Adapted complex structures

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Contents

Part I

Adapted complex structures classically

Chapter 1

Preliminaries

This dissertation consists of two main parts. The first part contains five chapters that can be read independently of each other. The first chapter contains no new results, only an introduction and collection of frequently used notations. The second chapter investigates symmetries of certain Stein manifolds and is based on the paper [Sz95]. The third chapter studies the adapted complex structures of compact normal Riemann homogeneous spaces and based on the paper [Sz98]. Chapter four is on the problem of existence of a geodesic flow invariant complex (or more generally involutive) structure. The chapter is based on two papers. Section 4.1 is based on [Sz99] and Section 4.2 is based on [Sz01]. Chapter five is devoted to two related problems. Section 5.1 is concerned with the problem of generalization of Chevalley's extension problem to Weyl group equivariant maps. This is based on the joint paper with Ádám Korányi [KSz]. Section 5.2 is about the problem of existence of hyperkahler metrics on (co)tangent bundles and is based on the joint paper with Andrew Dancer [DSz].

The whole second part of the dissertation is motivated by the problem of uniqueness in geometric quantization. The chapters here are related to each other and the next builts on the results of the previous. Section 6.1 is based on the joint paper with László Lempert [LSz12]. Section 6.2 is an introduction to geometric quantization, including some unpublished results of mine [Sz-prep] in Section 6.2.5 and 6.2.6. Chapter 7,8, and Section 9.1, 9.2 and 9.4 are based on the joint paper with László Lempert [LSz14]. Section 9.3 is based on the paper [Sz17].

1.1 Introduction

The notion that connects the different problems in this dissertation is the one in the title: adapted complex structures. Although this term appeared first time in our paper [LSz91], the equivalent notion of Monge-Ampère models was the subject of my PhD thesis [Sz90] written at Notre Dame University. The results of my thesis were published in the papers [LSz91, Sz91], but since they are important for this dissertation as well, we shortly summarize them in this introductory part together with some historical background.

Let X^n be a complex manifold of dimension n and $u : X \to \mathbb{R}$ a twice differentiable plurisubharmonic function. The complex, homogeneous, Monge-

Ampère equation for u is

$$(\partial\overline{\partial}u)^n = 0, \tag{1.1.1}$$

or, in local coordinates $z_1, \ldots, z_n$ on X,

$$\det\,(\partial^2 u/\partial z_j \partial \overline{z}_k) = 0.$$

When $n = 1$, the above equations reduce to Laplace's equation $\Delta u = 0$, and the Monge-Ampère equation is the most natural extension of the Laplace equation to higher dimensional complex manifolds. It first appeared in a paper by Bremermann, [Br]. For an extensive reference about work on the Monge-Ampère equation see a survey paper by Bedford [Bed2].

The question we address here is the following. To what extent is a solution u of (1.1.1), or even X, determined if certain global conditions are imposed on u? We shall consider plurisubharmonic solutions $u(z)$ of (1.1.1) that go to infinity as $z \in X$ diverges in X, or, more precisely, such that for any $c \in \mathbb{R}$

$$\{z \in X : u(z) \leq c\} \qquad \text{is compact.} \tag{1.1.2}$$

In this case u is called an exhaustion function of X. A little more generally we shall also consider bounded exhaustion functions u, i.e. when (1.1.2) is required to hold only for $c < \sup u < \infty$. In this generality there are too many solutions u. For example if $X = Y \times Z$ with Y compact and Z Stein, any smooth plurisubharmonic exhaustion function v on Z defines a solution $u(y, z) = v(z)$. To eliminate such examples we assume X itself is Stein.

Stein manifolds are generalizations of domains of holomorphy in $\mathbb{C}^n$. They admit plenty of nonconstant holomorphic functions so they are the natural objects to do function theory on them. They can be characterized as those complex manifolds that can be realized as a closed complex submanifold of $\mathbb{C}^N$ for some N. Another characterization, due to Grauert, says that X is Stein iff there is a $\tau : X \to [0, \infty)$ strictly plurisubharmonic exhaustion function.

On a fixed Stein manifold there are many such exhaustions. One may wonder if for a given X one could choose a specific τ that is canonically attached to X. Perhaps having such a special exhaustion makes it possible to characterize such an X among Stein manifolds. Possible such characterizations are interesting, since as one knows, the Riemann mapping theorem fails in higher dimensions.

It often happens in mathematics that the existence of a global solution of a certain differential equation helps to classify the underlying manifold. For instance having constant sectional curvature of a Riemannian metric.

The idea here is to try to use the (1.1.1) Monge-Ampère equation to classify Stein manifolds. Then however, it turns out that there are no everywhere smooth exhaustion functions u that would solve (1.1.1) ([LSz91, Theorem 1.1]). This naturally leads us to admit u that have some type of singularities on a set $M \subset X$. In fact, as [LSz91, Theorem 1.1] shows, this set M must be related to the minimum set of u. By a theorem of Harvey and Wells [HW] such a minimum set must be totally real and so its real dimension is at most n.

That a global condition, type of singularity, and the Monge-Ampère equation may uniquely characterize complex manifolds X and functions u on them was first observed by Stoll. In [Sto] he considered the situation when M reduces to a point, in which case the natural (minimal) singularity to prescribe is a logarithmic pole. He proved that in this case X is biholomorphic to $\mathbb{C}^n$ and u is equivalent to $|z|^2$. See also [Bu3, Wo].

Later, Patrizio and Wong considered the other extreme, when the singular set M is an n real dimensional manifold. Here the natural (minimal) singularity to assume is a "square root singularity", (cf. [PW]). They conjectured that in certain cases the mere knowledge of the differentiable manifold M determines X and u (viz. when M is diffeomorphic to a simply connected compact symmetric space of rank 1). They could settle this conjecture only under the assumption that a more precise information about the singular behavior of u is available. Also, they constructed examples of X and u with M a compact symmetric space of rank 1, or a torus. Apart from that, the main contribution of [PW] is the description of the rich geometry that is determined by u. That there is an interesting geometry associated with a solution of the Monge-Ampère equation was first discovered by Stoll and Burns.

Further examples of solutions of the Monge-Ampère equation with square root singularity were found by Lempert [L2]. In those examples M is a hyperbolic manifold, and the function u is a *bounded* exhaustion function of X.

The primary objects of study of the paper [LSz91] are unbounded exhaustion functions u on Stein manifolds X that satisfy (1.1.1) and have square root singularity along a smooth manifold M, $\dim_{\mathbb{R}} M = \dim_{\mathbb{C}} X$. A Kähler metric on X (cf. also (1.2.11)) and its restriction, a Riemannian metric on M is introduced. It is proved that X and u are determined (up to biholomorphism) by the metric on M (even when u is bounded). This extends the result of [PW] that applies when M with the metric above becomes a compact symmetric space of rank 1. This result can also be regarded as defining canonical complexifications of Riemannian manifolds. Such canonical complexifications were called in [LSz91] adapted complex structures (cf. also section 1.2.2). It was shown in [LSz91] that they are equivalent to a solution of (1.1.1) with a square root type singularity.

Guillemin and Stenzel ([GS1, GS2]) investigated related problems. They work on cotangent bundles of Riemannian manifolds. Although their formal definitions are different from the ones in [LSz91], they recover the same complex manifolds X and functions u as in [LSz91].

It is also proved in [LSz91] that when u is unbounded, the metric on M must be nonnegatively curved. From this it follows that when M is diffeomorphic to a torus, X and u are almost uniquely determined: they must be one of the examples found by Patrizio and Wong.

However, the original conjecture of Patrizio and Wong does not hold. There is a 1-parameter family of inequivalent examples with singularity set diffeomorphic to the sphere S^2 [Sz91]. Furthermore it was proved in [Sz91] that for a compact Riemannian symmetric space M of any rank, the adapted complex structure exists on TM and for arbitrary compact, real-analytic Riemannian manifold it exists in a neighborhood of the zero section in TM.

The notion of adapted complex structures was later extended to Koszul connections ([Bi, Sz04]), Finsler metrics [DK] and magnetic flows [HK2].

On the other hand, the adapted complex structure of a Riemannian manifold turns out to be just one member in a natural family of Kähler structures ([LSz12] and section 6.1). This is the family that respects the symmetries of TM (generated by the geodesic flow and fiberwise dilations). They are parametrized by $s \in \mathbb{C}\backslash\mathbb{R}$ and are positive Kähler when $\mathrm{Im}\, s > 0$ and negative for $\mathrm{Im}\, s < 0$. The Kähler manifolds thus obtained constitute the fibers of a holomorphic fibration over $\mathbb{C}\backslash\mathbb{R}$, and the adapted complex structure of section 1.2.2 ([GS1, LSz91]) corresponds to the fiber over $s = i$. It is possible to extend the fibration to a

fibration over $\mathbb{C}$; however, the fibers over $\mathbb{R}$ will be real polarized rather than Kähler. Thus one is led to the notion of adapted polarizations, of which an adapted complex structure is just an extreme example. These are discussed in chapter 6.

The papers [FMMN1, FMMN2] consider a one (real) parameter family of Kähler structures on the cotangent bundle of a compact Lie group, that degenerates to a real polarization; this family is then used to explain geometrically the so called Bargmann–Segal–Hall transformation of [Hal1, Hal2]. The papers themselves make no explicit connection with adapted complex structures, but the family considered there is the restriction of our family of adapted polarizations to the positive imaginary axis. Recently [HK] pointed out that for a general closed real analytic Riemannian manifold the original adapted complex structure is the analytic continuation to i of a real family of real polarizations. The novelty of our approach is first that all those Kähler structures and real polarizations can be derived from one principle; second that these structures, taken together, constitute a fiber bundle (Theorem 6.1.11).

This fiber bundle plays an essential role in chapter 8 and section 9.3 in the study of the problem of uniqueness in geometric quantization using the family of adapted Kähler structures to perform geometric quantization.

At its simplest, geometric quantization (cf. section 6.2) is about associating with a Riemannian manifold M a Hermitian line bundle $L \to X$ and a Hilbert space H (called the quantum Hilbert space) of its sections. In Kähler quantization, L is a holomorphic Hermitian line bundle and H consists of all square integrable holomorphic sections of L. One often knows how to find L, except that its construction involves choices, so that one really has to deal with a family $L_s \to X_s$ of line bundles and Hilbert spaces H_s, parametrized by the possible choices $s \in S$.

The *problem of uniqueness* is to find canonical unitary maps $H_s \to H_t$ (resp. $H_s^{corr} \to H_t^{corr}$) corresponding to different choices $s \neq t \in S$—or rather projective unitary maps, the natural class of maps, since only the projectivized Hilbert spaces have a physical meaning. This problem is a fundamental issue in geometric quantization.

There are various solutions to this problem, the first the Stone–von Neumann theorem [St1, vN1], long predating geometric quantization. It applies whenever two Hilbert spaces carry irreducible representations of the canonical commutation relations; if so, there is a unitary map, unique up to a scalar factor, that intertwines the two representations. However, the Hilbert spaces that geometric quantization supplies do not carry such representations unless the manifold to be quantized is an affine space. In geometric quantization there is the Blattner–Kostant–Sternberg pairing [Bl1, Bl2, Ko2], which sometimes gives rise to the sought for unitary map, but even in simple cases it may fail to do so [Ra2].

In the early 1990s Hitchin in [Hi] and Axelrod, Della Pietra, and Witten in [ADW] considered a situation when the possible choices s form a complex manifold S. One has to be careful with what "possible choices" mean. The choices in question are Kähler structures on TM, compatible with the canonical symplectic form. If literally all such Kähler structures were considered, uniqueness would be too much to hope for; it can be reasonably expected only if a preferred family of Kähler structures, those dictated by the symmetries of the problem, is used.

[Hi] and [ADW] proposed to view the H_s as fibers of a holomorphic Hilbert bundle $H \to S$, introduce a connection on H, and use parallel transport to identify the fibers H_s and H_t.

To see how parallel transport along a path from s to t depends on the path, they computed the curvature of the connection. The curvature turned out to be a scalar operator. Hence [ADW, Hi] concluded that parallel transport is, up to a scalar factor, independent of the path, and yields the required identification $H_s \approx H_t$. Hitchin quantized compact phase spaces, his Hilbert spaces were finite dimensional and his reasoning is mathematically rigorous. [ADW] is bolder, quantizes noncompact and even infinite dimensional manifolds (affine spaces and their quotients). This leads to infinite dimensional Hilbert spaces and worse, and the paper, from a mathematical perspective, is not fully satisfactory, even when the manifolds to be quantized are finite dimensional (cf. section 6.2.1).

The general set up is as follows. Consider a holomorphic submersion $\pi\colon Y \to S$ of complex manifolds with fibers $\pi^{-1}s = Y_s \subset Y$, which are complex submanifolds. Let ν be a smooth form on Y that restricts to a volume form on each Y_s, and let $(E, h^E) \to Y$ be a Hermitian holomorphic vector bundle. We assume that $\dim Y$, $\dim S$, and rk $E < \infty$, Finally, let H_s denote the Hilbert space of holomorphic L^2–sections u of $E|Y_s$, L^2 in the sense that $\int_{Y_s} h^E(u)\nu < \infty$.

The quantization procedure in [ADW] leads to a very special case of this set up. There the line bundles $(E|Y_s, h^E)$ can be smoothly identified and the Hilbert spaces H_s^{prQ} of all L^2 sections of $E|Y_s$ can be considered as fibers of a trivial Hilbert bundle $H^{prQ} \to S$. This is done quite naturally, because [ADW] forgoes the half–form correction. In each fiber of $H^{prQ} \to S$ sits a subspace H_s, and [ADW, bottom of p. 801] asserts that the H_s form a subbundle $H \subset H^{prQ}$. The paper offers no justification for this, nor an explanation of what is meant by a subbundle.

When affine symplectic spaces are quantized, all the above issues can be settled satisfactorily. One can either use the formulas in [W, Section 9.9], attributed to Rawnsley, or the results of Kirwin and Wu, [KW]. The first is based on the BKS pairing, the second on the Bargmann–Segal transformation.

A connection, closely related to the one in [ADW], and its parallel transport are studied in [FMMN1, FMMN2]. These papers go beyond affine spaces. They consider a one real parameter family of polarizations of the cotangent bundle of a compact Lie group, a connection on the bundle of the corresponding quantum Hilbert spaces, and express parallel transport through Hall's generalization of the Bargmann–Segal transformation [Hal1, Hal2]. This again justifies the definition of the connection a posteriori, but says little about the uniqueness problem that has not been known since [Hal2].

While it is certainly pleasing to realize that the BKS pairing and the Bargmann–Segal and Fourier transformations can be interpreted geometrically as a result of parallel transport, justifying [ADW] through [W, Section 9.9] and [KW] beats the original purpose of the connection: if both the pairing and the Bargmann–Segal transformation already identify the spaces H_s, why bother defining the connection and studying its parallel transport? Put it differently: will the connection proposed in [ADW] shed any light on the uniqueness problem when the BKS pairing fails to provide the unitary identifications and no explicit integral transformation like that of Bargmann–Segal is available? This is the question that we address and partially answer in chapters 7, 8 and section 9.3.

Most of these chapters revolve around the general set up described above, a holomorphic submersion $\pi\colon Y \to S$, a Hermitian holomorphic vector bundle $E \to Y$, and the Hilbert spaces H_s of holomorphic L^2–sections of $E|Y_s$. The spaces H_s form what we call a Hilbert field $p : H \to S$, where H simply means the disjoint union of $\{H_s\}_{s\in S}$. We ask whether one can endow H with the structure of a Hilbert bundle and a connection on the bundle; furthermore, whether the connection induces a path independent parallel transport. That is, we are trying to understand the direct image of E under π. We emphasize that π is not assumed to be proper. If it is, Grauert's theorem [Gr] describes the holomorphic structure of the direct image, and many papers, including [Be3, BP, BF, BGS, MT1, MT2, Ts] reveal some aspects of its Hermitian structure; the most recent related work seems to be [Sch]. However, the chief difficulties we encounter here arise when π is not proper. Berndtsson in [Be1, Be2, Be3] already studied the curvature of certain improper direct images, and in [Be4] gave a striking application.

It may seem futile to consider completely general $Y \to S$ and E, as the spaces H_s in general will not form a bundle and in fact will not have any extra structure at all. Still, certain constructions are always possible, and it is only this generality that guarantees that the constructions to be performed are natural. In favorable cases the constructions lead to what we call smooth and analytic fields of Hilbert spaces. These fields are analogous to Hermitian Hilbert bundles with a connection, but the notion is quite a bit weaker. Chapter 7 is devoted to fields of Hilbert spaces; the main results Theorems 7.1.7, 7.1.11, 7.4.2 and Corollaries 7.1.8, 7.1.12) say that if an analytic field of Hilbert spaces has zero, resp. central curvature, then it is equivalent to a Hermitian Hilbert bundle with a flat, resp. projectively flat, connection.

In chapter 8 we turn to the direct image problem and discuss the constructions that, in favorable cases, endow the direct image with the structure of a smooth field of Hilbert spaces. We also provide criteria for this to happen, and express the curvature of the field in terms of the geometry of Y and E.

Finally, in chapter 9 we test the general results obtained so far against geometric quantization of a compact Riemannian manifold M, when quantization is based on the family of adapted Kähler structures. The scheme leads to a direct image problem. In many cases the direct image is an analytic field of Hilbert spaces, and in some cases, namely for group manifolds, the field is even flat, hence parallel transport provides the natural identification of the quantum Hilbert spaces corresponding to different Kähler structures, i.e. in this case quantization is unique. In section 9.3 we prove that among compact irreducible Riemannian symmetric spaces precisely the group manifolds are those for which quantization is unique.

The ideas in [ADW, Hi] in the context of Kähler, or "almost Kähler" quantization of *compact* symplectic manifolds (N, ω) have been taken up in several papers.Viña [Viñ] computed the curvature of a natural connection on the family of quantum Hilbert spaces corresponding to (certain) complex structures on N compatible with ω, and found that in general the curvature was nonzero. Foth and Uribe [FU] replaced the prequantum line bundle $L \to N$ by higher powers L^k and computed the curvature of the resulting connection. Even in the semiclassical limit $k \to \infty$ the curvature did not tend to zero. However, Charles [Char] proved that if the quantization scheme includes the half–form correction, in the semiclassical limit the curvature does tend to zero.

1.2 Frequently used notations

1.2.1 Geometry of the tangent bundle.

Let (M, g) be a complete, smooth n-dimensional (pseudo-)Riemannian manifold, TM its tangent bundle, $\pi : TM \to M$ the bundle projection map and g the (pseudo-)Riemannian metric on M.

When $\gamma : \mathbb{R} \to M$ is a nonconstant geodesic, for any point $w \in \gamma_*(T\mathbb{R})$, the dimension of $T_w(TM)$ and the dimension of the vector space of Jacobi fields along γ is the same: $2n$. In fact there is a natural isomorphism between these two vector spaces, as we describe this below.

The image $T\mathbb{R} \setminus \mathbb{R}$ under the induced map $\gamma_* : T\mathbb{R} \to TM$ is a two dimensional surface. As γ runs through all the nonconstant geodesics in M, the surfaces $\gamma_*(T\mathbb{R} \setminus \mathbb{R})$ define a foliation of $TM \setminus M$, the so called *Riemann foliation.*

For a $\gamma : \mathbb{R} \to M$ geodesic, a *parallel vector field* ξ *along* γ_* is a vector field along the map $\gamma_* : T\mathbb{R} \to TM$ (i.e. a section of the pullback bundle $(\gamma_*)^*(TM)$), such that there exists a smooth family $\gamma_t : \mathbb{R} \to M$ of geodesics with $\gamma_0 = \gamma$ and

$$\left.\frac{d}{dt}\right|_{t=0} \gamma_{t*} = \xi.$$

For an $m \in M$, $0_m \in T_mM$ denotes the zero vector. The correspondence $m \leftrightarrow 0_m$ identifies the manifold M with the zero section in TM and gives rise to an identification of T_mM and the tangent space of the zero section at 0_m.

Let $\sigma \in \mathbb{R}$ and ξ a parallel vector field along γ_*. Then

$$\xi(\sigma) := \xi(0_\sigma) = \left.\frac{d}{dt}\gamma_{t*}(0_\sigma)\right|_{t=0} = \left.\frac{d}{dt}\gamma_t(\sigma)\right|_{t=0},$$

i.e. $\xi|_{\mathbb{R}}$ is a Jacobi field along γ. Parallel vector fields can be thought of as canonical extensions of Jacobi fields.

For $s \in \mathbb{R}$ let $N_s : TM \to TM$ be the map

$$N_s(z) = sz. \tag{1.2.1}$$

When $s \neq 0$, N_s is a diffeomorphism. Denote by $\phi_s : TM \to TM$ the geodesic flow. According to [LSz91, Proposition 6.1], a vector field ξ along $\gamma_* : T\mathbb{R} \to TM$ is parallel iff

$$N_{s*}\xi = \xi, \quad \phi_{s*}\xi = \xi \qquad s \in \mathbb{R}. \tag{1.2.2}$$

Since any point $z \in TM \setminus M$ determines a unique geodesic $\gamma_z : \mathbb{R} \to M$ with $\dot{\gamma}_z(0) = z$, it follows that the three vector spaces: Jacobi fields along γ_z, parallel vector fields along γ_{z*} and $T_w(TM)$ (for any $w \in \gamma_{z*}(T\mathbb{R})$) naturally correspond to each other. This is fundamental for the adapted complex structure and so for this dissertation.

The canonical 1-form ϑ on TM is defined by

$$\vartheta(v) := g(z, \pi_* v), \quad v \in T_z(TM) \tag{1.2.3}$$

and

$$\omega := -d\vartheta \tag{1.2.4}$$

is the canonical symplectic form on TM. Then

$$N_s^*\omega = s\omega, \quad \phi_s^*\omega = \omega \qquad s \in \mathbb{R}. \tag{1.2.5}$$

When ($z \in TM$), the Levi-Cività connection of g determines a splitting of $T_z(TM)$, into vertical and horizontal subspaces. For $z \in TM$ and $v \in T_{\pi(z)}M$, $v_z^H \in T_z(TM)$ (resp. $v_z^V \in T_z(TM)$) denotes the horizontal (resp. vertical) lift of v to z. Then for $v, z \in T_pM$ and $s \neq 0$ we have

$$(N_s)_* v_z^H = v_{sz}^H, \quad (N_s)_* v_z^V = s v_{sz}^V. \tag{1.2.6}$$

Now let γ be a unit speed geodesic. Let $z = \tau_0 \dot\gamma(\sigma_0)$, $v = \dot\gamma(\sigma_0)$. Then we have

$$(\gamma_*)_* \left(\frac{\partial}{\partial \sigma}\bigg|_{(\sigma_0,\tau_0)} \right) = v_z^H, \qquad (\gamma_*)_* \left(\frac{\partial}{\partial \tau}\bigg|_{(\sigma_0,\tau_0)} \right) = v_z^V. \tag{1.2.7}$$

Let now $\gamma : \mathbb{R} \to M$ be a unit speed geodesic, and $\xi_1, \dots, \xi_n, \eta_1 \dots, \eta_n$ parallel vector fields along γ_*. We call $(\xi_1, \dots, \xi_n, \eta_1 \dots, \eta_n)$ a *symplectic frame*, if there exists a real number σ_0 and an orthonormal basis

$$v_1, \dots, v_{n-1}, v_n = \dot\gamma(\sigma_0) \in T_{\gamma(\sigma_0)}M,$$

such that (with $v := \partial_{\sigma_0} \in T_{\sigma_0}\mathbb{R}$) for any $1 \le j \le n$, $\xi_j(v)$ is the horizontal and $\eta_j(v)$ is the vertical lift of v_j to $z := \gamma_*(v) = \dot\gamma(\sigma_0)$.

This condition is equivalent to the following: the Jacobi fields $\xi_j|_{\mathbb{R}}, \eta_j|_{\mathbb{R}}$ satisfy ($'$ means covariant derivative along γ) the initial conditions

$$\xi_j(\sigma_0) = v_j, \qquad \xi_j'(\sigma_0) = 0, \quad 1 \le j \le n,$$

$$\eta_j(\sigma_0) = 0, \qquad \eta_j'(\sigma_0) = v_j, \quad 1 \le j \le n.$$

In particular the set of those real numbers σ, where $\xi_1(\sigma), \dots, \xi_n(\sigma) \in T_{\gamma(\sigma)}M$ are linearly dependent, is a discrete subset of $\mathbb{R}$, denoted by S. Hence there exists a smooth matrix valued map $\varphi = (\varphi_{jk})$, defined on $\mathbb{R} \setminus S$, such that

$$\eta_k(\sigma) = \sum_{j=1}^n \varphi_{jk}(\sigma)\xi_j(\sigma), \qquad \sigma \in \mathbb{R} \setminus S, \quad 1 \le k \le n. \tag{1.2.8}$$

Proposition 1.2.1 (Szőke, [Sz95])**.** *Let (M^n, g) be a Riemannian manifold. Let γ be a unit speed geodesic and $\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_n$ be a symplectic frame along γ_*. Let $\sigma \in \mathbb{R}$, $0 \neq \tau \in \mathbb{R}$ and $w = \tau\partial_\sigma \in (T_\sigma\mathbb{R})$. Then the 2n-tuple of vectors $\{\xi_j(\gamma_*(w)), \dots, \eta_j(\gamma_*(w))\}_{j=1}^n$ forms a symplectic base in the symplectic vector space*

$$\left(T_{\gamma_*(w)}(TM), (1/\tau)\omega|_{\gamma_*(w)}\right),$$

i.e. for every $1 \le j, k \le n$,

$$\omega(\xi_j, \xi_k)(\gamma_* w) = 0 = \omega(\eta_j, \eta_k)(\gamma_* w), \quad \omega(\xi_j, \eta_k)(\gamma_* w) = \tau\delta_{jk}.$$

Proof. The orbit of a fixed point of the leaf $\gamma_*(T\mathbb{R} \setminus \mathbb{R})$, under repeated applications of N_s and ϕ_t is the whole leaf. Therefore, according to (1.2.2) and (1.2.5), it is enough to check our statement in one particular point z of $\gamma_*(T\mathbb{R} \setminus \mathbb{R})$. We can assume that z is the point where our frame is the horizontal resp. vertical lift of an orthonormal base of $T_{\pi(z)}M$, i.e. $z = \gamma_*(v)$ with $v := \partial_{\sigma_0} \in T_{\sigma_0}\mathbb{R}$). Choose a Riemannian normal coordinate system around the point $\pi(z)$, such that $v_j = \partial/\partial q_j$.

With this choice we have

$$\xi_j(v) = \frac{\partial}{\partial q_j} \quad \text{and} \quad \eta_j(v) = \frac{\partial}{\partial p_j}, \text{and} \quad \omega|_z = \sum_j dq_j \wedge dp_j.$$

□

1.2.2 Adapted complex structures.

Every real-analytic manifold M^n posseses a complexification $\mathbb{C}M$ (see [Shu], [WB]), that means the following: $\mathbb{C}M$ is a complex manifold containing M and $\mathbb{C}M$ is equipped with an antiholomorphic involution $\sigma : \mathbb{C}M \to \mathbb{C}M$ whose fixed point set is precisely M. Its construction is roughly as follows: one takes the gluing maps of M between different real-analytic coordinate charts and holomorphically extend these maps to some domain in $\mathbb{C}^n$. Using these holomorphic maps we can glue together these new domains and obtain $\mathbb{C}M$. The natural conjugation map on $\mathbb{C}^n$ induces an antiholomorphic involution $\sigma : \mathbb{C}M \to \mathbb{C}M$ with fixed point set M. This complexification is unique only as a germ of complex manifold along M. To get a canonical complexification one needs some extra structure on our original manifold. For example one can take a Riemannian metric.

There are two approaches to construct a canonical complexification out of a metric. Although these two methods are very different, they lead to equivalent complex structures. One approach is the method of Guillemin and Stenzel [GS1]. They work on the cotangent bundle T^*M. The energy function "generates" the complex structure, in an apropriate neighborhood of the zero section, with the help of the canonical one form. The antiholomorphic involution here is the map of T^*M that multiplies each element in T^*M with negative one.

The other approach grew out of studying global solutions of the complex homogeneous Monge-Ampére equation (see (1.1.1)) on Stein manifolds (cf. [Bu2, Bu3, Sto, PW, LSz91, Sz91]). This method also starts with a Riemannian metric and certain complex curves play a fundamental role. These curves correspond to geodesics on the original Riemannian manifold. The resulting complex structure lives on the tangent (rather than the cotangent) bundle (or on open subset of it) and is the so called *adapted complex structure* ([LSz91, Sz91], Definition 1.2.2).

When $M = \mathbb{R}$, there is a natural identification $T\mathbb{R} \cong \mathbb{C}$, given by ($\sigma$ denotes the coordinate on $\mathbb{R}$)

$$T_\sigma\mathbb{R} \ni \tau\frac{\partial}{\partial\sigma} \longleftrightarrow \sigma + i\tau \in \mathbb{C}. \tag{1.2.9}$$

This identification equipes $T\mathbb{R}$ with a complex structure that is fixed in the first part of this dissertation.

Let (M, g) be Riemannian and $0 < r \leq \infty$. Let T^rM be defined by

$$T^rM = \{v \in TM \mid g(v, v) < r^2\}.$$

When $r = \infty$, T^rM simply means TM. We call r the *radius* of the tube T^rM.

Definition 1.2.2. *Let (M, g) be a complete Riemannian manifold. Let D be a domain in TM containing the zero section. A complex structure J_A, defined on D, is called adapted if for every geodesic $\gamma : \mathbb{R} \to M$, the map γ_* is holomorphic on $(\gamma_*)^{-1}(D) \subset T\mathbb{R}$, where $T\mathbb{R}$ is endowed with the complex structure from* (1.2.9).

The special case $D = T^r M$ with the adapted complex structure is also called a *Grauert tube*. In the early days D was chosen to be of $T^r M$ with some $r > 0$ (see [Sz90, LSz91, Sz91]), but since then more general domains have turned out to be of importance (cf. [FHW]).

From the definition one easily sees that the zero section in $T^r M$ is a maximal dimensional totally real submanifold. According to [LSz91, Theorem 5.6] the function

$$\rho : TM \to \mathbb{R}, \quad \rho(v) := g(v, v) \tag{1.2.10}$$

is strictly plurisubharmonic w.r.t. this complex structure thus it is a potential function for a Kähler metric κ_g, defined by

$$\kappa_g(V, W) = -i\partial\bar{\partial}\rho(JV \wedge \overline{W}), \qquad V, W \in T_z(TM) \otimes \mathbb{C}, \quad z \in T^r M. \tag{1.2.11}$$

It was shown in [LSz91, Corollary 5.5], that $i\partial\bar{\partial}\rho = \omega$, hence the Kähler form of κ_g is precisely ω, the canonical symplectic form (see (1.2.4)) on the tangent bundle. Together with (1.2.11), this implies that when we restrict the metric κ_g to the zero section, we get back the original metric g. It was also proved in [LSz91, Theorem 5.6 and Proposition 3.9], that M is a totally geodesic submanifold of $(T^r M, \kappa_g)$ (see also [PW]). These properties indicate that κ_g is indeed a natural (Kähler) extension of g. When M is compact, strict plurisubharmonicity of ρ in virtue of Grauert's theorem implies that $T^r M$ is in fact a Stein manifold.

It is always possible to express the almost complex tensor J_A of the adapted complex structure in terms of Jacobi fields and analytic continuation (see [LSz91] or Theorem 2.2.3), but on a symmetric space we can do better. The Jacobi equation can be solved explicitely and we get an explicit formula for J_A as follows.

Denote by $\mathcal{R}$ the curvature tensor of the metric and let $z \in T_p M$. The operator

$$R_z(.) = \mathcal{R}(., z)z \tag{1.2.12}$$

is the curvature operator (or Jacobi operator) associated to z. Let $\mathring{\mathrm{T}}M := TM \setminus M$. Then [Sz2, Theorem 2.5] with a little bit of calculations implies :

Proposition 1.2.3. *Let (M, g) be a compact symmetric space. Let $z \in \mathring{T}M$. Then J_A maps the horizontal subspace at z to the vertical subspace and vice versa. More precisely let $v_n = z/\|z\|$ and $v_1, v_2, \ldots, v_n$ an orthonormal basis of $T_{\pi z}M$ consisting of eigenvectors of the Jacobi operator R_{v_n} with eigenvalue $\Lambda_j, j = 1, \ldots, n$. Let $h(x) := x \coth(x)$. Let $(v_j)_z^H$ (resp. $(v_j)_z^V$) be the horizontal (resp. vertical) lift of v_j to the point $z, j = 1, \ldots, n$. Then*

$$J_A(v_j)_z^H = h(\sqrt{\Lambda_j}\|z\|)(v_j)_z^V. \tag{1.2.13}$$

ie. with the positive, real-analytic function $t(x) := h(\sqrt{x})$, The matrix of J_A in the horizontal and vertical splitting $T_z(TM) = H_z \oplus V_z$ is:

$$J_A|_z = \begin{bmatrix} 0 & -(t(R_z))^{-1} \\ t(R_z) & 0 \end{bmatrix}. \tag{1.2.14}$$

Chapter 2

Automorphisms of certain Stein manifolds

2.1 The main results.

Let (M, g) be a compact Riemannian manifold and $0 < r \leq \infty$. The principal results of this chapter are as follows.

We call $r \leq \infty$ the *maximal radius* if adapted complex structure exists on $T^r M$ but it does not exists on any other tube $T^s M$ with $s > r$.

In what follows, it is sometimes important whether r is maximal or not. The advantage of a non-maximal radius is that in this case $T^r M$ is a relatively compact subdomain of a Stein manifold ($T^s M$, for some $s > r$) with smooth, strictly pseudoconvex boundary.

Theorem 2.1.1 (Szőke [Sz95]). *Let (M, g) and (N, h) be n-dimensional compact Riemannian manifolds and $0 < r, s < \infty$. Assume that adapted complex structures exist on $T^r M$ and $T^s N$. Let κ_g and κ_h be the induced Kähler metrics. Suppose*

$$\Phi : (T^r M, \kappa_g) \longrightarrow (T^s N, \kappa_h). \tag{2.1.1}$$

is a biholomorphic isometry. Then $r = s$. Let f be the restriction of Φ to M. Then f maps M isometrically onto N and the induced map f_* agrees with Φ on $T^r M$.

Later Burns and Hind [BH] proved that the theorem remains valid if $\Phi-$ is only a biholomorphism.

In section 2.4 we treat automorphisms (only biholomorphic selfmaps, without the isometry condition) of the complex manifold $T^r M$, when r is finite. As a corollary of a result of N. Mok ([Mo]), we show

Theorem 2.1.2 (Szőke, [Sz95]). *Let (M, g) be a compact Riemannian manifold. Assume that an adapted complex structure exists on $T^r M$ for some $0 < r < \infty$. Then*

(a) Aut$(T^r M)$ is a compact Lie group.

(b) If M is orientable, or the universal cover is compact, then for any $0 < s < S \leq r$, the complex manifolds $T^s M$ and $T^S M$ are not biholomorphic.

In section 2.5 we prove a similar rigidity result as Theorem 2.1.1 above when $r = \infty$.

Theorem 2.1.3 (Szőke [Sz95]). *Let (M, g) and (N, h) be compact Riemannian manifolds. Assume the adapted complex structure exist on TM and TN, and κ_g, κ_h are the induced Kähler metrics. Suppose that $H^1(M, \mathbb{R}) = 0$. Let*

$$\Phi : (TM, \kappa_g) \longrightarrow (TN, \kappa_h), \tag{2.1.2}$$

be a biholomorphic isometry. Then Φ maps M diffeomorphically onto N, the restriction map

$$f := \Phi|_M : (M, g) \longrightarrow (N, h),$$

is an isometry and $\Phi \equiv f_*$.

The isometry condition is important in the theorem, biholomorphism in itself is not enough. As the example 2.5.3 shows that $Aut(T(T^n))$ is infinite dimensional for the compact flat torus T^n.

Section 2.6 treats the isometry group action on the tangent bundle of a Riemannian manifold.

Theorem 2.1.4 (Szőke [Sz95]). *Let (M, g) be a compact Riemannian manifold that admits an adapted complex structure on its entire tangent bundle. Denote by G the isometry group of (M, g). Consider G as a transformation group, acting on TM by the induced action. This G-action extends to a group action of $G_{\mathbb{C}}$ (the complexification of G) on TM and this action is almost effective and holomorphic.*

2.2 Calculating the metric κ_g.

In this section we are going to give explicit formulas for κ_g using symplectic frames. First we need to recall some more notation.

Denote by $M^n_{\mathbb{C}}$ the set of $n \times n$ complex matrices. For a $Z \in M^n_{\mathbb{C}}$, $Z^\top$ denotes the transpose of Z. For a real matrix X, $X > 0$ indicates, that X is symmetric and positive definite.

The subset of $M^n_{\mathbb{C}}$,

$$\mathcal{H}^n = \{Z \in M^n_{\mathbb{C}} \mid Z = Z^\top, \quad \mathrm{Im}\ Z > 0\}$$

is called the *Siegel upper half plane*. In particular $\mathcal{H}^1$ is the ordinary upper half plane, that we also denote by $\mathbb{C}^+$.

Let (V, ω) be a symplectic vector space. A complex structure $J : V \to V$ is said to *calibrate* the symplectic form ω, if the bilinear form $\omega(u, Jv)$, $u, v \in V$ is symmetric and positive definite. We will denote the set of calibrating complex structures on (V, ω) by $\mathcal{J}_\omega$.

Proposition 2.2.1 (Szőke, [Sz95]). *Let (V^{2n}, ω) be a symplectic vector space. Then $\mathcal{J}_\omega$ can be identified with $\mathcal{H}^n$ as follows. Fix a symplectic base $u_1, \ldots, u_n$, $v_1, \ldots, v_n$. If $J \in \mathcal{J}_\omega$, then the n-tuples $\{u_j\}_{j=1}^n$ and $\{v_j\}_{j=1}^n$ both provide a $\mathbb{C}$ basis of the complex vector space (V, J). Denote by $Z := (f_{kl})$ the transition matrix, i.e.*

$$v_k = \sum_l f_{lk} u_l, \qquad k = 1, \ldots, n. \tag{2.2.1}$$

Then $Z \in \mathcal{H}^n$. And vice versa, assume that $Z = Re\ Z + iIm\ Z \in \mathcal{H}^n$. Then declaring $\{u_j\}_{j=1}^n$ and $\{v_j\}_{j=1}^n$ to be a $\mathbb{C}$ basis with transition matrix Z (as in (2.2.1)*), we define a complex structure $J_Z : V \to V$ which calibrates ω, and can be expressed as*

$$J_Z u_k = \sum_{j=1}^n (Im\ Z)^{-1}_{jk}\,[v_j - \sum_{l=1}^n (Re\ Z_{lj})u_l]. \tag{2.2.2}$$

The matrix of the symmetric, positive definite bilinear form $\omega(., J.)$ in the base u_j, v_k is

$$\begin{pmatrix} [Im\ Z]^{-1} & [Im\ Z]^{-1} Re\ Z \\ Re\ Z[Im\ Z]^{-1} & Re\ Z[Im\ Z]^{-1} Im\ Z + Im\ Z \end{pmatrix}$$

Proof. The proof is an easy calculation, left to the reader. □

Proposition 2.2.2 (Szőke, [Sz95])**.** *Let X and Y be complex manifolds and $\epsilon > 0$. Suppose we have a smooth map $f : (-\epsilon, \epsilon) \times X \to Y$ and for every fixed $-\epsilon < t < \epsilon$, the map $f_t(.) := f(t, .) : X \to Y$ is holomorphic. Let*

$$\xi = df_t/dt|_{t=0}\,.$$

This is a section of $f_0^* TY_0$. Then $\xi^{1,0}$ is a holomorphic section of $f_0^* T^{1,0}Y$.

Proof. (cf. [LSz91, Prop. 5.1, p. 698]) The statement is local, therefore we can assume $X = D_1 \subset \mathbb{C}^n$, $Y = D_2 \subset \mathbb{C}^m$ and $f : (-\epsilon, \epsilon) \times D_1 \to D_2$. We have to show that $df/dt|_{t=0}$ is holomorphic. But

$$\bar\partial_\zeta \left(\frac{df}{dt}\bigg|_{t=0} (t, \zeta) \right) = \frac{d}{dt}\bigg|_{t=0} \left(\bar\partial_\zeta f(t, \zeta)\right) \equiv 0.$$

□

Armed with the last two propositions, we are now ready to prove the main theorem of this section.

Theorem 2.2.3 (Szőke, [Sz95])**.** *Let (M, g) be a Riemannian manifold and $0 < r \le \infty$. Assume that adapted complex structure exists on $T^r M$. Let γ be a unit speed geodesic and $(\xi_1, \dots \xi_n, \eta_1, \dots, \eta_n)$ be a symplectic frame along γ_*. Let $D_r := \{\zeta = \sigma + i\tau \in \mathbb{C} \mid |\tau| < r\}$. Denote by $S \subset \mathbb{R}$ the discrete set of points $\sigma \in \mathbb{R}$, for which the vectors $\xi_1(\sigma), \dots, \xi_n(\sigma) \in T_{\gamma(\sigma)}M$ are linearly dependent and by φ_{jk} the smooth functions on $\mathbb{R} \setminus S$, such that* (1.2.8) *holds. Then there exist meromorphic functions $f_{jk} : D_r \to \mathbb{C} \cup \{\infty\}$, $1 \le j, k \le n$, which are holomorphic on $D_r \setminus S$ and $f_{jk}(\sigma) = \varphi_{jk}(\sigma)$, for $\sigma \in \mathbb{R} \setminus S$. Let $\zeta = \sigma + i\tau \in D_r \setminus \mathbb{R}$. Then*

$$F(\zeta) := (f_{jk}(\zeta)) \in \mathcal{H}^n, \quad \text{if} \quad \tau > 0, \qquad \text{and} \quad -F(\zeta) \in \mathcal{H}^n, \quad \text{if} \quad \tau < 0. \tag{2.2.3}$$

Let $T\mathbb{R} \cong \mathbb{C}$ as in (1.2.9) *and $p = \gamma_* \zeta$. Then*

$$\eta_k^{1,0}(\gamma_*(\zeta)) = \sum_{j=1}^n f_{jk}(\zeta)\xi_j^{1,0}(\gamma_*(\zeta)), \qquad \zeta \in D_r \setminus S, \qquad k = 1, \dots, n. \tag{2.2.4}$$

Let $J_p : T_p(TM) \to T_p(TM)$ be the adapted complex structure. If $\tau > 0$, then J_p can be expressed as

$$J_p\xi_k(p) = \sum_{l=1}^{n}(Im\ F)^{-1}_{lk}(\zeta)\left[\eta_l(p) - \sum_{j=1}^{n}(Re\ f_{jl})(\zeta)\xi_j(p)\right], \tag{2.2.5}$$

and for the Kähler metric κ_g we have

$$\begin{aligned}\langle\xi_i(p),\xi_k(p)\rangle_{\kappa_g} &= ||p||_g(Im\ F)^{-1}_{ik}(\zeta),\\ \langle\xi_i(p),\eta_k(p)\rangle_{\kappa_g} &= ||p||_g\left[(Im\ F)^{-1}Re\ F\right]_{ik}(\zeta),\\ \langle\eta_i(p),\eta_k(p)\rangle_{\kappa_g} &= ||p||_g\left[Re\ F(Im\ F)^{-1}Re\ F + Im\ F\right]_{ik}(\zeta).\end{aligned} \tag{2.2.6}$$

If $\tau < 0$, then formulas (2.2.5) *and* (2.2.6) *are still valid if we replace $ImF(\zeta)$ by -$ImF(\zeta)$.*

Proof. (For (2.2.3) and (2.2.5) a slightly different proof was given in [LSz91].)

As we mentioned above, the Kähler form of κ_g is ω, the symplectic form of the tangent bundle. Thus for any $z \in T^rM \setminus M$, and $X, Y \in T_z(TM)$,

$$\langle X, Y\rangle_{\kappa_g} = -\omega(JX, Y) = ||z||_g\left[(1/||z||_g)\omega(X, JY)\right]. \tag{2.2.7}$$

This implies that the complex structure $J_z : T_z(TM) \to T_z(TM)$ calibrates the symplectic form $(1/||z||_g)\omega_z$. For $\tau > 0$ (resp. $\tau < 0$) (according to Proposition 1.2.1) $\{\xi_j(\gamma_*\zeta), \eta_k(\gamma_*\zeta)\}$ (resp. $\{\xi_j(\gamma_*\zeta), -\eta_k(\gamma_*\zeta)\}$) is a symplectic basis of

$$(T_p(TM), (1/||p||_g)\omega_p).$$

Then Proposition 2.2.1 tells us that

$$\{\xi_j^{1,0}(\gamma_*(\zeta))\}_{j=1}^n \qquad \text{resp.} \qquad \{\eta_j^{1,0}(\gamma_*(\zeta))\}_{j=1}^n$$

are both $\mathbb{C}$-bases of the vector space $T^{1,0}_{\gamma_*(\zeta)}(TM)$. If $\sigma \in \mathbb{R}\backslash S$, then $\{\xi_j(\gamma(\sigma))\}_{j=1}^n$ being an $\mathbb{R}$ basis of the vector space $T_{\gamma(\sigma)}M$, is also a $\mathbb{C}$ basis of $T^{1,0}_{\gamma(\sigma)}(TM)$. Therefore for any $\zeta \in D_r \setminus S$, there exists a matrix $F(\zeta) = (f_{jk}(\zeta))_{j,k=1}^n$ such that (2.2.4) holds. Then (1.2.8) gives the equality $(\varphi_{jk}(\sigma)) = (f_{jk}(\sigma)), \sigma \in \mathbb{R}\backslash S$. From Proposition 2.2.2 we know that the maps

$$\xi_j^{1,0}, \eta_j^{1,0} : D_r \longrightarrow T^{1,0}(T^rM), \qquad j, k = 1, \ldots, n$$

are all holomorphic. Hence F is holomorphic on $D_r \setminus S$ and meromorphic on D_r. The rest follows from Proposition 2.2.1, and (2.2.1).

□

2.3 Holomorphic isometries of tubes with finite radius.

In this section we only deal with tubes T^rM, with $0 < r < \infty$. The case $r = \infty$ will be treated separately in sections 2.5 and 2.6.

Proposition 2.3.1 ("Schwarz lemma", Szőke, [Sz95]). *Let (M^n, g) and (N^n, h) be compact Riemannian manifolds and $0 < r, s < \infty$. Assume that adapted complex structure exists on T^rM and T^sN. Let*

$$\Phi : T^rM \to T^sN$$

be a holomorphic map such that $\Phi(M) \subset N$. Then

$$r||\Phi(p)||_h \leq s||p||_g, \qquad p \in T^rM. \tag{2.3.1}$$

Proof. Define the functions u and v on T^rM by $u(p) := ||p||_g$ and $v(p) := ||\Phi(p)||_h$. Let η, ϵ, δ be small positive numbers and define c_η and w_ϵ by

$$c_\eta := \max\{v(p) \mid ||p||_g = r - \eta\} \qquad \text{and} \qquad w_\epsilon := \frac{c_\eta}{r-\eta}u + \epsilon.$$

Denote by $D_{\delta\eta} \subset T^rM$ the domain

$$D_{\delta\eta} := \{p \in T^rM \mid \delta < u(p) < r - \eta\}.$$

For fixed ϵ, η and small enough $\delta > 0$,

$$w_\epsilon|_{\partial D_{\delta\eta}} \geq v|_{\partial D_{\delta\eta}}.$$

It follows from [LSz91, Theorem 5.6], that u and v (and therefore w_ϵ as well) are plurisubharmonic functions and they satisfy the complex homogeneous Monge-Ampère equation on $T^rM \setminus M$. Applying the minimum principle of Bedford and Taylor (see [BT]) for the functions w_ϵ and v on the domain $D_{\delta\eta}$ we get

$$w_\epsilon(q) \geq v(q), \qquad \text{for} \quad q \in D_{\delta\eta}. \tag{2.3.2}$$

Because v goes to zero as we approach M, (2.3.2) also holds for any $q \in T^rM$ with $u(q) \leq r - \eta$. Letting ϵ go to zero yields

$$v(p) \leq \frac{c_\eta}{r-\eta}u(p) \leq \frac{s}{r-\eta}u(p), \qquad \text{when} \quad u(p) \leq r - \eta. \tag{2.3.3}$$

Fix now a point p in T^rM. Then for every small enough η (2.3.3) holds. Letting now $\eta \to 0$ we obtain (2.3.1). □

Theorem 2.3.2 (Szőke, [Sz95]). *Let (M^n, g), (N^n, h) be compact Riemannian manifolds and $0 < r, s < \infty$. Assume that adapted complex structures exist on T^rM and T^sN. Let*

$$\Phi : T^rM \to T^sN$$

be a biholomorphism, such that $\Phi(M) \subset N$. Then

$$f := \Phi|_M : (M, sg) \to (N, rh)$$

is an isometry, onto and $\Phi \equiv f_* : T^rM \to T^sN$.

Proof. The fact that Φ is a biholomorphism and that N is compact and connected gives that f is indeed onto. Denote by κ_g and κ_h the Kähler metrics on T^rM and T^sN, induced by the strictly plurisubharmonic Kähler potential

function $|| \, ||_g^2$ and $|| \, ||_h^2$ respectively (see (1.2.11)). Applying our (2.3.1) Schwarz lemma for both Φ and its inverse, we obtain

$$r^2||\Phi(p)||_h^2 = s^2||p||_g^2, \qquad p \in T^rM. \tag{2.3.4}$$

It follows easily from its definition that rescaling the metric does not change the induced complex structure, i.e. for any $\lambda > 0$, g and λg have the same adapted complex structures defined on the same tube except the radius is measured with different scales. Thus (2.3.4) yields, that

$$\Phi : (T^rM, \kappa_{sg}) \to (T^sN, \kappa_{rh})$$

is a biholomorphic isometry. This, together with the fact that along the zero section the metric κ_{sg} (resp. κ_{rh}) is just sg (resp. rh), (see the remarks after (1.2.11)), implies that

$$f : (M, sg) \longrightarrow (N, rh)$$

is indeed an isometry itself. Hence f_*, (see [LSz91]) and Φ are both biholomorphic and agree on the maximal dimensional totally real submanifold M. This implies that they must agree everywhere. □

2.3.1 Proof of Theorem 2.1.1

We can assume that $s \geq r$. Denote by ρ_1 and ρ_2 the norm-square functions on T^rM and T^sN respectively. Now (2.1.1) yields

$$\partial\bar{\partial}\rho_1 = \Phi^*\partial\bar{\partial}\rho_2 = \partial\bar{\partial}(\rho_2 \circ \Phi). \tag{2.3.5}$$

Let

$$\lambda := \rho_2 \circ \Phi - \rho_1 + r^2 - s^2.$$

It follows from (2.3.5) that λ is a bounded pluriharmonic function on T^rM.

Let $\gamma : \mathbb{R} \to M$ be an arbitrary unit speed geodesic, parametrized by arclength. Then $v := \lambda \circ \gamma_*$ is a bounded harmonic function on the domain $D := \{\sigma + i\tau \mid \sigma \in \mathbb{R}, |\tau| < r\}$. If $\zeta_n \in D$, $\zeta_n \to z_0 \in \partial D$, then $v(\zeta_n)$ must go to zero (Φ is a biholomorphism). This yields that v must vanish everywhere. This is true for every geodesic, thus λ must also vanish identically. Hence we obtain

$$||\Phi(p)||_h^2 = ||p||_g^2 + s^2 - r^2, \qquad p \in T^rM. \tag{2.3.6}$$

Φ is biholomorphic, so we can take a point $q \in T^rM$ with $||\Phi(q)||_h = 0$. Since we assumed $s \geq r$, (2.3.6) implies $s = r$ and thus (2.3.6) reads as

$$||\Phi(p)||_h^2 = ||p||_g^2, \qquad p \in T^rM.$$

Theorem 2.3.2 now yields our claim.

2.4 Biholomorphisms of tubes with finite radius.

Now that we completely described all the biholomorphic isometries of our tube, we would like to drop the condition of isometry and want to study the biholomorphism group of T^rM, that we denote by $\mathrm{Aut}(T^rM)$.

Theorem 2.4.1 (Szőke, [Sz95]). *Let X be a complex manifold. Suppose that X admits a strictly plurisubharmonic bounded exhaustion function. Then X is Kobayashi hyperbolic and $Aut(X)$ is a Lie group.*

Proof. The results in [Si, Corollary 5] and [Kob, Theorem V.2.1], together imply our statement.

□

Theorem 2.4.2 (Szőke, [Sz95]). *Let X^n be a complex manifold which admits a bounded strictly plurisubharmonic exhaustion function. Suppose that the n-th homology group, $H_n(X, \mathbb{Z})$ is finitely generated and nonzero. Then $Aut(X)$ is a compact Lie group. Furthermore if $f : X \to X$ is a holomorphic map which induces an isomorphism on $H_n(X, \mathbb{Z})$ and f is injective, then $f \in Aut(X)$.*

Proof. The theorem is essentially contained in [Mo, Theorem 1], , except that Mok works with manifolds with a stronger assumption than ours. Namely he assumes, in addition to our conditions, that X is hyperbolic in the sense of Carathéodory. But the only place in his proof where he uses this extra condition is to prove his Proposition 1.1. To get this proposition, in fact it suffices to know that X is a taut manifold, which property our X has by virtue of [Si, Corollary 5], and [Ba, Theorem 2] . □

Proof of Theorem 2.1.2. From [LSz91] we know that $\rho : T^rM \to \mathbb{R}$ (ρ is from (1.2.10)) is a bounded strictly plurisubharmonic exhaustion function. Thus according to Theorem 2.4.1, $\mathrm{Aut}(T^rM)$ is a Lie group. If M is orientable, then $H_n(T^rM, \mathbb{Z}) \cong H_n(M, \mathbb{Z}) \cong \mathbb{Z}$. Therefore the compactness of the automorphism group and (b) follows from Theorem 2.4.2. (When $S < r$ or the adapted complex structure extends to a strictly larger tube than T^rM, then we do not need to rely on Mok's theorem, it is enough to quote a much simplier fact [Bed1, Corollary 1.5]). The case when M is not necessarily orientable but its universal cover is compact, follows from the part we have already shown, by standard lifting arguments.

Now suppose that M is arbitrary. Denote by $\widehat{M}$ the double sheeted orientable cover of M and let $p : \widehat{M} \to M$ be the projection map. Let $\widehat{g}$ be the pull back of g onto $\widehat{M}$. Since M is a compact differentiable manifold, its fundamental group is finitely generated. In particular for a fixed base point $x_0 \in T^rM, \pi_1(T^rM, x_0)$ has only finitely many subgroups of index 2. Denote these groups by $G_1, \ldots, G_N$.

Let $\hat{x}_0 \in T^r\widehat{M}$, be also fixed, such that $p_*(\hat{x}_0) = x_0$. We can assume that $G_1 = p_*\pi_1(T^r\widehat{M}, \hat{x}_0)$. Now choosing any other base point $x_1 \in T^rM$, we can connect x_0 and x_1 by a curve χ, which induces an isomorphism between $\pi_1(T^rM, x_0)$, and $\pi_1(T^rM, x_1)$. This way we also identified the subgroups of $\pi_1(T^rM, x_1)$ with index 2 with the groups G_j. It is easy to see that this identification is independent of the choice of the curve χ, since the subgroups G_j are normal. Thus we can talk about the groups G_j independently of the base point.

Now if $\Phi \in \mathrm{Aut}(T^rM), y \in T^rM$, and $\Phi_* : \pi_1(T^rM, y) \to \pi_1(T^rM, \Phi(y))$ maps a group G_j to G_l, then it is not hard to see that for any other point $z \in T^rM$ the induced map $\Phi_* : \pi_1(T^rM, z) \to \pi_1(T^rM, \Phi(z))$ will also map G_j to G_l. Hence it makes sence to say that an automorphism Φ maps the group G_j to G_l.

Now if $\Phi_1, \Phi_2 \in \mathrm{Aut}(T^r M)$, maps the group G_1 to the same group G_l, then the automorphism $\Phi = \Phi_2^{-1} \circ \Phi_1$ can be lifted to an automorphism of $T^r \widehat{M}$, since Φ_* preserves $G_1 = p_* \pi_1(T^r \widehat{M}, \hat{x}_0)$. Denote by G the subgroup of $\mathrm{Aut}(T^r M)$ consisting of all elements that preserves the subgroup G_1. The above argument shows that G has finite index and the already proved part of (a) gives that G is compact. This implies that $\mathrm{Aut}(T^r M)$ is compact as well. $\square$

2.5 Holomorphic isometries between tangent bundles.

In this and section 2.6 we will be working with Riemannian manifolds (M, g), which admit adapted complex structures on their entire tangent bundle. The complex manifold TM will never be hyperbolic, unlike the tubes with finite radius (any geodesic γ in M induces a nontrivial holomorphic map $\gamma_* : T\mathbb{R} \simeq \mathbb{C} \to TM$). Hence we do not a priori know whether $\mathrm{Aut}(TM)$ is a Lie group or not. In fact this group is not always finite dimensional, as the following example shows.

Example 2.5.1 (Szőke, [Sz95])**.** *The tangent bundle TS^1 of the unit circle, equipped with the adapted complex structure induced by the standard metric on S^1, is biholomorphic to the punctured complex plane $\mathbb{C}^*$. Let $T^n = S^1 \times \cdots \times S^1$ the n-dimensional torus with the product metric. Then $T(T^n)$ with its adapted complex structure is biholomorphic to $\mathbb{C}^{*n} := \mathbb{C}^* \times \cdots \times \mathbb{C}^*$. For any holomorphic function $f : \mathbb{C} \to \mathbb{C}$,*

$$(\mathbb{C}^*)^n \longrightarrow (\mathbb{C}^*)^n \tag{2.5.1}$$

$$\Phi : (z_1, z_2, z_3, \ldots, z_n) \longmapsto (e^{f(z_1 z_2)} z_1, e^{-f(z_1 z_2)} z_2, z_3, \ldots, z_n) \tag{2.5.2}$$

is an element of $Aut(\mathbb{C}^{*n})$, showing the infinite dimensionality of $Aut(T(T^n))$.

Instead of a torus we can take any compact Lie group K different from the unit circle, $Aut(K_{\mathbb{C}})$) will be infinite dimensional [Sz98, Corollary 2.6]

Example 2.5.2 (Szőke, [Sz95])**.** *Suppose that Γ is a lattice in $\mathbb{R}^n$ such that the quotient manifold $M := \mathbb{R}^n/\Gamma$, $i = 1, 2$ is diffeomorphic to the n-torus. Denote by g_Γ the induced metric on M. Considering the lattice Γ as being in the totally real part of $\mathbb{C}^n = \mathbb{R}^n + i\mathbb{R}^n$, we can form the complex manifold $\mathbb{C}^n/\Gamma$. Since $\mathbb{C}^n = T\mathbb{R}^n$ carries the complex structure adapted to the Euclidean metric on $\mathbb{R}^n$, the underlying differentiable manifold of the complex manifold $\mathbb{C}^n/\Gamma$ will be TM and the complex structure on it is adapted to g_Γ.*

It is straightforward to check that all the complex manifolds $\mathbb{C}^n/\Gamma$ will be biholomorphic to $(\mathbb{C}^*)^n$, but the arising Riemannian manifolds (M, g_Γ), for different choices of Γ, are not all isometric.

This example shows that the analogue of Theorem 2.3.2 for tubes with infinite radius is false.

As a contrast to Example 2.5.2, we can prove now a rigidity theorem for tubes with infinite radius.

2.5.1 Proof of Theorem 2.1.3

As in the proof of Theorem 2.1.1, let ρ_1 and ρ_2 be the norm-square functions on TM and TN. (2.1.2) implies

$$\partial\bar{\partial}\rho_1 = \Phi^*\partial\bar{\partial}\rho_2 = \partial\bar{\partial}(\rho_2 \circ \Phi). \tag{2.5.3}$$

Let

$$\lambda := \rho_2 \circ \Phi - \rho_1. \tag{2.5.4}$$

According to (2.5.3), λ is a pluriharmonic function on TM. Since $H^1(M, \mathbb{R}) = 0$, we can find a holomorphic function $\Lambda : TM \to \mathbb{C}$, such that the imaginary part of Λ is the function λ.

Lemma 2.5.3 (Szőke, [Sz95])**.** *Let $\gamma : \mathbb{R} \to M$ be a unit speed geodesic. Then there exist $A, \beta_\gamma \in \mathbb{R}$, ($\beta_\gamma$ depends on γ) such that for every $z = \sigma + i\tau \in \mathbb{C}$,*

$$\rho_2 \circ \Phi(\gamma_* z) = \tau^2 + \beta_\gamma \tau + A \ = \rho_1(\gamma_*(z)) + \beta_\gamma \tau + A \tag{2.5.5}$$

Proof of Lemma 2.5.3. Let $x \in N$ and $q \in T_xN$. Denote by dist_{κ_g} and dist_{κ_h} the distance function for the metric κ_g and κ_h respectively. From [LSz91] or [PW] we know that

$$\mathrm{dist}_{\kappa_h}(q, x) = \mathrm{dist}_{\kappa_h}(q, N) = ||q||_h. \tag{2.5.6}$$

Let now m be an arbitrary point of M and $p \in T_mM$. Denote by $x \in N$ the image of the point $\Phi(p)$ under the projection map $\pi : TN \to N$. Then (2.5.6) implies

$$\begin{aligned} ||\Phi(p)||_h &= \mathrm{dist}_{\kappa_h}(\Phi(p), x) \\ &\leq \mathrm{dist}_{\kappa_h}(\Phi(p), \Phi(m)) + \mathrm{dist}_{\kappa_h}(\Phi(m), x) \\ &= \mathrm{dist}_{\kappa_g}(p, m) + \mathrm{dist}_{\kappa_h}(\Phi(m), x) \\ &\leq ||p||_g + \max_{a \in M, b \in N} \mathrm{dist}_{\kappa_h}(\Phi(a), b) = ||p||_g + C. \end{aligned} \tag{2.5.7}$$

Taking square of both sides of (2.5.7), we obtain

$$\rho_2 \circ \Phi(p) \leq \rho_1(p) + 2||p||_g C + C^2. \tag{2.5.8}$$

Since λ is pluriharmonic (see (2.5.3) and (2.5.4)) and γ_* is holomorphic, the function $v(z) := \lambda(\gamma_*(z))$ is harmonic on $\mathbb{C}$. The estimate (2.5.8) with $p = \gamma_*(z)$ gives

$$v(z) = \lambda(\gamma_*(z)) = \rho_2(\Phi(\gamma_*(z)) - \rho_1(\gamma_* z) \leq 2|\tau|C + C^2. \tag{2.5.9}$$

Harmonic functions on the complex plane with such growth condition can only be linear (see [SaZ, (10.13), p. 335]), hence there exist β_γ, A_γ such that

$$v(z) = \beta_\gamma \tau + A_\gamma. \tag{2.5.10}$$

Notice that A_γ is the value that the function λ takes along the curve γ. In particular λ is a constant function along any geodesic in M. This implies that A_γ does not depend on γ. (2.5.9) and (2.5.10) now imply our claim.

□

End of the proof of Theorem 2.1.3. Let now γ be any unit speed geodesic in M. It follows from (2.5.4) and (2.5.5) that the holomorphic function $\Lambda(\gamma_*(z))$ must be of the form

$$\Lambda(\gamma_*(z)) = \beta_\gamma z + iA + \tilde{A}_\gamma, \tag{2.5.11}$$

for some real number $\tilde{A}_\gamma$. By our assumption M is compact and hence the real part of Λ is bounded there. (2.5.11) implies that along the geodesic γ,

$$\text{Re } \Lambda(\gamma_*(\sigma)) = \beta_\gamma \sigma + \widetilde{A}_\gamma.$$

This yields $\beta_\gamma = 0$. Therefore (2.5.5) reads as

$$\rho_2(\Phi(\gamma_*(z))) = \rho_1(\gamma_*(z)) + A. \tag{2.5.12}$$

This is true for every unit speed geodesic. Thus for every $p \in TM$,

$$||\Phi(p)||_h^2 = ||p||_g^2 + A. \tag{2.5.13}$$

Plugging a point of M into (2.5.13) we obtain that A must be nonnegative. On the other hand Φ is onto and thus for some $p \in TM$ the left side of (2.5.13) must vanish. This gives that $A = 0$ and therefore Φ maps M diffeomorphically onto N. Φ was an isometry, thus its restriction to the zero section, which we will call f, will also be an isometry. Since the restriction of the Kähler metric κ_g (resp. κ_h) gives back the original metric g (resp. h) (see the remarks after (1.2.11)), $f : (M, g) \to (N, h)$ is also an isometry. The biholomorphisms Φ and f_* (see [LSz91]) agree on the maximal dimensional totally real subset M, hence they must agree everywhere. □

Remarks. If $H^1(M, \mathbb{R})$ is nontrivial, we can have other biholomorphic isometries besides the ones that come from isometries between M and N. For instance take (M, g_Γ) as in Example 2.5.2. In $\mathbb{C}^n$ any translation with a nonzero, purely imaginary vector is a holomorphic isometry, which descends to $T(M) = \mathbb{C}^n/\Gamma$ and does not preserve the zero section.

2.6 Complexifying the isometry group action.

From [PW, Sz91] one knows that the round metric on the n-dimensional sphere S^n induces its adapted complex structure on the entire tangent bundle and as a complex manifold TS^n is biholomorphic to the affine hyperquadric, Q^n

$$Q^n = \{(z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} \mid z_1^2 + \cdots + z_{n+1}^2 = 1\}.$$

Question: what is the group of biholomorphisms of $Q^n \cong TS^n$? Even though we do not know the answer to this question, it is clear that Q^n admits many biholomorphic selfmaps that do not arise from an isometry of S^n. Namely the complex orthogonal group $O(n+1, \mathbb{C})$ is a subgroup of $\text{Aut}(Q^n)$.

The main purpose of this section is to construct, for a given compact Riemannian manifold (M, g) with adapted complex structure on TM, elements of $\text{Aut}(TM)$ which are not of the form $\varphi_* : TM \to TM$, for some $\varphi \in \text{Isom}(M, g)$. First we need some preparatory lemmas and propositions.

Theorem 2.6.1 (Fatou [Koo]). *Let u be a positive harmonic function defined on $\mathbb{C}^+$. Then there exist a non-negative Borel measure μ on the real line and a non-negative real number α, such that $\int_{\mathbb{R}} 1/(1+t^2)d\mu(t)$ is finite and for any $\zeta = \sigma + i\tau \in \mathbb{C}^+$,*

$$u(\zeta) = \alpha\tau + \frac{1}{\pi}\int_{\mathbb{R}} \frac{\tau}{|w-t|^2} d\mu(t). \tag{2.6.1}$$

Define the domain $D_{\sqcup} \subset \mathbb{C}$ by

$$D_{\sqcup} := \{\zeta = \sigma + i\tau \in \mathbb{C} \mid |\sigma| < 1, 1 < \tau\}.$$

Lemma 2.6.2 (Szőke, [Sz95]). *There exists real numbers $0 < c_1 < 1 < c_2$ with the property that for any $\zeta = \sigma + i\tau \in D_{\sqcup}$ and $t \in \mathbb{R}$ we have*

$$\begin{aligned} \frac{c_1}{(1+t^2)}\frac{1}{\tau} &< \frac{\tau}{\tau^2 + (\sigma - t)^2} \\ \frac{c_2}{1+t^2} &> \frac{(\tau-1)^2}{[(\sigma-t)^2+1][(\sigma-t)^2+\tau^2]}. \end{aligned} \tag{2.6.2}$$

Proof. An easy calculation, left to the reader. □

Lemma 2.6.3 (Szőke, [Sz95]). *There exist positive real numbers c, C such that for every holomorphic function $f : \mathbb{C}^+ \to \mathbb{C}^+$, and $\zeta = \sigma + i\tau \in D_{\sqcup}$ we have*

$$c\left(\min_{\substack{|\sigma|\le 1\\ z=\sigma+i}} [Imf(z)]\right)\frac{1}{\tau} < Imf(\zeta) < C\left(\max_{\substack{|\sigma|\le 1\\ z=\sigma+i}} |f(z)|\right)\tau, \tag{2.6.3}$$

$$|Re\ f(\zeta)| \le C \max_{\substack{|\sigma|\le 1\\ z=\sigma+i}} |f(z)|. \tag{2.6.4}$$

Proof. Applying (2.6.1) to the imaginary part of f, for any $\zeta = \sigma + i\tau \in \mathbb{C}^+$ we get (for some nonnegative α and nonnegative Borel measure μ)

$$\text{Im } f(\sigma + i\tau) = \alpha\tau + \frac{1}{\pi}\int_{\mathbb{R}} \frac{\tau}{(\sigma-t)^2+\tau^2} d\mu(t). \tag{2.6.5}$$

In particular for $\zeta = i$, (2.6.5) gives

$$\text{Im } f(i) = \alpha + \frac{1}{\pi}\int_{\mathbb{R}} \frac{1}{1+t^2} d\mu(t). \tag{2.6.6}$$

Since μ is nonnegative, this yields that $0 \le \alpha \le \text{Im} f(i)$.

Applying the estimate (2.6.2) to the integrand in (2.6.5) and using (2.6.6), we obtain

$$\text{Im } f(\sigma+i\tau) > \alpha\tau + \frac{1}{\pi}\int_{\mathbb{R}} \frac{c_1}{\tau(1+t^2)} d\mu(t) = \frac{\alpha(\tau^2 - c_1)}{\tau} + c_1\frac{\text{Im} f(i)}{\tau} \ge \frac{\text{Im} f(i)}{\tau} c_1.$$

This proves the left side of (2.6.3). To prove the rest, we have to differentiate (2.6.5) with respect to ζ:

$$f'(\sigma + i\tau) = \alpha + \frac{1}{\pi}\int_{\mathbb{R}} \frac{1}{(\sigma - t + i\tau)^2} d\mu(t).$$

This yields, by changing the order of integration,

$$\begin{aligned} f(\zeta) &= f(\sigma+i) + \int_{\sigma+i}^{\zeta} f'(z)dz \\ &= f(\sigma+i) + i\alpha(\tau-1) + \frac{1}{\pi}\int_{\mathbb{R}} \frac{i(\tau-1)}{(\sigma-t+i)(\sigma-t+i\tau)} d\mu(t). \end{aligned} \tag{2.6.7}$$

Using (2.6.2) and (2.6.6), we can estimate the integral in (2.6.7) by above,

$$\left|\frac{1}{\pi}\int_{\mathbb{R}} \frac{i(\tau-1)}{[\sigma-t+i][\sigma-t+i\tau]} d\mu(t)\right| \leq \frac{\sqrt{c_2}}{\pi}\int_{\mathbb{R}} \frac{1}{1+t^2} d\mu(t) \leq \sqrt{c_2}\mathrm{Im}\ f(i).$$

Applying this estimate, taking real and imaginary parts of (2.6.7) and using the upper bound for α from (2.6.6) now yield our claims. □

Lemma 2.6.4 (Szőke, [Sz95])**.** *Let K be a compact topological space. Suppose that we have a continuous map*

$$F : K \times \mathbb{C}^+ \to \mathcal{H}^n.$$

Assume that for every $y \in K$ the map

$$F(y,.) : \mathbb{C}^+ \to \mathcal{H}^n$$

is holomorphic. Then there exists a constant $A > 0$, such that for every $\zeta = \sigma + i\tau \in D_{\sqcup}$, and $y \in K$ we have ($||\ ||$ denotes the matrix norm)

$$\begin{aligned} &||Re\ F(y,\zeta)|| \leq A, \\ &||Im\ F(y,\zeta)|| \leq A\tau, \\ &||(Im\ F)^{-1}(y,\zeta)|| < A\tau. \end{aligned} \tag{2.6.8}$$

Proof. Let $w \in \mathbb{R}^n$, $||w|| = 1$. Define $f_w : K \times \mathbb{C}^+ \to \mathbb{C}$, by

$$f_w(y,\zeta) := \langle F(y,\zeta)w, w\rangle.$$

Then the real (resp. imaginary) part of f_w is the function $(y,\zeta) \mapsto \langle \mathrm{Re}F(y,\zeta)w, w\rangle$ (resp. $\langle \mathrm{Im}F(y,\zeta)w, w\rangle$). In particular f_w maps into $\mathbb{C}^+$ and thus for every fixed $y \in K, \zeta \in D_{\sqcup}$, the estimates (2.6.3) and (2.6.4) hold.

Denote by L the compact set

$$\{(x,z,w) \in K \times \mathbb{C} \times \mathbb{R}^n \mid |\sigma| \leq 1, z = \sigma + i, ||w|| = 1\}.$$

Then for every $y \in K, \zeta \in D_{\sqcup}$ we have (using (2.6.4))

$$||\mathrm{Re}F(y,\zeta)|| = \max_{\substack{w\in\mathbb{R}^n \\ ||w||=1}} |\mathrm{Re}f_w(y,\zeta)| \leq C\left(\max_L |\langle F(x,z)w, w\rangle|\right).$$

This proves the first inequality in (2.6.8). Using (2.6.3) in a similar way, now yields the second and the third estimate. □

Proposition 2.6.5 (Szőke, [Sz95]). *Let (M, g) be a Riemannian manifold which admits an adapted complex structure on the entire tangent bundle. Let X be an element of the Lie algebra of the isometry group Isom(M, g). Denote by $X_\#$ the induced infinitesimal vector field on TM (by the action of Isom(M, g) on TM). Then $X_\#^{1,0}$ is holomorphic on TM.*

Proof. Isom(M, g) acts on TM by biholomorphisms (see [LSz91]). Let

$$\begin{aligned} f : \mathbb{R} \times TM &\longrightarrow TM \\ (t, p) &\longmapsto (\exp tX)_* p. \end{aligned} \tag{2.6.9}$$

Then $X_\# = df/dt|_{t=0}$. Proposition 2.2.2 now implies our claim. □

Theorem 2.6.6 (Szőke, [Sz95]). *Let (M, g) be a compact Riemannian manifold. Suppose that an adapted complex structure exists on the entire tangent bundle. Let κ_g be the induced Kähler metric. Let X be an element of the Lie algebra of the isometry group Isom(M, g). Denote by $X_\#$ the induced infinitesimal vector field on TM (by the action of Isom(M, g) on TM). Then there exist positive constants A_X and B_X such that*

$$||X_\#(p)||_{\kappa_g} \leq A_X ||p||_g + B_X, \tag{2.6.10}$$

for every $p \in TM$.

Proof. Let p be a point in TM with norm one. Let ϵ be a small, positive number. Denote by D_ϵ the ball in $\mathbb{R}^{2n-2}$ with radius ϵ. If ϵ is small enough, then we can choose a neighbourhood U_p of the point p in the unit sphere bundle of TM, such that U_p is diffeomorphic to $D_\epsilon \times (-\epsilon, \epsilon)$ (because of [Le, Theorem 1.5], we can assume that in fact we have a real analytic diffeomorphism) and under this diffeomorphism ψ, the curves (for every fixed point $y \in D_\epsilon$)

$$t \mapsto (y, t) \in D_\epsilon \times (-\epsilon, \epsilon),$$

correspond to the trajectories of the geodesic flow. Thus D_ϵ parametrizes the trajectories of the geodesic flow in the neighbourhood U_p.

For a fixed point $q \in U_p$, take all positive multiples of q, to get a half line in the tangent space $T_{\pi(q)}M$. If we do this for every point in U_p, the union of these half lines provides us a domain W_p in TM. With the help of the diffeomorphism ψ, we can get another diffeomorphism Ψ_1,

$$\Psi_1 : D := D_\epsilon \times (-\epsilon, \epsilon) \times (0, \infty) \longrightarrow W_p, \qquad (y\ , \sigma\ , \tau\ ) \longmapsto \tau\psi(y, \sigma).$$

Choose real analytic vector fields $\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n$ along the map $\psi(., 0) : D_\epsilon \to U_p$ (i.e. $\xi_j(\psi(y, 0)), \eta_k(\psi(y, 0))$ are elements of $T_{\psi(y,0)}TM$) such that for every point $y \in D_\epsilon$, $q := \psi(y, 0)$

$$\xi_1(q), \ldots, \xi_n(q), \quad \text{and} \quad \eta_1(q), \ldots \eta_n(q)$$

are the horizontal and vertical lifts of an orthonormal frame $v_1, \ldots, v_n \in T_{\pi(q)}M$, and $v_n = q$. Extend now the vector fields ξ_j, η_k to be defined on D and invariant with respect to the geodesic flow Φ_t $(-\epsilon < t < \epsilon)$ and to the N_s $(0 < s)$ actions (see (1.2.2)), and call these extended fields with the same name.

This way we obtained vector fields along the map $\Psi_1 : D \to TM$, such that for every $q = \psi(y, 0)$, the frame $\{\xi_j, \eta_j\}_{j=1}^n$ is a (restriction of a) symplectic frame along the leaf L_q of the Riemann foliation, defined by q (see (1.2.2) and the discussion above (1.2.8)).

Since $X_\#$ is coming from the isometry group action, the vector field $X_\#$ is parallel along every leaf of the Riemannian foliation. Moreover, according to Proposition 2.6.5, $X_\#^{1,0}$ is holomorphic. Therefore there exists smooth (in fact real analytic) functions

$$\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n : D_\epsilon \to \mathbb{R},$$

such that for every $(y, \sigma, \tau) \in D$,

$$X_\#(\Psi_1(y, \sigma, \tau)) = \sum_{j=1}^n \alpha_j(y)\xi_j(\Psi_1(y, \sigma, \tau)) + \sum_{k=1}^n \beta_k(y)\eta_k(\Psi_1(y, \sigma, \tau)). \quad (2.6.11)$$

For a point $y \in D_\epsilon$, denote by γ_y the unit speed geodesic, with initial datum $\dot\gamma_y(0) = \psi(y, 0)$. Thus for $(y, \sigma, \tau) \in D, \zeta = \sigma + i\tau$ we have that $\Psi_1(y, \sigma, \tau) = \gamma_{y*}(\zeta)$.

From Theorem 2.2.3 we obtain a map

$$F = (f_{jk}) : D_\epsilon \times \mathbb{C}^+ \longrightarrow \mathcal{H}^n,$$

such that

$$\eta_k^{1,0}(\gamma_{y*}(\zeta)) = \sum_{j=1}^n f_{jk}(y, \zeta)\xi_j^{1,0}(\gamma_{y*}\zeta),$$

and because of our choice, F is also real analytic in the subdomain $D \subset D_\epsilon \times \mathbb{C}^+$. From the same theorem it also follows that for every $y \in D_\epsilon$, the map

$$F(y, .) : \mathbb{C}^+ \to \mathcal{H}^n$$

is holomorphic. According to [Sh, Theorem 1], this implies that

$$F : D_\epsilon \times \mathbb{C}^+ \longrightarrow \mathcal{H}^n$$

is real analytic. (Continuity would actually be enough for our purposes.) Let

$$\alpha := (\alpha_1, \dots, \alpha_n), \beta := (\beta_1, \dots, \beta_n) : D_\epsilon \to \mathbb{R}^n.$$

Then (2.2.6) and (2.6.11) yield for any $\zeta \in \mathbb{C}^+$, $y \in D_\epsilon$, $p := \gamma_{y*}(\zeta)$

$$\begin{aligned}\langle X_\#, X_\#\rangle_{\kappa_g}(p) = ||p||_g\{&\langle(\text{Im } F)^{-1}(y, \zeta)\alpha(y), \alpha(y)\rangle \\ &+ 2\langle((\text{Im } F)^{-1}\text{Re } F)(y, \zeta)\alpha(y), \beta(y)\rangle \\ &+ \langle[(\text{Re } F(\text{Im } F)^{-1}\text{Re } F)(y, \zeta) \\ &+ \text{Im } F(y, \zeta)]\beta(y), \beta(y)\rangle\}.\end{aligned} \quad (2.6.12)$$

Using our estimate (2.6.8), for $K = \overline{D}_{\epsilon/2}$, we find a positive constant $A = A_K$, such that for any $\zeta = \sigma + i\tau \in D_\sqcup$, and $y \in K$,

$$\begin{aligned}\langle(\mathrm{Im}\ F)^{-1}(y,\zeta)\alpha(y),\alpha(y)\rangle_{\kappa_g} &\leq A(\sup_K ||\alpha||^2)\tau\\ |\langle[(\mathrm{Im}\ F)^{-1}\mathrm{Re}\ F](y,\zeta)\alpha(y),\beta(y)\rangle_{\kappa_g}| &\leq A^2(\sup_K ||\alpha||\sup_K ||\beta||)\tau\\ |\langle[\mathrm{Re}\ F(\mathrm{Im}\ F)^{-1}\mathrm{Re}\ F+\mathrm{Im}\ F](y,\zeta)\beta(y),\beta(y)\rangle_{\kappa_g}| &\leq (A^3\tau+A\tau)\sup_K ||\beta||^2.\end{aligned} \tag{2.6.13}$$

(2.6.12) and (2.6.13) shows that for some positive real number $\widetilde{A}$, and arbitrary $z \in \Psi_1(D_{\epsilon/2} \times (-\epsilon,\epsilon) \times (1,\infty))$, we have

$$\langle X_\#, X_\#\rangle_{\kappa_g}(z) \leq \tilde{A}||z||_g^2. \tag{2.6.14}$$

Since the unit sphere bundle in TM is compact, we can choose a finite cover of it with open subsets of the form $\psi(D_{\epsilon/2}\times(-\epsilon,\epsilon))$. Hence we can find a constant $\widetilde{A}$, such that the estimate (2.6.14) in fact holds for every point z of the tangent bundle with $||z|| \geq 1$, yielding our claim of (2.6.10). □

Corollary 2.6.7 (Szőke, [Sz95])**.** *Let (M,g) be a compact Riemannian manifold which admits an adapted complex structure on the entire tangent bundle. Let X be an element of the Lie algebra of Isom(M,g). Denote by $X_\#$ the induced infinitesimal vector field on TM. Then the flow of $JX_\#$ is complete.*

Proof. For the sake of brevity denote again by ρ the norm-square function on TM and by κ_g the induced Kähler metric. From [LSz91, Prop.3.2] or [PW] we know that

$$||grad\rho||_{\kappa_g} = \sqrt{\rho}.$$

Hence Theorem 2.6.6 yields

$$|(JX_\#)\rho(p)| = |\langle \mathrm{grad}\rho, JX_\#\rangle_{\kappa_g}(p)| \leq C_X\rho(p) + D_X, \tag{2.6.15}$$

for some positive constants C_X and D_X. Since ρ is an exhaustion function on TM, (2.6.15) implies (see [AM, Prop. 2.1.20],) that the flow of $JX_\#$ is indeed complete. □

To prove Theorem 2.1.4 we need one more ingredient.

Theorem 2.6.8 (Szőke, [Sz95])**.** *Let (M,g) be a Riemannian manifold. Assume that adapted complex structure exists on T^rM for some positive number r. Let $D_r := \{\zeta = \sigma + i\tau \in \mathbb{C} \mid |\tau| < r\}$. Suppose that we are given a function $h : T^rM \to \mathbb{C}$, that is real analytic along the zero section, and for every unit speed geodesic γ, the composition map $h \circ \gamma_* : D_r \to \mathbb{C}$ is holomorphic. Then h is holomorphic.*

Proof. In fact, this is implicitly contained in [Sz91]. [Le, Theorem 1.5] implies that g is real analytic. The proof of [Sz95, Proposition 3.2], shows that h must be holomorphic in an open neighbourhood of the zero section.

In order to prove that h is holomorphic everywhere, it suffices to show that h is real analytic on T^rM. We will use [Sh, Theorem 1] to achieve this.

Let p be a point of $T^rM \setminus M$ with norm one. Choose a small open neighbourhood U_p of p in the unit sphere bundle. We can assume that the Hamiltonian

flow can be straightened out in U_p, i.e. there exists an $\epsilon > 0$ and a real-analytic diffeomorphism

$$\psi : (-\epsilon, \epsilon) \times \mathbb{B}_\epsilon \to U_p,$$

($\mathbb{B}_\epsilon \subset \mathbb{R}^{2n-2}$ being the open ϵ-ball), such that the curves $t \mapsto \psi(t, x)$ are precisely the flow lines in U_p, for every $x \in \mathbb{B}_\epsilon$. Let

$$D_\epsilon = (-\epsilon, \epsilon) \times (0, r) \times \mathbb{B}_\epsilon.$$

Then the map

$$\begin{aligned} D_\epsilon &\longrightarrow T^r M \\ \Psi : (\sigma, \tau, x) &\longmapsto \tau\psi(t, x) \end{aligned} \tag{2.6.16}$$

is a real-analytic diffeomorphism onto its image. It is enough to show that the composition map $h \circ \Psi$ is real-analytic. But using the fact that h is holomorphic in a small neighbourhood of the zero section, we get that $h \circ \Psi$ is real-analytic in the region $|\sigma| < \epsilon, 0 < \tau < \epsilon, x \in \mathbb{B}_\epsilon$, and holomorphic in $\zeta = \sigma + i\tau$ for each $x \in \mathbb{B}_\epsilon$. [Sh, Theorem 1] now implies our claim.

□

We are now ready to prove the main result of this section.

2.6.1 Proof of Theorem 2.1.4

Let G_0 be the identity component of G. Since M is compact, G can have only finitely many components. Hence it is enough to prove our statement for the action of G_0. Denote by $\mathcal{A}(TM)$ the complex vector space of holomorphic vector fields on the complex manifold TM, i.e. $\mathcal{A}(TM)$ consists of vector fields V such that $V^{1,0}$ is a holomorphic section of $T^{1,0}TM$. In fact $\mathcal{A}(TM)$ becomes a complex Lie algebra if we take the obvious complex multiplication and Lie bracket= minus the ordinary Lie bracket of vector fields. The integrability of the almost complex tensor assures that $\mathcal{A}(TM)$ is indeed a complex Lie algebra. The reason for the sign convention is to make things compatible with induced infinitesimal generators. (See below.) Denote by $\mathfrak{g}$ the Lie algebra of G. From Proposition 2.6.5 we know that for any $X \in \mathfrak{g}$, the induced infinitesimal generator $X_\#$ on TM belongs to $\mathcal{A}(TM)$. The map

$$\begin{aligned} \mathfrak{g}_\mathbb{C} = \mathfrak{g} + i\mathfrak{g} &\longrightarrow \mathcal{A}(TM) \\ \delta : X + iY &\longmapsto X_\# + JY_\# \end{aligned} \tag{2.6.17}$$

is a $\mathbb{C}$ linear Lie-algebra monomorphism. (Lie-algebra homomorphism follows from our sign convention see for example [AM, Proposition 4.1.26].)

Corollary 2.6.7 tells us that every element of $\mathcal{L} := \delta(\mathfrak{g}_\mathbb{C})$ induces a one parameter group of diffeomorphisms of TM. It follows from Palais' work (see [Pa]), that there exists a unique, connected Lie group $\widehat{G}_0$, whose underlying group is a subgroup of the group of diffeomorphisms of TM, the Lie algebra of $\widehat{G}_0$ is $\mathcal{L}$, the map

$$\widehat{G}_0 \times TM \longrightarrow TM$$

is differentiable (i.e. $\widehat{G}_0$ is a connected Lie transformation group on TM, each element of $\widehat{G}_0$ different from the identity acts nontrivially on TM), and $\widehat{G}_0$

extends the G_0 action on TM. (Hence G_0 can also be considered a Lie subgroup of $\widehat{G}_0$.)

Since $\mathcal{L}$ is a complex Lie-algebra, the corresponding group $\widehat{G}_0$ will be a complex Lie group. Let $\widetilde{G}_0$ be the universal covering group of G_0. Then $T\widetilde{G}_0 \cong (\widetilde{G}_0)_{\mathbb{C}}$ will be the universal cover of $TG_0 \cong (G_0)_{\mathbb{C}}$. By a theorem in Lie theory, there exists a unique homomorphism,

$$\Delta : (\widetilde{G}_0)_{\mathbb{C}} \longrightarrow \widehat{G}_0,$$

with differential δ at the unit element.

Therefore Δ is a holomorphic covering map. Then (since $\widehat{G}_0$ extends the action of G_0)

$$\text{Ker } \Delta \supset K = \text{Ker } (\widetilde{G}_0 \to G_0) = \text{Ker } [(\widetilde{G}_0)_{\mathbb{C}} \to (G_0)_{\mathbb{C}}].$$

Thus we get a holomorphic covering map,

$$\widetilde{\Delta} : (G_0)_{\mathbb{C}} = (\widetilde{G}_0)_{\mathbb{C}}/K \longrightarrow \widehat{G}_0.$$

Hence $(G_0)_{\mathbb{C}}$ indeed acts on TM and since Ker $\widetilde{\Delta}$ is discrete (only the elements of Ker $\widetilde{\Delta}$ act trivially on TM), the action is almost effective. Now we have to show that the action is holomorphic.

Since the Lie-algebra of $\widehat{G}_0$ is $\mathcal{L} \subset \mathcal{A}(TM)$, all the elements of $\widehat{G}_0$ that belong to a 1-parameter subgroup, act by biholomorphisms on TM. But these elements generate the whole group. Hence $\widehat{G}_0$, and then of course $(G_0)_{\mathbb{C}}$ as well, acts on TM by biholomorphisms. This implies that the transformation map

$$\beta^{\mathbb{C}} : (G_0)_{\mathbb{C}} \times TM \longrightarrow TM,$$

is holomorphic in the second variable. Since $\beta^{\mathbb{C}}$ is smooth, in order to prove that it is holomorphic in all its variables, it suffices to show that for any point $p \in TM$, the map

$$\beta^{\mathbb{C}}_p : (G_0)_{\mathbb{C}} \to TM, \qquad (G_0)_{\mathbb{C}} \ni a \mapsto \beta^{\mathbb{C}}(a, p),$$

is holomorphic. From [Le, Theorem 1.5] we know that the metric on M is real-analytic and therefore the restricted transformation map

$$\beta := \beta^{\mathbb{C}}\big|_{G_0} : G_0 \times TM \longrightarrow TM,$$

is real-analytic and consequently $\beta^{\mathbb{C}}_p\big|_{G_0}$ as well. Since TM is a Stein manifold, we can think of $\beta^{\mathbb{C}}_p$ as a map going into $\mathbb{C}^N$ for some large N. Equipping G_0 with a two-sided invariant metric h, from Proposition 3.1.3 we know that TG_0 with the adapted complex structure of h is precisely $(G_0)_{\mathbb{C}}$. Hence, using Theorem 2.6.8, it suffices to prove that for any unit-speed geodesic $\gamma : \mathbb{R} \to G_0$, the composition map $\beta^{\mathbb{C}}_p \circ \gamma_*$ is holomorphic. Because of homogeneity, it suffices to check this for geodesics through the unit element, i.e. for 1-parameter subgroups of G_0. Let $X \in \mathfrak{g}$, $\gamma(\sigma) = \exp(\sigma X)$ and $\zeta = \sigma + i\tau$. The induced map is

$$\gamma_* : T\mathbb{R} \cong \mathbb{C} \ni \zeta \longmapsto \exp(\zeta X) \in (G_0)_{\mathbb{C}}.$$

Hence the composition map $\beta_p^{\mathbb{C}} \circ \gamma_*$ can also be written as a composition of the holomorphic maps,

$$\mathbb{C} \ni \sigma + i\tau \longmapsto \sigma X_\# + \tau J X_\# \in \mathcal{L},$$

and

$$\mathcal{L} \ni V \longmapsto \chi_V(1),$$

and thus itself is also holomorphic. Here χ is the trajectory of the vector field $V \in \mathcal{L}$ with initial condition $\dot{\chi}(0) = V(p)$. (This latter map is holomorphic since: solutions of an O.D.E. that depends holomorphically on some parameters, also depend holomorphically on the same parameters.)

Chapter 3

Compact, normal Riemannian homogeneous spaces

3.1 Adapted complex structures of compact, normal Riemannian homogeneous spaces

The main result of this chapter is the following theorem.

Theorem 3.1.1 (Szőke [Sz98])**.** *Let K be a compact Lie group equipped with a two sided invariant Riemannian metric g. Let L be a closed subgroup of K and g_n the induced Riemannian metric on $M = K/L$. Then the adapted complex structure of g_n exists on TM and with this complex structure TM is biholomorphic to $K_{\mathbb{C}}/L_{\mathbb{C}}$.*

Throughout this section K denotes a compact Lie group, L a closed subgroup of K, g a two sided invariant Riemannian metric on K and g_n the induced Riemannian metric on $M = K/L$ (the so called normal homogeneous Riemannian metric). Two sided invarian (or biinvariant) means that left and right translations in K are $g-$isometries.

It is well known that K can be imbedded into a unitary group $U(N)$ (for some large N) and from now on we tacitly always assume this. Denote by e the identity matrix. Following [BD], the complexified group $K_{\mathbb{C}}$ can be described as follows. The underlying differentiable manifold for $K_{\mathbb{C}}$ is just $K \times \kappa$, where $\kappa = T_eK$ is the Lie algebra of K. The group structure and the complex structure can be defined by pulling back with the imbedding

$$\begin{aligned} \Lambda : K \times \kappa &\longrightarrow \mathrm{GL}(N, \mathbb{C}) \\ (a, X) &\longmapsto a \exp(iX). \end{aligned} \tag{3.1.1}$$

For an element $a \in K$ denote by L_a the left translation $L_a : b \mapsto ab$. With the help of these diffeomorphisms, we can identify TK and $K_{\mathbb{C}}$:

$$\Delta : T_aK \ni Y \longmapsto (a, (L_a)_*^{-1}Y) \in K \times \kappa = K_{\mathbb{C}}. \tag{3.1.2}$$

Denote by $\mathfrak{l}$ the Lie algebra of L and $\mathfrak{m}$ the orthogonal complement, ie. $\kappa = \mathfrak{l} + \mathfrak{m}$. Using the left K action on itself, the subspace $\mathfrak{m}$ determines a subbundle $\mathfrak{M}$ of TK. Define the right L-action on $K \times \kappa$ by

$$(k, X)l := (kl, Ad(l^{-1})X). \tag{3.1.3}$$

Since the metric g is two-sided invariant, the subspace $\mathfrak{m}$ is Ad_L invariant and the quotient space of $K \times \kappa$ w.r.t. the L-action of (3.1.3) is the vector bundle $K \times_L \mathfrak{m}$.

Let $p : K \to K/L$ be the projection and p_* the induced map $p_* : TK \to T(K/L)$. The derivative of the right L-action on K gives a right L-action on $\mathfrak{M}$ and the restriction of Δ to $\mathfrak{M}$ gives an L-equivariant identification $\Delta_{\mathfrak{M}}$ between the bundles $\mathfrak{M}$ and $K \times \mathfrak{m}$. The quotient space $\mathfrak{M}/L$ is precisely $T(K/L)$ and the quotient map is just p_*. The map $\Delta_{\mathfrak{M}}$ descends to a map $\Delta_{K/L} : T(K/L) \to K \times_L \mathfrak{m}$ and the following diagram commutes.

$$\begin{array}{ccc} TK \supset \mathfrak{M} & \xrightarrow{\Delta_{\mathfrak{M}}} & K \times \mathfrak{m} \\ {\scriptstyle p_*}\downarrow & & \downarrow \\ T(K/L) & \xrightarrow{\Delta_{K/L}} & K \times_L \mathfrak{m}. \end{array}$$

The quotient map $p : (K, g) \to (K/L, g_n)$ is a Riemannian submersion and at any point $k \in K$ the horizontal subspace is precisely $\mathfrak{M}_k$.

After all these preliminaries we only need to recall a result of Mostow (see [Hei, Decomposition Theorem of Mostow in Sect. 3.1], cf. [Mo1, Mo2]).

Theorem 3.1.2. *The map*

$$\begin{aligned} \Phi : K \times_L \mathfrak{m} &\longrightarrow K_{\mathbb{C}}/L_{\mathbb{C}} \\ (k, v) &\mapsto k \exp(iv) L_{\mathbb{C}}. \end{aligned} \tag{3.1.4}$$

is a K-equivariant diffeomorphism.

Proof of Theorem 3.1.1

The result follows from the following more precise theorem.

Theorem 3.1.3 (Szőke, [Sz98])**.** *Let K be a compact Lie group and L a closed subgroup of K. Let g be a two-sided invariant Riemannian metric on K. Then $\Phi \circ \Delta_{K/L}$ is a biholomorphism between $T(K/L)$ (with the adapted complex structure to g_n) and $K_{\mathbb{C}}/L_{\mathbb{C}}$.*

The special case $L = \{e\}$ was already treated in [Sz95, Proposition 3.5].

Proof. **Step 1:**

Suppose $L = \{e\}$. We need to show that pulling back the complex structure of $GL(N, \mathbb{C})$ by $\Lambda \circ \Delta : TK \to K \times \kappa \to GL(N, \mathbb{C})$ is adapted to g. Let γ be an arbitrary geodesic through $b \in K$. Since g is two-sided invariant, γ can be written as $\gamma(\sigma) = b \exp(\sigma X)$, for some $X \in \kappa$. Since for any $\sigma + i\tau \in \mathbb{C}$,

$$(L_{\gamma(\sigma)})_*^{-1}(\tau \dot{\gamma}(\sigma)) = \tau \left.\frac{d}{dt}\right|_{t=\sigma} [\gamma(\sigma)^{-1}\gamma(t)] = \tau \left.\frac{d}{dt}\right|_{t=0} \{\exp[(t-\sigma)X]\} = \tau X,$$

the composition map $\Lambda \circ \Delta \circ \gamma_* : \mathbb{C} \to GL(N, \mathbb{C})$ is

$$\zeta := \sigma + i\tau \mapsto b \exp(\sigma X) \exp(i\tau X) = b \exp(\zeta X),$$

that is holomorphic in ζ. Hence the complex structure is indeed adapted.

Step 2:

Let now $\gamma : \mathbb{R} \to K/L$ be a g_n geodesic. Since the projection map $p : (K, g) \to (K/L, g_n)$ is a Riemannian submersion, γ can be lifted to a horizontal geodesic $\widetilde{\gamma} : \mathbb{R} \to K$ (see for example [Bes, p.245, 9.42]). Horizontal means that in fact $\widetilde{\gamma}_* : T\mathbb{R} \to \mathfrak{M}$, ie. $\widetilde{\gamma}(\sigma) = b \exp \sigma X$, where $X \in \mathfrak{m}$.

To show that the complex structure on $T(K/L)$, given by the pull-back w.r.t. $\Phi \circ \Delta_{K/L}$ is adapted to g_n, it is necessary and sufficient to check that the composition $\Phi \circ \Delta_{K/L} \circ \gamma_* : T\mathbb{R} \to K_{\mathbb{C}}/L_{\mathbb{C}}$ is holomorphic. This follows from Step 1, the discussion before Theorem 3.1.2, the holomorphicity of π and the commutativity of the diagram below. (We think of the group $K_{\mathbb{C}}$ as the complex subgroup of $GL(N, \mathbb{C})$, $K_{\mathbb{C}} = \Lambda(K \times \mathfrak{m})$.)

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "\widetilde{\gamma}_*"] \arrow[d, "id"'] & \mathfrak{M} \arrow[r] \arrow[d, "p_*"'] & K \times \mathfrak{m} \arrow[r, "\Lambda"] \arrow[d] & K_{\mathbb{C}} \arrow[d, "\pi"'] \\
\mathbb{C} \arrow[r, "\gamma_*"] & T(K/L) \arrow[r, "\Delta_{K/L}"] & K \times_L \mathfrak{m} \arrow[r, "\Phi"] & K_{\mathbb{C}}/L_{\mathbb{C}}.
\end{tikzcd}$$

$\square$

Chapter 4

Geodesic flow invariant involutive structures

An involutive structure on a smooth manifold X is simply a complex subbundle V of the complexified tangent bundle of X with the property: for every (local) sections Z_1, Z_2 of V their Lie bracket $[Z_1, Z_2]$ is again a (local) section of V. This notion is a natural generalization of foliations and complex and CR structures. Their local properties were treated by Treves in [Tr] in great details. Some global results were obtained by Hanges and Jacobowitz in [HJ] (general involutive structures over compact manifolds) and by Jacobowitz [J], Meziani [Mez] and Anbo Le [Le] (Mizohata structures). In the theory of geometric quantization a specific kind of involutive structure, the so called complex polarization (see section 4.2.4. and the book of Woodhouse [W]) plays a crucial role.

In this chapter we consider certain involutive structures on (parts of) TM, when M is a compact symmetric space. They arise as the limits of pushing forward the adapted complex structure on TM by a certain 1-parameter group of diffeomorphisms (see (4.0.1)). When the rank is 1, this limit structure is a genuine complex structure. In the higher rank cases it shows a more complicated behavior. We treat these cases separately in the next two sections.

In this chapter we use the following notation. The punctured tangent bundle of a Riemannian manifold (M, g) is the set $\mathring{\mathrm{T}}M := TM \setminus M$ (similarly $\mathring{\mathrm{T}}^*M$). Let $\Phi_\varepsilon : \mathring{\mathrm{T}}M \to \mathring{\mathrm{T}}M$ be the diffeomorphism

$$\Phi_\varepsilon(v) = \varepsilon \exp(\| v \|) \frac{v}{\|v\|}. \tag{4.0.1}$$

4.1 Rank-1 symmetric spaces

The main purpose of this section is to explore the relationship of two complex structures arising from different constructions on tangent bundles of compact rank-1 symmetric spaces. The first of these is the adapted complex structure J_A. The second comes from geometric quantization. Since in this section only the differential geometric nature of the latter complex structure matters, geometric quantization itself will be discussed only later, in part two of this dissertation.

The second kind of complex structure J_S lives on the punctured (co)tangent

bundle of compact, rank-1 symmetric spaces. (Since the metric identifies TM and T^*M, from this point of view it does not matter which bundle we take.)

For spheres it was Souriau who in [So2] identified the regularized Kepler manifold with the punctured (co)tangent bundle $\mathring{\mathrm{T}}^*S^n$. He also defined a complex manifold structure on this space, by showing that it is diffeomorphic to the singular affine hyperquadric Q_0 (cf. section 4.1.2).

Later it was observed by Rawnsley [Ra1], that the norm function is strictly plurisubharmonic with respect to J_S and thus it defines a Kähler metric on $\mathring{\mathrm{T}}S^n$. He also observed that the Kähler form of this metric is ω, the canonical symplectic form (cf. (1.2.4)) restricted to $\mathring{\mathrm{T}}S^n$. He also showed that J_S is invariant w.r.t. the normalized geodesic flow. Rawnsley used this in [Ra2] to quantize the geodesic flow of spheres.

Subsequently Furutani and Tanaka [FT] defined a Kähler structure on the punctured cotangent bundle of complex and quaternionic projective spaces. Their Kähler structure is also invariant w.r.t. the normalized geodesic flow. Furutani and Yoshizawa [FY] used this Kähler structure to quantize the geodesic flow on complex and quaternionic projective spaces.

Furutani and Tanaka described their Kähler structures in terms of matrices in the spirit of Lie groups. A more geometric description of the complex (Kähler) structure on $\mathring{\mathrm{T}}M$ (where M now can be either a sphere, a complex or quaternionic projective space or their quotient w.r.t. a discrete group of isometries) was given by Ii and Morikawa [IM]. We shall use Ii-Morikawa's description of these structures (see section 4.1.2 for more details).

4.1.1 The main result

The main result of this section is the following.

Theorem 4.1.1 (Szőke,[Sz99]). *Let (M, g) be a compact, rank-1 symmetric space. Then on $\mathring{T}M$ the following limit complex structure J_0 exits (Φ_ε is from* (4.0.1)*).*

$$\lim_{\varepsilon\to 0}(\Phi_\varepsilon)_* J_A = J_0. \tag{4.1.1}$$

The maps $N_s, s > 0$ (cf. 1.2.1) and the normalized geodesic flow are J_0 holomorphic. When M is the sphere, J_0 agrees with J_S and for the complex and quaternionic projective spaces it coincides with the structure studied in the papers ([FT, IM]).

Remarks. The theorem gives a unified treatment of all rank-1 symmetric spaces and includes the missing exceptional Cayley projective plane as well, that was not studied in the papers above.

Since the entire construction and the proof is compatible with taking quotients w.r.t. a discrete subgroup of the isometry group, the result of Theorem 4.1.1 is valid for such quotients as well.

4.1.2 Complex structures on $\mathring{\mathrm{T}}M$

Let $t \geq 0$. The complex $n-$dimensional quadric Q_t is defined by

$$Q_t = \{z \in \mathbb{C}^{n+1} \mid \sum_{j=1}^{n+1} z_j^2 = t\}. \tag{4.1.2}$$

Q_t is a complex submanifold when $t > 0$ and it has one singular point, the origin for $t = 0$. When $t > 0$, Q_t is biholomorphic to Q_1. The adapted complex structure of the round metric on the sphere S^n is defined on the whole tangent bundle TS^n and (TS^n, J_A) is biholomorphic to Q_1. This latter fact can be seen by taking the standard realization of TS^n

$$\{(e, X) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid \|e\| = 1, < e, X >= 0\}. \tag{4.1.3}$$

Then one can check that the complex structure on TS^n using the diffeomorphism $\delta : TS^n \to Q_1$

$$\delta(e, X) := \cosh(\|X\|)e + i\frac{\sinh(\|X\|)}{\|X\|}X \tag{4.1.4}$$

is adapted to the round metric. Consequently for all $t > 0$, $Q_t \setminus \mathbb{R}^n$ is also biholomorphic to $(\mathring{\mathrm{T}}S^n, J_A)$.

The complex structure J_S on $\mathring{\mathrm{T}}S^n$ is obtained by pulling back the complex structure of Q_0 by the diffeomorphism $\mathring{\mathrm{T}}S^n \to Q_0 \setminus \{0\}$ defined by

$$(e, X) \mapsto z = ||X||e + iX \tag{4.1.5}$$

Ii and Morikawa's ([IM]) description of J_S on $\mathring{\mathrm{T}}S^n$ is as follows. For $0 \neq z \in T_pS^n$, J_S maps horizontal vectors at z to vertical vectors according to the formula

$$J_S v_z^H = ||z|| v_z^V, \quad v \in T_pM. \tag{4.1.6}$$

Proposition 4.1.2 (Szőke,[Sz99])**.** *The complex manifolds* $(\mathring{T}S^n, J_A)$ *and* $(\mathring{T}S^n, J_S)$ *are not biholomorphic.*

Proof. The statement is equivalent to showing that $X = Q_0 \setminus \{0\}$ and $Y = Q_1 \setminus S^n$ are not biholomorphic. Suppose on the contrary that $\psi : Y \to X$ is a biholomorphism. We can apply a theorem of Hartogs (see [Sha]), to conclude that ψ has a holomorphic extension $\tilde{\psi} : Q_1 \to \mathbb{C}^{n+1}$ (the theorem we use here says that if N is a complex manifold, $L \subset N$ a real submanifold of real codimension at least 2 and $\psi : N \setminus L \to \mathbb{C}$ is holomorphic which does not extend holomorphically to a point $q \in L$, then in a neighborhood of q, L is a complex submanifold).

Since Q is defined as the zero set of a holomorphic function, therefore $\tilde{\psi}$ also maps into Q_0. Since ψ is biholomorphic and $\tilde{\psi}$ is holomorphic, $\tilde{\psi}(S^n)$ must be the point 0. But S^n is a maximal dimensional totally real submanifold in Q, hence $\tilde{\psi}$ must be constant, a contradiction. □

Despite of Proposition 4.1.2, J_A and J_S do have something to do with each other. The complex manifold $Q_0 \setminus \mathbb{R}^n = Q_0 \setminus \{0\}$ is in some sense the limit of the complex manifolds $Q_t \setminus \mathbb{R}^n$ when t goes to zero. This gives the idea to try to push forward J_A by a family of diffeomorphisms in such a way that the limit of these push forwards is J_S. This indeed can be done as our Theorem 4.1.1 shows, but before we go on with the proof, we give another description of the complex structures studied in [FT, IM, Ra1, So2].

Let $(M, < .,. >)$ be a Riemannian manifold. Then $R_z = \mathcal{R}(., z)z$ is the Jacobi operator, where $\mathcal{R}$ denotes the curvature tensor. R_z is self-adjoint and z always belongs to its kernel. When M is a compact rank-1 symmetric space,

R_z is positive semidefinite and its kernel is one dimensional, spanned by z. We shall need a modified operator $\hat{R}_z : T_pM \to T_pM$, defined as follows

$$\hat{R}_z(X) =< X, z > z + R_z(X) =< X, z > z + \mathcal{R}(X, z)z. \tag{4.1.7}$$

$\hat{R}_z$ now is positive definite and we get an almost complex tensor $J_0 : T_z(TM) \to T_z(TM)$ in the horizontal and verical decomposition $T_z(TS^n) = H_z + V_z$, by the formula

$$J_0 = \begin{bmatrix} 0 & -\left(\sqrt{\hat{R}_z}\right)^{-1} \\ \sqrt{\hat{R}_z} & 0 \end{bmatrix}. \tag{4.1.8}$$

Examples.

1. Let (M, g) have constant curvature 1. Then the curvature tensor $\mathcal{R}$ has a particularly simple form $\mathcal{R}(X, Y)Z =< Y, Z > X- < X, Z > Y$.

   Therefore in this case $\hat{R}_z(X) =< z, z > X$. This implies that for $z \neq 0$ the complex structure J_S of (4.1.6), agrees with J_0.

2. Let now (M, g, J) be a Kähler manifold of constant holomorphic sectional curvature 4. Then the curvature tensor has the following form

$$\begin{aligned} \mathcal{R}(X, Y)Z =& < Y, Z > X- < X, Z > Y- < Y, JZ > JX \\ &+ < X, JZ > JY + 2 < X, JY > JZ. \end{aligned} \tag{4.1.9}$$

   This yields

$$\widetilde{\mathcal{R}}_z v =< z, z > v + 3 < v, Jz > Jz. \tag{4.1.10}$$

   From this it follows:

$$\sqrt{\widetilde{\mathcal{R}}_z v} = \frac{< v, Jz >}{\|z\|} Jz + \|z\|v, \quad z \neq 0. \tag{4.1.11}$$

   The complex structure on $\mathring{\mathrm{T}}M$ studied by Furutani and Tanaka in [FT] is defined by (see [IM])

$$J_1(v_z^H) = \begin{cases} 2\|z\|v_z^V, & \text{if } v = \alpha Jz, \text{ for some } \alpha \in \mathbb{R} \\ \|z\|v_z^V, & \text{if } v \perp Jz. \end{cases} \tag{4.1.12}$$

   It follows from formula (4.1.11) that J_1 in fact has the form of (4.1.8).

3. Let (M, g, I_1, I_2, I_3) be a quaternion Kähler manifold of constant Q-sectional curvature 4. Then

$$\begin{aligned} \mathcal{R}(X, Y)Z =& < Y, Z > X- < X, Z > Y \\ &- \sum_{i=1}^{3} < Y, I_iZ > I_iX+ < X, I_iZ > I_iY + 2 < X, I_iY > I_iZ \end{aligned} \tag{4.1.13}$$

   This yields

$$\widetilde{\mathcal{R}}_z v =< z, z > v + \sum_{i=1}^{3} < v, I_iz > I_iz. \tag{4.1.14}$$

Thus

$$\sqrt{\widetilde{\mathcal{R}}_z}v = \|z\|v + \sum_{i=1}^{3} \frac{< v, I_i z >}{\|z\|} I_i z. \tag{4.1.15}$$

The complex structure of Furutani-Tanaka is defined on $\mathring{\mathrm{T}}M$ by (see [IM])

$$J_2 v_z^H = \begin{cases} 2\|z\| v_z^V, & \text{if } v = \sum_{i=1}^3 \alpha_i I_i z \text{ for some } \alpha_i \in \mathbb{R} \\ \|z\| v_z^V, & \text{if } v \perp I_i z, i = 1, 2, 3. \end{cases} \tag{4.1.16}$$

Comparing this to formula (4.1.15) yields that J_2 again has the form (4.1.8).

Proof of Theorem 4.1.1. We shall prove that $\lim_{\varepsilon\to 0}(\Phi_\varepsilon)_* J_A$ exists and has the form of (4.1.8).

Step 1.

The diffeomorphism Φ_ε maps any leaf of the Riemann foliation (cf. section 1.2.1) onto itself. First we shall prove the existence of the limit complex structure along a leaf.

Let $\gamma : \mathbb{R} \to M$ be a unit speed geodesic. Denote by S the upper half plane. Then $\gamma_* : S \to TM$, $(\sigma + i\tau) \mapsto \tau\dot{\gamma}(\sigma)$ parametrizes a leaf. Define $\psi_\varepsilon : S \to S$ by $\psi_\varepsilon(\sigma + i\tau) = \sigma + i\varepsilon \exp\tau$. Then ψ_ε is a diffeomorphism of S and the following diagram commutes:

$$\begin{array}{ccc} S & \xrightarrow{\gamma_*} & (TM, J_A) \\ {\scriptstyle\psi_\varepsilon}\downarrow & & {\scriptstyle\Phi_\varepsilon}\downarrow \\ S & \xrightarrow{\gamma_*} & (TM, J_A). \end{array} \tag{4.1.17}$$

Denote by J_+ the complex structure tensor on S. Since γ_* is holomorphic, we get

$$(\Phi_\varepsilon)_* J_A = (\Phi_\varepsilon)_*(\gamma_*)_* J_+ = (\gamma_*)_*(\psi_\varepsilon)_* J_+. \tag{4.1.18}$$

Now let $\sigma_1 + i\tau_1 \in S$. It follows from the definition of ψ_ε, that

$$(\psi_\epsilon)_* \left(\partial\sigma|_{\sigma_1 + i\tau_1} \right) = \partial\sigma|_{\sigma_1 + i\varepsilon \exp\tau_1} \tag{4.1.19}$$

and

$$(\psi_\epsilon)_* \left(\partial\tau|_{\sigma_1 + i\tau_1} \right) = \varepsilon e^{\tau_1} \partial\tau|_{\sigma_1 + i\varepsilon \exp\tau_1}. \tag{4.1.20}$$

This implies that the push forward complex structure at a fixed point $\sigma_0 + i\tau_0$ can be computed as follows

$$\begin{aligned} [(\psi_\varepsilon)_* J_+] \left(\partial\sigma|_{(\sigma_0 + i\tau_0)} \right) &= (\psi_\varepsilon)_* \left(J_+|_{(\sigma_0 + i\log(\tau_0/\varepsilon))} \left(\partial\sigma|_{(\sigma_0 + i\log(\tau_0/\varepsilon))} \right) \right) \\ &= (\psi_\varepsilon)_* \left(\partial\tau|_{(\sigma_0 + i\log(\tau_0/\varepsilon)} \right) \\ &= \varepsilon \exp(\log(\tau_0/\varepsilon)\ \partial\tau|_{(\sigma_0 + i\tau_0)} \\ &= \tau_0\ \partial\tau|_{(\sigma_0 + i\tau_0)} \end{aligned} \tag{4.1.21}$$

Let $Y = \dot{\gamma}(\sigma_0)$ and $X = \tau_0 Y$. Recall from (1.2.7) that

$$(\gamma_*)_* \left(\partial\sigma|_{(\sigma_0+i\tau_0)} \right) = Y_z^H, \quad (\gamma_*)_* \left(\partial\tau|_{(\sigma_0+i\tau_0)} \right) = Y_z^V. \tag{4.1.22}$$

Furthermore

$$\hat{R}_X(Y) =< Y, X > X, \tag{4.1.23}$$

hence

$$\sqrt{\hat{R}_X} Y = \frac{< Y, X >}{\|X\|} X = X. \tag{4.1.24}$$

So we get

$$J_0 \left((\gamma_*)_* \left(\partial\sigma|_{(\sigma_0+\tau_0)} \right) \right) = J_0 Y_z^H = X_X^V = (\gamma_*)_* \left(\tau_0 \partial\tau|_{(\sigma_0+i\tau_0)} \right). \tag{4.1.25}$$

(4.1.18), (4.1.21), (4.1.25) together imply that the image $\gamma_*(S)$ is a complex submanifold w.r.t. both structures, J_0 and $(\Phi_\varepsilon)_* J_A$ and in fact these two complex structures coincide.

Step 2.

There is another natural distribution on $\mathring{\mathrm{T}}M$ (cf. [LSz91]), denoted by $\mathcal{H}$, that at a point $z \in \mathring{\mathrm{T}}M$ is defined by ($\rho : TM \to \mathbb{R}$ is from 1.2.11 and ϑ the canonical 1-form (see (1.2.3))

$$\mathcal{H}_z := \ker \vartheta_z \cap \ker d\rho. \tag{4.1.26}$$

$\mathcal{H}$ and the Riemann foliation (cf. 1.2.1) are complementary distributions, ie. if $\mathcal{L}_z$ is the leaf of the foliation through z, then $T_z(\mathring{\mathrm{T}}M) = \mathcal{L}_z + \mathcal{H}_z$.

To complete the proof of the theorem we need to show the existence of the limit $\lim_{\varepsilon\to 0}(\Phi_\varepsilon)_* J_A$, when we restrict ourselves to the distribution $\mathcal{H}$.

Let $\varepsilon > 0$ and $p \in \mathring{\mathrm{T}}M$ be fixed, $\|p\| = c$, define $v_n = p/c$, $r = \log(c/\varepsilon)$, $q = (r/c)p$, $\lambda = (\varepsilon e^r)/r$. Then Φ_ε maps the hypersurface $S_r = \{u = r\}$ onto $S_c = \{u = c\}$.

As before denote by N_λ the diffeomorphism of TM that is multiplication by λ in the fibers. It is clear that

$$\Phi_\varepsilon|_{S_r} = N_\lambda|_{S_r}. \tag{4.1.27}$$

Choose eigenvectors $v_1, \ldots, v_{n-1}$ of R_{v_n} such that together with v_n they form an orthonormal basis for $T_{\pi(p)}M$. Recall from (1.2.13)

$$J_A(v_j)_q^H = h(\sqrt{\Lambda_j} r)(v_j)_q^V, \tag{4.1.28}$$

where $h(x) = x \coth x$. From (1.2.6) we know

$$(N_\lambda)_*(v_j)_q^H = (v_j)_p^H, \quad (N_\lambda)_*(v_j)_q^V = (c/r)(v_j)_p^V. \tag{4.1.29}$$

From (4.1.27), (4.1.28) and (4.1.29) we get

$$
\begin{aligned}
(\Phi_\varepsilon) * J_A(v_j)_p^H &= (\Phi_\varepsilon)_* \left(J_A((\Phi_\varepsilon)_*^{-1}(v_j)_p^H)\right) \\
&= (\Phi_\varepsilon)_*(J_A(v_j)_q^H) \\
&= (\Phi_\varepsilon)_* \left(h(\sqrt{\Lambda_j}r)(v_j)_q^V\right) \\
&= h(\sqrt{\Lambda_j}r)(c/r)(v_j)_p^V.
\end{aligned} \tag{4.1.30}
$$

Now $r \to \infty$ as $\varepsilon \to 0$. Hence $h(\sqrt{\Lambda_j}r)(c/r) = \sqrt{\Lambda_j}\coth(\sqrt{\Lambda_j}r)c \to \sqrt{\Lambda_j}c$. This together with (4.1.30) implies

$$
\lim_{\varepsilon\to 0}(\Phi_\varepsilon)_* J_A(v_j)_p^H = \sqrt{\Lambda_j}c(v_j)_p^V. \tag{4.1.31}
$$

On the other hand

$$
\begin{aligned}
\hat{R}_p v_j &=< v_j, p > p + R_p(v_j) \\
&=< v_j, cv_1 > cv_1 + c^2 R_{v_1} v_j \\
&= c^2 \Lambda_j v_j.
\end{aligned} \tag{4.1.32}
$$

This implies

$$
J_0(v_j)_p^H = c\sqrt{\Lambda_j}(v_j)_p^V. \tag{4.1.33}
$$

Finally (4.1.31) and (4.1.33) together imply that indeed $\lim_{\varepsilon\to 0}(\Phi_\varepsilon)_* J_A$ exists and equals to J_0. The examples after (4.1.8) show that J_0 in fact coincides with the complex structure studied by Furutani, Tanaka, Ii and Morikawa. The normalized geodesic flow was shown to be J_0-holomorphic in [IM].

Let now $s > 0$, $0 \neq p \in T_mM$ and $v \in T_mM$ be arbitrary. Define w by $w = \sqrt{\hat{R}_{sp}}v$. From the definition of $\hat{R}_{sp}$ we get that

$$
w = s\sqrt{\hat{R}_p}v. \tag{4.1.34}
$$

This together with the action (1.2.6) of $(N_s)_*$ and the definition (4.1.8) of J_0 yields:

$$
J_0|_{sp}\,(N_s)_* v_p^H = J_0|_{sp}\, v_{sp}^H = w_{sp}^V = (N_s)_* J_0 v_p^H, \tag{4.1.35}
$$

i.e. N_s is indeed J_0-holomorphic. □

4.2 Higher rank symmetric spaces

4.2.1 Preliminaries and the main results

Let (M, g) be a Riemannian manifold, $\pi : TM \to M$ its tangent bundle, $z \in TM$ and $m = \pi(z)$ $\mathcal{R}$ the curvature tensor of g and R_z the curvature operator (see 1.2.12). Recall the operator

$$
\hat{R}_z = g(., z)z + R_z \tag{4.2.1}
$$

from (4.1.7).

Let now (M, g) be a locally symmetric space whose universal cover is compact. When the rank is larger than 1, there is no analogous complex structure

to J_S defined apriori. We need to find out what it should be. The operator $\hat{R}_z$ is only positive semi-definite but at least it admits a square root. We introduce the following bundles over $\mathring{\mathrm{T}}M$ by defining their fibers at z is as follows

$$\begin{aligned} \mathcal{D}_z &= \{X_z^H - i(\sqrt{R}_z X)_z^V : X \in T_m M \otimes \mathbb{C}\}, \\ \mathcal{E}_z &= \{X_z^H - i(\sqrt{\hat{R}}_z X)_z^V : X \in T_m M \otimes \mathbb{C}\}. \end{aligned} \tag{4.2.2}$$

When the rank is 1, $\mathcal{E}$ is the (1,0) tangent bundle of the J_0 complex structure from formula (4.1.7). When the rank of the universal cover of M is larger than 1, the bundles $\mathcal{D}, \mathcal{E}$, are only continuous, since $\hat{R}_z$ is only semidefinite. $\mathcal{E}$ is not the bundle of (1,0) tangent vectors of any complex or CR structure. It is rather a kind of stratified involutive bundle containing a stratified CR structure and posessing an open and dense stratum. See more on these later in this chapter.

An isometry of an arbitrary Riemannian manifold always preserves the curvature tensor and its action on the tangent bundle commutes with horizontal and vertical lifts. Thus we get the following invariance property of these bundles.

Proposition 4.2.1. *Let φ be an isometry of (M, g). Denote by $\mathcal{G}$ any of the bundles $\mathcal{D}$ or $\mathcal{E}$. Then*

$$(\varphi_*)_* \mathcal{G} = \mathcal{G}$$

One of the aims of this section is to exhibit several examples of involutive structures over (mostly) noncompact manifolds arising naturally in geometry. These examples describe the singular behavior of complex structures at the boundary of their domains of definition (either at finite points or at infinity).

In section 4.2.3 we study homogeneous involutive structures associated to a compact symmetric space $M = U/K$. In particular we show that all the U orbits in $\mathring{\mathrm{T}}M$ possess CR and other homogeneous involutive structures.

Section 4.2.4 contains the main result: Theorem 4.2.11, where we show that the bundle $\mathcal{E}$ is a nonnegative complex polarization on an open and dense subdomain of $\mathring{\mathrm{T}}M$ and furthermore it is preserved by the normalized geodesic flow.

More precisely we show that there exists a kind of stratification (due to the walls of the Weyl chamber) of $\mathring{\mathrm{T}}M$ and on each stratum $\mathcal{E}$ is a real-analytic involutive bundle containing an integrable CR structure. The CR dimension varies from stratum to stratum. The CR structure is very closely related to the classic complex structures of A. Borel on adjoint orbits (see Section 4.2.3). The proof of the involutivity uses the appropriate results of section 4.2.3 concerning homogeneous bundles over orbits.

Finally in Section 4.2.5 we explain why one cannot get (by scaling) geodesic flow invariant genuine Kähler polarizations in the higher rank cases.

There is a certain dual notion to adapted complex structures, the so called adapted product structures. All the results in this section have analogous statements for these structures, but now M is a Riemannian symmetric space of noncompact type. For the details and the statements see [Sz01].

4.2.2 Limit structures

In section 4.1 (see also [Ag]) we studied the behavior of the adapted complex structure at "infinity" for a compact, rank-1 symmetric space.

In this section we extend our method to treat all compact symmetric spaces. We showed in Theorem 4.1.1 that $J_0 = \lim_{\varepsilon\to 0}(\Phi_\varepsilon)_* J_A$ exists, provided (M, g) is of rank-1. Furthermore this limit complex structure in the horizontal and vertical splitting has the matrix in block form (4.1.8).

When the rank is larger than 1, the kernel of the operator $\hat{R}_z$ is nontrivial at every point z and its dimension varies from point to point due to the presence of the walls of the Weyl chambers. Thus the matrix formula defining J_0 makes no sense.

The right approach here is: instead of working directly with the complex structure tensor, consider rather the corresponding bundle of $(1, 0)$ tangent vectors.

Theorem 4.2.2 (Szőke,[Sz01])**.** *Let (M, g) be a Riemannian manifold whose universal cover is a compact, globally symmetric space. Then (Φ_ε is from* (4.0.1)*)*

$$\exists \lim_{\varepsilon\to 0}(\Phi_\varepsilon)_*(T^{1,0}\mathring{T}M) = \mathcal{E}. \tag{4.2.3}$$

Proof. Let $z \in \mathring{\mathrm{T}}M$ be fixed, $\varepsilon > 0$ be arbitrary and let $c = \|z\|, m = \pi(z), r = \log(c/\varepsilon), q = rz/c$. Then $\|q\| = r$ and $\Phi_\varepsilon(q) = z$. Let $v_n := \frac{z}{\|z\|}$ and $v_1, \dots, v_n$ be an orthonormal basis of eigenvectors of R_{v_n} with corresponding eigenvalues $\Lambda_1, \dots, \Lambda_n$ ($\Lambda_j \geq 0$ since $\widetilde{M}$ is compact symmetric). The calculations in [Sz91, Theorem 2.5] yield that the adapted complex structure J at q is defined by the formulas

$$\begin{aligned} J_q(v_j)_q^H &= \sqrt{\Lambda_j}\|q\| \coth(\sqrt{\Lambda_j}\|q\|)(v_j)_q^V \\ J_q(v_j)_p^V &= -\frac{\tanh(\sqrt{\Lambda_j}\|q\|)}{\sqrt{\Lambda_j}\|q\|}(v_j)_q^H, \end{aligned} \tag{4.2.4}$$

$j = 1, \dots, n$. Now we are going to calculate the image of the (1,0) tangent bundle under the map Φ_ε. First notice that when we restrict this map to the level surface $\|.\|_g = r$, it agrees with the map N_λ (N is from (1.2.1)) with $\lambda = c/r$. This observation together with (1.2.6) implies

$$(\Phi_\varepsilon)_*((v_j)_q^H) = (v_j)_z^H, \tag{4.2.5}$$

$j = 1, \dots, n$, and

$$(\Phi_\varepsilon)_*((v_k)_q^V) = (c/r)(v_k)_z^V, \tag{4.2.6}$$

$k = 1, \dots, n-1$. It easily follows from the definition of Φ_ε that

$$(\Phi_\varepsilon)_*((v_n)_q^V) = c(v_n)_z^V. \tag{4.2.7}$$

Now from (4.2.4)-(4.2.7) we get

$$\begin{aligned} (\Phi_\varepsilon)_*(T_q^{1,0}TM)) = \big\{&\alpha_n((v_n)_z^H - ic(v_n)_z^V) + \beta_n(c(v_n)_z^V + i(v_n)_z^H) \\ &+ \sum_{j=1}^{n-1} \alpha_j \left((v_j)_z^H - ic\sqrt{\Lambda_j}\coth(\sqrt{\Lambda_j}r)(v_j)_z^V\right) \\ &+ \sum_{k=1}^{n-1} \beta_k \left(c\sqrt{\Lambda_j}\coth(\sqrt{\Lambda_j}r)(v_k)_z^V + i(v_k)_z^H\right) : \alpha_j, \beta_k \in \mathbb{R}\Big\} \end{aligned} \tag{4.2.8}$$

Now as $\varepsilon \to 0$, r will go to infinity. Hence

$$\lim_{\varepsilon\to 0} \sqrt{\Lambda_j}\coth(\sqrt{\Lambda_j} r) = \sqrt{\Lambda_j}, \quad j = 1, \ldots, n-1. \tag{4.2.9}$$

Now because of our choices we get $\hat{R}_z v_n = c^2 v_n, \hat{R}_z v_j = c^2 \Lambda_j v_j, j = 1, \ldots, n-1$. This together with (4.2.2), (4.2.9) and (4.2.8) implies our claim (4.2.3). □

Remarks. Let $\varphi(\varepsilon, t) : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ be a smooth function such that for each $\varepsilon > 0$, $\varphi(\varepsilon, .)$ is a monotone increasing diffeomorphism of $\mathbb{R}^+$. Denote its inverse by ψ_ε. Define the diffeomorphism $\Phi_{\varepsilon,\varphi} : \mathring{\mathrm{T}}M \to \mathring{\mathrm{T}}M$ by $\Phi_{\varepsilon,\varphi}(v) = (\varphi(\varepsilon, \|v\|)/\|v\|)v$. Following the same steps as above one can prove that the limit

$$\lim_{\varepsilon\to 0} (\Phi_{\varepsilon,\varphi})_* (T^{1,0}(\mathring{\mathrm{T}}M))$$

exists provided

$$\lim_{\varepsilon\to 0} \frac{1}{\psi'_\varepsilon(c)}, \qquad \lim_{\varepsilon\to 0} a\coth(a\psi_\varepsilon(c))$$

exist for each $c > 0, a \geq 0$.

Take for instance for $\varphi(\varepsilon, t) = \varepsilon t^\lambda, \lambda > 0$. When $\lambda = 1$, $\Phi_{\varepsilon,\varphi}(v) = N_\varepsilon v = \varepsilon v$ and one gets that

$$\lim_{\varepsilon\to 0} (N_\varepsilon)_* T^{1,0}(\mathring{\mathrm{T}}M) = \mathcal{D},$$

where $\mathcal{D}$ is from (4.2.2).

The slight drawback of this choice is the fact that every z is in the kernel of R_z and thus $\mathcal{D}_z \cap \overline{\mathcal{D}}_z$ is always nontrivial. Even in the rank-1 case $\mathcal{D}$ recovers only part of the complex structure studied by Souriau etc... , the complex structure along the leaves of the Riemann foliation is lost.

4.2.3 Homogeneous involutive structures

Recall that the homogeneous manifold $B = U/K$ is called reductive if the Lie algebra $\mathfrak{u}$ of U admits a direct sum decomposition $\mathfrak{u} = \mathfrak{k}_0 + \mathfrak{m}_0$, where $\mathfrak{k}_0$ is the Lie algebra of K and $\mathfrak{m}_0$ is an $Ad(K)$ invariant vector subspace of $\mathfrak{u}$. A typical example is when K is compact and $\mathfrak{m}_0$ is the orthogonal complement of $\mathfrak{k}_0$ w.r.t. any $Ad(K)$ invariant inner product on $\mathfrak{u}$.

Let $\pi : U \to U/K$ be the projection map, $b := [K]$ and e the unit element in U. The restriction of π_* to $\mathfrak{m}_0$ gives an isomorphism between $\mathfrak{m}_0$ and T_bB. With this isomorphism a choice of a left U homogeneous subbundle E of $T^{\mathbb{C}}B = TB \otimes \mathbb{C}$ is equivalent with the choice of an $Ad(K)$ invariant subspace $S \subset \mathfrak{m} = \mathfrak{m}_0 \otimes \mathbb{C}$ such that $\pi_* : S \cong E_b$ (the fiber over b). $Ad(K)$ invariance implies (and is equivalent to when K is connected)

$$[\mathfrak{k}_0, S] \subset S. \tag{4.2.10}$$

Proposition 4.2.3 (Szőke,[Sz01])**.** *Let $B = U/K$ be a reductive homogeneous manifold. Let $\mathfrak{k} = \mathfrak{k}_0 \otimes \mathbb{C}$. A left U homogeneous subbundle E of $T^{\mathbb{C}}B$ is involutive iff*

$$\pi_*^{-1}(E_b) \quad \textit{is a Lie subalgebra of} \quad \mathfrak{u} \otimes \mathbb{C} \tag{4.2.11}$$

and this condition is equivalent to

$$[S, S] \subset S + \mathfrak{k}. \tag{4.2.12}$$

Proof. From our assumptions we have

$$\pi_*^{-1}(E_b) = S \oplus \mathfrak{k}.$$

Using (4.2.10) we get that (4.2.11) and (4.2.12) are indeed equivalent. With the help of the left U action on itself the vector spaces $S, \mathfrak{m}, \mathfrak{k}, \pi_*^{-1}(E_b)$ generate left U homogeneous subbundles $\mathcal{S}, \mathfrak{M}, \mathfrak{K}, \mathcal{E}$ of $T^{\mathbb{C}}U$. Since the vector spaces are $Ad(K)$ invariant, the corresponding bundles are right K homogeneous as well. We also have

$$\mathcal{E} = \mathcal{S} \oplus \mathfrak{K}.$$

Since $\mathfrak{k}$ is a Lie algebra, $\mathfrak{K}$ is an involutive bundle and (4.2.11) is equivalent with the involutivity of $\mathcal{E}$.

The left U equivariant map $\pi_* : \mathfrak{M} \to T^{\mathbb{C}}B$ is a fiberwise isomorphism, similarly the map $\pi_*|_{\mathcal{S}} : \mathcal{S} \to E$. Thus any smooth section X of $T^{\mathbb{C}}B$ can be uniquely lifted to a smooth section $\check{X}$ of $\mathfrak{M}$ so that for $q \in U$, $\pi_*(\check{X}_q) = X_{\pi(q)}$. Thus for any vector fields X, Y on B

$$\pi_*[\check{X}, \check{Y}] = [X, Y]. \tag{4.2.13}$$

For a vector bundle $\mathcal{G}$ denote by $\Gamma(\mathcal{G})$ the smooth (local) sections of $\mathcal{G}$.

(4.2.13) implies that the involutivity of E is equivalent to: for any $X, Y \in \Gamma(E)$, $[\check{X}, \check{Y}] \in \Gamma(\mathcal{E})$. Those sections of $\mathcal{S}$ that are lifts of $\Gamma(E)$, locally span $\mathcal{S}$. Consequently the involutivity of E is the same thing as

$$[\Gamma(\mathcal{S}), \Gamma(\mathcal{S})] \subset \Gamma(\mathcal{E}).$$

On the other hand from (4.2.10) we get

$$[\Gamma(\mathfrak{K}), \Gamma(\mathcal{S})] \subset \Gamma(\mathcal{S}).$$

Since $\mathfrak{K}$ is always involutive we finally can conclude that E is involutive iff $\mathcal{E}$ is. □

Now let U be a connected Lie group and K a closed subgroup of U. Let K_0 be the identity component of K. Then the natural map $U/K_0 \to U/K, aK_0 \mapsto aK$ is a covering with sheet number $|K : K_0|$. Thus if we assume in addition that U/K is simply connected, we can conclude that the group K is connected. (We could also conclude this using the beginning part of the homotopy sequence of the fibration $U \to U/K$.)

In the rest of this section we assume that (M, g) is a compact, simply connected globally symmetric Riemannian manifold. We shall denote by U the identity component of the isometry group and by K the stabilizer subgroup of a fixed point of M. Thus $\mathfrak{u}$ is semisimple (cf. [He1]) and U and K are compact and connected (connectivity of K follows from the discussion above) and $M \leftrightarrow U/K$. The geodesic symmetry at $p = [K]$ gives rise to an involutive automorphism θ of $\mathfrak{u}$ with fixed points set $\mathfrak{k}_0$, the Lie algebra of K. The eigenspaces of θ decompose the Lie algebra $\mathfrak{u} = \mathfrak{k}_0 \oplus \mathfrak{p}_*$ where $\mathfrak{p}_*$ is the -1 eigenspace. Since θ is an automorphism, we get

$$[\mathfrak{k}_0, \mathfrak{p}_*] \subset \mathfrak{p}_* \quad \text{and} \quad [\mathfrak{p}_*, \mathfrak{p}_*] \subset \mathfrak{k}_0. \tag{4.2.14}$$

The negative of the Killing form of $\mathfrak{u}$ provides an $Ad(K)$ invariant inner product on $\mathfrak{p}_* \leftrightarrow T_pM$ and we shall assume that this gives the U invariant symmetric metric on M. We shall need this in Theorem 4.2.6 and Proposition 4.2.7. In general U/K has many U invariant symmetric metrics (i.e. when the $Ad(K)$ action on $\mathfrak{p}_*$ is not irreducible) that are not proportional. However the induced Riemannian connection (cf. [He1]) and thus the adapted complex structure on TM (cf. [Sz98]) is the same for all possible choices.

Let $\mathfrak{g} = \mathfrak{u} \otimes \mathbb{C}$. Introduce the following subspaces of $\mathfrak{g}$. Let $\mathfrak{k} = \mathfrak{k}_0 \otimes \mathbb{C}$, $\mathfrak{p}_0 = i\mathfrak{p}_*, \mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ (a semisimple Lie algebra, the noncompact dual to $\mathfrak{u}$), $\mathfrak{h}_{\mathfrak{p}_*} \subset \mathfrak{p}_*$ be a maximal abelian subspace, $\mathfrak{h}_{\mathfrak{p}_0} = i\mathfrak{h}_{\mathfrak{p}_*}$, $\mathfrak{h}_{\mathfrak{p}} = \mathfrak{h}_{\mathfrak{p}_0} \otimes \mathbb{C}$, $\mathfrak{h}_0 \subset \mathfrak{g}_0$ a maximal abelian subalgebra that contains $\mathfrak{h}_{\mathfrak{p}_0}$, $\mathfrak{h}_{\mathfrak{k}_0} = \mathfrak{h}_0 \cap \mathfrak{k}_0$, $\mathfrak{h}_{\mathfrak{k}} = \mathfrak{h}_{\mathfrak{k}_0} \otimes \mathbb{C}$, $\mathfrak{h}_* = (\mathfrak{h}_{\mathfrak{k}_0} \oplus \mathfrak{h}_{\mathfrak{p}_*})$ and $\mathfrak{h} = \mathfrak{h}_* \otimes \mathbb{C}$. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$.

Denote by Δ the corresponding nonzero roots. For an $\alpha \in \Delta$, the root space is

$$\mathfrak{g}^\alpha = \{v \in \mathfrak{g} \mid [H, v] = \alpha(H)v, \forall H \in \mathfrak{h}\}.$$

Let $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{h}_{\mathfrak{p}_0})$ be the set of restricted roots and let $C_{\mathfrak{p}_0} \subset \mathfrak{h}_{\mathfrak{p}_0}$ be an open Weyl chamber , i.e. a component of the complement (in $\mathfrak{h}_{\mathfrak{p}_0}$) of the kernels of the restricted roots.

Let now $0 \neq v \in \overline{C}_{\mathfrak{p}_0}$ (the closed Weyl chamber) be a fixed vector. Denote by L_v the centralizer of v in K that is $L_v = \{k \in K \mid Ad(k)v = v\}$. The group L_v will be connected since K is (see [He, Corollary 2.8, p. 287]). Its Lie algebra is

$$\mathfrak{l}_v = \{X \in \mathfrak{k}_0 \mid [v, X] = 0\}.$$

If $\alpha \in \Delta$ then $\alpha^\theta = \alpha(\theta.)$ is also a nonzero root and $\alpha(v) = 0$ iff $\alpha^\theta(v) = 0$ since $\theta(v) = -v$. The root space corresponding to α^θ is $\theta(\mathfrak{g}^\alpha)$. For a $w \in \mathfrak{g}$ denote by $\overline{w}$ its conjugate w.r.t. $\mathfrak{u} \subset \mathfrak{g} = \mathfrak{u} \otimes \mathbb{C}$. It is well known (cf. [He1, Lemma 3.1 p.257]), that $\overline{\mathfrak{g}^\alpha} = \mathfrak{g}^{-\alpha}$. Let

$$\mathcal{H}_v = \sum_{\substack{\alpha \in \Delta^+ \\ \alpha(v) \neq 0}} \mathfrak{g}^\alpha \qquad \text{and} \qquad \mathcal{G}_v = \sum_{\substack{\alpha \in \Delta \\ \alpha(v) = 0}} \mathfrak{g}^\alpha. \tag{4.2.15}$$

The root space decomposition of $\mathfrak{g}$ yields a θ invariant decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathcal{G}_v \oplus \mathcal{H}_v \oplus \overline{\mathcal{H}}_v.$$

Thus

$$\mathfrak{l}_v \otimes \mathbb{C} = (\mathfrak{h} \oplus \mathcal{G}_v) \cap \mathfrak{k} = \mathfrak{h}_{\mathfrak{k}} \oplus \mathcal{G}_v \cap \mathfrak{k}, \tag{4.2.16}$$

and so

$$\mathfrak{g} = \mathfrak{l}_v \otimes \mathbb{C} \oplus \mathcal{H}_v \oplus \overline{\mathcal{H}}_v \oplus \mathfrak{h}_{\mathfrak{p}} \oplus \mathcal{G}_v \cap \mathfrak{p}. \tag{4.2.17}$$

Let $B_v = U/L_v$ and $b = [L_v]$. The tangent space $T_b^{\mathbb{C}} B_v$ is now identified with

$$\mathfrak{m} = \mathcal{H}_v \oplus \overline{\mathcal{H}}_v \oplus \mathfrak{h}_{\mathfrak{p}} \oplus \mathcal{G}_v \cap \mathfrak{p}. \tag{4.2.18}$$

We can get a more precise description of $\mathcal{G}_v \cap \mathfrak{p}$ as follows. Namely from its definition it is obvious that $\alpha = \alpha^\theta$ iff α vanishes identically on $\mathfrak{h}_{\mathfrak{p}}$. Denote by $\Delta_{\mathfrak{p}}$ the set of those roots that do not vanish identically on $\mathfrak{h}_{\mathfrak{p}}$. One easily sees that for $\alpha \in \Delta_{\mathfrak{p}}$ we have $\alpha^\theta < 0$ iff $\alpha > 0$ and

$$\sum_{\alpha \in \Delta \setminus \Delta_{\mathfrak{p}}} \mathfrak{g}^\alpha \subset \mathfrak{k}.$$

Now pick a nonzero element $X_\alpha \in \mathfrak{g}^\alpha$ for each $\alpha \in \Delta_{\mathfrak{p}}^+$. Thus we get

$$\mathcal{G}_v \cap \mathfrak{p} = \{ \sum_{\substack{\alpha \in \Delta_{\mathfrak{p}}^+ \\ \alpha(v)=0}} a_\alpha (X_\alpha - \theta X_\alpha) : a_\alpha \in \mathbb{C}\}.$$

Hence

$$\dim_{\mathbb{C}} \mathcal{G}_v \cap \mathfrak{p} = \#\{\alpha \in \Delta_{\mathfrak{p}}^+ : \alpha(v) = 0\}. \tag{4.2.19}$$

Denote this number by c_v.

So far we purposefully did not specify a particular ordering on the duals of $\mathfrak{h}_{\mathfrak{p}_0}$ and $\mathfrak{h}_{\mathbb{R}} = \mathfrak{h}_{\mathfrak{p}_0} + i\mathfrak{h}_{\mathfrak{k}_0}$. We shall do that now. The choice of the Weyl chamber determines the set of the positive restricted roots, consequently the set of the simple restricted roots $\beta_1, \ldots, \beta_l$, where $l = dim_{\mathbb{R}} \mathfrak{h}_{\mathfrak{p}_0}$ is the rank of M. But we still have the freedom to decide the order among them. We make this choice to be compatible with v.

So renumber the simple restricted roots to get $\lambda_1, \ldots, \lambda_l$ so that for some $1 \leq j \leq l$ (depending on v), $\lambda_1(v) \neq 0, \ldots, \lambda_j(v) \neq 0, \lambda_{j+1}(v) = \cdots = \lambda_l(v) = 0$. Declaring that $\lambda_1 > \cdots > \lambda_l$ we get an ordering of the dual of $\mathfrak{h}_{\mathfrak{p}_0}$. Extending this to a compatible ordering of the dual of $\mathfrak{h}_{\mathbb{R}}$ we get the ordering we shall use. Denote by Δ^+ the corresponding set of positive roots and Σ^+ the set of positive restricted roots. Let

$$F_v = \{w \in \overline{C}_{\mathfrak{p}_0} \mid \lambda_1(w) \neq 0, \ldots, \lambda_j(w) \neq 0, \lambda_{j+1}(w) = \cdots = \lambda_l(w) = 0\}. \tag{4.2.20}$$

This is the face of the closed Weyl chamber that contains v. For instance when $v \in C_{\mathfrak{p}_0}$, $F_v = C_{\mathfrak{p}_0}$.

Proposition 4.2.4 (Szőke,[Sz01])**.** *Let $\alpha \in \Delta, \alpha(v) \neq 0$. Then $\alpha(v) > 0$ and $\alpha > 0$ are equivalent. Let $\alpha \in \Delta^+$, $\alpha(v) \neq 0$ and $\beta \in \Delta$ with $\beta(v) = 0$. Then α is positive on F_v and $\alpha + \beta > 0$.*

Proof. $\alpha(v) \neq 0$ implies $\lambda = \alpha|_{\mathfrak{h}_{\mathfrak{p}}} \in \Sigma$. Hence there are integers n_k all non-negative or nonpositive so that $\lambda = \sum_{k=1}^l n_k \lambda_k$ (cf. in [He1, Theorem 2.19, p.292.]). Therefore $0 \neq \lambda(v) = \sum_{k=1}^j n_k \lambda_k(v)$. Thus one of the coefficients $n_t, t = 1, \ldots, j$ must be non-zero. This shows the equivalence of $\alpha(v) > 0$ and $\alpha > 0$. It also shows that $\alpha > 0$ on F_v. Our last statement is obvious since from our assumptions $\nu := \beta|_{\mathfrak{h}_{\mathfrak{p}}} = \sum_{k=j+1}^l m_k \lambda_k$ and $\lambda = \sum_{k=1}^j n_k \lambda_k$, where n_k is a non-negative integer and at least one of them is different from zero. □

Proposition 4.2.5 (Szőke,[Sz01])**.** *Every element in F_v has the same centralizer in K.*

Proof. Let $w \in F_v$. in [He1, Theorem 2.19, p. 292] shows that a root vanishes on v iff it vanishes on w. Thus the Lie algebra of L_w and L_v is the same (cf. (4.2.16)). Since both groups and K are connected our statement follows. □

Now we are going to define involutive structures on the homogeneous manifold $B_v = U/L_v$. Let $\mathcal{N}_v$ be the complexification of the orthogonal complement of v in $\mathfrak{h}_{\mathfrak{p}_0}$. Introduce the following subspaces of $\mathfrak{m}$ (also thought of as subspaces of $T_b^{\mathbb{C}} B_v$).

$$\begin{aligned} Q_b &= \mathfrak{h}_{\mathfrak{p}} \oplus (\mathcal{G}_v \cap \mathfrak{p}) \qquad & S_b &= \mathcal{N}_v \oplus (\mathcal{G}_v \cap \mathfrak{p}), \\ A_b &= \mathcal{H}_v \oplus Q_b \qquad & E_b &= \mathcal{H}_v \oplus S_b. \end{aligned} \tag{4.2.21}$$

Theorem 4.2.6 (Szőke,[Sz01]). *The subspace $\mathcal{H}_v$ determines a left U homogeneous integrable CR structure on B_v. Let l be the rank of M. The CR codimension is $l + c_v$. When v is generic (i.e. $v \in C_{\mathfrak{p}_0}$), $c_v = 0$ and we recover the CR structure of R. Aguilar (cf. [Ag]). The subspaces Q_b, S_b, A_b and E_b define the U homogeneous involutive subbundles Q, S, A, E of $T^{\mathbb{C}}B_v$.*

Proof. To show that the subspaces really provide U homogeneous bundles, we need to check that condition (4.2.10) holds. The integrability of the CR structure and the involutivity of the bundles requires to check (4.2.12). This is the point where we need our special choice of ordering of the roots. To calculate the appropriate Lie brackets we need to recall the following facts.

Let α be a nonzero root. Since the Killing form $\mathcal{B}$ of U is nondegenerate we can define a vector H_α so that $\mathcal{B}(., H_\alpha) = \alpha(.)$. Since we use $\mathcal{B}$ to get our metric on M, v and H_α are orthogonal iff $\alpha(v) = 0$. We also have that $\mathfrak{h}_{\mathbb{R}} = \sum_{\alpha\in\mathbb{R}} \mathbb{R}H_\alpha = \mathfrak{h}_{\mathfrak{p}_0} + i\mathfrak{h}_{\mathfrak{k}_0}$ (cf. [He1, Lemma 3.2, p.259]).

The structure theory of the root systems tells us (cf. [He1]) that for any $\alpha, \beta \in \Delta$, the Lie bracket $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta]$ is $\mathbb{C}H_\alpha$ if $\alpha + \beta = 0$, is $\mathfrak{g}^{\alpha+\beta}$ if $\alpha + \beta \in \Delta$ and is zero otherwise. This immediately implies

$$[\mathcal{G}, \mathcal{G}] \subset \mathcal{G} + \mathcal{N}_v + \mathfrak{h}_{\mathfrak{k}}.$$

Using this, together with (4.2.14) and Proposition 4.2.4, it is a straightforward calculation that indeed in all required cases (4.2.10) and (4.2.12) hold. In particular we get that the U homogeneous CR structure defined by $\mathcal{H}_v$ is formally integrable. Since everything is real-analytic it is necessarily a genuinely integrable (cf. Section 4.2.4) CR structure.

□

We shall call the CR structure in the previous theorem the *standard CR* structure on the manifold U/L_v since in a way it is well known. The reason is as follows. Let

$$Z_v = \{a \in U : Ad(a)v = v\}$$

Its Lie algebra $\mathfrak{z}_v$ is then

$$\mathfrak{z}_v = \{X \in \mathfrak{u} : [v, X] = 0\}.$$

Thus

$$\mathfrak{z}_v \otimes \mathbb{C} = \mathfrak{h} \oplus \mathcal{G}_v.$$

Hence

$$\mathfrak{z}_v \otimes \mathbb{C} = \mathfrak{l}_v \otimes \mathbb{C} \oplus \mathfrak{h}_{\mathfrak{p}} \oplus \mathcal{G}_v \cap \mathfrak{p}$$

and

$$\mathfrak{g} = \mathfrak{z}_v \otimes \mathbb{C} \oplus \mathcal{H}_v \oplus \overline{\mathcal{H}}_v.$$

Thus the quotient space U/Z_v, that is just the adjoint U orbit of iv in $\mathfrak{u}$, is a complex manifold. The bundle of (1,0) tangent vectors is the U homogeneous vector bundle corresponding to the subspace $\mathcal{H}_v$. This complex structure was discovered by A. Borel in the fifties and was studied extensively from various points of views ever since (see for example [Bes]). Because of the construction the natural projection

$$U/L_v \longrightarrow U/Z_v$$

is a CR map.

Our next aim is to use the involutive bundles of Theorem 4.2.6 to obtain a clear picture of the bundles $\mathcal{D}, \mathcal{E}$ (cf. (4.2.2)). $B_v = U/L_v$ is of course the U-orbit of $iv \in \mathfrak{h}_{\mathfrak{p}_*} \subset \mathfrak{p}_* \leftrightarrow T_pM$ in TM. Now according to Proposition 4.2.5 each element in F_v has the same centralizer in K, so the next proposition describing the orbit structure of TM comes very naturally. We shall use this description to establish the connection between the involutive bundles of Theorem 4.2.6 and $\mathcal{D}, \mathcal{E}$. This will prove the involutivity of the latter bundles over the "cone-like" submanifolds $\beta(U/L_v \times F_v) \subset \mathring{T}M$, where β sends (aL_v, w) to $a_*(iw)$. Letting U act trivially on the second coordinate we get an U action on $(U/L_v) \times F_v$. Let $b = [L_v]$ and $Z_v = B_v \times F_v$.

Proposition 4.2.7 (Szőke,[Sz01])**.** *The map*

$$\beta : Z_v \longrightarrow TM$$

is a real-analytic U equivariant imbedding.

Proof. We adopt the proof from [He1, p. 294-295], to our situation. Proposition 4.2.5 shows that β is well defined. β is clearly real-analytic and U equivariant. Suppose that for $a, c \in U$ and $w_1, w_2 \in F_v$ we have $a_*(iw_1) = c_*(iw_2)$. Then $(c^{-1}a)_*(iw_1) = iw_2$. This implies that $c^{-1}a$ belongs to K. Since w_j both belong to the closed Weyl chamber $\overline{C}_{\mathfrak{h}_{\mathfrak{p}_0}}$, there must be an element in the Weyl group that also sends w_1 to w_2. But $\overline{C}_{\mathfrak{h}_{\mathfrak{p}_0}}$ is the space of K orbits therefore $c^{-1}a$ must be the identity and so $w_1 = w_2$ and thus $c^{-1}a \in L_v$ because of Proposition 4.2.5. Hence β is injective.

To show that it is an imbedding therefore it suffices to check the injectivity of β_* at $([L_v], w)$ where $w \in F_v$ is arbitrary.

Let Y be in $T_bB_v \cap \mathfrak{p}_*$ and $S \in T_wF_v$. Then $((\exp tY)b, w + tS)$ is a curve in Z_v with tangent vector (Y, S). Since $(\exp tY)_*(iw)$ is the parallel translation of iw along the geodesic defined by Y (cf. in [He1, Theorem 3.3, p.208]) we get

$$\beta_*(Y, S) = (Y)^H_{iw} + (S)^V_{iw},$$

where H stands for the horizontal and V for the vertical lift. Therefore β_* is injective on $(T_bB_v \cap \mathfrak{p}_*) \times T_wF_v$. Now suppose $Y \in T_bB_v \cap \mathfrak{k}_0$. Then $(\exp tY)_*(iw) = Ad(tY)(iw)$. Therefore

$$\beta_*(Y, S) = ([Y, iw])^V_{iw} + (S)^V_{iw}.$$

Using the Killing form $\mathcal{B}$ of U

$$\mathcal{B}([Y, iw], S) = \mathcal{B}(Y, [iw, S]) = 0,$$

since $iw, S \in \mathfrak{h}_{\mathfrak{p}}$. Thus $[Y, iw]$ and S are perpendicular. (Remember that our metric on M comes from $\mathcal{B}$.) Therefore $\beta_*(Y, S)$ can be zero only if $S = 0 = [Y, iw]$. We need to show that Y is zero. From (4.2.17) we get

$$(T_bB_v \otimes \mathbb{C}) \cap \mathfrak{k} = \mathcal{H}_v \oplus \overline{\mathcal{H}}_v \cap \mathfrak{k}.$$

Pick an element X_α from each one dimensional positive root space $\mathfrak{g}^\alpha$. Since a root α that does not vanish identically on $\mathfrak{h}_{\mathfrak{p}_0}$ is positive iff $\alpha^\theta < 0$, each

$Y \in (T_b B_v \otimes \mathbb{C}) \cap \mathfrak{k}$ can be uniquelly written as

$$Y = \sum_{\substack{\alpha \in \Delta^+ \\ \alpha(v) \neq 0}} a_\alpha (X_\alpha + \theta X_\alpha)$$

for some complex numbers a_α. Thus for $w \in F_v$

$$[Y, iw] = - \sum_{\substack{\alpha \in \Delta^+ \\ \alpha(v) \neq 0}} a_\alpha \alpha(iw)(X_\alpha - \theta X_\alpha)$$

and this is again a direct sum. Now from [He1, Theorem 2.19, p.292] we know that $\alpha \in \Delta^+, \alpha(v) \neq 0$ implies $\alpha(w) \neq 0$ and thus $\alpha(iw) \neq 0$. Hence $[Y, iw]$ can be zero iff Y itself is equal to zero giving that indeed β_* is injective. □

Denote by $\mathbb{R}_+ v$ the half line in $\mathfrak{h}_{\mathfrak{p}_0}$ consisting of the positive multiples of v. Let $G_v = (U/L_v) \times \mathbb{R}_+ v \subset Z_v$. At the point $q = ([L_v], sv), s > 0$ define a subspace of $T_q G_v \otimes \mathbb{C}$ by $L_q = \{(\lambda v, -i\lambda sv) \mid \lambda \in \mathbb{C}\}$ (in the first component v is thought of as a tangent vector of U/L_v at $[L_v]$ as an element of $\mathfrak{h}_{\mathfrak{p}}$). Using the U action we get a U homogeneous line bundle $\mathcal{L} \to G_v$.

Proposition 4.2.8 (Szőke,[Sz01])**.** *Denote by $pr : G_v \to U/L_v$ the projection map. The bundle $pr^* E \oplus \mathcal{L} \to G_v$ is involutive.*

Proof. The sum is really a direct sum since v as a tangent vector of U/L_v was left out in the definition of E (cf. Theorem 4.2.6). Since $\mathcal{L}$ is a line bundle and we saw that E is involutive it is enough to show $[\Gamma(\mathcal{L}), \Gamma(pr^* E)] \subset \Gamma(pr^* E)$ (where again Γ denotes (local) sections). The vector v defines a unique U invariant complex vector field X over U/L_v that at the point $[L_v]$ is v itself. We also think of X as a vector field over G_v that does not depend on the second component.

The standard trivializing section of $T(\mathbb{R}_+ v)$ determines a U invariant section γ of TG_v. Then $X - is\gamma$ is a global trivializing section of $\mathcal{L}$. Since obviously $\Gamma(pr^* E)$ and γ commute, we only need to show that $[X, \Gamma(E)] \subset \Gamma(E)$. Lift E to get the U homogeneous subbundle $\check{E} \to U$ of $\mathfrak{M}$ (where $\mathfrak{M} \to U$ is the U homogeneous bundle corresponding to $\mathfrak{m} \leftrightarrow T_b^{\mathbb{C}} B_v$ cf. (4.2.18). Lift X to get the left U invariant vector field X_v over U. This is the left invariant vector field determined by $v \in T_e^{\mathbb{C}} U = \mathfrak{g}$. It suffices to show $[\check{X}, \check{E}] \subset \check{E}$ that we have using U homogeneity and the fact that

$$\check{E}_e = \sum_{\substack{\alpha \in \Delta^+ \\ \alpha(v) \neq 0}} \mathfrak{g}^\alpha \oplus \mathcal{N}_v \oplus (\sum_{\substack{\alpha \in \Delta \\ \alpha(v) = 0}} \mathfrak{g}^\alpha) \cap \mathfrak{p}.$$

□

Proposition 4.2.9 (Szőke,[Sz01])**.** *Let now $\|v\| = 1$. Then β_* is an isomorphism between the bundles $pr^* E \oplus \mathcal{L}$ and $\mathcal{E}\mid_{\beta(G_v)}$ and between $pr^* A$ and $\mathcal{D}\mid_{\beta(Z_v)}$.*

Proof. Because of homogeneity it suffices to show this for the points $([L_v], sv)$. As before let $B_v = U/L_v$, $p = [K], z = isv \in \mathfrak{h}_{\mathfrak{p}_*} \subset T_p M$ and $b = [L_v]$. Denote by W the orthogonal complement of $\ker R_z$ in $T_p M$. Then rewriting their definition (cf. (4.2.2)) we get

$$\mathcal{D}_z = (\ker R_z \otimes \mathbb{C})^H_z \oplus \{\nu^H_z - i(\sqrt{R}_z\nu)^V_z \mid \nu \in W \otimes \mathbb{C}\}$$
$$\mathcal{E}_z = (\ker \hat{R}_z \otimes \mathbb{C})^H_z \oplus \mathbb{C}([z]^H_z - i(\|z\|z)^V_z) \oplus \{\nu^H_z - i(\sqrt{R}_z\nu)^V_z \mid \nu \in W \otimes \mathbb{C}\}. \tag{4.2.22}$$

As we saw in the proof of Proposition 4.2.7 , for any $Y \in T_bB_v \cap \mathfrak{p}_*$, $\beta_*(Y,0) = (Y)^H_z$. Therefore the definition immediately implies that $\beta_*\mathcal{L}$ is precisely the middle term in $\mathcal{E}_z$.

[He1, Lemma 2.9, p. 288] yields that $S_b = (\ker \hat{R}_z)\otimes\mathbb{C}$ and $Q_b = (\ker R_z)\otimes\mathbb{C}$ (where Q_b and S_b are from Theorem 4.2.6). Thus we can conclude that $\beta_*(Q_b) = [Q_b]^H_z$ and $\beta_*(S_b) = [S_b]^H_z$ will be precisely the first components in (4.2.22).

Let now $0 \neq \eta \in \mathfrak{g}^\alpha$, where $\alpha \in \Delta^+, \alpha(v) \neq 0$. $\eta = \eta^{\mathfrak{k}} + \eta^{\mathfrak{p}}$ denotes the components w.r.t. the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let $w = sv$. $\eta \in \mathfrak{g}^\alpha$ implies

$$[w, \eta^{\mathfrak{k}}] = \alpha(w)\eta^{\mathfrak{p}}, \quad [w, \eta^{\mathfrak{p}}] = \alpha(w)\eta^{\mathfrak{k}}.$$

$\eta^{\mathfrak{p}}$ cannot be zero because this would mean $\alpha(w) = 0$ contradicting to $\alpha|_{F_v} > 0$. Just as in the proof of Proposition 4.2.7, for any $Y \in \mathfrak{k} \cap T^{\mathbb{C}}_b(B)$, we have $\beta_*(Y) = ([Y,z])^H_z$. Hence we get

$$\beta_*\eta = (\eta^{\mathfrak{p}})^H_z + ([\eta^{\mathfrak{k}}, z])^V_z = (\eta^{\mathfrak{p}})^H_z + (-\alpha(iw)\eta^{\mathfrak{p}})^V_z \neq 0. \tag{4.2.23}$$

On the other hand

$$R_z\eta^{\mathfrak{p}} = -[z,[z,\eta^{\mathfrak{p}}]] = \alpha^2(w)\eta^{\mathfrak{p}}. \tag{4.2.24}$$

Since $\alpha \in \Delta^+$, Proposition 4.2.4 implies $\alpha(w) > 0$ i.e. $\sqrt{R}_z\eta^{\mathfrak{p}} = \alpha(w)\eta^{\mathfrak{p}}$. Thus (4.2.23) yields that $\beta_*\eta$ belongs to the second component of $\mathcal{D}_z$ and the third component of $\mathcal{E}_z$ in (4.2.22). Consequently the entire image $\beta_*\mathcal{H}_v$ ($\mathcal{H}_v$ is from (4.2.15)) is contained in this component and we can again refer to [He1, Lemma 2.9, p. 288] to conclude that this way we get all the elements of this component. □

Corollary 4.2.10 (Szőke,[Sz01])**.** *For each $0 \neq v \in \overline{C}_{\mathfrak{p}_0}$, the restriction of the bundles $\mathcal{D}$ and $\mathcal{E}$ to the manifold $\beta_*((U/L_v) \times F_v)$ are involutive.*

Proof. F_v is foliated by the $\mathbb{R}_+$ action. This induces a foliation on $Z_v = (U/L_v) \times F_v$. As a consequence of Proposition 4.2.9 $\beta^*\mathcal{E}$ is a smooth in fact real-analytic vector bundle. Proposition 4.2.8 tells us that this bundle is tangential to and involutive along each leaf of the foliation of D. Consequently $\beta^*\mathcal{E}$ is globally involutive. The involutivity of the bundle A (Theorem 4.2.6) and Proposition 4.2.9 together imply the involutivity of $\mathcal{D}|_{\beta(Z_v)}$. □

4.2.4 Polarizations

Let (M,g) be a Riemannian manifold. Recall that the canonical 1-form θ on TM is defined by $\theta(\xi) = g(\pi_*\xi, z)$, where $z \in TM, \xi \in T_z(TM)$ and $\pi : TM \to M$ is the projection map. The canonical symplectic form is then $\omega = -d\theta$. Let $\xi, \eta \in T_z(TM)$. The value $\omega(\xi,\eta)$ can be computed by the formula

$$\omega(\xi,\eta) = g(\pi_*\xi, K\eta) - g(\pi_*\eta, K\xi), \tag{4.2.25}$$

where $K : T(TM) \to TM$ is the connection map.

Denote by $\omega_{\mathbb{C}}$ the complex bilinear extension of ω. Let $E : TM \to \mathbb{R}$, $E(v) = g(v,v)/2$ and $\varphi(v) = \|v\|$. The corresponding Hamiltonian vector fields are ξ_E, ξ_φ and their flows ϕ_t and ψ_t.

Now let (M,g) be a locally symmetric space whose universal cover $\widetilde{M}$ is compact. Let $cov : \widetilde{M} \to M$ be the covering map. We continue to use the terminology of section 4.2.3. So $\widetilde{M} = U/K, p = [K], \mathfrak{h}_{\mathfrak{p}_*} \subset T_p\widetilde{M}$ maximal abelian, $0 \neq v \in \overline{C}_{\mathfrak{p}_0} \subset \mathfrak{h}_{\mathfrak{p}_0} = i\mathfrak{h}_{\mathfrak{p}_*}$. Let $z = cov(iv)$ and $D_v = cov_* \circ \beta((U/L_v) \times F_v) \subset TM$. When v is generic (i.e. $v \in C_{\mathfrak{p}_0}$) D_v is an open and dense subset of TM.

Theorem 4.2.11 (Szőke,[Sz01])**.** *The continuous bundles $\mathcal{D}, \mathcal{E} \to \mathring{T}M$ (cf.* (4.2.2) *) are N_λ, $\lambda > 0$ and $\psi_t, t \in \mathbb{R}$ invariant. Their fibers are $\omega_{\mathbb{C}}$ Lagrangian. For each $\alpha \in \mathcal{D}_z$ or $\mathcal{E}_z$, $-i\omega_{\mathbb{C}}(\alpha, \overline{\alpha}) \geq 0$. The restrictions $\mathcal{D}|_{D_v}$, $\mathcal{E}|_{D_v}$ are real-analytic involutive bundles. Let l be the rank of $\widetilde{M}$. The dimension*

$$dim_{\mathbb{C}}\mathcal{E}_q \cap \overline{\mathcal{E}}_q = dim_{\mathbb{C}}\mathcal{D}_q \cap \overline{\mathcal{D}}_q - 1 = dim_{\mathbb{R}} ker\hat{R}_q = dim_{\mathbb{R}} ker R_q - 1 = l - 1 + c_v$$

is constant for $q \in D_v$ (where c_v is from (4.2.19)*).*

Before the proof, a few words about its motivation. Let X^N be a smooth manifold and V a smooth complex subbundle of $\mathbb{C} \otimes TX$ of rank s. (Since the least possible smoothness in this paper does not play any role, for us smooth shall always mean C^∞.) The bundle V is called *integrable* if in some neighborhood of each point in X there are smooth complex valued functions $g_1, \ldots, g_{N-s}$ such that $dg_1 \wedge \cdots \wedge dg_{N-s} \neq 0$ and with the property: if Z is any smooth local section of V, then $Z \lrcorner dg_j = 0$ for each $1 \leq j \leq N-s$. Integrability implies involutivity and in the real-analytic category they are equivalent (see for instance [BER]). They are also equivalent in the case $N = 2n, s = n, V \cap \overline{V} = 0$. This is the celebrated Newlander-Nirenberg theorem (cf. [NN]), but there do exist many CR structures that are only formaly integrable (involutive) but not integrable (imbeddable) in the above stronger sense.

Definition 4.2.12. *(cf. [W]) Let (X, ω) be a symplectic manifold and V be a complex subbundle of $\mathbb{C} \otimes TX$. V is called a complex polarization of (X, ω) if for each $x \in X$, the fiber V_x is Lagrangian in $\mathbb{C} \otimes T_xX$, the dimension of $V_x \cap \overline{V}_x$ is constant and V is integrable.*

For a symplectic vector space (W, ω), a complex Lagrangian subspace P of $W_{\mathbb{C}} = \mathbb{C} \otimes W$ is called *nonnegative* (resp. *positive*) if for each $\alpha \in P$ (resp. $0 \neq \alpha \in P$) $-i\omega(\alpha, \overline{\alpha}) \geq 0$ (resp. > 0). When this property holds for each fiber of a polarization we get the notion of *nonnegative* (resp. *positive*) *polarizations*. The bundle of (1,0) tangent vectors of a Kähler manifold is always a positive polarization and vice versa: if V is a positive polarization on X then $V \cap \overline{V} = 0$ and V defines a complex structure J whose (1,0) tangent bundle is V and ω becomes the Kähler form on X. Thus Theorem 4.2.11 can also be phrased in this way: for a generic vector v, the bundle $\mathcal{E}|_{D_v}$ is a nonnegative complex polarization of (D_v, ω) and this polarization is invariant w.r.t. the normalized geodesic flow. This was in fact the main motivation of the theorem.

Proof of Theorem 4.2.11. It is enough to prove the theorem for the case when $M = \widetilde{M}$ is a simply connected globally symmetric space since all the constructions naturally factorize when we take coverings.

The statement that the fibers $\mathcal{D}_z, \mathcal{E}_z$ are $\omega_{\mathbb{C}}$-Lagrangian follows from the definition of $\mathcal{E}$, the fact that the operators $R_z, \hat{R}_z$ are positive semidefinite (in fact we only need that they are selfadjoint) and from formula (4.2.25).

If $\alpha = A_z^H - i(\sqrt{\hat{R}_z}A)_z^V + iB_z^H + (\sqrt{\hat{R}_z}B)_z^V \in \mathcal{E}_z$, (where A, B are from $T_{\pi(z)}M$) a direct calculation using formula (4.2.25) again shows

$$-i\omega_{\mathbb{C}}(\alpha, \overline{\alpha}) = 2g(A, \sqrt{\hat{R}_z}A) + 2g(B, \sqrt{\hat{R}_z}B) \geq 0.$$

Replacing $\hat{R}_z$ by R_z we get the statement for $\mathcal{D}_z$

Now let $\lambda > 0, X \in T_{\pi(z)}M$. Then $\hat{R}_{\lambda z}X = g(X, \lambda z)\lambda z + \mathcal{R}(X, \lambda z)\lambda z = \lambda^2 \hat{R}_z X$. Let $w = N_\lambda z = \lambda z$. Thus (1.2.6) implies

$$(N_\lambda)_*(\sqrt{\hat{R}_z}X)_z^V = (\lambda\sqrt{\hat{R}_z}X)_w^V = (\sqrt{\hat{R}_{\lambda z}}X)_w^V.$$

This yields that $\mathcal{D}, \mathcal{E}$ are indeed N_λ-invariant.

Now we shall show that they are ψ_t-invariant as well. Let $z \in \mathring{\mathrm{T}}M$. We need to prove that for any $X \in T_{\pi(z)}M$

$$\begin{aligned} &(\psi_t)_*(X_z^H - i(\sqrt{R_z}X)_z^V) \in \mathcal{D}_{\psi_t(z)} \\ &(\psi_t)_*(X_z^H - i(\sqrt{\hat{R}_z}X)_z^V) \in \mathcal{E}_{\psi_t(z)}. \end{aligned} \tag{4.2.26}$$

The definition of ψ_t yields that for any $q \in \mathring{\mathrm{T}}M$, $s = t/\|q\|$

$$\psi_t q = \phi_s q. \tag{4.2.27}$$

This implies that $\psi_t(\lambda q) = \lambda\psi_t(q)$ for any $\lambda > 0$. Therefore it suffices to check (4.2.26) when $\|z\| = 1$. Denote by γ the geodesic with initial condition $\dot{\gamma}(0) = z$. Let $w = \dot{\gamma}(t)$. Thus $(\psi_t)_* z_z^H = (\phi_t)_* z_z^H = w_w^H$. It is also easy to see that $(\psi_t)_*(z)_z^V = (w)_w^V$. All these together with $\sqrt{\hat{R}_z}z = z$, $\sqrt{R_z}z = 0, \sqrt{R_w}w = 0$ imply

$$(\psi_t)_*(z_z^H) = (w)_w^H \in \mathcal{D}_w,$$

$$(\psi_t)_*(z_z^H - i(\sqrt{\hat{R}_z}z)_z^V) = (w)_w^H - i(w)_w^V = (w)_w^H - i(\sqrt{\hat{R}_w}w)_w^V \in \mathcal{E}_w.$$

Now let $X \perp z$. Then $\sqrt{\hat{R}_z}X = \sqrt{R_z}X$. We want to show that (4.2.26) holds. The vectors X_z^H and $(\sqrt{R_z}X)_z^V$ are tangential to the level surface $\|.\| = 1$, where ϕ_t and ψ_t are identical. Now ϕ_t invariant vector fields along the curve $\dot{\gamma}$ and Jacobi fields J along the geodesic γ correspond to each other with the correspondence $\xi = (J)^H + (J')^V$, where $'$ denotes covariant derivative.

As before let $v_n = z, v_1, \ldots, v_n$ be an orthonormal basis of eigenvectors of R_{v_n} with eigenvalues Λ_j ($\Lambda_j \geq 0$). Let Ξ_j be parallel vector fields along γ with $\Xi_j(0) = v_j$. Since $X \perp z, X = \sum_{j=1}^{n-1} \alpha_j v_j$.

The space M is symmetric so the vector field

$$J_1(t) = \sum_{j=1}^{n-1} \alpha_j \cos(\sqrt{\Lambda_j}t)\Xi_j$$

is a Jacobi field along γ with initial conditions $J_1(0) = X, J_1'(0) = 0$.

Similarly

$$J_2(t) = \sum_{j=1}^{n-1} \alpha_j \sin(\sqrt{\Lambda}_j t)\Xi_j$$

is another Jacobi field with initial conditions $J_2(0) = 0, J_2'(0) = \sqrt{R}_z X$. From the definitions we obtain

$$J_1'(t) = -\sqrt{R}_w J_2(t), J_2'(t) = \sqrt{R}_w J_1(t).$$

All the above together yield

$$\begin{aligned}(\psi_t)_*(X_z^H - i(\sqrt{R}_z X)_z^V) =&(\phi_t)_* X_z^H - i(\phi_t)_*((\sqrt{R}_z X)_z^V)\\ =&(J_1(t))_w^H + (J_1'(t))_w^V - i(J_2(t))_w^H - i(J_2'(t))_w^V\\ =&(J_1(t))_w^H - (\sqrt{R}_w J_2(t))_w^V\\ &- i(J_2(t))_w^H - i(\sqrt{R}_w J_1(t))_w^V \in \mathcal{E}_w.\end{aligned} \tag{4.2.28}$$

Corollary 4.2.10 implies that the restrictions of $\mathcal{D}, \mathcal{E}$ to D_v are real-analytic and involutive. From (4.2.22) we get $\mathcal{E}_q \cap \overline{\mathcal{E}}_q = (ker\hat{R}_z \otimes \mathbb{C})_z^H$. As we said in the proof of Proposition 4.2.9, $ker\hat{R}_z \otimes \mathbb{C} = S_b$ and (4.2.22) yields our claim. □

4.2.5 Scalings

By *scaling* we mean a family of diffeomorphisms $\Phi_\varepsilon : \mathring{\mathrm{T}}M \to \mathring{\mathrm{T}}M$. We do not require Φ_ε to be onto but smaller ε should mean larger image and the union of the images for all possible ε should be $\mathring{\mathrm{T}}M$.

We impose the following symmetry conditions on the scalings. For each ε

$$\Phi_\varepsilon \text{ is } U \text{ equivariant,} \tag{4.2.29}$$

and

$$\Phi_\varepsilon \text{ preserves } C_{\mathfrak{p}_*} \text{and the fibers of } TM. \tag{4.2.30}$$

Theorem 4.2.13 (Szőke,[Sz01])**.** *Let (M, g) be a compact, simply connected symmetric space. Let l be its rank and U the identity component of its isometry group. Suppose we have a scaling $\Phi_\varepsilon : \mathring{T}M \to \mathring{T}M$ with symmetry conditions* (4.2.29) *and* (4.2.30)*. Assume that the limit bundle*

$$\mathcal{E} = \lim(\Phi_\varepsilon)_* T^{1,0}(\mathring{T}M)$$

exists (where (1,0) is w.r.t. the adapted complex structure) on the domain $\mathcal{D} \subset TM$ that is the union of all U orbits of an open Weyl chamber. Furthermore assume that $\mathcal{E}$ is invariant w.r.t. the normalized geodesic flow and the scalings N_λ. Then for any $z \in \mathcal{D}$

$$dim_{\mathbb{C}} \mathcal{E}_z \cap \overline{\mathcal{E}}_z \geq \frac{l-1}{2}.$$

For the proof see [Sz01, Theorem 7.1].

Chapter 5

Weyl group equivariant maps and hyperkähler metrics

5.1 Weyl group equivariant maps, the main results

In the theory of symmetric spaces a fundamental role is played by Chevalley's extension theorem [He2, p.299, p.340], [Har1]:

Suppose $\mathfrak{g}$ is a semisimple Lie algebra of non-compact type over $\mathbb{R}$, θ a Cartan involution, and $\mathfrak{g} = \mathfrak{k}+\mathfrak{p}$ the corresponding Cartan decomposition of $\mathfrak{g}$ and $\mathfrak{a} \subset \mathfrak{p}$ a maximal Abelian subspace. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, let K be its maximal compact subgroup. Then K acts on $\mathfrak{p}$ by the adjoint representation and W, the Weyl group, acts on $\mathfrak{a}$. The theorem states that every W-invariant polynomial on $\mathfrak{a}$ extends to a unique K-invariant polynomial on $\mathfrak{p}$. It is an immediate consequence that the two polynomial algebras in question are isomorphic.

This theorem remains true if "polynomial" is replaced by C^∞ or C^ω (see [Da], [He2, p.295], and the comments at the beginning of section 5.1.1.

It is a natural question to ask, whether analogous results hold for W-equivariant polynomial (resp. C^∞, C^ω) mappings. In this note we show that the answer is positive, and in fact a substantial part of the solution is already contained, somewhat indirectly in [Sol] and [Mi1, Mi2].

Theorem 5.1.1 (Korányi, Szőke, [KSz])**.** *Any W-equivariant polynomial (resp. C^∞, C^ω) map $\mathfrak{a} \to \mathfrak{a}$ can be extended to a K-equivariant polynomial (resp. C^∞, C^ω) map $\mathfrak{p} \to \mathfrak{p}$. The extension is unique.*

The result will be used in Proposition 5.2.8 in constructing hyperkähler metrics on the tangent bundle of compact hermitian symmetric spaces. Another immediate consequence of the extension theorem is the following.

Corollary 5.1.2 (Korányi, Szőke, [KSz])**.** *Let (M, g) be a symmetric space of compact or non-compact type, $m_0 \in M$, $\mathfrak{a} \subset T_{m_0}M$ a maximal flat subspace and*

W the Weyl group. Then every $W-$equivariant C^∞ (resp. C^ω) map φ from $\mathfrak{a}$ to $\mathfrak{a}$ extends uniquely to an isometry group equivariant C^∞ (resp. C^ω) map Φ from TM to TM. Φ is a (C^∞ or C^ω) diffeomorphism iff φ is.

Theorem 5.1.1 provokes the next natural question. Does this theorem describe all possible K-equivariant maps? This is answered by our next result, Theorem 5.1.3.

We call a K-equivariant map $F : \mathfrak{p} \to \mathfrak{p}$ *radial* if there exists a maximal Abelian subspace $\mathfrak{a}$ in $\mathfrak{p}$ that is mapped into itself by F. Since K acts transitively on the set of such $\mathfrak{a}$'s, a radial map necessarily maps every maximal Abelian subspace of $\mathfrak{p}$ into itself.

Assume now that $\mathfrak{g}$ is simple. We say that $\mathfrak{g}$ is of Hermitian type if $\mathfrak{p}$ has a $K-$invariant complex structure, i.e. if the associated symmetric space is Hermitian symmetric.

Theorem 5.1.3 (Korányi, Szőke, [KSz])**.** *Let $F : \mathfrak{p} \to \mathfrak{p}$ be a K-equivariant polynomial (resp C^∞ or C^ω) map. If $\mathfrak{g}$ is not of Hermitian type, then F is radial. Let $\mathfrak{g}$ be of Hermitian type and let I be the complex structure on $\mathfrak{p}$. If $F_j : \mathfrak{p} \to \mathfrak{p}$, $j = 1, 2$ are arbitrary K-equivariant radial polynomial (resp. C^∞ or C^ω) maps, then $F = F_1 + IF_2$ is a K-equivariant polynomial (resp. C^∞ or C^ω) map. Every K-equivariant polynomial (resp. C^∞ or C^ω) map $\mathfrak{p} \to \mathfrak{p}$ arises this way.*

In section 5.1.1 we discuss a structure theorem for W-invariant C^∞ or C^ω p-forms inspired by Solomon's similar result for W-invariant polynomial p-forms (cf. [Sol], [He2, p.363] , [Mi1]). This structure theorem is important for our purpose because W-invariant one forms are essentially the same thing as W-equivariant maps (see Proposition 5.1.8).

Theorem 5.1.1 is proved in section 5.1.2, and Theorem 5.1.3 is proved in section 5.1.4.

5.1.1 W-invariant p-forms.

Let E be an n-dimensional real vector space and W a finite reflection group on E. A theorem of Chevalley ([Che], [He2, Theorem 3.1, p.356]) says that there exist algebraically independent W-invariant real polynomials $j_1, \dots, j_n$, such that every W-invariant real polynomial on E is a polynomial of $j_1, \dots, j_n$. In other words, setting $\mathcal{J}(x) = (j_1(x), \dots, j_n(x))$, for every W-invariant polynomial f on E we have $f = \bar{f} \circ \mathcal{J}$ with some polynomial $\bar{f}$ on $\mathbb{R}^n$. The same statement is true when f (and $\bar{f}$) are in C^∞ [Sch], [Da], or in C^ω [Lu]. Note that these results immediately imply the C^∞ and C^ω analogues of the Chevalley extension theorem.

The above results describe the structure of the W-invariant polynomial (resp. C^∞, C^ω) functions, i.e. W-invariant 0-forms. There exists an analogous structure theorem for W-invariant p-forms, where $p > 0$, as well.

Proposition 5.1.4 (Korányi, Szőke, [KSz])**.** *Let $0 \in G \subset \mathbb{R}^{n-1}$ be open, $B = G \times (-\varepsilon, \varepsilon) \subset \mathbb{R}^n$, $f \in C^k(B)$ (resp. $C^\infty(B)$ or $C^\omega(B)$) and $f(x', 0) \equiv 0, x' \in G$. Then $F := \frac{f}{x_n} \in C^{k-1}(B)$ (resp. $C^\infty(B)$ or $C^\omega(B)$) and $F(x', 0) = \partial_{x_n} f(x', 0)$.*

Proof. Let $x = (x', x_n)$ be fixed and let $g : B \times [0, 1] \to \mathbb{R}$ be defined as $g(x', x_n, t) = f(x', tx_n)$. The Newton-Leibniz formula applied to the function $g(x, .)$ yields

$$f(x) = \int_0^1 (\partial_t g)(x,s)ds = \int_0^1 x_n \partial_{x_n} f(x', sx_n)ds.$$

□

Proposition 5.1.5 (Korányi, Szőke, [KSz]). *Let X be a C^∞-manifold and $h \in C^\infty(X,\mathbb{R})$. Let $Z(h)$ denote the zeroes of h. Suppose $dh(x) \neq 0$ if $x \in Z(h)$. Let now $f \in C^\infty(X,\mathbb{R})$, $f|_{Z(h)} \equiv 0$. Then $F := \frac{f}{h} \in C^\infty(X,\mathbb{R})$. If $X, h, f \in C^\omega$, then $F \in C^\omega$ as well.*

Proof. The implicit function theorem and Proposition 5.1.4 yield the statement. □

Proposition 5.1.6 (Korányi, Szőke, [KSz]). *Let X be a C^ω-manifold and $h_1, h_2, \ldots, h_r \in C^\omega(X,\mathbb{R})$. Let Z_j be the zero set of h_j and $Z = \cup_{j=1}^r Z_j$. Assume that for each j, $dh_j|_{Z_j} \neq 0$. Furthermore suppose that for every component M of Z_j, for each $k \neq j$, $h_k|_M \not\equiv 0$. Let $f \in C^\infty(X,\mathbb{R})$ (resp. $\in C^\omega(X,\mathbb{R})$) and $f|_Z \equiv 0$. Then*

$$F = \frac{f}{h_1 h_2 \ldots h_r} \in C^\infty(X,\mathbb{R}) \quad (\textit{resp.} \in C^\omega(X,\mathbb{R})).$$

Proof. We prove the statement by induction on r. For $r = 1$ this is Proposition 5.1.5. Suppose we proved the statement for $r-1$. Then

$$F_{r-1} = \frac{f}{h_1 h_2 \ldots h_{r-1}} \in C^\infty(X,\mathbb{R}) \quad (\text{resp.} \in C^\omega(X,\mathbb{R})).$$

Let M be an arbitrary component of Z_r. Then for $1 \leq j \leq r-1$, $h_j|_M$ is $\not\equiv 0$ and real-analytic. Therefore the interior of $Z_j \cap M$ in M is empty. Consequently the set

$$H := M \setminus (\bigcup_{j=1}^{r-1} (Z_j \cap M))$$

is open and dense in M. By our assumption $F_{r-1}|_H \equiv 0$. Hence $F_{r-1}|_M \equiv 0$, yielding that $F_{r-1}|_{Z_r} \equiv 0$. This together with Proposition 5.1.5 proves our claim. □

Let now $E, W, j_1, \ldots, j_n$ be as at the beginning of this section.

Theorem 5.1.7 (Structure Theorem I, (Korányi, Szőke, [KSz])). *Let α be a $W-$invariant polynomial, C^∞ or C^ω p-form ($p > 0$) on E. Then α can be expressed uniquely as*

$$\alpha = \sum_{i_1 < \cdots < i_p} \alpha_{i_1,\ldots,i_p} dj_{i_1} \wedge \cdots \wedge dj_{i_p}, \tag{5.1.1}$$

where $\alpha_{i_1,\ldots,i_p}$ are W-invariant polynomial, C^∞ or C^ω functions on E.

The polynomial case is Solomon's theorem [Sol]. The C^∞ case was proved by P. Michor [Mi1, Lemma 3.3]. His proof could be slightly simplified by quoting [He2, Lemma 3.7, p.361], that immediately yields formula (4) in [Mi1, p. 1636]. The real-analytic version follows the same line of reasoning as the C^∞ case. To

be more precise, let $\sigma_1, \ldots, \sigma_r$ be the reflections in W and $\beta_1 = 0, \ldots, \beta_r = 0$ the corresponding reflecting hyperplanes. Let $\pi = \prod_{i=1}^r \beta_i$. If a C^ω function g satisfies $g \circ \sigma = \det \sigma^{-1} g$ for each $\sigma \in W$, then g must vanish on the zero locus of π and Proposition 5.1.6 implies that $g = \pi h$, where h is a W-invariant C^ω function.

The rest of the proof of [Mi1, Lemma 3.3], goes through literally, C^∞ replaced by C^ω everywhere, proving the real-analytic version.

Remarks. 1). If D is a W invariant open subset of E and α is a W-invariant p-form over D, the same proof shows that α can be expressed by the same formula as in the Theorem, except the functions $\alpha_{i_1,\ldots,i_p}$ are defined only on D and their smoothness (i.e. C^∞, or real-analytic) agrees with that of α.

2). The structure theorem can be restated as follows: For every W-invariant polynomial, C^∞ or C^ω p-form α on E there exists a p-form $\bar{\alpha}$ of the same smoothness on $\mathbb{R}^n$ such that $\alpha = \mathcal{J}^* \bar{\alpha}$.

5.1.2 Equivariant maps.

Let E be a finite dimensional real vector space. As usual we identify the tangent space of E at all points with E. Let b be a non-degenerate symmetric bilinear form on E. Given a 1-form α on E (i.e. a section of T^*E) we associate to it the map $h_\alpha : E \to E$ defined by

$$b(h_\alpha(p), v) = \alpha_p(v) \qquad (\forall v \in E). \tag{5.1.2}$$

Clearly $\alpha \to h_\alpha$ is a bijection between 1-forms on E and maps of E to E.

If A is a linear transformation on E, we have $A_* v = Av$ under our identifications. If A is orthogonal with respect to b, then

$$(A^*\alpha)_p(v) = \alpha_{Ap}(Av) = b(h_\alpha(Ap), Av) = b(A^{-1} h_\alpha(Ap), v). \tag{5.1.3}$$

Comparison with (5.1.2) shows that

$$A^*\alpha = \alpha \iff A^{-1} \circ h_\alpha \circ A = h_\alpha, \tag{5.1.4}$$

i.e. α is A-invariant iff h_α is A-equivariant.

So we have proved the following:

Proposition 5.1.8 (Korányi, Szőke, [KSz])**.** *Let E be a finite dimensional real vector space equipped with a nondegenerate symmetric bilinear form b. Let G be a group acting on E by b-orthogonal transformations and let α be a 1-form on E. Then α is G-invariant iff the corresponding map $h_\alpha : E \to E$ is G-equivariant.*

Let now $f : E \to \mathbb{R}$ be a smooth function and A a linear transformation on E. Since the pull-back by a smooth map commutes with the exterior derivative and the origin is a fixed point of A we have

$$f \circ A \equiv f \quad \text{iff} \quad A^* df \equiv df. \tag{5.1.5}$$

We write ∇f for h_{df}. This is then just the classical notion of the gradient of f regarded as an E-valued function on E. Now (5.1.3), (5.1.4) and (5.1.5) together imply:

Proposition 5.1.9 (Korányi, Szőke, [KSz]). *Let E be a finite-dimensional real vector space equipped with a nondegenerate symmetric bilinear form b. Let G be a group acting on E by b-orthogonal transformations and $f : E \to \mathbb{R}$ a differentiable function. Then the following statements are equivalent.*

1. *f is G-invariant,*
2. *df is G-invariant,*
3. *∇f is G-equivariant.*

Let now $j_1 \dots j_n$ be as at the beginning of subsection 5.1.1. In light of Proposition 5.1.9, the maps $\nabla j_i : \mathfrak{a} \to \mathfrak{a}$ are W-equivariant. Furthermore as a consequence of the Structure Theorem of subsection 5.1.1 and of Proposition 5.1.8 we get:

Proposition 5.1.10 (Korányi, Szőke, [KSz]). *Let E^n be an $n-$dimensional real vector space and W a finite reflection group on E. Then every W-equivariant polynomial (resp. C^∞ or C^ω) map $\varphi : E \to E$ is of the form*

$$\varphi = \sum_{i=1}^{n} h_i \nabla j_i,$$

where each h_i is a W-invariant polynomial (resp. C^∞ or C^ω) function.

We now return to the case of a real semisimple Lie algebra $\mathfrak{g}$ with $\mathfrak{k}$, $\mathfrak{p}$, $\mathfrak{a}$, G, K, W, as in the introduction.

5.1.3 Proof of Theorem 5.1.1

Let $\tilde{j}_i$ be the K-invariant extension of j_i to $\mathfrak{p}$. In view of Proposition 5.1.10 and of the Chevalley extension theorem (and its C^∞ and C^ω versions cf. the beginning of subsection 5.1.1) it suffices to show that each ∇j_i extends to a K-equivariant polynomial map $\mathfrak{p} \to \mathfrak{p}$. To prove this, we note that for $H \in \mathfrak{a}$ the K-orbit O_H of H is orthogonal to $\mathfrak{a}$, since every tangent vector at H to O_H is of the form $[Z, H]$ with $Z \in \mathfrak{k}$, and for every $H' \in \mathfrak{a}$, $B([Z, H], H') = B(Z, [H, H']) = 0$ (where B is the Killing form). If H is a regular element, comparison of the Iwasawa and Bruhat decompositions shows that the codimension of O_H in $\mathfrak{p}$ equals dim $\mathfrak{a}$. Since $\tilde{j}_i$ is constant on O_H, it follows that

$$\nabla_{\mathfrak{p}} \tilde{j}_i(H) = (\nabla_{\mathfrak{a}} j_i)(H).$$

By continuity this is then true for all $H \in \mathfrak{a}$. By Proposition 5.1.9, $\nabla_{\mathfrak{p}} \tilde{j}_i$ is K-equivariant, so the proof is finished.

Recall from subsection 5.1 that a K-equivariant map $F : \mathfrak{p} \to \mathfrak{p}$ is called radial, if there exists a maximal Abelian subspace $\mathfrak{a}$ in $\mathfrak{p}$, that is carried into itself by F. Theorem 5.1.1 yields an isomorphism between the space of W-equivariant polynomial, C^∞ resp. C^ω maps and K-equivariant radial polynomial, C^∞ resp. C^ω maps. The question whether there are other kinds of K-equivariant maps, will be addressed in the next subsection.

A differential form α on $\mathfrak{p}$ is called horizontal if α vanishes on the tangent vectors of the K-orbits, ie. $\iota_{X^\sharp}\alpha = 0$ for all $X \in \mathfrak{k}$, where $X^\sharp$ denotes the

induced vector field on $\mathfrak{p}$. Clearly a K-invariant 1-form is horizontal iff the corresponding K-equivariant map is radial.

As in the proof of Theorem 5.1.1, let $\tilde{j}_i$ be the K-invariant extension of j_i to $\mathfrak{p}$.

Structure Theorem I, Proposition 5.1.9, Chevalley's extension theorem and its C^∞ and C^ω versions imply.

Theorem 5.1.11 (Structure Theorem II (Korányi, Szőke, [KSz]))**.** *A polynomial (resp, C^∞ or C^ω) horizontal p-form on $\mathfrak{p}$ is K-invariant iff α can be expressed as*

$$\alpha = \sum_{i_1<\cdots<i_p} \alpha_{i_1,\ldots,i_p} d\tilde{j}_{i_1} \wedge \cdots \wedge d\tilde{j}_{i_p},$$

where $\alpha_{i_1,\ldots,i_p}$ are K-invariant polynomials (resp. C^∞, or C^ω functions). The imbedding $\iota : \mathfrak{a} \to \mathfrak{p}$ yields an isomorphism between the space of horizontal K-invariant p-forms on $\mathfrak{p}$ and the space of W-invariant p-forms on $\mathfrak{a}$ (cf. [Mi1, 3.7 Theorem]).

5.1.4 Proof of Theorem 5.1.3

Let $\mathfrak{g}$ be a real simple Lie algebra of non-compact type, with θ, $\mathfrak{k}$, $\mathfrak{p}$, $\mathfrak{a}$, G, K, W, as in subsection 5.1. As usual, we write M, M' for the centralizer resp. the normalizer of $\mathfrak{a}$ in K. We set

$$p^M = \{v \in \mathfrak{p} \mid \mathrm{Ad}(k)v = v, \forall k \in M\}.$$

Lemma 5.1.12 (Korányi, Szőke, [KSz])**.** *If $\mathfrak{g}$ is not of Hermitian type, then*

$$p^M = \mathfrak{a}. \tag{5.1.6}$$

If $\mathfrak{g}$ is of Hermitian type,

$$p^M = \mathfrak{a} \oplus I\mathfrak{a}. \tag{5.1.7}$$

Proof. Denote by Σ the set of nonzero restricted roots with respect to $\theta, \mathfrak{a}$. Let $\mathfrak{g}_\lambda$ be the root space corresponding to $\lambda \in \Sigma$. The group M maps every root space $\mathfrak{g}_\lambda$ into itself. Let S denote the (possibly empty) set of all $\lambda \in \Sigma$ roots, such that M acts trivially on $\mathfrak{g}_\lambda$. M' (and so the Weyl group as well) acts on Σ and it is not hard to see that S is the union of full W orbits. The M action and θ commute on $\mathfrak{g}$ and $\theta\mathfrak{g}_\lambda = \mathfrak{g}_{-\lambda}$. This shows that $\lambda \in S$ iff $-\lambda \in S$.

Choose an ordering in the dual of $\mathfrak{a}$. Then

$$\mathfrak{p} = \mathfrak{a} \oplus (Id - \theta)(\bigoplus_{\lambda>0} \mathfrak{g}_\lambda). \tag{5.1.8}$$

Denote by S^+ the positive roots in S. (5.1.8) implies

$$p^M = \mathfrak{a} \oplus (Id - \theta)(\bigoplus_{\lambda\in S^+} \mathfrak{g}_\lambda). \tag{5.1.9}$$

It is well known that in case $\dim \mathfrak{g}_\lambda > 1$, M acts transitively on the unit sphere in $\mathfrak{g}_\lambda$. Therefore all root spaces in (5.1.9) are 1-dimensional.

Let B be the Killing form of $\mathfrak{g}$ (which is positive definite on $\mathfrak{p}$) and for a $\lambda \in \Sigma$, denote by $A_\lambda \in \mathfrak{a}$ the vector such that $\lambda(.) = B(., A_\lambda)$. Let

$$A'_\lambda := \frac{2}{\lambda(A_\lambda)} A_\lambda.$$

Our statements involve only the adjoint action of G and K. This is the same for any connected version of G. Therefore, in the following we may assume that G is contained in the simply connected version of its complexification. [He1, (4), p.322], then says, that $m_\lambda := \exp_{G_\mathbb{C}}(i\pi A'_\lambda) \in K$ and then obviously $m_\lambda \in M$ for each $\lambda \in \Sigma$.

Let $\lambda, \alpha \in \Sigma$ be arbitrary simple roots. Then

$$\alpha(A'_\lambda) = \frac{2\alpha(A_\lambda)}{\lambda(A_\lambda)} = n(\alpha, \lambda)$$

is the corresponding Cartan integer. Let $X_\alpha \in \mathfrak{g}_\alpha$ be a nonzero vector. Then

$$Ad(m_\lambda)X_\alpha = Ad(\exp_{G_\mathbb{C}}(i\pi A'_\lambda))X_\alpha = e^{i\pi\alpha(A'_\lambda)}X_\alpha = e^{i\pi n(\alpha,\lambda)}X_\alpha. \qquad (5.1.10)$$

(5.1.10) implies that M will certainly be nontrivial on $\mathfrak{g}_\alpha$ (i.e. $\alpha \notin S$) if there is a simple λ such that $n(\alpha, \lambda)$ is odd. This is the case if in the Dynkin diagram α is tied to some λ by a single or a triple tie and also if there is a λ tied to α by a double tie but α is shorter than λ (in which case $n(\alpha, \lambda) = -1$).

Assume now that Σ is reduced. The discussion above shows that S cannot contain any simple root, except possibly in the case C_l ($l \geq 2$, where $l = \dim \mathfrak{a}$), when there is one simple root, namely the longest one, to be called α with only a double tie to a shorter root, and in the case A_1, where we call α the only simple root. In all these cases $\mathfrak{g}$ is of Hermitian type.

One of the standard properties of root systems (cf. [Bo, p. 279]) is that every $W-$orbit in Σ contains a simple root. Therefore when Σ is reduced, S is either empty or $\mathfrak{g}$ is of Hermitian type and S may contain only one orbit, the $W-$orbit of the longest simple root.

Let now Σ be non-reduced, i.e. of type BC_l ($l \geq 2$) or A_1. Then using the table in [He1, p.532-533] , together with (5.1.10), the discussion afterward and the fact that $\lambda \in S$ implies $\dim \mathfrak{g}_\lambda = 1$, we can conclude that S cannot contain any simple root. Consequently S may only contain the one left out $W-$orbit, namely the orbit of longest roots, i.e. if β denotes the unique shortest simple root, S may contain the $W-$ orbit of 2β. In the non-Hermitian case, again checking the table in [He1, p. 532-533], one can see that $\dim \mathfrak{g}_{2\beta} > 1$, showing that in this case $S = \emptyset$. This proves (5.1.6).

When $\mathfrak{g}$ is of Hermitian type, I is in the center of $\mathrm{Ad}(K)$, whence it follows that p^M contains $\mathfrak{a} \oplus I\mathfrak{a}$. To see that p^M cannot be larger we observe that, by the discussion above, S may contain only the longest roots of the root systems C_l, or BC_l or A_1, with the roots having multiplicity one. The number of longest positive roots in these systems is l (resp. 1). So $\dim p^M$ cannot exceed $2 \dim \mathfrak{a}$, proving (5.1.7).

□

Armed with the result of Lemma 5.1.12 we can now prove Theorem 5.1.3. Let $F : \mathfrak{p} \to \mathfrak{p}$ be a K-equivariant map. Then F is in particular M-equivariant and thus maps p^M into itself.

If $\mathfrak{g}$ is not of hermitian type, (5.1.6) implies that F is radial.

If $\mathfrak{g}$ is of hermitian type, then $I = \mathrm{Ad}(k_0)$, where k_0 is in the center of K. In particular as a map, $I : \mathfrak{p} \to \mathfrak{p}$ is K-equivariant and linear.

Hence for any two K-equivariant radial maps F_j, the sum $F_1 + IF_2$ is also K-equivariant and its smoothness agrees with that of F_j.

Now let $F : \mathfrak{p} \to \mathfrak{p}$ be an arbitrary K-equivariant polynomial (resp. C^∞ or C^ω) map. The restriction of F to $\mathfrak{a}$ determines F completely. In light of (5.1.7) this restriction maps into $\mathfrak{a} \oplus I\mathfrak{a}$. Therefore it is of the form $f_1 + If_2$, where $f_j : \mathfrak{a} \to \mathfrak{a}$.

Since F is $K-$equivariant, in particular it is $M'-$equivariant as well. But k_0 is central in K. This yields that f_j are $M'-$, and consequently $W-$equivariant maps. Now from Theorem 5.1.1 we know that f_j extends to a K-equivariant map $F_j : \mathfrak{p} \to \mathfrak{p}$. The maps $G = F_1 + IF_2$ and F are K-equivariant and their restriction to $\mathfrak{a}$ is the same. Therefore $G \equiv F$.

Since a K-invariant 1-form α is horizontal iff the corresponding K-equivariant map h_α is radial, as an immediate corollary of Theorem 5.1.3 we get.

Corollary 5.1.13 (Korányi, Szőke, [KSz])**.** *Suppose α is a K-invariant polynomial (resp. C^∞ or C^ω) 1-form. If $\mathfrak{g}$ is not of hermitian type, then α is horizontal. If $\mathfrak{g}$ is of hermitian type, then there exist unique K-invariant horizontal polynomial (resp. C^∞ or C^ω) 1-forms β_j, $j = 1, 2$, such that $\alpha = \beta_1 + I\beta_2$ and every 1-form of this type is K-invariant.*

5.2 Hyperkähler metrics, the main result

A manifold X is called hypercomplex if X admits two integrable anticommuting almost complex structure I and J. Then for all $(x, y, z) \in \mathbb{R}^3$, so that $x^2 + y^2 + z^2 = 1$ holds, $xI + yJ + zK$ is also an integrable almost complex structure. This implies in particular that the real dimension of X is a multiple of four. X is called hyperkähler if it admits a metric g that is Kähler with respect to both complex stuctures I and J. Then g will be Kähler also for the $xI+yJ+zK$ complex structures as well. A Riemannian manifold (X^{4n}, g) is hyperkähler iff the holonomy group lies in $Sp(n)$. These manifolds are important in particular because they are automatically Ricci flat. These Riemannian manifolds as possible new geometries first appeared in Berger's classification of the holonomy groups. The first (complete) examples were constructed by Eguchi and Hanson [EH] on $T^*\mathbb{C}P^1-$ and Calabi [C] on $T^*\mathbb{C}P^n$. Later Burns [Bu2], using twistor methods, generalized these. He showed that for any compact, Hermitian symmetric space M, T^*M admits a complete hyperkähler metric. Further important examples are the ALE spaces, gravitational instantons, moduli spaces of solutions to certain gauge theory equations (instanton moduli spaces, monopole moduli spaces, etc.), Nakajima quiver varieties.

Let M be a complex manifold. Then T^*M inherits a natural complex structure. If M also has a Riemannian metric g, we can identify TM and T^*M and obtain a complex structure I on TM, with respect to which the zero section M is complex. This will be different from the standard complex structure on TM induced by that on M.

It is tempting to conjecture that I and J (the adapted complex structure of g) anticommute and so generate a hypercomplex structure. Along the zero section

in TM this is indeed the case. But even for complex projective spaces (with their canonical Kähler structure) this is never true away from M. The idea to remedy this is to look for a diffeomorphism $\phi : TM \to TM$ such that $\phi^* JI = -I\phi^* J$. At least for compact hermitian symmetric spaces of classical type this approach works. Looking for such a diffeomorphism that is equivariant with respect to the isometry action makes it essentially (up to a positive constant) unique. All this is the content of Theorem 5.2.9.

There even exists a metric on the manifold TM that makes the hypercomplex structure generated by $\phi^* J$ and I hyperkähler and we get our main result.

Theorem 5.2.1 (Dancer-Szőke, [DSz])**.** *Let M be a compact, hermitian symmetric space associated to a classical Lie group. Then TM admits a complete hyperkähler metric.*

In our method M needs to be of classical type only at one step, in Lemma 5.2.6.

Our theorem gives a new proof of Burns' result using the adapted complex structures. Yet another proof, using the method of symplectic reduction, was given by Biquart s Gauduchon [BG].

5.2.1 Symmetric spaces.

We now assume M is an irreducible Hermitian (hence Kähler) symmetric space. Our aim is to find a diffeomorphism $\phi : TM \to TM$ such that $\phi^* JI = -I\phi^* J$. Our strategy is to first consider a diffeomorphism of the tangent space at one point, equivariant with respect to the isotropy action, and then extend it to the whole tangent bundle by homogeneity.

We first review the Cartan theory for symmetric spaces, following Helgason [He1]. Let $M = U/K$ be a compact irreducible symmetric space with Cartan decomposition $\mathfrak{u} = \mathfrak{k} + \mathfrak{p}_*$, where $\mathfrak{u}$ and $\mathfrak{k}$ are the Lie algebras of U and K, and $\mathfrak{p}_*$ is an Ad K-invariant complement for $\mathfrak{k}$. If we fix a basepoint m in U/K, then we can identify $\mathfrak{p}_*$ with the tangent space at this point. Denote by $\mathcal{R}$ the curvature tensor on U/K, so if $X, Y, Z \in \mathfrak{p}_*$ we have $\mathcal{R}(X,Y)Z = -[[X,Y],Z]$. It follows that the Jacobi operator $R_X = \mathcal{R}(.,X)X$ is equal to $-(\mathrm{ad}_X)^2$.

Let $\mathfrak{h}_{\mathfrak{p}_*}$ be a maximal abelian subspace of $\mathfrak{p}_*$. Then the dimension r of $\mathfrak{h}_{\mathfrak{p}_*}$ is the *rank* of the symmetric space. Σ denotes the set of *restricted roots*. They are real valued linear functionals on $i\mathfrak{h}_{\mathfrak{p}_*}$, but we shall identify them with real valued linear functionals on $\mathfrak{h}_{\mathfrak{p}_*}$ in the obvious way. The kernel of a restricted root α is a hyperplane denoted by $L(\alpha)$. The connected components of the complement of the union of these hyperplanes are the *Weyl chambers*. The *Weyl group* is generated by the reflections in the hyperplanes $L(\alpha)$, and acts simply transitively on the set of Weyl chambers.

The closure $\bar{C}$ of any Weyl chamber C is a transversal for the action of K on $\mathfrak{p}_*$. $\bar{C}$ is a convex subset of $\mathfrak{h}_{\mathfrak{p}_*}$ bounded by a collection of hyperplanes $L(\alpha_j) = \mathrm{Ker}\alpha_j$, $(j = 1, \ldots, s)$. If $\mathcal{S}$ is a subset of $\{\alpha_1, \ldots, \alpha_s\}$ we let $L_{\mathcal{S}} = \cap\{L(\alpha) : \alpha \in \mathcal{S}\}$. If $\mathcal{S}$ is empty we take $L_{\mathcal{S}} = \mathfrak{h}_{\mathfrak{p}_*}$.

Lemma 5.2.2 (Dancer-Szőke, [DSz])**.** *Let f be a bijection of $\bar{C}$ onto itself which maps $\bar{C} \cap L_{\mathcal{S}}$ bijectively onto itself for each subset $\mathcal{S}$ of $\{\alpha_1, \ldots, \alpha_s\}$. Then we can extend f to a K-equivariant bijection of $\mathfrak{p}_*$ onto itself and hence to a U-equivariant bijection of TM onto itself.*

Proof. The argument of of [He1, Lemma 2.14 of Chapter VII] shows that the stabiliser of a point $x \in \mathfrak{h}_{\mathfrak{p}_*}$ for the action of U depends only on the set of roots vanishing at x. Taking the intersection of the U-stabiliser with K, we see that this conclusion also holds for the K-stabiliser. Equivalently, the K-stabiliser of x depends only on the set of hyperplanes $L(\alpha_j)$ which contain x.

Our hypotheses on f now imply that the K-stabilisers of x and $f(x)$ are identical for each x, so the equivariant extension statements follow easily. □

Finally we need to establish some lemmas which will enable us to calculate the derivative of a U-equivariant diffeomorphism of the tangent bundle of a symmetric space U/K. We shall use these results in subsection 5.2.2, when we study the pullback of the adapted complex structure by a diffeomorphism of $T(U/K)$.

Lemma 5.2.3 (Dancer-Szőke, [DSz])**.** *Let $M = U/K$ be a symmetric space, and ϕ a U-equivariant fiber preserving diffeomorphism of TM. Let m be the basepoint $[K]$ of M, and identify $\mathfrak{p}_*$ with T_mM. Suppose that ϕ restricts to a diffeomorphism of $\mathfrak{h}_{\mathfrak{p}_*}$ onto itself. Let $v \in \mathfrak{p}_*$ and let z be a nonzero element of $\mathfrak{h}_{\mathfrak{p}_*}$. We shall regard z as a point of TM. Let $v_z^H, v_{\phi(z)}^H$ denote the horizontal lifts of v to z and $\phi(z)$ respectively. Then*

$$\phi_* v_z^H = v_{\phi(z)}^H.$$

Proof. Let γ be the geodesic with $\gamma(0) = m$ and $\dot\gamma(0) = v$. As M is a symmetric space, γ is given by $\gamma(t) = \exp(tv)m$. Moreover, parallel transport along this geodesic is given by $Y \mapsto \exp(tv)_*Y$. (We are regarding $\exp(tv) \in U$ as defining a transformation of M). Therefore the vector field χ defined by $\chi(t) = \exp(tv)_*z$ is parallel along γ and satisfies $\chi(0) = z$. We deduce that $v_z^H = \dot\chi(0)$.

Now, using the equivariance of ϕ we have

$$\phi(\chi(t)) = \phi(\exp(tv)_*z) = \exp(tv)_*\phi(z),$$

so the vector field $t \mapsto \phi(\chi(t))$ is also parallel along γ. Differentiating $\phi(\chi(t))$ at $t = 0$ proves our claim. □

Lemma 5.2.4 (Dancer-Szőke, [DSz])**.** *Let M, ϕ, m, v, z be as in Lemma 5.2.3 and let R_x be the Jacobi operator associated to $x \in \mathfrak{h}_{\mathfrak{p}_*}$. Suppose that λ is a linear functional on $\mathfrak{h}_{\mathfrak{p}_*}$ and that*

$$R_x v = \lambda(x)^2 v \tag{5.2.1}$$

for all $x \in \mathfrak{h}_{\mathfrak{p}_*}$. Assume moreover that $\lambda(z) \neq 0$. Denote by v_z^V and $v_{\phi(z)}^V$ the vertical lifts of v to z and $\phi(z)$ respectively. Then

$$\phi_* v_z^V = \frac{\lambda(\phi(z))}{\lambda(z)} v_{\phi(z)}^V.$$

Proof. First observe that putting $x = x_1 + x_2$ in (5.2.1), and using the Jacobi identity and the fact that x_1 and x_2 commute, shows that $[[x_1, v], x_2] = \lambda(x_1)\lambda(x_2)v$ for all $x_1, x_2 \in \mathfrak{h}_{\mathfrak{p}_*}$.

Consider the curve in $\mathfrak{p}_* = T_mM$ defined by $\kappa : t \mapsto \mathrm{Ad}(t\Theta)z$, where $\Theta = \lambda(z)^{-2}[z, v]$. Now $[\Theta, z] = v$, from (5.2.1), so $\kappa'(0)$ is the vertical lift v_z^V of v to z. Using the equivariance of ϕ again, we have

$$\phi(\kappa(t)) = \phi(\mathrm{Ad}(t\Theta)z) = \mathrm{Ad}(t\Theta)\phi(z). \tag{5.2.2}$$

Differentiating (5.2.2) at $t = 0$ shows that $\phi_* v_z^V$ is the vertical lift to $\phi(z)$ of $[\Theta, \phi(z)]$, but by the discussion above

$$[\Theta, \phi(z)] = \frac{1}{\lambda(z)^2}[[z, v], \phi(z)] = \frac{\lambda(z)\lambda(\phi(z))}{\lambda(z)^2} v,$$

giving the required result. □

5.2.2 Anticommuting complex structures.

Let M be a Kähler manifold with complex structure I_0. Then I_0 induces a complex manifold structure on T^*M. The pullback of this, via the metric defines the complex structure I on TM, that in the $T_z(TM) = T_z^H \oplus T_z^V$ decomposition has the form

$$I = I_0 \oplus (-I_0). \tag{5.2.3}$$

For any compact hermitian symmetric space $M = U/K$ the adapted complex structure J is defined on the whole of TM. If ϕ is a diffeomorphism of TM we can pull back J to obtain a new complex structure. Our aim is to find a U-equivariant diffeomorphism of TM such that the complex structure I anticommutes with $\phi^* J$. We shall simplify the calculations by making a suitable choice of bases for the tangent space to TM at points z and $\phi(z)$, and by exploiting the equivariance of ϕ.

Definition 5.2.5. *Let $M = U/K$ be a compact irreducible hermitian symmetric space of rank r, with its complex structure defined by an endomorphism I_0 of $\mathfrak{p}_*$. We shall say that M satisfies condition $(*)$ if there exists a maximal abelian subspace $\mathfrak{h}_{\mathfrak{p}_*}$ in $\mathfrak{p}_*$, an orthonormal basis $e_1, \ldots, e_r$ for $\mathfrak{h}_{\mathfrak{p}_*}$, and an orthogonal direct sum decomposition*

$$\mathfrak{p}_* = \mathfrak{h}_{\mathfrak{p}_*} \oplus I_0\mathfrak{h}_{\mathfrak{p}_*} \oplus_{1\le j<k\le r} (\mathcal{V}_{jk} \oplus I_0\mathcal{V}_{jk}) \oplus_{k=1}^r \mathcal{Q}_k \tag{5.2.4}$$

satisfying the following conditions.

(i) Each space $\mathcal{Q}_k$ is I_0-invariant (and possibly zero).

(ii) If $x = \sum_{s=1}^r \lambda_s(x)e_s \in \mathfrak{h}_{\mathfrak{p}_*}$, and $v \in \mathcal{V}_{jk}, q \in \mathcal{Q}_k$, we have

$$R_x I_0 e_i = 4\lambda_i^2(x) I_0 e_i \quad (i = 1, \ldots, r), \tag{5.2.5}$$

$$R_x v = (\lambda_j(x) - \lambda_k(x))^2 v, \tag{5.2.6}$$

$$R_x I_0 v = (\lambda_j(x) + \lambda_k(x))^2 I_0 v, \tag{5.2.7}$$

$$R_x q = \lambda_k(x)^2 q. \tag{5.2.8}$$

for each j, k.

Lemma 5.2.6 (Dancer-Szőke, [DSz])**.** *Every compact irreducible hermitian symmetric space associated to one of the classical groups satisfies condition $(*)$.*

Proof. This is established by a case-by-case check, using the classification of compact hermitian symmetric spaces (cf. [He1].

(i) Complex Grassmannians $SU(a+b)/S(U(a)\times U(b))$ with $a \le b$.

The rank is a. Let

$$\mathfrak{p}_* = \left\{ \begin{pmatrix} 0 & -\bar{P}^T \\ P & 0 \end{pmatrix} \ : \ P \in M_{a\times b}(\mathbb{C}) \right\}$$

with I_0 defined by multiplication by $\sqrt{-1}$. We let $\mathfrak{h}_{\mathfrak{p}_*}$ be the subset of $\mathfrak{p}_*$ obtained by taking $P = (\Delta\ 0)$ where $\Delta \in M_{a\times a}(\mathbb{R})$ is diagonal. If we denote by E_{jk} the matrix with 1 in the jk position and 0 elsewhere, taking $P = E_{ii}$ ($i = 1, \ldots .a$) defines an orthonormal basis for $\mathfrak{h}_{\mathfrak{p}_*}$. For $1 \le j < k \le a$, $\mathcal{V}_{jk}$ is spanned over $\mathbb{R}$ by the two elements of $\mathfrak{p}_*$ obtained by taking $P = E_{jk} + E_{kj}$ and $P = \sqrt{-1}(E_{jk} - E_{kj})$. For $1 \le k \le a$ we obtain a basis for $\mathcal{Q}_k$ over $\mathbb{C}$ by taking $P = E_{kl}$ ($l = a+1, \ldots, b$).

(ii) $SO(2n)/U(n)$. Here the rank is $[\frac{n}{2}]$. Let

$$\mathfrak{p}_* = \left\{ \begin{pmatrix} Z & W \\ W & -Z \end{pmatrix} \ : \ Z, W \in M_{n\times n}(\mathbb{R}),\ Z^T = -Z, W^T = -W \right\}$$

The complex structure sends $(Z\ W)$ to $(-W Z)$. We choose $\mathfrak{h}_{\mathfrak{p}_*}$ to be the subspace of $\mathfrak{p}_*$ where $W = 0$ and Z belongs to the standard Cartan algebra of $\mathfrak{so}(n)$. The matrices where one of the 2×2 blocks on the diagonal of Z is

$$\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 \end{pmatrix},$$

and the other blocks are zero, form an orthonormal basis for $\mathfrak{h}_{\mathfrak{p}_*}$. In order to define the other spaces in the decomposition $(*)$ we must introduce some more notation. Let $\mathcal{A}$ be an $n\times n$ real skew-symmetric matrix. If n is even, we write $\mathcal{A}$ as

$$\begin{pmatrix} \mathcal{A}_{11} & \ldots & \mathcal{A}_{1r} \\ -\mathcal{A}_{12}^T & \ldots & \mathcal{A}_{2r} \\ \cdot & \ldots & \cdot \\ \cdot & \ldots & \cdot \\ -\mathcal{A}_{1r}^T & \ldots & \mathcal{A}_{rr} \end{pmatrix},$$

where each $\mathcal{A}_{jk}$ is a 2×2 matrix. If n is odd, we write $\mathcal{A}$ as

$$\begin{pmatrix} \mathcal{A}_{11} & \ldots & \mathcal{A}_{1r} & \mathcal{B}_1 \\ -\mathcal{A}_{12}^T & \ldots & \mathcal{A}_{2r} & \mathcal{B}_2 \\ \cdot & \ldots & \cdot & \cdot \\ \cdot & \ldots & \cdot & \cdot \\ -\mathcal{A}_{1r}^T & \ldots & \mathcal{A}_{rr} & \mathcal{B}_r \\ -\mathcal{B}_1^T & \ldots & -\mathcal{B}_r^T & 0 \end{pmatrix},$$

where the $\mathcal{A}_{jk}$ are as above and the $\mathcal{B}_j$ are 2×1 matrices.

For every 2×2 matrix Ψ, let E_{jk}^{Ψ} be the $n\times n$ matrix with $\mathcal{A}_{jk} = \Psi$ and all other $\mathcal{A}_{mq}$ ($m \le q$), as well as the matrices $\mathcal{B}_m$, equal to zero.

If Ω is a 2×1 matrix, let E_j^{Ω} be the $n\times n$ matrix with $\mathcal{B}_j = \Omega$ and all the other $\mathcal{B}_k$, as well as all the $\mathcal{A}_{mq}$, equal to zero.

Then for $1 \leq j < k \leq r$,

$$\mathcal{V}_{jk} = \left\{ \begin{pmatrix} E^{\Psi}_{jk} & E^{\chi}_{jk} \\ E^{\chi}_{jk} & -E^{\Psi}_{jk} \end{pmatrix} : \Psi = \begin{pmatrix} a & b \\ -b & a \end{pmatrix},\ \chi = \begin{pmatrix} c & d \\ d & -c \end{pmatrix},\ a,b,c,d \in \mathbb{R} \right\}.$$

Also, for $k = 1, \ldots, r$ we have

$$\mathcal{Q}_k = \left\{ \begin{pmatrix} E^{\Omega}_{k} & E^{\Xi}_{k} \\ E^{\Xi}_{k} & -E^{\Omega}_{k} \end{pmatrix} : \Omega, \Xi \in \mathbb{R}^2 \right\}.$$

The $\mathcal{Q}_k$ terms only occur if n is odd.

(iii) $Sp(n)/U(n)$.

The rank is n. Let

$$\mathfrak{p}_* = \left\{ \begin{pmatrix} Z_1 & Z_2 \\ Z_2 & -Z_1 \end{pmatrix} : Z_1, Z_2 \in iM_{n\times n}(\mathbb{R}), Z_1, Z_2 \in \mathfrak{u}(n) \right\}.$$

The complex structure sends $(Z_1\ Z_2)$ to $(-Z_2\ Z_1)$. We choose $\mathfrak{h}_{\mathfrak{p}_*}$ to be the subspace defined by taking Z_1 to be diagonal and Z_2 to be zero. Letting $Z_1 = iE_{jj}$ $(j = 1, \ldots, n))$ defines an orthonormal basis for $\mathfrak{h}_{\mathfrak{p}_*}$. We define a basis for $\mathcal{V}_{jk}$ over $\mathbb{R}$ by letting $Z_1 = i(E_{jk} + E_{kj})$, $Z_2 = 0$. There are no $\mathcal{Q}_k$ terms.

(iv) Quadrics $SO(n+2)/SO(n) \times SO(2)$, $(n \geq 2)$.

The rank is 2. Let

$$\mathfrak{p}_* = \left\{ \begin{pmatrix} 0 & -Z^T \\ Z & 0 \end{pmatrix} : Z \in M_{n\times 2}(\mathbb{R}) \right\},$$

and let $\mathfrak{h}_{\mathfrak{p}_*}$ be defined by taking Z to be of the form, where $(a, b \in \mathbb{R})$

$$\begin{pmatrix} a & b \\ b & a \\ 0 & 0 \\ \cdot & \cdot \\ \cdot & \cdot \\ 0 & 0 \end{pmatrix}.$$

We define an orthonormal basis for $\mathfrak{h}_{\mathfrak{p}_*}$ by taking $a = 1/\sqrt{2}$, $b = 0$ and $a = 0$, $b = 1/\sqrt{2}$. If we write Z as $(Z_1\ Z_2)$ where Z_1, Z_2 are column vectors, then the complex structure is defined by

$$I_0 : (Z_1\ Z_2) \mapsto (-Z_2\ Z_1)$$

As the rank is two, the only $\mathcal{V}_{jk}$ term which occurs is $\mathcal{V}_{12}$. A basis of $\mathcal{V}_{12}$ over $\mathbb{R}$ is defined by taking $Z = E_{k1} - E_{k2}$, $(k = 3, \ldots, n)$. There are no $\mathcal{Q}_k$ terms. $\square$

It is not known whether condition (*) also holds for the exceptional hermitian symmetric spaces. Condition (*) in Definition 5.2.5 will enable us to choose a good basis in which to do the calculations of the next theorem.

Theorem 5.2.7 (Dancer-Szőke, [DSz])**.** *Let $M = U/K$ be a compact hermitian symmetric space satisfying condition $(*)$. Let $\mathfrak{h}_{\mathfrak{p}_*}, e_1, \ldots, e_r$ be as in Definition 5.2.4 and let ϕ be a U-equivariant diffeomorphism of TM, restricting to*

a diffeomorphism of $\mathfrak{h}_{\mathfrak{p}_*}$ onto itself, which preserves each open Weyl chamber. Then $\phi^* J$ anticommutes with I if and only if there exists a positive constant s such that

$$\phi(z) = \frac{1}{2}\sum_{j=1}^{r} \sinh^{-1}(s\lambda_j)e_j, \quad z = \sum_{j=1}^{r} \lambda_j e_j \in \mathfrak{h}_{\mathfrak{p}_*} \tag{5.2.9}$$

Proof. We regard the coordinates λ_j on $\mathfrak{h}_{\mathfrak{p}_*}$ as defining real-valued linear functionals on this space. As discussed in subsubsection 5.2.1, to each λ_j we can associate a linear functional $\tilde{\lambda}_j$ on $\mathfrak{h}_{\mathfrak{p}_*}$ by setting $\tilde{\lambda}_j(w) = \lambda_j(-iw)$. From [He1, Corollary2.10 of Chapter VII] we see that the set of restricted roots is

$$\Sigma = \{\pm 2\tilde{\lambda}_m, \pm(\tilde{\lambda}_j - \tilde{\lambda}_k), \pm(\tilde{\lambda}_j + \tilde{\lambda}_k), \pm\tilde{\lambda}_m : 1 \le m \le r, 1 \le j < k \le r\},$$

so

$$C = \{x \in \mathfrak{h}_{\mathfrak{p}_*} : \lambda_1(x) > \ldots > \lambda_r(x) > 0\}$$

is an open Weyl chamber in $\mathfrak{h}_{\mathfrak{p}_*}$. The set of points conjugate to points in C by the action of U is an open and dense subset of TM. The action of U on TM is holomorphic with respect to both I and J, so it is sufficient to check anticommutation at points z of C. We shall choose special bases of $T_z(TM)$ and $T_{\phi(z)}(TM)$ with respect to which we shall calculate ϕ_*, I and $(\phi^* J)_z$.

For each pair (j,k) with $1 \le j < k \le r$ choose an orthonormal basis for $\mathcal{V}_{jk}$. Applying I_0 to these bases gives an orthonormal basis for each $I_0\mathcal{V}_{jk}$. Finally, for each k pick an orthonormal basis for $\mathcal{Q}_k$. Then the union of these bases, together with the elements $e_j, I_0 e_j \quad (j = 1, \ldots, r)$ forms an orthonormal basis for $\mathfrak{p}_*$. The horizontal and vertical lifts of this basis to z and $\phi(z)$ give bases for $T_z(TM)$ and $T_{\phi(z)}(TM)$ respectively.

From (5.2.3), Proposition 1.2.3 and Lemmas 5.2.3 and 5.2.4, we see that I and ϕ_* preserve horizontal and vertical spaces while J interchanges them. With respect to the decomposition into horizontal and vertical spaces, we have that $\phi_* : T_z(TM) \to T_{\phi(z)}(TM)$ is represented by a matrix of the form

$$\begin{pmatrix} Id & 0 \\ 0 & B \end{pmatrix}.$$

The maps $J_{\phi(z)} : T_{\phi(z)}(TM) \to T_{\phi(z)}(TM)$ and $I_z : T_z(TM) \to T_z(TM)$ are represented by

$$\begin{pmatrix} 0 & -A^{-1} \\ A & 0 \end{pmatrix}, \qquad \begin{pmatrix} I_0 & 0 \\ 0 & -I_0 \end{pmatrix},$$

respectively, for some A. It readily follows that the pulled back complex structure $\phi^* J$ at z commutes with I_z if and only if

$$I_0 A^{-1} B = A^{-1} B I_0. \tag{5.2.10}$$

We may regard A, B, I_0 as endomorphisms of $\mathfrak{p}_*$. The decomposition (5.2.4) of $\mathfrak{p}_*$ will be used to calculate A, B, I_0 explicitly and see when the anticommutation relation (5.2.10) holds. We shall see that each of the spaces $\mathcal{V}_{jk} \oplus I_0\mathcal{V}_{jk}, \mathcal{Q}_k$ and $\mathfrak{h}_{\mathfrak{p}_*} + I_0\mathfrak{h}_{\mathfrak{p}_*}$ is invariant under I_0, A and B so it is sufficient to work on each of these spaces separately. We shall denote by λ_k, ϕ_k the k th. component of z and $\phi(z)$ respectively, relative to the basis $e_1, \ldots, e_r$.

(i) Let us first study the space $\mathcal{Q}_k$. From (5.2.8), we see that $R_{\phi(z)/\|\phi(z)\|}$ is a scalar operator on $\mathcal{Q}_k$ with eigenvalue $\phi_k^2/\parallel \phi(z) \parallel^2$. Then (1.2.13) now shows that on $\mathcal{Q}_k$

$$A = \phi_k \coth(\phi_k) Id.$$

Moreover, by Lemma 5.2.4 we find that B is also a scalar operator with eigenvalue ϕ_k/λ_k. Therefore (5.2.10) holds automatically on $\mathcal{Q}_k$.

(ii) On the subspace $\hat{\mathfrak{h}} = \mathfrak{h}_{\mathfrak{p}_*} + I_0\mathfrak{h}_{\mathfrak{p}_*}$, let us use the basis $e_1, \ldots, e_r, I_0e_1, \ldots, I_0e_r$. Now, for any $x \in \mathfrak{h}_{\mathfrak{p}_*}$ we have $R_x e_j = 0$. Combining this with (5.2.5) and applying Proposition 1.2.3, we can calculate A. As before, we use Lemma 5.2.4 to find B, and we find that on $\hat{\mathfrak{h}}$

$$A^{-1} = \begin{pmatrix} Id & 0 \\ 0 & \mu \end{pmatrix}, \quad B = \begin{pmatrix} d\phi & 0 \\ 0 & \nu \end{pmatrix},$$

with respect to our basis. Here $d\phi$ is the derivative of $\phi : \mathfrak{h}_{\mathfrak{p}_*} \to \mathfrak{h}_{\mathfrak{p}_*}$ in the coordinates given by the basis $e_1, \ldots, e_r$. The matrices μ and ν are diagonal with entries $\tanh(2\phi_j)/2\phi_j$ and ϕ_j/λ_j respectively.

On $\hat{\mathfrak{h}}$ the anticommutation equation (5.2.10) is equivalent to

$$d\phi = \nu\mu$$

which in turn is equivalent to

$$2\lambda_i \frac{\partial \phi_i}{\partial \lambda_i} = \tanh(2\phi_i), \quad (i = 1, \ldots, r) \tag{5.2.11}$$

$$\frac{\partial \phi_i}{\partial \lambda_j} = 0, \quad \text{if } i \neq j. \tag{5.2.12}$$

These equations have solution

$$\phi_i = \frac{1}{2}\sinh^{-1}(s_i\lambda_i), \quad (i = 1, \ldots, r) \tag{5.2.13}$$

where $s_1, \ldots, p_r$ are constants. (5.2.13) only defines a diffeomorphism of $\mathfrak{h}_{\mathfrak{p}_*}$ preserving Weyl chambers and equivariant with respect to the Weyl group if the s_i must all be equal to some positive constant s. Equation (5.2.13) shows that the restriction of ϕ to $\mathfrak{h}_{\mathfrak{p}_*}$ must be of the form (5.2.9). As ϕ is U-equivariant, we see that the anticommutation condition determines ϕ up to a choice of a positive real number.

(iii) We shall now demonstrate the converse implication. In order to show that the anticommutation relation (5.2.10) holds, we must look at the spaces $P_{jk} = \mathcal{V}_{jk} \oplus I_0\mathcal{V}_{jk}$. We see that

$$I_0|_{P_{jk}} = \begin{pmatrix} 0 & -Id \\ Id & 0 \end{pmatrix}$$

Proceeding as before, we use Proposition 1.2.3 and (5.2.6)-(5.2.7) to calculate A^{-1} and find that

$$A^{-1}|_{P_{jk}} = \begin{pmatrix} \rho_1 Id & 0 \\ 0 & \rho_2 Id \end{pmatrix},$$

where

$$\rho_1 = \frac{\tanh(\phi_j - \phi_k)}{\phi_j - \phi_k}, \quad \rho_2 = \frac{\tanh(\phi_j + \phi_k)}{\phi_j + \phi_k}.$$

Lemma 5.2.4 tells us that

$$B|_{P_{jk}} = \begin{pmatrix} \sigma_1 Id & 0 \\ 0 & \sigma_2 Id \end{pmatrix},$$

where

$$\sigma_1 = \frac{\tanh(\phi_j - \phi_k)}{\lambda_j - \lambda_k}, \quad \sigma_2 = \frac{\tanh(\phi_j + \phi_k)}{\lambda_j + \lambda_k}.$$

It follows that the anticommutation condition on P_{jk} is equivalent to

$$\frac{\tanh(\phi_j + \phi_k)}{\lambda_j + \lambda_k} = \frac{\tanh(\phi_j - \phi_k)}{\lambda_j - \lambda_k}. \tag{5.2.14}$$

We have already seen that if $\phi_i = \frac{1}{2}\sinh^{-1}(s\lambda_i)$ for each i then the anticommutation relation holds on each $\mathcal{Q}_k$ and on $\hat{\mathfrak{h}}$. It is easy to check that if ϕ is given by (5.2.9) on C then (5.2.14) holds, so anticommutation holds on each P_{jk} also. It follows that I and $\phi^* J$ anticommutate at all points of the open Weyl chamber C, and hence everywhere on TM. □

Proposition 5.2.8 (Dancer-Szőke, [DSz])**.** *Let $M = U/K$ be a compact, irreducible Riemannian symmetric space. Let $\mathfrak{h}_{\mathfrak{p}_*}$ be a maximal abelian subspace of $\mathfrak{p}_*$ and $e_1, \ldots, e_r$ an orthonormal basis for $\mathfrak{h}_{\mathfrak{p}_*}$ and $s > 0$. Let $\phi : C \to C$ be defined by*

$$\phi(z) : \sum_{j=1}^{r} \lambda_j e_j \mapsto \frac{1}{2} \sum_{j=1}^{r} \sinh^{-1}(s\lambda_j) e_j, \tag{5.2.15}$$

Then ϕ and ϕ^{-1} extends uniquely as a U-equivariant real-analytic diffeomorphism $TM \to TM$.

Proof. Clearly this map extends to a bijection of the closure of C onto itself, which satisfies the hypotheses of Lemma 5.2.2. It therefore extends uniquely to a U-equivariant bijection ϕ of TM. Then the result follows from Corollary 5.1.2. □

Geatti and Iannuzzi uses the diffeomorphism ϕ in a recent paper [GI] to treat the noncompact Hermitian symmetric spaces. The original proof of Proposition 5.2.8 in [DSz] used the classification of compact hermitian symmetric spaces and also assumed that M was of classical type. Note that ϕ is the identity on the zero section of TM, because the map (5.2.9) fixes the origin in $\bar{C}$. Combining Proposition 5.2.8 and Theorem 5.2.7, we obtain the next theorem.

Theorem 5.2.9. *Let $M = U/K$ be a compact irreducible Hermitian symmetric space satisfying condition $(*)$. Then the map* (5.2.15) *extends to a U-equivariant fibre-preserving diffeomorphism ϕ of TM, equal to the identity on M, such that I anticommutes with the pullback of J by ϕ.*

Proof of Theorem 5.2.1. The adapted complex structure for a product of manifolds is just the product of the individual adapted complex structures. This together with the fact that ϕ is equivariant and I, J are invariant with respect to the isometry group of M, implies that we can assume that M is irreducible.

It follows from Theorem 5.2.9 that I and $\phi^* J$ generate a hypercomplex structure on TM. Now, the symplectic form ω on TM and the complex structure

I define a complex-symplectic form $\omega_{\mathbb{C}} = \omega_2 + i\omega_3$, where ω_2, ω_3 are real-valued two forms. If we define a tensor $\hat{g}$ by $\hat{g}(X, Y) = -\omega_2((\phi^* J)X, Y)$ then, using our expression for $\phi^* J$, and Propositon 1.2.1, we can check that $\hat{g}$ defines a Riemannian metric, hermitian with respect to I and $\phi^* J$. We therefore have a hyperhermitian structure, and as ω_2 and ω_3 are closed it is in fact a hyperkähler structure.

As the hyperkähler structure is U-invariant and the complex-symplectic form is the standard one, results of [BG] show that it coincides with that found in [Bu2] and [BG]. The metric is therefore complete. $\square$

Part II

The family of adapted complex structures

Chapter 6

Adapted complex structures and geometric quantization

6.1 New look at adapted complex structures, the main results

In this chapter we are going to show that the adapted complex structure of a Riemannian manifold is in fact just one member in a natural family of Kähler structures. This is the family of Kähler structures that respects the symmetries of the phase space N. To see this, it is advantageous to adhere to Souriau's philosophy ([So1]) and define the phase space N of a compact Riemannian manifold not as TM or T^*M but as the manifold of parametrized geodesics $x : \mathbb{R} \to M$. Any $t_0 \in \mathbb{R}$ induces a diffeomorphism $N \ni x \mapsto \dot{x}(t_0) \in TM$, and the pull back of the canonical symplectic form of $TM \approx T^*M$ is independent of t_0; we denote it by ω. We identify M with the submanifold of zero speed geodesics in N. Affine reparametrizations $t \mapsto a + bt$, $a, b \in \mathbb{R}$, act on N and define a right action of the Lie semigroup $\mathcal{A}$ of affine reparametrizations.

To simplify the notation here, we formulate the main results assuming that we work on the full phase space. This is the situation that will be important most of the time later on.

Given a complex manifold structure on $\mathcal{A}$, a complex structure on N is called adapted if for every $x \in N$ the orbit map $\mathcal{A} \ni \sigma \mapsto x\sigma \in N$ is holomorphic.We show that: an adapted complex structure on N can exist only if the initial compex structure on $\mathcal{A}$ is left invariant. Left invariant complex structures on $\mathcal{A}$ are parametrized by the points of $\mathbb{C} \setminus \mathbb{R}$. For each $s \in \mathbb{C} \setminus \mathbb{R}$ and corresponding left invariant complex structure $I(s)$ on $\mathcal{A}$, if an $I(s)$ adapted complex structure $J(s)$ exists on N, then this structure is unique and if $J(i)$ exists, then $J(s)$ also exists for all s in $s \in \mathbb{C} \setminus \mathbb{R}$. The points of the upper half plane (denoted from now on by S) correspond to $J(s)$ in which ω is a Kähler form. The original definition of adapted complex structures in Definition 1.2.2 corresponds to the parameter $s = i$.

The family of adapted complex structures $J(s)$, $s \in S$ on N can all be put

together to form a holomorphic fibration $\pi : Y \to S$, where the fibers $Y_s = \pi^{-1}s$ are biholomorphic to $(N, J(s))$. In fact, as a differentiable manifold, $Y = S \times N$, and the projection pr: $Y \to N$ realizes the biholomorphisms $Y_s \to (N, J(s))$.

(cf. Theorems 6.1.2, 6.1.4, 6.1.6, 6.1.10, 6.1.11 and Corollary 6.1.5.)

6.1.1 Polarized manifolds

A polarization of a smooth manifold N is given by a smooth, involutive, complex subbundle $P \subset \mathbb{C} \otimes TN$, of rank $m = (1/2) \dim N$. Involutivity means that the bracket of sections of P is again a section of P. This definition is more general than the one, say, in [W] (or Definition 4.2.12) but in our context this is the natural one. Sometimes even more general structures have to be considered, where the rank condition is omitted; these are the involutive manifolds.

A polarization is real if $\overline{P} = P$; it is equivalent to the datum of an m dimensional foliation of N. A polarization is complex if P and $\overline{P}$ are transverse; this one is equivalent to a complex structure on N. In the former case P consists of tangents to the leaves, in the latter P is the bundle of $(1, 0)$ vectors. A smooth map f of polarized manifolds $(M, Q) \to (N, P)$ is called polarized if $f_*Q \subset P$. As is well known, a polarized map f between two polarized manifolds, where both polarizations are complex, is equivalent with f being a holomorphic map.

Consider now a smooth manifold N on which a Lie semigroup G acts smoothly on the right. Fix a polarization of G.

Definition 6.1.1. *A polarization of N is called adapted (to the polarization of G) if for every $x \in N$ the map $G \ni g \mapsto xg \in N$ is polarized.*

6.1.2 The affine semigroup

This is the Lie semigroup $\mathcal{A}$ of affine transformations $\sigma : \mathbb{R} \to \mathbb{R}$, $t \mapsto \sigma t = a+bt$, where $a, b \in \mathbb{R}$. Here $a = a(\sigma), b = b(\sigma)$ serve as global coordinates on $\mathcal{A}$, and identify it with $\mathbb{R}^2$. With this single coordinate chart $\mathcal{A}$ becomes a Lie semigroup. The subset of $\mathcal{A}$ with $b > 0$ forms a Lie (sub)group, that we shall denote by $\mathcal{A}_+$. In the above coordinates $\mathcal{A}_+$ is identified, as a smooth manifold, with the upper half plane S.

Denote by L_σ, R_σ left and right translations of $\mathcal{A}$. By a left invariant polarization Q of $\mathcal{A}$ we mean one for which $L_\sigma \colon \mathcal{A} \to \mathcal{A}$ is polarized for every $\sigma \in \mathcal{A}$. In the identification $\mathcal{A} \cong \mathbb{R}^2$ invariance means that the fibers Q_σ, $\sigma \in \mathcal{A}$, are Euclidean translates of each other. Hence associating with a polarization Q of $\mathcal{A}$ the complex line $Q_{\mathrm{id}} \subset \mathbb{C} \otimes T_{\mathrm{id}}\mathcal{A}$ yields a bijection between the set of left invariant polarizations and $\mathbb{P}(\mathbb{C} \otimes T_{\mathrm{id}}\mathcal{A}) \approx \mathbb{CP}_1$. All left invariant polarizations but one can be obtained by the following construction. The (left) action of $\mathcal{A}$ on $\mathbb{R}$ extends to an affine action on $\mathbb{C}$. Fix $s \in \mathbb{C}$, let

$$f_s(\sigma) = \sigma s \qquad \text{and} \qquad Q(s) = (f_{s*})^{-1} T^{1,0}\mathbb{C}. \tag{6.1.1}$$

E.g., $Q(i)$ gives the usual complex structure on $\mathcal{A} \cong \mathbb{R}^2$, In general, for s nonreal, $Q(s)$ is a complex polarization. We shall denote the corresponding complex structure by $I(s)$.

For s real $Q(s)$ is a real polarization whose leaves are straight lines of slope $-1/s$. Equivalently, with $H_s = \{\sigma \in \mathcal{A} \colon \sigma s = s\}$, $s \in \mathbb{R}$, the leaves of $Q(s)$ are left translates of H_s. We write $Q(\infty)$ for the exceptional, real polarization,

whose leaves have slope 0. Now (6.1.1) and $f_{\sigma s} = f_s \circ R_\sigma$ together imply that $R_\sigma \colon (\mathcal{A}, Q(\sigma s)) \to (\mathcal{A}, Q(s))$ is a polarized map.

For an $s \in \mathbb{C}$, let $\sigma_s \in \mathcal{A}$ be the unique element with $f_i(\sigma_s) = \sigma_s(i) = s$.

The curves $\alpha(u)$, $\beta(u)$, $u \in \mathbb{R}$ with

$$\alpha(u) = \{t \mapsto u + t\}, \quad \beta(u) = \{t \mapsto e^u t\} \tag{6.1.2}$$

are 1-parameter subgroups in $\mathcal{A}_+$ and $\dot\alpha(0), \dot\beta(0)$ form a basis in its Lie algebra.

Define the homomorphism $\chi \colon \mathcal{A} \to \mathbb{R}$ by $\chi(\sigma) = b$ if $\sigma t = a + bt$, and let $\mathcal{A}^\rho = \{\sigma \in \mathcal{A} \colon |\chi(\sigma)| \le \rho\}$, $0 \le \rho \le \infty$. Thus $\mathcal{A}^\rho \subset \mathcal{A}$ is a normal sub–semigroup if $\rho \le 1$. Everything that we discussed in this subsection applies to $\mathcal{A}^1$ instead of $\mathcal{A}$ as well.

6.1.3 Riemannian manifolds

Let M be a complete Riemannian manifold, $\dim M = m > 0$. Define the phase space N of M not as TM or T^*M, but as the manifold of geodesics $x : \mathbb{R} \to M$.

Any $t_0 \in \mathbb{R}$ induces a diffeomorphism $N \ni x \mapsto \dot x(t_0) \in TM$, and the pull back of the canonical symplectic form on $TM \approx T^*M$ is independent of t_0 (see [W, Section 2.3]); it will be denoted by ω. It corresponds to the form $\sum dq_i \wedge dp_i$ (written in terms of the usual local coordinates on $TM \approx T^*M$.) Note the sign convention used for ω.

Elements of T_xN can be identified with Jacobi fields along x. If $(\ ,\ )$ denotes the Riemannian inner product on TM, then for Jacobi fields $\xi, \eta \in T_xN$

$$\omega(\xi, \eta) = (\xi(t), \eta'(t)) - (\eta(t), \xi'(t)), \qquad \text{for any } t \in \mathbb{R}, \tag{6.1.3}$$

where prime indicates Levi–Civita covariant differentiation, see [Kl, 3.1.14–17]. Further, let $L(x)$ denote the speed of a geodesic $x \in N$, squared (so L is twice the free Lagrangian). It will be convenient to associate with a point $q \in M$ the constant geodesic $\equiv q$. This identifies M with the submanifold of zero speed geodesics.

Composition of geodesics with affine transformations $\mathbb{R} \to \mathbb{R}$ defines a right action of $\mathcal{A}$ on N. We shall consider adapted polarizations on $\mathcal{A}^1$–invariant open subsets $X \subset N$. The domains on which the adapted complex structures of [GS1, LSz91] were defined correspond to the set of geodesics of speed $< r$, with $r \in (0, \infty]$ (but since then adapted complex structures on more general invariant sets have turned out to be of importance, see [FHW] and references there). Put $\Omega_x(\sigma) = A_\sigma(x) = x \circ \sigma$ for $x \in N, \sigma \in \mathcal{A}$. Then

$$A_\sigma^* \omega = \chi(\sigma)\omega. \tag{6.1.4}$$

When $\sigma t = a + t$, (6.1.4) expresses the fact that the geodesic flow preserves ω, and when $\sigma t = bt$, (6.1.4) holds because in the canonical identification $N \approx TM$ in local coordinates A_σ is given by $(q_j, p_j) \mapsto (q_j, bp_j)$. Since $\mathcal{A}$ is generated by translations and dilations, (6.1.4) holds for all $\sigma \in \mathcal{A}$.

By Definition 6.1.1, given a polarization Q of $\mathcal{A}^1$, a polarization P of X is adapted to Q if $\Omega_x \colon (\mathcal{A}^1, Q) \to (X, P)$ is polarized for every $x \in X$. If Q is left invariant and $0 < \rho < \infty$, this is equivalent to saying that $\Omega_x \colon (\mathcal{A}^\rho, Q) \to (X, P)$ is polarized for every $x \in X\mathcal{A}^{1/\rho}$.

Theorem 6.1.2 (Lempert, Szőke,[LSz12]). *(a) If a nonempty, $\mathcal{A}^1$–invariant open $X \subset N$ has a polarization P adapted to a polarization Q of $\mathcal{A}^1$, then Q is left invariant. If $Q = Q(s)$, $s \in \mathbb{R}$, then it determines P uniquely. P must be a real polarization. For an $s \in \mathbb{C} \setminus \mathbb{R}$, $Q = Q(s)$ determines P uniquely iff $Q = Q(i)$ does. In any case there is at most one complex polarization on X that is adapted to $Q(s)$.*

(b) If M is a closed analytic Riemannian manifold, then there is a $\mathcal{A}^1$–invariant open $X \subset N$, containg all zero speed geodesics, such that $X\mathcal{A}^{1/|Im\, s|}$ has a polarization $P(s)$ adapted to $(\mathcal{A}^1, Q(s))$, for every $s \in \mathbb{C}$. $P(s)$ will be a complex polarization when $s \in \mathbb{C} \setminus \mathbb{R}$ and a real polarization for $s \in \mathbb{R}$. The same is true if M is not necessarily closed, but modulo its isometry group it is compact.

Proof. Suppose (X, P) is adapted to Q. Fix a nonconstant geodesic $x \in X$; then $\Omega_x \colon (\mathcal{A}^1, Q) \to (X, P)$ is a polarized immersion. Since $\Omega_{x\circ\sigma} = \Omega_x \circ L_\sigma$ is also polarized for $\sigma \in \mathcal{A}^1$, it follows that $L_\sigma \colon (\mathcal{A}^1, Q) \to (\mathcal{A}^1, Q)$ is polarized, i.e., Q is left invariant.

Let now $Q = Q(s)$, $s \in \mathbb{C}$, $\sigma \in \mathcal{A}$ and $x \in X$. Consider the commutative diagram

$$\begin{array}{ccc} X\mathcal{A}^{1/|\chi(\sigma)|} & \xrightarrow{A_\sigma} & X \\ {\scriptstyle \Omega_x}\uparrow & & {\scriptstyle \Omega_x}\uparrow \\ \mathcal{A}^{1/|\chi(\sigma)|} & \xrightarrow{R_\sigma} & \mathcal{A}^1, \end{array}$$

and recall that $R_\sigma \colon (\mathcal{A}, Q(\sigma s)) \to (\mathcal{A}, Q(s))$ is polarized.

Now $A_\sigma \colon X\mathcal{A}^{1/|\chi(\sigma)|} \to X$ is a diffeomorphism if $\chi(\sigma) \neq 0$, and the diagram implies that A_σ pulls back any $Q(\sigma s)$–adapted polarization to a $Q(s)$–adapted polarization. Therefore uniqueness and existence for $s = i$ implies the same for any $s \in \mathbb{C}\backslash\mathbb{R}$. Concretely, if $\sigma s = i$, then $\chi(\sigma) = 1/\mathrm{Im}\, s$, and so if X admits a $Q(i)$–adapted polarization, then $X\mathcal{A}^{1/|\mathrm{Im}\, s|}$ will admit a $Q(s)$–adapted polarization. Also if the $Q(i)$–adapted polarization is complex, the $Q(s)$–adapted polarization is complex as well. Now uniqueness and existence of the adapted complex polarization for $s = i$ is the content of [LSz91, Theorem 4.2] and [Sz91, Theorem 2.2]. This proves part (a).

Finally, let $s \in \mathbb{R}$ and $\pi_s \colon N \to M$ be given by $\pi_s(x) = x(s)$. As said, the leaves of $Q(s)$ are left translates of the sub–semigroup $H_s \subset \mathcal{A}$. That P is adapted to $Q(s)$ means it is tangential to the orbits $\Omega_x(H_s)$. As x ranges over a fiber $\pi_s^{-1}y$, these orbits all pass through the constant geodesic $\equiv y$, that we denote $\overline{y}$. Their tangents at $\overline{y}$ form the vertical tangent space $T_{\overline{y}}(\pi_s^{-1}y)$, which then must agree with $P_{\overline{y}}$. Furthermore, the vector field generating the H_s–action, being tangent to the orbits, is a section of P. Since P is involutive, it must be invariant under the action of $H_s \cap \mathcal{A}^1$. But π_s is also invariant, hence for any $x \in X$ and $\overline{y}$ as above, $\pi_{s*}P_x = \pi_{s*}P_{\overline{y}} = 0$. Therefore P consists of the tangent spaces to the fibers of π_s, and is unique. It is straightforward that, conversely, the tangent spaces to the fibers form a polarization of N, adapted to $(\mathcal{A}^1, Q(s))$ (and to $(\mathcal{A}, Q(s))$). □

Remarks. It is tempting to claim in part (a) that the $Q(i)$ complex polarization determines any adapted polarization on X, not only the complex ones. But this is not known. It is easy to see that for any adapted $Q(i)$ polarization

P, $P|_M$ and $\bar{P}|_M$ will be transverse and so they remain transverse in a small neighborhood of M in X. That means, P will be a complex polarization in this small neighborhood, hence the uniqueness result [LSz91, Theorem 4.2] applies. The difficulty is that a $Q(i)$ adapted polarization P may not be complex in all of X, as for example in [Sz01, Theorem 1.2] and we do not know how to prove uniqueness for such adapted polarizations.

It is straightforward that, after the canonical identification $N \cong TM$, a complex polarization adapted to $Q(i)$ becomes the adapted complex structure of [LSz91, Definition 4.1]. More generally (but as a special case of Definition 6.1.1)

Definition 6.1.3. *Given a complex manifold structure on $\mathcal{A}^1$, a complex structure on X is adapted if for every $x \in X$ the orbit map $\mathcal{A}^1 \ni \sigma \mapsto x\sigma \in X$ is holomorphic.*

From Theorem 6.1.2 (a) adapted complex structure on X can exist only if the initial complex structure on $\mathcal{A}^1$ is left invariant. Recall from subsection6.1.2 that the left invariant complex structures on $\mathcal{A}^1$ are parametrized by points in $\mathbb{C}\backslash\mathbb{R}$ as follows. Each $\sigma \in \mathcal{A}$ extends to an affine map of $\mathbb{C}$. For fixed $s \in \mathbb{C}\backslash\mathbb{R}$, let $I(s)$ denote the pull back of the complex structure of $\mathbb{C}$ by the map $\mathcal{A}^1 \ni \sigma \mapsto \sigma s \in \mathbb{C}$. Then the structures $I(s)$ are all the left invariant complex structures on $\mathcal{A}^1$. From Theorem 6.1.2 we get.

Theorem 6.1.4 (Lempert, Szőke,[LSz12])**.** *Let M be a compact Riemannian manifold.*

(a) If on an open $\mathcal{A}^1$–invariant $X \subset N$ there is a complex structure adapted to $(\mathcal{A}^1, I(s))$, then this structure is unique. It will be denoted $J(s)$.

(b) If M is real analytic, then there is an $\mathcal{A}^1$–invariant open neighborhood X of $M \subset N$ such that $X\mathcal{A}^{1/|Im\, s|}$ has a complex structure $J(s)$ adapted to $(\mathcal{A}^1, I(s))$, for all $s \in \mathbb{C}\backslash\mathbb{R}$.

The proof of Theorem 6.1.2 also gave the following.

Corollary 6.1.5 (Lempert, Szőke,[LSz12])**.** *Let $s \in \mathbb{C}$ and $\sigma \in \mathcal{A}$. If a $\mathcal{A}^1$–invariant $X \subset N$ admits a $Q(s)$–adapted polarization $P(s)$, then $X\mathcal{A}^{1/|\chi(\sigma)|}$ admits a $Q(\sigma s)$–adapted polarization $P(\sigma s)$, and*

$$A_\sigma \colon (X\mathcal{A}^{1/|\chi(\sigma)|}, P(\sigma s)) \to (X, P(s))$$

is polarized, in fact a polarized isomorphism when $\chi(\sigma) \neq 0$.

We shall continue using $J(s)$ for the $I(s)$–adapted complex structure on $X \subset N$, whenever it exists. Recall that $L(x)$ denote the speed squared of a geodesic $x \in N$.

Theorem 6.1.6 (Lempert, Szőke,[LSz12])**.** *For $s \in \mathbb{C}\backslash\mathbb{R}$ let $\partial_s, \overline{\partial}_s$ denote the complex exterior derivations for the complex structure $J(s)$ on X (if this latter exists). The symplectic form ω on $X \subset N \cong TM$ is given by*

$$i\omega = (Im\ s)\overline{\partial}_s \partial_s L.$$

In particular, ω is a positive or negative $(1,1)$–form depending on whether $Im\ s > 0$ or < 0.

Proof. When $s = i$, the claims are in [LSz91, Corollary 5.5 and Theorem 5.6]. (Note that E and Ω there correspond to our $L/2$, resp. $-\omega$.) Hence the general case follows by Corollary 6.1.5, because $A_\sigma^* L = \chi(\sigma)^2 L$ and by (6.1.4), $A_\sigma^* \omega = \chi(\sigma)\omega$. □

In [HK] Hall and Kirwin consider a holomorphic family $P'(s)$ of polarizations of (neighborhoods of M in) TM. When s is real, $P'(s)$ is the pullback of the vertical polarization of TM by time s geodesic flow, and $P'(i)$ is the original adapted complex structure of [GS1, LSz91]. [HK] also studies the effect of fiberwise scaling on this family. After analytic continuation from real to complex σ in [HK, Theorem 3.3], a comparison with our Corollary 6.1.5 shows that under the canonical identification $N \to TM$, $P(s)$ corresponds to $P'(s)$.

6.1.4 The canonical bundle

In the set up of Section 6.1.3, let $s \in \mathbb{C} \setminus \mathbb{R}$ and consider a $\mathcal{A}^1$–invariant open $X \subset N$ that admits a $Q(s)$–adapted complex polarization $P(s)$. Recall that the corresponding complex structures were denoted by $I(s)$ and $J(s)$. The canonical bundle of $(X, J(s))$ is the holomorphic line bundle $K^s \to X$ of $(m, 0)$–forms. If $s = i$, we simply write K instead of K^i. K^s has a Hermitian metric h^{K^s} defined by

$$h^{K^s}(\Theta)\omega^m(x) = i^{m^2} m! \Theta \wedge \bar{\Theta}, \qquad \Theta \in K^s_x, \quad x \in X. \tag{6.1.5}$$

Sometimes we we just write shortly $|\alpha|^2$ instead of $h^{K^s}(\alpha, \alpha)$. Recall from 6.1.2 that $\sigma_s \in \mathcal{A}$ is the unique element with $\sigma_s(i) = s$. For simplicity assume that $X = N$. According to 6.1.5, the map $A_{\sigma_s} : (N, J(s)) \to (N, J(i))$ is a biholomorphism.

Proposition 6.1.7 (Lempert, Szőke,[LSz14])**.** *Assume M is oriented. Then the bundle K^s is holomorphically trivial.*

Proof. It is enough to deal with the case $s = i$. Since $(N, J(i))$ is a Stein manifold ([LSz91, Theorem 5.6]), by the Oka principle, see e.g. [Hö, pp. 144-145] , it suffices to show that K is smoothly trivial. M is a deformation retract in N, so we only need to show that $K|_M$ is trivial. Since M is oriented, the bundle $K_M \to M$ of real $m-$forms is trivial. But restricting a form in $K|_M$ to TM is an isomorphism $K|_M \approx \mathbb{C} \otimes K_M$ and we are done. □

Due to the proposition, when M is orientable, there is a Hermitian holomorphic (in fact trivial) line bundle (κ, h^κ) so that $\kappa \otimes \kappa \approx K$. Let now Θ be a trivializing section of K, i.e. a nowhere vanishing holomorphic (m,0) form on $(N, J(i))$ and θ the corresponding section of κ with $\theta \otimes \theta = \Theta$. Taking $\kappa^s := A^*_{\sigma_s}\kappa$, we have $\kappa^s \otimes \kappa^s \approx K^s$. Let $\Theta_s := A^*_{\sigma_s}\Theta$ and $\theta_s := A^*_{\sigma_s}\theta$.

Let $\mathcal{E} = \frac{\omega^m}{m!}$ be the Liouville volume form on N. From (6.1.4) we get

$$A^*_\sigma(\mathcal{E}) = \chi(\sigma)^m. \tag{6.1.6}$$

Proposition 6.1.8 (Lempert, Szőke,[LSz14])**.**

$$|\Theta_s|^2 = (Im\ s)^m |\Theta|^2 \circ A_{\sigma_s} \quad |\theta_s|^2 = (Im\ s)^{\frac{m}{2}} |\Theta| \circ A_{\sigma_s}.$$

Proof. Since $\chi(\sigma_s) = \mathrm{Im}\ s$, ((6.1.5) and (6.1.6) implies

$$\Theta_s \wedge \bar{\Theta}_s = A^*_{\sigma_s}(\Theta \wedge \bar{\Theta}) = A^*_{\sigma_s}(|\Theta|^2(-i)^{m^2}\mathcal{E}) = (-i)^{m^2}(\mathrm{Im}\ s)^m(|\Theta|^2 \circ A_{\sigma_s})\mathcal{E}$$

In light of ((6.1.5) this proves the first formula from which the second follows. □

We shall need the above propositions in subsection 6.2.4.

In the rest of this subsection we compute h^{K^s}, something that is needed for purposes of quantization. We start by recalling certain constructions and results from [LSz91, Sz95]. Denote by id$\in \mathcal{A}$ the identity transformation $\mathbb{R} \to \mathbb{R}$.

The action of $\mathcal{A}$ on N induces an action on TN and $\mathbb{C} \otimes TN$, denoted $(\xi, \sigma) \mapsto \xi\sigma$. Let $x \in X$. Any $\xi \in T_xX$ can be decomposed into $(1,0)$ and $(0,1)$ components with respect to the structure $P(s)$: $\xi = \xi^{1,0} + \xi^{0,1}$. If $J : TX \to TX$ denotes the complex structure operator for $P(s)$, then $\xi^{1,0} = (\xi - iJ\xi)/2$. The map $\sigma \mapsto (\xi\sigma)^{1,0}$ is holomorphic as a map $(\mathcal{A}^1, Q(s)) \to T^{1,0}(X, P(s))$ (in the sense that it has a holomorphic extension to a neighborhood of $\mathcal{A}^1$), see [LSz91, Proposition 5.1]

Now consider two m–tuples $\xi_1, \ldots$ and $\eta_1, \ldots \in T_xN$, and assume that the $\xi_j^{1,0}$ are linearly independent. Those σ for which $(\xi_j\sigma)^{1,0}$ are linearly dependent form a discrete subset $\Delta \subset \mathcal{A}$. For $\sigma \in \mathcal{A}^0 \setminus \Delta$ the $\xi_j\sigma$ are also independent. Since when $\sigma \in \mathcal{A}^0$, the vectors $\xi_j\sigma, \eta_j\sigma$ are tangential to the m–dimensional manifold of zero speed geodesics, on $\mathcal{A}^0 \setminus \Delta$ there is a smooth real $m \times m$–matrix valued function $\phi^0 = (\phi^0_{jk})$ such that

$$\eta_j\sigma = \sum_k \phi^0_{jk}(\sigma)\xi_k\sigma, \qquad \sigma \in \mathcal{A}^0 \setminus \Delta. \tag{6.1.7}$$

Further, there is a meromorphic $m \times m$–matrix valued function $\phi = (\phi_{jk})$ on $(\mathcal{A}^1, Q(s))$, with poles restricted to Δ, such that

$$(\eta_j\sigma)^{1,0} = \sum_k \phi_{jk}(\sigma)(\xi_k\sigma)^{1,0} \tag{6.1.8}$$

By (6.1.7) and (6.1.8), ϕ is the analytic continuation of ϕ^0.

Theorem 6.1.9 (Lempert, Szőke,[LSz12])**.** *Suppose $x \in X$ and $\xi_1, \ldots, \eta_m \in T_xX$ form a symplectic basis:*

$$\omega(\xi_j, \xi_k) = \omega(\eta_j, \eta_k) = 0, \quad \omega(\xi_j, \eta_k) = \delta_{jk}, \qquad 1 \le j, k \le m.$$

If ϕ is as in (6.1.8)*, then for $\Theta \in K^s_x$*

$$h^K(\Theta) = 2^m|\Theta(\xi_1, \ldots, \xi_m)|^2 \det Im\ \phi(id). \tag{6.1.9}$$

It is possible to derive (6.1.9) from [HK, (3.9)] and [KW, (3.9)], but the following proof is shorter and self contained.

Proof. With $\zeta_j = \sum_k \mathrm{Im}\,\phi_{jk}(\mathrm{id})\xi_k$

$$\begin{aligned}\Theta \wedge \bar{\Theta}(\zeta_1^{1,0}, \ldots, \zeta_m^{1,0}, \eta_1^{0,1}, \ldots, \eta_m^{0,1}) &= \Theta(\zeta_1^{1,0}, \ldots)\overline{\Theta(\eta_1^{1,0}, \ldots)} \\ &= \Theta(\zeta_1, \ldots)\overline{\Theta(\eta_1^{1,0}, \ldots)} = |\Theta(\xi_1, \ldots)|^2 \det \mathrm{Im}\,\phi(\mathrm{id}) \det \bar{\phi}(\mathrm{id}).\end{aligned} \tag{6.1.10}$$

Taking real parts in (6.1.8) gives $\eta_j = \sum_k \mathrm{Re}\ \phi_{jk}(\mathrm{id})\xi_k + J\zeta_j$. Thus

$$\begin{aligned}2\omega(\zeta_j^{1,0}, \eta_l^{0,1}) &= 2\omega(\zeta_j^{1,0}, \eta_l) = \omega(\zeta_j - iJ\zeta_j, \eta_l)\\ &= \omega\big(\sum_k \mathrm{Im}\,\phi_{jk}(\mathrm{id})\xi_k, \eta_l\big) - i\omega\big(\eta_j - \sum_k \mathrm{Re}\ \phi_{jk}(\mathrm{id})\xi_k, \eta_l\big)\\ &= \mathrm{Im}\,\phi_{jl}(\mathrm{id}) + i\mathrm{Re}\ \phi_{jl}(\mathrm{id}) = i\bar{\phi}_{jl}(\mathrm{id}),\end{aligned}$$

and

$$\begin{aligned}\omega^m(\zeta_1^{1,0}, \dots, \zeta_m^{1,0}, \eta_1^{0,1}, \dots, \eta_m^{0,1}) &= m! i^{m(m-1)} \det\big(\omega(\zeta_j^{1,0}, \eta_l^{0,1})\big)\\ &= m! i^{m^2} 2^{-m} \det \bar{\phi}(\mathrm{id}).\end{aligned}$$

Comparing this with (6.1.5) and (6.1.10) yields (6.1.9). □

6.1.5 The family of adapted polarizations

Finally we shall construct a polarized fibration $Z \to \mathbb{C}$ whose fibers represent the various $(X, P(s))$. With $s \in \mathbb{C}$, $x \in N$ consider the embeddings

$$i^x \colon \mathbb{C} \ni s \mapsto (s, x) \in \mathbb{C} \times N, \qquad j^s \colon N \ni x \mapsto (s, x) \in \mathbb{C} \times N. \tag{6.1.11}$$

Also, let $\pi \colon \mathbb{C} \times N \to \mathbb{C}$ denote the projection.

Theorem 6.1.10 (Lempert, Szőke,[LSz12])**.** *Suppose that a $\mathcal{A}^1$–invariant open $X \subset N$ admits the adapted polarization (complex structure) $P(i)$.*

(a) On $Z = \{(s, x) \in \mathbb{C} \times N \colon x \in X\mathcal{A}^{1/|Im\ s|}\}$ there is a unique polarization P such that the maps

$$i^x \colon ((i^x)^{-1} Z, T^{1,0}\mathbb{C}) \to (Z, P), \qquad j^s \colon (X\mathcal{A}^{1/|Im\ s|}, P(s)) \to (Z, P)$$

are polarized for all $s \in \mathbb{C}$, $x \in N$. With this P, $\pi \colon (Z, P) \to (\mathbb{C}, T^{1,0}\mathbb{C})$ is polarized, and $(Z \setminus \pi^{-1}\mathbb{R}, P) = Z_0$ is a complex manifold.

(b) Let ∂, $\bar{\partial}$ denote the complex exterior derivations on Z_0, and $\tilde{\omega}$ the pullback of ω along the map $(s, x) \to x$. Then

$$i\tilde{\omega} = \bar{\partial}\partial(LIm\ s) \qquad \text{on } Z_0, \tag{6.1.12}$$

$LIm\ s$ is plurisub/superharmonic if $Im\ s > 0$, resp. < 0, and satisfies the Monge–Ampère equation $\mathrm{rk}\ \bar{\partial}\partial(LIm\ s) = m$.

(c) Finally, endow $(\mathbb{C}, T^{1,0}\mathbb{C}) \times (X, P(i))$ with the product complex structure. Then the map $\Phi \colon Z \to \mathbb{C} \times X$ given by

$$\Phi(s, x) = (s, x \circ \sigma), \qquad \text{where} \quad \sigma i = f_i(\sigma) = s, \tag{6.1.13}$$

is polarized, and in fact restricts to a biholomorphism $Z_0 \to (\mathbb{C}\backslash\mathbb{R}) \times X$.

Proof. Since the range of i^x_* and j^s_* together span $T(\mathbb{C} \times N)$, the polarization P in question is unique, and must be given by

$$P_{(s,x)} = i^x_* T^{1,0}_s \mathbb{C} \oplus j^s_* P(s)_x, \quad (s, x) \in Z. \tag{6.1.14}$$

In view of Corollary 6.1.5 this formula defines a subbundle $P \subset \mathbb{C} \otimes TZ$. Our P has rank $m+1$ all right, but is it involutive? To decide, first we check that Φ in

6.1.13 is polarized, i.e. Φ_* maps P into $T^{1,0}(\mathbb{C}\times X)$. With notation introduced earlier

$$(\Phi\circ i^x)(s) = (s, (\Omega_x\circ f_i^{-1})(s)), \qquad (\Phi\circ j^s)(x) = (s, A_\sigma x).$$

Now $\Omega_x\colon (\mathcal{A}^{r/\sqrt{L(x)}}, Q(i)) \to (X, P(i))$ is holomorphic by the definition of $P(i)$ and by the observation preceding Theorem 6.1.2; also $f_i\colon (\mathcal{A}, Q(i)) \to \mathbb{C}$ is biholomorphic by the definition of $Q(i)$. Therefore $\Phi\circ i^x$ is holomorphic. Similarly $\Phi\circ j^s\colon (X\mathcal{A}^{1/|\mathrm{Im}\, s|}, P(s)) \to (\mathbb{C}\times X, T^{1,0}(\mathbb{C}\times X))$ is polarized by Corollary 6.1.5 (s there corresponds to i here, though). Putting these and 6.1.13 together, we see $\Phi_* P \subset T^{1,0}(\mathbb{C}\times X)$ indeed.

Since $T^{1,0}(\mathbb{C}\times X)$ is involutive and Φ is a diffeomorphism over Z_0, it follows that P is involutive over Z_0, which therefore is a complex manifold. By density, P is involutive over all of Z. That π is polarized is obvious, so (a) and (c) have been proved. As to 6.1.12, the two sides restricted to the fibers of π agree by Theorem 6.1.6. Tangents to the fibers of $(s,x)\to x$, the "horizontal fibers", constitute the kernel of $i\tilde{\omega}$; to finish the proof it will suffice to show the same for Ker $\bar\partial\partial(L\mathrm{Im}\, s)$. The restriction of $\bar\partial\partial(L\mathrm{Im}\, s)$ to the horizontal fibers is certainly 0, since L restricts to a constant and i^x is holomorphic; but that is not quite enough. It will be necessary to compute $\bar\partial\partial(L\mathrm{Im}\, s)$, that we do by pulling it back along Φ^{-1}.

If in 6.1.13 $\sigma t = a + bt$, then $b = \mathrm{Im}\, s$. Hence $(\Phi^{-1})^*(L\mathrm{Im}\, s) = L/\mathrm{Im}\, s$. (With a slight abuse of notation, L stands for both a function on N and its pull back along the projection $\mathbb{C}\times N\to N$. Also, $\mathrm{Im}\, s$ stands for a function on $\mathbb{C}\times N$.) On $(\mathbb{C}\setminus\mathbb{R})\times X$

$$\bar\partial\partial\frac{L}{\mathrm{Im}\, s} = \frac{\bar\partial\partial L}{\mathrm{Im}\, s} + \frac{d\bar s\wedge\partial L - \bar\partial L\wedge ds}{2i(\mathrm{Im}\, s)^2} + \frac{L d\bar s\wedge ds}{2(\mathrm{Im}\, s)^3}. \tag{6.1.15}$$

In computing ∂L, $\bar\partial L$, the operators corresponding to the complex structure $P(i)$ are to be used. Now $(\bar\partial\partial L)^{m+1}$ and $(\bar\partial\partial L)^m\wedge\partial L$ vanish, thus by 6.1.15, $\big(\bar\partial\partial(L/\mathrm{Im}\, s)\big)^{m+1}$ equals

$$\begin{aligned}(m+1)\Big(\frac{\bar\partial\partial L}{\mathrm{Im}\ s}\Big)^m\wedge\frac{L\, d\bar s\wedge ds}{2(\mathrm{Im}\ s)^3} - \binom{m+1}{2}\Big(\frac{\bar\partial\partial L}{\mathrm{Im}\ s}\Big)^{m-1}\wedge\frac{(d\bar s\wedge\partial L - \bar\partial L\wedge ds)^2}{4(\mathrm{Im}\ s)^4}\\ = \frac{(m+1)d\bar s\wedge ds}{4(\mathrm{Im}\ s)^{m+3}}\wedge\big(2L(\bar\partial\partial L)^m - m(\bar\partial\partial L)^{m-1}\wedge\bar\partial L\wedge\partial L\big).\end{aligned} \tag{6.1.16}$$

According to [LSz91, (5.20)], where $E = L/2$, the last expression vanishes. As $-i\bar\partial\partial(L/\mathrm{Im}\ s)$ is definite along the fibers of π, its signature is m pluses (or minuses) and one 0, and the same holds for $-i\bar\partial\partial(L\mathrm{Im}\ s)$ on Z_0. In particular, $(L\mathrm{Im}\ s)$ is plurisub/superharmonic, and because $\bar\partial\partial(L\mathrm{Im}\ s)$ vanishes on the horizontal fibers, its kernel consists of the tangents to the horizontal fibers. This then proves 6.1.12 and the rest of (b). □

The results that will be important in the quantization later on are the following parts of Theorem 6.1.10. The adapted complex structures $J(s)$ of Theorem 6.1.4 can all be put together to form a holomorphic fibration.

Theorem 6.1.11 (Lempert, Szőke,[LSz12]). *Suppose that an $\mathcal{A}^1$–invariant open $X \subset N$ admits a complex structure adapted to $I(i)$. Then on*

$$Z = \{(s,x) \in (\mathbb{C}\backslash\mathbb{R}) \times N \colon x \in X\mathcal{A}^{1/|Im\, s|}\}$$

there is a unique complex structure that restricts on each fiber $\{s\} \times X\mathcal{A}^{1/|Im\, s|}$ to the structure adapted to $I(s)$, and for each $x \in N$, the map $s \mapsto (s,x) \in Z$ is holomorphic where defined. The projection map $\pi : Z \to \mathbb{C}\backslash\mathbb{R}$ is then a holomorphic submersion. The pull back $\tilde{\omega}$ of ω along the projection pr: $(\mathbb{C} \setminus \mathbb{R}) \times N \to N$ satisfies

$$i\tilde{\omega} = \overline{\partial}\partial(L \operatorname{Im} s) \qquad \text{on } Z. \tag{6.1.17}$$

(Here, a little abusively, $L \operatorname{Im} s$ stands for the function $(s,x) \mapsto L(x)\operatorname{Im} s$.) Finally, if X is endowed with the $I(i)$ adapted complex structure $J(i)$, and $(\mathbb{C}\backslash\mathbb{R}) \times X$ with the product structure, then the map

$$Z \ni (s,x) \mapsto (s, x\sigma) \in (\mathbb{C}\backslash\mathbb{R}) \times X, \qquad \text{where } \sigma i = s, \tag{6.1.18}$$

is a biholomorphism. In particular, $Z \ni (s,x) \mapsto s \in \mathbb{C}$ is holomorphic.

6.2 Geometric quantization

Suppose an m–dimensional compact Riemannian manifold M is the classical configuration space of a mechanical system, the metric corresponding to twice the kinetic energy. The aim of geometric quantization is to construct a Hilbert space H, called the quantum Hilbert space, associated to this system in a canonical way. In this process, according to the prescriptions of Kostant and Souriau [Ko1, So1, W], one first passes to phase space N, a symplectic manifold with an exact symplectic form ω. As it is discussed in section 6.1.3, N is the manifold of parametrized geodesics in M. (In fact geometric quantization intends to quantize any symplectic manifold with an integral symplectic form, not only the phase spaces we are concerned with in this dissertation.)

The next step is the choice of a Hermitian line bundle $L \to N$ (so called "prequantum line bundle") with a Hermitian connection whose curvature is $-i\omega$. If M is simply connected, the bundle is unique up to a connection preserving Hermitian isomorphism. In any case, one such line bundle is obtained from a real 1–form a on N such that $da = -\omega$, by letting $L = N \times \mathbb{C} \to N$ to be the trivial line bundle with $h^L(x,\gamma) = |\gamma|^2$ the trivial metric on it. If sections are identified with functions $\psi \colon N \to \mathbb{C}$, the connection ∇^L is defined by

$$\nabla^L_\zeta \psi = \zeta\psi + ia(\zeta)\psi, \qquad \zeta \in \operatorname{Vect} N. \tag{6.2.1}$$

The first candidate for the searched for Hilbert space is the so called prequantum Hilbert space H^{prQ} consisting of the L^2 sections of L (where the volume form is the Liouville volume form $\omega^m/m!$). Based on physical principles H^{prQ} is considered to be too big, and the construction has to be modified.

For this purpose the next step is a choice of a polarization on N. With the help of the polarization one would like to take the quantum Hilbert space H as the Hilbert subspace of H^{prQ} consisting of the L^2–sections of L that are covariantly constant along the polarization.

Suppose first that the polarization is real and $N = TM$. The most obvious choice is the vertical polarization, given by the foliation defined by the fibers T_qM, $q \in M$. In this case a section being covariantly constant along the polarization simply means that it is constant along each leaf of the polarization. But such a section could never be in L^2 w.r.t. the Liouville volume form. After some technical steps (that we leave out since real polarizations play no role in the rest), finally one gets the quantum Hilbert space consisting of the sections of L that are covariantly constant along the foliation and L^2, but w.r.t. a different measure, that is the extension of the volume measure of M by zero to $N \setminus M$. Saying it differently, H is simply the space $L^2(M)$ where the measure is the volume measure of the Riemannian metric on M. Real polarizations will play no role in the rest of this disszertation.

Now suppose a complex polarization exists on N, or on an open subset X in N, in which ω becomes a Kähler form. This yields a holomorphic line bundle structure on $L|_X$. Covariantly constant along the polarization then means precisely that the section is holomorphic. Therefore H will consist of sections of $L|_X$ that are holomorphic and L^2 with respect to the Liouville volume form.

Along with bare quantization there is also quantization with half–form correction. This produces somewhat different quantum Hilbert spaces; the corrected Hilbert spaces often have cleaner mathematical properties and are in better agreement with observations.

In Kähler quantization described above, this means the following. One looks at the canonical bundle $K_X \to X$ (the bundle of $(m, 0)$–forms) and fixes a line bundle $\kappa \to X$ such that $\kappa \otimes \kappa$ is isomorphic to K_X, along with an isomorphism $\kappa \otimes \kappa \to K_X$. κ inherits from K_X the structure of a Hermitian holomorphic line bundle. Such a κ does not necessarily exist and when it does it may not be unique. The corrected quantum Hilbert space H^{corr} then consists of holomorphic L^2–sections of $L \otimes \kappa$.

6.2.1 Problem of uniqueness

In certain situations an entire family (parametrized by a set S) of complex structures exists on N (or on an open subset X in N) resulting a family of quantum Hilbert spaces H_s (or H_s^{corr}). The question of **uniqueness** arises. Can one identify these Hilbert spaces in a canonical way? As it was mentioned in the introduction, when S is a complex manifold itself, [Hi] [ADW] proposed to view the H_s as fibers of a holomorphic Hilbert bundle $H \to S$, introduce a connection on H, and use parallel transport to identify the fibers H_s and H_t.

6.2.2 Quantization without half form correction

Since the smooth prequantum line bundle $L \to N$ is independent of $s \in S$, spaces H_s are all closed subspaces of the fixed Hilbert space H^{prQ}. To implement the idea of [ADW, Hi] above, it seems natural to try to view the family H_s as the fibers of a Hilbert subbundle $H \to S$ of the trivial bundle $S \times H^{prQ} \to S$, as [ADW, bottom of p. 801] suggests, but we are not aware of any general result that would guarantee such a statement. In particular it is not known whether, using the family of adapted complex structures, the family $\{H_s, s \in S\}$ forms a

subbundle in $S \times H^{prQ} \to S$ or not. The paper [ADW] provides no solution for this, nor an explanation of what is meant by a subbundle.

The next step is the definition of a connection on H, through its connection form [ADW, pp. 803–805]. (If H were a Hilbert subbundle of $S \times H^{prQ} \to S$, the quantum connection could arise from the canonical flat hermitian connection by orthogonal projection. But this in itself would not guarantee the (projective) flatness of H, ie. the uniqueness of quantization.)

In ordinary situations, connection and connection form determine each other once a (local) trivialization of the bundle, in this case H, is fixed. The situation at hand is not ordinary though, because no local trivialization of H is available a priori, and the connection form must refer to the trivialization of H^{prQ}. But this connection form and the connection ∇ it determines are also not ordinary. It is quite clear that if a smooth section of H^{prQ} is covariantly differentiated along a smooth vector field, the result in general will not be a section of H^{prQ}. The most one can hope for is that if a smooth section of H^{prQ} happens to take values in H, then its covariant derivative will be a smooth section of H; [ADW, last paragraph on p. 803] verifies this, but only under the implicit assumption that the derivative is a smooth section of H^{prQ}. In fact, at this point it is conceivable that zero is the only section of H^{prQ} that can be differentiated. Accepting, nevertheless, that ∇ can be applied to a large space of sections, its curvature can be computed, and turns out to be a multiple of the identity operator (on each fiber H_s). This raises a couple of questions: knowing that an out–of–ordinary connection ∇ is projectively flat, will its parallel transport be independent, up to a scalar, of the path? Even more fundamentally, does ∇ determine a parallel transport?

This is the question that we address and partially answer in chapters 7, 8 and 9.

6.2.3 Half form correction

When the half-form correction is included in the quantization process, a further difficulty arises in implementing the idea of [ADW]. The prequantum hermitian line bundles $L \otimes \kappa_s$ will now also depend on $s \in S$. Hence the Hilbert space H_s^{prQ}, the space of its L^2 sections, will depend on s as well, forming a family of Hilbert spaces of which the corrected quantum Hilbert spaces form a family of Hilbert subspaces. So the first step would be to produce a Hilbert bundle structure on H_s^{prQ}. $\{H_s^{prQ}, s \in S\}$ form a Hilbert bundle.

The main purpose of the next section is to demonstrate that using the family of adapted complex structures for geometric quantization, there are at least two natural, inequivalent ways to make $\{H_s^{prQ}\}$ a Hilbert bundle. The topology on these bundles is the same, but their smooth structure (and therefore the set of smooth sections) are different. This is the content of Theorem 6.2.2. This shows that we need other ways to handle this problem. See the remarks after that Theorem.

6.2.4 The field of prequantum Hilbert spaces

Recall some notations from section 6.1.3): S is the upper half plane, M^m is a simply connected, compact $m-$dimensional Riemannian manifold, the phase

space N is the manifold of parametrized geodesics, ω is the canonical symplectic form, $\mathcal{E} = \frac{\omega^m}{m!}$ the Liouville volume form, $da = -\omega$, where a is a real $1-$form.

Since M is simply connected, the prequantum Hermitian line bundle $L \to N$ with hermitian connection whose curvature is $-i\omega$ is unique. It is the trivial line bundle $L = N \times \mathbb{C} \to N$ with the trivial metric $h^L(x, \gamma) = |\gamma|^2$ on it. Let $\vartheta(x) := (x, 1)$, $x \in N$. Then the sections of L have the form $f\vartheta$, where $f : N \to \mathbb{C}$ and the connection formula (6.2.1) on L simplifies to

$$\nabla^L_\zeta \vartheta = ia(\zeta),$$

where ζ is any (complex) vector field on N.

We assume that the adapted complex structure $J(i)$ exists on N. Then from Corollary 6.1.5 we know that for each $s \in S$ the adapted complex structure $J(s)$ also exists, furthermore $(N, J(s))$ and $(N, J(i))$ are biholomorphic. For each $s \in S$ the symplectic form ω is a Kähler form on $(N, J(s))$. (Theorem 6.1.6). Hence L becomes a holomorphic line bundle. From Proposition 6.1.7 we know that the canonical bundle $K \to (N, J(i))$ is holomorphically trivial. Let Θ be a nowhere vanishing holomorphic (m,0) form on $(N, J(i))$. Let (κ, h^κ) be the holomorphic (in fact trivial) line bundle so that $\kappa \otimes \kappa \approx K$ and θ the corresponding holomorphic section of κ with $\theta \otimes \theta = \Theta$. Let $\kappa^s := A^*_{\sigma_s}\kappa$. Then $\kappa^s \otimes \kappa^s \approx K^s$. $\Theta_s := A^*_{\sigma_s}\Theta$ and $\theta_s := A^*_{\sigma_s}\theta$ are also nowhere vanishing sections of K^s resp. κ^s.

Since $\vartheta \otimes \theta_s$ is a nowhere vanishing section of $L \otimes \kappa^s$, the half-form corrected prequantum Hilbert space H^{prQ}_s corresponding to the Kähler manifold $(N, \omega, J(s))$, $s \in S$ is the Hilbert space of L^2 sections of the bundle $L \otimes \kappa^s$, i.e.

$$H^{prQ}_s = \{f_s\vartheta \otimes \theta_s \mid f_s : N \to \mathbb{C}, \int_N |f_s|^2|\theta_s|^2\mathcal{L} < \infty\}.$$

Let

$$H^{prQ} := \cup^*_{s\in S} H^{prQ}_s \tag{6.2.2}$$

be the disjoint union of these Hilbert spaces. Except the natural projection map $p : H^{prQ} \to S$, for which each fiber $p^{-1}(s)$ is a Hilbert space, there is no further structure defined yet on the set H^{prQ}. This is an example that we call (cf. Definition 7.1.1) a field of Hilbert spaces, that is simply a a map $p\colon H \to S$ of sets, with each fiber $H_s = p^{-1}s$ endowed with the structure of a Hilbert space.

6.2.5 A unitary representation of $\mathcal{A}_+$

Keeping the notations of the previous section, let $L^2(N, \mathcal{E})$ be the L^2 space w.r.t. the Liouville volume form and $U(L^2(N, \mathcal{E}))$ the unitary self maps of this Hilbert space. Induced by the right action of $\mathcal{A}$ on N, we get the vector fields $\mathcal{X}, \mathcal{Y}$ on N, corresponding to $\dot\alpha(0)$ and $\dot\beta(0)$ (cf. 6.1.2).

Theorem 6.2.1. *With $\sigma \in \mathcal{A}_+$, the map*

$$\begin{array}{rcl} \rho(\sigma) : L^2(N, \mathcal{E}) & \to & L^2(N, \mathcal{E}) \\ f & \mapsto & \chi(\sigma)^{\frac{m}{2}} f \circ A_\sigma \end{array} \tag{6.2.3}$$

is unitary and yields a unitary representation

$$\rho : \mathcal{A}_+ \to U(L^2(N, \mathcal{E})).$$

The map

$$\varrho : \mathcal{A}_+ \times L^2(N, \mathcal{E}) \longrightarrow L^2(N, \mathcal{E}), \tag{6.2.4}$$

defined by $\varrho(\sigma, f) := \rho(\sigma)f$ *is continuous, but not differentiable.*

Proof. (6.1.6) implies

$$\int_N |\rho(\sigma)f|^2\mathcal{E} = \int_N \chi(\sigma)^m |f \circ A_\sigma|^2\mathcal{E} = \int_N A_\sigma^*(|f|^2\mathcal{E}) = \int_N |f|^2\mathcal{E}$$

hence $\rho(\sigma)$ in (6.2.3) is unitary. Also if $\sigma, \sigma' \in \mathcal{A}_+$,

$$\rho(\sigma\sigma')f = \chi(\sigma\sigma')^{\frac{m}{2}} f \circ A_{\sigma\sigma'} =$$

$$\chi(\sigma\sigma')^{\frac{m}{2}} f \circ (A_{\sigma'} \circ A_\sigma) = \chi(\sigma)^{\frac{m}{2}} (\rho(\sigma')f) \circ A_\sigma = \rho(\sigma)(\rho(\sigma')f)$$

shows that ρ is a representation.

We prove the continuity of ϱ in two steps.

First step: Let $g \in L^2(N, \mathcal{E})$ and $\sigma \in \mathcal{A}_+$ be fixed. Assume that g is continuous with compact support. We want to show that, if $f \in L^2(N, \mathcal{E})$ is close to g and $\sigma' \in \mathcal{A}_+$ is close to σ', then $\rho(\sigma')f$ is close to $\rho(\sigma)g$. In the rest of the proof norm always refers to the L^2 norm w.r.t. the volume form $\mathcal{E}$.

$$\|\rho(\sigma')f - \rho(\sigma)g\| \leq \|\rho(\sigma')f - \rho(\sigma')g\| + \|\rho(\sigma')g - \rho(\sigma)g\| =$$

$$= \|f - g\| + \|\rho(\sigma')g - \rho(\sigma)g\|,$$

since $\rho(\sigma')$ is unitary. We rewrite the second term.

$$\begin{aligned} C^2 := \|\rho(\sigma')g - \rho(\sigma)g\|^2 &= \int_N |\chi(\sigma')^{\frac{m}{2}} g \circ A_{\sigma'} - \chi(\sigma)^{\frac{m}{2}} g \circ A_\sigma|^2\mathcal{E} = \\ &= \int_N \chi(\sigma)^m (|\chi(\sigma'\sigma^{-1}))^{\frac{m}{2}} g \circ A_{\sigma^{-1}\sigma'} - g|^2 \circ A_\sigma)\mathcal{E} = \\ &= \int_N |(\chi(\sigma'\sigma^{-1}))^{\frac{m}{2}} g \circ A_{\sigma^{-1}\sigma'} - g|^2\mathcal{E}, \end{aligned} \tag{6.2.5}$$

because $\rho(\sigma)$ is also unitary. Thus the triangle inequality implies

$$C \leq \left(\int_N ((\chi(\sigma'\sigma^{-1}))^{\frac{m}{2}} - 1)^2 |g \circ A_{\sigma^{-1}\sigma'}|^2\mathcal{E} \right)^{\frac{1}{2}} + \left(\int_N |g \circ A_{\sigma^{-1}\sigma'} - g|^2\mathcal{E} \right)^{\frac{1}{2}}.$$

Denote the two terms here by I and II. Using the unitarity of $\rho(\sigma^{-1}\sigma')$ we get

$$I = \chi(\sigma\sigma'^{-1})^{\frac{m}{2}} \left|\chi(\sigma'\sigma^{-1})^{\frac{m}{2}} - 1\right| \|g\|$$

If σ' is close to σ, $\chi(\sigma'\sigma^{-1})$ is near to 1 and so I is close to zero. Also in this case $A_{\sigma^{-1}\sigma'}$ is close to the identity diffeomorphism of N. Because of our choice g is uniformly continuous on its support. Consequently II is also near to zero. All these imply the continuity of ϱ.

Second step: Let now $g \in L^2(N, \mathcal{E})$ be arbitrary. Choose a g_1 near g that is continuous with compact support. Let $f \in L^2(N, \mathcal{E})$. Then

$$\begin{aligned}\|\rho(\sigma')f - \rho(\sigma)g\| &\leq \|\rho(\sigma')f - \rho(\sigma')g\| + \|\rho(\sigma')g - \rho(\sigma')g_1\| + \\ &+ \|\rho(\sigma')g_1 - \rho(\sigma)g_1\| + \|\rho(\sigma)g_1 - \rho(\sigma)g\| \\ &= \|f - g\| + \|g - g_1\| + \|\rho(\sigma')g_1 - \rho(\sigma)g_1\| + \|g_1 - g\|\end{aligned}$$

and applying the first step to g_1 we get the continuity of ϱ at (σ, g).

To justify that ϱ is not differentiable, it suffices to show that its partial derivative with respect to the σ variable does not exist.

Consider the 1-parameter subgroups α, β in $\mathcal{A}_+$ from (6.1.2) and let $g \in L^2(N, \mathcal{E})$ be arbitrary. Then $\varrho(\alpha(s), g) = g \circ A_{\alpha(s)}$, $\varrho(\beta(s), g) = e^{\frac{sm}{2}} g \circ A_{\beta(s)}$. So we would get

$$\left.\frac{d}{ds}\right|_{s=0} \varrho(\alpha(s), g) = \mathcal{X}g, \quad \left.\frac{d}{ds}\right|_{s=0} \varrho(\beta(s), g) = \frac{m}{2} g + \mathcal{Y}g.$$

For a generic g neither $\mathcal{X}g$, nor $\mathcal{Y}g$ exists and even if it does it is not necessarily square integrable. $\square$

6.2.6 Nonequivalent smooth structures on H^{prQ}

Now the map

$$\begin{array}{ccc} \mathcal{L} := S \times L^2(N, \mathcal{E}) & \xrightarrow{A} & H^{prQ} \\ (s, f) & \longmapsto & \frac{f}{|\theta_s|} \vartheta \otimes \theta_s \end{array}$$

is a fiber preserving bijection and its restriction to each fiber is unitary. Therefore pushing forward $\mathcal{L}$ with A, equips H^{prQ} with a smooth (in fact) trivial Hilbert bundle structure. The canonical flat hermitian connection on $\mathcal{L}$ yields an orthogonal connection on H^{prQ}.

When M is a compact Lie group equipped with a biinvariant metric, this is the Hilbert bundle structure with hermitian connection chosen by C. Florentino, P. Matias, J. Mourão and J. Nunes in their papers [FMMN1, FMMN2], except that they do not consider the full parameter space S, only the positive imaginary axes.

But this is not the only possible natural way to equip H^{prQ} with a Hilbert bundle structure. Let $\psi_s = f_s \vartheta \otimes \theta_s \in H^{prQ}_s$. Then from Proposition 6.1.8 we get

$$\|\psi_s\|^2_{L^2} = \int_N |f_s|^2 |\theta_s|^2 \mathcal{E} = (\text{Im } s)^{\frac{m}{2}} \int_N |f_s|^2 (|\Theta| \circ A_{\sigma_s}) \mathcal{E} =$$

$$= (\text{Im } s)^{-\frac{m}{2}} \int_N A^*_{\sigma_s}(|f_s|^2 \circ A^{-1}_{\sigma_s} |\Theta|) \mathcal{E} = (\text{Im } s)^{-\frac{m}{2}} \int_N (|f_s|^2 \circ A^{-1}_{\sigma_s}) |\Theta| \mathcal{E}$$

Therefore the map

$$\begin{array}{ccc} H^{prQ} & \xrightarrow{B} & \mathcal{L} = S \times L^2(N, \mathcal{E}) \\ f_s \vartheta \otimes \theta_s & \longmapsto & (s, (\text{Im } s)^{-\frac{m}{4}} f_s \circ A^{-1}_{\sigma_s} \sqrt{|\Theta|}) \end{array}$$

is a fiber preserving bijection whose restriction to each fiber is unitary. By pulling back $\mathcal{L}$ with B, the Hilbert field $p : H^{prQ} \to S$ inherits another smooth (in fact trivial) Hilbert bundle structure. We claim that as a smooth bundle, this is different from the one we obtained with the help of the map A earlier.

Theorem 6.2.2. *The Hilbert bundle structures on $p : H^{prQ} \to S$ obtained by the maps A and B are isomorphic as continuous Hilbert bundles but their smooth structures are different.*

Proof. Using Proposition 6.1.8 we calculate the map $B \circ A : \mathcal{L} \to \mathcal{L}$ to be

$$(s, f) \mapsto (s, (\mathrm{Im}\ \mathrm{s})^{-\frac{m}{2}} f \circ A_{\sigma_s}^{-1}) = (s, \varrho((\sigma_s)^{-1}) f).$$

It follows from Theorem 6.2.1 that the fiberwise unitary map $B \circ A$ is a homeomorphism but it is not differentiable. □

In light of Theorem 6.2.2, the idea in section 6.2.1 to make H^{corr} a Hilbert subbundle of H^{prQ} doesn't work. This difficulty led us in [LSz14] to introduce the notion of smooth and analytic fields of Hilbert spaces generalizing Hilbert bundles with a hermitian connection. The next chapter treats these objects in details. In chapter 8 we show that direct images of holomorphic vector bundles under a nonproper map naturaly produce such objects. Finally in chapter 9 we return to the uniqueness problem of quantization using the family of adapted complex structures. The holomorphic submersion of Theorem 6.1.11 plays a fundamental role in translating things into a direct image problem that produces the fields of Hilbert spaces whose flatness means uniqueness does hold in quantization.

Chapter 7

Fields of Hilbert spaces

7.1 Hilbert bundles and fields of Hilbert spaces

7.1.1 Hilbert bundles

Since this notion is used rather liberally in the subject, it will be reviewed here to fix the terminology. Given Banach spaces X, Y over the reals and $U \subset X$ open, a map $f\colon U \to Y$ is C^1 if

$$df(x;\xi) = \lim_{t\to 0} \frac{f(x+t\xi) - f(x)}{t} \tag{7.1.1}$$

exists and defines a continuous function $U \times X \to Y$. If df is C^1 one says f is C^2, and so on. Smooth maps are the ones that are C^n for all n. A Banach manifold is a Hausdorff space M with an open cover $\mathfrak{U}$ and homeomorphisms φ_U of $U \in \mathfrak{U}$ on open subsets $V_U \subset X_U$ of Banach spaces; the compositions $\varphi_{U'} \circ \varphi_U^{-1}$ should be smooth where defined. C^n–maps between Banach manifolds M, M' are defined using the charts $\varphi_U, \varphi'_{U'}$. The set of C^n maps $M \to M'$ is denoted $C^n(M; M')$, and when $M' = \mathbb{C}$, simply $C^n(M)$, with $n = \infty$ corresponding to smooth maps.

A smooth (always complex) Hilbert bundle is a smooth map $p\colon H \to S$ of Banach manifolds, each fiber $p^{-1}s,\ s \in S$, endowed with the structure of a complex vector space; for each $s \in S$ there should exist a neighborhood $U \subset S$, a complex Hilbert space X, and a smooth map (local trivialization) $F\colon p^{-1}U \to X$, whose restriction to each fiber $p^{-1}t,\ t \in U$, is linear, and such that $p \times F : p^{-1}U \to U \times X$ is diffeomorphic. A subset $K \subset H$ is a subbundle if above U, X, and F can be chosen so that $F(K \cap p^{-1}t) = Y$ for every $t \in U$, where $Y \subset X$ is a closed subspace. In this case $K \to S$ inherits the structure of a Hilbert bundle. Smooth sections of a Hilbert bundle and the sum $H' \oplus H''$ of Hilbert bundles $H', H'' \to S$ are defined as in finite dimensions. The space of smooth sections is denoted $C^\infty(S, H)$.

A (smooth) Hermitian metric on a Hilbert bundle $H \to S$ is a function $h\colon H \oplus H \to \mathbb{C}$; it is required that the local trivializations $F\colon p^{-1}U \to X$ discussed above can be chosen so that $h(u,v) = \langle F(u), F(v)\rangle$ for $u, v \in p^{-1}t,\ t \in U$, where $\langle , \rangle$ stands for the inner product of X. Our convention is that $\langle , \rangle$ and so h are $\mathbb{C}$–linear in the first argument.

Let Vect S denote the Lie algebra of smooth complex vector fields on S. (In all that follows S will be finite dimensional, so we need not worry about how exactly vector fields are defined in infinite dimensions.) The action of $\xi \in \operatorname{Vect} S$ on Banach valued functions $f\colon U \to Y$, $U \subset S$ open, is denoted ξf. A connection ∇ on a Hilbert bundle $H \to S$ associates with every $\xi \in \operatorname{Vect} S$ a linear map $\nabla_\xi\colon C^\infty(S,H) \to C^\infty(S,H)$. It is required that for every local trivialization $F\colon p^{-1}U \to X$ there should exist a smooth map $A\colon \mathbb{C} \otimes TU \to \operatorname{End} X$, linear on the fibers $\mathbb{C} \otimes T_sU$, such that on U

$$F(\nabla_\xi \varphi) = \xi F(\varphi) + A(\xi)F(\varphi), \qquad \varphi \in C^\infty(S,H).$$

Here $\operatorname{End} X$ is the Banach space of continuous linear operators on X, endowed with the operator norm. Thus A is an $\operatorname{End} X$ valued form on U, the connection form of ∇ in the given local trivialization. The connection is flat, resp. projectively flat, if in some neighborhood of every $s \in S$ there is a trivialization in which the connection form is 0, resp. takes values in scalar operators. These are equivalent to requiring that the curvature operator $\nabla_\xi\nabla_\eta - \nabla_\eta\nabla_\xi - \nabla_{[\xi,\eta]}$ should be 0, resp. multiplication by a function $r(\xi,\eta) \in C^\infty(S)$. If H has a Hermitian metric h, ∇ is said to be Hermitian if

$$\xi h(\varphi,\psi) = h(\nabla_\xi\varphi,\psi) + h(\varphi,\nabla_{\overline{\xi}}\psi), \qquad \xi \in \operatorname{Vect} S, \quad \varphi,\psi \in C^\infty(S,H).$$

Holomorphic Hilbert bundles are defined analogously. When X, Y are complex Banach spaces, and $U \subset X$ is open, $f\colon U \to Y$ is holomorphic if $df(x;\xi)$ defined in (7.1.1) is not only continuous but also complex linear in $\xi \in X$. This implies $f \in C^\infty(U;Y)$. Given the notion of holomorphy, complex manifolds and holomorphic Hilbert bundles over them are defined as their smooth counterparts, except "smooth" is replaced by "holomorphic" throughout.

7.1.2 Fields of Hilbert spaces.

In most respects, Hilbert bundles behave very much like finite rank bundles. However, the type of direct images discussed in the Introduction are rarely Hilbert bundles, and even when they are, it is impossible to prove this directly. Fields of Hilbert spaces are looser structures that direct images are more likely to be. We proceed to define them and formulate the main results that connect these weaker structures with Hilbert bundles.

Definition 7.1.1. *A field of Hilbert spaces is a map $p\colon H \to S$ of sets, with each fiber $H_s = p^{-1}s$ endowed with the structure of a Hilbert space.*

This, of course, is such a weak notion that it borders the useless. Something that goes for it is that any direct image considered in the Introduction has this structure. We shall presently see variants of this notion, with more structure. For the time being, note that one can talk about sections of a field of Hilbert spaces: these are maps $\varphi\colon S \to H$ with $\varphi(s) \in H_s$. Sections constitute a module over the ring of all functions $S \to \mathbb{C}$ in an obvious way. The inner products on the fibers, taken together, define a function

$$h\colon H \oplus H \to \mathbb{C}, \qquad \text{where} \qquad H \oplus H = \coprod_{s\in S} H_s \oplus H_s.$$

If $v \in H$, we also write $h(v)$ for $h(v,v)$ (and we do likewise with Hermitian metrics on Hilbert bundles). By the restriction of $H \to S$ to a subset $U \subset S$ is meant the field $H|U = p^{-1}U \xrightarrow{p} U$ of Hilbert spaces.

Definition 7.1.2. *Let S be a smooth manifold. A smooth structure on a field $H \to S$ of Hilbert spaces is given by specifying a set Γ^∞ of sections of H, closed under addition and under multiplication by elements of $C^\infty(S)$, and linear operators $\nabla_\xi \colon \Gamma^\infty \to \Gamma^\infty$ for each $\xi \in \mathit{Vect}\, S$, such that for $\xi, \eta \in \mathit{Vect}\, S$, $f \in C^\infty(S)$, $\varphi, \psi \in \Gamma^\infty$*

$$\nabla_{\xi+\eta} = \nabla_\xi + \nabla_\eta,\ \nabla_{f\xi} = f\nabla_\xi,\ \nabla_\xi(f\varphi) = (\xi f)\varphi + f\nabla_\xi\varphi; \tag{7.1.2}$$

$$h(\varphi,\psi) \in C^\infty(S) \text{ and } \xi h(\varphi,\psi) = h(\nabla_\xi\varphi,\psi) + h(\varphi,\nabla_{\bar\xi}\psi); \tag{7.1.3}$$

$$\{\varphi(s) \colon \varphi \in \Gamma^\infty\} \subset H_s \text{ is dense, for all } s \in S. \tag{7.1.4}$$

The collection ∇ of the operators ∇_ξ is called a connection on H.—The analogous, but cruder notion of "continuous field of Hilbert spaces" was invented by Godement in 1951; and even earlier von Neumann introduced what now are called "measurable fields of Hilbert spaces", [Di, Go, vN2]. In addition to these, the definition above was motivated by a suggestion of Berndtsson, made in 2005 in an email, that the bundle–like objects that arise from direct images should be studied through a dense family of their sections, rather than through local trivializations.

For brevity, fields of Hilbert spaces (with a smooth structure) will be called (smooth) Hilbert fields. Fix a smooth Hilbert field $H \to S$. Henceforward S will always be assumed finite dimensional.

Lemma 7.1.3 (Lempert, Szőke [LSz14])**.** *If $\varphi, \psi \in \Gamma^\infty$ agree in a neighborhood of some $s \in S$, then so do $\nabla_\xi\varphi$ and $\nabla_\xi\psi$.*

Proof. Let $f \in C^\infty(S)$ be 0 near s and 1 in a neighborhood of supp $(\varphi - \psi)$. Then near s

$$\nabla_\xi\varphi - \nabla_\xi\psi = \nabla_\xi(f(\varphi-\psi)) = (\xi f)(\varphi-\psi) + f\nabla_\xi(\varphi-\psi) = 0.$$

□

For this reason, if $U \subset S$ is open, the Hilbert field $H|U \to U$ has a natural smooth structure given by $\Gamma^\infty|U = \{\varphi|U \colon \varphi \in \Gamma^\infty\}$ and ∇_U defined by restriction.

The curvature R of $H \to S$ is defined by

$$R(\xi,\eta)\varphi = (\nabla_\xi\nabla_\eta - \nabla_\eta\nabla_\xi - \nabla_{[\xi,\eta]})\varphi, \qquad \xi,\eta \in \mathrm{Vect}\, S,\ \varphi \in \Gamma^\infty,$$

and H is called flat if $R = 0$, i.e., $R(\xi,\eta)\varphi = 0$ for all ξ, η, φ.

Lemma 7.1.4. *(Lempert, Szőke [LSz14])*

(i) $R(\xi,\eta)\varphi(s)$ depends only on $\xi(s), \eta(s)$, and $\varphi(s)$, hence induces a densely defined operator on H_s, denoted $R(\xi(s),\eta(s))$.

(ii) The adjoint of $R(\xi(s),\eta(s))$ is an extension of $-R(\bar\xi(s),\bar\eta(s))$. In particular, the adjoint is densely defined, and so $R(\xi(s),\eta(s))$ is closable.

Proof. From its definition one checks that $R(\xi,\eta)$ is $C^\infty(S)$–bilinear in ξ,η. Any ξ that vanishes at s can be written $\sum f_j\xi_j$ with $f_j(s)=0$, whence $R(\xi,\eta)\varphi(s)=0$ follows if $\xi(s)=0$; and similarly if $\eta(s)=0$. This implies that as far as ξ and η are concerned, $R(\xi,\eta)\varphi(s)$ depends only on $\xi(s),\eta(s)$. Next apply (7.1.3) repeatedly, to obtain for $\varphi,\psi\in\Gamma^\infty$

$$0=(\xi\eta-\eta\xi-[\xi,\eta])h(\varphi,\psi)=h(R(\xi,\eta)\varphi,\psi)+h(\varphi,R(\overline{\xi},\overline{\eta})\psi). \tag{7.1.5}$$

By the density condition (7.1.4) the rest of (i) and also (ii) follow. □

Our main concern will be flat fields and bundles. The following is a key definition:

Definition 7.1.5. *A trivialization of a smooth Hilbert field $H\to S$ is a map $T\colon H\to V$, with V a Hilbert space, such that $T|H_s$ is unitary, $s\in S$, and for $\varphi\in\Gamma^\infty$, $\xi\in \mathit{Vect}\,S$*

$$T\varphi\in C^\infty(S;V)\quad \textit{and}\qquad T(\nabla_\xi\varphi)=\xi T\varphi. \tag{7.1.6}$$

If $H\to S$ has a trivialization, it is flat, but to prove the converse more needs to be assumed, namely that H is *analytic*.

7.1.3 Analytic Hilbert fields

Let $H\to S$ be a smooth Hilbert field over a (real) analytic manifold S. Write $\mathrm{Vect}^\omega S\subset \mathrm{Vect}\,S$ for the Lie algebra of analytic vector fields.

Definition 7.1.6. *(i) A section $\varphi\in\Gamma^\infty$ is analytic if for any compact $C\subset S$ and any finite set Ξ of vector fields, analytic in a neighborhood of C, there is an $\varepsilon>0$ such that*

$$\sup\frac{\varepsilon^n}{n!}\,h(\nabla_{\xi_n}\dots\nabla_{\xi_1}\varphi)(s)^{1/2}<\infty, \tag{7.1.7}$$

where the sup *is taken over $n=0,1,\dots,\xi_j\in\Xi$, and $s\in C$. The set of analytic sections is denoted $\Gamma^\omega\subset\Gamma^\infty$.*

(ii) $H\to S$ is an analytic Hilbert field if $\{\varphi(s)\colon\varphi\in\Gamma^\omega\}\subset H_s$ is dense for all $s\in S$.

If $H\to S$ is analytic and $U\subset S$ is open, then clearly $H|U$ is also analytic.

Theorem 7.1.7 (Lempert, Szőke [LSz14])**.** *Let $H\to S$ be an analytic Hilbert field over a connected base S.*

(i) If $T\colon H\to V$ and $T'\colon H\to V'$ are trivializations, then $T'=\tau T$ with a unitary $\tau\colon V\to V'$.

(ii) If S is simply connected and H is flat, then H has a trivialization.

Corollary 7.1.8 (Lempert, Szőke [LSz14])**.** *Let $H\to S$ be a flat analytic Hilbert field. Then there are a Hermitian Hilbert bundle $K\to S$ with a flat connection ∇^K and a map $F\colon H\to K$, unitary between the fibers H_s,K_s, such that for $\varphi\in\Gamma^\infty$ and $\xi\in\mathit{Vect}\,S$*

$$F\varphi\in C^\infty(S,K)\qquad\textit{and}\qquad F(\nabla_\xi\varphi)=\nabla^K_\xi F\varphi.$$

Moreover, if $K'\to S$ is another flat Hermitian Hilbert bundle and $F'\colon H\to K'$ is like F, then $F'\circ F^{-1}\colon K\to K'$ is a connection preserving isometric isomorphism.

The proof of the Corollary is left to the reader.

Proof of Theorem 7.1.7. (i)Let $\| \ \|$ denote the norm of V. Iterating (7.1.7) gives

$$T(\nabla_{\xi_n} \dots \nabla_{\xi_1} \varphi) = \xi_n \dots \xi_1 T\varphi. \tag{7.1.8}$$

This implies that $T\varphi\colon S \to V$ is analytic when $\varphi \in \Gamma^\omega$. Indeed, it can be assumed that $S \subset \mathbb{R}^d$ is open. Let $\Xi \subset \mathrm{Vect}^\omega S$ consist of coordinate vector fields $\partial_1, \dots, \partial_d$. If $C \subset S$ is compact, then by (7.1.7) and (7.1.8)

$$\sup \frac{\varepsilon^n}{n!} \|\xi_n \dots \xi_1 T\varphi\| < \infty,$$

the sup over $n = 0, 1, \dots, \xi_j \in \Xi,\ s \in C$, so that $T\varphi$ is analytic. Similarly, $T'\varphi$ is also analytic.

Now fix $s_0 \in S$ and define a unitary map $\tau = T'(T|H_{s_0})^{-1}\colon V \to V$. If $\varphi \in A$ and $\xi_1, \dots, \xi_n \in \mathrm{Vect}\, S$, then at s_0

$$\xi_n \dots \xi_1 T'\varphi = T'\nabla_{\xi_n} \dots \nabla_{\xi_1}\varphi = \tau T \nabla_{\xi_n} \dots \nabla_{\xi_1}\varphi = \xi_n \dots \xi_1 \tau T\varphi.$$

Since the derivatives of $\tau T\varphi$ and $T'\varphi$ agree at s_0, $\tau T\varphi = T'\varphi$ everywhere. By density $\tau T = T'$ then follows. □

The proof of the existence part is harder, and the details will take up sections 7.2, 7.3, and 7.4. For the time being we note that in Theorem 7.1.7 the analyticity assumption cannot be relaxed to mere smoothness. The following example emerged in a conversation with Larry Brown.

Example 7.1.9 (Lempert, Szőke [LSz14])**.** *There is a flat smooth Hilbert field $H \to \mathbb{R}^d$ that cannot be trivialized.*

Indeed, let $U \subset \mathbb{R}^d$ be open, X a positive dimensional Hilbert space, and $H_s = X$ if $s \in U$, $H_s = \{0\}$ if $s \in \mathbb{R}^d \setminus U$. Then $H = \coprod_{s \in \mathbb{R}^d} H_s \to \mathbb{R}^d$ is a Hilbert field, whose sections can be identified with functions $\varphi\colon \mathbb{R}^d \to X$, vanishing outside U. Let

$$\Gamma^\infty = \{\varphi \in C^\infty(\mathbb{R}^d; X)\colon\ \mathrm{supp}\ \varphi \subset U\},$$

and $\nabla_\xi \varphi = \xi\varphi$. This defines a smooth structure on H, which is flat but cannot be trivialized unless $U = \emptyset$ or $\mathbb{R}^d$.

The example is not as artificial as it may seem. Hilbert fields like it do arise as direct images of holomorphic vector bundles under improper submersions, see 8.3.3.

7.1.4 Projective flatness

A smooth Hilbert field $H \to S$ with positive dimensional fibers is called projectively flat if the curvature operator $R(\xi, \eta)\colon \Gamma^\infty \to \Gamma^\infty$ is multiplication by a function $r(\xi, \eta)\colon S \to \mathbb{C}$. In this case one also says that the curvature is central.

The function $r(\xi, \eta)$ is necessarily smooth, because for $\varphi, \psi \in \Gamma^\infty$

$$r(\xi, \eta) h(\varphi, \psi) = h(R(\xi, \eta)\varphi, \psi) \in C^\infty(S).$$

It is pure imaginary when ξ, η are real, since $R(\xi, \eta)$ is skew–symmetric (Lemma 7.1.4). Like $R(\xi(s), \eta(s))$, $r(\xi, \eta)(s)$ depends only on $\xi(s)$ and $\eta(s)$. Hence r is a 2–form, in fact a closed 2–form, as one computes directly from the definitions (the point is that R satisfies the Bianchi identity).

As with bundles, a simple twisting will reduce projectively flat smooth Hilbert fields $H \to S$ to flat ones. Suppose r is not only closed but exact. There is a smoothly trivial Hermitian line bundle $L \to S$ with Hermitian connection ∇^L whose curvature is $-r$, see e.g. [W, Proposition (8.3.1)]. The twisted Hilbert field

$$L \otimes H = \coprod_{s \in S} L_s \otimes H_s \to S$$

has an obvious smooth structure given by $\Gamma^\infty_{L \otimes H} = \{\lambda \otimes \varphi \colon \lambda \in C^\infty(S, L), \varphi \in \Gamma^\infty\}$,

$$\nabla^{L \otimes H}_\xi (\lambda \otimes \varphi) = (\nabla^L_\xi \lambda) \otimes \varphi + \lambda \otimes \nabla_\xi \varphi;$$

and one computes that $L \otimes H$ has zero curvature. If H is analytic, so will be $L \otimes H$.

Definition 7.1.10. *A projective trivialization of a smooth Hilbert field $H \to S$ is a map $T \colon H \to V$, with V a Hilbert space, such that $T|H_s$ is unitary for $s \in S$, and with some 1–form a on S, for all $\varphi \in \Gamma^\infty$, $\xi \in \mathit{Vect}\, S$*

$$T\varphi \in C^\infty(S; V), \qquad T(\nabla_\xi \varphi) = \xi T\varphi + a(\xi) T\varphi.$$

If H has a projective trivialization, then it is projectively flat, its curvature $R(\xi, \eta)$ being multiplication by $da(\xi, \eta)$. Further, if T is a projective trivialization, then $T' = f \cdot T$ will be another one, with any $f \in C^\infty(S)$, $|f| \equiv 1$. The corresponding 1–form is $a' = a - df/f$.

In view of the above twisting construction, one can deduce from Theorem 7.1.7:

Theorem 7.1.11 (Lempert, Szőke [LSz14])**.** *Let $H \to S$ be an analytic Hilbert field over a connected base S.*

(i) If $T \colon H \to V$ and $T' \colon H \to V'$ are projective trivializations, then $T' = f \cdot (\tau T)$, with $f \in C^\infty(S)$ and $\tau \colon V \to V'$ unitary.

(ii) Suppose the curvature $R(\xi, \eta)$ of H is multiplication by $r(\xi, \eta)$, and r is exact. If S is simply connected, then H has a projective trivialization.

The significance of Theorems 7.1.7 and 7.1.11 for the uniqueness problem is the following. Suppose $H \to S$ is a (projectively) flat analytic Hilbert field, S is connected and simply connected (and $H^2(S, \mathbb{R}) = 0$). Then the trivializations in Theorem 7.1.7, resp. 7.1.11, provide a way to identify the fibers of H canonically (resp. canonically up to a scalar factor).

Theorem 7.1.11 in turn implies

Corollary 7.1.12 (Lempert, Szőke [LSz14])**.** *Let $H \to S$ be a projectively flat analytic Hilbert field. There are a Hermitian Hilbert bundle $K \to S$ with a projectively flat connection ∇^K and a fibered map $F \colon H \to K$, fiberwise unitary, such that for $\varphi \in \Gamma^\infty$ and $\xi \in \mathit{Vect}\, S$*

$$F\varphi \in C^\infty(S, K), \qquad F(\nabla_\xi \varphi) = \nabla^K_\xi F\varphi.$$

Moreover, if $K' \to S$ and $F' \colon H \to K'$ are like K and F, then $F' \circ F^{-1} \colon K \to K'$ is a connection preserving isometric isomorphism.

7.2 Fundamentals of analysis in Hilbert fields

Fix a smooth Hilbert field $H \to S$.

7.2.1 Completion

Let $U \subset S$ be open and φ_j a sequence of sections of $H|U$. We say that φ_j converges to a section φ (almost) everywhere or (locally) uniformly if $h(\varphi_j - \varphi) \to 0$ in the corresponding sense. The following is obvious:

Lemma 7.2.1 (Lempert, Szőke [LSz14])**.** *If $\varphi_j \to \varphi$ and $\psi_j \to \psi$ in any of the four senses indicated, then $\varphi_j + \psi_j \to \varphi + \psi$ and $h(\varphi_j, \psi_j) \to h(\varphi, \psi)$.*

Denote by $\Gamma^0(U)$ the $C(U)$–module of those sections φ of $H|U$ that are locally uniform limits on U of $\varphi_j \in \Gamma^\infty$. Further, denote by $\Gamma^1(U)$ the $C^1(U)$–submodule of those $\varphi \in \Gamma^0(U)$ for which there are $\varphi_j \in \Gamma^\infty$ such that $\varphi_j|U \to \varphi$ locally uniformly, and for every $\xi \in \operatorname{Vect} U$

$$\nabla_\xi \varphi_j|U \text{ converges locally uniformly.} \tag{7.2.1}$$

Clearly, it suffices to require (7.2.1) for ξ in a family $\Xi \subset \operatorname{Vect} S$ that spans $\mathbb{C} \otimes TU$.

Lemma 7.2.2 (Lempert, Szőke [LSz14])**.** *The limit in* (7.2.1) *depends only on φ, not on φ_j.*

Proof. With $\psi \in \Gamma^\infty$

$$\xi h(\varphi_j, \psi) = h(\nabla_\xi \varphi_j, \psi) + h(\varphi_j, \nabla_{\overline{\xi}} \psi).$$

As $j \to \infty$, the right side tends to a continuous limit, locally uniformly on U, therefore so does the left hand side. It follows that $h(\varphi, \psi) \in C^1(U)$ and

$$\xi h(\varphi, \psi) = \lim h(\nabla_\xi \varphi_j, \psi) + h(\varphi, \nabla_{\overline{\xi}} \psi).$$

Hence the limit here is independent of φ_j, and the density assumption (7.1.4) implies the claim. □

If $\varphi \in \Gamma^1(U)$ and φ_j are as above, put $\nabla^U_\xi \varphi = \lim \nabla_\xi \varphi_j|U \in \Gamma^0(U)$. The operator $\nabla^U_\xi : \Gamma^1(U) \to \Gamma^0(U)$ has the properties described in (7.1.2) and (7.1.3) (except that only $h(\varphi, \psi) \in C^1(U)$ is guaranteed for $\varphi, \psi \in \Gamma^1(U)$). In what follows, we will drop the superscript U and just write $\nabla_\xi \colon \Gamma^1(U) \to \Gamma^0(U)$.

The $C^n(U)$–modules $\Gamma^n(U)$ for $n \in \mathbb{N}$ can now be defined inductively: $\varphi \in \Gamma^n(U)$ if $\varphi, \nabla_\xi \varphi \in \Gamma^{n-1}(U)$ for all $\xi \in \operatorname{Vect} U$. The $C^\infty(U)$–module $\Gamma^\infty(U) = \bigcap_n \Gamma^n(U) \supset \Gamma^\infty|U$ together with $\nabla|\Gamma^\infty(U)$ define a smooth structure on the Hilbert field $H|U$. Given $\xi_1, \xi_2, \ldots \in \operatorname{Vect} S$ and a compact $C \subset U$,

$$\|\varphi\|_{C,\xi_1,\ldots,\xi_m} = \max_C h(\nabla_{\xi_m} \ldots \nabla_{\xi_1} \varphi)^{1/2}$$

is a seminorm on $\Gamma^\infty(U)$ and $\Gamma^n(U)$, provided $m \le n$. These seminorms turn $\Gamma^\infty(U)$ and $\Gamma^n(U)$ into locally convex topological vector spaces. The spaces are in fact Fréchet, because countably many seminorms suffice to define the topology, and because they will be complete, as one shows by a simple diagonal argument for $n = 1$ and by induction for $n > 1$. The operation of $C^\infty(U)$, $C^n(U)$ on $\Gamma^\infty(U)$, $\Gamma^n(U)$, given by $(f, \varphi) \mapsto f\varphi$ is continuous, so these spaces are continuous modules.

7.2.2 Sobolev norms

Fix a smooth volume form λ on S and a finite $\Xi \subset \operatorname{Vect} S$ that spans the tangent bundle of S. If $\varphi \in \Gamma^n(S)$, put

$$\|\varphi\|_n^2 = \sum \int_S h(\nabla_{\xi_m} \dots \nabla_{\xi_1} \varphi)\lambda \le \infty, \tag{7.2.2}$$

where the sum is over $0 \le m \le n$ and $\xi_j \in \Xi$. The Sobolev "norm" $|| \ ||_n$ depends on the choice of λ and Ξ, but if a compact $C \subset S$ is fixed, for sections supported in C different choices lead to equivalent norms.

Lemma 7.2.3 (Lempert, Szőke [LSz14])**.** *Given a compact $C \subset S$, there is a constant α such that with $d = \dim S$ and $\varphi \in \Gamma^d(S)$*

$$\max_C h(\varphi) \le \alpha \|\varphi\|_d^2.$$

This is weaker than the usual Sobolev inequality, where d could be replaced by any $n > d/2$, but it is still useful.

Proof. A partition of unity will reduce to the case when $S = \mathbb{R}^d$, $\lambda = dx_1 \wedge dx_2 \wedge \dots$, Ξ consists of $\xi_j = \partial/\partial x_j$, $j = 1, \dots, d$, and φ is compactly supported. Since $f(x) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \dots (\xi_d \dots \xi_1 f)\lambda$ for compactly supported $f \in C^d(S)$,

$$\sup_S |f| \le \int_S |\xi_d \dots \xi_1 f|\lambda.$$

Putting $f = h(\varphi)$ and repeatedly using Leibniz's rule (7.1.3), the Lemma follows. □

7.2.3 Analyticity

This subsection revolves around the notion of uniform analyticity. Consider a smooth Hilbert field $H \to S$ over an analytic base S.

Definition 7.2.4. *Let $C \subset S$ be compact and F and A families of functions, resp. sections of H, each smooth in a neighborhood of C. Then F, resp. A, is uniformly analytic on C if, given a finite family Ξ of vector fields, analytic in a neighborhood of C, there is an $\varepsilon > 0$ such that for $f \in F$, resp. $\varphi \in A$,*

$$\sup \frac{\varepsilon^n}{n!} |\xi_n \dots \xi_1 f(s)| < \infty, \quad \textit{resp.} \ \sup \frac{\varepsilon^n}{n!} h(\nabla_{\xi_n} \dots \nabla_{\xi_1} \varphi)(s)^{1/2} < \infty, \tag{7.2.3}$$

the sup *over $n = 0, 1, \dots$, $\xi_j \in \Xi$, and $s \in C$. A family $A \subset \Gamma^\omega$ is uniformly analytic if it is uniformly analytic on every compact $C \subset S$.*

Lemma 7.2.5 (Lempert, Szőke [LSz14])**.** *Let F be a family of functions analytic in a neighborhood of a compact $C \subset S$.*

(i) If F is finite, then it is uniformly analytic on C.

(ii) If F is uniformly analytic on C, $F' \subset F$ is finite, and Ξ is a finite family

of vector fields, analytic in a neighborhood of C, then there are constants a, depending only on F, and b, depending only on F', such that for $f_j \in F'$, $\xi_j \in \Xi$

$$\max_C |\xi_n \dots \xi_1(f_m \cdots f_1)| \le n! a^n b^m.$$

In particular, polynomials of elements of F also form a uniformly analytic family on C.

Proof. (i) It suffices to prove for $F = \{f\}$ a singleton. If on a neighborhood of C there are analytic coordinates $x_1, \dots, x_d$ and $\Xi = \{\partial/\partial x_j : j = 1, \dots, d\}$, then (7.2.3) is the definition of analyticity of f. If the vector fields in Ξ are linear combinations of $\partial/\partial x_j$ with analytic coefficients, then (7.2.3) follows from [Ne, Theorem 2 and Corollary 3.1]. Indeed, by Theorem 2 the family $\{\partial/\partial x_j : j = 1, \dots, d\}$ "analytically dominates" Ξ; when this is fed into Corollary 3.1, the conclusion becomes the first estimate in (7.2.3). Finally, an arbitrary C is the union of finitely many C_i, each contained in a coordinate neighborhood, so that F is indeed uniformly analytic.

(ii) By assumption there is a $\beta > 0$, depending only on F', such that

$$\max_C |\xi_n \dots \xi_1 f| \le n! \varepsilon^{-n} \beta \qquad \text{for } f \in F'. \tag{7.2.4}$$

Introduce the following notation for $I = \{i_1 < i_2 < \dots < i_k\}$:

$$\xi_I = \xi_{i_k} \dots \xi_{i_1}, \qquad \nabla_I = \nabla_{\xi_{i_k}} \dots \nabla_{\xi_{i_1}}, \qquad f^I = f_{i_k} \cdots f_{i_1}. \tag{7.2.5}$$

Then $\xi_n \dots \xi_1(f_m \cdots f_1) = \sum (\xi_{J_m} f_m)(\xi_{J_{m-1}} f_{m-1}) \cdots (\xi_{J_1} f_1)$, the sum taken over all partitions $J_1 \sqcup \dots \sqcup J_m = \{1, \dots, n\}$. By (7.2.4)

$$\max_C |\xi_n \cdots \xi_1(f_m \cdots f_1)| \le \sum \varepsilon^{-|J_m| - \dots - |J_1|} \beta^m |J_m|! \cdots |J_1|! =$$

$$\sum_{k_1 + \dots + k_m = n} \varepsilon^{-n} \beta^m k_m! \cdots k_1! \, \frac{n!}{k_1!} \cdots k_m!,$$

where the multinomial coefficient counts the number of partitions with $|J_i| = k_i \ge 0$. There are

$$\binom{n+m-1}{m-1} \le 2^{n+m}$$

terms in the last sum, which then is $\le n!(2/\varepsilon)^n (2\beta)^m$. □

Lemma 7.2.6 (Lempert, Szőke [LSz14])**.** *Let $C \subset S$ be compact, F a family of functions, uniformly analytic on C, Ξ a finite set of vector fields, analytic in a neighborhood of C, and Z a finite set of linear combinations of elements of Ξ, with analytic coefficients. Suppose $\varepsilon > 0$, $A \subset \Gamma^\infty$, and for every $\varphi \in A$*

$$\sup \frac{\varepsilon^n}{n!} \, h(\nabla_{\xi_n} \dots \nabla_{\xi_1} \varphi)(s)^{1/2} < \infty, \tag{7.2.6}$$

the sup *taken over $n = 0, 1, \dots, \xi_i \in \Xi$, and $s \in C$. Then there is a $\delta > 0$ such that for every ψ in the vector space spanned by $f \nabla_{\eta_m} \dots \nabla_{\eta_1} \varphi$, where $f \in F$, $m = 0, 1, \dots, \eta_j \in \Xi$, and $\varphi \in A$,*

$$\sup \frac{\delta^n}{n!} \, h(\nabla_{\zeta_n} \dots \nabla_{\zeta_1} \psi)(s)^{1/2} < \infty, \tag{7.2.7}$$

the sup *taken over $n = 0, 1, \dots, \zeta_j \in Z$, and $s \in C$.*

Proof. First, assume that each $\zeta_j \in \Xi$. It suffices to deal with ψ of form $\psi = f\varphi$, where $f \in F$ and $\varphi \in A$, because $\varphi' = \nabla_{\eta_m} \dots \nabla_{\eta_1}\varphi$ also satisfies (7.2.6) with ε replaced by any $\varepsilon' < \varepsilon$. Using notation (7.2.5)

$$\nabla_{\zeta_n} \dots \nabla_{\zeta_1}(f\varphi) = \sum (\zeta_I f)\nabla_J \varphi,$$

the sum is over partitions $I \sqcup J = \{1, \dots, n\}$. It can be assumed that the ε's in the first estimate in (7.2.3) and in (7.2.6) are the same. Denoting by α a number that dominates both suprema, on C

$$\begin{aligned} h(\nabla_{\zeta_n} \dots \nabla_{\zeta_1}(f\varphi))^{1/2} &\le \sum_{I,J} \alpha\varepsilon^{-|I|}|I|!\alpha\varepsilon^{-|J|}|J|! = \sum_{k=0}^{n} \alpha^2\varepsilon^{-n}k!(n-k)!\binom{n}{k} \\ &\le (n+1)!\alpha^2\varepsilon^{-n}. \end{aligned}$$

It follows that (7.2.7) holds with $\delta = \varepsilon/2$.

Second, assume that $\psi = \varphi \in A$. There is a finite family F' of functions analytic in a neighborhood of C such that each $\zeta \in Z$ is a sum of vector fields of form $f\xi$, $f \in F'$, $\xi \in \Xi$. It suffices to check (7.2.7) when the ζ_j are of form $f_j\xi_j$, $f_j \in F$, $\xi_j \in \Xi$. We prove by induction that

$$\nabla_{f_n\xi_n} \dots \nabla_{f_1\xi_1}\varphi = \sum f^{I_k}(\xi_{J_k} f^{I_{k-1}}) \cdots (\xi_{J_2} f^{I_1})\nabla_{J_1}\varphi, \tag{7.2.8}$$

where in the sum $\coprod_1^k I_i = \coprod_1^k J_j = \{1, \dots, n\}; J_j \neq \emptyset$; and each partition $\coprod J_j$ occurs at most once. Suppose this is true for $n-1$, i.e.,

$$\nabla_{f_{n-1}\xi_{n-1}} \dots \nabla_{f_1\xi_1}\varphi = \sum f^{I_k}(\xi_{J_k} f^{I_{k-1}}) \cdots \nabla_{J_1}\varphi. \tag{7.2.9}$$

Applying $\nabla_{f_n\xi_n} = f_n\nabla_{\xi_n}$, each term on the right gives rise to

$$\begin{aligned} &f_n\xi_n f^{I_k}(\xi_{J_k} f^{I_{k-1}}) \cdots \nabla_{J_1}\varphi + f^{nI_k}(\xi_{nJ_k} f^{I_{k-1}}) \cdots \nabla_{J_1}\varphi + \\ &+ f^{nI_k}(\xi_{J_k} f^{I_{k-1}})(\xi_{nJ_{k-1}} f^{I_{k-2}}) \cdots \nabla_{J_1}\varphi + \dots \\ &+ f^{nI_k}(\xi_{J_k} f^{I_{k-1}}) \cdots (\xi_{J_2} f^{I_1})\nabla_{nJ_1}\varphi, \end{aligned}$$

where nI and nJ stand for $\{n\} \cup I$ and $\{n\} \cup J$. Every term here is indeed of form $f^{I'_l}(\xi_{J'_l} f^{I'_{l-1}}) \cdots \nabla_{J'_1}\varphi$, the $J'_j \neq \emptyset$ partition $\{1, \dots, n\}$, and in (7.2.3) no partition is repeated. Moreover, knowing $\{J'_l, \dots, J'_1\}$, the unique $\{J_k, \dots, J_1\}$ in (7.2.9) can be located that gave rise to it. Thus (7.2.8) is verified.

Choose a, b as in Lemma 7.2.5, and let α denote the supremum in (7.2.6). It can be assumed that $\varepsilon a = 1$. If in (7.2.8) the partitions $\coprod J_j$ are grouped according to the cardinalities $|J_j| = n_j > 0$, each group will contain at most $n!/(n_1! \cdots n_k!)$ partitions. Hence

$$\begin{aligned} h(\nabla_{f_n\xi_n} \dots \nabla_{f_1\xi_1}\varphi)^{1/2} &\le \sum a^{|J_k|+\dots+|J_2|} b^{|I_k|+\dots+|I_1|} \alpha\varepsilon^{-|J_1|}|J_k|! \cdots |J_1|! \\ &= \alpha \sum_{n_1+n_2+\dots=n} a^n b^n n_1! \cdots n_k! \, \frac{n!}{n_1!} \cdots n_k!. \end{aligned}$$

The last sum has 2^{n-1} terms, which means that $\delta = 1/(2ab)$ satisfies (7.2.7).

Thus (7.2.7) has been proved in two special cases. By combining the two, the Lemma is obtained in general. □

Corollary 7.2.7 (Lempert, Szőke [LSz14]). *To prove that $A \subset \Gamma^\infty$ is uniformly analytic on C, it suffices to check Definition 7.2.4 for a single $\Xi \subset \mathit{Vect}^\omega S$, as long as Ξ spans $\mathbb{C} \otimes TS$.*

The next result will not be needed until section 8.4. Briefly, it says that an analytic Hilbert bundle with an analytic connection gives rise to an analytic Hilbert field; and the same for Banach bundles and Banach fields. Let $(\mathfrak{B}, \| \ \|)$ be a Banach space and $A\colon \mathbb{C} \otimes TS \to \operatorname{End} \mathfrak{B}$ an analytic map, linear on each $\mathbb{C} \otimes T_s S$. Thus A is a connection form, and determines a connection D on functions $f \in C^\infty(S; \mathfrak{B})$:

$$D_\xi f = \xi f + A(\xi) f \in C^\infty(S; \mathfrak{B}), \qquad \xi \in \operatorname{Vect} S.$$

In other words, D is a connection on the trivial bundle $S \times \mathfrak{B} \to S$.

Lemma 7.2.8 (Lempert, Szőke [LSz14]). *Given a finite $\Xi \subset \mathit{Vect}^\omega S$, a compact $C \subset S$, and an analytic $f\colon S \to \mathfrak{B}$, there is an $\varepsilon > 0$ such that*

$$\sup \ \frac{\varepsilon^n}{n!} \ \|D_{\xi_n} \dots D_{\xi_1} f(s)\| < \infty, \tag{7.2.10}$$

the sup taken over $n = 0, 1, \dots, \xi_j \in \Xi$, and $s \in C$.

Proof. First consider the complex version, where S is a complex manifold, A and f are holomorphic, and Ξ consists of holomorphic vector fields of type $(1,0)$. Since the issue is local, S can be taken an open subset of $\mathbb{C}^d$, and it can be assumed without losing generality that each $\xi \in \Xi$ has length < 1. There are a $\delta_0 > 0$ and a neighborhood $U \subset S$ of C such that each vector field $\xi \in \Xi$ has a flow $g^t_\xi = g^t$ defined on U, for complex time t, $|t| < \delta_0$. This means that g^t maps U biholomorphically into $\mathbb{C}^d$, $g^t s$ depends holomorphically on (s,t), $g^0 = \mathrm{id}_U$, and $\partial g^t s / \partial t = \xi(g^t s)$ (in particular, ξ is holomorphic on $g^t U$). Next define holomorphic functions $P^t_\xi = P^t\colon U \to \operatorname{End} \mathfrak{B}$, $|t| < \delta_0$, by the initial value problem

$$\partial P^t(s)/\partial t = P^t(s) A(\xi(g^t s)), \qquad P^0(s) = \mathrm{id}_{\mathfrak{B}}, \ s \in U.$$

Then $P^t(s)$ is holomorphic in (s,t), and for $f \in C^\infty(U; \mathfrak{B})$

$$\partial\big(P^t(s) f(g^t s)\big)/\partial t = (\partial P^t(s)/\partial t) \ f(g^t s) + P^t(s)(\xi f)(g^t s) = D_\xi f(s),$$

when $t = 0$. Using this with $\xi = \xi_j \in \Xi$ and iterating, for $s \in U$

$$D_{\xi_n} \dots D_{\xi_1} f(s) = \left. \frac{\partial^n P^{t_1 \dots t_n}(s) f(g^{t_1}_{\xi_1} \dots g^{t_n}_{\xi_n})}{\partial t_1 \dots \partial t_n} \right|_{t_1 = \dots = t_n = 0}, \tag{7.2.11}$$

where

$$P^{t_1 \dots t_n}(s) = P^{t_n}_{\xi_n}(s) P^{t_{n-1}}_{\xi_{n-1}}(g^{t_n}_{\xi_n} s) \dots P^{t_1}_{\xi_1}(g^{t_2}_{\xi_2} \dots g^{t_n}_{\xi_n} s). \tag{7.2.12}$$

Choose a positive $\delta < \min\{\delta_0, \ \operatorname{dist}(C, \partial U)\}$. Since each ξ_j has length < 1, it follows by induction that if $|t_1| + \dots + |t_n| \le \delta$, then $g^{t_n}_{\xi_1} \dots g^{t_n}_{\xi_n}(C)$ is inside the $(|t_1| + \dots + |t_n|)$–neighborhood of C, in particular, inside U. Choose $a > 0$ so that

$$\|f(s)\|, \ \|P^t_\xi(s)\|_{\operatorname{End} \mathfrak{B}} < a, \qquad \text{when } \xi \in \Xi, \ |t| \le \delta,$$

and s is in the δ–neighborhood of C. Then $\|P^{t_1\dots t_n}(s)f(g_{\xi_1}^{t_1}\dots g_{\xi_n}^{t_n}s)\| < a^{n+1}$ for $s \in C$ and $|t_j| \le \delta/n$, in view of (7.2.12). By Cauchy's estimate (7.2.11) indeed implies

$$\|D_{\xi_n}\dots D_{\xi_1}f(s)\| \le a^{n+1}(n/\delta)^n \le n!a(ae^2/\delta)^n.$$

The lemma, as stated for real analytic objects, follows from the complex analytic version by passing to a complexification of S and extending to it A, f, and $\xi \in \Xi$ holomorphically. □

7.3 Horizontal sections in Hilbert fields

The trivialization claimed in Theorem 7.1.7(ii) depends on the existence of a large supply of horizontal sections, whose properties we will investigate in this section.

7.3.1 Horizontal sections

Let $p : H \to S$ be a smooth Hilbert field. If $U \subset S$ is open, a section $\varphi \in \Gamma^1(U)$ satisfying $\nabla_\xi\varphi = 0$ for all $\xi \in \operatorname{Vect} U$ is called horizontal. A horizontal section is automatically in $\Gamma^\infty(U)$. Of course, it suffices to verify $\nabla_\xi\varphi = 0$ for a family of ξ's that span each tangent space T_sU.

Lemma 7.3.1 (Lempert, Szőke [LSz14])**.** *If U is connected and $\varphi, \psi \in \Gamma^\infty(U)$ are horizontal, then $h(\varphi, \psi)$ is constant.*

Proof. Indeed, $\xi h(\varphi, \psi) = h(\nabla_\xi\varphi, \psi) + h(\varphi, \nabla_{\overline{\xi}}\psi) = 0$. □

Lemma 7.3.2 (Lempert, Szőke [LSz14])**.** *Given $s \in S$, the set*

$$\{\theta(s) \,:\, \theta \in \Gamma^\infty(S) \text{ is horizontal}\}$$

is closed in H_s.

Proof. We can assume S connected. If $\theta_j \in \Gamma^\infty(S)$ are horizontal for $j = 1, 2, \dots$, and $\theta_j(s) \to v \in H_s$, then Lemma 7.3.1 implies θ_j is a Cauchy sequence in $\Gamma^0(S)$, hence by horizontality also in $\Gamma^\infty(S)$. The limit $\theta \in \Gamma^\infty(S)$ is clearly horizontal, and $\theta(s) = v$. □

Lemma 7.3.3 (Lempert, Szőke [LSz14])**.** *Suppose S is simply connected and each $s \in S$ has a neighborhood U_s such that through every $v \in H|U_s$ there passes a horizontal section of $H|U_s$. Then through every $v \in H$ there passes a horizontal section of H.*

Proof. Consider open subsets $U \subset S$ and horizontal $\theta \in \Gamma^\infty(U)$. The sets $\theta(U) \subset H$ for all such pairs (U, θ) form a basis of a topology on H, and with this topology $p\colon H \to S$ is a covering map.

Indeed, the sets $\theta(U)$ cover H by assumption. Further, if $v \in \theta'(U') \cap \theta''(U'')$, and $V \subset U' \cap U''$ is a connected neighborhood of pv, then Lemma 7.3.1 implies $h(\theta'|V - \theta''|V)$ is constant, hence 0. Therefore $v \in \theta'(V) \subset \theta'(U') \cap \theta''(U'')$; this is all that is needed for the collection $\theta(U)$ to be a basis of a topology. Next with any connected U_s as in the assumption let

$$W = \{\theta \in \Gamma^\infty(U_s)\colon \theta \text{ is horizontal}\},$$

endowed with the discrete topology. Using Lemma 7.3.1 and the assumption one checks that the map

$$U_s \times W \ni (t, \theta) \mapsto \theta(t) \in H|U_s$$

is a homeomorphism. Thus p is a covering map.

But the covering $p\colon H \to S$ is trivial, because S is simply connected. Since sections of p are the same as horizontal $\theta \in \Gamma^\infty(S)$, the lemma follows. □

7.3.2 Flat analytic Hilbert fields

Now let $H \to S$ be a flat analytic Hilbert field over a simply connected base.

Lemma 7.3.4 (Lempert, Szőke [LSz14])**.** *Through every $v \in H$ there passes a horizontal section of H.*

The proof to be given is rather simpler than the one in the first version of the paper. The simplification was inspired by an idea of Dat Tran, who proposed formula (7.3.2) below, when $\dim S = 1$, to construct horizontal sections out of analytic sections.

Proof. Assume first that there is a uniformly analytic subspace $A \subset \Gamma^\infty$ that is dense in Γ^∞ in the topology of $\Gamma^\infty(S)$. Fix a relatively compact open $U \subset S$ so that analytic coordinates $x_1, \dots, x_d$ exist in a neighborhood of $\overline{U}$ and

$$U = \{s \in S\colon |x_j(s)| < 1,\ j = 1, \dots, d\}.$$

Set $\Xi = \{\eta_j = \partial/\partial x_j,\ j = 1, \dots, d\}$. As A is uniformly analytic, there is an $\varepsilon > 0$ such that for $\varphi \in A$

$$\sup\ \frac{\varepsilon^n}{n!} \max_{\overline{U}} h(\nabla_{\xi_n} \dots \nabla_{\xi_1} \varphi)^{1/2} < \infty, \tag{7.3.1}$$

the sup over $n = 0, 1, \dots$ and $\xi_j \in \Xi$. We will use multiindex notation: if $I = (i_1, \dots, i_d)$ is a nonnegative multiindex, and $y = (y_1, \dots, y_d)$, then

$$|I| = i_1 + \ldots + i_d, \quad I! = i_1! \cdots i_d!, \quad y^I = y_1^{i_1} \cdots y_d^{i_d}, \quad \nabla^I = \nabla_{\eta_1}^{i_1} \dots \nabla_{\eta_d}^{i_d}.$$

Since H is flat, it does not matter in which order we apply the operators ∇_{η_j} in the last expression. Given $\varphi \in A$ and $t \in U$, define

$$\theta = \sum_I \big(x(t) - x\big)^I \nabla^I \varphi / I!, \tag{7.3.2}$$

the sum over all nonnegative multiindices I. In view of (7.3.1) the series is termwise dominated by

$$\begin{aligned} &\sum_{n=0}^\infty \sum_{|I|=n} \frac{\big|\big(x(t) - x\big)^I\big| h(\nabla^I \varphi)^{1/2}}{I!} \\ &\quad \le \text{const} \sum_{n=0}^\infty \sum_{|I|=n} \big|\big(x(t) - x\big)^I\big| \binom{n}{I} \varepsilon^{-n} = \text{const} \sum_{n=0}^\infty \Big(\sum_{j=1}^d \frac{|x_j(t) - x_j|}{\varepsilon} \Big)^n, \end{aligned} \tag{7.3.3}$$

hence converges locally uniformly on $V_t = \{\tau \in U : \sum_j |x_j(t) - x_j(\tau)| < \varepsilon\}$. Similarly, taking covariant derivatives gives series that converge locally uniformly on V_t, whence $\theta \in \Gamma^\infty(V_t)$. In particular, ∇_{η_k} applied to (7.3.2) produces a series in which for each multiindex $J = (j_1, \dots, j_d)$ the coefficient of $\big(x(t) - x\big)^J$ is

$$\frac{\nabla^{(j_1,\dots,j_k+1,\dots,j_d)}\varphi}{J!} - \frac{\nabla^{(j_1,\dots,j_k+1,\dots,j_d)}\varphi}{J!} = 0;$$

which means that θ is horizontal. Since for the θ obtained in this way the values $\theta(t) = \varphi(t) \in H_t$ form a dense set, by Lemma 7.3.2 through any $v \in H_t$ there passes a horizontal section of $H|V_t$. Letting $U_s = \{t \in U : \sum_j |x_j(s) - x_j(t)| < \varepsilon/2\}$, $s \in U$, it follows that through any $v \in H|U_s$ there passes a horizontal section of $H|U_s$. But then by Lemma 7.3.3 through v there even passes a horizontal section of H, as claimed.

Without the assumption on the uniformly analytic subspace A we can argue as follows. Embed S as an analytic submanifold of some $\mathbb{R}^k$ and fix a finite $\Xi \subset \mathrm{Vect}^\omega S$ that spans $\mathbb{C} \otimes TS$. Let $S' \subset S$ be a relatively compact, simply connected, open subset. Given $\varepsilon > 0$, let $B_\varepsilon \subset \Gamma^\infty|S'$ consist of those $\varphi \in \Gamma^\infty|S'$ for which

$$\sup \frac{\varepsilon^n}{n!}\, h(\nabla_{\xi_n} \dots \nabla_{\xi_1}\varphi)(s)^{1/2} < \infty,$$

the sup taken over $n = 0, 1, \dots, \xi_j \in \Xi$, and $s \in S'$. Let furthermore $A_\varepsilon \subset \Gamma^\infty|S'$ be the vector space spanned by $\psi = f\nabla_{\xi_m} \dots \nabla_{\xi_1}\varphi$, where f is the restriction to S' of a polynomial on $\mathbb{R}^k$, $m = 0, 1, \dots, \xi_j \in \Xi$, and $\varphi \in B_\varepsilon$. By Lemma 7.2.5, Lemma 7.2.6 and by Corollary 7.2.7, A_ε is uniformly analytic on S'. Finally, for $s \in S'$ let

$$H_s^\varepsilon = \overline{\{\psi(s) \colon \psi \in A_\varepsilon\}}, \qquad \text{and} \qquad \Gamma_\varepsilon^\infty = \overline{A}_\varepsilon \cap \Gamma^\infty|S', \tag{7.3.4}$$

the first closure taken in H_s, the second in $\Gamma^\infty(S')$; and let $H^\varepsilon = \coprod_{s \in S'} H_s^\varepsilon$.

Now $H^\varepsilon \to S'$ is a subfield of $H|S' \to S'$ and $\Gamma_\varepsilon^\infty$ a $C^\infty(S')$–module of its sections. Since $\nabla_\xi \Gamma_\varepsilon^\infty \subset \Gamma_\varepsilon^\infty$ for $\xi \in \mathrm{Vect}\, S'$, $\Gamma_\varepsilon^\infty$ defines a smooth structure on the Hilbert field $H^\varepsilon \to S'$. The subspace $A_\varepsilon \subset \Gamma_\varepsilon^\infty$ being uniformly analytic on S', by (7.3.4) the first part of this proof gives that through every $v \in H^\varepsilon$ there passes a horizontal section $\theta \in \Gamma_\varepsilon^\infty(S') \subset \Gamma^\infty(S')$. Since $\bigcup_{\varepsilon > 0} H_s^\varepsilon$ is dense in H_s for $s \in S'$, Lemma 7.3.2 implies that through every $v \in H|S'$ there passes a horizontal $\theta \in \Gamma(S')$. Lemma 7.3.4 in complete generality then follows from Lemma 7.3.3. □

7.4 Trivializing Hilbert fields

In this section we fix a flat analytic Hilbert field $p \colon H \to S$ over a connected and simply connected base, and after some preparation prove Theorem 7.1.7(ii), in fact in a more precise form.

Lemma 7.4.1 (Lempert, Szőke [LSz14]). *Let V be a Hilbert space with inner product $(\ |\ )$ and $f \in C^{n-1}(S; V)$, $n = 1, 2, \dots$. If for every $\xi \in \mathrm{Vect}\, S$ there is an $f_\xi \in C^{n-1}(S; V)$ such that*

$$(f|\theta) \in C^n(S; V) \quad \textit{and} \qquad \xi(f|\theta) = (f_\xi|\theta), \qquad \theta \in V,$$

then $f \in C^n(S; V)$ and $\xi f = f_\xi$.

Proof. We can assume $S = \mathbb{R}^d$. If $\chi \in C^n(S)$ is compactly supported, then $\chi * f$, $\chi * f_\xi \in C^n(S;V)$. With a constant vector field ξ and $\theta \in V$

$$(\xi(\chi * f)|\theta) = \xi(\chi * (f|\theta)) = \chi * \xi(f|\theta) = \chi * (f_\xi|\theta) = (\chi * f_\xi|\theta),$$

whence $\xi(\chi * f) = \chi * f_\xi$. Choose a sequence of $\chi = \chi_k$ that approximate the Dirac measure at 0. Then $\chi_k * f \to f$ and $\chi_k * f_\xi \to f_\xi$ in $C^{n-1}(S;V)$. Furthermore

$$\xi(\chi_k * f) - \xi(\chi_l * f) = \chi_k * f_\xi - \chi_l * f_\xi \to 0, \qquad \text{as } k, l \to \infty,$$

also in $C^{n-1}(S;V)$. Thus $\chi_k * f$ is a Cauchy sequence even in $C^n(S;V)$, whence the claim. □

We are now ready to prove the existence part of Theorem 7.1.7, in the following stronger form:

Theorem 7.4.2 (Lempert, Szőke [LSz14])**.** *Let S be a connected and simply connected analytic manifold and $H \to S$ a flat analytic Hilbert field. There are a Hilbert space V and a map $T\colon H \to V$, unitary on each fiber H_s, such that a section φ of H is in $\Gamma^n(S)$ if and only if $T\varphi \in C^n(S;V)$, $n = 0, 1, \ldots$. Moreover*

$$\xi T\varphi = T\nabla_\xi \varphi \qquad \textit{if} \qquad \xi \in \textit{Vect}\, S,\ \varphi \in \Gamma^1(S).$$

Proof. Let V be the vector space of horizontal sections in $\Gamma^\infty(S)$. By Lemma 7.3.1 $h(\varphi, \psi)$ is constant if $\varphi, \psi \in V$. Denote this constant by $(\varphi|\psi)$; it is an inner product that turns V into a pre–Hilbert space. Given $s \in S$, the map $V \ni \theta \mapsto \theta(s) \in H_s$ is linear, isometric, and, by Lemma 7.3.4, surjective. In particular, V is a Hilbert space. The inverse maps $H_s \to V$, put together, define a fiberwise unitary map $T\colon H \to V$. Composition by T induces a bijection between sections of H and functions $S \to V$. By the definition of T, if $\theta \in \Gamma^\infty(S)$ is horizontal, i.e., $\theta \in V$, then $T\theta\colon S \to V$ is the constant map $\equiv \theta$.

To verify the properties of T, assume $S = \mathbb{R}^d$ with coordinates $x_1, \ldots, x_d$. Suppose $T\varphi = P = \sum \theta_J x^J$ is a V–valued polynomial, $\theta_J \in V$. Then $\varphi = \sum x^J \theta_J \in \Gamma^\infty(S)$, and

$$T(\nabla_{\xi_m} \ldots \nabla_{\xi_1} \varphi) = \xi_n \ldots \xi_1 P, \qquad \xi_j \in \operatorname{Vect} S. \tag{7.4.1}$$

Suppose next that $T\varphi \in C^n(S;V)$. There is a sequence P_k of V–valued polynomials tending to $T\varphi$ in the C^n–topology. If $P_k = T\varphi_k$, then (7.4.1) shows that φ_k is a Cauchy sequence in $\Gamma^n(S)$. Also, $\varphi_k \to \varphi$ pointwise, hence $\varphi \in \Gamma^n(S)$; and $\nabla_\xi \varphi_k \to \nabla_\xi \varphi$ pointwise, if $n \geq 1$. Therefore

$$\xi T\varphi = \lim \xi T\varphi_k = \lim T\nabla_\xi \varphi_k = T(\nabla_\xi \varphi).$$

The converse implication will be proved by induction on n. Start with $\varphi \in \Gamma^0(S)$. Given $s_0 \in S$, let $\theta = T\varphi(s_0)$, so that $\theta(s_0) = \varphi(s_0)$. Also $\theta = T\theta(s)$ for $s \in S$, hence $\|\ \ \|$ denoting the norm on V

$$\|T\varphi(s) - T\varphi(s_0)\| = \|T\varphi(s) - T\theta(s)\| = h(\varphi(s) - \theta(s))^{1/2} \to 0$$

as $s \to s_0$. In other words, $T\varphi$ is continuous.

Suppose now that $n \geq 1$ and that $\psi \in \Gamma^{n-1}(S)$ implies $T\psi \in C^{n-1}(S;V)$. If $\varphi \in \Gamma^n(S)$ then $f = T\varphi \in C^{n-1}(S;V)$. For any $\theta \in V$ and $\xi \in \operatorname{Vect} S$

$$(f|\theta) = h(\varphi, \theta) \in C^n(S) \qquad \text{and} \qquad \xi(f|\theta) = h(\nabla_\xi \varphi, \theta) = (f_\xi|\theta),$$

with $f_\xi = T\nabla_\xi \varphi \in C^{n-1}(S;V)$. By Lemma 7.4.1 $f \in C^n(S;V)$, and the proof is complete. $\square$

Chapter 8

Direct images as fields of Hilbert spaces

In this section we fix a surjective holomorphic submersion $\pi\colon Y \to S$ of finite dimensional complex manifolds; a smooth form ν on Y that restricts to a volume form on each fiber $Y_s = \pi^{-1}s$, $s \in S$ [1]; and a Hermitian holomorphic vector bundle $(E, h^E) \to Y$ of finite rank. Write E_s for $E|Y_s$, and let H_s denote the Hilbert space of holomorphic L^2–sections of E_s, with

$$h(u,v) = \int_{Y_s} h^E(u,v)\nu, \qquad u, v \in H_s, \tag{$*$}$$

the inner product. The spaces H_s together form a Hilbert field $H \to S$. The main question is under what conditions can H be endowed with a natural smooth structure. In section 8.1, under a mild condition on E we construct a $C^\infty(S)$–module Γ^∞ of sections of H and a Hermitian connection ∇ on it. Whether Γ^∞ and ∇ indeed turn H into a smooth Hilbert field depends on whether Γ^∞ is dense in every fiber H_s, as required by (7.1.4). In section 8.2 we formulate geometric and analytic conditions that imply (7.1.4). The geometric condition bears on the fibration $Y \to S$, and in practice is easy to verify. Among the analytic conditions the most fearsome concerns the Bergman projection of E_s and its smoothness as s varies. In the next chapter we will see that in direct images that arise in quantization the geometric condition is always satisfied, and often the analytic condition can be verified, too.

8.1 Basic constructions

8.1.1 Notation

In addition to H_s it will be convenient to introduce the spaces K_s, consisting of smooth L^2–sections of E_s. They constitute a field of pre–Hilbert spaces $K = \coprod_{s\in S} K_s \to S$; the inner products on K will still be denoted by h, defined by the same formula $(*)$ as for H. Sections φ of K are in one to one correspondence

[1] In all that follows, only the restrictions $\nu|Y_s$ will matter, so one could as well take ν to be a relative form on the fibration. The form ν will be called a relative volume form.

with sections Φ of E that are smooth and L^2 on each Y_s, the correspondence being $\Phi(y) = \varphi(\pi y)(y)$, for $y \in Y$. Write $\Phi = \hat{\varphi}$ or $\varphi = \check{\Phi}$ to indicate φ and Φ correspond.

A lift of a smooth vector field $\xi \in \operatorname{Vect} S$ is a vector field $\hat{\xi} \in \operatorname{Vect} Y$ such that $\pi_* \hat{\xi}(y) = \xi(\pi(y))$ for $y \in Y$. If ξ is of type $(1,0)$ or $(0,1)$, the lift $\hat{\xi}$ should also be. In spite of what is perhaps suggested by the notation, $\hat{\xi}$ is not determined by ξ. Lifts of $\xi \equiv 0$ are the *vertical* vector fields.

The Chern connection on (E, h^E) will be denoted ∇^E. That is, ∇^E is Hermitian, and if $\zeta \in \operatorname{Vect} Y$ is of type $(0,1)$, then in any holomorphic local trivialization of E $\nabla_\zeta \Phi$ can be computed by applying $\overline{\partial}_\zeta = i_\zeta \overline{\partial}$ to the components of Φ. In particular ∇^E_ζ depends only on the holomorphic structure of E, not on h^E, when ζ is of type $(0,1)$. The curvature of ∇^E will be denoted R^E.

On a general complex manifold X, $\operatorname{Vect}' X$ and $\operatorname{Vect}'' X$ will stand for the space of smooth $(1,0)$, resp. $(0,1)$, vector fields.

8.1.2 Continuous sections

Let us say that a section φ of H or K is continuous if $\hat{\varphi}$ is a continuous section of E and $h(\varphi) \in C(S)$.

Lemma 8.1.1 (Lempert, Szőke [LSz14])**.** *If φ, ψ are continuous sections of H or K, then $h(\varphi, \psi) \in C(S)$ and $\varphi + \psi$ is also a continuous section.*

Proof. The second claim is an obvious consequence of the first, which in turn is a special case of the following: if Φ, Ψ are continuous sections of E and $\int_{Y_s} h^E(\Phi)\nu$, $\int_{Y_s} h^E(\Psi)\nu < \infty$ depend continuously on $s \in S$, then $\int_{Y_s} h^E(\Phi, \Psi)\nu$ is also continuous in s. This latter is clear if $Y = S \times X \to S$ is a trivial fibration, $E \to Y$ is also trivial, and Φ, Ψ are compactly supported. The case of a general Y, E, but Φ, Ψ still compactly supported, follows from this by a partition of unity. If Φ, Ψ are arbitrary, $s_0 \in S$, and $\varepsilon > 0$, choose a compact $C \subset Y_{s_0}$ so that $\int_{Y_{s_0} \setminus C} (h^E(\Phi) + h^E(\Psi))\nu < \varepsilon$. Let $f \colon Y \to [0,1]$ be continuous and compactly supported, $f = 1$ on C. By what we know already, as $s \to s_0$

$$\int_{Y_s} f^2 h^E(\Phi, \Psi)\nu = \int_{Y_s} h^E(f\Phi, f\Psi)\nu \to \int_{Y_{s_0}} f^2 h^E(\Phi, \Psi)\nu. \tag{8.1.1}$$

On the other hand

$$0 \le \int_{Y_s} h^E(\Phi)\nu - \int_{Y_s} f^2 h^E(\Phi)\nu \to \int_{Y_{s_0}} \big(h^E(\Phi) - h^E(f\Phi)\big)\nu \le \int_{Y_{s_0} \setminus C} h^E(\Phi)\nu < \varepsilon,$$

and similarly for Ψ, whence

$$\Big| \int_{Y_s} (1 - f^2) h^E(\Phi, \Psi)\nu \Big|^2 \le \int_{Y_s} (1 - f^2) h^E(\Phi)\nu \int_{Y_s} (1 - f^2) h^E(\Psi)\nu < \varepsilon^2 \tag{8.1.2}$$

if s is sufficiently close to s_0. Putting (8.1.1) and (8.1.2) together finishes the proof. □

It follows that continuous sections of H and K form a $C(S)$–module; write Γ^0 for the former.

8.1.3 Smooth sections

Let $\xi \in \mathrm{Vect}''S$, $\hat{\xi} \in \mathrm{Vect}''Y$ its lift, and $\varphi, \psi \in \Gamma^0$.

Definition 8.1.2. *If $\hat{\varphi} \in C^1(Y, E)$ and $\nabla^E_{\hat{\xi}} \hat{\varphi} = \hat{\psi}$, write $\nabla_\xi \varphi = \psi$.*

Lemma 8.1.3 (Lempert, Szőke [LSz14])**.** *Given φ, there is at most one such ψ, and ψ is independent of the lift $\hat{\xi}$.*

Proof. Uniqueness is obvious because $\hat{\psi}$ determines ψ; independence follows because two lifts differ by a vertical $(0,1)$–field, which annihilates $\hat{\varphi}$. □

Let

$$\Gamma^{\overline{\partial}} = \{\varphi \in \Gamma^0 \colon \nabla_\xi \varphi \in \Gamma^0 \text{ exists for all } \xi \in \mathrm{Vect}''S\},$$

a $C^1(S)$–submodule of Γ^0.

Definition 8.1.4. *Given $\xi \in \mathrm{Vect}'S$ and $\varphi, \psi \in \Gamma^0$, $\nabla_\xi \varphi = \psi$ means that*

$$\xi h(\varphi, \theta) = h(\psi, \theta) + h(\varphi, \nabla_{\overline{\xi}} \theta), \qquad \theta \in \Gamma^{\overline{\partial}},$$

in the weak sense (or "in the sense of distributions").

To ensure that $\nabla_\xi \varphi$ is unique, we introduce

Hypothesis 8.1.5. *$\{\theta(s) \colon \theta \in \Gamma^{\overline{\partial}}\} \subset H_s$ is dense for $s \in S$,*

and we will assume it throughout this section. We define Γ^1 as the set of those $\varphi \in \Gamma^{\overline{\partial}}$ for which $\nabla_\xi \varphi$ exists for all $\xi \in \mathrm{Vect}'S$. If $\varphi \in \Gamma^1$ and $\xi \in \mathrm{Vect}\, S$, define

$$\nabla_\xi \varphi = \nabla_{\xi^{1,0}} \varphi + \nabla_{\xi^{0,1}} \varphi,$$

with $\xi^{1,0}$ and $\xi^{0,1}$ the $(1,0)$ and $(0,1)$ components of ξ. Thus Γ^1 is a $C^\infty(S)$–module, and $\nabla_\xi \colon \Gamma^1 \to \Gamma^0$ has the usual properties of covariant differentiation ($\nabla_\xi \varphi$ is $C^\infty(S)$–linear in ξ; in φ it is $\mathbb{C}$–linear and satisfies the Leibniz rule). The spaces $\Gamma^n \subset \Gamma^1$ for $n = 2, 3, \ldots$ are defined inductively: $\varphi \in \Gamma^n$ if $\nabla_\xi \varphi \in \Gamma^{n-1}$ for every $\xi \in \mathrm{Vect}\, S$. Finally, $\Gamma^\infty = \bigcap_n \Gamma^n$.

Lemma 8.1.6 (Lempert, Szőke [LSz14])**.** *If $\varphi \in \Gamma^0$ and $\nabla_{\xi_n} \ldots \nabla_{\xi_1} \varphi \in \Gamma^0$ for all n and $\xi_1, \ldots \in \mathrm{Vect}''S$ (in particular, if $\varphi \in \Gamma^\infty$), then $\hat{\varphi} \in C^\infty(Y, E)$.*

Proof. Any $\zeta \in \mathrm{Vect}\, Y$ is a linear combination of lifted vector fields $\hat{\xi}$ with smooth coefficients. It follows that $\nabla^E_{\hat{\zeta}_n} \ldots \nabla^E_{\hat{\zeta}_1} \hat{\varphi}$ is continuous for all n and $\zeta_1, \ldots \in \mathrm{Vect}''Y$, and the regularity of $\overline{\partial}$ implies $\hat{\varphi}$ is smooth. □

Lemma 8.1.7 (Lempert, Szőke [LSz14])**.** *If $\varphi, \psi \in \Gamma^1$ and $\xi \in \mathrm{Vect}\, S$, then $h(\varphi, \psi) \in C^1(S)$ and*

$$\xi h(\varphi, \psi) = h(\nabla_\xi \varphi, \psi) + h(\varphi, \nabla_{\overline{\xi}} \psi). \tag{8.1.3}$$

Proof. If $\xi \in \mathrm{Vect}'S$ then (8.1.3) holds by definition, at least weakly. Taking conjugates:

$$\overline{\xi} h(\psi, \varphi) = h(\psi, \nabla_\xi \varphi) + h(\nabla_{\overline{\xi}} \psi, \varphi),$$

so that (8.1.3) holds weakly for $(0,1)$ fields as well; hence for all $\xi \in \mathrm{Vect}\, S$. Thus all weak derivatives $\xi h(\varphi, \psi)$ are continuous, whence $h(\varphi, \psi) \in C^1(S)$. □

So ∇ is a Hermitian connection, and by induction, $h(\varphi, \psi) \in C^n(S)$ if $\varphi, \psi \in \Gamma^n$. Therefore Γ^∞ and ∇ (restricted to Γ^∞) have all the attributes of a smooth structure, except possibly the density property (7.1.4). The density issue will be addressed in the next section.

8.1.4 Symmetries

Since the construction above was natural, the effect of symmetries on the direct image is easy to understand. Suppose $g\colon Y \to Y$ is a biholomorphism that leaves each Y_s and $\nu|Y_s$ invariant, and lifts to a holomorphic automorphism g_E of (E, h^E). Composition with g_E defines a unitary operator on each H_s, and so an automorphism of the Hilbert field $H \to S$, denoted g_H. Assuming the construction in 8.1.2, 8.1.3 does endow H with a smooth structure, it is straightforward that Γ^∞ is invariant under composition with g_H, and $\nabla_\xi(g_H\varphi) = g_H\nabla_\xi\varphi$.

If G is a compact group of such biholomorphisms, and the lifts satisfy $g_E g'_E = (gg')_E$, then the g_H define an action of G on H. Given an irreducible representation of G on a vector space V and χ its character, the linear span of all invariant subspaces of H_s, resp. Γ^∞, that are isomorphic to V form the χ–isotypical subspace $H_s^\chi \subset H_s$, resp. $\Gamma^\infty_\chi \subset \Gamma^\infty$ (see [BD, III.5]). Thus $H_\chi = \coprod_{s\in S} H_s^\chi$ is a Hilbert subfield of H and Γ^∞_χ is a $C^\infty(S)$–module of its sections. It is straightforward that $\nabla_\xi\Gamma^\infty_\chi \subset \Gamma^\infty_\chi$ for $\xi \in \operatorname{Vect} S$.

Lemma 8.1.8 (Lempert, Szőke [LSz14])**.** *If the direct image H is a smooth Hilbert field, then $\Gamma^\infty_\chi \subset \Gamma^\infty$ and $\nabla^\chi = \nabla|\Gamma^\infty_\chi$ endow H_χ with a smooth Hilbert field structure. The curvature $R_\chi(\xi,\eta)$ of H_χ is the restriction of the curvature $R(\xi,\eta)$ of H. If H is analytic, then so is H_χ.*

Proof. Let dg denote Haar measure on G, of total mass 1. If $\varphi \in \Gamma^\infty$, resp. Γ^ω, then $\psi = \int_G \overline{\chi}(g) g_H\varphi\, dg/\chi(e) \in \Gamma^\infty_\chi$, resp. Γ^ω_χ. In fact, $\varphi \mapsto \psi$ is a projection $\Gamma^\infty \to \Gamma^\infty_\chi$. The corresponding fact is in [He2, IV, Lemma 1.7] for isotypical subspaces of locally convex spaces like $C^\infty(Y,E)$, hence it holds also for Γ^∞. This implies that $\{\psi(s)\colon \psi \in \Gamma^\infty_\chi\} \subset (H_\chi)_s$ is dense, and therefore H_χ is smooth; analyticity is dealt with likewise. The relation between R and R_χ follows directly from the definitions. □

8.1.5 Direct image in the smooth category

The above tentative construction of a smooth Hilbert field structure on the direct image depended strongly on holomorphy, and would be impossible in the smooth category. Suppose $Y \to S$ is a submersion of smooth manifolds, ν is a smooth form on Y restricting to a volume form on each fiber Y_s, and $(E, h^E) \to Y$ is a smooth complex vector bundle with a Hermitian connection ∇^E. The spaces $\overline{K}_s$ of L^2–sections of $E|Y_s$ form a Hilbert field $\overline{K} \to S$, and it is possible to define the module Γ^0 of its continuous sections, similarly to what was done in 8.1.2 But it is not possible to go further to define Γ^1 and a connection in a natural way. This even applies to the holomorphic category, if in 8.1.3, instead of holomorphic L^2 sections H_s one considers all L^2–sections $\overline{K}_s$ (or smooth L^2–sections K_s). The space Γ^1 and ∇ can be defined only if the submersion $Y \to S$ is given more structure, for example a connection (a smooth subbundle of TY, complementary to the vertical subbundle). In the direct image problems originating in geometric quantization, to be considered in the next chapter, there are at least two equally natural candidates for such a connection. This means that on the Hilbert field $\overline{K}$ there are two natural, and different, (tentative) smooth structures. For this reason it is best not to try to explain the smooth structure of H through $\overline{K}$ (as is done in [ADW, FMMN1, FMMN2]), by invoking a more–or–less natural connection on $Y \to S$, but rather define

the smooth structure of H directly, relying only on the structures that Y and E naturally have.

8.2 The density issue

In this section we will subject $Y \to S$ and $E \to Y$ to geometric and analytic conditions to ensure the direct image field $H \to S$ is indeed smooth.

8.2.1 Complete vector fields

The geometric condition involves vector fields that are complete in a certain sense. Let M be an m–dimensional smooth manifold. A continuous m–form ω on M induces a Borel measure, denoted $|\omega|$: if in a coordinate patch $\omega = f dx_1 \wedge dx_2 \wedge \ldots$, then $|\omega| = |f| dx_1 dx_2 \ldots$. Suppose now M is oriented. Let $\mathcal{L}_\xi$ stand for Lie derivative.

Definition 8.2.1. *A vector field $\xi \in \mathit{Vect}\, M$ is integrally complete if the following holds. Suppose ω is an m–form of class C^1 on M. If $|\omega|$ and $|\mathcal{L}_\xi \omega|$ are finite measures, then $\int\limits_M \mathcal{L}_\xi \omega = 0$.*

Lemma 8.2.2 (Lempert, Szőke [LSz14])**.** *Any of the assumptions below implies $\xi \in \mathit{Vect}\, M$ is integrally complete:*

(i) $\overline{\xi}$ is integrally complete;

(ii) ξ is real and complete;

(iii) there are compactly supported C^1–functions $a_k \colon M \to [0,1]$ such that for every compact $C \subset M$, $\quad a_k|C \equiv 1$ for large enough k, and $\sup_{x,k} |\xi a_k(x)| < \infty$;

(iv) M is a complete Riemannian manifold and the length $|\xi(x)|$ grows linearly ($= O(1 + \mathit{dist}(x, x_0))$).

Proof. (i) This is so because $\mathcal{L}_{\overline{\xi}} \overline{\omega} = \overline{\mathcal{L}_\xi \omega}$.

(ii) Completeness means ξ has a global flow g_t, $t \in \mathbb{R}$. If $|\mathcal{L}_\xi \omega|$ is finite, then $\int\limits_M \mathcal{L}_\xi \omega = \int\limits_M g_t^* \mathcal{L}_\xi \omega$ for any t. Hence

$$\int\limits_M \mathcal{L}_\xi \omega = \int\limits_0^1 \int\limits_M g_t^* \mathcal{L}_\xi \omega dt = \int\limits_M \int\limits_0^1 \frac{d}{dt} (g_t^* \omega) dt = \int\limits_M g_1^* \omega - \int\limits_M \omega = 0.$$

(iii) If ξ is compactly supported, then $\mathrm{Re}\,, \xi$ and $\mathrm{Im}\, \xi$ are complete, and (ii) implies the claim. If ω, instead of ξ, is compactly supported, $\int\limits_M \mathcal{L}_\xi \omega = 0$ still follows for we are free to modify ξ outside supp ω to make it compactly supported. For a general ω as in Definition
refD:711, $a_k \omega$ is compactly supported, so as $k \to \infty$

$$0 = \int\limits_M \mathcal{L}_\xi a_k \omega = \int\limits_M (\xi a_k) \omega + \int\limits_M a_k \mathcal{L}_\xi \omega \to \int\limits_M \mathcal{L}_\xi \omega.$$

(iv) By smoothing the Lipschitz function $\mathrm{dist}(\cdot, x_0)$ one obtains a real $f \in C^1(M)$ such that $|f(x) - \mathrm{dist}(x, x_0)| \le 1$ and $|\mathrm{grad}\, f(x)| \le 2$. Let furthermore

$\alpha_k\colon \mathbb{R} \to [0,1]$ be C^1–functions such that

$$\alpha_k(t) = \begin{cases} 1, & \text{if } t \le k \\ 0, & \text{if } t \ge 2k \end{cases}, \qquad |\alpha_k'(t)| \le 2/k \text{ for all } t.$$

Then $a_k = \alpha_k \circ f$ satisfies the conditions in (iii) and the claim follows. □

8.2.2 The conditions

Returning to the vector bundle $E \to Y$, for $s \in S$, let $B_s\colon L^2(E_s) \to H_s$ denote the Bergman projection (orthogonal projection on H_s). If Φ is such a section of E that $\Phi|Y_s \in L^2(E_s)$, then using notation introduced in 8.1.1, sections $B\Phi$ of E and $\check{B}\Phi$ of H can be defined by

$$(B\Phi)|Y_s = B_s(\Phi|Y_s) \qquad \text{and} \qquad \check{B}\Phi = (B\Phi)\check{}.$$

If $\zeta \in \operatorname{Vect} Y$, then $\operatorname{div}\zeta = \operatorname{div}_\nu\zeta$ will denote the smooth function on Y satisfying

$$(\mathcal{L}_\zeta \nu)|Y_s = (\operatorname{div}\zeta)\nu|Y_s, \qquad s \in S.$$

Consider the following conditions on $Y \to S$, resp. $E \to Y$:

(G) *There is a family $\Xi \subset \mathit{Vect}'S$ that spans the bundle $T^{1,0}S$, and each $\xi \in \Xi$ has an integrally complete lift $\xi^c \in \mathit{Vect}'Y$.*

This is a geometric condition. To formulate the analytic condition, we fix Ξ and the lifts ξ^c of $\xi \in \Xi$ once and for all. If $\overline{\eta} \in \Xi$ then η^c denotes the conjugate of $\overline{\eta}^c$.

(A) *There is a subspace $\mathcal{A}_E \subset C^\infty(Y,E)$ with the following properties. If $\Phi \in \mathcal{A}_E$ then*

(A1) $\int_{Y_s} h^E(\Phi)\nu \in \mathbb{R}$ depends continuously on $s \in S$; and

(A2) if $\xi \in \Xi$ and $\eta = \overline{\xi}$, then $(\mathit{div}\,\xi^c)\Phi$, $\nabla^E_{\xi^c}\Phi$, $\nabla^E_{\eta^c}\Phi$, and $B\Phi \in \mathcal{A}_E$. Further,

(A3) if $u \in H_s$ and $\varepsilon > 0$, then there is a $\Phi \in \mathcal{A}_E$ such that $\int_{Y_s} h^E(\Phi - u)\nu < \varepsilon$.

Theorem 8.2.3 (Lempert, Szőke [LSz14])**.** *If (G) and (A) hold, then so does Hypothesis 8.1.5, and Γ^∞, ∇ defined in 8.1.3 endow $H \to S$ with the structure of a smooth Hilbert field.*

Proof. If $\Phi \in \mathcal{A}_E$ then (A1–2) imply $\check{B}\Phi \in \Gamma^0$, and (A2) implies $\Psi = \nabla^E_{\eta^c} B\Phi \in \mathcal{A}_E$ when $\overline{\eta} \in \Xi$. Now Ψ is holomorphic along the fibers Y_s. Indeed, if $\zeta \in \operatorname{Vect}''Y$ is vertical,

$$\nabla^E_\zeta \Psi = \nabla^E_\zeta \nabla^E_{\eta^c} B\Phi = \nabla^E_{\eta^c}\nabla^E_\zeta B\Phi + \nabla^E_{[\zeta,\eta^c]} B\Phi, \tag{8.2.1}$$

because the curvature of ∇^E is of type $(1,1)$. Furthermore, $[\zeta,\eta^c] = -\mathcal{L}_{\eta^c}\zeta \in \operatorname{Vect}''Y$ is also vertical, because η^c is a lifted vector field. Since $B\Phi$ is fiberwise holomorphic, (8.2.1) vanishes. Thus Ψ is fiberwise holomorphic and so $\nabla_\eta \check{B}\Phi = \check{\Psi} \in \Gamma^0$. This being true when $\overline{\eta} \in \Xi$, $\check{B}\Phi \in \Gamma^{\overline{\partial}}$ follows as Ξ spans. But (A3) implies

$$\{(\check{B}\Phi)(s)\colon \Phi \in \mathcal{A}_E\} \subset H_s \text{ is dense}, \tag{8.2.2}$$

hence Hypothesis 8.1.5 holds. To complete the proof we need □

Lemma 8.2.4 (Lempert, Szőke [LSz14]). *If $\Phi \in \mathcal{A}_E$ then $\check{B}\Phi \in \Gamma^\infty$ and for $\xi \in \Xi$*

$$\nabla_\xi \check{B}\Phi = \check{B}(\nabla^E_{\xi^c}\Phi + \Phi\, div \xi^c). \tag{8.2.3}$$

Granting the lemma we are done, since among the requirements for a smooth Hilbert field (2.2.1–2) were already verified in subsection 6.3, and (7.1.4) follows from (8.2.2) and the lemma.

8.2.3 The proof of Lemma 8.2.4.

This will take some preparation.

Lemma 8.2.5 (Lempert, Szőke [LSz14]). *Let λ be a smooth, compactly supported form on S, of top degree, and $f\colon Y \to \mathbb{C}$ Borel measurable. If either $f \geq 0$ and $\lambda \geq 0$, or f is integrable with respect to the measure $|\nu \wedge \pi^*\lambda|$, then $g(s) = \int_{Y_s} f\nu$ exists for a.e. $s \in S$, and*

$$\int_S g\lambda = \int_Y f\nu \wedge \pi^*\lambda. \tag{8.2.4}$$

Proof. If $\pi\colon Y = S \times X \to S$ is trivial, and ν is pulled back from a form on X, then the claim is a special case of the Fubini–Tonnelli theorem. If π is still trivial but ν is arbitrary, then one can factorize $\nu = a\nu_0$ with $a\colon Y \to (0,\infty)$ smooth and ν_0 pulled back from X, so that this case follows from the previous. Since a general submersion $Y \to S$ is locally (in Y) trivial, (8.2.4) still follows if f is compactly supported. Failing that, choose a sequence $b_k\colon Y \to [0,1]$ of compactly supported smooth functions that converge monotonically to 1. Writing (8.2.4) for $b_k f$ and letting $k \to \infty$, the claim follows in general. □

Lemma 8.2.6. *(Lempert, Szőke [LSz14]) If $\Phi, \Psi \in \mathcal{A}_E$ then $g(s) = \int_{Y_s} h^E(\Phi,\Psi)\nu$ is a smooth function on S, and with $\xi \in \Xi$, $\eta = \overline{\xi}$*

$$(\xi g)(s) = \int_{Y_s} h^E(\nabla^E_{\xi^c}\Phi + \Phi\ \mathrm{div}\,\xi^c, \Psi)\nu + \int_{Y_s} h^E(\Phi, \nabla^E_{\eta^c}\Psi)\nu. \tag{8.2.5}$$

Proof. Let $J(s)$ stand for the right hand side of (8.2.5). By Lemma 8.1.1 and conditions (A1–2), J is continuous. That (8.2.5) holds in the *weak* sense means that for any compactly supported smooth form λ on S, of top degree,

$$\int_S g\mathcal{L}_\xi\lambda + \int_S J\lambda = 0.$$

The left hand side here is, in view of Lemma 8.2.5,

$$\int_Y h^E(\Phi,\Psi)\nu \wedge \mathcal{L}_{\xi^c}\pi^*\lambda + \int_Y \{h^E(\nabla^E_{\xi^c}\Phi + \Phi\,\mathrm{div}\,\xi^c, \Psi) + h^E(\Phi, \nabla^E_{\eta^c}\Psi)\}\nu \wedge \pi^*\lambda$$
$$= \int_Y \mathcal{L}_{\xi^c}\{h^E(\Phi,\Psi)\nu \wedge \pi^*\lambda\},$$

which is indeed 0, since ξ^c is integrally complete. Upon interchanging Φ and Ψ, and conjugating, (8.2.5) in the weak sense also follows if $\overline{\xi} \in \Xi$. Since all these weak derivatives are continuous, $g \in C^1(S)$, and (8.2.5) holds in the pointwise sense. From here $g \in C^n(S)$ follows by induction, taking condition (A2) into account. □

The same proof also gives

Lemma 8.2.7 (Lempert, Szőke [LSz14])**.** (8.2.5) *holds in the weak sense even if instead of $\Psi \in \mathcal{A}_E$ we assume $\Psi = \hat{\theta}$, where $\theta \in \Gamma^{\overline{\partial}}$.*

Proof of Lemma 8.2.4. Write ψ for the right hand side of (8.2.3) and let $\eta = \overline{\xi}$. By (A1–2), $\psi \in \Gamma^0$. If $\theta \in \Gamma^{\overline{\partial}}$ then

$$h(\check{B}\Phi, \theta)(s) = \int\limits_{Y_s} h^E(B\Phi, \hat{\theta})\nu = \int\limits_{Y_s} h^E(\Phi, \hat{\theta})\nu,$$

$$h(\psi, \theta)(s) = \int\limits_{Y_s} h^E(B(\nabla^E_{\xi^c}\Phi + \Phi \operatorname{div} \xi^c), \hat{\theta})\nu = \int\limits_{Y_s} h^E(\nabla^E_{\xi^c}\Phi + \Phi \operatorname{div} \xi^c, \hat{\theta})\nu,$$

$$h(\check{B}\Phi, \nabla_\eta\theta)(s) = \int\limits_{Y_s} h^E(B\Phi, \nabla^E_{\eta^c}\hat{\theta})\nu = \int\limits_{Y_s} h^E(\Phi, \nabla^E_{\eta^c}\hat{\theta}),$$

because $\hat{\theta}$ and $\nabla^E_{\eta^c}\hat{\theta} = (\nabla_\eta\theta)\hat{}$ are holomorphic on Y_s. Hence Lemma 8.2.7 gives

$$\xi h(\check{B}\Phi, \theta) = h(\psi, \theta) + h(\check{B}\Phi, \nabla_{\overline{\xi}}\theta)$$

in the weak sense, which is the formula that defines $\psi = \nabla_\xi \check{B}\Phi$. This proves (8.2.3). We have already seen in the proof of Theorem refT:721 that $\check{B}\Phi \in \Gamma^{\overline{\partial}}$, now we can conclude $\check{B}\Phi \in \Gamma^1$. From here $\check{B}\Phi \in \Gamma^n$ for all n is proved by induction, using (8.2.3). □

8.3 Curvature

In this section we assume conditions (G) and (A) of subsection 8.2.2 and compute the curvature R of the direct image Hilbert field. The simpler the structure of $\pi\colon Y \to S$ and E, the more transparent the expression of R will be.

8.3.1 Generalities

Let $A = \{\check{B}\Phi\colon \Phi \in \mathcal{A}_E\}$. By Lemma 8.2.4, $A \subset \Gamma^\infty$. If $\varphi \in A$ then $\hat{\varphi} = B\hat{\varphi}$, hence by Lemma 8.2.4 and by Definition 8.1.2

$$\nabla_\xi\varphi = \check{B}(\nabla^E_{\hat{\xi}}\hat{\varphi} + \hat{\varphi} \operatorname{div} \hat{\xi}), \qquad \nabla_\eta\varphi = (\nabla^E_{\hat{\eta}}\hat{\varphi})\check{} \tag{8.3.1}$$

provided $\xi \in \Xi$, $\hat{\xi} = \xi^c$, and $\eta \in \operatorname{Vect}'' S$. The first formula will also hold for arbitrary $\xi \in \operatorname{Vect}' S$, except the lift $\hat{\xi}$ will have to be chosen carefully. If $\xi = \sum f_j\xi_j$, a locally finite sum with $\xi_j \in \Xi$ and $f_j \in C^\infty(S)$, a correct lift is $\hat{\xi} = \sum(\pi^* f_j)\xi_j^c$. In principle (8.3.1) allows for the computation of $R(\xi, \eta)\varphi = (\nabla_\xi\nabla_\eta - \nabla_\eta\nabla_\xi - \nabla_{[\xi,\eta]})\varphi$ when $\xi \in \operatorname{Vect}' S$ and $\eta \in \operatorname{Vect}'' S$ (and $\varphi \in A$). These are the only nonzero components of R:

Lemma 8.3.1. *(Lempert, Szőke [LSz14])* $R(\xi_1,\xi_2) = R(\eta_1,\eta_2) = 0$ *if* $\xi_j \in \mathit{Vect}'S$, $\eta_j \in \mathit{Vect}''S$.

Proof. Since $R^E(\hat\eta_1,\hat\eta_2) = 0$, Definition 8.1.2 gives $R(\eta_1,\eta_2) = 0$. It follows that $h\big(R(\xi_1,\xi_2)\varphi,\psi\big) = -h\big(\varphi, R(\bar\xi_1,\bar\xi_2)\psi\big) = 0$ by (7.1.5), and $R(\xi_1,\xi_2) = 0$. □

The computation of $R(\xi,\eta)$ depends, predictably, on understanding commutators, specifically $\nabla^E_{\hat\eta}B - B\nabla^E_{\hat\eta}$.

Lemma 8.3.2 (Lempert, Szőke [LSz14])**.** *(i) Let* $C_\eta = \nabla^E_{\hat\eta}B - B\nabla^E_{\hat\eta}$. *If* $\overline{\eta} \in \Xi$ *then* $(C_\eta\Phi)|Y_s \in H_s$ *depends only on* $\Phi|Y_s$ *for* $\Phi \in \mathcal{A}_E$ *and* $s \in S$.

(ii) Defining $\check C_\eta\colon \mathcal{A}_E \to \Gamma^\infty$ *by* $\check C_\eta\Phi = (C_\eta\Phi)\check{}$,

$$R(\xi,\eta)\varphi = \check B\big(R^E(\hat\xi,\hat\eta) + \nabla^E_{[\hat\xi,\hat\eta]} - (\hat\eta\operatorname{div}\hat\xi)\big)\hat\varphi - \check C_\eta(\nabla^E_{\hat\xi} + \operatorname{div}\hat\xi)\hat\varphi, \tag{8.3.2}$$

if $\xi \in \mathit{Vect}'S$, $\eta \in \mathit{Vect}''S$, $[\xi,\eta] = 0$, *and* $\varphi \in A$. *The lift* $\hat\xi$ *should be chosen as* $\sum(\pi^*f_j)\xi_j^c$ *if* $\xi = \sum f_j\xi_j$ *with* $\xi_j \in \Xi$, $f_j \in C^\infty(S)$, *and similarly for* $\hat\eta$.

In (8.3.2) and in various curvature formulas below $\hat\eta\operatorname{div}\hat\xi$, $\operatorname{div}\hat\xi$, etc. stand for the operators of multiplication by the corresponding function.

Proof. If $\Psi \in \mathcal{A}_E$ then $g(s) = \int_{Y_s} h^E(B\Psi, \Phi - B\Phi)\nu = 0$. Therefore $\overline{\eta}g = 0$ and by Lemma 8.2.6

$$\begin{aligned}\int_{Y_s} h^E(\nabla^E_{\overline{\eta}^c}B\Psi + (B\Psi)\operatorname{div}\overline{\eta}^c, \Phi - B\Phi)\nu &= \int_{Y_s} h^E(B\Psi, \nabla^E_{\eta^c}B\Phi - \nabla^E_{\eta^c}\Phi)\nu \\ &= \int_{Y_s} h^E(B\Psi, \nabla^E_{\eta^c}B\Phi - B\nabla^E_{\eta^c}\Phi)\nu = h(\check B\Psi, \check C_\eta\Phi)(s).\end{aligned} \tag{8.3.3}$$

Since the first term in (8.3.3) depends only on (Ψ and) $\Phi|Y_s$, so does the last, and (i) follows by condition (A3). (ii) in turn follows by substituting (8.3.1) in the formula $R(\xi,\eta)\varphi = \nabla_\xi\nabla_\eta\varphi - \nabla_\eta\nabla_\xi\varphi$ and commuting B past $\nabla^E_{\hat\eta}$. □

8.3.2 Special cases.

Suppose that, in addition to ∇^E, E admits another connection ∇', and each $\xi \in \Xi$ has a lift $\xi^h \in \mathrm{Vect}'Y$ such that if $\Phi \in C^\infty(Y,E)$ is holomorphic on the fibers Y_s, then so is $\nabla'_{\xi^h}\Phi$. The connection ∇' need not be Hermitian or of type $(1,0)$. Thus $\nabla^E = \nabla' + a$, with a an $\operatorname{End}E$–valued 1–form, and $\xi^c - \xi^h = \beta(\xi)$ is vertical.

Lemma 8.3.3 (Lempert, Szőke [LSz14])**.** *In addition to the assumptions and notation above, suppose* $a(\xi^h)\Psi \in \mathcal{A}_E$ *and* $\nabla^E_{\beta(\xi)}\Psi \in \mathcal{A}_E$ *when* $\xi \in \Xi$ *and* $\Psi \in \mathcal{A}_E$. *Then on* $\mathcal{A}_E$

$$(I - B)\nabla^E_{\hat\xi}B = (I - B)(\nabla^E_{\beta(\xi)} + a(\xi^h))B, \tag{8.3.4}$$

$$C_{\overline{\xi}} = [(I - B)(\nabla^E_{\beta(\xi)} + a(\xi^h) + \operatorname{div}\hat\xi)B]^*, \tag{8.3.5}$$

if $\hat{\xi} = \xi^c$. *Here the operator* $\nabla^E_{\beta(\xi)} + a(\xi^h) + \operatorname{div}\hat{\xi}$ *is considered fiberwise, defined on the dense subspaces* $\{\Psi(s)\colon \Psi \in \mathcal{A}_E\} \subset L^2(E_s)$, *and* * *means adjoint.*

Again, (8.3.4) *and* (8.3.5) *also hold for locally finite combinations* $\xi = \sum f_j\xi_j$ *of* $\xi_j \in \Xi$ *with* $f_j \in C^\infty(S)$, *if* $\hat{\xi}, \xi^h$, *and* $\beta(\xi)$ *are defined by* $\sum(\pi^* f_j)\xi^c_j$, $\sum(\pi^* f_j)\xi^h_j$, *and* $\sum(\pi^* f_j)\beta(\xi_j)$.

Proof. (8.3.4) follows because

$$\nabla^E_{\xi^c} = \nabla^E_{\beta(\xi)} + a(\xi^h) + \nabla'_{\xi^h}, \tag{8.3.6}$$

and $I - B$ annihilates fiberwise holomorphic sections. Since $B^*_s = B_s$, (8.3.3) can be rewritten, setting $\overline{\eta} = \xi$,

$$\int_{Y_s} h^E\big((I-B)(\nabla^E_{\hat{\xi}} + \operatorname{div}\hat{\xi})B\Psi, \Phi\big)\nu = \int_{Y_s} h^E(B\Psi, C_\eta\Phi)\nu = \int_{Y_s} h^E(\Psi, C_\eta\Phi)\nu,$$

because $C_\eta\Phi|Y_s \in H_s$. Substituting (8.3.4) on the left, (8.3.5) follows. □

Putting (8.3.2), (8.3.4) and (8.3.5) together gives

$$\begin{aligned}\big(R(\xi,\eta)\varphi\big)\hat{} &= B\big(R^E(\hat{\xi},\hat{\eta}) + \nabla^E_{[\hat{\xi},\hat{\eta}]} - (\hat{\eta}\operatorname{div}\hat{\xi})\big)\hat{\varphi}\\ &\quad - [(I-B)(\nabla^E_{\beta(\overline{\eta})} + a(\overline{\eta}^h) + \operatorname{div}\hat{\overline{\eta}})B]^*(\nabla^E_{\beta(\xi)} + a(\xi^h) + \operatorname{div}\hat{\xi})\hat{\varphi},\end{aligned} \tag{8.3.7}$$

provided $[\xi,\eta] = 0$ and $\varphi \in A$.

The connection ∇' and the lifts ξ^h can be found if $Y \to S$ is an open subfibration of a trivial fibration $S \times X \to S$, and E is the restriction to Y of a bundle pulled back from a bundle $F \to X$. Indeed, the pull back of any connection on F can serve as ∇', if ξ^h denotes the horizontal lift of ξ. A simplification occurs if $Y = S \times X \to S$ itself is trivial. Then condition (G) is satisfied if Ξ consists of all compactly supported $\xi \in \operatorname{Vect}'S$, and $\xi^c = \xi^h$ is the horizontal lift. That ξ^c is integrally complete follows from Lemma 8.2.2(iv) ($S \times X$ is to be endowed with a complete product metric). In this case $\beta(\overline{\eta}) = 0$ and after a little manipulation (8.3.7) becomes

$$\begin{aligned}R(\xi,\eta)\varphi &= \check{B}\big(R^E(\xi^h,\eta^h) - (\eta^h\operatorname{div}\xi^h)\big)\hat{\varphi}\\ &\quad - \check{B}(a(\overline{\eta}^h) + \operatorname{div}\overline{\eta}^h)^*(I-B)(a(\xi^h) + \operatorname{div}\xi^h)\hat{\varphi}.\end{aligned} \tag{8.3.8}$$

In (8.3.8) the adjoint can be computed pointwise, on each E_y. For example, $(\operatorname{div}\overline{\eta}^h)^* = \operatorname{div}\eta^h$.

Finally, suppose that $Y = S \times X \to S$ is trivial, (E, h^E) is pulled back from a bundle $(F, h^F) \to X$, and condition (A) in 8.2.2 holds. Choosing $\xi^h = \xi^c$ the horizontal lift of $\xi \in \operatorname{Vect} S$, one can take $\nabla' = \nabla^E$. This gives $R^E(\xi^h,\eta^h) = 0$ and $a = 0$, so (8.3.8) becomes

$$R(\xi,\eta)\varphi = -\check{B}(\eta^h\operatorname{div}\xi^h)\hat{\varphi} - \check{B}(\operatorname{div}\eta^h)(I-B)(\operatorname{div}\xi^h)\hat{\varphi}, \tag{8.3.9}$$

provided $\xi \in \operatorname{Vect}'S$, $\eta \in \operatorname{Vect}''S$, $[\xi,\eta] = 0$, and $\varphi \in A$.

8.3.3 A smooth and flat nontrivial Hilbert field [LSz3]

We illustrate the material in sections 8.1 - 8.3 by the following example. Let S be any connected complex manifold, $Y = S \times \mathbb{C}$, $\pi : Y \to S$ the projection, and $(E, h) \to Y$ the trivial Hermitian line bundle. Let $\rho : S \to [0, \infty)$ be smooth, and define the relative volume form ν by

$$i(1 + \rho(s)|x|^2)^{-2} d\bar{x} \wedge dx, \qquad s \in S, \quad x \in \mathbb{C}.$$

Write S^+ for the set where $\rho > 0$. It is easy to check that the fibers H_s of the direct image Hilbert field consist of the constant functions when $s \in S^+$, while $H_s = 0$ for other $s \in S$. Theorem 8.2.3 can be used to show that the direct image is in fact a smooth Hilbert field. For this Ξ is chosen to consist of all compactly supported $\xi \in \mathrm{Vect}' S$, and ξ^c the horizontal lift of $\xi \in \Xi$. If $\mathcal{A}_E$ consists of finite sums of sections of form $f(s)(1+\rho(s)|x|^2)^{-k}$, where $f \in C^\infty(S)$ is supported in S^+ and $k = 0, 1, \ldots$, it is not hard to verify that the conditions of the theorem are satisfied, and the direct image is indeed a smooth Hilbert field. Further, the curvature of the field can be computed, e.g., using (8.3.9), and if ρ is the modulus of a holomorphic function squared, it turns out to be 0.

Therefore in this case the direct image Hilbert field H is smooth and flat, but unless ρ vanishes identically or nowhere, it cannot be trivialized.

8.4 An example

8.4.1 A good family of holomorphic sections

Here we discuss direct image problems for which conditions (G) and (A) of 8.2.2 can be verified. As a result, the direct image Hilbert fields are smooth, resp. analytic. The analysis of direct image problems in geometric quantization will be based on these examples.

Let $(F, h^F) \to X$ be a Hermitian holomorphic vector bundle and ν_0 a smooth volume form on X. With a complex manifold S let $Y = S \times X$, $\Lambda \in C^\infty(Y)$, and $\pi\colon S \times X \to S$, $\mathrm{pr}\colon S \times X \to X$ the projections. Consider the direct image Hilbert field $H \to S$ of the pulled back bundle $(E, h^E) = \mathrm{pr}^*(F, h^F)$, using the relative volume form $\nu = e^\Lambda \mathrm{pr}^* \nu_0$. For simplicity assume $\Lambda(s, x) = a(s)L(x) + b(s)$, with $a < 0$, $L > 0$. If $\xi^h \in \mathrm{Vect}\, Y$ denotes the horizontal lift of $\xi \in \mathrm{Vect}\, S$, then

$$\mathrm{div}\, \xi^h = \xi^h \Lambda, \qquad \xi^h \Lambda(s, x) = L(x)\xi a(s) + \xi b(s). \tag{8.4.1}$$

Given $t \in \mathbb{R}$, let W^t be the Hilbert space of measurable sections v of F such that

$$h^t(v) = \int_X h^F(v) e^{tL} \nu_0 < \infty, \tag{8.4.2}$$

and $V^t \subset W^t$ the subspace of holomorphic sections.

Lemma 8.4.1 (Lempert, Szőke [LSz14]). *Let $\{V_i\}_{i \in I}$ be a collection of vector spaces, each consisting of certain holomorphic sections of F. Assume that for $t < 0$*

(i) each $V_i \subset V^t$, and the norms $(h^t)^{1/2}$ for different t are all equivalent on

V_i;

(ii) if $t + 2\tau < 0$ and $v \in V_i$, the Bergman projection of W^t maps $e^{\tau L}v$ into V_i;

(iii) $\sum_{i\in I} V_i$ is dense in V^t.

Then the direct image $H \to S$ of $E \to Y$ is a smooth Hilbert field. If a, b are analytic, then H is analytic, too.

The hypothesis is satisfied if L is bounded and the collection consists of a single space, namely V^t for any $t < 0$. In section 9.2 the lemma will be applied with L unbounded, but F will admit a large group of symmetries, and for the isotypical subspaces V_i the hypothesis can be verified.

Proof. (a) The V_i can be assumed complete in the norms $(h^t)^{1/2}$, $t < 0$. Assumption (ii) implies that for $v \in V_i$ the Bergman projection of W^t maps $L^n e^{\tau L}v$ into V_i, if $t+2\tau < 0$ and $n = 0, 1, \ldots$. This can be proved by induction as follows. When $n = 0$, the claim is just (ii). For any n

$$\frac{L^n e^{\alpha' L} - L^n e^{\alpha L}}{\alpha' - \alpha} \to L^{n+1} e^{\alpha L}, \qquad \text{as } \alpha' \to \alpha < 0,$$

uniformly on X. Hence if $t + 2\tau < 0$ then

$$\frac{L^n e^{\tau' L} - L^n e^{\tau L}}{\tau' - \tau}\, v \to L^{n+1} e^{\tau L} v \qquad \text{as } \tau' \to \tau$$

in W^t. Applying Bergman projection to both sides provides the induction step.

For $s \in S$, let $P_{i,n}(s)\colon V_i \to V_i$ denote the Toeplitz operator of multiplication by L^n followed by Bergman projection in the space $L^2(F, e^{\Lambda(s,\cdot)}\nu_0)$. As we have seen, $P_{i,n}(s)$ indeed maps into V_i. It follows from Lemma 8.4.2 below that $P_{i,n}\colon S \to \operatorname{End} V_i$ is smooth, and even analytic if a is. Here $\operatorname{End} V_i$ is the Banach space of operators on V_i, endowed with the operator norm coming from any $h^\tau, \tau < 0$.

That H is smooth will follow from Theorem refT:721. To satisfy condition (G), Ξ is taken to consist of all compactly supported $\xi \in \operatorname{Vect}' S$ and $\xi^c = \xi^h$ is the horizontal lift. As to condition (A), if f is a function on S such that $f(s)$, for $s \in S$, is a smooth section of F that is in $L^2(F, e^{\Lambda(s,\cdot)}\nu_0)$, define sections $\sigma(f)$, $\check{\sigma}(f)$ of E and K (cf. 8.1.1) by

$$\sigma(f)(s, x) = f(s)(x) \qquad \text{and} \qquad \check{\sigma}(f)(s) = \sigma(f)|\{s\} \times X, \tag{8.4.3}$$

so that $\check{\sigma}(f) = \sigma(f)\check{}$. Let $\mathcal{A}_i$ consist of linear combinations of sections of E of form

$$\Psi = \sigma(L^n f), \qquad \text{where } n = 0, 1, \ldots \text{ and } f \in C^\infty(S; V_i), \tag{8.4.4}$$

and let $\mathcal{A}_E = \sum_{i\in I} \mathcal{A}_i$. As the inclusion $V_i \subset C^\infty(X, F)$ is continuous, $\mathcal{A}_E \subset C^\infty(Y, E)$. It is easy to check that it satisfies conditions (A1–3). First, with Ψ in (8.4.4)

$$\int_{\{s\}\times X} h^E(\Psi)\nu = \int_X h^F(f(s)) L^{2n} e^{a(s)L+b(s)}\nu_0 < \infty$$

by (i), and depends continuously on s by the Dominated Convergence Theorem. In view of Lemma 8.1.1 it follows that all $\Phi \in \mathcal{A}_E$ satisfy (A1). Also, for $\xi \in \operatorname{Vect} S$

$$\Psi \operatorname{div} \xi^h = \sigma\big(L^{n+1}(\xi a)f\big) + \sigma\big(L^n(\xi b)f\big) \in \mathcal{A}_i,$$
$$\nabla^E_{\xi^c}\Psi = \sigma(L^n \xi f) \in \mathcal{A}_i, \qquad \text{and} \qquad B\Psi = \sigma(P_{i,n} f) \in \mathcal{A}_i,$$

the latter because $P_{i,n}$ is smooth. Hence condition (A2) is satisfied, and so is (A3), in view of (iii). Therefore $H \to S$ is indeed smooth.

(b) Suppose a, b are analytic. Since $\sigma(v)(s,x) = v(x)$ for $v \in V_i$, all one needs to prove is that $\check{\sigma}(v) \in \Gamma^\omega$; then the analyticity of H will follow in view of (iii). This means that given a finite $\Xi_0 \subset \operatorname{Vect}^\omega S$, the derivatives $\nabla_{\xi_n} \dots \nabla_{\xi_1} \check{\sigma}(v)$ have to be estimated for $\xi_j \in \Xi_0$ as in (7.1.7), cf. also Corollary 7.2.7. For $f \in C^\infty(S; V_i)$ and $\xi, \overline{\eta} \in \operatorname{Vect}' S$, by (8.3.1) and (8.4.1)

$$\nabla_\xi \check{\sigma}(f) = \check{\sigma}\{\xi f + (\xi a) P_{i,1} f + (\xi b) f\}, \qquad \nabla_\eta \check{\sigma}(f) = \check{\sigma}(\eta f). \tag{8.4.5}$$

Defining $D_\xi f = \xi f + (\xi a) P_{i,1} f + (\xi b) f$ and $D_\eta f = \eta f$, then extending D_ζ by linearity to all $\zeta \in \operatorname{Vect} S$, (8.4.5) simplifies:

$$\nabla_\zeta \check{\sigma}(f) = \check{\sigma}(D_\zeta f), \qquad \zeta \in \operatorname{Vect} S.$$

Here $D_\zeta \colon C^\infty(S; V_i) \to C^\infty(S; V_i)$ is a connection of the type discussed in Lemma 7.2.8. Iterating:

$$\nabla_{\xi_n} \dots \nabla_{\xi_1} \check{\sigma}(f) = \check{\sigma}(D_{\xi_n} \dots D_{\xi_1} f),$$

and the estimate (7.2.10) indeed implies that $\check{\sigma}(v) \in \Gamma^\omega$. □

It remains to show that $P_{i,n}$ is smooth. In the situation of Lemma 8.4.1 (assuming V_i complete), fix $t < 0$ and with $\tau < t/2$, consider the Toeplitz operator $Q_i(\tau) \colon V_i \to V_i$ that is multiplication by $e^{(\tau - t)L}$ followed by Bergman projection in W^t. (Again, $Q_i(\tau)$ indeed maps into V_i by assumption (ii) of Lemma 8.4.1.) Multiplication by $e^{(\tau-t)L}$, as an operator $V_i \to W^t$, is an analytic function of $\tau < t/2$, so that $Q_i \colon (-\infty, t/2) \to \operatorname{End} V_i$ is also analytic. Since

$$\int_X h^F(Q_i(\tau)v, v) e^{tL} \nu_0 = \int_X h^F(v) e^{\tau L} \nu_0 = h^\tau(v), \qquad v \in V_i,$$

Lemma 8.4.1(i) implies that the self–adjoint operator $Q_i(\tau)$ on (V_i, h^t) has a bounded inverse. Hence $Q_i^{-1} \colon (-\infty, t/2) \to \operatorname{End} V_i$ is also analytic. Smoothness and analyticity of $P_{i,n}$ therefore follow from

Lemma 8.4.2 (Lempert, Szőke [LSz14]). *$P_{i,n}(s) = Q_i^{-1}(a(s)) Q_i^{(n)}(a(s))$ when $a(s) < t/2$.*

Proof. By the definition of $Q_i(\tau)$, for $v, w \in V_i$

$$\int_X h^F(e^{(\tau-t)L} v, w) e^{tL} \nu_0 = \int_X h^F(Q_i(\tau)v, w) e^{tL} \nu_0. \tag{8.4.6}$$

Differentiating n times with respect to τ, and using (8.4.6) again

$$\begin{aligned}\int_X h^F(e^{(\tau-t)L}L^n v, w)e^{tL}\nu_0 &= \int_X h^F(Q_i^{(n)}(\tau)v, w)e^{tL}\nu_0\\ &= \int_X h^F(Q_i(\tau)Q_i(\tau)^{-1}Q_i^{(n)}(\tau)v, w)e^{tL}\nu_0\\ &= \int_X h^F(e^{(\tau-t)L}Q_i(\tau)^{-1}Q_i^{(n)}(\tau)v, w)e^{tL}\nu_0.\end{aligned}$$

Hence, putting $\tau = a(s)$

$$\int_X h^F(L^n v, w)e^{\Lambda(s,\cdot)}\nu_0 = \int_X h^F(Q_i(\tau)^{-1}Q_i^{(n)}(\tau)v, w)e^{\Lambda(s,\cdot)}\nu_0,$$

and the claim follows. □

8.4.2 Curvature.

Under the assumptions of Lemma 8.4.1 the curvature of H can be expressed very simply. Put $P_i(s) = e^{b(s)}Q_i(a(s))$ (also a Toeplitz operator, with symbol $e^{\Lambda(s,\cdot)-tL}$); from Lemma 8.4.2 $P_i^{-1}\xi P_i = (\xi a)P_{i,1} + \xi b$, so that by (8.4.5)

$$\nabla_\xi \check{\sigma}(f) = \check{\sigma}(\xi f + (P_i^{-1}\xi P_i)f) \quad \text{for} \quad \xi \in \mathrm{Vect}'S \quad \text{and} \quad f \in C^\infty(S, V_i).$$

Hence if $\eta \in \mathrm{Vect}''S$ and $[\xi, \eta] = 0$, then for $v \in V_i$

$$R(\xi,\eta)\check{\sigma}(v) = -\nabla_\eta\nabla_\xi\check{\sigma}(v) = -\check{\sigma}\big(\eta(P_i^{-1}\xi P_i)v\big). \tag{8.4.7}$$

Theorem 8.4.3 (Lempert, Szőke [LSz14])**.** *Let $t < 0$. In the situation of Lemma 8.4.1, in order that on $S_t = \{s \in S : a(s) < t/2\}$ the curvature R of H be zero, resp. central (see 7.1.4), it is sufficient and necessary that for $s \in S_t$ and $\xi \in \mathrm{Vect}'S_t$, $\eta \in \mathrm{Vect}''S_t$ the operators*

$$\overline{\partial}(P_i^{-1}\partial P_i)\big(\xi(s), \eta(s)\big)\colon V_i \to V_i, \qquad i \in I,$$

should be zero, resp. multiples $r\,\mathrm{id}_{V_i}$ of the identity, r independent of i.

Proof. The necessity is obvious from (8.4.7). As to sufficiency, the assumption implies that for each $s \in S_t$ the operator $R(\xi(s), \eta(s))\colon H_s \to H_s$ agrees with a multiple of id_{H_s} on a dense subset of H_s. Therefore the closure of $R(\xi(s), \eta(s))$, which exists by Lemma 7.1.4, is a multiple of id_{H_s}, whence R is indeed zero, resp. central. □

Chapter 9

Quantizing the family of adapted Kähler structures

9.1 Quantization

In this section M will be a compact, real-analytic Riemannian manifold and N denotes the manifold of its parametrized geodesics. Unless otherwise stated, $X \subset N$ will be an $\mathcal{A}^1$–invariant open subset as in Theorem 6.1.4(b), on which the adapted complex structure $J(i)$ exists, or more generally, any open subset of N contained in an $\mathcal{A}^1$–invariant open subset of N on which $J(i)$ exists.

9.1.1 Quantization without half form correction

According to Theorem 6.1.6 for each $s \in \mathbb{C}\backslash\mathbb{R}$ the symplectic form ω is of type $(1,1)$ in the structure $J(s)$. Hence there is a Hermitian holomorphic line bundle $E_s \to (X\mathcal{A}^{1/|\operatorname{Im} s}, J(s))$, with curvature $-i\omega$, as discussed in section 6.2; it is unique, if X (i.e., M) is simply connected. The quantum Hilbert space H_s consists of holomorphic L^2–sections of E_s. By Theorem 6.1.6 E_s is positively or negatively curved according to the sign of $\operatorname{Im} s$, and the two types behave very differently. Positively curved bundles tend to have an ample supply of holomorphic L^2–sections; negatively curved ones tend to have few. For example, when $M = S^1$ is quantized, the adapted complex structures exist on all of N, and $X = N$ is a possible choice. If $\operatorname{Im} s < 0$, zero will be the only holomorphic L^2–section of E_s. This suggests that when $\operatorname{Im} s < 0$, the quantum Hilbert space should be the L^2–cohomology group of E_s ($\overline{\partial}$–cohomology) in degree $(0,m)$, an idea that first appeared in [Va]. We shall not pursue this line here, though, and henceforward restrict ourselves to s lying in the upper half plane $S \subset \mathbb{C}$ and instead of Z of Theorem 6.1.11, we will work with

$$Y = \{(s,x) \in S \times N\colon \quad x \in X\mathcal{A}^{1/\operatorname{Im} s}\} \subset Z.$$

Thus Y inherits a complex manifold structure from Z. As before, the projection $Y \to S$ will be denoted π, the projection $S \times N \to N$ by pr, and $Y_s = \pi^{-1}s$. There is a Hermitian holomorphic line bundle $E \to Y$ whose curvature is

$$-i\tilde{\omega}|Y = -\overline{\partial}\partial(L \operatorname{Im} s) = i\, d(\partial - \overline{\partial})(iL \operatorname{Im} s/2),$$

where $\tilde{\omega} = \mathrm{pr}^*\omega$. It is constructed as the prequantum line bundle in section 6.2. As a smooth bundle, $E = Y \times \mathbb{C} \to Y$, the metric is $h^E(y,\gamma) = |\gamma|^2$, and the connection, viewed as if acting on functions $\psi\colon Y \to \mathbb{C}$, is

$$\nabla^E_\zeta \psi = \zeta\psi + \psi(\zeta^{0,1} - \zeta^{1,0})(L\,\mathrm{Im}\,s)/2, \tag{9.1.1}$$

where $\zeta^{1,0}, \zeta^{0,1}$ are the $(1,0)$ and $(0,1)$ components of $\zeta \in \mathrm{Vect}\,Y$. The holomorphic structure of E is determined by declaring a section ψ holomorphic if $\nabla^E_\zeta \psi = 0$ for $\zeta \in \mathrm{Vect}''Y$; by (9.1.1) this means $-2\zeta\psi = \psi\zeta(L\,\mathrm{Im}\,s)$. For example, the section ψ_0 corresponding to $e^{-L\mathrm{Im}\,s/2}$ is holomorphic, and its Hermitian length squared is

$$h^E(\psi_0(s,x)) = e^{-L(x)\mathrm{Im}\,s}, \qquad (s,x) \in Y. \tag{9.1.2}$$

In particular, E is holomorphically trivial.

For $s \in S$ the bundles $E_s = E|Y_s$ are the prequantum line bundles for the Kähler manifold $(Y_s, J(s), \tilde{\omega}|Y_s)$. This means that the spaces H_s of their holomorphic L^2–sections are the fibers of a direct image Hilbert field $H \to S$ of the type studied in chapter 8. The relative volume form ν there is now $\tilde{\omega}^m/m!$. To solve the uniqueness problem therefore one must decide if the construction in section 8.1 indeed endows $H \to S$ with a smooth structure; whether this structure is in fact analytic; and whether it is projectively flat. These questions will be partially answered in section 9.2 and 9.3. For the time being, we derive a rather general formula for the curvature of the direct image, under the assumptions in section 8.3.

Define a metric h_0 on E by

$$h^E_0((s,x),\gamma) = |\gamma|^2 e^{L(x)\mathrm{Im}\,s}, \quad (s,x) \in Y, \gamma \in \mathbb{C}.$$

In view of (9.1.2) $h^E_0(\psi_0) \equiv 1$, whence (E, h^E_0) is trivial as a Hermitian holomorphic line bundle. Since

$$\int_{Y_s} h^E(\psi)\,\frac{\tilde{\omega}^m}{m!} = \int_{Y_s} h^E_0(\psi)\nu, \qquad \text{where } \nu = \frac{\tilde{\omega}^m}{m!}\,e^{-L\mathrm{Im}\,s}, \tag{9.1.3}$$

the Hilbert field $H \to S$ is also the direct image of (E, h^E_0), provided the relative volume form ν from (9.1.3) is used. Furthermore, by Theorem 6.1.11 the fibration $Y \to S$ is isomorphic to the trivial fibration $S \times X \to S$, where X is considered with the complex structure $J(i)$: the inverse of the map (6.1.18) provides the isomorphism $\Psi\colon S \times X \to Y$. Thus $H \to S$ is also the direct image of the trivial Hermitian holomorphic line bundle

$$(E', h^{E'}_0) = \Psi^*(E, h^E_0) \to S \times X,$$

using the relative volume form $\nu' = \Psi^*\nu$. To compute ν', note that in (6.1.18) if $\sigma t = a + bt$, then $\sigma i = s$ means $b = \mathrm{Im}\ s$, hence $L(x\sigma^{-1}) = L(x)/(\mathrm{Im}\,s)^2$. Therefore

$$\Psi^*(L\,\mathrm{Im}\,s) = L/\mathrm{Im}\,s.$$

Here, mildly abusively, $L\,\mathrm{Im}\,s$ on the left stands for the function $Y \ni (s,x) \mapsto L(x)\mathrm{Im}\,s$, while $L/\mathrm{Im}\,s$ on the right stands for the function $S \times X \ni (s,x) \mapsto$

$L(x)/\mathrm{Im}\, s$. Because of (6.1.17) it follows that when restricted to $\{s\} \times X$,

$$i\Psi^*\tilde{\omega} = \overline{\partial}\partial L/\mathrm{Im}\, s = i\tilde{\omega}/\mathrm{Im}\, s, \tag{9.1.4}$$

$$\nu' = \Psi^*\nu = (\mathrm{Im}\, s)^{-m} e^{-L/\mathrm{Im}\, s}\tilde{\omega}^m/m!. \tag{9.1.5}$$

Knowing the restriction of ν' to each $\{s\} \times X$ determines the structure of the direct image. It also determines $\mathrm{div}\, \xi^h = \mathrm{div}_{\nu'}\xi^h$ and $\mathrm{div}\, \eta^h$ for the horizontal lift of, say, $\xi = \partial/\partial s$ and $\eta = \partial/\partial\overline{s}$:

$$\mathrm{div}\, \xi^h = \frac{im}{2\mathrm{Im}\ s} - \frac{iL}{2(\mathrm{Im}\ s)^2}, \qquad \mathrm{div}\, \eta^h = -\frac{im}{2\mathrm{Im}\ s} + \frac{iL}{2(\mathrm{Im}\ s)^2}.$$

Hence (8.3.9) gives

Lemma 9.1.1 (Lempert, Szőke [LSz14])**.** *If condition (A) of section 8.2.2 holds, then the curvature of H is given, for $\varphi \in A$, by*

$$4R\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial \overline{s}}\right)\varphi = \check{B}\left(\frac{LBL}{(\mathrm{Im}\ s)^4} - \frac{L^2}{(\mathrm{Im}\, s)^4} + \frac{2L}{(\mathrm{Im}\ s)^3} - \frac{m}{(\mathrm{Im}\ s)^2}\right)\hat{\varphi}. \tag{9.1.6}$$

Here (and in (9.1.11)*) L stands for the operator of multiplication with the function L, and LBL means the product of three operators.*

9.1.2 The half–form correction.

Let $\Omega = \bigwedge^m T^{*1,0}Y \to Y$ be the holomorphic vector bundle of $(m,0)$–forms, Ω^0 the subbundle of those forms that vanish on each Y_s, and $K_\pi = \Omega/\Omega^0$ the relative canonical bundle, a holomorphic line bundle. Elements of a fiber $(K_\pi)_y$ are in one–to–one correspondence with $(m,0)$–forms on $T_yY_{\pi(y)}$. Thus $K_\pi|Y_s$ is (canonically isomorphic to) the canonical bundle of Y_s and the Kähler metric on Y_s induces a Hermitian metric h^{K_π} on K_π by the formula

$$h^{K_\pi}(\alpha)\tilde{\omega}^m|Y_s = i^{m^2} m!\alpha \wedge \overline{\alpha}, \qquad \alpha \in K_\pi|Y_s. \tag{9.1.7}$$

Lemma 9.1.2 (Lempert, Szőke [LSz14])**.** *If M is orientable, then K_π is smoothly trivial.*

Proof. Let $\sigma_0 \in \mathcal{A}^1$ be the zero map $\mathbb{R} \to \mathbb{R}$. Since the semigroup $\mathcal{A}^1$ is connected and acts fiberwise on Y, $\sigma_0 Y = S \times M$ is a deformation retract of $Y \subset N$. On the other hand, $\{i\} \times M$ is a deformation retract of $S \times M$. The upshot is that it suffices to prove that $K_\pi|\{i\} \times M$, or $K_X|M$, is trivial. Let K_M denote the bundle of real m–forms on M, trivial by assumption. Restricting a form in $K_X|M$ to TM is an isomorphism $K_X|M \approx \mathbb{C} \otimes K_M$, hence $K_X|M$ is indeed trivial. □

Assuming therefore that M is orientable, there is a smoothly trivial Hermitian holomorphic line bundle (κ, h^κ) so that $\kappa \otimes \kappa \approx K_\pi$. If M is simply connected, then κ (and the isomorphism $\kappa \otimes \kappa \to K_\pi$) are unique, up to a certain natural notion of equivalence. In any case, we fix κ. The restrictions $\kappa|Y_s$ are the half–form bundles of the fibers Y_s, and the spaces of holomorphic L^2–sections of $E \otimes \kappa|Y_s$ form the corrected Hilbert field $H^{\mathrm{corr}} \to S$.

If Y is a Stein manifold—and one can always find X so that Y is Stein—, the smooth triviality of κ implies it is holomorphically trivial, by the Oka

principle, see e.g. [Hö, pp. 144–145]. In all the examples to work out, κ will be trivial. In this case the correction can be implemented not by changing the bundle E to $E \otimes \kappa$, but by modifying the relative volume form ν. Suppose θ_0 is a nowhere zero holomorphic section of κ. Tensoring with θ_0 induces an isomorphism between the spaces of holomorphic sections of $E|Y_s$ and $E \otimes \kappa|Y_s$. For a section ψ of $E|Y_s$

$$\int_{Y_s} h^{E\otimes\kappa}(\psi \otimes \theta_0)\, \frac{\tilde{\omega}^m}{m!} = \int_{Y_s} h_0^E(\psi)\nu, \qquad \text{where } \nu = \frac{\tilde{\omega}^m}{m!}\, e^{-L\operatorname{Im} s} h^\kappa(\theta_0), \quad (9.1.8)$$

and $h_0^E = h^E e^{L\operatorname{Im} s}$ is the flat metric from section 9.1.1. This shows that the corrected Hilbert field $H^{\mathrm{corr}} \to S$ is the direct image of E itself but with relative volume form ν given in (9.1.8).

It is also the direct image of the flat bundle $(E', h_0^{E'}) = \Psi^*(E, h_0^E) \to S \times X$, the pull back of E along the biholomorphism $\Psi\colon S \times X \to Y$, as in section 9.1.1, but this time the relative volume form

$$\nu' = \Psi^*\nu = (\operatorname{Im} s)^{-m} e^{-L/\operatorname{Im} s} \tilde{\omega}^m \Psi^* h^\kappa(\theta_0)/m! \quad (9.1.9)$$

is to be used. If choices are made with care, the factor $\Psi^* h^\kappa(\theta_0)$ above can be represented more explicitly. Start with a nowhere zero holomorphic section Θ of K_X, the canonical bundle of X (endowed with the complex structure $J(i)$). Choose the half–form bundle κ_X of X so that it has a holomorphic section θ whose square is Θ. The pull back of K_X along $\mathrm{pr}\colon S \times X \to X$ will be identified with $\Psi^* K_\pi$ as a holomorphic line bundle, and similarly $\mathrm{pr}^*\kappa_X$ with $\Psi^*\kappa$. Finally, pick θ_0 so that $\Psi^*\theta_0 = \mathrm{pr}^*\theta$, and let $\Theta_0 = \theta_0 \otimes \theta_0$. From (9.1.7), restricted to $\{s\} \times X$,

$$\Psi^*(h^{K_\pi}(\Theta_0)\tilde{\omega}^m) = i^{m^2} m! \Psi^*(\Theta_0 \wedge \overline{\Theta}_0).$$

Using (9.1.4), the left hand side is

$$\Psi^* h^{K_\pi}(\Theta_0)\Psi^*\tilde{\omega}^m = \Psi^* h^{K_\pi}(\Theta_0)\tilde{\omega}^m (\operatorname{Im} s)^{-m},$$

while the right hand side is

$$i^{m^2} m!\ \mathrm{pr}^*(\Theta \wedge \overline{\Theta}) = h^{K_X}(\Theta)\tilde{\omega}^m,$$

where the metric h^{K_X} on K_X is defined by $h^{K_X}(\alpha)\omega^m = i^{m^2} m! \alpha \wedge \overline{\alpha}$, $\alpha \in K_X$. It follows that $\Psi^* h^\kappa(\theta_0) = \Psi^* h^{K_\pi}(\Theta_0)^{1/2} = h^{K_X}(\Theta)^{1/2}(\operatorname{Im} s)^{m/2}$.

Substituting into (9.1.9):

$$\nu' = (\operatorname{Im} s)^{-m/2} e^{-L/\operatorname{Im} s} h^{K_X}(\Theta)^{1/2} \tilde{\omega}^m/m!, \quad (9.1.10)$$

where again $h^{K_X}(\Theta)$ and L are used both for functions on X and for their pull back to $S \times X$. From (9.1.10) $\operatorname{div} \xi^h$ can be computed for $\xi \in \operatorname{Vect} S$, and (8.3.9) gives a formula for the corrected curvature:

Lemma 9.1.3 (Lempert, Szőke [LSz14])**.** *If condition (A) of subsection 8.2.2 holds, then the curvature of the corrected direct image field is given, for $\varphi \in A$, by*

$$4R\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial \overline{s}}\right)\varphi = \check{B}\left(\frac{LBL}{(Im\ s)^4} - \frac{L^2}{(Im\ s)^4} + \frac{2L}{(Im\ s)^3} - \frac{m}{2(Im\ s)^2}\right)\hat{\varphi}. \quad (9.1.11)$$

Seemingly this differs from the uncorrected curvature (9.1.6) by a central term only, but the difference is more important than that: the Bergman projections in (9.1.6) and (9.1.11) refer to differently weighted Bergman spaces.

The analysis in this chapter proved the following:

Theorem 9.1.4 (Lempert, Szőke [LSz14]). *Consider the adapted Kähler quantizations of an m–dimensional compact Riemannian manifold M, as described in this chapter. The resulting field of quantum Hilbert spaces can also be obtained as the direct image of a trivial Hermitian holomorphic line bundle over $S \times X$, with relative volume form $e^{\Lambda} pr^* \nu_0$, where $pr\colon S \times X \to X$ is the projection and*

$$\Lambda(s,x) = -L(x)/\operatorname{Im} s - m \log \operatorname{Im} s, \qquad \nu_0 = \omega^m/m! \tag{9.1.12}$$

for bare quantization, and

$$\Lambda(s,x) = -L(x)/\operatorname{Im} s - (m/2) \log \operatorname{Im} s, \qquad \nu_0 = h^{K_X}(\Theta)^{1/2} \omega^m/m! \tag{9.1.13}$$

for half–form corrected quantization. Here $X \subset N$ is open, contained in an $\mathcal{A}^1$–invariant open subset of N on which the complex structure adapted to $(\mathcal{A}^1, I(i))$ exists, and $h^{K_X}(\Theta)^{1/2}$ is the norm of a nonvanishing holomorphic section Θ of K_X (assumed to exist).

This implies

Corollary 9.1.5 (Lempert, Szőke [LSz14]). *If L is bounded on X, then the resulting field of quantum Hilbert spaces, corrected or not, is analytic.*

Proof. In view of the assumptions, (9.1.12) and (9.1.13) this follows from Lemma 8.4.1. Indeed, W^t, V^t of the lemma are independent of $t \in \mathbb{R}$, and with $I = \{i\}$ a singleton, $V_i = V^t$ satisfies the hypotheses of the lemma. □

9.2 Groups and homogeneous spaces.

The main emphasis of this section is on quantizing Riemannian manifolds that are Lie groups, using the family of adapted Kähler structures. The resulting fields of quantum Hilbert spaces, corrected or not, are analytic; the corrected fields are flat, while the uncorrected ones are in general not even projectively flat. Some of the analysis applies to certain homogeneous spaces as well, and sections 9.2.1, 9.2.2 are written in this generality.

9.2.1 Normal homogeneous spaces.

Suppose on a compact Riemannian manifold M a compact Lie group G acts on the left by isometries. The induced action on the manifold N of geodesics preserves each adapted complex structure. Assume the action on M is transitive, and fix a point $o \in M$. The group has a left invariant Riemannian metric so that the map $G \ni g \mapsto go \in M$ is a Riemannian submersion. Denoting by $G_o \subset G$ the isotropy subgroup of o, M can be isometrically identified with G/G_o. Write $\mathfrak{g}$ and $\mathfrak{g}_o \subset \mathfrak{g}$ for the Lie algebra of G and G_o, and let $\mathfrak{p} \subset \mathfrak{g}$ be the orthogonal complement of $\mathfrak{g}_o$. Let exp stand for the exponential map $\mathfrak{g} \to G$ (and later also for the exponential map $\mathbb{C} \otimes \mathfrak{g} \to G^{\mathbb{C}}$ of the complexified group).

We assume that M is a normal homogeneous space, which means that the metric on G can be chosen biinvariant. This has three consequences. First, the geodesics in M are of form $t \mapsto g(\exp t\zeta)o$, with $g \in G$ and $\zeta \in \mathfrak{p}$ (because $t \mapsto g \exp t\zeta$ are the geodesics in G that are orthogonal to the fibers of the projection $G \to G/G_o$). Second, the adapted Kähler structures $J(s)$ exist on all of N; third, the action of G on N extends to a holomorphic action of the complexified group $G^{\mathbb{C}}$ on $(N, J(s))$. The isotropy group of (the constant geodesic $\equiv$) o in N is the complexification $G_o^{\mathbb{C}} \subset G^{\mathbb{C}}$ of G_o, so that $(N, J(s))$ is $G^{\mathbb{C}}$–equivariantly biholomorphic to $G^{\mathbb{C}}/G_o^{\mathbb{C}}$. This is proved in [Sz98] for $s = i$, and follows for general s from Theorem 6.1.11. The construction in [Sz98, Theorem 2.2], transcribed from TM to N, gives the following description of the equivariant biholomorphism $\Psi\colon (N, J(s)) \to G^{\mathbb{C}}/G_o^{\mathbb{C}}$. Any geodesic $x\colon \mathbb{R} \to M \approx G/G_o \subset G^{\mathbb{C}}/G_o^{\mathbb{C}}$ can be continued to a holomorphic map $\mathbb{C} \to G^{\mathbb{C}}/G_o^{\mathbb{C}}$, also denoted x; then $\Psi(x) = x(s)$. That is, if $x(t) = g(\exp t\zeta)o$, then

$$\Psi(x) = g(\exp s\zeta)G_o^{\mathbb{C}} \in G^{\mathbb{C}}/G_o^{\mathbb{C}}. \tag{9.2.1}$$

The map $G^{\mathbb{C}} \ni g \mapsto go \in N$ will be denoted q.

The upshot of all this is that it is possible to quantize M by the procedure described in 9.1.1 and 9.1.2, by taking $X = N$. However, it will be instructive to be more general, and allow $X \subset N$ to be an arbitrary connected G–invariant neighborhood of $M \subset N$.

Theorem 9.2.1 (Lempert, Szőke, [LSz14])**.** *The resulting field of quantum Hilbert spaces, corrected or not, is analytic.*

This will follow from Lemma 8.4.1 and Theorem 9.1.4, upon decomposing the quantum Hilbert spaces into G–isotypical summands. However, in the corrected version the factor $h^{K_X}(\Theta)$ in (9.1.13) has to be evaluated first. Let $P\colon \mathbb{C} \otimes \mathfrak{g} \to \mathbb{C} \otimes \mathfrak{p}$ denote projection along $\mathbb{C} \otimes \mathfrak{g}_o$.

Lemma 9.2.2 (Lempert, Szőke [LSz14])**.** *K_X has a $G^{\mathbb{C}}$–invariant holomorphic section Θ whose restriction to $\bigwedge^m TM$ is the Riemannian volume form of M. Further, let $\zeta \in \mathfrak{p}$, $\gamma \in G$, and $x(t) = \gamma(\exp t\zeta)o$ be a geodesic. Consider the operators on $\mathbb{C} \otimes \mathfrak{p}$*

$$\begin{aligned} A_1(t,\zeta) &= P(e^{-t\operatorname{ad}\zeta} + \frac{1 - e^{-t\operatorname{ad}\zeta}}{2\operatorname{ad}\zeta}\ P\operatorname{ad}\zeta)|\mathbb{C} \otimes \mathfrak{p}, \\ A_2(t,\zeta) &= P\ \frac{1 - e^{-t\operatorname{ad}\zeta}}{\operatorname{ad}\zeta}|\mathbb{C} \otimes \mathfrak{p}, \end{aligned} \tag{9.2.2}$$

where $(1 - e^{-t\operatorname{ad}\zeta})/\operatorname{ad}\zeta$ is defined by its power series. Then

$$h^{K_X}(\Theta)(x) = i^m \det\bigl(A_2^*(i,\zeta)A_1(i,\zeta) - A_1^*(i,\zeta)A_2(i,\zeta)\bigr). \tag{9.2.3}$$

Proof. It can be assumed that $X = N$. Let $\lambda \in (K_X)_o$ restrict to the Riemannian volume form. Then $g^*\lambda = \lambda$ for $g \in G_o$, and by analytic continuation also for $g \in G_o^{\mathbb{C}}$. This implies that if $x \in N$, and $g \in G^{\mathbb{C}}$ is such that $gx = o$, then $g^*\lambda$ is independent of which g is chosen; therefore $\Theta(x) = g^*\lambda$ defines the section sought.

Next, $h^{K_X}(\Theta)$ can be computed in the following way according to [LSz12, Theorem 5]. Take a symplectic basis $\xi_1, \dots, \xi_m, \eta_1, \dots, \eta_m$ of T_xN, i.e.,

$$\omega(\xi_j, \xi_k) = \omega(\eta_j, \eta_k) = 0, \quad \omega(\xi_j, \eta_k) = \delta_{jk}. \tag{9.2.4}$$

Denoting the induced action of Σ on TN by $(\xi, \sigma) \mapsto \xi\sigma$, there is a smooth $m \times m$ matrix valued function $\phi^0 = (\phi^0_{jk})$ on Σ^0 minus a discrete set such that

$$\eta_j\sigma = \sum_k \phi^0_{jk}(\sigma)\xi_k\sigma.$$

This ϕ^0 has a meromorphic continuation ϕ to a neighborhood of $(\Sigma^1, I(i))$, holomorphic near $\sigma = \mathrm{id}$ (in fact, on all of $\Sigma \backslash \Sigma^0$). Then

$$h^{K_X}(\Theta)(x) = 2^m |\Theta(\xi_1, \dots, \xi_m)|^2 \det \Im\phi(\mathrm{id}). \tag{9.2.5}$$

To prove that this agrees with (9.2.3), by G–invariance it can be assumed that $\gamma = \mathrm{id}$ so that $x(0) = o$. The Jacobi fields $\xi_1, \dots, \eta_m$ will be constructed as follows. If $\tau \in \mathfrak{g}$ and $g \in G$, write $g\tau, \tau g \in T_gG$ for the left, resp. right, translate of τ. When $\tau \in \mathfrak{g}_o$ then $g\tau \perp g\mathfrak{p}$, so that for any $\tau \in \mathfrak{g}$ we have $q_*g\tau = q_*gP\tau$. Let $\zeta_1, \dots, \zeta_m \in \mathfrak{p}$ be an orthonormal basis, and consider the vector fields along $x\colon \mathbb{R} \to M$ given by

$$\begin{aligned} \xi_j(t) &= q_*(\exp t\zeta)A_1(t, \zeta)\zeta_j \\ &= q_*(\exp t\zeta)\big(e^{-t\mathrm{ad}\,\zeta} + \frac{1 - e^{-t\mathrm{ad}\,\zeta}}{2\mathrm{ad}\,\zeta}\ P\mathrm{ad}\,\zeta\big)\zeta_j, \\ \eta_j(t) &= q_*(\exp t\zeta)A_2(t, \zeta)\zeta_j = q_*(\exp t\zeta)\frac{1 - e^{-t\mathrm{ad}\,\zeta}}{\mathrm{ad}\,\zeta}\zeta_j. \end{aligned} \tag{9.2.6}$$

Here η_j is the Jacobi field corresponding to the geodesic variation $y_u(t) = q\exp t(\zeta + u\zeta_j)$, according to the formula for the differential of the exponential map, see [He1, Chapter II, Theorem 1.7].

In ξ_j the term $q_*(\exp t\zeta)e^{-t\mathrm{ad}\,\zeta}\zeta_j = q_*(\zeta_j \exp t\zeta)$ is the Jacobi field corresponding to the variation $x_u(t) = q(\exp u\zeta_j)(\exp t\zeta)$. The other term is the same as $\eta_j(t)/2$, except that ζ_j is replaced by $P(\mathrm{ad}\,\zeta)\zeta_j \in \mathfrak{p}$, so it is also a Jacobi field. The upshot is that both ξ_j, η_j are Jacobi fields, ξ_j, $\eta_j \in T_xN$. From (9.2.6) $\xi_j(0) = q_*\zeta_j$, $\eta_j(0) = 0$, and $\eta_j'(0) = q_*\zeta_j$; hence when $t = 0$

$$\xi_j'(t) = q_*(\exp t\zeta)'\zeta_j + q_*\big(dA_1(t, \zeta)/dt\big)\zeta_j. \tag{9.2.7}$$

According to [GHL, 3.55] the first term on the right is the projection of a covariant derivative on G; namely, of the left invariant extension of ζ_j, in the direction ζ. This covariant derivative, in turn, is $[\zeta, \zeta_j]/2$, see [GHL, 2.90]. As the last term in (9.2.7) is $q_*(-\mathrm{ad}\,\zeta + P\mathrm{ad}\,\zeta/2)\zeta_j$,

$$\xi_j'(0) = q_*\left([\zeta, \zeta_j]/2 - [\zeta, \zeta_j] + P[\zeta, \zeta_j]/2\right) = 0.$$

Hence by (6.1.3) ξ_j, η_j form a symplectic basis of T_xN. From (9.2.6)

$$\eta_j(t) = \sum_k \psi_{jk}(t)\xi_k(t), \quad t \in \mathbb{R},$$

where $\psi(t) = (\psi_{jk}(t))$ is the matrix of $A_2(t,\zeta)A_1(t,\zeta)^{-1}$ in the basis $\zeta_1,\dots,\zeta_m$; by [LSz91, Proposition 6.11] and by analytic continuation it is symmetric, for any $t \in \mathbb{C}$.

Suppose $\sigma \in \Sigma^0$ is a constant map $\sigma t \equiv a \in \mathbb{R}$. Then $\xi_j\sigma \in T_{x\sigma}N$ agrees with $\xi_j(a) \in T_{x(a)}M \subset T_{x\sigma}N$, hence the matrix $\phi^0(\sigma)$ equals $\psi(a) = \psi(\sigma i)$. The map $(\Sigma, I(i)) \ni \sigma \mapsto \sigma i \in \mathbb{C}$ being holomorphic, $\phi(\mathrm{id}) = \psi(i)$ follows. As this matrix is symmetric,

$$\det \Im\phi(\mathrm{id}) = (2i)^{-m} \det\bigl(A_2(i,\zeta)A_1(i,\zeta)^{-1} - A_1^*(i,\zeta)^{-1}A_2^*(i,\zeta)\bigr). \tag{9.2.8}$$

Similarly, $\Theta\bigl((\xi_1\sigma)^{1,0},\dots,(\xi_m\sigma)^{1,0}\bigr)$ is a holomorphic function of σ, because each $(\xi_j\sigma)^{1,0} \in T^{1,0}X$ is, see [LSz91, Proposition 5.1]. When $\sigma \in \Sigma^0$ as above,

$$\Theta\bigl((\xi_1\sigma)^{1,0},\dots,(\xi_m\sigma)^{1,0}\bigr) = \Theta^0(\xi_1\sigma,\dots,\xi_m\sigma) = \det A_1(\sigma i,\zeta),$$

hence by analytic continuation to $\sigma = \mathrm{id}$

$$\det A_1(i,\zeta) = \Theta(\xi_1^{1,0},\dots,\xi_m^{1,0}) = \Theta(\xi_1,\dots,\xi_m).$$

Substituting this and (9.2.8) into (9.2.5), (9.2.3) follows. □

Proof of Theorem 9.2.1. We will apply Lemma 8.4.1 and Theorem 9.1.4. The Hilbert field in question is the direct image of the trivial Hermitian holomorphic line bundle on $S \times X$, using a relative volume form $\nu = e^{\Lambda}\mathrm{pr}^*\nu_0$. Here, by (9.1.12) and (9.1.13)

$$\Lambda(s,x) = -L(x)/\Im s - m\log\Im s, \quad \nu_0 = \omega^m/m!, \quad \text{resp.} \tag{9.2.9}$$

$$\Lambda(s,x) = -L(x)/\Im s - (m/2)\log\Im s, \quad \nu_0 = h^{K_X}(\Theta)^{1/2}\omega^m/m!, \tag{9.2.10}$$

for bare, resp. corrected quantization, $h^{K_X}(\Theta)$ given in (9.2.3). In both cases ν_0 is G–invariant. It follows that G acts unitarily on each Hilbert space $W^T = L^2(X, e^{TL}\nu_0)$, $T \in \mathbb{R}$, and on its subspace V^T of holomorphic functions: the action of $g \in G$ on $v \in W^T$ is $gv = (g^{-1})^*v$ (pull back by g^{-1}). The same formula also defines an action of G on $\mathcal{O}(X)$, and the isotypical subspaces $V_\chi \subset \mathcal{O}(X)$ corresponding to irreducible characters χ of G will play the role of the spaces V_i in Lemma 8.4.1. Accordingly, the conditions of the lemma have to be verified.

Since $M \subset X$ is maximally real, $\mathcal{O}(X) \ni v \mapsto v|M$ maps V_χ injectively in the χ–isotypical subspace of $L^2(M)$. By the Peter–Weyl theorem this latter is finite dimensional, and therefore so is V_χ. The restriction $G \to \mathrm{GL}(V_\chi)$ of the G–representation on $\mathcal{O}(X)$ extends to a holomorphic representation $\rho\colon G^{\mathbb{C}} \to \mathrm{GL}(V_\chi)$. Functions $v \in V_\chi$ can be estimated pointwise as follows. In a fixed orthonormal basis $v_1,\dots,v_n$ of V_χ, ρ is given by a matrix (ρ_{jk}). Let $g \in G$, $\zeta \in \mathfrak{p}$, and $x \in X$ be given by $x(t) = g(\exp t\zeta)o$. Since $g\exp i\zeta$ acts on $G^{\mathbb{C}}/G_o^{\mathbb{C}}$ by left multiplication, formula (9.2.1) for the $G^{\mathbb{C}}$ equivariant biholomorphism $N \to G^{\mathbb{C}}/G_o^{\mathbb{C}}$ shows that $x = g(\exp i\zeta)o$, if $o \in M$ is identified with the constant geodesic $\equiv o$. If $v = \sum \alpha_k v_k$ then $\rho\bigl((g\exp i\zeta)^{-1}\bigr)v = \sum_{jk}\rho_{jk}\bigl((g\exp i\zeta)^{-1}\bigr)\alpha_k v_j$, and

$$|v(x)| = \sum \rho_{jk}\bigl((g\exp i\zeta)^{-1}\bigr)\alpha_k v_j(o) \le c_1 e^{c_2|\zeta|} = c_1 e^{c_2\sqrt{L(x)}}, \tag{9.2.11}$$

because the operator norm of $\rho(g)$ is 1 and of $\rho((\exp i\zeta)^{-1})$ is $\leq e^{c_2|\zeta|}$. Similarly, from (9.2.2) and (9.2.3) $h^{K_X}(\Theta)^{1/2}(x) \leq c_3 e^{c_4\sqrt{L(x)}}$. Finally, phase space integrals and volumes can be easily computed by first integrating along the fibers and then over the base, see e.g. [Chav, Theorem 5.2, p.227]. This gives

$$\int\limits_{\{x\in N:\ \sqrt{L(x)}<r\}} \omega^m/m! = \sigma_m r^m \mathrm{Vol}\ M,$$

σ_m denoting the volume of the unit ball in $\mathbb{R}^m$. Putting all this together, if $T<0$

$$\int\limits_X |v|^2 e^{TL}\nu_0 \leq \int\limits_N c' e^{c\sqrt{L}+TL}\ \frac{\omega^m}{m!} = c'' \int\limits_0^\infty e^{c\sqrt{r}+Tr} dr^m < \infty,$$

whether ν_0 is given in (9.2.9) or (9.2.10), so that $V_\chi \subset V^T$. Since $\dim V_\chi < \infty$, the norms $(h^T)^{1/2}$ are equivalent on V_χ, which proves assumption (i) of Lemma 8.4.1. Since both multiplication by $e^{\tau L}$ and Bergman projection in W^T are G–equivariant, (ii) of the lemma is satisfied; and (iii) is also, because the V_χ are the isotypical subspaces of V^T as well, and their span is dense (see [He2, IV. Lemma 1.9]). Hence Theorem 9.2.1 indeed follows from Lemma 8.4.1. □

9.2.2 Curvature.

According to 8.4.2, the curvature of the direct image can be computed from certain Toeplitz operators. Continuing with the set up and the notation in 9.2.1, if $\tau<0$ is fixed, for $a(s)<\tau/2$ the Toeplitz operators $P_\chi(s)\colon V_\chi \to V_\chi$ in question are multiplication by $e^{\Lambda(s,\cdot)-\tau L}$, followed by orthogonal projection in $L^2(X, e^{\tau L}\nu_0)$. Here $\Lambda(s,\cdot) = a(s)L + b(s)$ and ν_0 are given in (9.2.9), resp. (9.2.10). Often $P_\chi(s)$ turns out to be a scalar operator, and can be computed from a character integral. Let $\mathfrak{p}_X$ consist of those $\zeta \in \mathfrak{p}$ for which the geodesic $t \mapsto (\exp t\zeta)o$ is in X; this is an open subset of $\mathfrak{p}$.

Lemma 9.2.3 (Lempert, Szőke [LSz14])**.** *Suppose* $\dim V_\chi > 0$ *and* $P_\chi(s)$ *is a scalar operator* $p_\chi(s)\, id_{V_\chi}$*. Then*

$$p_\chi(s) = \int\limits_{\mathfrak{p}_X}\int\limits_{G_o} e^{a(s)|\zeta|^2+b(s)}\chi(g_o\exp(-2i\zeta))\, d_o g_o\, d\mu(\zeta), \tag{9.2.12}$$

where $d_o g_o$ *is normalized Haar measure on* G_o*; for bare quantization* μ *is a suitable translation invariant measure on* $\mathfrak{p}$*—possibly depending on* χ *but not on* s*—, while for corrected quantization* μ *is the invariant measure multiplied by (cf.* (9.2.2)*)*

$$|\det\big(A_2^*(i,\zeta)A_1(i,\zeta) - A_1^*(i,\zeta)A_2(i,\zeta)\big)|^{1/2}. \tag{9.2.13}$$

Proof. The holomorphic function $G^{\mathbb{C}} \ni g \mapsto \chi(g^{-1}) \in \mathbb{C}$ is in the χ–isotypical subspace of the left regular representation of $G^{\mathbb{C}}$ on $\mathcal{O}(G^{\mathbb{C}})$, because the corresponding matrix elements are. Therefore $\tilde{v} \in \mathcal{O}(G^{\mathbb{C}})$ given by

$$\tilde{v}(g) = \int\limits_{G_o} \chi(g^{-1}g_o)\, d_o g_o \tag{9.2.14}$$

is also in the isotypical subspace. Since $\tilde{v}$ is invariant under translations by G_o, hence also by $G_o^{\mathbb{C}}$, it descends to a $v \in V_\chi$. Now $v \not\equiv 0$. Indeed, the projection of any $w \in L^2(M)$ on the χ–isotypical subspace is $\dim \chi \int_G \chi(g^{-1}) gw\, dg$. Take a $u \in V_\chi$ with $u(o) \neq 0$ (a suitable translate of any $u' \in V_\chi \setminus \{0\}$ will have this property). The projection of $u|M$ is of course itself, so

$$0 \neq \int_G \chi(g^{-1}) u(g^{-1}o)\, dg = \int_{G \times G_o} \chi(g^{-1}) u(g^{-1} g_o^{-1} o)\, dg\, d_o g_o \tag{9.2.15}$$

$$= \int_G \Big(\int_{G_o} \chi(g^{-1} g_o)\, d_o g_o \Big) u(g^{-1}o)\, dg. \tag{9.2.16}$$

Hence (9.2.14) shows that $\tilde{v} \not\equiv 0$ and $v \not\equiv 0$. Next

$$\int_X e^{a(s)L + b(s)} v \overline{v} \nu_0 = \int_X (P_\chi(s) v) \overline{v} e^{\tau L} \nu_0 = \int_X p_\chi(s) |v|^2 e^{\tau L} \nu_0, \qquad \text{and} \tag{9.2.17}$$

$$p_\chi(s) = \int_X e^{a(s)L + b(s)} |v|^2 \nu_0 \Big/ \int_X e^{\tau L} |v|^2 \nu_0. \tag{9.2.18}$$

As L and ν_0 are G–invariant, the first integral in (9.2.17) is

$$\int_X \Big(\int_G e^{aL+b} |\gamma v|^2 d\gamma \Big) \nu_0, \tag{9.2.19}$$

the phase space integral of a G–invariant function. Let $N_o = q \exp i\mathfrak{p} \subset N$ consist of geodesics x such that $x(0) = o$, $X_o = X \cap N_o = q \exp i\mathfrak{p}_X$, and let dx, resp. $d\zeta$, be the translation invariant measure on $N_o \approx T_o M$, resp. $\mathfrak{p}$, normalized by the metric. When $\nu_0 = \omega^m / m!$, again by (9.2.19) equals

$$\begin{aligned} &\mathrm{Vol}\ (M) \int_{X_o} \int_G e^{aL(x)+b} |\gamma v(x)|^2 d\gamma\, dx \\ &\quad = \mathrm{Vol}\ (M) \int_{\mathfrak{p}_X} \int_G e^{a|\zeta|^2 + b} |v\big(\gamma^{-1} (\exp i\zeta) o\big)|^2 d\gamma\, d\zeta. \end{aligned} \tag{9.2.20}$$

With the half–form correction included, in view of (9.2.3), (9.2.10) the integrand on the right of (9.2.20) has to be multiplied by (9.2.13), to yield, in both cases

$$\int_X e^{aL+b} |v|^2 \nu_0 = \mathrm{Vol}\ (M) \int_{\mathfrak{p}_X} e^{a|\zeta|^2 + b} \int_G |\tilde{v}(\gamma^{-1} \exp i\zeta)|^2 d\gamma\, d\mu(\zeta). \tag{9.2.21}$$

Next we compute the inner integral on the right. If $g = \gamma \exp i\zeta$ with $\gamma \in G$ and $\zeta \in \mathfrak{g}$, write $g^* = (\exp i\zeta)\gamma^{-1}$, so that the map $g \mapsto g^*$ is antiholomorphic. When $g_1, g_2 \in G$,

$$\int_G \chi(g_1 \gamma) \overline{\chi(g_2 \gamma)}\, d\gamma = \int_G \chi(g) \chi(g^{-1} g_1 g_2^{-1})\, dg = \chi(g_1 g_2^{-1}) / \dim \chi,$$

see [BD p.83, Proposition 4.16]. The last expression is $\chi(g_1 g_2^*)/\dim\chi$, hence

$$\int_G \chi(g_1\gamma)\overline{\chi(g_2\gamma)}\,d\gamma = \chi(g_1 g_2^*)/\dim\chi, \qquad g_1, g_2 \in G^{\mathbb{C}}$$

by analytic continuation. As χ is a class function, with $\zeta \in \mathfrak{p}$ and $g = \exp i\zeta$ therefore

$$\int_G |\tilde{v}(\gamma^{-1}g)|^2 d\gamma = \int_{G\times G_o\times G_o} \chi(g_1 g^{-1}\gamma)\overline{\chi(g_2 g^{-1}\gamma)}\; d\gamma\, d_o g_1\, d_o g_2$$

$$= \int_{G_o\times G_o} \chi(g_1 g^{-1}(g_2 g^{-1})^*)\, d_o g_1\, d_o g_2/\dim\chi$$

$$= \int_{G_o\times G_o} \chi(g_2^{-1} g_1 (g^* g)^{-1})\, d_o g_1\, d_o g_2/\dim\chi = \int_{G_o} \chi(g_o \exp(-2i\zeta))\, d_o g_o/\dim\chi.$$

Substituting this into (9.2.21) and then into (9.2.17), the lemma is obtained, if one notes that the second integral in (9.2.17) is independent of s, and one subsumes all the constants into $d\mu$. □

9.2.3 Group manifolds.

Theorem 9.2.4 (Lempert, Szőke, [LSz14])**.** *Suppose M is a compact Lie group G with a biinvariant metric. If M is quantized by the family of adapted Kähler structures $(N, J(s))$, $Im\, s > 0$, and the half–form correction is included, then the resulting field of quantum Hilbert spaces H^{corr} is flat.*

Proof. The results of 9.2.1 and 9.2.2 apply with G_o the trivial group and $\mathfrak{p}_X = \mathfrak{p} = \mathfrak{g}$. Since P in (9.2.2) is the identity, one computes $A_1(i,\zeta) = (1 + e^{-i\mathrm{ad}\,\zeta})/2$ and $A_2(i,\zeta) = (1 - e^{-i\mathrm{ad}\,\zeta})/\mathrm{ad}\,\zeta$, so that

$$i^m \det\big(A_2^*(i,\zeta)A_1(i,\zeta) - A_1^*(i,\zeta)A_2(i,\zeta)\big) = \det(2\sin\mathrm{ad}\,\zeta/\mathrm{ad}\,\zeta) > 0$$

in view of (9.2.3). The isotypical subspaces of $L^2(M)$ are invariant under the left–right action of $G\times G$ and are irreducible as $G\times G$ representations. It follows that the V_χ are also irreducible. As both L and ν_0 are $G\times G$–invariant, the Toeplitz operators $P_\chi(s) : V_\chi \to V_\chi$ are $G\times G$–equivariant, whence multiples of the identity by Schur's lemma. Which multiple, is given by Lemma 9.2.3:

$$p_\chi(s) = \int_{\mathfrak{g}} e^{a(s)|\zeta|^2 + b(s)} \chi(\exp(-2i\zeta)) \left(\det \frac{2\sin\mathrm{ad}\,\zeta}{\mathrm{ad}\,\zeta}\right)^{1/2} d\zeta, \tag{9.2.22}$$

$d\zeta$ denoting a suitable translation invariant measure on $\mathfrak{g}$. In light of Theorem 8.4.3 all we have to show is that $\log p_\chi$ is harmonic.

Let $T \subset G$ be a maximal torus, $\mathfrak{t} \subset \mathfrak{g}$ its Lie algebra with orthogonal complement $\mathfrak{t}^\perp$, and W the Weyl group. The integral in (9.2.22) can be reduced to $\mathfrak{t}$. The map

$$G/T \times \mathfrak{t} \ni (gT, \tau) \mapsto \mathrm{Ad}\,(g)\tau \in \mathfrak{g} \tag{9.2.23}$$

is generically a $|W|$–fold covering, and by computing its differential, one can relate the pullback of $d\zeta$ to the product of the G–invariant measure on G/T

and the translation invariant measure on $\mathfrak{t}$. The pullback measure turns out to be $|\det \operatorname{ad}\tau|\mathfrak{t}^\perp|$ times an invariant product measure. The computation is the same as for Weyl's formula, see e.g. [BD, IV. (1.8)]. If R denotes the set of (nonzero) roots, the eigenvalues of $\operatorname{ad}\tau|\mathfrak{t}^\perp$ are $i\alpha(\tau)$, $\alpha \in R$, and as the negative of each root is also a root, the factor above is $\prod_{\alpha\in R}\alpha(\tau)$. Thus the integral of any Ad G–invariant $f \in L^1(\mathfrak{g})$ can be computed upon pulling back by (9.2.23):

$$\int_{\mathfrak{g}} f(\zeta)d\zeta = \int_{\mathfrak{t}} f(\tau)\prod_{\alpha\in R}\alpha(\tau)d\tau, \tag{9.2.24}$$

with $d\tau$ a suitable translation invariant measure. Denoting by $R^+ \subset R$ a choice of positive roots, the constituents in (9.2.22) restrict to $\mathfrak{t}$ as

$$\left(\det\frac{2\sin\operatorname{ad}\tau}{\operatorname{ad}\tau}\right)^{1/2} = \left(\prod_{\alpha\in R}\frac{2\sin i\alpha(\tau)}{i\alpha(\tau)}\right)^{1/2} = \prod_{\alpha\in R^+}\frac{2\operatorname{sh}\alpha(\tau)}{\alpha(\tau)},$$
$$\chi(\exp(-2i\tau)) = \sum_{w\in W} e^{2\lambda(w\tau)}\det w\Big/\prod_{\alpha\in R^+}\operatorname{sh}\alpha(\tau),$$

this latter by Weyl's character and denominator formulas, see [Kn, Theorem 5.113]. Here $\lambda\colon \mathfrak{t}\to\mathbb{R}$ is a linear form, the highest weight of χ plus $\sum_{\alpha\in R^+}\alpha/2$, and $\det w = \pm 1$ is the determinant of $w\colon \mathfrak{t}\to\mathfrak{t}$. Further, $\prod_{\alpha\in R^+}\alpha(w\tau) = \det w\prod_{\alpha\in R^+}\alpha(\tau)$. This is obvious for reflections $w\in W$ that change the sign of one positive root and permute the others, and it follows in general because W is generated by such reflections, see [BD, V. (4.6) Corollary and (4.10) Lemma]. Therefore by (9.2.22) and (9.2.24)

$$\begin{aligned} p_\chi &= 2^{|R^+|}\int_{\mathfrak{t}} e^{a|\tau|^2+b}\sum_{w\in W} e^{2\lambda(w\tau)}\det w\prod_{\alpha\in R^+}\alpha(\tau)\,d\tau \\ &= |W|2^{|R^+|}\int_{\mathfrak{t}} e^{a|\tau|^2+b}\,e^{2\lambda(\tau)}\prod_{\alpha\in R^+}\alpha(\tau)\,d\tau. \end{aligned}$$

Denoting by $\lambda^*\in\mathfrak{t}$ the dual of $\lambda\in\mathfrak{t}^*$ with respect to the inner product on $\mathfrak{t}$, the substitution $\tau\to\tau/\sqrt{-a}-\lambda^*/a$ transforms the last integral into

$$(-a)^{-(\dim\mathfrak{t})/2}\,e^{b-|\lambda^*|^2/a}\int_{\mathfrak{t}} e^{-|\tau|^2}\prod_{\alpha\in R^+}\alpha\big(\tau/\sqrt{-a}-\lambda^*/a\big)\,d\tau. \tag{9.2.25}$$

Lemma 9.2.5 (Lempert, Szőke [LSz14])**.** *The function $\prod_{\alpha\in R_+}\alpha$ is harmonic on $\mathfrak{t}$.*

Accepting this for the moment, by the mean value theorem the integral in (9.2.25) is

$$\int_{\mathfrak{t}} e^{-|\tau|^2}\prod_{\alpha\in R^+}\alpha(-\lambda^*/a)\,d\tau = \pi^{(\dim\mathfrak{t})/2}(-a)^{-|R^+|}\prod_{\alpha\in R^+}\alpha(\lambda^*).$$

Now $a(s) = -1/\Im s$ and $b(s) = -(m/2)\log\Im s$. Since $\mathbb{C}\otimes\mathfrak{g}$ is the direct sum of $\mathbb{C}\otimes\mathfrak{t}$ and the one dimensional root spaces $\mathfrak{g}_\alpha$, $\alpha\in R$, it follows that $m = \dim\mathfrak{t} + 2|R^+|$, and (9.2.3), (9.2.25) give

$$p_\chi(s) = \text{const}\,(\Im s)^{|R^+|+(\dim\mathfrak{t}-m)/2}\,e^{|\lambda^*|^2\Im s} = \text{const}\,e^{|\lambda^*|^2\Im s},$$

with the constant depending on χ but not on s. Hence $\overline{\partial}\partial \log p_\chi = 0$, and H^{corr} is flat by Theorem 8.4.3. $\square$

Proof of Lemma 9.2.5. See [He2, Chapter III], immediately after Corollary 3.8. Alternatively, the lemma can be deduced from Weyl's denominator formula

$$\prod_{\alpha\in R^+} \operatorname{sh}\alpha(\tau) = \sum_{w\in W} e^{\rho(w\tau)} \det w, \quad \rho = \sum_{\alpha\in R^+} \alpha/2.$$

The right hand side is manifestly an eigenfunction of the Laplacian Δ. Hence $\prod_{\alpha\in R^+}\alpha(\tau)$, the lowest term in the homogeneous expansion of the left hand side, must be annihilated by Δ. $\square$

In [Hu, Lemma 3.3] Huebschmann already computed the integral in (9.2.22), and in fact the integrals in (9.2.19) when $X = G^{\mathbb{C}}$, by somewhat different means.

Without the half–form correction little changes formally: in the integrand in (9.2.22) the last factor is omitted, which leads to

$$p_\chi = \text{ const} \int_{\mathfrak{t}} e^{a|\tau|^2+b+2\lambda(\tau)} \prod_{\alpha\in R^+} \frac{\alpha(\tau)^2}{\operatorname{sh}\alpha(\tau)}\, d\tau. \tag{9.2.26}$$

When G is commutative, the uncorrected integral is the same as the corrected, except that now $b(s) = -m\log\Im s$, so that

$$\overline{\partial}\partial \log p_\chi(s) = \frac{m d\overline{s}\wedge ds}{8(\Im s)^2}.$$

This is still independent of χ, and H is projectively flat. However, with a noncommutative G matters are altogether different. For example, if $G = \mathrm{SU}(2)$,

$$T = \{\mathrm{diag}(e^{it}, e^{-it})\colon t\in\mathbb{R}\}, \quad \mathfrak{t} = \{\tau = i\mathrm{diag}(t,-t)\colon t\in\mathbb{R}) \subset \mathfrak{su}(2),$$

the roots are $\alpha(\tau) = \pm 2t$, of which we take $2t$ as positive. In (9.2.26) the possible λ are $\lambda(\tau) = (k+1)t$, $k = 0, 1, \ldots$. Hence p_χ is constant times

$$\int_{\mathbb{R}} e^{at^2+b}\, \frac{e^{2(k+1)t}}{\operatorname{sh} 2t}\, t^2 dt = \int_{\mathbb{R}} e^{at^2+b}\, \frac{e^{2(k+1)t} - e^{-2(k+1)t}}{e^{2t} - e^{-2t}}\, t^2 dt$$

$$= \int_{\mathbb{R}} e^{at^2+b} \sum_{j=0}^{k} e^{2(k-2j)t} t^2 dt = (\Im s)^{-3/2} \sum_{j=0}^{k} e^{(k-2j)^2\Im s}\big(1 + 2(k-2j)^2\Im s\big).$$

Now $\overline{\partial}\partial\log p_\chi$ depends on χ, i.e. on k. Indeed, write $u_k(s)$ for the last sum above. Comparing the cases of $k = 0$ and a general k one sees that $\overline{\partial}\partial\log p_\chi$ is independent of k only if $\log u_k$ is harmonic. But $\log u_k(s)$ is a function of $\Im s$, and not a linear function at that; hence it is not harmonic. Therefore by Theorem 8.4.3 the uncorrected direct image is not projectively flat.

9.2.4 A variant [LSz14].

Even if the adapted Kähler structures of a compact group $M = G$ exist on the entire space N of its geodesics, quantization can be based on any open $X \subset N$.

A little calculation shows that, in general, the resulting field of quantum Hilbert spaces will not be projectively flat. For example, let $G = S^1$, $r > 0$, and let X consist of geodesics of speed $< r$. From Lemma 9.2.3

$$p_\chi = \text{const} \int_{-r}^{r} e^{a\zeta^2+b} e^{2k\zeta} d\zeta,$$

$k \in \mathbb{Z}$ parametrizing the irreducible characters of S^1. The substitution $\zeta = r - t/(k + ar)$ evaluates the integral as

$$\frac{e^{ar^2+2kr+b}}{k+ar} \int_0^{2r(k+ar)} e^{at^2/(k+ar)^2-2t}\, dt \sim \frac{e^{ar^2+2kr+b}}{2(k+ar)}$$

when $k \to \infty$, and similar asymptotics hold for the s–derivatives of the integral. Hence

$$\overline{\partial}\partial \log p_\chi(s) = \overline{\partial}\partial\big(a(s)r^2 + b(s)\big) + (r/k)\overline{\partial}\partial(1/\Im s) + o(1/k).$$

This again depends on k, so by Theorem 8.4.3 the fields H and H^{corr} are not projectively flat.

9.3 Compact symmetric spaces. The main results

Let (M^m, g) be an m-dimensional, compact, irreducible, simply connected, Riemannian symmetric space. Then (see [He1]) M is isometric to U/K, where U is a compact, connected, simply connected, semisimple Lie group and K is the fixed point set (automatically connected) of a nontrivial involution $\theta : U \to U$. The metric on U/K is induced from a biinvariant metric on U. Furthermore either U is simple or has the form $U = G \times G$ where G is simple, $\theta(g_1, g_2) = (g_2, g_1)$ and K is the diagonal in $G \times G$. In the latter case M is isometric to G equipped with a biinvariant metric.

Let $\mathfrak{u}$ be the Lie algebra of U, $\mathfrak{u}_\mathbb{C}$ its complexification and $U_\mathbb{C}$ the simply connected complex Lie group with Lie algebra $\mathfrak{u}_\mathbb{C}$.

As we saw in section 3.1 $U_\mathbb{C}$ is biholomorphic with the tangent bundle TU, the latter is equipped with the adapted complex structure of a biinvariant metric on U (see Theorem 3.1.1). This is the complex structure that corresponds to the parameter i from Section 6.1.2.

θ induces a Lie algebra involution $\theta_* : \mathfrak{u} \to \mathfrak{u}$. Then $\mathfrak{u} = \mathfrak{k} + \mathfrak{p}_*$, where $\mathfrak{k} = \{X \in \mathfrak{u} : \theta_*(X) = X\}$ and $\mathfrak{p}_* = \{X \in \mathfrak{u} : \theta_*(X) = -X\}$. Here $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{p}_*$ can be identified with $T_{[K]}M$.

Let $\mathfrak{p}_0 = i\mathfrak{p}_*$, $\mathfrak{g}_0 = \mathfrak{k} + \mathfrak{p}_0$ and denote by G_0 the analytic subgroup of $U_\mathbb{C}$ with Lie algebra $\mathfrak{g}_0$. Then G_0 is closed in $U_\mathbb{C}$ and $K \subset G_0$. Let $\theta_\mathbb{C}$ be the holomorphic extension of θ to $U_\mathbb{C}$. Then $\theta_\mathbb{C}|_{G_0}$ is a Cartan involution on G_0 with fixed point set K. The corresponding symmetric space $X = G_0/K$ is the noncompact dual of U/K.

Let $\mathfrak{a}_* \subset \mathfrak{p}_*$ be a maximal Abelian subspace. Its dimension $r := \dim \mathfrak{a}_*$ is the rank of M. Let $\mathfrak{a}_0 := i\mathfrak{a}_*$ and $\mathfrak{h}_0 \subset \mathfrak{g}_0$ be a maximal Abelian subalgebra

containing $\mathfrak{a}_0$. The complexification of $\mathfrak{h}_0$ (resp. of $\mathfrak{a}_0$) is $\mathfrak{h}$ (resp. $\mathfrak{a}$). Let Δ be the set of nonzero roots corresponding to $(\mathfrak{u}_{\mathbb{C}}, \mathfrak{h})$ and Σ the set of restricted roots.

Let $\mathfrak{h}_{\mathfrak{k}_0} = \mathfrak{h}_0 \cap \mathfrak{k}$ and $\mathfrak{h}_{\mathbb{R}} = \mathfrak{a}_0 + i\mathfrak{h}_{\mathfrak{k}_0}$. The roots are real valued on $\mathfrak{h}_{\mathbb{R}}$. Choose a compatible ordering in the dual spaces of $\mathfrak{a}_0$ and $\mathfrak{h}_{\mathbb{R}}$. This yields an ordering of Δ and Σ. Let ρ_Δ be half the sum of the positive roots and ρ its restriction to $\mathfrak{a}$, i.e. $\rho = (1/2) \sum\limits_{\alpha \in \Sigma^+} m_\alpha \alpha$, where m_α is the multiplicity of α. $\mathfrak{a}_+ \subset \mathfrak{a}_0$ denotes the positive Weyl chamber

$$\mathfrak{a}_+ := \{H \in \mathfrak{a}_0 : \alpha(H) > 0, \forall \alpha \in \Sigma^+\}.$$

The classification of compact, irreducible Riemannian symmetric spaces shows that the restricted root system together with the multiplicity function determines the symmetric space uniquely (see [He1]). In particular

Proposition 9.3.1. *A compact, simply connected Riemannian symmetric space M is isometric to a compact, simply connected Lie group equipped with a biinvariant metric if and only if Σ is a reduced root system and each m_α is equal to* 2.

(See [Lo, Theorem 4.4, p.82]).

Our main result is.

Theorem 9.3.2 (Szőke, [Sz17])**.** *Let (M, g) be a compact, irreducible, simply connected, Riemannian symmetric space. Assume the corrected field of quantum Hilbert spaces $H^{corr} \to S$ obtained from the family of adapted Kähler structures is projectively flat. Then M is isometric to a group manifold, i.e. a compact, connected, simple, simply connected Lie group equipped with a biinvariant metric.*

The special case when (M, g) is a round sphere was proved in [LSz14, Theorem 12.1.1] and the rank-1 case in [LSz15, Theorem 1.1].

The main scheme of the proof of the theorem is based on the rank-1 case [LSz15], but the situation here is much more complicated.

Writing M in the form $M = U/K$, each irreducible $K-$spherical representation of U gives rise to a certain integral on the positive Weyl chamber (9.3.7). The integrand involves the corresponding $K-$spherical function and it also depends on a real parameter τ, that takes arbitrary positive values. Projective flatness of the Hilbert field H^{corr} is expressed as a simple relation among these integrals (Theorem 9.3.4). Since the explicit value of these integrals is not known, one needs other ways to test projective flatness.

The idea in the rank-1 case ([LSz15]) was to tend with τ to zero resp. to ∞, calculate the asymptotic behavior of our integrals and compare the information obtained this way with the relation that holds among the integrals corresponding to different spherical representations. In the rank-1 situation $K-$spherical functions are quite explicit, they reduce to hypergeometric polynomials, greatly simplifying the situation.

In the higher rank case, the basic idea is the same, but the situation is more involved. We still want to calculate the asymptotic behavior of those (now multivariable) integrals as $\tau \to 0$, resp. $\tau \to \infty$. The $\tau \to 0$ asymptotic needs a multivariable version of Watson's lemma. The spherical functions now

correspond to multivariable Jacobi polynomials associated to the restricted root system ([Hec, HO1, HO2, HS]) and they are much more complicated functions to calculate with.

The key observation here is that despite this, their main contribution to the asymptotic behavior of our integrals (when $\tau \to \infty$) is simple. The Jacobi polynomials are actually exponential polynomials, where each term corresponds to a weight of the given $K-$spherical representation. The main contribution comes only from one term, that corresponds to the highest weight. We even know the coefficient of this term, it is Harish-Chandra's $\mathbf{c}-$function. This is the content of Proposition 9.3.8 and Theorem 9.3.16. As a consequence, projective flatness implies that a certain numerical quantity $Q(\delta)$ (see (9.3.27)) associated to every irreducible $K-$spherical representation δ, that involves only the usual Γ function, the restricted root system, the multiplicities and the highest weight of δ, in fact is independent of the representation (Theorem 9.3.19).

Finally the question, for which spaces will this be true, can be translated to a problem about abstract root systems with multiplicities. This problem is treated in Theorem 9.3.23, after which the proof of Theorem 9.3.2 easily follows.

We prove Theorem 9.3.2 in Section 9.3.9. Recall that group manifolds were treated in section 9.2.3 , where in 9.2.4 it was shown that whenever M is isometric to a compact, simply connected Lie group with a biinvariant metric, the field $H^{corr} \to S$ is flat. Theorem 9.3.2 and Theorem 9.2.4 together yields the following result.

Theorem 9.3.3 (Szőke, [Sz17])**.** *Let (M, g) be a compact, irreducible, simply connected, Riemannian symmetric space. Then the corrected field of quantum Hilbert spaces $H^{corr} \to S$ is projectively flat if and only if M is isometric to a group manifold. In the latter case the field $H^{corr} \to S$ is even flat.*

We remark here, that it is not known whether H^{corr} is a Hilbert bundle or not, when M is a compact symmetric space, but not a group manifold.

9.3.1 Flatness and projective flatness

At least for some normal homogeneous spaces, namely for symmetric spaces the computations outlined in 9.2.1 and 9.2.2 can be made concrete enough to determine if the curvature of the associated field of quantum Hilbert spaces is central or not. Consider a compact, simply connected, irreducible Riemannian symmetric space (M^m, g), given in the form $M = U/K$ as in Sect. 9.3

This fits into the framework of 9.2.1 and 9.2.2. To quantize M the family of adapted Kähler structures on all of N will be used and $H^{corr} \to S$ is the corresponding field of quantum Hilbert spaces obtained. (S being the complex upper half plane). We will only treat half–form corrected quantization.

U acts on $(N, J(i))$ by biholomorphisms and this action induces a representation $\hat{\pi}$ on $\mathcal{O}(N, J(i))$, by the formula $av = (a^{-1})^* v$ (pull back by a^{-1}), where $a \in U$, $v \in \mathcal{O}(N, J(i))$. The same formula defines a unitary representation π on $L^2(M)$. The restrictions $V_\chi|_M$ of the isotypical subspaces of $\hat{\pi}$ are precisely the isotypical subspaces of π and the latter are well known to be finite dimensional. Since M is a maximal dimensional, totally real submanifold in N, we get that V_χ are also finite dimensional.

The restrictions of $\hat{\pi}$ to the isotypical subspaces V_χ (or equivalently the restrictions of π to $V_\chi|_M$) are irreducible, they are precisely the irreducible

$K-$spherical representations of U ([He2, Chap. V, Theorem 4.3]). Therefore from now on we use the spherical representations δ themselves instead of their character χ, to label the objects (unlike in section 9.2.2 or [LSz14]), for example V_δ will replace V_χ.

Flatness of the field $H^{corr} \to S$ can be understood in terms of certain Toeplitz operators $P_\delta(s)$ on V_δ (cf. section 9.2.2. They are U-equivariant, whence according to Schur's lemma, have the form $P_\delta(s) = p_\delta(s) Id_{V_\delta}$ with an appropriate function p_δ. $H^{corr} \to S$ is flat (resp. projectively flat) if and only if $\bar\partial\partial \log p_\delta(s) = 0$ for all δ (resp. $\bar\partial\partial \log p_\delta(s)$ is independent of δ), see 8.4.3, ([LSz14, Theorem 9.2.1]).

In our situation according to 9.2.3, ([LSz14, Lemma 11.2.1]) $p_\delta(s)$ depends only on $\tau = \mathrm{Im} s$ and has the specific form

$$p_\delta(s) = C c_\delta \tau^{-m/2} q_\delta(\tau), \tag{9.3.1}$$

where m is the dimension of the space M, C is some constant, c_δ a constant for each representation δ and q_δ an appropriate function (see (9.3.7) for the precise form). As one easily sees, a factor like $C\tau^{-m/2}$ that depends only on $\tau = \mathrm{Im} s$ but not on δ does not affect the condition for projective flatness. In our case, in light of (9.3.1), the above mentioned characterization of (projective) flatness takes the form.

Theorem 9.3.4.

(a) $H^{corr} \to S$ is flat iff for each δ, $\log(p_\delta(s))$ is harmonic.

(b) $H^{corr} \to S$ is projectively flat iff for each δ there exist constants $A_\delta > 0$, B_δ with $q_\delta(\tau) = A_\delta e^{B_\delta \tau} q_{\delta_0}$, where δ_0 denotes the trivial representation.

Since we cannot compute q_δ explicitly, we cannot check directly whether condition (b) in Theorem 9.3.4 holds or not. Therefore we shall apply the following strategy to prove Theorem 9.3.2 We shall investigate the asymptotic behavior of $q_\delta(\tau)$ as τ tends to 0 and to infinity. From the $\tau \to 0$ asymptotics we shall determine the values of A_δ and B_δ dictated by condition (b) in Theorem 9.3.4. Then do the same as $\tau \to \infty$ and obtain possible different values for A_δ and B_δ. If the values for A_δ or B_δ do not match as $\tau \to 0$ and as $\tau \to \infty$, we can conclude that the Hilbert field is not projectively flat.

It turns out, that B_δ does not help in determining the projective flatness of $H^{corr} \to S$, for all rank-1 symmetric spaces the two asymptotics give the same value for B_δ (see Remark 9.3.6, after Theorem 9.3.18). Theorem 9.3.2 is proved by showing that the $\tau \to 0$ asymptotics yields $A_\delta = 1$ for all δ (see Theorem 9.3.14), on the other hand the $\tau \to \infty$ asymptotic shows that if the coefficient A_δ is independent of δ, the restricted root system of M must be reduced and all multiplicities of the roots are equal to two (see Section 9.3.6 and 9.3.8). But these properties characterize compact Lie groups among compact Riemannian symmetric spaces (see [Lo]]) and Theorem 9.3.2 will follow.

9.3.2 The function $q_\delta(\tau)$

Now to implement the plan in Sect. 9.3.1, we need to recall first of all the precise form of $p_\delta(s)$ (see (9.2.12), $\tau =$Ims).

$$p_\delta(s) = \frac{c_\delta}{\tau^{m/2}} \int_{\mathfrak{p}_*} \int_K e^{-\frac{|\zeta|^2}{\tau}} \chi_\delta(k \exp(-2i\zeta)) dk\, \mu d\zeta, \tag{9.3.2}$$

where c_δ is independent of s, dk is normalized Haar measure on K, $d\zeta$ is the Lebesgue measure on $\mathfrak{p}_*$ induced by the metric, χ_δ the character of δ and μ is expressed through the operators A_1, A_2 in (9.2.2).

Now for symmetric spaces $[\mathfrak{k}, \mathfrak{p}_*] \subset \mathfrak{p}_*$ and $[\mathfrak{p}_*, \mathfrak{p}_*] \subset \mathfrak{k}$. Therefore if $\zeta \in \mathfrak{p}_*$ then $P\mathrm{ad}\zeta|\mathbb{C} \otimes \mathfrak{p}_* = 0$,

$$A_1(i,\zeta) = \cos \mathrm{ad}\zeta|\mathbb{C} \otimes \mathfrak{p}_*, \quad A_2(i,\zeta) = i(\sin \mathrm{ad}\zeta)/\mathrm{ad}\zeta|\mathbb{C} \otimes \mathfrak{p}_*, \qquad \text{and}$$
$$i^m \det\big(A_2^*(i,\zeta)A_1(i,\zeta) - A_1^*(i,\zeta)A_2(i,\zeta)\big) = \det\big((\sin 2\mathrm{ad}\zeta)/\mathrm{ad}\zeta|\mathbb{C} \otimes \mathfrak{p}_*\big) > 0.$$

Hence in (9.3.2) $\mu = \sqrt{\eta}$, where

$$\eta(\zeta) = \det\left(\frac{\sin 2\mathrm{ad}\zeta}{\mathrm{ad}\zeta}\bigg|_{\mathbb{C}\otimes\mathfrak{p}_*}\right). \tag{9.3.3}$$

The function $f_\delta(g) = \int_K \chi_\delta(kg^{-1})dk$, $g \in U$ is known as the $K-$spherical function ([Har1, Har2]), corresponding to the representation δ, see [He2, Theorem 4.2, p.417]. We denote by the same letter the holomorphic extension of f_δ to the complexified group $U_\mathbb{C}$. Thus we can rewrite (9.3.2) as an integral over $\mathfrak{p}_0$ and we get

$$p_\delta(s) = \frac{c_\delta}{\tau^{m/2}} \int\limits_{\mathfrak{p}_0} e^{-\frac{|H|^2}{\tau}} f_\delta(\exp(2H))\sqrt{\eta(-iH)}\, dH. \tag{9.3.4}$$

Every restricted root $\alpha \in \Sigma$ is real valued on $\mathfrak{a}_0$. Furthermore the operator ad_H^2, $H \in \mathfrak{p}_0$, is symmetric, has zero eigenvalue with multiplicity $r = \dim \mathfrak{a}_0$ and $\alpha(H)^2$ with multiplicity m_α. Thus from (9.3.3) and the identity $\sin i2z/iz = \mathrm{sh}2z/z$ we get

$$\eta(-iH) = 2^r \prod_{\alpha\in\Sigma^+} \left(\frac{\mathrm{sh}(2\alpha(H))}{\alpha(H)}\right)^{m_\alpha}. \tag{9.3.5}$$

Let $C(\mathfrak{a}_0) := \{k \in K : Ad(k)\zeta = \zeta,\ \forall \zeta \in \mathfrak{a}_0\}$ be the centralizer of $\mathfrak{a}_0$ in K. Recall the following integral formula for the generalized polar coordinate map

$$\Phi : (K/C(\mathfrak{a}_0)) \times \mathfrak{a}_0 \to \mathfrak{p}_0, \quad \Phi(kC(\mathfrak{a}_0), H) := Ad(k)H,$$

Theorem 9.3.5. *Let $f \in L^1(\mathfrak{p}_0)$ be an $Ad(K)$ invariant function. Then*

$$\int\limits_{\mathfrak{p}_0} f(H)dH = c \int\limits_{\mathfrak{a}_+} f(H) \prod_{\alpha\in\Sigma^+} \alpha(H)^{m_\alpha} dH,$$

where c is some constant, independent of f and $\mathfrak{a}_+ \subset \mathfrak{a}_0$ denotes the positive Weyl chamber.

(See [He2, Theorem 5.17, p.195].)

Proposition 9.3.6 (Lempert, Szőke [LSz15])**.** *The function $f_\chi \circ \exp$ is Ad_K invariant on the Lie algebra $\mathfrak{u}_\mathbb{C}$ of $U_\mathbb{C}$.*

Proof. For any $k, k_0 \in K, \zeta \in \mathfrak{u}_\mathbb{C}$

$$\chi(k\exp(-\mathrm{Ad}(k_0)\zeta)) = \chi(kk_0\exp(-\zeta)k_0^{-1}) = \chi(k_0^{-1}kk_0\exp(-\zeta)).$$

Thus

$$f_\chi(\exp(\mathrm{Ad}(k_0)\zeta)) = \int_K \chi(k_0^{-1}kk_0\exp(-\zeta))dk = f_\chi(\exp(\zeta)).$$

□

Proposition 9.3.7 (Lempert, Szőke [LSz15]). *Let $F \in \mathcal{O}(\mathbb{C})$ be an even function and $v \in \mathfrak{p}_*$. Then $F(\mathrm{ad}(v))$ (defined by its power series) maps $\mathbb{C} \otimes \mathfrak{p}_*$ into itself and $\det(F(\mathrm{ad}(v))|_{\mathbb{C}\otimes\mathfrak{p}_*})$ is an Ad_K invariant function.*

Proof. For every k in K, $\mathrm{Ad}(k)$ is in $\mathrm{Aut}(\mathfrak{u})$. Thus for every $v \in \mathfrak{u}$, $l = 0, 1, \dots$

$$(\mathrm{ad}(\mathrm{Ad}(k)v))^l = \mathrm{Ad}(k) \circ (\mathrm{ad}(v))^l \circ \mathrm{Ad}(k)^{-1}.$$

Hence

$$F(\mathrm{ad}(\mathrm{Ad}(k)v)) = \mathrm{Ad}(k) \circ F(\mathrm{ad}(v)) \circ \mathrm{Ad}(k)^{-1}.$$

Since $\mathfrak{p}_*$ is both $\mathrm{Ad}(k)$ and $(\mathrm{ad}(v))^{2l}$ invariant ($l = 0, 1 \dots$), the statement follows. □

From Proposition 9.3.6 and Proposition 9.3.7 we know that $f_\delta \circ \exp$ and η are Ad_K invariant on $\mathfrak{p}_0$. Thus Theorem9.3.5, (9.3.4) and (9.3.5) yields the following formula.

$$p_\delta(s) = 2^r c_\delta c \tau^{-m/2} \int_{\mathfrak{a}_+} e^{-\frac{|H|^2}{\tau}} f_\delta(\exp(2H)) \prod_{\alpha\in\Sigma^+} (\alpha(H)\mathrm{sh}(2\alpha(H)))^{\frac{m_\alpha}{2}}\, dH. \tag{9.3.6}$$

In the special case when M is isometric to a compact Lie group G, let $U = G \times G$ and K be the diagonal in U. Then the $K-$spherical functions will have the form $f_\delta = \chi_\delta/d(\delta)$, where δ is an irreducible representation of G, χ_δ its character and $d(\delta)$ denotes its dimension ([He2, p.407]). Thus, as we saw in section 9.2.4, f_δ is given by the Weyl character formula and since all $m_\alpha = 2$ the terms $\mathrm{sh}(2\alpha(H))$ cancel out the Weyl denominator and we end up essentially integrating the product of a Gaussian and a harmonic polynomial. This yielded that $\log(p_\delta(s)) = c_1 + c_2 \mathrm{Im} s$, that is a harmonic function and we got that the field $H^{corr} \to S$ was flat.

To treat the other symmetric spaces, we introduce the essential part of p_δ as a function of $\tau > 0$:

$$q_\delta(\tau) := \int_{\mathfrak{a}_+} e^{-\frac{|H|^2}{\tau}} f_\delta(\exp(2H)) \prod_{\alpha\in\Sigma^+} (\alpha(H)\mathrm{sh}(2\alpha(H)))^{\frac{m_\alpha}{2}}\, dH. \tag{9.3.7}$$

9.3.3 Spherical functions

In order to be able to handle the integral in (9.3.7), we shall need another description of spherical functions. Let $\delta : U \to GL(V)$ be an irreducible $K-$spherical representation. We can endow V with a scalar product $\langle .,. \rangle$ that makes δ unitary. Let $v_K \in V$ be a $K-$fixed vector with unit length. Then the spherical function f_δ corresponding to δ is ([He2, Theorem 3.7, p.414])

$$f_\delta(g) := \langle \delta(g)v_K, v_K \rangle, \quad g \in U.$$

Since δ extends holomorphically to the complexified group $U_{\mathbb{C}}$, the same formula yields the holomorphic extension of f_δ to $U_{\mathbb{C}}$.

We would like to obtain some formula for the function $f_\delta \circ \exp$, occurring in (9.3.7), when we restrict it to the Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{u}_{\mathbb{C}}$.

Let $\Lambda(\delta)$ be the set of weights of δ and for a weight μ, W_μ the corresponding weight space.

The weight spaces give an orthogonal direct decomposition of V, thus

$$v_K = \sum_{\mu \in \Lambda(\delta)} w_\mu, \quad w_\mu \in W_\mu,$$

where $\|v_K\| = 1$ implies $\sum \|w_\mu\|^2 = 1$.

Let $H \in \mathfrak{h}$. Then (cf. [Vr])

$$f_\delta(\exp 2H) = \langle \exp(\delta_* 2H) v_K, v_K \rangle = \sum_{\mu \in \Lambda(\delta)} e^{2\mu(H)} \langle w_\mu, w_\mu \rangle. \tag{9.3.8}$$

Later on we shall need to figure out which term in (9.3.8) has the dominating contribution when (9.3.8) is plugged into the formula (9.3.7) of q_δ. It is no surprise that the term corresponding to the highest weight will play this role. Theorem 9.3.16 will give the precise answer. That theorem will be based on Theorem 9.3.15, a general result on asymptotics of integrals of the form (9.3.7), where the function f_δ is replaced by an exponential of a linear function, like the terms in (9.3.8). The result of Theorem 9.3.15 shall explain why we need Proposition 9.3.8.

Let λ be the highest weight of δ. Then $\dim W_\lambda = 1$. Let $v_\lambda \in W_\lambda$ with $\|v_\lambda\| = 1$. Thus $w_\lambda = a_\lambda v_\lambda$ with $a_\lambda = \langle v_K, v_\lambda \rangle$. From the first formula in [He2, p.538], we know that $a_\lambda \neq 0$ and

$$\langle w_\lambda, w_\lambda \rangle = |a_\lambda|^2 = \mathbf{c}(-i\lambda - i\rho), \tag{9.3.9}$$

where $\rho = \rho_\Delta|_{\mathfrak{a}_0}$ is half the sum of the positive restricted roots with multiplicity, $X = G/K$ the noncompact dual symmetric space and $\mathbf{c}$ is the corresponding Harish-Chandra's $\mathbf{c}$–function of G ([Har1, Har2], [He2, (8), p.538]).

Proposition 9.3.8 (Szőke [Sz17])**.** *Let $\mu \in \Lambda(\delta)$, $\mu \neq \lambda$. Then*

$$\| (\mu + \rho_\Delta)|_{\mathfrak{a}_0} \| < \| (\lambda + \rho_\Delta)|_{\mathfrak{a}_0} \|.$$

Proof. We follow the steps of the proof of [He2, Theorem. 1.3, p.498], that is the same statement without taking restrictions to $\mathfrak{a}_0$. First we show that

$$(\lambda - \mu)|_{\mathfrak{a}_0} \not\equiv 0. \tag{9.3.10}$$

Since λ is the highest weight of a $K-$spherical representation, the Cartan-Helgason theorem ([He2, Theorem 4.1 (1), p.535]) implies

$$\lambda|_{i\mathfrak{h}_{\mathfrak{k}_0}} \equiv 0.$$

Thus if (9.3.10) does not hold, we would get $\langle \lambda - \mu, \lambda \rangle = 0$ and then

$$\langle \mu, \mu \rangle = \langle \mu - \lambda, \mu - \lambda \rangle + \langle \lambda, \lambda \rangle > \langle \lambda, \lambda \rangle, \tag{9.3.11}$$

since $\mu \neq \lambda$. But (9.3.11) contradicts the fact that for all weights μ, $\|\mu\| \leq \|\lambda\|$ (see [He2, Theorem 1.3 (7), p.498]) and so (9.3.10) is proved.

We need to show

$$C := \|\lambda + \rho\|^2 - \| \mu|_{\mathfrak{a}_0} + \rho\|^2 > 0.$$

But

$$C = \|\lambda\|^2 - \| \mu|_{\mathfrak{a}_0} \|^2 + 2\langle \lambda - \mu|_{\mathfrak{a}_0}, \rho\rangle \geq \|\lambda\|^2 - \|\mu\|^2 + 2\langle \lambda - \mu|_{\mathfrak{a}_0}, \rho\rangle. \quad (9.3.12)$$

And since $\|\lambda\| \geq \|\mu\|$, it suffices to show that the last term in (9.3.12) is positive.

Let $\alpha_1, \ldots, \alpha_l$ be a basis of the roots, compatible with Σ, i.e. for $1 \leq j \leq r$ $\alpha_j|_{\mathfrak{a}_0} \in \Sigma^+$ forming a basis of Σ. Since μ is a weight, $\exists n_j \in \mathbb{Z}_+$ with $\mu = \lambda - \sum_1^l n_j \alpha_j$. Now (9.3.10) implies that $\exists j$ with $1 \leq j \leq r$ and $n_j > 0$. Proposition 9.3.9 below shows that $\langle \alpha_j|_{\mathfrak{a}_0}, \rho\rangle > 0$ for $1 \leq j \leq r$. Hence

$$\langle \lambda - \mu|_{\mathfrak{a}_0}, \rho\rangle = \sum_1^r n_j \langle \alpha_j|_{\mathfrak{a}_0}, \rho\rangle > 0,$$

thus indeed $C > 0$. □

Proposition 9.3.9 (Szőke [Sz17]). *Let $\alpha_1, \ldots, \alpha_r \in \Sigma^+$ be a basis of the restricted roots Σ with multiplicities m_{α_j}. Then*

$$\langle \rho, \alpha_j\rangle = (m_{\alpha_j}/2 + m_{2\alpha_j})\langle \alpha_j, \alpha_j\rangle, \quad j = 1 \ldots, r \quad (9.3.13)$$

where $m_{2\alpha_j}$ is meant to be zero if $2\alpha_j$ is not a root.

Proof. Let $\Sigma_j^+ = \Sigma^+ \setminus \{\alpha_j, 2\alpha_j\}$, $\rho_j = \frac{1}{2} \sum_{\alpha \in \Sigma_j^+} m_\alpha \alpha$ and S_{α_j} the reflection on $\mathfrak{a}_0$, corresponding to α_j. As is well known ([He1, ChVII, Sect. 3, Lemma 2.21]) S_{α_j} permutes the elements of Σ_j^+, hence $S_{\alpha_j}\rho_j = \rho_j$. From their definitions we get

$$\rho = \rho_j + \frac{m_{\alpha_j}\alpha_j + m_{2\alpha_j} 2\alpha_j}{2}.$$

Thus

$$S_{\alpha_j}\rho = \rho - m_{\alpha_j}\alpha_j - m_{2\alpha_j} 2\alpha_j.$$

Since S_{α_j} is an orthogonal transformation, we obtain

$$\langle \rho, \alpha_j\rangle = \langle S_{\alpha_j}\rho, S_{\alpha_j}\alpha_j\rangle = \langle \rho - m_{\alpha_j}\alpha_j - m_{2\alpha_j} 2\alpha_j, -\alpha_j\rangle$$

and (9.3.13) follows. □

9.3.4 $\tau \to 0$ asymptotics, a multivariable Watson lemma

Proposition 9.3.10 (Szőke [Sz17]). *Let $0 < \tau$, $0 < h$, $D \subset \mathbb{R}^n$ be a domain that is a homogeneous cone ($\xi \in D, 0 < r$ implies $r\xi \in D$), $G := D \cap S^{n-1}$ (where S^{n-1} is the unit sphere in $\mathbb{R}^n$) and Q an h-homogeneous (for all $\xi \in D, 0 < r$, $Q(r\xi) = r^h Q(\xi)$) continuous function defined on $\overline{D}$. Then*

$$\int_D e^{\frac{-\|H\|^2}{\tau}} Q(H) dH = \frac{\Gamma(\frac{n+h}{2})}{2} \left(\int_G Q(\xi) d\xi \right) \tau^{\frac{n+h}{2}},$$

where Γ denotes the usual gamma function.

Proof. Using polar coordinates and the homogeneity of Q we get

$$\int_D e^{\frac{-\|H\|^2}{\tau}} Q(H)dH = \int_0^\infty \int_G e^{\frac{-r^2}{\tau}} r^{n+h-1} Q(\xi)d\xi dr.$$

Substituting $r = \sqrt{\tau t}$ yields the formula.

□

Proposition 9.3.11 (Szőke [Sz17])**.** *Let $\delta, \tau_0 > 0$, $D \subset \mathbb{R}^n$ be a domain, $D_\delta := D \cap \{\|H\| \geq \delta\}$ and g a Lebesgue measurable function on D with*

$$C := \int_D e^{-\frac{\|H\|^2}{\tau_0}} |g(H)|dH < \infty.$$

Then for every $0 < \tau < \tau_0$

$$\int_{D_\delta} e^{\frac{-\|H\|^2}{\tau}} |g(H)|dH \leq Ce^{\delta^2(\frac{1}{\tau_0} - \frac{1}{\tau})}.$$

Proof. Let $\delta \leq \|H\|$. Then

$$e^{\|H\|^2(\frac{1}{\tau_0} - \frac{1}{\tau})} \leq e^{\delta^2(\frac{1}{\tau_0} - \frac{1}{\tau})}.$$

Thus

$$\int_{D_\delta} e^{\frac{-\|H\|^2}{\tau}} |g(H)|dH = \int_{D_\delta} e^{\frac{-\|H\|^2}{\tau_0}} |g(H)| e^{\|H\|^2(\frac{1}{\tau_0} - \frac{1}{\tau})} dH \leq$$
$$\leq \int_{D_\delta} e^{\frac{-\|H\|^2}{\tau_0}} |g(H)| e^{\delta^2(\frac{1}{\tau_0} - \frac{1}{\tau})} dH.$$

□

Theorem 9.3.12 (Szőke [Sz17])**.** *Let $0 < a \leq \infty$, G be a domain in S^{n-1} (unit sphere), $0 < d$,*

$$G_a := \{r\xi : 0 < r < a, \xi \in G\}$$

and Q a $d-$homogeneous continuous function defined on $\overline{G}_a$. Suppose $f \in C(G_a)$ that is C^∞ in a neighborhood of the origin. Assume that for some $0 < \tau_0$ the function $e^{-\|H\|^2/\tau_0} Q(H)f(H)$ is in $L^1(G_a)$. For $0 < \tau < \tau_0$ let $\Phi(\tau)$ be defined by

$$\Phi(\tau) = \int_{G_a} e^{-\frac{\|H\|^2}{\tau}} Q(H)f(H)dH.$$

Then Φ admits an asymptotic series expansion around 0:

$$\Phi(\tau) \sim \sum_{j=0}^{\infty} \frac{\Gamma(\frac{n+d+j}{2})}{2} \int_G QP_j d\xi \; \tau^{\frac{n+d+j}{2}}, \quad \tau \to 0,$$

where P_j is the $j-th$ homogeneous polynomial term of the Taylor series of f around the origin.

Proof. We follow the scheme of the proof of Watson's lemma in one variable (cf. [Mu]). Let $0 < \delta \leq a$ be so small that f is C^∞ in a neighborhood of the ball $\overline{\mathbb{B}}^n_\delta(0)$. Then $G_a \cap \mathbb{B}^n_\delta(0) = G_\delta$ and with $h(\tau, H) = e^{-\|H\|^2/\tau} Q(H) f(H)$

$$\Phi(\tau) = \int\limits_{G_a \cap \{\|H\| \geq \delta\}} h(\tau, H) dH + \int\limits_{G_\delta} h(\tau, H) dH =: \Phi_1(\tau) + \Phi_2(\tau).$$

With $g(H) = Q(H)f(H)$ and $C = \int\limits_{G_a} e^{-\frac{\|H\|^2}{\tau_0}} |Q(H)f(H)| dH$, Proposition 9.3.11 implies

$$|\Phi_1(\tau)| \leq C e^{\frac{\delta^2}{\tau_0}} e^{-\frac{\delta^2}{\tau}} = o(\tau^n), \quad \tau \to 0,$$

for all $n \in \mathbb{N}$. The Taylor formula with remainder term yields

$$f(H) = \sum_{j=0}^{N} P_j(H) + f_N(H), \quad \|H\| \leq \delta, \quad |f_N(H)| \leq C_N \|H\|^{N+1}, \quad (9.3.14)$$

where P_j is a $j-$homogeneous polynomial and C_N an appropriate constant. Thus

$$\Phi_2(\tau) = \sum_{j=0}^{N} \int\limits_{G_\delta} e^{-\frac{\|H\|^2}{\tau}} Q(H) P_j(H) dH + \int\limits_{G_\delta} e^{-\frac{\|H\|^2}{\tau}} Q(H) f_N(H) dH$$

and

$$\int\limits_{G_\delta} e^{-\frac{\|H\|^2}{\tau}} Q(H) P_j(H) dH = \int\limits_{G_\infty} e^{-\frac{\|H\|^2}{\tau}} Q(H) P_j(H) dH - \int\limits_{G_\infty \cap \{\|H\| \geq \delta\}} e^{-\frac{\|H\|^2}{\tau}} Q(H) P_j(H) dH.$$

In light of Proposition 9.3.10 the first integral on the right hand side is equal to

$$\frac{\Gamma(\frac{n+d+j}{2})}{2} \left(\int\limits_G Q P_j d\xi \right) \tau^{\frac{n+d+j}{2}},$$

and Proposition 9.3.11 yields with $g = QP_j$, that the second integral is $o(\tau^n)$ for all $n \in \mathbb{N}$. Homogeneity of Q implies $|Q(H)| \leq K\|H\|^d$ with some $K > 0$. Then from (9.3.14) and Proposition 9.3.10 we get

$$\left| \int\limits_{G_\delta} e^{-\frac{\|H\|^2}{\tau}} Q(H) f_N(H) dH \right| \leq C_N K \int\limits_{G_\infty} e^{-\frac{\|H\|^2}{\tau}} \|H\|^{N+d+1} dH =$$
$$= Vol(G) C_N K \frac{\Gamma(\frac{n+d+N+1}{2})}{2} \tau^{\frac{n+d+N+1}{2}},$$

finishing the proof of the theorem.

□

9.3.5 Determining A_δ and B_δ from $\tau \to 0$

Let us get back to the symmetric space situation. Suppose $(M^m = U/K, g)$ is a compact, simply connected, irreducible, Riemannian symmetric space as in Section 9.3. As before, m is the dimension of M. Let δ be an irreducible $K-$spherical representation and f_δ the corresponding spherical function. Then

$$f_\delta(\exp(2H)) = 1 + R_1(H) + R_2(H) + \dots, \quad H \in \mathfrak{a}_0, \tag{9.3.15}$$

where R_j is the $j-$th homogeneous polynomial term of the Taylor series. Since $f_\delta \circ \exp$ is Ad_K invariant on $\mathfrak{p}_0$ (see [LSz15, Proposition 2.1]), it is Weyl group invariant on $\mathfrak{a}_0$. Therefore every R_j is Weyl group invariant as well. Since M is irreducible, the Weyl group acts irreducible on $\mathfrak{a}_0$, thus $R_1 \equiv 0$ and R_2 must be of the form

$$R_2(H) = b_\delta \|H\|^2, \tag{9.3.16}$$

with some $b_\delta \in \mathbb{R}$. (9.3.16) is true because up to a constant scalar, $\|H\|^2$ is the only Weyl group invariant quadratic polynomial on $\mathfrak{a}_0$. One can see this either as a corollary of Schur's lemma, or as a corollary of Chevalley's theorem (see [Hum, Sect. 3.5, 3.7]). For the trivial representation δ_0, $f_{\delta_0} \equiv 1$ and $b_{\delta_0} = 0$.

Proposition 9.3.13 (Szőke [Sz17])**.** *Assume that the rank of M is 1 and λ is the highest weight of δ. Then*

$$b_\delta = \frac{2(\|\lambda + \rho\|^2 - \|\rho\|^2)}{m}.$$

Proof. If Σ is nonreduced, $\Sigma^+ = \{\beta, \beta/2\}$ and $\Sigma^+ = \{\beta\}$ in the reduced case. The corresponding multiplicities are m_β and $m_{\beta/2}$, where our convention is that the latter is zero when Σ is reduced. Let $H_0 \in \mathfrak{a}_+$ with $\|H_0\| = 1$. Then $\beta(H_0) = \|\beta\|$. Recall that Gauss' hypergeometric functions are given by

$$F(a,b,c,z) := 1 + \frac{ab}{c} z + \dots + \frac{a(a+1)\dots(a+k-1)b(b+1)\dots(b+k-1)}{k!c(c+1)\dots(c+k-1)} z^k + \dots$$

where $a, b, c \in \mathbb{C}$, $c \notin \mathbb{Z}_- = \{0, -1, -2, \dots\}$. The series converges at least in the unit disk. If $n \in \mathbb{Z}_+$, $b = -n$, $A \in \mathbb{C} \setminus \mathbb{Z}_-$, and $a = A + n$, then F is a polynomial (in z) of degree n.

According to [He2, Theorem 4.1(ii), p. 535, and Sect. 3, p. 542] the highest weight of δ has the form $\lambda = n_\delta \beta$, where $n_\delta \in \mathbb{Z}_+$. Let

$$a_\delta := \frac{1}{2} m_{\beta/2} + m_\beta + n_\delta, \quad c_\delta := \frac{m_{\beta/2} + m_\beta + 1}{2} = \frac{m}{2}.$$

Denote by F_δ the hypergeometric function (polynomial in this case), corresponding to these parameters

$$F_\delta(x) = F(a_\delta, -n_\delta, c_\delta, x).$$

According to [He2, formula (25), p.543], the spherical function f_δ can be expressed as

$$f_\delta(\exp(2H)) = F_\delta(-\mathrm{sh}^2(\beta(H))), \quad H \in \mathfrak{a}_0.$$

Hence

$$f_\delta(\exp(2H)) = 1 + \frac{a_\delta n_\delta}{c_\delta} \|\beta\|^2 \|H\|^2 + o(\|H\|^2).$$

Thus $b_\delta = \frac{a_\delta n_\delta}{c_\delta}\|\beta\|^2$. Now $\rho = \frac{1}{2}(m_{\beta/2}\beta/2 + m_\beta\beta)$, hence

$$a_\delta n_\delta \|\beta\|^2 = 2\langle\rho,\lambda\rangle + \|\lambda\|^2,$$

and our statement follows. □

The $\tau \to 0$ asymptotics yields the following values for A_δ, B_δ in Theorem 9.3.4 (b).

Theorem 9.3.14 (Szőke [Sz17])**.** *Suppose the corrected field of quantum Hilbert spaces $H^{corr} \to S$ is projectively flat. Then for every irreducible $K-$spherical representation δ,*

$$A_\delta = 1, \quad B_\delta = \frac{m}{2} b_\delta.$$

Proof. Easy calculation shows that

$$F(H) := \prod_{\alpha\in\Sigma^+}\left(\frac{\mathrm{sh}(2\alpha(H))}{\alpha(H)}\right)^{\frac{m_\alpha}{2}} = 1 + \sum_{\alpha\in\Sigma^+}\frac{m_\alpha}{3}\alpha^2(H) + \dots \tag{9.3.17}$$

From (9.3.16) and (9.3.17) we obtain that in the homogeneous polynomial series expansion of

$$f_\delta(\exp(2H))F(H) = 1 + P_2^\delta(H) + P_3^\delta(H) + \dots,$$

the quadratic term is

$$P_2^\delta(H) = b_\delta\|H\|^2 + \sum_{\alpha\in\Sigma^+}\frac{m_\alpha}{3}\alpha^2(H) = b_\delta\|H\|^2 + P_2^{\delta_0}(H). \tag{9.3.18}$$

Now $Q(H) := \prod_{\alpha\in\Sigma^+}\alpha(H)^{m_\alpha}$ is a homogeneous polynomial of degree

$$d = \sum_{\alpha\in\Sigma^+} m_\alpha = m - r,$$

where $r = \dim\mathfrak{a}_0$ is the rank of M. Applying Theorem 9.3.12 with f, Q, $a = \infty$, $G = \mathfrak{a}_+ \cap S^{r-1}$ we obtain

$$q_\delta(\tau) = \frac{\Gamma(\frac{m}{2})}{2}\int_G Q(\xi)d\xi\ \tau^{\frac{m}{2}} + \frac{\Gamma(\frac{m+2}{2})}{2}\int_G Q(\xi)P_2^\delta(\xi)d\xi\ \tau^{\frac{m+2}{2}} + o(\tau^{\frac{m+2}{2}}). \tag{9.3.19}$$

Hence we get $\int_G Q(\xi)d\xi > 0$, since the restricted roots are positive on the Weyl chamber $\mathfrak{a}_+$. Now writing out (9.3.19) for both δ and the trivial representation δ_0, comparing the coefficients of the $\tau^{\frac{m}{2}}$ term in the asymptotic series and using Theorem 9.3.4 (b) we obtain $A_\delta = 1$. Then comparing the coefficients of the $\tau^{\frac{m+2}{2}}$ as well, we obtain

$$B_\delta\frac{\Gamma(\frac{m}{2})}{2}\int_G Q(\xi)d\xi = \frac{\Gamma(\frac{m+2}{2})}{2}\int_G Q(\xi)(P_2^\delta(\xi) - P_2^{\delta_0}(\xi))d\xi. \tag{9.3.20}$$

From (9.3.18) we get $P_2^\delta(\xi) - P_2^{\delta_0}(\xi) = b_\delta\|\xi\|^2 = b_\delta$, since G is part of the unit sphere. Thus (9.3.20) yields $B_\delta = \frac{m}{2}b_\delta$.

□

9.3.6 Asymptotics at infinity

The following setting is motivated by the system of restricted roots of a compact Riemannian symmetric space.

Let $(Z, \langle .,. \rangle)$ be a Euclidean space of dimension r and $\Sigma^+ \subset Z^*$ a finite set so that

$$Z_+ := \{H \in Z \mid \alpha(H) > 0, \forall \alpha \in \Sigma^+\}$$

is nonempty. For each $\alpha \in \Sigma^+$, let $m_\alpha > 0$ be given and define

$$m := r + \sum_{\alpha \in \Sigma^+} m_\alpha, \quad \rho := \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha.$$

For a linear functional $l : Z \to \mathbb{R}$, define $A_l \in Z$ by $l(H) =< A_l, H >$, $H \in Z$. Then $\langle l, L \rangle := \langle A_l, A_L \rangle$, $l, L \in Z^*$, defines an inner product on Z^*. Let $f : Z_+ \to \mathbb{R}$ be any measurable function. Assuming the integral below is finite, introduce the following function, defined for $\tau > 0$.

$$q(\tau, f) = \int_{Z_+} e^{-\frac{\|H\|^2}{\tau}} f(H) \prod_{\alpha \in \Sigma^+} (\alpha(H) \mathrm{sh}(2\alpha(H)))^{\frac{m_\alpha}{2}} \, dH. \tag{9.3.21}$$

With $\mu \in Z^*$, let $I_\mu(\tau) := q(\tau, e^{2\mu})$. Even though it is impossible to calculate precisely this integral (except in some special cases), it is possible to determine the order of its magnitude as $\tau \to \infty$, and that suffices for our purposes.

Theorem 9.3.15 (Szőke [Sz17])**.** *For any $\mu \in Z^*$*

$$I_\mu(\tau) = \begin{cases} 2^{r-m} \pi^{\frac{r}{2}} \prod_{\alpha \in \Sigma^+} \langle \mu + \rho, \alpha \rangle^{\frac{m_\alpha}{2}} \tau^{\frac{m}{2}} e^{\tau \|\mu+\rho\|^2} (1 + o(1)), & A_{\mu+\rho} \in Z_+ \\ \tau^{\frac{m}{2}} e^{\tau \|\mu+\rho\|^2} o(1), & A_{\mu+\rho} \in Z \setminus Z_+ \end{cases}$$

as $\tau \to \infty$.

Proof. Factoring out $e^{m_\alpha \alpha(H)}$ from the product for each $\alpha \in \Sigma^+$, we get

$$I_\mu(\tau) = 2^{r-m} \int_{Z_+} e^{-\frac{\|H\|^2}{\tau} + 2\mu(H) + 2\rho(H)} \prod_{\alpha \in \Sigma^+} \alpha(H)^{\frac{m_\alpha}{2}} (1 - e^{-4\alpha(H)})^{\frac{m_\alpha}{2}} dH$$

Now

$$-\|H\|^2/\tau + 2\mu(H) + 2\rho(H) = -\|H/\sqrt{\tau} - \sqrt{\tau} A_{\mu+\rho}\|^2 + \tau \|\mu + \rho\|^2.$$

Thus

$$I_\mu(\tau) = \frac{e^{\tau \|\mu+\rho\|^2}}{2^{m-r}} \int_{Z_+} e^{-\|H/\sqrt{\tau} - \sqrt{\tau} A_{\mu+\rho}\|^2} \prod_{\alpha \in \Sigma^+} \alpha(H)^{\frac{m_\alpha}{2}} (1 - e^{-4\alpha(H)})^{\frac{m_\alpha}{2}} dH$$

Let $\Phi_\tau(Y)$ be the affine linear change of coordinates in Z defined by

$$\Phi_\tau(Y) := \sqrt{\tau} Y + \tau A_{\mu+\rho}.$$

Then $\det \Phi'_\tau = \tau^{\frac{r}{2}}$ and with $H = \Phi_\tau(Y)$,

$$\alpha(H) = \alpha(\sqrt{\tau} Y + \tau A_{\mu+\rho}) = \tau \alpha(Y/\sqrt{\tau} + A_{\mu+\rho}).$$

Using the coordinate change Φ_τ the integral I_μ is transformed to

$$I_\mu(\tau) = \frac{\tau^{\frac{m}{2}} e^{\tau\|\mu+\rho\|^2}}{2^{m-r}} \int\limits_{\Phi^{-1}(Z_+)} e^{-\|Y\|^2} \prod_{\alpha\in\Sigma^+} \alpha(\Phi_\tau(Y)/\tau)^{\frac{m_\alpha}{2}} (1-e^{-4\alpha(\Phi_\tau(Y))})^{\frac{m_\alpha}{2}} dY$$

Let $\chi_\tau(Y)$ be the characteristic function of the set $\Phi^{-1}(Z_+)$ and let

$$g_\tau(Y) := \chi_\tau(Y) \prod_{\alpha\in\Sigma^+} \alpha(Y/\sqrt{\tau} + A_{\mu+\rho})^{\frac{m_\alpha}{2}} (1 - e^{-4\alpha(\Phi_\tau(Y))})^{\frac{m_\alpha}{2}},$$

that is now defined on the entire space Z and

$$I_\mu(\tau) = \frac{\tau^{\frac{m}{2}} e^{\tau\|\mu+\rho\|^2}}{2^{m-r}} \int\limits_Z e^{-\|Y\|^2} g_\tau(Y) dY.$$

We want to show that the integral here has a limit as $\tau \to \infty$. First we prove this for the function $g_\tau(Y)$.

Claim. *For all $Y \in Z$*

$$\lim_{\tau\to\infty} g_\tau(Y) = \begin{cases} \prod\limits_{\alpha\in\Sigma^+} \langle \mu+\rho, \alpha\rangle^{\frac{m_\alpha}{2}}, & A_{\mu+\rho} \in Z_+ \\ 0, & A_{\mu+\rho} \in Z \setminus Z_+ \end{cases}$$

Proof of the Claim. First let $A_{\mu+\rho} \in Z_+$. Then $\alpha(A_{\mu+\rho}) > 0$, for all Σ^+. Let $Y \in Z$ be arbitrary. Then with an appropriate τ_0, $\alpha(\sqrt{\tau}Y + \tau A_{\mu+\rho}) > 0$ holds for every $\tau \geq \tau_0$. Thus $Y \in \Phi_\tau^{-1}(Z_+)$ and so $\chi_\tau(Y) = 1$ for $\tau \geq \tau_0$. Also

$$\lim_{\tau\to\infty} \alpha(Y/\sqrt{\tau} + A_{\mu+\rho}) = \alpha(A_{\mu+\rho}) = \langle \alpha, \mu+\rho\rangle > 0$$

and hence $\lim\limits_{\tau\to\infty} \alpha(\Phi_\tau Y) = \infty$. All these together prove our claim in this case.

Now let $A_{\mu+\rho} \in Z \setminus Z_+$. Suppose there is an $\alpha \in \Sigma^+$ with $\alpha(A_{\mu+\rho}) < 0$. Then for all Y in Z there exists some $\tau_0 > 0$ so that for every $\tau \geq \tau_0$, $\alpha(\sqrt{\tau}Y + \tau A_{\mu+\rho}) < 0$ and consequently $Y \notin \Phi_\tau^{-1}(Z_+)$ implying $\chi_\tau(Y) = 0 = g_\tau(Y)$.

Now assume there is at least one $\alpha \in \Sigma^+$ with $\alpha(A_{\mu+\rho}) = 0$ and $\beta(A_{\mu+\rho}) \geq 0$ for all $\beta \in \Sigma^+$. Denote by Σ_{+0} those $\beta \in \Sigma^+$, for which $\beta(A_{\mu+\rho}) = 0$.

Let $Y \in Z$. If there exists a $\beta \in \Sigma_{+0}$ with $\beta(Y) \leq 0$, then $\beta(\sqrt{\tau}Y + \tau A_{\mu+\rho}) \leq 0$ and so $\chi_\tau(Y) = 0 = g_\tau(Y)$ for all $\tau > 0$.

Suppose that for all $\beta \in \Sigma_{+0}$, $\beta(Y) > 0$. Then for all $\tau > 0$ and $\beta \in \Sigma_{+0}$, $\beta(\sqrt{\tau}Y) = \beta(\Phi_\tau Y) > 0$ and so $0 < 1 - e^{-4\beta(\Phi_\tau(Y))} < 1$. Also just as before: with an appropriate τ_0, $\beta(\sqrt{\tau}Y + \tau A_{\mu+\rho}) > 0$ holds for every $\tau \geq \tau_0$ and $\beta \in \Sigma^+ \setminus \Sigma_{+0}$. Thus for all $\tau \geq \tau_0$, $\Phi_\tau(Y) \in Z_+$ hence

$$\chi_\tau(Y) = 1 \quad \text{and} \quad 0 < \prod_{\alpha\in\Sigma^+} (1 - e^{-4\alpha(\Phi_\tau(Y)})^{\frac{m_\alpha}{2}} < 1.$$

But

$$\lim_{\tau\to\infty} \prod_{\alpha\in\Sigma^+} (\alpha(Y/\sqrt{\tau} + A_{\mu+\rho}))^{\frac{m_\alpha}{2}} = \prod_{\alpha\in\Sigma^+} (\alpha(A_{\mu+\rho}))^{\frac{m_\alpha}{2}} = 0,$$

proving that $\lim\limits_{\tau\to\infty} g_\tau(Y) = 0$.

□

Now to finish the proof of the theorem we estimate $g_\tau(Y)$. By its definition $g_\tau(Y)$ vanishes outside of the set $\Phi^{-1}(Z_+)$.

Hence the trivial estimate yields

$$|g_\tau(Y)| \leq \prod_{\alpha\in\Sigma^+} \|\alpha\|^{\frac{m_\alpha}{2}} (\|Y\| + \|A_{\mu+\rho}\|)^{\frac{m_\alpha}{2}} =: C.$$

Valid for all $Y \in Z$ and $\tau \geq 1$. Thus $Ce^{-\|Y\|^2}$ is an integrable majorant of $g_\tau(Y)$ for all $\tau \geq 1$. Using Lebesgue's dominated convergence theorem together with our claim and the fact that $\int\limits_Z e^{-\|Y\|^2} dY = \pi^{\frac{r}{2}}$ finishes the proof of the theorem.

□

Back to symmetric spaces again, let $(M^m = U/K, g)$ be a compact, irreducible, simply connected, Riemannian symmetric space, δ an irreducible unitary $K-$spherical representation of U with highest weight λ. $\mathbf{c}$ denotes Harish-Chandra's $\mathbf{c}-$function associated to the dual symmetric space $X = G/K$ and q_δ is from (9.3.7).

Theorem 9.3.16 (Szőke [Sz17])**.**

$$q_\delta(\tau) = 2^{r-m}\pi^{\frac{r}{2}}\mathbf{c}(-i\lambda - i\rho)\left(\prod_{\alpha\in\Sigma^+} \langle\lambda+\rho,\alpha\rangle^{\frac{m_\alpha}{2}}\right)\tau^{\frac{m}{2}} e^{\tau\|\lambda+\rho\|^2}(1+o(1)),$$

as $\tau \to \infty$.

Proof. It follows from the Cartan-Helgason theorem ([He2, Theorem 4.1, p.535]), that $A_\lambda \in \overline{\mathfrak{a}}_+$. But then Proposition 9.3.9 implies with $l = \lambda + \rho$, that $A_l \in \mathfrak{a}_+$. Thus if μ is a weight of δ, different from λ, Proposition 9.3.8 and Theorem 9.3.15 (with $Z = \mathfrak{a}_0$ and Σ^+ the set of positive restricted roots) yields $I_\mu(\tau) = I_\lambda(\tau)o(1)$, as $\tau \to \infty$. Now using (9.3.8) for the spherical function corresponding to δ we get

$$q_\delta(\tau) = \sum_{\mu\in\Lambda(\delta)} \langle w_\mu, w_\mu\rangle I_\mu(\tau) \tag{9.3.22}$$

The discussion above implies, that $I_\lambda(\tau)$ dominates all the other terms in (9.3.22). Therefore (9.3.9) and Theorem 9.3.15 finish the proof.

□

Since $\mathbf{c}(-i\rho) = 1$, Theorem 9.3.4 and Theorem 9.3.16 yield Theorem 9.3.17.

Theorem 9.3.17 (Szőke [Sz17])**.** *If the corrected field of quantum Hilbert spaces* $H^{corr} \to S$ *is projectively flat, then for every irreducible* $K-$*spherical representation* δ *with highest weight* λ*,*

$$A_\delta = \frac{\mathbf{c}(-i\lambda - i\rho)\prod\limits_{\alpha\in\Sigma^+}\langle\lambda+\rho,\alpha\rangle^{\frac{m_\alpha}{2}}}{\prod\limits_{\alpha\in\Sigma^+}\langle\rho,\alpha\rangle^{\frac{m_\alpha}{2}}}. \tag{9.3.23}$$

and

$$B_\delta = \|\lambda+\rho\|^2 - \|\rho\|^2. \tag{9.3.24}$$

Denote by Σ_0 the set of indivisible restricted roots, i.e. those $\alpha \in \Sigma$, for which $c\alpha \in \Sigma$ implies $c = \pm 1, \pm 2$. Let $\Sigma_0^+ = \Sigma_0 \cap \Sigma^+$. As before, for an $\alpha \in \Sigma$ we take $m_{2\alpha} = 0$ if $2\alpha \notin \Sigma$ and $\alpha_0 = \alpha/\langle \alpha, \alpha \rangle$. Now combining Theorem 9.3.14 with Theorem 9.3.17 we get.

Theorem 9.3.18 (Szőke [Sz17])**.** *Assume the corrected field of quantum Hilbert spaces $H^{corr} \to S$ is projectively flat. Let δ be an irreducible $K-$spherical representation with highest weight λ. Then A_δ must be equal to* 1*, hence the quantity*

$$\mathbf{c}(-i\lambda - i\rho) \prod_{\alpha \in \Sigma_0^+} \langle \lambda + \rho, \alpha_0 \rangle^{\frac{m_\alpha + m_{2\alpha}}{2}} \tag{9.3.25}$$

is independent of δ and

$$\|\lambda + \rho\|^2 - \|\rho\|^2 = \frac{m}{2} b_\delta, \tag{9.3.26}$$

where b_δ is from (9.3.16)*.*

Remarks. 1) Proposition 9.3.13 shows that when $M = U/K$ is any compact, irreducible, simply connected Riemannian symmetric space of rank-1, (9.3.26) holds for every irreducible $K-$spherical representation of U. Thus the constants B_δ from Theorem 9.3.4 (b) do not help in deciding whether the field $H^{corr} \to S$ is projectively flat or not. It is not clear whether (9.3.26) should always hold for the higher rank symmetric spaces as well, regardless of projective flatness.

2) If M is isometric to a compact Lie group U equipped with a biinvariant metric, we know from [LSz14, Theorem 11.3.1], that $H^{corr} \to S$ is flat. Also it is well known in this case, that for all $\alpha \in \Sigma$, $m_\alpha = 2$ and $m_{2\alpha} = 0$ (i.e. Σ is reduced). Now with

$$\pi(\nu) := \prod_{\alpha \in \Sigma^+} \langle \nu, \alpha \rangle, \quad \nu \in \mathfrak{a}_0^*,$$

we have

$$\mathbf{c}(\nu) = \frac{\pi(\rho)}{\pi(i\nu)}$$

(see [He2, p. 447]) and the quantity in (9.3.25) is equal to $\pi(\rho)$, indeed independent of δ.

Next we express condition (9.3.25) purely in terms of the root system Σ and its multiplicities.

Theorem 9.3.19 (Szőke [Sz17])**.** *Let δ be an irreducible $K-$spherical representation with highest weight λ. Suppose the corrected field of quantum Hilbert spaces $H^{corr} \to S$ is projectively flat. Then the quantity*

$$Q(\delta) := \prod_{\alpha \in \Sigma_0^+} \frac{\Gamma(\frac{1}{4} m_\alpha + \frac{1}{2}\langle \lambda + \rho, \alpha_0 \rangle)\Gamma(\langle \lambda + \rho, \alpha_0 \rangle)\langle \lambda + \rho, \alpha_0 \rangle^{\frac{m_\alpha + m_{2\alpha}}{2}}}{\Gamma(\frac{1}{2} m_\alpha + \langle \lambda + \rho, \alpha_0 \rangle)\Gamma(\frac{1}{4} m_\alpha + \frac{1}{2} m_{2\alpha} + \frac{1}{2}\langle \lambda + \rho, \alpha_0 \rangle)} \tag{9.3.27}$$

is independent of δ.

If $m_{2\alpha} = 0$ and $m_\alpha = 2$ for all $\alpha \in \Sigma_0^+$, then it is obvious that $Q(\delta)$ is in fact independent of δ. This is the group manifold case.

Proof. The Gindikin-Karpelevič formula expresses Harish-Chandra's $\mathbf{c}-$function as a meromorphic function on $\mathfrak{a}_{\mathbb{C}}^*$ (see [He2, p.447]),

$$\mathbf{c}(\nu) = c_0 \prod_{\alpha\in\Sigma_0^+} \frac{2^{\langle -i\nu,\alpha_0\rangle}\Gamma(\langle i\nu,\alpha_0\rangle)}{\Gamma(\frac{1}{2}(\frac{1}{2}m_\alpha + 1 + \langle i\nu,\alpha_0\rangle))\Gamma(\frac{1}{2}(\frac{1}{2}m_\alpha + m_{2\alpha} + \langle i\nu,\alpha_0\rangle))}. \tag{9.3.28}$$

Here the constant c_0 is determined by $\mathbf{c}(-i\rho) = 1$. Using the duplication formula

$$\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma(z+\frac{1}{2}),$$

from (9.3.28) we get

$$c(-i\lambda - i\rho) = c_1 \prod_{\alpha\in\Sigma_0^+} \frac{\Gamma(\frac{1}{2}(\frac{1}{2}m_\alpha + \langle\lambda+\rho,\alpha_0\rangle))\Gamma(\langle\lambda+\rho,\alpha_0\rangle)}{\Gamma(\frac{1}{2}m_\alpha + \langle\lambda+\rho,\alpha_0\rangle)\Gamma(\frac{1}{2}(\frac{1}{2}m_\alpha + m_{2\alpha} + \langle\lambda+\rho,\alpha_0\rangle))}, \tag{9.3.29}$$

where

$$c_1 = c_0 \prod_{\alpha\in\Sigma_0^+} \frac{2^{m_\alpha/2}}{2\sqrt{\pi}}.$$

From (9.3.25) and (9.3.29) we see (since $(2\alpha)_0 = \alpha_0/2$), that the quantity in (9.3.25) does not depend on δ iff $Q(\delta)$ is independent of δ. □

9.3.7 Γ-related functions

Here we take a closer look at the functions appearing in (9.3.27) to find out which compact symmetric spaces have the property that $Q(\delta)$ (from (9.3.27)) is independent of δ.

Let $0 < a$, $0 \le b, c, d$ be given constants, $P := \{z \in \mathbb{C} : 0 < \text{Re z}\}$ and

$$F(z,a,b,c,d) := \frac{\Gamma(cz+a+b)\Gamma(2cz+2a)(2cz+2a)^{2b+d}}{\Gamma(2cz+2a+2b)\Gamma(cz+a+b+d)}, \tag{9.3.30}$$

considered as a function of z, where Γ denotes the usual Γ function.

Proposition 9.3.20 (Szőke [Sz17])**.** *$F(z,a,b,c,d)$ is a bounded holomorphic function in a neighborhood of $\overline{P}$.*

Proof. Since Γ is zero free and holomorphic in P, F will be holomorphic in a neighborhood of $\overline{P}$. The substitution $w = cz$ shows that it is enough to prove boundedness when $c = 1$. Let $0 < A$ be arbitrary. From

$$\Gamma(w+A) \sim w^A\Gamma(w), \quad w\to\infty, \quad w\in P,$$

(see [Re, p.59]) we get

$$F(w,a,b,1,d) \sim \frac{w^{a+b}(2w)^{2a}(2w+2a)^{2b+d}}{(2w)^{2a+2b}w^{a+b+d}} \sim 2^d, \quad w\to\infty, \quad w\in P,$$

showing the boundedness of F. □

Let $0 < a_j, c_j$, $0 \le b_j, d_j$, $j = 1,\dots,N$ and $G(z) := \prod_{j=1}^N F(z,a_j,b_j,c_j,d_j)$.

Proposition 9.3.21 (Szőke [Sz17]). *Assume that for some s, $\frac{a_s}{c_s} < \frac{a_j}{c_j}$, for all $j \neq s$ and there exists a constant $D \neq 0$ with $G(n) = D$ for all $n \in \mathbb{Z}_+$. Then $2b_s + d_s = 1$.*

Proof. After renumbering we can assume that $s = 1$. From Proposition 9.3.20 we know that G is a bounded holomorphic function in a neighborhood of $\overline{P}$. In light of Carlson's theorem ([Ti, p.186]), our assumptions imply that $G \equiv D$ and so

$$(2c_1 z + 2a_1)^{2b_1+d_1} \prod_{j=2}^{N} (2c_j z + 2a_j)^{2b_j+d_j} \equiv D \prod_{j=1}^{N} \frac{\Gamma(2c_j z + 2a_j + 2b_j)\Gamma(c_j z + a_j + b_j + d_j)}{\Gamma(c_j z + a_j + b_j)\Gamma(2c_j z + 2a_j)}. \quad (9.3.31)$$

Since $a_1/c_1 < a_j/c_j$, $1 < j$ and because Γ is zero free and holomorphic in $\mathbb{C} \setminus \{0, -1, -2, \dots\}$ and has first order poles in the nonpositive integers, the right hand side is holomorphic in a neighborhood U of $\{\mathrm{Re} z \geq -\frac{a_1}{c_1}\}$ and has a simple zero at $-\frac{a_1}{c_1}$. Furthermore $\prod_{j=2}^{N} (2c_j z + 2a_j)^{2b_j+d_j}$ is holomorphic and zero free in U. Hence $(2c_1 z + 2a_1)^{2b_1+d_1}$ should extend holomorphically to a neighborhood of $z_0 := -\frac{a_1}{c_1}$, with a first order zero at z_0. But this happens iff $2b_1 + d_1 = 1$. □

9.3.8 Root systems

Let $(Z, \langle .,. \rangle)$ be an $r-$dimensional Euclidean space. For $0 \neq \alpha \in Z$ let $\alpha_0 = \alpha / \langle \alpha, \alpha \rangle$.

Let $R \subset Z$ be a (possible nonreduced) root system. Choose a basis $\alpha_1, \dots, \alpha_r$ of R and let R^+ be the set of positive roots, $\mathbb{Z}_+ := \{0, 1, 2, \dots\}$.

$$P_+ := \{\gamma \in Z : \langle \gamma, \alpha_0 \rangle \in \mathbb{Z}_+, \forall \alpha \in R^+\}. \quad (9.3.32)$$

According to the Cartan-Helgason theorem ([He2, Theorem 4.1, p.535, Corollary 4.2, p.538]), when $Z = \mathfrak{a}_0^*$ and $R = \Sigma$ the set of restricted roots of a compact, simply connected Riemannian symmetric space $M = U/K$, the highest weights of the irreducible K-spherical representations of U are precisely the elements of P_+.

A multiplicity function on R is a map $m : R \to \mathbb{R}$, denoted by $\alpha \mapsto m_\alpha$ such that $m_{w\alpha} = m_\alpha$ for every Weyl group element w. Let $\rho := \frac{1}{2} \sum_{\alpha \in R_+} m_\alpha \alpha$. Denote by R_0 the set of indivisible roots and $R_0^+ = R^+ \cap R_0$. Inspired by the formula (9.3.27) for $Q(\delta)$, we define the analogous function for $\mu \in P_+$ as follows.

$$Q(\mu) := \prod_{\alpha \in R_0^+} \frac{\Gamma(\frac{1}{4} m_\alpha + \frac{1}{2}\langle \mu + \rho, \alpha_0 \rangle)\Gamma(\langle \mu + \rho, \alpha_0 \rangle)\langle \mu + \rho, \alpha_0 \rangle^{\frac{m_\alpha + m_{2\alpha}}{2}}}{\Gamma(\frac{1}{2} m_\alpha + \langle \mu + \rho, \alpha_0 \rangle)\Gamma(\frac{1}{4} m_\alpha + \frac{1}{2} m_{2\alpha} + \frac{1}{2}\langle \mu + \rho, \alpha_0 \rangle)} \quad (9.3.33)$$

(9.3.36) below shows that this is a well defined quantity when all multiplicities are positive. Denote by R_* the set of unmultipliable roots in R. A basis

$\beta_1, \ldots, \beta_r$ of R_* can be obtained by taking $\beta_j = \alpha_j$ if $2\alpha_j \notin R$ and $\beta_j = 2\alpha_j$ if $2\alpha_j \in R$. Define $\mu_j \in Z$, $j = 1, \ldots, r$ by

$$\langle \mu_j, \beta_{k,0} \rangle = \delta_{jk}, \quad j, k = 1, \ldots, r. \tag{9.3.34}$$

Then

$$\mu \in P_+ \quad \text{if and only if} \quad \mu = \sum_{j=1}^{r} n_j \mu_j \quad \text{with} \quad n_j \in \mathbb{Z}_+ \tag{9.3.35}$$

([He3, Proposition 4.23, p.150]).

Proposition 9.3.22 (Szőke [Sz17])**.** *Suppose that $0 < m_\alpha$ for all $\alpha \in R$. Then*

$$0 < \langle \rho, \alpha \rangle \quad \textit{and} \quad 0 \leq \langle \mu, \alpha \rangle \quad \forall \alpha \in R^+, \forall \mu \in P_+. \tag{9.3.36}$$

For a fixed $1 \leq j \leq r$, let $R_j^+ := \{\alpha \in R_0^+ : 0 < \langle \mu_j, \alpha_0 \rangle\}$. Then

$$\frac{\langle \rho, \alpha_{j,0} \rangle}{\langle \mu_j, \alpha_{j,0} \rangle} < \frac{\langle \rho, \alpha_0 \rangle}{\langle \mu_j, \alpha_0 \rangle}, \quad \forall \alpha \in R_j^+, \quad \alpha \neq \alpha_j. \tag{9.3.37}$$

Proof. The proof of Proposition 9.3.9 also works here, showing the first part of (9.3.36). The second part follows from (9.3.34) and (9.3.35). If $\alpha_j \neq \alpha \in R_j^+$, then $\alpha = \sum_1^r n_s \alpha_s$ with $n_s \in \mathbb{Z}_+$. From (9.3.34) we have $0 < \langle \mu_j, \alpha_j \rangle$ and

$$0 < \langle \mu_j, \alpha \rangle = n_j \langle \mu_j, \alpha_j \rangle. \tag{9.3.38}$$

Hence $0 < n_j$. Since α is indivisible and is different from α_j, there must be at least one more s with $0 < n_s$. (9.3.36) then implies

$$\langle \rho, n_j \alpha_j \rangle < \langle \rho, \alpha \rangle. \tag{9.3.39}$$

Now in light of (9.3.38), if we divide (9.3.39) by $n_j \langle \mu_j, \alpha_j \rangle$ we get (9.3.37). □

We call a multiplicity function $m : R \to \mathbb{R}$ *geometric* if it takes only positive integer values and satisfies the following property: if $\alpha \in R$ and m_α is odd, then $2\alpha \notin R$. For $\alpha \in R$ we use the convention as before: $m_{2\alpha} = 0$ if 2α is not a root. If $R = \Sigma$, a restricted root system of a compact, Riemannian symmetric space, its multiplicity function is geometric in this sense, see [Ar, Proposition 2.3] or [He1, p.530, 4F].

Theorem 9.3.23 (Szőke [Sz17])**.** *Let R be an irreducible root system with a geometric multiplicity function m. Suppose $Q(\mu)$, $\mu \in P_+$ is independent of μ ($Q(\mu)$ is from (9.3.33)). Then R is reduced and for all $\alpha \in R$, $m_\alpha = 2$.*

Proof. Let $\beta_j \in R$, $\mu_j \in Z$ as in (9.3.34) and fix a j with $1 \leq j \leq r$. From (9.3.34) we have $n\mu_j \in P_+$ for all $n \in \mathbb{Z}_+$. Now let

$$H_j(z) := \prod_{\alpha \in R_0^+} F\left(z, \frac{\langle \rho, \alpha_0 \rangle}{2}, \frac{m_\alpha}{4}, \frac{\langle \mu_j, \alpha_0 \rangle}{2}, \frac{m_{2\alpha}}{2}\right),$$

where F is from (9.3.30). Then from (9.3.33) we get

$$Q(n\mu_j) = H_j(n), \quad \forall n \in \mathbb{Z}_+.$$

By our assumption on Q, $H_j(n)$ will be independent of n. For any values of the parameters a, b, d, the function $F(z, a, b, 0, d)$ from (9.3.30) is always a nonzero constant. Thus if we leave out from the definition of H_j all those terms that correspond to a root $\alpha \in R_0^+$ with $\langle \mu_j, \alpha_0 \rangle = 0$, the result is still a function that is a nonzero constant on the nonnegative integers. Let $R_j^+ := \{\alpha \in R_0^+ : \langle \mu_j, \alpha_0 \rangle > 0\}$ be as in Proposition 9.3.22 and

$$G_j(z) := \prod_{\alpha \in R_j^+} F\left(z, \frac{\langle \rho, \alpha_0 \rangle}{2}, \frac{m_\alpha}{4}, \frac{\langle \mu_j, \alpha_0 \rangle}{2}, \frac{m_{2\alpha}}{2}\right).$$

Then we still have that $G_j(n)$ is a nonzero constant when $n \in \mathbb{Z}_+$. This together with (9.3.37) and Proposition 9.3.21 implies

$$m_{\alpha_j} + m_{2\alpha_j} = 2. \tag{9.3.40}$$

Since m is a geometric multiplicity function, (9.3.40) yields $m_{\alpha_j} = 2$ and $m_{2\alpha_j} = 0$. Thus $2\alpha_j$ is not a root. Since R_0, R and m are Weyl group invariant, this yields that R is reduced and $m \equiv 2$.

□

9.3.9 Proof of Theorem 9.3.2

If (M, g) is an irreducible, simply connected, compact, Riemannian symmetric space, the set of restricted roots Σ in $\mathfrak{a}_0^*$ forms an irreducible root system with a geometric multiplicity function. In light of Theorem 9.3.19 and Theorem 9.3.33, projective flatness of $H^{corr} \to S$ implies Σ is reduced and all the multiplicities are equal to 2. But these conditions characterize compact Lie groups among compact Riemannian symmetric spaces ([Lo, Theorem 4.4, p.82]).

9.4 Factoring out symmetries [LSz14]

The above computations together with subsection 9.2.3 throw some light on the problem of reduction in quantization. Suppose a mechanical system, with classical configuration space a Riemannian manifold M, admits a group K of symmetries. Thus K acts on M by isometries. The question is how to reduce the corresponding quantum Hilbert space, i.e., how to factor out the symmetries. Should one first construct the quantum Hilbert space $\mathcal{H}^{corr}$ of M, on which K acts unitarily, and then pass to the subspace $\mathcal{H}^{corr,K}$ of fixed vectors; or rather quantize the quotient M/K (assumed to be a manifold)?

Suppose $M = U$ is a compact Lie group with biinvariant metric, $K \subset U$ is a closed subgroup, that acts on M by left translations, and the quantum Hilbert spaces are constructed from the adapted Kähler structures. In the first method of reduction, the field $H^{corr} \to S$ of corrected quantum Hilbert spaces for M is flat, hence so is the subfield $H^{corr,K} \to S$ of fixed vectors, by Theorem 9.2.4 and Lemma 8.1.8. Therefore the quantum Hilbert spaces $H_s^{corr,K}$, $s \in S$, are canonically isomorphic. On the other hand Theorem 9.3.2 shows, that in the second method of reduction, when U/K is a compact, irreducible Riemannian symmetric space that is not a group manifold, the field of quantum Hilbert spaces for U/K will not be projectively flat, and the quantum Hilbert spaces

corresponding to different adapted Kähler structures will not be (projectively) canonically isomorphic. This suggests that the first method of reduction is favored over the second.

Bibliography

[AM] R. Abraham, J. Marsden: *Foundations of Mechanics* 2nd ed., Benjamin- Cummings, 1987

[Ag] R.M. Aguilar: *Symplectic reduction and the homogeneous complex Monge-Ampére equation*, Ann. Glob. Anal. Geom. **19**, 2001, 327-353

[Ar] S. Araki: *On root systems and an infinitesimal classification of irreducible symmetric spaces*, J. Math. Osaka City Univ. **13**, 1962, 1-34

[ADW] S. Axelrod, S. Della Pietra, E. Witten: *Geometric quantization of Chern–Simons gauge theory*, J. Diff. Geo., **33**, 1991, 787–902

[BER] M.S. Baouendi, P. Ebenfelt, L.P. Rothschild: *Real submanifolds in complex space and their mappings*, PUP, 1999

[Ba] T. Barth: *Taut and tight complex manifolds*, Proc. AMS, **24**, 429-431, 1970

[Bed1] E. Bedford: *On the automorphism group of a Stein manifold*, Math. Ann.,**266**, 1983, 215-227

[Bed2] E. Bedford: *Survey of Pluri-Potentian theory*, in: Several Complex variables, Proc. of Mittag-Leffler Inst. 1987-1988, PUP, 1993, 48-97

[BT] E. Bedford, B.A. Taylor: *The Dirichlet problem for a complex Monge-Ampère equation* Invent. Math., **37**, 1-44, 1976

[Be1] B. Berndtsson: *Curvature of vector bundles and subharmonicity of Bergman kernels*, manuscript (2005), arxiv: math.CV/0505470

[Be2] B. Berndtsson: *Subharmonicity properties of the Bergman kernel and some other functions associated to pseudoconvex domains*, Ann. Inst. Fourier (Grenoble), **56**, 2006, 1633–1662

[Be3] B. Berndtsson: *Curvature of vector bundles associated to holomorphic fibrations*, Ann. of Math. (2), **169**, 2009, 531–560

[Be4] B. Berndtsson: *The openness conjecture for plurisubharmonic functions* manuscript (2013), arxiv:1303.5781

[BP] B. Berndtsson, M. Păun: *Bergman kernels and the pseudoeffectivity of relative canonical bundles*, Duke Math. J., **145**, 2008, 341–378

[Bes] A.L. Besse: *Einstein manifolds*, Springer-Verlag, 1987

[Bi] R. Bielawski: *Complexification and hypercomplexification of manifolds with a linear connection*, Internat. J. Math., **14**, 2003, 813–824

[BG] O. Biquard, P. Gauduchon: *Hyperkähler metrics on cotangent bundles of hermitian symmetric spaces* Geometry and Physics (Aarhus 1995), 287-298, Lect. Notes in Pure and Appl. Math., 184, Dekker, New York, 1997

[BF] J.-M. Bismut, D. Freed: *The analysis of elliptic families I–II*, Comm. Math. Phys., **106**, 1986, 159–176, **107**, 1986, 103–163

[BGS] J.-M. Bismut, H. Gillet, C. Soulé: *Analytic torsion and holomorphic determinant bundles I–III*, Comm. Math. Phys., **115**, 1988, 49–78, 79–126, 301–351

[Bl1] R.J. Blattner: *Quantization and representation theory*, Proc. Symp. Pure Math., **26**, 147–165, AMS, Providence, 1973

[Bl2] R.J. Blattner: *The meta–linear geometry of non–real polarizations*, Lecture Notes in Math., **570**, 1975, 11–45, Springer, Berlin, 1977

[Bo] N. Bourbaki: *Groupes et algébres de Lie*, Chapitres 4, 5 et 6, Hermann, Paris 1968

[Br] H. Bremermann: *On a generalized Dirichlet problem for plurisubharmonic functions and pseudoconvex domains. Characterization of Shilov boundaries*, Trans. Am. Math. Soc. **91**, 1956, 246-276

[BD] T. Bröcker, T. tom Dieck: *Representations of compact Lie groups*, Springer, New York, 1985

[Bu1] D. Burns: *On the uniqueness and characterization of Grauert tubes*, Complex Analysis and Geometry, eds:V. Ancona, E. Ballico and A. Silva, Marcel Dekker, 1995

[Bu2] D. Burns: *Some examples of the twistor construction*, "Contributions to several complex variables, in honour of Wilhelm Stoll" eds. A. Howard and P.-M. Wong, Friedr. Vieweg, Braunschweig, 1986

[Bu3] D. Burns: *Curvatures of Monge-Ampère foliations and parabolic manifolds*, Annals of Math., **115**, 1982, 348-373

[BHH] D. Burns, S. Halverscheid, R. Hind: *The geometry of Grauert tubes and complexification of symmetric spaces*, Duke Math. J., **118**, 2003, 465-491

[BH] D. Burns, R. Hind: *Symplectic geometry and the uniqueness of Grauert tubes* Duke Math. J., **118**, 2003, 465-491

[C] E. Calabi: *Métriques kähleriennes et fibres holomorphes*, Ann. Ecol. Norm. Sup., **12**, 1979, 269-294

[Char] L. Charles: *Semi-classical properties of geometric quantization with metaplectic correction*, Comm.Math. Phys., **270**, 2007, 445–480

[Chav] I. Chavel: *Riemannian geometry: A modern introduction*, Cambridge U. Press, Cambridge, 1996

[Che] C. Chevalley: *Invariants of finite groups generated by reflections*, Amer. J. Math., **77**, 1955, 778-782

[Da] J. Dadok: *On the C^∞ Chevalley's theorem* Adv. in Math., **44**, 1982, 121-131

[Di] J. Dixmier: *Les C^*–algèbres et leurs représentations*, 2ème éd., Gauthier–Villars, Paris, 1969

[DSz] A. Dancer, R. Szőke: *Symmetric spaces, adapted complex structures and hyperkähler structures*, Quart. J. Math. Oxford **48**, 1997, 27-38

[DK] T. Duchamp, M. Kalka: *Singular Monge-Ampère foliations*, Math. Ann., **325**, 2003, 187-209

[EH] T. Eguchi, A.J. Hanson: *Asymptotically flat self-dual solutions to Euclidean gravity*, Phys. Lett., **237**, 1978, 249-251

[Fa] J. Faraut: *Espaces Hilbertiens invariant de fonctions holomophes* Semin. Congr. Vol.7. Soc. Math. de France, Paris, 2003, 101-167

[FHW] G. Fels, A. Huckleberry, J. Wolf: *Cycle spaces of flag domains. A complex geometric viewpoint*, Birkhäuser, MA, 2006

[FMMN1] C. Florentino, P. Matias, J. Mourão, J.P. Nunes: *Geometric quantization, complex structures and the coherent state transform*, J. Funct. Anal., **221**, 2005, 303–322

[FMMN2] C. Florentino, P. Matias, J. Mourão, J.P. Nunes: *On the BKS pairing for Kähler quantizations of the cotangent bundle of a Lie group*, J. Funct. Anal., **234**, 2006, 180–198

[Fo] G.B. Folland: *Harmonic analysis in phase space*, Princeton U. Press, Princeton, 1989

[FU] T. Foth, A. Uribe: *The manifold of compatible almost complex structures and geometric quantization*, Comm. Math. Phys., **274**, 2007, 357379

[FT] K. Furutani, R. Tanaka: *A Kähler structure on the punctured cotangent bundle of complex and quaternion projective spaces and its application to geometric quantization I*, J. Math. Kyoto Univ., **34**, 1994, 719-737

[FY] K. Furutani, S. Yoshizawa: *A Kähler structure on the punctured cotangent bundle of complex and quaternion projective spaces and its application to geometric quantization II*, Jap. J. Math., **21**, 1995, 355-392

[GHL] S. Gallot, D. Hulin, J. Lafontaine: *Riemannian geometry*, Springer, Berlin, 1987

[GI] L. Geatti, A. Iannuzzi: *The adapted hyper-Kähler structure on the crown domain*, arXiv:1711.01605

[Go] R. Godement: *Sur la théorie des représentations unitaires*, Ann. of Math., (2), **53**, 1951, 68–124

[Gr] H. Grauert: *Ein Theorem der analytischen Garbentheorie und die Modulräume komplexer Strukturen*, Inst. Hautes Études Sci. Publ. Math., No 5, 1960

[GS1] V. Guillemin, M.B. Stenzel: *Grauert tubes and the homogeneous Monge–Ampère equation I* J. Diff. Geom, **34**, 1991, 561–570

[GS2] V. Guillemin, M.B. Stenzel: *Grauert tubes and the homogeneous Monge–Ampère equation II*, J. Diff. Geom, **35**, 1992, 627-641

[Hal1] B.C. Hall: *The Segal–Bargmann "coherent state" transform for compact Lie groups*, J. Funct. Anal., **122**, 1994, 103–151

[Hal2] B.C. Hall: *Geometric quantization and the generalized Segal–Bargmann transform for Lie groups of compact type*, Comm. Math. Phys., **226**, 2002, 233–268

[HK] B.C. Hall, W.D. Kirwin: *Adapted complex structures and the geodesic flow*, Math. Ann., **350**, 2011, 455-474

[HK2] B.C. Hall, W.D. Kirwin: *Complex structures adapted to magnetic flows* J. Geometry Phys., **90**, 2015, 111-131

[HJ] N. Hanges, H. Jacobowitz: *Involutive structures on compact manifolds*, Amer. J. Math., **117**, 1995, 491-522

[Har1] Harish-Chandra: *Spherical functions on a semisimple Lie group I*, Amer. J. Math, **80**, 2, 1958, 241-310

[Har2] Harish-Chandra: *Spherical functions on a semisimple Lie group II*, Amer. J. Math, **80**, 3, 1958, 553-613

[HW] R. Harvey, R. Wells: *Zero sets of nonnegative strictly plurisubharmonic functions*, Math. Ann.,**201**, 1973, 165-170

[Hec] G. J. Heckmann: *Root systems and hypergeometric functions II*, Comp. Math, **64**, 1987, 353-373

[HO1] G. J. Heckmann, E. M. Opdam: *Root systems and hypergeometric functions I*, Comp. Math, **64**, 1987, 329-3352

[HO2] G. J. Heckmann, E. M. Opdam: *Jacobi polynomials and hypergeometric functions associated with root systems*, preprint, 2015, http://www.math.ru.nl/∼heckman/

[HS] G. J. Heckmann, H. Schlicktkrull: *Harmonic analysis and special functions on symmetric spaces*, Perspectives in Mathematics, Vol.16, Academic Press, San Diego, 1994

[Hei] P.Heinzner: *Geometric invariant theory on Stein spaces* Math. Ann.,**289**, 1991, 631–662

[He1] S. Helgason: *Differential geometry, Lie groups, and symmetric spaces*, Amer. Math. Soc., Providence, 2001

[He2] S. Helgason: *Groups and geometric analysis, integral geometry, invariant differential operators and spherical functions*, Amer. Math. Soc., Providence, 2002

[He3] S. Helgason: *Geometric Analysis on Symmetric Spaces*, Amer. Math. Soc., Providence, 1994

[Hi] N. Hitchin: *Flat connections and geometric quantization*, Comm. Math. Phys., **131**, 1990, 347–380

[Hö] L. Hörmander: *An introduction to complex analysis in several complex variables*, 3rd ed., North Holland, Amsterdam, 1991

[Hue] J. Huebschmann: *Kirillov's character formula, the holomorphic Peter-Weyl theorem, and the Blattner-Kostant-Sternberg pairing*, J. Geom. Phys., **58**, 2008, 833–848

[Hum] J. E. Humphreys: *Reflection groups and Coxeter groups*, CUP, 1990

[IM] K. Ii, T. Morikawa: *Kähler structures on tangent bundles of Riemannian manifolds of constant positive curvature*, 1999 Bull. Yamagata Univ. Natur. Sci., **14**, 1999, 141-154

[J] H. Jacobowitz: *Global Mizohata structures*, J. Geom. Anal., **3**, 1993, 153-193

[KW] W.D. Kirwin, S. Wu: *Geometric quantization, parallel transport, and the Fourier transform*, Comm. Math. Phys., **266**, 2006, 577–594

[Kl] W. Klingenberg: *Riemannian geometry*, Walter de Gruyter, Berlin, 1982

[Kn] A.W. Knapp: *Lie groups beyond an introduction*, 2nd ed., Birkhäuser, Boston, 2002

[Kob] S. Kobayashi: *Hyperbolic manifolds and holomorphic mappings*, Marcel Dekker, New York, 1970

[Koo] P. Koosis: *Introduction to H_p spaces*, Lond. Math. Soc. Lecture Note Ser. 40. Cambridge, Cambridge University Press, 1980

[KSz] A. Korányi, R. Szőke: *On Weyl group equivariant maps*, Proc. Am. Math. Soc., **134**, 3449-3456, 2006

[Ko1] B. Kostant: *Quantization and unitary representations I.*, Lectures in modern analysis and applications III, Lecture Notes in Math., **170**, Springer, Berlin, 1970, 87–208

[Ko2] B. Kostant: *Symplectic spinors*, Symposia Mathematica XIV, 139–152, Academic Press, London, 1974

[Le] A. Le: *CR circle bundles and Mizohata structures*, Amer. J. Math., **120**, 251-287, 1998

[L1] L. Lempert: *Complex structures on the tangent bundle of Riemannian manifolds*, Complex Analysis and Geometry, Plenum Press, New York, N.Y., 1993, 235–251

[L2] L. Lempert: *Elliptic and hyperbolic tubes* in: Several Complex Variables. Proc. of the Mittag-Leffler Inst. 1987-88, PUP, 1993, 440-456

[LSz91] L. Lempert, R. Szőke: *Global solutions of the homogeneous complex Monge–Ampère equation and complex structures on the tangent bundle of Riemannian manifolds*, Math. Ann., **290**, 1991, 689–712

[LSz12] L. Lempert, R. Szőke: *A new look at adapted complex structures*, Bull. Lond. Math. Soc., **44**, 2012, 367–374

[LSz14] L. Lempert, R. Szőke: *Direct images, fields of Hilbert spaces, and geometric quantization*, Comm. Math. Phys., **327**, 2014, 49-99

[LSz15] L. Lempert, R. Szőke: *Curvature of fields of quantum Hilbert spaces*, Q. J. Math., **66**, 2015, 645–657,

[Lo] O. Loos: *Symmetric spaces II: compact spaces and classification*, Benjamin, New York, Amsterdam, 1969

[Lu] L. Luna: *Fonctions differentiables invariantes sous l'operation d'un groupe reductif* Ann. Ins. Fourier, Grenoble, **26**, 1976, 33-49

[Mez] A. Meziani: *Mizohata structures on S^2: Automorphisms and standardness*, Contemp. Math., **205**, 235-246, 1997

[Mi1] P. Michor: *Basic differential forms for actions of Lie groups*, Proc. AMS., **124**, 1996, 1633-1642

[Mi2] P. Michor: *Basic differential forms for actions of Lie groups.II*, Proc. AMS., **125**, 1997, 2175-2177

[Mo] N. Mok: *Rigidity of holomorphic self-mappings and the automorphism groups of hyperbolic Stein spaces*, Math. Ann., **266**, 1984, 433-447

[MT1] C. Mourougane, S. Takayama: *Hodge metrics and positivity of direct images* J. Reine Angew. Math., **606**, 2007, 167–178

[MT2] C. Mourougane, S. Takayama: *Hodge metrics and the curvature of higher direct images* Ann. Sci. Éc. Norm. Sup.(4), **41**, 2008, 905–924

[Mo1] G.D. Mostow: *Some new decomposition theorems for semi-simple groups*, Memoirs of the AMS, **4**, 1955, 31–54

[Mo2] G.D. Mostow: *On covariant fiberings of Klein spaces*, Am.J.Math.,**77**, 1955, 247–278

[Mu] J.D. Murray: *Asymptotic analysis*, Springer-Verlag, New York, 1984

[Ne] E. Nelson: *Analytic vectors*, Ann. of Math. (2), **70**, 1959, 572–615

[vN1] J. von Neumann: *Die Eindeutigkeit der Schrödingerschen Operatoren*, Math. Ann., **104**, 1931, 570–578

[vN2] J. von Neumann: *On rings of operators. Reduction theory*, Ann. of Math. (2), **50**, 1949, 401–485

[NN] A. Newlander, L. Nirenberg: *Complex coordinates in almost complex manifolds*, Ann. of Math., **65**, 1957, 391-404

[OW] G. Ólafsson, K. Wiboonton: *The Segal-Bargmann transform on compact symmetric spaces and their direct limits*, Progress in Mathematics, Vol 306: Lie groups: Structure, actions, and representations, In Honor of J. A. Wolf on the occasion of his 75th birthday, eds A. Huckleberry, I. Penkov, G. Zuckerman, Birkhäuser, 2013

[Pa] R. Palais: *A global formulation of the Lie theory of transformation groups*, Mem. of AMS., **22**, 1957

[PW] G. Patrizio, P.M. Wong: *Stein manifolds with compact symmetric center*, Math. Ann., **289**, 1991, 355-382

[Ra1] J.-H. Rawnsley: *Coherent states and Kähler manifolds*, Quart. J. Math. Oxford, **28**, 1977, 403-415

[Ra2] J.-H. Rawnsley: *A nonunitary pairing of polarizations for the Kepler problem*, Trans. Amer. Math. Soc., **250**, 1979, 167–180

[Re] R. Remmert: *Classical topics in complex function theory*, Springer-Verlag, New York, 1998

[SaZ] S. Saks, A. Zygmund: *Analytic functions*, Warsawa-Wroclaw, 1952

[Sch] G. Schumacher: *Positivity of relative canonical bundles and applications*, Invent. Math., **190**, 2012, 1-56

[Schw] G.W. Schwarz: *Smooth functions invariant under the action of a compact Lie group* Topology, **14**, 1975, 63-68

[Sha] B.-V. Shabat: *Introduction to complex analysis. PartII. Functions of several variables*, Translationsof Mathematical Monographs 110, AMS, Providence, RI, 1992

[Sh] B. Shiffman: *Separate analyticity and Hartogs theorems*, Ind. Univ. Math. J., **38**, 1989, 943-957

[Shu] H. B. Shutrick: *Complex extensions*, Quart. J. Math., **9**, 1958, 189-201

[Si] N. Sibony: *A class of hyperbolic manifolds*, Recent developments in SCV, Princeton, Princeton University Press, 1981

[Sol] L. Solomon: *Invariants of finite reflection groups*, Nagoya Math. J., **22**, 1963, 57-64

[So1] J.-M. Souriau: *Structure des systèmes dynamiques*, Dunod, Paris, 1970

[So2] J.-M. Souriau: *Sur la variété de Kepler*, Symposia Math., **14**, Academic Press, London, 1974, 343-360

[Ste] M. Stenzel: *The Segal-Bargmann transform on a symmetric space of compact type*, J. Funct. Anal., **195**, 1999, 44-58

[Sto] W. Stoll: *The characterization of strictly parabolic manifolds*, Ann. Scuola. Norm. Sup. Piza, **7**, 1980, 87-154

[St1] M.H. Stone: *Linear transformations in Hilbert space III*, Proc. Nat. Acad. Sci. USA, **16**, 1930, 172–175

[St2] M.H. Stone: *On one–parameter unitary groups in Hilbert space*, Ann. of Math. (2), **33**, 1932, 643–648

[Sz90] R. Szőke: *Monge-Ampére models*, Ph-D. thesis, Notre Dame Univ., 1990

[Sz91] R. Szőke: *Complex structures on tangent bundles of Riemannian manifolds*, Math. Ann., **291**, 1991, 409–428

[Sz95] R. Szőke: *Automorphisms of certain Stein manifolds*, Math. Z., **219**, 1995, 357–385

[Sz98] R. Szőke: *Adapted complex structures and Riemannian homogeneous spaces*, Ann. Polon. Math., **LXX**, 1998, 215–220

[Sz99] R. Szőke: *Adapted complex structures and geometric quantization*, Nagoya Math. J., **154**, 171-183, 1999

[Sz01] R. Szőke: *Involutive structures on the tangent bundle of symmetric spaces*, Math. Ann., **319**, 2001, 319-348

[Sz04] R. Szőke: *Canonical complex structures associated to connections and complexifications of Lie groups*, Math. Ann.**328**, 2004, 553–591

[Sz17] R. Szőke: *Quantization of compact Riemannian symmetric spaces*, J. Geom. Phys., **119**, 2017, 286-303

[Sz-prep] R. Szőke: *Smooth structures on the field of prequantum Hilbert spaces*, manuscript

[Th1] T. Thiemann: *Reality conditions inducing transforms for quantum gauge field theory and quantum gravity*, Classical and Quantum Gravity, **13**, 1996, 1383-1403

[Th2] T. Thiemann: *Gauge field theory coherent states I. General properties*, Class. Quant. Grav., **18**, 2001, 2025-2064

[Ti] E. C. Titchmarsh: *The theory of functions*, 2nd ed., OUP, London, 1939

[Tr] F. Treves: *Hypo-analytic structures. Local theory*, PUP, Princeton N.J., 1992

[Ts] H. Tsuji: *Curvature semipositivity of relative pluricanonical systems*, manuscript, 2007, arxiv:math. AG/0703729

[Va] I. Vaisman: *Geometric quantization on the space of differential forms*, Rend. Sem. Mat. Univ. Politec. Torino, **39**, 1981, 139–152

[Vil] N.J. Vilenkin: *Special functions and the theory of group representations*, Translations of Mathematical Monographs, **22**, AMS, Providence, 1968

[Viñ] A. Viña: *Identification of Kähler quantizations and the Berry phase*, J. Geom. Phys., **36**, 223–250, 2000

[Vr] L. Vretare: *Elementary spherical functions on symmetric spaces*, Math. Scand., **39**, 1976, 343-358

[Wo] P.M. Wong: *Geometry of the homogeneous Monge-Ampére equation* Invent. Math. 67, 1982, 261-274

[W] N.M.J. Woodhouse: *Geometric quantization*, 2nd ed., Clarendon Press, Oxford, 1992

[WB] H. Whitney, F. Bruhat: *Quelques propriétés fondamentales des ensembles analytiques réels*, Comm. Math. Helv., **33**, 1959, 132-160