AN ENCOUNTER BETWEEN SET THEORY AND ANALYSIS

DISSERTATION

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Contents

Introduction

The aim of this dissertation is to describe certain connections between two seemingly far away fields of mathematics, namely set theory and analysis. This connection is classical in the first chapter of the dissertation, which is concerned with descriptive set theory, an area traditionally involving both set theory and analysis. However, in the second chapter we present numerous instances of this connection that are of rather surprising nature. Sometimes questions about Hausdorff measures turn out to be independent of the usual Zermelo-Fraenkel Axioms (ZFC) of mathematics, sometimes the solutions of questions concerning Hausdorff measures require set theoretical techniques, and in one instance a purely set theoretical question is answered with the help of Hausdorff measures.

Let us now briefly outline the main results, and say a few words about the organisation of the dissertation. First, in Chapters 1 and 2 we describe our results, and also briefly indicate some key ingredients of the proofs. The formal proofs are all postponed to Chapter 3.

Chapter 1 is concerned with descriptive set theory, which is the study of definable (open, closed, G_δ, F_σ, Borel, etc.) subsets of $\mathbb{R}^n$, or more generally, of Polish spaces. (A topological space is Polish, if it has a countable dense subset, and its topology can be induced by a complete metric. For the basic notions and terminology of the dissertation we refer the reader to the next chapter.)

Section 1.1 presents the solution to and old problem of M. Laczkovich. In the 1970s he posed the following problem: Let $\mathcal{B}_1(X)$ denote the set of Baire class 1 functions defined on an uncountable Polish space X equipped with the pointwise ordering. (A function is of Baire class 1 if it is the pointwise limit of continuous functions.)

Characterise the order types of the linearly ordered subsets of $\mathcal{B}_1(X)$.

The main result of the section is a complete solution to this problem.

We prove that a linear order is isomorphic to a linearly ordered family of Baire class 1 functions iff it is isomorphic to a subset of the following linear order that we call $([0,1]^{<\omega_1}_{\searrow 0}, <_{altlex})$, where $[0,1]^{<\omega_1}_{\searrow 0}$ is the set of strictly decreasing transfinite sequences of reals in $[0,1]$ with last element 0, and $<_{altlex}$, the so called *alternating lexicographical ordering*, is defined as follows: if $(x_\alpha)_{\alpha\leq\xi}, (x'_\alpha)_{\alpha\leq\xi'} \in [0,1]^{<\omega_1}_{\searrow 0}$ are distinct, and δ is the minimal ordinal where the two sequences differ then we say that

$$(x_\alpha)_{\alpha\leq\xi} <_{altlex} (x'_\alpha)_{\alpha\leq\xi'} \iff (\delta \text{ is even and } x_\delta < x'_\delta) \text{ or } (\delta \text{ is odd and } x_\delta > x'_\delta).$$

Using this characterisation we easily reprove all the known results and answer all the known open questions of the topic. The material of this section can be found in [32].

Section 1.2 presents solutions to problems of J. Mycielski and D. H. Fremlin. Polish groups have recently been playing a central role in modern descriptive set theory. Let G be an abelian Polish group, e.g. a separable Banach space. A subset $A \subset G$ is called *Haar null (in the sense of Christensen)* if there exists a Borel set $B \supset A$ and a Borel probability measure μ on G such that $\mu(B+g)=0$ for every $g \in G$. The term *shy* is also commonly used for Haar null, and co-Haar null sets are often called *prevalent*.

Answering an old question of J. Mycielski we show that if G is *not* locally compact then there exists a Borel Haar null set that is not contained in any G_δ Haar null set. We also show that G_δ can be replaced by any other class of the Borel hierarchy, which implies that the so called additivity of the σ-ideal of Haar null sets is ω_1.

The definition of a *generalised Haar null set* is obtained by replacing the Borelness of B in the above definition by universal measurability. We give an example of a generalised Haar null set that is not Haar null, more precisely we construct a coanalytic generalised Haar null set without a Borel Haar null hull. This solves Problem GP from Fremlin's problem list. Actually, all our results readily generalise to all Polish groups that admit a two-sided invariant metric.

We also answer one half of Problem FC from Fremlin's list, which asked if we can simply leave out the Borel set B from the definition of Haar null sets. Fremlin noted that the answer was in the negative under the Continuum Hypothesis, but we provide a ZFC counterexample in $\mathbb{R}$. The material of this section can be found in [33] and [29].

Section 1.3 solves a problem posed by the author and M. Laczkovich. In 1990 Kechris and Louveau developed the theory of three very natural ranks on the Baire class 1 functions. A rank is a function assigning countable ordinals to certain objects, typically measuring their complexity. We extend this theory to the case of Baire class ξ functions, and generalise most of the results from the Baire class 1 case. We also show that their assumption of the compactness of the underlying space can be eliminated. As an application, we solve a problem concerning the so called solvability cardinals of systems of difference equations, arising from the theory of geometric decompositions. We also indicate that certain other very natural generalisations of the ranks of Kechris and Louveau surprisingly turn out to be bounded in ω_1, and also that all ranks satisfying some natural properties coincide for bounded functions. The material of this section can be found in [24].

Section 1.4 investigates the four versions of the following problem. Does there exist a monotone (wrt. inclusion) map that assigns a Borel/G_δ hull to every negligible/measurable subset of $[0,1]$? (A *hull* of $A \subset [0,1]$ is a set H containing A such that $\lambda^*(H) = \lambda^*(A)$.) We prove that all versions are independent of ZFC. We also answer a question of Z. Gyenes and D. Pálvölgyi which asks if monotone hulls can be defined for every chain (wrt. inclusion) of measurable sets. We also comment on the problem of hulls of all subsets of $[0,1]$. The material of this section can be found in [27].

Now we turn to Chapter 2.

Section 2.1 deals with certain problems from the theory of cardinal invariants of the continuum. One of the most often cited areas of set theory is that of the so called Cichoń Diagram. This diagram does not only describe all the ZFC inequalities between the ten most commonly used cardinal invariants, but it is also known that every assignment of the cardinals ω_1 and ω_2 to these invariants that is permitted by the diagram is indeed consistent with ZFC.

The problem we consider in this section is how to fit the cardinal invariants of the nullsets of Hausdorff measures into the diagram. Let $0 < r < n$, and $\mathcal{N}_n^r$ be the σ-ideal of sets in $\mathbb{R}^n$ of r-dimensional Hausdorff measure zero. D. H. Fremlin determined the position of the cardinal invariants of this σ-ideal in the Cichoń Diagram, see the figure after Definition 2.1.1 below. This required proving numerous inequalities, but the hard and more useful questions are tipically if the inequalities can be strict in certain models. For one of the remaining ones Fremlin posed this as an open question in his monograph [38]. We answer this by showing that consistently $\mathrm{cov}(\mathcal{N}_n^r) > \mathrm{cov}(\mathcal{N})$. We also prove that the remaining two inequalities can be strict. The proofs use the technique of forcing.

To demonstrate how this result can be used outside of the theory of cardinal invariants, we solve the following problem of P. Humke and M. Laczkovich [47]. Is it consistent that there is an ordering of the reals in which all proper initial segments are Lebesgue null but for every ordering of the reals there is a proper initial segment that is not null with respect to the 1/2-dimensional Hausdorff measure? We determine the values of the cardinal invariants of the Cichoń Diagram as well as the invariants of the nullsets of Hausdorff measures in one of the models of ZFC mentioned in the previous paragraph, and as an application we answer this question of Humke and Laczkovich affirmatively. The material of this section can be found in [31].

In Section 2.2 we answer a question of Darji and Keleti by proving that there exists a compact set $C_{EK} \subset \mathbb{R}$ (first considered by Erdős and Kakutani) of measure zero such that for every nonempty perfect set $P \subset \mathbb{R}$ there exists $x \in \mathbb{R}$ such that $(C_{EK} + x) \cap P$ is uncountable. Using this C_{EK} we answer a question of Gruenhage by showing that it is consistent with ZFC (as it follows e.g. from $\mathrm{cof}(\mathcal{N}) < 2^\omega$) that less than 2^ω many translates of a compact set of measure zero can cover $\mathbb{R}$. The material of this section can be found in [30].

In Section 2.3 we show that the Continuum Hypothesis implies that for every $0 < d_1 \leq d_2 < n$ the measure spaces $\left(\mathbb{R}^n, \mathcal{M}_{\mathcal{H}^{d_1}}, \mathcal{H}^{d_1}\right)$ and $\left(\mathbb{R}^n, \mathcal{M}_{\mathcal{H}^{d_2}}, \mathcal{H}^{d_2}\right)$ are isomorphic, where $\mathcal{H}^d$ is d-dimensional Hausdorff measure and $\mathcal{M}_{\mathcal{H}^d}$ is the σ-algebra of measurable sets with respect to $\mathcal{H}^d$. This is motivated by the well-known question (circulated by D. Preiss and sometimes attributed to B. Weiss as well, and later solved in the negative by A. Máthé) whether such an isomorphism exists if we replace measurable sets by Borel sets. The material of this section can be found in [21].

Section 2.4 investigates the related question whether every continuous function (or the generic continuous function in the sense of Baire category) is Hölder continuous (or is of bounded variation) on a set of positive Hausdorff dimension. We proved some nonzero upper estimates for these dimensions, which later turned out to be sharp. The material of this section can be found in [21].

Section 2.5 solves a problem of R. D. Mauldin by showing that the set of Liouville numbers is either null or non-σ-finite with respect to every translation invariant Borel measure on $\mathbb{R}$, in particular, with respect to every Hausdorff measure $\mathcal{H}^g$ with gauge function g. We also indicate that some other simply defined Borel sets like the set of non-normal or some Besicovitch-Eggleston numbers, as well as all Borel subgroups of $\mathbb{R}$ that are not F_σ possess the above property. We discuss that, apart from some trivial cases, the Borel class, Hausdorff or packing dimension of a Borel set with no such measure on it can be arbitrary. The material of this section can be found in [23].

Section 2.6 considers a question of J. Zapletal. He asked if all the forcing notions considered in his monograph [80] are homogeneous. We prove that e.g. the σ-ideal consisting of Borel sets of σ-finite 2-dimensional Hausdorff measure in $\mathbb{R}^3$ is non-homogeneous. This is a surprising instance when Hausdorff measures are used to answer a set theoretical question! The material of this section can be found in [31].

Basic definitions, notation and terminology

In this chapter we collect the notions and definitions that show up multiple times throughout the dissertation. However, to keep the sections somewhat self-contained, occasionally we will recall these notions and definitions when we feel it is necessary. The notions that are only used in a single section are defined within the section in question.

For descriptive set theory the basic reference is [51], for set theory it is [55] and [50], and for geometric measure theory (mainly Hausdorff measures) it is [67] and [35].

Throughout the dissertation, let $X = (X, \tau) = (X, \tau(X))$ be a *Polish space*, that is, a separable and completely metrisable topological space.

We define the *ξth additive, multiplicative and ambiguous Borel classes* of X, in notation $\mathbf{\Sigma}^0_\xi(X)$, $\mathbf{\Pi}^0_\xi(X)$ and $\mathbf{\Delta}^0_\xi(X)$, respectively, as follows.

Let, by transfinite recursion,

$$\mathbf{\Sigma}^0_1(X) = \tau = \text{the class of open sets},$$

for every $1 \leq \xi$ let

$$\mathbf{\Pi}^0_\xi(X) = \{X \setminus H : H \in \mathbf{\Sigma}^0_\xi(X)\},$$

and for or every $1 < \xi$ let

$$\mathbf{\Sigma}^0_\xi(X) = \{\cup_{i=1}^\infty H_i : \forall i \ H_i \in \cup_{\eta<\xi} \mathbf{\Pi}^0_\eta(X)\}.$$

Finally, for every $1 \leq \xi$ let

$$\mathbf{\Delta}^0_\xi(X) = \mathbf{\Sigma}^0_\xi(X) \cap \mathbf{\Pi}^0_\xi(X).$$

We typically simply write $\mathbf{\Sigma}^0_\xi$, $\mathbf{\Pi}^0_\xi$ and $\mathbf{\Delta}^0_\xi$, when there is no danger of confusion. It is well known that this hierarchy, called the *Borel hierarchy*, stabilises at $\mathbf{\Sigma}^0_{\omega_1}(X) = \mathbf{\Pi}^0_{\omega_1}(X) =$ the class of Borel sets.

For example, $A \in \mathbf{\Delta}^0_2$ iff A is F_σ (countable union of closed sets) and G_δ (countable intersection of open sets) at the same time.

Let us now turn to the hierarchy of functions. For $f : X \to \mathbb{R}$ we define $f \in \mathcal{B}_0(X)$ if f is continuous, and for $1 < \xi$ we say that $f \in \mathcal{B}_\xi(X)$ if there exists a sequence $(f_i)_{i=1}^\infty \subset \cup_{\eta<\xi} \mathcal{B}_\eta(X)$ converging pointwise to f. In such a case we also say that f is of Baire class ξ. Again, we typically drop X from the notation. It is well known that this hierarchy, called the *Baire hierarchy*, stabilises at $\mathcal{B}_{\omega_1} =$ the class of Borel measurable functions.

For a real valued function f on X and a real number c, we let $\{f < c\} = \{x \in X : f(x) < c\}$. We use the notations $\{f > c\}$, $\{f \leq c\}$, $\{f \geq c\}$, $\{f = c\}$ and $\{f \neq c\}$ analogously. It is well-known that a function is of Baire class ξ iff the inverse image of every open set is in $\mathbf{\Sigma}^0_{\xi+1}$ iff $\{f < c\}$ and $\{f > c\}$ are in $\mathbf{\Sigma}^0_{\xi+1}$ for every $c \in \mathbb{R}$.

Specifically, a function f is of Baire class 1 iff it is the pointwise limit of continuous functions iff the preimage of every open set under f is in $\mathbf{\Sigma}^0_2$ iff $\{f < c\}$ and $\{f > c\}$ are in $\mathbf{\Sigma}^0_2$ for every $c \in \mathbb{R}$. This easily implies that a characteristic function χ_A is of Baire class 1 if and only if $A \in \mathbf{\Delta}^0_2$. The above equivalent definitions also imply that semi-continuous functions are of Baire class 1.

A set is called *analytic* if it is the continuous image of a Borel subset of a Polish space. A set is *coanalytic* if its complement is analytic. A set is called *universally measurable* if it is measurable with respect to the completion of every Borel probability measure. Analytic and coanalytic sets are known to be universally measurable.

Let us now fix a compatible metric d on X. The symbol $\mathcal{K}(X)$ will stand for the set of the nonempty compact subsets of X endowed with the Hausdorff metric, that is, for $K_1, K_2 \in \mathcal{K}(X)$

$$d_H(K_1, K_2) = \inf\{\varepsilon : K_1 \subset U_\varepsilon(K_2),\ K_2 \subset U_\varepsilon(K_1)\},$$

where $U_\varepsilon(H) = \{x \in X : \exists y \in H\ d(x,y) < \varepsilon\}$. It is well known that if X is Polish then so is $\mathcal{K}(X)$, and the compactness of X is equivalent to the compactness of $\mathcal{K}(X)$.

Every ordinal is identified with the set of its predecessors, in particular, $2 = \{0, 1\}$. The cardinality of the continuum is denoted by 2^ω and also by $\mathfrak{c}$.

For a set H we denote the closure, cardinality and complement of H by $\overline{H}$, $|H|$ and H^c, respectively.

Let $\mathcal{I}$ be a *σ-ideal* on a Polish space X, that is, a nonempty collection of subsets of X closed under subsets and countable unions, and not containing X. The two most important classical examples are $\mathcal{N}$, the class of Lebesgue nullsets, and $\mathcal{M}$, the class of meagre sets (a set is nowhere dense if it is not dense in any nonempty open set, and a set is meagre if it is the countable union of nowhere dense sets). Moreover, one also often encounters the σ-ideal $\mathcal{K}$ of subsets of the irrational numbers that are coverable by countably many compact subsets of the irrationals. For a σ-ideal let us define the four main cardinal invariants as follows.

$$\begin{aligned}
\mathrm{add}(\mathcal{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subset \mathcal{I}, \textstyle\bigcup \mathcal{A} \notin \mathcal{I}\},\\
\mathrm{cov}(\mathcal{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subset \mathcal{I}, \textstyle\bigcup \mathcal{A} = X\},\\
\mathrm{non}(\mathcal{I}) &= \min\{|H| : H \subset X, H \notin \mathcal{I}\},\\
\mathrm{cof}(\mathcal{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subset \mathcal{I},\ \forall I \in \mathcal{I}\ \exists A \in \mathcal{A}, I \subset A\}.
\end{aligned}$$

These invariants are called the *additivity, covering number, uniformity, and cofinality of $\mathcal{I}$*, respectively.

The *r-dimensional Hausdorff measure* of a set $A \subset \mathbb{R}^n$ is

$$\mathcal{H}^r(A) = \lim_{\delta \to 0+} \mathcal{H}^r_\delta(A), \text{ where}$$

$$\mathcal{H}^r_\delta(A) = \inf\left\{\sum_{k=0}^{\infty} (\mathrm{diam}(A_k))^r : A \subset \bigcup_{k=0}^{\infty} A_k,\ \forall k\ \mathrm{diam}(A_k) \leq \delta\right\}.$$

The *Hausdorff dimension* of a set A is defined by

$$\dim_H(A) = \inf\{r : \mathcal{H}^r(A) = 0\}.$$

If $g : [0,\infty) \to [0,\infty)$ is a nondecreasing function with $g(0) = 0$ then we may also define the *generalised Hausdorff measure with gauge function* g, in symbol, $\mathcal{H}^g$ such that in the above definition we replace $(\mathrm{diam}(A_k))^r$ with $g(\mathrm{diam}(A_k))$.

A subset of a Polish space is called *perfect* if it is closed and has no isolated points. Nonempty perfect sets are of cardinality continuum.

Chapter 1

Descriptive set theory

1.1 Linearly ordered families of Baire class 1 functions

Let $\mathcal{F}(X)$ be a class of real valued functions defined on a Polish space X, e.g. $\mathcal{C}(X)$, the set of continuous functions. The natural partial ordering on this space is the pointwise ordering $<_p$, that is, we say that $f <_p g$ if for every $x \in X$ we have $f(x) \leq g(x)$ and there exists at least one x such that $f(x) < g(x)$. If we would like to understand the structure of this partially ordered set (poset), the first step is to describe its linearly ordered subsets.

For example, if $X = [0,1]$ and $\mathcal{F}(X) = \mathcal{C}([0,1])$ then it is a well known result that the possible order types of the linearly ordered subsets of $\mathcal{C}([0,1])$ are the real order types (that is, the order types of the subsets of the reals). Indeed, a real order type is clearly representable by constant functions, and if $\mathcal{L} \subset \mathcal{C}([0,1])$ is a linearly ordered family of continuous functions then (by continuity) $f \mapsto \int_0^1 f$ is a *strictly* monotone map of $\mathcal{L}$ into the reals.

The next natural class to look at is the class of Lebesgue measurable functions. However, it is not hard to check that the assumption of measurability is rather meaningless here. Indeed, if $\mathcal{L}$ is a linearly ordered family of *arbitrary* real functions and $\varphi : \mathbb{R} \to \mathbb{R}$ is a map that maps the Cantor set onto $\mathbb{R}$ and is zero outside of the Cantor set then $f \mapsto f \circ \varphi$ is a strictly monotone map of $\mathcal{L}$ into the class of Lebesgue measurable functions.

Therefore it is more natural to consider the class of Borel measurable functions. However, P. Komjáth [53] proved that it is already independent of ZFC whether the class of Borel measurable functions contains a strictly increasing transfinite sequence of length ω_2.

The next step is therefore to look at subclasses of the Borel measurable functions, namely the Baire hierarchy. Komjáth actually also proved that in his above mentioned result the set of Borel measurable function can be replaced by the set of Baire class 2 functions. This explains why the Baire class 1 case seems to be the most interesting one. We note that Baire class 1 functions play a central role in various branches of mathematics, most notably in Banach space theory, see e.g. [1] or [45].

Back in the 1970s M. Laczkovich [59] posed the following problem:

Problem 1.1.1. *Characterise the order types of the linearly ordered subsets of* $(\mathcal{B}_1(X), <_p)$.

We will use the following notation:

Definition 1.1.2. Let $(P, <_P)$ and $(Q, <_Q)$ be two posets. We say that P is *embeddable into* Q, in symbols $(P, <_P) \hookrightarrow (Q, <_Q)$, if there exists a map $\Phi : P \to Q$ such that for every $p, q \in P$ if $p <_P q$ then $\Phi(p) <_Q \Phi(q)$. (Note that an embedding may not be 1-to-1 in general. However, an embedding of a *linearly* ordered set *is* 1-to-1.) If $(L, <_L)$ is a linear ordering and $(L, <_L) \hookrightarrow (Q, <_Q)$ then we also say that L *is representable in* Q.

Whenever the ordering of a poset $(P, <_P)$ is clear from the context we will use the notation $P = (P, <_P)$. Moreover, when Q is not specified, the term "representable" will refer to representability in $\mathcal{B}_1(X)$.

The earliest result that is relevant to Laczkovich's problem is due to Kuratowski. He showed that for any Polish space X we have $\omega_1, \omega_1^* \not\hookrightarrow \mathcal{B}_1(X)$, or in other words, there is no ω_1-long strictly increasing or decreasing sequence of Baire class 1 functions (see [56, §24. III.2.]).

It seems conceivable at first sight that this is the only obstruction, that is, every linearly ordered set that does not contain ω_1-long strictly increasing or decreasing sequences is representable in $\mathcal{B}_1(\mathbb{R})$. First, answering a question of Gerlits and Petruska, this conjecture was consistently refuted by P. Komjáth [53] who showed that no Suslin line is representable in $\mathcal{B}_1(\mathbb{R})$. (A *Suslin line* is a nonseparable linearly ordered set containing no uncountable family of disjoint non-degenerate intervals. The existence of a Suslin line is independent of ZFC.) Komjáth's short and elegant proof uses the very difficult set theoretical technique of forcing. Laczkovich [60] asked if a forcing-free proof exists.

The author and Steprāns [28] improved upon the above example. On the one hand we proved that consistently Kuratowski's result is a characterisation for order types of cardinality strictly less than the continuum (and the continuum is large). On the other hand we strengthened Komjáth's result by constructing in ZFC a linearly ordered set L not containing Suslin lines or ω_1-long strictly increasing or decreasing sequences such that L is not representable in $\mathcal{B}_1(X)$.

Among other results, I [22] proved that if X and Y are both uncountable σ-compact or both non-σ-compact Polish spaces then for every linearly ordered set L we have $L \hookrightarrow \mathcal{B}_1(X) \iff L \hookrightarrow \mathcal{B}_1(Y)$. Then I asked if the same holds if X is an uncountable σ-compact and Y is a non-σ-compact Polish space. Moreover, I also asked whether the same linearly ordered sets can be embedded into the set of *characteristic* functions in $\mathcal{B}_1(X)$ as into $\mathcal{B}_1(X)$. Notice that a characteristic function χ_A is of Baire class 1 if and only if A is simultaneously F_σ and G_δ (denoted by $A \in \mathbf{\Delta}^0_2(X)$). Moreover, $\chi_A <_p \chi_B \iff A \subsetneqq B$, hence the above question is equivalent to whether $L \hookrightarrow (\mathcal{B}_1(X), <_p)$ implies $L \hookrightarrow (\mathbf{\Delta}^0_2(X), \subsetneqq)$. I also asked if *duplications* and completions of representable orders are themselves representable, where the duplication of L is $L \times \{0, 1\}$ ordered lexicographically.

Our main result in this section is a complete answer to Problem 1.1.1 and consequently answers to all the above mentioned questions. The solution proceeds by constructing a *universal* linearly ordered set for $\mathcal{B}_1(X)$, that is, a linear order that is representable in $\mathcal{B}_1(X)$ such that every representable linearly ordered set is embeddable into it. Of course such a linear order only provides a useful characterisation if it is sufficiently simple combinatorially to work with. We demonstrate this by indicating new, simpler proofs of the known theorems (including a forcing-free proof of Komjáth's theorem), and also by answering the above mentioned open questions.

The universal linear ordering can be defined as follows.

Definition 1.1.3. Let $[0,1]^{<\omega_1}_{\searrow 0}$ be the set of strictly decreasing well-ordered transfinite sequences in $[0,1]$ with last element zero. Let $\bar{x} = (x_\alpha)_{\alpha\leq\xi}, \bar{x}' = (x'_\alpha)_{\alpha\leq\xi'} \in [0,1]^{<\omega_1}_{\searrow 0}$ be distinct and let δ be the minimal ordinal such that $x_\delta \neq x'_\delta$. We say that

$$(x_\alpha)_{\alpha\leq\xi} <_{altlex} (x'_\alpha)_{\alpha\leq\xi'} \iff (\delta \text{ is even and } x_\delta < x'_\delta) \text{ or } (\delta \text{ is odd and } x_\delta > x'_\delta).$$

Now we can formulate the main result of the section.

Theorem 1.1.4. *Let X be an uncountable Polish space. Then the following are equivalent for a linear ordering $(L,<)$:*

(1) $(L,<) \hookrightarrow (\mathcal{B}_1(X), <_p)$,

(2) $(L,<) \hookrightarrow ([0,1]^{<\omega_1}_{\searrow 0}, <_{altlex})$.

In fact, $(\mathcal{B}_1(X), <_p)$ *and* $([0,1]^{<\omega_1}_{\searrow 0}, <_{altlex})$ *are embeddable into each other.*

Using this theorem one can reduce every question concerning the linearly ordered subsets of $\mathcal{B}_1(X)$ to a purely combinatorial problem. We were able to answer all of the known such questions and we reproved easily the known theorems as well. The most important results are:

- Answering another question of Laczkovich, we give a new, forcing free proof of Komjáth's theorem.
- The class of ordered sets representable in $\mathcal{B}_1(X)$ does not depend on the uncountable Polish space X.
- There exists an embedding $(\mathcal{B}_1(X), <_p) \hookrightarrow (\mathbf{\Delta}^0_2(X), \subsetneqq)$, hence a linear ordering is representable by Baire class 1 functions iff it is representable by Baire class 1 *characteristic* functions.
- The duplication of a representable linearly ordered set is representable. More generally, countable lexicographical products of representable sets are representable.
- There exists a linearly ordered set that is representable in $\mathcal{B}_1(X)$ but none of its completions are representable.

About the proofs. The proof of the main result consists of two parts. First we prove that there exists an embedding $\mathcal{B}_1(X) \hookrightarrow [0,1]^{<\omega_1}_{\searrow 0}$ building heavily on a method of Kechris and Louveau [52] on how to write every bounded Baire class 1 function as an alternating series of a decreasing transfinite sequence of upper semi-continuous functions. Unfortunately for us, they only consider the case of compact Polish spaces, while it is of crucial importance in our proof to use their theorem for arbitrary Polish spaces. Moreover, their proof seems to contain a slight error. Hence it was unavoidable to reprove and generalise their result. Then the second part is $[0,1]^{<\omega_1}_{\searrow 0} \hookrightarrow \mathcal{B}_1(X)$, which is also a delicate and long argument somewhat building on a former construction in [28]. For the proof of the main result see Section 3.1, for the proofs of the above listed corollaries consult [32].

1.2 Haar null sets in Polish groups

Polish groups have recently been playing a central role in modern descriptive set theory. This section is concerned with an analogue of Lebesgue null sets in this setting.

Throughout this section, let G be an abelian Polish group, that is, an abelian topological group whose topology is Polish. The group operation will be denoted by $+$ and the neutral element by 0. It is a well-known result of Birkhoff and Kakutani that any metrisable group admits a left invariant metric [6, 1.1.1], which is clearly two-sided invariant for abelian groups. Moreover, it is also well-known that a two-sided invariant metric on a Polish group is complete [6, 1.2.2]. Hence from now on let d be a fixed complete two-sided invariant metric on G. For the ease of notation we will restrict our attention to abelian groups, but we remark that all our results easily generalise to all Polish groups admitting a two-sided invariant metric.

If G is locally compact than there exists a Haar measure on G, that is, a regular invariant Borel measure that is finite for compact sets and positive for nonempty open sets. This measure, which is unique up to a positive multiplicative constant, plays a fundamental role in the study of locally compact groups. Unfortunately, it is known that non-locally compact Polish groups admit no Haar measure. However, the notion of a Haar nullset has a very well-behaved generalisation. The following definition was invented by Christensen [12], and later rediscovered by Hunt, Sauer and Yorke [49]. (Actually, Christensen's definition was what we call generalised Haar null below, but this subtlety will only play a role later.)

Definition 1.2.1. A set $A \subset G$ is called *Haar null* if there exists a Borel set $B \supset A$ and a Borel probability measure μ on G such that $\mu(B+g) = 0$ for every $g \in G$.

Note that the term *shy* is also commonly used for Haar null, and co-Haar null sets are often called *prevalent*.

Christensen showed that the Haar null sets form a σ-ideal, and also that in locally compact groups a set is Haar null iff it is of measure zero with respect to the Haar measure. During the last two decades Christensen's notion has been very useful in studying exceptional sets in diverse areas such as analysis, functional analysis, dynamical systems, geometric measure theory, group theory, and descriptive set theory.

Therefore it is very important to understand the fundamental properties of this σ-ideal, such as the Fubini properties, ccc-ness, and all other similarities and differences between the locally compact and the general case.

One such example is the following very natural question, which was Problem 1 in Mycielski's celebrated paper [69] more than 25 years ago, and was also discussed e.g. in [20], [4] and [75].

Question 1.2.2. *(J. Mycielski) Let G be a Polish group. Can every Haar null set be covered by a G_δ Haar null set?*

It is easy to see using the regularity of Haar measure that the answer is in the affirmative if G is locally compact.

The first main goal of the present section is to answer this question.

Theorem 1.2.3. *If G is a non-locally compact abelian Polish group then there exists a (Borel) Haar null set $B \subset G$ that cannot be covered by a G_δ Haar null set.*

Actually, the proof will immediately yield that G_δ can be replaced by any other class of the Borel hierarchy. As usual, $\mathbf{\Pi}^0_\xi$ stands for the ξth multiplicative class of the Borel hierarchy.

Theorem 1.2.4. *If G is a non-locally compact abelian Polish group and $1 \leq \xi < \omega_1$ then there exists a (Borel) Haar null set $B \subset G$ that cannot be covered by a $\mathbf{\Pi}^0_\xi$ Haar null set.*

It was pointed out to us by Sz. Głąb, see e.g. [8, Proposition 5.2] that an easy but very surprising consequence of this theorem is the following.

Corollary 1.2.5. *If G is a non-locally compact abelian Polish group then the additivity of the σ-ideal of Haar null sets is ω_1.*

In order to be able to formulate the next question we need to introduce a slightly modified notion of Haar nullness. Numerous authors actually use the following weaker definition, in which B is only required to be universally measurable. (A set is called *universally measurable* if it is measurable with respect to every Borel probability measure. Borel measures are identified with their completions.)

Definition 1.2.6. A set $A \subset G$ is called *generalised Haar null* if there exists a universally measurable set $B \supset A$ and a Borel probability measure μ on G such that $\mu(B + g) = 0$ for every $g \in G$.

In most applications A is actually Borel, so it does not matter which of the above two definitions we use. Still, it is of substantial theoretical importance to understand the relation between the two definitions. The next question is from Fremlin's problem list [39].

Question 1.2.7. *(D. H. Fremlin, Problem GP) Is every generalised Haar null set Haar null? In other words, can every generalised Haar null set be covered by a Borel Haar null set?*

Dougherty [20, p.86] showed that under the Continuum Hypothesis or Martin's Axiom the answer is in the negative in every non-locally compact Polish group with a two-sided invariant metric. Later Banakh [4] proved the same under slightly different set theoretical assumptions. Dougherty uses transfinite induction, and Banakh's proof is basically an existence proof using that the cofinality of the σ-ideal of generalised Haar null sets is greater than the continuum in some models, hence these examples are clearly very far from being Borel.

The second main goal of the section is to answer Fremlin's problem in ZFC.

Recall that a set is *analytic* if it is the continuous image of a Borel set, and *coanalytic* if its complement is analytic. Analytic and coanalytic sets are known to be universally measurable. Since Solecki [75] proved that every analytic generalised Haar null set is contained in a Borel Haar null set, the following result is optimal.

Theorem 1.2.8. *Not every generalised Haar null set is Haar null. More precisely, if G is a non-locally compact abelian Polish group then there exists a coanalytic generalised Haar null set $P \subset G$ that cannot be covered by a Borel Haar null set.*

We also answer one half of Problem FC from Fremlin's list, which essentially asked the following.

Problem 1.2.9. *Can one simply leave out the Borel set B from the definition of Haar null sets?*

Fremlin noted that the answer was in the negative under the Continuum Hypothesis. The third main theorem of the section provides in ZFC a counterexample in $\mathbb{R}$.

Theorem 1.2.10. *Problem 1.2.9 has a negative answer in $\mathbb{R}$, that is, there exist $X \subset \mathbb{R}$ with $\lambda(X) > 0$ and a Borel probability measure μ such that $\mu(X+t) = 0$ for every $t \in \mathbb{R}$.*

For more results concerning fundamental properties and applications of Haar null sets in non-locally compact groups see e.g. [2], [3], [15], [18], [19], [29], [46], [66], [76], [78].

About the proofs. Surprisingly, the proofs of Theorem 1.2.4 and Theorem 1.2.8 are essentially identical. The hard part is to show that the potential hulls are not Haar null. On the one hand we use uniformisation results and other tricks to solve the toy problem when the potential witness measures are restricted to be just the 1-dimensional Lebesgue measure on the y-axis in the plane, and second we apply a nice result of Solecki to find pairwise disjoint translates of all compact sets, and mimic the toy problem in each. See Section 3.2 or [33].

As for Theorem 1.2.10, the key idea is to find sufficiently 'thin' Cantor sets supporting our witness measure μ, and surprisingly this thinness is understood in the sense of small fractal dimension! An interesting instance of fractals popping up unexpectedly. See Section 3.3 or [29]

1.3 Ranks on the Baire class ξ functions

It is well-known that f is of Baire class 1 iff it is the pointwise limit of a sequence of continuous functions iff the inverse image of every open set is F_σ iff there is a point of continuity relative to every nonempty closed set [51]. Baire class 1 functions play a central role in various branches of mathematics, most notably in Banach space theory, see e.g. [1] or [45]. A fundamental tool in the analysis of Baire class 1 functions is the theory of ranks, that is, maps assigning countable ordinals to Baire class 1 functions, typically measuring their complexity. In their seminal paper [52], Kechris and Louveau systematically investigated three very important ranks, called α, β and γ, on the Baire class 1 functions. We only spell out the rather technical definitions in Chapter 3, and only note here that they correspond to above three equivalent definitions of Baire class 1 functions. One can easily see that the theory has no straightforward generalisation to the case of Baire class ξ functions.

Hence the following very natural but somewhat vague question arises.

Question 1.3.1. *Is there a natural extension of the theory of Kechris and Louveau to the case of Baire class ξ functions?*

There is actually a very concrete version of this question that was raised by the author and Laczkovich in [26]. In order to be able to formulate this we need some preparation. For $\theta, \theta' < \omega_1$ let us define the relation $\theta \lesssim \theta'$ if $\theta' \leq \omega^\eta \implies \theta \leq \omega^\eta$ for every $1 \leq \eta < \omega_1$ (we use ordinal exponentiation here). Note that $\theta \leq \theta'$ implies $\theta \lesssim \theta'$, while $\theta \lesssim \theta'$, $\theta' > 0$ implies $\theta \leq \theta' \cdot \omega$. We will also use the notation $\theta \approx \theta'$ if $\theta \lesssim \theta'$ and $\theta' \lesssim \theta$. Then $\approx$ is an equivalence relation. Let us denote the set of Baire class ξ functions defined on $\mathbb{R}$ by $\mathcal{B}_\xi(\mathbb{R})$. The characteristic function of a set H is denoted by χ_H. Define the translation map $T_t : \mathbb{R} \to \mathbb{R}$ by $T_t(x) = x + t$ for every $x \in \mathbb{R}$.

Question 1.3.2. *[26, Question 6.7] Is there a map $\rho : \mathcal{B}_\xi(\mathbb{R}) \to \omega_1$ such that*

- *ρ is unbounded in ω_1, moreover, for every nonempty perfect set $P \subset \mathbb{R}$ and ordinal $\zeta < \omega_1$ there is a function $f \in \mathcal{B}_\xi(\mathbb{R})$ such that f is 0 outside of P and $\rho(f) \geq \zeta$,*
- *ρ is translation-invariant, i.e., $\rho(f \circ T_t) = \rho(f)$ for every $f \in \mathcal{B}_\xi(\mathbb{R})$ and $t \in \mathbb{R}$,*
- *ρ is essentially linear, i.e., $\rho(cf) \approx \rho(f)$ and $\rho(f + g) \lesssim \max\{\rho(f), \rho(g)\}$ for every $f, g \in \mathcal{B}_\xi(\mathbb{R})$ and $c \in \mathbb{R} \setminus \{0\}$,*
- *$\rho(f \cdot \chi_F) \lesssim \rho(f)$ for every closed set $F \subset \mathbb{R}$ and $f \in \mathcal{B}_\xi(\mathbb{R})$?*

The problem is not formulated in this exact form in [26], but a careful examination of the proofs there reveals that this is what we need for the results to go through. Actually, there are numerous equivalent formulations, for example we may simply replace $\lesssim$ by $\leq$ (indeed, just replace ρ satisfying the above properties by $\rho'(f) = \min\{\omega^\eta : \rho(f) \leq \omega^\eta\}$). However, it turns out, as it was already also the case in [52], that $\lesssim$ is more natural here.

Their original motivation came from the theory of paradoxical geometric decompositions (like the Banach-Tarski paradox, Tarski's problem of circling the square, etc.). It has turned out that the solvability of certain systems of difference equations plays a key role in this theory.

Definition 1.3.3. Let $\mathbb{R}^\mathbb{R}$ denote the set of functions from $\mathbb{R}$ to $\mathbb{R}$. A *difference operator* is a mapping $D : \mathbb{R}^\mathbb{R} \to \mathbb{R}^\mathbb{R}$ of the form

$$(Df)(x) = \sum_{i=1}^{n} a_i f(x + b_i),$$

where a_i and b_i are fixed real numbers.

Definition 1.3.4. A *difference equation* is a functional equation

$$Df = g,$$

where D is a difference operator, g is a given function and f is the unknown.

Definition 1.3.5. A *system of difference equations* is

$$D_i f = g_i \quad (i \in I),$$

where I is an arbitrary set of indices.

It is not very hard to show that a system of difference equations is solvable iff every *finite* subsystem is solvable. But if we are interested in continuous solutions then this result is no longer true. However, if every *countable* subsystem of a system has a continuous solution the the whole system has a continuous solution as well. This motivates the following definition, which has turned out to be a very useful tool for finding necessary conditions for the existence of certain solutions.

Definition 1.3.6. Let $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ be a class of real functions. The *solvability cardinal* of $\mathcal{F}$ is the minimal cardinal $sc(\mathcal{F})$ with the property that if every subsystem of size less than $sc(\mathcal{F})$ of a system of difference equations has a solution in $\mathcal{F}$ then the whole system has a solution in $\mathcal{F}$.

It was shown in [26] that the behavior of $sc(\mathcal{F})$ is rather erratic. For example, $sc(\text{polynomials}) = 3$ but $sc(\text{trigonometric polynomials}) = \omega_1$, $sc(\{f : f \text{ is continuous}\}) = \omega_1$ but $sc(\{f : f \text{ is Darboux}\}) = (2^\omega)^+$, and $sc(\mathbb{R}^{\mathbb{R}}) = \omega$.

It is also proved in that paper that $\omega_2 \leq sc(\{f : f \text{ is Borel}\}) \leq (2^\omega)^+$, therefore if we assume the Continuum Hypothesis then $sc(\{f : f \text{ is Borel}\}) = \omega_2$. Moreover, they obtained that $sc(\mathcal{B}_\xi) \leq (2^\omega)^+$ for every $2 \leq \xi < \omega_1$, and asked if $sc(\mathcal{B}_\xi) \geq \omega_2$. We noted that a positive answer to Question 1.3.2 would yield a positive answer here.

For more information on the connection between ranks, solvability cardinals, systems of difference equations, liftings, and paradoxical decompositions consult [26], [58], [57] and the references therein.

In order to be able to answer the above questions we need to address one more problem. This is slightly unfortunate for us, but Kechris and Louveau have only worked out their theory in compact metric spaces, while it is really essential for our purposes to be able to apply the results in arbitrary Polish spaces.

Question 1.3.7. *Does the theory of Kechris and Louveau generalise from compact metric spaces to arbitrary Polish spaces?*

Now the main result of the section is that the answer to all the above questions is in the affirmative.

Theorem 1.3.8. *The answers to Question 1.3.1, Question 1.3.2 and Question 1.3.7 are all in the affirmative.*

Corollary 1.3.9. *Let $2 \leq \xi < \omega_1$. Then $sc(\mathcal{B}_\xi) \geq \omega_2$, and hence if we assume the Continuum Hypothesis then $sc(\mathcal{B}_\xi) = \omega_2$.*

Moreover, we propose numerous very natural ranks on the Baire class ξ functions, using simply that these functions are limits of elements of the smaller classes, which surprisingly turn out to be bounded in ω_1!

Also, we prove that if a rank has certain natural properties then it coincides with the ranks α, β and γ of Kechris and Louveau on the bounded Baire class 1 functions. We also indicate how one could generalise this to the bounded Baire class ξ case.

About the proofs. The key idea is to apply topology refinement methods. Namely, for a Baire class ξ function f on a Polish space (X, τ) there is a finer

Polish topology $\tau' \subset \mathbf{\Sigma}^0_\xi(\tau)$ such that f is of Baire class 1 with respect to τ'. This allows us to fix a rank on the Baire class 1 functions and obtain a new one by taking the minimum of the ranks of the Baire class 1 functions obtained in this way. We actually define four ranks on every $\mathcal{B}_\xi$, but two of these turn out to be essentially equal, and the resulting three ranks are very good analogues of the original ranks of Kechris and Louveau.

Topology refinements do not preserve compactness, hence it was essential to extend the results of Kechris and Louveau to the non-compact case. See Section 3.4 or [24].

1.4 Can we assign the Borel hulls in a monotone way?

Let us denote by $\mathcal{N}, \mathcal{L}, \mathcal{B}$ and $\mathcal{G}_\delta$ the class of Lebesgue negligible, Lebesgue measurable, Borel and G_δ subsets of $[0,1]$, respectively. Let $\lambda(A)$ stand for the Lebesgue measure of A, or, if A is nonmeasurable, the Lebesgue outer measure of A.

Definition 1.4.1. A set $H \subset [0,1]$ is a *hull* of $A \subset [0,1]$, if $H \supset A$ and $\lambda(H) = \lambda(A)$.

By regularity, every set has a Borel, even a G_δ hull. It is then very natural to ask whether 'a bigger set has a bigger hull'. (For the two original motivations of the problem see below.)

Definition 1.4.2. Let $\mathcal{D}$ and $\mathcal{H}$ be two subclasses of $\mathcal{P}([0,1])$ (usually $\mathcal{D}$ is $\mathcal{N}$ or $\mathcal{L}$, and $\mathcal{H}$ is $\mathcal{B}$ or $\mathcal{G}_\delta$). If there exists a map $\varphi : \mathcal{D} \to \mathcal{H}$ such that

1. $\varphi(D)$ is a hull of D for every $D \in \mathcal{D}$,
2. $D \subset D'$ implies $\varphi(D) \subset \varphi(D')$,

then we say that *a monotone $\mathcal{H}$ hull operation on $\mathcal{D}$ exists.*

The four questions we address in this section are the following.

Question 1.4.3. *Let $\mathcal{D}$ be either $\mathcal{N}$ or $\mathcal{L}$, and let $\mathcal{H}$ be either $\mathcal{B}$ or $\mathcal{G}_\delta$. Does there exist a monotone $\mathcal{H}$ hull operation on $\mathcal{D}$?*

The problem was originally motivated by the following question.

Question 1.4.4. *(Z. Gyenes and D. Pálvölgyi [42]) Suppose that $\mathcal{C} \subset \mathcal{L}$ is a chain of sets, i.e. for every $C, C' \in \mathcal{C}$ either $C \subset C'$ or $C' \subset C$ holds. Does there exist a monotone $\mathcal{B}/G_\delta$ hull operation on $\mathcal{C}$?*

Remark 1.4.5. Another motivation for our set of problems is that it seems to be very closely related to the huge theory of so called liftings. A map $l : \mathcal{L} \to \mathcal{L}$ is called a *lifting* if it preserves $\emptyset$, finite unions and complement, moreover, it is constant on the equivalence classes modulo nullsets, and also it maps each equivalence class to one of its members. Note that liftings are clearly monotone. For a survey of this theory see the chapter by Strauss, Macheras and Musiał in [44], or the chapter by Fremlin in [43], or Fremlin [38]. Note that the existence of Borel liftings is known to be independent of ZFC, but the existence of a lifting with range in a fixed Borel class is not known to be consistent.

The next two theorems show that all four versions of our problem are independent of ZFC.

Theorem 1.4.6. *It is consistent with ZFC that there is no monotone Borel hull operation on $\mathcal{N}$.*

Theorem 1.4.7. *It is consistent with ZFC that there is a monotone $\mathcal{G}_\delta$ hull operation on $\mathcal{L}$.*

From the latter result and the proof of the former one one readily obtains the following.

Corollary 1.4.8. *The question of Gyenes and Pálvölgyi is also independent of ZFC.*

We also remark here that the results (and proofs) of this section remain valid if we replace $[0,1]$ by $\mathbb{R}$, or by $\mathbb{R}^n$, or more generally, by any uncountable Polish space endowed with a nonzero continuous σ-finite Borel measure. Moreover, one can show using the so called density topology that the existence of hulls for measurable sets is equivalent to the existence of hulls for all sets.

We would also like to mention that this line of research has been continued in two papers of S. Shelah [71, 37], who is arguably the greatest set theorist alive.

About the proofs. The negative statement holds in the so called Cohen model, the positive one is a rather involved construction using the Continuum Hypothesis. See Sections 3.5 and 3.6 or [27].

Chapter 2

Set theory and Hausdorff measures

Now we start describing other types of connections between set theory and analysis.

2.1 Cardinal invariants of Hausdorff measures

Our next problem concerns fitting the cardinal invariants of the σ-ideal of nullsets of the Hausdorff measures into the Cichoń Diagram. For more information on this diagram consult [5].

Definition 2.1.1. Let $0 < r < n$ and let

$$\mathcal{N}_n^r = \{H \subset \mathbb{R}^n : \mathcal{H}^r(H) = 0\}.$$

D. H. Fremlin [38, 534B] showed that the picture is as follows. An arrow $\kappa \to \lambda$ means $\kappa \leq \lambda$.

$$\begin{tikzcd}
\mathrm{cov}(\mathcal{N}) \arrow[r] & \mathrm{cov}(\mathcal{N}_n^r) \arrow[r] & \mathrm{non}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{N}_n^r) \arrow[r, phantom, "="] & \mathrm{cof}(\mathcal{N}) \\
 & & \mathfrak{b} \arrow[u] \arrow[r] & \mathfrak{d} \arrow[u] & & \\
\mathrm{add}(\mathcal{N}) \arrow[uu] \arrow[r, phantom, "="] & \mathrm{add}(\mathcal{N}_n^r) \arrow[uu] \arrow[r] & \mathrm{add}(\mathcal{M}) \arrow[u] \arrow[r] & \mathrm{cov}(\mathcal{M}) \arrow[u] \arrow[r] & \mathrm{non}(\mathcal{N}_n^r) \arrow[uu] \arrow[r] & \mathrm{non}(\mathcal{N}) \arrow[uu]
\end{tikzcd}$$

All but three arrows (= inequalities) are known to be strict in the appropriate models (see e.g. [5] for the inequalities not involving $\mathcal{N}_n^r$ and [73] for $\mathrm{non}(\mathcal{N}_n^r) < \mathrm{non}(\mathcal{N})$). Fremlin, addressing one of these three questions, asked the following.

Question 2.1.2. *[38, 534Z, Problem (a)] Does* $\mathrm{cov}(\mathcal{N}) = \mathrm{cov}(\mathcal{N}_n^r)$ *hold in ZFC?*

The next theorem answers this question in the negative.

Theorem 2.1.3. *It is consistent with ZFC that* $\mathrm{cov}(\mathcal{N}) = \omega_1$ *and* $\mathrm{cov}(\mathcal{N}_n^r) = \omega_2$.

The next two theorems handle the two remaining open questions concerning the consistent strictness of the inequalities in the above extended Chichoń Diagram.

Theorem 2.1.4. *It is consistent with ZFC that* $\mathrm{cov}(\mathcal{N}_n^r) = \omega_1$ *and* $\mathrm{non}(\mathcal{M}) = \omega_2$.

Theorem 2.1.5. *It is consistent with ZFC that* $\mathrm{cov}(\mathcal{M}) = \omega_1$ *and* $\mathrm{non}(\mathcal{N}_n^r) = \omega_2$.

About the proofs. All three models are constructed by the method of forcing. The first one is a 'Zapletal-style' iterated forcing with the σ-ideal $\mathcal{I}^r_{n,\sigma-fin}$ to be defined in the next section (see Section 3.7 or [31]), the second model is the so called Laver model (see Section 3.8 or [31]), and the third one is a 'Rosłanowski-Shelah-type' creature forcing (see Section 3.9 or [31]).

As an application, we answer a problem that was formulated in a recent preprint of P. Humke and M. Laczkovich [47]. Working on certain generalisations of results of Sierpiński and of Erdős they isolated the following definition.

Definition 2.1.6. For a σ-ideal $\mathcal{I}$ on $\mathbb{R}$ let us abbreviate the following statement as

$$(*)_{\mathcal{I}} \iff \exists \text{ an ordering of } \mathbb{R} \text{ such that all proper initial segments are in } \mathcal{I}.$$

Using this notation our problem can be formulated as follows.

Question 2.1.7. *[47] Does* $(*)_{\mathcal{N}}$ *imply* $(*)_{\mathcal{N}_1^{1/2}}$*?*

We remark here that the answer is obviously true in some models of ZFC, e.g. if the Continuum Hypothesis holds, since in that case there exists an ordering of $\mathbb{R}$ such that all proper initial segments are countable.

The following is easy to see and is also shown in [47].

Claim 2.1.8. $\mathrm{add}(\mathcal{I}) = \mathrm{cov}(\mathcal{I}) \implies (*)_{\mathcal{I}} \implies \mathrm{cov}(\mathcal{I}) \leq \mathrm{non}(\mathcal{I})$.

Hence it suffices to answer the following question affirmatively.

Question 2.1.9. *Is it consistent with ZFC that* $\mathrm{add}(\mathcal{N}) = \mathrm{cov}(\mathcal{N})$ *and* $\mathrm{cov}(\mathcal{N}_1^{1/2}) > \mathrm{non}(\mathcal{N}_1^{1/2})$*?*

Our next theorem provides the answer.

Theorem 2.1.10. *It is consistent with ZFC that* $\mathrm{add}(\mathcal{N}) = \mathrm{cov}(\mathcal{N})$ *and* $\mathrm{cov}(\mathcal{N}_1^{1/2}) > \mathrm{non}(\mathcal{N}_1^{1/2})$.

About the proofs. The proof is the calculation of the values of the cardinal invariants of the extended Cichoń Diagram in the first model mentioned above. See Section 3.7 or [31].

2.2 Less than 2^ω many translates of a compact nullset may cover the real line

In this section we are interested in some variants of the cardinal invariant $\mathrm{cov}(\mathcal{N})$. There are two natural ways to modify this definition. (See [5] Chapters 2.6 and 2.7.) First, $\mathrm{cov}^*(\mathcal{N})$ is the least cardinal κ for which it is possible to cover $\mathbb{R}$ by κ many translates of some nullset. In other words, $\mathrm{cov}^*(\mathcal{N}) = \min\{|A| \mid A \subset \mathbb{R}, \exists N \in \mathcal{N}, A + N = \mathbb{R}\}$, where $A+N = \{a+n \mid a \in A, n \in N\}$. The other possible modification is $\mathrm{cov}(c\mathcal{N})$, that is the least cardinal κ for which it is possible to cover $\mathbb{R}$ by κ many compact nullsets. (At this point we depart from the terminology of [5] as this notion is denoted by $\mathrm{cov}(\mathcal{E})$ there. Moreover, $c\mathcal{N}$ is not a σ-ideal, so we should actually consider the sets that can be covered by countably many compact nullsets, but this causes no difference here.) It can be found in these two chapters of this monograph that both $\mathrm{cov}^*(\mathcal{N}) < 2^\omega$ and $\mathrm{cov}(c\mathcal{N}) < 2^\omega$ are consistent with ZFC.

G. Gruenhage posed the natural question whether $\mathrm{cov}^*(c\mathcal{N}) < 2^\omega$ is also consistent, that is, whether we can consistently cover $\mathbb{R}$ by less than continuum many translates of a compact nullset.

The main goal of this section is to answer this question in the affirmative via an answer (in ZFC) to a question of U. B. Darji and T. Keleti that is also interesting in its own right.

We remark here that under CH (the Continuum Hypothesis, or more generally under $\mathrm{cov}(\mathcal{N}) = 2^\omega$) the real line obviously cannot be covered by less than 2^ω many nullsets. Therefore it is consistent that the type of covering we are looking for does not exist.

So the interesting case is when the consistent inequality $\mathrm{cov}^*(\mathcal{N}) < 2^\omega$ holds. The nullset in this statement can obviously be chosen to be G_δ. So the content of Gruenhage's question actually is whether this can be an F_σ or closed or compact nullset. We formulate the strongest version.

Question 2.2.1. *(G. Gruenhage) Is it consistent that there exists a compact set $C \subset \mathbb{R}$ of Lebesgue measure zero and $A \subset \mathbb{R}$ of cardinality less than 2^ω such that $C + A = \mathbb{R}$?*

For example Gruenhage showed that no such covering is possible if C is the usual ternary Cantor set (see [16] and for another motivation of this question see [41]).

Working on this question Darji and Keleti [16] introduced the following notion.

Definition 2.2.2. (U. B. Darji - T. Keleti) Let $C \subset \mathbb{R}$ be arbitrary. A set $P \subset \mathbb{R}$ is called a *witness for C* if P is nonempty perfect and for every translate $C + x$ of C we have that $(C + x) \cap P$ is countable.

Obviously, if there is a witness P for C then less than 2^ω many translates of C cannot cover P (since nonempty perfect sets are of cardinality continuum), so they cannot cover $\mathbb{R}$ as well. Motivated by a question of R. D. Mauldin, who asked what can be said if C is of Hausdorff dimension strictly less than 1, Darji and Keleti proved the following. (For the definition of packing dimension see [67] or [35].)

Theorem 2.2.3. *(U. B. Darji - T. Keleti) If $C \subset \mathbb{R}$ is a compact set of packing dimension* $\dim_p(C) < 1$ *then there is a witness for C, and consequently less than 2^ω translates of C cannot cover $\mathbb{R}$.*

They posed the following question, an affirmative answer to which would also answer the original question of Gruenhage in the negative.

Question 2.2.4. *(U. B. Darji - T. Keleti) Is there a witness for every compact set $C \subset \mathbb{R}$ of Lebesgue measure zero?*

We will answer this question in the negative, which still leaves the original question of Gruenhage open.

The following set is fairly well known in geometric measure theory, as it is probably the most natural example of a compact set of measure zero but of Hausdorff and packing dimension 1. It was investigated for example by Erdős and Kakutani [34].

Definition 2.2.5. Denote

$$C_{EK} = \left\{ \sum_{n=2}^{\infty} \frac{d_n}{n!} \;\middle|\; \forall n \ d_n \in \{0, 1, \ldots, n-2\} \right\}.$$

Think of d_n as digits with "increasing base"; then all but countably many $x \in [0, 1]$ have a unique expansion

$$x = \sum_{n=2}^{\infty} \frac{x_n}{n!},$$

where $x_n \in \{0, 1, \ldots, n-1\}$ for every $n = 2, 3, \ldots$

Theorem 2.2.6. *For every nonempty perfect set $P \subset \mathbb{R}$ there exists a translate $C_{EK} + x$ of the compact nullset C_{EK} such that $(C_{EK} + x) \cap P$ is uncountable.*

Then we will show that using the same ideas it is also possible to give an affirmative answer to Gruenhage's question. Recall that $\mathcal{N}$ is the ideal of measure zero sets, *cof*($\mathcal{N}$) is the minimal cardinality of a family $\mathcal{F} \subset \mathcal{N}$ for which every nullset is contained in some member of $\mathcal{F}$, and also that there is a model of ZFC in which $\mathrm{cof}(\mathcal{N}) < \mathfrak{c}$, see [5, p. 388].

Theorem 2.2.7. *$\mathbb{R}$ can be covered by cof($\mathcal{N}$) many translates of C_{EK}, consequently there is a model of ZFC in which $\mathbb{R}$ can be covered by less than continuum many translates of a compact nullset.*

We remark that building on our ideas A. Máthé [62] also answered Mauldin's question.

About the proofs. The key point was to realise that Gruenhage's problem has to do with fractal dimension. After that we had to find a compact Lebesgue nullset of full Hausdorff dimension, namely the above C_{EK}, and use set theoretic techniques to prove the result. This technique was the theory of so called *slaloms.* See Sections 3.10 and 3.11 or [30].

2.3 Are all Hausdorff measures isomorphic?

The following problem was circulated by D. Preiss, while it is unclear, who actually asked this first, see also [13], where the question is attributed to Preiss. (Sometimes the problem is under the names of D. Preiss and B. Weiss.) Let $\mathcal{B}$ denote the σ-algebra of Borel subsets of $\mathbb{R}^n$. By *isomorphism* of two measure spaces we mean a bijection f such that both f and f^{-1} are measurable set and measure preserving.

Question 2.3.1. *Let $0 < d_1 < d_2 < n$. Are the measure spaces $\left(\mathbb{R}^n, \mathcal{B}, \mathcal{H}^{d_1}\right)$ and $\left(\mathbb{R}^n, \mathcal{B}, \mathcal{H}^{d_2}\right)$ isomorphic?*

An equally natural question is whether such an isomorphism exists if we replace Borel sets by measurable sets with respect to Hausdorff measures. Denote by $\mathcal{M}_d$ the σ-algebra of measurable sets with respect to $\mathcal{H}^d$, in the usual sense of Carathéodory.

Question 2.3.2. *Let $0 < d_1 < d_2 < n$. Are the measure spaces $\left(\mathbb{R}^n, \mathcal{M}_{d_1}, \mathcal{H}^{d_1}\right)$ and $\left(\mathbb{R}^n, \mathcal{M}_{d_2}, \mathcal{H}^{d_2}\right)$ isomorphic?*

The main result of the section is the following affirmative answer to this question assuming the Continuum Hypothesis.

Theorem 2.3.3. *Under the Continuum Hypothesis, for every $0 < d_1 \leq d_2 < n$ the measure spaces $\left(\mathbb{R}^n, \mathcal{M}_{d_1}, \mathcal{H}^{d_1}\right)$ and $\left(\mathbb{R}^n, \mathcal{M}_{d_2}, \mathcal{H}^{d_2}\right)$ are isomorphic.*

We do not know if the assumption of the Continuum Hypothesis can be dropped in Theorem 2.3.3. However, it is rather unlikely as the following remark shows. As above, denote by $\mathcal{N}_d$ the σ-ideal of negligible sets with respect to $\mathcal{H}^d$. If it were known that in some model of set theory $\operatorname{non}(\mathcal{N}_{d_1}) \neq \operatorname{non}(\mathcal{N}_{d_2})$ held, it would be proven that the Continuum Hypothesis cannot be dropped in Theorem 2.3.3. However, so far the above statement is only known to hold in some model when $d_2 = n$ (see [73]).

We note here that Question 2.3.1 was answered by A. Máthé in the negative [63], first for certain values of d_1 and d_2 using our methods outlined in the next section, then in general using different techniques.

About the proofs. The proof is a rather involved transfinite construction, where the key point was to make sure that certain strange sets are measurable. See Section 3.12 or [21].

2.4 Regular restrictions of functions

In this section, motivated by Question 2.3.1, we consider the following set of problems.

Question 2.4.1. *Can we find for every $f : [0, 1] \to \mathbb{R}$ continuous/Borel/typical continuous (in the Baire category sense, see e.g. [10]) function a set of positive Hausdorff dimension on which the function agrees with a function of bounded variation/Lipschitz/Hölder continuous function?*

For example it is clear that showing that every Borel function is Hölder continuous of some suitable exponent on a set of sufficiently large Hausdorff dimension would answer Question 2.3.1 in the negative. The other versions are less closely related to our problem, however, they are of independent interest. We prove the following two results.

Theorem 2.4.2. *Fix $0 < \alpha \leq 1$. A typical continuous function is not Hölder continuous of exponent α on any set of Hausdorff dimension larger than $1 - \alpha$.*

Theorem 2.4.3. *A typical continuous function does not agree with any function of bounded variation on any set of Hausdorff dimension larger than $\frac{1}{2}$.*

The second theorem is motivated by an analogous result of Humke and Laczkovich [48], who proved that a typical continuous function is not monotonic on any set of positive Hausdorff dimension. So one would expect the same for functions of bounded variation.

However, A. Máthé showed in [64] that both these results are sharp!

About the proofs. See Sections 3.13 and 3.14 or [21].

2.5 Borel sets which are null or non-σ-finite for every translation invariant measure

In many branches of mathematics a standard tool is that 'nicely defined' sets admit natural probability measures. For example, limit sets in the theory of Iterated Function Systems or Conformal Dynamics as well as self-similar sets in Geometric Measure Theory are usually naturally equipped with an invariant Borel measure, very often with a Hausdorff or packing measure. In many situations the sets in consideration are unbounded, for example periodic, so we cannot hope for an invariant probability measure. Similarly, the trajectories of the Brownian motion are of positive σ-finite $\mathcal{H}^g$-measure with probability 1, where the gauge function g is $g(t) = t^2 \log\log \frac{1}{t}$ in case of planar Brownian motion and $g(t) = t^2 \log \frac{1}{t} \log\log\log \frac{1}{t}$ in dimension 3 and higher. Therefore the natural notion to work with is that of an invariant Borel measure that is positive and σ-finite on our set.

It is natural to ask if there is some sort of unified theory behind the existence of these measures, for example, one is tempted to ask if every Borel subset of $\mathbb{R}^n$ of some 'regular structure' is positive and σ-finite for some generalised Hausdorff measure, or at least for some invariant Borel measure. In particular, R. D. Mauldin ([68], [14] and see [11] and [7] for partial and related results) formulated this question about a specific well-known set of very nice structure; the set of Liouville numbers, denoted by L:

Definition 2.5.1.

$$L = \left\{ x \in \mathbb{R} \setminus \mathbb{Q} : \forall n \in \mathbb{N}\ \exists p, q \in \mathbb{N}\ (q \geq 2) \text{ such that } \left| x - \frac{p}{q} \right| < \frac{1}{q^n} \right\}.$$

Question 2.5.2. *(R. D. Mauldin) Is there a translation invariant Borel measure on $\mathbb{R}$ such that the set of Liouville numbers is of positive and σ-finite measure?*

Note that we of course do not require that the measure be σ-finite on $\mathbb{R}$. Not only because Hausdorff measures are non-σ-finite on $\mathbb{R}$, but also because it is well-known that every σ-finite translation invariant Borel measure on the real line is a constant multiple of Lebesgue measure.

As we will answer this question in the negative, we introduce a definition.

Definition 2.5.3. A nonempty Borel set $B \subset \mathbb{R}$ is said to be *immeasurable* if it is either null or non-σ-finite for every translation invariant Borel measure on $\mathbb{R}$.

The main result of this section is an answer to Mauldin's question.

Theorem 2.5.4. *The set of Liouville numbers is immeasurable.*

Moreover, we also showed using various methods that there are other well-known 'nice' immeasurable sets. Specifically, the set of non-normal numbers, the complement of the set of so called Besicovitch-Eggleston numbers, $BE(1,0)$ (one of the Besicovitch-Eggleston classes itself) are all immeasurable. One of the main tools is that every Borel but not F_σ additive subgroup of $\mathbb{R}$ is immeasurable. Using this we also show that there are immeasurable sets of arbitrary Borel class (except of course open, as sets of positive Lebesgue measure are obviously not immeasurable). Similarly, we provide examples of immeasurable sets of arbitrary Hausdorff or packing dimension.

We note here that it is not only the regular structure of the sets considered here that makes it difficult to prove immeasurability. Even it is highly nontrivial to construct some immeasurable set. The two papers [61] and [17] containing the two known examples are entirely devoted to the constructions of the immeasurable sets.

We remark here that numerous questions left open or raised in our paper has been answered by A. Máthé [65].

About the proofs. It is only the proof that involves set theory here. We show, using transfinite induction, that every nonempty G_δ Lebesgue nullset with a dense set of periods is immeasurable. See Section 3.15 or [23].

2.6 Homogeneity of forcing notions and ideals

In order to be able to formulate the problem this section deals with, we need some definitions. For more information on forcing one can consult [55] or [50].

Definition 2.6.1. A notion of forcing $\mathbb{P}$ is called *homogeneous* if for every $p \in \mathbb{P}$ the restriction of $\mathbb{P}$ below p (i. e. $\{q \in \mathbb{P} : q \leq p\}$) is forcing equivalent to $\mathbb{P}$.

In his monograph [80] J. Zapletal poses the following problem.

Problem 2.6.2. ([80, Question 7.1.3.]) *"Prove that some of the forcings presented in this book are not homogeneous."*

In fact, we will work with the following closely related notion, see [80, Definition 2.3.7.].

Definition 2.6.3. A σ-ideal $\mathcal{I}$ on a Polish space X is *homogeneous* if for every Borel set B there is a Borel measurable function $f : X \to B$ such that $I \in \mathcal{I}$ implies $f^{-1}(I) \in \mathcal{I}$.

Zapletal writes “In all cases encountered in this book the homogeneity of the forcing and the underlying ideal always come together”. The aim of this section is to show that an interesting example of ideals discussed in his book is actually non-homogeneous.

Let us define the following σ-ideal consisting of sets of σ-finite r-dimensional Hausdorff measure.

Definition 2.6.4.

$$\mathcal{I}^r_{n,\sigma-fin} = \{H \subset \mathbb{R}^n : \exists H_k \subset \mathbb{R}^n,\ \cup_{k\in\omega} H_k = H,\ \mathcal{H}^r(H_k) < \infty \text{ for every } k \in \omega\}.$$

The next theorem solves this version of the problem.

Theorem 2.6.5. *The σ-ideal $\mathcal{I}^r_{n,\sigma-fin}$ is non-homogeneous.*

About the proof. The proof is a surprisingly simple application of a theorem of A. Máthé. I believe the reason why this question was open for so long is that this may be the first instance when fractal geometry is applied to answer a purely set theoretical problem! See 3.16 or [31].

Chapter 3

Proofs

3.1 Proof of Theorem 1.1.4

3.1.1 Preliminaries

Our terminology will mostly follow [51] and [77].

$USC(X)$ stands for the set of *upper semicontinuous* functions, that is, the set of functions f for which for every $r \in \mathbb{R}$ the set $f^{-1}((-\infty, r))$ is open in X. It is easy to see that the infimum of USC functions is also USC.

If $\mathcal{F}(X)$ is a class of real valued functions then we will denote by $b\mathcal{F}(X)$ and $\mathcal{F}^+(X)$ the set of bounded and nonnegative functions in $\mathcal{F}(X)$, respectively.

Recall that $\mathcal{K}(X)$ will stand for the set of the nonempty compact subsets of X endowed with the Hausdorff metric. It is well known (see [51, Section 4.F]) that if X is Polish then so is $\mathcal{K}(X)$. Moreover, the compactness of X is equivalent to the compactness of $\mathcal{K}(X)$.

Also recall that we denote the *ξth additive and multiplicative Borel classes* of a Polish space X by $\mathbf{\Sigma}^0_\xi(X)$ and $\mathbf{\Pi}^0_\xi(X)$, respectively. We will also use the notation $\mathbf{\Delta}^0_\xi(X) = \mathbf{\Sigma}^0_\xi(X) \cap \mathbf{\Pi}^0_\xi(X)$. We call a set A *ambiguous*, if $A \in \mathbf{\Delta}^0_2(X)$. Sometimes the following equivalent definition is also used for the first Baire class: $f \in \mathcal{B}_1(X) \iff$ the preimage of every open set under f is in $\mathbf{\Sigma}^0_2(X)$ (see [51, 24.10]). This easily implies that a characteristic function χ_A is of Baire class 1 if and only if $A \in \mathbf{\Delta}^0_2(X)$. The above equivalent definition also implies that USC functions are of Baire class 1.

In this section, somewhat unusually, because of the heavy use of subscripts the x-section of a set $H \subset X \times Y$ will be denoted by $H^x = \{y : (x, y) \in H\}$.

For a function $f : X \to \mathbb{R}$ *the subgraph of* f is the set $sgr(f) = \{(x, r) \in X \times \mathbb{R} : r \leq f(x)\}$. Notice that a function is USC if and only if its subgraph is closed.

Let $(P, <_p)$ be a poset. We follow the notation of S. Todorčević [77] letting

$$\sigma P = \{F : \alpha \to P \mid \alpha \text{ is an ordinal, } F \text{ is strictly increasing}\},$$

that is, σP is the set of well-ordered sequences in P. We will use the notation $\sigma^* P$ for the reverse well-ordered sequences, i. e.,

$$\sigma^* P = \{F : \alpha \to P \mid \alpha \text{ is an ordinal, } F \text{ is strictly decreasing}\}.$$

Then $\sigma^*[0,1]$ is the set of strictly decreasing well-ordered transfinite sequences of reals in $[0,1]$.

For a poset P, if $\bar{p} \in \sigma^* P$ and the domain of $\bar{p}$ is ξ then we will write $\bar{p}$ as $(p_\alpha)_{\alpha<\xi}$, where $p_\alpha = \bar{p}(\alpha)$. We will call the ordinal ξ the length of $\bar{p}$, in symbols $l(\bar{p})$.

Let H and H' be two subsets of the linearly ordered set $(L, <_L)$. We will say that $H \leq_L H'$ or $H <_L H'$ if for every $h \in H$ and $h' \in H'$ we have $h \leq_L h'$ or $h <_L h'$, respectively. A set $H \subset L$ is called *convex* if for every $a, b \in H$ and $c \in L$ with $a <_L c <_L b$ we have $c \in H$. An *interval* is a set of the form $[a, \infty)(= \{c : a \leq_L c\})$, $(a, b]$ or (a, b) etc. for some $a, b \in L$. We say that a linearly ordered set is *densely ordered* if it contains no neighboring points, while it is said to be *separable* if it has a countable subset which intersects every nonempty open interval. Finally, L is *nowhere separable*, if $(a, b) \cap L$ is not separable for every $a, b \in L$.

Now if $\bar{p}, \bar{p}' \in \sigma^* P$ and $\bar{p} \not\subset \bar{p}', \bar{p}' \not\subset \bar{p}$ then there exists a minimal ordinal δ such that $p_\delta \neq p'_\delta$. This ordinal is denoted by $\delta(\bar{p}, \bar{p}')$.

Le α be a successor ordinal, then $\alpha - 1$ will stand for its predecessor. Now, since every ordinal α can be uniquely written in the form $\alpha = \gamma + n$ where γ is limit and n is finite, we let $(-1)^\alpha = (-1)^n$ and refer to the parity of n as the parity of α.

A poset $(T, <_T)$ is called a tree if for every $t \in T$ the ordering $<_T$ restricted to the set $\{s : s <_T t\}$ is a well-ordering. We denote by $Lev_\alpha(T)$ the αth level of T, that is, the set $\{t \in T : <_T |_{\{s : s <_T t\}}$ has order type $\alpha\}$. If $t \in T$ with $t \in Lev_\alpha(T)$ it can be identified with an enumeration (i. e., a strictly increasing bijection) $e_t : \alpha + 1 \to \{s \in T : s \leq_T t\}$. So, for an ordinal β we can talk about $t|_\beta$ which is the map $e_t|_\beta$. In particular, if α is a successor (note that $t \in Lev_\alpha(T)$), we will denote by $t|_\alpha$ the predecessor of t.

An *α-chain* C is a subset of a tree such that $<_T |_C$ is a well-ordering in type α, whereas an *antichain* is a set that consists of $\leq_T$-incomparable elements. A set $D \subset T$ is called *dense* if for every $t \in T$ there exists a $p \in D$ such that $t \leq_T p$. A set is called *open* if if for every $p \in D$ we have $\{t \in T : t \geq_T p\} \subset D$.

A tree $(T, <_T)$ of cardinality $\aleph_1$ is called an Aronszajn tree, if for every $\alpha < \omega_1$ we have $|Lev_\alpha(T)| \leq \aleph_0$ and T contains no ω_1-chains. An Aronszajn tree is called a Suslin tree if it contains no uncountable antichains.

A Suslin line is a linearly ordered set that is ccc (it contains no uncountable pairwise disjoint collection of nonempty open intervals) but not separable.

We will call a poset $(P, <_P)$ $\mathbb{R}$-special ($\mathbb{Q}$-special) if there exists an embedding $P \hookrightarrow \mathbb{R}$ ($P \hookrightarrow \mathbb{Q}$).

Every ordinal is identified with the set of its predecessors, in particular, $2 = \{0, 1\}$.

3.1.2 $\mathcal{B}_1(X) \hookrightarrow ([0,1]^{<\omega_1}_{\searrow 0}, <_{altlex})$

Recall that

$$[0,1]^{<\omega_1}_{\searrow 0} = \{\bar{x} \in \sigma^*[0,1] : \min \bar{x} = 0\}$$

and also that for $\bar{x} = (x_\alpha)_{\alpha \leq \xi}, \bar{x}' = (x'_\alpha)_{\alpha \leq \xi'} \in [0,1]^{<\omega_1}_{\searrow 0}$ distinct and $\delta = \delta(\bar{x}, \bar{x}')$ we say that

$$(x_\alpha)_{\alpha \leq \xi} <_{altlex} (x'_\alpha)_{\alpha \leq \xi'} \iff (\delta \text{ is even and } x_\delta < x'_\delta) \text{ or } (\delta \text{ is odd and } x_\delta > x'_\delta).$$

Theorem 3.1.1. *Let X be a Polish space. Then $\mathcal{B}_1(X) \hookrightarrow [0,1]^{<\omega_1}_{\searrow 0}$.*

In order to prove the theorem we have to make some preparation. We will use results of Kechris and Louveau [52]. They developed a method to decompose a Baire class 1 function into a sum of a transfinite alternating series, which is analogous to the well known Hausdorff-Kuratowski analysis of $\mathbf{\Delta}^0_2$ sets.

First we define the generalised sums.

Definition 3.1.2. ([52]) Suppose that $(f_\beta)_{\beta<\alpha}$ is a pointwise decreasing sequence of nonnegative bounded USC functions for an ordinal $\alpha < \omega_1$. Let us define the *generalised alternating sum* $\sum^*_{\beta<\alpha}(-1)^\beta f_\beta$ by induction on α as follows:

$$\Sigma^*_{\beta<0}(-1)^\beta f_\beta = 0$$

and

$$\Sigma^*_{\beta<\alpha}(-1)^\beta f_\beta = \Sigma^*_{\beta<\alpha-1}(-1)^\beta f_\beta + (-1)^{\alpha-1} f_{\alpha-1}$$

if α is a successor and

$$\Sigma^*_{\beta<\alpha}(-1)^\beta f_\beta = \sup\{\Sigma^*_{\gamma<\beta}(-1)^\gamma f_\gamma : \beta < \alpha, \beta \text{ even}\}$$

if $\alpha > 0$ is a limit.

Every nonnegative bounded Baire class 1 function can be canonically decomposed into such a sum. For this we need the notion of upper regularisation.

Definition 3.1.3. ([52]) Let $f : X \to \mathbb{R}$ be a nonnegative bounded function. The *upper regularisation* of f is defined as

$$\hat{f} = \inf\{g : f \leq_p g, g \in \mathrm{USC}(X)\}.$$

Note that $\hat{f}$ is USC, since the infimum of USC functions is USC. Also, clearly $\hat{f} = f$ if f is USC.

Definition 3.1.4. ([52]) Let

$$g_0 = f, f_0 = \widehat{g_0},$$

if α is a successor then let

$$g_\alpha = f_{\alpha-1} - g_{\alpha-1}, f_\alpha = \widehat{g}_\alpha,$$

if $\alpha > 0$ is a limit then let

$$g_\alpha = \inf_{\substack{\beta<\alpha \\ \beta \text{ even}}} g_\beta \text{ and } f_\alpha = \widehat{g}_\alpha.$$

Now if there exists a minimal ξ_f such that $f_{\xi_f} \equiv f_{\xi_f+1}$ then let $\Phi(f) = (f_\alpha)_{\alpha\leq\xi_f}$.

Note that we need some results of Kechris and Louveau for arbitrary Polish spaces, however in [52] the authors proved the theorems only in the compact Polish case, although the proofs still work for the general case as well. Unfortunately, in our proof the non-σ-compact statement plays a significant role, hence we must check the validity of their results on such spaces. The results used are summarised in Proposition 3.1.5 and the proof can be found in Section 3.1.5. Notice that the original proof seems to contain a small error, but it can be corrected without essential new ideas.

Proposition 3.1.5. *([52]) Let X be a Polish space and $f \in b\mathcal{B}_1^+(X)$. Then $\Phi(f)$ is defined, $\Phi(f) \in \sigma^* bUSC^+$ and we have*

(1) $f = \sum_{\beta<\alpha}^* (-1)^\beta f_\beta + (-1)^\alpha g_\alpha$ *for every* $\alpha \le \xi_f$,

(2) $f_{\xi_f} \equiv 0$,

(3) $f = \sum_{\alpha<\xi_f}^* (-1)^\alpha f_\alpha$.

Proof. See Section 3.1.5. □

Proposition 3.1.6. *Let X be a Polish space and $f_0, f_1 \in b\mathcal{B}_1^+(X)$. Suppose that $f_0 <_p f_1$ and let $\Phi(f_0) = (f_\alpha^0)_{\alpha\le\xi_{f_0}}$ and $\Phi(f_1) = (f_\alpha^1)_{\alpha\le\xi_{f_1}}$. Then $\Phi(f_0) \neq \Phi(f_1)$ and if $\delta = \delta(\Phi(f_0), \Phi(f_1))$ then $f_\delta^0 <_p f_\delta^1$ if δ is even and $f_\delta^0 >_p f_\delta^1$ if δ is odd.*

Proof.

First notice that if $f_0 \neq f_1$ then by (3) of Proposition 3.1.5 we have that $\Phi(f_0) \neq \Phi(f_1)$.

Let $(g_\beta^0)_{\beta\le\xi_{f_0}}$ and $(g_\beta^1)_{\beta\le\xi_{f_1}}$ be the appropriate sequences (used in Definition 3.1.4 with $\widehat{g_\beta^i} = f_\beta^i$).

We show by induction on β that for every even ordinal $\beta \le \delta$ we have $g_\beta^0 \le_p g_\beta^1$ and for every odd ordinal $\beta \le \delta$ we have $g_\beta^0 \ge_p g_\beta^1$.

For $\beta = 0$ by definition $g_0^0 = f_0$ and $g_0^1 = f_1$, so $g_0^0 \le_p g_0^1$.

Suppose that we are done for every $\gamma < \beta$.

- for limit β we have that
$$g_\beta^0 = \inf_{\substack{\gamma<\beta \\ \gamma \text{ even}}} g_\gamma^0$$
so by the inductive hypothesis obviously $g_\beta^0 \le_p g_\beta^1$.

- if β is an odd ordinal, since $\beta - 1 < \delta$ we have $f_{\beta-1}^0 = f_{\beta-1}^1$ so
$$g_\beta^0 = f_{\beta-1}^0 - g_{\beta-1}^0 \ge_p f_{\beta-1}^0 - g_{\beta-1}^1 = f_{\beta-1}^1 - g_{\beta-1}^1 = g_\beta^1$$
by $\beta - 1$ being even and using the inductive hypothesis.

- if β is an even successor, the calculation is similar, using that $g_{\beta-1}^0 \ge_p g_{\beta-1}^1$ we obtain
$$g_\beta^0 = f_{\beta-1}^0 - g_{\beta-1}^0 \le_p f_{\beta-1}^0 - g_{\beta-1}^1 = f_{\beta-1}^1 - g_{\beta-1}^1 = g_\beta^1.$$

Consequently, the induction shows that $g_\delta^0 \le_p g_\delta^1$ if δ is even and $g_\delta^0 \ge_p g_\delta^1$ if δ is odd. Therefore, since $\widehat{g_\delta^i} = f_\delta^i$ we have that $f_\delta^0 \le_p f_\delta^1$ if δ is even and $f_\delta^0 \ge_p f_\delta^1$ if δ is odd. But by the definition of δ it is clear that $f_\delta^0 \neq f_\delta^1$, hence $f_\delta^0 <_p f_\delta^1$ if δ is even and $f_\delta^0 >_p f_\delta^1$ if δ is odd. This finishes the proof of Proposition 3.1.6. □

Now to finish the proof of Theorem 3.1.1 we need the following folklore lemma.

Lemma 3.1.7. *There exists an order preserving embedding $\Psi_0 : USC^+(X) \hookrightarrow [0,1]$ where the image of the function $f \equiv 0$ is 0. In particular, there is no uncountable strictly monotone transfinite sequence in $USC^+(X)$.*

Proof. Fix a countable basis $\{B_n : n \in \omega\}$ of $X \times [0,\infty)$. Assign to each $f \in USC^+(X)$ the real

$$r_f = 1 - \sum_{B_n \cap sgr(f) = \emptyset} 2^{-n-1}.$$

If $f <_p g$ then $sgr(f) \subsetneqq sgr(g)$ so, as the subgraph of an USC function is a closed set, there exists an $n \in \omega$ such that B_n is an open neighborhood of a point in $sgr(g) \setminus sgr(f)$. Thus, $\{n : B_n \cap sgr(f) = \emptyset\} \supsetneqq \{n : B_n \cap sgr(g) = \emptyset\}$. Consequently, $r_f < r_g$. □Proof. [Proof of Theorem 3.1.1] Let $\Psi : \sigma^* USC^+(X) \to \sigma^*[0,1]$ be the map that applies the above Ψ_0 to every coordinate of the sequences in $\sigma^* USC^+(X)$. Thus, Ψ is order preserving coordinate-wise.

Clearly, $h(x) = \frac{1}{\pi} \arctan(x) + 1$ is an order preserving homeomorphism from $\mathbb{R}$ to $(0,1)$ and for $f \in \mathcal{B}_1(X)$ let $H(f) = h \circ f$. Composing the functions in $\mathcal{B}_1(X)$ with h we still have Baire class 1 functions and this does not effect the pointwise ordering. Thus, H is an order preserving map from $\mathcal{B}_1(X)$ into $b\mathcal{B}_1^+(X)$.

Let $\Theta = \Psi \circ \Phi \circ H$. Notice that as $H : \mathcal{B}_1(X) \to b\mathcal{B}_1^+(X)$, $\Phi : b\mathcal{B}_1^+(X) \to \sigma^* bUSC^+(X)$ and $\Psi : \sigma^* USC(X) \to \sigma^*[0,1]$, the map Θ is well defined.

Now, by Lemma 3.1.7 we have that Ψ_0 maps the constant zero function to zero and by (2) of Proposition 3.1.5 we have that for every function f its Φ image ends with the constant zero function. Thus, the Θ image of every function f ends with zero. Therefore, Θ maps into $[0,1]^{<\omega_1}_{\searrow 0}$.

If $f_0 <_p f_1$ are Baire class 1 functions then clearly $H(f_0) <_p H(f_1)$ hence by Proposition 3.1.6 we have that if $\delta = \delta(\Phi(H(f_0)), \Phi(H(f_1)))$, then $\Phi(H(f_0))(\delta) <_p \Phi(H(f_1))(\delta)$ if δ is even and $\Phi(H(f_0))(\delta) >_p \Phi(H(f_1))(\delta)$ if δ is odd. Since Ψ is order preserving coordinate-wise, we obtain that Θ is an order preserving embedding of $\mathcal{B}_1(X)$ into $([0,1]^{<\omega_1}_{\searrow 0}, <_{altlex})$, which finishes the proof of the theorem. □

3.1.3 $([0,1]^{<\omega_1}_{\searrow 0}, <_{altlex}) \hookrightarrow \mathcal{B}_1(X)$

Theorem 3.1.8. *The linearly ordered set $([0,1]^{<\omega_1}_{\searrow 0}, <_{altlex})$ can be represented by $\mathbf{\Delta}^0_2$ subsets of $\mathcal{K}([0,1]^2)$ ordered by inclusion.*

Proof. First we define a map $\Psi : [0,1]^{<\omega_1}_{\searrow 0} \to \mathcal{K}([0,1]^2)$, basically assigning to each sequence its closure (as a subset of the unit interval). However, such a map cannot distinguish between continuous sequences and sequences omitting a limit point. To remedy this we place a line segment on each limit point contained in the sequence.

Let $\bar{x} \in [0,1]^{<\omega_1}_{\searrow 0}$, with $\bar{x} = (x_\alpha)_{\alpha \le \xi}$. Now let

$$\Psi(\bar{x}) = \overline{\{(x_\alpha, 0) : \alpha \le \xi\}} \cup$$

$$\bigcup \{\{x_\alpha\} \times [0, x_\alpha - x_{\alpha+1}] : \text{ if } 0 < \alpha < \xi \text{ and } x_\alpha = \inf\{x_\beta : \beta < \alpha\}\}.$$

Lemma 3.1.9. *$\Psi(\bar{x})$ is a compact set for every $\bar{x} \in [0,1]^{<\omega_1}_{\searrow 0}$.*

PROOF. Clearly, it is enough to show that if $(p_n, q_n) \to (p, q)$ is a convergent sequence such that for every n we have

$$(p_n, q_n) \in \tag{3.1.1}$$

$$\bigcup \{\{x_\alpha\} \times [0, x_\alpha - x_{\alpha+1}] : \text{ if } 0 < \alpha < \xi \text{ and } x_\alpha = \inf\{x_\beta : \beta < \alpha\}\}$$

then $(p, q) \in \Psi(\bar{x})$.

Obviously, $p_n = x_{\alpha_n}$ for some ordinals α_n. First, if the sequence x_{α_n} is eventually constant, then there exists an α such that $p = x_\alpha$ and except for finitely many n's by (3.1.1) we have $q_n \in [0, x_\alpha - x_{\alpha+1}]$. So $(p, q) \in \{x_\alpha\} \times [0, x_\alpha - x_{\alpha+1}] \subset \Psi(\bar{x})$.

Now if the sequence $(x_{\alpha_n})_{n\in\omega}$ is not eventually constant, since the sequence $(x_\alpha)_{\alpha\leq\xi}$ is strictly decreasing and well-ordered then (passing to a subsequence of $(x_{\alpha_n})_{n\in\omega}$ if necessary) we can suppose that $(x_{\alpha_n})_{n\in\omega}$ is a strictly decreasing sequence.

Using the fact that $(x_{\alpha_n})_{n\in\omega}$ is a strictly decreasing subset of $(x_\alpha)_{\alpha\leq\xi}$ we obtain that $x_{\alpha_n} - x_{\alpha_n+1} \leq x_{\alpha_n} - p$. Hence from (3.1.1) we obtain

$$0 \leq q_n \leq x_{\alpha_n} - x_{\alpha_n+1} \leq x_{\alpha_n} - p \to 0$$

so $q_n = 0$. Therefore,

$$(p, q) = (\lim_{n\to\infty} x_{\alpha_n}, 0) \in \overline{\{(x_\alpha, 0) : \alpha \leq \xi\}} \subset \Psi(\bar{x}).$$

□

Now we define a decreasing sequence of subsets of $\mathcal{K}([0,1]^2)$ for each $\bar{x} = (x_\alpha)_{\alpha\leq\xi}$ and $\alpha \leq \xi$ as follows:

$$\mathcal{H}^{\bar{x}}_\alpha = \{\Psi(\bar{z}) : \bar{z}|_\alpha = \bar{x}|_\alpha, z_\alpha \leq x_\alpha\}. \tag{3.1.2}$$

We will use the following notations for an even ordinal $\alpha \leq \xi$:

$$\mathcal{K}^{\bar{x}}_\alpha = \overline{\mathcal{H}^{\bar{x}}_\alpha} (= \overline{\{\Psi(\bar{z}) : \bar{z}|_\alpha = \bar{x}|_\alpha, z_\alpha \leq x_\alpha\}}), \tag{3.1.3}$$

and if $\alpha + 1 \leq \xi$ then

$$\mathcal{L}^{\bar{x}}_\alpha = \overline{\mathcal{H}^{\bar{x}}_{\alpha+1}} (= \overline{\{\Psi(\bar{z}) : \bar{z}|_{\alpha+1} = \bar{x}|_{\alpha+1}, z_{\alpha+1} \leq x_{\alpha+1}\}}). \tag{3.1.4}$$

Finally, if $\alpha = \xi$ then let $\mathcal{L}^{\bar{x}}_\alpha = \emptyset$. So $\mathcal{K}^{\bar{x}}_\alpha$ and $\mathcal{L}^{\bar{x}}_\alpha$ is defined for every even $\alpha \leq \xi$.

Notice that the sequence $(\overline{\mathcal{H}^{\bar{x}}_\alpha})_{\alpha\leq\xi}$ is a decreasing sequence of closed sets.

To each $\bar{x} = (x_\alpha)_{\alpha\leq\xi}$ let us assign

$$\mathcal{A}^{\bar{x}} = \bigcup_{\alpha\leq\xi, \alpha \text{ even}} (\mathcal{K}^{\bar{x}}_\alpha \setminus \mathcal{L}^{\bar{x}}_\alpha).$$

By [51, 22.27], since $\mathcal{A}^{\bar{x}}$ is a transfinite difference of a decreasing sequence of closed sets, we have $\mathcal{A}^{\bar{x}} \in \mathbf{\Delta}^0_2(\mathcal{K}([0,1]^2))$.

To overcome some technical difficulties we prove the following lemma.

Lemma 3.1.10. *Let $\bar{z} \in [0,1]^{<\omega_1}_{\searrow 0}$ and β be an ordinal such that $\beta + 1 \leq l(\bar{z})$.*

(1) If $K \in \overline{\mathcal{H}^{\bar{z}}_{\beta+1}}$, β is a limit ordinal, $\inf\{z_\gamma : \gamma < \beta\} = z_\beta$ and $l(\bar{z}) > \beta + 1$ then $(z_\beta, z_\beta - z_{\beta+1}) \in K$.

(2) If $K \in \overline{\mathcal{H}^{\bar{z}}_{\beta}}$ and β is a successor then $(z_{\beta-1}, 0) \in K$.

(3) If $K \in \overline{\mathcal{H}^{\bar{z}}_{\beta}}$, β is a limit ordinal and $\inf\{z_\gamma : \gamma < \beta\} > z_\beta$ or β is a successor then

(a) $K \cap ((z_\beta, \inf\{z_\gamma : \gamma < \beta\}) \times [0,1]) = \emptyset$

(b) $K \cap (\{\inf\{z_\gamma : \gamma < \beta\}\} \times (0,1]) = \emptyset$

(notice that if β is a successor then $\inf\{z_\gamma : \gamma < \beta\} = z_{\beta-1}$).

Proof. For (1) and (2) just notice that by equation (3.1.2) whenever $\Psi(\bar{w}) \in \mathcal{H}^{\bar{z}}_{\beta}$ ($\mathcal{H}^{\bar{z}}_{\beta+1}$, respectively) then $\Psi(\bar{w})$ contains the point $(z_{\beta-1}, 0)$ (the point $(z_\beta, z_\beta - z_{\beta+1})$). Consequently, every compact set which is in the closure of $\mathcal{H}^{\bar{z}}_{\beta}$ (or $\mathcal{H}^{\bar{z}}_{\beta+1}$) contains the point $(z_{\beta-1}, 0)$ (the point $(z_\beta, z_\beta - z_{\beta+1})$).

In order to see (3) first observe the following: if $U \subset [0,1]^2$ is a relatively open set and $L \in \mathcal{K}([0,1]^2)$ then the set $S = \{K \in \mathcal{K}([0,1]^2) : K \cap U = L \cap U\}$ is closed: if $K_n \to K$ with $K_n \in S$ then $K \cap U \supset L \cap U$ is obvious. Now if there was an $x \in (K \cap U) \setminus L$ then there would be a (relatively) open set $V \subset U \setminus L$ with $x \in V$. But then by $x \in K$ we would have that $K_n \cap V \neq \emptyset$ for a large enough n, contradicting $K_n \cap V = L \cap V = \emptyset$, so S is indeed closed.

Now, by the definition of $\mathcal{H}^{\bar{z}}_{\beta}$ for every $\bar{w}$ such that $\Psi(\bar{w}) \in \mathcal{H}^{\bar{z}}_{\beta}$ we have

$$\psi(\bar{w}) \cap ((z_\beta, 1] \times [0,1]) = \psi(\bar{z}) \cap ((z_\beta, 1] \times [0,1]).$$

Thus, since the set $((z_\beta, 1] \times [0,1])$ is relatively open in $[0,1]^2$ we have

$$\overline{\mathcal{H}^{\bar{z}}_{\beta}} \subset \{K \in \mathcal{K}([0,1]^2) : K \cap ((z_\beta, 1] \times [0,1]) = \psi(\bar{z}) \cap ((z_\beta, 1] \times [0,1])\},$$

as the latter set is closed and contains $\mathcal{H}^{\bar{z}}_{\beta}$. In particular, for every $K \in \overline{\mathcal{H}^{\bar{z}}_{\beta}}$ we get that

$$K \cap ((z_\beta, \inf\{z_\gamma : \gamma < \beta\}) \times [0,1]) = \emptyset$$

and

$$K \cap (\{\inf\{z_\gamma : \gamma < \beta\}\} \times (0,1]) = \emptyset$$

hold, which proves the lemma. $\square$

In order to show that $\bar{x} \mapsto \mathcal{A}^{\bar{x}}$ is an embedding it is enough to prove the following claim.

Main Claim. If $\bar{x} <_{altlex} \bar{y}$ then $\mathcal{A}^{\bar{x}} \subsetneq \mathcal{A}^{\bar{y}}$.

To verify this we have to distinguish two cases.

Case 1. $\delta = \delta(\bar{x}, \bar{y})$ is even.

Then $x_\delta < y_\delta$ and $\delta + 1 < l(\bar{y})$. We will show the following lemma.

Lemma 3.1.11. $\mathcal{K}^{\bar{x}}_{\delta} \subsetneq \mathcal{K}^{\bar{y}}_{\delta} \setminus \mathcal{L}^{\bar{y}}_{\delta}$.

Proof. [Proof of Lemma 3.1.11.] From $x_\delta < y_\delta$ we have

$$\{\Psi(\bar{z}) : \bar{z}|_\delta = \bar{x}|_\delta, z_\delta \le x_\delta\} \subset \{\Psi(\bar{z}) : \bar{z}|_\delta = \bar{x}|_\delta, z_\delta \le y_\delta\}$$

so $\mathcal{K}^{\bar{x}}_{\delta} \subset \mathcal{K}^{\bar{y}}_{\delta}$.

First, we prove that
$$\mathcal{K}^{\bar{x}}_{\delta} \subset \mathcal{K}^{\bar{y}}_{\delta} \setminus \mathcal{L}^{\bar{y}}_{\delta}. \tag{3.1.5}$$
Here we have to separate two subcases.

SUBCASE 1. δ is a limit ordinal and $y_\delta = \inf\{y_\alpha : \alpha < \delta\}$.

On the one hand, using (1) of Lemma 3.1.10 (with $\bar{z} = \bar{y}$ and $\beta = \delta$) we obtain that for every $K \in \mathcal{L}^{\bar{y}}_{\delta}(= \overline{\mathcal{H}^{\bar{y}}_{\delta+1}})$ we have $(y_\delta, y_\delta - y_{\delta+1}) \in K$.

On the other hand, from (3b) of Lemma 3.1.10 (with $\bar{z} = \bar{x}$ and $\beta = \delta$) we have that for every $K \in \mathcal{K}^{\bar{x}}_{\delta}(= \overline{\mathcal{H}^{\bar{x}}_{\delta}})$ we have $K \cap (\{\inf\{x_\alpha : \alpha < \delta\}\} \times (0, 1]) = \emptyset$. In particular, as $y_\delta = \inf\{y_\alpha : \alpha < \delta\} = \inf\{x_\alpha : \alpha < \delta\}$, we have $(y_\delta, y_\delta - y_{\delta+1}) \notin K$. So we obtain $\mathcal{K}^{\bar{x}}_{\delta} \cap \mathcal{L}^{\bar{y}}_{\delta} = \emptyset$, hence by $\mathcal{K}^{\bar{x}}_{\delta} \subset \mathcal{K}^{\bar{y}}_{\delta}$ we have $\mathcal{K}^{\bar{x}}_{\delta} \subset \mathcal{K}^{\bar{y}}_{\delta} \setminus \mathcal{L}^{\bar{y}}_{\delta}$.

SUBCASE 2. δ is a limit and $y_\delta < \inf\{y_{\delta'} : \delta' < \delta\}$ or δ is a successor.

Using (2) of Lemma 3.1.10 (with $\bar{z} = \bar{y}$ and $\beta = \delta + 1$) we obtain that every $K \in \mathcal{L}^{\bar{y}}_{\delta}(= \overline{\mathcal{H}^{\bar{y}}_{\delta+1}})$ contains the point $(y_\delta, 0)$. From (3a) of Lemma 3.1.10 (with $\bar{z} = \bar{x}$, $\beta = \delta$) we have that for every $K \in \mathcal{K}^{\bar{x}}_{\delta}(= \overline{\mathcal{H}^{\bar{x}}_{\delta}})$ the set $K \cap ((x_\delta, \inf\{x_\alpha : \alpha < \delta\}) \times [0, 1])$ is empty. But $y_\delta \in (x_\delta, \inf\{x_\alpha : \alpha < \delta\})$ so $\mathcal{K}^{\bar{x}}_{\delta} \cap \mathcal{L}^{\bar{y}}_{\delta} = \emptyset$. This finishes the proof of equation (3.1.5).

Second, in order to prove $\mathcal{K}^{\bar{x}}_{\delta} \neq \mathcal{K}^{\bar{y}}_{\delta} \setminus \mathcal{L}^{\bar{y}}_{\delta}$ let $\bar{w}$ be such that $\bar{w}|_\delta = \bar{x}|_\delta$, $x_\delta, y_{\delta+1} < w_\delta < y_\delta$ and $w_{\delta+1} = 0$. Clearly, $\Psi(\bar{w}) \in \mathcal{K}^{\bar{y}}_{\delta}$.

By (3a) of Lemma 3.1.10 (used for $\bar{z} = \bar{x}$ and $\beta = \delta$) we have that $\Psi(\bar{w}) \in \mathcal{K}^{\bar{x}}_{\delta}(= \overline{\mathcal{H}^{\bar{x}}_{\delta}})$ would imply $\Psi(\bar{w}) \cap ((x_\delta, \inf\{x_\alpha : \alpha < \delta\}) \times [0, 1]) = \emptyset$, but $(w_\delta, 0) \in (x_\delta, y_\delta) \times [0, 1]$ and $\inf\{x_\alpha : \alpha < \delta\} = \inf\{y_\alpha : \alpha < \delta\} \geq y_\delta$ which is a contradiction. Hence $\Psi(\bar{w}) \notin \mathcal{K}^{\bar{x}}_{\delta}$.

Now we prove $\Psi(\bar{w}) \notin \mathcal{L}^{\bar{y}}_{\delta}$. Suppose the contrary, then using (3a) of Lemma 3.1.10 (with $\bar{z} = \bar{y}$ and $\beta = \delta+1$) one can obtain that for every $K \in \mathcal{L}^{\bar{y}}_{\delta}(= \overline{\mathcal{H}^{\bar{y}}_{\delta+1}})$ the set $K \cap ((y_{\delta+1}, y_\delta) \times [0, 1])$ is empty. But clearly $(w_\delta, 0) \in \Psi(\bar{w}) \cap ((y_{\delta+1}, y_\delta) \times [0, 1])$, a contradiction. So $\Psi(\bar{w}) \notin \mathcal{L}^{\bar{y}}_{\delta}$.

Thus, it follows that $\Psi(\bar{w}) \in (\mathcal{K}^{\bar{y}}_{\delta} \setminus \mathcal{L}^{\bar{y}}_{\delta}) \setminus \mathcal{K}^{\bar{x}}_{\delta}$. From this and from equation (3.1.5) we can conclude Lemma 3.1.11. $\square$

Now we prove the Main Claim in Case 1. If δ' is even and $\delta' < \delta$, the definitions (3.1.3) and (3.1.4) of $\mathcal{K}^{\bar{y}}_{\delta'}$ and $\mathcal{L}^{\bar{y}}_{\delta'}$ depend only on $(x_\alpha)_{\alpha \leq \delta'+1}$ so
$$\mathcal{K}^{\bar{x}}_{\delta'} = \mathcal{K}^{\bar{y}}_{\delta'} \tag{3.1.6}$$
and
$$\mathcal{L}^{\bar{x}}_{\delta'} = \mathcal{L}^{\bar{y}}_{\delta'}. \tag{3.1.7}$$
Now, from Lemma 3.1.11 we have $\mathcal{A}^{\bar{x}} \subset \mathcal{A}^{\bar{y}}$, since for every $K \in \mathcal{A}^{\bar{x}}$ we have either $K \in \mathcal{K}^{\bar{x}}_{\delta'} \setminus \mathcal{L}^{\bar{x}}_{\delta'} = \mathcal{K}^{\bar{y}}_{\delta'} \setminus \mathcal{L}^{\bar{y}}_{\delta'}$ for some $\delta' < \delta$ or $K \in \mathcal{K}^{\bar{x}}_{\delta}$.

Moreover, we claim that using Lemma 3.1.11 one can prove that $\mathcal{A}^{\bar{x}} \subsetneq \mathcal{A}^{\bar{y}}$. From the definition of $\mathcal{A}^{\bar{x}}$, from the fact that the sequence $(\mathcal{H}^{\bar{x}}_{\alpha})_{\alpha \leq \xi} = (\mathcal{K}^{\bar{x}}_0, \mathcal{L}^{\bar{x}}_0, \mathcal{K}^{\bar{x}}_1, \mathcal{L}^{\bar{x}}_1, \dots)$ is decreasing and from equations (3.1.6) and (3.1.7) follows that
$$(\mathcal{K}^{\bar{x}}_{\delta})^c \cap \mathcal{A}^{\bar{x}} = \bigcup_{\delta' < \delta,\ \delta' \text{ even}} \mathcal{K}^{\bar{x}}_{\delta'} \setminus \mathcal{L}^{\bar{x}}_{\delta'} = \bigcup_{\delta' < \delta,\ \delta' \text{ even}} \mathcal{K}^{\bar{y}}_{\delta'} \setminus \mathcal{L}^{\bar{y}}_{\delta'} = (\mathcal{K}^{\bar{y}}_{\delta})^c \cap \mathcal{A}^{\bar{y}}$$
So $\mathcal{A}^{\bar{x}} \subset (\mathcal{K}^{\bar{y}}_{\delta})^c \cup \mathcal{K}^{\bar{x}}_{\delta}$. Hence, if $K \in (\mathcal{K}^{\bar{y}}_{\delta} \setminus \mathcal{L}^{\bar{y}}_{\delta}) \setminus \mathcal{K}^{\bar{x}}_{\delta}$ then
$$K \in \mathcal{K}^{\bar{y}}_{\delta} \setminus \mathcal{L}^{\bar{y}}_{\delta} \subset \mathcal{A}^{\bar{y}}$$

and

$$K \notin (\mathcal{K}^{\bar{y}}_{\delta})^c \cup \mathcal{K}^{\bar{x}}_{\delta} \supset \mathcal{A}^{\bar{x}}$$

so indeed, we obtain that the containment is strict, hence we are done with Case 1.

Case 2. $\delta = \delta(\bar{x}, \bar{y})$ is odd.

Then $x_\delta > y_\delta$ and $\delta + 1 < l(\bar{x})$.

Notice that as the length of $\bar{x}$ is larger than $\delta + 1$, the sets $\mathcal{K}^{\bar{x}}_{\delta+1}$ and $\mathcal{L}^{\bar{x}}_{\delta+1}$ are defined.

Now for every even $\delta' \leq \delta - 1$ the definition of $\mathcal{K}^{\bar{x}}_{\delta'}$ and $\mathcal{K}^{\bar{y}}_{\delta'}$ depend only on $(x_\alpha)_{\alpha \leq \delta'} = (y_\alpha)_{\alpha \leq \delta'}$. Thus for every even $\delta' \leq \delta - 1$

$$\mathcal{K}^{\bar{x}}_{\delta'} = \mathcal{K}^{\bar{y}}_{\delta'} \tag{3.1.8}$$

and also for every even $\delta' < \delta - 1$

$$\mathcal{L}^{\bar{x}}_{\delta'} = \mathcal{L}^{\bar{y}}_{\delta'}. \tag{3.1.9}$$

We will show the following:

Lemma 3.1.12. *(1)* $\mathcal{K}^{\bar{x}}_{\delta-1} \setminus \mathcal{L}^{\bar{x}}_{\delta-1} \subset \mathcal{K}^{\bar{y}}_{\delta-1} \setminus \mathcal{L}^{\bar{y}}_{\delta-1}$

(2) $\mathcal{K}^{\bar{x}}_{\delta+1} \subset \mathcal{K}^{\bar{y}}_{\delta-1} \setminus \mathcal{L}^{\bar{y}}_{\delta-1}$.

Proof. [Proof of Lemma 3.1.12.]

It is easy to prove (1): from equation (3.1.8) we get $\mathcal{K}^{\bar{x}}_{\delta-1} = \mathcal{K}^{\bar{y}}_{\delta-1}$. Moreover, $\mathcal{L}^{\bar{x}}_{\delta-1} \supset \mathcal{L}^{\bar{y}}_{\delta-1}$, since

$$\mathcal{L}^{\bar{x}}_{\delta-1} = \overline{\{\Psi(\bar{z}) : \bar{z}|_\delta = \bar{x}|_\delta, z_\delta \leq x_\delta\}} \supset \overline{\{\Psi(\bar{z}) : \bar{z}|_\delta = \bar{y}|_\delta, z_\delta \leq y_\delta\}} = \mathcal{L}^{\bar{y}}_{\delta-1}$$

holds by $x_\delta > y_\delta$.

Now we show (2). First, $\mathcal{K}^{\bar{x}}_{\delta+1} \subset \mathcal{K}^{\bar{x}}_{\delta-1} = \mathcal{K}^{\bar{y}}_{\delta-1}$, using that the sequence $(\mathcal{K}^{\bar{x}}_{\alpha})_{\alpha \leq \delta+1}$ is decreasing.

So it is suffices to show that $\mathcal{K}^{\bar{x}}_{\delta+1} \cap \mathcal{L}^{\bar{y}}_{\delta-1} = \emptyset$. Using (3a) of Lemma 3.1.10 (with $\bar{z} = \bar{y}$ and $\beta = \delta$) we obtain that for every $K \in \mathcal{L}^{\bar{y}}_{\delta-1} (= \overline{\mathcal{H}^{\bar{y}}_{\delta}})$, we have $K \cap ((y_\delta, y_{\delta-1}) \times [0, 1]) = \emptyset$.

However, by (2) of Lemma 3.1.10 (used with $\bar{z} = \bar{x}$ and $\beta = \delta + 1$) if $K \in \mathcal{K}^{\bar{x}}_{\delta+1} (= \overline{\mathcal{H}^{\bar{x}}_{\delta+1}})$ then $(x_\delta, 0) \in K$. Therefore, $x_\delta \in (y_\delta, y_{\delta-1})$ implies that the intersection $\mathcal{K}^{\bar{x}}_{\delta+1} \cap \mathcal{L}^{\bar{y}}_{\delta-1}$ must be empty. So we are done with the lemma. $\square$

Now we prove the Main Claim in Case 2. By definition of $\mathcal{A}^{\bar{x}}$ and by the fact that the sequence $(\mathcal{H}^{\bar{x}}_{\alpha})_{\alpha \leq \xi} = (\mathcal{K}^{\bar{x}}_0, \mathcal{L}^{\bar{x}}_0, \mathcal{K}^{\bar{x}}_1, \mathcal{L}^{\bar{x}}_1, \dots)$ is decreasing we have that if $K \in \mathcal{A}^{\bar{x}}$ then either $K \in \mathcal{K}^{\bar{x}}_{\delta'} \setminus \mathcal{L}^{\bar{x}}_{\delta'} = \mathcal{K}^{\bar{y}}_{\delta'} \setminus \mathcal{L}^{\bar{y}}_{\delta'}$ for some even $\delta' < \delta - 1$ or $K \in \mathcal{K}^{\bar{x}}_{\delta-1} \setminus \mathcal{L}^{\bar{x}}_{\delta-1}$ or $K \in \mathcal{K}^{\bar{x}}_{\delta+1}$. Hence using equations (3.1.8) and (3.1.9) and Lemma 3.1.12 we obtain

$$\mathcal{A}^{\bar{x}} \subset \mathcal{A}^{\bar{y}}. \tag{3.1.10}$$

In order to show that $\mathcal{A}^{\bar{x}} \neq \mathcal{A}^{\bar{y}}$ it is enough to find a $\bar{w}$ such that

$$\Psi(\bar{w}) \in \mathcal{K}^{\bar{y}}_{\delta-1} \setminus \mathcal{L}^{\bar{y}}_{\delta-1} \subset \mathcal{A}^{\bar{y}} \tag{3.1.11}$$

and

$$\Psi(\bar{w}) \notin \mathcal{K}^{\bar{x}}_{\delta+1} \cup (\mathcal{L}^{\bar{x}}_{\delta-1})^c \supset \mathcal{A}^{\bar{x}}. \tag{3.1.12}$$

Take $\bar{w}|_\delta = \bar{y}|_\delta$ and w_δ such that $x_{\delta+1}, y_\delta < w_\delta < x_\delta$ and $w_{\delta+1} = 0$.

Now, in order to see (3.1.11) clearly $\Psi(\bar{w}) \in \mathcal{K}^{\bar{y}}_{\delta-1}$. On the other hand if $K \in \mathcal{L}^{\bar{y}}_{\delta-1}(= \overline{\mathcal{H}^{\bar{y}}_{\delta}})$ by (3a) of Lemma 3.1.10 (with $\bar{z} = \bar{y}$ and $\beta = \delta$) we have $K \cap ((y_\delta, y_{\delta-1}) \times [0,1]) = \emptyset$. But $y_\delta < w_\delta < x_\delta < x_{\delta-1} = y_{\delta-1}$, so $(w_\delta, 0) \in \Psi(\bar{w}) \cap ((y_\delta, y_{\delta-1}) \times [0,1])$. Therefore, $\Psi(\bar{w}) \notin \mathcal{L}^{\bar{y}}_{\delta-1}$.

In order to prove (3.1.12) it is obvious that $\Psi(\bar{w}) \in \mathcal{L}^{\bar{x}}_{\delta-1}$. Now using again (3a) of Lemma 3.1.10 (with $\bar{z} = \bar{x}$ and $\beta = \delta + 1$) we obtain that whenever $K \in \mathcal{K}^{\bar{x}}_{\delta+1}(= \overline{\mathcal{H}^{\bar{x}}_{\delta+1}})$ then $K \cap ((x_{\delta+1}, x_\delta) \times [0,1]) = \emptyset$. However, $w_\delta \in (x_{\delta+1}, x_\delta)$ hence $(w_\delta, 0) \in \Psi(\bar{w}) \cap ((x_{\delta+1}, x_\delta) \times [0,1])$, so $\Psi(\bar{w}) \notin \mathcal{K}^{\bar{x}}_{\delta+1}$.

So we can conclude that $\mathcal{A}^{\bar{x}} \neq \mathcal{A}^{\bar{y}}$. Thus, using equation (3.1.10) we can finish the proof of the Main Claim in Case 2 and hence we obtain Theorem 3.1.8 as well. $\square$

3.1.4 The main theorem

Theorem 3.1.13. *(Main Theorem) Let X be an uncountable Polish space. Then the following are equivalent for a linear ordering $(L, <)$:*

(1) $(L, <) \hookrightarrow (\mathcal{B}_1(X), <_p)$

(2) $(L, <) \hookrightarrow ([0,1]^{<\omega_1}_{\searrow 0}, <_{altlex})$

(3) $(L, <) \hookrightarrow (\mathbf{\Delta}^0_2(X), \subsetneqq)$

In fact, $([0,1]^{<\omega_1}_{\searrow 0}, <_{altlex})$, $(\mathbf{\Delta}^0_2(X), \subsetneqq)$ *and* $(\mathcal{B}_1(X), <_p)$ *are embeddable into each other.*

Proof.

$(\mathcal{B}_1(X), <_p) \hookrightarrow ([0,1]^{<\omega_1}_{\searrow 0}, <_{altlex})$: Theorem 3.1.1.

$([0,1]^{<\omega_1}_{\searrow 0}, <_{altlex}) \hookrightarrow (\mathbf{\Delta}^0_2(X), \subsetneqq)$: we proved in Theorem 3.1.8 that $([0,1]^{<\omega_1}_{\searrow 0}, <_{altlex}) \hookrightarrow (\mathbf{\Delta}^0_2(\mathcal{K}([0,1]^2)), \subsetneqq)$. Now, [22, Theorem 1.2] states that the class of linear orderings representable in $\mathbf{\Delta}^0_2$ coincide for all uncountable σ-compact Polish spaces. Therefore, if C is the Cantor space then $([0,1]^{<\omega_1}_{\searrow 0}, <_{altlex}) \hookrightarrow (\mathbf{\Delta}^0_2(C), \subsetneqq)$. If X is an uncountable Polish space then there exists a continuous injection $h : C \to X$. Now, since $h(C)$ is a closed set in X we have that $A \mapsto h(A)$ is an inclusion-preserving embedding $(\mathbf{\Delta}^0_2(C), \subsetneqq) \hookrightarrow (\mathbf{\Delta}^0_2(X), \subsetneqq)$. Consequently, $([0,1]^{<\omega_1}_{\searrow 0}, <_{altlex}) \hookrightarrow (\mathbf{\Delta}^0_2(X), \subsetneqq)$.

$(\mathbf{\Delta}^0_2(X), \subsetneqq) \hookrightarrow (\mathcal{B}_1(X), <_p)$: if A is a $\mathbf{\Delta}^0_2$ set then χ_A is a Baire class 1 function and $A \mapsto \chi_A$ is an embedding $(\mathbf{\Delta}^0_2(X), \subsetneqq) \hookrightarrow (\mathcal{B}_1(X), <_p)$. $\square$

3.1.5 Proof of Proposition 3.1.5

Proposition 3.1.5. ([52]) Let X be a Polish space and $f \in b\mathcal{B}^+_1(X)$. Then $\Phi(f)$ is defined, $\Phi(f) \in \sigma^* bUSC^+$ and we have

(1) $f = \sum^*_{\beta<\alpha}(-1)^\beta f_\beta + (-1)^\alpha g_\alpha$ for every $\alpha \leq \xi_f$,

(2) $f_{\xi_f} \equiv 0$,

(3) $f = \sum^*_{\alpha<\xi_f}(-1)^\alpha f_\alpha$.

Proof. First we show that $\Phi(f)$ is defined and $\Phi(f) \in \sigma^* bUSC^+$. In order to prove this, we will show the following lemma.

Lemma 3.1.14. *The functions g_α and f_α (assigned to f in Definition 3.1.4) are bounded nonnegative and the sequence (f_α) is decreasing.*

Proof.

It follows trivially from the definition of the upper regularisation that if g is an arbitrary function then

$$g \text{ is bounded} \Rightarrow \widehat{g} \text{ exists, is bounded and } \widehat{g} \geq_p g. \tag{3.1.13}$$

Now we prove the statement of the lemma by induction on α. If $\alpha = 0$ then $g_0 = f$ and $f_0 = \widehat{f}$, hence from $f \in b\mathcal{B}_1^+(X)$ and (3.1.13) clearly follows that g_0 and f_0 are bounded nonnegative functions.

If α is a successor then by definition $g_\alpha = \widehat{g_{\alpha-1}} - g_{\alpha-1}$ so by the second part of (3.1.13) we have $g_\alpha \geq_p 0$. Moreover, since $g_{\alpha-1}$ is bounded $\widehat{g_{\alpha-1}}$ is also bounded. Thus, g_α is the difference of two bounded functions, therefore it is also bounded. Therefore, by (3.1.13) f_α exists (notice that we have defined the upper regularisation only for bounded functions) and also bounded and nonnegative.

Now we show that the sequence (f_α) is also decreasing. By the nonnegativity of $g_{\alpha-1}$ we have $f_{\alpha-1} - g_{\alpha-1} \leq_p f_{\alpha-1}$, so

$$f_\alpha = \widehat{f_{\alpha-1} - g_{\alpha-1}} \leq_p \widehat{f}_{\alpha-1} = f_{\alpha-1}.$$

For limit α we have

$$g_\alpha = \inf\{g_\beta : \beta < \alpha \text{ and } \beta \text{ is even}\}, \tag{3.1.14}$$

so clearly $g_\alpha \geq_p 0$ and g_α is bounded. Hence using again (3.1.13) we obtain that f_α is bounded and nonnegative.

Now for every β we have $g_\beta \leq_p f_\beta$. Therefore, if β is an even ordinal and $\beta < \alpha$ then by (3.1.14) we have

$$g_\alpha \leq_p g_\beta \leq_p f_\beta,$$

so $f_\alpha = \widehat{g_\alpha} \leq_p \widehat{f_\beta} = f_\beta$. But if β is odd, then $\beta+1$ is even and $\beta+1 < \alpha$. Using (3.1.14) we obtain $g_\alpha \leq_p g_{\beta+1}$ hence by the definition of f_α and $f_{\beta+1}$ and the inductive hypothesis we have $f_\alpha \leq_p f_{\beta+1} \leq_p f_\beta$. This finishes the proof of the lemma. □

Clearly, by the definition of upper regularisation, the functions f_α are upper semicontinuous. Therefore, by Lemma 3.1.14 we obtain that (f_α) is a decreasing sequence of nonnegative USC functions, so it must stabilise for some countable ordinal ξ_f (Lemma 3.1.7). Therefore, for every function in $f \in b\mathcal{B}_1^+(X)$ we have that $\Phi(f)$ is defined and $\Phi(f) \in \sigma^* bUSC^+(X)$.

Now we need the following lemma.

Lemma 3.1.15. *Let $(f_\alpha)_{\alpha<\xi} \in \sigma^* USC^+$. Then $\sum^*_{\alpha<\xi} (-1)^\alpha f_\alpha$ is a Baire class 1 function.*

Proof. We prove the lemma by induction on ξ.

First, if ξ is a successor just use that Baire class 1 functions are closed under addition and subtraction.

Second, if ξ is a limit, by definition of the alternating sums we have that

$$\Sigma^*_{\alpha<\xi}(-1)^\alpha f_\alpha = \sup\{\Sigma^*_{\beta<\alpha}(-1)^\beta f_\beta : \alpha < \xi, \alpha \text{ even}\}.$$

For even $\alpha < \xi$ we have

$$\Sigma^*_{\beta<\alpha}(-1)^\beta f_\beta = \Sigma^*_{\beta<\alpha+1}(-1)^\beta f_\beta - f_\alpha. \tag{*}$$

Again, for even α

$$\Sigma^*_{\beta<\alpha}(-1)^\beta f_\beta + f_\alpha - f_{\alpha+1} = \Sigma^*_{\beta<\alpha+2}(-1)^\beta f_\beta$$

so since the sequence $(f_\alpha)_{\alpha<\xi}$ is decreasing the sequence $(\sum^*_{\beta<\alpha}(-1)^\beta f_\beta)_{\alpha \text{ even}}$ is increasing. Similarly, the sequence $(\sum^*_{\beta<\alpha+1}(-1)^\beta f_\beta)_{\alpha \text{ even}}$ is decreasing. Notice that if $(r_\beta)_{\beta<\alpha}$ and $(t_\beta)_{\beta<\alpha}$ are decreasing transfinite sequences of non-negative reals such that $r_\beta - t_\beta$ is increasing, then

$$\sup\{r_\beta - t_\beta : \beta < \alpha\} = \inf\{r_\beta : \beta < \alpha\} - \inf\{t_\beta : \beta < \alpha\}.$$

Therefore, applying $(*)$ and these facts we have

$$\sup\{\Sigma^*_{\beta<\alpha}(-1)^\beta f_\beta : \alpha < \xi \text{ even}\} =$$

$$\inf\{\Sigma^*_{\beta<\alpha+1}(-1)^\beta f_\beta : \alpha < \xi \text{ even}\} - \inf\{f_\alpha : \alpha < \xi \text{ even}\}.$$

The infimum of USC functions is also USC, hence the right-hand side of the equation is the difference of the infimum of a countable family of Baire class 1 functions and a USC function. Therefore, $\sup\{\sum^*_{\beta<\alpha}(-1)^\beta f_\beta : \alpha < \xi \text{ even}\}$ is the infimum of a countable family of Baire class 1 functions. Moreover, by the inductive hypothesis, this function is also the supremum of a countable family of Baire class 1 functions. Now, using the fact that a function is Baire class 1 if and only if the preimage of every open set is $\mathbf{\Sigma}^0_2(X)$ it is easy to see that if a function h is the infimum of a countable family of Baire class 1 functions then for every $a \in \mathbb{R}$ we have that $h^{-1}((-\infty, a))$ is in $\mathbf{\Sigma}^0_2(X)$. Similarly, if h is the supremum of a countable family of Baire class 1 functions then the sets $h^{-1}((a, \infty))$ are also in $\mathbf{\Sigma}^0_2(X)$. But this implies that a function that is both an infimum and a supremum of countable families of Baire class 1 functions is also Baire class 1.

So, as an infimum and supremum of countable families of Baire class 1 functions, the function $\sup\{\sum^*_{\beta<\alpha}(-1)^\beta f_\beta : \alpha < \xi \text{ even}\}$ is also a Baire class 1 function, which completes the inductive proof. □

Now we prove (1) of the Proposition by induction on α.

For $\alpha = 0$ this is clear. If α is a successor, then $g_{\alpha-1} = f_{\alpha-1} - g_\alpha$, so

$$f = \Sigma^*_{\beta<\alpha-1}(-1)^\beta f_\beta + (-1)^{\alpha-1} g_{\alpha-1} =$$

$$\Sigma^*_{\beta<\alpha-1}(-1)^\beta f_\beta + (-1)^{\alpha-1}(f_{\alpha-1} - g_\alpha) = \Sigma^*_{\beta<\alpha}(-1)^\beta f_\beta + (-1)^\alpha g_\alpha.$$

For limit α notice that we have by induction for every even $\beta < \alpha$

$$f = \Sigma^*_{\gamma<\beta}(-1)^\gamma f_\gamma + g_\beta.$$

Then, using that $(f_\beta)_{\beta<\alpha}$ is decreasing, the sequence $(\Sigma^*_{\gamma<\beta}(-1)^\gamma f_\gamma)_{\beta \text{ even}}$ is increasing, so $(g_\beta)_{\beta \text{ even}}$ is decreasing as their sum is constant f.

Notice that if $(r_\beta)_{\beta<\alpha}$ is an increasing and $(t_\beta)_{\beta<\alpha}$ is a decreasing transfinite sequence of nonnegative reals such that $r_\beta + t_\beta = c$ is constant, then

$$c = \sup\{r_\beta + t_\beta : \beta < \alpha\} = \sup\{r_\beta : \beta < \alpha\} + \inf\{t_\beta : \beta < \alpha\}.$$

So

$$f = \sup_{\beta \text{ even}, \beta<\alpha} \left(\Sigma^*_{\gamma<\beta}(-1)^\gamma f_\gamma + g_\beta\right) =$$

$$\sup_{\beta \text{ even}, \beta<\alpha} \Sigma^*_{\gamma<\beta}(-1)^\gamma f_\gamma + \inf_{\beta \text{ even}, \beta<\alpha} g_\beta = \Sigma^*_{\beta<\alpha}(-1)^\beta f_\beta + g_\alpha,$$

where the last equality follows from the definition of $\sum^*_{\beta<\alpha}(-1)^\beta f_\beta$ and g_α.

This proves the induction hypothesis, so we have (1).

After rearranging the equality in (1) we have that

$$(-1)^{\alpha+1} g_\alpha = \Sigma^*_{\beta<\alpha}(-1)^\beta f_\beta - f.$$

By Lemma 3.1.15 we have that the sum on the right-hand side of the equation is a Baire class 1 function, therefore g_α is also Baire class 1. We have that $f_{\xi_f+1} \equiv f_{\xi_f}$, so by Definition 3.1.4 we have $\widehat{\widehat{g_{\xi_f}} - g_{\xi_f}} = \widehat{g_{\xi_f}}$. Hence in order to prove (2) it is enough to show the following claim.

Claim. *If g is a nonnegative, bounded Baire class 1 function such that $\widehat{g} = \widehat{\widehat{g} - g}$ then $g \equiv 0$.*

Proof. [Proof of the Claim.] Suppose the contrary. Then there exists an $\varepsilon > 0$ such that $\{x : g(x) > \varepsilon\} \neq \emptyset$. Let $K = \overline{\{x : g(x) > \varepsilon\}}$. Since g is a Baire class 1 function we have that there exists an open set V such that

$$\varepsilon > osc(g, K \cap V) \quad (= \sup_{x,y \in K\cap V} |g(x) - g(y)|)$$

and $K \cap V$ is not empty (see [51, 24.15]).

The function $\limsup_{y\to x} g(y)$ (here in the lim sup we do not exclude those sequences which contain x) is USC. Therefore, by definition $\widehat{g} \leq_p \limsup g$. Hence letting $h = \widehat{g} - g$ we have that

$$h \leq_p \limsup(g) - g. \tag{3.1.15}$$

Now, we claim that

$$(\limsup(g) - g)|_{V\cap K} \leq \varepsilon. \tag{3.1.16}$$

Suppose the contrary, then there is $x \in V \cap K$ such that $(\limsup_{y\to x} g(x)) - g(x) > \varepsilon$. Consequently, there exists a sequence $y_n \to x$, such that $\lim_{n\to\infty} g(y_n) > g(x) + \varepsilon$. Using the nonnegativity of g and the fact that $g|_{K^c} \leq \varepsilon$ we get that $y_n \in K \cap V$ except for finitely many n's. But then $osc(g, K \cap V) > \varepsilon$, a contradiction. So we have (3.1.16) and using (3.1.15) we obtain

$$h|_{V\cap K} \leq \varepsilon. \tag{3.1.17}$$

Observe now that if for a bounded function f and an open set U we have that $f|_U \leq \varepsilon$, then $\widehat{f}|_U \leq \varepsilon$ (clearly, if $|f| < K$ then the function $K \cdot \chi_{U^c} + \varepsilon \cdot \chi_U$ is an USC upper bound of f).

By the above observation used for g on K^c we have that $\widehat{g}|_{K^c} \leq \varepsilon$, in particular from $h = \widehat{g} - g \leq_p \widehat{g}$ we obtain that $h|_{K^c} \leq \varepsilon$. Then from (3.1.17) we get $h|_V \leq \varepsilon$. So finally, using the above observation for h and V we obtain $\widehat{h}|_V \leq \varepsilon$.

The set $\{x : g(x) > \varepsilon\}$ is dense in K, hence there exists an $x_0 \in V \cap \{x : g(x) > \varepsilon\}$. On the one hand $\widehat{g}(x_0) \geq g(x_0) > \varepsilon$, on the other by $x \in V$ we get $\widehat{h}(x_0) \leq \varepsilon$. This contradicts the assumption that $\widehat{g} = \widehat{h}$. $\square$

So we have proved (2) of Proposition 3.1.5.

(3) easily follows from Lemma 3.1.14, (1), (2) since $0 \leq g_{\xi_f} \leq f_{\xi_f} \equiv 0$. This finishes the proof of the proposition. $\square$

3.2 Proof of Theorems 1.2.4 and 1.2.8

3.2.1 Notation and basic facts

The following notions and facts can all be found in [51].

Let $\mathcal{F}(G)$ denote the family of closed subsets of G equipped with the so called Effros Borel structure. Let $\mathcal{K}(G)$ be the family of compact subsets of G equipped with the Hausdorff metric. Then $\mathcal{K}(G)$ is a Borel subset of $\mathcal{F}(G)$ and the inherited Borel structure on $\mathcal{K}(G)$ coincides with the one given by the Hausdorff metric.

Let us denote by $\mathcal{P}(G)$ the set of Borel probability measures on G, where by Borel probability measure we mean the completion of a probability measure defined on the Borel sets. These measures form a Polish space equipped with the weak*-topology. For $\mu \in \mathcal{P}(G)$ we denote by $\operatorname{supp}(\mu)$ the support of μ, i.e. the minimal closed subset F of G so that $\mu(F) = 1$. Let $\mathcal{P}_c(G) = \{\mu \in \mathcal{P}(G) : \operatorname{supp}(\mu) \text{ is compact}\}$.

Recall that $\mathbf{\Pi}^0_\xi$ stands for the ξth multiplicative level of the Borel hierarchy, $\mathbf{\Delta}^1_1$, $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$ denote the classes of Borel, analytic and coanalytic sets, respectively. For a Polish space X, $\mathbf{\Pi}^0_\xi(X)$, $\mathbf{\Delta}^1_1(X)$ etc. denote the collections of subsets of X in the appropriate classes. Symbols Γ and Λ will denote one of the above mentioned classes, and $\check{\Lambda} = \{A^c : A \in \Lambda\}$.

For a set $H \subset X \times Y$ we define its x-section as $H_x = \{y \in Y : (x, y) \in H\}$, and similarly if $H \subset X \times Y \times Z$ then $H_{x,y} = \{z \in Z : (x, y, z) \in H\}$, etc. For a function $f \colon X \times Y \to Z$ the x-section is the function $f_x \colon Y \to Z$ defined by $f_x(y) = f(x, y)$. We will sometimes also write $f_x = f(x, \cdot)$.

For $A, B \subset G$ let $d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$ and $A + B = \{a + b : a \in A, b \in B\}$. Let us denote by $B(g, r)$ and $\bar{B}(g, r)$ the open and closed ball centered at g of radius r.

Remark 3.2.1. Note that in the definition of Haar null sets certain authors actually require that the measure μ, which we will often refer to as a *witness measure*, has compact support. This is quite important if the underlying group is non-separable. However, in our case this would make no difference, since in a Polish space for every Borel probability measure there exists a compact set with positive measure [51, 17.11], and then restricting the measure to this set and normalising yields a witness with a compact support. Therefore we may suppose throughout the proofs that our witness measures have compact support.

3.2.2 A function with a surprisingly thick graph

Throughout the proofs, let $\Gamma = \mathbf{\Delta}^1_1$ and $\Lambda = \mathbf{\Pi}^0_\xi$ for some $1 \le \xi < \omega_1$, or let $\Gamma = \mathbf{\Pi}^1_1$ and $\Lambda = \mathbf{\Delta}^1_1$.

The following result will be the starting point of our constructions. For a fixed measure μ statement 2. below describes the following strange phenomenon: There exists a Borel graph of a function in a product space such that every G_δ cover of the graph has a vertical section of positive measure.

Theorem 3.2.2. *Let $\Gamma = \mathbf{\Delta}^1_1$ and $\Lambda = \mathbf{\Pi}^0_\xi$ for some $1 \le \xi < \omega_1$, or let $\Gamma = \mathbf{\Pi}^1_1$ and $\Lambda = \mathbf{\Delta}^1_1$. Then there exists a partial function $f : \mathcal{P}_c(G) \times 2^\omega \to G$ with* $\mathrm{graph}(f) \in \Gamma$ *satisfying the following properties: $\forall \mu \in \mathcal{P}_c(G)$*

1. $(\forall x \in 2^\omega)\,[(\mu, x) \in \mathrm{dom}(f) \Rightarrow f(\mu, x) \in \mathrm{supp}(\mu)]$,
2. $(\forall S \in \Lambda(2^\omega \times G))\,[(\mathrm{graph}(f_\mu) \subset S \Rightarrow (\exists x \in 2^\omega)(\mu(S_x) > 0)]$.

Before the proof we need several technical lemmas.

Lemma 3.2.3. *$\mathcal{P}_c(G)$ is a Borel subset of $\mathcal{P}(G)$.*

Proof. The map $\mu \mapsto \mathrm{supp}(\mu)$ between $\mathcal{P}(G)$ and $\mathcal{F}(G)$ is Borel (see [51, 17.38]) and $\mathcal{P}_c(G)$ is the preimage of $\mathcal{K}(G)$ under this map. □

Lemma 3.2.4. *Let X be a Polish space and $C \subset \mathcal{P}_c(G) \times X \times G$ with $C \in \Gamma$. Then $\{(\mu, x) : \mu(C_{\mu,x}) > 0\} \in \Gamma$.*

Proof. Let first $\Gamma = \mathbf{\Delta}^1_1$. If Y is a Borel space and $C \subset Y \times G$ is a Borel set then the map $\varphi\colon Y \times \mathcal{P}_c(G) \to [0,1]$ defined by $\varphi(y, \mu) = \mu(C_y)$ is Borel ([51, 17.25]). Using this for $Y = \mathcal{P}_c(G) \times X$ we obtain that the map $\psi\colon \mathcal{P}_c(G) \times X \to [0,1]$ given by $\psi(\mu, x) = \varphi((\mu, x), \mu) = \mu(C_{\mu,x})$ is also Borel. Then $\{(\mu, x) : \mu(C_{\mu,x}) > 0\} = \psi^{-1}((0,1])$, hence Borel.

For $\Gamma = \mathbf{\Pi}^1_1$ this is simply a special case of [51, 36.23]. □

Lemma 3.2.5. *The set $\{(\mu, g) : g \in \mathrm{supp}(\mu)\} \subset \mathcal{P}_c(G) \times G$ is Borel.*

Proof. As mentioned above, the map $\mu \mapsto \mathrm{supp}(\mu)$ is Borel between $\mathcal{P}(G)$ and $\mathcal{F}(G)$, hence its restriction to $\mathcal{P}_c(G)$ is also Borel.

Let $E = \{(K, g) : K \in \mathcal{K}(G),\ g \in K\}$, which clearly is a closed subset of $\mathcal{K}(G) \times G$. If we denote by $\Psi : \mathcal{P}_c(G) \times G \to \mathcal{K}(G) \times G$ the Borel map defined by $(\mu, g) \mapsto (\mathrm{supp}(\mu), g)$ then we obtain that $\{(\mu, g) : g \in \mathrm{supp}(\mu)\} = \Psi^{-1}(E)$ is Borel. □

Let us now prove Theorem 3.2.2.

Proof. Let $U \in \Gamma(2^\omega \times 2^\omega \times G)$ be universal for the $\check{\Lambda}$ subsets of $2^\omega \times G$, that is, for every $A \in \check{\Lambda}(2^\omega \times G)$ there exists an $x \in 2^\omega$ such that $U_x = A$ (for the existence of such a set see [51, 22.3, 26.1]). Notice that $\check{\Lambda} \subset \Gamma$. Let

$$U' = \mathcal{P}_c(G) \times U.$$

Define

$$U'' = \{(\mu, x, g) \in \mathcal{P}_c(G) \times 2^\omega \times G : (\mu, x, x, g) \in U' \text{ and } \mu(U'_{\mu,x,x}) > 0\},$$

then $U'' \in \Gamma$ using that the map $(\mu, x, g) \mapsto (\mu, x, x, g)$ is continuous and by Lemma 3.2.4. Let

$$U''' = \{(\mu, x, g) \in U'' : g \in \operatorname{supp}(\mu)\},$$

then $U''' \in \Gamma$ by Lemma 3.2.5. Clearly,

$$U'''_{\mu,x} = \begin{cases} U'_{\mu,x,x} \cap \operatorname{supp}(\mu) & \text{if } \mu(U'_{\mu,x,x}) > 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

Since for all (μ, x) the section $U'''_{\mu,x}$ is either empty or has positive μ measure, by the 'large section uniformisation theorem' [51, 18.6] and the coanalytic uniformisation theorem [51, 36.14] there exists a partial function f with $\operatorname{graph}(f) \in \Gamma$ such that $\operatorname{dom}(f) = \{(\mu, x) \in \mathcal{P}_c(G) \times 2^\omega : \mu(U'_{\mu,x,x}) > 0\}$ and $\operatorname{graph}(f) \subset U'''$.

We claim that this f has all the required properties.

First, by the definition of U''', clearly $f(\mu, x) \in \operatorname{supp}(\mu)$ holds whenever $(\mu, x) \in \operatorname{dom}(f)$, hence Property 1. of Theorem 3.2.2 holds.

Let us now prove Property 2. Assume towards a contradiction that there exists $\mu \in \mathcal{P}_c(G)$ and $S \in \Lambda(2^\omega \times G)$ such that $\operatorname{graph}(f_\mu) \subset S$ and $\mu(S_x) = 0$ for every $x \in 2^\omega$. Define $B = (2^\omega \times G) \setminus S$. By the universality of U there exists $x \in 2^\omega$ such that $U_x = U'_{\mu,x} = B$. Now, for every $y \in 2^\omega$ the section B_y is of positive (actually full) μ measure, in particular $\mu(U'_{\mu,x,x}) > 0$, and therefore $(\mu, x) \in \operatorname{dom}(f)$ and

$$f(\mu, x) \in U'''_{\mu,x} \subset U''_{\mu,x} = U'_{\mu,x,x} = B_x.$$

However, $f(\mu, x) \in S_x = G \setminus B_x$, a contradiction. $\square$

3.2.3 Translating the compact sets apart

This section heavily builds on ideas of Solecki [74], [75]. The main point is that if G is non-locally compact then one can apply a translation (chosen in a Borel way) to every compact subset of G so that the resulting translates are disjoint. (For technical reasons we will need to consider continuum many copies of each compact set and also to 'blow them up' by a fixed compact set C.)

Proposition 3.2.6. *Let $C \in \mathcal{K}(G)$ be fixed. Then there exists a Borel map $t : \mathcal{K}(G) \times 2^\omega \times 2^\omega \to G$ so that*

1. *if $(K, x, y) \neq (K', x', y')$ are elements of $\mathcal{K}(G) \times 2^\omega \times 2^\omega$ then*

$$(K - C + t(K, x, y)) \cap (K' - C' + t(K', x', y')) = \emptyset$$

2. *for every $K \in \mathcal{K}(G)$ and $y \in 2^\omega$ the map $t(K, \cdot, y)$ is continuous.*

Proof. We use Solecki's arguments [74], [75], which he used for different purposes, with some modifications. However, for the sake of completeness, we repeat large parts of his proofs.

Fix an increasing sequence of finite sets $Q_k \subset G$ with $0 \in Q_0$ such that $\cup_{k\in\omega} Q_k$ is dense in G.

Lemma 3.2.7. *For every $\epsilon > 0$ there exists $\delta > 0$ and a sequence $\{g_k\}_{k\in\omega} \subset B(0,\epsilon)$ such that for every distinct $k, k' \in \omega$*

$$d(Q_k + g_k, Q_{k'} + g_{k'}) \geq \delta.$$

Proof. Since G is not locally compact, there exists $\delta > 0$ and a countably infinite set $S \subset B(0,\epsilon)$ such $d(s,s') \geq 2\delta$ for every distinct $s, s' \in S$.

Now we define g_k inductively as follows. Suppose that we are done for $i < k$. If for every $s \in S$ there are $a \in Q_k$, $i < k$ and $b \in Q_i$ with $d(a+s, b+g_i) < \delta$ then there is a pair s, s' of distinct members of S with the same a, i and b. But then

$$d(s,s') = d(a+s, a+s') \leq d(a+s, b+g_i) + d(b+g_i, a+s') < 2\delta,$$

a contradiction. Hence we can let $g_k = s$ for an appropriate $s \in S$. □

It is easy to see that using the previous lemma repeatedly we can inductively fix ϵ_n, $\delta_n < \varepsilon_n$ and sequences $\{g^n_k\}_{k\in\omega}$ such that for every $n \in \omega$

- $\{g^n_k\}_{k\in\omega} \subset B(0,\varepsilon_n)$,
- $d(Q_k + g^n_k, Q_{k'} + g^n_{k'}) \geq 2\delta_n$ for every distinct $k, k' \in \omega$,
- $\sum_{m>n} \epsilon_m < \frac{\delta_n}{3}$.

Note that the second property implies that for every $n \in \omega$ the function $k \mapsto g^n_k$ is injective. Note also that $\varepsilon_n \to 0$ and hence $\delta_n \to 0$, moreover, $\sum \delta_n$ is also convergent.

Let us also fix a Borel injection $c : \mathcal{K}(G) \times 2^\omega \times 2^\omega \to \omega^\omega$ such that for each K and y the map $c(K,\cdot,y)$ is continuous. (E.g. fix a Borel injection $c_1 : \mathcal{K}(G) \to 2^\omega$ and continuous injection $c_2 : 2^\omega \times 2^\omega \times 2^\omega \to \omega^\omega$ and let $c(K,x,y) = c_2(c_1(K),x,y)$.)

Our goal now is to define $t(K,x,y)$, so let us fix a triple (K,x,y). First we define a sequence $\{h_n = h_n(K,x,y)\}_{n\in\omega}$ with $h_n \in \{g^n_k\}_{k\in\omega}$ as follows. Suppose that we are given h_i for $i < n$. By the density of $\cup_k Q_k$ we have $G = \cup_k(Q_k + B(0,\delta_n/2))$. Since $K - C$ is compact, there exists a minimal index $k_n(K,x,y)$ so that

$$K - C + \sum_{i<n} h_i \subset Q_{k_n(K,x,y)} + B(0,\delta_n/2).$$

Fix an injective map $\phi : \omega \times \omega \to \omega$ with $\phi(i,j) \geq i$ for every $i \in \omega$ and let

$$h_n = g^n_{\phi(k_n(K,x,y),c(K,x,y)(n))} \tag{3.2.1}$$

and

$$t(K,x,y) = \sum_{n\in\omega} h_n. \tag{3.2.2}$$

We claim that this function has the required properties.

First, it is well defined, that is, the sum is convergent since $h_n \in B(0,\epsilon_n)$, and hence for all $n \in \omega$

$$\sum_{m>n} h_m \in \bar{B}(0,\delta_n/3). \tag{3.2.3}$$

In order to prove 1. of the Proposition, let us now fix $(K,x,y) \neq (K',x',y')$. Then there exists an $n \in \omega$ such that $c(K,x,y)(n) \neq c(K',x',y')(n)$. By the injectivity of ϕ and of the sequence $k \mapsto g^n_k$ and also by (3.2.1) we obtain that $h_n(K,x,y) \neq h_n(K',x',y')$. Denote by h_i and h'_i the elements $h_i(K,x,y)$ and $h_i(K',x',y')$, respectively. Set

$$k = \phi(k_n(K,x,y), c(K,x,y)(n)) \text{ and } k' = \phi(k_n(K',x',y'), c(K',x',y')(n)).$$

The condition $\phi(i,j) \geq i$ implies $k \geq k_n(K,x,y)$, hence $Q_k \supset Q_{k_n(K,x,y)}$ and similarly $k' \geq k_n(K',x',y')$, so $Q_{k'} \supset Q_{k_n(K',x',y')}$. Therefore, by the definition of k_n,

$$K - C + \sum_{i<n} h_i \in Q_k + B(0,\delta_n/2) \text{ and } K' - C + \sum_{i<n} h'_i \in Q_{k'} + B(0,\delta_n/2),$$

hence

$$K-C+\sum_{i\leq n} h_i \in Q_k+h_n+B(0,\delta_n/2) \text{ and } K'-C+\sum_{i\leq n} h'_i \in Q_{k'}+h'_n+B(0,\delta_n/2).$$

Thus, using the triangle inequality and the second property of the g^n_k we obtain

$$d(K - C + \sum_{i\leq n} h_i, K' - C + \sum_{i\leq n} h'_i) \geq d(Q_k + h_n, Q_{k'} + h'_n) - 2\cdot\frac{\delta_n}{2} =$$

$$= d(Q_k + g^n_k, Q_{k'} + g^n_{k'}) - \delta_n \geq 2\delta_n - \delta_n = \delta_n.$$

From this, using (3.2.3), we obtain $d(K-C+t(K,x,y), K'-C+t(K',x',y')) \geq \delta_n - 2\frac{\delta_n}{3} = \frac{\delta_n}{3} > 0$, which proves 1.

What remains to show is that t is a Borel map and for every K and y the map $t(K,\cdot,y)$ is continuous. But (3.2.3) shows that the series defining t in (3.2.2) is uniformly convergent, so the next lemma finishes the proof.

Lemma 3.2.8. *For every $n \in \omega$ the map h_n is Borel and for every K and y the map $h_n(K,\cdot,y)$ is continuous.*

Proof. We will actually prove more by induction on n. Define $f_n\colon \mathcal{K}(G) \times 2^\omega \times 2^\omega \to \mathcal{K}(G)$ by

$$f_n(K,x,y) = K - C + \sum_{i<n} h_i(K,x,y). \tag{3.2.4}$$

We claim that the maps f_n, k_n and h_n are Borel and for every K and y the maps $f_n(K,\cdot,y)$, $k_n(K,\cdot,y)$ and $h_n(K,\cdot,y)$ are locally constant.

Note that if a function takes its values from a discrete set than locally constant is equivalent to continuous.

First we prove that the maps are Borel. Suppose that we are done for $i < n$. Let us check that f_n is Borel. Put $\eta : (K,x,y) \mapsto (K, \sum_{i<n} h_i(K,x,y))$ and $\psi : (K,g) \mapsto K - C + g$, then $f_n = \psi \circ \eta$. Moreover, η is Borel by induction, and ψ is easily seen to be continuous, hence f_n is Borel.

Next we show that k_n is Borel. Since $\operatorname{ran}(k_n) \subset \omega$, we need to check that for every fixed $m \in \omega$ the set $B = \{(K,x,y)\colon k_n(K,x,y) = m\}$ is Borel. By the definition of $k_n(K,x,y)$, clearly

$$B = \{(K,x,y)\colon f_n(K,x,y) \subset U \text{ and } f_n(K,x,y) \not\subset V\},$$

where $U = Q_m + B(0, \delta_n/2)$ and $V = Q_{m-1} + B(0, \delta_n/2)$ are fixed open sets.

Set $\mathcal{U}_W = \{L \in \mathcal{K}(G) : L \subset W\}$, which is open in $\mathcal{K}(G)$ for every open set $W \subset G$. Then clearly

$$B = f_n^{-1}(\mathcal{U}_U) \setminus f_n^{-1}(\mathcal{U}_V),$$

hence Borel.

Since the functions $k \mapsto g_k^n$ and ϕ defined on countable sets are clearly Borel, the Borelness of k_n and c imply by (3.2.1) that h_n is also Borel.

In order to prove that f_n, k_n and h_n are locally constant in the second variable, fix K and y and suppose that we are done for $i < n$. Then (3.2.4) shows that f_n is locally constant in the second variable by induction. This easily implies using the definition of k_n that k_n is also is locally constant in the second variable. But from this, and from the fact that $c(K, \cdot, y)(n) \colon 2^\omega \to \omega$ is continuous, hence locally constant, it is also clear using (3.2.1) that h_n is also locally constant in the second variable, which finishes the proof of the Lemma. $\square$

Therefore the proof of the Proposition is also complete. $\square$

3.2.4 Putting the ingredients together

Now we are ready to prove our main results, which are summarised in the following theorem.

Theorem 3.2.9. *Let $\Gamma = \mathbf{\Delta}^1_1$ and $\Lambda = \mathbf{\Pi}^0_\xi$ for some $1 \leq \xi < \omega_1$, or let $\Gamma = \mathbf{\Pi}^1_1$ and $\Lambda = \mathbf{\Delta}^1_1$. If G is a non-locally compact abelian Polish group then there exists a (generalised, in the case of $\Gamma = \mathbf{\Pi}^1_1$) Haar null set $E \in \Gamma(G)$ that is not contained in any Haar null set $H \in \Lambda(G)$.*

Proof. Let f be given by Theorem 3.2.2.

Denote the Borel map $\mu \mapsto \operatorname{supp}(\mu)$ by $\operatorname{supp} : \mathcal{P}_c(G) \to \mathcal{K}(G)$. Let us also fix a Borel bijection $c : \mathcal{P}_c(G) \to 2^\omega$ (which we think of as a coding map) and a continuous probability measure ν on G with compact support C containing 0 (compactly supported continuous measures exist on every Polish space without isolated points, since such spaces contain copies of 2^ω). Let $t : \mathcal{K}(G) \times 2^\omega \times 2^\omega \to G$ be the map from Proposition 3.2.6 with the C fixed above, and define the map $\Psi \colon \mathcal{P}_c(G) \times 2^\omega \times G \to G$ by

$$\Psi(\mu, x, g) = g + t(\operatorname{supp}(\mu), x, c(\mu)). \tag{3.2.5}$$

Finally, define $E = \Psi(\operatorname{graph}(f))$.

Claim 3.2.10. $E \in \Gamma$.

Proof. Ψ is clearly a Borel map. We claim that it is injective on $D = \{(\mu, x, g) : \mu \in \mathcal{P}_c(G), g \in \operatorname{supp}(\mu)\}$, which is Borel by Lemma 3.2.3 and Lemma 3.2.5. Let $(\mu, x, g) \neq (\mu', x', g')$ be elements of D, we need to check that Ψ takes distinct values on them. The case $(\mu, x) = (\mu', x')$ is obvious, while the case $(\mu, x) \neq (\mu', x')$ follows from Property 1. in Proposition 3.2.6, since $\Psi(\mu, x, g) \in \operatorname{supp}(\mu) - C + t(\operatorname{supp}(\mu), x, c(\mu))$ (recall that $g \in \operatorname{supp}(\mu)$ and $0 \in C$). Therefore Ψ is a Borel isomorphism on D. By $\operatorname{graph}(f) \subset D$ this implies that $E = \Psi(\operatorname{graph}(f))$ is in Γ (for $\Gamma = \mathbf{\Delta^1_1}$ see [51, 15.4], for $\Gamma = \mathbf{\Pi}^1_1$ notice that by [51, 25.A] a Borel isomorphism takes analytic sets to analytic sets, hence coanalytic sets to coanalytic sets). □

Claim 3.2.11. *E is Haar null (generalised Haar null in the case of $\Gamma = \mathbf{\Pi}^1_1$).*

Proof. We prove that ν is witnessing this fact. Actually, we prove more: $|C \cap (E+g)| \leq 1$ for every $g \in G$, or equivalently $|(C+g) \cap E| \leq 1$ for every $g \in G$. So let us fix $g \in G$.

$$E = \Psi(\mathrm{graph}(f)) = \{\Psi(\mu, x, f(\mu, x)) : (\mu, x) \in \mathrm{dom}(f)\} =$$

$$\{f(\mu, x) + t(\mathrm{supp}(\mu), x, c(\mu)) : (\mu, x) \in \mathrm{dom}(f)\},$$

hence the elements of E are of the form $g^{\mu,x} = f(\mu, x) + t(\mathrm{supp}(\mu), x, c(\mu))$. This element $g^{\mu,x}$ is clearly in $A^{\mu,x} = \mathrm{supp}(\mu) + t(\mathrm{supp}(\mu), x, c(\mu))$ by Property 1. of Theorem 3.2.2, and the sets $A^{\mu,x}$ form a pairwise disjoint family as (μ, x) ranges over $\mathrm{dom}(f)$, by Property 2. of Proposition 3.2.6. Hence it suffices to show that $C + g$ can intersect at most one $A^{\mu,x}$. But it can actually intersect at most one set of the form $K + t(K, x, y)$, since otherwise g would be in the intersection of two distinct sets of the form $K - C + t(K, x, y)$, contradicting Property 2. of Proposition 3.2.6. □

Claim 3.2.12. *There is no Haar null set $H \in \Lambda$ containing E.*

Suppose that $H \in \Lambda$ is such a set. Then by Remark 3.2.1 there exists a probability measure μ with compact support witnessing this fact. The section map $\Psi_\mu = \Psi(\mu, \cdot, \cdot)$ is continuous by (3.2.5) and Property 2. of Proposition 3.2.6. Now let $S = \Psi_\mu^{-1}(H)$, then $S \in \Lambda(2^\omega \times G)$.

It is easy to check that $\mathrm{graph}(f_\mu) \subset S$, and therefore, using Theorem 3.2.2, there exists $x \in 2^\omega$ such that $\mu(S_x) > 0$. By the definition of S we have that $\Psi(\mu, x, S_x) \subset \Psi_\mu(S) \subset H$. But $\Psi(\mu, x, \cdot) : G \to G$ is a translation, so a translate of H contains S_x, which is of positive μ measure, contradicting that H is Haar null with witness μ. □

This concludes the proof. □

3.3 Proof of Theorem 1.2.10

The heart of the proof of this result is the following theorem, which is based on ideas from [16]. For the definition and basic properties of packing dimension, denoted by $\dim_p H$, see [35] or [67].

Theorem 3.3.1. *Let $K' \subset \mathbb{R}$ be a Cantor set with $\dim_p K' < 1$ and let $T \subset \mathbb{R}$ be such that $|T| < \mathfrak{c}$. Then $K' + T$ contains no measurable set of positive measure.*

Proof. Suppose on the contrary that $K' + T$ contains a measurable set P of positive measure. We may assume that P is compact. By throwing away all portions (i.e. relatively open nonempty subsets) of measure zero, we may also assume that every portion of P is of positive measure. In particular, P has no isolated points. The idea of the proof will be to construct a Cantor set $P' \subset P$ such that $P' \cap (K' + r)$ is finite for every $r \in \mathbb{R}$. This clearly suffices, since a Cantor set is of cardinality continuum and hence less than continuum many translates of K' cannot cover P', let alone P.

Let N be a positive integer and let us define F_N to be the set of N-tuples that can be covered by a translate of K', that is,

$$F_N = \{(x_0, \dots, x_{N-1}) \in \mathbb{R}^N : \exists t \in \mathbb{R} \text{ such that } \{x_0, \dots, x_{N-1}\} \subset K' + t\}.$$

An easy compactness argument shows that F_N is closed. Reformulating the definition one can easily check that

$$F_N = (K')^N + \mathbb{R}(1, \dots, 1),$$

where $(1, \dots, 1)$ is a vector of N coordinates, and the operations are Minkowski sum and Minkowski product. It is easy to see that F_N is a Lipschitz image of $(K')^N \times \mathbb{R}$, and using that Lipschitz images do not increase packing dimension as well as $\dim_p(A \times B) \leq \dim_p A + \dim_p B$ and $\dim_p \mathbb{R} = 1$ we obtain

$$\dim_p F_N \leq N \dim_p K' + 1.$$

If we choose N large enough, actually if $N > \frac{1}{1-\dim_p K'}$, then $N \dim_p K' + 1 < N$, hence

$$\dim_p F_N < N.$$

Let us fix such an N.

Lemma 3.3.2. *Let $J_i \subset \mathbb{R}$ $(i < N)$ be closed intervals such that* $\operatorname{int} J_i \cap P \neq \emptyset$ *$(i < N)$. Then there are disjoint closed intervals $I_i \subset J_i$ $(i < N)$ such that* $\operatorname{int} I_i \cap P \neq \emptyset$ $(i < N)$ *and*

$$\prod_{i<N} (I_i \cap P) \cap F_N = \emptyset.$$

Proof. Since every portion (i. e., nonempty relatively open subset) of P is of positive measure, we obtain $\lambda^N \left(\prod_{i<N} (\operatorname{int} J_i \cap P)\right) > 0$, hence $\dim_p \left(\prod_{i<N} (\operatorname{int} J_i \cap P)\right) = N > \dim_p F_N$. Therefore $\left(\prod_{i<N} (\operatorname{int} J_i \cap P)\right) \setminus F_N \neq \emptyset$, and, since F_N is closed, $\prod_{i<N} (\operatorname{int} J_i \cap P)$ contains a nonempty relatively open set avoiding F_N. This open set contains a basic open set, so there are open intervals $J'_i \subset \operatorname{int} J_i$ $(i < N)$ intersecting P such that $\prod_{i<N} (J'_i \cap P) \cap F_N = \emptyset$.

Finally, since P has no isolated points, it is easy to shrink every J'_i to a closed interval I_i such that they become disjoint but their interiors still meet P. This finishes the proof of the lemma. $\square$

Now we return to the proof of the theorem. All that remains is to construct P'. We will actually prove

$$|P' \cap (K' + r)| < N \text{ for every } r \in \mathbb{R}. \tag{3.3.1}$$

We construct a usual Cantor scheme, where the k^{th} level $\mathcal{L}_k$ will have the following properties for all $k \in \omega$.

(1) $\mathcal{L}_k$ consist of N^k many disjoint closed intervals,

(2) $\forall I \in \mathcal{L}_{k+1} \exists I' \in \mathcal{L}_k : I \subset I'$,

(3) $\forall I \in \mathcal{L}_k$ there are N many $I' \in \mathcal{L}_{k+1}$ with $I' \subset I$,

(4) $\forall I \in \mathcal{L}_k : \operatorname{int} I \cap P \neq \emptyset$,

(5) $\forall I \in \mathcal{L}_k : \operatorname{diam} I \leq \frac{1}{k+1}$,

(6) If $I_0, \dots, I_{N-1} \in \mathcal{L}_k$ are distinct then $\prod_{i<N}(I_i \cap P) \cap F_N = \emptyset$.

(Note that the intervals in (6) are not necessarily subsets of the same $I' \in \mathcal{L}_{k-1}$.) Assume first that such a Cantor scheme exists, and define

$$P' = \bigcap_{k \in \omega} \bigcup \mathcal{L}_k.$$

It is easy to see that P' is a Cantor set ([51]), while the closedness of P, (4) and (5) imply $P' \subset P$. Let $x_0, \dots, x_{N-1}$ be N distinct points in P'. Clearly, there is a k and distinct intervals $I_0, \dots, I_{N-1} \in \mathcal{L}_k$ such that $x_i \in I_i$ $(i < N)$. Then (6) shows that $\{x_0, \dots, x_{N-1}\}$ cannot be covered by a translate of K', which proves (3.3.1).

Finally, let us prove by induction that such a Cantor scheme exists. Let $\mathcal{L}_0 = \{I\}$, where I is an arbitrary closed interval of length at most 1 whose interior meets P. Assume that $\mathcal{L}_k$ have already been constructed with the required properties. Let $\mathcal{L}'_{k+1}$ be a family of disjoint closed intervals of length at most $\frac{1}{k+2}$ whose interiors meet P such that each $I \in \mathcal{L}_k$ contains N members of $\mathcal{L}'_{k+1}$. Then recursively shrinking these intervals by applying Lemma 3.3.2 $\binom{N^{k+1}}{N}$ times to all the possible N-tuples of distinct intervals we obtain $\mathcal{L}_{k+1}$ satisfying all assumptions. This concludes the proof of the theorem. □

Theorem 3.3.3. *Let $K \subset \mathbb{R}$ be a Cantor set with $\dim_p K < 1/2$. Then there exists $X \subset \mathbb{R}$ with $\lambda(X) > 0$ such that $|K \cap (X + t)| \leq 1$ for every $t \in \mathbb{R}$.*

Proof. As above, let us enumerate the Borel sets of Lebesgue measure zero as $\{Z_\alpha : \alpha < \mathfrak{c}\}$. Since $K - K$ is a Lipschitz image of $K \times K$, we obtain $\dim_p(K - K) < 1$. At stage α let us pick an

$$x_\alpha \in \mathbb{R} \setminus \left(((K - K) + \{x_\beta : \beta < \alpha\}) \cup Z_\alpha\right).$$

This is indeed possible by the above theorem applied to $K' = K - K$. Set

$$X = \{x_\alpha : \alpha < \mathfrak{c}\}.$$

Then $x_\alpha \notin Z_\alpha$ shows that $\lambda(X) > 0$. We still have to check that $|K \cap (X + t)| \leq 1$ for every $t \in \mathbb{R}$. Let $x_\alpha, x_\beta \in X$ with $\alpha > \beta$, and let us assume $x_\alpha + t, x_\beta + t \in K$. Then $t \in K - x_\beta$, $x_\alpha \in K - (K - x_\beta) = (K - K) + x_\beta$, contradicting the choice of x_α. □

Finally, from this we easily obtain Theorem 1.2.10.

Proof. Let K be any Cantor set with $\dim_p K < 1/2$ (e.g. the "middle-α Cantor set" is such a set for $\alpha > 1/2$). Let X be as in the previous theorem. Also, let μ be any atomless Borel probability measure on K. Then by the previous theorem $\mu(X + t) = 0$ for every $t \in \mathbb{R}$. □

3.4 Proof of Theorem 1.3.8

3.4.1 Preliminaries

Most of the following notations and facts can be found in [51].

Throughout the section let (X,τ) be an uncountable *Polish* space. We denote a compatible, complete metric for (X,τ) by d.

Recall that a *Polish group* is a topological group whose topology is Polish, and also that a set H is *ambiguous* if $H \in \mathbf{\Delta}^0_2$.

If τ' is a topology on X then we denote the family of real valued functions defined on X that are of Baire class ξ with respect to τ' by $\mathcal{B}_\xi(\tau')$. In particular, $\mathcal{B}_\xi = \mathcal{B}_\xi(\tau)$. If Y is another Polish space (whose topology is clear from the context) then we also use the notation $\mathcal{B}_\xi(Y)$ for the family of Baire class ξ functions defined on Y. Similarly, $\mathbf{\Sigma}^0_\xi(\tau')$ and $\mathbf{\Sigma}^0_\xi(Y)$ are both the set of $\mathbf{\Sigma}^0_\xi$ subsets, with respect to τ', and in Y, respectively. We use the analogous notation for all the other pointclasses.

Recall that a nonempty perfect subset of a Polish space with the subspace topology is an uncountable Polish space.

Recall that a function is of Baire class ξ iff the inverse image of every open set is in $\mathbf{\Sigma}^0_{\xi+1}$ iff $\{f < c\}$ and $\{f > c\}$ are in $\mathbf{\Sigma}^0_{\xi+1}$ for every $c \in \mathbb{R}$. Moreover, the family of Baire class ξ functions is closed under uniform limits.

For a set $H \subset X \times Y$ and an element $x \in X$ we denote the x-section of H by $H_x = \{y \in Y : (x,y) \in H\}$.

If $\mathcal{H}$ is a family of sets then

$$\mathcal{H}_\sigma = \left\{ \bigcup_{n\in\mathbb{N}} H_n : H_n \in \mathcal{H} \right\} \text{ and } \mathcal{H}_\delta = \left\{ \bigcap_{n\in\mathbb{N}} H_n : H_n \in \mathcal{H} \right\}.$$

For $\theta, \theta' < \omega_1$ we use the relation $\theta \lesssim \theta'$ if $\theta' \le \omega^\eta \implies \theta \le \omega^\eta$ for every $1 \le \eta < \omega_1$ (we use ordinal exponentiation here). Note that $\theta \le \theta'$ implies $\theta \lesssim \theta'$ and $\theta \lesssim \theta'$, $\theta' > 0$ implies $\theta \le \theta' \cdot \omega$. We write $\theta \approx \theta'$ if $\theta \lesssim \theta'$ and $\theta' \lesssim \theta$. Then $\approx$ is an equivalence relation. For every ordinal θ we have $2\theta < \theta + \omega$, and since ω^η is a limit ordinal for every $\eta \ge 1$ we obtain that $2\theta \approx \theta$ for every ordinal θ.

A rank $\rho : \mathcal{B}_\xi \to \omega_1$ is called *additive* if $\rho(f+g) \le \max\{\rho(f), \rho(g)\}$ for every $f, g \in \mathcal{B}_\xi$. It is called *linear* if it is additive and $\rho(cf) = \rho(f)$ for every $f \in \mathcal{B}_\xi$ and $c \in \mathbb{R} \setminus \{0\}$. If X is a Polish group then the left and right translation operators are defined as $L_{x_0}(x) = x_0 \cdot x$ $(x \in X)$ and $R_{x_0}(x) = x \cdot x_0$ $(x \in X)$. A rank $\rho : \mathcal{B}_\xi \to \omega_1$ is called *translation-invariant* if $\rho(f \circ L_{x_0}) = \rho(f \circ R_{x_0}) = \rho(f)$ for every $f \in \mathcal{B}_\xi$ and $x_0 \in X$. We say that it is *essentially additive*, *essentially linear*, and *essentially translation-invariant* if the corresponding inequalities and equations hold with $\lesssim$ and $\approx$. Moreover, ρ is additive, essentially additive etc. *for bounded functions*, if the corresponding relations hold whenever f and g are bounded.

Let $(F_\eta)_{\eta<\lambda}$ be a (not necessarily strictly) decreasing sequence of sets. Let us assume that $F_0 = X$ and that the sequence is *continuous*, that is, $F_\eta = \bigcap_{\theta<\eta} F_\theta$ for every limit η and if λ is a limit then $\bigcap_{\eta<\lambda} F_\eta = \emptyset$. We also use the convention that $F_\eta = \emptyset$ if $\eta \ge \lambda$. We say that a set H is the *transfinite difference of* $(F_\eta)_{\eta<\lambda}$ if $H = \bigcup_{\substack{\eta<\lambda \\ \eta \text{ even}}} (F_\eta \setminus F_{\eta+1})$. It is well-known that a set is in $\mathbf{\Delta}^0_{\xi+1}$ iff it is a transfinite difference of $\mathbf{\Pi}^0_\xi$ sets see e.g. [51, 22.27]. We have to point out here that the monograph [51] does *not* assume that the decreasing sequences are continuous, but when proving that every set in $\mathbf{\Delta}^0_{\xi+1}$ has a representation as a transfinite difference they actually construct continuous sequences, hence this issue causes no difficulty here.

The set of sequences of length k whose terms are elements of the set $\{0, \dots, n-1\}$ is denoted by n^k. For $s \in n^k$ we denote the i-th term of s by $s(i)$. If $l \in \{0, \dots, n-1\}$ then $s^\wedge l$ denotes the sequence in n^{k+1} whose first k terms agree with those of s and whose $k+1$st term is l.

3.4.2 Ranks on the Baire class 1 functions without compactness

In this section we summarise some results concerning ranks on the Baire class 1 functions, following the work of Kechris and Louveau. We do not consider most results in this section as original, we basically just carefully check that the results of Kechris and Louveau hold without the assumption of compactness of X. This is inevitable, since they assumed compactness throughout their paper but we will need these results in Section 3.4.7 for arbitrary Polish spaces.

A notable exception is Theorem 3.4.33 stating that the three ranks essentially coincide for bounded Baire class 1 functions. Since our highly non-trivial proof for the case of general Polish spaces required completely new ideas, we consider this result as original in the non-compact case.

The definitions of the ranks will use the notion of a *derivative operation.*

Definition 3.4.1. A *derivative* on the closed subsets of X is a map $D : \mathbf{\Pi}^0_1(X) \to \mathbf{\Pi}^0_1(X)$ such that $D(A) \subset A$ and $A \subset B \Rightarrow D(A) \subset D(B)$ for every $A, B \in \mathbf{\Pi}^0_1(X)$.

Definition 3.4.2. For a derivative D we define the *iterated derivatives* of the closed set F as follows:

$$\begin{aligned} D^0(F) &= F, \\ D^{\eta+1}(F) &= D(D^\eta(F)), \\ D^\eta(F) &= \bigcap_{\theta<\eta} D^\theta(F) \text{ if } \eta \text{ is a limit.} \end{aligned}$$

Definition 3.4.3. Let D be a derivative. The *rank* of D is the smallest ordinal η, such that $D^\eta(X) = \emptyset$, if such ordinal exists, ω_1 otherwise. We denote the rank of D by $\mathrm{rk}(D)$.

Remark 3.4.4. In all our applications D satisfies $D(F) \subsetneq F$ for every nonempty closed set F, and since in a Polish space there is no strictly decreasing sequence of closed sets of length ω_1 (see e.g. [51, 6.9]), the rank of a derivative is always a countable ordinal.

Proposition 3.4.5. *If the derivatives D_1 and D_2 satisfy $D_1(F) \subset D_2(F)$ for every closed subset $F \subset X$ then* $\mathrm{rk}(D_1) \le \mathrm{rk}(D_2)$.

Proof. It is enough to prove that $D_1^\eta(X) \subset D_2^\eta(X)$ for every ordinal η. We prove this by transfinite induction on η. For $\eta = 0$ this is obvious, since $D_1^0(X) = D_2^0(X) = X$. Now suppose this holds for η and we prove it for $\eta+1$. Since $D_1^\eta(X) \subset D_2^\eta(X)$ and D_1 is a derivative, we have $D_1(D_1^\eta(X)) \subset D_1(D_2^\eta(X))$. Using this observation and the condition of the proposition for the closed set $D_2^\eta(X)$, we have $D_1^{\eta+1}(X) = D_1(D_1^\eta(X)) \subset D_1(D_2^\eta(X)) \subset D_2(D_2^\eta(X)) = D_2^{\eta+1}(X)$.

For limit η the claim is an easy consequence of the continuity of the sequences, hence the proof is complete. □

Proposition 3.4.6. *Let $n \geq 1$ and D, $D_0, \dots, D_{n-1}$ be derivative operations on the closed subsets of X. Suppose that they satisfy the following conditions for arbitrary closed sets F and F':*

$$D(F) \subset \bigcup_{k=0}^{n-1} D_k(F), \tag{3.4.1}$$

$$D(F \cup F') \subset D(F) \cup D(F'). \tag{3.4.2}$$

Then for these derivatives

$$\mathrm{rk}(D) \lesssim \max_{k<n} \mathrm{rk}(D_k). \tag{3.4.3}$$

Proof. We will prove by induction on η that

$$D^{\omega^\eta}(F) \subset \bigcup_{k=0}^{n-1} D_k^{\omega^\eta}(F) \tag{3.4.4}$$

for every closed set F. It is easy to see that proving (3.4.4) is enough, since if η is an ordinal satisfying $\mathrm{rk}(D_k) \leq \omega^\eta$ for every $k < n$ then we have $\mathrm{rk}(D) \leq \omega^\eta$.

Now we prove (3.4.4). The case $\eta = 0$ is exactly (3.4.1). For limit η the statement is obvious, since the sequences are decreasing and continuous. Hence, it remains to prove (3.4.4) for $\eta+1$ if it holds for η. For this it is enough to show that for every $m \in \omega$

$$D^{\omega^\eta \cdot m \cdot n}(F) \subset \bigcup_{k=0}^{n-1} D_k^{\omega^\eta \cdot m}(F), \tag{3.4.5}$$

indeed,

$$D^{\omega^{\eta+1}}(F) = \bigcap_{m\in\omega} D^{\omega^\eta \cdot m \cdot n}(F) \subset \bigcap_{m\in\omega} \left(\bigcup_{k=0}^{n-1} D_k^{\omega^\eta \cdot m}(F) \right),$$

hence $x \in D^{\omega^{\eta+1}}(F)$ implies that without loss of generality $x \in D_0^{\omega^\eta \cdot m}(F)$ for infinitely many m, but the sequence $D_0^{\omega^\eta \cdot m}(F)$ is decreasing, hence $x \in \bigcap_{m\in\omega} D_0^{\omega^\eta \cdot m}(F) = D_0^{\omega^{\eta+1}}(F)$.

Now we prove (3.4.5). Let $F_\emptyset = F$, and for $m \in \mathbb{N}$, $s \in n^m$ and $k < n$ let

$$F_{s^\wedge k} = D_k^{\omega^\eta}(F_s).$$

It is enough that for $m \geq 1$

$$D^{\omega^\eta \cdot m}(F) \subset \bigcup_{s \in n^m} F_s, \tag{3.4.6}$$

since it is easy to see that

$$\bigcup_{s\in n^{m\cdot n}} F_s \subset \bigcup_{k=0}^{n-1} \bigcup \{F_s : s \in n^{m\cdot n} \text{ and } |\{i : s(i) = k\}| \geq m\},$$

yielding (3.4.5), as

$$\bigcup \{F_s : s \in n^{m\cdot n} \text{ and } |\{i : s(i) = k\}| \geq m\} \subset D_k^{\omega^\eta\cdot m}(F).$$

It remains to prove (3.4.6) by induction on m. For $m = 1$, this is only the induction hypothesis of (3.4.4) for η. By supposing (3.4.6) for m, we have

$$D^{\omega^\eta\cdot(m+1)}(F) = D^{\omega^\eta}\left(D^{\omega^\eta\cdot m}(F)\right) \subset D^{\omega^\eta}\left(\bigcup_{s\in n^m} F_s\right) \subset$$
$$\subset \bigcup_{s\in n^m} D^{\omega^\eta}(F_s) \subset \bigcup_{s\in n^{m+1}} F_s,$$

where we used (3.4.2) ω^η many times for the second containment, and for the last one we used the induction hypothesis, that is (3.4.4) for η. This finishes the proof. □

3.4.3 The separation rank of Baire class 1 functions

This rank was first introduced by Bourgain [9].

Definition 3.4.7. Let A and B be two subsets of X. We associate a derivative with them by

$$D_{A,B}(F) = \overline{F \cap A} \cap \overline{F \cap B}. \tag{3.4.7}$$

It is easy to see that $D_{A,B}(F)$ is closed, $D_{A,B}(F) \subset F$ and $D_{A,B}(F) \subset D_{A,B}(F')$ for every pair of sets A and B and every pair of closed sets $F \subset F'$, hence $D_{A,B}$ is a derivative. We use the notation $\alpha(A, B) = \mathrm{rk}(D_{A,B})$.

Definition 3.4.8. The *separation rank* of a Baire class 1 function f is defined as

$$\alpha(f) = \sup_{\substack{p<q\\ p,q\in\mathbb{Q}}} \alpha(\{f \leq p\}, \{f \geq q\}). \tag{3.4.8}$$

Remark 3.4.9. Actually,

$$\alpha(f) = \sup_{\substack{x<y\\ x,y\in\mathbb{R}}} \alpha(\{f \leq x\}, \{f \geq y\}),$$

since if $x < p < q < y$ then $\alpha(\{f \leq x\}, \{f \geq y\}) \leq \alpha(\{f \leq p\}, \{f \geq q\})$, since any set $H \in \mathbf{\Delta}_2^0(X)$ separating the level sets $\{f \leq p\}$ and $\{f \geq q\}$ also separates $\{f \leq x\}$ and $\{f \geq y\}$.

Proposition 3.4.10. *If f is a Baire class 1 function then $\alpha(f) < \omega_1$.*

Proof. From the definition of the rank and Remark 3.4.4 it is enough to prove that for any pair of rational numbers $p < q$ and nonempty closed set $F \subset X$, $D_{A,B}(F) \subsetneq F$, where $A = \{f \leq p\}$ and $B = \{f \geq q\}$. Since f is of Baire class 1, it has a point of continuity restricted to F, hence A and B cannot be both dense in F. Consequently, $D_{A,B}(F) = \overline{F \cap A} \cap \overline{F \cap B} \subsetneq F$, proving the proposition. □

Next we prove that $\alpha(A, B) < \omega_1$ iff A and B can be separated by a transfinite difference of closed sets.

Definition 3.4.11. If the sets A and B can be separated by a transfinite difference of closed sets then let $\alpha_1(A, B)$ denote the length of the shortest such sequence, otherwise let $\alpha_1(A, B) = \omega_1$. We define the *modified separation rank* of a Baire class 1 function f as

$$\alpha_1(f) = \sup_{\substack{p<q \\ p,q \in \mathbb{Q}}} \alpha_1(\{f \leq p\}, \{f \geq q\}). \tag{3.4.9}$$

Proposition 3.4.12. *Let A and B two subsets of X. Then*

$$\alpha(A, B) \leq \alpha_1(A, B) \leq 2\alpha(A, B), \textit{ hence } \alpha(A, B) \approx \alpha_1(A, B).$$

Proof. For the first inequality we can assume that $\alpha_1(A, B) < \omega_1$, so A and B can be separated by a transfinite difference of closed sets. Let $(F_\eta)_{\eta<\lambda}$ be such a sequence, where $\lambda = \alpha_1(A, B)$. Now we have

$$A \subset \bigcup_{\substack{\eta<\lambda \\ \eta \text{ even}}} (F_\eta \setminus F_{\eta+1}) \subset B^c.$$

It is enough to prove that $D^\eta_{A,B}(X) \subset F_\eta$ for every η. We prove this by induction. For $\eta = 0$ this is obvious, since $D^0_{A,B}(X) = F_0 = X$.

Now suppose that $D^\eta_{A,B}(X) \subset F_\eta$. We will show that $D^{\eta+1}_{A,B}(X) = \overline{D^\eta_{A,B}(X) \cap A} \cap \overline{D^\eta_{A,B}(X) \cap B} \subset F_{\eta+1}$. If η is even then

$$D^\eta_{A,B}(X) \setminus F_{\eta+1} \subset F_\eta \setminus F_{\eta+1} \subset B^c,$$

hence $D^\eta_{A,B}(X) \cap B \subset F_{\eta+1}$. Since $F_{\eta+1}$ is closed, we obtain $\overline{D^\eta_{A,B}(X) \cap B} \subset F_{\eta+1}$, hence $D^{\eta+1}_{A,B} \subset F_{\eta+1}$. If η is odd then $F_\eta \setminus F_{\eta+1}$ is disjoint from $\bigcup_{\substack{\eta<\lambda \\ \eta \text{ even}}} (F_\eta \setminus F_{\eta+1})$, hence $F_\eta \setminus F_{\eta+1} \subset A^c$, and an argument analogous to the above one yields $\overline{D^\eta_{A,B}(X) \cap A} \subset F_{\eta+1}$, hence $D^{\eta+1}_{A,B} \subset F_{\eta+1}$.

If η is limit and $D^\theta_{A,B}(X) \subset F_\theta$ for every $\theta < \eta$ then $D^\eta_{A,B}(X) \subset F_\eta$ because the sequences $D^\eta_{A,B}(X)$ and F_η are continuous.

For the second inequality we suppose that $\alpha(A, B) < \omega_1$, that is, the sequence $D^\eta_{A,B}(X)$ terminates at the empty set at some countable ordinal. Let

$$F_{2\eta} = D^\eta_{A,B}(X), \quad F_{2\eta+1} = \overline{D^\eta_{A,B}(X) \cap B}.$$

Clearly, $F_0 = X$ and $F_{2\eta} \supseteq F_{2\eta+1}$ for every η. It is easily seen from the definition of $D^{\eta+1}_{A,B}(X)$ that $F_{2\eta+1} \supseteq F_{2\eta+2}$ for every η. Moreover, the sequence

$F_{2\eta} = D^{\eta}_{A,B}(X)$ is continuous. This implies that the sequence formed by the F_η's is decreasing and continuous.

Now we show that the transfinite difference of this sequence separates A and B.

Every ring of the form $F_{2\eta} \setminus F_{2\eta+1}$ is disjoint from B, so we only need to prove that A is contained in the union of these rings. We show that A is disjoint from the complement of this union by proving that

$$(F_{2\eta+1} \setminus F_{2\eta+2}) \cap A = \left(\overline{D^{\eta}_{A,B}(X) \cap B} \setminus D^{\eta+1}_{A,B}(X)\right) \cap A = \emptyset$$

for every η. From the definition of the derivative, $D^{\eta+1}_{A,B}(X) = \overline{D^{\eta}_{A,B}(X) \cap A} \cap \overline{D^{\eta}_{A,B}(X) \cap B}$. Using the fact that $D^{\eta}_{A,B}(X)$ is closed, for a point $x \in A \cap \overline{D^{\eta}_{A,B}(X) \cap B}$ we have $x \in \overline{D^{\eta}_{A,B}(X) \cap A}$, hence $x \in D^{\eta+1}_{A,B}(X)$. □

Remark 3.4.13. It is claimed in [52] that if X is compact and $\alpha(A, B) = \lambda + n$ with λ limit and $0 < n \in \omega$ then $\alpha_1(A, B)$ is either $\lambda + 2n$ or $\lambda + 2n - 1$. However, this does not seem to be true. For a counterexample, let X be the $2n$+1-dimensional cube in $\mathbb{R}^{2n+1}$. Let $A = (F_0 \setminus F_1) \cup (F_2 \setminus F_3) \cup \cdots \cup (F_{2n} \setminus F_{2n+1})$, where F_i is a $(2n + 1 - i)$-dimensional face of X, and $F_{i+1} \subset F_i$ for $i \le 2n$. Let $B = X \setminus A$. The definition of A shows that $\alpha_1(A, B) \le 2n + 2$.

Now $D^0_{A,B}(X) = X = F_0$, and by induction, $D^i_{A,B}(X) = F_i$ for $0 \le i \le 2n+1$, since $D^i_{A,B}(X) = D(D^{i-1}_{A,B}(X)) = D_{A,B}(F_{i-1}) = \overline{F_{i-1} \cap A} \cap \overline{F_{i-1} \cap B} = F_i$. Now we have $D^{2n+2}_{A,B}(X) = D_{A,B}(D^{2n+1}_{A,B}(X)) = D_{A,B}(F_{2n+1}) = \emptyset$, proving that in this case $\alpha(A, B) = 2n + 2$. Using Proposition 3.4.12 this shows that $\alpha_1(A, B) = \alpha(A, B) = 2n + 2$.

We leave the proof of the following corollary to the reader.

Corollary 3.4.14. *If f is a Baire class 1 function then*

$$\alpha(f) \le \alpha_1(f) \le 2\alpha(f), \textit{ hence } \alpha(f) \approx \alpha_1(f).$$

Corollary 3.4.15. *If f is a Baire class 1 function then $\alpha_1(f) < \omega_1$.*

Proof. It is an easy consequence of the previous corollary and Proposition 3.4.10. □

3.4.4 The oscillation rank of Baire class 1 functions

This rank was investigated by numerous authors, see e.g. [45].

First, we define the oscillation of a function, then turn to the oscillation rank.

Definition 3.4.16. The *oscillation* of a function $f : X \to \mathbb{R}$ at a point $x \in X$ restricted to a closed set $F \subset X$ is

$$\omega(f, x, F) = \inf \left\{ \sup_{x_1, x_2 \in U \cap F} |f(x_1) - f(x_2)| : U \text{ open}, x \in U \right\}. \tag{3.4.10}$$

Definition 3.4.17. For each $\varepsilon > 0$ consider the derivative defined by

$$D_{f,\varepsilon}(F) = \{x \in F : \omega(f, x, F) \ge \varepsilon\}. \tag{3.4.11}$$

It is obvious that $D_{f,\varepsilon}(F)$ is closed, $D_{f,\varepsilon}(F) \subset F$ and $D_{f,\varepsilon}(F) \subset D_{f,\varepsilon}(F')$ for every function $f : X \to \mathbb{R}$, every $\varepsilon > 0$ and every pair of closed sets $F \subset F'$, hence $D_{f,\varepsilon}$ is a derivative. Let us denote the rank of $D_{f,\varepsilon}$ by $\beta(f,\varepsilon)$.

Definition 3.4.18. The *oscillation rank* of a function f is

$$\beta(f) = \sup_{\varepsilon>0} \beta(f,\varepsilon). \tag{3.4.12}$$

Proposition 3.4.19. *If f is a Baire class 1 function then $\beta(f) < \omega_1$.*

Proof. Using Remark 3.4.4, it is enough to prove $D_{f,\varepsilon}(F) \subsetneq F$ for every $\varepsilon > 0$ and every nonempty closed set $F \subset X$. And this is easy, since f restricted to F is continuous at a point $x \in F$, and thus $x \notin D_{f,\varepsilon}(F)$, hence $D_{f,\varepsilon}(F) \subsetneq F$. □

3.4.5 The convergence rank of Baire class 1 functions

Now we turn to the convergence rank following Zalcwasser [79] and Gillespie and Hurwitz [40].

Definition 3.4.20. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of real valued continuous functions on X. The *oscillation* of this sequence at a point x restricted to a closed set $F \subset X$ is

$$\omega((f_n)_{n\in\mathbb{N}}, x, F) = \inf_{\substack{x\in U\\ U \text{ open}}} \inf_{N\in\mathbb{N}} \sup \{|f_m(y) - f_n(y)| : n,m \geq N,\ y \in U \cap F\}. \tag{3.4.13}$$

Definition 3.4.21. Consider a sequence $(f_n)_{n\in\mathbb{N}}$ of real valued continuous functions, and for each $\varepsilon > 0$, define a derivative as

$$D_{(f_n)_{n\in\mathbb{N}},\varepsilon}(F) = \{x \in F : \omega((f_n)_{n\in\mathbb{N}}, x, F) \geq \varepsilon\}. \tag{3.4.14}$$

It is easy to see that $D_{(f_n)_{n\in\mathbb{N}},\varepsilon}(F)$ is closed, $D_{(f_n)_{n\in\mathbb{N}},\varepsilon}(F) \subset F$ and $D_{(f_n)_{n\in\mathbb{N}},\varepsilon}(F) \subset D_{(f_n)_{n\in\mathbb{N}},\varepsilon}(F')$ for every sequence of continuous functions $(f_n)_{n\in\mathbb{N}}$, every $\varepsilon > 0$ and every pair of closed sets $F \subset F'$, hence $D_{(f_n)_{n\in\mathbb{N}},\varepsilon}$ is a derivative. Let us denote the rank of $D_{(f_n)_{n\in\mathbb{N}},\varepsilon}$ by $\gamma((f_n)_{n\in\mathbb{N}},\varepsilon)$.

Definition 3.4.22. For a Baire class 1 function f let the *convergence rank* of f be defined by

$$\gamma(f) = \min\left\{ \sup_{\varepsilon>0} \gamma((f_n)_{n\in\mathbb{N}},\varepsilon) : \forall n\ f_n \text{ is continuous and } f_n \to f \text{ pointwise} \right\}. \tag{3.4.15}$$

Proposition 3.4.23. *If f is a Baire class 1 function then $\gamma(f) < \omega_1$.*

Proof. It suffices to show that $D_{(f_n)_{n\in\mathbb{N}},\varepsilon}(F) \subsetneq F$ for every $\varepsilon > 0$, every nonempty closed set $F \subset X$ and every sequence of pointwise convergent continuous functions $(f_n)_{n\in\mathbb{N}}$. Suppose the contrary, then for every N the set $G_N = \{x \in F : \exists n,m \geq N\ |f_n(x) - f_m(x)| > \frac{\varepsilon}{2}\}$ is dense in F. It is also open in F, hence by the Baire category theorem there is a point $x \in F$ such that $x \in G_N$ for every $N \in \mathbb{N}$, hence the sequence $(f_n)_{n\in\mathbb{N}}$ does not converge at x, contradicting our assumption. □

3.4.6 Properties of the Baire class 1 ranks

Theorem 3.4.24. *If f is a Baire class 1 function then $\alpha(f) \leq \beta(f) \leq \gamma(f)$.*

PROOF. For the first inequality, it is enough to prove that for every $p, q \in \mathbb{Q}$, $p < q$ we can find $\varepsilon > 0$ such that $\alpha(\{f \leq p\}, \{f \geq q\}) \leq \beta(f, \varepsilon)$. Let $A = \{f \leq p\}$, $B = \{f \geq q\}$ and $\varepsilon = p - q$. Using Proposition 3.4.5 it suffices to show that $D_{A,B}(F) \subset D_{f,\varepsilon}(F)$ for every $F \in \mathbf{\Pi}^0_1(X)$. If $x \in F \setminus D_{f,\varepsilon}(F)$ then x has a neighbourhood U such that $\sup_{x_1,x_2 \in U \cap F} |f(x_1) - f(x_2)| < \varepsilon = p - q$, hence U cannot intersect both A and B. So $x \notin D_{A,B}(F)$, proving the first inequality.

For the second inequality, let $(f_n)_{n\in\mathbb{N}}$ be a sequence of continuous functions converging pointwise to a function f. It is enough to show that $\beta(f, \varepsilon) \leq \gamma((f_n)_{n\in\mathbb{N}}, \varepsilon/3)$. As in the first paragraph we show that $D_{f,\varepsilon}(F) \subset D_{(f_n)_{n\in\mathbb{N}},\varepsilon/3}(F)$ for every $F \in \mathbf{\Pi}^0_1(X)$. It is enough to show that if $x \in F \setminus D_{(f_n)_{n\in\mathbb{N}},\varepsilon/3}(F)$ then $x \notin D_{f,\varepsilon}(F)$. For such an x there is a neighborhood U of x and an $N \in \mathbb{N}$ such that for all $n, m \geq N$ and $x' \in F \cap U$, $|f_n(x') - f_m(x')| < \varepsilon/3$. Letting $m \to \infty$ we get $|f_n(x') - f(x')| \leq \varepsilon/3$ for all $n \geq N$ and $x' \in F \cap U$. Let $V \subset U$ be a neighborhood of x for which $\sup_V f_N - \inf_V f_N < \varepsilon/6$. Now for every $x', x'' \in V \cap F$ we have

$$|f(x') - f(x'')| \leq |f_N(x') - f_N(x'')| + 2\frac{\varepsilon}{3} < \frac{5}{6}\varepsilon < \varepsilon,$$

showing that $x \notin D_{f,\varepsilon}(F)$. $\square$

Proposition 3.4.25. *If X is a Polish group then the ranks α, β and γ are translation invariant.*

PROOF. Note first that for a Baire class 1 function f and $x_0 \in X$ the functions $f \circ L_{x_0}$ and $f \circ R_{x_0}$ are also of Baire class 1. Since the topology of a topological group is translation invariant, and the the definitions of the ranks depend only on the topology of the space, the proposition easily follows. $\square$

Theorem 3.4.26. *The ranks are unbounded in ω_1, actually unbounded already on the characteristic functions.*

We postpone the proof, since later we will prove the more general Theorem 3.4.42.

Proposition 3.4.27. *If f is continuous then $\alpha(f) = \beta(f) = \gamma(f) = 1$.*

PROOF. In order to prove $\alpha(f) = 1$, consider the derivative $D_{\{f\leq p\},\{f\geq q\}}$, where $p < q$ is a pair of rational numbers. Since the level sets $\{f \leq p\}$ and $\{f \geq q\}$ are disjoint closed sets, $D_{\{f\leq p\},\{f\geq q\}}(X) = \emptyset$.

For $\beta(f) = 1$, note that a continuous function f has oscillation 0 at every point restricted to every set, hence $D_{f,\varepsilon}(X) = \emptyset$ for every $\varepsilon > 0$.

And finally for $\gamma(f) = 1$ consider the sequence of continuous functions $(f_n)_{n\in\mathbb{N}}$, for which $f_n = f$ for every $n \in \mathbb{N}$. It is easy to see that $\omega((f_n)_{n\in\mathbb{N}}, x, F) = 0$ for every point $x \in X$ and every closed set $F \subset X$. Now we have that $D_{(f_n)_{n\in\mathbb{N}},\varepsilon}(X) = \emptyset$ for every $\varepsilon > 0$, hence $\gamma(f) = 1$. $\square$

Theorem 3.4.28. *If f is a Baire class 1 function and $F \subset X$ is closed then $\alpha(f \cdot \chi_F) \le 1 + \alpha(f)$, $\beta(f \cdot \chi_F) \le 1 + \beta(f)$ and $\gamma(f \cdot \chi_F) \le 1 + \gamma(f)$.*

PROOF. First we prove the statement for the ranks α and β. Let D be a derivative either of the form $D_{A,B}$ or of the form $D_{f,\varepsilon}$ where $A = \{f \le p\}$ and $B = \{f \ge q\}$ for a pair of rational numbers $p < q$ and $\varepsilon > 0$. Let $\overline{D}$ be the corresponding derivative for the function $f \cdot \chi_F$, i.e. $\overline{D} = D_{A',B'}$ or $\overline{D} = D_{f\cdot\chi_F,\varepsilon}$, where $A' = \{f \cdot \chi_F \le p\}$ and $B' = \{f \cdot \chi_F \ge q\}$.

Since the function $f \cdot \chi_F$ is constant 0 on the open set $X \setminus F$, it is easy to check that in both cases $\overline{D}(X) \subset F$. And since the functions f and $f \cdot \chi_F$ agree on F, we have by transfinite induction that $\overline{D}^{1+\eta}(X) \subset D^{\eta}(X)$ for every countable ordinal η, implying that $\alpha(f \cdot \chi_F) \le 1 + \alpha(f)$ and also $\beta(f \cdot \chi_F) \le 1 + \beta(f)$.

Now we prove the statement for γ. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of continuous functions converging pointwise to f with $\sup_{\varepsilon>0} \gamma((f_n)_{n\in\mathbb{N}}, \varepsilon) = \gamma(f)$. Let $g_n(x) = 1 - \min\{1, n \cdot d(x, F)\}$ and set $f_n'(x) = f_n(x) \cdot g_n(x)$. It is easy to check that for every n the function f_n' is continuous and $f_n' \to f \cdot \chi_F$ pointwise. For every $x \in X \setminus F$ there is a neighborhood of x such that for large enough n the function f_n' is 0 on this neighborhood, hence $D_{(f_n')_{n\in\mathbb{N}},\varepsilon}(X) \subset F$ for every $\varepsilon > 0$. From this point on the proof is similar to the previous cases, since the sequences of functions $(f_n)_{n\in\mathbb{N}}$ and $(f_n')_{n\in\mathbb{N}}$ agree on F, hence, by transfinite induction $D^{1+\eta}_{(f_n')_{n\in\mathbb{N}},\varepsilon}(X) \subset D^{\eta}_{(f_n)_{n\in\mathbb{N}},\varepsilon}(X)$ for every $\varepsilon > 0$. From this we have $\gamma((f_n')_{n\in\mathbb{N}}, \varepsilon) \le 1 + \gamma((f_n)_{n\in\mathbb{N}}, \varepsilon)$ for every $\varepsilon > 0$, hence $\gamma(f \cdot \chi_F) \le 1 + \gamma(f)$. Thus the proof of the theorem is complete. $\square$

Theorem 3.4.29. *The ranks β and γ are essentially linear.*

PROOF. It is easy to see that $\beta(cf) = \beta(f)$ and $\gamma(cf) = \gamma(f)$ for every $c \in \mathbb{R} \setminus \{0\}$, hence it suffices to show that β and γ are essentially additive.

First we consider a modification of the definition of the rank β as follows. Let β_0 be the rank obtained by simply replacing $\sup_{x_1,x_2\in U\cap F} |f(x_1) - f(x_2)|$ in (3.4.10) by $\sup_{x_1\in U\cap F} |f(x) - f(x_1)|$ in the definition of β. Clearly, $\beta_0(f, \varepsilon) \le \beta(f, \varepsilon) \le \beta_0(f, \varepsilon/2)$, hence actually $\beta_0 = \beta$. Therefore it is sufficient to prove the theorem for β_0.

To prove the theorem for β_0, let $D_0 = D_{f,\varepsilon/2}$, $D_1 = D_{g,\varepsilon/2}$ and $D = D_{f+g,\varepsilon}$ (we use here the derivatives defining β_0). We show that the conditions of Proposition 3.4.6 hold for these derivatives.

For condition (3.4.1), let $x \in D_{f+g,\varepsilon}(F)$. Since $\omega(f + g, x, F) \ge \varepsilon$, we have $\omega(f, x, F)$ or $\omega(g, x, F) \ge \varepsilon/2$, hence $x \in D_{f,\varepsilon/2}(F) \cup D_{g,\varepsilon/2}(F)$.

Condition (3.4.2) is similar, let $x \in (F \cup F') \setminus (D_{f+g,\varepsilon}(F) \cup D_{f+g,\varepsilon}(F'))$. Since $x \notin D_{f+g,\varepsilon}(F)$, there is a neighbourhood U of x with $|(f + g)(x) - (f + g)(x')| < \varepsilon' < \varepsilon$ for $x' \in U \cap F$. And similarly, there is a neighborhood U' with $|(f + g)(x) - (f + g)(x')| < \varepsilon'' < \varepsilon$ for $x' \in U' \cap F'$. Now the neighborhood $U \cap U'$ shows that $\omega(f + g, x, F \cup F') < \varepsilon$, proving that $x \notin D_{f+g,\varepsilon}(F \cup F')$.

The proposition yields that $\beta_0(f + g, \varepsilon) \lesssim \max\{\beta_0(f, \varepsilon/2), \beta_0(g, \varepsilon/2)\}$, hence $\beta_0(f + g) \lesssim \max\{\beta_0(f), \beta_0(g)\}$. This proves the statement for β_0, hence for β.

For γ, we do the same, prove the conditions of the proposition for $D_0 = D_{(f_n)_{n\in\mathbb{N}},\varepsilon/2}$, $D_1 = D_{(g_n)_{n\in\mathbb{N}},\varepsilon/2}$ and $D = D_{(f_n+g_n)_{n\in\mathbb{N}},\varepsilon}$, and use the conclusion of the proposition to finish the proof.

For condition (3.4.1), let $x \in F \setminus \big(D_{(f_n)_{n\in\mathbb{N}},\varepsilon/2}(F) \cup D_{(g_n)_{n\in\mathbb{N}},\varepsilon}(F)\big)$. Now we can choose a common open set $x \in U$ and a common $N \in \mathbb{N}$ such that for all $n, m \geq N$ and $y \in U \cap F$ we have $|f_n(y) - f_m(y)| \leq \varepsilon' < \varepsilon/2$ and $|g_n(y) - g_m(y)| \leq \varepsilon' < \varepsilon/2$ (again, with a common $\varepsilon' < \varepsilon/2$). But from this we have $|(f_n + g_n)(y) - (f_m + g_m)(y)| \leq 2\varepsilon' < \varepsilon$ for all $n, m \geq N$ and $y \in U \cap F$, so $x \notin D_{(f_n+g_n)_{n\in\mathbb{N}},\varepsilon}(F)$, yielding (3.4.1).

For (3.4.2) let $x \in (F \cup F') \setminus \left(D_{(f_n+g_n)_{n\in\mathbb{N}},\varepsilon}(F) \cup D_{(f_n+g_n)_{n\in\mathbb{N}},\varepsilon}(F')\right)$. For this x we have a neighborhood U of x, $N \in \mathbb{N}$ and $\varepsilon' < \varepsilon$, such that $|(f_n + g_n)(y) - (f_m + g_m)(y)| \leq \varepsilon'$ for every $n, m \geq N$ and $y \in U \cap F$. Similarly, we can find a neighbourhood U', $N' \in \mathbb{N}$ and $\varepsilon'' < \varepsilon$, such that $|(f_n+g_n)(y)-(f_m+g_m)(y)| \leq \varepsilon''$ for every $n, m \geq N'$ and $y \in U' \cap F'$. From this, $\omega((f_n+g_n)_{n\in\mathbb{N}}, x, F \cup F') \leq \max\{\varepsilon', \varepsilon''\} < \varepsilon$, hence $x \notin D_{(f_n+g_n)_{n\in\mathbb{N}},\varepsilon}(F \cup F')$.

Therefore the proof of the theorem is complete. $\square$

Remark 3.4.30. The analogous result does not hold for the rank α. To see this note first that $\alpha(A, A^c)$ can be arbitrarily large below ω_1 when A ranges over $\boldsymbol{\Delta}^0_2(X)$. This is a classical fact and we prove a more general result in Corollary 3.4.43.

First we check that for every $A \in \boldsymbol{\Delta}^0_2(X)$ the characteristic function χ_A can be written as the difference of two upper semicontinuous (usc) functions. Indeed, let $(K_n)_{n\in\omega}$ and $(L_n)_{n\in\omega}$ be increasing sequences of closed sets with $A = \bigcup_n K_n$ and $A^c = \bigcup_n L_n$, and let

$$f_0 = \begin{cases} 0 & \text{on } K_0 \cup L_0, \\ -n & \text{on } (K_n \cup L_n) \setminus (K_{n-1} \cup L_{n-1}) \text{ for } n \geq 1 \end{cases}$$

and

$$f_1 = \begin{cases} 0 & \text{on } L_0, \\ -1 & \text{on } (K_0 \cup L_1) \setminus L_0, \\ -n & \text{on } (K_{n-1} \cup L_n) \setminus (K_{n-2} \cup L_{n-1}) \text{ for } n \geq 2. \end{cases}$$

Then f_0 and f_1 are usc functions with $\chi_A = f_0 - f_1$.

Now we complete the remark by showing that $\alpha(f) \leq 2$ for every usc function f. For $p < q$ let $A = \{f \leq p\}$ and $B = \{f \geq q\}$. Then B is closed, so $D_{A,B}(X) = \overline{X \cap A} \cap \overline{X \cap B} = \overline{X \cap A} \cap B \subset B$. Hence $D^2_{A,B}(X) \subset D_{A,B}(B) = \overline{A \cap B} \cap B = \emptyset \cap B = \emptyset$.

Proposition 3.4.31. *If the sequence of Baire class 1 functions f_n converges uniformly to f then $\beta(f) \leq \sup_n \beta(f_n)$.*

Proof. If $|f - f_n| < \varepsilon/3$ then $|\omega(f, x, F) - \omega(f_n, x, F)| \leq \frac{2}{3}\varepsilon$ for every x and F. Therefore $D_{f,\varepsilon}(F) \subset D_{f_n,\varepsilon/3}(F)$ for every F, which in turn implies $\beta(f, \varepsilon) \leq \beta(f_n, \varepsilon/3)$, from which the proposition easily follows. $\square$

Proposition 3.4.32. *If the sequence of Baire class 1 functions f_n converges uniformly to f then $\gamma(f) \lesssim \sup_n \gamma(f_n)$.*

Proof. By taking a subsequence we can suppose that $|f_n(x) - f(x)| \leq \frac{1}{2^n}$ for every $n \in \mathbb{N}$ and every $x \in X$. With $g_n(x) = f_n(x) - f_{n-1}(x)$ we have $|g_n(x)| \leq \frac{3}{2^n}$, hence $\sum_{n=1}^{\infty} g_n(x)$ is uniformly convergent, and $f(x) = f_0(x) + \sum_{n=1}^{\infty} g_n(x)$. Using Theorem 3.4.29 we have $\gamma(g_n) \lesssim \max\{\gamma(f_n), \gamma(f_{n-1})\}$, hence $\sup_n \gamma(g_n) \lesssim \sup_n \gamma(f_n)$. It is enough to prove that for $g = \sum_{n=1}^{\infty} g_n$ we have $\gamma(g) \lesssim \sup_n \gamma(g_n)$, since Theorem 3.4.29 yields $\gamma(f) \lesssim \max\{\gamma(f_0), \gamma(g)\}$.

Now for every $n \in \mathbb{N}$ let $(\varphi_n^k)_{k\in\mathbb{N}}$ be a sequence of continuous functions converging pointwise to g_n with $\sup_{\varepsilon>0} \gamma((\varphi_n^k)_{k\in\mathbb{N}}, \varepsilon) = \gamma(g_n)$. It is easy to see that we can suppose $|\varphi_n^k(x)| \le \frac{3}{2^n}$ for every $n \in \mathbb{N}$ and $k \in \mathbb{N}$, since by replacing $(\varphi_n^k)_{k\in\mathbb{N}}$ with $\left(\max\left(\min\left(\varphi_n^k, \frac{3}{2^n}\right), -\frac{3}{2^n}\right)\right)_{k\in\mathbb{N}}$ we have a sequence of continuous functions satisfying this, and the sequence is still converging pointwise to g_n, while $\gamma((\varphi_n^k)_{k\in\mathbb{N}}, \varepsilon)$ is not increased.

Let $\phi_k = \sum_{n=0}^k \varphi_n^k$. We show that $(\phi_k)_{k\in\mathbb{N}}$ converges pointwise to g and also that $\gamma(g) \le \sup_{\varepsilon>0} \gamma((\phi_k)_{k\in\mathbb{N}}, \varepsilon) \lesssim \sup_n \sup_{\varepsilon>0} \gamma((\varphi_n^k)_{k\in\mathbb{N}}, \varepsilon) = \sup_n \gamma(g_n)$, which finishes the proof. To prove pointwise convergence, let $\varepsilon > 0$ be arbitrary and fix $K \in \mathbb{N}$ with $\frac{6}{2^K} < \varepsilon$. For $k > K$ we have

$$|\phi_k(x) - g(x)| = \left|\sum_{n=0}^{k} \varphi_n^k(x) - g(x)\right| \le \left|\sum_{n=0}^{K} \varphi_n^k(x) - g(x)\right| + \left|\sum_{n=K+1}^{k} \varphi_n^k(x)\right|,$$

where the first term of the last expression tends to $\left|\sum_{n=0}^K g_n(x) - g(x)\right| \le \frac{3}{2^K}$, while the second is at most $\frac{3}{2^K}$. Hence $\limsup_{k\to\infty} |\phi_k(x) - g(x)| \le 2\frac{3}{2^K} < \varepsilon$ for every $\varepsilon > 0$, showing that $\phi_k(x) \to g(x)$.

Now fix an $\varepsilon > 0$ and $K \in \mathbb{N}$ as before, it is enough to show that $\gamma((\phi_k)_{k\in\mathbb{N}}, 3\varepsilon) \lesssim \sup_n \sup_{\varepsilon>0} \gamma((\varphi_n^k)_{k\in\mathbb{N}}, \varepsilon)$.

For any $x \in X$ and $k, l > K$ we have

$$\begin{aligned} |\phi_k(x) - \phi_l(x)| &= \left|\sum_{n=0}^{k} \varphi_n^k(x) - \sum_{n=0}^{l} \varphi_n^l(x)\right| \\ &\le \sum_{n=0}^{K} \left|\varphi_n^k(x) - \varphi_n^l(x)\right| + \left|\sum_{n=K+1}^{k} \varphi_n^k(x)\right| + \left|\sum_{n=K+1}^{l} \varphi_n^l(x)\right|. \end{aligned} \tag{3.4.16}$$

As before, the sum of the last two terms is at most ε. We want to use Proposition 3.4.6 for the derivatives $D = D_{(\phi_k)_{k\in\mathbb{N}}, 3\varepsilon}$ and $D_n = D_{(\varphi_n^k)_{k\in\mathbb{N}}, \frac{\varepsilon}{K+1}}$ for $n \le K$. To check condition (3.4.1), let $x \in F \setminus \bigcup_{n=0}^K D_{(\varphi_n^k)_{k\in\mathbb{N}}, \frac{\varepsilon}{K+1}}(F)$. Then we have a neighborhood U of x and an $N \in \mathbb{N}$ such that $\left|\varphi_n^k(y) - \varphi_n^l(y)\right| < \frac{\varepsilon}{K+1}$ for every $n \le K$, every $y \in U \cap F$ and every $k, l \ge N$. This observation and (3.4.16) yields that $|\phi_k(y) - \phi_l(y)| \le 2\varepsilon$ for every $y \in U \cap F$ and $k, l \ge N$ showing that $x \notin D_{(\phi_k)_{k\in\mathbb{N}}, 3\varepsilon}(F)$.

Condition (3.4.2) is similar, and it can be seen as in the proof of Theorem 3.4.29. Now Proposition 3.4.6 gives

$$\gamma((\phi_k)_{k\in\mathbb{N}}, 3\varepsilon) \lesssim \max_{n\le K} \gamma\left((\varphi_n^k)_{k\in\mathbb{N}}, \frac{\varepsilon}{K+1}\right) \le \sup_n \sup_{\varepsilon>0} \gamma((\varphi_n^k)_{k\in\mathbb{N}}, \varepsilon),$$

completing the proof. □

Theorem 3.4.33. *If f is a bounded Baire class 1 function then $\alpha(f) \approx \beta(f) \approx \gamma(f)$.*

Proof. Using Theorem 3.4.24, it is enough to prove that $\gamma(f) \lesssim \alpha(f)$. First, we prove the theorem for characteristic functions.

Lemma 3.4.34. *Suppose that $A \in \mathbf{\Delta}_2^0$. Then $\gamma(\chi_A) \lesssim \alpha(\chi_A)$.*

Proof. In order to prove this, first we have to produce a sequence of continuous functions converging pointwise to χ_A.

For this let $(F_\eta)_{\eta<\lambda}$ be a continuous transfinite decreasing sequence of closed sets, so that
$$A = \bigcup_{\substack{\eta<\lambda \\ \eta \text{ even}}} (F_\eta \setminus F_{\eta+1})$$
and $\lambda \approx \alpha(\chi_A)$ given by Corollary 3.4.14. We can assume that the last element of the sequence $(F_\eta)_{\eta<\lambda}$ is $\emptyset$, hence every $x \in X$ is contained in a unique set of the form $F_\eta \setminus F_{\eta+1}$.

For each $k \in \omega$ and $\eta < \lambda$ let $f_\eta^k : X \to [0,1]$ be a continuous function so that $f_\eta^k | F_\eta \equiv 1$, and whenever $x \in X$ and $d(x, F_\eta) \geq \frac{1}{k+1}$ then $f_\eta^k(x) = 0$. Such a function exists by Urysohn's lemma, since the sets F_η and $\{x \in X : d(x, F_\eta) \geq \frac{1}{k+1}\}$ are disjoint closed sets.

Now let (η_n) be an enumeration of λ in type $\leq \omega$. Let us define
$$f_k = \sum_{\substack{n \leq k \\ \eta_n \text{ even}}} f_{\eta_n}^k - f_{\eta_n+1}^k.$$

Since the functions f_k are finite sums of continuous functions, they are continuous. We claim that $f_k \to \chi_A$ as $k \to \infty$.

To see this, first let $x \in X$ be arbitrary. Then there exists a unique m so that $x \in F_{\eta_m} \setminus F_{\eta_m+1}$. Choose $k \in \omega$ so that $k \geq m$ and $d(x, F_{\eta_m+1}) \geq \frac{1}{k+1}$.

Then if $x \in A$ then η_m even and
$$f_k(x) = \sum_{\substack{n \leq k \\ \eta_n \text{ even}}} f_{\eta_n}^k(x) - f_{\eta_n+1}^k(x) =$$
$$= \left(\sum_{\substack{n \leq k \\ \eta_n \text{ even} \\ \eta_n < \eta_m}} f_{\eta_n}^k(x) - f_{\eta_n+1}^k(x) \right) + \left(\sum_{\substack{n \leq k \\ \eta_n \text{ even} \\ \eta_n > \eta_m}} f_{\eta_n}^k(x) - f_{\eta_n+1}^k(x) \right) + f_{\eta_m}^k(x) - f_{\eta_m+1}^k(x).$$

The first sum is clearly 0 since $f_{\eta_n}^k \equiv 1$ on F_{η_m} if $\eta_m > \eta_n$. This is also true for the second one, since if $d(x, F_{\eta_n}) \geq \frac{1}{k+1}$ then $f_{\eta_n}^k(x) = 0$. Finally, $f_{\eta_m}(x) = 1$ and $f_{\eta_m+1}(x) = 0$, so $f_k(x) = 1$.

If $x \notin A$ then η_m is odd and
$$f_k(x) = \sum_{\substack{n \leq k \\ \eta_n \text{ even}}} f_{\eta_n}^k(x) - f_{\eta_n+1}^k(x) =$$
$$= \sum_{\substack{n \leq k \\ \eta_n \text{ even} \\ \eta_n < \eta_m}} f_{\eta_n}^k(x) - f_{\eta_n+1}^k(x) + \sum_{\substack{n \leq k \\ \eta_n \text{ even} \\ \eta_n > \eta_m}} f_{\eta_n}^k(x) - f_{\eta_n+1}^k(x).$$

Now the previous argument gives $f_k(x) = 0$.

So $f_k \to \chi_A$ holds. Next we prove by induction on η that for every $\eta < \lambda$ and every $\varepsilon > 0$ we have
$$D^\eta_{(f_k)_{k\in\mathbb{N}},\varepsilon}(X) \subset F_\eta.$$

This will clearly complete the proof.

For $\eta = 0$ we have

$$D^0_{(f_k)_{k\in\mathbb{N}},\varepsilon}(X) = X = F_0.$$

If η is a limit ordinal, the statement is clear, since the sequence of derivatives as well as $(F_\eta)_{\eta<\lambda}$ are continuous.

Now let $\eta = \theta + 1$ and $D^\theta_{(f_k)_{k\in\mathbb{N}},\varepsilon}(X) \subset F_\theta$. For some m we have $\theta = \eta_m$. Let $x \in F_{\eta_m} \setminus F_{\eta_m+1}$. Then it is enough to prove that $x \notin D^\eta_{(f_k)_{k\in\mathbb{N}},\varepsilon}(X)$. Let k be so that $d(x, F_{\eta_m+1}) \geq \frac{2}{k+1}$.

If $d(x,y) < \frac{1}{k+1}$ and $y \in D^\theta_{(f_k)_{k\in\mathbb{N}},\varepsilon}(X)$, then $y \in F_{\eta_m} \setminus F_{\eta_m+1}$. From this, $l_1, l_2 \geq k$ implies that $f^{l_1}_\eta(y) = f^{l_2}_\eta(y) = 1$ if $\eta \leq \eta_m$, and $f^{l_1}_\eta(y) = f^{l_2}_\eta(y) = 0$ if $\eta > \eta_m$. Hence $f_{l_1}(y) - f_{l_2}(y) = 0$.

So the sequence f_k is eventually constant on a relative neighbourhood of x in F_{η_m}. Therefore $x \notin D^\eta_{(f_k)_{k\in\mathbb{N}},\varepsilon}(X)$, which finishes the proof. $\square$

Next we prove that $\gamma(f) \lesssim \alpha(f)$ for every step function f. We still need the following lemma.

Lemma 3.4.35. *If A and B are ambiguous sets then*

$$\alpha\left(\chi_{A\cap B}\right) \lesssim \max\left\{\alpha\left(\chi_A\right), \alpha\left(\chi_B\right)\right\}.$$

Proof. It is enough to prove this for β since the previous lemma and Theorem 3.4.24 yields that the ranks essentially agree on characteristic functions. Theorem 3.4.29 gives $\beta(\chi_A + \chi_B) \lesssim \max\{\beta(\chi_A), \beta(\chi_B)\}$, hence it suffices to prove that $\beta\left(\chi_{A\cap B}\right) \leq \beta(\chi_A + \chi_B)$. But this easily follows, since one can readily check that for every $\varepsilon < 1$ and F we have $D_{\chi_{A\cap B},\varepsilon}(F) \subset D_{\chi_A+\chi_B,\varepsilon}(F)$, finishing the proof. $\square$

Now let f be a step function, so $f = \sum_{i=1}^n c_i\chi_{A_i}$, where the A_i's are disjoint ambiguous sets covering X, and we can also suppose that the c_i's form a strictly increasing sequence of real numbers.

Lemma 3.4.36. $\max_i\{\alpha(\chi_{A_i})\} \lesssim \alpha(f)$.

Proof. Let $H_i = \bigcup_{j=1}^i A_j$. By the definition of the rank α, for every i we have

$$\alpha(H_i, H_i^c) \leq \alpha(f). \tag{3.4.17}$$

This shows that $\alpha(\chi_{A_1}) \lesssim \alpha(f)$, and together with the previous lemma, for $i > 1$

$$\begin{aligned}\alpha(\chi_{A_i}) &= \alpha(\chi_{H_i\setminus H_{i-1}}) = \alpha(\chi_{H_i\cap H_{i-1}^c}) \lesssim \max\{\alpha(\chi_{H_i}), \alpha(\chi_{H_{i-1}^c})\}\\ &= \max\{\alpha(H_i, H_i^c), \alpha(H_{i-1}, H_{i-1}^c)\} \leq \alpha(f),\end{aligned}$$

where the last but one inequality follows from the above lemma and the last inequality from (3.4.17). $\square$

Now we have

$$\gamma(f) \lesssim \max_i\{\gamma(\chi_{A_i})\} \approx \max_i\{\alpha(\chi_{A_i})\} \lesssim \alpha(f),$$

where we used Theorem 3.4.29, this theorem for characteristic functions and Lemma 3.4.36, proving the theorem for step functions.

In particular, $\alpha(f) \leq \beta(f) \leq \gamma(f)$ (Theorem 3.4.24) gives the following corollary.

Corollary 3.4.37. *If $f = \sum_{i=1}^{n} c_i \chi_{A_i}$, where the A_i's are disjoint ambiguous sets covering X and the c_i's are distinct then*

$$\alpha(f) \approx \max_i \{\alpha(\chi_{A_i})\}$$

and similarly for β and γ.

Now let f be an arbitrary bounded Baire class 1 function.

Lemma 3.4.38. *There is a sequence f_n of step functions converging uniformly to f, satisfying $\sup_n \alpha(f_n) \lesssim \alpha(f)$.*

Proof. Let $p_{n,k} = k/2^n$ for all $k \in \mathbb{Z}$ and $n \in \mathbb{N}$. The level sets $\{f \leq p_{n,k}\}$ and $\{f \geq p_{n,k+1}\}$ are disjoint $\mathbf{\Pi}^0_2$ sets, hence they can be separated by a $H_{n,k} \in \mathbf{\Delta}^0_2(X)$ (see e.g. [51, 22.16]). We can choose $H_{n,k}$ to satisfy $\alpha_1(H_{n,k}, H^c_{n,k}) \leq 2\alpha(f)$ using Proposition 3.4.12.

Since f is bounded, for fixed n there are only finitely many $k \in \mathbb{Z}$ for which $H_{n,k+1} \setminus H_{n,k} \neq \emptyset$. Set

$$f_n = \sum_{k \in \mathbb{Z}} p_{n,k} \cdot \chi_{H_{n,k+1} \setminus H_{n,k}}.$$

Now for each n, f_n is a step function with $|f - f_n| \leq 2^{n-1}$. Hence $f_n \to f$ uniformly. Since the level sets of a function f_n are of the form $H_{n,k}$ or $H^c_{n,k}$ for some $k \in \mathbb{Z}$, we have $\alpha(f_n) \leq 2\alpha(f)$, proving the lemma. □

Let f_n be a sequence of step functions given by this lemma. Using Proposition 3.4.32 and this theorem for step functions, we have $\gamma(f) \lesssim \sup_n \gamma(f_n) \lesssim \sup_n \alpha(f_n) \lesssim \alpha(f)$, completing the proof. □

We have seen above that α is not essentially additive on the Baire class 1 functions but β and γ are, therefore α cannot essentially coincide with β or γ.

Proposition 3.4.39. *If the sequence of Baire class 1 functions f_n converges uniformly to f then $\alpha(f) \lesssim \sup_n \alpha(f_n)$.*

Proof. If f is bounded (hence without loss of generality the f_n are also bounded) this is an easy consequence of Theorem 3.4.33 and Proposition 3.4.31.

For an arbitrary function g let $g' = \arctan \circ g$. It is easy to show that $\alpha(g') = \alpha(g)$ using Remark 3.4.9.

If the functions f and f_n are given such that $f_n \to f$ uniformly then $f'_n \to f'$ uniformly, and these are bounded functions, so we have $\alpha(f) = \alpha(f') \lesssim \sup_n \alpha(f'_n) = \sup_n \alpha(f_n)$. □

3.4.7 Well-behaved ranks on the Baire class ξ functions

In this section we finally show that there actually exist ranks with very nice properties. Two of these ranks will answer Question 1.3.1 and Question 1.3.2. Throughout the section, let $1 \leq \xi < \omega_1$ be fixed.

First we need a result concerning unboundedness.

Definition 3.4.40. Let A and B be disjoint $\mathbf{\Pi}^0_{\xi+1}$ sets. Then they can be separated by a $\mathbf{\Delta}^0_{\xi+1}$ set (see e.g. [51, 22.16]). Since every $\mathbf{\Delta}^0_{\xi+1}$ set is the transfinite difference of $\mathbf{\Pi}^0_\xi$ sets, A and B can be separated by the transfinite difference of such a sequence. Let $\alpha_\xi(A,B)$ denote the length of the shortest such sequence.

Definition 3.4.41. Let f be a Baire class ξ function, and $p < q \in \mathbb{Q}$. Then $\{f \leq p\}$ and $\{f \geq q\}$ are disjoint $\mathbf{\Pi}^0_{\xi+1}$ sets. Let the *separation rank* of f be

$$\alpha_\xi(f) = \sup_{\substack{p<q \\ p,q\in\mathbb{Q}}} \alpha_\xi(\{f \leq p\}, \{f \geq q\}).$$

Note that this really extends the definition of α_1.

Theorem 3.4.42. *For every $1 \leq \xi < \omega_1$ the rank α_ξ is unbounded in ω_1 on the characteristic Baire class ξ functions.*

Proof. Let $\mathcal{U} \in \mathbf{\Pi}^0_\xi(2^\omega \times X)$ be a universal set for $\mathbf{\Pi}^0_\xi(X)$ sets, that is, for every $F \subset X$, $F \in \mathbf{\Pi}^0_\xi(X)$ there exists a $y \in 2^\omega$ such that $\mathcal{U}^y = F$. For the existence of such a set see [51, 22.3]. Let us use the notation $\Gamma_\zeta(X)$ for the the family of sets $H \subset X$ satisfying $\alpha_\xi(H, H^c) < \zeta$. From [51, 22.27] we have $\Gamma_\zeta(X) \subset \mathbf{\Delta}^0_{\xi+1}(X)$. We will show that there exists a $\mathbf{\Delta}^0_{\xi+1}$ set for every $\zeta < \omega_1$ which is universal for the family of Γ_ζ sets. Since X is uncountable, there is a continuous embedding of 2^ω into X ([51, 6.5]), hence no universal set exists in $2^\omega \times X$ for the family of $\mathbf{\Delta}^0_{\xi+1}(X)$ sets (easy corollary of [51, 22.7]). This implies for every $\zeta < \omega_1$ that $\Gamma_\zeta \neq \mathbf{\Delta}^0_{\xi+1}$, hence the rank is really unbounded.

Let $p : \zeta \times \mathbb{N} \to \mathbb{N}$ be a bijection. For $\eta < \zeta$ and $y \in 2^\omega$ we define $\phi(y)_\eta \in 2^\omega$ by $\phi(y)_\eta(n) = y(p(\eta, n))$. First we check that for a fixed $\eta < \zeta$ the map $y \mapsto \phi(y)_\eta$ is continuous. Let $U = \{x \in 2^\omega : x(0) = i_0, \dots, x(n) = i_n\}$ be a set from the usual basis of 2^ω. The preimage of U is the set $\{y \in 2^\omega : \forall k \leq n\ \phi(y)_\eta(k) = i_k\} = \{y \in 2^\omega : \forall k \leq n\ y(p(\eta,k)) = i_k\}$, which is a basic open set, too. Now $\mathcal{U}_\eta = \{(y,x) : (\phi(y)_\eta, x) \in \mathcal{U}\}$ is a continuous preimage of a $\mathbf{\Pi}^0_\xi$ set, hence $\mathcal{U}_\eta \in \mathbf{\Pi}^0_\xi(2^\omega \times X)$ (see [51, 22.1]). Let

$$\begin{aligned} \mathcal{U}' = \{(y,x) \in 2^\omega \times X :{}& \text{the smallest ordinal } \eta \text{ such that } (y,x) \notin \mathcal{U}_\eta \text{ is odd,} \\ & \text{if such an } \eta \text{ exists, or no such } \eta \text{ exists and } \zeta \text{ is odd}\}. \end{aligned}$$

Now we check that $\mathcal{U}' \in \mathbf{\Delta}^0_{\xi+1}(2^\omega \times X)$. Let $\mathcal{V}_\eta = \bigcap_{\theta<\eta} \mathcal{U}_\theta$, then these sets form a continuous decreasing sequence of $\mathbf{\Pi}^0_\xi$ sets and it is easy to see that $\mathcal{U}'^c$ is the transfinite difference of the sequence $(\mathcal{V}_\eta)_{\eta<\zeta+1}$, hence $\mathcal{U}'^c \in \mathbf{\Delta}^0_{\xi+1}$, proving that $\mathcal{U}' \in \mathbf{\Delta}^0_{\xi+1}$, since the family of $\mathbf{\Delta}^0_{\xi+1}$ sets is closed under complements (see [51, 22.1]).

Now we show that $\mathcal{U}'$ is universal. For a set $H \in \Gamma_\zeta(X)$ there is a sequence $(z_\eta)_{\eta<\zeta}$ in 2^ω, such that H is the transfinite difference of the sets $\mathcal{U}^{z_\eta}$. For every sequence $(z_\eta)_{\eta<\zeta}$ we can find $y \in 2^\omega$ such that $\phi(y)_\eta = z_\eta$. Namely $y : p(\eta, n) \mapsto z_\eta(n)$ makes sense (since p is a bijection), and works. Consequently, for H there is $y \in 2^\omega$, such that H is the transfinite difference of the sets $\mathcal{U}^{z_\eta} = \mathcal{U}^{\phi(y)_\eta} = (\mathcal{U}_\eta)^y$. It is easy to see that if H is the transfinite difference of the sequence $((\mathcal{U}_\eta)^y)_{\eta<\zeta}$ then

$$\begin{aligned} H = \{x \in X :{}& \text{the smallest ordinal } \eta \text{ such that } x \notin (\mathcal{U}_\eta)^y \text{ is odd,} \\ & \text{if such an } \eta \text{ exists, or no such } \eta \text{ exists and } \zeta \text{ is odd}\}, \end{aligned}$$

hence $H = \mathcal{U}'^y$. $\square$

Corollary 3.4.43. *For every $1 \leq \xi < \omega_1$, every nonempty perfect set $P \subset X$ and every ordinal $\zeta < \omega_1$ there is a characteristic function $\chi_A \in \mathcal{B}_\xi(X)$ with $A \subset P$ and $\alpha_\xi(\chi_A) \geq \zeta$.*

Proof. Since P is nonempty perfect, it is an uncountable Polish space with the subspace topology, hence the rank α_ξ is unbounded on the characteristic Baire class ξ functions defined on P by the previous theorem. Hence we can take a characteristic function $f' \in \mathcal{B}_\xi(P)$ with $\alpha_\xi(f') \geq \zeta$, and set

$$f(x) = \begin{cases} f'(x) & \text{if } x \in P \\ 0 & \text{if } x \in X \setminus P. \end{cases}$$

It is easy to see that $f \in \mathcal{B}_\xi(X)$, hence it is enough to prove that $\alpha_\xi(f) \geq \zeta$.

For this, it is enough to prove that $\alpha_\xi(\{f' \leq p\}, \{f' \geq q\}) \leq \alpha_\xi(\{f \leq p\}, \{f \geq q\})$ for every pair of rational numbers $p < q$. For this, let $H \in \mathbf{\Delta}^0_{\xi+1}(X)$ where $\{f \leq p\} \subset H \subset \{f \geq q\}^c$ and H is the transfinite difference of the sets $(F_\eta)_{\eta<\lambda}$ with $\lambda = \alpha_\xi(\{f \leq p\}, \{f \geq q\})$ and $F_\eta \in \mathbf{\Pi}^0_\xi(X)$ for every $\eta < \lambda$.

Let $H' = P \cap H$ and for every $\eta < \lambda$ let $F'_\eta = P \cap F_\eta$. It is easy to see that H' separates the level sets $\{f' \leq p\}$ and $\{f' \geq q\}$ and H' is the transfinite difference of the sets $(F'_\eta)_{\eta<\lambda}$. And since $H' \in \mathbf{\Delta}^0_{\xi+1}(P)$ and $F'_\eta \in \mathbf{\Pi}^0_\xi(P)$ for every $\eta < \lambda$ ([51, 22.A]), we have the desired inequality $\alpha_\xi(\{f' \leq p\}, \{f' \geq q\}) \leq \alpha_\xi(\{f \leq p\}, \{f \geq q\})$. Thus the proof is complete. $\square$

Let again f be of Baire class ξ. Let

$$T_{f,\xi} = \{\tau' : \tau' \supseteq \tau \text{ Polish}, \tau' \subset \mathbf{\Sigma}^0_\xi(\tau), f \in \mathcal{B}_1(\tau')\}.$$

So $T_{f,\xi}$ is the set of those Polish refinements of the original topology that are subsets of the $\mathbf{\Sigma}^0_\xi$ sets turning f into a Baire class 1 function.

Remark 3.4.44. Clearly, $T_{f,1} = \{\tau\}$ for every Baire class 1 function f.

In order to show that the ranks we are about to construct are well-defined, we need the following proposition.

Proposition 3.4.45. *$T_{f,\xi} \neq \emptyset$ for every Baire class ξ function f.*

Proof. By the previous remark we may assume $\xi \geq 2$. For every rational p the level sets $\{f \leq p\}$ and $\{f \geq p\}$ are $\mathbf{\Pi}^0_{\xi+1}$ sets, hence they are countable intersections of $\mathbf{\Sigma}^0_\xi$ sets. In turn, these $\mathbf{\Sigma}^0_\xi$ sets are countable unions of sets from $\bigcup_{\eta<\xi} \mathbf{\Pi}^0_\eta(\tau)$. Clearly, $\bigcup_{\eta<\xi} \mathbf{\Pi}^0_\eta(\tau) \subset \mathbf{\Delta}^0_\xi$ for $\xi \geq 2$. By Kuratowski's theorem [51, 22.18], there exists a Polish refinement $\tau' \subset \mathbf{\Sigma}^0_\xi(\tau)$ of τ for which all these countable many $\mathbf{\Delta}^0_\xi$ sets are in $\mathbf{\Delta}^0_1(\tau')$. Then for every rational p the level sets are now $\mathbf{\Pi}^0_2(\tau')$ sets, and the same holds for irrational numbers too, since these level sets can be written as countable intersection of rational level sets, proving $T_{f,\xi} \neq \emptyset$. $\square$

We now define a rank on the Baire class ξ functions starting from an arbitrary rank on the Baire class 1 functions.

Definition 3.4.46. Let ρ be a rank on the Baire class 1 functions. Then for a Baire class ξ function f let

$$\rho^*_\xi(f) = \min_{\tau' \in T_{f,\xi}} \rho_{\tau'}(f), \tag{3.4.18}$$

where $\rho_{\tau'}(f)$ is the ρ-rank of f in the topology τ'.

Remark 3.4.47. From Remark 3.4.44 it is clear that $\rho^*_1 = \rho$ for every ρ.

Proposition 3.4.48. *Let ρ and η be ranks on the Baire class 1 functions. If $\rho = \eta$, or $\rho \le \eta$, or $\rho \approx \eta$, or $\rho \lesssim \eta$ then $\rho^*_\xi = \eta^*_\xi$, or $\rho^*_\xi \le \eta^*_\xi$, or $\rho^*_\xi \approx \eta^*_\xi$, or $\rho^*_\xi \lesssim \eta^*_\xi$, respectively. Moreover, the same implications hold relative to the bounded Baire class 1 functions.*

Proof. The statement for $=$ and $\le$ is immediate from the definitions, and the case of $\approx$ obviously follows from the case $\lesssim$, so it suffices to prove this latter case only. So assume $\rho \lesssim \eta$ (or $\rho \lesssim \eta$ on the bounded Baire class 1 functions). Choose an optimal $\tau' \in T_{f,\xi}$ for η, that is, $\eta^*_\xi(f) = \eta_{\tau'}(f)$. Then $\rho^*_\xi(f) \le \rho_{\tau'}(f) \lesssim \eta_{\tau'}(f) = \eta^*_\xi(f)$, completing the proof. □

Then the following two corollaries are immediate from Theorem 3.4.24, and Theorem 3.4.33.

Corollary 3.4.49. $\alpha^*_\xi \le \beta^*_\xi \le \gamma^*_\xi$.

Corollary 3.4.50. *$\alpha^*_\xi(f) \approx \beta^*_\xi(f) \approx \gamma^*_\xi(f)$ for every bounded Baire class ξ function f.*

Note that by repeating the argument of Remark 3.4.30 one can show that α^*_ξ differs from β^*_ξ and γ^*_ξ.

Theorem 3.4.51. *If X is a Polish group then the ranks α^*_ξ, β^*_ξ and γ^*_ξ are translation invariant.*

Proof. Note first that for a Baire class ξ function f and $x_0 \in X$ the functions $f \circ L_{x_0}$ and $f \circ R_{x_0}$ are also of Baire class ξ. We prove the statement only for the rank α^*_ξ, because an analogous argument works for the ranks β^*_ξ and γ^*_ξ.

Let f be a Baire class ξ function and $x_0 \in X$, first we prove that $\alpha^*_\xi(f) \ge \alpha^*_\xi(f \circ R_{x_0})$. Let $\tau' \in T_{f,\xi}$ be arbitrary and consider the topology $\tau'' = \{U \cdot x_0^{-1} : U \in \tau'\}$. The map $\phi : x \mapsto x \cdot x_0^{-1}$ is a homeomorphism between the spaces (X, τ') and (X, τ''), satisfying $f(x) = (f \circ R_{x_0})(\phi(x))$. From this it is clear that $\tau'' \in T_{f \circ R_{x_0},\xi}$ and since the definition of the rank α depends only on the topology of the space, we have $\alpha_{\tau'}(f) = \alpha_{\tau''}(f \circ R_{x_0})$. Since $\tau' \in T_{f,\xi}$ was arbitrary, the fact that $\alpha^*_\xi(f) \ge \alpha^*_\xi(f \circ R_{x_0})$ easily follows.

Repeating the argument with the function $f \circ R_{x_0}$ and element x_0^{-1}, we have $\alpha^*_\xi(f \circ R_{x_0}) \ge \alpha^*_\xi(f \circ R_{x_0} \circ R_{x_0^{-1}}) = \alpha^*_\xi(f)$, hence $\alpha^*_\xi(f) = \alpha^*_\xi(f \circ R_{x_0})$. For the function $f \circ L_{x_0}$ we can do same using the topology $\tau'' = \{x_0^{-1} \cdot U : U \in \tau'\}$ and the homeomorphism $\phi : x \mapsto x_0^{-1} \cdot x$, yielding $\alpha^*_\xi(f) = \alpha^*_\xi(f \circ L_{x_0})$. This finishes the proof. □

Theorem 3.4.52. *If f is a Baire class ξ function and $F \subset X$ is a closed set then $f \cdot \chi_F$ is of Baire class ξ, and $\alpha^*_\xi(f \cdot \chi_F) \le 1 + \alpha^*_\xi(f)$, $\beta^*_\xi(f \cdot \chi_F) \le 1 + \beta^*_\xi(f)$ and $\gamma^*_\xi(f \cdot \chi_F) \le 1 + \gamma^*_\xi(f)$.*

Proof. Examining the level sets of the function $f \cdot \chi_F$, it is easy to check that it is of Baire class ξ.

Now let $\tau' \in T_{f,\xi}$ be arbitrary. Clearly, $f \cdot \chi_F$ is of Baire class 1 with respect to τ', and by Proposition 3.4.28 we have $\alpha_{\tau'}(f \cdot \chi_F) \leq 1 + \alpha_{\tau'}(f)$ for every $\tau' \in T_{f,\xi}$, hence $\alpha^*_\xi(f \cdot \chi_F) \leq 1 + \alpha^*_\xi(f)$. The other two inequalities follow similarly. □

Proposition 3.4.53. *If f is a Baire class ζ function with $\zeta < \xi$ then $\alpha^*_\xi(f) = \beta^*_\xi(f) = \gamma^*_\xi(f) = 1$.*

Proof. Using Proposition 3.4.27, it is enough to show that there exists a topology $\tau' \in T_{f,\xi}$ such that $f : (X, \tau') \to \mathbb{R}$ is continuous, and this is clear from [51, 24.5]. □

Next we prove a useful lemma, and then investigate further properties of the ranks α^*_ξ, β^*_ξ and γ^*_ξ.

Lemma 3.4.54. *For every n let τ_n be a Polish refinement of τ with $\tau_n \subset \mathbf{\Sigma}^0_\xi(\tau)$. Then there exists a common Polish refinement τ' of the τ_n's also satisfying $\tau' \subset \mathbf{\Sigma}^0_\xi(\tau)$.*

Proof. The case $\xi = 1$ is again trivial, so we may assume $\xi \geq 2$. Take a base $\{G^k_n : k \in \mathbb{N}\}$ for τ_n. Since these sets are in $\mathbf{\Sigma}^0_\xi(\tau)$, they can be written as the countable unions of sets from $\bigcup_{\eta<\xi} \mathbf{\Pi}^0_\eta(\tau)$. Clearly, $\bigcup_{\eta<\xi} \mathbf{\Pi}^0_\eta(\tau) \subset \mathbf{\Delta}^0_\xi$ for $\xi \geq 2$. As above, by Kuratowski's theorem [51, 22.18], we have a Polish topology τ', for which these countably many $\mathbf{\Delta}^0_\xi(\tau)$ sets are in $\mathbf{\Delta}^0_1(\tau')$ satisfying $\tau' \subset \mathbf{\Sigma}^0_\xi(\tau)$. This τ' works. □

Lemma 3.4.55. *If $\tau' \subset \tau''$ are two Polish topologies with $f \in \mathcal{B}_1(\tau')$ then $f \in \mathcal{B}_1(\tau'')$, moreover, $\beta_{\tau'}(f) \geq \beta_{\tau''}(f)$ and $\gamma_{\tau'}(f) \geq \gamma_{\tau''}(f)$.*

Proof. To prove that $f \in \mathcal{B}_1(\tau'')$ note that the level sets $\{f < c\}, \{f > c\} \in \mathbf{\Sigma}^0_2(\tau')$, hence $\{f < c\}, \{f > c\} \in \mathbf{\Sigma}^0_2(\tau'')$, so $f \in \mathcal{B}_1(\tau'')$.

Now recall the definition of the derivative defining β:

$$\omega(f, x, F) = \inf \left\{ \sup_{x_1, x_2 \in U \cap F} |f(x_1) - f(x_2)| : U \text{ open}, \, x \in U \right\},$$

$$D_{f,\epsilon}(F) = \{x \in F : \omega(f, x, F) \geq \epsilon\}.$$

Let us now fix f and $\varepsilon > 0$ and let us denote the derivative $D_{f,\epsilon}$ with respect to the topology τ' by $D_{\tau'}$, and with respect to the topology τ'' by $D_{\tau''}$. By Proposition 3.4.5 it is enough to prove that $D_{\tau''}(F) \subset D_{\tau'}(F)$ for every closed set $F \subset X$.

For this it is enough to show that $\omega_{\tau''}(f, x, F) \leq \omega_{\tau'}(f, x, F)$ for every $x \in F$ where $\omega_{\tau'}(f, x, F)$ is the oscillation with respect to the topology τ'. And this is clear, since in the case of τ'', the infimum in the definition goes through more open set containing x, hence the resulting oscillation will be less.

For the rank γ, we proceed similarly. First we recall the definition of γ:

$$\omega((f_n)_{n\in\mathbb{N}}, x, F) = \inf_{\substack{x \in U \\ U \text{ open}}} \inf_{N \in \mathbb{N}} \sup \{|f_m(y) - f_n(y)| : n, m \geq N, \; y \in U \cap F\},$$

$$D_{(f_n)_{n\in\mathbb{N}}, \varepsilon}(F) = \{x \in F : \omega((f_n)_{n\in\mathbb{N}}, x, F) \geq \varepsilon\},$$

$$\gamma(f) = \min \left\{ \sup_{\varepsilon > 0} \gamma((f_n)_{n\in\mathbb{N}}, \varepsilon) : \forall n \; f_n \text{ is continuous and } f_n \to f \text{ pointwise} \right\}.$$

Let us fix a sequence $(f_n)_{n\in\mathbb{N}}$ of τ'-continuous (hence also τ''-continuous) functions converging pointwise to f, and also fix $\varepsilon > 0$. Let us denote the derivative $D_{(f_n)_{n\in\mathbb{N}},\varepsilon}$ with respect to τ' by $D_{\tau'}$ and with respect to τ'' by $D_{\tau''}$. Again, by Proposition 3.4.5 it is enough to prove that $D_{\tau''}(F) \subset D_{\tau'}(F)$ for every closed set $F \subset X$. And similarly to the previous case it is enough to prove that the oscillation $\omega((f_n)_{n\in\mathbb{N}}, x, F)$ with respect to the topology τ'' is at most the oscillation with respect to τ', but this is clear, since, as before, the infimum goes through more open set in the case of τ''. $\square$

Theorem 3.4.56. *The ranks β^*_ξ and γ^*_ξ are essentially linear.*

Proof. We only consider β^*_ξ, since the proof for the rank γ^*_ξ is completely analogous.

It is easy to see that $\beta^*_\xi(cf) = \beta^*_\xi(f)$ for every $c \in \mathbb{R} \setminus \{0\}$, hence it suffices to show that β^*_ξ is essentially additive.

For f and g let τ_f and τ_g be such that $\beta_{\tau_f}(f) = \beta^*_\xi(f)$ and $\beta_{\tau_g}(g) = \beta^*_\xi(g)$. Using Lemma 3.4.54 we have a common refinement τ' of τ_f and τ_g with $\tau' \subset \mathbf{\Sigma}^0_\xi(\tau)$. Now $f, g \in \mathcal{B}_1(\tau')$, so $f + g \in \mathcal{B}_1(\tau')$, hence $\tau' \in T_{f+g,\xi}$. Therefore $\beta^*_\xi(f+g) \le \beta_{\tau'}(f+g)$. By Lemma 3.4.55 we have that $\beta_{\tau'}(f) \le \beta_{\tau_f}(f)$ (in fact equality holds), and similarly for g. But $\beta_{\tau'}$ is additive by Theorem 3.4.29, so

$$\beta^*_\xi(f+g) \le \beta_{\tau'}(f+g) \lesssim \max\{\beta_{\tau'}(f), \beta_{\tau'}(g)\} \le \max\{\beta_{\tau_f}(f), \beta_{\tau_g}(g)\} = \max\{\beta^*_\xi(f), \beta^*_\xi(g)\}.$$

$\square$

Theorem 3.4.57. *If f is a Baire class ξ function then*

$$\alpha^*_\xi(f) \le \alpha_\xi(f) \le 2\alpha^*_\xi(f), \text{ hence } \alpha^*_\xi(f) \approx \alpha_\xi(f).$$

Proof. For $\xi = 1$ the claim is an easy consequence of the definition of the two ranks and Corollary 3.4.14. From now on, we suppose that $\xi \ge 2$.

For the first inequality, for every pair of rationals $p < q$ pick a sequence $(F^\zeta_{p,q})_{\zeta<\alpha_\xi(f)} \subset \mathbf{\Pi}^0_\xi(X)$, whose transfinite difference separates the level sets $\{f \le p\}$ and $\{f \ge q\}$.

Every $\mathbf{\Pi}^0_\xi(X)$ set is the intersection of countably many $\mathbf{\Delta}^0_\xi$ sets, hence $F^\zeta_{p,q} = \bigcap_n H^\zeta_{p,q,n}$, with $H^\zeta_{p,q,n} \in \mathbf{\Delta}^0_\xi$. By Kuratowski's theorem [51, 22.18], there is a finer Polish topology $\tau' \subset \mathbf{\Sigma}^0_\xi(\tau)$, for which $H^\zeta_{p,q,n} \in \mathbf{\Delta}^0_1(\tau')$ for every p, q, n and $\zeta < \alpha_\xi(f)$, hence $F^\zeta_{p,q} \in \mathbf{\Pi}^0_1(\tau')$.

This means that the level sets of f can be separated by transfinite differences of closed sets with respect to τ', hence they can be separated by sets in $\mathbf{\Delta}^0_2(\tau')$. Then it is easy to see that for every $c \in \mathbb{R}$ the level sets $\{f \le c\}$ and $\{f \ge c\}$ are countable intersections of $\mathbf{\Delta}^0_2(\tau')$ sets, hence they are $\mathbf{\Pi}^0_2(\tau')$ sets, proving that $f \in \mathcal{B}_1(\tau')$. Moreover, $\alpha_{1,\tau'}(f) \le \alpha_\xi(f)$ easily follows from the construction (here $\alpha_{1,\tau'}$ is the rank α_1 with respect to τ'). And by Corollary 3.4.14 we have $\alpha^*_\xi \le \alpha_{\tau'}(f) \le \alpha_{1,\tau'}(f) \le \alpha_\xi(f)$, proving the first inequality of the theorem.

For the second inequality, take a topology τ' with $\alpha_{\tau'}(f) = \alpha^*_\xi(f)$. Again, by Corollary 3.4.14, we have $\alpha_{1,\tau'}(f) \le 2\alpha_{\tau'}(f) = 2\alpha^*_\xi(f)$.

It remains to prove that $\alpha_\xi(f) \le \alpha_{1,\tau'}(f)$. A τ'-closed set is $\mathbf{\Pi}^0_\xi$ with respect to τ. Therefore, if $(F_\eta)_{\eta<\zeta}$ is a decreasing continuous sequence of τ'-closed sets whose transfinite difference separates $\{f \le p\}$ and $\{f \ge q\}$ then the same sequence is a decreasing continuous sequence of sets from $\mathbf{\Pi}^0_\xi(\tau)$, proving $\alpha_\xi(f) \leq \alpha_{1,\tau'}(f)$. $\square$

Corollary 3.4.58. *α_ξ and α^*_ξ are essentially linear for bounded functions for every ξ.*

PROOF. $\alpha_\xi \approx \alpha^*_\xi$ by the previous theorem, $\alpha^*_\xi \approx \beta^*_\xi$ for bounded functions by Corollary 3.4.50, and β^*_ξ is essentially linear by Theorem 3.4.56. $\square$

From Corollary 3.4.37 we can obtain the appropriate statement for the ranks $\alpha^*_\xi, \beta^*_\xi$ and γ^*_ξ.

Proposition 3.4.59. *If $f = \sum_{i=1}^n c_i \chi_{A_i}$, where the A_i's are disjoint $\mathbf{\Delta}^0_{\xi+1}$ sets covering X and the c_i's are distinct then*

$$\alpha^*_\xi(f) \approx \max_i \{\alpha^*_\xi(\chi_{A_i})\},$$

and similarly for β^*_ξ and γ^*_ξ.

PROOF. The additivity of α^*_ξ implies $\alpha^*_\xi(f) \lesssim \max_i\{\alpha^*_\xi(\chi_{A_i})\}$. For the other inequality let τ' be a topology for which f is Baire class 1. Then the characteristic functions χ_{A_i} are also Baire class 1, and hence by Corollary 3.4.37 we obtain $\alpha_{\tau'}(f) \approx \max_i\{\alpha_{\tau'}(\chi_{A_i})\}$. But by the definition of α^*_ξ for every such topology $\alpha^*_\xi(\chi_{A_i}) \leq \alpha_{\tau'}(\chi_{A_i})$, therefore $\max_i\{\alpha^*_\xi(\chi_{A_i})\} \leq \max_i\{\alpha_{\tau'}(\chi_{A_i})\} \approx \alpha_{\tau'}(f)$. Then choosing τ' so that $\alpha_{\tau'}(f) = \alpha^*_\xi(f)$ the proof is complete.

$\square$

Theorem 3.4.60. *The ranks α^*_ξ, β^*_ξ and γ^*_ξ are unbounded in ω_1. Moreover, for every nonempty perfect set $P \subset X$ and ordinal $\zeta < \omega_1$ there exists a characteristic function $\chi_A \in \mathcal{B}_\xi(X)$ with $A \subset P$ such that $\alpha^*_\xi(\chi_A), \beta^*_\xi(\chi_A), \gamma^*_\xi(\chi_A) \geq \zeta$.*

PROOF. In order to prove the theorem, by Corollary 3.4.49 it suffices to prove the statement for α^*_ξ. Moreover, instead of $\alpha^*_\xi(\chi_A) \geq \zeta$ it suffices to obtain $\alpha^*_\xi(\chi_A) \gtrsim \zeta$. And this is clear from Theorem 3.4.57 and Corollary 3.4.43. $\square$

Proposition 3.4.61. *If f_n, f are Baire class ξ functions and $f_n \to f$ uniformly then $\beta^*_\xi(f) \leq \sup_n \beta^*_\xi(f_n)$.*

PROOF. For every n let $\tau_n \in T_{f_n,\xi}$ with $\beta_{\tau_n}(f_n) = \beta^*_\xi(f_n)$. Using Lemma 3.4.54, let τ' be their common refinement satisfying $\tau' \subset \mathbf{\Sigma}^0_\xi(\tau)$, where τ is the original topology. Note that $f_n \in \mathcal{B}_1(\tau')$ for every n, and the Baire class 1 functions are closed under uniform limits [51, 24.4], hence $\tau' \in T_{f,\xi}$. Then by Proposition 3.4.31 and Lemma 3.4.55 we have

$$\beta^*_\xi(f) \leq \beta_{\tau'}(f) \leq \sup_n \beta_{\tau'}(f_n) \leq \sup_n \beta_{\tau_n}(f_n) = \sup_n \beta^*_\xi(f_n).$$

$\square$

Proposition 3.4.62. *If f_n, f are Baire class ξ functions and $f_n \to f$ uniformly then $\alpha^*_\xi(f) \lesssim \sup_n \alpha^*_\xi(f_n)$ and $\gamma^*_\xi(f) \lesssim \sup_n \gamma^*_\xi(f_n)$.*

PROOF. Repeat the previous argument but apply Proposition 3.4.39 and Proposition 3.4.32 instead of Proposition 3.4.31. $\square$

3.5 Proof of Theorem 1.4.6

Proof. We need two well-known facts. Firstly, $\mathrm{non}(\mathcal{N}) = \omega_2$ in this model [5]. Secondly, in this model there is no strictly increasing (wrt. inclusion) sequence of Borel sets of length ω_2 (this is proved in [54], see also [25]).

Assume that $\varphi : \mathcal{N} \to \mathcal{B}$ is a monotone hull operation. Choose $H = \{x_\alpha : \alpha < \mathrm{non}(\mathcal{N})\} \notin \mathcal{N}$, and consider $\varphi(\{x_\beta : \beta < \alpha\})$ for $\alpha < \mathrm{non}(\mathcal{N})$. This is an increasing ω_2 long sequence of Borel sets, which cannot stabilise, since then H would be contained in a nullset. But then we can select a strictly increasing subsequence of length ω_2, a contradiction. $\square$

3.6 Proof of Theorem 1.4.7

Let us denote by $A\Delta B$ the symmetric difference of A and B.

Lemma 3.6.1. *Assume the Continuum Hypothesis. Then there exists a monotone map $\psi : \mathcal{L} \to \mathcal{G}_\delta$ such that $\lambda(M\Delta\psi(M)) = 0$ for every $M \in \mathcal{L}$ and that $\lambda(M\Delta M') = 0$ implies $\psi(M) = \psi(M')$ for every $M, M' \in \mathcal{L}$.*

Proof. Let us say that $M, M' \in \mathcal{L}$ are *equivalent*, if $\lambda(M\Delta M') = 0$. Denote by $[M]$ the equivalence class of M and by $\mathcal{L}/\mathcal{N}$ the set of classes. We say that $[M_1] \leq [M_2]$ if there are $M_1' \in [M_1]$ and $M_2' \in [M_2]$ such that $M_1' \subset M_2'$.

It is sufficient to define $\Psi : \mathcal{L}/\mathcal{N} \to \mathcal{G}_\delta$ so that $[M] \leq [M']$ implies $\Psi([M]) \subset \Psi([M'])$ for every $M, M' \in \mathcal{L}$, and that $\Psi([M]) \in [M]$ for every $M \in \mathcal{L}$.

Enumerate $\mathcal{L}/\mathcal{N}$ as $\{[M_\alpha] : \alpha < \omega_1\}$. For every $\alpha < \omega_1$ define

$$\Psi([M_\alpha]) = \bigcap_{\substack{\beta<\alpha \\ [M_\beta]\geq[M_\alpha]}} \Psi([M_\beta]) \;\cap\; \Big(\text{a } G_\delta \text{ hull of } \bigcup_{\substack{\gamma<\alpha \\ [M_\gamma]\leq[M_\alpha]}} \Psi([M_\gamma]) \cup M_\alpha\Big).$$

It is not hard to check that this is a G_δ set such that $[M_\gamma] \leq [M_\alpha] \leq [M_\beta]$ implies $\Psi([M_\gamma]) \subset \Psi([M_\alpha]) \subset \Psi([M_\beta])$, and that $\Psi([M_\alpha]) \in [M_\alpha]$, hence the construction works. $\square$

The following lemma is the only result we can prove for $\mathcal{B}$ but not for $\mathcal{G}_\delta$.

Lemma 3.6.2. *Assume the Continuum Hypothesis. Then there exists a monotone hull operation $\varphi : \mathcal{N} \to \mathcal{B}$ such that*

1. *$\varphi(N \cup N') \subset \varphi(N) \cup \varphi(N')$ for every $N, N' \in \mathcal{N}$ (subadditivity),*
2. *$\bigcup\{\varphi(N) : N \subset B,\ N \in \mathcal{N}\} \setminus B \in \mathcal{N}$ for every $B \in \mathcal{B}$.*

Proof. Let $\{N_\alpha : \alpha < \omega_1\}$ be a cofinal family in $\mathcal{N}$, that is, $\forall N \in \mathcal{N}\ \exists \alpha < \omega_1$ such that $N \subset N_\alpha$. (Such a family exists, since there are continuum many Borel nullsets.) For every $\alpha < \omega_1$ define, using transfinite recursion, $A_\alpha =$ a G_δ hull of $(\bigcup_{\beta<\alpha} A_\beta \cup N_\alpha)$. Clearly, $\{A_\alpha : \alpha < \omega_1\}$ is a cofinal increasing sequence of G_δ sets.

Set $A_\alpha^* = A_\alpha \setminus \bigcup_{\beta<\alpha} A_\beta$. Enumerate $\mathcal{B}$ as $\{B_\alpha : \alpha < \omega_1\}$ and for every $\alpha < \omega_1$ define the countable set

$$\mathcal{B}_\alpha = \Big\{ \bigcup_{i=0}^{n} B_{\beta_i} : n \in \mathbb{N},\ \beta_i < \alpha\ (0 \leq i \leq n)\Big\}.$$

Note that every $\mathcal{B}_\alpha$ is closed under finite unions.

Now define
$$\varphi(N) = \bigcup_{\alpha \le \alpha_N} \Big(A_\alpha^* \cap \bigcap_{\substack{B \in \mathcal{B}_\alpha \\ N \cap A_\alpha^* \subset B}} B \Big).$$

This is clearly a disjoint union. It is easy to see that φ is a monotone Borel hull operation (note that $\varphi(N) \subset A_{\alpha_N}$).

For every $\alpha < \omega_1$ define $\varphi_\alpha(N) = A_\alpha^* \cap \varphi(N)$ $(N \in \mathcal{N})$. In order to check subadditivity, let $N, N' \in \mathcal{N}$. We may assume $\alpha_N \le \alpha_{N'}$, so clearly $\alpha_{N \cup N'} = \alpha_{N'}$. It suffices to check that each φ_α is subadditive. If $\alpha > \alpha_N$ then actually $\varphi_\alpha(N \cup N') = \varphi_\alpha(N')$, so we are done. Suppose now $\alpha \le \alpha_N$. Let $x \in A_\alpha^*$ such that $x \notin \varphi(N) \cup \varphi(N')$. Then there exist $B \supseteq N \cap A_\alpha^*$ and $B' \supseteq N' \cap A_\alpha^*$ in $\mathcal{B}_\alpha$ such that $x \notin B, B'$. But then $B \cup B' \in \mathcal{B}_\alpha$ witnesses that $x \notin \varphi(N \cup N')$ since $x \notin B \cup B' \supseteq (N \cup N') \cap A_\alpha^*$.

Finally, to prove *2* it is sufficient to show that $N \subset B_\alpha$ implies $\varphi(N) \setminus B_\alpha \subset A_\alpha$ for every $N \in \mathcal{N}$ and $\alpha < \omega_1$. So let $x \in \varphi_\beta(N)$ for some $\beta > \alpha$. We have to show $x \in B_\alpha$. But this simply follows from the definition of φ since $B_\alpha \in \mathcal{B}_\beta$. $\square$

Lemma 3.6.3. *Assume that there exists a monotone map $\psi : \mathcal{L} \to \mathcal{B}$ such that $\lambda(M \Delta \psi(M)) = 0$ for every $M \in \mathcal{L}$ and also that there exists a monotone hull operation $\varphi : \mathcal{N} \to \mathcal{B}$ such that*

1. *$\varphi(N \cup N') \subset \varphi(N) \cup \varphi(N')$ for every $N, N' \in \mathcal{N}$,*
2. *$\bigcup\{\varphi(N) : N \subset B, N \in \mathcal{N}\} \setminus B \in \mathcal{N}$ for every $B \in \mathcal{B}$.*

Then φ can be extended to a monotone hull operation $\varphi^* : \mathcal{L} \to \mathcal{B}$.

Proof. We may assume that $\psi(N) = \emptyset$ for every $N \in \mathcal{N}$ (by redefining ψ on $\mathcal{N}$ to be constant $\emptyset$, if necessary).

Define
$$\varphi^*(M) = \psi\big(M\big) \cup \varphi\big(M \setminus \psi(M)\big) \cup \varphi\Big(\bigcup_{\substack{N \subset \psi(M) \\ \emptyset \ne N \in \mathcal{N}}} \varphi(N) \setminus \psi(M) \Big).$$

Clearly $\varphi^*(M) \in \mathcal{B}$. As the union of first two terms contains M, we obtain $M \subset \varphi^*(M)$. Moreover, $\varphi^*(M)$ is a hull of M, since the first term is equivalent to M and the last two terms are nullsets. It is also easy to see that φ^* extends φ.

We still have to check monotonicity of φ^*. First we prove
$$N' \in \mathcal{N},\ M' \in \mathcal{L},\ N' \subset \psi(M') \Rightarrow \varphi(N') \subset \varphi^*(M'). \tag{3.6.1}$$

Indeed, the case $N' = \emptyset$ is trivial to check, otherwise
$$\varphi(N') \subset \bigcup_{\substack{N \subset \psi(M') \\ \emptyset \ne N \in \mathcal{N}}} \varphi(N) \subset \Big(\bigcup_{\substack{N \subset \psi(M') \\ \emptyset \ne N \in \mathcal{N}}} \varphi(N) \setminus \psi(M') \Big) \cup \psi(M') \subset$$
$$\subset \varphi\Big(\bigcup_{\substack{N \subset \psi(M') \\ \emptyset \ne N \in \mathcal{N}}} \varphi(N) \setminus \psi(M') \Big) \cup \psi(M') \subset \varphi^*(M'),$$

which proves (3.6.1).

Let now $M \subset M'$ be arbitrary elements of $\mathcal{L}$. We need to show that all three terms of $\varphi^*(M)$ are contained in $\varphi^*(M')$.

Firstly, $\psi(M) \subset \psi(M')$.

Secondly, define $N' = \big(M \setminus \psi(M)\big) \cap \psi(M')$. Using the subadditivity of φ and then (3.6.1) we obtain

$$\begin{aligned}\varphi\big(M \setminus \psi(M)\big) \subset \varphi\Big(\big(M \setminus \psi(M)\big) \cap \psi(M')\Big) \cup \varphi\Big(\big(M \setminus \psi(M)\big) \setminus \psi(M')\Big) \subset \\ \subset \varphi\big(N'\big) \cup \varphi\big(M' \setminus \psi(M')\big) \subset \varphi^*(M').\end{aligned}$$

Thirdly, let

$$N' = \Big(\bigcup_{\substack{N \subset \psi(M)\\ \emptyset \neq N \in \mathcal{N}}} \varphi(N) \setminus \psi(M)\Big) \cap \psi(M').$$

Using the subadditivity of φ and then (3.6.1) we obtain

$$\begin{aligned}&\varphi\Big(\bigcup_{\substack{N \subset \psi(M)\\ \emptyset \neq N \in \mathcal{N}}} \varphi(N) \setminus \psi(M)\Big) \subset \\ &\varphi\bigg(\Big(\bigcup_{\substack{N \subset \psi(M)\\ \emptyset \neq N \in \mathcal{N}}} \varphi(N) \setminus \psi(M)\Big) \cap \psi(M')\bigg) \cup \varphi\bigg(\Big(\bigcup_{\substack{N \subset \psi(M)\\ \emptyset \neq N \in \mathcal{N}}} \varphi(N) \setminus \psi(M)\Big) \setminus \psi(M')\bigg) \subset \\ &\qquad\qquad \subset \varphi(N') \cup \varphi\Big(\bigcup_{\substack{N \subset \psi(M')\\ \emptyset \neq N \in \mathcal{N}}} \varphi(N) \setminus \psi(M')\Big) \subset \varphi^*(M').\end{aligned}$$

This concludes the proof. □

The following lemma was pointed out to us by J. Swaczyna.

Lemma 3.6.4. *Assume the Continuum Hypothesis. Then there exists a monotone hull operation $\varrho : \mathcal{B} \to \mathcal{G}_\delta$.*

Proof. Enumerate $\mathcal{B}$ as $\mathcal{B} = \{B_\alpha : \alpha < \omega_1\}$. By transfinite induction define

$$\varrho(B_\alpha) = \left(\text{a } G_\delta \text{ hull of } \left[\bigcup_{\substack{\beta<\alpha\\ B_\beta \subset B_\alpha}} \varrho(B_\beta) \cup B_\alpha\right]\right) \cap \bigcap_{\substack{\beta<\alpha\\ B_\beta \supset B_\alpha}} \varrho(B_\beta).$$

It is straightforward to check that this works. □

Now we are ready to prove Theorem 1.4.7.

Proof. Lemma 3.6.1 and Lemma 3.6.2 show that the requirements of Lemma 3.6.3 can be satisfied, so we obtain a monotone Borel hull operation on $\mathcal{L}$. Composing this with ϱ from the previous lemma finishes the proof. □

3.7 Proof of Theorems 2.1.3 and 2.1.10

Recall that the symbol $\mathcal{I}^r_{n,\sigma-fin}$ stands for the σ-ideal consisting of sets of σ-finite r-dimensional Hausdorff measure in $\mathbb{R}^n$.

Lemma 3.7.1. $\mathrm{cov}(\mathcal{I}^r_{n,\sigma-fin}) = \mathrm{cov}(\mathcal{N}^r_n)$.

Proof. The inequality $\mathrm{cov}(\mathcal{I}^r_{n,\sigma-fin}) \leq \mathrm{cov}(\mathcal{N}^r_n)$ is clear by $\mathcal{N}^r_n \subset \mathcal{I}^r_{n,\sigma-fin}$. In order to prove the opposite inequality let $\{I_\alpha\}_{\alpha<\mathrm{cov}(\mathcal{I}^r_{n,\sigma-fin})}$ be a cover of $\mathbb{R}^n$ by sets of σ-finite $\mathcal{H}^r$-measure. We can assume that they are actually of finite $\mathcal{H}^r$-measure, and also that they are Borel (even G_δ). By the Isomorphism Theorem of Measures [51, Thm. 17.41] a Borel set of finite $\mathcal{H}^r$-measure can be covered by $\mathrm{cov}(\mathcal{N})$ many $\mathcal{H}^r$-nullsets. Therefore $\mathbb{R}^n$ can be covered by $\mathrm{cov}(\mathcal{I}^r_{n,\sigma-fin}) \cdot \mathrm{cov}(\mathcal{N})$ many $\mathcal{H}^r$-nullsets. But $\mathcal{I}^r_{n,\sigma-fin} \subset \mathcal{N}$ implies $\mathrm{cov}(\mathcal{I}^r_{n,\sigma-fin}) \geq \mathrm{cov}(\mathcal{N})$, hence $\mathbb{R}^n$ can be covered by $\mathrm{cov}(\mathcal{I}^r_{n,\sigma-fin})$ many $\mathcal{H}^r$-nullsets, proving $\mathrm{cov}(\mathcal{N}^r_n) \leq \mathrm{cov}(\mathcal{I}^r_{n,\sigma-fin})$. $\square$

The following theorem describes the values of all the cardinal invariants of the extended Cichoń diagram in a specific model of ZFC.

Theorem 3.7.2. *It is consistent with ZFC that* $\mathrm{cov}(\mathcal{N}) = \mathfrak{d} = \mathrm{non}(\mathcal{N}) = \omega_1$ *and* $\mathrm{cov}(\mathcal{N}^r_n) = \mathfrak{c} = \omega_2$.

Proof. Most ingredients of this proof are actually present in [80]. Let V be a ground model satisfying the Continuum Hypothesis, and let W be obtained by the countable support iteration of $\mathbb{P}_{\mathcal{I}^r_{n,\sigma-fin}}$ of length ω_2. Since the forcing $\mathbb{P}_{\mathcal{I}^r_{n,\sigma-fin}}$ is proper by [80, 4.4.2] and adds a generic real avoiding the Borel members of $\mathcal{I}^r_{n,\sigma-fin}$ coded in V, we obtain that $\mathrm{cov}(\mathcal{I}^r_{n,\sigma-fin}) = \mathfrak{c} = \omega_2$ in W. Hence, $\mathrm{cov}(\mathcal{N}^r_n) = \mathfrak{c} = \omega_2$ in W by Lemma 3.7.1. By [80, 4.4.8] $\mathbb{P}_{\mathcal{I}^r_{n,\sigma-fin}}$ adds no splitting reals, hence no Random reals, and this is well-known to be preserved by the iteration, thus the Borel nullsets coded in V cover the reals of W, therefore $\mathrm{cov}(\mathcal{N}) = \omega_1$ in W. Moreover, by [80, Ex. 3.6.4] $\mathcal{I}^r_{n,\sigma-fin}$ is polar, which is preserved by the iteration, therefore it preserves outer Lebesgue measure, hence the ground model is not null, thus $\mathrm{non}(\mathcal{N}) = \omega_1$ in W. Finally, the forcing is ω^ω-bounding by [80, 4.4.8], hence the same holds for the iteration, therefore $\mathfrak{d} = \omega_1$ in W. $\square$

The following are immediate.

Corollary 3.7.3. *Consistently* $\mathrm{cov}(\mathcal{N}) < \mathrm{cov}(\mathcal{N}^r_n)$*, answering Fremlin's question.*

Corollary 3.7.4. *The answer to Question 2.1.9 is affirmative, hence so is the answer to the question of Humke and Laczkovich.*

3.8 Proof of Theorem 2.1.4

Proof. Let W be the Laver model, that is, the model obtained by iteratively adding ω_2 Laver reals with countable support over a model V satisfying the Continuum Hypothesis, see [5] for the definitions and basic properties of this model. For example, it is well-known that $\mathrm{non}(\mathcal{M}) = \omega_2$ in this model.

On the other hand, W satisfies the so called Laver property, an equivalent form of which is the following:

If $0 < r < n$ and $x \in \prod_{k\in\omega} 2^{kn} \cap W$ then there is

$$T \in \prod_{k\in\omega} [2^{kn}]^{2^{k\frac{r}{2}}} \cap V$$

such that $x(k) \in T(k)$ for all $k \in \omega$. This follows from [5, Lemma 6.3.32] by letting $f(k) = 2^{kn}$, $S(k) = \{x(k)\}$, and using and arbitrary positive rational number $s < \frac{r}{2}$.

The following argument takes place in W. For every $k \in \omega$ let ψ_k be a bijection from 2^{kn} to the set of all cubes of the form

$$\left[\frac{j_0}{2^k}, \frac{j_0+1}{2^k}\right] \times \ldots \times \left[\frac{j_{n-1}}{2^k}, \frac{j_{n-1}+1}{2^k}\right],$$

where $j_i \in 2^k$ for each $i \in n$.

For every $T \in \prod_{k\in\omega} [2^{kn}]^{2^{k\frac{r}{2}}}$ define

$$N_T = \bigcap_{k\in\omega} \bigcup_{j\in T(k)} \psi_k(j).$$

First we show that $N_T \in \mathcal{N}^r_n$. Note that the diameter of a cube of side-length $\frac{1}{2^k}$ is $\sqrt{n}\frac{1}{2^k}$. Clearly, for every $k \in \omega$ we have $\mathcal{H}^r_\infty(N_T) \leq \mathcal{H}^r_\infty\left(\bigcup_{j\in T(k)} \psi_k(j)\right) \leq |T(k)| \left(\sqrt{n}\frac{1}{2^k}\right)^r = 2^{k\frac{r}{2}} \left(\sqrt{n}\frac{1}{2^k}\right)^r = \sqrt{n}^r 2^{-k\frac{r}{2}}$, which tends to 0 as k tends to ∞, therefore $\mathcal{H}^r_\infty(N_T) = 0$ and consequently, (as it is easy to check that $\mathcal{H}^r(H) = 0$ iff $\mathcal{H}^r_\infty(H) = 0$), $\mathcal{H}^r(N_T) = 0$.

Next we finish the proof by showing that $\{N_T : T \in \prod_{k\in\omega} [2^{kn}]^{2^{k\frac{r}{2}}} \cap V\}$ is a cover of $[0,1]^n$ (note that $|V| = \omega_1$ in W, and also that if ω_1 members of $\mathcal{N}^r_n$ cover the unit cube then the same holds for $\mathbb{R}^n$, hence this implies $\mathrm{cov}(\mathcal{N}^r_n) = \omega_1$). So let $z \in [0,1]^n$, then there exists $x \in \prod_{k\in\omega} 2^{kn}$ such that $z \in \psi_k(x(k))$ for each $k \in \omega$. Let $T \in \prod_{k\in\omega} [2^{kn}]^{2^{k\frac{r}{2}}} \cap V$ be such that $x(k) \in T(k)$ for all $k \in \omega$, then it is easy to check that $z \in N_T$, finishing the proof. $\square$

3.9 Proof of Theorem 2.1.5

Now we turn to the consistency of $\mathrm{cov}(\mathcal{M}) < \mathrm{non}(\mathcal{N}^r_n)$. First we need some preparation.

For each $k \in \omega$ let $M_k \in \omega$ be so large that

$$2^k \left(\frac{\sqrt{n}}{M_k}\right)^r < \frac{1}{2^k}. \tag{3.9.1}$$

Definition 3.9.1. Let C_k be the set of all cubes of the form

$$\left[\frac{j_0}{M_k}, \frac{j_0+1}{M_k}\right] \times \ldots \times \left[\frac{j_{n-1}}{M_k}, \frac{j_{n-1}+1}{M_k}\right],$$

where $j_i \in M_k$ for each $i \in n$. Let $\mathbb{C}_k$ consist of all sets that can be written as the union of 2^k elements of C_k.

Lemma 3.9.2. *For every partition $\mathbb{C}_k = \bigcup_{i\in 2^k} X_i$ there is some $i \in 2^k$ such that $\cup X_i = [0,1]^n$.*

PROOF. Otherwise, pick $x_i \notin \cup X_i$ and cubes $Q_i \in C_k$ containing x_i, then $\bigcup_{i \in 2^k} Q_i \in \mathbb{C}_k$ belongs to one of the X_i, yielding a contradiction. $\square$

Definition 3.9.3. Now we define the *norm function* $\nu \colon \bigcup_{k \in \omega} \mathcal{P}(\mathbb{C}_k) \to \omega$ as follows. For $X \subset \mathbb{C}_k$ define $\nu(X) \geq 1$ if $\cup X = [0,1]^n$ and define $\nu(X) \geq j+1$ if for every partition $X = X_0 \cup X_1$ there is $i \in 2$ such that $\nu(X_i) \geq j$.

Lemma 3.9.4. $\nu(\mathbb{C}_k) \geq k+1$.

PROOF. Otherwise, we could iteratively split $\mathbb{C}_k$ into pieces so that at stage m we have a partition into 2^m many sets each with norm at most $k-m$, hence eventually we could have a partition into 2^k many sets none of which covers $[0,1]^n$, contradicting the previous lemma. $\square$

Lemma 3.9.5. *If $X \subset \mathbb{C}_k$ and $\nu(X) \geq j$ and $y \in [0,1]^n$ then $\nu(\{H \in X : y \in H\}) \geq j-1$.*

PROOF. We may assume $j > 1$. Let $X_0 = \{H \in X : y \in H\}$ and $X_1 = \{H \in X : y \notin H\}$. Then either $\nu(X_0) \geq j-1 \geq 1$ or $\nu(X_1) \geq j-1 \geq 1$. But note that $\nu(X_1) \ngeq 1$ since $y \notin \cup X_1$. $\square$

In this section a *finite sequence* will mean a function defined on a natural number, the *length* of the sequence t, denoted by $|t|$ is simply $\mathrm{dom}(t)$. Moreover, a *tree* will mean a set of finite sequences closed under initial segments. Then for $t, s \in T$ we have $t \subset s$ iff s end-extends t and this partial order is indeed a tree in the usual sense. For a $t \in T$ let us denote by $\mathrm{succ}_T(t)$ the set of immediate successors of t in T.

Now let us define the following forcing notion.

Definition 3.9.6. Let $T \in \mathbb{P}$ iff

(1) T is a nonempty tree,

(2) for every $t \in T$ and $k < |t|$ we have $t(k) \in \mathbb{C}_k$,

(3) for every $t \in T$ we have $\mathrm{succ}_T(t) \neq \emptyset$,

(4) for every $t \in T$ there exists $s \in T$, $s \supset t$ with $|\mathrm{succ}_T(s)| > 1$,

(5) for every $K \in \omega$ the set $\{t \in T : |\mathrm{succ}_T(t)| > 1 \text{ and } \nu(\mathrm{succ}_T(t)) \leq K\}$ is finite.

If $T, T' \in \mathbb{P}$ then define

$$T \leq_{\mathbb{P}} T' \iff T \subset T'.$$

We will usually simply write $\leq$ for $\leq_{\mathbb{P}}$. Clearly, $1_{\mathbb{P}}$ is the set of all finite sequences satisfying (2).

Remark 3.9.7. A $t \in T$ with $|\mathrm{succ}_T(t)| > 1$ is called a *branching node*. For $t \in T$ define $T[t] = \{s \in T : s \subset t \text{ or } s \supset t\}$. It is easy to see that if $t \in T \in \mathbb{P}$ then $T[t] \in \mathbb{P}$ and $T[t] \leq T$.

Remark 3.9.8. Forcing notions of this type are discussed in [72] in great generality. However, in order to keep the section relatively self-contained we also include the rather standard proofs here, but note that most of the techniques below can already be found in [72].

Lemma 3.9.9. $\mathbb{P}$ *is proper.*

PROOF. Let $\mathfrak{M}$ be a countable elementary submodel, and recall that $T \in \mathbb{P}$ is $\mathfrak{M}$*-generic* if for every dense open subset $D \subset \mathbb{P}$ with $D \in \mathfrak{M}$ we have $T \Vdash$ " $\dot{G} \cap D \cap \mathfrak{M} \neq \emptyset$ ", where $\dot{G}$ is a name for the generic filter. Also recall that properness means that whenever a condition $T \in \mathfrak{M}$ is given then there exists an $\mathfrak{M}$-generic $T' \leq T$. We construct this T' by a so called fusion argument.

Let the sequence $D_0, D_1, \ldots$ enumerate the dense open subsets of $\mathbb{P}$ that are in $\mathfrak{M}$. During the construction we make sure that all objects we pick (t, s, t'_s, s', L_k, L_k^+ S_s, etc.) are in $\mathfrak{M}$. The whole construction, and hence T', will typically not be in $\mathfrak{M}$.

We define the set of branching notes of T' 'level-by level' as follows. Let $t \in T$ be a branching node with $\nu(\mathrm{succ}_T(t)) > 0$ and set $L_0 = \{t\}$. Also define $L_0^+ = \mathrm{succ}_T(t)$. Moreover, for every $s \in L_0^+$ also fix a $S_s \leq T[s]$ with $S_s \in D_0$ (this is possible, since D_0 is dense). This finishes the 0th step of the fusion.

Now, if L_k, L_k^+, and for every $s \in L_k^+$ a condition $S_s \leq T[s]$ have already been defined then for every $s \in L_k^+$ we pick a $t'_s \in S_s$ with $\nu(\mathrm{succ}_{S_s}(t'_s)) > k+1$. Let $L_{k+1} = \{t'_s : s \in L_k^+\}$, and define $L_{k+1}^+ = \bigcup_{s \in L_k^+} \mathrm{succ}_{S_s}(t'_s)$. Now, for every $s' \in L_{k+1}^+$ pick a $S_{s'} \leq S_s[s']$ with $S_{s'} \in D_{k+1}$. This finishes the $k+1$st step of the fusion.

Finally, define T' as the closure of $\bigcup_{k \in \omega} L_k$ under initial segments (this is the same as the closure of $\bigcup_{k \in \omega} L_k^+$ under initial segments). It is easy to check that $T' \in \mathbb{P}$ and $T' \leq T$. It remains to show that T' is $\mathfrak{M}$-generic. So let $k \in \omega$ be fixed, and we need to show that $T' \Vdash$ " $\dot{G} \cap D_k \cap \mathfrak{M} \neq \emptyset$ ".

Before the proof let us make three remarks. First, it is easy to see from the construction that if $T'' \leq T'$ then for every $k \in \omega$ there exists $s \in L_k^+ \cap T''$. Second, it can also be seen from the construction that $T'[s] \leq S_s$ for every s. Third, if $S \in D \cap \mathfrak{M}$ then obviously $S \Vdash$ " $\dot{G} \cap D \cap \mathfrak{M} \neq \emptyset$ ", since $S \Vdash$ " $S \in \dot{G} \cap D \cap \mathfrak{M}$ ".

Now we prove $T' \Vdash$ " $\dot{G} \cap D_k \cap \mathfrak{M} \neq \emptyset$ ". We prove this by showing that for every $T'' \leq T'$ there exists $T''' \leq T''$ forcing this. Let $T'' \leq T'$ be given. Then, by the above remark there exists $s \in L_k^+ \cap T''$. Set $T''' = T''[s]$, then clearly $T''' \leq T''$. Finally, $T''' = T''[s] \leq T'[s] \leq S_s \in D_k \cap \mathfrak{M}$, hence $S_s \Vdash$ " $\dot{G} \cap D_k \cap \mathfrak{M} \neq \emptyset$ ", hence T''' forces the same, finishing the proof. $\square$

Lemma 3.9.10. *If* $(H_k)_{k \in \omega} \in \Pi_{k \in \omega} \mathbb{C}_k$ *then* $\mathcal{H}^r(\bigcap_{m \in \omega} \bigcup_{k \geq m} H_k) = 0$.

PROOF. For every $m \in \omega$, using (3.9.1), we have

$$\mathcal{H}^r_\infty \left(\bigcap_{m \in \omega} \bigcup_{k \geq m} H_k \right) \leq \mathcal{H}^r_\infty \left(\bigcup_{k \geq m} H_k \right) \leq \sum_{k \geq m} \mathcal{H}^r_\infty (H_k) \leq$$

$$\leq \sum_{k \geq m} 2^k \left(\frac{\sqrt{n}}{M_k} \right)^r \leq \sum_{k \geq m} \frac{1}{2^k} = \frac{1}{2^{m-1}},$$

hence $\mathcal{H}^r_\infty(\bigcap_{m \in \omega} \bigcup_{k \geq m} H_k) = 0$, therefore $\mathcal{H}^r(\bigcap_{m \in \omega} \bigcup_{k \geq m} H_k) = 0$. $\square$

Remark 3.9.11. In the usual way, by slight abuse of notation, the generic filter G can be thought of as a sequence $G = (G_k)_{k\in\omega} \in \Pi_{k\in\omega}\mathbb{C}_k$. What we will formally need is that if a generic filter G is given, then $\bigcap_{T\in G} T$ defines such a sequence, hence G_k makes sense.

Lemma 3.9.12. *If G is a generic filter over a ground model V then $V[G] \models V \cap [0,1]^n \subset \bigcap_{m\in\omega}\bigcup_{k\geq m} G_k$.*

Proof. Fix $y \in V \cap [0,1]^n$. In order to show that $1_{\mathbb{P}} \Vdash$ " $V \cap [0,1]^n \subset \bigcap_{m\in\omega}\bigcup_{k\geq m} \dot{G}_k$ " we show that for every $T \in \mathbb{P}$ there is $T' \leq T$ forcing this. So let T be given, and define T' as follows. Starting from the root of T, we recursively thin out T such that for every $t \in T$ with $\nu(\mathrm{succ}_T(t)) \geq 1$ we cut off all the nodes $s \in \mathrm{succ}_T(t)$ with $y \notin s(|s|-1)$. One can easily check using Lemma 3.9.5 that $T' \in \mathbb{P}$ and $T' \leq T$. So it suffices to show that for every $m \in \omega$ we have $T' \Vdash$ " $y \in \bigcup_{k\geq m} \dot{G}_k$ ". Hence let $T'' \leq T'$ be given, we need to find $T''' \leq T''$ forcing this. Pick $t \in T''$ with $|t| \geq m$ and $\nu(\mathrm{succ}_{T''}(t)) \geq 1$. This implies that the successors of t were thinned out, hence $y \in s(|s|-1)$ for every $s \in \mathrm{succ}_{T''}(t)$. Fix such an s, and define $T''' = T''[s]$. Then $T''' = T''[s] \Vdash$ " $\dot{G}_{|s|-1} = s(|s|-1) \ni y$ ", finishing the proof. $\square$

Lemma 3.9.13. *$\mathbb{P}$ is ω^ω-bounding.*

Proof. For $f, g \in \omega^\omega$ we write $f \leq g$ if $f(n) \leq g(n)$ for every $n \in \omega$. Let $\dot{f} \in \omega^\omega$ be a name. We claim that $1_{\mathbb{P}} \Vdash$ " $\exists g \in V \cap \omega^\omega$ such that $\dot{f} \leq g$ ". It suffices to show that for every T there exists $T' \leq T$ and $g \in V \cap \omega^\omega$ such that $T' \Vdash$ " $\dot{f} \leq g$ ". We will construct this T' by a fusion argument similar to that of Lemma 3.9.9.

Let $t \in T$ be a node with $\nu(\mathrm{succ}_T(t)) > 0$ and set $L_0 = \{t\}$. Also define $L_0^+ = \mathrm{succ}_T(t)$. Moreover, for every $s \in L_0^+$ also fix a $S_s \leq T[s]$ and $m_s \in \omega$ such that $S_s \Vdash$ " $\dot{f}(0) = m_s$ " (this is possible by the basic properties of forcing). This finishes the 0th step of the fusion.

Now, if L_k, L_k^+, and for every $s \in L_k^+$ a condition $S_s \leq T[s]$ have already been defined then for every $s \in L_k^+$ we pick a $t'_s \in S_s$ with $\nu(\mathrm{succ}_{S_s}(t'_s)) > k+1$. Let $L_{k+1} = \{t'_s : s \in L_k^+\}$, and define $L_{k+1}^+ = \bigcup_{s\in L_k^+} \mathrm{succ}_{S_s}(t'_s)$. Now, for every $s' \in L_{k+1}^+$ pick a $S_{s'} \leq S_s[s']$ and $m_{s'} \in \omega$ with $S_{s'} \Vdash$ " $\dot{f}(k+1) = m_{s'}$ ". This finishes the $k+1$st step of the fusion.

Finally, define T' as the closure of $\bigcup_{k\in\omega} L_k^+$ under initial segments. It is easy to check that $T' \in \mathbb{P}$ and $T' \leq T$. Define $g(k) = \max\{m_s : s \in L_k^+\}$ (the maximum exists, since this set is finite). It remains to show that $T' \Vdash$ " $\dot{f} \leq g$ ". So let $k \in \omega$ be fixed, and let $T'' \leq T'$ be given. Pick $s \in T'' \cap L_k^+$, and define $T''' = T''[s]$. Then $T''' = T''[s] \leq S_s \Vdash$ " $\dot{f}(k) = m_s$ ", hence $T''' \Vdash$ " $\dot{f}(k) \leq g(k)$ ", finishing the proof. $\square$

Theorem 3.9.14. *It is consistent with ZFC that* $\mathrm{cov}(\mathcal{M}) < \mathrm{non}(\mathcal{N}_n^r)$.

Proof. Let V be a model satisfying the Continuum Hypothesis, and let V_{ω_2} be the model obtained by an ω_2-long countable support iteration of $\mathbb{P}$. Let $(V_\alpha)_{\alpha\leq\omega_2}$ denote the intermediate models. Since $\mathbb{P}$ is proper and adds a real, by standard arguments the continuum is ω_2 in V_{ω_2}.

On the one hand, $\mathbb{P}$ is ω^ω-bounding, hence so is its iteration. Therefore the iteration adds no Cohen reals, hence the meagre Borel sets coded in V cover $V_{\omega_2} \cap \mathbb{R}^n$, hence $\mathrm{cov}(\mathcal{M}) = \omega_1$.

On the other hand, if $H \in V_{\omega_2}, |H| = \omega_1$ then, by a standard reflection argument, $H \subset V_\alpha$ for some $\alpha < \omega_2$. Hence, by Lemma 3.9.12 and Lemma 3.9.10 we have $V_{\alpha+1} \models \mathcal{H}^r(H) = 0$. Therefore, since $\mathcal{H}^r(H) = 0$ means the existence of certain covers, and by absoluteness the corresponding covers exist in V_{ω_2}, we obtain $V_{\omega_2} \models \mathcal{H}^r(H) = 0$. Hence, $\mathrm{non}(\mathcal{N}_n^r) = \omega_2$. Therefore the proof is complete. □

3.10 Proof of Theorem 2.2.6

Proof. We will show that there exists a nonempty perfect set $Q \subset P$ and $y \in \mathbb{R}$ such that $Q + y \subset C_{EK}$. This is clearly sufficient, as then for $x = -y$ we get $Q \subset (C_{EK} + x) \cap P$, so this intersection is uncountable. First we will construct Q via a dyadic tree of basic intervals, then we will construct the "digits" of y.

By translating P if necessary we can assume that P intersects $(0, \frac{1}{5!})$. Instead of P we may as well work with any nonempty perfect subset of it, for if we find the set Q inside this subset then this Q also works for P. So we may find a nonempty perfect subset of P in $[0, \frac{1}{5!}]$ and therefore we can assume that P itself is inside $[0, \frac{1}{5!}]$. Moreover, as P is uncountable and the endpoints of the basic intervals form a countable set, we can find a nonempty perfect subset of P that is disjoint from the set of endpoints (we used here twice the well known fact that every uncountable Borel set contains a nonempty perfect set). Therefore we can assume that P itself is disjoint from the endpoints.

Now we recursively pick an increasing sequence of levels $(l_k)_{k=0}^\infty$ and for every k choose a set $\mathcal{I}_{l_k} \subset \mathcal{B}_{l_k}$ of size 2^k such that

1. for each $I \in \mathcal{I}_{l_k}$ there are exactly two intervals in $\mathcal{I}_{l_{k+1}}$ (at level l_{k+1}) that are contained in I, the so called *successors* of I
2. for each $I \in \mathcal{I}_{l_k}$ we have $I \cap P \neq \emptyset$
3. $l_k \geq 2^{k+2} + 1$.

The recursion is carried out as follows. Fix $p_0 \in P$. Let $l_0 = 5$, and at level 5 we pick the (unique) basic interval I containing p_0 in its interior. Let $\mathcal{I}_{l_0} = \mathcal{I}_5 = \{I\}$. The recursion step is as follows. As P is disjoint from the endpoints of the basic intervals, each interval $I \in \mathcal{I}_{l_k}$ (at level l_k) contains some point $p_I \in P$ in its interior by condition 2. As P is nonempty perfect, we can choose a distinct point $p'_I \in P$ in I. We can find a large enough n such that the 2^{k+1} distinct points p_I and p'_I $(I \in \mathcal{I}_{l_k})$ are all separated by $\mathcal{B}_n$. Define

$$l_{k+1} = \max\{n, 2^{k+3} + 1\}.$$

Clearly, condition 3 is also satisfied.

Let $\mathcal{I}_{l_{k+1}}$ be the subcollection of $\mathcal{B}_{l_{k+1}}$ consisting of the 2^{k+1} basic intervals containing all the points p_I and p'_I $(I \in \mathcal{I}_{l_k})$. This recursion clearly provides a system of intervals satisfying the required properties.

Now we can define

$$Q = \bigcap_{k=0}^{\infty} \bigcup \mathcal{I}_{l_k}.$$

Let us extend this tree of intervals to the intermediate levels in the natural way, that is, for every $I \in \mathcal{I}_{l_k}$ and successor $J \in \mathcal{I}_{l_{k+1}}$ and every $n \in (l_k, l_{k+1})$ let us add to the tree the unique basic interval of level n that is contained in I and contains J. For $n = 2, 3, 4, 5$ define $\mathcal{I}_n = \{[0, \frac{1}{n!}]\}$. Hence we get $\mathcal{I}_n$ for every $n = 2, 3, \dots$ so that

$$\bigcap_{n=2}^{\infty} \bigcup \mathcal{I}_n = \bigcap_{k=0}^{\infty} \bigcup \mathcal{I}_{l_k} = Q.$$

Our next goal is to define $y = \sum_{n=2}^{\infty} \frac{y_n}{n!}$ so that $Q + y \subset C_{EK}$. Define $y_2 = y_3 = y_4 = y_5 = 0$. For every $n \geq 6$ there exists k such that $l_k < n \leq l_{k+1}$. Clearly, the size of $\mathcal{I}_n$ is 2^{k+1}, and $Q \subset \cup \mathcal{I}_n$. This means that there are at most 2^{k+1} possible values for q_n, where $q \in Q$ and $q = \sum_{n=2}^{\infty} \frac{q_n}{n!}$ (we do not have to worry about nonunique expansions, as $Q \subset P$ so Q is disjoint from the endpoints of the basic intervals). For every such q_n there are at most two values of m such that $q_n + m \in \{n-2, n-1\}$. Hence altogether there are at most $2 \cdot 2^{k+1}$ such "bad" values, so if $n - 1 > 2 \cdot 2^{k+1}$ then we can fix a $y_n \in \{0, 1, \dots, n-2\}$ such that $q_n + y_n \notin \{n-2, n-1\}$ for every possible q_n. But our requirement on n and k, namely $n - 1 > 2 \cdot 2^{k+1}$, is clearly satisfied as $n > l_k \geq 2^{k+2} + 1$ by condition 3.

So we can define

$$y = \sum_{n=2}^{\infty} \frac{y_n}{n!}$$

so that for every $n \geq 6$ we have $y_n \in \{0, 1, \dots, n-2\}$ and that for every $q \in Q$ with $q = \sum_{n=2}^{\infty} \frac{q_n}{n!}$ we have $q_n + y_n \notin \{n-2, n-1\}$. We claim that $Q + y \subset C_{EK}$, which will complete the proof. Fix $q \in Q$ with $q = \sum_{n=2}^{\infty} \frac{q_n}{n!}$, then

$$q + y = \sum_{n=2}^{\infty} \frac{q_n + y_n}{n!} = \sum_{n=2}^{\infty} \frac{q_n + y_n - n\varepsilon_n}{n!} + \sum_{n=2}^{\infty} \frac{n\varepsilon_n}{n!},$$

where ε_n is the "carried digit", so

$$\varepsilon_n = \begin{cases} 0 & \text{if } q_n + y_n \leq n-1 \\ 1 & \text{otherwise.} \end{cases}$$

Continuing the above calculation we get

$$q + y = \sum_{n=2}^{\infty} \frac{q_n + y_n - n\varepsilon_n}{n!} + \sum_{n=2}^{\infty} \frac{\varepsilon_n}{(n-1)!} = \sum_{n=2}^{\infty} \frac{q_n + y_n - n\varepsilon_n}{n!} + \sum_{n=1}^{\infty} \frac{\varepsilon_{n+1}}{n!} =$$

$$= \varepsilon_2 + \sum_{n=2}^{\infty} \frac{q_n + y_n - n\varepsilon_n + \varepsilon_{n+1}}{n!} = \sum_{n=2}^{\infty} \frac{q_n + y_n - n\varepsilon_n + \varepsilon_{n+1}}{n!},$$

since $\varepsilon_2 = 0$ by e.g. $y_2 = 0$. We now check that for every $n \geq 2$ the numerator $q_n + y_n - n\varepsilon_n + \varepsilon_{n+1} \in \{0, 1, \dots, n-2\}$, which shows that $q + y \in C_{EK}$. For $n < 6$ this is clear, as $y_n = 0$ and also $q_n = 0$ by the assumption $P \subset [0, \frac{1}{5!}]$.

For $n \geq 6$ recall that $q_n \leq n-1$, $y_n \leq n-2$, so $q_n + y_n \leq 2n-3$ and also that $q_n + y_n \notin \{n-2, n-1\}$. We separate the cases $\varepsilon_n = 0$ and $\varepsilon_n = 1$. If $\varepsilon_n = 0$, then $q_n + y_n \leq n-1$, but then also $q_n + y_n \leq n-3$. Therefore $q_n + y_n - n\varepsilon_n + \varepsilon_{n+1} = q_n + y_n + \varepsilon_{n+1} \leq n-2$, and we are done. On the other hand, if $\varepsilon_n = 1$, then $q_n + y_n - n\varepsilon_n \leq n-3$, so $q_n + y_n - n\varepsilon_n + \varepsilon_{n+1} \leq n-2$, so this case is also done. This completes the proof. $\square$

3.11 Proof of Theorem 2.2.7

Proof. It is clearly sufficient to cover the unit interval.

Fix $f : \omega \setminus \{0,1\} \to \omega \setminus \{0\}$. A set of the form $S = \Pi_{n=2}^{\infty} A_n$, where $A_n \subset \{0, 1, \ldots, n-1\}$ is of cardinality at most $f(n)$ for every n is called an *f-slalom.* Suppose $\lim_{n\to\infty} f(n) = \infty$, $f(2) = f(3) = f(4) = f(5) = 1$ and also that $f(n) < \frac{n-1}{2}$ for every $n \geq 6$. Combining [5, Thm 2.3.9] and [41, Thm 2.10] we obtain a cover of $\Pi_{n=2}^{\infty}\{0, 1, \ldots, n-1\}$ by $\operatorname{cof}(\mathcal{N})$ many f-slaloms $\{S_\alpha \mid \alpha < \operatorname{cof}(\mathcal{N})\}$. (We actually obtain a cover of ω^ω first, which is a larger space, so it is trivial to restrict this cover to get a cover of $\Pi_{n=2}^{\infty}\{0, 1, \ldots, n-1\}$. Moreover, [5] works with inclusion mod finite, but that makes no difference, as we can replace each slalom by countably many slaloms to get around this difficulty.) For a slalom S define

$$S^* = \left\{ \sum_{n=2}^{\infty} \frac{s_n}{n!} \,\middle|\, (s_n)_{n=2}^{\infty} \in S \right\}.$$

Clearly $\{S_\alpha^* \mid \alpha < \operatorname{cof}(\mathcal{N})\}$ covers the unit interval. The following lemma will complete the proof of the theorem.

Lemma 3.11.1. *Let f be as above and S be an f-slalom. Than there exists $y \in \mathbb{R}$ such that $S^* + y \subset C_{EK}$.*

Proof. The proof is based on the ideas used in Theorem 2.2.6. S^* plays the role of Q. Our goal is to define

$$y = \sum_{n=2}^{\infty} \frac{y_n}{n!}$$

with $y_n \in \{0, 1, \ldots, n-2\}$ so that for every $(s_n)_{n=2}^{\infty} \in S$ and for every $n \geq 6$ we have $s_n + y_n \notin \{n-2, n-1\}$. But this is clearly possible by our assumptions on f, as there are at most $f(n) < \frac{n-1}{2}$ possibilities for s_n, hence there are two consecutive values excluded, and so we can find a suitable $y_n \in \{0, 1, \ldots, n-2\}$.

Then by the same calculation as in the last part of the proof of Theorem 2.2.6 we check that

$$\sum_{n=2}^{\infty} \frac{s_n + y_n}{n!} \in C_{EK}.$$

This completes the proof of the lemma. $\square$

Hence for every $\alpha < \operatorname{cof}(\mathcal{N})$ there exists y_α such that $S_\alpha^* + y_\alpha \subset C_{EK}$, but than for $x_\alpha = -y_\alpha$ we have $S_\alpha^* \subset C_{EK} + x_\alpha$, so we obtain a cover of the unit interval by $\operatorname{cof}(\mathcal{N})$ many translates of C_{EK} and therefore the proof of the theorem is also complete. $\square$

3.12 Proof of Theorem 2.3.3

First we present some lemmas we use in the sequel. Probably most of them are well-known, however, we could not find suitable references, so we could not avoid including them. Let λ denote one-dimensional Lebesgue measure, then $\mathcal{M}_\lambda$ is the class of Lebesgue measurable sets. A *Borel isomorphism* is a bijection such that images and preimages of Borel sets are Borel.

Lemma 3.12.1. *Let $d > 0$, $B \subset \mathbb{R}^n$ be Borel such that $0 < \mathcal{H}^d(B) < \infty$ and let $I = [0, \mathcal{H}^d(B)]$. Then there exists an isomorphism f between the measure spaces $(B, \mathcal{M}_{\mathcal{H}^d}, \mathcal{H}^d)$ and $(I, \mathcal{M}_\lambda, \lambda)$ that is also a Borel isomorphism.*

Proof. We may clearly assume $\mathcal{H}^d(B) = 1$. It is stated in [51, 12.B] that every Borel set, hence B is a standard Borel space (that is, B is Borel isomorphic to some Polish space). Hence we can apply [51, 17.41] which states that every continuous (that is, singletons are of measure zero) probability measure on a standard Borel space is isomorphic by a Borel isomorphism to Lebesgue measure on the unit interval. $\square$

Lemma 3.12.2. *Let $B \subset \mathbb{R}^n$ be a Borel set of infinite but σ-finite $\mathcal{H}^d$-measure. Then there exists an isomorphism f between the measure spaces $(B, \mathcal{M}_{\mathcal{H}^d}, \mathcal{H}^d)$ and $([0,\infty), \mathcal{M}_\lambda, \lambda)$ that is also a Borel isomorphism.*

Proof. Define a system $\mathcal{A}$ of pairwise disjoint Borel subsets of B by transfinite recursion, such that $\mathcal{H}^d(A) = 1$ for every $A \in \mathcal{A}$. By σ-finiteness this procedure stops at some countable ordinal, so $\mathcal{A}$ is countable. Put $m = \mathcal{H}^d(B \setminus \cup\mathcal{A})$. As we can always find a Borel subset of $\mathcal{H}^d$-measure 1 inside a Borel set of $\mathcal{H}^d$-measure at least 1 (see e.g. [67, 8.20]), we obtain $m < 1$. In particular, $\mathcal{A}$ is infinite, say $\mathcal{A} = \{A_0, A_1, \dots\}$. By the previous lemma $B \setminus \cup_{i=0}^\infty A_i$ is isomorphic to the interval $[-m, 0)$ (or more precisely to $[-m, 0]$, but these two intervals are isomorphic again by the previous lemma), and A_i is isomorphic to $[i, i+1)$ for every i. Therefore B is clearly isomorphic to $[-m, \infty)$ which is obviously isomorphic to $[0, \infty)$. $\square$

Lemma 3.12.3. *Let $n \in \mathbb{N}$, $d \geq 0$ and $H \subset \mathbb{R}^n$ be arbitrary. Then the following statements are equivalent:*

(i) *H is $\mathcal{H}^d$-measurable,*

(ii) *$\mathcal{H}^d(B) = \mathcal{H}^d(B \cap H) + \mathcal{H}^d(B \cap H^C)$ for every Borel set $B \subset \mathbb{R}^n$ with $0 < \mathcal{H}^d(B) < \infty$,*

(iii) *$H \cap B$ is $\mathcal{H}^d$-measurable for every Borel set $B \subset \mathbb{R}^n$ with $0 < \mathcal{H}^d(B) < \infty$,*

(iv) *For every Borel set $B \subset \mathbb{R}^n$ with $0 < \mathcal{H}^d(B) < \infty$, we have $H \cap B = A \cup N$, where A is Borel and N is $\mathcal{H}^d$-negligible.*

Proof. (i) $\iff$ (ii) By definition, the set H is $\mathcal{H}^d$-measurable in the sense of Carathéodory if and only if $\mathcal{H}^d(X) = \mathcal{H}^d(X \cap H) + \mathcal{H}^d(X \cap H^C)$ for every $X \subset \mathbb{R}^n$. As outer measures are subadditive, this is equivalent to $\mathcal{H}^d(X) \geq \mathcal{H}^d(X \cap H) + \mathcal{H}^d(X \cap H^C)$ for every $X \subset \mathbb{R}^n$. Once this inequality fails to hold, using the Borel regularity of Hausdorff measures (see e.g. [67, 4.5]), there is a

Borel set $B \supset X$ such that $\mathcal{H}^d(B) < \mathcal{H}^d(X \cap H) + \mathcal{H}^d(X \cap H^C)$. Therefore B is of finite measure, moreover, $\mathcal{H}^d(B) < \mathcal{H}^d(B \cap H) + \mathcal{H}^d(B \cap H^C)$, in particular, B is clearly of positive measure.

(ii) $\iff$ (iii) The third condition obviously implies the second one as $\mathcal{H}^d$ is additive on measurable sets. So suppose (ii). It is enough to show that $\mathcal{H}^d(A) \geq \mathcal{H}^d(A \cap (H \cap B)) + \mathcal{H}^d(A \cap (H \cap B)^C)$ for every Borel set $A \subset \mathbb{R}^n$ with $0 < \mathcal{H}^d(A) < \infty$. We can assume that $\mathcal{H}^d(A \cap B) > 0$, otherwise the above inequality clearly holds, as the first term on the right hand side is 0, while the second term is not greater than the left hand side. Now

$$\mathcal{H}^d(A) \geq \mathcal{H}^d(A \cap B) + \mathcal{H}^d(A \cap B^C) \geq$$

$$\mathcal{H}^d((A \cap B) \cap H) + \mathcal{H}^d((A \cap B) \cap H^C) + \mathcal{H}^d(A \cap B^C) \geq$$

$$\mathcal{H}^d(A \cap (H \cap B)) + \mathcal{H}^d(A \cap (H \cap B)^C),$$

where the first inequality holds as B is measurable, the second one is (ii) applied to $A \cap B$, and the third one follows from the subadditivity of $\mathcal{H}^d$ since $A \cap (H \cap B)^C$ is the (disjoint) union of $(A \cap B) \cap H^C$ and $A \cap B^C$.

(iii) $\iff$ (iv) As negligible sets are measurable, the fourth condition implies the third, while we immediately obtain the other direction if we apply Lemma 3.12.1 to B. $\square$

Remark 3.12.4. We could also assume that the above B is compact, but we will not need this fact.

The following lemma is essentially [36, 2.5.10]. Recall that CH abbreviates the Continuum Hypothesis.

Lemma 3.12.5. *Let $0 < d < n$ and suppose CH holds. Then there exists a disjoint family $\{B_\alpha : \alpha < \omega_1\}$ of Borel subsets of $\mathbb{R}^n$ of finite $\mathcal{H}^d$-measure, such that a set $H \subset \mathbb{R}^n$ is $\mathcal{H}^d$-measurable iff $H \cap B_\alpha$ is $\mathcal{H}^d$-measurable for every $\alpha < \omega_1$.*

PROOF. Let $\{A_\alpha : \alpha < \omega_1\}$ be an enumeration of the Borel subsets of $\mathbb{R}^n$ of positive finite $\mathcal{H}^d$-measure, and put $B_\alpha = A_\alpha \setminus (\cup_{\beta<\alpha} A_\beta)$. These are clearly pairwise disjoint Borel sets of finite $\mathcal{H}^d$-measure. The other direction being trivial we only have to verify that if $H \subset \mathbb{R}^n$ is such that $H \cap B_\alpha$ is $\mathcal{H}^d$-measurable for every $\alpha < \omega_1$, then H itself is $\mathcal{H}^d$-measurable. By Lemma 3.12.3 we only have to show that $H \cap A_\alpha$ is $\mathcal{H}^d$-measurable for every $\alpha < \omega_1$. But $A_\alpha = \cup_{\beta \leq \alpha} B_\beta$, therefore $H \cap A_\alpha = \cup_{\beta \leq \alpha}(H \cap B_\alpha)$, which is clearly $\mathcal{H}^d$-measurable, which completes the proof. $\square$

Lemma 3.12.6. *Let $0 < d < n$ and suppose CH holds. Then there exists a disjoint family $\{S_\alpha : \alpha < \omega_1\}$ of Borel subsets of $\mathbb{R}^n$ of infinite but σ-finite $\mathcal{H}^d$-measure, such that a set $H \subset \mathbb{R}^n$ is $\mathcal{H}^d$-measurable iff $H \cap S_\alpha$ is $\mathcal{H}^d$-measurable for every $\alpha < \omega_1$.*

PROOF. First we check that uncountably many B_α of Lemma 3.12.5 are of positive $\mathcal{H}^d$-measure. Otherwise, as $\mathbb{R}^n$ is not σ-finite and as by [67, 8.20] every Borel set of infinite $\mathcal{H}^d$-measure contains a Borel set of $\mathcal{H}^d$-measure 1, we could find a Borel set of positive and finite measure that is disjoint from these B_α.

But this set was enumerated as A_α for some α, moreover $A_\alpha = \cup_{\beta\leq\alpha} B_\beta$. Since A_α is disjoint from all B_β of positive measure, by CH, countably many zerosets cover A_α, so it is a zeroset, a contradiction.

This obviously implies that for some integer N uncountably many B_α are of measure at least $\frac{1}{N}$. Now we can recursively define a partition of the set of countable ordinals into countable intervals $\{I_\alpha : \alpha < \omega_1\}$ such that for every α the $\mathcal{H}^d$-measure of $\cup_{\xi\in I_\alpha} B_\xi$ is infinite. On the other hand, this measure is clearly σ-finite. Now we check that $S_\alpha = \cup_{\xi\in I_\alpha} B_\xi$ works. Every S_α is clearly a Borel subsets of $\mathbb{R}^n$ of infinite but σ-finite $\mathcal{H}^d$-measure. Now we have to show that the last statement of the lemma holds, namely that $\mathcal{H}^d$-measurability "reflects" on the sets S_α. One direction is trivial, so in order to prove the other one let us assume that $H \cap S_\alpha$ is $\mathcal{H}^d$-measurable for every $\alpha < \omega_1$. We have to show that H is $\mathcal{H}^d$-measurable. But this is obvious by the previous lemma, as every B_α can be covered by some S_β, hence $H \cap B_\alpha = B_\alpha \cap (H \cap S_\beta)$ which is measurable. $\square$

Lemma 3.12.7. *Assume CH. A set $H \subset \mathbb{R}^n$ is of σ-finite $\mathcal{H}^d$-measure iff it can be covered by countably many of the above S_α.*

Proof. One direction is trivial. For the other one we can assume that $H \subset \mathbb{R}^n$ is of finite $\mathcal{H}^d$-measure. There exists a Borel set B of positive finite $\mathcal{H}^d$-measure containing H. This set was enumerated in Lemma 3.12.5 as A_α for some α, so by CH it can be covered by countably many B_α hence also by countably many S_α. $\square$

Now we are able to complete the proof of Theorem 2.3.3.

Proof. Find two partitions $\{S_\alpha^{d_1} : \alpha < \omega_1\}$ and $\{S_\alpha^{d_2} : \alpha < \omega_1\}$ of $\mathbb{R}^n$ as in Lemma 3.12.6. By Lemma 3.12.2 find isomorphisms $f_\alpha : S_\alpha^{d_1} \to S_\alpha^{d_2}$ for every α. Define $f = \cup_{\alpha<\omega_1} f_\alpha$. We have to check that f is an isomorphism. First we prove that f preserves measurable sets. Suppose $H \in \mathcal{M}_{\mathcal{H}^{d_1}}$. By Lemma 3.12.6 it is sufficient to show that $f(H) \cap S_\alpha^{d_2} \in \mathcal{M}_{\mathcal{H}^{d_2}}$ for every α. But $f(H) \cap S_\alpha^{d_2} = f_\alpha(H \cap S_\alpha^{d_1})$ which is $\mathcal{H}^{d_2}$-measurable as f_α is an isomorphism. Similarly, f^{-1} also preserves measurable sets. Now we show that f preserves measure. (Again, the same argument works for f^{-1}.) As mentioned in the Introduction it is enough to show this for measurable sets. First, f preserves non-σ-finiteness since it clearly preserves the property that characterises σ-finiteness in Lemma 3.12.7, so we can restrict ourselves to σ-finite sets. But such a set is partitioned by the countably many $S_\alpha^{d_1}$ that cover it, and the countably many isomorphisms f_α preserve measure, so the proof is complete. $\square$

Remark 3.12.8. If $d = 0$ then all subsets of $\mathbb{R}^n$ are measurable, while if $d = n$ then $\mathcal{H}^d$ is σ-finite, therefore the theorem cannot be extended to these cases.

As we already mentioned in the Introduction, it is unknown whether CH can be dropped from the theorem, but the paper [73] is a huge step in this direction.

3.13 Proof of Theorem 2.4.2

Proof. Denote by $\mathcal{C}[0,1]$ the Banach space of continuous real-valued functions defined on $[0,1]$. We say that $g \in \mathcal{C}[0,1]$ is Hölder continuous of exponent α

and constant K if $|g(x)-g(y)| \leq K|x-y|^\alpha$ for every $x, y \in [0,1]$. Length of an interval I is denoted by $|I|$.

First we show that it is sufficient to prove that for all integers $N, M > 0$ the set

$$D(N,M) = \{f \in \mathcal{C}[0,1] : \mathcal{H}_\infty^{1-\alpha+\frac{1}{N}}(\{f=g\}) < \frac{1}{M}$$

$$\text{for every Hölder function } g \text{ of exponent } \alpha \text{ and constant } 1\}$$

contains a dense open subset of $\mathcal{C}[0,1]$. Indeed, this implies that the set $D = \cap_{N=1}^\infty \cap_{M=1}^\infty D(N,M)$ contains a dense G_δ set, hence is residual, which in turn implies that the typical continuous function does not agree with a Hölder continuous function of exponent α and constant 1 on any set of Hausdorff dimension larger than $1-\alpha$. Moreover, the map $f \mapsto Kf$ is a homeomorphism of $\mathcal{C}[0,1]$, hence $KD = \{Kf : f \in D\}$ is residual for every K. Therefore $\cap_{K=1}^\infty KD$ is also residual, and so the typical continuous function does not agree with a Hölder continuous function of exponent α on any set of Hausdorff dimension larger than $1-\alpha$.

Now we show that we can assume $0 < \alpha < 1$. Indeed, if the statement of the theorem holds for $1-\frac{1}{L}$ for every L, then intersecting the corresponding sequence of residual subsets of $\mathcal{C}[0,1]$ we obtain the case $\alpha = 1$.

Now what remains to be proven is that $D(N,M)$ contains a dense open set. Let $f_0 \in \mathcal{C}[0,1]$ and $r_0 > 0$ be given. The closed ball centred at f_0 and of radius r_0 is denoted by $B(f_0, r_0)$. We have to find $f_1 \in \mathcal{C}[0,1]$ and $r_1 > 0$ such that $B(f_1, r_1) \subset B(f_0, r_0) \cap D(N,M)$. By uniform continuity of f_0, for a large enough integer m, the inequality $|x-y| < \frac{2}{m}$ implies $|f_0(x) - f_0(y)| < \frac{r_0}{5}$. The exact value of m will be chosen later. Let k be another positive integer to be fixed later. (For those who like to see the explicit choice of the constants in advance, k and m will be chosen so that, in addition to the above requirement concerning uniform continuity, the inequalities (3.13.3) and (3.13.4) below are satisfied. The straightforward calculation that this choice is indeed possible can be found in the last paragraph of the proof.)

Now we define the piecewise constant function $\bar{f}_1$ as follows. For a pair of integers $0 \leq i < m$ and $0 \leq j < k$ set

$$\bar{f}_1(x) = f_0(\frac{i}{m}) + j\frac{r_0}{5k} \qquad \text{for} \qquad \frac{i}{m} + \frac{j}{mk} \leq x < \frac{i}{m} + \frac{j+1}{mk}.$$

Put $\bar{f}_1(1) = f_0(1)$. Now choose a finite system $\mathcal{I}$ of pairwise disjoint open intervals covering 1 and all numbers of the form $\frac{i}{m} + \frac{j}{mk}$ such that

$$\sum_{I \in \mathcal{I}} |I|^{1-\alpha+\frac{1}{N}} < \frac{1}{2M}. \tag{3.13.1}$$

We can clearly choose a continuous $f_1 \in \mathcal{C}[0,1]$ which is linear on each $I \in \mathcal{I}$, agrees with $\bar{f}_1$ outside $\cup\mathcal{I}$ and satisfies $f_1(0) = f_0(0)$ and $f_1(1) = f_0(1)$. Denote the supremum-norm of a not necessarily continuous function f by $||f||$. One can easily check that $||f_0 - \bar{f}_1|| \leq \frac{2}{5}r_0$ and that $||\bar{f}_1 - f_1|| \leq \frac{2}{5}r_0$, hence $||f_0 - f_1|| \leq \frac{4}{5}r_0$. So if we put $r_1 = \frac{r_0}{20k} \leq \frac{r_0}{5}$ then $B(f_1, r_1) \subset B(f_0, r_0)$. Now we claim that the inequality

$$\left(\frac{2}{mk}\right)^\alpha < \frac{r_0}{10k} \tag{3.13.2}$$

implies that for every $f \in B(f_1, r_1)$, every Hölder continuous function g of exponent α and constant 1 and every fixed $0 \le i < m$, there is at most one $0 \le j < k$ such that $\left(\{f = g\} \cap \left(\frac{i}{m} + \frac{j}{mk}, \frac{i}{m} + \frac{j+1}{mk}\right)\right) \setminus \cup\mathcal{I} \neq \emptyset$. Indeed, by the concavity of the function x^α it is enough to check that for this fixed i the graph of f and g cannot meet over two consecutive intervals of the k intervals of length $\frac{1}{mk}$, but this is clear from the fact that the value of $\bar{f}_1$ 'jumps' by $\frac{r_0}{5k}$, moreover $\frac{r_0}{5k} - 2r_1 = \frac{r_0}{10k}$, and from (3.13.2).

This means that $\{f = g\}$ can be covered by the elements of $\mathcal{I}$ and by m intervals of length $\frac{1}{mk}$. Before we use this fact to estimate $\mathcal{H}_\infty^{1-\alpha+\frac{1}{N}}(\{f = g\})$, we need some more preparations.

By rearranging (3.13.2) we obtain

$$k < \left(\frac{r_0}{10 \cdot 2^\alpha}\right)^{\frac{1}{1-\alpha}} m^{\frac{\alpha}{1-\alpha}},$$

and we will also make sure that even the following holds.

$$\frac{1}{2}\left(\frac{r_0}{10 \cdot 2^\alpha}\right)^{\frac{1}{1-\alpha}} m^{\frac{\alpha}{1-\alpha}} < k < \left(\frac{r_0}{10 \cdot 2^\alpha}\right)^{\frac{1}{1-\alpha}} m^{\frac{\alpha}{1-\alpha}} \tag{3.13.3}$$

The last condition we need for the estimation of the Hausdorff capacity is that

$$m\left(\frac{1}{mk}\right)^{1-\alpha+\frac{1}{N}} < \frac{1}{2M}. \tag{3.13.4}$$

Indeed, together with (3.13.1) this implies that $\mathcal{H}_\infty^{1-\alpha+\frac{1}{N}}(\{f = g\}) < \frac{1}{M}$ for every $f \in B(f_1, r_1)$ and every Hölder continuous function g of exponent α and constant 1. Using the left hand side of (3.13.3) we obtain

$$m\left(\frac{1}{mk}\right)^{1-\alpha+\frac{1}{N}} < \left(\frac{1}{2}\left(\frac{r_0}{10 \cdot 2^\alpha}\right)^{\frac{1}{1-\alpha}}\right)^{-(1-\alpha+\frac{1}{N})} m^{1-(1-\alpha+\frac{1}{N})-\frac{\alpha(1-\alpha+\frac{1}{N})}{1-\alpha}}, \tag{3.13.5}$$

and as the exponent

$$1 - (1 - \alpha + \frac{1}{N}) - \frac{\alpha(1 - \alpha + \frac{1}{N})}{1 - \alpha} = -\frac{1}{(1-\alpha)N} < 0,$$

we can choose a large enough m such that (3.13.4) holds, and then we can fix k according to (3.13.3), so the proof is complete. □

3.14 Proof of Theorem 2.4.3

PROOF. The proof is similar to that of Theorem 2.4.2. As usual, total variation of a function g is denoted by

$$Var(g) = \sup\left\{\sum_{i=1}^{n} |g(x_i) - g(x_{i-1})| : n \in \mathbb{N},\ 0 = x_0 < x_1 < \cdots < x_n = 1\right\}.$$

It is sufficient to prove that for all integers $N, M > 0$ the set

$$\left\{f \in \mathcal{C}[0,1] : \mathcal{H}_\infty^{\frac{1}{2}+\frac{1}{N}}(\{f = g\}) < \frac{1}{M} \text{ for every } g \text{ with } Var(g) \le 1\right\} \tag{3.14.1}$$

contains a dense open set. Given f_0 and r_0 define $\bar{f}_1$ as in Theorem 2.4.2 with m and k yet unspecified. (In fact, we will choose $m = k$ so that $|x-y| < \frac{2}{m}$ implies $|f_0(x) - f_0(y)| < \frac{r_0}{5}$ and so that the right hand side of (3.14.3) below is smaller than $\frac{1}{2M}$.)

Choose $\mathcal{I}$ such that

$$\sum_{I \in \mathcal{I}} |I|^{\frac{1}{2}+\frac{1}{N}} < \frac{1}{2M}, \tag{3.14.2}$$

and define f_1 and r_1 as before.

Now for every $f \in B(f_1, r_1)$, g as in (3.14.1), and fixed $0 \le i < m$, denote l_i the number of intervals over which the graph of f and g meet outside $\cup\mathcal{I}$; that is,

$$l_i = \#\left\{ j : \left(\{f = g\} \cap \left(\frac{i}{m} + \frac{j}{mk}, \frac{i}{m} + \frac{j+1}{mk} \right) \right) \setminus \cup\mathcal{I} \neq \emptyset \right\}.$$

As the 'jumps' of $\bar{f}_1$ are of height $\frac{r_0}{5k}$, moreover $\frac{r_0}{5k} - 2r_1 = \frac{r_0}{10k}$, it is easily seen that $Var(g) \ge \sum_{i=1}^{m-1} \left(\frac{r_0}{10k}(l_i - 1) \right) = \frac{r_0}{10k} \left(\sum_{i=1}^{m-1} l_i - m \right)$. Thus $Var(g) \le 1$ implies that $\sum_{i=1}^{m-1} l_i$; that is, the number intervals of length $\frac{1}{mk}$ needed to cover $\{f = g\} \setminus \cup\mathcal{I}$ is at most $m + \frac{10k}{r_0}$.

Choose $k = m$. Then $\mathcal{H}_\infty^{\frac{1}{2}+\frac{1}{N}}(\{f = g\})$ can be estimated by (3.14.2) and

$$\left(m + \frac{10m}{r_0} \right) \left(\frac{1}{m^2} \right)^{\frac{1}{2}+\frac{1}{N}} = \left(\frac{10}{r_0} + 1 \right) m^{-\frac{2}{N}}, \tag{3.14.3}$$

which is smaller than $\frac{1}{2M}$ if m is large enough. □

3.15 Proof of Theorem 2.5.4

It is well-known and easy to check that L is of Lebesgue measure zero, dense, G_δ and periodic mod $\mathbb{Q}$ (that is, $L + q = L$ for every $q \in \mathbb{Q}$). Hence our theorem is a corollary to the following.

Theorem 3.15.1. *Let $B \subset \mathbb{R}$ be a nonempty G_δ set of Lebesgue measure zero. Assume that $\{t \in \mathbb{R} : B + t \subset B\}$ is dense in $\mathbb{R}$. Then B is immeasurable.*

The proof of Theorem 2.5.4 will be based on two lemmas. The first one is reminiscent of [70], where similar results are proved for finite measures using more complicated methods.

Lemma 3.15.2. *Let B be a Borel set of Lebesgue measure zero and μ a Borel measure on $\mathbb{R}$ for which B is positive and σ-finite. Then*

(i) $\mu(B \cap (B + t)) = 0$ *for* λ*-a.e.* t,

(ii) there exists a Borel set $B' \subset B$ *with* $\mu(B') > 0$ *and* $\operatorname{int}(B' - B') = \emptyset$,

(iii) there exists a compact set $C \subset B$ *with* $\mu(C) > 0$ *and* $\operatorname{int}(C - C) = \emptyset$

Lemma 3.15.3. *Let B be a dense G_δ set such that $\{t \in \mathbb{R} : B + t \subset B\}$ is dense in $\mathbb{R}$, and $C \subset B$ be a compact set with $\operatorname{int}(C - C) = \emptyset$. Then there are uncountably many (in fact, continuum many) disjoint translates of C inside B.*

It is easy to see that applying Lemma 3.15.3 to C of Lemma 3.15.2 *(iii)* yields Theorem 2.5.4.

In the rest of the section we prove the two lemmas.

PROOF. (Lemma 3.15.2) *(i)* Let $\mu : \mathcal{S} \to [0, \infty]$ be the given measure, where $\mathcal{S}$ is a σ-algebra of subsets of $\mathbb{R}$ containing all Borel sets. Define a new measure μ_B by

$$\mu_B(S) = \mu(B \cap S) \text{ for every } S \in \mathcal{S}.$$

μ_B is clearly a σ-finite Borel measure on $\mathbb{R}$. Define

$$\widetilde{B} = \{(x, y) \in \mathbb{R}^2 : x + y \in B\}.$$

This is clearly a Borel set, hence $\mu_B \times \lambda$-measurable. As both measures are σ-finite, we can apply the Fubini theorem to $\widetilde{B}$. Vertical sections of $\widetilde{B}$ are of the form

$$\{y \in \mathbb{R} : x + y \in B\} = \{y \in \mathbb{R} : y \in B - x\} = B - x,$$

therefore are all of Lebesgue measure zero. By Fubini, $[\mu_B \times \lambda](\widetilde{B}) = 0$, and so λ-a.e. horizontal section of $\widetilde{B}$ is of μ_B-measure zero. A horizontal section is $\{x \in \mathbb{R} : x + y \in B\} = B - y$, therefore for λ-a.e. y we obtain $0 = \mu_B(B - y) = \mu(B \cap (B - y))$. Replacing y by $-t$ yields the result.

(ii) By *(i)* we can choose a countable dense set $D \subset \mathbb{R}$ such that $\mu(B \cap (B + d)) = 0$ for every $d \in D$. Define

$$B' = B \setminus \bigcup_{d \in D} (B + d).$$

It is easy to check that $\mu(B') = \mu(B) > 0$ and $D \cap (B' - B') = \emptyset$, so the proof of *(ii)* is complete.

(iii) It is sufficient to find a compact set $C \subset B'$ of positive μ-measure. Since $B' \subset B$, B' is σ-finite for μ. Let $B' = \cup_{n=0}^{\infty} S_n$, where $S_n \in \mathcal{S}$ and $\mu(S_n) < \infty$. Since $\mu(B') > 0$, there exists $S = S_n \subset B'$ such that $0 < \mu(S) < \infty$. Define

$$\mu_S(A) = \mu(S \cap A) \text{ for every Borel set } A \subset \mathbb{R}.$$

Note that in contrast with the above μ_B, this time the measure is defined only for Borel sets.

μ_S is clearly a finite measure on the Borel sets, hence inner regular w.r.t. compact sets [51, Thm 17.11]. Apply this to B', a Borel set with $\mu_S(B') = \mu(S \cap B') = \mu(S) > 0$, and obtain a compact set $C \subset B'$ with $\mu_S(C) = \mu(S \cap C) > 0$. So $\mu(C) > 0$ follows, and the proof of Lemma 3.15.2 is complete. $\square$

PROOF. (Lemma 3.15.3) Let

$$T = \{t \in \mathbb{R} : C + t \subset B\}.$$

B is G_δ, so there are open sets U_n such that $B = \cap_{n=0}^{\infty} U_n$. Clearly $C + t \subset \cap_{n=0}^{\infty} U_n$ iff $C + t \subset U_n$ holds for every $n \in \mathbb{N}$. Therefore $T = \cap_{n=0}^{\infty} G_n$, where $G_n = \{t \in \mathbb{R} : C + t \subset U_n\}$. As C is compact and U_n is open, G_n is open. Note that G_n is also dense by our assumption.

It is clearly sufficient to construct a Cantor set $P \subset T$ with the property that $(C+p_0) \cap (C+p_1) = \emptyset$ holds for every pair of distinct points $p_0, p_1 \in P$. We define P via a usual Cantor scheme as follows. Let 2^n stand for the set of 0–1 sequences of length n. We define nondegenerate compact intervals I_s for every $n \in \mathbb{N}$ and $s \in 2^n$ by induction on n (we also make sure that at level n all intervals are of length at most $\frac{1}{n}$). Fix $I_\emptyset$ such that $I_\emptyset \subset G_0$. Once I_s is already defined for some $s \in 2^n$, we pick $x \in I_s \cap G_{n+1}$. As G_{n+1} is open dense and $C-C$ is nowhere dense we can find $y \in [I_s \cap G_{n+1}] \setminus [(C-C)+x]$. This ensures $(C+x) \cap (C+y) = \emptyset$, as otherwise $c_0 + x = c_1 + y$ for some $c_0, c_1 \in C$, so $y = (c_0 - c_1) + x$, a contradiction. By compactness we can find disjoint $I_{s^\frown 0}, I_{s^\frown 1} \subset I_s \cap G_{n+1}$ such that $x \in I_{s^\frown 0}$, $y \in I_{s^\frown 1}$ and $(C + I_{s^\frown 0}) \cap (C + I_{s^\frown 1}) = \emptyset$.

Now define

$$P = \bigcap_{n=0}^{\infty} \bigcup_{s \in 2^n} I_s.$$

Then clearly P is a Cantor set, $P \subset T = \cap_{n=0}^{\infty} G_n$ as $I_s \subset G_n$ for every n and $s \in 2^n$. Moreover, if $p_0, p_1 \in P$, $p_0 \neq p_1$ then there are n and $s \in 2^n$ such that $p_0 \in I_{s^\frown 0}$ and $p_1 \in I_{s^\frown 1}$ (or the other way around), but then $(C+p_0) \cap (C+p_1) = \emptyset$ holds since $(C + I_{s^\frown 0}) \cap (C + I_{s^\frown 1}) = \emptyset$. This completes the proof of Lemma 3.15.3. $\square$

3.16 Proof of Theorem 2.6.5

Proof. Let $B \subset \mathbb{R}^3$ be an arbitrary Borel set with $\dim_H(B) = \frac{5}{2}$. Let $f : \mathbb{R}^3 \to B$ be an arbitrary Borel map. Then [67, Theorem 1.4] states that for every Borel set $A \subset \mathbb{R}^n$, Borel map $f : A \to \mathbb{R}^m$ and $0 \leq d \leq 1$ there exists a Borel set $D \subset A$ such that $\dim_H D = d \cdot \dim_H A$ and $\dim_H f(D) \leq d \cdot \dim_H f(A)$. Applying this with $n = m = 3$, $A = \mathbb{R}^3$, and $d = \frac{11}{15}$ we obtain that there exists a Borel set $D \subset \mathbb{R}^3$ with $\dim_H(D) = \frac{11}{5}$ such that $\dim_H(f(D)) \leq \frac{11}{15} \cdot \frac{5}{2} = \frac{11}{6}$. Then $\dim_H(D) > 2$ and $\dim_H(f(D)) < 2$, therefore $f(D) \in \mathcal{I}^2_{3,\sigma-fin}$, but $f^{-1}(f(D)) \supset D \notin \mathcal{I}^2_{3,\sigma-fin}$. Since f was arbitrary, the choice $I = f(D)$ shows that $\mathcal{I}^2_{3,\sigma-fin}$ is not homogeneous. $\square$

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