**Answers to the questions raised by Máté Gerencsér, Ph.D. in his referee report**

First of all I would like to thank Máté Gerencsér, Ph.D. for his thorough referee report. The questions raised in the report are answered as follows.

**Q1.** *What is the practical motivation behind studying weak convergence for infinite-dimensional systems? What is the typical $g$ that appears?*

The motivation behind considering weak convergence is that in many cases one is not interested in the solution itself but rather a functional of the solution. Typical examples include moments; that is, $g(x) = \|x\|^p$, $p \geq 1$, or a smooth approximation of an indicator function such as, for example, $g(x) = \Phi(M(x - \delta))$, where $\Phi(\cdot)$ denotes the cumulative distribution function for the standard normal distribution. In this case, $g$ approximates the indicator function of $(\delta, \infty)$ when $M > 0$ is large enough. For functionals that may depend on the orbit of the solution one is often interested in statistical quantities such as covariances, or higher order statistics, as in Corollary 2.3.8 of the dissertation.

**Q2.** *Let us consider only, say, parabolic equations, where Markovianity is not an problem. Is there an example where the bounds and/or the strategy in the linear additive case can be used in either a nonlinear or a multiplicative (w.r.t. the noise) equation?*

The approach to weak error analysis via Kolmogorov's equations has been applied to various nonlinear problems. The key extra ingredient is a Malliavin integration by parts to deal with the term that contains the unbounded operator. This is the very term that can be eliminated in the linear additive case using the special structure of the solution but it will be present in the nonlinear setting. The first paper that introduced the additional Malliavin integration by parts technique (semilinear equation with multiplicative white noise, time discretization in 1D) is [4]. Further examples, for the heat equation, are [5] (semilinear equation with additive coloured noise, time discretization), and [1] (semilinear equation with additive and multiplicative noise, finite element space discretization).

**Q3.** *When applying the error representation formula, the bounds used on the derivatives of $u$ are the ones that immediately follow from differentiating (1.1.21) (or (3.2.4)). Are there more sophisticated regularity results available for the Kolmogorov equation considered here that could relax certain assumptions? For instance could analogues of the finite* *dimensional bounds*

$$\partial_t u + \Delta u = 0, \quad u(T,x) = g(x) \implies |u_{xx}(T-t,x)| \le t^{-\frac{1}{2}} \sup_y |g'(y)|$$

*relax the condition $g \in C^2$?*

The situation in infinite dimensions is much less favourable. Even for the simplest (forward) Kolmogorov equation

$$\partial_t u(t,x) = \frac{1}{2}\mathrm{Tr}\,[QD^2 u(t,x)],\ t > 0, x \in H; \quad u(0,x) = g(x),\ x \in H,$$

it is known that if $\dim H = \infty$, then $g$ being bounded and uniformly continuous does not imply that $u$ is continuously differentiable. In fact, $x \to u(t,x)$ might not even be Lipschitz continuous, see [6]. This phenomenon transfers to the weak error as well. It is shown in [3] that if $g$ is Lipschitz continuous, then the weak rate, in general, is not better than the strong rate. Therefore, it is not expected that one can significantly reduce the regularity assumptions on $g$ and still observe a weak rate that is twice the strong rate. A small reduction in the regularity requirement could be possible such as, for example, instead of asking for $g \in C^2$ only ask for $g'$ to be Lipschitz continuous. But this would not be a major relaxation of the regularity requirements on $g$.

**Q4.** *The variation of constants formula (2.1.2) together with the smoothing effect (2.1.1) readily implies some Sobolev regularity of the solution. Could this spatial regularity be used to allow $f$ to lose regularity, i.e. map from $H \to H^{-\delta'}$ for some $\delta' > 0$ (perhaps different from $\delta$)?*

There are existence and regularity results for stochastic Volterra equations under the assumptions that $f : \dot{H}^{r-1+\frac{1}{\rho}} \to \dot{H}^{r-1}$ for some $r < 1$, see [2]. Therefore, it is very likely that one can relax the assumptions on $f$ accordingly. The assumptions on $f$ in the dissertation were put in place to accommodate the situation when $f$ is a Nemytskii operator. When $f$ involves a gradient, then the suggested relaxed regularity assumption could indeed be useful.

**Q5.** *How important is the role of boundary conditions? In some examples Dirichlet, in others Neumann is chosen, is there a particular reason for these choices?*

From the point of view of the abstract theory the particular choice of the boundary condition is not important as long as the linear operator with that particular choice satisfies the desired abstract spectral properties. From a modelling point of view different boundary conditions correspond to different physical situations. For example, in case of the Cahn-Hilliard equation homogeneous Neumann boundary condition means conservation of mass. This is often used in the literature. For a Volterra intergro-differential equation arising in viscoelasticity, assuming that the unknown function is the particle velocity, homogenous Dirichlet boundary condition represent the so-called no-slip boundary condition.

## References


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