $|S_0| = 1$ or $S = S_0^1$, where S_0 is either a non-trivial commutative nil Δ-semigroup or a two-element right zero semigroup.*

Proof. Let S be a semilattice decomposable $\mathcal{RGC}_n$-commutative Δ-semigroup such that S_1 is a group (with an identity element e). By Lemma 3.1.6, eS is a two-sided ideal of S and $eS \cap S_1 \neq \emptyset$, $eS \cap S_0 \neq \emptyset$. As S is a Δ-semigroup and S_1 is a group, we get $eS = S$. Hence e is a left identity element of S. By Theorem 3.3.6 and Theorem 1.3.7, S_0 has an idempotent element.

First consider the case when S_0 has a zero element. Then S_0 is a nil semigroup, because it is archimedean.

Assume $|S_1| = 1$. Let a be an arbitrary element of S_0. Assume $ae \neq a$. As S is an $\mathcal{R}$-commutative semigroup, there is an element $s \in S^1$ such that $a = ea = aes$. From the assumption $ae \neq a$ it follows that $s \in S_0$. Then $a = aes = as$, because e is a left identity element of S. Thus $a = as^i$ for every positive integer i. As S_0 is a nil semigroup and $s \in S_0$, we get $a = 0$. Therefore, $|S_0| = 1$ or e is a (two-sided) identity element of S and $|S_0| > 1$. We can consider the second case. Then $S = S_0^1$. It is easy to see that S_0 is $\mathcal{R}$-commutative. Then, by Corollary 3.1.5, S_0 is a non-trivial commutative nil semigroup.

Suppose $|S_1| > 1$. By Theorem 1.3.3, S_1^0 and therefore S_1 are Δ-semigroups. As S_1 is archimedean, it is isomorphic to a non-trivial subgroup of a quasicyclic p-group (p is a prime). Let H be the least non-trivial subgroup of S_1. Assume $xHy \cap H \neq \emptyset$ for some $x, y \in S^1$. Then $x, y \notin S_0$ and so $xHy \subseteq H$. Thus H is a normal complex (Definition 1.1.1) in S and so there is a congruence β of S such that H is a β-class. It is evident that β is not a Rees congruence on S. Then, by Theorem 1.3.11, there is an element $a \in S$ such that the β-classes are I_a, the one-element subsets in the complement of $J(a)$ and some classification of J_a. If $a \in S_0$, then $J(a) \subseteq S_0$ which means that H contains only one element. But this is a contradiction. Consequently $a \in S_1$. Then $J(a) = S$ and $I_a = S_0$. Let $b \in S_0$ be an arbitrary element. Then $(b, 0) \in \beta$ and so there are elements $x_i, y_i \in S^1$ and $p_i, q_i \in H$ $(i = 1, 2, \dots n)$ such that

$$b = x_1p_1y_1,\ x_1q_1y_1 = x_2p_2y_2, \dots, x_{n-1}q_{n-1}y_{n-1} = x_np_ny_n,\ x_nq_ny_n = 0.$$

Applying Lemma 3.4.4 for y_i, we get $p_iy_i = y_iu_i$ and $q_iy_i = y_iv_i$ for some $u_i, v_i \in S_1$. Then

$$b = x_1y_1u_1,\ x_1y_1v_1 = x_2y_2u_2, \dots, x_{n-1}y_{n-1}v_{n-1} = x_ny_nu_n,\ x_ny_nv_n = 0.$$

As $u_i, v_i \in S_1$, we have $J(x_iy_iu_i) = J(x_iy_iv_i)$ for every i. Then

$$J(b) = J(x_1y_1u_1) = J(x_1y_1v_1) = \cdots = J(x_ny_nu_n) = J(x_ny_nv_n) = J(0).$$

Consequently $b = 0$. Thus $|S_0| = 1$.

Next, consider the case when S_0 has no a zero element. Then, by Lemma 3.4.5, S_0 is a right zero semigroup ($|S_0| \geq 2$). It is easy to see that $\eta = \{(a, b) \in S \times S : (\forall r \in S_0)\ ra = rb\}$ is a congruence on S whose restriction to S_0 is the identity relation of S_0. As S is a Δ-semigroup, S_0 is a dense ideal of S and so η is the identity relation on S. Let $g \in S_1$ and $r \in S_0$ be arbitrary elements. As S is $\mathcal{GC}_n$-commutative,

$$g^nrg^{n+1} = g^{n+1}rg^n$$

from which we get

$$erg = gre$$

(e is the identity element of S_1). Then

$$rerg = rgre.$$

As $re, rg \in S_0$ and S_0 is a right zero semigroup, we have

$$rg = re$$

(for every $r \in S_0$) and so

$$(g, e) \in \eta.$$

Then $g = e$ which means that $S_1 = \{e\}$. Thus S is a band. By Theorem 1.3.18, $S = S_0^1$ and $|S_0| = 2$. The lemma is proved. □

Lemma 3.4.7 *([Nag98]) If S is an $\mathcal{RGC}_n$-commutative Δ-semigroup such that S_1 is a right zero semigroup of order two and S_0 is a nil semigroup, then $|S_0| = 1$.*

Proof. Let $S_1 = \{u, v\}$. As uS is a right ideal of S and S is $\mathcal{R}$-commutative, we get that uS is an ideal of S. As $uS \cap S_1 \neq \emptyset$, $uS \cap S_0 \neq \emptyset$ and S_1 is simple, we have $uS = S$. Thus u is a left identity element of S. Similarly, v is a left identity element of S. It is easy to see that

$$\tau_u = \{(a, b) \in S \times S :\ au = bu\}$$

is a congruences on S such that $(u, v) \in \tau_u$. As S is a Δ semigroup, $\tau_u \subseteq \varrho_{S_0}$ or $\varrho_{S_0} \subseteq \tau_u$, where ϱ_{S_0} denotes the Rees congruence of S modulo S_0. As $(u, v) \in \tau_u$ and $(u, v) \notin \varrho_{S_0}$, we have $\varrho_{S_0} \subseteq \tau_u$. Hence, $(a, 0) \in \tau_u$ and so $au = 0$ for every $a \in S_0$, where 0 is the zero element of S_0. Let $a \in S_0$ be an arbitrary element. As S is $\mathcal{R}$-commutative, there is an element $s \in S^1$ such that $a = ua = aus = 0$. Therefore $|S_0| = 1$. □

Theorem 3.4.8 *([JN03]) A semilattice decomposable $\mathcal{RGC}_n$-commutative semigroup is a Δ-semigroup if and only if it is isomorphic to either G^0 or F or R^0 or R^1 or N^1, where G is a non-trivial subgroup of a quasicyclic p-group (p is a prime), F is a two-element semilattice, R a right zero semigroup of order 2 and N is a non-trivial commutative nil Δ-semigroup.*

Proof. Let S be a semilattice decomposable $\mathcal{RGC}_n$-commutative Δ-semigroup. Then, by Theorem 1.3.3, the Rees factor semigroup S/S_0 is a Δ-semigroup. From this it follows that S_1 is a semilattice indecomposable $\mathcal{RGC}_n$-commutative Δ-semigroup. Then, by Theorem 3.4.3, S_1 is isomorphic to either a non-trivial subgroup of a quasicyclic p-group (p is a prime) or a commutative nil semigroup whose ideals are chain ordered with respect to inclusion or a right zero semigroup of order 2.

If S_1 is a non-trivial subgroup G of a quasicyclic p-group (p is a prime), then $|S_0| = 1$ by Lemma 3.4.6, and so $S = G^0$.

If S_1 is a commutative nil Δ-semigroup, $S_0 \cup \{f\}$ is an ideal of S, where f is the zero of S_1. The Rees congruence on S modulo $S_0 \cup \{f\}$ is comparable with the least semilattice congruence on S in only that case when $|S_1| = 1$. Hence, by Lemma 3.4.6, S is a two-element semilattice or $S = S_0^1$ where S_0 is either a non trivial commutative nil Δ-semigroup or a two-element right zero semigroup.

If S_1 is a right zero semigroup of order 2, S_0 is a proper ideal of S and so, by Theorem 3.3.6 and Theorem 1.3.7, S_0 has an idempotent element. By Theorem 3.3.4, S_0 is an ideal extension of a direct product D of a commutative group G and a right zero semigroup R by a commutative nil semigroup N. As $D^2 = D$ and D is an ideal of S_0, we get that D is also an ideal of S (Theorem 1.1.4). If $|D| = 1$, then S_0 is isomorphic to N. Then, by Lemma 3.4.7, $|N| = 1$ and so $S = S_1^0$. If $|D| > 1$, then $|G| = 1$ by Theorem 1.3.7, and so $D = R$. Let ϕ denote the canonical homomorphism of S onto the Rees factor semigroup of S modulo R. It is easy to see that $\phi(S)$ is an $\mathcal{RGC}_n$-commutative Δ-semigroup. $\phi(S)$ is semilattice decomposable, $\phi(S_1) = S_1$ and $\phi(S_0) = N$. Then, as above, $|N| = 1$ and so $S_0 = R$. As S_1 and S_0 are right zero semigroups, S is a (semilattice decomposable) band. As $|S_1| > 1$, we have $|R| = 1$ by Theorem 1.3.18 and so $S = S_1^0$. □

By Theorem 3.4.3 and Theorem 3.4.8, we can formulate our main result on $\mathcal{RGC}_n$-commutative Δ-semigroups.

Theorem 3.4.9 *([JN03]) A semigroup S is an $\mathcal{RGC}_n$-commutative Δ-semigroup if and only if it satisfies one of the following conditions.*

(i) S is isomorphic to either G or G^0 , where G is a non-trivial subgroup of a quasicyclic p-group (p is a prime).

(ii) S is isomorphic to a two-element semilattice.

(iii) S is isomorphic to R or R^0 or R^1, where R is a two-element right zero semigroup.

(iv) S is isomorphic to a commutative nil semigroup with chain ordered principal ideals.

(v) S is isomorphic to N^1, where N is a non trivial commutative nil semigroup with chain ordered principal ideals. □

In the next theorem we characterize the $\mathcal{RC}$-commutative Δ-semigroups.

Theorem 3.4.10 *([Nag92]) A semigroup S is an $\mathcal{RC}$-commutative Δ-semigroup if and only if it satisfies one of the following conditions.*

(i) S is isomorphic to either G or G^0, where G is a non-trivial subgroup of a quasicyclic p-group (p is a prime).

(ii) S is isomorphic to a two-element semilattice.

(iii) S is isomorphic to either R or R^0, where R is a two-element right zero semigroup.

(iv) S is isomorphic to a commutative nil semigroup with chain ordered principal ideals.

(v) S is isomorphic to N^1, where N is a non-trivial commutative nil semigroup whit chain ordered principal ideals.

Proof. Let S be an $\mathcal{RC}$-commutative Δ-semigroup. Then it is a $\mathcal{RGC}_n$-commutative Δ-semigroup for every positive integer n. Then S is isomorphic to one of the semigroups listed in Theorem 3.3.4. As a conditionally commutative monoid is commutative, $S \cong R^1$ is impossible, because R is a two-element right zero semigroup. As the semigroups listed in the theorem are $\mathcal{RC}$-commutative Δ-semigroups, the theorem is proved. $\square$

Chapter 4

Permutative semigroups

A semigroup S is called a permutative semigroup if there is an integer $n \geq 2$ and there is a non-identity permutation σ of $\{1, 2, \ldots, n\}$ such that, for every $x_1, x_2, \ldots, x_n \in S$, we have $x_1 x_2 \ldots x_n = x_{\sigma(1)} x_{\sigma(2)} \ldots x_{\sigma(n)}$. In this chapter we deal with permutative semigroups. The chapter contains three sections.

In the first section we deal with the semilattice decomposition of permutative semigroups. It is known (see [Nor88]) that every permutative semigroup is a semilattice of archimedean semigroups. We show that every permutative archimedean semigroup containing at least one idempotent element is an ideal extension of a rectangular abelian group by a nil semigroup. We also show that every permutative archimedean semigroup without idempotent element has a non-trivial commutative group homomorphic image.

In the second section of this chapter we deal with the permutative Δ-semigroups. The main result of this section is that every permutative Δ-semigroup is medial.

In the third section of this chapter we examine the permutative congruence permutable semigroups. Using also the results of the second section, we show that every permutative congruence permutable semigroup is medial. Especially, every strictly permutative congruence permutable semigroup is commutative.

4.1 Semilattice decomposition of permutative semigroups

Definition 4.1.1 *A semigroup S is called a permutative semigroup if there is an integer $n \geq 2$ and there is a non-identity permutation σ of $\{1, 2, \ldots, n\}$ such that, for every $x_1, x_2, \ldots, x_n \in S$, the equation $x_1 x_2 \ldots x_n = x_{\sigma(1)} x_{\sigma(2)} \ldots x_{\sigma(n)}$ is satisfied.*

Theorem 4.1.2 *([Nor88, Corollary 1.4]) Every permutative semigroup is a semilattice of archimedean semigroups.* □

Semigroups satisfying the identity $axyb = ayxb$ form a subclass of the class of all permutative semigroups; these semigroups are called medial semigroups. The next theorem proved by M.S. Putcha and A. Yakub ([PY71]) shows that the medial semigroups play a special role in the class of all permutative semigroups. This theorem will be used in this chapter several times.

Theorem 4.1.3 *([PY71, Theorem 1]) If S is a permutative semigroup, then there is a positive integer k such that, for all $u, v \in S^k$ and all $a, b \in S$, we have $uabv = ubav$. In particular, S^k is a medial semigroup.* □

Using also the previous theorem, we prove the next result on permutative simple semigroups.

Theorem 4.1.4 *A semigroup is simple and permutative if and only if it a rectangular abelian group.*

Proof. Let S be a simple permutative semigroup. Then, by Theorem 4.1.3, S is a simple medial semigroup. As every medial semigroup is weakly exponential, it follows from Theorem 2.1.4 that S is a rectangular abelian group.

It is easy to see that every rectangular abelian group is simple and satisfies the (non-identity) permutation identity $x_1x_2x_3x_4 = x_1x_3x_2x_4$. Thus the converse assertion is obvious. □

The following theorem is on permutative archimedean semigroups containing at least one idempotent element.

Theorem 4.1.5 *([NJ04]) Every permutative archimedean semigroup containing at least one idempotent element is an ideal extension of a rectangular abelian group by a nil semigroup.* □

Proof. Let S be a permutative archimedean semigroup containing at least one idempotent element. By Theorem 1.1.12, S is an ideal extension of a simple subsemigroup K containing an idempotent element by a nil semigroup. By Theorem 4.1.4, K is a rectangular abelian group. □

The next lemma will be used in the proof of Theorem 4.1.7 in which the permutative archimedean semigroups without idempotent elements will be examined.

Lemma 4.1.6 *([NJ04]) If a is an arbitrary element of a permutative semigroup S, then*

$$S_a = \{x \in S : \ a^i x a^j = a^h \text{ for some positive integers } i,\ j,\ k\}$$

is the smallest reflexive unitary subsemigroup of S that contains a.

Proof. Let S be a permutative semigroup. Then, by Theorem 4.1.3, there is a positive integer k such that $uabv = ubav$ for every $u, v \in S^k$ and every $a, b \in S$. Let a be an arbitrary element of S. It is clear that $a \in S_a$. To show that S_a is a subsemigroup of S, let $x, y \in S_a$ be arbitrary elements. Then

$$a^i x a^j = a^h \text{ and } a^m y a^n = a^t$$

for some positive integers i, j, h, m, n, t. We can suppose that

$$i, n \geq k.$$

Then

$$a^{h+t} = a^i x a^j a^m y a^n = a^i x y a^{j+m+n}$$

and so $xy \in S_a$. To show that S_a is left unitary, assume $x, xy \in S_a$ for some $x, y \in S$. Then

$$a^i x a^j = a^h \text{ and } a^m x y a^n = a^t$$

for some positive integers i, j, h, m, n, t. We can suppose that

$$m \geq j \text{ and } i, n \geq k.$$

Then

$$a^{i+t} = a^i a^m x y a^n = a^i x a^m y a^n = a^i x a^j a^{(m-j)} y a^n = a^{h+m-j} y a^n.$$

Hence $y \in S_a$. We can prove, in a similar way, that $y, xy \in S_a$ implies $x \in S_a$. Thus S_a is an unitary subsemigroup of S. S_a is reflexive, because it is unitary and

$$(xy)^3 = x(yx)^2 y = xy^2 x^2 y = xy(yx)xy$$

holds in S. Let B be a unitary subsemigroup of S such that $a \in B$. Then, for an arbitrary element $x \in S_a$, there are positive integers i, j, k such that

$$a^i x a^j = a^k \in B.$$

Then $x \in B$ and so $S_a \subseteq B$. $\square$

Theorem 4.1.7 *([NJ04]) Every permutative archimedean semigroup without idempotent element has a non-trivial commutative group homomorphic image.*

Proof. Let S be a permutative archimedean semigroup without idempotent element. Assume $S_a \neq S$ for some $a \in S$. Then the principal right congruence $\mathcal{R}_{S_a}$ of S defined by the reflexive unitary subsemigroup S_a is a group congruence on S (see Theorem 1.1.3) and so the factor semigroup $S/\mathcal{R}_{S_a}$ is a non-trivial group homomorphic image of S. Suppose $S_a = S$ for all $a \in S$. Then, for any $a \in S$, $S_{a^2} = S$ and so $a \in S_{a^2}$. Then there are positive integers i, j, h such that we have $a^{2i} a a^{2j} = a^{2h}$, that is, $a^{2(i+j)+1} = a^{2h}$. One of the exponents is even, the other is odd. Thus the order of a is finite and so S contains an idempotent element. This contradicts the assumption that S has no idempotent element. $\square$

4.2 Permutative Δ-semigroups

In this section we prove that every permutative Δ-semigroup is medial. First we deal with permutative, archimedean Δ-semigroups. First of all, we prove three lemmas that will be used in the proof of Theorem 4.2.7 below.

Recall that a semigroup S is called idempotent if $S^2 = S$.

Lemma 4.2.1 *([NJ04]) Every nilpotent Δ-semigroup is finite cyclic. Every non-nilpotent, nil permutative Δ-semigroup is idempotent. Hence any permutative nil Δ-semigroup is medial.*

Proof. First, suppose that S is a non-idempotent nil Δ-semigroup. Let $a, b \in S - S^2$. Since the ideals of S are totally ordered, we may assume without loss of generality that $S^1bS^1 \subseteq S^1aS^1$. If $b \neq a$, then $b = sat$, where either s or t is in S, contradicting $b \notin S^2$. Hence $b = a$ and so $S - S^2 = \{a\}$. Let $k > 1$ be an arbitrary integer. If $c \in S^{k-1} - S^k$, then $c = c_1c_2 \cdots c_{k-1}$ for some $c_i \in S - S^2$. Hence $c = a^{k-1}$.

If S is nilpotent, then $S^j = \{0\}$ for some least positive integer j and, by the above, $S = \{a, a^2, \ldots, a^j = 0\}$. Clearly such a semigroup is medial.

If S is non-idempotent and nil, but non-nilpotent, then $S^j \neq \{0\}$ for all $j \geq 1$. Let n be any positive integer such that $a^n = 0$. Let $b \in S^{3n} - \{0\}$, $b = b_1b_2 \cdots b_{3n}$ say. Since $a \notin S^2$, $a \notin S^1b_iS^1$ unless $a = b_i$ for each i. By the total ordering on ideals of S, for each i, there are elements $s_i, t_i \in S^1$ such that $b_i = s_iat_i$. Now, for some index $i < n$, $t_is_{i+1} \in S^m - \{0\}$ for every $m > 0$, for otherwise, the product

$$b = (s_1at_1)(s_2at_2) \cdots (s_nat_n) \cdots (s_{2n}at_{2n}) \cdots (s_{3n}at_{3n})$$

involves the power a^n. Similarly, an element t_js_{j+1} has the same property for some index $j \geq 2n$.

If S is also permutative, then there exists k such that S^k is medial. Therefore if $n \geq k$, all the terms between t_is_{i+1} and t_js_{j+1} in the product for b may be commuted, yielding a term a^n, contradicting $b \neq 0$. Thus the second statement in the lemma is proven. By Theorem 4.1.3, every idempotent, permutative semigroup is medial. □

A semigroup is called a left [right] commutative semigroup if it satisfies the identity $xya = yxa$ [$axy = ayx$].

Proposition 4.2.2 *([NJ04]) If S is a left or right commutative nil Δ-semigroup, then it is commutative.*

Proof. We need only consider the identity $abx = bax$. Let $\rho = \{(a, b) \in S \times S : as = bs \text{ for all } s \in S\}$. It is well known that ρ is a congruence on S; from the identity it follows that S/ρ is commutative.

By Theorem 1.3.9, ρ is the Rees ideal congruence modulo the ideal $I = [0]_\rho$, which is the left annihilator of S. Thus if $a \in S$, either $aS = 0$ or $[a]_\rho = \{a\}$.

Now let $a, b \in S, a \neq b$. If $a, b, ab \notin I$, then since S/ρ is commutative, $ab = ba$. If $a, b \in I$, then $ab = ba = 0$.

If $a, b \notin I$ then, since the principal ideals of S are totally ordered, without loss of generality $a = xby$ for some $x, y \in S^1$. Since $a \notin I$, $x, y \notin I$. By the first case above, x, b, y commute. Hence $ab = ba$.

Without loss of generality, the remaining case is where $a \in I, b \notin I$. As above, $a = xby$ for some $x, y \in S^1$. If $y \neq 1$, then $xby = bxy$. Thus we may assume that either $a = bx$ or $a = xb$ for some $x \in S$. If $x \notin I$, then by the above result $bx = xb$ and so $ab = ba$. Thus we may assume $x \in I$. Now we may similarly write $x = bx_1$ or $x = x_1 b$ for some $x_1 \in S$. If $x_1 \notin I$ then, again similarly, $bx_1 = x_1 b$ and so $a = b^2 x_1$ or $a = x_1 b^2$, whence $ab = ba$. If $x_1 \in I$, continue this process by writing $x_1 = bx_2$ or $x_1 = x_2 b$. By induction, either some $x_i \notin I$ and then $ab = ba$, or for all i there exists x_i such that $a = b^{i+1} x_i$ or $a = x_i b^{i+1}$. But S is nil, so it follows that $a = 0$, completing the proof. □

Theorem 4.2.3 *([NJ04]) If S is a medial nil Δ-semigroup, then S is commutative.*

Proof. Again, let ρ be the congruence $\{(a, b) \in S \times S : as = bs \text{ for all } s \in S\}$. From the medial identity it is clear that S/ρ is right commutative. Since it is again a nil Δ-semigroup, it is commutative, by the previous proposition. Let $I_L = [0]_\rho$. Let λ be the dual congruence, so that S/λ is also commutative. Let $I_R = [0]_\lambda$. As in the proof of the proposition, for each $a \in S$, either $[a]_\rho = I_L$ or $[a]_\rho = \{a\}$, and dually.

Since the ideals of S are totally ordered, without loss of generality $I_L \subseteq I_R$. Let $a, b \in S$. If $a, b \notin I_L$, then precisely as in the third and fourth paragraphs of the proof of the previous proposition, $ab = ba$. Otherwise, without loss of generality, $a \in I_L$, so $ab = 0$. But also $a \in I_R$, so $ba = 0$. □

Lemma 4.2.4 *([NJ04]) Let S be a permutative semigroup with a dense ideal R that is a right zero semigroup. If R is non-trivial, then S/R is nilpotent.*

Proof. Suppose S satisfies the identity $x_1 x_2 \cdots x_n = x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)}$ for some $n > 1$, where σ is a non-trivial permutation. Then $\sigma(n) = n$ since, otherwise, if r, s are distinct members of R, substituting $r = x_n$ and $s = x_{\sigma(n)}$ (and substituting arbitrarily for any other variables) yields $r = s$. Let i be least such that $\sigma(j) = j$ for $i \leq j \leq n$. Clearly $i > 2$. Let $r \in R$ and substitute $x_{i-1} = r$. Then $rx_i \cdots x_n = rwx_i \cdots x_n$ for every $r \in R$, where w is a non-empty word in $\{x_1, x_2, \ldots, x_{i-2}\}$. It is easy to see that $\eta = \{(a, b) \in S \times S : (\forall r \in R)\ ra = rb\}$ is a congruence on S such that the restriction $\eta|_R$ of η to R equals id_R. As R is a dense ideal of S, we have $\eta = id_S$. As $(x_i \cdots x_n, wx_i \cdots x_n) \in \eta$, we get that $x_i \cdots x_n = wx_i \cdots x_n$ is an identity satisfied in S. Now by choosing for any one of the variables in w an element of R, it follows that $x_i \cdots x_n \in R$ for all $x_i, \ldots, x_n \in S$. Thus $S^{n-i+1} \in R$; equivalently, $(S/R)^{n-i+1} = \{0\}$. □

The next lemma will be used in the proof of Lemma 4.2.6.

Lemma 4.2.5 *(Lemma 3.1 of [BC80]) No Δ-semigroup can contain an ideal that is itself an ideal extension of a non-trivial right (or left) zero semigroup by a non-trivial nil semigroup that is finite cyclic.*

Lemma 4.2.6 *([NJ04]) No permutative Δ-semigroup can be an ideal extension of a non-trivial right (or left) zero semigroup by a non-trivial nil semigroup.*

Proof. Suppose such a semigroup S exists, with non-trivial right zero ideal R. Let α be a congruence on S such that the restriction of α to R is the identity relation on R. Then $\alpha \cap \varrho_R = id_S$, where ϱ_R denotes the Rees congruence on S defined by the ideal R of S. As S is a Δ-semigroup and $|R| > 1$, we get $\alpha = id_S$. Thus R is a dens ideal of S. By Lemma 4.2.4, S/R is nilpotent. Since S/R is also a Δ-semigroup, it is finite cyclic. Then Lemma 4.2.5 applies. □

Theorem 4.2.7 *([NJ04]) Every permutative, archimedean Δ-semigroup is either (a) simple, whence a group or a left or right zero semigroup, or (b) nil. In any case, every such semigroup is medial.*

Proof. Let S be such a semigroup. If S is simple, then S is a rectangular abelian group by Theorem 4.1.4, and so (a) is satisfied by Corollary 1.3.17.

If S is not simple, then S contains an idempotent element by Theorem 4.1.7 and Theorem 1.3.7. By Theorem 4.1.5, Theorem 1.3.7 and Corollary 1.3.17, S is an ideal extension of a right or left zero semigroup K by a non-trivial nil semigroup. By Lemma 4.2.6, $|K| = 1$, that is, S is a non-trivial nil semigroup. The mediality now follows by Lemma 4.2.1. □

Finally, we may consider the general permutative case. We can formulate the main theorem of this section.

Theorem 4.2.8 *([NJ04]) Every permutative Δ-semigroup is medial.*

Proof. Let S be such a semigroup. The archimedean case is covered by the preceding result. We have seen that the alternative case is when S is a semilattice of two archimedean semigroups S_1 and S_0 with $S_0S_1 \subseteq S_0$. By Theorem 1.3.3, S_1^0 and so S_1 is an archimedean Δ-semigroup. It is clear that S_1 is permutative. Then S_1 is either a group or a two-element right or left zero semigroup (see also Theorem 1.3.12). In all three cases $S^2 \cap S_0 \neq \emptyset$ and $S_1 \subseteq S^2$. As the ideals S_0 and S^2 of S are comparable, we have $S^2 = S$, that is, S is idempotent. By Theorem 4.1.3, S is a medial semigroup. □

4.3 Permutative congruence permutable semigroups

In the previous section we proved that every permutative Δ-semigroup is medial. Using also this fact, in this section we generalize this result. We prove that every permutative congruence permutable semigroup is medial. First we prove the following lemma.

Lemma 4.3.1 *([Nag05]) Every permutative congruence permutable nil semigroup is commutative.*

Proof. Let S be a permutative congruence permutable nil semigroup. By Theorem 1.3.10, S is a Δ-semigroup. By Theorem 4.2.8, every permutative Δ-semigroup is medial. By Theorem 4.2.3, every medial nil Δ-semigroup is commutative. Thus S is a commutative semigroup. □

Theorem 4.3.2 *([Nag05]) Every permutative congruence permutable semigroup is medial or an ideal extension of a rectangular band by a non-trivial commutative nil semigroup.*

Proof. Let S be a permutative congruence permutable semigroup. By Theorem 4.1.2, S is a semilattice of permutative archimedean semigroups. As every homomorphic image of a congruence permutable semigroup is congruence permutable, and a congruence permutable semilattice has at most two elements, we have that S is either archimedean or a semilattice of two permutative archimedean subsemigroups S_1 and S_0 such that $S_0S_1 \subseteq S_0$.

Assume that S is archimedean. If S is simple, then $S^k = S$ for every positive integer k and so, by Theorem 4.1.3, S is a medial semigroup. If S has a proper ideal, then it has no a non-trivial group homomorphic image by Remark 1.2.10. Then, by Theorem 4.1.7, S has an idempotent element and so, by Theorem 4.1.5, S is an ideal extension of a rectangular abelian group K by a non-trivial permutative nil semigroup N. By Theorem 1.2.4 and Lemma 4.3.1, N is commutative. K is the direct product of a rectangular band B and a commutative group G. As G is a homomorphic image of the proper ideal K of S, Remark 1.2.10 implies $|G| = 1$. Thus S is an ideal extension of the rectangular band B by the non-trivial commutative nil semigroup N.

Now assume that S is a semilattice of two archimedean subsemigroups S_1 and S_0 such that $S_0S_1 \subseteq S_0$. By Theorem 1.2.7, S_1 is simple. As $S^2 \cap S_1 \neq \emptyset$ and $S^2 \cap S_0 \neq \emptyset$, we have $S^2 = S$, because S^2 is an ideal of S, the ideals of S form a chain with respect to inclusion, $S^2 \cap S_1$ is an ideal of S_1 and S_1 is simple. Thus $S^k = S$ for every positive integer k. Then, by Theorem 4.1.3, S is a medial semigroup. □

Theorem 4.3.3 *Every permutative congruence permutable semigroup is medial.* □

Proof. A. Deák ([Dea06]) and P.R. Jones ([Jon06]) proved that if a permutative congruence permutable semigroup S is an ideal extension of a rectangular band by a non-trivial commutative nil semigroup, then S is medial. This result and Theorem 4.3.2 together imply the assertion of the theorem. □

Definition 4.3.4 *A semigroup S is called strictly permutative if it satisfies a permutation identity $x_1x_2 \dots x_n = x_{\sigma(1)}x_{\sigma(2)} \dots x_{\sigma(n)}$ in which $\sigma(1) \neq 1$ and $\sigma(n) \neq n$.*

Theorem 4.3.5 *A semigroup is strictly permutative and simple if and only if it is a commutative group.*

Proof. Let S be a simple semigroup. Assume that S satisfies an identity $x_1x_2\dots x_n = x_{\sigma(1)}x_{\sigma(2)}\dots x_{\sigma(n)}$ for a permutation σ of $\{1, 2, \dots, n\}$ with condition $\sigma(1) \neq 1$ and $\sigma(n) \neq n$. By Theorem 4.1.4, S is a direct product of a left zero semigroup L, a commutative group G and a right zero semigroup R. Let $i_0 \in L$ and $j_0 \in R$ be arbitrary fixed elements. For arbitrary $i \in L$ and $j \in R$, let $x_1 = (i_0, e, j)$, $x_n = (i, e, j_0)$ and (if $n > 2$, then) $x_2 = \dots = x_{n-1} = (i, e, j)$, where e is the identity element of G. Then $x_1x_2\cdots x_n = (i_0, e, j_0)$ and $x_{\sigma(1)}x_{\sigma(2)}\cdots x_{\sigma(n)} = (i, e, j)$. From this it follows that $i = i_0$ and $j = j_0$ for every $i \in L$ and $j \in R$. Thus $|L| = |R| = 1$ and so S is isomorphic to the commutative group G. The converse assertion is obvious.□

Theorem 4.3.6 *([Nag05]) Every strictly permutative congruence permutable semigroup is commutative.*

Proof. Let S be a congruence permutable semigroup such that S satisfies an identity $x_1x_2\dots x_n = x_{\sigma(1)}x_{\sigma(2)}\dots x_{\sigma(n)}$ for a permutation σ of $\{1, 2, \dots, n\}$ with condition $\sigma(1) \neq 1$ and $\sigma(n) \neq n$.

First consider the case when S is archimedean. If S is simple, then S is a commutative group by Theorem 4.3.5. Assume that S is not simple. Then S contains an idempotent element by Theorem 1.2.8 and Theorem 4.1.7. Then, by Theorem 4.1.5 and Theorem 4.3.5, S is an ideal extension of an abelian group G by a permutative nil semigroup N. As an ideal extension of a group is a retract extension, G is a homomorphic image of S. By Theorem 1.2.8, $|G| = 1$ and so S is a permutative congruence permutable nil semigroup. Then, by Lemma 4.3.1, N is commutative.

Next, suppose that S is a semilattice of two archimedean subsemigroups S_1 and S_0 such that $S_0S_1 \subseteq S_0$. Then, by Theorem 1.2.7, S_1 is simple. By Theorem 4.3.5, S_1 is an abelian group. Let e be the identity element of S_1. Then, for every element $s \in S$,

$$se = (se^{n-1})e = ese = e(e^{n-1}s) = es.$$

Thus $Se = eS$ is an ideal of S such that $Se \cap S_1 \neq \emptyset, Se \cap S_0 \neq \emptyset$. As the ideals of a congruence permutable semigroups form a chain with respect to the inclusion, $Se = eS = S$ and so e is the identity element of S. Thus $S = S^2$ and so, by Theorem 4.1.3, S is a medial semigroup. As e is the identity element of S, it follows that $ab = eabe = ebae = ba$ for every $a, b \in S$, and so S is commutative. □

Corollary 4.3.7 *([Nag05]) Every strictly permutative Δ-semigroup is commutative.*

Proof. As every Δ-semigroup is congruence permutable, our assertion follows from Theorem 4.3.6. □

Chapter 5

Medial semigroups

In this chapter, which is a continuation of Chapter 4, we deal with the medial semigroups. The chapter contains three sections.

The first section contains results on the semilattice decomposition of medial semigroups and on medial archimedean semigroups. We prove that every medial semigroup is a left and right Putcha semigroup and so a semilattice of medial archimedean semigroups. Moreover, a semigroup is a medial archimedean semigroup containing at least one idempotent element if and only if it is a retract extension of a rectangular abelian group by a medial nil semigroup. We also show that every medial archimedean semigroup without idempotent element has a non-trivial commutative group homomorphic image.

In Chapter 4 it was proved that every permutative Δ-semigroup is medial. In the second section of this chapter we describe the medial Δ-semigroups. W.A. Etterbeek in his PhD dissertation [Ett70] dealt with the medial Δ-semigroups, but the proof of Theorem 3.45 of the dissertation is false and so he gave an incorrect list for medial Δ-semigroups in Theorem 3.49. In Theorem 3.45 it was asserted that if $S = S_0 \cup \{e\}$ is a right commutative Δ-semigroup such that S_0 is a nil semigroup and e is a right identity element of S, then S is necessarily commutative. The Example of my paper [Nag00] shows that this assertion is false. It is easy to see that if S is a semigroup which can be obtained from a zero semigroup $\{0, a\}$ by adjunction an idempotent element e such that e is a right identity element of S and a left annihilator of $\{0, a\}$, then S satisfies the condition of Theorem 3.45 of [Ett70], but S is not commutative. In our paper [NJ04], we revisited the results of Etterbeek. In this section we present the results of [NJ04]. We give a correct list of medial Δ-semigroups. We show that a semigroup S is a medial Δ-semigroup if and only if it satisfies one of the following conditions: (*i*) S is a commutative Δ-semigroup; (*ii*) S is isomorphic to either R or R^0, where R is a two-element right zero semigroup; (*iii*) S is isomorphic to the semigroup $Z = \{0, e, a\}$, obtained by adjoining to a zero semigroup $\{0, a\}$ an idempotent element e that is both a right identity element of Z and a left annihilator of $\{0, a\}$; (*iv*) S is isomorphic to the dual of a semigroup of type (*ii*) or (*iii*).

In Chapter 4 it was proved that every permutative congruence permutable semigroup is medial. In the third section of this chapter we deal with the medial congruence permutable semigroups. In [BC81] B. Bonzini and A. Cherubini defined three kinds of congruence permutable semigroups (first kind, second kind, third kind), and showed that every medial non-archimedean congruence permutable semigroup is one of them. In this section we concentrate our attention on medial congruence permutable semigroups of the first kind. We define the notion of the left (right) reflection of semigroups, and show that the medial congruence permutable semigroups of the first kind can be obtained from the non-archimedean commutative congruence permutable semigroups applying both of the left reflection and the right reflection.

5.1 Semilattice decopmosition of medial semigroups

Definition 5.1.1 *A semigroup is called a medial semigroup if it satisfies the identity $xaby = xbay$.*

Theorem 5.1.2 *([Nag01, Theorem 9.2]) Every medial semigroup is a left and right Putcha semigroup.*

Proof. Let S be a medial semigroup and $a, b \in S$ be arbitrary elements with $b \in aS^1$, that is, $b = ax$ for some $x \in S^1$. Then

$$b^2 = (ax)^2 = a^2x^2,$$

that is,

$$b^2 \in a^2S^1.$$

Hence S is a left Putcha semigroup. We can prove, in a similar way, that S is a right Putcha semigroup. □

Theorem 5.1.3 *([Nag01], [Chr69]) Every medial semigroup is a semilattice of medial archimedean semigroups.*

Proof. By Theorem 5.1.2, Lemma 1.1.16 and Theorem 1.1.11, our assertion is obvious. □

The next theorem is a consequence of the proof of Theorem 4.1 of [Chr69].

Theorem 5.1.4 *([Chr69]) A semigroup is medial and simple if and only if it is a rectangular abelian group.* □

In [Chr69], J.L. Chrislock proved that a medial semigroup is archimedean and contains at least one idempotent element if and only if it is an ideal extension of a rectangular abelian group by a nil semigroup. In the next two theorems we prove a little bit more.

Theorem 5.1.5 *If a semigroup S is a retract extension of a medial semigroup by a medial semigroup with a zero, then S is medial.*

Proof. Let S be a semigroup which is a retract extension of a medial semigroup K by a medial semigroup Q with a zero 0. It is clear that $S = K \cup Q^*$, where $Q^* = Q - \{0\}$. Let φ be retract homomorphism of S onto K. Let $a, x, y, b \in S$ be arbitrary elements.

If $axyb, ayxb \notin K$, then $a, x, y, b \in Q^*$ and $axyb = ayxb$ in Q. Thus $axyb = ayxb$ in S.

Next consider the case when one of the products $axyb$ and $ayxb$ is in K. We can suppose $axyb \in K$. The investigation of the other case is similar.

If $a, x, y, b \notin K$. Then $axyb = 0$ in Q and so $ayxb = 0$ in Q. Thus $ayxb \in K$ and so

$$axyb = \varphi(axyb) = \varphi(a)\varphi(x)\varphi(y)\varphi(b) = \varphi(a)\varphi(y)\varphi(x)\varphi(b) = \varphi(ayxb) = ayxb.$$

If one of a, x, y, b is in K, then $ayxb \in K$ because K is an ideal of S. Hence (as in the previous case) $axyb = ayxb$ is satisfied in S. □

Theorem 5.1.6 *([Nag01]) A semigroup is a medial archimedean semigroup containing at least one idempotent element if and only if it is a retract extension of a rectangular abelian group by a medial nil semigroup.* □

Proof. Let S be a medial archimedean semigroup containing at least one idempotent element. Then, by Theorem 5.1.2, S is a left and a right Putcha semigroup. By Theorem 1.1.19 and Theorem 5.1.4, S is a retract extension of a rectangular abelian group by a nil semigroup N. It is clear that N is medial.

Conversely, assume that a semigroup S is a retract extension of a rectangular abelian group K by a medial nil semigroup N. As K is completely simple, Theorem 1.1.19 implies that S is an archimedean semigroup containing at least one idempotent. As K and N are medial semigroups, Theorem 5.1.5 implies that S is a medial semigroup. □

Theorem 5.1.7 *([Nag01, Theorem 9.11]) Every medial archimedean semigroup without idempotent element has a non-trivial group homomorphic image.*

Proof. As every medial semigroup is weakly exponential, our assertion follows from Theorem 2.1.8. □

5.2 Medial Δ-semigroups

In Chapter 4 we proved that every permutative Δ-semigroup is medial. In this section we describe the medial Δ-semigroups. First we prove a theorem about the medial Δ-semigroups which is deduced from the next theorem (presented by Trotter in [Tro76]).

Theorem 5.2.1 *([Tro76]) A semigroup S is an exponential Δ-semigroup if and only if one of the following satisfied.*

(i) $S \cong G$ *or* G^0*, where* G *is a non-trivial subgroup of a quasicyclic* p*-group.*

(ii) $S \cong F$*, where* F *is a two-element semilattice.*

(iii) $S \cong R$ *or* R^0 *or* R^1*, where* R *is a two-element right zero semigroup.*

(iv) $S \cong L$ *or* L^0 *or* L^1*, where* L *is a two-element left zero semigroup.*

(v) S *is an exponential nil semigroup whose principal ideals are chain ordered by inclusion.*

(vi) S *is an exponential T1 or a T2R or a T2L semigroup.*

Our first result on medial Δ-semigroups is the following (which was published in [Nag01]).

Theorem 5.2.2 *([NJ04]) A semigroup S is a medial Δ-semigroup if and only if it satisfies one of the following conditions.*

(i) S *is isomorphic to* G *or* G^0*, where* G *is a non-trivial subgroup of a quasicyclic* p*-group (*p *is a prime).*

(ii) S *is a two-element semilattice.*

(iii) S *is isomorphic to either* R *or* R^0*, where* R *is a two-element right zero semigroup.*

(iv) S *is isomorphic to either* L *or* L^0*, where* L *is a two-element left zero semigroup.*

(v) S *is a medial nil* Δ*-semigroup (that is, the principal ideals form a chain with respect to inclusion).*

(vi) S *is a medial T1 semigroup (if* S *has an identity, then it is commutative).*

Proof. Let S be a medial Δ-semigroup. Then S is an exponential Δ-semigroup and so it is isomorphic one of the semigroups listed in Theorem 5.2.1.

It is obvious that a medial monoid is commutative. Thus the cases $S \cong R^1$ (in (*iii*) of Theorem 5.2.1) and $S \cong L^1$ (in (*iv*) of Theorem 5.2.1) are impossible. Moreover, if S is a T1 semigroup (see (*vi*) of Theorem 5.2.1) with an identity element, then S is commutative)

The proof will be complete if we show that the case is impossible when S is a T2RL semigroup or a T2R semigroup. Assume, in an indirect way, that S is a medial $T2L$ semigroup. Then it is a semilattice of a two-element left zero semigroup $L = \{u, v\}$ and a non-trivial nil semigroup S_0. Using also the fact that u and v are idempotent elements, it is easy to verify that

$$\tau_u = \{(a, b) \in S \times S : \; ua = ub\}$$

and

$$\tau_v = \{(a,b) \in S \times S : \ va = vb\}$$

are congruences of S such that $(u,v) \in \tau_u$ and $(u,v) \in \tau_v$. As S is a Δ-semigroup, we have $\rho_{S_0} \in \tau_u$ and $\rho_{S_0} \in \tau_v$, where ρ_{S_0} denotes the Rees congruence of S modulo S_0. Thus $(a,0) \in \tau_u$ and $(a,0) \in \tau_v$ for every $a \in S_0$, that is, $ua = va = 0$ for every $a \in S_0$. Let

$$I = \{a \in S : \ au = av\}.$$

It is easy to see that I is a left ideal of S. We show that I is also a right ideal of S. Let $a \in I$ and $s \in S$ be arbitrary elements. Then

$$asu = asuu = ausu = avsu = asvu = asv$$

and so $as \in I$. Hence I is an ideal of S. It is clear that $u, v \in I$. As S is a Δ-semigroup, and $u, v \notin S_0$, we have $I = S$. Thus $au = av$ for every $a \in S_0$. Let β be the following equivalence on S.

$$\beta = \{(a,b) \in S \times S : \ a = b \text{ or } a, b \in S_1\}.$$

As $ua = va$ and $au = av$ for every $a \in S_0$, we have that β is a congruence on S. It is clear that $\beta \cap \rho_{S_0} = id_S$, where ρ_{S_0} is the Rees congruence on S determined by the ideal S_0 of S. As S is a Δ-semigroup, either $\beta \subseteq \rho_{S_0}$ or $\rho_{S_0} \subseteq \beta$ and so either $\beta = id_S$ or $\rho_{S_0} = id_S$. As $u \neq v$, we would have only $\rho_{S_0} = id_S$. Hence $|S_0| = 1$ which is a contradiction. Thus S is not a T2L semigroup. Dually, S is not a T2R semigroup. Thus the theorem is proved. □

In the next, we shall refine Theorem 5.2.2. We shall first show that every medial, nil Δ-semigroup is commutative; and then that every medial T1 semigroup is either commutative or is isomorphic to the semigroup Z or its dual, where $Z = \{0, e, a\}$, obtained by adjoining to a zero semigroup $\{0, a\}$ an idempotent element e that is both a right identity element of Z and a left annihilator of $\{0, a\}$. The proof of Theorem 5.2.5 is then complete.

We now turn to T1 semigroups.

Lemma 5.2.3 *[Tro76, Lemma 3.3], [Nag01, Theorem 1.58] Let $S = N \cup \{e\}$ be any T1 semigroup. Then every ideal of N is also an ideal of S (and so N is also a Δ-semigroup).* □

Theorem 5.2.4 *([NJ04]) Let $S = N \cup \{e\}$ be a medial T1 semigroup. Then N is a commutative Δ-semigroup and S satisfies one of the following conditions.*

(1) e acts as an identity element for N and S itself is commutative.

(2) e acts as a right identity and a left annihilator for N and S is isomorphic to the semigroup $Z = \{0, e, a\}$, obtained by adjoining to a zero semigroup $\{0, a\}$ an idempotent element e that is both a right identity element of Z and a left annihilator of $\{0, a\}$.

(3) the dual of the previous case.

Proof. That N is commutative is immediate from Lemma 5.2.3 and Theorem 4.2.3.

Now suppose that S is any T1 semigroup for which N is commutative. We show first that for any $a \in N$, either $ea = a$ or $ea = 0$. (The dual statement obviously also holds.) Since N^1aN^1 is an ideal of N, then Lemma 5.2.3 implies that it is also an ideal of S, whence it contains ea. Hence, if $ea \neq a$, then $ea = at$ for some $t \in N$. Then $ea = eat = eat^n$ for each n and, since $t \in N$, $ea = 0$.

Next suppose that $ea = a$ for some nonzero $a \in N$. Let $b \in N$. Either $b = ax$ or $a = bx$, for some $x \in S^1$. In the former case, $eb = eax = ax = b$; in the latter case, suppose $eb = 0$: then $ea = ebx = 0$, a contradiction, so that again $eb = b$. Hence e is either a left identity for S or a left annihilator for N. Clearly the dual statement also holds.

Notice, however, that if N is nonzero, then e cannot be both a left and a right annihilator for N. For in that event, given $a \in N - \{0\}$, $S^1aS^1 \subset S^1eS^1$, so $a = set$ for some $s, t \in S^1$. Both s and t cannot belong to N, for then $se = et = 0$. But otherwise, either $a = ea$ or $a = ae$, contradicting the assumption.

Thus e is either an identity for S, or is a right identity for S and a left annihilator for N, or is a left identity for S and a right annihilator for N. In the second of those three cases, let $a, b \in N$. Then $ab = (ae)b = a(eb) = 0$, that is, N is a null semigroup. But every subset of N that contains 0 is an ideal, so $|N| \leq 2$. When $N = \{0\}$, e actually acts as an identity and so S falls under (1). Otherwise, $N = \{a, 0\}$, say, where $ae = a, ee = e$ and all other products are 0. Clearly, the third case is dual. □

Finally, we can formulate the main theorem on medial Δ-semigroups.

Theorem 5.2.5 *A semigroup S is a medial Δ-semigroup if and only if it satisfies one of the following conditions.*

(i) S is a commutative Δ-semigroup.

(ii) S is isomorphic to either R or R^0, where R is a two-element right zero semigroup.

(iii) S is isomorphic to the semigroup $Z = \{0, e, a\}$, obtained by adjoining to a zero semigroup $\{0, a\}$ an idempotent element e that is both a right identity element of Z and a left annihilator of $\{0, a\}$.

(iv) S is isomorphic to the dual of a semigroup of type (ii) or (iii). □

Proof. By Theorem 5.2.2 and Theorem 5.2.4 it is obvious. □

5.3 Medial congruence permutable semigroups

In the previous chapter it was proved that every permutative congruence permutable semigroup is medial. In this section we deal with the medial congruence permutable semigroups.

The medial congruence permutable semigroups are examined in [BC81]. Using the terminology of [BC81], a semigroup S is called a semigroup of type a if it is a semilattice of a nil semigroup S_0 and a rectangular group $S_1 = L \times G \times R$ with $|L| \leq 2$, $|R| \leq 2$ (L is a left zero semigroup, G is a group, R is a right zero semigroup). A semigroup S of type a is called of

(1) the first kind if $a \in S_1 a S_1$ for every $a \in S$,

(2) the second kind if $a \in S_1 a$ and $aS_1 = \{0\}$ for every $a \in S_0$,

(3) the third kind if $a \in aS_1$ and $S_1 a = \{0\}$ for every $a \in S_0$.

By Corollary 1.2 and Theorem 3.4 of [BC81], semigroup S is a medial congruence permutable semigroup if and only if it satisfies one of the following conditions.

(1) S is a commutative nil semigroup whose principal ideals form a chain with respect to inclusion.

(2) S is a rectangular abelian group $L \times G \times R$, with $|L| \leq 2$, $|R| \leq 2$ (L is a left zero semigroup, G is an abelian group, R is a right zero semigroup).

(3) S is a medial congruence permutable semigroup of the first or the second or the third kind.

In [Bon83] a construction is given for medial congruence permutable semigroups of the second [the third] kind. In [Nag08], we dealt with medial congruence permutable semigroups of the first kind. We showed that they can be obtained from the commutative non-archimedean congruence permutable semigroups. In this section of the dissertation we present the results of [Nag08]. First of all we note that if S is a non-archimedean commutative congruence permutable semigroup, then S is of type a, because it is a semilattice of a commutative group G and a commutative nil semigroup. Moreover, the identity element of G is the identity element of S, and so S is of the first kind.

Let S be a medial congruence permutable semigroup of the first kind. Then S is a semilattice of a nil semigroup S_0 and a rectangular abelian group $S_1 = L \times G \times R$ with $|L| \leq 2$, $|R| \leq 2$ (L is a left zero semigroup, G is an abelian group, R is a right zero semigroup). It is obvious that S_1 is a rectangular band $L \times R$ of disjoint subgroups $G_{ij} = \{i\} \times G \times \{j\}$ ($i \in L$, $j \in R$), and the idempotent elements of S_1 are the identity elements $e_{ij} = (i, e, j)$ of G_{ij} (here e denotes the identity element of G).

Introduce the following notation: for an element t of a non-empty set T containing at most two elements, let $\tilde{t} = t$ if $|T| = 1$, and let $\tilde{t} \in T - \{t\}$ if $|T| = 2$.

Lemma 5.3.1 *([Nag08]) Let S be a medial congruence permutable semigroup of the first kind. Then, for every $a \in S$, $i \in L$ and $j \in R$, we have*

(1) $e_{ij}a = e_{i\tilde{j}}a$,

(2) $ae_{ij} = ae_{\tilde{i}j}$.

Proof. As S is a medial semigroup, for every $a \in S$, $i \in L$ and $j \in R$, we have

$$e_{ij}a = e_{ij}e_{i\tilde{j}}e_{ij}a = e_{ij}e_{ij}e_{i\tilde{j}}a = e_{i\tilde{j}}a$$

and

$$ae_{ij} = ae_{ij}e_{\tilde{i}j}e_{ij} = ae_{\tilde{i}j}e_{ij}e_{ij} = ae_{\tilde{i}j}.$$

□

Introduce the following notations. For arbitrary $i \in L$ and $j \in R$, let

$$A_i = e_{ij}S = e_{i\tilde{j}}S$$

and

$$B_j = Se_{ij} = Se_{\tilde{i}j}.$$

It is clear that

$$A_i = G_{ij} \cup G_{i\tilde{j}} \cup e_{ij}S_0$$

and

$$B_j = G_{ij} \cup G_{\tilde{i}j} \cup S_0e_{ij}.$$

A semigroup is said to be *left* [*right*] *commutative* if it satisfies the identity $abc = bac$ [$abc = acb$].

Lemma 5.3.2 *([Nag08]) Let S be a medial congruence permutable semigroup of the first kind. Then A_i ($i \in L$) and B_j ($j \in R$) are left and right commutative subsemigroups of S, respectively.*

Proof. It is clear that e_{ij} is a left identity elements of A_i. Then, for arbitrary elements $a, x, y \in A_i$,

$$xya = e_{ij}xya = e_{ij}yxa = yxa.$$

Hence A_i is left commutative. The proof of the assertion for B_j is similar. □

Lemma 5.3.3 *([Nag08]) Let S be a medial congruence permutable semigroup of the first kind. Then*

$$S = A_i \cup A_{\tilde{i}} = B_j \cup B_{\tilde{j}} \ (i \in L, j \in R).$$

Moreover, $A_i \cap A_{\tilde{i}}$ and $B_j \cap B_{\tilde{j}}$ ($i \in L, j \in R$) are ideals of S.

Proof. Let S be a medial congruence permutable semigroup of the first kind. Then, for every $a \in S$, there is an element $e_{ij} \in E(S_1)$ such that

$$a = e_{ij}a \in A_i.$$

Thus

$$S = A_i \cup A_{\tilde{i}} \quad (i \in L).$$

Similarly,

$$S = B_j \cup B_{\tilde{j}} \quad (j \in R).$$

It is clear that $A_i \cap A_{\tilde{i}} \neq \emptyset$ is a right ideal of S. Let $s \in S$, $a \in A_i \cap A_{\tilde{i}}$ be arbitrary elements. Then

$$e_{t,k}a = a$$

for every $t \in L, k \in R$. Assume $s \in A_i$ (and so $s = e_{ij}s$ for every $j \in R$). As A_i is a subsemigroup of S, $sa \in A_i$. As S is of the first kind,

$$a = at$$

for an element $t \in S_1$. Thus, for arbitrary $j \in R$,

$$e_{\tilde{i}j}sa = e_{\tilde{i}j}sat = e_{\tilde{j}i}ast = ast = e_{ij}ast = e_{ij}sat = sa,$$

that is,

$$sa \in A_{\tilde{i}}.$$

Thus

$$sa \in A_i \cap A_{\tilde{i}}.$$

Hence $A_i \cap A_{\tilde{i}}$ is an ideal of A_i. We can similarly prove that $A_i \cap A_{\tilde{i}}$ is an ideal of $A_{\tilde{i}}$. Hence $A_i \cap A_{\tilde{i}}$ is an ideal of S. The proof of the assertion that $B_j \cap B_{\tilde{j}}$ is an ideal of S is similar. $\square$

Lemma 5.3.4 *([Nag08]) If f is an idempotent element of a medial semigroup S, then*

$$\lambda_f = \{(x, y) \in S \times S : \ fx = fy\}$$

and

$$\rho_f = \{(x, y) \in S \times S : \ xf = yf\}$$

are congruences on S.

Proof. It is clear that λ_f is a right congruence. Let x, y, s be arbitrary elements of S such that $(x, y) \in \lambda_f$ Then

$$fsx = ffsx = fsfx = fsfy = ffsy = fsy$$

and so

$$(sx, sy) \in \lambda_f.$$

Hence λ_f is a congruence on S. The proof is similar for ρ_f. $\square$

Lemma 5.3.5 *([Nag08]) Let S be a medial congruence permutable semigroup of the first kind. Then, for every $i \in L$ and $j \in R$,*

(1) $\lambda_{e_{i\tilde{j}}} = \lambda_{e_{ij}} = \lambda_{e_{\tilde{i}j}} = \lambda_{e_{\tilde{i}\tilde{j}}}$,

(2) $\rho_{e_{\tilde{i}j}} = \rho_{e_{ij}} = \rho_{e_{i\tilde{j}}} = \rho_{e_{\tilde{i}\tilde{j}}}$.

Proof. By Lemma 5.3.1,

$$\lambda_{e_{i\tilde{j}}} = \lambda_{e_{ij}}$$

and

$$\lambda_{e_{\tilde{i}j}} = \lambda_{e_{\tilde{i}\tilde{j}}}.$$

We show that $\lambda_{e_{ij}} = \lambda_{e_{\tilde{i}j}}$. Assume $(a, b) \in \lambda_{e_{ij}}$ for some $a, b \in S$. Then

$$e_{ij}a = e_{ij}b$$

and so

$$e_{\tilde{i}j}a = e_{\tilde{i}j}e_{ij}a = e_{\tilde{i}j}e_{ij}b = e_{\tilde{i}j}b.$$

Then

$$(a, b) \in \lambda_{e_{\tilde{i}j}}.$$

Thus

$$\lambda_{e_{ij}} \subseteq \lambda_{e_{\tilde{i}j}}.$$

Similarly,

$$\lambda_{e_{\tilde{i}j}} \subseteq \lambda_{e_{ij}}.$$

Thus

$$\lambda_{e_{ij}} = \lambda_{e_{\tilde{i}j}}$$

and so (1) is satisfied. The proof of (2) is similar. □

Introduce the following notations: let

$$\rho = \rho_{e_{ij}} \text{ and } \lambda = \lambda_{e_{ij}}$$

for some (for all) $i \in L$ and $j \in R$.

Lemma 5.3.6 *([Nag08]) Let S be a medial congruence permutable semigroup of the first kind. Then, for every $i \in L$ and $j \in R$, $A_i \cong S/\lambda$ and $B_j \cong S/\rho$.*

Proof. Let $[a]_\lambda$ denote the λ-class of S containing the element a of S. We show that $[a]_\lambda = (E(S_1))a$. Assume $(x, y) \in \lambda$ for some $x, y \in A_i$. As e_{ij} is a left identity element of A_i, we have

$$x = e_{ij}x = e_{ij}y = y.$$

Thus

$$\lambda|A_i = id_{A_i},$$

where $\lambda|A_i$ is the restriction of λ to A_i and id_{A_i} is the identity relation of A_i. Let $a \in S$ be an arbitrary element. Then, by Lemma 5.3.3,

$$S = A_i \cup A_{\tilde{i}},$$

and so there is an element $i \in L$ such that $a \in A_i$. As

$$e_{ij}a = e_{ij}e_{\tilde{i}j}a \quad (j \in R),$$

we have

$$(a, e_{\tilde{i}j}a) \in \lambda.$$

Thus

$$[a]_\lambda = \{a, e_{\tilde{i}j}a\}.$$

Since

$$a = e_{ij}a = e_{i\tilde{j}}a$$

and

$$e_{\tilde{i}\tilde{j}}a = e_{\tilde{i}\tilde{j}}e_{\tilde{i}j}e_{\tilde{i}\tilde{j}}a = e_{\tilde{i}\tilde{j}}e_{\tilde{i}\tilde{j}}e_{\tilde{i}j}a = e_{\tilde{i}j}a,$$

we get

$$[a]_\lambda = \{a, e_{\tilde{i}j}a\} = (E(S_1))a.$$

This result implies that

$$|A_i \cap [a]_\lambda| = 1$$

for every $a \in S$. Let Φ_i denote the mapping of S/λ to A_i defined by

$$\Phi_i : [a]_\lambda \ \mapsto A_i \cap [a]_\lambda.$$

Then Φ_i is bijective. As

$$(A_i \cap [a]_\lambda)(A_i \cap [b]_\lambda) \in A_i \cap [ab]_\lambda,$$

we get

$$\Phi_i(a)\Phi_i(b) = (A_i \cap [a]_\lambda)(A_i \cap [b]_\lambda) = A_i \cap [ab]_\lambda = \Phi_i(ab)$$

which means that Φ_i is a homomorphism. Thus Φ_i is an isomorphism of S/λ onto A_i. The proof of $B_j \cong S/\rho$ is similar. □

Corollary 5.3.7 *([Nag08]) Let S be a medial congruence permutable semigroup of the first kind. Then, for every $i \in L$ and $j \in R$, $\phi_i :\ a \mapsto a' = e_{\tilde{i}j}a$ $(a \in A_i)$ and $\psi_j :\ b \mapsto b' = be_{i\tilde{j}}$ $(b \in B_j)$ are isomorphisms of A_i and B_j onto $A_{\tilde{i}}$ and $B_{\tilde{j}}$, respectively.*

Proof. If S is a medial semigroup, then

$$eaeb = eab$$

and

$$aebe = abe$$

for every $a, b \in S$ and every idempotent element e of S. Thus ϕ_i and ψ_j are homomorphisms. ϕ_i and ψ_j map S onto $A_{\tilde{i}}$ and $B_{\tilde{j}}$, respectively. Moreover, $ker\phi_i = \lambda$ and $ker\psi_j = \rho$. Thus, by the proof of the previous lemma, ϕ_i and ψ_j are isomorphisms of A_i and B_j onto $A_{\tilde{i}}$ and $B_{\tilde{j}}$, respectively. □

Construction Let S be a semigroup and I be an ideal of S. Let $\phi : s \mapsto s'$ be an isomorphism of S onto a semigroup $(S'; +)$ such that $S \cap S' = I$ and ϕ leaves the elements of I fixed. (We note that, for every $a, b \in I$, $a + b = a' + b' = (ab)' = ab$.) On the set $S'' = S \cup S'$ we define an operation $*$ as follows. Let $*$ be an extension of both of the operations of S and S'. For arbitrary $x \in S$ and $y' \in S'$, let $x * y' = xy$ and $y' * x = (yx)'$. The groupoid $(S''; *)$ will be called a *left reflection* of S (with respect to I) and will be denoted by $(S; I, *)_l$. The *right reflection* of S is the dual of the left reflection, which will be denoted by $(S; I, *)_r$. More precisely, the operation $*$ in a right reflection of S is defined by $x * y' = (xy)'$ and $y' * x = yx$. If $I \neq S$, then the left and right reflection of S will be called proper.

Introduce the following notation. Let S'' be a left or right reflection of a semigroup S. Then, for an element $s \in S$, let s'' denote s or s'.

Lemma 5.3.8 *([Nag08]) The left reflection [right reflection] of any semigroup is a semigroup.*

Proof. Let $S'' = (S; I, *)_l$ be a left reflection of a semigroup S. Let $a'', b'' \in S''$ $(a, b \in S)$ be arbitrary elements. Then, for arbitrary $c \in S$,

$$c * (a'' * b'') = c * (ab)'' = c(ab) = (ca)b = (ca) * b'' = (c * a'') * b''.$$

and

$$c' * (a'' * b'') = c' * (ab)'' = (c(ab))' = ((ca)b)' = (ca)' * b'' = (c' * a'') * b''.$$

Thus the operation $*$ is associative on S''. The proof of the dual is similar. □

Lemma 5.3.9 *([Nag08]) If $x_1, \dots, x_n$ are arbitrary elements of a semigroup S then, in a left [right] reflection of S, we have*

$$x_1 * x_2'' * \cdots * x_n'' = x_1 x_2 \dots x_n \text{ and } x_1' * x_2'' \cdots * x_n'' = (x_1 x_2 \dots x_n)'$$

$$[x_1'' * \cdots * x_{n-1}'' * x_n = x_1 \dots x_{n-1} x_n \text{ and } x_1'' * \cdots * x_{n-1}'' * x_n' = (x_1 \dots x_{n-1} x_n)'].$$

Proof. It is obvious. □

Lemma 5.3.10 *([Nag08]) A left [right] reflection of a commutative semigroup is right [left] commutative.*

Proof. Let $S'' = (S; I, *)_l$ be a left reflection of a commutative semigroup S. Since S and S' are commutative semigroups and, for every $x, y, z \in S$,

$$x * (y * z') = x * (yz) = x * (zy) = x * (zy)' = x * (z' * y),$$

$$x' * (y * z') = x' * (yz) = x' * (zy) = (x(zy))' = x' * (zy)' = x *' (z' * y),$$

then the semigroup S'' is right commutative. The proof of the dual assertion is similar □

Lemma 5.3.11 *([Nag08]) A left reflection $S'' = (S; I, *)_l$ [a right reflection $S'' = (S; I, *)_r$] of a non-archimedean commutative congruence permutable semigroup S is a right [left] commutative congruence permutable semigroup of the first kind. If S'' is a proper reflection of S, then S'' is not left [right] commutative.*

Proof. Let $S'' = (S; I, *)_l$ be a left reflection of a non-archimedean commutative congruence permutable semigroup $S = G \cup S_0$. By Lemma 5.3.10, S'' is right commutative. If S'' is a proper reflection of S (that is, $I \neq S$), then $I \subseteq S_0$, because the ideals of a congruence permutable semigroup form a chain with respect to inclusion (see Theorem 1.2.2). This implies that $e \neq e'$, where e is the identity element of G. Then

$$e * e' * e' = e \neq e' = e' * e * e'$$

and so S'' is not left commutative. To show that S'' is congruence permutable, we can suppose that $I \neq S$ and so $I \subseteq S_0$. It is easy to see that S'' is a semilattice of the left abelian group $L = G \cup G'$ and the nil semigroup $S''_0 = S_0 \cup S'_0$. Thus S'' is a semigroup of type a. Let $a \in S$ be an arbitrary element. Then

$$S''_1 * a'' * S''_1 = (G \cup G') * a'' * (G \cup G') = GaG \cup (GaG)'.$$

As S is of the first kind, $a \in GaG$ and so

$$a'' \in S''_1 * a'' * S''_1.$$

Thus S'' is a semigroup of the first kind. As $S = G \cup S_0$ is a medial semigroup of the first kind, [BC81, Lemma 3.3] implies that

$$\rho = \{(a, b) \in S \times S : \; GaG = GbG\}$$

is the least congruence of S which has G as a class, and moreover, for every $a \in S$,

$$[a]_\rho = GaG.$$

As S is congruence permutable, [BC81, Theorem 3.5] implies that S/ρ is a Δ-semigroup. As S'' is a medial semigroup of the first kind, [BC81, Lemma 3.3] implies that

$$\rho'' = \{(a'', b'') \in S'' \times S'' : \; S''_1 * a * S''_1 = S''_1 * b * S''_1\}$$

is the least congruence of S'' which has S''_1 as a class, and moreover, for every $a \in S$,

$$[a'']_{\rho''} = S''_1 * a'' * S''_1.$$

As $S''_1 * a * S''_1 = GaG \cup (GaG)'$, we have

$$S \cap [a]_{\rho''} = [a]_\rho \cup ([a]_\rho)'.$$

Moreover,

$$S'' = \cup_{a \in S}[a]_{\rho''}.$$

From these results it follows that the mapping φ of S''/ρ'' into S/ρ defined by

$$\varphi : \; [a'']_{\rho''} \mapsto [a]_\rho$$

is an isomorphism of S''/ρ'' onto S/ρ. As S/ρ is a Δ-semigroup, [BC81, Theorem 3.5] implies that S'' is congruence permutable. □

Theorem 5.3.12 *([Nag08]) A semigroup is a right [left] commutative congruence permutable semigroup of the first kind if and only if it is isomorphic to a left [right] reflection of a non-archimedean commutative congruence permutable semigroup.*

Proof. By Lemma 5.3.10, a left reflection of a non-archimedean commutative congruence permutable semigroup is a right commutative congruence permutable semigroup of the first kind.

Conversely, assume that F is a right commutative congruence permutable semigroup of first kind. Then $F = S_1 \cup S''_0$ (S_1 is a left abelian group, S''_0 is a commutative nil semigroup).

If S_1 is a group, then the identity element of G is an identity element of F (because F is of the first kind) and so F is commutative. In this case F is isomorphic to the left reflection $(F; F, *)_l$ of F.

Assume that S_1 is not a group. Then it is a left zero semigroup of two disjoint isomorphic subgroups G_i and $G_{\bar{i}}$ ($i \in L$). Let e_i denote the identity element of G_i. By Lemma 5.3.1 and Lemma 5.3.3, $Fe_i = Fe_{\bar{i}} = F$ and so e_i is an identity element of $A_i = e_iF$ ($i \in L$). Thus $A_i = G_i \cup e_iS_0$ is a non-archimedean commutative subsemigroup of F. As S is of the first kind, Lemma 5.3.3 implies that

$$F = A_i \cup A_{\bar{i}}.$$

By Corollary 1,

$$\phi_i : \; a \mapsto e_{\bar{i}}a \; (a \in A_i)$$

is an isomorphism of A_i onto $A_{\bar{i}}$ which leaves the elements of $I = A_i \cap A_{\bar{i}}$ fixed. By Lemma 5.3.3, I is an ideal of F. If a' denotes $\psi_i(a)$, then, for arbitrary $a, b \in A_i$, we have

$$ab' = ae_{\bar{i}}b = ab$$

and

$$b'a = e_{\bar{i}}be_{\bar{i}}a = e_{\bar{i}}(ba) = (ba)'.$$

Thus F is a left reflection of the non-archimedean commutative congruence permutable semigroup A_i. The proof of the dual assertion is similar. $\square$

Lemma 5.3.13 *([Nag08]) A left [right] reflection of a left [right] commutative semigroup S is medial.*

Proof. Let $S'' = (S; I, *)_l$ be a left reflection of a left commutative congruence permutable semigroup $S = S_1 \cup S_0$. By Lemma 5.3.8, S'' is a semigroup. From Lemma 5.3.9, it follows that, for every $a, b, x, y \in S$,

$$a * (x'' * y'') * b'' = a(xy)b = a(yx)b = a * (y'' * x'') * b''$$

and

$$a' * (x'' * y'') * b'' = (a(xy)b)' = (a(yx)b)' = a' * (y'' * x'') * b''.$$

Thus S'' is a medial semigroup. $\square$

Lemma 5.3.14 *([Nag08]) A left reflection $S'' = (S; I, *)_l$ [a right reflection $S'' = (S; I, *)_r$] of a left [right] commutative congruence permutable semigroup S of the first kind is a medial congruence permutable semigroup of the first kind. If S'' is a proper reflection of S, then S'' is not left [right] commutative.*

Proof. Let $S'' = (S; I, *)_l$ be a left reflection of a left commutative congruence permutable semigroup $S = S_1 \cup S_0$ of the first kind. By Lemma 5.3.13, S'' is a medial semigroup. If S'' is a proper reflection of S (that is, $I \neq S$), then $I \subseteq S_0$ (see Theorem 1.2.2) and $e \neq e'$ for an idempotent element of S_1. Then

$$e * e' * e = e^3 = e \neq e' = (e^3)' = e' * e * e,$$

which shows that S'' is not left commutative. To show that S'' is congruence permutable, we can suppose that $I \neq S$. As S is congruence permutable, $I \subseteq S_0$. Thus S'' is a semigroup of type a. Let $a'' \in S''$ be an arbitrary element (recall that, for $a \in S$, a'' denotes a or a'). Then

$$S_1''a''S_1'' = (S_1 \cup S_1') * a'' * S_1'' = S_1 * a'' * S_1'' \cup S_1' * a'' * S_1'' = (S_1aS_1) \cup (S_1aS_1)'.$$

As $a \in S_1aS_1$, we get $a' \in (S_1aS_1)'$ and so

$$a'' \in S_1'' * a'' * S_1''.$$

Thus S'' is a semigroup of the first kind. By [BC81, Lemma 3.3],

$$\rho = \{(a, b) \in S \times S : \ S_1aS_1 = S_1bS_1\}$$

is the least congruence of S which has S_1 as a class, and moreover, for every $a \in S$,

$$[a]_\rho = S_1aS_1.$$

As S is congruence permutable, [BC81, Theorem 3.5] implies that S/ρ is a Δ-semigroup. As S'' is a medial semigroup of the first kind, [BC81, Lemma 3.3] implies that

$$\rho'' = \{(a'', b'') \in S'' \times S'' : \ S_1'' * a'' * S_1'' = S_1'' * b'' * S_1''\}$$

is the least congruence of S'' which has S_1'' as a class, and moreover, for every $a'' \in S''$,

$$[a'']_{\rho''} = S_1'' * a'' * S_1'' = (S_1 a S_1) \cup (S_1 a S_1)' = [a]_\rho \cup ([a]_\rho)'.$$

Thus $[a]_{\rho''} = [a']_{\rho''}$ for every $a \in S$ and so

$$F = \cup_{a \in S} [a]_{\rho''}.$$

Thus the mapping φ of S''/ρ'' into S/ρ defined by

$$\varphi : \ [a]_{\rho''} \mapsto [a]_\rho$$

is an isomorphism of S''/ρ'' onto S/ρ. Thus S'' is a medial semigroup of the first kind such that S''/ρ'' is a Δ-semigroup. Then, by [BC81, Theorem 3.5], S'' is congruence permutable. The proof of the dual assertion is similar. $\square$

Theorem 5.3.15 *([Nag08]) A semigroup is a medial congruence permutable semigroup of the first kind if and only if it is*

(1) a left reflection of a left commutative congruence permutable semigroup of the first kind or

(2) a right reflection of a right commutative congruence permutable semigroup of the first kind.

Proof. By Lemma 5.3.11, both a left reflection of a left commutative congruence permutable semigroup of the first kind and a right reflection of a right commutative congruence permutable semigroup of the first kind are medial congruence permutable semigroups of the first kind.

Conversely, assume that F is a medial congruence permutable semigroup of the first kind. Then $F = S_1 \cup S_0''$ ($S_1 = (L \times G \times R)$ is a rectangular abelian group with $|L| \leq 2, |R| \leq 2$, S_0'' is a commutative nil semigroup).

If $|L| = |R| = 1$, then S_1 is a group whose identity element is the identity element of F. Then F is commutative. As F is a left and right reflection of itself, (1) and (2) are satisfied for F.

Assume $|L| = 2$ and $|R| = 1$. Then S_1 is a disjoint union of two isomorphic subgroups G_i and $G_{\bar{i}}$ ($i \in L$). Let e_i denote the identity element of G_i ($i \in L$). It is clear that e_i and $e_{\bar{i}}$ are right identity elements of F. Then, for arbitrary elements $a, b, c \in F$,

$$abc = abce_i = acbe_i = acb$$

for every $a, b, c \in F$. Thus F is right commutative. As F is a right reflection of itself, (2) is satisfied for F.

Assume $|L| = 1$ and $|R| = 2$. As in the previous part, we can prove that F is left commutative, and so (1) is satisfied for F.

Assume $|L| = |R| = 2$. By Lemma 5.3.2, A_i $(i \in L)$ is a left commutative subsemigroup of F. By Lemma 5.3.6, $A_i \cong S/\lambda$, and so A_i is congruence permutable. Moreover, $A_{\tilde{i}}$ is an isomorphic copy of A_i ($\phi_i : a \mapsto a' = e_{\tilde{i}j}a$ is the corresponding isomorphism by Corollary 5.3.7), and $I = A_i \cap A_{\tilde{i}}$ is an ideal of A_i. The isomorphism ϕ_i leaves the elements of I fixed. By Lemma 5.3.3

$$F = A_i \cup A_{\tilde{i}} \ (i \in L).$$

Since, for arbitrary $a, b \in A_i$,

$$ab' = e_{i,j}ae_{\tilde{i}j}b = e_{ij}e_{\tilde{i}j}ab = e_{ij}ab = ab$$

and

$$b'a = e_{\tilde{i}j}be_{ij}a = e_{\tilde{i}j}e_{ij}ba = e_{\tilde{i}j}ba = (ba)',$$

then F is a left reflection of the left commutative congruence permutable semigroup A_i $(i \in L)$. Hence (1) is satisfied.

We note that F is also a right reflection of the right commutative congruence permutable semigroup B_j $(= Fe_{ij} = Fe\tilde{ij})$. Thus F also satisfies (2). $\square$

Chapter 6

Finite Putcha semigroups

In this chapter we examine finite congruence permutable Putcha semigroups. The chapter contains two sections.

In the first section we describe the finite congruence permutable archimedean semigroups. We show that the finite archimedean congruence permutable semigroups are exactly the finite cyclic nilpotent semigroups and the finite completely simple congruence permutable semigroups.

In the second section we deal with the finite congruence permutable non-archimedean Putcha semigroups. We show that if S is a finite non-archimedean congruence permutable Putcha semigroup, then it is a semilattice of a completely simple semigroup $S_1 = M(I, G, J; P)$ with $|I|, |J| \leq 2$ and a semigroup S_0 such that $S_1 S_0 \subseteq S_0$ and S_0 is an ideal extension of a completely simple semigroup by a nilpotent semigroup. We only focus for the case when S_1 is a group. We prove that, in this case, S_0 is either (i) a completely simple semigroup or (ii) a non-trivial zero semigroup such that the identity element of the group S_1 is a right identity element of S and $SN = \{0\}$ or (iii) a dual of the previous case or (iv) an ideal extension of a completely simple semigroup K by a non-trivial nilpotent semigroup such that the identity element of the group S_1 is the identity element of the factor semigroup S/K. We deal with only that cases when S_1 is a group and S_0 is a non-trivial zero semigroup. We give a construction, and show that a finite semigroup S is a congruence permutable semigroup which is a semilattice of a group G and a non-trivial zero semigroup N such that the identity element of G is a right identity element of S and $SN = \{0\}$ if and only if S is isomorphic to a semigroup defined by this construction. We also characterize finite congruence permutable semigroups S which are semilattice of a group and a non-trivial nilpotent semigroup such that the identity element of G is the identity element of S. In both investigations we use [PP80, Lemma 3] several times.

6.1 Finite archimedean congruence permutable semigroups

It is known that every finite semigroup has a kernel K which is completely simple. Moreover, every finite nil semigroup is nilpotent. Thus we have the following lemma.

Lemma 6.1.1 *([DN10]) A finite semigroup is archimedean if and only if it is an ideal extension of a completely simple semigroup by a nilpotent semigroup.*□

Theorem 6.1.2 *([DN10]) A finite semigroup is an archimedean congruence permutable semigroup if and only if it is either a cyclic nilpotent semigroup or a congruence permutable completely simple semigroup.*

Proof. Let S be a finite congruence permutable archimedean semigroup. By Lemma 6.1.1, S is an ideal extension of a completely simple semigroup K (K is the kernel of S) by the nilpotent semigroup $N = S/K$. By Theorem 1.2.4 and Lemma 4.2.1, N is a cyclic nilpotent semigroup. If $|K| = 1$, then S is isomorphic to N. Consider the case when $|K| > 1$. We show that $S = K$. Assume, in an indirect way, that $K \neq S$. By Theorem 1.1.5, K is isomorphic to a Rees matrix semigroup $\mathcal{M}(G; I, J; P)$. As K is completely simple, the Green's equivalences $\mathcal{R}$ and $\mathcal{L}$ are congruences on K. We note that two elements (i, g, j) and (λ, h, μ) of K are in $\mathcal{R}$-relation if and only if $i = \lambda$; they are in $\mathcal{L}$-relation if and only if $j = \mu$. Consider the equivalence relation $\mathcal{R} \cup \iota_S$ on S. We show that it is a congruence on S. Assume $(a, b) \in \mathcal{R} \cup \iota_S$ for some $a, b \in S$. We can suppose that $a \neq b$. Then $a, b \in K$ and $ax = b$, $by = a$ for some $x, y \in K$. Let $s \in S$ be an arbitrary element. Then $sax = sb$ and $sby = sa$. As $sa, sb, x \in K$, we have $(sa, sb) \in \mathcal{R}$ and so $(sa, sb) \in \mathcal{R} \cup \iota_S$. Consequently $\mathcal{R} \cup \iota_S$ is a right congruence on S. To show that $\mathcal{R} \cup \iota_S$ is a left congruence, assume that $a = (i, g, j)$, $b = (\lambda, h, \mu)$, $as = (i^*, g^*, j^*)$ and $bs = (\lambda^*, h^*, \mu^*)$. As $(a, b) \in \mathcal{R}$, we have $i = \lambda$. Then

$$(i^*, g^*, j^*) = as = (by)s = b(ys) = (\lambda, h, \mu)(ys) = (i, h, \mu)(ys)$$

and

$$(\lambda^*, h^*, \mu^*) = bs = (ax)s = a(xs) = (i, g, j)(xs).$$

From this it follows that $i^* = i = \lambda^*$ and so $(as, bs) \in \mathcal{R}$. Hence $(as, bs) \in \mathcal{R} \cup \iota_S$. Thus $\mathcal{R} \cup \iota_S$ is a left congruence on S. Hence $\mathcal{R} \cup \iota_S$ is a congruence on S. Similarly, $\mathcal{L} \cup \iota_S$ is a congruence on S. It is clear that the kernels of the quotient semigroups are, respectively, the left zero semigroup $K/\mathcal{R}$ and the right zero semigroup $K/\mathcal{L}$. By Theorem 1.2.8, $K/\mathcal{R}$ or $K/\mathcal{L}$ is non-trivial. By symmetry, it can be assumed without loss of generality that K is a non-trivial right zero semigroup. Let $a \in S - K$ and let $f = a^n \in K$, so $fa^i = f$ for all positive integers i and $xf = f$ for all $x \in S$. Let $b \in K$, $b \neq f$. Applying Theorem 1.2.11, b is related to f under the congruence ρ on S generated by (a, f), so there exists a sequence of elementary ρ-transitions from b to f that begins either $b = sat \mapsto sft$ or $b = sft \mapsto sat$ ($s, t \in S^1$), where the right hand side is distinct from b. In addition, since $b = bb$ and $f = bf$, we can assume without loss of generality that $s = bs \in K$. If $t = 1$, then $b = sa$ (since $b \neq bf = f$); otherwise, since K is right zero, $t \notin K$, so $t = a^i$ for some $i < n$ and $b = sa^{i+1}$, since again $b \neq sfa^i = f$. In either case, $b = ca$ for some $c \in K$, $c \neq f$. Now the same argument applies to c and iterating the argument leads to $b = xa^n = xf = f$, a contradiction. Thus the first part of the theorem is proved. As the converse is obvious, the theorem is proved. □

6.2 Finite non-archimedean congruence permutable Putcha semigroups

First of all we remark that a completely simple semigroup $\mathcal{M} = (G; I, J; P)$ is congruence permutable if and only if $|I| \leq 2, |J| \leq 2$ (see [BC93]).

Lemma 6.2.1 *([DN10]) If S is a finite non-archimedean Putcha congruence permutable semigroup, then it is a semilattice of a completely simple semigroup $S_1 = M(G; I, J; P)$ such that $|I|, |J| \leq 2$ and a semigroup S_0 such that $S_1 S_0 \subseteq S_0$ and S_0 is an ideal extension of a completely simple semigroup K by a nilpotent semigroup.*

Proof. Let S be a finite congruence permutable non-archimedean Putcha-semigroup. Then, by Theorem 1.2.4 and Theorem 1.2.5, S is a semilattice of two archimedean semigroups S_0 and S_1 such that $S_0 S_1 \subseteq S_0$. As the Rees factor $S_1^0 = S/S_0$ is congruence permutable by Theorem 1.2.4, S_1 is a congruence permutable archimedean semigroup. By Theorem 1.2.7 and Theorem 6.1.2, S_1 is completely simple. Then S_1 is a Rees matrix semigroup $S_1 = \mathcal{M}(G; I, J; P)$ and $|I|, |J| \leq 2$ by the remark before Lemma 6.2.1. By Lemma 6.1.1, S_0 is an ideal extension of a completely simple semigroup K by the nilpotent Rees factor semigroup $N = S_0/K$. □

We deal with only that case when S_1 is a group.

Lemma 6.2.2 *([DN10]) If a finite congruence permutable semigroup S is a semilattice of a group G and a nilpotent semigroup N such that $NG \subseteq N$, then the identity element of G is a left identity element or a right identity element of S.*

Proof. Let $a \in N$ be an arbitrary element. Then $J(a) \subseteq J(e)$, where e denotes the identity element of G. Then there are elements $x, y \in S^1$ such that $a = xey$. So $N = eN \cup Ne \cup NeN$. Since N is an ideal, $Ne \cup NeN$ and $eN \cup NeN$ are ideals of S and so, by hypothesis, one is included in the other. Suppose $eN \subseteq Ne \cup NeN$, so that

$$N = Ne \cup NeN = Ne \cup (Ne)(eN) \subseteq Ne \cup (Ne)(Ne \cup NeN) \subseteq Ne \cup (Ne)^2 N.$$

Inductively, $N \subseteq Ne \cup (Ne)^i N$ for all positive integers i, and since N is nilpotent, $N = Ne$, as required. In case $Ne \subseteq eN \cup NeN$, we get $N = eN$. □

Lemma 6.2.3 *([DN10]) Let S be a finite congruence permutable semigroup which is a semilattice of a group G and a nilpotent semigroup N such that $GN \subseteq N$. Let e denote the identity element of G. If $Ne = N$, then $eN = \{0\}$ or $eN = N$. Similarly, if $eN = N$, then $Ne = \{0\}$ or $Ne = N$.*

Proof: By the symmetry, we deal with only the first assertion of the lemma. Assume $N = Ne$. Then $SeN = eN \cup NeN$, which is an ideal of S. If $SeN = N$, then $N = eN \cup NeN$ from which we get $N = eN$ as in the proof of Lemma 6.2.2. If $SeN \neq N$, then consider the equivalence

$$\alpha = \{(a, b) \in S \times S : \ ea = eb\}.$$

It is obvious that α is a right congruence. Let a, b, s be arbitrary elements of S such that $(a, b) \in \alpha$. As e is a right identity element of S, we get

$$sa = (se)a = s(ea) = s(eb) = (se)b = sb$$

and so

$$e(sa) = e(sb).$$

Thus α is also a left congruence of S, and so it is a congruence of S. Let $x \in N$ be an arbitrary element. As $(x, ex) \in \alpha$ and $ex \in SeN$, by Theorem 1.2.11, the ideal SeN is contained by the α-class of x, and so $(0, x) \in \alpha$, that is, $0 = e0 = ex$. Hence $eN = \{0\}$. Thus the lemma is proved. □

Lemma 6.2.4 *([DN10]) Let S be a finite non-archimedean congruence permutable semigroup which is a semilattice of a group G and an archimedean semigroup S_0 such that $GS_0 \subseteq S_0$. Then S_0 is either*

(1) completely simple,

(2) or a non-trivial zero semigroup N such that the identity element of G is a right identity element of S and $SN = \{0\}$,

(3) or a non-trivial zero semigroup N such that the identity element of G is a left identity element of S and $NS = \{0\}$,

(4) or an ideal extension of a completely simple semigroup K by a non-trivial nilpotent semigroup N such that the identity element of G is an identity element of the factor semigroup S/K.

Proof. By Lemma 6.1.1, S_0 is an ideal extension of a completely simple semigroup K by the Rees factor semigroup $N = S_0/K$ which is nilpotent. If $S_0 = K$, then (1) is satisfied.

Assume $S_0 \neq K$. As $K = K^2$ is an ideal of S_0 and S_0 is an ideal of S, we have that K is an ideal of S (see Theorem 1.1.4). Consider the Rees factor semigroup S/K which is a semilattice of G and a nilpotent semigroup which is isomorphic to the non-trivial semigroup $N = S_0/K$. By Lemma 6.2.2, the identity element of G is the right identity element or the left identity element of S/K.

First consider the case when the identity element e of G is the right identity element but not a left identity element of S/K. Then, by Lemma 6.2.3, $eS_0 \subseteq K$. Now without loss of generality, if K is non-trivial it can be assumed to be either left zero or right zero, but the two cases must be treated separately because of the asymmetry of the hypothesis on S. In either case, let $a \in S - K - G$, such that $f = a^2 \in K$, and suppose $b \in K$, $b \neq f$. By Theorem 1.2.11, b is related to f under the congruence ρ on S generated by (a, f), so there exists a sequence of elementary ρ-transitions from b to f that begins either $b = sat \mapsto sft$ or $b = sft \mapsto sat$ $(s, t \in S^1)$, where the right hand side is distinct from b. First suppose that K is right zero. Then again $t \notin K$. If $t \in N$, then $at \in K$ and so $at = a(at) = ft$, giving $sat = sft$, a contradiction. So $t \in G$ and therefore $b = be$. Hence $K = Ke$. As in the proof of Theorem 6.1.2, without loss of generality, $s \in K$ and so $s = se$. Also $ea \in K$. Then

$$sa = (se)a = s(ea) = ea = a(ea) = (ae)a = a^2 = f = sf,$$

again giving the contradiction $sat = sft$. Next suppose K is left zero. Now, without loss of generality, $t \in K$ and $s \notin K$. If $s \neq 1$, then since $S \subseteq K$, $sa = sasa = sa^2 = sf$ (since $sas = sa$), a contradiction. So $s = 1$ and since $b \neq f = ft$, $b = at$. But $t \in K$ and $t \neq f$ (since $af = f$) so similarly $t = at'$ for some t', yielding the contradiction $b = a^2t' = ft' = f$. From this it follows, that $|K| = 1$ and so $S_0 = N$. Let $a \in S_0 = N$ be arbitrary. As $eN = \{0\}$ and $ea \in eN$, we get $ea = 0$ and so, for every $s \in S$, $sa = sea = 0$. Thus $SN = \{0\}$ and so (2) is satisfied.

If the identity element of G is a left identity element but not a right identity element of S/K, then (3) (the dual of (2)) is satisfied.

If the identity element of G is the identity element of S/K, then (4) is satisfied. Thus the lemma is proved. $\square$

Remark 6.2.5 *If* $|S_0| = 1$ *is satisfied in case* (1) *of Lemma 6.2.4, then* S *is a group with a zero adjoined and so* S *is congruence permutable.*

Remark 6.2.6 *Condition* (4) *of Lemma 6.2.4 has two subcases:*

(4a): $|K| = 1$ *and so* S_0 *is a non-trivial nilpotent semigroup such that the identity element of* G *is an identity element of* S*.*

(4b): $|K| > 1$*, but* $K \neq S_0$*.*

We describe only that finite congruence permutable non-archimedean Putcha semigroups which are semilattice of a group G and a semigroup S_0 with $GS_0 \subseteq S_0$, where S_0 satisfies either condition (2) or condition (3) of Lemma 6.2.4 or condition $(4a)$ of Remark 6.2.6.

When the identity element of G is a right identity element of S

In this section we deal with only the right side case, but the main theorem (Theorem 6.2.14) will be formulated for both right and left cases. First we prove two lemmas.

For a non-trivial nil semigroup N, let $N^* = N - \{0\}$, where 0 is the zero of N.

Lemma 6.2.7 *([DN10]) Let S be a finite congruence permutable semigroup which is a semilattice of a group G and a non-trivial zero semigroup N such that the identity element of G is a right identity element of S and $SN = \{0\}$. Then $aG = N^*$ for every $a \in N^*$.*

Proof. Let $a \in N^*$ be arbitrary. If $ag = 0$ for some $g \in G$, then $a = ae = agg^{-1} = 0$ which is a contradiction. Thus $aG \subseteq N^*$. As S is finite and the ideals of S form a chain, there is an element $b \in N^*$ such that $S^1bS^1 = N$. Thus $N = S^1bS^1 = (S \cup 1)b(G \cup N \cup 1) = bG \cup bN = bG \cup \{0\}$. Thus $bG = N^*$. From this it follows that, for an arbitrary $a \in N^*$ and some $x \in G$, $aG = bxG = bG = N^*$. $\square$

Lemma 6.2.8 *([DN10]) If an arbitrary semigroup S is a semilattice of a group G and a non-trivial null semigroup N such that $GN = \{0\}$ and $aG = N^*$ for every $a \in N^*$, then, for every non-universal congruence α of S, $[0]_\alpha$ is either $\{0\}$ or N, and $[g]_\alpha \subseteq G$ for every $g \in G$.*

Proof. If $g \in [0]_\alpha$ for some $g \in G$, then $G \subseteq [0]_\alpha$ and so $N^* = aG \subseteq [0]_\alpha$, where $a \in N^*$ is an arbitrary element. Then $[0]_\alpha = S$. If α is not a universal congruence of S, then $[0]_\alpha \subseteq N$. Assume $[0]_\alpha \neq \{0\}$. Then there is an element $a \in N^*$ such that $a \in [0]_\alpha$, and so $N^* = aG \subseteq [0]_\alpha$. Hence $[0]_\alpha = N$.

Assume $(a, g) \in \alpha$ for some $a \in N, g \in G$ and a non-universal congruence α of S. Then $(ea, g) \in \alpha$, where e is the identity element of G. As $ea = 0$, we get $(0, g) \in \alpha$ and so $(0, h) \in \alpha$ for every $h \in G$ and so α is the universal congruence of S by the above. It is a contradiction. Thus $[g]_\alpha \subseteq G$ for every $g \in G$. $\square$

Remark 6.2.9 *If a semigroup S satisfies the conditions of Lemma 6.2.7, then N^* is a right G-set and G acts on N^* transitively.*

By Remark 6.2.9, we need an information on the congruence lattice of transitive group actions. We can use Lemma 3 of the paper [PP80] published by P.P. Pálfy and P. Pudlák.

Lemma 6.2.10 *([PP80, Lemma 3]) If X is a right G-set such that the group G acts on X transitively, then the congruence lattice $Con(X)$ of the G-set X is isomorphic to the interval $[Stab_G(x), G]$ of the subgroup lattice of G for every $x \in X$, where $Stab_G(x) = \{g \in G : \ xg = x\}$.*

Remark 6.2.11 *We remark that the corresponding isomorphisms (in Lemma 6.2.10) are*

$$\phi : \alpha \ \mapsto H_\alpha = \{g \in G : \ xg \ \alpha \ x\} \ (\alpha \in Con(X))$$

and

$$\psi : H \ \mapsto \ \alpha_H = \{(xg, xh) \in A \times A : \ Hg = Hh\} \ (H \in [Stab_G(x), G])$$

(which are inverses of each other).

By the previous lemma, we can formulate the following result.

Lemma 6.2.12 *([DN10]) Let X be a right G-set such that the group G acts on X transitively. Let $x \in X$ be an arbitrary fixed element. Then $\alpha \circ \beta = \beta \circ \alpha$ is satisfied for some congruences $\alpha, \beta \in Con(X)$ if and only if $H_\alpha H_\beta = H_\beta H_\alpha$ is satisfied for $H_\alpha, H_\beta \in [Stab_G(x), G]$. Thus the congruences of the G-set X commute with each other if and only if $HK = KH$ is satisfied for every subgroups H and K of G containing $Stab_G(x)$.* □

The following construction plays an important role in our investigation.

Construction 6.2.13 *([DN10]) Let G be a finite group and M be a subgroup of G such that $HK = KH$ is satisfied for all subgroups H, K of G containing M. Let N^* denote the right quotient set G/M, that is, the set of all right cosets Mg $(g \in G)$ of G defined by M. Let $S = G \cup N^* \cup \{0\}$, where 0 is a symbol not contained in $G \cup N^*$. On S we define an operation as follows. For arbitrary $g, h \in G$, let gh be the original product of g and h in G; let $0g = 0$ for every $g \in G$. If $a \in N$, then let $sa = 0$ for arbitrary $s \in S$. For arbitrary $g \in G$ and arbitrary $Mh \in N^*$, let $(Mh)g = M(hg)$. It is easy to check that S is a semigroup.*

The main theorem of this section is the following.

Theorem 6.2.14 *([DN10]) A finite semigroup S is a congruence permutable semigroup which is a semilattice of a group G and a non-trivial zero semigroup such that the identity element of G is a right [left] identity element of S and $SN = \{0\}$ [$NS = \{0\}$] if and only if it is isomorphic to a semigroup defined in Construction 6.2.13 [the dual of Construction 6.2.13].*

Proof. First of all we show that the semigroup S defined in Construction 6.2.13, is a congruence permutable semigroup. It is clear that S is a semilattice of the group G and the non-trivial zero semigroup $N = N^* \cup \{0\}$ such that $SN = \{0\}$ and the identity element e of G is a right identity element of S. Moreover, $(Mg)G = N^*$ for all $Mg \in N^*$. Thus N^* is a right G-set and G acts on N^* tranisitively. By Lemma 6.2.10, the congruence lattice $Con(N^*)$ of the G-set N^* is isomorphic to the interval $[Stab_G(M), G]$, where $Stab_G(M) = \{g \in G; Mg = M\} = M$. Let α be a non-universal congruence of S. Then, by Lemma 6.2.8,

$[g]_\alpha \subseteq G$ for every $g \in G$ and $[0]_\alpha$ is either $\{0\}$ or N. As N^* is a right G-set and G acts on N^* transitively, moreover the restriction α^* of α to N^* is in $Con(N^*)$, there is a subgroup $H_{\alpha^*} \in [M, G]$ which determines α on N^*.

Let α and β be arbitrary congruences of S. We show that $\alpha \circ \beta = \beta \circ \alpha$. We can suppose that α and β are not the universal relations of S. Assume $(b, c) \in \alpha \circ \beta$ for arbitrary elements b and c of S. Then there is an element $x \in S$ such that $(b, x) \in \alpha$, $(x, c) \in \beta$. We have two cases.
Case 1: $x \in G$. Then, by Lemma 6.2.8, $b, c \in G$. As every group is congruence permutable, there is an element $y \in G$ such that $(b, y) \in \beta$ and $(y, c) \in \alpha$. Hence $(b, c) \in \beta \circ \alpha$.
Case 2: $x \in N$. Then, by Lemma 6.2.8, $b, c \in N$. If $[0]_\alpha = N$ or $[0]_\beta = N$, then $(b, c) \in \alpha$ or $(b, c) \in \beta$ and so $(b, c) \in \beta \circ \alpha$. Consider the case when $[0]_\alpha = [0]_\beta = \{0\}$. Then N^* is saturated by both α and β. If $x = 0$, then $b = c = 0$ and so $(b, c) \in \beta \circ \alpha$. If $x \in N^*$, then $b, c \in N^*$. If α^* and β^* denote the restriction of α and β to N^*, respectively, then $H_{\alpha^*}, H_{\beta^*} \supseteq M$. As $H_{\alpha^*}H_{\beta^*} = H_{\beta^*}H_{\alpha^*}$, we get $\alpha^* \circ \beta^* = \beta^* \circ \alpha^*$ by Lemma 6.2.12. Hence $(b, c) \in \beta \circ \alpha$.

Thus we have $(b, c) \in \beta \circ \alpha$ in both cases. Consequently, $\alpha \circ \beta \subseteq \beta \circ \alpha$. By the symmetry, we get $\alpha \circ \beta = \beta \circ \alpha$. Thus S is a congruence permutable semigroup.

Conversely, assume that S is a congruence permutable semigroup which is a semilattice of a group G and a non-trivial zero semigroup N such that the identity element of G is a right identity element of S and $SN = \{0\}$. Then $aG = N^*$ for every $a \in N^*$ by Lemma 6.2.7. Thus N^* is a right G-set and G acts on N^* transitively. Fix an element a in N^* and consider $G_a = Stab_G(a) = \{g \in G : \ ag = a\}$. It is easy to check that $ag = ah$ for some $g, h \in G$ if and only if $G_a g = G_a h$. Thus $|N^*| = |G : G_a|$. Let ϕ be the bijection of N^* to the factor set G/G_a defined by $\phi(b) = G_a g$ if $b = ag$. It is clear that ϕ is well defined. Moreover, for all $g, h \in G$, $(G_a g)h = G_a(gh)$ implies $(\phi(b))h = \phi(bh)$. If we identify every $b \in N^*$ with $\phi(b)$, then N^* can be considered as the set of all right cosets of G defined by G_a, and the operation on S is defined as in the Construction 6.2.13. Let H and K be arbitrary subgroups of G containing the subgroup G_a. Let $\alpha'_H = \alpha_H \cup 1_S$ and $\alpha'_K = \alpha_K \cup 1_S$, where $\alpha_H = \psi(H)$ and $\alpha_K = \psi(K)$ are congruences of the right G-set N^* defined by H and K, respectively (for ψ, we refer to Remark 6.2.11). It is easy to see that α'_H and α'_K are congruences of S. As S is congruence permutable, they commute with each other from which we get $\alpha_H \circ \alpha_K = \alpha_K \circ \alpha_H$. Hence $HK = KH$ by Lemma 6.2.12. Thus the theorem is proved. □

When the identity element of G is the identity element of S

Lemma 6.2.15 *([DN10]) If S is a finite congruence permutable semigroup which is a semilattice of a group G and a non-trivial nilpotent semigroup N of nilpotency degree t such that the identity element of G is an identity element of S, then $GaG = N^i - N^{i+1}$ is satisfied for every $i = 1, \ldots, t-1$ and every $a \in N^i - N^{i+1}$.*

Proof. Let $i \in \{1, \dots, t-1\}$ and $a \in N^i - N^{i+1}$ be arbitrary. As the identity element of G is the identity element of S, the ideal of S generated by a equals SaS. It is easy to see that N^i and N^{i+1} are ideals of S. As $a \in N^i - N^{i+1}$ and the ideals of S form a chain with respect to inclusion, we have $N^{i+1} \subseteq SaS \subseteq N^i$. As S is finite, there is an element $b \in N^i - N^{i+1}$ such that $SbS = N^i$. Since

$$N^i = SbS = GbG \cup GbN \cup NbG \cup NbN \subseteq GbG \cup N^{i+1}$$

and

$$GbG \cap N^{i+1} = \emptyset,$$

we get

$$GbG = N^i - N^{i+1}.$$

Thus $a = gbh$ for some $g, h \in G$. Hence

$$GaG = GgbhG = GbG = N^i - N^{i+1}.$$

□

Lemma 6.2.16 *([DN10]) Let the finite semigroup S be a semilattice of a group G and a non-trivial nilpotent semigroup N of nilpotency degree t such that, for every $a \in N^i - N^{i+1}$ $(i = 1, \dots t-1)$, $GaG = N^i - N^{i+1}$ is satisfied. Then the ideals of S are $S, N, N^2, \dots N^t = \{0\}$.*

Proof. It is clear that $S, N, N^2, \dots, N^t = \{0\}$ are ideals of S. Let I be an arbitrary ideal of S. Let j be the least positive integer such that $I \cap N^j \neq \emptyset$. If $a \in I \cap (N^j - N^{j+1})$, then $N^j - N^{j+1} = GaG \subseteq I$. Let $b \in N^{j+1} - N^{j+2}$ supposing that $N^{j+1} \neq \{0\}$. There are elements $x_1, \dots, x_{j+1} \in N - N^2$ such that $b = x_1 \dots x_{j+1}$. It is clear that $x_1 \dots x_j \in N^j - N^{j+1} \subseteq I$ and so $b \in I$ which implies that $N^{j+1} - N^{j+2} \subseteq I$. Continouing this procedure, we get that $N^j = I \cap N$. If $I \cap G = \emptyset$, then $I = N^j$. Assume that $I \cap G \neq \emptyset$. Then $G \subseteq I$. Moreover, for all $i = 1, \dots, t-1$, and every $a \in N^i - N^{i+1}$, we have $N^i - N^{i+1} = GaG \subseteq I$ which implies $I = S$. Thus the lemma is proved. □

Lemma 6.2.17 *([DN10]) Let S be a finite semigroup which is a semilattice of a group G and a nilpotent semigroup N of nilpotency degree t such that, for every $i \in \{1, \dots, t-1\}$ and for some (and so for every) $a \in N^i - N^{i+1}$, $GaG = N^i - N^{i+1}$ is satisfied. Then, for every non-universal congruence α of S, $[0]_\alpha = N^j$ for some positive integer $j = 1, \dots, t$ and $[g]_\alpha \subseteq G$ for every $g \in G$, moreover $[a]_\alpha \subseteq N^i - N^{i+1}$ for every $a \in N^i - N^{i+1}$ $(i = 1, \dots j-1)$.*

Proof. Let α be a non-universal congruence of S. If $(g, a) \in \alpha$ for some $g \in G$ and $a \in N$, then $(g^t, a^t) \in \alpha$. As $a^t = 0$, and $(ugv, 0) \in \alpha$ for all $u, v \in G$, we get $G \subseteq [0]_\alpha$. Let $a \in N$ be an arbitrary element. Then $(gah, 0) \in \alpha$ for all $g, h \in G$ and so $N^i - N^{i+1} \subseteq [0]_\alpha$ for everi $i = 1, \dots t-1$. Thus $S = [0]_\alpha$ which is a contradiction. Hence $[g]_\alpha \subseteq G$ for every $g \in G$. By Lemma 6.2.16, the ideals of S are $S, N, N^2, \dots N^{t-1}, N^t = \{0\}$. Then there is a least positive integer $j \in \{1, 2, \dots t\}$ such that $[0]_\alpha = N^j$. If $j = 1$ or $j = 2$, then the assertion is true for α. Assume $j \geq 3$. Let $a \in N^{j-1} - N^j$ be arbitrary. It is clear that $(a, b) \notin \alpha$ for every $b \in N^j$. Assume $(a, b) \in \alpha$ for some $b \in N^{k-1} - N^k$ for some $k < j$. There are elements $x_1, \dots, x_{j-1} \in N - N^2$ such that

$$a = x_1 \dots x_{k-1} \dots x_{j-1}.$$

It is clear that

$$x_1 \dots x_{j-2} \in N^{j-2},$$

$$x_1 \dots x_{j-3} \in N^{j-3} - N^{j-2},$$

and finally,

$$c = x_1 \dots x_{k-1} \in N^{k-1} - N^k = GbG.$$

Then $c = gbh$ for some $g, h \in G$. Thus $(c, gah) \in \alpha$. As $gah \in N^{j-1} - N^j$, $d = gahxk \dots x_{j-1} \in N^j$ and so $a = cx_{k+1} \dots x_{j-1}$ implies $(a, d) \in \alpha$ which is impossible. Hence $[a]_\alpha \subseteq N^{j-1} - N^j$. Thus the lemma is proved. □

For an arbitrary group G, let G^* denote the dual of G, that is, $xy = u$ in G^* if and only if $yx = u$ in G.

Theorem 6.2.18 *([DN10]) Let S be a finite semigroup which is a semilattice of a group G and a non-trivial nilpotent semigroup N of nilpotency degree t such that the identity element of G is the identity element of S. Then S is congruence permutable if and only if, for all $i = 1, \dots, t-1$, there is an element a_i in $N^i - N^{i+1}$ such that $Ga_iG = N^i - N^{i+1}$, and $HK = KH$ is satisfied for all subgroups $H, K \supseteq G_{a_i} = \{(g, h) \in G^* \times G : \ ga_ih = a_i\}$.*

Proof. Let S be a finite semigroup which is a semilattice of a group G and a nilpotent semigroup N of nilpotency degree t such that the identity element of G is the identity element of S. First assume that S is congruence permutable. Let $i \in \{1, \dots, t-1\}$ be arbitrary. Then, for every $a_i \in N^i - N^{i+1}$, $Ga_iG = N^i - N^{i+1}$ is satisfied by Lemma 6.2.15. It is a matter of checking to see that this result implies that $N^i - N^{i+1}$ is a right $(G^* \times G)$-set ($a(g, h) = gah$ for every $a \in N^*$ and every $(g, h) \in G^* \times G$) and $G^* \times G$ acts on $N^i - N^{i+1}$ transitively. Let $G_{a_i} = Stab_{G^* \times G}(a_i) = \{(g, h) \in G^* \times G : \ ga_ih = a_i\}$. By Lemma 6.2.10 the congruence lattice $Con(N^i - N^{i+1})$ of the right $(G^* \times G)$-set $N^i - N^{i+1}$ is isomorphic to $[Stab_{G^* \times G}(a_i), G^* \times G]$. The corresponding isomorphisms $\phi: \ \alpha_i \mapsto H_{\alpha_i}$ ($\alpha_i \in Con(N^i - N^{i+1})$ and $\psi: \ H \mapsto \alpha_H^{(i)}$ ($H \in [Stab_{G^* \times G}(a_i), G^* \times G]$) defined as in Remark 6.2.11. Let H be an arbitrary subgroup of $G^* \times G$ containing the subgroup G_{a_i}. Let α'_H be the relation of S defined by $(a, b) \in \alpha'_H$ if and only if $a = b$ or $a, b \in N^{i+1}$ or $a, b \in N^i - N^{i+1}$ and $(a, b) \in \alpha_H^{(i)}$. It is clear that α'_H is an equivalence relation. We show that it is a congruence of S. Assume $(a, b) \in \alpha'_H$ for some $a, b \in S$. We can suppose that $a \neq b$. If $a, b \in N^{i+1}$, then $sa, sb, as, bs \in N^{n+1}$ and so $(sa, sb) \in \alpha'_H$ and $(as, bs) \in \alpha'_H$. Consider the case when $a, b \in N^i - N^{i+1}$. Then $(a, b) \in \alpha_H^{(i)}$ and so, for every $x \in G$, we have $(a(e, x), b(e, x)) \in \alpha_H^{(i)}$ and

$(a(x,e),b(x,e)) \in \alpha_H^{(i)}$, because $\alpha_H^{(i)}$ is a congruence of the right $(G^* \times G)$-set $N^i - N^{i+1}$. Thus $(ax,bx) \in \alpha_H^{(i)}$ and $(xa,xb) \in \alpha_H^{(i)}$. Hence $(ax,bx) \in \alpha'_H$ and $(xa,xb) \in \alpha'_H$. If $u \in N$, then $ua, ub, au, bu \in N^{i+1}$ and so $(au,bu) \in \alpha'_H$ and $(ua,ub) \in \alpha'_H$. Consequently, α'_H is a congruence on S. Let H and K be arbitrary subgroups of $G^* \times G$ containing the subgroup G_{a_i}. Let α'_H and α'_K be the congruences of S defined by H and K (see above). As S is congruence permutable, $\alpha'_H \circ \alpha'_K = \alpha'_K \circ \alpha'_H$ from which we get $\alpha_H^{(i)} \circ \alpha_K^{(i)} = \alpha_K^{(i)} \circ \alpha_H^{(i)}$. Then $HK = KH$ by Lemma 6.2.12. Thus the necessity of the permutability of S is proved.

Conversely, assume that, for all $i = 1,\dots,t-1$, there is an element a_i in $N^i - N^{i+1}$ such that $Ga_iG = N^i - N^{i+1}$, and $HK = KH$ is satisfied for all subgroups $H, K \supseteq G_{a_i} = \{(g,h) \in G^* \times G : \ ga_ih = a_i\}$. We note that, from $Ga_iG = N^i - N^{i+1}$, it follows that $GaG = N^i - N^{i+1}$ for every $a \in N^i - N^{i+1}$. Thus $N^i - N^{i+1}$ is a right $(G^* \times G)$-set and $G^* \times G$ acts on $N^i - N^{i+1}$ transitively. By Lemma 6.2.10, the congruence lattice $Con(N^i - N^{i+1})$ of the $(G^* \times G)$-set $N^i - N^{i+1}$ is isomorphic to $[Stab_{G^* \times G}(a_i), G^* \times G]$ (for the corresponding isomorphisms we refer to Remark 6.2.11). By Lemma 6.2.16, the ideals of S are $S, N, N^2, \dots, N^t$. Let α be a non-universal congruence on S. Then, by Lemma 6.2.17, $[0]_\alpha = N^j$ for some positive integer $j \in \{1, \dots t\}$, $[g]_\alpha \subseteq G$ for every $g \in G$, and $[a]_\alpha \subseteq N^i - N^{i+1}$ for every $a \in N^i - N^{i+1}$ $(i = 1, \dots j-1)$. Let α_i denote the restriction of α to $N^i - N^{i+1}$, and let

$$H_\alpha^{(i)} = \phi(\alpha_i) = \{(g,h) \in G^* \times G : \ a_i(g,h) \ \alpha_i \ a_i\}$$

$(a_i \in N^i - N^{i+1},\ i = 1, \dots t-1)$. $H_\alpha^{(i)}$ is a subgroup of $G^* \times G$ and $G_{a_i} \subseteq H_\alpha^{(i)}$. Let β be an arbitrary non-universal congruence on S. As $G_{a_i} \subseteq H_\beta^{(i)}$, we have $H_\alpha^{(i)} H_\beta^{(i)} = H_\beta^{(i)} H_\alpha^{(i)}$. As α_i and β_i are in the congruence lattice $Con(N^i - N^{i+1})$ of the right $(G^* \times G)$-set $N^i - N^{i+1}$, we have $\alpha_i \circ \beta_i = \beta_i \circ \alpha_i$. We show that $\alpha \circ \beta = \beta \circ \alpha$. Assume $(a,b) \in \alpha \circ \beta$ for some $a, b \in S$. Then there is an element $c \in S$ such that $(a,c) \in \alpha$ and $(c,b) \in \beta$. If $c \in G$, then $a, b \in G$ by Lemma 6.2.17. As every group is congruence permutable, we get $(a,b) \in \beta \circ \alpha$. By Lemma 6.2.16 and Lemma 6.2.17, $[0]_\alpha \subseteq [0]_\beta$ or $[0]_\beta \subseteq [0]_\alpha$. Assume $[0]_\alpha \subseteq [0]_\beta$. If $c \in [0]_\beta$, then $a, b \in [0]_\beta$ and so $(a,b) \in \beta$, $(b,b) \in \alpha$ implies $(a,b) \in \beta \circ \alpha$. Assume $c \notin [0]_\beta$, $c \in N^i - N^{i+1}$. Then, by Lemma 6.2.17, $a, b \in N^i - N^{i+1}$. Thus $(a,b) \in \alpha_i \circ \beta_i = \beta_i \circ \alpha_i$ (see also Lemma 6.2.12). Thus $(a,b) \in \beta \circ \alpha$. The proof of $(a,b) \in \beta \circ \alpha$ is similar in that case when $[0]_\beta \subseteq [0]_\alpha$. Thus $\alpha \circ \beta \subseteq \beta \circ \alpha$. The proof of $\beta \circ \alpha \subseteq \alpha \circ \beta$ is similar. Thus $\alpha \circ \beta = \beta \circ \alpha$. Hence S is congruence permutable. □

Chapter 7

An application for semigroup algebras

In this section we deal with a semigroup algebraic problem in which the congruence permutable semigroups are in the centre. For an ideal J of a semigroup algebra $\mathbb{F}[S]$, let ϱ_J denote the congruence on the semigroup S which is the restriction of the congruence on $\mathbb{F}[S]$ defined by the ideal J. We show that if S is a semilattice or a rectangular band, then the mapping $\varphi_{\{S;\mathbb{F}\}}:\ J \mapsto \varrho_J$ is a $\circ$-homomorphism if and only if S is congruence permutable.

7.1 The general case

Let S be a semigroup and $\mathbb{F}$ be a field. For an arbitrary congruence α on S, let $\mathbb{F}[\alpha]$ denote the kernel of the extended canonical homomorphism $\mathbb{F}[S] \to \mathbb{F}[S/\alpha]$. By Lemma 5 of Chapter 4 of [Okn91], for every semigroup S and every field $\mathbb{F}$, the mapping $\varphi_{\{S;\mathbb{F}\}}:\ J \mapsto \varrho_J$ is a surjective homomorphism of the semilattice $(Con(\mathbb{F}[S]); \wedge)$ onto the semilattice $(Con(S); \wedge)$ such that $\varrho_{\mathbb{F}[\alpha]} = \alpha$ for every congruence α on S. As a homomorphic image of a semigroup is also a semigroup, and $\alpha \circ \beta = \alpha \vee \beta$ is satisfied for every congruences α and β of a congruence permutable semigroup, the assertions of the following lemma are obvious.

Lemma 7.1.1 *([NZ16]) If S is a semigroup such that, for a field $\mathbb{F}$, $\varphi_{\{S;\mathbb{F}\}}:\ J \to \varrho_J$ is a homomorphism of the semigroup $(Con(\mathbb{F}[S]); \circ)$ into the semigroup $(\mathcal{B}_S; \circ)$, then S is a congruence permutable semigroup. Moreover, if S is a congruence permutable semigroup, then $\varphi_{\{S;\mathbb{F}\}}$ is a homomorphism of $(Con(\mathbb{F}[S]); \circ)$ onto the semigroup $(Con(S); \circ)$ if and only if $\varphi_{\{S;\mathbb{F}\}}$ is a homomorphism of the semilattice $(Con(\mathbb{F}[S]); \vee)$ onto the semilattice $(Con(S); \vee)$, that is, $ker_{\varphi_{\{S;\mathbb{F}\}}}$ is $\vee$-compatible.*

The next example shows that the converse of the first assertion of Lemma 7.1.1 is not true, in general; for a congruence permutable semigroup S, the condition "$\varphi_{\{S;\mathbb{F}\}}$ is a homomorphism of $(Con(\mathbb{F}[S]);\circ)$ onto the semigroup $(Con(S);\circ)$" depends on the field $\mathbb{F}$.

Example Let C_4, $\mathbb{F}_3$ and $\mathbb{F}_2$ denote the cyclic group of order 4, the 3-element field, and the 2-element field, respectively. It is known that every group is a congruence permutable semigroup. Denote the elements of C_4 by $1, a, a^2, a^3$ (1 is the identity element). It is easy to see that $I = \mathrm{Span}(1+a^2, a+a^3)$ and $J = \mathrm{Span}(1+a, a+a^2, a^2+a^3)$ are ideals of $\mathbb{F}_3[C_4]$. Moreover, $\varphi_{\{C_4;\mathbb{F}_3\}}(I) = \varrho_I = \iota_{C_4}$, and $\varphi_{\{C_4;\mathbb{F}_3\}}(J) = \varrho_J = \alpha_{C_2}$, where α_{C_2} denotes the congruence on C_4 defined by $C_2 = \{1, a^2\}$. From this it follows that

$$\varphi_{\{C_4;\mathbb{F}_3\}}(I) \vee \varphi_{\{C_4;\mathbb{F}_3\}}(J) = \varrho_I \vee \varrho_J \neq \omega_{C_4} = \varrho_{(I+J)} = \varrho_{(I\vee J)} = \varphi_{\{C_4;\mathbb{F}_3\}}(I \vee J).$$

Thus $ker_{\varphi_{\{C_4;\mathbb{F}_3\}}}$ is not $\vee$-compatible and so $\varphi_{\{C_4;\mathbb{F}_3\}}$ is not a homomorphism of $Con(\mathbb{F}_3[C_4]);\circ)$ onto the semigroup $(Con(C_4);\circ)$.

It is a matter of checking to see that the ideals of $\mathbb{F}_2[C_4]$ are $\{0\}$, $\mathbb{F}_2[C_4]$ and

$$\mathbb{F}_2[\omega_{C_4}] = \{0, 1+a+a^2+a^3, 1+a^2, a+a^3, 1+a, a+a^2, a^2+a^3, 1+a^3\},$$

$$\mathbb{F}_2[\alpha_{C_2}] = \{0, 1+a+a^2+a^3, 1+a^2, a+a^3\},$$

$$Span(1+a+a^2+a^3) = \{0, 1+a+a^2+a^3\}.$$

Thus $Con(\mathbb{F}_2[C_4])$ is the next:

$$\begin{array}{c} \mathbb{F}_2[C_4] \\ | \\ \mathbb{F}_2[\omega_{C_4}] \\ | \\ \mathbb{F}_2[\alpha_{C_2}] \\ | \\ Span(1+a+a^2+a^3) \\ | \\ \{0\} \end{array}$$

It is easy to see that $ker_{\varphi_{\{C_4;\mathbb{F}_2\}}}$ is $\vee$-compatible and so $\varphi_{\{C_4;\mathbb{F}_2\}}$ is a homomorphism of $Con(\mathbb{F}_2[C_4]);\circ)$ onto the semigroup $(Con(C_4);\circ)$.

By Lemma 7.1.1 and the above Example, it is a natural idea to find all couples $(S,\mathbb{F})$ of congruence permutable semigroups S and fields $\mathbb{F}$, for which the mapping $\varphi_{\{S;\mathbb{F}\}}$ is a homomorphism of the semigroup $(\mathrm{Con}(\mathbb{F}[S]),\circ)$ onto the semigroup $(Con(S);\circ)$. In the next we show that if S is an arbitrary congruence permutable semilattice or an arbitrary congruence permutable rectangular band, then $\varphi_{\{S;\mathbb{F}\}}$ satisfies the previous condition for an arbitrary field $\mathbb{F}$.

7.2 Semilattices

Theorem 7.2.1 *([NZ16]) Let S be a congruence permutable semilattice. Then, for an arbitrary field $\mathbb{F}$, $\varphi_{\{S;\mathbb{F}\}}$ is a homomorphism of the semigroup* $(\mathrm{Con}(\mathbb{F}[S]), \circ)$ *onto the semigroup $(Con(S); \circ)$.*

Proof. Assume that S is a congruence permutable semilattice. Then, by [Ham75, Lemma 2], $|S| \leq 2$. We can consider the case when $|S| = 2$. Let

$$S = \{e, f\} \quad (e \neq f).$$

Then

$$e^2 = e, \quad \text{and} \quad f^2 = f.$$

We can suppose that

$$ef = fe = e.$$

It is clear that S has two congruences: ι_S and ω_S.

Let $\mathbb{F}$ be an arbitrary field. It is easy to see that

$$J_e = \{\alpha e : \ \alpha \in \mathbb{F}\} \quad \text{and} \quad J_{e-f} = \{\alpha(e - f) : \ \alpha \in \mathbb{F}\}$$

are proper ideals of $\mathbb{F}[S]$. As

$$dim(J_e) = dim(J_{e-f}) = 1,$$

the ideals J_e and J_{e-f} are minimal ideals of $\mathbb{F}[S]$. We show that the ideals of $\mathbb{F}[S]$ are

$$\{0\}, J_e, J_{e-f} \quad \text{and} \quad \mathbb{F}[S].$$

Let $J \neq \{0\}$ be a proper ideal of $\mathbb{F}[S]$. Clearly $dim(J) = 1$. Let

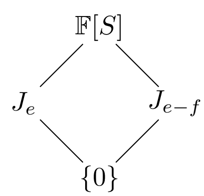
$$A = \alpha e + \beta f \in J$$

be a non-zero element. Then

$$(\alpha + \beta)e = e(\alpha e + \beta f) = eA \in J.$$

If $\alpha + \beta \neq 0$, then $J = J_e$. If $\alpha + \beta = 0$, then $J = J_{e-f}$.

Thus the ideals of $\mathbb{F}[S]$ are $\{0\}$, J_e, J_{e-f}, $\mathbb{F}[S]$. So $Con(\mathbb{F}[S])$ is the next:



It is a matter of checking to see that the $ker_{\varphi_{\{S;\mathbb{F}\}}}$-classes of $Con(\mathbb{F}[S])$ are $\{\{0\}, J_e\}$ and $\{J_{e-f}, \mathbb{F}[S]\}$. It is easy to see that $ker_{\varphi_{\{S;\mathbb{F}\}}}$ is $\vee$-compatible and so, by Lemma 7.1.1, $\varphi_{\{S;\mathbb{F}\}}$ is a homomorphism of the semigroup $(\mathrm{Con}(\mathbb{F}[S]), \circ)$ onto the semigroup $(Con(S); \circ)$. $\square$

Corollary 7.2.2 *([NZ16]) Let S be a semilattice. Then, for a field $\mathbb{F}$, $\varphi_{\{S;\mathbb{F}\}}$ is a homomorphism of the semigroup* $(\mathrm{Con}(\mathbb{F}[S]), \circ)$ *into the relation semigroup* $(\mathcal{B}_S; \circ)$ *if and only if S is congruence permutable.*

Proof. It is obvious by Lemma 7.1.1 and Theorem 7.2.1. □

7.3 Rectangular bands

Theorem 7.3.1 *([NZ16]) Let $S = L \times R$ be a congruence permutable rectangular band (L is a left zero semigroup, R is a right zero semigroup). Then, for an arbitrary field $\mathbb{F}$, $\varphi_{\{S;\mathbb{F}\}}$ is a homomorphism of the semigroup* $(\mathrm{Con}(\mathbb{F}[S]), \circ)$ *onto the semigroup* $(Con(S); \circ)$.

Proof. Let $\mathbb{F}$ be an arbitrary field and $S = L \times R$ be a congruence permutable rectangular band. As a rectangular band satisfies the identity $axyb = ayxb$, that is, every rectangular band is a medial semigroup, [BC81, Corollary 1.2] implies $|L| \leq 2$ and $|R| \leq 2$.

First consider the case when $|L| = 1$. Then S is isomorphic to the right zero semigroup R, and $|S| \leq 2$. We can suppose that $|S| = 2$. Let $S = \{e, f\}$ ($e \neq f$). The congruences of S are ι_S and ω_S. We show that the ideals of $\mathbb{F}[S]$ are

$$\{0\}, J_{e-f} = \mathbb{F}[\omega_S] = \{\alpha(e-f) :\ \alpha \in \mathbb{F}\} \quad \text{and} \quad \mathbb{F}[S].$$

Let $J \neq \{0\}$ be an arbitrary ideal. Assume that there is an element

$$0 \neq \alpha_0 e + \beta_0 f \in J$$

for which

$$\alpha_0 + \beta_0 \neq 0$$

is satisfied. Then

$$(\alpha_0 + \beta_0)e = (\alpha_0 e + \beta_0 f)e \in J$$

and so

$$e = \frac{1}{\alpha_0 + \beta_0}(\alpha_0 + \beta_0)e \in J$$

from which we get $f = ef \in J$. Consequently

$$J = \mathbb{F}[S].$$

Next, consider the case when

$$\alpha_0 + \beta_0 = 0$$

is satisfied for every

$$A = \alpha_0 e + \beta_0 f \in J.$$

Then $\beta = -\alpha_0$ and so

$$A = \alpha_0 e + \beta_0 f = \alpha_0 e - \alpha_0 f = \alpha_0(e - f) \in J_{e-f}.$$

Consequently,

$$J \subseteq J_{e-f}.$$

As $dim(J_{e-f}) = 1$, the ideal J_{e-f} is minimal. Hence

$$J = J_{e-f}.$$

Thus the ideals of $\mathbb{F}[S]$ are $\{0\}$, J_{e-f} and $\mathbb{F}[S]$, indeed. So $Con(\mathbb{F}[S])$ is

$$\begin{array}{c} \mathbb{F}[S] \\ | \\ J_{e-f} \\ | \\ \{0\} \end{array}$$

It is a matter of checking to see that the $ker_{\varphi_{\{S;\mathbb{F}\}}}$-classes of $Con(\mathbb{F}[S])$ are $\{\{0\}\}$ and $\{J_{e-f}, \mathbb{F}[S]\}$. It is easy to see that $ker_{\varphi_{\{S;\mathbb{F}\}}}$ is $\vee$-compatible and so, by Lemma 7.1.1, $\varphi_{\{S;\mathbb{F}\}}$ is a homomorphism of the semigroup $(\mathrm{Con}(\mathbb{F}[S]), \circ)$ onto the semigroup $(Con(S); \circ)$.

If $|R| = 1$, then S is a left zero semigroup, and $|S| \leq 2$. We can prove, as in the previous part of the proof, that $ker_{\varphi_{\{S;\mathbb{F}\}}}$ is $\vee$-compatible.

Next, consider the case when $|L| = |R| = 2$. Let

$$L = \{a_1, a_2\}, \quad R = \{b_1, b_2\}.$$

Let α_L and α_R denote the kernels of the projection homomorphisms $S \mapsto L$ and $S \mapsto R$, respectively. The α_L-classes of S are

$$\{(a_1, b_1); (a_1, b_2)\} \quad \text{and} \quad \{(a_2, b_1); (a_2, b_2)\}.$$

The α_R-classes of S are

$$\{(a_1, b_1); (a_2, b_1)\} \quad \text{and} \quad \{(a_1, b_2); (a_2, b_2)\}.$$

It is easy to see that the congruences of S are ι_S, α_L, α_R and ω_S. We show that the ideals of $\mathbb{F}[S]$ are

$$\mathbb{F}[S], \mathbb{F}[\omega_S] = \{\sum_{i,j=1}^{2} \alpha_{i,j}(a_i, b_j) : \sum_{i,j=1}^{2} \alpha_{i,j} = 0\},$$

$$J_L = \mathbb{F}[\alpha_L], J_R = \mathbb{F}[\alpha_R], J_L \cap J_R, \{0\}.$$

We note that

$$dim(\mathbb{F}[\omega_S]) = 3, dim(J_L) = dim(J_R) = 2.$$

First we show that $J \subseteq \mathbb{F}[\omega_S]$ or $J = \mathbb{F}[S]$ for every ideal J of $\mathbb{F}[S]$. Let J be an arbitrary ideal of $\mathbb{F}[S]$. Assume that there is an element

$$A = \alpha_{1,1}(a_1, b_1) + \alpha_{1,2}(a_1, b_2) + \alpha_{2,1}(a_2, b_1) + \alpha_{2,2}(a_2, b_2) \in J$$

such that $A \notin \mathbb{F}[\omega_S]$, that is $\sum_{i,j=1}^{2} \alpha_{i,j} \neq 0$. Let $i, j \in \{1, 2\}$ be arbitrary elements. Then

$$(\sum_{i,j=1}^{2} \alpha_{i,j})(a_i, b_j) = (a_i, b_1)A(a_1, b_j) \in J.$$

As $\sum_{i,j=1}^{2} \alpha_{i,j} \neq 0$, we get $(a_i, b_j) \in J$ from which it follows that $S \subseteq J$. Consequently, $J = \mathbb{F}[S]$. Thus $\mathbb{F}[\omega_S]$ is the only maximal ideal of $\mathbb{F}[S]$.

Next we show that $J_L \cap J_R$ is the only ideal of $\mathbb{F}[S]$ whose dimension is 1. Let

$$A = \alpha_{1,1}(a_1, b_1) + \alpha_{1,2}(a_1, b_2) + \alpha_{2,1}(a_2, b_1) + \alpha_{2,2}(a_2, b_2) \in J_L \cap J_R$$

be an arbitrary element. As

$$(a_1, b_1)\ \alpha_L\ (a_1, b_2) \quad \text{and} \quad (a_2, b_1)\ \alpha_L\ (a_2, b_2),$$

we have

$$\alpha_{1,2} = -\alpha_{1,1} \quad \text{and} \quad \alpha_{2,2} = -\alpha_{2,1}.$$

As

$$(a_1, b_1)\ \alpha_R\ (a_2, b_1) \quad \text{and} \quad (a_1, b_2)\ \alpha_R\ (a_2, b_2),$$

we have

$$\alpha_{2,1} = -\alpha_{1,1} \quad \text{and} \quad \alpha_{2,2} = -\alpha_{1,2}.$$

Thus

$$A = \alpha_{1,1}((a_1, b_1) - (a_1, b_2) - (a_2, b_1) + (a_2, b_2))$$

for some $\alpha \in \mathbb{F}$. Consequently, the ideal $J_L \cap J_R$ is generated by

$$(a_1, b_1) - (a_1, b_2) - (a_2, b_1) + (a_2, b_2).$$

Hence the dimension of $J_L \cap J_R$ is 1.

To show that $J_L \cap J_R$ is the only ideal of $\mathbb{F}[S]$ whose dimension is 1, consider an ideal J of $\mathbb{F}[S]$ generated by an element

$$0 \neq B = \alpha_{1,1}(a_1, b_1) + \alpha_{1,2}(a_1, b_2) + \alpha_{2,1}(a_2, b_1) + \alpha_{2,2}(a_2, b_2).$$

Then $J \subset \mathbb{F}[\omega_S]$ and

$$(a_1, b_1)B = (\alpha_{1,1} + \alpha_{2,1})(a_1, b_1) + (\alpha_{1,2} + \alpha_{2,2})(a_1, b_2) \in J.$$

Thus there is a coefficient $\xi \in \mathbb{F}$ such that

$$(a_1, b_1)B = \xi B.$$

Assume $\xi \neq 0$. Then $\alpha_{2,1} = \alpha_{2,2} = 0$ and so

$$B = \alpha_{1,1}(a_1, b_1) + \alpha_{1,2}(a_1, b_2).$$

From

$$(a_2, b_2)B = \alpha_{1,1}(a_2, b_1) + \alpha_{1,2}(a_2, b_2) \in J$$

we can conclude that $\alpha_{1,1} = \alpha_{1,2} = 0$ and so $B = 0$. This is a contradiction. Hence $\xi = 0$. Thus

$$B = \alpha_{1,1}(a_1, b_1) + \alpha_{1,2}(a_1, b_2)\alpha_{1,1}(a_2, b_1) - \alpha_{1,2}(a_2, b_2).$$

As

$$B(a_1, b_1) = (\alpha_{1,1} + \alpha_{1,2}((a_1, b_1) - (a_2, b_1)) \in J,$$

we get $B(a_1, b_1) = \tau B$ for some $\tau \in \mathbb{F}$. Assume $\tau \neq 0$. Then $\alpha_{1,2} = 0$ and so

$$B = \alpha_{1,1}(a_1, b_1) - \alpha_{1,1}(a_2, b_1).$$

From

$$B(a_2, b_2) = \alpha_{1,1}((a_1, b_2) - (a, 2, b_2)) \in J$$

we can conclude that $\alpha_{1,1} = 0$ and so $B = 0$. This is a contradiction. Hence $\tau = 0$. Thus $\alpha_{1,2} = -\alpha_{1,1}$ and so

$$B = \alpha_{1,1}((a_1, b_1) - (a_1, b_2) - (a_2, b_1) + (a_2, b_2)) \in J_L \cap J_R.$$

As $J \neq \{0\}$ and $J_L \cap J_R$ is a minimal ideal of $\mathbb{F}[S]$, we get

$$J = J_L \cap J_R,$$

that is, $J_L \cap J_R$ is the only ideal of $\mathbb{F}[S]$ whose dimension is 1.

As $dim(J_R + J_L) > dim J_R$ and $\mathbb{F}[\omega_S] \supseteq J_R + J_L$, we have

$$J_R + J_L = \mathbb{F}[\omega_S].$$

Let J be an arbitrary ideal of $\mathbb{F}[S]$ which differs from all of the ideals $\mathbb{F}[S], \mathbb{F}[\omega_S], J_L, J_R, J_L \cap J_R, \{0\}$. Then $J \subset \mathbb{F}[\omega_S]$ and $dim(J) = 2$.

If $J \cap J_L = \{0\}$, then $dim(J + J_L) = 4$ which contradicts $J + J_L \subseteq \mathbb{F}[\omega_S]$. Hence $dim(J \cap J_L) = 1$ and so $J_L \cap J_R = J \cap J_L$. From this we get $J \cap J_R = J_L \cap J_R$. Recall that $C = (a_1, b_1) - (a_1, b_2) - (a_2, b_1) + (a_2, b_2)$ generates the ideal $J_L \cap J_R$. Let A be an arbitrary element of $J - (J_L \cap J_R)$. Then A and C are linearly independent. So

$$A = \alpha(a_1, b_1) + \beta(a_1, b_2) + \gamma(a_2, b_1) + (-\alpha - \beta - \gamma)(a_2, b_2).$$

If $\alpha = -\gamma$, then $A \in J_R$ which is a contradiction. Thus $\alpha \neq -\gamma$. Then

$$(a_1, b_1)A = (\alpha + \gamma)((a_1, b_1) - (a_1, b_2)) \in J_L.$$

As J is an ideal and $A \in J$ we have

$$(a_1, b_1)A \in J \cap J_L = J_L \cap J_R.$$

It means $\alpha = -\gamma$ which is also contradiction. Thus $Con(\mathbb{F}[S])$ is the next:

$$
\begin{tikzcd}[column sep=small]
 & \mathbb{F}[S] \arrow[d, dash] & \\
 & \mathbb{F}[\omega_S] \arrow[dl, dash] \arrow[dr, dash] & \\
J_L \arrow[dr, dash] & & J_R \arrow[dl, dash] \\
 & J_L \cap J_R \arrow[d, dash] & \\
 & \{0\} &
\end{tikzcd}
$$

It is a matter of checking to see that the $ker_{\varphi_{\{S;\mathbb{F}\}}}$-classes of $Con(\mathbb{F}[S])$ are $\{\{0\}, J_L \cap J_R\}, \{J_L\}, \{J_R\}$ and $\{\mathbb{F}[\omega_S], \mathbb{F}[S]\}$. It is easy to see that $ker_{\varphi_{\{S;\mathbb{F}\}}}$ is $\vee$-compatible and so, by Lemma 7.1.1, $\varphi_{\{S;\mathbb{F}\}}$ is a homomorphism of the semigroup $(\mathrm{Con}(\mathbb{F}[S]), \circ)$ onto the semigroup $(Con(S); \circ)$. $\square$

Corollary 7.3.2 *([NZ16]) Let $S = L \times R$ be a rectangular band. Then, for a field $\mathbb{F}$, $\varphi_{\{S;\mathbb{F}\}}$ is a homomorphism of the semigroup $(\mathrm{Con}(\mathbb{F}[S]), \circ)$ into the relation semigroup $(\mathcal{B}_S; \circ)$ if and only if S is congruence permutable.*

Proof. It is obvious by Lemma 7.1.1 and Theorem 7.3.1. $\square$

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