dc_1520_18

Asymptotic Properties of Multiplicative Arithmetic Functions of One and Several Variables

DSc dissertation

László Tóth

University of Pécs

2018

Contents

Chapter 1

Introduction

1.1 Multiplicative functions

Various asymptotic properties of multiplicative arithmetic functions, i.e., nonzero functions $f : \mathbb{N} \to \mathbb{C}$ satisfying $f(mn) = f(m)f(n)$, provided that $(m, n) = 1$, are well known in the literature. It is one of the main objectives of elementary and analytic number theory to deduce asymptotic formulas with sharp error terms for sums $\sum_{n\leqslant x} f(n)$, where $f(n)$ is a special multiplicative function or it is belonging to a certain class of such functions.

For example, the Dirichlet divisor problem consists in finding the infimum of exponents θ such that the formula

$$\sum_{n\leqslant x} \tau(n) = x\log x + (2\gamma - 1)x + O(x^{\theta+\varepsilon}), \tag{1.1}$$

holds for every $\varepsilon > 0$. It is known that $1/4 \leqslant \theta \leqslant 131/416 \doteq 0.314903$. More exactly, the best error term in (1.1) up to date is $O(x^{131/416}(\log x)^{26947/8320})$, due to Huxley [46].[1]

More generally, for positive integers $a_1 \leqslant \cdots \leqslant a_k$ consider the generalized divisor function $\tau(a_1, \ldots, a_k; n) := \sum_{d_1^{a_1}\cdots d_k^{a_k}=n} 1$ and let $\Delta(a_1, \ldots, a_k; x)$ stand for the remainder term in the related asymptotic formula, i.e.,

$$\sum_{n\leqslant x} \tau(a_1, \ldots, a_k; n) = H(a_1, \ldots, a_k; x) + \Delta(a_1, \ldots, a_k; x),$$

where $H(a_1, \ldots, a_k; x)$ is the main term. See, e.g., the book by Krätzel [56, Ch. 6]. In the case $a_1 = \cdots = a_k = 1$ we have the Piltz divisor function $\tau_k(n)$, and let $\Delta_k(x)$ denote, as usual, the corresponding error term (Piltz divisor problem).

[1]In a recent preprint of 13 September 2017, Bourgain and Watt [12] proved the better result $\theta \leqslant 517/1648 \doteq 0.313713$. The same error term is valid for the Gauss circle problem.

The squarefree divisor problem goes back to the work of Mertens (1874). Let $\tau^{(2)}(n) = 2^{\omega(n)}$ denote the number of squarefree divisors of n. One has

$$\sum_{n\leqslant x} \tau^{(2)}(n) = \frac{6}{\pi^2}x\left(\log x + 2\gamma - 1 - \frac{2\zeta'(2)}{\zeta(2)}\right) + O(R(x)), \tag{1.2}$$

with $R(x) \ll x^{1/2}\delta(x)$, where

$$\delta(x) := \exp(-c(\log x)^{3/5}(\log\log x)^{-1/5}), \tag{1.3}$$

c being a positive constant. See Suryanarayana and Siva Rama Prasad [90]. If the Riemann hypothesis (RH) is true, then $R(x) \ll x^{4/11+\varepsilon}$, due to Baker [6].

Another example, we quote here is the asymptotic formula

$$\sum_{n\leqslant x} \phi(n) = \frac{3}{\pi^2}x^2 + O\left(x(\log x)^{2/3}(\log\log x)^{4/3}\right), \tag{1.4}$$

concerning Euler's function $\phi(n)$, with the best error term known to date, due to Walfisz [120, Satz 1, p. 144]. The formula

$$\sum_{n\leqslant x} \frac{1}{\phi(n)} = A\left(\log x + \gamma - B\right) + O\left(x^{-1}(\log x)^{2/3}\right), \tag{1.5}$$

where

$$A = \frac{\zeta(2)\zeta(3)}{\zeta(6)} = \frac{315\zeta(3)}{2\pi^4}, \qquad B = \sum_p \frac{\log p}{p^2 - p + 1}, \tag{1.6}$$

with the weaker error term $O\left(x^{-1}\log x\right)$ goes back to the work of Landau. See the book by De Koninck and Ivíc [26, Th. 1.1]. The error term in (1.5) was obtained by Sita Rama Chandra Rao [81].

Now consider the class $\mathcal{W}$ of multiplicative functions $f : \mathbb{N} \to [0,1]$. According to a celebrated result of Wirsing [123], if f is in the class $\mathcal{W}$, then the mean value

$$M(f) := \lim_{x\to\infty} \frac{1}{x}\sum_{n\leqslant x} f(n)$$

exists and

$$M(f) = \prod_p \left(1 - \frac{1}{p}\right)\sum_{\nu=0}^{\infty} \frac{f(p^\nu)}{p^\nu},$$

with the convention that the product is zero provided that the series $\sum_p \frac{1-f(p)}{p}$ diverges.

Other type of results are concerning the maximal order of certain multiplicative functions. For example, the following useful theorem on the maximal order of a class of prime-independent multiplicative functions was proved by Suryanarayana and Sita Rama

Chandra Rao [89]: Let f be a positive function satisfying $f(n) = O(n^{\beta})$ for some fixed $\beta > 0$. Let F be the multiplicative function with $F(p^{\nu}) = f(\nu)$ for every prime power p^{ν} ($\nu \geqslant 1$). Then

$$\limsup_{n\to\infty} \frac{\log F(n) \log\log n}{\log n} = \sup_{m\geqslant 1} \frac{\log f(m)}{m}.$$

This applies to the function $F(n) = \tau(n)$ and gives

$$\limsup_{n\to\infty} \frac{\log \tau(n) \log\log n}{\log n} = \log 2, \tag{1.7}$$

which is a well known result. The same formula is true for $\tau(n)$ replaced by $\tau^{(2)}(n)$. If $F(n) = \tau^{(e)}(n)$, the number of exponential divisors of n, then we obtain

$$\limsup_{n\to\infty} \frac{\log \tau^{(e)}(n) \log\log n}{\log n} = \frac{\log 2}{2},$$

proved earlier by Erdős. See [87, Th. 6.2].

Ramanujan [77] derived pointwise convergent series representations of arithmetic functions with respect to the sums $c_q(n)$, now called Ramanujan sums. For example, let $\sigma(n)$ denote the sum of divisors of n. For every fixed $n \in \mathbb{N}$,

$$\frac{\sigma(n)}{n} = \zeta(2) \sum_{q=1}^{\infty} \frac{c_q(n)}{q^2} \tag{1.8}$$

$$= \frac{\pi^2}{6} \left(1 + \frac{(-1)^n}{2^2} + \frac{2\cos(2\pi n/3)}{3^2} + \frac{2\cos(\pi n/2)}{4^2} + \cdots \right),$$

which shows how the values of $\sigma(n)/n$ fluctuate harmonically about their mean value $\pi^2/6$.

Delange [30] proved the following general theorem concerning such expansions, called Ramanujan (or Ramanujan-Fourier) expansions of arithmetic functions. Let $f : \mathbb{N} \to \mathbb{C}$ be an arithmetic function. Assume that

$$\sum_{n=1}^{\infty} 2^{\omega(n)} \frac{|(\mu * f)(n)|}{n} < \infty. \tag{1.9}$$

Then for every $n \in \mathbb{N}$ we have the absolutely convergent Ramanujan expansion

$$f(n) = \sum_{q=1}^{\infty} a_q c_q(n),$$

where the coefficients a_q are given by

$$a_q = \sum_{m=1}^{\infty} \frac{(\mu * f)(mq)}{mq} \quad (q \in \mathbb{N}).$$

Delange also pointed out how this result can be formulated for multiplicative functions f. By Wintner's theorem condition (1.9) ensures that the mean value $M(f)$ exists and $a_1 = M(f)$.

A nonzero function $f : \mathbb{N}^k \to \mathbb{C}$ is said to be multiplicative if $f(m_1n_1, \ldots, m_kn_k) = f(m_1, \ldots, m_k)f(n_1, \ldots, n_k)$, provided that $(m_1 \cdots m_k, n_1 \cdots n_k) = 1$. Therefore, if f is multiplicative, then it is determined by the values $f(p^{\nu_1}, \ldots, p^{\nu_k})$, where p is prime and $\nu_1, \ldots, \nu_k \in \mathbb{N} \cup \{0\}$. More exactly, $f(1, \ldots, 1) = 1$ and for any $n_1, \ldots, n_k \in \mathbb{N}$,

$$f(n_1, \ldots, n_k) = \prod_p f(p^{\nu_p(n_1)}, \ldots, p^{\nu_p(n_k)}).$$

If the case $k = 1$ this reduces to the usual multiplicativity. Some simple examples of multiplicative functions of k variables are $(n_1, \ldots, n_k)$ and $[n_1, \ldots, n_k]$. Among other examples of such functions we mention $s(n_1, \ldots, n_k)$ and $c(n_1, \ldots, n_k)$, representing the total number of subgroups and the number of cyclic subgroups, respectively, of the group $(\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}, +)$. Let ϱ_r denote the characteristic function of the set of ordered r-tuples $(n_1, \ldots, n_r) \in \mathbb{N}^r$ such that $n_1, \ldots, n_r$ are pairwise relatively prime. Then ϱ_r is a multiplicative function of r variables and it satisfies

$$\sum_{d_1 \mid n_1, \ldots, d_r \mid n_r} \varrho_r(d_1, \ldots, d_r) = \tau(n_1 \cdots n_r) \quad (n_1, \ldots, n_r \in \mathbb{N}). \tag{1.10}$$

A detailed study of multiplicative functions of several variables was carried out by Vaidyanathaswamy [119] more than eighty-five years ago. However, the paper [119] includes algebraic and arithmetic properties, essentially. Even to the present day, there are only a few asymptotic results in the literature for multiplicative functions of several variables. My paper [108] is a survey on this topic.

The mean value of a function $f : \mathbb{N}^k \to \mathbb{C}$ is

$$M(f) := \lim_{x_1, \ldots, x_k \to \infty} \frac{1}{x_1 \cdots x_k} \sum_{n_1 \leqslant x_1, \ldots, n_k \leqslant x_k} f(n_1, \ldots, n_k),$$

provided that this limit exists. As a generalization of Wintner's theorem (valid in the one variable case), Ushiroya [116, Th. 1] proved the next result: If f is a function of k variables, not necessary multiplicative, such that

$$\sum_{n_1, \ldots, n_k = 1}^{\infty} \frac{|(\mu_k * f)(n_1, \ldots, n_k)|}{n_1 \cdots n_k} < \infty,$$

then the mean value $M(f)$ exists and

$$M(f) = \sum_{n_1, \ldots, n_k = 1}^{\infty} \frac{(\mu_k * f)(n_1, \ldots, n_k)}{n_1 \cdots n_k},$$

where $*$ denotes the Dirichlet convolution defined by

$$(f*g)(n_1,\ldots,n_k) = \sum_{d_1\mid n_1,\ldots,d_k\mid n_k} f(d_1,\ldots,d_k)g(n_1/d_1,\ldots,n_k/d_k),$$

and $\mu_k(n_1,\ldots,n_k) = \mu(n_1)\cdots\mu(n_k)$ is the Möbius function of k variables (the inverse of the constant 1 function under $*$).

For multiplicative functions the above result was formulated by us [108, Prop. 19] as follows (see Ushiroya [116, Th. 4] for the same result in a slightly different form and for its proof): Let $f:\mathbb{N}^k \to \mathbb{C}$ be a multiplicative function. Assume that

$$\sum_p \sum_{\substack{\nu_1,\ldots,\nu_k=0\\ \nu_1+\cdots+\nu_k\geqslant 1}}^{\infty} \frac{|(\mu_k*f)(p^{\nu_1},\ldots,p^{\nu_k})|}{p^{\nu_1+\cdots+\nu_k}} < \infty.$$

Then the mean value $M(f)$ exists and

$$M(f) = \prod_p \left(1-\frac{1}{p}\right)^k \sum_{\nu_1,\ldots,\nu_k=0}^{\infty} \frac{f(p^{\nu_1},\ldots,p^{\nu_k})}{p^{\nu_1+\cdots+\nu_k}}.$$

We are not aware of more general mean value results concerning the several variables case. Asymptotic formulas for sums of type $\sum_{n_1,\ldots,n_k\leqslant x} f(n_1,\ldots,n_k)$, with certain special functions f, were derived by Balazard, Naimi, Pétermann [8] and de la Bretèche [28] using analytic methods. For example, in paper [8] the authors use an effective Perron inversion formula in k variables to prove (by a very complicated process) that

$$\sum_{n_1,\ldots,n_k\leqslant x} \frac{\mu(n_1)\cdots\mu(n_k)}{[n_1,\ldots,n_k]} = P_k(\log x) + O(\delta(x)),$$

where $P_k(t)$ is a polynomial in t and $\delta(x)$ is defined by (1.3). They also prove that $P_k(t)$ is identically zero, when k is odd (in the case $k=1$ this is equivalent to the prime number theorem).

1.2 Summary of the main results

This thesis is based on my research work in the past 16 years. I collected the results which I consider the most relevant, related to those presented in Section 1.1. They were published in my papers [96, 97, 98, 100, 101, 103, 104, 105, 107, 109, 110, 111, 112, 113] and [38] (joint work with Mario Hampejs), [42] (with Titus Hilberdink), [71] (with Werner Georg Nowak), [93] (with Marius Tărnăuceanu), [114] (with Eduard Wirsing), [115] (with Wenguang Zhai). I also present the significant preceding results and those obtained ulterior in the literature. Group theoretical, combinatorial and computational aspects are pointed out, as well.

Chapter 2 is concerning multiplicative functions of one variable. In Section 2.1 we present asymptotic formulas valid for wide classes of multiplicative functions. Theorem 2.1.1 applies to certain multiplicative functions f such that $f(n)$ depends only on the ℓ-full kernel of n, where $\ell \geqslant 2$ is a fixed integer. It can be used to deduce asymptotic formulas for the r-th powers ($r \in \mathbb{N}$) of the following special functions: the exponential divisor function $\tau^{(e)}(n)$ (Theorem 2.1.3); the function $a(n)$, representing the number of non-isomorphic abelian groups of order n (Corollary 2.1.4); the exponential analogue of the Euler function (Theorem 2.4.1). Our result for $\sum_{n\leqslant x} a(n)^2$ improves the error term given by Zhang, Lü and Zhai [124]. We also notice a result (Remark 2.1.7), which applies to multiplicative functions f such that $f(p) = k$ for every prime p, where $k \in \mathbb{N}$ is fixed, and the values $f(p^\nu)$ "are not too large" for prime powers p^ν with $\nu \geqslant 2$.

Section 2.2 includes asymptotic formulas for alternating sums $\sum_{n\leqslant x}(-1)^{n-1}f(n)^{-1}$, where $f(n)$ are certain multiplicative functions. In particular, we consider the cases of $f(n) = \phi(n)$ (Corollary 2.2.2) $f(n) = \sigma(n)$ (Corollary 2.2.3), $f(n) = \tau(n)$ (Theorem 2.2.4) and $f(n) = \sigma^{**}(n)$, denoting the sum of bi-unitary divisors of n (Theorem 2.2.7). Our results improve the error terms obtained by Bordellès and Cloitre [11].

Our method of Section 2.2 requires estimates of the coefficients of the reciprocals of some formal power series. If the coefficients of the original power series are positive and log-convex, then a result of Kaluza [52] can be used. We prove a new explicit Kendall-type inequality (Proposition 2.2.5) for reciprocals of power series, which can be applied in some other cases.

In Section 2.3 we present easily applicable results concerning the maximal order of certain multiplicative functions, such as $\sigma(n)$, $\sigma^{(e)}(n)$ and $P^{(e)}(n)$, the latter being the exponential analog of the gcd-sum function.

Section 2.4 is devoted to the study of functions defined by exponential divisors. In particular, Theorems 2.4.2 and 2.4.3 are results for the exponential Möbius function $\mu^{(e)}(n)$, while Theorem 2.4.4 is concerning the function $t^{(e)}(n)$, defined as the number of exponentially squarefree exponential divisors of n.

In Section 2.5 we discuss properties of the gcd-sum function (Pillai's function) and its analogs associated with exponential divisors and regular integers (mod n), respectively.

Section 2.6 is concerning certain weighted averages of the Ramanujan sums, involving logarithms, binomial coefficients and the Gamma function, as weights. I also present (see Section 4.6) a *simpler proof* of a related identity due to Alkan [1].

It is the aim of Section 2.7 to give *short direct proofs* for the number of solutions of the quadratic congruence $x_1^2 + \cdots + x_k^2 \equiv n \pmod r$, obtained by Cohen [21], and to point out some new related asymptotic formulas. Theorems 2.7.1, 2.7.2 and 2.7.3 can be considered as analogs of Dirichlet's formula (1.1), the squarefree divisor problem (1.2) and the Gauss circle problem, respectively.

One of the most important problems of combinatorial group theory is to determine the number of subgroups of a finite group. This is completely settled in the literature for finite abelian groups, by reducing the problem to p-groups. Instead of p-groups, we consider in Section 2.8 the group $(\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}, +)$, where $n_1, \ldots, n_k \in \mathbb{N}$ and the functions $s(n_1, \ldots, n_k)$ and $c(n_1, \ldots, n_k)$, denoting the total number of its subgroups and the number of its cyclic subgroups, respectively. These are multiplicative functions of k variables. The functions $s(n, \ldots, n)$ and $c(n, \ldots, n)$ are multiplicative in n, as functions of a single variable.

Theorem 2.8.1 gives a compact formula for $c(n_1, \ldots, n_k)$. We investigate the cases $k = 2$ and $k = 3$ and give complete representations of the subgroups of $\mathbb{Z}_m \times \mathbb{Z}_n$ (Theorem 2.8.2) and $\mathbb{Z}_m \times \mathbb{Z}_n \times \mathbb{Z}_r$ (Theorem 2.8.5). As applications, we deduce simple formulas for the number of subgroups and establish asymptotic formulas for related multiplicative functions of one variable.

In Chapter 2 we use the *convolution method* to establish asymptotic formulas for sums $\sum_{n \leqslant x} f(n)$. This requires to write the function f as $f = g * h$, the Dirichlet convolution of the functions g and h. If $g(n)$ is "small enough" and there is a "good" formula for $\sum_{n \leqslant x} h(n)$, then we can deduce an asymptotic formula with a sharp error term for $\sum_{n \leqslant x} f(n)$.

Most of the error terms of our formulas are unconditional, but for some of them we assume the Riemann hypothesis (RH). Many of the error terms we obtain are related to the Dirichlet divisor problem (1.1), the squarefree divisor problem (1.2) or other similar remarkable problems.

In Chapter 3 we investigate multiplicative functions of several variables. We deduce asymptotic formulas with sharp error terms for the characteristic function of the set of r-tuples of positive integers with k-wise relatively prime components (Section 3.1), for $f(n_1 \cdots n_r)$ and $f([n_1, \ldots, n_k])$ with certain functions f (Sections 3.2 and 3.3). For $k \geqslant 3$ the error term concerning r-tuples of positive integers with k-wise relatively prime components improves the result by Hu [43]. Our results of Section 3.2 generalize and refine the result $\sum_{m,n,q \leqslant x} [m, n, q]^r \sim c_r x^{3(r+1)}$, valid for $r \in \mathbb{N}$, with a certain constant c_r, obtained by Fernández and Fernández [34]. The asymptotic formulas included in Section 3.3 refine and generalize a result of Lelechenko [59] deduced for the sum $\sum_{m,n \leqslant x} \tau(1, 2; mn)$, by using the complex integration method.

In order to establish the asymptotic formulas for multiplicative functions $F(n_1, \ldots, n_k)$, we elaborated some details of the *convolution method in the several variables case*, which seems to be the *most natural approach*. In order to obtain and to apply a convolutional identity it is necessary a careful study of the corresponding multiple Dirichlet series and Euler products given by

$$\sum_{n_1, \ldots, n_r = 1}^{\infty} \frac{F(n_1, \ldots, n_r)}{n_1^{s_1} \cdots n_r^{s_r}} = \prod_p \sum_{\nu_1, \ldots, \nu_r = 0}^{\infty} \frac{F(p^{\nu_1}, \ldots, p^{\nu_r})}{p^{\nu_1 s_1 + \cdots + \nu_r s_r}},$$

but we use only *elementary arguments* (do not utilize analytic continuation and contour integration). The difficulty consists in estimating some intermediate multiple sums of the type

$$\sum_{\substack{n_1\leqslant x,\dots,n_t\leqslant x\\ n_{t+1}>x,\dots,n_k>x}} \psi(n_1,\dots,n_k),$$

where ψ is a certain multiplicative function of k variables.

In Section 3.4, based on our paper [71], we investigate the multiplicative function $s(m,n)$, representing the total number of subgroups of the group $(\mathbb{Z}_m \times \mathbb{Z}_n, +)$. We obtain asymptotic formulas for the sum $\sum_{m,n\leqslant x} s(m,n)$ (Theorem 3.4.2) and for the corresponding sum restricted to $(m,n)>1$, i.e., concerning the groups $\mathbb{Z}_m\times\mathbb{Z}_n$ having rank two (Theorem 3.4.5). The method we use to prove Theorem 3.4.2 is the *hyperbola method* adopted to this function of two variables. In paper [71] we proved Theorem 3.4.5 by analytic arguments, namely by using Perron's formula in one variable. However, I present here the sketch of an *elementary proof* by using the Busche-Ramanujan identity for the divisor function (see Section 3.5).

We derive in Section 3.5 two new generalizations of the Busche-Ramanujan identities. Namely, we consider the values of a specially multiplicative function for products of several arbitrary integers (Theorem 3.5.1). Then we deduce formulas for the convolution of several arbitrary completely multiplicative functions (Theorem 3.5.2). The proofs use arguments concerning formal Dirichlet series of arithmetic functions of several variables and properties of symmetric polynomials of several variables.

Finally, in Section 3.6 we obtain results on the Ramanujan-Fourier expansions of arithmetic functions of several variables. Our results generalize those of Delange [30] and Ushiroya [118].

1.3 Notations

Throughout the dissertation we use standard notations. Some of them are fixed below. Further notations will be explained by their first appearance.

• $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ denote the set positive integers, integers, real and complex numbers, respectively;

• $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ is the set of residue classes modulo n ($n\in\mathbb{N}$);

• the prime power factorization of $n\in\mathbb{N}$ is $n=\prod_p p^{\nu_p(n)}$, the product being over the primes p, where all but a finite number of the exponents $\nu_p(n)$ are zero;

• $(n_1,\dots n_k)$ and $\gcd(n_1,\dots n_k)$ denote the greatest common divisor of $n_1,\dots,n_k\in\mathbb{N}$;

• $[n_1,\dots,n_k]$ and $\operatorname{lcm}(n_1,\dots,n_k)$ denote the least common multiple of $n_1,\dots,n_k\in\mathbb{N}$;

• id is the function $\operatorname{id}(n)=n$ ($n\in\mathbb{N}$);

• $\tau(n)$ is the number of divisors of n, $\sigma(n)$ is the sum of divisors of n, $\sigma_s(n)$ is the sum of s-th powers of the divisors of n;

• $\phi(n)$ is Euler's totient function, $\phi_s(n)=n^s\prod_{p\mid n}(1-1/p^s)$ is the Jordan function of order s;

- $\mu(n)$ is the Möbius function, $\psi(n)$ is the Dedekind function given by $\psi(n) = n\prod_{p|n}(1+1/p)$, $\kappa(n) = \prod_{p|n} p$ is the squarefree kernel of n;
- $\omega(n)$ denotes the number of distinct prime factors of n, $\Omega(n) = \sum_p \nu_p(n)$ is the number of prime power divisors of n;
- $\tau^{(2)}(n) = 2^{\omega(n)}$ is the number of squarefree divisors of n;
- $\tau_k(n)$ is the Piltz divisor function, representing the number of ways n can be written as a product of k factors;
- $\tau^{(e)}(n) = \prod_{p^\nu || n} \tau(\nu)$ is the number of exponential divisors of n;
- $\sigma^{(e)}(n) = \prod_{p^\nu || n} \sum_{d|\nu} p^d$ denotes the sum of exponential divisors of n;
- $a(n)$ represents the number of non-isomorphic abelian groups of order n;
- $P(n) = \sum_{k=1}^{n}(k,n)$ is the gcd-sum function (Pillai's function);
- $P(n)$ also denotes the number of unrestricted partitions of n;
- $c_q(n) = \sum_{1\leqslant k\leqslant q,(k,q)=1} \exp(2\pi ikn/q)$ are the Ramanujan sums;
- $*$ is the Dirichlet convolution of arithmetic functions;
- ζ is the Riemann zeta function;
- $\gamma \doteq 0.577215$ is Euler's constant;
- $G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \doteq 0.915956$ is the Catalan constant;
- $\sum_p$ and $\prod_p$ are sums and products over the primes;
- the O ($\ll$), o, Ω and $\sim$ notations are used in the usual way, for the first one the implied constant may depend on certain parameters;

Chapter 2

Results for multiplicative functions of one variable

2.1 Average orders

Let $\tau(n)$ denote the number of divisors of n. Ramanujan [76] stated without proof that the estimate

$$\sum_{n\leqslant x} \tau(n)^2 = x(A(\log x)^3 + B(\log x)^2 + C\log x + D) + O(x^{3/5+\varepsilon}) \tag{2.1}$$

holds for any real $\varepsilon > 0$, with $A = \pi^{-2}$ and certain constants B, C, D. By using analytic methods, Wilson [122] proved Ramanujan's claim and generalized it by showing that for any integer $r \geqslant 2$ one has

$$\sum_{n\leqslant x} \tau(n)^r = xP_{2^r-1}(\log x) + O(x^{\frac{2^r-1}{2^r+2}+\varepsilon}),$$

where $P_{2^r-1}(t)$ is a polynomial of degree $2^r - 1$ in t with leading coefficient

$$C_r = \frac{1}{(2^r-1)!} \prod_p \left(1 - \frac{1}{p}\right)^{2^r} \sum_{\nu=0}^{\infty} \frac{(\nu+1)^r}{p^\nu}.$$

Note that in the case $r = 2$, Wilson's error term is better than the one stated by Ramanujan.

Now consider $\tau^{(e)}(n)$, denoting the number of exponential divisors of n. See Section 2.4. The function $\tau^{(e)}$ is multiplicative and $\tau^{(e)}(p^\nu) = \tau(\nu)$ for every prime power p^ν ($\nu \geqslant 1$). Wu [121] showed, improving an earlier result of Subbarao [87], that

$$\sum_{n\leqslant x} \tau^{(e)}(n) = A_1 x + B_1 x^{1/2} + O(x^{2/9}\log x), \tag{2.2}$$

where

$$A_1 := \prod_p \left(1 + \sum_{\nu=2}^{\infty} \frac{\tau(\nu) - \tau(\nu-1)}{p^\nu}\right),$$

$$B_1 := \prod_p \left(1 + \sum_{\nu=5}^{\infty} \frac{\tau(\nu) - \tau(\nu-1) - \tau(\nu-2) + \tau(\nu-3)}{p^{\nu/2}}\right).$$

The error term in (2.2) is strongly related to the error term $\Delta(1, 2; x)$ on the divisor function $\tau(1, 2; n) = \sum_{ab^2=n} 1$. It can be sharpened into $O(x^{1057/4785+\varepsilon})$. See [121, Remark, p. 135].

An asymptotic formula for the function $f(n) = \tau^{(e)}(n)^r$ with any integer $r \geqslant 1$ follows from the following general result concerning certain multiplicative functions f such that $f(n)$ depends only on the ℓ-full kernel of n, where $\ell \geqslant 2$ is a fixed integer. Let $\Delta_{k,\ell}(x) := \Delta((1, \underbrace{\ell, \ell, \ldots, \ell}_{k-1}); x)$ denote the error term of the corresponding generalized divisor problem.

Theorem 2.1.1 (**Tóth [98], [103, Th. 2]**). *Let $f : \mathbb{N} \to \mathbb{C}$ be a multiplicative arithmetic function. Assume that*

i) $f(p) = f(p^2) = \cdots = f(p^{\ell-1}) = 1$, $f(p^\ell) = k$ for every prime p, where $\ell, k \geqslant 2$ are fixed integers,

ii) $f(p^\nu) \ll 2^{\nu/(\ell+1)}$ $(\nu \to \infty)$ uniformly for the primes p.

Then

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \zeta(s)\zeta^{k-1}(\ell s)V(s),$$

absolutely convergent for $\Re(s) > 1$, where the Dirichlet series $V(s)$ is absolutely convergent for $\Re(s) > 1/(\ell+1)$.

Furthermore, suppose that $\Delta_{k,\ell} \ll x^{\alpha_{k,\ell}}(\log x)^{\beta_{k,\ell}}$, with $1/(\ell+1) < \alpha_{k,\ell} < 1/\ell$. Then

$$\sum_{n \leqslant x} f(n) = \widetilde{C}_f x + x^{1/\ell} P_{f,k-2}(\log x) + R_f(x), \tag{2.3}$$

where $P_{f,k-2}$ is a polynomial of degree $k-2$,

$$\widetilde{C}_f := \prod_p \left(1 + \sum_{\nu=\ell}^{\infty} \frac{f(p^\nu) - f(p^{\nu-1})}{p^\nu}\right),$$

and $R_f(x) \ll x^{\alpha_{k,\ell}}(\log x)^{\beta_{k,\ell}}$ (is the same).

Remark 2.1.2. For every $k, \ell \geqslant 2$, $\Delta_{k,\ell}(x) \ll x^{u_{k,\ell}+\varepsilon}$, where $u_{k,\ell} := \frac{2k-1}{3+(2k-1)\ell} \in (\frac{1}{\ell+1}, \frac{1}{\ell})$. See [56, Th. 6.10]. Therefore, $R_f(x) \ll x^{u_{k,\ell}+\varepsilon}$ is valid, as well.

Applying Theorem 2.1.1 and Remark 2.1.2 to the function $f(n) = \tau^{(e)}(n)^r$ with $\ell = 2$, $k = 2^r$, we deduce the following result.

Theorem 2.1.3 (Tóth [98, Eq. (4)]). *Let $r \geqslant 1$ be a fixed integer. The asymptotic formula*

$$\sum_{n \leqslant x} \tau^{(e)}(n)^r = A_r x + x^{1/2} Q_{2^r-2}(\log x) + O(x^{u_r+\varepsilon})$$

holds for every $\varepsilon > 0$, where

$$A_r := \prod_p \left(1 + \sum_{a=2}^{\infty} \frac{\tau(a)^r - \tau(a-1)^r}{p^a}\right),$$

Q_{2^r-2} is a polynomial of degree $2^r - 2$ and $u_r := \frac{2^{r+1}-1}{2^{r+2}+1}$.

Theorem 2.1.1 applies to other special functions, as well. For example, consider the function $a(n)$, representing the number of non-isomorphic abelian groups of order n. The function $a(n)$ is multiplicative and for every prime power p^ν ($\nu \geqslant 1$), $a(p^\nu) = P(\nu)$ is the number of unrestricted partitions of ν. Thus, for every prime p, $a(p) = 1$, $a(p^2) = 2$, $a(p^3) = 3$, $a(p^4) = 5$, $a(p^5) = 7$, etc. An asymptotic formula for the sum $\sum_{n \leqslant x} a(n)$ was obtained for the first time by Erdős and Szekeres [32]. The corresponding error term was investigated by several authors. See, e.g., [49, Ch. 14], [56, Ch. 7] for historical surveys. It is known that

$$\sum_{n \leqslant x} a(n) = A_1 x + A_2 x^{1/2} + A_3 x^{1/3} + R(x),$$

where $A_j := \prod_{k=1, k \neq j}^{\infty} \zeta(k/j)$ ($j = 1, 2, 3$), and the best result for the error term is $R(x) \ll x^{1/4+\varepsilon}$ for every $\varepsilon > 0$, proved by O. Robert and P. Sargos [78].

The asymptotic behavior of the sum $\sum_{n \leqslant x} 1/a(n)$ was investigated by Nowak [70]. An asymptotic formula for the quadratic moment of the function a, i.e., for $\sum_{n \leqslant x} a(n)^2$ was given by Zhang, Lü and Zhai [124].

Let $\Delta_r(x) := \Delta((1, \underbrace{2, 2, \ldots, 2}_{2^r-1}); x)$. For the r-th moment of the function $a(n)$ we have the next result.

Corollary 2.1.4 (Tóth [103, Th. 1]). *Let $r \geqslant 2$ be a fixed integer. Assume that $\Delta_r(x) \ll x^{\alpha_r}(\log x)^{\beta_r}$, with $1/3 < \alpha_r < 1/2$. Then*

$$\sum_{n \leqslant x} a(n)^r = C_r x + x^{1/2} S_{2^r-2}(\log x) + R_r(x),$$

where

$$C_r := \prod_p \left(1 + \sum_{\nu=2}^{\infty} \frac{P(\nu)^r - P(\nu-1)^r}{p^\nu}\right),$$

S_{2^r-2} is a polynomial of degree $2^r - 2$ and $R_r(x) \ll x^{\alpha_r}(\log x)^{\beta_r}$ (is the same). The estimate $R_r(x) \ll x^{u_r}$ holds true, where u_r is given in Theorem 2.1.3.

Remark 2.1.5. According to a result of Krätzel [57],

$$\Delta_2(x) = \Delta((1,2,2,2);x) \ll x^{45/127}(\log x)^5,$$

where $45/127 \doteq 0.354330 \in (1/3, 1/2)$, hence the same is the remainder term for $\sum_{n\leqslant x} a(n)^2$. This improves $R_2(x) \ll x^{96/245+\varepsilon}$ with $96/245 \doteq 0.391836$, obtained in [124] by reducing the error term to the Piltz divisor problem concerning $\Delta_3(x)$.

Remark 2.1.6. Referring to our paper [98], Lelechenko [60, Th. 4] pointed out that the error term $R_f(x) \ll x^{u_{k,\ell}+\varepsilon}$ given in Remark 2.1.2 can be improved by using another result included in the book by Krätzel [56, Th. 6.8]. Namely, one can take $u_{k,\ell} = \frac{1}{\ell+1-\theta_{k-1}}$, where θ_t is an exponent such that $\Delta_t(x) \ll x^{\theta_t+\varepsilon}$ in the Piltz divisor problem. Since $\theta_t(x) \leqslant \frac{t-1}{t+2}$ holds true for $t \geqslant 4$, see Titchmarsh [94, Th. 12.3], it follows that $u_{k,\ell} \leqslant \frac{k+1}{\ell(k+1)+3} \in (1/(\ell+1), 1/\ell)$ for $k \geqslant 5$. In particular, in the case $r \geqslant 3$ the error terms of Theorem 2.1.3 and Corollary 2.1.4 can be improved by taking $u_r = \frac{2^r+1}{2^{r+1}+5}$.

Remark 2.1.7. An elementary proof of the asymptotic formula

$$\sum_{n\leqslant x} \tau(n)^2 \sim Ax(\log x)^3 \qquad (x\to\infty),$$

appears in several places. See, for example, Nathanson [69, Th. 7.8]. Although this and related questions were investigated by several authors, we are not aware even of elementary proofs for the asymptotic formula

$$\sum_{n\leqslant x} \tau(n)^r \sim C_r x(\log x)^{2^r-1} \qquad (x\to\infty),$$

valid for any integer $r \geqslant 2$. In the joint work with Luca [62] we gave a minimal elementary proof of the following more general result:

Let $k \in \mathbb{N}$ and let $f : \mathbb{N} \to \mathbb{C}$ be a multiplicative function satisfying the following properties:

(i) $f(p) = k$ for every prime p,

(ii) $f(p^\nu) = \nu^{O(1)}$ for every prime p and every integer $\nu \geqslant 2$, where the constant implied by the O symbol is uniform in p.

Then

$$\sum_{n\leqslant x} \frac{f(n)}{n} = \frac{1}{k!}C_f(\log x)^k + D_f(\log x)^{k-1} + O((\log x)^{k-2})$$

and

$$\sum_{n\leqslant x} f(n) = \frac{1}{(k-1)!}C_f x(\log x)^{k-1} + O(x(\log x)^{k-2}),$$

where

$$C_f = \prod_p \left(1-\frac{1}{p}\right)^k \sum_{\nu=0}^{\infty} \frac{f(p^\nu)}{p^\nu}$$

and D_f is another constant depending on f. In the case $f(n) = \tau(n)^r$ with $r \in \mathbb{N}$, this applies by selecting $k = 2^r$. Also see Martin [63, Prop. A.3] for a similar result proved by using analytic arguments.

2.2 Alternating sums concerning multiplicative functions

In what follows we consider certain alternating sums

$$\sum_{n \leqslant x} (-1)^{n-1} \frac{1}{f(n)}, \tag{2.4}$$

where $f(n)$ is a nonvanishing multiplicative function. The exposition of this section is based on our paper [112].

Bordellès and Cloitre [11] established asymptotic formulas with error terms for sums (2.4), where f belongs to a class of multiplicative arithmetic functions, including Euler's function $\phi(n)$, the sum-of-divisors function $\sigma(n)$ and the Dedekind function $\psi(n)$. It seems that there are no other results in the literature for alternating sums of type (2.4). Using a different approach, also based on the convolution method, we show that for many classical multiplicative arithmetic functions f, estimates with sharp error terms for the alternating sum (2.4) can be deduced by using known results for

$$\sum_{n \leqslant x} \frac{1}{f(n)}.$$

If $f(n)$ is multiplicative, then the function $(-1)^{n-1} \frac{1}{f(n)}$ is also multiplicative. Furthermore,

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{f(n) n^s} = \left(\sum_{n=1}^{\infty} \frac{1}{f(n) n^s} \right) \left(2 \left(\sum_{\nu=0}^{\infty} \frac{1}{f(2^\nu) 2^{\nu s}} \right)^{-1} - 1 \right),$$

but to obtain sharp error terms we need some auxiliary results.

Assume that f is a nonzero complex-valued multiplicative function. Consider the formal power series

$$S_f(x) := \sum_{\nu=0}^{\infty} a_\nu x^\nu,$$

where $a_\nu = f(2^\nu)$ $(\nu \geqslant 0)$, $a_0 = f(1) = 1$. Note that $S_f(x)$ is the Bell series of the function f for the prime $p = 2$. Let

$$\overline{S}_f(x) := \sum_{\nu=0}^{\infty} b_\nu x^\nu$$

be its formal reciprocal power series. Here the coefficients b_ν are given by $b_0 = 1$ and

$$\sum_{j=0}^{\nu} a_j b_{\nu-j} = 0 \quad (\nu \geqslant 1). \tag{2.5}$$

If both series $S_f(x)$ and $\overline{S}_f(x)$ converge for an $x \in \mathbb{C}$, then $S_f(x)\overline{S}_f(x) = 1$. In particular, if r_f and $\overline{r}_f$ are the radii of convergence of $S_f(x)$, respectively $\overline{S}_f(x)$, then $S_f(x)\overline{S}_f(x) = 1$ for every $x \in \mathbb{C}$ such that $|x| < \min(r_f, \overline{r}_f)$.

Theorem 2.2.1 (**Tóth [112, Prop. 7]**). *Let f be a nonvanishing multiplicative function. Assume that*

(i) there exist constants D_f and E_f such that

$$\sum_{n\leqslant x} \frac{1}{f(n)} = D_f(\log x + E_f) + O\left(x^{-1}R_{1/f}(x)\right), \tag{2.6}$$

where $1 \ll R_{1/f}(x) = o(x)$ as $x \to \infty$, and $R_{1/f}(x)$ is nondecreasing;

(ii) the radius of convergence of the series $S_{1/f}(x)$ is $r_{1/f} > 1$;

(iii) the coefficients b_ν of the reciprocal power series $\overline{S}_{1/f}(x)$ satisfy $b_\nu \ll M^\nu$ as $\nu \to \infty$, where $0 < M < 1$ is a real number.

Then

$$\sum_{n\leqslant x} (-1)^{n-1}\frac{1}{f(n)} = D_f\left(\left(\frac{2}{S_{1/f}(1)} - 1\right)(\log x + E_f) + 2(\log 2)\frac{S'_{1/f}(1)}{S_{1/f}(1)^2}\right) + O\left(T_{1/f}(x)\right), \tag{2.7}$$

where

$$T_{1/f}(x) = \begin{cases} x^{-1}R_{1/f}(x), & \text{if } 0 < M < \frac{1}{2}; \\ x^{-1}R_{1/f}(x)\log x, & \text{if } M = \frac{1}{2}; \\ x^{\log M/\log 2}\max(\log x, R_{1/f}(x)), & \text{if } \frac{1}{2} < M < 1. \end{cases} \tag{2.8}$$

By taking $f(n) = \phi(n)$ and $M = 1/2$ we deduce the following result, which is the "alternating version" of formula (1.5).

Corollary 2.2.2 (**Tóth [112, Th. 17]**).

$$\sum_{n\leqslant x} (-1)^{n-1}\frac{1}{\phi(n)} = -\frac{A}{3}\left(\log x + \gamma - B - \frac{8}{3}\log 2\right) + O\left(x^{-1}(\log x)^{5/3}\right), \tag{2.9}$$

where γ is Euler's constant, A and B being the constant defined by (1.6).

The result (2.9) improves the error term $O\left(x^{-1}(\log x)^3\right)$ obtained by Bordellès and Cloitre [11, Cor. 4, (i)].

The following asymptotic formula is due to Sita Ramaiah and Suryanarayana [83, Cor. 4.1]:

$$\sum_{n\leqslant x}\frac{1}{\sigma(n)} = E\left(\log x+\gamma+F\right)+O\left(x^{-1}(\log x)^{2/3}(\log\log x)^{4/3}\right), \tag{2.10}$$

where γ is Euler's constant,

$$E=\prod_p \alpha(p), \qquad F=\sum_p \frac{(p-1)^2\beta(p)\log p}{p\alpha(p)},$$

$$\alpha(p) = \left(1-\frac{1}{p}\right)\sum_{\nu=0}^{\infty}\frac{1}{\sigma(p^\nu)} = 1-\frac{(p-1)^2}{p}\sum_{j=1}^{\infty}\frac{1}{(p^j-1)(p^{j+1}-1)},$$

$$\beta(p) = \sum_{j=1}^{\infty}\frac{j}{(p^j-1)(p^{j+1}-1)}.$$

If $f(n)=\sigma(n)$ (and $M=1/2$ again), then Theorem 2.2.1 gives

Corollary 2.2.3 (Tóth [112, Th. 23]).

$$\sum_{n\leqslant x}(-1)^{n-1}\frac{1}{\sigma(n)} = E\left(\left(\frac{2}{K}-1\right)(\log x+\gamma+F)+2(\log 2)\frac{K'}{K^2}\right) \tag{2.11}$$

$$+O\left(x^{-1}(\log x)^{5/3}(\log\log x)^{4/3}\right),$$

where

$$K=\sum_{j=0}^{\infty}\frac{1}{2^{j+1}-1}, \qquad K'=\sum_{j=1}^{\infty}\frac{j}{2^{j+1}-1}.$$

The result (2.11) improves the error term $O\left(x^{-1}(\log x)^4\right)$ obtained by Bordellès and Cloitre [11, Cor. 4, (v)]. Here $K \doteq 1.606695$ is the Erdős-Borwein constant, known to be irrational.

Our next formula is the analog of a result of Ramanujan [76, Eq. (7)] for the sum $\sum_{n\leqslant x}\frac{1}{\tau(n)}$. See Wilson [122, Sect. 3] for its proof.

Theorem 2.2.4 (Tóth [112, Th. 27]).

$$\sum_{n\leqslant x}(-1)^{n-1}\frac{1}{\tau(n)} = x\sum_{t=1}^{N}\frac{B_t}{(\log x)^{t-1/2}}+O\left(\frac{x}{(\log x)^{N+1/2}}\right), \tag{2.12}$$

valid for every real $x\geqslant 2$ and every fixed integer $N\geqslant 1$, where B_t $(1\leqslant t\leqslant N)$ are computable constants. In particular,

$$B_1=\frac{1}{\sqrt{\pi}}\left(\frac{1}{\log 2}-1\right)\prod_p\left(\sqrt{p^2-p}\ \log\left(\frac{p}{p-1}\right)\right).$$

To obtain the error terms of (2.11) and (2.12) we need to use the following result of Kaluza [52] on reciprocal power series: Let $\sum_{\nu=0}^{\infty} a_\nu x^\nu$ be a power series such that $a_\nu > 0$ $(\nu \geqslant 0)$ and the sequence $(a_\nu)_{\nu\geqslant 0}$ is log-convex, that is $a_\nu^2 \leqslant a_{\nu-1}a_{\nu+1}$ $(\nu \geqslant 1)$. Then for the coefficients b_ν of the (formal) reciprocal power series $\sum_{\nu=0}^{\infty} b_\nu x^\nu$ one has $b_0 = 1/a_0 > 0$ and

$$-\frac{1}{a_0^2} a_\nu \leqslant b_\nu \leqslant 0 \quad \text{for all } \nu \geqslant 1.$$

Another result on estimates of the coefficients of reciprocal power series is Kendall's renewal theorem. See Berenhaut, Allen, and Fraser [10, Th. 1.1]. An explicit form of Kendall's theorem was proved in [10, Th. 1.2]. However, it cannot be used for our purposes. We prove the following new explicit Kendall-type inequality, which can be applied to some functions we deal with.

Proposition 2.2.5 (Tóth [112, Prop. 12]). *Assume that $\sum_{\nu=0}^{\infty} a_\nu x^\nu$ is a power series such that $a_0 = 1$ and $|a_\nu| \leqslant Aq^\nu$ $(\nu \geqslant 1)$ for some absolute constants $A, q > 0$. Then for the coefficients b_ν of the reciprocal power series one has*

$$|b_\nu| \leqslant Aq^\nu (A+1)^{\nu-1} \quad (\nu \geqslant 1).$$

Corollary 2.2.6 (Tóth [112, Cor. 13]). *Let f be a positive multiplicative function such that*

(i) asymptotic formula (2.6) is valid with $1 \ll R_{1/f}(x) = o(x)$ as $x \to \infty$;

(ii) $1/f(2^\nu) \leqslant Aq^\nu$ $(\nu \geqslant 1)$, where $A, q > 0$ are fixed real constants satisfying $M := q(A+1) < 1$.

Then the asymptotic formula (2.7) holds for $\sum_{n\leqslant x} (-1)^{n-1} \frac{1}{f(n)}$, with error term (2.8).

Let $\sigma^{**}(n)$ denote the sum of bi-unitary divisors of n. Recall that a divisor d of n is a unitary divisor if $(d, n/d) = 1$. A divisor d of n is called a bi-unitary divisor if the greatest common unitary divisor of d and n/d is 1. The function σ^{**} is multiplicative and for every prime power p^ν $(\nu \geqslant 1)$,

$$\sigma^{**}(p^\nu) = \begin{cases} \sigma(p^\nu), & \text{if } \nu \text{ is odd;} \\ \sigma(p^\nu) - p^{\nu/2}, & \text{if } \nu \text{ is even.} \end{cases}$$

Sitaramaiah and Subbarao [82, Th. 3.2] established an asymptotic formula for the sum $\sum_{n\leqslant x} 1/\sigma^{**}(n)$. By using Corollary 2.2.6 we show

Theorem 2.2.7 (Tóth [112, Th. 50]).

$$\sum_{n\leqslant x} (-1)^{n-1} \frac{1}{\sigma^{**}(n)} = A_1^{**} \log x + B_1^{**} + O(x^c (\log x)^{14/3} (\log\log x)^{4/3}),$$

where A_1^{**}, B_1^{**} are explicit constants and $c = (\log 9/10)/(\log 2) \doteq -0.152003$.

Our paper [112] also includes asymptotic formulas for the sums $\sum_{n\leqslant x}(-1)^{n-1}/f(n)$, where $f(n)$ is replaced by $\psi(n)$ (Dedekind function), $P(n)$ (gcd-sum function) and other special multiplicative functions. Furthermore, we derived formulas for $\sum_{n\leqslant x}(-1)^{n-1}f(n)$. For example [112, Th. 15] states that

$$\sum_{n\leqslant x}(-1)^{n-1}\phi(n) = \frac{1}{\pi^2}x^2 + O\left(x(\log x)^{2/3}(\log\log x)^{4/3}\right),$$

which may be compared to (1.4).

2.3 Maximal orders

In this section we present easily applicable theorems for determining

$$L = L(f) := \limsup_{n\to\infty} \frac{f(n)}{\log\log n},$$

where f are certain nonnegative real-valued multiplicative functions. Let denote

$$\varrho(p) = \varrho(f,p) := \sup_{\nu\geqslant 0} f(p^\nu),$$

for the primes p, and consider the product

$$R = R(f) := \prod_p \left(1-\frac{1}{p}\right)\varrho(p).$$

We formulate the conditions for lower and upper estimates for L separately. Note that $\varrho(p) \geqslant f(p^0) = 1$ for all p.

Theorem 2.3.1 (Tóth and Wirsing [114, Th. 1]). *Let f be a nonnegative real-valued multiplicative function. Assume that $\varrho(p) < \infty$ for all primes p and that the product R converges unconditionally (i.e. irrespectively of order), improper limits being allowed. Then*

$$L \leqslant e^\gamma R. \tag{2.13}$$

The next result uses a different assumption.

Theorem 2.3.2 (Tóth and Wirsing [114, Th. 2]). *Let f be any nonnegative real-valued multiplicative function. Suppose that $\varrho(p) < \infty$ for all p and that the product R converges, improper limits being allowed, and that*

$$\varrho(p) \leqslant 1 + o\left(\frac{\log p}{p}\right).$$

Then (2.13) *holds.*

To establish $e^\gamma R$ also as the lower limit more information is required. Namely, the suprema $\varrho(p)$ must be sufficiently well approximated by not too large powers of p.

Theorem 2.3.3 (**Tóth and Wirsing [114, Th. 3]**). *Let f be a nonnegative real-valued multiplicative function. Suppose that*

(i) $\varrho(p) < \infty$ for all primes p,

(ii) for each prime p there is an exponent $e_p = p^{o(1)} \in \mathbb{N}$ such that

$$\prod_p f(p^{e_p})\varrho(p)^{-1} > 0,$$

and

(iii) the product R converges, improper limits being allowed.

Then

$$L \geqslant e^\gamma R.$$

Corollary 2.3.4 (**Tóth and Wirsing [114, Cor. 1]**). *Let f be a nonnegative real-valued multiplicative arithmetic function such that for each prime p,*

(i) $\varrho(p) \leqslant (1-1/p)^{-1}$, and

(ii) there is an exponent $e_p = p^{o(1)} \in \mathbb{N}$ satisfying $f(p^{e_p}) \geqslant 1 + 1/p$.

Then

$$L = e^\gamma R,$$

that is the maximal order of $f(n)$ is $e^\gamma R \log\log n$.

Corollary 2.3.4 applies, for example, to the functions $f(n) = \sigma(n)$ and $f(n) = 1/\phi(n)$ and we recover the well known results of Gronwall (1913) and Landau (1909). Furthermore, let $f(n) = \sigma^{(e)}(n)$, the sum of exponential divisors of n. See Section 2.4. Then we obtain that

$$\limsup_{n\to\infty} \frac{\sigma^{(e)}(n)}{n \log\log n} = \frac{6}{\pi^2} e^\gamma,$$

which is a result of Fabrykowski and Subbarao [33]. Another application is Theorem 2.5.5. Further applications were given by Apostol [3, Prop. 4], Apostol [4, Prop. 7], Apostol and Petrescu [5, Prop. 7] and by myself [99, Th. 7, 8].

2.4 Arithmetic functions associated with exponential divisors

Let $d = \prod_p p^{\nu_p(d)}$ be a divisor of the integer $n = \prod_p p^{\nu_p(n)}$. Then d is called an *exponential divisor* of n, if $\nu_p(d) \mid \nu_p(n)$ for every prime p. Notation: $d \mid_e n$. This concept was introduced by Subbarao [87]. According to the definition, $1 \mid_e 1$, but 1 is not an exponential divisor of $n > 1$. The smallest exponential divisor of $n > 1$ is its squarefree kernel $\kappa(n) = \prod_{p|n} p$.

Let $\tau^{(e)}(n)$ and $\sigma^{(e)}(n)$ denote the number and the sum of exponential divisors of n, respectively. The function $\tau^{(e)}$ is called the *exponential divisor function*, already quoted in Section 2.1. The integer $n = \prod_p p^{\nu_p(n)}$ is called *exponentially squarefree* if all the exponents $\nu_p(n) \geqslant 1$ are squarefree. By convention, 1 is also exponentially squarefree. Let $q^{(e)}$ denote the characteristic function of exponentially squarefree integers. Properties of these and related functions were investigated by several authors. See Cao and Zhai [16], Lelechenko [60, 61], Pétermann and Wu [75], Smati and Wu [85], Subbarao [87], Wu [121].

Two integers $m, n > 1$ have common exponential divisors if and only if they have the same prime factors and in this case, i.e., for $m = \prod_{i=1}^r p_i^{a_i}$, $n = \prod_{i=1}^r p_i^{b_i}$, $a_i, b_i \geqslant 1$ $(1 \leqslant i \leqslant r)$, the greatest common exponential divisor of m and n is

$$(m,n)_{(e)} := \prod_{i=1}^r p_i^{(a_i,b_i)}.$$

Here $(1,1)_{(e)} = 1$ by convention and $(1,n)_{(e)}$ does not exist for $n > 1$.

2.4.1 Exponential Euler function

The integers $m, n > 1$ are called *exponentially coprime*, if they have the same prime factors and $(a_i, b_i) = 1$ for every $1 \leqslant i \leqslant r$, with the notation of above. In this case $(m,n)_{(e)} = \kappa(m) = \kappa(n)$. 1 and 1 are considered to be exponentially coprime. 1 and $n > 1$ are not exponentially coprime.

For $n = \prod_{i=1}^r p_i^{a_i} > 1$ with $a_i \geqslant 1$ $(1 \leqslant i \leqslant r)$, denote by $\phi^{(e)}(n)$ the number of integers $\prod_{i=1}^r p_i^{c_i}$ such that $1 \leqslant c_i \leqslant a_i$ and $(c_i, a_i) = 1$ for $1 \leqslant i \leqslant r$, and let $\phi^{(e)}(1) = 1$. Thus $\phi^{(e)}(n)$ counts the number of divisors d of n such that d and n are exponentially coprime. The function $\phi^{(e)}(n)$ is called the *exponential Euler function*, it is multiplicative and for every prime power p^ν $(\nu \geqslant 1)$, $\phi^{(e)}(p^\nu) = \phi(\nu)$, where ϕ is Euler's function.

As another consequence of Theorem 2.1.1 and Remark 2.1.2, by selecting $f(n) = \phi^{(e)}(n)^r$ with $\ell = 3$, $k = 2^r$, we have the following result.

Theorem 2.4.1 (Tóth [96, Th. 1], [98, Eq. (6)]). *Let $r \geqslant 1$ be an integer. Then*

$$\sum_{n \leqslant x} \phi^{(e)}(n)^r = B_r x + x^{1/3} T_{2^r-2}(\log x) + O(x^{t_r+\varepsilon}), \tag{2.14}$$

for every $\varepsilon > 0$, where

$$B_r := \prod_p \left(1 + \sum_{a=3}^\infty \frac{\phi(a)^r - \phi(a-1)^r}{p^a}\right),$$

T_{2^r-2} is a polynomial of degree $2^r - 2$ and $t_1 = 1/5$, $t_r := \frac{2^{r+1}-1}{3 \cdot 2^{r+1}}$ for $r \geqslant 2$.

In the case $r = 1$, Pétermann [74, Th. 1] improved the error term in (2.14) into $O(x^{1/5}\log x)$. Cao and Zhai [17] obtained new results on the four dimensional divisor problem of (a,b,c,c) type, where $1 \leqslant a \leqslant b < c$ are fixed integers. As an application, they proved in [17, Th. 3] the following more precise asymptotic formula:

$$\sum_{n\leqslant x} \phi^{(e)}(n) = B_1 x + B_2 x^{1/3} + D_1 x^{1/5}\log x + D_2 x^{1/5} + +O(x^{18/95+\varepsilon}), \tag{2.15}$$

where $18/95 \doteq 0.189473 < 1/5$. They also showed that the error term in (2.15) is $\Omega(x^{1/8})$.

In the case $r \geqslant 3$ the error term of (2.14) can be improved by taking $t_r = \frac{2^r+1}{3(2^r+2)}$, as shown by Lelechenko [60]. See Remark 2.1.6.

2.4.2 Exponential Möbius function

The *exponential convolution* of the arithmetic functions f and g is defined by

$$(f \odot g)(n) = \sum_{b_1 c_1 = a_1} \cdots \sum_{b_r c_r = a_r} f(p_1^{b_1}\cdots p_r^{b_r}) g(p_1^{c_1}\cdots p_r^{c_r}),$$

where $n = p_1^{a_1}\cdots p_r^{a_r}$.

The convolution $\odot$ is commutative, associative and has the identity element μ^2. Furthermore, a function f has an inverse with respect to $\odot$ if and only if $f(1) \neq 0$ and $f(p_1\cdots p_s) \neq 0$ for any distinct primes $p_1,\ldots,p_s$. The inverse with respect to $\odot$ of the constant 1 function is called the *exponential Möbius function* and is denoted by $\mu^{(e)}$. Hence for every $n \geqslant 1$,

$$\sum_{d|_e n} \mu^{(e)}(d) = \mu^2(n).$$

Here $\mu^{(e)}(1) = 1$ and for $n = p_1^{a_1}\cdots p_r^{a_r} > 1$,

$$\mu^{(e)}(n) = \mu(a_1)\cdots\mu(a_r).$$

Observe that $|\mu^{(e)}(n)| = 1$ or 0, according as n is exponentially squarefree or not. Wu [121, Th. 2] deduced, improving a result by Subbarao [87] that

$$\sum_{n\leqslant x} |\mu^{(e)}(n)| = C_1 x + O(x^{1/4}\delta(x)),$$

where $\delta(x)$ is defined by (1.3) and

$$C_1 = \prod_p \left(1 + \sum_{a=4}^{\infty} \frac{\mu^2(a) - \mu^2(a-1)}{p^a}\right).$$

I showed that the corresponding error term can be improved on the assumption of the Riemann hypothesis (RH).

Theorem 2.4.2 (Tóth [97, Th. 3]). *If RH is true, then*

$$\sum_{n\leqslant x} |\mu^{(e)}(n)| = C_1 x + O(x^{1/5+\varepsilon}),$$

for every $\varepsilon > 0$.

Later on, Cao and Zhai [16, Th. 1] proved the following more precise asymptotic formula: If RH is true, then

$$\sum_{n\leqslant x} |\mu^{(e)}(n)| = C_1 x + C_2 x^{1/5} + O(x^{38/193+\varepsilon}), \tag{2.16}$$

where C_2 is a computable constant and $38/193 \doteq 0.196891$. Cao and Zhai [16, Th. 2] also proved that the error term in (2.16) is $\Omega(x^{1/8})$. Shevelev [80] investigated the asymptotic density of S-exponential numbers defined as the positive integers such that all exponents in their prime power factorization are in S, where S is a fixed subset of $\mathbb{N}$. If S is the set of squarefree numbers, then one reobtains the exponentially squarefree integers.

For the function $\mu^{(e)}(n)$ we have the next result.

Theorem 2.4.3 (Tóth [97, Th. 2]). *(i) The Dirichlet series of* $\mu^{(e)}$ *is of the form*

$$\sum_{n=1}^{\infty} \frac{\mu^{(e)}(n)}{n^s} = \frac{\zeta(s)}{\zeta^2(2s)} U(s), \quad \Re s > 1,$$

where $U(s) := \sum_{n=1}^{\infty} \frac{u(n)}{n^s}$ *is absolutely convergent for* $\Re s > 1/5$.

(ii)

$$\sum_{n\leqslant x} \mu^{(e)}(n) = Kx + O(x^{1/2} \exp(-c(\log x)^{9/25-\delta})), \tag{2.17}$$

for any $\delta > 0$, *where* $c > 0$ *is a constant and*

$$K = \prod_p \left(1 + \sum_{a=2}^{\infty} \frac{\mu(a) - \mu(a-1)}{p^a}\right).$$

(iii) Assume RH. Let $1/4 < r < 1/3$ *be an exponent such that* $D(x) := \sum_{n\leqslant x} \mu^2(n) - x/\zeta(2) = O(x^{r+\varepsilon})$ *for every* $\varepsilon > 0$. *Then the error term in (ii) is* $O(x^{(2-r)/(5-4r)+\varepsilon})$ *for every* $\varepsilon > 0$.

The best known value – to our knowledge – of r is $r = 17/54 \doteq 0.314814$, obtained by Jia [50]. See also Pappalardi [73]. Therefore the error term in (2.17), assuming RH, is $O(x^{91/202+\varepsilon})$ for every $\varepsilon > 0$, where $91/202 \doteq 0.450495$.

By using analytic arguments (Perron's formula and complex integration), Cao and Zhai [16, Th. 1] improved the error term (2.17) under RH into $O(x^{37/94+\varepsilon})$, where $37/94 \doteq 0.393617$, and showed in [16, Th. 2] that this error term is $\Omega(x^{1/4})$.

2.4.3 The function $t^{(e)}(n)$

I introduced and investigated in paper [97] the functions $t^{(e)}(n)$ and $\kappa^{(e)}(n)$, denoting the number of exponentially squarefree exponential divisors of n and the maximal exponentially squarefree exponential divisor of n, respectively. These are the exponential analogues of the functions representing the number of squarefree divisors of n (i.e. $\theta(n) = 2^{\omega(n)}$) and the maximal squarefree divisor of n (the squarefree kernel $\kappa(n) = \prod_{p|n} p$), respectively.

The functions $t^{(e)}(n)$ and $\kappa^{(e)}(n)$ are multiplicative and for $n = p_1^{a_1} \cdots p_r^{a_r} > 1$,

$$t^{(e)}(n) = 2^{\omega(a_1)} \cdots 2^{\omega(a_r)},$$

$$\kappa^{(e)}(n) = p_1^{\kappa(a_1)} \cdots p_r^{\kappa(a_r)}.$$

Here I discuss only the function $t^{(e)}(n)$. Note that for every prime p, $t^{(e)}(p) = 1$, $t^{(e)}(p^2) = t^{(e)}(p^3) = t^{(e)}(p^4) = t^{(e)}(p^5) = 2, t^{(e)}(p^6) = 4$, etc. We have

Theorem 2.4.4 (Tóth [97, Th. 4]). *(i) The Dirichlet series of $t^{(e)}$ is of form*

$$\sum_{n=1}^{\infty} \frac{t^{(e)}(n)}{n^s} = \zeta(s)\zeta(2s)V(s), \quad \Re s > 1,$$

where $V(s) = \sum_{n=1}^{\infty} \frac{v(n)}{n^s}$ is absolutely convergent for $\Re s > 1/4$.

(ii)

$$\sum_{n \leqslant x} t^{(e)}(n) = C_1 x + C_2 x^{1/2} + O(x^{1/4+\varepsilon}), \tag{2.18}$$

for every $\varepsilon > 0$, where C_1, C_2 are constants given by

$$C_1 := \prod_p \left(1 + \frac{1}{p^2} + \sum_{a=6}^{\infty} \frac{2^{\omega(a)} - 2^{\omega(a-1)}}{p^a}\right),$$

$$C_2 := \zeta(1/2) \prod_p \left(1 + \sum_{a=4}^{\infty} \frac{2^{\omega(a)} - 2^{\omega(a-1)} - 2^{\omega(a-2)} + 2^{\omega(a-3)}}{p^{a/2}}\right).$$

Pétermann [74, Th. 1] improved the error term in (2.18) into $O(x^{1/4})$. Cao and Zhai [16, Th. 1] obtained that under RH it is $O(x^{3728/15469+\varepsilon})$, where $3728/15469 \doteq 0.240998 < 1/4$. Cao and Zhai [16, Th. 2] also proved that the error term in (2.18) is $\Omega(x^{1/6})$.

By generalizing results of Rényi and Ivić, Zurita [128] established asymptotic formulas for the sums

$$\sum_{\substack{n \leqslant x \\ \Omega(n) - \omega(n) = q}} f(n),$$

where $f(n)$ are certain multiplicative functions, which apply, among others, to the functions $f(n) = \tau^{(e)}(n)$, $\phi^{(e)}(n)$, $t^{(e)}(n)$, $\sigma^{(e)}(n)/n$, $P(n)/n$, $P^{(e)}(n)/n$, $a(n)$. Here $P^{(e)}(n)$ is the exponential analog of the gcd-sum function $P(n)$. See Section 2.5.2.

2.5 Gcd-sum functions

2.5.1 Gcd-sum function

The *gcd-sum function*, also called Pillai's arithmetical function is defined by

$$P(n) = \sum_{k=1}^{n} (k, n).$$

The function $P(n)$ is multiplicative and for every $n \in \mathbb{N}$,

$$P(n) = \sum_{d|n} d\, \phi(n/d) = \sum_{d|n} d\tau(d)\mu(n/d).$$

Hence the arithmetic mean of $(1, n), \ldots, (n, n)$ is given by

$$A(n) := \frac{P(n)}{n} = \sum_{d|n} \frac{\phi(d)}{d} = \tau(n) \prod_{p^\nu || n} \left(1 - \frac{\nu/(\nu+1)}{p}\right), \tag{2.19}$$

which is „close" to $\tau(n)$.

Various properties, generalizations and analogs of the function $P(n)$ were investigated by several authors. See my survey paper [101] and my subsequent papers [102, 106].

Chidambaraswamy and Sitaramachandrarao [20] proved the following result:

$$\sum_{n \leqslant x} P(n) = \frac{x^2}{2\zeta(2)} \left(\log x + 2\gamma - \frac{1}{2} - \frac{\zeta'(2)}{\zeta(2)}\right) + \mathcal{O}(x^{1+\theta+\varepsilon}), \tag{2.20}$$

where θ is the exponent appearing in the Dirichlet divisor problem (1.1). They also proved that

$$\limsup_{n \to \infty} \frac{\log A(n) \log\log n}{\log n} = \log 2, \tag{2.21}$$

which is well known for the function $\tau(n)$ instead of $A(n)$. See (1.7).

Our next result is an asymptotic formula for the quadratic moment of the function $A(n)$. Let α_4 be the exponent appearing in the Piltz divisor problem for $\tau_4(n)$. It is known that $\alpha_4 \leqslant 1/2$ (result of Hardy and Littlewood) and it is conjectured that $\alpha_4 = 3/8$, cf. Titchmarsh [94, Ch. 12].

Theorem 2.5.1 (Tóth [101, Th. 1]). *i) For any $\varepsilon > 0$,*

$$\sum_{n \leqslant x} A(n)^2 = x(C_1 \log^3 x + C_2 \log^2 x + C_3 \log x + C_4) + O(x^{1/2+\varepsilon}), \tag{2.22}$$

where

$$C_1 = \frac{1}{\pi^2} \prod_p \left(1 + \frac{1}{p^3} - \frac{4}{p(p+1)}\right),$$

C_2, C_3, C_4 are constants given in terms of the constants appearing in the asymptotic formula (2.1) *for* $\sum_{n\leqslant x}\tau(n)^2$.

ii) Assume that $\alpha_4 < 1/2$. *Then the error term in* (2.22) *is* $O(x^{1/2}\delta(x))$, *where* $\delta(x)$ *is defined by* (1.3).

iii) If RH is true, then the error term in (2.22) *is* $O(x^{(2-\alpha_4)/(5-4\alpha_4)}\lambda(x))$, *where*

$$\lambda(x) := \exp((\log x)^{1/2}(\log\log x)^{14}).$$

Remark 2.5.2. Let $M(x) = \sum_{n\leqslant x}\mu(n)$ denote the Mertens function. The error term of iii) comes from the estimate $M(x) \ll \sqrt{x}\,\lambda(x)$, the best up to now, valid under RH, due to Soundararajan [86].[1]

Remark 2.5.3. Later, by using the analytic method (properties of the zeta function), Zhang and Zhai [127, Th. 1] established the following asymptotic formula, where $k \geqslant 1$ is a fixed integer:

$$\sum_{n\leqslant x} A(n)^k = xQ_{2^k-1}(\log x) + O(x^{\beta_k+\varepsilon}),$$

where $Q_{2^k-1}(t)$ is a polynomial of degree 2^k-1 in t and $\beta_2 = 1/2$, $\beta_3 = 5/8$, $\beta_4 = 7/9$, $\beta_5 = 31/36$, $\beta_6 = 207/224$, $\beta_j = 1 - 2^{-2j/3}/50$ $(j \geqslant 7)$.

Note that in the case $k = 2$ this is the same error as in i) of Theorem 2.5.1.

The next formula was proved by Chen and Zhai [19, Th. 4], sharpening my result [101, Th. 6], which is recovered for $N = 0$:

$$\sum_{n\leqslant x}\frac{1}{P(n)} = \sum_{j=0}^{N}\frac{K_j}{(\log x)^{j-1/2}} + O\left(\frac{1}{(\log x)^{N+1/2}}\right), \tag{2.23}$$

valid for every real $x \geqslant 2$ and every fixed integer $N \geqslant 0$ where K_j $(0 \leqslant j \leqslant N)$ are computable constants,

$$K_0 = \frac{2}{\sqrt{\pi}}\prod_p\left(1-\frac{1}{p}\right)^{1/2}\sum_{\nu=0}^{\infty}\frac{1}{P(p^\nu)}.$$

Referring to my similar formulas [101, Th. 6] concerning the functions $P^{(e)}(n)$ and $\widetilde{P}(n)$, to be defined in the next Sections, Chen and Zhai [19, Th. 4] obtained results analogous to (2.23).

[1] Balazard and Roton showed in their preprint [9] that under RH the slightly better estimate $M(x) \ll \sqrt{x}\exp((\log x)^{1/2}(\log\log x)^{5/2+\varepsilon})$ holds.

2.5.2 Exponential analog of the gcd-sum function

In paper [96] I introduced the function

$$P^{(e)}(n) = \sum_{\substack{j=1 \\ \kappa(j)=\kappa(n)}}^{n} (j,n)_{(e)},$$

representing an analog of Pillai's function $P(n) = \sum_{j=1}^{n}(j,n)$. The function $P^{(e)}(n)$, called the exponential gcd-sum function, is also multiplicative and for every prime power p^{ν} ($\nu \geqslant 1$),

$$P^{(e)}(p^{\nu}) = \sum_{t=1}^{\nu} p^{(t,\nu)} = \sum_{d\mid\nu} p^{d}\phi(\nu/d),$$

so here $P^{(e)}(p) = p$, $P^{(e)}(p^2) = p + p^2$, $P^{(e)}(p^3) = 2p + p^3$, $P^{(e)}(p^4) = 2p + p^2 + p^4$, etc.

Theorem 2.5.4 (Tóth [96, Th. 3]).

$$\sum_{n\leqslant x} P^{(e)}(n) = C_4 x^2 + O(x(\log x)^{5/3}), \tag{2.24}$$

where the constant C_4 *is given by*

$$C_4 = \frac{1}{2}\prod_{p}\left(1 + \sum_{a=2}^{\infty}\frac{P^{(e)}(p^a) - pP^{(e)}(p^{a-1})}{p^{2a}}\right).$$

Theorem 2.5.5 (Tóth [96, Th. 4]).

$$\limsup_{n\to\infty}\frac{P^{(e)}(n)}{n\log\log n} = \frac{6}{\pi^2}e^{\gamma}. \tag{2.25}$$

It is a simple consequence of (2.25) that the error term in (2.24) is $\Omega(x\log\log x)$. Pétermann [74, Th. 2] proved the stronger result that it is $\Omega_{\pm}(x\log\log x)$.

2.5.3 A gcd-sum function involving regular integers (mod n)

An integer k is called a *regular integer* (mod n), if there exists an integer x such that $k^2x \equiv k$ (mod n), i.e., the residue class of k is a regular element (in the sense of J. von Neumann) of the ring $\mathbb{Z}_n$ of residue classes (mod n). In general, an element k of a ring R is said to be (von Neumann) regular if there is an $x \in R$ such that $k = kxk$. If every $k \in R$ has this property, then R is called a *von Neumann regular ring.*

Let $n > 1$ be an integer with prime factorization $n = p_1^{\nu_1}\cdots p_r^{\nu_r}$. It can be shown that $k \geqslant 1$ is regular (mod n) if and only if for every $i \in \{1,\ldots,r\}$ either $p_i \nmid k$ or $p_i^{\nu_i} \mid k$. These integers occur in the literature also in another context. It is said that an integer k

possesses a *weak order* (mod n) if there exists an integer $m \geqslant 1$ such that $k^{m+1} \equiv k$ (mod n). Then the weak order of k is the smallest m with this property. It turns out that k is regular (mod n) if and only if k possesses a weak order (mod n). See my paper [99].

Let $\operatorname{Reg}_n = \{k : 1 \leqslant k \leqslant n,\ k \text{ is regular (mod } n)\}$ and let $\varrho(n) = \# \operatorname{Reg}_n$ denote the number of regular integers k (mod n) such that $1 \leqslant k \leqslant n$. This function is multiplicative and $\varrho(p^\nu) = \phi(p^\nu) + 1 = p^\nu - p^{\nu-1} + 1$ for every prime power p^ν ($\nu \geqslant 1$), where ϕ is the Euler function. The average order of the function $\varrho(n)$ was considered by Joshi [51]. One has

$$\sum_{n \leqslant x} \varrho(n) = \frac{1}{2} A x^2 + R(x),$$

where

$$A = \prod_p \left(1 - \frac{1}{p^2(p+1)}\right) = \zeta(2) \prod_p \left(1 - \frac{1}{p^2} - \frac{1}{p^3} + \frac{1}{p^4}\right) \doteq 0.8815$$

is the so called quadratic class-number constant, and $R(x) = O(x \log^3 x)$, given in [51] using elementary arguments. This was improved into $R(x) = O(x \log x)$ by Herzog and Smith [41], using analytic methods. Also, $R(x) = \Omega_{\pm}(x \sqrt{\log\log x})$, see [41].

In the paper [100] I introduced the function

$$\widetilde{P}(n) := \sum_{k \in \operatorname{Reg}_n} (k, n),$$

which is another analog of the gcd-sum function $P(n)$, discussed above. I showed that the function $\widetilde{P}(n)$ is multiplicative and for every $n \in \mathbb{N}$,

$$\widetilde{P}(n) = \sum_{\substack{de=n \\ (d,e)=1}} d\phi(e) \tag{2.26}$$

$$= n \prod_{p \mid n} \left(2 - \frac{1}{p}\right) = 2^{\omega(n)} n \prod_{p \mid n} \left(1 - \frac{1}{2p}\right), \tag{2.27}$$

which is ,,close” to $2^{\omega(n)} n$.

Let $\psi(n) = n \prod_{p \mid n} (1 + 1/p)$ denote the Dedekind function and let

$$\alpha(n) = \sum_{p \mid n} \frac{\log p}{p-1}, \quad \beta(n) = \sum_{p \mid n} \frac{\log p}{p^2 - 1}.$$

Theorem 2.5.6 (Tóth [100, Th. 2]). *We have*

$$\sum_{n \leqslant x} \widetilde{P}(n) = \frac{x^2}{2\zeta(2)} (K_1 \log x + K_2) + O(x^{3/2} \delta(x)), \tag{2.28}$$

where the constants K_1 and K_2 are given by

$$K_1 := \sum_{n=1}^{\infty} \frac{\mu(n)}{n\psi(n)} = \prod_p \left(1 - \frac{1}{p(p+1)}\right),$$

$$K_2 := K_1 \left(2\gamma - \frac{1}{2} - \frac{2\zeta'(2)}{\zeta(2)}\right) - \sum_{n=1}^{\infty} \frac{\mu(n)(\log n - \alpha(n) + 2\beta(n))}{n\psi(n)},$$

and $\delta(x)$ is given by (1.3).

If RH is true, then the error term of (2.28) *is $O(x^{(7-5\theta)/(5-4\theta)}\eta(x))$, where θ is the exponent in the Dirichlet divisor problem* (1.1) *and*

$$\eta(x) := \exp\left(B(\log x)(\log\log x)^{-1}\right),$$

with a positive constant B.

Theorem 2.5.7 (Tóth [100, Th. 1]). *The minimal order of $\widetilde{P}(n)$ is $3n/2$ and the maximal order of $\log(\widetilde{P}(n)/n)$ is $\log 2 \log n/\log\log n$.*

Zhang and Zhai [125] pointed out that the estimate of $\sum_{n\leqslant x} \widetilde{P}(n)$ is closely related to the squarefree divisor problem. They showed that under RH the error term in (2.28) is $O(x^{15/11+\varepsilon})$, due to the result of Baker [6], quoted in the Introduction.

De Koninck and Kátai [27] introduced two wide classes of arithmetic functions, $\mathcal{R}$ and $\mathcal{U}$, the first of which includes the function $P(n)/n$, and the second includes $\widetilde{P}(n)/n$. They deduced asymptotic formulas for $\sum_{n\leqslant x} R(n)$, $\sum_{n\leqslant x} U(n)$ and $\sum_{p\leqslant x} R(p-1)$, $\sum_{p\leqslant x} U(p-1)$, where $R \in \mathcal{R}$, $U \in \mathcal{U}$.

Zhang and Zhai [126] improved the error term for $\sum_{n\leqslant x} U(n)$ and also deduced a short interval result for $\sum_{x\leqslant n\leqslant x+y} U(n)$.

2.6 Weighted averages of Ramanujan sums

Alkan [1] considered the weighted average of the Ramanujan sums $c_k(j)$ defined by

$$S_r(k) := \frac{1}{k^{r+1}} \sum_{j=1}^{k} j^r c_k(j) \quad (r \in \mathbb{N}), \tag{2.29}$$

being motivated by the use of (2.29) in proving exact formulas for certain mean square averages of special values of L-functions. He showed that for every $k, r \in \mathbb{N}$,

$$S_r(k) = \frac{\phi(k)}{2k} + \frac{1}{r+1} \sum_{m=1}^{\lfloor r/2 \rfloor} \binom{r+1}{2m} B_{2m} \prod_{p|k} \left(1 - \frac{1}{p^{2m}}\right), \tag{2.30}$$

leading to the asymptotic formula

$$\sum_{k\leqslant x} S_r(k) = \left(\frac{3}{\pi^2} + \frac{1}{r+1} \sum_{m=1}^{\lfloor r/2 \rfloor} \binom{r+1}{2m} \frac{B_{2m}}{\zeta(2m+1)} \right) x + O(\log x),$$

where B_m ($m \geqslant 0$) are the Bernoulli numbers. See [1, Eq. 2.19, Th. 1]. The same identity (2.30) and the same proof, based on Hölder's evaluation of the Ramanujan sums were given by Alkan also in [2, Proof of Th. 1].

In paper [107] I presented a simpler proof of identity (2.30), see Section 4.6, and I established identities for other weighted averages of the Ramanujan sums with weights concerning logarithms, values of arithmetic functions for gcd's, the Gamma function, the Bernoulli polynomials and binomial coefficients.

Proposition 2.6.1 (Tóth [107, Prop. 2, 4, 5]). *For every $k \in \mathbb{N}$,*

$$\frac{1}{k} \sum_{j=1}^{k} (\log j) c_k(j) = \Lambda(k) + \sum_{d\mid k} \frac{\mu(d)}{d} \log(d!), \tag{2.31}$$

where Λ is the von Mangoldt function,

$$\frac{1}{\phi(k)} \sum_{j=1}^{k} (\log \Gamma(j/k)) c_k(j) = \frac{1}{2} \sum_{p\mid k} \frac{\log p}{p-1} - \frac{\log 2\pi}{2} \quad (k \geqslant 2), \tag{2.32}$$

$$\frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j} c_k(j) = \sum_{d\mid k} \mu(k/d) \sum_{\ell=1}^{d} (-1)^{\ell k/d} \cos^k(\ell\pi/d). \tag{2.33}$$

Note the symmetry property $\binom{k}{j} c_k(j) = \binom{k}{k-j} c_k(k-j)$ $(0 \leqslant j \leqslant k)$.

Generalizations and analogues of identities (2.31), (2.32) and (2.33) were investigated recently by Ikeda, Kiuchi and Matsuoka [48], Kiuchi [53, 54, 55], Namboothiri [67]. For example, Namboothiri [67] considered Cohen's generalization of the Ramanujan sums, defined by

$$c_k^{(s)}(j) = \sum_{\substack{m=1 \\ (m,k^s)_s=1}}^{n} \exp(2\pi i j m/n^s), \tag{2.34}$$

where $(m, k^s)_s$ is the greatest common s-power divisor of m and k^s ($s \in \mathbb{N}$). Note that moments of averages of the function (2.34) were studied in another recent paper by Robles and Roy [79]. Furthermore, Kiuchi [53] investigated asymptotic properties of the sum

$$\sum_{k\leqslant x} \frac{1}{k^{r+1}} \sum_{j=1}^{k} j^r \gcd(j,k) \quad (r \in \mathbb{N}).$$

2.7 Counting solutions of quadratic congruences in several variables

Let $k \in \mathbb{N}$, $n \in \mathbb{Z}$ be fixed and let $N_k(n,r)$ denote the number of incongruent solutions of the quadratic congruence

$$x_1^2 + \cdots + x_k^2 \equiv n \pmod{r}.$$

The function $r \mapsto N_k(n,r)$ is multiplicative. Therefore, it is sufficient to consider the case $r = p^s$, a prime power. Identities for $N_k(n,p^s)$ can be derived using Gauss and Jacobi sums. Much less known is that for k even and r odd, $N_k(n,r)$ can be expressed in a compact form using the Ramanujan sums $c_q(n)$. Furthermore, for k odd, r odd and $\gcd(n,r) = 1$, $N_k(n,r)$ can be given in terms of the Möbius μ function and the Jacobi symbol. For example, one has the following identities:

If $k \equiv 0 \pmod{4}$, r is odd and $n \in \mathbb{Z}$, then

$$N_k(n,r) = r^{k-1} \sum_{d \mid r} \frac{c_d(n)}{d^{k/2}},$$

and if $k \equiv 1 \pmod{4}$, r is odd, $n \in \mathbb{Z}$, $\gcd(n,r) = 1$, then

$$N_k(n,r) = r^{k-1} \sum_{d \mid r} \frac{\mu^2(d)}{d^{(k-1)/2}} \left(\frac{n}{d}\right). \tag{2.35}$$

These are special cases of the identities deduced by Cohen [21]. The proofs given in [21] are lengthy and use Cohen's previous work, although in Section 7 of [21] a direct approach using finite Fourier sums is also described.

In my paper [110], including more historical remarks, I gave short direct proofs of the above and related identities. Slightly more generally, I considered — as Cohen did — the quadratic congruence $a_1x_1^2 + \cdots + a_kx_k^2 \equiv n \pmod{r}$, where $a_1, \ldots, a_k \in \mathbb{Z}$. For the proofs it was needed to express $N_k(n,r)$ by a trigonometric sum and to use the evaluation of the Gauss quadratic sum. The proofs are quite simple if k is even and more involved if k is odd. I present in Section 4.7 the sketch of the proofs of identity (2.35) and of the following result: Assume that $k, r \in \mathbb{N}$ are odd and $n = 0$. Then

$$N_k(0,r) = r^{k-1} \sum_{d^2 \mid r} \frac{\phi(d)}{d^{k-1}}. \tag{2.36}$$

I also established asymptotic formulas — not given in the literature, as far as I know — for the sums $\sum_{r \leqslant x} N_k(n,r)$, taken over all integers r with $1 \leqslant r \leqslant x$, in the cases $(k,n) = (1,0), (1,1), (2,0), (2,1), (3,0), (3,1), (4,0), (4,1)$. Note that the mean values of the functions $r \mapsto N_k(n,r)/n^{k-1}$ were investigated by Cohen [23], but only over the odd values of r.

I point out the asymptotics for $N_1(0,r)$, which can be considered as an analog of Dirichlet's formula (1.1); $N_1(1,r)$, related to the squarefree divisor problem; and $N_2(0,r)$, related to the Gauss circle problem.

Theorem 2.7.1 (Tóth [110, Prop. 28]). *We have*

$$\sum_{r\leqslant x} N_1(0,r) = \frac{3}{\pi^2} x\log x + cx + O(x^{2/3}),$$

where $c = \frac{3}{\pi^2}\left(3\gamma - 1 - \frac{2\zeta'(2)}{\zeta(2)}\right)$.

Theorem 2.7.2 (Tóth [110, Prop. 30]).

$$\sum_{r\leqslant x} N_1(1,r) = \frac{6}{\pi^2} x\log x + c_1 x + O(x^{1/2}\delta(x)),$$

where $c_1 = \frac{6}{\pi^2}\left(2\gamma - 1 - \frac{\log 2}{2} - \frac{2\zeta'(2)}{\zeta(2)}\right)$ *and* $\delta(x)$ *is defined by* (1.3)*. If RH is true, then the error term is* $O(x^{4/11+\varepsilon})$ *for every* $\varepsilon > 0$.

Theorem 2.7.3 (Tóth [110, Prop. 34]).

$$\sum_{r\leqslant x} N_2(0,r) = \frac{\pi}{8G} x^2 + O(x^{547/416}(\log x)^{26947/8320})$$

where G *is the Catalan constant.*[2]

Remark 2.7.4. See the paper by Finch, Martin, and Sebah [36] for asymptotic formulas on the number of solutions of the higher degree congruences $x^\ell \equiv 0 \pmod n$ and $x^\ell \equiv 1 \pmod n$, respectively, where $\ell \in \mathbb{N}$. The error terms of our Theorems 2.7.1 and 2.7.2 are better than those of [36] applied to $\ell = 2$.

2.8 Counting subgroups of finite abelian groups

Let G be a finite abelian group of order n. Let $s(G)$ and $c(G)$ denote the total number of subgroups of G and the number of its cyclic subgroups, respectively. Also, let $G = \times_p G_p$ be the primary decomposition of G, where $|G_p| = p^{\nu_p(n)}$ (p prime). Then

$$s(G) = \prod_p s(G_p), \qquad c(G) = \prod_p c(G_p). \tag{2.37}$$

[2] A better error term is $O(x^{2165/1648})$ by using a recent improvement by Bourgain and Watt [12] for the Gauss circle problem. See the Introduction.

Therefore, the problem of counting the subgroups of G reduces to p-groups. Formulas for the total number of subgroups of a given type were established by Delsarte (1948), Yeh (1948), Shokuev (1972), Bhowmik (1996), and others. See, e.g., the monograph by Butler [13]. One of these formulas is in terms of the Gaussian coefficients. Let $G_{(p)}$ be a p-group of type $\lambda = (\lambda_1, \ldots, \lambda_r)$, with $\lambda_1 \geqslant \ldots \geqslant \lambda_r \geqslant 1$, where λ is a partition of $|\lambda| = \lambda_1 + \cdots + \lambda_r$. Then the number $s_\mu(G_{(p)})$ of subgroups of type μ ($\mu \subseteq \lambda$) of $G_{(p)}$ is

$$s_\mu(G_{(p)}) = \prod_{j=1}^{\lambda_1} p^{\mu'_{j+1}(\lambda'_j - \mu'_j)} \begin{bmatrix} \lambda'_j - \mu'_{j+1} \\ \mu'_j - \mu'_{j+1} \end{bmatrix}_p,$$

where λ' and μ' are the conjugates (according to the Ferrers diagrams) of λ and μ, respectively. Hence $s_\mu(G_{(p)})$ is a polynomial in p, with integer coefficients, depending only on λ and μ (Hall polynomials).

Therefore, the total number of the subgroups of $G_{(p)}$ is

$$s_{p^k}(G_{(p)}) = \sum_{0 \leqslant k \leqslant |\lambda|} \sum_{\substack{\mu \subseteq \lambda \\ |\mu| = k}} s_\mu(G_{(p)}), \tag{2.38}$$

and it is rather complicate to apply this (and the other formulas) to compute numerically the total number of subgroups of a p-group. Also, it is difficult to find the coefficients of the polynomials in p representing the number of subgroups of a p-group, even in the case of small rank. There are more simple formulas for the number of cyclic subgroups of p-groups.

Recall that a finite abelian group of order > 1 has rank t if it is isomorphic to

$$\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_t}, \quad \text{with} \quad n_1, \ldots, n_t \in \mathbb{N} \setminus \{1\}, \quad n_j \mid n_{j+1} \quad (1 \leqslant j \leqslant t-1), \tag{2.39}$$

which is the invariant factor decomposition of the given group. Here the number t is uniquely determined and represents the minimal number of generators of the given group.

Instead of p-groups I prefer to consider the group $G := \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$, where $n_1, \ldots, n_k$ are arbitrary positive integers, so this group being more general than (2.39). Let $s(n_1, \ldots, n_k)$ and $c(n_1, \ldots, n_k)$ stand for the total number of subgroups of G and the number of its cyclic subgroups, respectively. For $c(n_1, \ldots, n_k)$ we have the following compact formula.

Theorem 2.8.1 (Tóth [104, Th. 1]). *For any* $n_1, \ldots, n_k \in \mathbb{N}$,

$$c(n_1, \ldots, n_k) = \sum_{d_1 \mid n_1, \ldots, d_k \mid n_k} \frac{\phi(d_1) \cdots \phi(d_k)}{\phi([d_1, \ldots, d_k])}. \tag{2.40}$$

According to (2.37), the functions $s(n_1, \ldots, n_k)$ and $c(n_1, \ldots, n_k)$ are multiplicative functions of k variables. Note that for $c(n_1, \ldots, n_k)$ this property is a direct consequence of formula (2.40). Therefore, the functions $s(n,\dots,n)$ and $c(n,\dots,n)$ are multiplicative in n, as functions of a single variable.

It seems to be difficult to find asymptotic properties of these functions, valid for every $k \in \mathbb{N}$. In this section we consider the cases $k = 2$ and $k = 3$ and establish results for some related one variable multiplicative functions. An asymptotic formula for $\sum_{m,n\leqslant x} s(m,n)$ will be presented in Section 3.4.

2.8.1 Subgroups of rank two groups

Consider the group $\mathbb{Z}_m \times \mathbb{Z}_n$, where $m,n \in \mathbb{N}$. Note that $\mathbb{Z}_m \times \mathbb{Z}_n$ is isomorphic to $\mathbb{Z}_{(m,n)} \times \mathbb{Z}_{[m,n]}$, and it has rank two, assuming that $(m,n) > 1$. In paper [109] I gave the following representation of the subgroups of $\mathbb{Z}_m \times \mathbb{Z}_n$ by using Goursat's lemma for groups. For every $m,n \in \mathbb{N}$ let

$$J_{m,n} := \left\{(a,b,c,d,\ell) \in \mathbb{N}^5 : a \mid m, b \mid a, c \mid n, d \mid c, \frac{a}{b} = \frac{c}{d}, \ell \leqslant \frac{a}{b}, \gcd\left(\ell, \frac{a}{b}\right) = 1\right\}. \tag{2.41}$$

For $(a,b,c,d,\ell) \in J_{m,n}$ define

$$K_{a,b,c,d,\ell} := \left\{\left(i\frac{m}{a}, i\ell\frac{n}{c} + j\frac{n}{d}\right) : 0 \leqslant i \leqslant a-1, 0 \leqslant j \leqslant d-1\right\}. \tag{2.42}$$

Theorem 2.8.2 (Tóth [109, Th. 3.1]). *Let $m,n \in \mathbb{N}$.*

i) The map $(a,b,c,d,\ell) \mapsto K_{a,b,c,d,\ell}$ is a bijection between the set $J_{m,n}$ and the set of subgroups of $(\mathbb{Z}_m \times \mathbb{Z}_n, +)$.

ii) The invariant factor decomposition of the subgroup $K_{a,b,c,d,\ell}$ is

$$K_{a,b,c,d,\ell} \simeq \mathbb{Z}_{(b,d)} \times \mathbb{Z}_{[a,c]}, \tag{2.43}$$

where $(b,d) \mid [a,c]$.

iii) The order of the subgroup $K_{a,b,c,d,\ell}$ is ad and its exponent is $[a,c]$.

iv) The subgroup $K_{a,b,c,d,\ell}$ is cyclic if and only if $(b,d) = 1$.

Figure 1 represents the subgroup $K_{6,2,18,6,1}$ of $\mathbb{Z}_{12} \times \mathbb{Z}_{18}$. It has order 36 and is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_{18}$.

According to Theorem 2.8.2, the total number $s(m,n)$ of subgroups of $\mathbb{Z}_m \times \mathbb{Z}_n$ can be obtained by counting the elements of the set $J_{m,n}$. We have the following simple compact formulas.

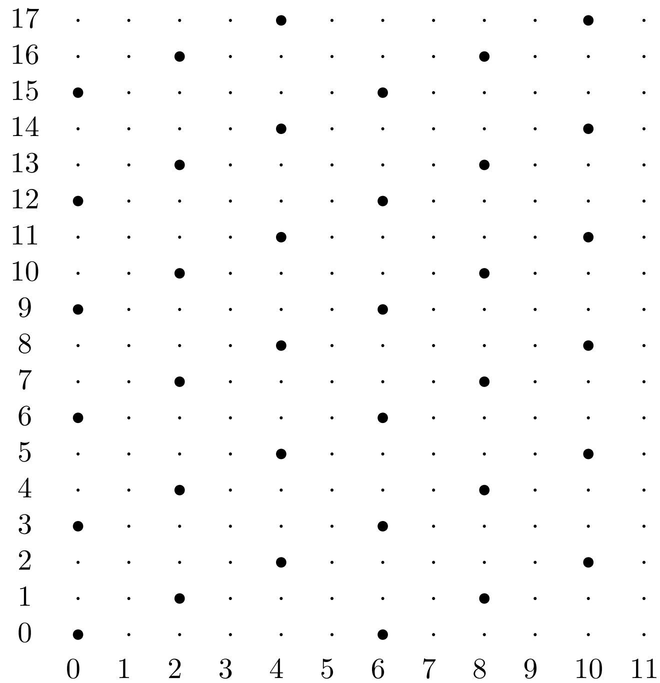
Theorem 2.8.3 (Tóth [109, Th. 4.1]). *For every $m,n \in \mathbb{N}$, $s(m,n)$ is given by*

$$s(m,n) = \sum_{i\mid m, j\mid n} (i,j) \tag{2.44}$$

$$= \sum_{t\mid (m,n)} \phi(t)\tau\left(\frac{m}{t}\right)\tau\left(\frac{n}{t}\right). \tag{2.45}$$

Note that (2.44) is a special case of an identity representing the total number of subgroups of a class of groups formed as cyclic extensions of cyclic groups, deduced by Calhoun [15] and having a laborious proof. Remark also, that by (2.45), $s(n,n) = \sum_{d|n} \tau(d)\psi(n/d)$ ($n \in \mathbb{N}$), which quickly leads to an asymptotic formula for $\sum_{n\leqslant x} s(n,n)$. A similar formula for $\sum_{n\leqslant x} c(n,n)$ can be deduced, as well. We do not go into details.

Figure 1. The subgroup $K_{6,2,18,6,1}$ of $\mathbb{Z}_{12} \times \mathbb{Z}_{18}$



It was proved in our paper [93, Prop. 3.2] that the sum of exponents of the subgroups of $\mathbb{Z}_m \times \mathbb{Z}_n$ is $\sigma(m)\sigma(n)$ for every $m, n \in \mathbb{N}$. Therefore, the arithmetic mean of exponents of the subgroups of $\mathbb{Z}_m \times \mathbb{Z}_n$ is $\sigma(m)\sigma(n)/s(m,n)$. Now consider the case $m = n$. Let $AE(n)$ stand for the arithmetic mean of exponents of the subgroups of $\mathbb{Z}_n \times \mathbb{Z}_n$. We have

$$AE(n) = \frac{\sigma(n)^2}{s(n,n)},$$

which is multiplicative in n.

Theorem 2.8.4 (Tărnăuceanu and Tóth [93, Prop. 3.3]). *We have*

$$\sum_{n\leqslant x} AE(n) = \frac{C}{2}x^2 + O\left(x \log^3 x\right), \tag{2.46}$$

where

$$C := \prod_p \left(1 - \frac{1}{p}\right) \sum_{\nu=0}^{\infty} \frac{(p^{\nu+1}-1)^2}{p^{2\nu}(p^{\nu+2} + p^{\nu+1} - (2\nu+3)p + 2\nu + 1)}.$$

2.8.2 Subgroups of rank three groups

Now we consider the subgroups of the group $\mathbb{Z}_m \times \mathbb{Z}_n \times \mathbb{Z}_r$, which has rank three when $(m, n, r) > 1$.

Theorem 2.8.5 (Hampejs and Tóth [38, Th. 2.1]). *Let $m, n, r \in \mathbb{N}$. The subgroups of the group $\Gamma = \mathbb{Z}_m \times \mathbb{Z}_n \times \mathbb{Z}_r$ can be represented as follows.*

(i) Choose $a, b, c \in \mathbb{N}$ such that $a \mid m, b \mid n, c \mid r$.

(ii) Compute $A := \gcd(a, n/b)$, $B := \gcd(b, r/c)$, $C := \gcd(a, r/c)$.

(iii) Compute

$$X := \frac{ABC}{\gcd(a(r/c), ABC)}.$$

(iv) Let $s := at/A$, where $0 \leqslant t \leqslant A - 1$.

(v) Let

$$v := \frac{bX}{B \gcd(t, X)} w, \text{ where } 0 \leqslant w \leqslant B \gcd(t, X)/X - 1.$$

(vi) Find a solution u_0 of the linear congruence

$$(r/c)u \equiv rvs/(bc) \pmod{a}.$$

(vii) Let $u := u_0 + az/C$, where $0 \leqslant z \leqslant C - 1$.

(viii) Consider

$$U_{a,b,c,t,w,z} := \langle (a, 0, 0), (s, b, 0), (u, v, c) \rangle$$

$$= \{(ia + js + ku, jb + kv, kc) : 0 \leqslant i \leqslant n/a - 1, 0 \leqslant j \leqslant n/b - 1, 0 \leqslant k \leqslant n/c - 1\}.$$

Then $U_{a,b,c,t,w,z}$ is a subgroup of order $mnr/(abc)$ of Γ. Moreover, there is a bijection between the set of 6-tuples (a, b, c, t, w, z) satisfying the conditions (i)-(viii) and the set of subgroups of Γ.

Next we give a formula for the number of subgroups of Γ.

Theorem 2.8.6 (Hampejs and Tóth [38, Th. 2.2]). *For every $m, n, r \in \mathbb{N}$ the total number of subgroups of the group $\mathbb{Z}_m \times \mathbb{Z}_n \times \mathbb{Z}_r$ is given by*

$$s(m, n, r) = \sum_{a|m,\, b|n,\, c|r} \frac{ABC}{X^2} P(X), \tag{2.47}$$

with the notation of Theorem 2.8.5, where $P(n)$ is the gcd-sum function.

If one of m, n, r is 1, then formula (2.47) reduces to (2.44).

I remark that in the joint paper with Tărnăuceanu [93, Cor. 2.2] we proved, based on formula (2.38), that the total number of subgroups of the rank three p-group $\mathbb{Z}_{p^{\lambda_1}} \times \mathbb{Z}_{p^{\lambda_2}} \times \mathbb{Z}_{p^{\lambda_3}}$, with $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant 1$, is given by the following polynomial:

$$\frac{F(p)}{(p^2-1)^2(p-1)},$$

where

$$\begin{aligned}F(p) =&(\lambda_3+1)(\lambda_1-\lambda_2+1)p^{\lambda_2+\lambda_3+5}+2(\lambda_3+1)p^{\lambda_2+\lambda_3+4}\\&-2(\lambda_3+1)(\lambda_1-\lambda_2)p^{\lambda_2+\lambda_3+3}-2(\lambda_3+1)p^{\lambda_2+\lambda_3+2}\\&+(\lambda_3+1)(\lambda_1-\lambda_2-1)p^{\lambda_2+\lambda_3+1}-(\lambda_1+\lambda_2-\lambda_3+3)p^{2\lambda_3+4}\\&-2p^{2\lambda_3+3}+(\lambda_1+\lambda_2-\lambda_3-1)p^{2\lambda_3+2}\\&+(\lambda_1+\lambda_2+\lambda_3+5)p^2+2p-(\lambda_1+\lambda_2+\lambda_3+1).\end{aligned}$$

This formula was also obtained by Oh [72, Cor. 2.2], using different arguments.

The final result of this chapter is an asymptotic formula for the number of subgroups $s(n,n,n)$ of $\mathbb{Z}_n^3$. The values of this function for $1 \leqslant n \leqslant 50$, obtained by Theorem 2.8.6 and using the software Mathematica, are given in Table 1.

Table 1. Values of $s(n) := s(n,n,n)$ for $1 \leqslant n \leqslant 50$

n	$s(n)$	n	$s(n)$	n	$s(n)$	n	$s(n)$	n	$s(n)$
1	1	11	268	21	3248	31	1988	41	3448
2	16	12	3612	22	4288	32	22308	42	51968
3	28	13	368	23	1108	33	7504	43	3788
4	129	14	1856	24	22456	34	9856	44	34572
5	64	15	1792	25	2607	35	7424	45	28480
6	448	16	4387	26	5888	36	57405	46	17728
7	116	17	616	27	5776	37	2816	47	4516
8	802	18	7120	28	14964	38	12224	48	122836
9	445	19	764	29	1744	39	10304	49	9009
10	1024	20	8256	30	28672	40	51328	50	41712

Define the multiplicative function h by

$$s(n,n,n) = \sum_{d\mid n} d^2\tau(d)h(n/d) \quad (n \in \mathbb{N}) \tag{2.48}$$

and let $H(z) = \sum_{n=1}^{\infty} h(n)n^{-z}$ be the Dirichlet series of h.

Theorem 2.8.7 (Hampejs and Tóth [38, Th. 2.3]). *For every $\varepsilon > 0$,*

$$\sum_{n\leqslant x} s(n,n,n) = \frac{x^3}{3}\left(H(3)(\log x + 2\gamma - 1) + H'(3)\right) + O(x^{2+\theta+\varepsilon}), \tag{2.49}$$

where θ is the exponent in the Dirichlet divisor problem (1.1).

Chapter 3

Results for multiplicative functions of several variables

3.1 Counting r-tuples of positive integers with k-wise relatively prime components

Let $r \geqslant k \geqslant 2$ be fixed integers. The positive integers $n_1, \ldots, n_r$ are called k-wise relatively prime if any k of them are relatively prime, that is $(n_{i_1}, \ldots, n_{i_k}) = 1$ for every $1 \leqslant i_1 < \cdots < i_k \leqslant r$. In particular, in the case $k = 2$ the integers are pairwise relatively prime and for $k = r$ they are mutually relatively prime.

Let $\mathcal{S}_{r,k}$ denote the set of r-tuples of positive integers with k-wise relatively prime components and let $\varrho_{r,k}$ stand for its characteristic function. What is the asymptotic density

$$d_{r,k} = \lim_{x\to\infty} \frac{1}{x^r} \sum_{n_1,\ldots,n_r \leqslant x} \varrho_{r,k}(n_1, \ldots, n_r)$$

of the set $\mathcal{S}_{r,k}$? Heuristically, the probability that a positive integer is divisible by a fixed prime p is $1/p$, hence the probability that given r positive integers exactly j of them are divisible by p is

$$\binom{r}{j} \frac{1}{p^j} \left(1 - \frac{1}{p}\right)^{r-j}$$

and the probability that they are k-wise relatively prime is

$$P_{r,k} = \prod_p \sum_{j=0}^{k-1} \binom{r}{j} \frac{1}{p^j} \left(1 - \frac{1}{p}\right)^{r-j}. \tag{3.1}$$

Note that for every $r \geqslant k \geqslant 2$,

$$c \prod_{p>r-1} \left(1 - \frac{(r-1)^2}{p^2}\right) \leqslant P_{r,2} \leqslant P_{r,k} \leqslant P_{r,r} = \prod_p \left(1 - \frac{1}{p^r}\right),$$

with some constant $c > 0$ (depending on r), hence the infinite product (3.1) converges. Some approximate values of $P_{r,k}$ are shown by Table 2.

Table 2. Approximate values of $P_{r,k}$ for $2 \leqslant k \leqslant r \leqslant 8$

$P_{r,k}$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
$r=2$	0.607						
$r=3$	0.286	0.831					
$r=4$	0.114	0.584	0.923				
$r=5$	0.040	0.357	0.768	0.964			
$r=6$	0.013	0.195	0.576	0.873	0.982		
$r=7$	0.004	0.097	0.394	0.734	0.930	0.991	
$r=8$	0.001	0.045	0.247	0.573	0.837	0.962	0.995

If $k = r$, then it is well known that $d_{r,r} = P_{r,r} = 1/\zeta(r)$ is the correct value of the corresponding asymptotic density. The case $k = 2$ was treated by the author [95], proving by an inductive approach that

$$\sum_{n_1,\ldots,n_r \leqslant x} \varrho_{r,2}(n_1,\ldots,n_r) = d_{r,2}x^r + O\left(x^{r-1}(\log x)^{r-1}\right), \tag{3.2}$$

where $d_{r,2} = P_{r,2}$.

In the case $k = 2$, the asymptotic density was also deduced by Cai and Bach [14, Th. 3.3] using probabilistic arguments. J. Hu [43, 44] proved that $d_{r,k} = P_{r,k}$ for every $r \geqslant k \geqslant 2$. In fact, by generalizing the method of [95] it was shown in [43] that

$$\sum_{n_1,\ldots,n_r \leqslant x} \varrho_{r,k}(n_1,\ldots,n_r) = P_{r,k}x^r + O\left(x^{r-1}(\log x)^{\delta_{r,k}}\right), \tag{3.3}$$

where $\delta_{r,k} = \max\left\{\binom{r-1}{j} : 1 \leqslant j \leqslant k-1\right\}$. For $k = 2$ the asymptotic formula (3.3) reduces to (3.2).

Referring to our paper [95], de Reyna and Heyman [29] considered modified pairwise coprimality conditions and by using certain graph representations they obtained asymptotic formulas similar to (3.3). Probabilistic aspects of pairwise coprimality were investigated by Fernández and Fernández [35].

It is the goal of the present Section to use a method, which differs from all approaches mentioned above and which seems to be the most natural, to establish the asymptotic formula (3.3) with a better error term. More exactly, we take into account that the function $\varrho_{r,k}(n_1,\ldots,n_r)$ is multiplicative, viewed as an arithmetic function of r variables. Therefore, its multiple Dirichlet series can be expressed as an Euler product and an explicit formula can be given for it. Then we use the convolution method to obtain the desired asymptotic formula by elementary arguments.

We prove the following results. Let $e_j(x_1,\ldots,x_r)$ denote the elementary symmetric polynomials in $x_1,\ldots,x_r$ of degree j ($j \geqslant 0$).

Theorem 3.1.1 (Tóth [111, Th. 2.1]). *Let $r \geqslant k \geqslant 2$ and let $s_i \in \mathbb{C}$ $(1 \leqslant i \leqslant r)$. If $\Re s_i > 1$ $(1 \leqslant i \leqslant r)$, then*

$$\sum_{n_1,\ldots,n_r=1}^{\infty} \frac{\varrho_{r,k}(n_1,\ldots,n_r)}{n_1^{s_1}\cdots n_r^{s_r}} = \zeta(s_1)\cdots\zeta(s_r)D_{r,k}(s_1,\ldots,s_r),$$

where

$$D_{r,k}(s_1,\ldots,s_r) = \prod_p \left(1 - \sum_{j=k}^{r}(-1)^{j-k}\binom{j-1}{k-1}e_j(p^{-s_1},\ldots,p^{-s_r})\right)$$

is absolutely convergent if $\Re(s_{i_1}+\cdots+s_{i_j}) > 1$ for every $1 \leqslant i_1 < \ldots < i_j \leqslant r$ with $k \leqslant j \leqslant r$.

Theorem 3.1.2 (**Tóth [111, Th. 2.2]**). *If $r \geqslant k \geqslant 2$, then*

$$\sum_{n_1,\ldots,n_r \leqslant x} \varrho_{r,k}(n_1,\ldots,n_r) = A_{r,k}x^r + O\left(R_{r,k}(x)\right),$$

where

$$A_{r,k} = \prod_p \left(1 - \sum_{j=k}^{r}(-1)^{j-k}\binom{r}{j}\binom{j-1}{k-1}\frac{1}{p^j}\right) \tag{3.4}$$

and

$$R_{r,k}(x) = \begin{cases} x^{r-1}, & \text{if } r \geqslant k \geqslant 3, \\ x^{r-1}(\log x)^{r-1}, & \text{if } r \geqslant k = 2. \end{cases} \tag{3.5}$$

For $k \geqslant 3$ the error term $R_{r,k}(x)$ is better than in (3.3), obtained by J. Hu [43]. Note also that $A_{r,k} = P_{r,k}$, given by (3.1), which follows by certain properties of the binomial coefficients.

3.2 The average value of the least common multiple of k positive integers

Consider the greatest common divisor $(n_1,\ldots,n_k)$ of $n_1,\ldots,n_k \in \mathbb{N}$. It is easy to see that for any arithmetic function f we have the identity

$$\sum_{n_1,\ldots,n_k \leqslant x} f((n_1,\ldots,n_k)) = \sum_{d \leqslant x}(\mu * f)(d)\left\lfloor \frac{x}{d}\right\rfloor^k, \tag{3.6}$$

which leads to asymptotic formulas for this sum. For example, if $f(n) = n$ and $k \geqslant 3$, then we have

$$\sum_{n_1,\ldots,n_k \leqslant x} (n_1,\ldots,n_k) = \frac{\zeta(k-1)}{\zeta(k)}x^k + O(R_k(x)),$$

where $R_3(x) = x^2 \log x$ and $R_k(x) = x^{k-1}$ for $k \geqslant 4$. The case $f(n) = n$, $k = 2$ can be treated separately by writing

$$\sum_{m,n \leqslant x} (m,n) = 2 \sum_{m \leqslant n \leqslant x} (m,n) - \sum_{n \leqslant x} n$$

$$= 2 \sum_{n \leqslant x} (\mu * \mathrm{id}\,\tau)(n) - \frac{x^2}{2} + O(x),$$

giving, by using elementary arguments, the formula

$$\sum_{m,n \leqslant x} (m,n) = \frac{x^2}{\zeta(2)} \left(\log x + 2\gamma - \frac{1}{2} - \frac{\zeta(2)}{2} - \frac{\zeta'(2)}{\zeta(2)} \right) + O(x^{1+\theta+\varepsilon}), \tag{3.7}$$

valid for every $\varepsilon > 0$, where γ is Euler's constant and θ is the exponent appearing in the Dirichlet divisor problem (1.1). Here (3.7) is equivalent to formula (2.20), concerning the gcd-sum function.

For the least common multiple of k positive integers there is no formula similar to (3.6). However, in the case $k = 2$, the lcm of the integers $m, n \in \mathbb{N}$ can be written using their gcd as $[m,n] = mn/(m,n)$, which enables to establish the following asymptotic formula, valid for any $r \in \mathbb{N}$:

$$\sum_{m,n \leqslant x} [m,n]^r = \frac{\zeta(r+2)}{\zeta(2)} \cdot \frac{x^{2(r+1)}}{(r+1)^2} + O(x^{2r+1} (\log x)^{2/3} (\log\log x)^{4/3}),$$

which is a consequence of the result (1.4) of Walfisz on $\sum_{n \leqslant x} \phi(n)$.

The above and related results go back to the work of Cesàro [18], Cohen[22], Diaconis and Erdős [31], Tanigawa and Zhai [92], Ikeda and Matsuoka [47], and others.

The result

$$\sum_{m,n,q \leqslant x} [m,n,q]^r \sim c_r \frac{x^{3(r+1)}}{(r+1)^3} \quad (x \to \infty),$$

valid for $r \in \mathbb{N}$, without any error term and with a computable constant c_r given in an implicit form, was obtained by J. L. Fernández and P. Fernández [34, Th. 3(b)]. Their proof is by an ingenious method based on the identity $[m,n,q](m,n)(m,q)(n,q) = mnq(m,n,q)$ $(m,n,q \in \mathbb{N})$ and using the dominated convergence theorem. As far as we know, there are no other asymptotic results in the literature for the sum

$$\sum_{n_1,\ldots,n_k \leqslant x} f([n_1,\ldots,n_k]), \tag{3.8}$$

in the case $k \geqslant 3$, where f is an arithmetic function. It seems that the method of [34] can not be extended for $k \geqslant 3$, even in the case $f(n) = n^r$. Also, it is not possible to reduce the estimation of the sum (3.8) to sums of a single variable, like in (3.6).

In this Section we deduce an asymptotic formula with remainder term for the sum (3.8), where $k \geqslant 2$ and f belongs to a large class of multiplicative arithmetic functions.

Let $r \in \mathbb{R}$ be a fixed number. Let $\mathcal{A}_r$ denote the class of complex valued multiplicative arithmetic functions satisfying the following properties: there exist real constants C_1, C_2 such that

$$|f(p) - p^r| \leqslant C_1 p^{r-1/2} \quad \text{for every prime } p, \tag{i}$$

and

$$|f(p^\nu)| \leqslant C_2 p^{\nu r} \quad \text{for every prime power } p^\nu \text{ with } \nu \geqslant 2. \tag{ii}$$

Note that conditions (i) and (ii) imply that

$$|f(p^\nu)| \leqslant C_3 p^{\nu r} \quad \text{for every prime power } p^\nu \text{ with } \nu \geqslant 1, \tag{iii}$$

where $C_3 = \max(C_1 + 1, C_2)$.

For example, the following functions belong to the class $\mathcal{A}_r$: $f(n) = n^r, \sigma(n)^r, \phi(n)^r, \sigma^{(e)}(n)^r$ ($r \in \mathbb{R}$), $f(n) = \sigma_r(n) = \sum_{d|n} d^r$ ($r \in \mathbb{R}$ with $r \geqslant 1/2$). Furthermore, if f is a bounded multiplicative function such that $f(p) = 1$ for every prime p, then $f \in \mathcal{A}_0$. In particular, $\mu^2 \in \mathcal{A}_0$.

Theorem 3.2.1 (Hilberdink and Tóth [42, Th. 2.1]). *Let $k \geqslant 2$ be a fixed integer and let $f \in \mathcal{A}_r$ be a function, where $r > -1$ is real. Then for every $\varepsilon > 0$,*

$$\sum_{n_1,\ldots,n_k \leqslant x} f([n_1,\ldots,n_k]) = C_{f,k} \frac{x^{k(r+1)}}{(r+1)^k} + O\left(x^{k(r+1)-\frac{1}{2}\min(r+1,1)+\varepsilon}\right), \tag{3.9}$$

and

$$\sum_{n_1,\ldots,n_k \leqslant x} \frac{f([n_1,\ldots,n_k])}{(n_1\cdots n_k)^r} = C_{f,k} x^k + O\left(x^{k-\frac{1}{2}\min(r+1,1)+\varepsilon}\right), \tag{3.10}$$

where

$$C_{f,k} = \prod_p \left(1 - \frac{1}{p}\right)^k \sum_{\nu_1,\ldots,\nu_k=0}^{\infty} \frac{f(p^{\max(\nu_1,\ldots,\nu_k)})}{p^{(r+1)(\nu_1+\cdots+\nu_k)}}.$$

Formula (3.9) shows that the average order of $f([n_1,\ldots,n_k])$ is $C_{f,k}(n_1\cdots n_k)^r$, in the sense that

$$\sum_{n_1,\ldots,n_k \leqslant x} f([n_1,\ldots,n_k]) \sim \sum_{n_1,\ldots,n_k \leqslant x} C_{f,k}(n_1\cdots n_k)^r \quad (x \to \infty).$$

From (3.10) we deduce that

$$\lim_{x\to\infty} \frac{1}{x^k} \sum_{n_1,\ldots,n_k \leqslant x} \frac{f([n_1,\ldots,n_k])}{(n_1\cdots n_k)^r} = C_{f,k},$$

representing the mean value of the function $f([n_1,\ldots,n_k])/(n_1\cdots n_k)^r$.

Theorem 3.2.2 (Hilberdink and Tóth [42, Th. 2.2]). *Let $k \geqslant 2$ be a fixed integer and let $f \in \mathcal{A}_r$ be a function, where $r \geqslant 0$ is real. Then for every $\varepsilon > 0$,*

$$\sum_{n_1,\ldots,n_k \leqslant x} f\left(\frac{[n_1,\ldots,n_k]}{(n_1,\ldots,n_k)}\right) = D_{f,k}\frac{x^{k(r+1)}}{(r+1)^k} + O\left(x^{k(r+1)-\frac{1}{2}+\varepsilon}\right), \tag{3.11}$$

where

$$D_{f,k} = \prod_p \left(1-\frac{1}{p}\right)^k \sum_{\nu_1,\ldots,\nu_k=0}^{\infty} \frac{f(p^{\max(\nu_1,\ldots,\nu_k)-\min(\nu_1,\ldots,\nu_k)})}{p^{(r+1)(\nu_1+\cdots+\nu_k)}}.$$

Corollary 3.2.3 (Hilberdink and Tóth [42, Cor. 1]). *Let $k \geqslant 3$ and $r > -1$ be a real number. Then for every $\varepsilon > 0$,*

$$\sum_{n_1,\ldots,n_k \leqslant x} [n_1,\ldots,n_k]^r = C_{r,k}\frac{x^{k(r+1)}}{(r+1)^k} + O\left(x^{k(r+1)-\frac{1}{2}\min(r+1,1)+\varepsilon}\right),$$

and

$$\sum_{n_1,\ldots,n_k \leqslant x} \left(\frac{[n_1,\ldots,n_k]}{n_1\cdots n_k}\right)^r = C_{r,k}x^k + O\left(x^{k-\frac{1}{2}\min(r+1,1)+\varepsilon}\right),$$

where

$$C_{r,k} = \prod_p \left(1-\frac{1}{p}\right)^k \sum_{\nu_1,\ldots,\nu_k=0}^{\infty} \frac{p^{r\max(\nu_1,\ldots,\nu_k)}}{p^{(r+1)(\nu_1+\cdots+\nu_k)}}.$$

In particular,

$$C_{r,3} = \zeta(r+2)\zeta(2r+3)\prod_p \left(1-\frac{3}{p^2}+\frac{2}{p^3}+\frac{2}{p^{r+2}}-\frac{3}{p^{r+3}}+\frac{1}{p^{r+5}}\right), \tag{3.12}$$

$$C_{r,4} = \zeta(r+2)\zeta(2r+3)\zeta(3r+4)\prod_p \left(1-\frac{6}{p^2}+\frac{8}{p^3}-\frac{3}{p^4}+\frac{5}{p^{r+2}}-\frac{12}{p^{r+3}}+\frac{6}{p^{r+4}}+\frac{4}{p^{r+5}}\right.$$
$$\left.-\frac{3}{p^{r+6}}+\frac{3}{p^{2r+3}}-\frac{4}{p^{2r+4}}-\frac{6}{p^{2r+5}}+\frac{12}{p^{2r+6}}-\frac{5}{p^{2r+7}}+\frac{3}{p^{3r+5}}-\frac{8}{p^{3r+6}}+\frac{6}{p^{3r+7}}-\frac{1}{p^{3r+9}}\right). \tag{3.13}$$

Note that our method does not work in the case $r = -1$, that is for the sum

$$S_k(x) := \sum_{n_1,\ldots,n_k \leqslant x} \frac{1}{[n_1,\ldots,n_k]}.$$

By using different arguments, we proved (unpublished) that $S_k(x) \asymp (\log x)^{2^k-1}$ for every $k \geqslant 2$, this sum being related to $\sum_{n\leqslant x} \tau(n)^k/n$. However, we are not able to obtain an asymptotic formula for $S_k(x)$.

Among other special cases of the above results we consider here only the function $\sigma \in \mathcal{A}_1$.

Corollary 3.2.4 (Hilberdink and Tóth [42, Cor. 3]). *Let $k \geqslant 2$. Then for every $\varepsilon > 0$,*

$$\sum_{n_1,\ldots,n_k \leqslant x} \sigma([n_1,\ldots,n_k]) = C_{\sigma,k} \frac{x^{2k}}{2^k} + O\left(x^{2k-1/2+\varepsilon}\right),$$

and

$$\sum_{n_1,\ldots,n_k \leqslant x} \frac{\sigma([n_1,\ldots,n_k])}{n_1 \cdots n_k} = C_{\sigma,k} x^k + O\left(x^{k-1/2+\varepsilon}\right),$$

where

$$C_{\sigma,k} = \prod_p \left(1 - \frac{1}{p}\right)^k \sum_{\nu_1,\ldots,\nu_k=0}^{\infty} \frac{\sigma(p^{\max(\nu_1,\ldots,\nu_k)})}{p^{2(\nu_1+\cdots+\nu_k)}}.$$

In particular,

$$C_{\sigma,2} = \zeta(3)\zeta(4) \prod_p \left(1 + \frac{1}{p^2} - \frac{2}{p^3} - \frac{2}{p^5} + \frac{2}{p^6}\right).$$

3.3 Multivariable averages of divisor functions

Consider the divisor function $\tau(1,k;n) = \sum_{ab^k=n} 1$, where $k \in \mathbb{N}$ is a fixed integer. For $k = 1$ this is the divisor function $\tau(n)$. For $k \geqslant 2$,

$$\sum_{n \leqslant x} \tau(1,k;n) = \zeta(k)x + \zeta(1/k)x^{1/k} + O(x^{\theta_k+\varepsilon}), \tag{3.14}$$

where $1/(2(k+1)) \leqslant \theta_k \leqslant 1/(k+2)$, which can be improved. See, e.g., the book by Krätzel [56, Ch. 5]. Note that $\theta_2 \leqslant \frac{1057}{4785} \doteq 0.220898$, which is a result of Graham and Kolesnik [37].

Lelechenko [59] proved, using a multidimensional Perron formula and the complex integration method that

$$\sum_{m,n \leqslant x} \tau(1,2;mn) = A_2 x^2 + B_2 x^{3/2} + O(x^{10/7+\varepsilon}), \tag{3.15}$$

where A_2, B_2 are constants and $10/7 \doteq 1.428571$. He noted that in the case $k \geqslant 3$ the same method does not furnish the expected asymptotic formula

$$\sum_{m,n \leqslant x} \tau(1,k;mn) = A_k x^2 + B_k x^{1+1/k} + O(x^{\alpha_k+\varepsilon}), \tag{3.16}$$

since the obtained error term is larger than $x^{4/3}$, even under the Riemann hypothesis, and absorbs the term $x^{1+1/k}$. It is also noted in paper [59] that formula (3.15) remains valid for the exponential divisor function $\tau^{(e)}$, instead of $\tau(1,2;\cdot)$, due to the fact that $\tau^{(e)}(p^\nu) = \tau(1,2;p^\nu)$ for $\nu \in \{1,2,3,4\}$.

It is the goal of the present Section to improve the error term of (3.15) and to deduce formula (3.16) with a sharp error term. More generally, we derive asymptotic formulas for the sums $\sum_{n_1,\ldots,n_r \leqslant x} \tau(1,k;n_1\cdots n_r)$ and $\sum_{n_1,\ldots,n_r\leqslant x} \tau(1,k;[n_1,\ldots,n_r])$, where $k \geqslant 1$ and $r \geqslant 2$ are fixed integers.

Furthermore, we deduce similar asymptotic formulas concerning the divisor functions $\tau^{(e)}(n)$ and $\tau^{(2)}(n) = 2^{\omega(n)}$. Our approach is based on the study of multiple Dirichlet series and the convolution method. Note that Theorem 3.2.1 does not apply for the divisor functions investigated in the present Section.

Theorem 3.3.1 (**Tóth and Zhai [115, Th. 3.1]**). *If $k, r \geqslant 2$, then*

$$\sum_{n_1,\ldots,n_r\leqslant x} \tau(1,k;n_1\cdots n_r) = A_{r,k}x^r + B_{r,k}x^{r-1+1/k} + O(x^{r-1+\theta_k+\varepsilon}), \tag{3.17}$$

$$\sum_{n_1,\ldots,n_r\leqslant x} \tau(1,k;[n_1,\ldots,n_r]) = C_{r,k}x^r + D_{r,k}x^{r-1+1/k} + O(x^{r-1+\theta_k+\varepsilon}), \tag{3.18}$$

for every $\varepsilon > 0$, where θ_k is the exponent in the error term of formula (3.14) *and $A_{r,k}$, $B_{r,k}$, $C_{r,k}$, $D_{r,k}$ are computable constants. Here*

$$A_{r,k} := \prod_p \left(1-\frac{1}{p}\right)^r \sum_{\nu_1,\ldots,\nu_r=0}^\infty \frac{\lfloor(\nu_1+\cdots+\nu_r)/k\rfloor+1}{p^{\nu_1+\cdots+\nu_r}},$$

$$C_{r,k} := \prod_p \left(1-\frac{1}{p}\right)^r \sum_{\nu_1,\ldots,\nu_r=0}^\infty \frac{\lfloor\max(\nu_1,\ldots,\nu_r)/k\rfloor+1}{p^{\nu_1+\cdots+\nu_r}}.$$

Theorem 3.3.2 (**Tóth and Zhai [115, Th. 3.2]**). *If $r \geqslant 2$, then*

$$\sum_{n_1,\ldots,n_r\leqslant x} \tau^{(e)}(n_1\cdots n_r) = K_r x^r + L_r x^{r-1/2} + O(x^{r-1+\theta_2+\varepsilon}),$$

$$\sum_{n_1,\ldots,n_r\leqslant x} \tau^{(e)}([n_1,\ldots,n_r]) = K'_r x^r + L'_r x^{r-1/2} + O(x^{r-1+\theta_2+\varepsilon}),$$

for every $\varepsilon > 0$, where θ_2 is defined by (3.14) *and K_r, L_r, K'_r, L'_r are computable constants. Here*

$$K_r = \prod_p \left(1-\frac{1}{p}\right)^r \left(1 + \sum_{\substack{\nu_1,\ldots,\nu_r=0\\ \nu_1+\cdots+\nu_r\geqslant 1}}^\infty \frac{\tau(\nu_1+\cdots+\nu_r)}{p^{\nu_1+\cdots+\nu_r}}\right),$$

$$K'_r = \prod_p \left(1-\frac{1}{p}\right)^r \left(1 + \sum_{\substack{\nu_1,\ldots,\nu_r=0\\ \nu_1+\cdots+\nu_r\geqslant 1}}^\infty \frac{\tau(\max(\nu_1,\ldots,\nu_r))}{p^{\nu_1+\cdots+\nu_r}}\right).$$

Our multivariable asymptotic formulas regarding the divisor functions τ and $\tau^{(2)}$ are special cases of the following general convolution result.

Theorem 3.3.3 (**Tóth and Zhai [115, Th. 3.3]**). *Let $r \geqslant 2$ and let $h : \mathbb{N}^r \to \mathbb{C}$, $g : \mathbb{N}^r \to \mathbb{C}$, $f_j : \mathbb{N} \to \mathbb{C}$ $(1 \leqslant j \leqslant r)$ be arithmetic functions such that*

$$h(n_1, \ldots, n_r) = \sum_{d_1 m_1 = n_1, \ldots, d_r m_r = n_r} g(d_1, \ldots, d_r) f_1(m_1) \cdots f_r(m_r)$$

for every $n_1, \ldots, n_r \in \mathbb{N}$. Assume that

(i) there exist constants $0 < b_j < a_j$ $(1 \leqslant j \leqslant r)$ such that

$$F_j(x) := \sum_{n \leqslant x} f_j(n) = x^{a_j} P_j(\log x) + O(x^{b_j}) \quad (1 \leqslant j \leqslant r),$$

where $P_j(u)$ are polynomials in u of degrees δ_j, with leading coefficients K_j $(1 \leqslant j \leqslant r)$,

(ii) the Dirichlet series

$$G(s_1, \ldots, s_r) := \sum_{n_1, \ldots, n_r = 1}^{\infty} \frac{g(n_1, \ldots, n_r)}{n_1^{s_1} \cdots n_r^{s_r}}$$

is absolutely convergent for $(s_1, \ldots, s_r) = (a_1 - \varepsilon, \ldots, a_{j-1} - \varepsilon, b_j - \varepsilon, a_{j+1} - \varepsilon, \ldots, a_r - \varepsilon)$ for sufficiently small $\varepsilon > 0$ and $1 \leqslant j \leqslant r$.

Then the asymptotic formula

$$\sum_{n_1, \ldots, n_r \leqslant x} h(n_1, \ldots, n_r) = x^{a_1 + \cdots + a_r} Q(\log x) + O(x^{a_1 + \cdots + a_r - \Delta} (\log x)^{\delta_1 + \cdots + \delta_r})$$

holds, where $Q(u)$ is a polynomial in u of degree $\delta_1 + \cdots + \delta_r$, with leading coefficient $K_1 \cdots K_r G(a_1, \ldots, a_r)$ and $\Delta = \min_{1 \leqslant j \leqslant r}(a_j - b_j)$.

Theorem 3.3.4 (**Tóth and Zhai [115, Th. 3.4]**). *If $r \geqslant 2$, then*

$$\sum_{n_1, \ldots, n_r \leqslant x} \tau(n_1 \cdots n_r) = x^r P_r(\log x) + O(x^{r-1+\theta+\varepsilon}),$$

$$\sum_{n_1, \ldots, n_r \leqslant x} \tau([n_1, \ldots, n_r]) = x^r Q_r(\log x) + O(x^{r-1+\theta+\varepsilon}),$$

for every $\varepsilon > 0$, where θ is the exponent in the Dirichlet divisor problem (1.1), *$P_r(t)$ and $Q_r(t)$ are polynomials in t of degree r having the leading coefficients*

$$K_{P,r} := \prod_p \left(1 - \frac{1}{p}\right)^{r-1} \left(1 + \frac{r-1}{p}\right) \tag{3.19}$$

and

$$K_{Q,r} := \prod_p \left(1 - \frac{1}{p}\right)^{2r} \sum_{\nu_1, \ldots, \nu_r = 0}^{\infty} \frac{\max(\nu_1, \ldots, \nu_r) + 1}{p^{\nu_1 + \cdots + \nu_r}},$$

respectively.

Note that the constant $K_{P,r}$ defined by (3.19) equals the asymptotic density of the set of r-tuples of positive integers with pairwise relatively prime components. See Section 3.1.

Furthermore, in the case $r = 2$,

$$K_{Q,2} = \zeta(2) K_{P,3} = \zeta(2) \prod_p \left(1 - \frac{1}{p}\right)^2 \left(1 + \frac{2}{p}\right).$$

Now consider the function $\tau^{(2)}(n)$. Note that $\tau^{(2)}(n_1 \cdots n_r) = \tau^{(2)}([n_1, \ldots, n_r])$ for every $n_1, \ldots, n_r \in \mathbb{N}$.

Theorem 3.3.5 (Tóth and Zhai [115, Th. 3.5]). *If $r \geqslant 2$, then*

$$\sum_{n_1,\ldots,n_r \leqslant x} \tau^{(2)}(n_1 \cdots n_r) = x^r P_r^*(\log x) + O(x^{r-1/2+\varepsilon}),$$

where $P_r^*(t)$ is a polynomials in t of degree r having the leading coefficient

$$K_{P*,r} := \prod_p \left(1 - \frac{1}{p}\right)^r \left(2 - \left(1 - \frac{1}{p}\right)^r\right).$$

3.4 The average number of subgroups of the groups $\mathbb{Z}_m \times \mathbb{Z}_n$

Let $s(m, n)$ denote, as earlier, the total number of subgroups of the group $(\mathbb{Z}_m \times \mathbb{Z}_n, +)$. Based on formula (2.45) we obtain the following results.

Theorem 3.4.1 (Nowak and Tóth [71, Th. 2.1]). *For every $z, w \in \mathbb{C}$ with $\Re z > 1, \Re w > 1$,*

$$\sum_{m,n=1}^{\infty} \frac{s(m,n)}{m^z n^w} = \frac{\zeta^2(z)\zeta^2(w)\zeta(z+w-1)}{\zeta(z+w)}. \tag{3.20}$$

Theorem 3.4.2 (Nowak and Tóth [71, Th. 2.2]). *For every fixed $\varepsilon > 0$,*

$$\sum_{m,n \leqslant x} s(m,n) = x^2 \sum_{r=0}^{3} A_r (\log x)^r + O\left(x^{\frac{1117}{701}+\varepsilon}\right), \tag{3.21}$$

where $1117/701 \doteq 1.593437$, A_r $(0 \leqslant r \leqslant 3)$ are constants,

$$A_3 = \frac{1}{3\zeta(2)} = \frac{2}{\pi^2}, \quad A_2 = \frac{1}{\zeta(2)} \left(3\gamma - 1 - \frac{\zeta'(2)}{\zeta(2)}\right),$$

$$A_1 = \frac{1}{\zeta(2)}\left(8\gamma^2 - 6\gamma - 2\gamma_1 + 1 - 2(3\gamma-1)\frac{\zeta'(2)}{\zeta(2)} + 2\left(\frac{\zeta'(2)}{\zeta(2)}\right)^2 - \frac{\zeta''(2)}{\zeta(2)}\right),$$

with $\gamma_1 = \lim_{x\to\infty}\left(\sum_{n\leqslant x}\frac{\log n}{n} - \frac{(\log x)^2}{2}\right)$.

Remark 3.4.3. In fact, the error term $O\left(x^{\frac{1117}{701}+\varepsilon}\right)$ can be replaced by $O\left(x^{\frac{3-\theta}{2-\theta}+\varepsilon}\right)$, where θ is the exponent in Dirichlet's divisor problem. The O-term stated above is given by the hitherto sharpest result due to Huxley.

Remark 3.4.4. The constant A_0 can be constructed from the proof below. It is quite complicated and hardly accessible to numerical evaluation, since they involve *inter alia* the infinite series $\sum_{k=1}^{\infty}\tau(k)\Delta(k)k^{-2}$, where $\Delta(x)$ the error term in Dirichlet's divisor problem.

Our next result is concerning the number of rank two subgroups of the groups $\mathbb{Z}_m \times \mathbb{Z}_n$ with $m, n \leqslant x$. We need some further notations. For $K \in \mathbb{N}$ and $s \in \mathbb{C}$ let

$$F_K(s) := \prod_{p^{\nu_p(K)}\,||\,K}\left(1 - \eta_p(K)\,p^{-s}\right), \quad \text{where} \quad \eta_p(K) := \frac{\nu_p(K)}{\nu_p(K)+1},$$

$$\alpha_0(K) := F_K(1) = \prod_{p^{\nu_p(K)}\,||\,K}\left(1 - \eta_p(K)\,p^{-1}\right),$$

$$\alpha_1(K) := F_K'(1) = \sum_{p^*\,|\,K}\frac{\eta_{p^*}(K)}{p^*}\log p^* \prod_{p^{\nu_p(K)}\,||\,K,\ p\neq p^*}\left(1 - \frac{\eta_p(K)}{p}\right), \tag{3.22}$$

and let

$$\beta_0(K) := \tau(K)\alpha_0(K), \quad \beta_1(K) := \tau(K)\left(\alpha_0(K)(2\gamma-1) + \alpha_1(K)\right). \tag{3.23}$$

Theorem 3.4.5 (Nowak and Tóth [71, Th. 2.5]). *For every fixed* $\varepsilon > 0$,

$$S^{(2)}(x) := \sum_{\substack{m,n\leqslant x\\ \gcd(m,n)>1}} s(m,n) = x^2\left(\sum_{r=0}^{3} C_r(\log x)^r\right) + O\left(x^{\frac{1117}{701}+\varepsilon}\right), \tag{3.24}$$

where $C_3 = A_3$, $C_r = A_r - b_r$ $(0 \leqslant r \leqslant 2)$ *with* A_r $(0 \leqslant r \leqslant 3)$ *defined in Theorem 3.4.2 and* b_r $(0 \leqslant r \leqslant 2)$ *given by*

$$b_2 = \sum_{K=1}^{\infty}\mu(K)\left(\frac{\beta_0(K)}{K}\right)^2 = \prod_p\left(1 - \frac{4}{p^2} + \frac{4}{p^3} - \frac{1}{p^4}\right),$$

$$b_1 = \sum_{K=1}^{\infty}\frac{2\mu(K)}{K^2}\beta_0(K)\left(\beta_1(K) - \beta_0(K)\log K\right),$$

$$b_0 = \sum_{K=1}^{\infty}\frac{\mu(K)}{K^2}\left(\beta_1(K) - \beta_0(K)\log K\right)^2,$$

using the notation of (3.22) *and* (3.23).

In the paper [71] we proved similar formulas for $\sum_{m,n\leqslant x} c(m,n)$ and $\sum_{m,n\leqslant x,(m,n)>1} c(m,n)$, where $c(m,n)$ stands for the number of cyclic subgroups of the group $(\mathbb{Z}_m \times \mathbb{Z}_n, +)$.

Referring to our paper [71], Ushiroya [117] investigated the existence of the limit

$$\lim_{x\to\infty} \frac{1}{x^2(\log x)^{k-1}} \sum_{m,n\leqslant x} f(m,n),$$

where $f(m,n)$ is a multiplicative function and k is a fixed positive integer.

3.5 Generalizations of the Busche-Ramanujan identities

Let g and h be two completely multiplicative arithmetic functions and let $f = g * h$. Then f is called specially multiplicative function or quadratic function. The Busche-Ramanujan identities state that for every $m,n \in \mathbb{N}$,

$$f(mn) = \sum_{d\mid \gcd(m,n)} f\left(\frac{m}{d}\right) f\left(\frac{n}{d}\right) \mu(d)g(d)h(d) \tag{3.25}$$

and

$$f(m)f(n) = \sum_{d\mid \gcd(m,n)} f\left(\frac{mn}{d^2}\right) g(d)h(d). \tag{3.26}$$

For example, these identities hold true for the functions $f(n) = \tau(n), \sigma(n), \sigma_k(n) = \sum_{d\mid n} d^k$ and the function $R_1(n) = R(n)/4$, where $R(n)$ represents the number of ordered pairs $(x,y) \in \mathbb{Z}^2$ such that $n = x^2 + y^2$. For history and discussion, as well as for generalizations and analogs of the Busche-Ramanujan identities we refer to the papers [7, 39, 40, 64, 65, 119].

Identities (3.25) and (3.26) can be well understood by using functions of two variables. For example, by selecting the σ function, the common analytic version of (3.25) and (3.26) is the formula

$$\sum_{n_1,n_2=1}^{\infty} \frac{\sigma(n_1n_2)}{n_1^{s_1} n_2^{s_2}} = \frac{\zeta(s_1)\zeta(s_1-1)\zeta(s_2)\zeta(s_2-1)}{\zeta(s_1+s_2-1)} \quad (\Re s_1, \Re s_2 > 2),$$

which also explains the equivalence of the identities of type (3.25) and (3.26).

We prove the following generalizations of the Busche-Ramanujan identities.

Theorem 3.5.1 (Tóth [105, Th. 3.1]). *Let g and h be two completely multiplicative functions and let $f = g * h$. Let ψ_f be the multiplicative function of r ($r \in \mathbb{N}$) variables defined as follows. For every prime p and every $\nu_1, \ldots, \nu_r \in \mathbb{N} \cup \{0\}$ set*

$$\psi_f(p^{\nu_1}, \ldots, p^{\nu_r}) = \begin{cases} 1, & \nu_1 = \cdots = \nu_r = 0, \\ (-1)^{j-1} g(p)h(p)f(p^{j-2}), & \nu_1, \ldots, \nu_r \in \{0,1\}, \\ & j := \nu_1 + \cdots + \nu_r \geqslant 2, \\ 0, & \textit{otherwise}. \end{cases}$$

Furthermore, let ψ_f^{-1*} be the inverse under the r variables convolution of the function ψ_f. Then for every $n_1, \ldots, n_r \in \mathbb{N}$ ($r \in \mathbb{N}$),

$$f(n_1 \cdots n_r) = \sum_{a_1 | n_1, \ldots, a_r | n_r} f\left(\frac{n_1}{a_1}\right) \cdots f\left(\frac{n_r}{a_r}\right) \psi_f(a_1, \ldots, a_r), \tag{3.27}$$

and

$$f(n_1) \cdots f(n_r) = \sum_{a_1 | n_1, \ldots, a_r | n_r} f\left(\frac{n_1 \cdots n_r}{a_1 \cdots a_r}\right) \psi_f^{-1*}(a_1, \ldots, a_r). \tag{3.28}$$

We remark that for the divisor function the analytic version of Theorem 3.5.1 is the following identity:

$$\sum_{n_1, \ldots, n_r = 1}^{\infty} \frac{\tau(n_1 \cdots n_r)}{n_1^{s_1} \cdots n_r^{s_r}}$$

$$= \zeta^2(s_1) \cdots \zeta^2(s_r) \prod_p \left(1 + \sum_{j=2}^{r} (-1)^{j-1}(j-1) \sum_{1 \leqslant i_1 < \ldots < i_j \leqslant r} \frac{1}{p^{s_{i_1} + \cdots + s_{i_j}}}\right), \tag{3.29}$$

which is, in fact, the special case for $k = 2$ of Theorem 3.1.1, taking into account formula (1.10). The special cases of (3.29) corresponding to $r = 2$ and $r = 3$ were pointed out by Kurokawa and Ochiai [58, Sect. 6].

Theorem 3.5.2 (Tóth [105, Th. 3.2]). *Let $f_1, \ldots, f_k$ be completely multiplicative functions ($k \in \mathbb{N}$) and let $F = f_1 * \cdots * f_k$. Let ϑ_F be the multiplicative function of two variables defined as follows. For every prime p and every $\nu_1, \nu_2 \in \mathbb{N} \cup \{0\}$ set*

$$\vartheta_F(p^{\nu_1}, p^{\nu_2}) = \begin{cases} 1, & \nu_1 = \nu_2 = 0, \\ (-1)^{\nu_1 + \nu_2 - 1} e_{\nu_1 + \nu_2}(f_1(p), \ldots, f_k(p)), & \nu_1, \nu_2 \geqslant 1, \nu_1 + \nu_2 \leqslant k, \\ 0, & \textit{otherwise}, \end{cases}$$

where $e_d(x_1, \ldots, x_k)$ represents the elementary symmetric polynomial in $x_1, \ldots, x_k$ of degree d. Furthermore, let ϑ_F^{-1*} denote the inverse under the two variables convolution of *the function ϑ_F. Then for every $n_1, n_2 \in \mathbb{N}$,*

$$F(n_1 n_2) = \sum_{a_1 \mid n_1,\, a_2 \mid n_2} F\left(\frac{n_1}{a_1}\right) F\left(\frac{n_2}{a_2}\right) \vartheta_F(a_1, a_2), \tag{3.30}$$

and

$$F(n_1)F(n_2) = \sum_{a_1 \mid n_1,\, a_2 \mid n_2} F\left(\frac{n_1 n_2}{a_1 a_2}\right) \vartheta_F^{-1*}(a_1, a_2). \tag{3.31}$$

Theorems 3.5.1 and 3.5.2 reduce in the cases $r = 2$, respectively $k = 2$ to the Busche-Ramanujan identities (3.25) and (3.26).

Next we give the result of Theorem 3.5.2, applied to the Piltz divisor function $\tau_k(n)$.

Corollary 3.5.3 (Tóth [105, Cor. 3.4]). *Let $k \in \mathbb{N}$. For every $n_1, n_2 \in \mathbb{N}$,*

$$\tau_k(n_1 n_2) = \sum_{a_1 \mid n_1,\, a_2 \mid n_2} \tau_k\left(\frac{n_1}{a_1}\right) \tau_k\left(\frac{n_2}{a_2}\right) \vartheta_k(a_1, a_2),$$

where the multiplicative function ϑ_k is defined for every prime p and every $\nu_1, \nu_2 \in \mathbb{N} \cup \{0\}$ by

$$\vartheta_k(p^{\nu_1}, p^{\nu_2}) = \begin{cases} 1, & \nu_1 = \nu_2 = 0, \\ (-1)^{\nu_1+\nu_2-1}\binom{k}{\nu_1+\nu_2}, & \nu_1, \nu_2 \geqslant 1, \nu_1 + \nu_2 \leqslant k, \\ 0, & \text{otherwise.} \end{cases}$$

3.6 Ramanujan expansions of arithmetic functions of several variables

In this section we prove the following general result. Let $c_q(n)$ denote the Ramanujan sums.

Theorem 3.6.1 (Tóth [113, Th. 2]). *Let $f : \mathbb{N}^k \to \mathbb{C}$ be an arithmetic function ($k \in \mathbb{N}$). Assume that*

$$\sum_{n_1,\ldots,n_k=1}^{\infty} 2^{\omega(n_1)+\cdots+\omega(n_k)} \frac{|(\mu_k * f)(n_1, \ldots, n_k)|}{n_1 \cdots n_k} < \infty. \tag{3.32}$$

Then for every $n_1, \ldots, n_k \in \mathbb{N}$,

$$f(n_1, \ldots, n_k) = \sum_{q_1,\ldots,q_k=1}^{\infty} a_{q_1,\ldots,q_k} c_{q_1}(n_1) \cdots c_{q_k}(n_k), \tag{3.33}$$

where

$$a_{q_1,\ldots,q_k} = \sum_{m_1,\ldots,m_k=1}^{\infty} \frac{(\mu_k * f)(m_1 q_1, \ldots, m_k q_k)}{m_1 q_1 \cdots m_k q_k}, \tag{3.34}$$

the series (3.33) being absolutely convergent.

We remark that according to the generalized Wintner theorem, quoted in the Introduction, under conditions of Theorem 3.6.1, the mean value $M(f)$ exists and $a_{1,\ldots,1} = M(f)$.

For multiplicative functions f, condition (3.32) is equivalent to

$$\sum_{n_1,\ldots,n_k=1}^{\infty} \frac{|(\mu_k * f)(n_1,\ldots,n_k)|}{n_1\cdots n_k} < \infty \tag{3.35}$$

and to

$$\sum_{p} \sum_{\substack{\nu_1,\ldots,\nu_k=0 \\ \nu_1+\cdots+\nu_k\geqslant 1}}^{\infty} \frac{|(\mu_k * f)(p^{\nu_1},\ldots,p^{\nu_k})|}{p^{\nu_1+\cdots+\nu_k}} < \infty. \tag{3.36}$$

We deduce the following result:

Corollary 3.6.2 (Tóth [113, Cor. 1]). *Let $f : \mathbb{N}^k \to \mathbb{C}$ be a multiplicative function ($k \in \mathbb{N}$). Assume that condition* (3.35) *or* (3.36) *holds. Then for every $n_1,\ldots,n_k \in \mathbb{N}$ one has the absolutely convergent expansion* (3.33)*, and the coefficients can be written as*

$$a_{q_1,\ldots,q_k} = \prod_{p} \sum_{\nu_1\geqslant \nu_p(q_1),\ldots,\nu_k\geqslant \nu_p(q_k)} \frac{(\mu_k * f)(p^{\nu_1},\ldots,p^{\nu_k})}{p^{\nu_1+\cdots+\nu_k}}.$$

Theorem 3.6.1 and Corollary 3.6.2 generalize the results by Delange [30] and Ushiroya [118], obtained in the one variable and the two variables cases, respectively. The method given in [118] to compute the coefficients in the case of special functions is complicated.

We present a simple approach to deduce k dimensional versions of the identities concerning special multiplicative functions, like (1.8) for the sigma function and

$$\frac{\phi(n)}{n} = \frac{1}{\zeta(2)} \sum_{q=1}^{\infty} \frac{\mu(q) c_q(n)}{\phi_2(q)}, \tag{3.37}$$

valid for $n \in \mathbb{N}$, due to Ramanujan [77]. For example, we have the next identities.

Corollary 3.6.3 (Tóth [113, Cor. 3]). *For every $n_1,\ldots,n_k \in \mathbb{N}$ the following series are absolutely convergent:*

$$\frac{\sigma_s((n_1,\ldots,n_k))}{(n_1,\ldots,n_k)^s} = \zeta(s+k) \sum_{q_1,\ldots,q_k=1}^{\infty} \frac{c_{q_1}(n_1)\cdots c_{q_k}(n_k)}{Q^{s+k}} \quad (s \in \mathbb{R}, s+k>1),$$

$$\frac{\sigma((n_1,\ldots,n_k))}{(n_1,\ldots,n_k)} = \zeta(k+1) \sum_{q_1,\ldots,q_k=1}^{\infty} \frac{c_{q_1}(n_1)\cdots c_{q_k}(n_k)}{Q^{k+1}} \quad (k \geqslant 1) \tag{3.38}$$

$$\tau((n_1,\ldots,n_k)) = \zeta(k) \sum_{q_1,\ldots,q_k=1}^{\infty} \frac{c_{q_1}(n_1)\cdots c_{q_k}(n_k)}{Q^{k}} \quad (k \geqslant 2), \tag{3.39}$$

where $Q = [q_1,\ldots,q_k]$.

Corollary 3.6.4 (Tóth [113, Cor. 8]). *For every $n_1, \ldots, n_k \in \mathbb{N}$ the following series are absolutely convergent:*

$$\frac{\phi_s((n_1,\ldots,n_k))}{(n_1,\ldots,n_k)^s} = \frac{1}{\zeta(s+k)} \sum_{q_1,\ldots,q_k=1}^{\infty} \frac{\mu(Q)c_{q_1}(n_1)\cdots c_{q_k}(n_k)}{\phi_{s+k}(Q)} \quad (s \in \mathbb{R}, s+k > 1),$$

$$\frac{\phi((n_1,\ldots,n_k))}{(n_1,\ldots,n_k)} = \frac{1}{\zeta(k+1)} \sum_{q_1,\ldots,q_k=1}^{\infty} \frac{\mu(Q)c_{q_1}(n_1)\cdots c_{q_k}(n_k)}{\phi_{k+1}(Q)} \quad (k \geqslant 1). \tag{3.40}$$

For $k = 1$ identities (3.38) and (3.40) recover (1.8) and (3.37), respectively. However, (3.39) does not have a direct one dimensional analog. The identity of Ramanujan for the divisor function, namely,

$$\tau(n) = -\sum_{q=1}^{\infty} \frac{\log q}{q} c_q(n) \quad (n \in \mathbb{N})$$

can not be obtained by the same approach, since the mean value $M(\tau)$ does not exist.

In my paper [113] I also considered expansions of arithmetic functions of several variables with respect to the unitary Ramanujan sums $c_q^*(n)$, defined by

$$c_q^*(n) = \sum_{\substack{1 \leqslant k \leqslant q \\ (k,q)_* = 1}} \exp(2\pi i k n/q) \quad (q, n \in \mathbb{N}),$$

where $(k,q)_*$ stands for the greatest divisor of k, which is a unitary divisor of q. I showed that the above properties on expansions of functions of one and several variables using classical and unitary Ramanujan sums, respectively, run parallel. However, there are identities concerning the sums $c_q^*(n)$, which do not have simple counterparts in the classical case. For example, consider the Piltz divisor function τ_m. Let $m \geqslant 2$ and $k \geqslant 2$. Then for any $n_1, \ldots, n_k \in \mathbb{N}$ we have ([113, Cor. 9])

$$\tau_m((n_1,\ldots,n_k)) = \zeta(k)^{m-1} \sum_{q_1,\ldots,q_k=1}^{\infty} \frac{\tau_{m-1}(Q)\phi_k(Q)^{m-1}}{Q^{km}} c_{q_1}^*(n_1)\cdots c_{q_k}^*(n_k).$$

Chapter 4

Proofs of the results of Chapter 2

4.1 Proofs of the results of Section 2.1

Proof of Theorem 2.1.1. For an integer $\ell \geqslant 1$ let $\mu_\ell(n) = \mu(m)$ or 0, according as $n = m^\ell$ or not, where μ is the Möbius function. The function μ_ℓ is multiplicative and for any prime power p^ν ($\nu \geqslant 1$),

$$\mu_\ell(p^\nu) = \begin{cases} -1, & \text{if } \nu = \ell, \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, for an integer $h \geqslant 1$ define the multiplicative function $\mu_\ell^{(h)}$ in terms of the Dirichlet convolution by

$$\mu_\ell^{(h)} = \underbrace{\mu_\ell * \mu_\ell * \cdots * \mu_\ell}_{h}.$$

It is easy to see that for any integers $h, \ell \geqslant 1$ and any prime power p^ν ($\nu \geqslant 1$),

$$\mu_\ell^{(h)}(p^\nu) = \begin{cases} (-1)^j \binom{h}{j}, & \text{if } \nu = j\ell, \quad 1 \leqslant j \leqslant h, \\ 0, & \text{otherwise.} \end{cases}$$

Let $V(s) := \sum_{n=1}^\infty v(n)/n^s$. We obtain the desired Dirichlet series representation by taking $v = f * \mu * \mu_\ell^{(k-1)}$. Here v is also multiplicative and direct computations show that $v(p^\nu) = 0$ for any $1 \leqslant \nu \leqslant \ell$. For $\nu \geqslant \ell + 1$,

$$v(p^\nu) = \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} \left(f(p^{\nu - j\ell}) - f(p^{\nu - j\ell - 1}) \right),$$

leading to the absolute convergence of $V(s)$ for $\Re(s) > 1/(\ell + 1)$. Now the asymptotic formula (2.3) follows from the representation

$$f(n) = \sum_{ab=n} \tau(1, \underbrace{\ell, \ell, \ldots, \ell}_{k-1}; a) v(b)$$

and from the assumption on $\Delta_{k,\ell}$ by the convolution method. □

Proof of Theorem 2.1.3. Apply Theorem 2.1.1 for the function $f(n) = \tau^{(e)}(n)^r$ with $\ell = 2$ and $k = 2^r$. According to Remark 2.1.2, the error term is $O(x^{u_r+\varepsilon})$. □

Proof of Corollary 2.1.4. Apply Theorem 2.1.1 by choosing $f(n) = a(n)^r$, $k = 2^r$ and $\ell = 2$. Note that $P(\nu) < e^{\pi\sqrt{2\nu/3}}$ $(\nu \geqslant 1)$, see e.g., [68, p. 236], thus condition ii) is verified. □

4.2 Proofs of the results of Section 2.2

Proof of Theorem 2.2.1. If f is a multiplicative function, then

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{f(n)}{n^s} = \left(\sum_{n=1}^{\infty}\frac{f(n)}{n^s}\right)\left(2\left(\sum_{\nu=0}^{\infty}\frac{f(2^\nu)}{2^{\nu s}}\right)^{-1} - 1\right), \tag{4.1}$$

formally or in case of convergence. It follows from (4.1) that the convolution identity

$$(-1)^{n-1}f(n) = \sum_{dj=n} h_f(d)f(j) \quad (n \geqslant 1) \tag{4.2}$$

holds, where the function h_f is multiplicative, $h_f(p^\nu) = 0$ if $p > 2$, $\nu \geqslant 1$ and $h_f(2^\nu) = 2b_\nu$ $(\nu \geqslant 1)$, $h_f(1) = 2b_0 - 1 = 1$, with the notation (2.5). Note that, according to (4.2) and (4.1),

$$\sum_{n=1}^{\infty}\frac{h_f(n)}{n^s} = \frac{2}{S_f(1/2^s)} - 1, \tag{4.3}$$

provided that both $S_f(1/2^s)$ and $\overline{S}_f(1/2^s)$ converge. By differentiating,

$$\sum_{n=1}^{\infty}\frac{h_f(n)\log n}{n^s} = -\frac{\log 2}{2^{s-1}}\cdot\frac{S_f'(1/2^s)}{S_f(1/2^s)^2}, \tag{4.4}$$

assuming that $|1/2^s| < \min(r_f, \overline{r}_f)$.

Using (4.2), with $1/f$ instead of f, we deduce that

$$\begin{aligned}\sum_{n\leqslant x}(-1)^{n-1}\frac{1}{f(n)} &= \sum_{d\leqslant x} h_{1/f}(d)\sum_{j\leqslant x/d}\frac{1}{f(j)}\\ &= \sum_{d\leqslant x} h_{1/f}(d)\left(D_f\left(\log\frac{x}{d} + E_f\right) + O\left((x/d)^{-1}R_{1/f}(x/d)\right)\right)\\ &= D_f(\log x + E_f)\sum_{d\leqslant x} h_{1/f}(d) - D_f\sum_{d\leqslant x} h_{1/f}(d)\log d\\ &\quad + O\left(x^{-1}R_{1/f}(x)\sum_{d\leqslant x} d|h_{1/f}(d)|\right).\end{aligned}$$

That is,

$$\sum_{n\leqslant x}(-1)^{n-1}\frac{1}{f(n)} = D_f(\log x + E_f)\sum_{d=1}^{\infty} h_{1/f}(d) + O\left(\log x \sum_{d>x}|h_{1/f}(d)|\right)$$

$$-D_f\sum_{d=1}^{\infty} h_{1/f}(d)\log d + O\left(\sum_{d>x}|h_{1/f}(d)|\log d\right) + O\left(x^{-1}R_{1/f}(x)\sum_{d\leqslant x} d|h_{1/f}(d)|\right). \quad (4.5)$$

Note that $\min(r_{1/f},\overline{r}_{1/f}) > 1$ by conditions (ii) and (iii). By using (4.3) and (4.4) for $s=0$,

$$\sum_{d=1}^{\infty} h_{1/f}(d) = \frac{2}{S_{1/f}(1)} - 1,$$

$$\sum_{d=1}^{\infty} h_{1/f}(d)\log d = -2(\log 2)\frac{S'_{1/f}(1)}{S_{1/f}(1)^2}.$$

Furthermore,

$$\sum_{d>x}|h_{1/f}(d)| = \sum_{d=2^\nu>x}|h_{1/f}(2^\nu)| \ll \sum_{2^\nu>x}|b_\nu| \ll \sum_{2^\nu>x} M^\nu \ll x^{\log M/\log 2},$$

$$\sum_{d>x}|h_{1/f}(d)|\log d \ll \sum_{2^\nu>x}\nu|b_\nu| \ll \sum_{2^\nu>x}\nu M^\nu \ll x^{\log M/\log 2}\log x,$$

$$\sum_{d\leqslant x} d|h_{1/f}(d)| = \sum_{2^\nu\leqslant x} 2^\nu|b_\nu| \ll \sum_{\nu\leqslant \log x/\log 2}(2M)^\nu,$$

where the latter sum is bounded if $0 < M < 1/2$, it is $\ll \log x$ if $M = 1/2$, and is $\ll x^{1+\log M/\log 2}$ if $1/2 < M < 1$.

Inserting these into (4.5), the proof is complete. □

Proof of Corollary 2.2.2. Apply Theorem 2.2.1 for $f=\phi$ and use the asymptotic formula (1.5). Here

$$S_{1/\phi}(x) = \sum_{\nu=0}^{\infty}\frac{x^\nu}{\phi(2^\nu)} = 1 + \sum_{\nu=1}^{\infty}\frac{x^\nu}{2^{\nu-1}} = \frac{2+x}{2-x} \quad (|x|<2),$$

$$\overline{S}_{1/\phi}(x) = \frac{1-x/2}{1+x/2} = 1 + \sum_{\nu=1}^{\infty}(-1)^\nu\frac{x^\nu}{2^{\nu-1}} \quad (|x|<2);$$

hence $b_\nu \ll 2^{-\nu}$ and choose $M=1/2$. Use also that $S_{1/\phi}(1)=3$ and $S'_{1/\phi}(1)=4$. □

Proof of Corollary 2.2.3. Apply Theorem 2.2.1 for $f=\sigma$, using the formula (2.10). Now

$$S_{1/\sigma}(x) = \sum_{\nu=0}^{\infty}\frac{x^\nu}{\sigma(2^\nu)} = \sum_{\nu=0}^{\infty}\frac{x^\nu}{2^{\nu+1}-1}$$

and $S_{1/\sigma}(1) = K$. Note that $S'_{1/\psi}(1) = K'$.

The coefficients b_ν of the reciprocal power series are $b_0 = 1$, $b_1 = -\frac{1}{3}$, $b_2 = -\frac{2}{63}$, $b_3 = -\frac{8}{945}$, etc. Observe that the sequence $\left(\frac{1}{2^{\nu+1}-1}\right)_{\nu \geqslant 0}$ is log-convex. Therefore, according to the cited result of Kaluza,

$$-\frac{1}{2^{\nu+1}-1} \leqslant b_\nu \leqslant 0 \quad (\nu \geqslant 1),$$

which shows that $b_\nu \ll 2^{-\nu}$ and we can choose $M = 1/2$. □

Proof of Corollary 2.2.4. The formula

$$\sum_{n \leqslant x} \frac{1}{\tau(n)} = x \sum_{j=1}^{N} \frac{A_j}{(\log x)^{j-1/2}} + O\left(\frac{x}{(\log x)^{N+1/2}}\right),$$

valid for every real $x \geqslant 2$ and every fixed integer $N \geqslant 1$, where A_j $(1 \leqslant j \leqslant N)$ are computable constants,

$$A_1 = \frac{1}{\sqrt{\pi}} \prod_p \left(\sqrt{p^2 - p}\, \log\left(\frac{p}{p-1}\right)\right)$$

was stated by Ramanujan [76, Eq. (7)]. See Wilson [122, Sect. 3] for its proof.

Now

$$S_{1/\tau}(x) = \sum_{\nu=0}^{\infty} \frac{1}{\tau(2^\nu)} x^\nu = \sum_{\nu=0}^{\infty} \frac{1}{\nu+1} x^\nu = -\frac{\log(1-x)}{x} \qquad (|x| < 1)$$

and the reciprocal power series is

$$\overline{S}_{1/\tau}(x) = -\frac{x}{\log(1-x)} = \sum_{\nu=0}^{\infty} b_\nu x^\nu,$$

where $b_0 = 1$, $b_1 = -1/2$, $b_2 = -1/12$, $b_3 = -1/24$, etc. Note that the sequence $\left(\frac{1}{\nu+1}\right)_{\nu \geqslant 0}$ is log-convex. According to the result of Kaluza (this example was considered in [52]),

$$-\frac{1}{\nu+1} \leqslant b_\nu \leqslant 0 \qquad (\nu \geqslant 1).$$

This shows, using (4.2), that

$$(-1)^{n-1} \frac{1}{\tau(n)} = \sum_{dj=n} h_{1/\tau}(d) \frac{1}{\tau(j)} \qquad (n \geqslant 1),$$

where the function $h_{1/\tau}$ is multiplicative, $h_{1/\tau}(2^\nu) \ll \frac{1}{\nu}$ as $\nu \to \infty$ and $h_{1/\tau}(p^\nu) = 0$ for every prime $p > 2$ and $\nu \geqslant 1$.

Hence

$$T(x) := \sum_{n\leqslant x}(-1)^{n-1}\frac{1}{\tau(n)} = \sum_{d\leqslant x/2} h_{1/\tau}(d) \sum_{j\leqslant x/d}\frac{1}{\tau(j)} + \sum_{x/2<d\leqslant x} h_{1/\tau}(d)$$

$$= \sum_{d\leqslant x/2} h_{1/\tau}(d)\left(\frac{x}{d}\sum_{j=1}^{N}\frac{A_j}{(\log(x/d))^{j-1/2}} + O\left(\frac{x/d}{(\log(x/d))^{N+1/2}}\right)\right) + \sum_{x/2<d\leqslant x} h_{1/\tau}(d)$$

$$= x\sum_{j=1}^{N}\frac{A_j}{(\log x)^{j-1/2}}\sum_{d\leqslant x/2}\frac{h_{1/\tau}(d)}{d(1-\frac{\log d}{\log x})^{j-1/2}} + O\left(\frac{x}{(\log x)^{N+1/2}}\sum_{d\leqslant x/2}\frac{|h_{1/\tau}(d)|}{d(1-\frac{\log d}{\log x})^{N+1/2}}\right)$$
$$+ \sum_{x/2<d\leqslant x} h_{1/\tau}(d).$$

Here the last term is small:

$$\sum_{x/2<d\leqslant x} h_{1/\tau}(d) \ll \sum_{d=2^\nu\leqslant x}|h_{1/\tau}(2^\nu)| \ll \sum_{\nu\leqslant \log x/\log 2}\frac{1}{\nu} \ll \log\log x.$$

Using the power series expansion

$$(1+x)^t = \sum_{j=0}^{\infty}\binom{t}{j}x^j \qquad (x,t\in\mathbb{R}, |x|<1),$$

we deduce

$$\sum_{d\leqslant x/2}\frac{|h_{1/\tau}(d)|}{d(1-\frac{\log d}{\log x})^{N+1/2}} = \sum_{d\leqslant x/2}\frac{|h_{1/\tau}(d)|}{d}\left(1+O\left(\frac{\log d}{\log x}\right)\right)$$
$$= \sum_{d=2^\nu\leqslant x/2}\frac{|h_{1/\tau}(2^\nu)|}{2^\nu} + O\left(\frac{1}{\log x}\sum_{d=2^\nu\leqslant x/2}\frac{|h_{1/\tau}(2^\nu)|}{2^\nu}\log 2^\nu\right)$$
$$\ll \sum_{2^\nu\leqslant x/2}\frac{1}{\nu 2^\nu} + \frac{1}{\log x}\sum_{2^\nu\leqslant x/2}\frac{1}{2^\nu} \ll 1.$$

Therefore, the remainder term of above is

$$O\left(\frac{x}{(\log x)^{N+1/2}}\right).$$

Furthermore,

$$\sum_{d\leqslant x/2} \frac{h_{1/\tau}(d)}{d(1-\frac{\log d}{\log x})^{j-1/2}} = \sum_{d\leqslant x/2} \frac{h_{1/\tau}(d)}{d} \sum_{\ell=0}^{\infty}(-1)^{\ell}\binom{-j+1/2}{\ell}\left(\frac{\log d}{\log x}\right)^{\ell}$$
$$= \sum_{\ell=0}^{\infty}(-1)^{\ell}\binom{-j+1/2}{\ell}\frac{1}{(\log x)^{\ell}} \sum_{d\leqslant x/2} \frac{h_{1/\tau}(d)}{d}(\log d)^{\ell}$$
$$= \sum_{\ell=0}^{\infty}(-1)^{\ell}\binom{-j+1/2}{\ell}\frac{1}{(\log x)^{\ell}} \left(K_{\ell} + O\left(\frac{(\log x)^{\ell-1}}{x}\right)\right),$$

where for every $\ell \geqslant 0$ the series

$$K_{\ell} := \sum_{d=1}^{\infty} \frac{h_{1/\tau}(d)}{d}(\log d)^{\ell} = \sum_{\substack{d=2^{\nu}\\ \nu\geqslant 0}} \frac{h_{1/\tau}(2^{\nu})}{2^{\nu}}(\log 2^{\nu})^{\ell}$$

is absolutely convergent, since $|h_{1/\tau}(2^{\nu})| \ll \frac{1}{\nu}$, and

$$\sum_{d>x/2} \frac{|h_{1/\tau}(d)|}{d}(\log d)^{\ell} = \sum_{d=2^{\nu}>x/2} \frac{|h_{1/\tau}(2^{\nu})|}{2^{\nu}}(\log 2^{\nu})^{\ell} \ll \sum_{\nu>\log x/\log 2} \frac{\nu^{\ell-1}}{2^{\nu}} \ll \frac{(\log x)^{\ell-1}}{x}.$$

We deduce that

$$T(x) = x\sum_{j=1}^{N} \frac{A_j}{(\log x)^{j-1/2}} \sum_{\ell=0}^{\infty}(-1)^{\ell}\binom{-j+1/2}{\ell}\frac{1}{(\log x)^{\ell}}\left(K_{\ell} + O\left(\frac{(\log x)^{\ell-1}}{x}\right)\right)$$
$$+O\left(\frac{x}{(\log x)^{N+1/2}}\right)$$
$$= x\sum_{t=1}^{N} \frac{1}{(\log x)^{t-1/2}} \sum_{j=1}^{N}(-1)^{t-j}\binom{-j+1/2}{t-j}A_j K_{t-j} + O\left(\frac{x}{(\log x)^{N+1/2}}\right).$$

The proof is complete by denoting

$$B_t = \sum_{j=1}^{N}(-1)^{t-j}\binom{-j+1/2}{t-j}A_j K_{t-j},$$

where $B_1 = A_1 K_0 = A_1(\frac{1}{\log 2} - 1)$ by (4.3) (applied to $s = 1$). □

Proof of Proposition 2.2.5. Let $\sum_{\nu=0}^{\infty} a_{\nu}x^{\nu}$ be a power series such that $a_0 = 1$. It is known that for the coefficients b_{ν} of the reciprocal power series $\sum_{\nu=0}^{\infty} b_{\nu}x^{\nu}$ one has $b_0 = 1$ and

$$b_{\nu} = \sum_{\substack{t_1,\ldots,t_{\nu}\geqslant 0\\ t_1+2t_2+\cdots+\nu t_{\nu}=\nu}} (-1)^{t_1+\cdots+t_{\nu}}\binom{t_1+\cdots+t_{\nu}}{t_1,\ldots,t_{\nu}}a_1^{t_1}\cdots a_{\nu}^{t_{\nu}} \quad (\nu \geqslant 1), \tag{4.6}$$

where $\binom{t_1+\cdots+t_\nu}{t_1,\ldots,t_\nu}$ are the multinomial coefficients. Now fix $\nu \geqslant 1$. By grouping the terms in (4.6) according to the values $k = t_1 + \cdots + t_\nu$, where $1 \leqslant k \leqslant \nu$, we have

$$b_\nu = \sum_{k=1}^{\nu} (-1)^k \sum_{\substack{t_1,\ldots,t_\nu \geqslant 0 \\ t_1+2t_2+\cdots+\nu t_\nu = \nu \\ t_1+\cdots+t_\nu = k}} \binom{t_1+\cdots+t_\nu}{t_1,\ldots,t_\nu} a_1^{t_1}\cdots a_\nu^{t_\nu}. \tag{4.7}$$

Observe that

$$\sum_{\substack{t_1,\ldots,t_\nu \geqslant 0 \\ t_1+2t_2+\cdots+\nu t_\nu = \nu \\ t_1+\cdots+t_\nu = k}} \binom{t_1+\cdots+t_\nu}{t_1,\ldots,t_\nu} a_1^{t_1}\cdots a_\nu^{t_\nu} = \sum_{\substack{j_1,\ldots,j_k \geqslant 1 \\ j_1+\cdots+j_k = \nu}} a_{j_1}\cdots a_{j_k}, \tag{4.8}$$

which follows by starting with its right-hand side and denoting by $t_1, \ldots, t_\nu$ the number of values $j_1, \ldots, j_k$ which are equal to $1, \ldots, \nu$, respectively.

Now, identities (4.7) and (4.8) give

$$b_\nu = \sum_{k=1}^{\nu} (-1)^k \sum_{\substack{j_1,\ldots,j_k \geqslant 1 \\ j_1+\cdots+j_k = \nu}} a_{j_1}\cdots a_{j_k} \quad (\nu \geqslant 1). \tag{4.9}$$

By identity (4.9) and the assumption $|a_\nu| \leqslant Aq^\nu$ $(\nu \geqslant 1)$ we have

$$|b_\nu| \leqslant \sum_{k=1}^{\nu} A^k \sum_{\substack{j_1,\ldots,j_k \geqslant 1 \\ j_1+\ldots+j_k = \nu}} q^{j_1+\cdots+j_k} = q^\nu \sum_{k=1}^{\nu} A^k \sum_{\substack{j_1,\ldots,j_k \geqslant 1 \\ j_1+\ldots+j_k = \nu}} 1$$

$$= q^\nu \sum_{k=1}^{\nu} A^k \binom{\nu-1}{k-1} = Aq^\nu (A+1)^{\nu-1},$$

as asserted. □

Proof of Corollary 2.2.6. This follows from Theorem 2.2.1 and Proposition 2.2.5. Note that the radius of convergence of the series $S_{1/f}(x)$ is > 1 by condition (ii). □

Proof of Theorem 2.2.7. Sitaramaiah and Subbarao [82, Th. 3.2] established that

$$\sum_{n \leqslant x} \frac{1}{\sigma^{**}(n)} = A^{**} \log x + B^{**} + O(x^{-1}(\log x)^{14/3}(\log\log x)^{4/3}),$$

where A^{**}, B^{**} are certain explicit constants. Here the sequence $\left(\frac{1}{\sigma^{**}(2^\nu)}\right)_{\nu \geqslant 0}$ is not log-convex, therefore the result of Kaluza can not be used. But it is easy to check that

$$\frac{1}{\sigma^{**}(2^\nu)} \leqslant \frac{4}{5} \cdot \frac{1}{2^\nu} \quad (\nu \geqslant 1),$$

hence Corollary 2.2.6 can be applied with $A = 4/5$, $q = 1/2$, where $M = q(A+1) = 9/10 < 1$. □

4.3 Proofs of the results of Section 2.3

Proof of Theorem 2.3.1. An arbitrary $n = \prod p^{\nu_p}$ we write as $n = n_1 n_2$ with $n_1 := \prod_{p \leqslant \log n} p^{\nu_p}$. The Mertens formula $\prod_{p \leqslant x}(1-1/p)^{-1} \sim e^{\gamma} \log x$ and the definition of $\varrho(p)$ imply

$$f(n_1) = \prod_{p \leqslant \log n} f(p^{\nu_p}) \leqslant \prod_{p \leqslant \log n} \varrho(p) = \prod_{p \leqslant \log n} \left(1 - \frac{1}{p}\right)^{-1} \cdot \prod_{p \leqslant \log n} \left(1 - \frac{1}{p}\right) \varrho(p),$$

$$f(n_1) \leqslant (1 + o(1))\, e^{\gamma} R \log\log n \qquad \text{as } n \to \infty\,. \tag{4.10}$$

Let a denote the number of prime divisors in n_2. Then $a \leqslant \log n / \log\log n$. There is nothing to prove if $R = \infty$, so let $R < \infty$. Using the unconditional convergence,

$$\begin{aligned} f(n_2) &\leqslant \prod_{p|n,\, p>\log n} \left(1 - \frac{1}{p}\right) \varrho(p) \cdot \prod_{p|n,\, p>\log n} \left(1 - \frac{1}{p}\right)^{-1} \\ &\leqslant (1 + o(1)) \cdot \left(1 - \frac{1}{\log n}\right)^{-a}, \end{aligned}$$

that is,

$$f(n_2) \leqslant (1 + o(1))\; e^{O(1/\log\log n)} \;\to\; 1\,. \tag{4.11}$$

Combining (4.10) and (4.11) finishes the proof. □

Proof of Theorem 2.3.2. There is no change in the estimation of $f(n_1)$. For n_2 we have

$$f(n_2) \leqslant \left(1 + o\left(\frac{\log\log n}{\log n}\right)\right)^{\frac{\log n}{\log\log n}} = 1 + o(1)\,.$$

□

Proof of Theorem 2.3.3. We treat the case of proper convergence only. There is nothing to prove if $R = 0$ and the changes for $R = \infty$ are obvious. For given ε take P so large that

$$\prod_{p>P} f(p^{e_p}) \varrho(p)^{-1} \geqslant 1 - \varepsilon \tag{4.12}$$

and choose exponents k_p for the $p \leqslant P$ such that

$$\prod_{p \leqslant P} f(p^{k_p}) \geqslant (1 - \varepsilon) \prod_{p \leqslant P} \varrho(p)\,. \tag{4.13}$$

Keeping P and the k_p fixed let x tend to infinity and consider

$$n(x) \;:=\; \prod_{p \leqslant P} p^{k_p} \prod_{P<p \leqslant x} p^{e_p}\,.$$

Now on the one hand, using (4.12) and (4.13) we see

$$f(n(x))\prod_{p\leqslant x}\left(1-\frac{1}{p}\right)\geqslant(1-\varepsilon)\prod_{p\leqslant x}\left(1-\frac{1}{p}\right)\varrho(p)\cdot\prod_{P<p\leqslant x}f(p^{e_p})\varrho(p)^{-1}$$
$$\geqslant(1-\varepsilon)^2(1+o(1))R$$

and with Mertens' formula again

$$f(n(x))\geqslant(1-\varepsilon)^2(1+o(1))R\,e^{\gamma}\log x\,. \tag{4.14}$$

On the other hand, since $e_p=p^{o(1)}$, we have

$$\log n(x)\leqslant\sum_{p\leqslant P}k_p\log p\ +\sum_{P<p\leqslant x}e_p\log p\leqslant x^{o(1)}\sum_{p\leqslant x}\log p=x^{1+o(1)}\,,$$

and therefore

$$\log\log n(x)\leqslant(1+o(1))\log x\,.$$

Together with (4.14) this yields the lower bound

$$\limsup_{x\to\infty}\frac{f(n(x))}{\log\log n(x)}\geqslant(1-\varepsilon)^2R\,e^{\gamma}$$

with arbitrary $\varepsilon>0$. □

Proof of Corollary 2.3.4. Apply Theorems 2.3.1 (or 2.3.2) and 2.3.3. □

4.4 Proofs of the results of Section 2.4

4.4.1 Proofs for Section 2.4.1

Proof of Theorem 2.4.1. Apply Theorem 2.1.1 by selecting $f(n)=\phi^{(e)}(n)^r$ with $\ell=3$ and $k=2^r$. If $r\geqslant 2$, then use the error term given by Remark 2.1.2. In the case $r=1$, use that

$$\sum_{n\leqslant x}\tau(1,3;n)=\zeta(3)x+\zeta(1/3)x^{1/3}+O(x^{1/5}),$$

cf. Krätzel [56, pp. 196–199]. □

4.4.2 Proofs for Section 2.4.2

Proof of Theorem 2.4.2. Use the convolution method by taking $|\mu^{(e)}(n)| = (q_4 * w)(n)$, in terms of the Dirichlet convolution $*$, where q_4 denotes the characteristic function of the 4-free integers. Apply the estimate

$$\sum_{n\leqslant x} q_4(n) = \frac{x}{\zeta(4)} + O\left(x^{1/5+\varepsilon}\right),$$

valid under RH, according to the result of Montgomery and Vaughan [66]. □

Note that in our paper [97] we formulated a general theorem, namely [97, Th. 6], which is a variant of Theorem 2.1.1 and which applies directly to the function $|\mu^{(e)}(n)|$.

Proof of Theorem 2.4.3. (i) Let $\mu_2(n) = \mu(m)$ or 0, according as $n = m^2$ or not, and let $E_2(n) = 1$ or 0, according as $n = m^2$ or not. The given identity is verified for $\mu^{(e)} = \mu^2 * \mu_2 * u$, equivalent to $u = \mu^{(e)} * \lambda * E_2$, in terms of the Dirichlet convolution, where λ is the Liouville function. It is easy to check that $u(p) = u(p^2) = u(p^3) = u(p^4) = 0$, $|(\lambda * E_2)(p^a)| \leqslant a$ for every prime power p^a with $a \geqslant 1$, hence $|u(p^b)| \leqslant 1 + \sum_{a=1}^{b} |(\lambda * E_2)(p^a)| < b^2$ for every prime power p^b with $b \geqslant 5$. We obtain that the Dirichlet series of the function u is absolutely convergent for $\Re s > 1/5$.

(ii) According to (i), $\sum_{n\leqslant x} \mu^{(e)}(n) = \sum_{n\leqslant x} u(n) S(x/n)$, where

$$S(x) := \sum_{nd^2\leqslant x} \mu^2(n)\mu(d).$$

First we estimate the sum $S(x)$. Let $\varrho = \varrho(x)$ such that $0 < \varrho < 1$ to be defined later. If $nd^2 \leqslant x$, then both $n > \varrho^{-2}$ and $d > \varrho\sqrt{x}$ can not hold good in the same time, therefore

$$S(x) = \sum_{\substack{nd^2\leqslant x\\ d\leqslant \varrho\sqrt{x}}} \mu^2(n)\mu(d) + \sum_{\substack{nd^2\leqslant x\\ n\leqslant \varrho^{-2}}} \mu^2(n)\mu(d) - \sum_{\substack{d\leqslant \varrho\sqrt{x}\\ n\leqslant \varrho^{-2}}} \mu^2(n)\mu(d) = S_1(x) + S_2(x) - S_3(x),$$

say. We use the following estimates of A. Walfisz [120],

$$M(x) := \sum_{n\leqslant x} \mu(n) = O(x\delta(x)),$$

$$E(x) := \sum_{n\leqslant x} \mu^2(n) = \frac{x}{\zeta(2)} + O(x^{1/2}\delta(x)).$$

Note that $\delta(x)$, defined by (1.3), is decreasing and $x^{\varepsilon}\delta(x)$ is increasing for every $\varepsilon > 0$. By partial summation,

$$R(x) := \sum_{n>x} \frac{\mu(n)}{n^2} = O(x^{-1}\delta(x)).$$

Here

$$S_1(x) = \sum_{d\leqslant \varrho\sqrt{x}} \mu(d)E(x/d^2) = \frac{x}{\zeta(2)} \sum_{d\leqslant \varrho\sqrt{x}} \frac{\mu(d)}{d^2} + O\left(x^{1/2} \sum_{d\leqslant \varrho\sqrt{x}} \frac{\delta(x/d^2)}{d}\right)$$

$$= \frac{x}{\zeta(2)}\left(\frac{1}{\zeta(2)} - R(\varrho\sqrt{x})\right) + O\left(x^{1/2}\delta(\varrho^{-2}) \sum_{d\leqslant\sqrt{x}} \frac{1}{d}\right)$$

$$= \frac{x}{\zeta^2(2)} + O\left(\varrho^{-1}x^{1/2}\delta(\varrho\sqrt{x})\right) + O\left(x^{1/2}\delta(\varrho^{-2})\log x\right),$$

$$S_2(x) = \sum_{n\leqslant \varrho^{-2}} \mu^2(n)M((x/n)^{1/2}) = O\left(\sum_{n\leqslant\varrho^{-2}} (x/n)^{1/2}\delta((x/n)^{1/2})\right)$$

$$= O\left(\delta(\varrho\sqrt{x})x^{1/2} \sum_{n\leqslant\varrho^{-2}} \frac{1}{\sqrt{n}}\right) = O\left(\varrho^{-1}x^{1/2}\delta(\varrho\sqrt{x})\right),$$

$$S_3(x) = M(\varrho\sqrt{x})E(\varrho^{-2}) = O\left(\varrho^{-1}x^{1/2}\delta(\varrho\sqrt{x})\right).$$

We obtain that

$$S(x) = \frac{x}{\zeta^2(2)} + O\left(\varrho^{-1}x^{1/2}\delta(\varrho\sqrt{x})\right) + O\left(x^{1/2}\delta(1/\varrho^2)\log x\right).$$

Take $\varrho = \exp(-(\log x)^\beta)$, where $0 < \beta < 1$. Then $\varrho\sqrt{x} = \exp(\frac{1}{2}(\log x) - (\log x)^\beta) \geqslant \exp(\frac{1}{4}(\log x)) = x^{1/4}$ for sufficiently large x. Hence $\delta(\varrho\sqrt{x}) \leqslant \delta(x^{1/4}) \ll \delta_B(x)$ with a suitable constant $B > 0$.

For $\beta < 3/5$ we obtain $\varrho^{-1}\delta(\varrho\sqrt{x}) \ll \exp((\log x)^\beta - B(\log x)^{3/5}(\log\log x)^{-1/5}) \ll \delta_C(x)$ with a suitable constant $C > 0$.

If $\eta < 3/5$, then $\delta_A(x) \ll \exp(-A(\log x)^\eta)$ and obtain that $\delta(\varrho^{-2}) \ll \exp(-A(2(\log x)^\beta)^\eta) = \exp(-D(\log x)^{\beta\eta})$ with a suitable $D > 0$, where $\beta\eta < 9/25$.

Therefore,

$$S(x) = \frac{x}{\zeta^2(2)} + O\left(x^{1/2}\exp(-c(\log x)^\Delta\right),$$

where $\Delta < 9/25$ and $c > 0$ are constants. Now,

$$\sum_{n\leqslant x} \mu^{(e)}(n) = \sum_{n\leqslant x} u(n)S(x/n)$$

$$= \sum_{n\leqslant x} u(n)\left(\frac{x}{\zeta^2(2)n} + O\left((x/n)^{1/2}\exp(-c(\log(x/n))^\Delta)\right)\right)$$

$$= \frac{x}{\zeta^2(2)} \sum_{n\leqslant x} \frac{u(n)}{n} + O\left(x^{1/2} \sum_{n\leqslant x} \frac{|u(n)|}{n^{1/2}} \exp(-c(\log(x/n))^{\Delta}) \right),$$

where, using that $x^{\varepsilon}\exp(-c(\log x)^{\Delta})$ is increasing for any $\varepsilon > 0$, the O-term is

$$O\left(x^{1/2} \sum_{n\leqslant x} \frac{|u(n)|}{n^{1/2}} (\frac{x}{n})^{-\varepsilon} (\frac{x}{n})^{\varepsilon} \exp(-c(\log(x/n)^{\Delta})) \right)$$

$$= O\left(x^{1/2} x^{\varepsilon} \exp(-c(\log x)^{\Delta}) x^{-\varepsilon} \sum_{d\leqslant x} \frac{|u(n)|}{n^{1/2-\varepsilon}} \right)$$

$$= O\left(x^{1/2}\exp(-c(\log x)^{\Delta}) \right),$$

for $1/2 - \varepsilon > 1/5$. Furthermore,

$$\sum_{n\leqslant x} \frac{u(n)}{n} = U(1) + O\left(\sum_{n>x} \frac{|u(n)|}{n} \right),$$

with $U(1) = \zeta^{-2}(2)m(\mu^{(e)})$ and $\sum_{n>x} \frac{u(n)}{n} = O\left(x^{-3/5} \sum_{n>x} \frac{|u(n)|}{n^{2/5}} \right) = O(x^{-3/5})$, which finishes the proof of (ii).

(iii) Assume RH. We use that, see [94],

$$M(x) := \sum_{n\leqslant x} \mu(n) = O\left(x^{1/2}\omega(x) \right),$$

where $\omega(x) := \exp(A(\log x)(\log\log x)^{-1})$, A being a positive constant, which gives by partial summation,

$$R(x) := \sum_{n>x} \frac{\mu(n)}{n^2} = O(x^{-3/2}\omega(x)).$$

Suppose that $D(x) := \sum_{n\leqslant x} \mu^2(n) - x/\zeta(2) = O(x^{r+\varepsilon})$ for every $\varepsilon > 0$, where $1/4 < r < 1/3$. Then we obtain by similar computations that

$$S_1(x) = \frac{x}{\zeta^2(2)} + O\left(\varrho^{-3/2} x^{1/4} \omega(\varrho\sqrt{x}) \right) + O\left(x^{1/2} \varrho^{1-2(r+\varepsilon)} \right),$$

$$S_2(x) = O\left(\varrho^{-3/2} x^{1/4} \omega(\varrho\sqrt{x}) \right), \qquad S_3(x) = O\left(\varrho^{-3/2} x^{1/4} \omega(\varrho\sqrt{x}) \right),$$

Therefore

$$S(x) = \frac{x}{\zeta^2(2)} + O\left(\varrho^{-3/2} x^{1/4} \omega(\varrho\sqrt{x}) \right) + O\left(x^{1/2} \varrho^{1-2(r+\varepsilon)} \right).$$

Choose $\varrho = x^{-t}$, $t > 0$. Then $\varrho^{-3/2}x^{1/4} = x^{(6t+1)/4}$, $\varrho\sqrt{x} = x^{1/2-t} < x$, hence $\omega(\varrho\sqrt{x}) < \omega(x) \ll x^{\varepsilon}$ for every $\varepsilon > 0$ and obtain

$$S(x) = \frac{x}{\zeta^2(2)} + O\left(x^{(6t+1)/4+\varepsilon} \right) + O\left(x^{1/2-t(1-2r)+\varepsilon} \right).$$

Take $(6t + 1)/4 = 1/2 - t(1 - 2r)$, this gives $t = 1/(10 - 8r)$ leading to the common value $(2 - r)/(5 - 4r) + \varepsilon$ of the exponents. $\square$

4.4.3 Proofs for Section 2.4.3

Proof of Theorem 2.4.4. The proof is similar to the proof of Theorem 2.1.1.

(i) To obtain the given identity, let $f = \mu_2 * \mu$, where $\mu_2(n) = \mu(m)$ or 0, according as $n = m^2$ or not, and let $v = t^{(e)} * f$. Here both f and v are multiplicative and it is easy to check that $f(p) = f(p^2) = -1, f(p^3) = 1, f(p^a) = 0$ for each $a \geqslant 4$, and $v(p) = v(p^2) = v(p^3) = 0$, $v(p^a) = 2^{\omega(a)} - 2^{\omega(a-1)} - 2^{\omega(a-2)} + 2^{\omega(a-3)}$ for $a \geqslant 4$.

(ii) According to (i), $t^{(e)} = v * \tau(1,2,\cdot)$, where $\tau(1,2,n) = \sum_{ab^2=n} 1$ for which

$$\sum_{n\leqslant x} \tau(1,2,n) = \zeta(2)x + \zeta(1/2)x^{1/2} + O(x^{1/4}),$$

cf. [56, p. 196–199]. Therefore,

$$\sum_{n\leqslant x} t^{(e)}(n) = \sum_{d\leqslant x} v(d) \sum_{e\leqslant x/d} \tau(1,2,e)$$

and we obtain the given result by usual estimates. □

4.5 Proofs of the results of Section 2.5

4.5.1 Proofs for Section 2.5.1

Proof of Theorem 2.5.1. i) First we show that

$$A^2(n) = \sum_{de=n} \tau^2(d)g(e), \tag{4.15}$$

where g is multiplicative and $g(p) = -4/p + 1/p^2$, $g(p^a) = 4(-1)^a/p$ for any prime p and $a \geqslant 2$.

By the multiplicativity of the involved functions it is enough to verify (4.15) for prime powers p^a ($a \geqslant 1$). We have

$$\sum_{de=p^a} \tau^2(d)g(e) = \sum_{j=1}^{a-1} \tau^2(p^{j-1})g(p^{a-j+1}) + \tau^2(p^{a-1})g(p) + \tau^2(p^a)$$

$$= \sum_{j=1}^{a-1} j^2(-1)^{a-j+1}\frac{4}{p} + a^2(-4/p + 1/p^2) + (a+1)^2$$

$$= (-1)^a\frac{4}{p}\sum_{j=1}^{a-1}(-1)^{j-1}j^2 + \frac{a^2}{p^2} - \frac{4a^2}{p} + (a+1)^2 = (a+1-a/p)^2 = A^2(p^a).$$

Here the Dirichlet series of g is given by

$$G(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s} = \prod_p \left(1 + \frac{1}{p^{s+2}} - \frac{4}{p(p^s+1)}\right),$$

which is absolutely convergent for $s \in \mathbb{C}$ with $\Re s > 0$. Therefore, for any $\varepsilon > 0$,

$$\sum_{n \leqslant x} g(n) = O(x^{\varepsilon}), \quad \sum_{n > x} \frac{g(n)}{n} = O(x^{-1+\varepsilon}).$$

We need the next formula of Ramanujan, already quoted as (2.1):

$$\sum_{n \leqslant x} \tau^2(n) = x(a \log^3 x + b \log^2 x + c \log x + d) + O(x^{1/2+\varepsilon}), \tag{4.16}$$

where $a = 1/\pi^2, b, c, d$ are constants.

By (4.15) and (4.16) we obtain

$$\sum_{n \leqslant x} A^2(n) = \sum_{d \leqslant x} g(d) \sum_{e \leqslant x/d} \tau^2(e)$$

$$= ax \sum_{d \leqslant x} \frac{g(d)}{d} \log^3(x/d) + bx \sum_{d \leqslant x} \frac{g(d)}{d} \log^2(x/d) + cx \sum_{d \leqslant x} \frac{g(d)}{d} \log(x/d) + dx \sum_{d \leqslant x} \frac{g(d)}{d}$$

$$+O\left(x^{1/2+\varepsilon} \sum_{d \leqslant x} \frac{|g(d)|}{d^{1/2+\varepsilon}}\right).$$

Now formula (2.22) follows by usual estimates with the constants

$$C_1 = aG(1), \quad C_2 = 3aG'(1) + bG(1), \quad C_3 = 3aG''(1) + 2bG'(1) + cG(1),$$

$$C_4 = aG'''(1) + bG''(1) + cG'(1) + dG(1),$$

where G', G'', G''' are the derivatives of G.

ii) Assume that $\alpha_4 < 1/2$. We use that in this case the error term for $\sum_{n \leqslant x} \tau^2(n)$ in (4.16) is $O(x^{1/2}\delta(x))$, as it was proved by Suryanarayana and Sitaramachandra Rao [88]. We obtain, applying that $x^{\varepsilon}\delta(x)$ is increasing, that the error term for $\sum_{n \leqslant x} A^2(n)$ is

$$\ll \sum_{d \leqslant x} |g(d)|(x/d)^{1/2}\delta(x/d) = \sum_{d \leqslant x} |g(d)|(x/d)^{1/2-\varepsilon}(x/d)^{\varepsilon}\delta(x/d)$$

$$\ll x^{1/2-\varepsilon}(x^{\varepsilon}\delta(x)) \sum_{d \leqslant x} \frac{|g(d)|}{d^{1/2-\varepsilon}} \ll x^{1/2}\delta(x).$$

iii) Assume RH. Then we apply that the error term of (4.16) is $O(x^{(2-\alpha_4)/(5-4\alpha_4)}\lambda(x))$, cf. [88, Lemma 2.4, Th. 3.2], where $\sum_{n \leqslant x} \mu(n) \ll x^{1/2}\lambda(x)$ according to the result of Soundararajan [86]. Using that $\lambda(x)$ is increasing, we obtain the given error term. $\square$

4.5.2 Proofs for Section 2.5.2

Proof of Theorem 2.5.4. The Dirichlet series of the function $P^{(e)}(n)$ is

$$\sum_{n=1}^{\infty} \frac{P^{(e)}(n)}{n^s} = \prod_p \left(1 + \sum_{a=1}^{\infty} \sum_{d|a} \frac{p^d \phi(a/d)}{p^{as}}\right)$$

$$= \prod_p \left(1 + \sum_{j=1}^{\infty} \phi(j) \sum_{d=1}^{\infty} \frac{1}{p^{d(js-1)}}\right) = \prod_p \left(1 + \sum_{j=1}^{\infty} \frac{\phi(j)}{p^{js-1}-1}\right)$$

$$= \frac{\zeta(s-1)\zeta(2s-1)}{\zeta(3s-2)} W(s) \quad (\Re s > 2),$$

where

$$W(s) := \prod_p \left(1 + \frac{(p^{s-1}-1)(p^{2s-1}-1)}{p^{3s-2}-1} \sum_{j=3}^{\infty} \frac{\phi(j)}{p^{js-1}-1}\right),$$

which is absolutely convergent for $\Re s > 3/4$.

Therefore, $P^{(e)} = h * w$, where

$$h(n) = \sum_{ab^2c^3=n} abc^2\mu(c),$$

and obtain the desired result by using the estimate

$$\sum_{mn^2 \leqslant x} mn = \frac{1}{2}\zeta(3)x^2 + O(x(\log x)^{2/3})$$

due to Pétermann and Wu [75, Th. 1]. □

Proof of Theorem 2.5.5. Apply Corollary 2.3.4 for the function $f(n) = P^{(e)}(n)/n$, where $f(p^a) \leqslant (p + p^2 + \cdots + p^a)p^{-a} < (1 - 1/p)^{-1}$ for every $a \geqslant 1$ and $f(p^2) = 1 + 1/p$, hence we can choose $e_p = 2$ for all p. Moreover, $\varrho(p) = 1 + 1/p$ for all p. □

4.5.3 Proofs for Section 2.5.3

Proof of Theorem 2.5.6. We need the following auxiliary results. Let $\omega(n,k)$ denote the number of distinct prime factors of n which do not divide k. Also, let $\tau^{(2)}(n,k)$ denote the number of unitary divisors of n which are relatively prime to k. We have $\tau^{(2)}(n,k) = 2^{\omega(n,k)}$.

Another representation of the function $\widetilde{P}$ is given by

$$\widetilde{P}(n) = \sum_{de=n} \mu(d)e \cdot 2^{\omega(e,d)} \quad (n \geqslant 1). \tag{4.17}$$

Indeed, by (2.26),

$$\widetilde{P}(n)=\sum_{\substack{de=n\\(d,e)=1}} d\phi(e)=\sum_{\substack{de=n\\(d,e)=1}} d\sum_{ab=e}\mu(a)b=\sum_{\substack{dab=n\\(d,a)=1\\(d,b)=1}} d\mu(a)b$$

$$=\sum_{ac=n}\mu(a)c\sum_{\substack{bd=c\\(b,d)=1\\(d,a)=1}} 1=\sum_{ac=n}\mu(a)c\,\tau^{(2)}(c,a).$$

Let $\sigma_s'(n)$ be the sum of s-th powers of the square-free divisors of n.

Lemma 4.5.1. *For every $\varepsilon>0$,*

$$\sum_{n\leqslant x}2^{\omega(n,k)}n=\frac{kx^2}{2\zeta(2)\psi(k)}\left(\log x+\alpha(k)-2\beta(k)+2\gamma-\frac{1}{2}-\frac{2\zeta'(2)}{\zeta(2)}\right)$$
$$+O\left(\sigma'_{-1+\varepsilon}(k)\sigma'_{-\theta}(k)x^{3/2}\delta(x)\right), \tag{4.18}$$

If RH is true, then $x^{3/2}\delta(x)$ in the error term of (4.18) can be replaced by $x^{(7-5\theta)/(5-4\theta)}\eta(x)$.

Lemma 4.5.1 follows by partial summation from the similar result for $\sum_{n\leqslant x}2^{\omega(n,k)}$, proved in [91, Th. 4.3, 5.2].

Now, by (4.17) and Lemma 4.5.1,

$$\sum_{n\leqslant x}\widetilde{P}(n)=\sum_{d\leqslant x}\mu(d)\sum_{e\leqslant x/d}2^{\omega(e,d)}e$$

$$=\frac{x^2}{2\zeta(2)}\left(\sum_{d\leqslant x}\frac{\mu(d)}{d\psi(d)}\left(\log x+2\gamma-\frac{1}{2}-\frac{2\zeta'(2)}{\zeta(2)}\right)-\sum_{d\leqslant x}\frac{\mu(d)(\log d-\alpha(d)+2\beta(d))}{d\psi(d)}\right)$$
$$+O\left(\sum_{d\leqslant x}\sigma'_{-1+\varepsilon}(d)\sigma'_{-\theta}(d)(x/d)^{3/2}\delta(x/d)\right).$$

For every $\varepsilon>0$ and x sufficiently large, $x^{\varepsilon}\delta(x)$ is increasing, therefore

$$(x/d)^{3/2}\delta(x/d)=(x/d)^{3/2-\varepsilon}(x/d)^{\varepsilon}\delta(x/d)\leqslant(x/d)^{3/2-\varepsilon}x^{\varepsilon}\delta(x)=x^{3/2}\delta(x)/d^{3/2-\varepsilon}.$$

Furthermore, it is enough to use the inequalities $\sigma'_{-1+\varepsilon}(d)\leqslant\tau(d)$ (for $\varepsilon<1$) and $\sigma'_{-\theta}(d)\leqslant\tau(d)$ and then obtain the given formula by using $\alpha(n)=O(\log n)$, $\beta(n)=O(1)$ and the well known estimates

$$\sum_{d>x}\frac{1}{d^2}\ll\frac{1}{x},\quad \sum_{d>x}\frac{\log d}{d^2}\ll\frac{\log x}{x}.$$

Assume RH. Then in the error term use the property that $\eta(x)$ is increasing, so $\eta(x/d)\leqslant\eta(x)$ for $d\geqslant 1$. □

Proof of Theorem 2.5.7. Note that for every $n \geqslant 1$ we have $\widetilde{P}(n) \leqslant P(n)$, with equality iff n is square-free, and $2^{\omega(n)}\phi(n) \leqslant \widetilde{P}(n) \leqslant 2^{\omega(n)}n$, with equality iff $n = 1$.

From (2.27) we have $\widetilde{P}(n) \geqslant n(3/2)^{\omega(n)} \geqslant 3n/2$ for every $n \geqslant 1$, with equality for $n = 2^{\nu}$ ($\nu \geqslant 1$), giving the minimal order of $\widetilde{P}(n)$.

For the maximal order take into account (2.21), where the limsup is attained for a sequence of square-free integers (more exactly for $n_k = \prod_{k/\log^2 k<p\leqslant k} p$, $k \to \infty$), see [20, Theorem 4.1], and use $\widetilde{P}(n) \leqslant P(n)$ for every $n \geqslant 1$, with equality iff n is square-free. □

4.6 Proofs of the results of Section 2.6

Simple proof of identity (2.30). By using the formula

$$c_k(j) = \sum_{d\mid \gcd(k,j)} d\,\mu(k/d) \quad (k, j \in \mathbb{N}) \tag{4.19}$$

we obtain that

$$S_r(k) = \frac{1}{k^{r+1}} \sum_{j=1}^{k} j^r \sum_{d\mid \gcd(k,j)} d\,\mu(k/d) = \frac{1}{k^{r+1}} \sum_{d\mid k} d^{r+1}\mu(k/d) \sum_{m=1}^{k/d} m^r$$

$$= \sum_{d\mid k} \frac{\mu(d)}{d^{r+1}} \sum_{m=1}^{d} m^r.$$

It is well known that for every $n, r \in \mathbb{N}$,

$$\sum_{j=1}^{n} j^r = \frac{1}{r+1} \sum_{m=0}^{r} (-1)^m \binom{r+1}{m} B_m n^{r+1-m}$$

$$= \frac{n^r}{2} + \frac{1}{r+1} \sum_{m=0}^{\lfloor r/2\rfloor} \binom{r+1}{2m} B_{2m} n^{r+1-2m}.$$

We deduce

$$S_r(k) = \sum_{d\mid k} \frac{\mu(d)}{d^{r+1}} \left(\frac{d^r}{2} + \frac{1}{r+1} \sum_{m=0}^{\lfloor r/2\rfloor} \binom{r+1}{2m} B_{2m} d^{r+1-2m} \right)$$

$$= \frac{1}{2} \sum_{d\mid k} \frac{\mu(d)}{d} + \frac{1}{r+1} \sum_{m=0}^{\lfloor r/2\rfloor} \binom{r+1}{2m} B_{2m} \sum_{d\mid k} \frac{\mu(d)}{d^{2m}},$$

giving (2.30) by using the elementary convolutional identities on $\phi(k)$ and $\phi_{2m}(k)$. □

Proof of Proposition 2.6.1. Using identity (4.19) we have

$$\frac{1}{k}\sum_{j=1}^{k}(\log j)c_k(j) = \frac{1}{k}\sum_{j=1}^{k}(\log j)\sum_{d\mid\gcd(k,j)} d\mu(k/d)$$

$$= \sum_{d\mid k}(d/k)\mu(k/d)\sum_{m=1}^{k/d}\log(dm) = \sum_{d\mid k}\frac{\mu(d)}{d}\sum_{m=1}^{d}\log(mk/d)$$

$$= \sum_{d\mid k}\mu(d)\log(k/d) + \sum_{d\mid k}\frac{\mu(d)}{d}\log(d!),$$

where the first sum is $\mu * \log = \Lambda$, which completes the proof of (2.31).

Next apply that

$$\prod_{k=1}^{n}\Gamma(k/n) = \frac{(2\pi)^{(n-1)/2}}{\sqrt{n}} \qquad (n \in \mathbb{N}),$$

which is a consequence of the Gauss multiplication formula. We deduce

$$\sum_{j=1}^{k}(\log\Gamma(j/k))c_k(j) = \sum_{j=1}^{k}(\log\Gamma(j/k))\sum_{d\mid\gcd(k,j)} d\mu(k/d)$$

$$= \sum_{d\mid k} d\mu(k/d)\sum_{m=1}^{k/d}\log\Gamma(md/k) = \sum_{d\mid k}(k/d)\mu(d)\sum_{m=1}^{d}\log\Gamma(m/d)$$

$$= \sum_{d\mid k}(k/d)\mu(d)\log\frac{(2\pi)^{(d-1)/2}}{\sqrt{d}}$$

$$= \log(2\pi)\sum_{d\mid k}\frac{k}{d}\mu(d)\frac{d-1}{2} - \frac{1}{2}\sum_{d\mid k}\frac{k}{d}\mu(d)\log d$$

$$= \log(2\pi)\left(\frac{k}{2}\sum_{d\mid k}\mu(d) - \frac{1}{2}\sum_{d\mid k}\frac{k}{d}\mu(d)\right) - \frac{k}{2}\sum_{d\mid k}\frac{\mu(d)}{d}\log d,$$

where the first sum is zero for $k > 1$ and the second sum is $\phi(k)$. Now the use of the known identity

$$\sum_{d\mid k}\frac{\mu(d)}{d}\log d = -\frac{\phi(k)}{k}\sum_{p\mid k}\frac{\log p}{p-1},$$

completes the proof of (2.32).

Finally, we need another known formula, namely

$$\sum_{k=0}^{\lfloor n/r\rfloor} \binom{n}{kr} = \frac{2^n}{r} \sum_{j=1}^{r} \cos^n(j\pi/r)\cos(nj\pi/r) \qquad (n,r \in \mathbb{N}),$$

cf., e.g., Comtet [24, p. 84]. We conclude

$$\sum_{j=0}^{k} \binom{k}{j} c_k(j) = \sum_{j=0}^{k} \binom{k}{j} \sum_{d\mid \gcd(k,j)} d\mu(k/d) = \sum_{d\mid k} d\mu(k/d) \sum_{m=0}^{k/d} \binom{k}{dm}$$

$$= 2^k \sum_{d\mid k} \mu(k/d) \sum_{\ell=1}^{d} \cos^k(\ell\pi/d)\cos(k\ell\pi/d),$$

where the last factor is $(-1)^{\ell k/d}$. This gives (2.33). □

4.7 Proofs of the results of Section 2.7

Sketch of proof of formulas (2.35) and (2.36). Let $e(x) = \exp(2\pi i x)$ and consider the Gauss quadratic sum defined by

$$S(\ell,r) = \sum_{j=1}^{r} e(\ell j^2/r) \quad (\ell, r \in \mathbb{N}, \gcd(\ell,r)=1).$$

For every $k, r \in \mathbb{N}$ and $n \in \mathbb{Z}$,

$$N_k(n,r) = r^{k-1} \sum_{d\mid r} \frac{1}{d^k} \sum_{\substack{\ell=1 \\ (\ell,d)=1}}^{d} e(-\ell n/d) S(\ell,d)\cdots S(\ell,d).$$

Now let r be odd. Using that for $\ell \in \mathbb{N}$ such that $\gcd(\ell,r)=1$,

$$S(\ell,r) = \begin{cases} \left(\frac{\ell}{r}\right)\sqrt{r}, & \text{if } \ r \equiv 1 \pmod 4; \\ i\left(\frac{\ell}{r}\right)\sqrt{r}, & \text{if } \ r \equiv -1 \pmod 4), \end{cases}$$

we deduce

$$N_k(n,r) = r^{k-1} \sum_{d\mid r} \frac{i^{k(d-1)^2/4}}{d^{k/2}} \sum_{\substack{\ell=1 \\ (\ell,d)=1}}^{d} \left(\frac{\ell}{d}\right)^k e(-\ell n/d).$$

Assume in what follows that also k is odd and for $\gcd(n,r)=1$ consider the sum

$$T(n,r) := \sum_{\substack{j=1 \\ \gcd(j,r)=1}}^{r} \left(\frac{j}{r}\right) e(jn/r).$$

For r odd the Jacobi symbol $j \mapsto \left(\frac{j}{r}\right)$ is a real character (mod r) and $T(n,r) = \left(\frac{n}{r}\right) T(1,r)$ holds if $\gcd(n,r)=1$. See, e.g., [45, Ch. 7].

i) If r is squarefree, then $j \mapsto \left(\frac{j}{r}\right)$ is a primitive character (mod r). Thus, $T(1,r)=\sqrt{r}$ for $\left(\frac{-1}{r}\right)=1$ and $T(1,r)=i\sqrt{r}$ for $\left(\frac{-1}{r}\right)=-1$ ([45, Th. 7.5.8]), that is

$$T(n,r) = \begin{cases} \left(\frac{n}{r}\right)\sqrt{r}, & \text{if } \ r\equiv 1 \pmod 4; \\ i\left(\frac{n}{r}\right)\sqrt{r}, & \text{if } \ r \equiv -1 \pmod 4. \end{cases} \tag{4.20}$$

ii) We show that if r is not squarefree, then $T(1,r)=0$. Here r can be written as $r=p^2s$, where p is a prime and by putting $j=ks+q$,

$$T(1,r) = \sum_{q=1}^{s} \sum_{k=0}^{p^2-1} \left(\frac{ks+q}{r}\right) e((ks+q)/r)$$

$$= \sum_{q=1}^{s} \left(\frac{q}{s}\right) e(q/(p^2s)) \sum_{k=0}^{p^2-1} e(k/p^2) = 0,$$

since the inner sum is zero.

We deduce that for $k=2m+1$ ($m \geqslant 0$), $r\in \mathbb{N}$ is odd, $n\in\mathbb{Z}$ such that $\gcd(n,r)=1$,

$$N_{2m+1}(n,r) = r^{2m} \sum_{d\mid r} \frac{\mu^2(d)}{d^m}\left(\frac{(-1)^m n}{d}\right). \tag{4.21}$$

Now if $k=4m+1$, then (4.21) gives (2.35).

Furthermore, we need to evaluate

$$V(r) := T(0,r) = \sum_{\substack{j=1 \\ \gcd(j,r)=1}}^{r} \left(\frac{j}{r}\right).$$

If $r=t^2$ is a square, then $\left(\frac{j}{r}\right)=\left(\frac{j}{t^2}\right)=1$ for every j with $\gcd(j,r)=1$ and deduce that $V(r)=\phi(r)$.

Now assume that r is not a square. Then, since r is odd, there is a prime $p>2$ such that $r=p^\nu s$, where ν is odd and $\gcd(p,s)=1$. Let c be a quadratic nonresidue (mod p) and consider the simultaneous congruences $x\equiv c \pmod p$, $x \equiv 1 \pmod s$. By the Chinese remainder theorem there exists a solution $x=j_0$ satisfying

$$\left(\frac{j_0}{r}\right) = -1,$$

Hence

$$V(r) = \sum_{\substack{j=1 \\ \gcd(j,r)=1}}^{r} \left(\frac{jj_0}{r}\right) = - \sum_{\substack{j=1 \\ \gcd(j,r)=1}}^{r} \left(\frac{j}{r}\right) = -V(r),$$

giving that $V(r) = 0$.

Therefore, if $r \in \mathbb{N}$ is odd, then

$$V(r) = \begin{cases} \phi(r), & \text{if } r \text{ is a square;} \\ 0, & \text{otherwise,} \end{cases}$$

which leads to (2.36). □

Proof of Theorem 2.7.1. Consider the congruence $x^2 \equiv 0 \pmod r$. It follows from general results of our paper [110] and can be deduced also directly that for the number $N_1(0, r)$ of its solutions one has $N_1(0, p^\nu) = p^{\lfloor \nu/2 \rfloor}$ for every prime power p^ν ($\nu \geqslant 1$). This leads to the Dirichlet series representation

$$\sum_{r=1}^{\infty} \frac{N_1(0,r)}{r^s} = \frac{\zeta(2s-1)\zeta(s)}{\zeta(2s)}. \tag{4.22}$$

By the identity (4.22) we infer that for every $r \in \mathbb{N}$,

$$N_1(0,r) = \sum_{a^2 b^2 c = r} \mu(a) b.$$

Now using Dirichlet's hyperbola method we have

$$E(x) := \sum_{b^2 c \leqslant x} b = \sum_{b \leqslant x^{1/3}} b \sum_{c \leqslant x/b^2} 1 + \sum_{c \leqslant x^{1/3}} \sum_{b \leqslant (x/c)^{1/2}} b - \sum_{b \leqslant x^{1/3}} b \sum_{c \leqslant x^{1/3}} 1,$$

which gives by the trivial estimate (i.e., $|x - \lfloor x \rfloor| < 1$),

$$E(x) = \frac{1}{2} x \log x + \frac{1}{2}(3\gamma - 1)x + O(x^{2/3}).$$

Now,

$$\sum_{r \leqslant x} N_1(0,r) = \sum_{a \leqslant x^{1/2}} \mu(a) E(x/a^2)$$

and elementary computations complete the proof. □

Proof of Theorem 2.7.2. It is known, that for the number $N_1(1, r)$ of solutions of the congruence $x^2 \equiv 1 \pmod r$ one has $N_1(1, p^\nu) = 2$ for every prime $p > 2$ and every $\nu \geqslant 1$, $N_1(1, 2) = 1$, $N_1(1, 4) = 2$, $N_1(1, 2^\nu) = 4$ for every $\nu \geqslant 3$ (sequence A060594 in [84]). The Dirichlet series representation

$$\sum_{r=1}^{\infty} \frac{N_1(1,r)}{r^s} = \frac{\zeta^2(s)}{\zeta(2s)} \left(1 - \frac{1}{2^s} + \frac{2}{2^{2s}}\right) \tag{4.23}$$

shows that estimating the sum $\sum_{r\leqslant x} N_1(1,r)$ is closely related to the squarefree divisor problem concerning the function $\tau^{(2)}(n) = 2^{\omega(n)}$. Here

$$\sum_{n=1}^{\infty} \frac{\tau^{(2)}(n)}{n^s} = \frac{\zeta^2(s)}{\zeta(2s)}. \tag{4.24}$$

By the identities (4.23) and (4.24) it follows that for every $r \in \mathbb{N}$,

$$N_1(1,r) = \sum_{ab=r} \tau^{(2)}(a)h(b),$$

where the multiplicative function h is defined by

$$h(p^\nu) = \begin{cases} -1, & p=2, \nu=1; \\ 2, & p=2,\nu=2; \\ 0, & \text{otherwise.} \end{cases}$$

Now the convolution method and the result (1.2) for $\sum_{n\leqslant x} \tau^{(2)}(n)$ conclude the proof. If RH is true, then the estimate $R(x) \ll x^{4/11+\varepsilon}$ due to Baker [6] can be used. □

Proof of Theorem 2.7.3. Here $N_2(0,r)$ is the number of solutions of the congruence $x^2 + y^2 \equiv 0 \pmod r$. It is the sequence A086933 in [84]), and for r odd it is given by identity [110, Eq. (21)]

$$N_2(0,r) = r\sum_{d\mid r} (-1)^{(d-1)/2} \frac{\phi(d)}{d}.$$

Furthermore, $N_2(0,2^\nu) = 2^\nu$ $(\nu \geqslant 1)$, cf. [110, Prop. 25]. We deduce that for every prime power p^ν $(\nu \geqslant 1)$,

$$N_2(0,p^\nu) = \begin{cases} p^\nu(\nu+1-\nu/p), & p\equiv 1 \pmod 4, \nu \geqslant 1; \\ p^\nu, & p \equiv -1 \pmod 4, \nu \text{ even}; \\ p^{\nu-1}, & p\equiv -1 \pmod 4, \nu \text{ odd}; \\ 2^\nu, & p=2, \nu\geqslant 1. \end{cases}$$

We obtain the Dirichlet series representation

$$\sum_{r=1}^{\infty} \frac{N_2(0,r)}{r^s} = \zeta(s-1) \prod_{p>2} \left(1 - \frac{(-1)^{(p-1)/2}}{p^s}\right) \left(1 - \frac{(-1)^{(p-1)/2}}{p^{s-1}}\right)^{-1}$$

$$= \zeta(s-1)L(s-1,\chi)L(s,\chi)^{-1},$$

where $L(s,\chi)$ is the Dirichlet series of χ, the nonprincipal character (mod 4). Therefore,

$$N_2(0,r) = (\mathrm{id} \cdot (\mathbf{1} * \chi) * \mu\chi)(r) \quad (r \in \mathbb{N}),$$

where $\mathbf{1}(n) = 1$, $\mathrm{id}(n) = n$. Here $4(\mathbf{1} * \chi)(n) = r_2(n)$ is the number of ways n can be written as a sum of two squares. This shows that the sum $\sum_{r \leqslant x} N_2(0, r)$ is closely related to the Gauss circle problem. According to the asymptotic formula due to Huxley [46],

$$\sum_{n \leqslant x} r_2(n) = \pi x + O(x^a (\log x)^b), \tag{4.25}$$

where $a = 131/416 \doteq 0.314903$ and $b = 26947/8320$.

We deduce that

$$\sum_{r \leqslant x} N_2(0, r) = \frac{1}{4} \sum_{d \leqslant x} \mu(d) \chi(d) \sum_{n \leqslant x/d} n r_2(n).$$

Now partial summation on (4.25) and usual estimates give the result. □

4.8 Proofs of the results of Section 2.8

Proof of Theorem 2.8.1. Let $o(x)$ denote the order of a group element x. Let $G = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$ in short. Let $x_i \in \mathbb{Z}_{n_i}$ such that $o(x_i) = n_i$ $(1 \leqslant i \leqslant k)$. Then $G \simeq \{x = (x_1^{i_1}, \ldots, x_k^{i_k}) : 1 \leqslant i_1 \leqslant n_1, \ldots, 1 \leqslant i_k \leqslant n_k\}$. Let $n = [n_1, \ldots, n_k]$.

Using elementary properties of the order of elements in a group and of the gcd and lcm of integers, respectively we deduce that for every $x = (x_1^{i_1}, \ldots, x_k^{i_k}) \in G$,

$$o(x) = \mathrm{lcm}(o(x_1^{i_1}), \ldots, o(x_k^{i_k})) = \mathrm{lcm}\left(\frac{o(x_1)}{\gcd(o(x_1), i_1)}, \ldots, \frac{o(x_k)}{\gcd(o(x_k), i_k)}\right)$$

$$= \mathrm{lcm}\left(\frac{n_1}{\gcd(n_1, i_1)}, \ldots, \frac{n_k}{\gcd(n_k, i_k)}\right) = \mathrm{lcm}\left(\frac{n}{\gcd(n, i_1 n/n_1)}, \ldots, \frac{n}{\gcd(n, i_k n/n_k)}\right)$$

$$= \frac{n}{\gcd(i_1 n/n_1, \ldots, i_k n/n_k, n)}.$$

Assume that $o(x) = \delta$, where $\delta \mid n$ is fixed. Then $\gcd(i_1 n/n_1, \ldots, i_k n/n_k, n) = n/\delta$.

Write $i_1 n/n_1 = j_1 n/\delta, \ldots, i_k n/n_k = j_k n/\delta$. Then $\gcd(j_1, \ldots, j_k, \delta) = 1$ and $j_1 = \delta i_1/n_1$, ..., $j_k = \delta i_k/n_k$ are integers, that is $\delta i_1 \equiv 0 \pmod{n_1}$, ..., $\delta i_k \equiv 0 \pmod{n_k}$. We obtain, as solutions of these linear congruences, that $i_1 = t_1 n_1/\gcd(\delta, n_1)$ with $1 \leqslant t_1 \leqslant \gcd(\delta, n_1)$, ..., $i_k = t_k n_k/\gcd(\delta, n_k)$ with $1 \leqslant t_k \leqslant \gcd(\delta, n_k)$.

Therefore, $o_\delta(n_1, \ldots, n_k)$, denoting the number of elements of order δ in G is the number of ordered k-tuples $(t_1, \ldots, t_k)$ satisfying the conditions

(i) $1 \leqslant t_1 \leqslant \gcd(\delta, n_1), \ldots, 1 \leqslant t_k \leqslant \gcd(\delta, n_k)$ and

(ii) $\gcd\left(t_1 \frac{\delta}{\gcd(\delta, n_1)}, \ldots, t_k \frac{\delta}{\gcd(\delta, n_k)}, \delta\right) = 1$.

Using the familiar formula $\sum_{d|n}\mu(d)=1$ for $n=1$ and 0 otherwise, this can be written as

$$o_\delta(n_1,\dots,n_k)=\sum_{e|\delta}\mu(e)\sum_{\substack{t_1\leqslant \gcd(\delta,n_1)\\ e|t_1\frac{\delta}{\gcd(\delta,n_1)}}}1\cdots\sum_{\substack{t_k\leqslant \gcd(\delta,n_k)\\ e|t_k\frac{\delta}{\gcd(\delta,n_k)}}}1.$$

Here the linear congruence $t_1\frac{\delta}{\gcd(\delta,n_1)}\equiv 0 \pmod{e}$ has $\gcd(e,\frac{\delta}{\gcd(\delta,n_1)})$ solutions in t_1 (mod e), and (mod $\gcd(\delta,n_1)$) it has exactly

$$\frac{\gcd(\delta,n_1)}{e}\gcd\left(e,\frac{\delta}{\gcd(\delta,n_1)}\right)=\gcd\left(\gcd(\delta,n_1),\frac{\delta}{e}\right)=\gcd\left(\frac{\delta}{e},n_1\right)$$

solutions (similar for the indexes $2,\dots,k$). Hence,

$$o_\delta(n_1,\dots,n_k)=\sum_{e|\delta}\mu(e)\gcd(\delta/e,n_1)\cdots\gcd(\delta/e,n_k)$$

$$=\sum_{e|\delta}\mu(\delta/e)\gcd(e,n_1)\cdots\gcd(e,n_k),$$

Now using the identity $\sum_{d|n}\phi(d)=n$,

$$o_\delta(n_1,\dots,n_k)=\sum_{e|\delta}\mu(\delta/e)\sum_{d_1|\gcd(e,n_1)}\phi(d_1)\cdots\sum_{d_k|\gcd(e,n_k)}\phi(d_k)$$

$$=\sum_{d_1|n_1,\dots,d_k|n_k}\phi(d_1)\cdots\phi(d_k)\sum_{ab\,\mathrm{lcm}(d_1,\dots,d_k)=\delta}\mu(a),$$

where the inner sum is 0 unless $\mathrm{lcm}(d_1,\dots,d_k)=\delta$, and in this case it is 1. This gives

$$o_\delta(n_1,\dots,n_k)=\sum_{\substack{d_1|n_1,\dots,d_k|n_k\\ \mathrm{lcm}(d_1,\dots,d_k)=\delta}}\phi(d_1)\cdots\phi(d_k). \tag{4.26}$$

Let $c_\delta(n_1,\dots,n_r)$ denote the number of cyclic subgroups of order δ ($\delta\mid n$) of the group G. Since a cyclic subgroup of order δ has $\phi(\delta)$ generators,

$$c_\delta(n_1,\dots,n_r)=\frac{o_\delta(n_1,\dots,n_r)}{\phi(\delta)}. \tag{4.27}$$

Now (2.40) follows immediately from (4.26) and (4.27) by

$$c(n_1,\dots,n_r)=\sum_{\delta|n}c_\delta(n_1,\dots,n_r).$$

$\square$

4.8.1 Proofs for Section 2.8.1

Proof of Theorem 2.8.2. We use Goursat's lemma for groups, which can be stated as follows:

Let G and H be arbitrary groups. Then there is a bijection between the set S of all subgroups of $G \times H$ and the set T of all 5-tuples (A, B, C, D, Ψ), where $B \trianglelefteq A \leqslant G$, $D \trianglelefteq C \leqslant H$ and $\Psi : A/B \to C/D$ is an isomorphism (here $\leqslant$ denotes subgroup and $\trianglelefteq$ denotes normal subgroup). More precisely, the subgroup corresponding to (A, B, C, D, Ψ) is

$$K = \{(g, h) \in A \times C : \Psi(gB) = hD\}. \tag{4.28}$$

Apply Goursat's lemma for the groups $G = \mathbb{Z}_m$ and $H = \mathbb{Z}_n$. Let $|A| = a$, $|B| = b$, $|C| = c$, $|D| = d$, where $a \mid m$, $b \mid a$, $c \mid n$, $d \mid c$. Writing explicitly the corresponding subgroups and quotient groups we deduce

$$A = \langle m/a \rangle = \left\{0, \frac{m}{a}, 2\frac{m}{a}, \ldots, (a-1)\frac{m}{a}\right\} \leqslant \mathbb{Z}_m,$$

$$B = \langle m/b \rangle = \left\{0, \frac{m}{b}, 2\frac{m}{b}, \ldots, (b-1)\frac{m}{b}\right\} \leqslant A,$$

$$A/B = \left\langle \frac{m}{a} + B \right\rangle = \left\{B, \frac{m}{a} + B, 2\frac{m}{a} + B, \ldots, \left(\frac{a}{b} - 1\right)\frac{m}{a} + B\right\},$$

and similarly

$$C = \langle n/c \rangle = \left\{0, \frac{n}{c}, 2\frac{n}{c}, \ldots, (c-1)\frac{n}{c}\right\} \leqslant \mathbb{Z}_n,$$

$$D = \langle n/d \rangle = \left\{0, \frac{n}{d}, 2\frac{n}{d}, \ldots, (d-1)\frac{n}{d}\right\} \leqslant C,$$

$$C/D = \left\langle \frac{n}{c} + D \right\rangle = \left\{D, \frac{n}{c} + D, 2\frac{n}{c} + D, \ldots, \left(\frac{c}{d} - 1\right)\frac{n}{c} + D\right\}.$$

Now, in the case $a/b = c/d$ the values of the isomorphisms $\Psi : A/B \to C/D$ are

$$\Psi\left(i\frac{m}{a} + B\right) = i\ell\frac{n}{c} + D, \qquad 0 \leqslant i \leqslant \frac{a}{b} - 1,$$

where $1 \leqslant \ell \leqslant a/b$, $\gcd(\ell, a/b) = 1$. Using (4.28) we deduce that the corresponding subgroup is

$$K = \left\{\left(i\frac{m}{a}, k\frac{n}{c}\right) \in A \times C : \Psi\left(i\frac{m}{a} + B\right) = k\frac{n}{c} + D\right\}$$

$$= \left\{\left(i\frac{m}{a}, k\frac{n}{c}\right) : 0 \leqslant i \leqslant a-1, 0 \leqslant k \leqslant c-1, i\ell\frac{n}{c} + D = k\frac{n}{c} + D\right\},$$

where the last condition is equivalent, in turn, to $kn/c \equiv i\ell n/c \pmod{n/d}$, $k \equiv i\ell \pmod{c/d}$, and finally $k = i\ell + jc/d$, $0 \leqslant j \leqslant d-1$. Hence,

$$K = \left\{\left(i\frac{m}{a}, \left(i\ell + j\frac{c}{d}\right)\frac{n}{c}\right) : 0 \leqslant i \leqslant a-1, 0 \leqslant j \leqslant d-1\right\},$$

and the proof of the representation formula is complete.

ii-iii) It is clear from (2.42) that $|K_{a,b,c,d,\ell}| = ad = bc$. Next we deduce the exponent of $K_{a,b,c,d,\ell}$. According to (2.42) the subgroup $K_{a,b,c,d,\ell}$ is generated by the elements $(0, n/d)$ and $(m/a, \ell n/c)$. Here the order of $(0, n/d)$ is d. To obtain the order of $(m/a, \ell n/c)$ note the following properties:

(1) $m \mid r(m/a)$ if and only if $m/\gcd(m, m/a) \mid r$ if and only if $a \mid r$, and the least such $r \in \mathbb{N}$ is a,

(2) $n \mid t(\ell n/c)$ if and only if $n/\gcd(n, \ell n/c) \mid t$ if and only if $c/\gcd(\ell, c) \mid t$, and the least such $t \in \mathbb{N}$ is $c/\gcd(\ell, c)$.

Therefore the order of $(m/a, \ell n/c)$ is $\operatorname{lcm}(a, c/\gcd(\ell, c))$. We deduce that the exponent of $K_{a,b,c,d,\ell}$ is

$$\operatorname{lcm}\left(d, \operatorname{lcm}\left(a, \frac{c}{\gcd(\ell, c)}\right)\right) = \operatorname{lcm}\left(d, a, \frac{c}{\gcd(\ell, c)}\right)$$

$$= \operatorname{lcm}\left(\frac{ac}{ac/d}, \frac{ac}{c}, \frac{ac}{a\gcd(\ell, c)}\right) = \frac{ac}{\gcd(ac/d, c, a\gcd(\ell, c))}$$

$$= \frac{ac}{\gcd(c, a\gcd(c/d, \gcd(\ell, c)))} = \frac{ac}{\gcd(c, a\gcd(\ell, \gcd(c/d, c)))}$$

$$= \frac{ac}{\gcd(c, a\gcd(\ell, c/d))} = \frac{ac}{\gcd(a, c)} = \operatorname{lcm}(a, c),$$

using that $\gcd(\ell, c/d) = 1$, cf. (2.41).

Now $K_{a,b,c,d,\ell}$ is a subgroup of the abelian group $\mathbb{Z}_m \times \mathbb{Z}_n$ having rank $\leqslant 2$. Therefore, $K_{a,b,c,d,\ell}$ has also rank $\leqslant 2$. That is, $K_{a,b,c,d,\ell} \simeq \mathbb{Z}_u \times \mathbb{Z}_v$ for unique u and v, where $u \mid v$ and $uv = ad$. Hence the exponent of $K_{a,b,c,d,\ell}$ is $\operatorname{lcm}(u, v) = v$. We obtain $v = \operatorname{lcm}(a, c)$ and $u = ad/\operatorname{lcm}(a, c) = \gcd(b, d)$. This gives (2.43).

iv) Clear from ii). □

Proof of Theorem 2.8.3. We have

$$s(m, n) = |J_{m,n}| = \sum_{\substack{a|m \\ b|a}} \sum_{\substack{c|n \\ d|c}} \sum_{a/b=c/d=e} \phi(e). \tag{4.29}$$

Let $m = ax$, $a = by$, $n = cz$, $c = dt$. Then, by the condition $a/b = c/d = e$ we have $y = t = e$. Rearranging the terms of (4.29),

$$s(m, n) = \sum_{bxe=m} \sum_{dze=n} \phi(e) = \sum_{\substack{ix=m \\ jz=n}} \sum_{\substack{be=i \\ de=j}} \phi(e) \tag{4.30}$$

$$= \sum_{\substack{i|m \\ j|n}} \sum_{e|\gcd(i,j)} \phi(e) = \sum_{\substack{i|m \\ j|n}} \gcd(i, j),$$

finishing the proof of (2.44). To obtain the formula (2.45) write (4.30) as follows:

$$s(m,n) = \sum_{\substack{ek=m\\ e\ell=n}} \phi(e) \sum_{\substack{bx=k\\ dz=\ell}} 1 = \sum_{\substack{ek=m\\ e\ell=n}} \phi(e)\tau(k)\tau(\ell)$$

$$= \sum_{e\mid \gcd(m,n)} \phi(e)\tau\left(\frac{m}{e}\right)\tau\left(\frac{n}{e}\right).$$

□

Proof of Theorem 2.8.4. Let $f(n) := AE(n)/n = \sum_{d\mid n} g(d)$ $(n \in \mathbb{N})$, that is $g = \mu * f$ in terms of the Dirichlet convolution. Here $g(p^\nu) = f(p^\nu) - f(p^{\nu-1})$ for every prime power p^ν $(\nu \in \mathbb{N})$. Note that

$$AE(p^k) = \frac{(\sum_{i=0}^k p^i)^2}{\sum_{i=0}^k (2i+1)p^{k-i}} \quad (k \geqslant 0),$$

so if we denote $S = \sum_{i=0}^{\nu-1} p^i$ and $T = \sum_{i=0}^{\nu-1}(2i+1)p^{\nu-1-i}$, we have

$$g(p^\nu) = \frac{(Sp+1)^2}{p^\nu(Tp+2\nu+1)} - \frac{S^2}{p^{\nu-1}T} = \frac{2STp+T-pS^2(2\nu+1)}{p^\nu T(Tp+2\nu+1)}.$$

Since $S \leqslant T$ and $T \leqslant (2\nu-1)S$, we have

$$\begin{aligned}|g(p^\nu)| &< \frac{\max\{2STp+T-pS^2(2\nu-1), pS^2(2\nu+1)-2STp\}}{p^\nu T(Tp+2\nu+1)}\\ &\leqslant \frac{1}{p^\nu}\max\left\{\frac{STp+T}{T(Tp+1)}, \frac{pS^2(2\nu-1)}{T^2p}\right\}\\ &\leqslant \frac{1}{p^\nu}\max\{1, 2\nu-1\} = \frac{2\nu-1}{p^\nu},\end{aligned}$$

valid for every prime power p^ν $(\nu \in \mathbb{N})$. Hence, $|g(n)| \leqslant \tau(n^2)/n$ for every $n \geqslant 1$.

We deduce that

$$\sum_{n\leqslant x} f(n) = \sum_{de\leqslant x} g(d) = \sum_{d\leqslant x} g(d) \sum_{e\leqslant x/d} 1 = \sum_{d\leqslant x} g(d)\,(x/d + O(1))$$

$$= x\sum_{d=1}^{\infty} \frac{g(d)}{d} + O\left(x\sum_{d>x}\frac{|g(d)|}{d}\right) + O\left(\sum_{d\leqslant x}|g(d)|\right)$$

$$= Cx + O\left(x\sum_{d>x}\frac{\tau(d^2)}{d^2}\right) + O\left(\sum_{d\leqslant x}\frac{\tau(d^2)}{d}\right),$$

where in the main term the coefficient of x is C, by Euler's product formula. Using that $\sum_{n\leqslant x}\tau(n^2) = cx\log^2 x + O(x\log x)$ with a certain constant c, partial summation shows that $\sum_{n>x}\tau(n^2)/n^2 = O((\log^2 x)/x)$, $\sum_{n\leqslant x}\tau(n^2)/n = O(\log^3 x)$. Therefore,

$$\sum_{n\leqslant x} f(n) = Cx + O\left(\log^3 x\right). \tag{4.31}$$

Now (2.46) follows from (4.31) by partial summation. □

4.8.2 Proofs for Section 2.8.2

We need the next general result regarding the subgroups of the group $G\times\mathbb{Z}_q$, where $(G,+)$ is an arbitrary finite Abelian group. For a subgroup H of G (notation $H\leqslant G$) consider the congruence relation ϱ_H on G defined for $x,x'\in G$ by $x\varrho_H x'$ if $x-x'\in H$.

Lemma 4.8.1. *For a finite Abelian group $(G,+)$ and $q\in\mathbb{N}$ let*

$$I_{G,q} := \{(H,\alpha,d) : H\leqslant G, \alpha\in\mathcal{S}_H, d\mid q \text{ and } (q/d)\alpha\in H\},$$

where $\mathcal{S}_H$ is a complete system of representants of the equivalence classes determined by ϱ_H. For $(H,\alpha,d)\in I_{G,q}$ define

$$V_{H,\alpha,d} := \{(k\alpha+\beta, kd) : 0\leqslant k\leqslant q/d-1, \beta\in H\}.$$

Then $V_{H,\alpha,d}$ is a subgroup of order $(q/d)\#H$ of $G\times\mathbb{Z}_q$ and the map $(H,\alpha,d)\mapsto V_{H,\alpha,d}$ is a bijection between the set $I_{G,q}$ and the set of subgroups of $G\times\mathbb{Z}_q$.

Proof. Let V be a subgroup of $G\times\mathbb{Z}_q$. Consider the natural projection $\pi_2 : G\times\mathbb{Z}_q\to\mathbb{Z}_q$ given by $\pi_2(x,y)=y$. Then $\pi_2(V)$ is a subgroup of $\mathbb{Z}_q$ and there is a unique divisor d of q such that $\pi_2(V)=\langle d\rangle := \{kd : 0\leqslant k\leqslant q/d-1\}$. Let $\alpha\in G$ such that $(\alpha,d)\in V$.

Furthermore, consider the natural inclusion $\iota_1 : G\to G\times\mathbb{Z}_q$ given by $\iota_1(x)=(x,0)$. Then $\iota_1^{-1}(V)=H$ is a subgroup of G. We show that $V=\{(k\alpha+\beta,kd) : k\in\mathbb{Z},\beta\in H\}$. Indeed, for every $k\in\mathbb{Z}$ and $\beta\in H$, $(k\alpha+\beta,kd)=k(\alpha,d)+(\beta,0)\in V$. On the other hand, for every $(u,v)\in V$ one has $v\in\pi_2(V)$ and hence there is $k\in\mathbb{Z}$ such that $v=kd$. We obtain $(u-k\alpha,0)=(u,v)-k(\alpha,d)\in V$, thus $\beta:=u-k\alpha\in\iota_1^{-1}(V)=H$.

Here a necessary condition is that $(q/d)\alpha\in H$ (obtained for $k=q/d$, $\beta=0$). Clearly, if this is verified, then for the above representation of V it is enough to take the values $0\leqslant k\leqslant q/d-1$.

Conversely, every $(H,\alpha,d)\in I_{G,q}$ generates a subgroup $V_{H,\alpha,d}$ of order $(q/d)\#H$ of $G\times\mathbb{Z}_q$. Furthermore, for fixed $H\leqslant G$ and $d\mid q$ we have $V_{H,\alpha,d}=H_{H,\alpha',d}$ if and only if $\alpha\varrho_H\alpha'$. This completes the proof. □

In the case $G=\mathbb{Z}_m$ (and with $q=n$) Lemma 4.8.1 can be stated as follows:

Lemma 4.8.2. *For every $m, n \in \mathbb{N}$ let*

$$I_{m,n} := \{(a,b,s) \in \mathbb{N}^2 \times \mathbb{N} \cup \{0\} : a \mid m, b \mid n, 0 \leqslant s \leqslant a-1 \text{ and } a \mid (n/b)s\}$$

and for $(a,b,s) \in I_{m,n}$ define

$$\begin{aligned} V_{a,b,s} &:= \langle (a,0),(b,s) \rangle \\ &= \{(ia+js, jb) : 0 \leqslant i \leqslant m/a-1, 0 \leqslant j \leqslant n/b-1\}. \end{aligned} \tag{4.32}$$

Then $V_{a,b,s}$ is a subgroup of order $\frac{mn}{ab}$ of $\mathbb{Z}_m \times \mathbb{Z}_n$ and the map $(a,b,s) \mapsto V_{a,b,s}$ is a bijection between the set $I_{m,n}$ and the set of subgroups of $\mathbb{Z}_m \times \mathbb{Z}_n$.

Note that $a \mid (n/b)s$ holds if and only if $a/\gcd(a,n/b) \mid s$. That is, for $s \in I_{m,n}$ we have

$$s = \frac{at}{A}, \quad 0 \leqslant t \leqslant A-1, \tag{4.33}$$

where $A = \gcd(a, n/b)$, notation given in Theorem 2.8.5. This quickly leads to formula (2.44) regarding the number $s(m,n)$ of subgroups of $\mathbb{Z}_m \times \mathbb{Z}_n$, namely

$$s(m,n) = \sum_{a \mid m, b \mid n} \sum_{0 \leqslant t \leqslant A-1} 1 = \sum_{a \mid m, b \mid n} \gcd(a,b).$$

deduced by different arguments.

Proof of Theorem 2.8.5. Apply Lemma 4.8.1 for $G = \mathbb{Z}_m \times \mathbb{Z}_n$ and with $q = r$. For the subgroups $V = V_{a,b,s}$ given by Lemma 4.8.2 a complete system of representants of the equivalence classes determined by ϱ_V is $\mathcal{S}_{a,b} = \{0,1,\ldots,a-1\} \times \{0,1,\ldots,b-1\}$. Indeed, the elements of $\mathcal{S}_{a,b}$ are pairwise incongruent with respect to V, and for every $(x,y) \in \mathbb{Z}_m \times \mathbb{Z}_n$ there is a unique $(x',y') \in \mathcal{S}_{a,b}$ such that $(x,y) - (x',y') \in V$. Namely, let

$$(x_1,y_1) = (x,y) - \lfloor y/b \rfloor (s,b), \qquad (x',y') = (x_1,y_1) - \lfloor x_1/a \rfloor (a,0).$$

We obtain that the subgroups of $\mathbb{Z}_m \times \mathbb{Z}_n \times \mathbb{Z}_r$ are of the form

$$U = U_{H,\alpha,c} = \{(k\alpha + \beta, kc) : 0 \leqslant k \leqslant r/c-1, \beta \in V\},$$

where $c \mid r$ and $\alpha = (u,v) \in \mathcal{S}_{a,b}$ such that $(r/c)\alpha \in V$.

Now using (4.32) we deduce

$$\begin{aligned} U &= U_{a,b,s,\alpha,c} \\ &= \{(ia+js+ku, jb+kv, kc) : 0 \leqslant i \leqslant n/a-1, 0 \leqslant j \leqslant n/b-1, 0 \leqslant k \leqslant n/c-1\}, \end{aligned}$$

where (4.33) holds and $(r/c)(u,v) \in V$. From the latter condition we deduce that there are i_0, j_0 such that

$$(r/c)u = i_0 a + j_0 s, \qquad (r/c)v = j_0 b. \tag{4.34}$$

The second condition of (4.34) holds if $b \mid (r/c)v$, that is $b/\gcd(b, r/c) \mid v$. Let

$$v = \frac{bv_1}{B}, \quad 0 \leqslant v_1 \leqslant B-1, \tag{4.35}$$

where $B = \gcd(b, r/c)$. Also, $j_0 = rv/(bc)$ and inserting this into the first equation of (4.34) we obtain $(r/c)u \equiv rvs/(bc) \pmod a$. This linear congruence in u has a solution u_0 if and only if

$$\gcd(a, r/c) \mid \frac{rvs}{bc} \tag{4.36}$$

and its all solutions are $u = u_0 + az/C$ with $0 \leqslant z \leqslant C-1$ with $C = \gcd(a, r/c)$.

Substituting (4.35) and (4.33) into (4.36) we obtain

$$\gcd(a, r/c) \mid \frac{rab}{\gcd(ab, n)\gcd(bc, r)} v_1 t,$$

that is

$$\gcd(ab, n)\gcd(ac, r)\gcd(bc, r) \mid abcrv_1 t,$$

equivalent to

$$\frac{\gcd(ab, n)\gcd(ac, r)\gcd(bc, r)}{\gcd(abcr, \gcd(ab, n)\gcd(ac, r)\gcd(bc, r))} \mid v_1 t,$$

and to

$$X \mid v_1 t, \tag{4.37}$$

where X is defined in the statement of Theorem 2.8.5. Note that $X \mid B$ (indeed, $A \mid a$, $C \mid (r/c)$ and the property follows from $X = B/\gcd((a/A)(r/c)/C), B)$.

Let t be fixed. We obtain from (4.37) that v_1 is of the form $v_1 = Xw/\gcd(t, X)$, where $0 \leqslant w \leqslant B\gcd(t, X)/X - 1$. Also, from (4.35), $v = bXw/B\gcd(t, X)$. Collecting the conditions on a, b, c, t, w, z in terms of A, B, C, X finishes the proof. □

Proof of Theorem 2.8.6. According to Theorem 2.8.5, the number of subgroups of Γ is

$$s(m, n, r) = \sum_{a|m, b|n, c|r} \sum_{0 \leqslant t \leqslant A-1} \sum_{0 \leqslant w \leqslant B\gcd(t,X)/X-1} \sum_{0 \leqslant z \leqslant C-1} 1$$

$$= \sum_{a|m, b|n, c|r} C \sum_{0 \leqslant t \leqslant A-1} \frac{B}{X}\gcd(t, X) = \sum_{a|m, b|n, c|r} \frac{BC}{X} \sum_{1 \leqslant t \leqslant A} \gcd(t, X).$$

Here $X \mid A$ (similar to $X \mid B$ shown above), hence the inner sum is $(A/X)P(X)$ and we obtain formula (2.47). □

Proof of Theorem 2.8.7. Let $s(n) = s(n,n,n)$. According to (2.48), the function $s(n)$ can be expressed in terms of the Dirichlet convolution $*$ as $s = E_2\tau * h$, where $E_2(n) = n^2$ $(n \in \mathbb{N})$. Therefore, $h = s * (\mu * \mu)E_2$. We obtain that $h(p) = 2p + 4$, $h(p^2) = 5p + 7$, $h(p^3) = 8p + 10$ and

$$h(p^\nu) = s(p^\nu) - 2p^2 s(p^{\nu-1}) + p^4 s(p^{\nu-2}) \quad (\nu \geqslant 2). \tag{4.38}$$

For the gcd-sum function $P(n)$ one has $P(n) \leqslant n\tau(n)$ $(n \in \mathbb{N})$, cf. (2.19). Hence

$$s(n) = \sum_{a,b,c|n} \gcd(a(r/c), ABC)P(X)/X \leqslant \sum_{a,b,c|n} a(r/c)\tau(X)$$

$$\leqslant n^2\tau(n) \sum_{a,b,c|n} 1 = n^2(\tau(n))^4$$

for every $n \in \mathbb{N}$ and for every prime power p^ν $(\nu \in \mathbb{N})$,

$$s(p^\nu) \leqslant p^{2\nu}(\nu+1)^4. \tag{4.39}$$

Now from (4.38) and (4.39) we deduce that for every prime power p^ν $(\nu \geqslant 2)$,

$$0 < h(p^\nu) \leqslant 2p^{2\nu}(\nu+1)^4. \tag{4.40}$$

From the Euler product formula

$$H(z) = \prod_p \left(1 + \frac{2p+4}{p^z} + \frac{5p+7}{p^{2z}} + \frac{8p+10}{p^{3z}} + \sum_{\nu=4}^{\infty} \frac{h(p^\nu)}{p^{\nu z}}\right)$$

and from (4.40) we obtain that $H(z)$ is absolutely convergent for $z \in \mathbb{C}$ with $4(\mathrm{Re}(z) - 2 - \varepsilon) > 1$, i.e., for $\mathrm{Re}(z) > 9/4 + \varepsilon$ with an arbitrary $\varepsilon > 0$.

Furthermore, by partial summation we obtain from (1.1) that

$$\sum_{n \leqslant x} n^2\tau(n) = \frac{1}{3}x^3 \log x + \frac{1}{3}\left(2\gamma - \frac{1}{3}\right)x^3 + O(x^{2+\theta+\varepsilon}). \tag{4.41}$$

Now

$$\sum_{n \leqslant x} s(n) = \sum_{d \leqslant x} h(d) \sum_{e \leqslant x/d} e^2\tau(e),$$

and inserting (4.41) we get

$$\sum_{n \leqslant x} s(n) = \frac{x^3 \log x}{3} \sum_{d \leqslant x} \frac{h(d)}{d^3} - \frac{x^3}{3} \sum_{d \leqslant x} \frac{h(d)\log d}{d^3} + \frac{x^3}{3}\left(2\gamma - \frac{1}{3}\right) \sum_{d \leqslant x} \frac{h(d)}{d^3}$$

$$+O\left(x^{2+\theta+\varepsilon} \sum_{d \leqslant x} \frac{|h(d)|}{d^{2+\theta+\varepsilon}}\right),$$

where the last term is $O\left(x^{2+\theta+\varepsilon}\right)$. This gives the asymptotic formula (2.49). □

Chapter 5

Proofs of the results of Chapter 3

5.1 Proofs of the results of Section 3.1

Proof of Theorem 3.1.1. We use the polynomial identity (proved in our paper [111])

$$\sum_{j=0}^{k-1} \sum_{1\leqslant i_1<\ldots<i_j\leqslant r} x_{i_1}\cdots x_{i_j} \prod_{\substack{\ell=1\\ \ell\neq i_1,\ldots,i_j}}^{r} (1-x_\ell) = 1-\sum_{j=k}^{r}(-1)^{j-k}\binom{j-1}{k-1} e_j(x_1,\ldots,x_r), \quad (5.1)$$

where on the left hand side the term for $j=0$ is considered to be $\prod_{\ell=1}^{r}(1-x_\ell)$.

Note that the left hand side of (5.1) is a symmetric polynomial in $x_1,\ldots,x_r$ and the right hand side shows how it can be written as a polynomial of the elementary symmetric polynomials.

The function $(n_1,\ldots,n_r)\mapsto \varrho_{r,k}(n_1,\ldots,n_r)$ is multiplicative. Also, by its definition, for every $\nu_1,\ldots,\nu_r\geqslant 0$,

$$\varrho_{r,k}(p^{\nu_1},\ldots,p^{\nu_r}) = \begin{cases} 1, & \text{if there are at most } k-1 \text{ values } \nu_i\geqslant 1,\\ 0, & \text{otherwise.}\end{cases}$$

Therefore, the Dirichlet series of $\varrho_{r,k}(n_1,\ldots,n_r)$ can be expanded into the Euler product:

$$\sum_{n_1,\ldots,n_r=1}^{\infty} \frac{\varrho_{r,k}(n_1,\ldots,n_r)}{n_1^{s_1}\cdots n_r^{s_r}} = \prod_p \sum_{\nu_1,\ldots,\nu_r=0}^{\infty} \frac{\varrho_{r,k}(p^{\nu_1},\ldots,p^{\nu_r})}{p^{\nu_1 s_1+\cdots+\nu_r s_r}}$$

$$= \prod_p \left(1+\sum_{j=1}^{k-1}\sum_{1\leqslant i_1<\ldots<i_j\leqslant r}\sum_{\nu_{i_1},\ldots,\nu_{i_j}=1}^{\infty} \frac{1}{p^{\nu_{i_1}s_{i_1}+\cdots+\nu_{i_j}s_{i_j}}}\right)$$

$$= \prod_p \left(1+\sum_{j=1}^{k-1}\sum_{1\leqslant i_1<\ldots<i_j\leqslant r} \frac{1}{p^{s_{i_1}}}\left(1-\frac{1}{p^{s_{i_1}}}\right)^{-1}\cdots \frac{1}{p^{s_{i_j}}}\left(1-\frac{1}{p^{s_{i_j}}}\right)^{-1}\right)$$

$$= \zeta(s_1)\cdots\zeta(s_r)\prod_p \left(\prod_{\ell=1}^r \left(1-\frac{1}{p^{s_\ell}}\right) + \sum_{j=1}^{k-1} \sum_{1\leqslant i_1<\ldots<i_j\leqslant r} \frac{1}{p^{s_{i_1}+\cdots+s_{i_j}}} \prod_{\substack{\ell=1\\ \ell\ne i_1,\ldots,i_j}}^r \left(1-\frac{1}{p^{s_\ell}}\right)\right)$$

$$= \zeta(s_1)\cdots\zeta(s_r)\prod_p \left(1-\sum_{j=k}^r (-1)^{j-k}\binom{j-1}{k-1} e_j(p^{-s_1},\ldots,p^{-s_r})\right),$$

by using identity (5.1) for $x_1 = p^{-s_1},\ldots,x_r = p^{-s_r}$ in the last step. □

Proof of Theorem 3.1.2. According to Theorem 3.1.1, for every $n_1,\ldots,n_r \in \mathbb{N}$,

$$\varrho_{r,k}(n_1,\ldots,n_r) = \sum_{d_1\mid n_1,\ldots,d_r\mid n_r} \psi_{r,k}(d_1,\ldots,d_r), \tag{5.2}$$

where

$$\sum_{n_1,\ldots,n_r=1}^{\infty} \frac{\psi_{r,k}(n_1,\ldots,n_r)}{n_1^{s_1}\cdots n_r^{s_r}} = D_{r,k}(s_1,\ldots,s_r).$$

The function $\psi_{r,k}$ is also multiplicative, symmetric in the variables and for any prime powers $p^{\nu_1},\ldots,p^{\nu_r}$,

$$\psi_{r,k}(p^{\nu_1},\ldots,p^{\nu_r}) = \begin{cases} 1, & \nu_1=\ldots=\nu_r=0,\\ (-1)^{j-k+1}\binom{j-1}{k-1}, & \nu_1,\ldots,\nu_r\in\{0,1\},\ j:=\nu_1+\ldots+\nu_r\geqslant k,\\ 0, & \text{otherwise.}\end{cases} \tag{5.3}$$

Note that $\psi_{r,k}(p^{\nu_1},\ldots,p^{\nu_r}) = 0$ provided that $\nu_i \geqslant 2$ for at least one $1\leqslant i\leqslant r$, or $\nu_1,\ldots,\nu_r \in \{0,1\}$ and $\nu_1+\ldots+\nu_r < k$. For $k\geqslant 2$ one has $\psi_{r,k}(p,1,\ldots,1)=0$ and for $k\geqslant 3$ one has $\psi_{r,k}(p,p,1,\ldots,1)=0$, where p is any prime.

From (5.2) we deduce

$$\sum_{n_1,\ldots,n_r\leqslant x} \varrho_{r,k}(n_1,\ldots,n_r) = \sum_{d_1,\ldots,d_r\leqslant x} \psi_{r,k}(d_1,\ldots,d_r)\left\lfloor\frac{x}{d_1}\right\rfloor\cdots\left\lfloor\frac{x}{d_r}\right\rfloor$$

$$= \sum_{d_1,\ldots,d_r\leqslant x} \psi(d_1,\ldots,d_r)\left(\frac{x}{d_1}+O(1)\right)\cdots\left(\frac{x}{d_r}+O(1)\right)$$

$$= x^r \sum_{d_1,\ldots,d_r\leqslant x} \frac{\psi_{r,k}(d_1,\ldots,d_r)}{d_1\cdots d_r} + Q_{r,k}(x), \tag{5.4}$$

with

$$Q_{r,k}(x) \ll \sum_{u_1,\ldots,u_r} x^{u_1+\cdots+u_r} \sum_{d_1,\ldots,d_r\leqslant x} \frac{|\psi_{r,k}(d_1,\ldots,d_r)|}{d_1^{u_1}\cdots d_r^{u_r}},$$

where the first sum is over $u_1,\dots,u_r \in \{0,1\}$ such that at least one u_i is 0. Let $u_1,\dots,u_r$ be fixed and assume that $u_r = 0$. Since $(x/d_i)^{u_i} \leqslant x/d_i$ for every i, we have

$$A := x^{u_1+\cdots+u_r} \sum_{d_1,\dots,d_r\leqslant x} \frac{|\psi_{r,k}(d_1,\dots,d_r)|}{d_1^{u_1}\cdots d_r^{u_r}} \leqslant x^{r-1} \sum_{d_1,\dots,d_r\leqslant x} \frac{|\psi_{r,k}(d_1,\dots,d_r)|}{d_1\cdots d_{r-1}}$$

Assume that $k \geqslant 3$. Then

$$A \leqslant x^{r-1} \sum_{d_1,\dots,d_r=1}^{\infty} \frac{|\psi_{r,k}(d_1,\dots,d_r)|}{d_1\cdots d_{r-1}} \ll x^{r-1},$$

since the series $D_{r,k}(1,\dots,1,0)$ is absolutely convergent for $k \geqslant 3$ by Theorem 3.1.1. We obtain that

$$Q_{r,k}(x) \ll x^{r-1} \quad (k \geqslant 3). \tag{5.5}$$

If $k = 2$, then

$$A \leqslant x^{r-1} \prod_{p\leqslant x} \sum_{\nu_1,\dots,\nu_r=0}^{\infty} \frac{|\psi_{r,2}(p^{\nu_1},\dots,p^{\nu_r})|}{p^{\nu_1+\cdots+\nu_{r-1}}} \tag{5.6}$$

$$= x^{r-1} \prod_{p\leqslant x} \left(1 + \frac{r-1}{p} + \frac{c_2}{p^2} + \cdots + \frac{c_{r-1}}{p^{r-1}}\right),$$

by (5.3), where $c_2,\dots,c_{r-1}$ are certain positive integers, using also that we have p in the denominator if and only if $\nu_r = 1$ and exactly one of $\nu_1,\dots,\nu_{r-1}$ is 1, the rest being 0, which occurs $r-1$ times. We deduce that

$$A \ll x^{r-1} \prod_{p\leqslant x} \left(1 + \frac{1}{p}\right)^{r-1} \ll x^{r-1}(\log x)^{r-1}$$

by Mertens' formula. This shows that

$$Q_{r,2}(x) \ll x^{r-1}(\log x)^{r-1}. \tag{5.7}$$

Furthermore, for the main term of (5.4) we have

$$\sum_{d_1,\dots,d_r\leqslant x} \frac{\psi_{r,k}(d_1,\dots,d_r)}{d_1\cdots d_r}$$

$$= \sum_{d_1,\dots,d_r=1}^{\infty} \frac{\psi_{r,k}(d_1,\dots,d_r)}{d_1\cdots d_r} - \sum_{\varnothing\neq I\subseteq\{1,\dots,r\}} \sum_{\substack{d_i>x,\, i\in I\\ d_j\leqslant x,\, j\notin I}} \frac{\psi_{r,k}(d_1,\dots,d_r)}{d_1\cdots d_r}, \tag{5.8}$$

where the series is convergent by Theorem 3.1.1 and its sum is $D_{r,k}(1,\dots,1) = A_{r,k}$, given by (3.4).

Let I be fixed and assume that $I = \{1, 2, \ldots, t\}$, that is $d_1, \ldots, d_t > x$ and $d_{t+1}, \ldots, d_r \leqslant x$, where $t \geqslant 1$. We estimate the sum

$$B := \sum_{\substack{d_1,\ldots,d_t > x\\ d_{t+1},\ldots,d_r \leqslant x}} \frac{|\psi_{r,k}(d_1, \ldots, d_r)|}{d_1 \cdots d_r}$$

by distinguishing the following cases:

Case i) $k \geqslant 3$, $t \geqslant 1$:

$$B < \frac{1}{x} \sum_{d_1,\ldots,d_r=1}^{\infty} \frac{|\psi_{r,k}(d_1, \ldots, d_r)|}{d_2 \cdots d_r} \ll \frac{1}{x},$$

since the series is convergent by Theorem 3.1.1.

Case ii) $k = 2$, $t \geqslant 3$: if $0 < \varepsilon < 1/2$, then

$$B = \sum_{\substack{d_1,\ldots,d_t > x\\ d_{t+1},\ldots,d_r \leqslant x}} \frac{|\psi_{r,2}(d_1, \ldots, d_r)| d_1^{\varepsilon-1/2} \cdots d_t^{\varepsilon-1/2}}{d_1^{1/2+\varepsilon} \cdots d_t^{1/2+\varepsilon} d_{t+1} \cdots d_r}$$

$$< x^{t(\varepsilon-1/2)} \sum_{d_1,\ldots,d_r=1}^{\infty} \frac{|\psi_{r,2}(d_1, \ldots, d_r)|}{d_1^{1/2+\varepsilon} \cdots d_t^{1/2+\varepsilon} d_{t+1} \cdots d_r} \ll x^{t(\varepsilon-1/2)},$$

since the series is convergent (for $t \geqslant 1$). Using that $t(\varepsilon - 1/2) < -1$ for $0 < \varepsilon < (t-2)/(2t)$, here we need $t \geqslant 3$, we obtain $B \ll \frac{1}{x}$.

Case iii) $k = 2$, $t = 1$: Let $d_1 > x$, $d_2, \ldots, d_r \leqslant x$ and consider a prime p. If $p \mid d_i$ for an $i \in \{2, \ldots, r\}$, then $p \leqslant x$. If $p \mid d_1$ and $p > x$, then $p \nmid d_i$ for every $i \in \{2, \ldots, r\}$ and $\psi_{r,2}(d_1, \ldots, d_r) = 0$ by its definition (5.3). Hence it is enough to consider the primes $p \leqslant x$. We deduce

$$B < \frac{1}{x} \sum_{\substack{d_1 > x\\ d_2,\ldots,d_r \leqslant x}} \frac{|\psi_{r,2}(d_1, \ldots, d_r)|}{d_2 \cdots d_r}$$

$$\leqslant \frac{1}{x} \prod_{p \leqslant x} \sum_{\nu_1,\ldots,\nu_r=0}^{\infty} \frac{|\psi_{r,2}(p^{\nu_1}, \ldots, p^{\nu_r})|}{p^{\nu_2+\cdots+\nu_r}} \ll \frac{1}{x} (\log x)^{r-1},$$

similar to the estimate of (5.6).

Case iv) $k = 2$, $t = 2$: We split the sum B into two sums, namely

$$B = \sum_{\substack{d_1 > x, d_2 > x\\ d_3,\ldots,d_r \leqslant x}} \frac{|\psi_{r,2}(d_1, \ldots, d_r)|}{d_1 \cdots d_r}$$

$$= \sum_{\substack{d_1 > x^{3/2}, d_2 > x \\ d_3, \ldots, d_r \leqslant x}} \frac{|\psi_{r,2}(d_1, \ldots, d_r)|}{d_1 \cdots d_r} + \sum_{\substack{x^{3/2} \geqslant d_1 > x, d_2 > x \\ d_3, \ldots, d_r \leqslant x}} \frac{|\psi_{r,2}(d_1, \ldots, d_r)|}{d_1 \cdots d_r} =: B_1 + B_2,$$

say, where

$$B_1 = \sum_{\substack{d_1 > x^{3/2}, d_2 > x \\ d_3, \ldots, d_r \leqslant x}} \frac{|\psi_{r,2}(d_1, \ldots, d_r)|}{d_1^{1/3} d_2 \cdots d_r} \frac{1}{d_1^{2/3}}$$

$$< \frac{1}{x} \sum_{d_1, \ldots, d_r = 1}^{\infty} \frac{|\psi_{r,2}(d_1, \ldots, d_r)|}{d_1^{1/3} d_2 \cdots d_r} \ll \frac{1}{x},$$

since the series is convergent. Furthermore,

$$B_2 < \frac{1}{x} \sum_{\substack{x^{3/2} \geqslant d_1, d_2 > x \\ d_3, \ldots, d_r \leqslant x}} \frac{|\psi_{r,2}(d_1, \ldots, d_r)|}{d_1 d_3 \cdots d_r},$$

where $d_1 \leqslant x^{3/2}$, $d_2 > x$, $d_3, \ldots, d_r \leqslant x$. Consider a prime p. If $p \mid d_i$ for an $i \in \{1, 3, \ldots, r\}$, then $p \leqslant x^{3/2}$. If $p \mid d_2$ and $p > x^{3/2}$, then $p \nmid d_i$ for every $i \in \{1, 3, \ldots, r\}$ and $\psi_{r,2}(d_1, \ldots, d_r) = 0$ by its definition. Hence it is enough to consider the primes $p \leqslant x^{3/2}$. We deduce, cf. the estimate of (5.6),

$$B_2 < \frac{1}{x} \prod_{p \leqslant x^{3/2}} \sum_{\nu_1, \ldots, \nu_r = 0}^{\infty} \frac{|\psi_{r,2}(p^{\nu_1}, \ldots, p^{\nu_r})|}{p^{\nu_1 + \nu_3 + \cdots + \nu_r}} \ll \frac{1}{x} (\log x^{3/2})^{r-1} \ll \frac{1}{x} (\log x)^{r-1}.$$

Hence, given any $t \geqslant 1$, we have $B \ll \frac{1}{x}$ for $k \geqslant 3$ and $B \ll \frac{1}{x} (\log x)^{r-1}$ for $k = 2$. Therefore, by (5.8),

$$\sum_{d_1, \ldots, d_r \leqslant x} \frac{\psi_{r,k}(d_1, \ldots, d_r)}{d_1 \cdots d_r} = A_{r,k} + O(R_{r,k}(x)) \tag{5.9}$$

with the notation (3.4) and (3.5).

The proof is complete by putting together (5.4), (5.5), (5.7) and (5.9). □

5.2 Proofs of the results of Section 3.2

We need the following lemmas.

Lemma 5.2.1. *If $k \geqslant 2$ and $f \in \mathcal{A}_r$ with $r > -1$ real, then*

$$L_{f,k}(z_1, \ldots, z_k) := \sum_{n_1, \ldots, n_k = 1}^{\infty} \frac{f([n_1, \ldots, n_k])}{n_1^{z_1} \cdots n_k^{z_k}} = \zeta(z_1 - r) \cdots \zeta(z_k - r) H_{f,k}(z_1, \ldots, z_k),$$

where the multiple Dirichlet series $H_{f,k}(z_1,\ldots,z_k)$ *is absolutely convergent for*

$$\Re z_1,\ldots,\Re z_k > A := \begin{cases} r+\frac{1}{2}, & \text{if } r\geqslant 0,\\ \frac{r+1}{2}, & \text{if } -1<r<0. \end{cases} \tag{5.10}$$

Proof. If f is a multiplicative function of a single variable, then the arithmetic function of k variables $f([n_1,\ldots,n_k])$ is multiplicative. It follows that

$$L_{f,k}(z_1,\ldots,z_k) = \prod_p \sum_{\nu_1,\ldots,\nu_k=0}^{\infty} \frac{f(p^{\max(\nu_1,\ldots,\nu_k)})}{p^{\nu_1 z_1+\cdots+\nu_k z_k}} \tag{5.11}$$

Case I. Assume that $r\geqslant 0$. Grouping the terms of the sum in (5.11) according to the values $\nu_1+\cdots+\nu_k$ we have

$$L_{f,k}(z_1,\ldots,z_k) = \prod_p \left(1+\frac{f(p)}{p^{z_1}}+\cdots+\frac{f(p)}{p^{z_k}}+\sum_{\nu_1+\cdots+\nu_k\geqslant 2}\frac{f(p^{\max(\nu_1,\ldots,\nu_k)})}{p^{\nu_1 z_1+\cdots+\nu_k z_k}}\right). \tag{5.12}$$

Let $\Re z_1,\ldots,\Re z_k\geqslant\delta>r$. By using condition (i) from the definition of the class $\mathcal{A}_r$,

$$\frac{f(p)}{p^{z_j}} = \frac{1}{p^{z_j-r}} + O\left(\frac{1}{p^{\delta-r+1/2}}\right) \quad (1\leqslant j\leqslant k).$$

Also, by condition (iii) following the definition of the class $\mathcal{A}_r$ and by using that $r\geqslant 0$ we deduce that

$$\left|\frac{f(p^{\max(\nu_1,\ldots,\nu_k)})}{p^{\nu_1 z_1+\cdots+\nu_k z_k}}\right| \leqslant C_3\frac{p^{r\max(\nu_1,\ldots,\nu_k)}}{p^{\delta(\nu_1+\cdots+\nu_k)}} \leqslant C_3\frac{1}{p^{(\delta-r)(\nu_1+\cdots+\nu_k)}}.$$

Thus the sum in (5.12) over $\nu_1+\cdots+\nu_k\geqslant 2$ is $O\left(p^{-2(\delta-r)}\right)$. We obtain

$$L_{f,k}(z_1,\ldots,z_k)\zeta^{-1}(z_1-r)\cdots\zeta^{-1}(z_k-r)$$

$$= \prod_p \left(1-\frac{1}{p^{z_1-r}}\right)\cdots\left(1-\frac{1}{p^{z_k-r}}\right)\left(1+\frac{1}{p^{z_1-r}}+\cdots+\frac{1}{p^{z_k-r}}+O\left(\frac{1}{p^{\delta-r+1/2}}\right)\right.$$

$$\left.+O\left(\frac{1}{p^{2(\delta-r)}}\right)\right) = \prod_p\left(1+O\left(\frac{1}{p^{\delta-r+1/2}}\right)+O\left(\frac{1}{p^{2(\delta-r)}}\right)\right),$$

since $\Re z_j\geqslant\delta$ $(1\leqslant j\leqslant k)$, where the terms $\pm\frac{1}{p^{z_j-r}}$ $(1\leqslant j\leqslant k)$ cancel out. Here the latter product converges absolutely when $\delta-r+1/2>1$ and $2(\delta-r)>1$, that is, for $\delta>r+1/2$.

Case II. Assume that $-1 < r < 0$. Now we group the terms of the sum in (5.11) according to the values $\max(\nu_1, \ldots, \nu_k)$:

$$L_{f,k}(z_1, \ldots, z_k) = \prod_p \left(1 + \sum_{\max(\nu_1,\ldots,\nu_k)=1} \frac{f(p)}{p^{\nu_1 z_1 + \cdots + \nu_k z_k}} + \sum_{\max(\nu_1,\ldots,\nu_k)\geqslant 2} \frac{f(p^{\max(\nu_1,\ldots,\nu_k)})}{p^{\nu_1 z_1 + \cdots + \nu_k z_k}}\right). \tag{5.13}$$

Let $\Re z_1, \ldots, \Re z_k \geqslant \delta \geqslant 0$. Consider the sum in (5.13) over $\max(\nu_1, \ldots, \nu_k) = 1$ and suppose that $\nu_i = 1$ for m $(1 \leqslant m \leqslant k)$ distinct values of i. If $m = 1$, then by condition (i) from the definition of the class $\mathcal{A}_r$ we have

$$\frac{f(p)}{p^{z_j}} = \frac{1}{p^{z_j - r}} + O\left(\frac{1}{p^{\delta - r + 1/2}}\right) \quad (1 \leqslant j \leqslant k).$$

If $m \geqslant 2$, then

$$\left|\frac{f(p)}{p^{\nu_1 z_1 + \cdots + \nu_k z_k}}\right| \leqslant \frac{(C_1 + 1)p^r}{p^{m\delta}} = O\left(\frac{1}{p^{2\delta - r}}\right).$$

This shows that the sum in (5.13) over $\max(\nu_1, \ldots, \nu_k) = 1$ is

$$\frac{1}{p^{z_1 - r}} + \cdots + \frac{1}{p^{z_k - r}} + O\left(\frac{1}{p^{\delta - r + 1/2}}\right) + O\left(\frac{1}{p^{2\delta - r}}\right).$$

Furthermore, by condition (ii) we deduce that for $\max(\nu_1, \ldots, \nu_k) \geqslant 2$,

$$\left|\frac{f(p^{\max(\nu_1,\ldots,\nu_k)})}{p^{\nu_1 z_1 + \cdots + \nu_k z_k}}\right| \leqslant C_2 \frac{p^{r \max(\nu_1,\ldots,\nu_k)}}{p^{\delta(\nu_1 + \cdots + \nu_k)}} \leqslant C_2 \frac{1}{p^{(\delta - r)\max(\nu_1,\ldots,\nu_k)}}$$

$(\delta \geqslant 0)$ and it follows that the sum in (5.13) over $\max(\nu_1, \ldots, \nu_k) \geqslant 2$ is $O\left(p^{-2(\delta - r)}\right) = O\left(p^{-(2\delta - r)}\right)$, since $r < 0$.

We obtain that

$$L_{f,k}(z_1, \ldots, z_k) = \prod_p \left(1 + \frac{1}{p^{z_1 - r}} + \cdots + \frac{1}{p^{z_k - r}} + O\left(\frac{1}{p^{\delta - r + 1/2}}\right) + O\left(\frac{1}{p^{2\delta - r}}\right)\right)$$

and

$$\begin{aligned} &L_{f,k}(z_1, \ldots, z_k)\zeta^{-1}(z_1 - r) \cdots \zeta^{-1}(z_k - r) \\ &= \prod_p \left(1 - \frac{1}{p^{z_1 - r}}\right) \cdots \left(1 - \frac{1}{p^{z_k - r}}\right) \prod_p \left(1 + \frac{1}{p^{z_1 - r}} + \cdots + \frac{1}{p^{z_k - r}}\right. \\ &\left. + O\left(\frac{1}{p^{\delta - r + 1/2}}\right) + O\left(\frac{1}{p^{2\delta - r}}\right)\right) \\ &= \prod_p \left(1 + O\left(\frac{1}{p^{\delta - r + 1/2}}\right) + O\left(\frac{1}{p^{2\delta - r}}\right)\right), \end{aligned}$$

since $\Re z_j \geqslant \delta$ $(1 \leqslant j \leqslant k)$, where the terms $\pm \frac{1}{p^{z_j - r}}$ $(1 \leqslant j \leqslant k)$ cancel out, similar to Case I. Here the latter product converges absolutely when $\delta - r + 1/2 > 1$ and $2\delta - r > 1$, that is, for $\delta > (r+1)/2 > 0$. □

Lemma 5.2.2. *If $k \geqslant 2$ and $f \in \mathcal{A}_r$ with $r \geqslant 0$, then*

$$\overline{L}_{f,k}(z_1, \ldots, z_k) := \sum_{n_1,\ldots,n_k=1}^{\infty} \frac{f\left(\frac{[n_1,\ldots,n_k]}{(n_1,\ldots,n_k)}\right)}{n_1^{z_1} \cdots n_k^{z_k}} = \zeta(z_1 - r) \cdots \zeta(z_k - r) \overline{H}_{f,k}(z_1, \ldots, z_k),$$

where the multiple Dirichlet series $\overline{H}_{f,k}(z_1, \ldots, z_k)$ is absolutely convergent for $\Re z_1, \ldots, \Re z_k > r + 1/2$.

Proof. Similar to the proof of Lemma 5.2.1, Case I. If f is multiplicative, then the function $f([n_1, \ldots, n_k]/(n_1, \ldots, n_k))$ is also multiplicative and we have

$$\overline{L}_{f,k}(z_1, \ldots, z_k) = \prod_p \sum_{\nu_1,\ldots,\nu_k=0}^{\infty} \frac{f(p^{\max(\nu_1,\ldots,\nu_k)-\min(\nu_1,\ldots,\nu_k)})}{p^{\nu_1 z_1 + \cdots + \nu_k z_k}}$$

$$= \prod_p \left(1 + \frac{f(p)}{p^{z_1}} + \cdots + \frac{f(p)}{p^{z_k}} + \sum_{\nu_1+\cdots+\nu_k \geqslant 2} \frac{f(p^{\max(\nu_1,\ldots,\nu_k)-\min(\nu_1,\ldots,\nu_k)})}{p^{\nu_1 z_1 + \cdots + \nu_k z_k}}\right). \tag{5.14}$$

If $\Re z_1, \ldots, \Re z_k \geqslant \delta > r$, then it follows that

$$\left| \frac{f(p^{\max(\nu_1,\ldots,\nu_k)-\min(\nu_1,\ldots,\nu_k)})}{p^{\nu_1 z_1 + \cdots + \nu_k z_k}} \right| \leqslant C \frac{p^{r(\max(\nu_1,\ldots,\nu_k)-\min(\nu_1,\ldots,\nu_k))}}{p^{\delta(\nu_1+\cdots+\nu_k)}} \leqslant C \frac{1}{p^{(\delta-r)(\nu_1+\cdots+\nu_k)}},$$

thus the sum in (5.14) over $\nu_1 + \cdots + \nu_k \geqslant 2$ is $O\left(p^{-2(\delta-r)}\right)$. Furthermore, we use the same arguments as in the previous proof. □

Proof of Theorem 3.2.1. From Lemma 5.2.1 we deduce the convolutional identity

$$f([n_1, \ldots, n_k]) = \sum_{j_1 d_1 = n_1, \ldots, j_k d_k = n_k} j_1^r \cdots j_k^r h_{f,k}(d_1, \ldots, d_k),$$

where

$$\sum_{n_1,\ldots,n_k=1}^{\infty} \frac{h_{f,k}(n_1, \ldots, n_k)}{n_1^{z_1} \cdots n_k^{z_k}} = H_{f,k}(z_1, \ldots, z_k).$$

Therefore

$$\sum_{n_1,\ldots,n_k \leqslant x} f([n_1, \ldots, n_k]) = \sum_{j_1 d_1 \leqslant x, \ldots, j_k d_k \leqslant x} j_1^r \cdots j_k^r h_{f,k}(d_1, \ldots, d_k)$$

$$= \sum_{d_1,\ldots,d_k \leqslant x} h_{f,k}(d_1, \ldots, d_k) \sum_{j_1 \leqslant x/d_1} j_1^r \cdots \sum_{j_k \leqslant x/d_k} j_k^r$$

$$= \sum_{d_1,\ldots,d_k\leqslant x} h_{f,k}(d_1,\ldots,d_k)\left(\frac{x^{r+1}}{(r+1)d_1^{r+1}}+O(\frac{x^R}{d_1^R})\right)\cdots\left(\frac{x^{r+1}}{(r+1)d_k^{r+1}}+O(\frac{x^R}{d_k^R})\right),$$

where $R := \max(r,0)$. We deduce that

$$\sum_{n_1,\ldots,n_k\leqslant x} f([n_1,\ldots,n_k]) = \frac{x^{k(r+1)}}{(r+1)^k}\sum_{d_1,\ldots,d_k\leqslant x}\frac{h_{f,k}(d_1,\ldots,d_k)}{d_1^{r+1}\cdots d_k^{r+1}} + S_{k,r}(x), \tag{5.15}$$

with

$$S_{k,r}(x) \ll \sum_{u_1,\ldots,u_k} x^{u_1+\cdots+u_k}\sum_{d_1,\ldots,d_k\leqslant x}\frac{|h_{f,k}(d_1,\ldots,d_k)|}{d_1^{u_1}\cdots d_k^{u_k}}, \tag{5.16}$$

where the first sum is over $u_1,\ldots,u_k \in \{r+1,R\}$ such that at least one u_i is R. Let $u_1,\ldots,u_k$ be fixed and assume that $u_i = R$ for t ($1\leqslant t\leqslant k$) values of i, we take the first t values of i. Then $x^{u_1+\cdots+u_k}$ times the inner sum of (5.16) is, using the notation A given by (5.10),

$$\begin{aligned}
&\ll x^{(k-t)(r+1)+tR}\sum_{d_1,\ldots,d_k\leqslant x}\frac{|h_{f,k}(d_1,\ldots,d_k)|}{d_1^R\cdots d_t^R d_{t+1}^{r+1}\cdots d_k^{r+1}}\\
&= x^{(k-t)(r+1)+tR}\sum_{d_1,\ldots,d_k\leqslant x}\frac{|h_{f,k}(d_1,\ldots,d_k)|d_1^{A-R+\varepsilon}\cdots d_t^{A-R+\varepsilon}}{d_1^{A+\varepsilon}\cdots d_t^{A+\varepsilon}d_{t+1}^{r+1}\cdots d_k^{r+1}}\\
&\leqslant x^{(k-t)(r+1)+tR}x^{t(A-R+\varepsilon)}\sum_{d_1,\ldots,d_k=1}^{\infty}\frac{|h_{f,k}(d_1,\ldots,d_k)|}{d_1^{A+\varepsilon}\cdots d_t^{A+\varepsilon}d_{t+1}^{r+1}\cdots d_k^{r+1}}\\
&= x^{k(r+1)-t(r+1-A)+t\varepsilon}H_{f,k}(A+\varepsilon,\ldots,A+\varepsilon,r+1,\ldots,r+1)\\
&\ll x^{k(r+1)-t(r+1-A)+t\varepsilon},
\end{aligned}$$

since the latter series is convergent by Lemma 5.2.1. Using that $r+1-A = \frac{1}{2}\min(r+1,1) > 0$, the obtained error is maximal for $t = 1$ giving

$$O\left(x^{k(r+1)-\frac{1}{2}\min(r+1,1)+\varepsilon}\right).$$

Furthermore, for the sum in the main term of (5.15) we have

$$\sum_{d_1,\ldots,d_k\leqslant x}\frac{h_{f,k}(d_1,\ldots,d_k)}{d_1^{r+1}\cdots d_k^{r+1}}$$

$$= \sum_{d_1,\ldots,d_k=1}^{\infty}\frac{h_{f,k}(d_1,\ldots,d_k)}{d_1^{r+1}\cdots d_k^{r+1}} - \sum_{\varnothing\neq I\subseteq\{1,\ldots,k\}}\sum_{\substack{d_i>x,\, i\in I\\ d_j\leqslant x,\, j\notin I}}\frac{h_{f,k}(d_1,\ldots,d_k)}{d_1^{r+1}\cdots d_k^{r+1}}, \tag{5.17}$$

where the series is convergent by Lemma 5.2.1, and its sum is $H_{f,k}(r+1,\dots,r+1)$.

Let I be fixed and assume that $I=\{1,2,\dots,s\}$, that is $d_1,\dots,d_s>x$ and $d_{t+1},\dots,d_k\leqslant x$, where $s\geqslant 1$. We deduce, by noting that $A-(r+1)=-\frac{1}{2}\min(r+1,1)<0$,

$$\sum_{\substack{d_1,\dots,d_s>x\\ d_{s+1},\dots,d_k\leqslant x}} \frac{|h_{f,k}(d_1,\dots,d_k)|}{d_1^{r+1}\cdots d_k^{r+1}}$$

$$= \sum_{\substack{d_1,\dots,d_s>x\\ d_{s+1},\dots,d_k\leqslant x}} \frac{|h_{f,k}(d_1,\dots,d_k)|d_1^{A-(r+1)+\varepsilon}\cdots d_s^{A-(r+1)+\varepsilon}}{d_1^{A+\varepsilon}\cdots d_s^{A+\varepsilon}d_{s+1}^{r+1}\cdots d_k^{r+1}}$$

$$\leqslant x^{s(A-(r+1)+\varepsilon)} \sum_{d_1,\dots,d_k=1}^{\infty} \frac{|h_{f,k}(d_1,\dots,d_k)|}{d_1^{A+\varepsilon}\cdots d_s^{A+\varepsilon}d_{s+1}^{r+1}\cdots d_k^{r+1}}$$

$$= x^{s(A-(r+1)+\varepsilon)} H_{f,k}(A+\varepsilon,\dots,A+\varepsilon,r+1,\dots,r+1)$$

$$\ll x^{-\frac{s}{2}\min(r+1,1)+s\varepsilon},$$

the latter series (the same as before) being convergent, and the obtained error is maximal for $s=1$ giving, according to (5.15) and (5.17), the same error

$$O\left(x^{k(r+1)-\frac{1}{2}\min(r+1,1)+\varepsilon}\right).$$

This proves asymptotic formula (3.9) with the constant $C_{f,k}=H_{f,k}(r+1,\dots,r+1)$. Here, according to Lemma 5.2.1,

$$C_{f,k}=\prod_p \left(1-\frac{1}{p}\right)^k \sum_{\nu_1,\dots,\nu_k=0}^{\infty} \frac{f(p^{\max(\nu_1,\dots,\nu_k)})}{p^{(r+1)(\nu_1+\cdots+\nu_k)}}.$$

The proof of (3.10) is similar, based on Lemma 5.2.1 and the convolutional identity

$$\frac{f([n_1,\dots,n_k])}{(n_1\cdots n_k)^r} = \sum_{j_1d_1=n_1,\dots,j_kd_k=n_k} \frac{h_{f,k}(d_1,\dots,d_k)}{d_1^r\cdots d_k^r},$$

which implies that

$$\sum_{n_1,\dots,n_k\leqslant x} \frac{f([n_1,\dots,n_k])}{(n_1\cdots n_r)^r} = \sum_{d_1,\dots,d_k\leqslant x} \frac{h_{f,k}(d_1,\dots,d_k)}{d_1^r\cdots d_k^r} \sum_{j_1\leqslant x/d_1} 1 \cdots \sum_{j_k\leqslant x/d_k} 1.$$

□

Proof of Theorem 3.2.2. Formula (3.11) is obtained by using Lemma 5.2.2, in exactly the same way as (3.9) (here $r\geqslant 0$ and $R=\max(r,0)=r$), with the constant $D_{f,k}=\overline{H}_{f,k}(r+1,\dots,r+1)$. □

Proof of Corollary 3.2.3. Apply Theorem 3.2.1 for $f(n) = n^r$. Here

$$C_{r,3} = \prod_p \left(1 - \frac{1}{p}\right)^3 \sum_{a,b,c=0}^{\infty} \frac{p^{r\max(a,b,c)}}{p^{(r+1)(a+b+c)}}$$

$$= \prod_p \left(1 - \frac{1}{p}\right)^3 (6S_1 + 3S_2 + 3S_3 + S_4),$$

with

$$S_1 = \sum_{0 \leqslant a < b < c} \frac{p^{rc}}{p^{(r+1)(a+b+c)}}, \quad S_2 = \sum_{0 \leqslant a = b < c} \frac{p^{rc}}{p^{(r+1)(2a+c)}},$$

$$S_3 = \sum_{0 \leqslant a < b = c} \frac{p^{rc}}{p^{(r+1)(a+2c)}}, \quad S_4 = \sum_{0 \leqslant a = b = c} \frac{p^{rc}}{p^{(r+1)3c}},$$

which gives (3.12). Formula (3.13) for the constant $C_{r,4}$ can be computed in a similar manner. □

Proof of Corollary 3.2.4. Apply Theorem 3.2.1 for $f(n) = \sigma(n)$ with $r = 1$. □

5.3 Proofs of the results of Section 3.3

We need the following auxiliary results:

Lemma 5.3.1. *Let $r \geqslant 2$, $k \geqslant 1$ and let $s_1, \ldots, s_r \in \mathbb{C}$ with $\Re s_j > 1$ $(1 \leqslant j \leqslant r)$. Then*

$$\sum_{n_1,\ldots,n_r=1}^{\infty} \frac{\tau(1,k;n_1\cdots n_r)}{n_1^{s_1}\cdots n_r^{s_r}} = \zeta(s_1)\zeta(ks_1)\cdots\zeta(s_r)\zeta(ks_r)F_{r,k}(s_1,\ldots,s_r),$$

where

$$F_{r,k}(s_1,\ldots,s_r) = \sum_{n_1,\ldots,n_r=1}^{\infty} \frac{f_{r,k}(n_1,\ldots,n_r)}{n_1^{s_1}\cdots n_r^{s_r}}$$

is absolutely convergent provided that $\Re s_j > 0$ $(1 \leqslant j \leqslant r)$ and $\Re(s_j + s_\ell) > 1$ $(1 \leqslant j < \ell \leqslant r)$.

Proof of Lemma 5.3.1. The function $n \mapsto \tau(1,k;n)$ is multiplicative and $\tau(1,k;p^\nu) = \lfloor \nu/k \rfloor + 1$ for every prime power p^ν $(\nu \geqslant 0)$. The function $(n_1,\ldots,n_r) \mapsto \tau(1,k;n_1\cdots n_r)$ is multiplicative, viewed as a function of r variables. Therefore, its multiple Dirichlet series can be expanded into an Euler product. We obtain

$$\sum_{n_1,\ldots,n_r=1}^{\infty} \frac{\tau(1,k;n_1\cdots n_r)}{n_1^{s_1}\cdots n_r^{s_r}} = \prod_p \sum_{\nu_1,\ldots,\nu_r=0}^{\infty} \frac{\tau(1,k;p^{\nu_1+\cdots+\nu_r})}{p^{\nu_1 s_1+\cdots+\nu_r s_r}}$$

$$= \prod_p \sum_{\nu_1,\dots,\nu_r=0}^{\infty} \frac{\lfloor(\nu_1+\cdots+\nu_r)/k\rfloor+1}{p^{\nu_1 s_1+\cdots+\nu_r s_r}} = \zeta(s_1)\zeta(ks_1)\cdots\zeta(s_r)\zeta(ks_r)F_{r,k}(s_1,\dots,s_r),$$

where

$$F_{r,k}(s_1,\dots,s_r)$$

$$= \prod_p \left(1-\frac{1}{p^{s_1}}\right)\left(1-\frac{1}{p^{ks_1}}\right)\cdots\left(1-\frac{1}{p^{s_r}}\right)\left(1-\frac{1}{p^{ks_r}}\right) \sum_{\nu_1,\dots,\nu_r=0}^{\infty} \frac{\lfloor(\nu_1+\cdots+\nu_r)/k\rfloor+1}{p^{\nu_1 s_1+\cdots+\nu_r s_r}}$$

$$= \prod_p \left(1-\frac{1}{p^{s_1}}-\frac{1}{p^{ks_1}}+\frac{1}{p^{(k+1)s_1}}\right)\cdots\left(1-\frac{1}{p^{s_r}}-\frac{1}{p^{ks_r}}+\frac{1}{p^{(k+1)s_r}}\right)$$

$$\times \sum_{\nu_1,\dots,\nu_r=0}^{\infty} \frac{\lfloor(\nu_1+\cdots+\nu_r)/k\rfloor+1}{p^{\nu_1 s_1+\cdots+\nu_r s_r}}$$

$$= \prod_p \left(1+\sum_{\substack{\nu_1,\dots,\nu_r=0\\ \#A(\nu_1,\dots,\nu_r)\geqslant 2}}^{\infty} \frac{c_{\nu_1,\dots,\nu_r}}{p^{\nu_1 s_1+\cdots+\nu_r s_r}}\right)$$

with some coefficients $c_{\nu_1,\dots,\nu_r}$, where $A(\nu_1,\dots,\nu_r) := \{j : 1 \leqslant j \leqslant r, \nu_j \neq 0\}$. Here the coefficient c of $1/p^{\ell s_j}$ (the case $\nu_t = 0$ for all $t \neq j$ and $\nu_j = \ell$) is zero for every $1 \leqslant j \leqslant r$ and $\ell \geqslant 1$. Indeed,

$$c = \begin{cases} \left(\lfloor\frac{\ell}{k}\rfloor+1\right)-\left(\lfloor\frac{\ell-1}{k}\rfloor+1\right)=0, & \text{if } 1\leqslant \ell\leqslant k-1,\\ \left(\lfloor\frac{k}{k}\rfloor+1\right)-\left(\lfloor\frac{k-1}{k}\rfloor+1\right)-1=0, & \text{if } \ell = k,\\ \left(\lfloor\frac{\ell}{k}\rfloor+1\right)-\left(\lfloor\frac{\ell-1}{k}\rfloor+1\right)-\left(\lfloor\frac{\ell-k}{k}\rfloor+1\right)+\left(\lfloor\frac{\ell-k-1}{k}\rfloor+1\right)=0, & \text{if } \ell\geqslant k+1.\end{cases}$$

Hence a sufficient condition of absolute convergence of $F_{r,k}(s_1,\dots,s_r)$ is that $\Re s_j > 0$ $(1 \leqslant j \leqslant r)$ and $\Re(s_j + s_\ell) > 1$ $(1 \leqslant j < \ell \leqslant r)$. Note that another sufficient condition for absolute convergence is $\Re s_j > 1/2$ $(1 \leqslant j \leqslant r)$, which can not be used in the proof of the corresponding asymptotic formula. □

Lemma 5.3.2. *Let $r \geqslant 2$, $k \geqslant 1$ and let $s_1,\dots,s_r \in \mathbb{C}$ with $\Re s_j > 1$ $(1 \leqslant j \leqslant r)$. Then*

$$\sum_{n_1,\dots,n_r=1}^{\infty} \frac{\tau(1,k;[n_1,\dots,n_r])}{n_1^{s_1}\cdots n_r^{s_r}} = \zeta(s_1)\zeta(ks_1)\cdots\zeta(s_r)\zeta(ks_r)G_{r,k}(s_1,\dots,s_r),$$

where

$$G_{r,k}(s_1,\dots,s_r) = \sum_{n_1,\dots,n_r=1}^{\infty} \frac{g_{r,k}(n_1,\dots,n_r)}{n_1^{s_1}\cdots n_r^{s_r}}$$

is absolutely convergent provided that $\Re s_j > 0$ $(1 \leqslant j \leqslant r)$ and $\Re(s_j+s_\ell) > 1$ $(1 \leqslant j < \ell \leqslant r)$.

Proof of Lemma 5.3.2. Similar to the proof of Lemma 5.3.1. The function $(n_1,\ldots,n_r)\mapsto \tau(1,k;[n_1,\ldots,n_r])$ is also multiplicative. Its multiple Dirichlet series can be written as

$$\sum_{n_1,\ldots,n_r=1}^{\infty} \frac{\tau(1,k;[n_1,\ldots,n_r])}{n_1^{s_1}\cdots n_r^{s_r}} = \prod_p \sum_{\nu_1,\ldots,\nu_r=0}^{\infty} \frac{\tau(1,k;p^{\max(\nu_1,\ldots,\nu_r)})}{p^{\nu_1 s_1+\cdots+\nu_r s_r}}$$

$$= \prod_p \sum_{\nu_1,\ldots,\nu_r=0}^{\infty} \frac{\lfloor \max(\nu_1,\ldots,\nu_r)/k\rfloor +1}{p^{\nu_1 s_1+\cdots+\nu_r s_r}} = \zeta(s_1)\zeta(ks_1)\cdots\zeta(s_r)\zeta(ks_r)G_{r,k}(s_1,\ldots,s_r),$$

where

$$G_{r,k}(s_1,\ldots,s_r)$$
$$= \prod_p \left(1-\frac{1}{p^{s_1}}-\frac{1}{p^{ks_1}}+\frac{1}{p^{(k+1)s_1}}\right)\cdots\left(1-\frac{1}{p^{s_r}}-\frac{1}{p^{ks_r}}+\frac{1}{p^{(k+1)s_r}}\right)$$
$$\times \sum_{\nu_1,\ldots,\nu_r=0}^{\infty} \frac{\lfloor \max(\nu_1,\ldots,\nu_r)/k\rfloor +1}{p^{\nu_1 s_1+\cdots+\nu_r s_r}}$$
$$= \prod_p \left(1+\sum_{\substack{\nu_1,\ldots,\nu_r=0\\ \#A(\nu_1,\ldots,\nu_r)\geqslant 2}}^{\infty} \frac{c'_{\nu_1,\ldots,\nu_r}}{p^{\nu_1 s_1+\cdots+\nu_r s_r}}\right)$$

since here the coefficient c' of $1/p^{\ell s_j}$ equals the coefficient c of $1/p^{\ell s_j}$ in $F_{r,k}(s_1,\ldots,s_r)$, which vanishes, as explained in the proof of Lemma 5.3.1. □

Concerning the exponential divisor function $\tau^{(e)}$ we have

Lemma 5.3.3. *Let $r\geqslant 2$ and let $s_1,\ldots,s_r\in\mathbb{C}$ with $\Re s_j>1$ $(1\leqslant j\leqslant r)$. Then*

$$\sum_{n_1,\ldots,n_r=1}^{\infty} \frac{\tau^{(e)}(n_1\cdots n_r)}{n_1^{s_1}\cdots n_r^{s_r}} = \zeta(s_1)\zeta(2s_1)\cdots\zeta(s_r)\zeta(2s_r)T_r(s_1,\ldots,s_r),$$

and

$$\sum_{n_1,\ldots,n_r=1}^{\infty} \frac{\tau^{(e)}([n_1,\ldots n_r])}{n_1^{s_1}\cdots n_r^{s_r}} = \zeta(s_1)\zeta(2s_1)\cdots\zeta(s_r)\zeta(2s_r)V_r(s_1,\ldots,s_r),$$

where the Dirichlet series

$$T_r(s_1,\ldots,s_r) = \sum_{n_1,\ldots,n_r=1}^{\infty} \frac{t(n_1,\ldots,n_r)}{n_1^{s_1}\cdots n_r^{s_r}}$$

and

$$V_r(s_1,\ldots,s_r) = \sum_{n_1,\ldots,n_r=1}^{\infty} \frac{v(n_1,\ldots,n_r)}{n_1^{s_1}\cdots n_r^{s_r}}$$

are both absolutely convergent if $\Re s_j>1/5$ $(1\leqslant j\leqslant r)$ and $\Re(s_j+s_\ell)>1$ $(1\leqslant j<\ell\leqslant r)$.

Proof of Lemma 5.3.3. Similar to the proof of Lemma 5.3.1. Since $\tau^{(e)}(p^\nu) = \tau(\nu) = \tau(1,2;p^\nu)$ for $\nu \in \{1,2,3,4\}$, in $T_r(s_1,\dots,s_r)$ and $V_r(s_1,\dots,s_r)$ the coefficients of the terms $1/p^{\ell s_j}$ will be zero for every $1 \leqslant j \leqslant r$ and $\ell \in \{1,2,3,4\}$. However, $\tau^{(e)}(p^5) = \tau(5) = 2 \neq 3 = \tau(1,2;p^5)$, therefore the coefficients of the terms $1/p^{5s_j}$ will not vanish (they are -1 for every $1 \leqslant j \leqslant r$). Hence for absolute convergence it is necessary that $\Re s_j > 1/5$ $(1 \leqslant j \leqslant r)$. Together with $\Re(s_j + s_\ell) > 1$ $(1 \leqslant j < \ell \leqslant r)$, these are necessary and sufficient conditions for absolute convergence. □

Lemma 5.3.4. *Let $r \geqslant 2$ and let $s_1,\dots,s_r \in \mathbb{C}$ with $\Re s_j > 1$ $(1 \leqslant j \leqslant r)$. Then*

$$\sum_{n_1,\dots,n_r=1}^{\infty} \frac{\tau^{(2)}(n_1\cdots n_r)}{n_1^{s_1}\cdots n_r^{s_r}} = \zeta^2(s_1)\cdots\zeta^2(s_r) H_r(s_1,\dots,s_r),$$

where

$$H_r(s_1,\dots,s_r) = \prod_p \left(1-\frac{1}{p^{s_1}}\right)\cdots\left(1-\frac{1}{p^{s_r}}\right)\left(2-\left(1-\frac{1}{p^{s_1}}\right)\cdots\left(1-\frac{1}{p^{s_r}}\right)\right)$$

is absolutely convergent if $\Re s_j > 1/2$ $(1 \leqslant j \leqslant r)$.

Proof of Lemma 5.3.4. The function $(n_1,\dots,n_r) \mapsto \tau^{(2)}(n_1\cdots n_r)$ is multiplicative. Its multiple Dirichlet series can be written as

$$\sum_{n_1,\dots,n_r=1}^{\infty} \frac{\tau^{(2)}(n_1\cdots n_r)}{n_1^{s_1}\cdots n_r^{s_r}} = \prod_p \sum_{\nu_1,\dots,\nu_r=0}^{\infty} \frac{\tau^{(2)}(p^{\nu_1+\cdots+\nu_r})}{p^{\nu_1 s_1+\cdots+\nu_r s_r}}$$

$$= \prod_p \left(1 + \sum_{\substack{\nu_1,\dots,\nu_r=0\\ \nu_1+\cdots+\nu_r\geqslant 1}}^{\infty} \frac{2}{p^{\nu_1 s_1+\cdots+\nu_r s_r}}\right) = \prod_p \left(2\left(1-\frac{1}{p^{s_1}}\right)^{-1}\cdots\left(1-\frac{1}{p^{s_r}}\right)^{-1} - 1\right)$$

$$= \zeta^2(s_1)\cdots\zeta^2(s_r) H_r(s_1,\dots,s_r),$$

where in $H_r(s_1,\dots,s_r)$ the coefficients of $1/p^{s_j}$ are zero $(1 \leqslant j \leqslant r)$. □

Proof of Theorem 3.3.1. We prove formula (3.17). According to Lemma 5.3.1,

$$\tau(1,k;n_1\cdots n_r) = \sum_{d_1 m_1 = n_1,\dots,d_r m_r = n_r} f_{r,k}(d_1,\dots,d_r)\tau(1,k;m_1)\cdots\tau(1,k;m_r) \tag{5.18}$$

for every $n_1,\dots,n_r \in \mathbb{N}$, where $f_{r,k}$ is a multiplicative function and symmetric in the variables. Therefore,

$$S_{r,k}(x) := \sum_{n_1,\dots,n_r \leqslant x} \tau(1,k;n_1\cdots n_r) = \sum_{d_1,\dots,d_r \leqslant x} f_{r,k}(d_1,\dots,d_r) \prod_{j=1}^{r} \sum_{m_j \leqslant x/d_j} \tau(1,k;m_j).$$

For $k \geqslant 2$ we deduce by (3.14) that

$$S_{r,k}(x) = \sum_{d_1,\ldots,d_r \leqslant x} f_{r,k}(d_1,\ldots,d_r) \prod_{j=1}^{r} \left(\zeta(k)\frac{x}{d_j} + \zeta(1/k)\left(\frac{x}{d_j}\right)^{1/k} + O(\left(\frac{x}{d_j}\right)^{\theta_k+\varepsilon}) \right). \tag{5.19}$$

Here the main term will be

$$M_{r,k}(x) := (\zeta(k))^r x^r \sum_{d_1,\ldots,d_r \leqslant x} \frac{f_{r,k}(d_1,\ldots,d_r)}{d_1\cdots d_r}$$

$$= (\zeta(k))^r x^r \sum_{d_1,\ldots,d_r=1}^{\infty} \frac{f_{r,k}(d_1,\ldots,d_r)}{d_1\cdots d_r} + R_{r,k}(x) = (\zeta(k))^r x^r F_{r,k}(1,\ldots,1) + R_{r,k}(x),$$

where $F_{r,k}(1,\ldots,1)$ is convergent and its value is by (5.18),

$$F_{r,k}(1,\ldots,1) = \prod_p \left(1-\frac{1}{p}\right)^r \left(1-\frac{1}{p^k}\right)^r \sum_{\nu_1,\ldots,\nu_r=0}^{\infty} \frac{\tau(1,k;p^{\nu_1+\cdots+\nu_r})}{p^{\nu_1+\cdots+\nu_r}}$$

$$= (\zeta(k))^{-r} \prod_p \left(1-\frac{1}{p}\right)^r \sum_{\nu_1,\ldots,\nu_r=0}^{\infty} \frac{\lfloor(\nu_1+\cdots+\nu_r)/k\rfloor+1}{p^{\nu_1+\cdots+\nu_r}},$$

while

$$R_{r,k}(x) \ll x^r {\sum}' \frac{|f_{r,k}(d_1,\ldots,d_r)|}{d_1\cdots d_r},$$

$\sum'$ meaning that $d_1,\ldots,d_r \leqslant x$ does not hold. That is, there is at least one t such that $d_t > x$. We can assume, without restricting the generality, that $t=1$. We obtain that

$$\sum_{d_1>x}{}' \frac{|f_{r,k}(d_1,\ldots,d_r)|}{d_1\cdots d_r} = \sum_{d_1>x}{}' \frac{|f_{r,k}(d_1,\ldots,d_r)|}{d_1^{\varepsilon}d_2\cdots d_r} \cdot \frac{1}{d_1^{1-\varepsilon}}$$

$$\ll \frac{1}{x^{1-\varepsilon}} \sum_{d_1,\ldots,d_r=1}^{\infty} \frac{|f_{r,k}(d_1,\ldots,d_r)|}{d_1^{\varepsilon}d_2\cdots d_r} \ll \frac{1}{x^{1-\varepsilon}},$$

since the latter series is $F_{r,k}(\varepsilon,1,\ldots,1)$, which converges by Lemma 5.3.1. This gives $R_{r,k}(x) \ll x^{r-1+\varepsilon}$ and

$$M_{r,k}(x) = A_{k,r}x^r + O(x^{r-1+\varepsilon}). \tag{5.20}$$

By multiplying in (5.19) the terms $\zeta(k)\frac{x}{d_j}$ $(1 \leqslant j \leqslant r-1)$ and $\zeta(1/k)(\frac{x}{d_r})^{1/k}$ we have

$$(\zeta(k))^{r-1}\zeta(1/k)x^{r-1+1/k} \sum_{d_1,\ldots,d_r \leqslant x} \frac{f_{r,k}(d_1,\ldots,d_r)}{d_1\cdots d_{r-1}d_r^{1/k}}$$

$$= (\zeta(k))^{r-1}\zeta(1/k)x^{r-1+1/k}\sum_{d_1,\ldots,d_r=1}^{\infty}\frac{f_{r,k}(d_1,\ldots,d_r)}{d_1\cdots d_{r-1}d_r^{1/k}} + T_{r,k}(x)$$

$$= (\zeta(k))^{r-1}\zeta(1/k)x^{r-1+1/k}F_{r,k}(1,\ldots,1,1/k) + T_{r,k}(x),$$

where $F_{r,k}(1,\ldots,1,1/k)$ is convergent and

$$T_{r,k}(x) \ll x^{r-1+1/k}\sum{}' \frac{|f_{r,k}(d_1,\ldots,d_r)|}{d_1\cdots d_{r-1}d_r^{1/k}}$$

with $\sum'$ as above. There are two cases. Case I: Assuming that $d_r > x$ we deduce

$$\sum_{d_r>x}{}' \frac{|f_{r,k}(d_1,\ldots,d_r)|}{d_1\cdots d_{r-1}d_r^{1/k}} = \sum_{d_r>x}{}' \frac{|f_{r,k}(d_1,\ldots,d_r)|}{d_1\cdots d_{r-1}d_r^{\varepsilon}}\cdot\frac{1}{d_r^{1/k-\varepsilon}}$$

$$\ll \frac{1}{x^{1/k-\varepsilon}}\sum_{d_1,\ldots,d_r=1}^{\infty}\frac{|f_{r,k}(d_1,\ldots,d_r)|}{d_1\cdots d_{r-1}d_r^{\varepsilon}} \ll \frac{1}{x^{1/k-\varepsilon}}.$$

Case II: If $d_r \leqslant x$, then there is a $t \in \{1,\ldots,r-1\}$ such that $d_t > x$. We deduce by taking $t = 1$,

$$\sum_{\substack{d_1>x\\ d_r\leqslant x}}{}' \frac{|f_{r,k}(d_1,\ldots,d_r)|}{d_1 d_2\cdots d_{r-1}d_r^{1/k}} = \sum_{\substack{d_1>x\\ d_r\leqslant x}}{}' \frac{|f_{r,k}(d_1,\ldots,d_r)|}{d_1^{\varepsilon}d_2\cdots d_{r-1}d_r}\cdot\frac{d_r^{1-1/k}}{d_1^{1-\varepsilon}}$$

$$\ll \frac{x^{1-1/k}}{x^{1-\varepsilon}}\sum_{d_1,\ldots,d_r=1}^{\infty}\frac{|f_{r,k}(d_1,\ldots,d_r)|}{d_1^{\varepsilon}d_2\cdots d_{r-1}d_r} \ll \frac{1}{x^{1/k-\varepsilon}}.$$

Hence

$$T_{r,k}(x) \ll x^{r-1+\varepsilon},$$

the same error as in (5.20).

All the terms obtained by multiplying in (5.19) $\zeta(k)\frac{x}{d_j}$ ($j \in \{1,\ldots,r\}\backslash\{t\}$) and $\zeta(1/k)(\frac{x}{d_t})^{1/k}$ are of the same size and give together

$$B_{r,k}x^{r-1+1/k} + O(x^{r-1+\varepsilon}), \tag{5.21}$$

where

$$B_{r,k} = r(\zeta(k))^{r-1}\zeta(1/k)F_{r,k}(1,\ldots,1,1/k).$$

Now, if in (5.19) we take an error term, say $O((\frac{x}{d_r})^{\theta_k+\varepsilon})$, then we have to consider $\zeta(k)\frac{x}{d_j}$ ($1 \leqslant j \leqslant r-1$) to obtain, by multiplying, the largest term, which is

$$\ll x^{r-1+\theta_k+\varepsilon}\sum_{d_1,\ldots,d_r\leqslant x}\frac{|f_{r,k}(d_1,\ldots,d_r)|}{d_1\cdots d_{r-1}d_r^{\theta_k+\varepsilon}}$$

$$\ll x^{r-1+\theta_k+\varepsilon} \sum_{d_1,\ldots,d_r=1}^{\infty} \frac{|f_{r,k}(d_1,\ldots,d_r)|}{d_1\cdots d_{r-1}d_r^{\theta_k+\varepsilon}},$$

giving the error

$$x^{r-1+\theta_k+\varepsilon}, \tag{5.22}$$

since the involved series is convergent.

Therefore, (3.17) follows by (5.20), (5.21) and (5.22).

The proof of (3.18) is by similar arguments, based on Lemma 5.3.2. □

Proof of Theorem 3.3.2. Similar to the proof of Theorem 3.3.1, by selecting $k = 2$, using Lemma 5.3.3 and the fact that the behavior of $\tau^{(e)}$ is similar to $\tau(1,2;n)$, as explained before. □

Proof of Theorem 3.3.3. For each $1 \leqslant j \leqslant r$, we write by condition (i),

$$F_j(x) = M_j(x) + E_j(x),$$

where

$$M_j(x) = x^{a_j} P_j(\log x), \quad E_j(x) = O(x^{b_j}).$$

Then we have

$$\begin{aligned}\sum_{n_1,\ldots,n_r\leqslant x} h(n_1\cdots n_r) &= \sum_{d_1,\ldots,d_r\leqslant x} g(d_1,\ldots,d_r)\prod_{j=1}^{r} F_j(x/d_j) \qquad (5.23)\\ &= \sum_{d_1,\ldots,d_r\leqslant x} g(d_1,\ldots,d_r)\prod_{j=1}^{r} \left(M_j(x/d_j)+E_j(x/d_j)\right).\end{aligned}$$

It is easy to see that we can write

$$\prod_{j=1}^{r}\left(M_j(x/d_j)+E_j(x/d_j)\right) = \prod_{j=1}^{r} M_j(x/d_j) + \eta(x;d_1,\ldots,d_r), \tag{5.24}$$

$$\eta(x;d_1,\ldots,d_r) \ll \sum_{j=1}^{r}\left(\frac{x}{d_j}\right)^{b_j} \prod_{\substack{1\leqslant k\leqslant r\\ k\neq j}} \left(\frac{x}{d_k}\right)^{a_k} \times (\log x)^{\delta_1+\cdots+\delta_r}.$$

Let

$$L_j(x) := x^{a_1+\cdots+a_{j-1}+b_j+a_{j+1}+\cdots+a_r} \quad (1\leqslant j\leqslant r).$$

The contribution of $\eta(x;d_1,\ldots,d_r)$ is

$$\begin{aligned}&\ll \sum_{d_1,\ldots,d_r\leqslant x} |g(d_1,\ldots,d_r)| \times |\eta(x;d_1,\cdots,d_r)| \qquad (5.25)\\ &\ll (\log x)^{\delta_1+\cdots+\delta_r}\sum_{j=1}^{r} L_j(x) \sum_{d_1,\ldots,d_r\leqslant x} \frac{|g(d_1,\ldots,d_r)|}{d_1^{a_1}\cdots d_{j-1}^{a_{j-1}} d_j^{b_j} d_{j+1}^{a_{j+1}}\cdots d_r^{a_r}}\\ &\ll x^{a_1+\cdots+a_r-\Delta}(\log x)^{\delta_1+\cdots+\delta_r},\end{aligned}$$

where we used condition (ii).

Now we evaluate the sum

$$M(x) := \sum_{d_1,\dots,d_r\leqslant x} g(d_1,\dots,d_r)\prod_{j=1}^{r} M_j\left(\frac{x}{d_j}\right).$$

Since $M_j(u) = x^{a_j}P_j(u)$ with $P_j(u)$ a polynomial in u of degree δ_j, we have

$$\prod_{j=1}^{r} M_j\left(\frac{x}{d_j}\right) = \frac{x^{a_1+\dots+a_r}}{d_1^{a_1}\cdots d_r^{a_r}}\sum_{\ell=0}^{\delta_1+\dots+\delta_r} C_\ell(\log d_1,\dots,\log d_r)(\log x)^\ell,$$

where

$$C_\ell(\log d_1,\dots,\log d_r) = \sum_{j_1,\dots,j_r} c(j_1,\dots,j_r)(\log d_1)^{j_1}\cdots(\log d_r)^{j_r},$$

the sum being over $0\leqslant j_t\leqslant \delta_t$ $(1\leqslant t\leqslant r)$. So we have

$$\begin{aligned} M(x) &= x^{a_1+\dots+a_r}\sum_{\ell=0}^{\delta_1+\dots+\delta_r}(\log x)^\ell \sum_{d_1,\dots,d_r\leqslant x}\frac{g(d_1,\dots,d_r)C_\ell(\log d_1,\dots,\log d_r)}{d_1^{a_1}\cdots d_r^{a_r}} \qquad (5.26)\\ &= x^{a_1+\dots+a_r}\sum_{\ell=0}^{\delta_1+\dots+\delta_r} d_\ell(\log x)^\ell \\ &\quad + \quad x^{a_1+\dots+a_r}\sum_{\ell=0}^{\delta_1+\dots+\delta_r}(\log x)^\ell {\sum_{d_1,\dots,d_r}}' \frac{g(d_1,\dots,d_r)C_\ell(\log d_1,\dots,\log d_r)}{d_1^{a_1}\cdots d_r^{a_r}}, \end{aligned}$$

where

$$d_\ell := \sum_{d_1,\dots,d_r=1}^{\infty}\frac{g(d_1,\dots,d_r)C_\ell(\log d_1,\dots,\log d_r)}{d_1^{a_1}\cdots d_r^{a_r}}$$

and where $\sum'$ means that there is at least one j $(1\leqslant j\leqslant r)$ such that $d_j > x$. Without loss of generality, we suppose $d_r > x$.

Suppose $\varepsilon > 0$ is sufficiently small and we have $\log n \ll n^\varepsilon$. Thus

$$\begin{aligned} &x^{a_1+\dots+a_r}\sum_{\ell=0}^{\delta_1+\dots+\delta_r}(\log x)^\ell {\sum_{d_1,\dots,d_r}}' \frac{g(d_1,\dots,d_r)C_\ell(\log d_1,\dots,\log d_r)}{d_1^{a_1}\cdots d_r^{a_r}} \qquad (5.27)\\ &\ll x^{a_1+\dots+a_r}\sum_{\ell=0}^{\delta_1+\dots+\delta_r}(\log x)^\ell \sum_{d_1,\dots,d_{r-1}=1}^{\infty}\sum_{d_r>x}\frac{|g(d_1,\dots,d_r)|d_1^{\delta_1\varepsilon}\cdots d_r^{\delta_r\varepsilon}}{d_1^{a_1}\cdots d_r^{a_r}} \end{aligned}$$

$$
\begin{aligned}
&\ll x^{a_1+\cdots+a_r} \sum_{\ell=0}^{\delta_1+\cdots+\delta_r} (\log x)^\ell \sum_{d_1,\ldots,d_{r-1}=1}^{\infty} \sum_{d_r>x} \frac{|g(d_1,\ldots,d_r)| d_1^{\delta_1\varepsilon} \cdots d_r^{\delta_r\varepsilon}}{d_1^{a_1}\cdots d_{r-1}^{a_{r-1}} d_r^{b_r}} \times \frac{1}{d_r^{a_r-b_r}} \\
&\ll x^{a_1+\cdots+a_r-(a_r-b_r)} \sum_{\ell=0}^{\delta_1+\cdots+\delta_r} (\log x)^\ell \sum_{d_1,\ldots,d_r=1}^{\infty} \frac{|g(d_1,\ldots,d_r)|}{d_1^{a_1-\delta_1\varepsilon}\cdots d_{r-1}^{a_{r-1}-\delta_{r-1}\varepsilon} d_r^{b_r-\delta_r\varepsilon}} \\
&\ll x^{a_1+\cdots+a_r-(a_r-b_r)} (\log x)^{\delta_1+\cdots+\delta_r} \ll x^{a_1+\cdots+a_r-\Delta}(\log x)^{\delta_1+\cdots+\delta_r},
\end{aligned}
$$

by using condition (ii) again.

From (5.26) and (5.27) we get

$$
M(x) = x^{a_1+\cdots+a_r} \sum_{\ell=0}^{\delta_1+\cdots+\delta_r} d_\ell (\log x)^\ell + O(x^{a_1+\cdots+a_r-\Delta}(\log x)^{\delta_1+\cdots+\delta_r}). \tag{5.28}
$$

Now Theorem 3.3.3 follows from (5.23), (5.24), (5.25) and (5.28). □

Proof of Theorem 3.3.4. Apply Theorem 3.3.3 in the case $f_j(n)=\tau(n)$, $a_j=1$, $b_j=\theta+\varepsilon$, $\delta_j=1$ ($1\leqslant j\leqslant r$) by using Lemma 5.3.1 and Lemma 5.3.2. □

Proof of Theorem 3.3.5. Apply Theorem 3.3.3 in the case $f_j(n)=\tau^{(2)}(n)$, $a_j=1$, $b_j=1/2+\varepsilon$, $\delta_j=1$ ($1\leqslant j\leqslant r$) by using Lemma 5.3.4. □

5.4 Proofs of the results of Section 3.4

Proof of Theorem 3.4.1. Applying formula (2.45) we deduce for $\Re z, \Re w>1$,

$$
\sum_{m,n=1}^{\infty} \frac{s(m,n)}{m^z n^w} = \sum_{d,a,b=1}^{\infty} \frac{\phi(d)\tau(a)\tau(b)}{(da)^z (db)^w} = \sum_{d=1}^{\infty} \frac{\phi(d)}{d^{z+w}} \sum_{a=1}^{\infty} \frac{\tau(a)}{a^z} \sum_{b=1}^{\infty} \frac{\tau(b)}{b^w},
$$

and using the familiar formulas for the latter Dirichlet series we obtain (3.20). □

We need the following result.

Lemma 5.4.1. *For $m,n\in\mathbb{N}$ let*

$$
T(m,n) := \sum_{\ell \mid \gcd(m,n)} \ell\, \tau\left(\frac{m}{\ell}\right) \tau\left(\frac{n}{\ell}\right),
$$

i.e.,

$$
\sum_{m,n=1}^{\infty} \frac{T(m,n)}{m^z n^w} = \zeta^2(z)\zeta^2(w)\zeta(z+w-1)
$$

for $\Re z, \Re w > 1$. Then for an arbitrary fixed $\varepsilon > 0$,

$$S(x) := \sum_{m,n \leqslant x} T(m,n) = x^2 \left(\sum_{r=0}^{3} c_r (\log x)^r \right) + O\left(x^{\frac{1117}{701}+\varepsilon} \right) .$$

Here $c_3 = \frac{1}{3}$, $c_2 = 3\gamma - 1$, $c_1 = 8\gamma^2 - 6\gamma - 2\gamma_1 + 1$.

The constant c_0 can be constructed from the proof below, but is not accessible to numerical evaluation for the reason described in Remark 3.4.4.

Proof of Lemma 5.4.1. For x large, let $1 < y < x$ be a positive real parameter at our disposal, and put $z := \frac{x}{y}$. Further, write $M := \max(j,k)$ for short. Then,

$$\begin{aligned} S(x) = \sum_{\substack{\ell M \leqslant x \\ \ell,j,k \in \mathbb{N}}} \ell\, \tau(j)\tau(k) &= \left\{ \sum_{\substack{\ell M \leqslant x \\ \ell \leqslant y}} + \sum_{\substack{\ell M \leqslant x \\ M \leqslant z}} - \sum_{\substack{\ell M \leqslant x \\ \ell \leqslant y, M \leqslant z}} \right\} \ell\, \tau(j)\tau(k) \\ &=: S_1(x) + S_2(x) - S_3(x) , \end{aligned} \tag{5.29}$$

say. Let θ_0 be the infimum of the exponents θ in the Dirichlet divisor problem (1.1). In what follows let θ be an arbitrary fixed real greater than θ_0.[1]. Then, firstly,

$$\begin{aligned} S_3(x) &= \sum_{\ell \leqslant y} \ell \sum_{j,k \leqslant z} \tau(j)\tau(k) \\ &= \left(\tfrac{1}{2}y^2 + O(y)\right)\left(z\log z + (2\gamma-1)z + O(z^\theta)\right)^2 \\ &= \tfrac{1}{2}x^2\log^2 z + (2\gamma-1)x^2\log z + \tfrac{1}{2}(2\gamma-1)^2x^2 \\ &+ O\left(\frac{x^2}{y}\log^2 x\right) + O\left(x^{1+\theta}y^{1-\theta}\right) . \end{aligned} \tag{5.30}$$

Secondly,

$$\begin{aligned} S_1(x) &= \sum_{\ell \leqslant y} \ell \left(\left(\frac{x}{\ell}\log\frac{x}{\ell} + (2\gamma-1)\frac{x}{\ell} \right)^2 + O\left(\left(\frac{x}{\ell}\right)^{1+\theta} \right) \right) \\ &= \sum_{\ell \leqslant y} \ell \left(\frac{x}{\ell}\log\frac{x}{\ell} + (2\gamma-1)\frac{x}{\ell} \right)^2 + O\left(x^{1+\theta}y^{1-\theta}\right) . \end{aligned} \tag{5.31}$$

By a direct computation,

$$\begin{aligned} &\sum_{\ell \leqslant y} \ell \left(\frac{x}{\ell}\log\frac{x}{\ell} + (2\gamma-1)\frac{x}{\ell} \right)^2 \\ &= x^2 \sum_{\ell \leqslant y} \frac{\log^2 \ell}{\ell} - 2x^2(\log x + (2\gamma-1)) \sum_{\ell \leqslant y} \frac{\log \ell}{\ell} \\ &+ x^2(\log x + (2\gamma-1))^2 \sum_{\ell \leqslant y} \frac{1}{\ell} . \end{aligned} \tag{5.32}$$

[1]This arrangement implies that $O(x^\theta \log x)$ can be replaced throughout by $O(x^\theta)$, etc.

By Euler's summation formula, for $r = 0, 1, 2$,

$$\sum_{\ell \leqslant y} \frac{\log^r \ell}{\ell} = \frac{\log^{r+1} y}{r+1} + \gamma_r + O\left(\frac{\log^r y}{y}\right) .$$

Combining this with (5.32) and (5.31), we get

$$\begin{aligned} S_1(x) &= x^2 \bigl(\log^2 x(\log y + \gamma) \\ &- \log x \left(\log^2 y - 2(2\gamma - 1)\log y - 2\gamma(2\gamma-1) + 2\gamma_1\right) \\ &+ \frac{1}{3}\log^3 y - (2\gamma-1)\log^2 y \\ &+ (2\gamma-1)^2 \log y + \gamma(2\gamma-1)^2 - 4\gamma\gamma_1 + 2\gamma_1 + \gamma_2\bigr) \\ &+ O\left(\frac{x^2}{y}\log^2 x\right) + O\left(x^{1+\theta}y^{1-\theta}\right) . \end{aligned} \tag{5.33}$$

Finally, with $M := \max(j,k)$,

$$S_2(x) = \sum_{j,k \leqslant z} \tau(j)\tau(k) \sum_{\ell \leqslant \frac{x}{M}} \ell = \sum_{j,k\leqslant z} \tau(j)\tau(k)\left(\frac{x^2}{2M^2} + O\left(\frac{x}{M}\right)\right) . \tag{5.34}$$

The O-term here contributes overall

$$\ll x \sum_{j,k\leqslant z} \frac{\tau(j)\tau(k)}{\sqrt{jk}} \ll x \left(\sum_{j \leqslant z} \frac{\tau(j)}{\sqrt{j}}\right)^2 \ll xz \log^2 x = \frac{x^2}{y}\log^2 x . \tag{5.35}$$

Writing $S_2^*(x)$ for the main term in (5.34), we get

$$\begin{aligned} S_2^*(x) &= \frac{x^2}{2} \sum_{j,k\leqslant z} \frac{\tau(j)\tau(k)}{\max(j^2,k^2)} \\ &= x^2 \sum_{j \leqslant k \leqslant z} \frac{\tau(k)}{k^2}\tau(j) - \frac{x^2}{2}\sum_{k\leqslant z}\frac{\tau^2(k)}{k^2} =: x^2\left(R_1(z) - \tfrac{1}{2}R_2(z)\right) . \end{aligned} \tag{5.36}$$

Now

$$R_2(z) = \frac{\zeta^4(2)}{\zeta(4)} + O\left(\frac{\log^3 x}{z}\right) = \frac{5\pi^4}{72} + O\left(\frac{\log^3 x}{z}\right) . \tag{5.37}$$

Further,

$$\begin{aligned} R_1(z) &= \sum_{k\leqslant z} \frac{\tau(k)}{k^2}\left(k\log k + (2\gamma-1)k + \Delta(k)\right) \\ &= \sum_{k\leqslant z}\frac{\tau(k)}{k}(\log k + 2\gamma - 1) + \sum_{k\leqslant z}\frac{\tau(k)\Delta(k)}{k^2} . \end{aligned} \tag{5.38}$$

Here the last sum equals

$$\sum_{k=1}^{\infty} \frac{\tau(k)\Delta(k)}{k^2} + O\left(z^{\theta-1}\right) =: C_1 + O\left(z^{\theta-1}\right) .$$

Moreover, direct computations show that

$$\sum_{k\leqslant z} \frac{\tau(k)}{k}(\log k + 2\gamma - 1)$$

$$= \frac{1}{3}\log^3 z + \left(2\gamma - \tfrac{1}{2}\right)\log^2 z + 2\gamma(2\gamma-1)\log z + C_2 + O\left(z^{\theta-1}\right), \qquad (5.39)$$

where

$$C_2 := (2\gamma-1)^2 + \int_1^{\infty} \frac{\log u + 2(\gamma-1)}{u^2}\Delta(u)\,\mathrm{d}u .$$

Putting together (5.34) - (5.39), and recalling that $z = \frac{x}{y}$, we arrive at

$$\begin{aligned} S_2(x) = x^2 \Bigg(& \frac{1}{3}\log^3 z + \left(2\gamma - \tfrac{1}{2}\right)\log^2 z + 2\gamma(2\gamma-1)\log z \\ & + C_1 + C_2 - \frac{5\pi^4}{144}\Bigg) + O\left(\frac{x^2}{y}\log^2 x\right) + O\left(x^{1+\theta}y^{1-\theta}\right) . \end{aligned} \qquad (5.40)$$

Finally, using (5.30), (5.33), and (5.40) in (5.29), a direct calculation yields

$$S(x) = x^2\left(\sum_{r=0}^{3} c_r(\log x)^r\right) + O\left(\frac{x^2}{y}\log^2 x\right) + O\left(x^{1+\theta}y^{1-\theta}\right) ,$$

with c_1, c_2, c_3 as stated in Lemma 5.4.1. Balancing the two O-terms here, the optimal choice is $y = x^{\frac{1-\theta}{2-\theta}}$. This completes the proof of Lemma 5.4.1. □

Proof of Theorem 3.4.2. Use that (cf. Theorem 3.4.1),

$$s(m,n) = \sum_{d \mid \gcd(m,n)} \mu(d) T(m/d, n/d) \quad (m,n \in \mathbb{N}).$$

We deduce

$$\begin{aligned} \sum_{m,n\leqslant x} s(m,n) &= \sum_{d\leqslant x} \mu(d) \sum_{a,b\leqslant x/d} T(a,b) \\ &= \sum_{d\leqslant x} \mu(d) \left(\left(\frac{x}{d}\right)^2 \sum_{r=0}^{3} c_r \left(\log\frac{x}{d}\right)^r + O\left(\left(\frac{x}{d}\right)^{\frac{1117}{701}+\varepsilon}\right)\right) \\ &= x^2 V(x) + O\left(\sum_{d\leqslant x}\left(\frac{x}{d}\right)^{\frac{1117}{701}+\varepsilon}\right) , \end{aligned}$$

where the error term is $O\left(x^{\frac{1117}{701}+\varepsilon}\right)$ and

$$\begin{aligned}
V(x) &= \left(c_3\log^3 x + c_2\log^2 x + c_1\log x + c_0\right)\sum_{d\leqslant x}\frac{\mu(d)}{d^2} \\
&\quad - \left(3c_3\log^2 x + 2c_2\log x + c_1\right)\sum_{d\leqslant x}\frac{\mu(d)\log d}{d^2} \\
&\quad + (3c_3\log x + c_2)\sum_{d\leqslant x}\frac{\mu(d)\log^2 d}{d^2} - c_3\sum_{d\leqslant x}\frac{\mu(d)\log^3 d}{d^2} \\
&= \left(c_3\log^3 x + c_2\log^2 x + c_1\log x + c_0\right)\left(\frac{1}{\zeta(2)} + O\left(\frac{1}{x}\right)\right) \\
&\quad - \left(3c_3\log^2 x + 2c_2\log x + c_1\right)\left(\frac{\zeta'(2)}{\zeta^2(2)} + O\left(\frac{\log x}{x}\right)\right) \\
&\quad + (3c_3\log x + c_2)\left(\frac{2(\zeta'(2))^2 - \zeta''(2)\zeta(2)}{\zeta^3(2)} + O\left(\frac{\log^3 x}{x}\right)\right) \\
&\quad - c_3\left(c^* + O\left(\frac{\log^3 x}{x}\right)\right),
\end{aligned}$$

with a certain constant c^*, leading to the asymptotic formula (3.21). □

Sketch of proof of Theorem 3.4.5. By (2.45)

$$S^{(2)}(x) = \sum_{\substack{m,n\leqslant x\\ \gcd(m,n)>1}} s(m,n) = \sum_{m,n\leqslant x} s(m,n) - \sum_{\substack{m,n\leqslant x\\ \gcd(m,n)=1}} \tau(m)\tau(n). \tag{5.41}$$

Here

$$\begin{aligned}
T(x) := \sum_{\substack{m,n\leqslant x\\ \gcd(m,n)=1}} \tau(m)\tau(n) &= \sum_{m,n\leqslant x} \tau(m)\tau(n) \sum_{d\mid\gcd(m,n)} \mu(d) \\
&= \sum_{d\leqslant x}\mu(d)\left(\sum_{a\leqslant x/d}\tau(da)\right)^2.
\end{aligned} \tag{5.42}$$

Now, according to the Busche-Ramanujan identities for the divisor function (cf. Section 3.5),

$$\tau(mn) = \sum_{d\mid\gcd(m,n)} \mu(d)\tau\left(\frac{m}{d}\right)\tau\left(\frac{n}{d}\right) \tag{5.43}$$

for every $m, n \in \mathbb{N}$. Let K be a fixed integer. Using (5.43),

$$T_K(x) := \sum_{n \leqslant x} \tau(Kn) = \sum_{n \leqslant x} \sum_{d \mid \gcd(K,n)} \mu(d)\tau(K/d)\tau(n/d)$$

$$= \sum_{d \mid K} \mu(d)\tau(K/d) \sum_{j \leqslant x/d} \tau(j),$$

which can be computed by using Dirichlet's formula (1.1). We obtain

$$T_K(x) = \sum_{n \leqslant x} \tau(Kn) = \beta_0(K)x \log x + \beta_1(K)x + O(x^{\theta+\varepsilon}K^{\varepsilon}),$$

uniformly in K, where $\beta_0(K)$ and $\beta_1(K)$ are defined by (3.23). Therefore,

$$T(x) = \sum_{d \leqslant x} \mu(d)T_d(x)^2 = \sum_{d \leqslant x} \mu(d) \left(\beta_0(d)\frac{x}{d} \log \frac{x}{d} + \beta_1(d)\frac{x}{d} + O((\frac{x}{d})^{\theta+\varepsilon} d^{\varepsilon}) \right),$$

which gives, by direct computations,

$$T(x) = x^2(b_2 \log^2 x + b_1 \log x + b_0) + O(x^{1+\theta+\varepsilon}) \tag{5.44}$$

with the constants given in Theorem 3.4.5. Note that in our paper [71] formula (5.44) was obtained with the weaker error term $O(x^{4/3+\varepsilon})$.

By (5.41) and (5.42) we have

$$S^{(2)}(x) = \sum_{m,n \leqslant x} s(m,n) - T(x),$$

and obtain the final result from Theorem 3.4.2 and formula (5.44). Here the error for the first sum is $O(x^{1117/701+\varepsilon})$, with $1117/701 \doteq 1.593437$, so this will be the final error term of (3.24). □

5.5 Proofs of the results of Section 3.5

Proof of Theorem 3.5.1. Formula (3.27) is a direct consequence of the following identity concerning multiple Dirichlet series: If g and h are completely multiplicative functions and $f = g * h$, then

$$\sum_{n_1,\ldots,n_r=1}^{\infty} \frac{f(n_1 \cdots n_r)}{n_1^{s_1} \cdots n_r^{s_r}} = \left(\sum_{n_1=1}^{\infty} \frac{f(n_1)}{n_1^{s_1}} \right) \cdots \left(\sum_{n_r=1}^{\infty} \frac{f(n_r)}{n_r^{s_r}} \right)$$

$$\times \prod_p \left(1 + g(p)h(p) \sum_{j=2}^{r} (-1)^{j-1} f(p^{j-2}) \sum_{1 \leqslant i_1 < \ldots < i_j \leqslant r} \frac{1}{p^{s_{i_1}+\ldots+s_{i_j}}} \right). \tag{5.45}$$

Now we prove (5.45). Since the function $(n_1,\ldots,n_r)\mapsto f(n_1\cdots n_r)$ is multiplicative, we deduce

$$D := \sum_{n_1,\ldots,n_r=1}^{\infty} \frac{f(n_1\cdots n_r)}{n_1^{s_1}\cdots n_r^{s_r}} = \prod_p \sum_{\nu_1,\ldots,\nu_r=0}^{\infty} \frac{f(p^{\nu_1+\ldots+\nu_r})}{p^{\nu_1 s_1+\ldots+\nu_r s_r}}$$
$$= \prod_p \sum_{\nu_1,\ldots,\nu_r=0}^{\infty} \frac{x^{\nu_1+\ldots+\nu_r+1}-y^{\nu_1+\ldots+\nu_r+1}}{(x-y)p^{\nu_1 s_1+\ldots+\nu_r s_r}},$$

where $x=g(p)$ and $y=h(p)$, for short.

Therefore,

$$D = \prod_p \frac{1}{x-y}\left(x\sum_{\nu_1,\ldots,\nu_r=0}^{\infty}\left(\frac{x}{p^{s_1}}\right)^{\nu_1}\cdots\left(\frac{x}{p^{s_r}}\right)^{\nu_r} - y\sum_{\nu_1,\ldots,\nu_r=0}^{\infty}\left(\frac{y}{p^{s_1}}\right)^{\nu_1}\cdots\left(\frac{y}{p^{s_r}}\right)^{\nu_r}\right)$$
$$= \prod_p \frac{1}{x-y}\left(x\left(1-\frac{x}{p^{s_1}}\right)^{-1}\cdots\left(1-\frac{x}{p^{s_r}}\right)^{-1} - y\left(1-\frac{y}{p^{s_1}}\right)^{-1}\cdots\left(1-\frac{y}{p^{s_r}}\right)^{-1}\right)$$
$$= \prod_p \frac{1}{x-y}\left(1-\frac{x}{p^{s_1}}\right)^{-1}\left(1-\frac{y}{p^{s_1}}\right)^{-1}\cdots\left(1-\frac{x}{p^{s_r}}\right)^{-1}\left(1-\frac{y}{p^{s_r}}\right)^{-1}$$
$$\times\left(x\left(1-\frac{y}{p^{s_1}}\right)\cdots\left(1-\frac{y}{p^{s_r}}\right) - y\left(1-\frac{x}{p^{s_1}}\right)\cdots\left(1-\frac{x}{p^{s_r}}\right)\right)$$
$$= D(g,s_1)D(h,s_1)\cdots D(g,s_r)D(h,s_r)$$
$$\times\prod_p\left(1-xy\sum_{1\leqslant i<j\leqslant r}\frac{1}{p^{s_i+s_j}} + xy\,(x+y)\sum_{1\leqslant i<j<k\leqslant r}\frac{1}{p^{s_i+s_j+s_k}} - \ldots\right.$$
$$\left. +(-1)^{r-1}xy\left(x^{r-2}+x^{r-3}y+\ldots+xy^{r-3}+y^{r-2}\right)\frac{1}{p^{s_1+\ldots+s_r}}\right),$$

simplifying by $x-y$, which shows that the computations are valid also in the case when $x-y=0$, that is, $g(p)=h(p)$ for some primes p. This gives (5.45). Formula (3.28) is obtained by expressing the function $(n_1,\ldots,n_r)\mapsto f(n_1)\cdots f(n_r)$ from the convolutional identity (3.27). This completes the proof. □

Proof of Theorem 3.5.2. Let $e_d(x_1,\ldots,x_k)=\sum_{1\leqslant i_1<\ldots<i_d\leqslant k} x_{i_1}\cdots x_{i_d}$ be the elementary symmetric polynomial in $x_1,\ldots,x_k$ of degree d ($d\geqslant 0$). By convention, $e_0(x_1,\ldots,x_k)=1$ and $e_d(x_1,\ldots,x_k)=0$ ($d\geqslant k+1$). Furthermore, let $h_d(x_1,\ldots,x_k)=\sum_{1\leqslant i_1\leqslant\ldots\leqslant i_d\leqslant k} x_{i_1}\cdots x_{i_d}$ stand for the complete homogeneous symmetric polynomial in $x_1,\ldots,x_k$ of degree d ($d\geqslant 0$). By convention, $h_0(x_1,\ldots,x_k)=1$.

We will use the known polynomial identity

$$\sum_{d=0}^{n}(-1)^d e_d(x_1,\ldots,x_k)h_{n-d}(x_1,\ldots,x_k) = \begin{cases}1, & n=0,\\ 0, & n\geqslant 1.\end{cases} \tag{5.46}$$

We also need the representation (cf. [25])

$$h_d(x_1,\dots,x_k)=\sum_{i=1}^{k} x_i^{d+k-1}\prod_{\substack{j=1\\ j\ne i}}^{k}(x_i-x_j)^{-1} \quad (d\geqslant 0), \tag{5.47}$$

The connection between arithmetic functions and symmetric polynomials we utilize in the proof is the following: If $f_1,\dots,f_k$ are completely multiplicative functions of a single variable and $F=f_1*\cdots*f_k$, then for every prime power p^ν $(\nu\in\mathbb{N})$,

$$F(p^\nu)=h_\nu(x_1,\dots,x_k), \tag{5.48}$$

where $x_i=f_i(p)$ $(1\leqslant i\leqslant k)$.

Using (5.48) and (5.47) we deduce

$$\sum_{n_1,n_2=1}^{\infty}\frac{F(n_1n_2)}{n_1^{s_1}n_2^{s_2}}=\prod_p\sum_{\nu_1,\nu_2=0}^{\infty}\frac{F(p^{\nu_1+\nu_2})}{p^{\nu_1s_1+\nu_2s_2}}$$

$$=\prod_p\left(\sum_{\nu_1,\nu_2=0}^{\infty}\frac{1}{p^{\nu_1s_1+\nu_2s_2}}\sum_{i=1}^{k}x_i^{\nu_1+\nu_2+k-1}\prod_{\substack{j=1\\ j\ne i}}^{k}(x_i-x_j)^{-1}\right)$$

$$=\prod_p\left(\sum_{i=1}^{k}x_i^{k-1}\prod_{\substack{j=1\\ j\ne i}}^{k}(x_i-x_j)^{-1}\sum_{\nu_1,\nu_2=0}^{\infty}\left(\frac{x_i}{p^{s_1}}\right)^{\nu_1}\left(\frac{x_i}{p^{s_2}}\right)^{\nu_2}\right)$$

$$=\prod_p\left(\sum_{i=1}^{k}x_i^{k-1}\left(1-\frac{x_i}{p^{s_1}}\right)^{-1}\left(1-\frac{x_i}{p^{s_2}}\right)^{-1}\prod_{\substack{j=1\\ j\ne i}}^{k}(x_i-x_j)^{-1}\right)$$

$$=\prod_p\left(\prod_{\ell=1}^{k}\left(1-\frac{x_\ell}{p^{s_1}}\right)^{-1}\left(1-\frac{x_\ell}{p^{s_2}}\right)^{-1}\sum_{i=1}^{k}x_i^{k-1}\prod_{\substack{j=1\\ j\ne i}}^{k}(x_i-x_j)^{-1}\left(1-\frac{x_j}{p^{s_1}}\right)\left(1-\frac{x_j}{p^{s_2}}\right)\right)$$

$$=\left(\sum_{n_1=1}^{\infty}\frac{F(n_1)}{n_1^{s_1}}\right)\left(\sum_{n_2=1}^{\infty}\frac{F(n_r)}{n_2^{s_2}}\right)\prod_p Q_k(p^{-s_1},p^{-s_2}),$$

where $Q_k(u,v)$ is the polynomial in u and v, given by

$$Q_k(u,v)=\sum_{i=1}^{k}x_i^{k-1}\prod_{\substack{j=1\\ j\ne i}}^{k}(x_i-x_j)^{-1}(1-x_ju)(1-x_jv)=\sum_{m,n=0}^{k-1}c_{m,n}u^mv^n.$$

Here the coefficients $c_{m,n}$ $(1 \leqslant m, n \leqslant k-1)$ are given by

$$c_{m,n} = (-1)^{m+n} \sum_{i=1}^{k} x_i^{k-1} \prod_{\substack{j=1 \\ j \neq i}}^{k} (x_i - x_j)^{-1} e_m^{(i)}(x_1, \ldots, x_k) e_n^{(i)}(x_1, \ldots, x_k) \tag{5.49}$$

where $e_m^{(i)}(x_1, \ldots, x_k) = e_m(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k)$ $(1 \leqslant i \leqslant k)$. We will show that

$$c_{m,n} = \begin{cases} 1, & m = n = 0, \\ (-1)^{m+n-1} e_{m+n}(x_1, \ldots, x_k), & m, n \geqslant 1, m+n \leqslant k \\ 0, & \text{otherwise.} \end{cases}$$

To this end, note that $e_m^{(i)}(x_1, \ldots, x_k) = e_m(x_1, \ldots, x_k) - x_i e_{m-1}^{(i)}(x_1, \ldots, x_k)$ $(1 \leqslant m \leqslant k)$, which leads to the identity

$$e_m^{(i)}(x_1, \ldots, x_k) = \sum_{\ell=0}^{m} (-1)^\ell x_i^\ell e_{m-\ell}(x_1, \ldots, x_k).$$

Therefore, from (5.49) we deduce

$$c_{m,n} = (-1)^{m+n} \sum_{i=1}^{k} x_i^{k-1} \prod_{\substack{j=1 \\ j \neq i}}^{k} (x_i - x_j)^{-1} \sum_{\ell=0}^{m} (-1)^\ell x_i^\ell e_{m-\ell}(x_1, \ldots, x_k) \sum_{s=0}^{n} (-1)^s x_i^s e_{n-s}(x_1, \ldots, x_k)$$

$$= (-1)^{m+n} \sum_{\ell=0}^{m} \sum_{s=0}^{n} (-1)^{\ell+s} e_{m-\ell}(x_1, \ldots, x_k) e_{n-s}(x_1, \ldots, x_k) \sum_{i=1}^{k} x_i^{\ell+s+k-1} \prod_{\substack{j=1 \\ j \neq i}}^{k} (x_i - x_j)^{-1}$$

$$= (-1)^{m+n} \sum_{\ell=0}^{m} (-1)^\ell e_{m-\ell}(x_1, \ldots, x_k) \sum_{s=0}^{n} (-1)^s e_{n-s}(x_1, \ldots, x_k) h_{\ell+s}(x_1, \ldots, x_k), \tag{5.50}$$

using again (5.47). For $n = 0$ this gives, by (5.46),

$$c_{m,0} = (-1)^m \sum_{\ell=0}^{m} (-1)^\ell e_{m-\ell}(x_1, \ldots, x_k) h_\ell(x_1, \ldots, x_k) = \begin{cases} 1, & m = 0, \\ 0, & m \geqslant 1. \end{cases}$$

We deduce in the same way that $c_{0,n} = 0$ for $n \geqslant 1$. Now let $m, n \geqslant 1$. Then the inner sum in (5.50) is, by denoting $j = \ell + s$,

$$\sum_{j=\ell}^{n+\ell} (-1)^{j-\ell} e_{n+\ell-j}(x_1, \ldots, x_k) h_j(x_1, \ldots, x_k)$$

$$= (-1)^{\ell} \left(\sum_{j=0}^{n+\ell} (-1)^j e_{n+\ell-j}(x_1,\ldots,x_k) h_j(x_1,\ldots,x_k) \right.$$

$$\left. - \sum_{j=0}^{\ell-1} (-1)^j e_{n+\ell-j}(x_1,\ldots,x_k) h_j(x_1,\ldots,x_k) \right)$$

$$= (-1)^{\ell-1} \sum_{j=0}^{\ell-1} (-1)^j e_{n+\ell-j}(x_1,\ldots,x_k) h_j(x_1,\ldots,x_k), \tag{5.51}$$

since the first sum is zero, according to (5.46), where $n+\ell \geqslant 1+\ell \geqslant 1$ for every $\ell \geqslant 0$. For $\ell = 0$ (5.51) is zero (empty sum). We obtain

$$c_{m,n} = (-1)^{m+n-1} \sum_{\ell=1}^{m} e_{m-\ell}(x_1,\ldots,x_k) \sum_{j=0}^{\ell-1} (-1)^j e_{n+\ell-j}(x_1,\ldots,x_k) h_j(x_1,\ldots,x_k)$$

and regrouping the terms according to the values $t = \ell - j$,

$$c_{m,n} = (-1)^{m+n-1} \sum_{t=1}^{m} e_{n+t}(x_1,\ldots,x_k) \sum_{j=0}^{m-t} (-1)^j e_{m-t-j}(x_1,\ldots,x_k) h_j(x_1,\ldots,x_k),$$

where the inner sum is 0 for $t < m$ and it is 1 for $t = m$. Therefore,

$$c_{m,n} = (-1)^{m+n-1} e_{m+n}(x_1,\ldots,x_k),$$

which is zero for $m+n > k$. This finishes the proof of (3.30). Now (3.31) is obtained by expressing the function $(n_1,n_2) \mapsto F(n_1)F(n_2)$.

□

5.6 Proofs of the results of Section 3.6

Proof of Theorem 3.6.1. We use the following approach to quickly derive the Ramanujan expansions. For any $n_1,\ldots,n_k \in \mathbb{N}$ we have

$$f(n_1,\ldots,n_k) = \sum_{d_1 \mid n_1,\ldots,d_k \mid n_k} (\mu_k * f)(d_1,\ldots,d_k)$$

$$= \sum_{d_1,\ldots,d_k=1}^{\infty} \frac{(\mu_k * f)(d_1,\ldots,d_k)}{d_1 \cdots d_k} \sum_{q_1 \mid d_1} c_{q_1}(n_1) \cdots \sum_{q_k \mid d_k} c_{q_k}(n_k)$$

$$= \sum_{q_1,\ldots,q_k=1}^{\infty} c_{q_1}(n_1) \cdots c_{q_k}(n_k) \sum_{\substack{d_1,\ldots,d_k=1 \\ q_1 \mid d_1,\ldots,q_k \mid d_k}}^{\infty} \frac{(\mu_k * f)(d_1,\ldots,d_k)}{d_1 \cdots d_k},$$

giving expansion (3.33) with the coefficients (3.34), by denoting $d_1 = m_1q_1, \dots, d_k = m_kq_k$. The rearranging of the terms is justified by the absolute convergence of the multiple series, shown hereinafter:

$$\sum_{q_1,\dots,q_k=1}^{\infty} |a_{q_1,\dots,q_k}||c_{q_1}(n_1)|\cdots|c_{q_k}(n_k)|$$

$$\leqslant \sum_{\substack{q_1,\dots,q_k=1\\ m_1,\dots,m_k=1}}^{\infty} \frac{|(\mu_k * f)(m_1q_1,\dots,m_kq_k)|}{m_1q_1\cdots m_kq_k}|c^*_{q_1}(n_1)|\cdots|c^*_{q_k}(n_k)|$$

$$= \sum_{t_1,\dots,t_k=1}^{\infty} \frac{|(\mu_k * f)(t_1,\dots,t_k)|}{t_1\cdots t_k} \sum_{m_1q_1=t_1} |c_{q_1}(n_1)|\cdots \sum_{m_kq_k=t_k} |c_{q_k}(n_k)|$$

$$\leqslant n_1\cdots n_k \sum_{t_1,\dots,t_k=1}^{\infty} 2^{\omega(t_1)+\cdots+\omega(t_k)} \frac{|(\mu_k * f)(t_1,\dots,t_k)|}{t_1\cdots t_k} < \infty,$$

by using the inequality

$$\sum_{d|q} |c_d(n)| \leqslant 2^{\omega(q)} n \quad (n \in \mathbb{N})$$

and condition (3.32). □

Proof of Corollary 3.6.2. This is a direct consequence of Theorem 3.6.1 and the definition of multiplicative functions of k variables. □

Proof of Corollary 3.6.3. More generally, let $g : \mathbb{N} \to \mathbb{C}$ be an arithmetic function and let $k \in \mathbb{N}$. Assume that

$$\sum_{n=1}^{\infty} 2^{k\,\omega(n)} \frac{|(\mu * g)(n)|}{n^k} < \infty.$$

Then for every $n_1, \dots, n_k \in \mathbb{N}$,

$$g((n_1,\dots,n_k)) = \sum_{q_1,\dots,q_k=1}^{\infty} a_{q_1,\dots,q_k} c_{q_1}(n_1)\cdots c_{q_k}(n_k), \tag{5.52}$$

is absolutely convergent, where

$$a_{q_1,\dots,q_k} = \frac{1}{Q^k} \sum_{m=1}^{\infty} \frac{(\mu * g)(mQ)}{m^k},$$

with the notation $Q = [q_1, \dots, q_k]$. Indeed, apply Theorem 3.6.1 for $f(n_1,\dots,n_k) = g((n_1,\dots,n_k))$. The identity

$$g((n_1,\dots,n_k)) = \sum_{d|n_1,\dots,d|n_k} (\mu * g)(d),$$

shows that

$$(\mu_k * f)(n_1, \ldots, n_k) = \begin{cases} (\mu * g)(n), & \text{if } n_1 = \cdots = n_k = n, \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, if $\mu*g$ is completely multiplicative, then (5.52) holds for every $n_1, \ldots, n_k \in \mathbb{N}$ with coefficients

$$a_{q_1,\ldots,q_k} = \frac{(\mu * g)(Q)}{Q^k} \sum_{m=1}^{\infty} \frac{(\mu * g)(m)}{m^k}, \tag{5.53}$$

Now apply (5.53) to $g(n) = \sigma_s(n)/n^s$, where the function $(\mu*g)(n) = 1/n^s$ is completely multiplicative. □

Proof of Corollary 3.6.4. Apply identity (5.53) for $g(n) = \phi_s(n)/n^s$. Here $(\mu * g)(n) = \mu(n)/n^s$ and deduce that the coefficients are

$$a_{q_1,\ldots,q_k} = \frac{1}{Q^k} \sum_{m=1}^{\infty} \frac{\mu(mQ)}{m^{s+k}} = \frac{\mu(Q)}{Q^{s+k}} \sum_{\substack{m=1 \\ (m,Q)=1}}^{\infty} \frac{\mu(m)}{m^{s+k}} = \frac{\mu(Q)}{\zeta(s+k)\phi_{s+k}(Q)}.$$

□

Bibliography

[1] E. Alkan, Distribution of averages of Ramanujan sums, *Ramanujan J.* **29** (2012), 385–408.

[2] E. Alkan, Ramanujan sums and the Burgess zeta function, *Int. J. Number Theory* **8** (2012), 2069–2092.

[3] B. Apostol, Extremal orders of some functions connected to regular integers modulo n, *An. Stiint. Univ. Ovidius Constanta, Ser. Mat.* **21** (2013), 5–19.

[4] B. Apostol, Asymptotic properties of some functions related to regular integers modulo n, *Bull. Math. Soc. Sci. Math. Roumanie* **60 (108)** (2017), 221–231.

[5] B. Apostol and L. Petrescu, Extremal orders of certain functions associated with regular integers (mod n), *J. Integer Seq.* **16** (2013), Article 13.7.5, 13 pp.

[6] R. C. Baker, The square-free divisor problem II, *Quart. J. Math. (Oxford)* (2) **47** (1996), 133–146.

[7] N. Balasubramanian, On the Busche-Ramanujan identities, *Nieuw Arch. Wiskd.* IV. Ser. **15** (1997), 133–140.

[8] M. Balazard, M. Naimi, and Y.-F. S. Pétermann, Étude d'une somme arithmétique multiple liée à la fonction de Möbius, *Acta Arith.* **132** (2008), 245–298.

[9] M. Balazard and A. De Roton, Notes de lecture de l'article "Partial sums of the Möbius function" de Kannan Soundararajan (French), Preprint, 2008, 25 pp. http://arxiv.org/abs/0810.3587

[10] K. S. Berenhaut, E. E. Allen, and S. J. Fraser, Bounds on coefficients of reciprocals of formal power series with rapidly decreasing coefficients, *Discrete Dyn. Nat. Soc.* (2006), Article ID 40270.

[11] O. Bordellès and B. Cloitre, An alternating sum involving the reciprocal of certain multiplicative functions, *J. Integer Seq.* **16** (2013), Article 13.6.3, 12 pp.

[12] J. Bourgain and N. Watt, Mean square of zeta function, circle problem and divisor problem revisited, Preprint, 23 pp. https://arxiv.org/abs/1709.04340

[13] L. M. Butler, *Subgroup Lattices and Symmetric Functions*, Mem. Amer. Math. Soc., vol. 112, no. **539**, 1994.

[14] J.-Y. Cai and E. Bach, On testing for zero polynomials by a set of points with bounded precision, *Theoret. Comp. Sci.* **296** (2003), 15–25.

[15] W. C. Calhoun, Counting the subgroups of some finite groups, *Amer. Math. Monthly* **94** (1987), 54–59.

[16] X. Cao and W. Zhai, Some arithmetic functions involving exponential divisors, *J. Integer Seq.* **13** (2010), Article 10.3.7, 13 pp.

[17] X. Cao and W. Zhai, On the four-dimensional divisor problem of (a, b, c, c) type, *Funct. Approx. Comment. Math.* **49** (2013), 251–267.

[18] E. Cesàro, Ètude moyenne du plus grand commun diviseur de deux nombres, *Annali di Matematica Pura ed Applicata* **13** (1885), 235–250.

[19] S. Chen and W. Zhai, Reciprocals of the gcd-sum functions, *J. Integer Seq.* **14** (2011), Article 11.8.3, 13 pp.

[20] J. Chidambaraswamy and R. Sitaramachandrarao, Asymptotic results for a class of arithmetical functions, *Monatsh. Math.* **99** (1985), 19–27.

[21] E. Cohen, Rings of arithmetic functions. II: The number of solutions of quadratic congruences, *Duke Math. J.* **21** (1954), 9–28.

[22] E. Cohen, Arithmetical functions of a greatest common divisor. I, *Proc. Amer. Math. Soc.* **11** (1960), 164–171.

[23] E. Cohen, A generalization of Axer's theorem and some of its applications, *Math. Nachr.* **27** (1963/1964), 163–177.

[24] L. Comtet, *Advanced Combinatorics. The Art of Finite and Infinite Expansions*, D. Reidel Publishing Co., 1974.

[25] E. F. Cornelius, Jr., Identities for complete homogeneous symmetric polynomials, *JP J. Algebra Number Theory Appl.* **21** (2011), 109–116.

[26] J.-M. De Koninck and A. Ivić, *Topics in Arithmetical Functions*, North-Holland Mathematics Studies **43**, Notas de Matemática (72), North-Holland Publishing Company, XVII, 1980.

[27] J.-M. de Koninck and I. Kátai, Some remarks on a paper of L. Toth, *J. Integer Seq.* **13** (2010), no. 1, Article 10.1.2, 26 pp.

[28] R. de la Bretèche, Estimation de sommes multiples de fonctions arithmétiques, *Compos. Math.* **128** (2001), 261–298.

[29] J. A. de Reyna and R. Heyman, Counting tuples restricted by pairwise coprimality conditions, *J. Integer Seq.* **18** (2015), Article 15.10.4, 16 pp.

[30] H. Delange, On Ramanujan expansions of certain arithmetical functions, *Acta. Arith.* **31** (1976), 259–270.

[31] P. Diaconis and P. Erdős, On the distribution of the greatest common divisor, Technical Report No. 12, Department of Statistics, Stanford University, Stanford, 1977; Reprinted in *A festschrift for Herman Rubin, IMS Lecture Notes Monogr. Ser., Inst. Math. Statist.*, **45**, (2004), 56–61.

[32] P. Erdős and G. Szekeres, Über die Anzahl der Abelschen Gruppen gegebener Ordung und über ein verwandtes zahlentheoretisches Problem, *Acta Sci. Math. (Szeged)* **9** (1934-35), 95–102.

[33] J. Fabrykowski and M. V. Subbarao, The maximal order and the average order of the multiplicative function $\sigma^{(e)}(n)$, Théorie des nombres (Québec, PQ, 1987), 201–206, de Gruyter (Berlin – New York, 1989).

[34] J. L. Fernández and P. Fernández, On the probability distribution of the gcd and lcm of r-tuples of integers, Preprint, 2013, 24 pp. https://arxiv.org/abs/1305.0536

[35] J. L. Fernández and P. Fernández, Asymptotic normality and greatest common divisors, *Int. J. Number Theory* **11** (2015), 89–126.

[36] S. Finch, G. Martin, and P. Sebah, Roots of unity and nullity modulo n, *Proc. Amer. Math. Soc.* **138** (2010), 2729–2743.

[37] S. W. Graham and G. Kolesnik, On the difference between consecutive squarefree integers, *Acta Arith.* **49** (1988), 435–447.

[38] M. Hampejs and L. Tóth, On the subgroups of finite abelian groups of rank three, *Annales Univ. Sci. Budapest., Sect Comp.* **39** (2013), 111–124.

[39] P. Haukkanen, Classical arithmetical identities involving a generalization of Ramanujan's sum, *Ann. Acad. Sci. Fenn., Ser. A I, Diss.* **68** (1988), 69 pp.

[40] P. Haukkanen, An exponential Busche-Ramanujan identity, *Mathematica (Cluj)* **41 (64)** (1999) 177-185.

[41] J. Herzog and P. R. Smith, Lower bounds for a certain class of error functions, *Acta Arith.* **60** (1992), 289–305.

[42] T. Hilberdink and L. Tóth, On the average value of the least common multiple of k positive integers, *J. Number Theory* **169** (2016), 327–341.

[43] J. Hu, The probability that random positive integers are k-wise relatively prime, *Int. J. Number Theory* **9** (2013), no. 5, 1263–1271.

[44] J. Hu, The probability that random positive integers are 3-wise relatively prime, in vol. *Combinatorial and Additive Number Theory*, Selected papers based on the presentations at the conferences CANT 2011 and 2012, M. B. Nathanson (ed.), Springer, New York, 2014, 55–61.

[45] L. K. Hua, *Introduction to Number Theory*, Springer, 1982.

[46] M. N. Huxley, Exponential sums and lattice points III., *Proc. London Math. Soc.* **87** (2003), 591–609.

[47] S. Ikeda and K. Matsuoka, On the lcm-sum function, *J. Integer Seq.* **17** (2014), Article 14.1.7, 11 pp.

[48] S. Ikeda, I. Kiuchi, and K. Matsuoka, Sums of products of generalized Ramanujan sums, *J. Integer Seq.* **19** (2016), Article 16.2.7, 22 pp.

[49] A. Ivić, *The Riemann Zeta-Function*, John Wiley & Sons, 1985.

[50] Chao Hua Jia, The distribution of square-free numbers, *Sci. China Ser. A.* **36** (1993), 154–169.

[51] V. S. Joshi, Order-free integers (mod m), Number theory (Mysore, 1981), Lecture Notes in Math., 938, Springer, 1982. pp. 93–100.

[52] T. Kaluza, Über die Koeffizienten reziproker Potenzreihen, *Math. Z.* **28** (1928), 161–170.

[53] I. Kiuchi, Sums of averages of gcd-sum functions, *J. Number Theory* **176** (2017), 449–472.

[54] I. Kiuchi, Sums of averages of generalized Ramanujan sums, *J. Number Theory* **180** (2017), 310–348.

[55] I. Kiuchi, On sums of averages of generalized Ramanujan sums, *Tokyo J. Math.* **40** (2017), 255–275.

[56] E. Krätzel, *Lattice Points*, Kluwer, Dordrecht-Boston-London, 1988.

[57] E. Krätzel, New estimates in the four-dimensional divisor problem with applications, *Acta Math. Hung.* **126** (2010), 258–278.

[58] N. Kurokawa and H. Ochiai, A multivariable Euler product of Igusa type and its applications, *J. Number Theory* **129** (2009), 1919–1930.

[59] A. V. Lelechenko, Average number of squares dividing mn, *Visn. Odessk. Univ., Ser. Mat. Mekh.* **19** #2 (22) (2014), 52–65.

[60] A. V. Lelechenko, Exponential divisor functions, *Šiauliai Math. Semin.* **10(18)** (2015), 181–197.

[61] A. V. Lelechenko, Exponential and infinitary divisors, *Ukr. Math. J.* **68** (2017), no. 8, 1222–1237.

[62] F. Luca and L. Tóth, The rth moment of the divisor function: an elementary approach, *J. Integer Seq.* **20** (2017), Article 17.7.4, 8 pp.

[63] G. Martin, An asymptotic formula for the number of smooth values of a polynomial, *J. Number Theory* **93** (2002), 108–182.

[64] P. J. McCarthy, Busche-Ramanujan identities, *Amer. Math. Monthly* **67** (1960), 966–970.

[65] A. Mercier, Une représentation pour la série de Dirichlet engendrée par $f(n^r M)$, où f est multiplicative, *Colloq. Math.* **57** (1989), 353–359.

[66] H. L. Montgomery and R. C. Vaughan, The distribution of squarefree numbers, *Recent progress in analytic number theory, Vol. 1* (Durham, 1979), pp. 247–256, Academic Press, London-New York, 1981.

[67] K. V. Namboothiri, Certain weighted averages of generalized Ramanujan sums, *Ramanujan J.* **44** (2017), 531–547.

[68] W. Narkiewicz, *Number Theory*, World Scientific, Singapore, 1983.

[69] M. B. Nathanson, *Elementary Methods in Number Theory*, Graduate Texts in Mathematics, Vol. 195, Springer-Verlag, 2000.

[70] W. G. Nowak, On the average number of finite Abelian groups of a given order, *Ann. Sci. Math. Québec* **15** (1991), 193–202.

[71] W. G. Nowak and L. Tóth, On the average number of subgroups of the group $\mathbb{Z}_m \times \mathbb{Z}_n$, *Int. J. Number Theory* **10** (2014), no. 2, 363–374.

[72] Ju-Mok Oh, An explicit formula for the number of subgroups of a finite abelian p-group up to rank 3, *Commun. Korean Math. Soc.* **28** (2013), 649–667.

[73] F. Pappalardi, A survey on k-freeness, *Ramanujan Math. Soc. Lect. Notes Ser. 1,* Ramanujan Math. Soc., Mysore, 2005, pp. 71–88.

[74] Y.-F. S. Pétermann, Arithmetical functions involving exponential divisors: note on two papers by L. Tóth, *Ann. Univ. Sci. Budapest. Sect. Comput.* **32** (2010), 143–149.

[75] Y.-F. S. Pétermann and J. Wu, On the sum of exponential divisors of an integer, *Acta Math. Acad. Sci. Hung.* **77** (1997), 159–175.

[76] S. Ramanujan, Some formulæ in the analytic theory of numbers, *Messenger of Math.* **45** (1915), 81–84.

[77] S. Ramanujan, On certain trigonometric sums and their applications in the theory of numbers, *Trans. Cambridge Philos. Soc.* **22** (1918), 179–199.

[78] O. Robert and P. Sargos, Three-dimensional exponential sums with monomials, *J. Reine Angew. Math.* **591** (2006), 1–20.

[79] N. Robles and A. Roy, Moments of averages of generalized Ramanujan sums, *Monatsh. Math.* **182** (2017), 433–461.

[80] V. Shevelev, S-exponential numbers, *Acta Arith.* **175** (2016), no. 4, 385–395.

[81] R. Sita Rama Chandra Rao, On an error term of Landau, *Indian J. Pure Appl. Math.* **13** (1982), 882–885.

[82] V. Sitaramaiah and M. V. Subbarao, Asymptotic formulae for sums of reciprocals of some multiplicative functions, *J. Indian Math. Soc.* **57** (1991), 153–167.

[83] V. Sita Ramaiah and D. Suryanarayana, Sums of reciprocals of some multiplicative functions, *Math. J. Okayama Univ.* **21** (1979), 155–164.

[84] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences. http://oeis.org

[85] A. Smati and J. Wu, On the exponential divisor function, *Publ. Inst. Math. (Beograd) (N. S.)* **61** (1997), 21–32.

[86] K. Soundararajan, Partial sums of the Möbius function, *J. Reine Angew. Math.* **631** (2009), 141–152.

[87] M. V. Subbarao, On some arithmetic convolutions, in *The Theory of Arithmetic Functions*, Lecture Notes in Mathematics No. **251**, 247–271, Springer, 1972.

[88] D. Suryanarayana and R. Sitaramachandra Rao, On an asymptotic formula of Ramanujan, *Math. Scand.* **32** (1973), 258–264.

[89] D. Suryanarayana and R. Sita Rama Chandra Rao, On the true maximum order of a class of arithmetical functions, Math. J. Okayama Univ. **17** (1975), 95–101.

[90] D. Suryanarayana and V. Siva Rama Prasad, The number of k-free divisors of an integer, *Acta Arith.* **17** (1971), 345–354.

[91] D. Suryanarayana and V. Siva Rama Prasad, The number of k-free and k-ary divisors of m which are prime to n, *J. Reine Angew. Math.* **264** (1973), 56–75.

[92] Y. Tanigawa and W. Zhai, On the gcd-sum function, *J. Integer Seq.* **11** (2008), Article 08.2.3, 11 pp.

[93] M. Tărnăuceanu and L. Tóth, On the number of subgroups of a given exponent in a finite abelian group, *Publ. Inst. Math. Beograd* **101 (115)** (2017), 121–133.

[94] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, Oxford University Press, New York, 1986.

[95] L. Tóth, The probability that k positive integers are pairwise relatively prime, *Fibonacci Quart.* **40** (2002), no. 1, 13–18.

[96] L. Tóth, On certain arithmetic functions involving exponential divisors, *Ann. Univ. Sci. Budapest. Sect. Comput.* **24** (2004), 285–294.

[97] L. Tóth, On certain arithmetical functions involving exponential divisors. II., *Ann. Univ. Sci. Budapest. Sect. Comput.* **27** (2007), 155–166.

[98] L. Tóth, An order result for the exponential divisor function, *Publ. Math. Debrecen* **71** (2007), no. 1-2, 165–171.

[99] L. Tóth, Regular integers (mod n), *Annales Univ. Sci. Budapest., Sect. Comp.* **29** (2008), 263–275.

[100] L. Tóth, A gcd-sum function over regular integers modulo n, *J. Integer Seq.* **12** (2009), no. 2, Article 09.2.5, 8 pp.

[101] L. Tóth, A survey of gcd-sum functions, *J. Integer Seq.* **13** (2010), Article 10.8.1, 23 pp.

[102] L. Tóth, Weighted gcd-sum functions, *J. Integer Seq.* **14** (2011), Article 11.7.7, 10 pp.

[103] L. Tóth, A note on the number of abelian groups of a given order, *Math. Pannon.* **23** (2012), 157–160.

[104] L. Tóth, On the number of cyclic subgroups of a finite Abelian group, *Bull. Math. Soc. Sci. Math. Roumanie (N.S.)* **55(103)** (2012), 423–428.

[105] L. Tóth, Two generalizations of the Busche-Ramanujan identities, *Int. J. Number Theory* **9** (2013), 1301–1311.

[106] L. Tóth, Another generalization of the gcd-sum function, *Arab. J. Math.* **2** (2013), 313–320.

[107] L. Tóth, Averages of Ramanujan sums: Note on two papers by E. Alkan, *Ramanujan J.* **35** (2014), 149–156.

[108] L. Tóth, Multiplicative arithmetic functions of several variables: a survey, *Mathematics Without Boundaries*, Springer, New York, 2014, 483–514.

[109] L. Tóth, Subgroups of finite Abelian groups having rank two via Goursat's lemma, *Tatra Mt. Math. Publ.* **59** (2014), 93–103.

[110] L. Tóth, Counting solutions of quadratic congruences in several variables revisited, *J. Integer Seq.* **17** (2014), Article 14.11.6, 23 pp.

[111] L. Tóth, Counting r-tuples of positive integers with k-wise relatively prime components, *J. Number Theory* **166** (2016), 105–116.

[112] L. Tóth, Alternating sums concerning multiplicative arithmetic functions, *J. Integer Seq.* **20** (2017), Article 17.2.1, 41 pp.

[113] L. Tóth, Ramanujan expansions of arithmetic functions of several variables, *Ramanujan J.*, accepted, 2017. https://doi.org/10.1007/s11139-017-9944-z

[114] L. Tóth and E. Wirsing, The maximal order of a class of multiplicative arithmetical functions, *Ann. Univ. Sci. Budapest. Sect. Comput.* **22** (2003), 353–364.

[115] L. Tóth and W. Zhai, On multivariable averages of divisor functions, Preprint, 2017, https://arxiv.org/abs/1711.04257

[116] N. Ushiroya, Mean-value theorems for multiplicative arithmetic functions of several variables, *Integers* **12** (2012), 989–1002.

[117] N. Ushiroya, On some generalizations of mean value theorems for arithmetic functions of two variables, *JP J. Algebra Number Theory Appl.* **38** (2016), 151–184.

[118] N. Ushiroya, Ramanujan-Fourier series of certain arithmetic functions of two variables, *Hardy-Ramanujan J.* **39** (2016), 1–20.

[119] R. Vaidyanathaswamy, The theory of multiplicative arithmetic functions, *Trans. Amer. Math. Soc.* **33** (1931), 579–662.

[120] A. Walfisz, *Weylsche Exponentialsummen in der neueren Zahlentheorie*, Mathematische Forschungsberichte, XV, VEB Deutscher Verlag der Wissenschaften, 1963.

[121] J. Wu, Problème de diviseurs exponentiels et entiers exponentiellement sans facteur carré, *J. Théor. Nombres Bordeaux* **7** (1995), 133–141.

[122] B. M. Wilson, Proofs of some formulae enunciated by Ramanujan, *Proc. London Math. Soc.* **21** (1922), 235–255.

[123] E. Wirsing, Das asymptotische Verhalten von Summen über multiplikative Funktionen, II., *Acta Math. Acad. Sci. Hungar.* **18** (1967), 411–467.

[124] L. Zhang, M. Lü, and W. Zhai, On the mean value of $a^2(n)$, *Sci. Magna* **4** (2008), No. 4, 15–17.

[125] D. Zhang and W. Zhai, Mean values of a gcd-sum function over regular integers modulo n, *J. Integer Seq.* **13** (2010), no. 4, Article 10.4.7, 11 pp.

[126] D. Zhang and W. Zhai, Mean values of a class of arithmetical functions, *J. Integer Seq.* **14** (2011), no. 6, Article 11.6.5, 9 pp.

[127] D. Zhang and W. Zhai, On an open problem of Tóth, *J. Integer Seq.* **16** (2013), no. 6, Article 13.6.5, 8 pp.

[128] R. Zurita, Sur un probléme de Rényi et Ivić concernant les fonctions de diviseurs de Piltz, *Acta Arith.* **161** (2013), 69–100.

Index