Causality, Locality, and Probability in Quantum Theory

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Preface

In this volume I collected my main research results achieved in the past several years in the philosophical foundations of quantum theory. All these results are related to the question as to how the notion of causality, locality and probability can be implemented into quantum theory. The volume contains 10 of my recently published research papers on these subject issues. Although philosophy of physics is generally pursued as a team work, and indeed many of my papers are also produced by collaborating with various colleagues, in the present book I picked only papers written without collaboration. My intention was not to make up a self-contained monograph since all the results of this volume have already appeared or will appear in one or other of the books recently published with co-authors.

The main topics and principles analyzed in this volume are Bell's notion of local causality, the Common Cause Principle, the Causal Markov Condition, d-separation, Bell's inequalities and the EPR scenario. Each chapter of the volume is a different paper, with a separate abstract, introduction, bibliography and sometimes appendix. To make the volume coherent and to provide an overview of the general landscape I inserted an extra chapter, the Introduction, at the beginning of the book where I summarize the main themes and results of the subsequent chapters and their interdependence.

The chapters of the volume are the following papers:

Chapter 1. Gábor Hofer-Szabó, "Quantum mechanics as a representation of classical conditional probabilities," *Journal of Mathematical Physics* (submitted).

Chapter 2. Gábor Hofer-Szabó, "Three principles leading to the Bell inequalities," *Belgrade Philosophical Annual*, **29**, 57-66 (2016).

Chapter 3. Gábor Hofer-Szabó, "How man and nature shake hands: the role of no-conspiracy in physical theories," *Studies in the History and Philosophy of Modern Physics*, **57**, 89-97 (2017).

Chapter 4. Gábor Hofer-Szabó, "Relating Bell's local causality to the Causal Markov Condition," *Foundations of Physics* **45 (9)** 1110-1136 (2015).

Chapter 5. Gábor Hofer-Szabó, "Bell's local causality is a d-separation criterion," *Springer Proceedings in Mathematics and Statistics* (forthcoming).

Chapter 6. Gábor Hofer-Szabó, "Local causality and complete specification: a reply to Seevinck and Uffink," in U. Mäki et al. (eds), *Recent Developments in the Philosophy of Science: EPSA13* Helsinki, Springer Verlag, 209-226 (2015).

Chapter 7. Gábor Hofer-Szabó, "Noncommutative causality in algebraic quantum field theory," in M. C. Galavotti, D. Dieks, W. J. Gonzalez, S. Hartmann, Th. Uebel, M. Weber (eds.), *The Philosophy of Science in a European Perspective*, Vol. 5., 543-554 (2014).

Chapter 8. Gábor Hofer-Szabó, "On the relation between the probabilistic characterization of the common cause and Bell's notion of local causality," *Studies in the History and Philosophy of Modern Physics*, **49**, 32-41. (2015).

Chapter 9. Gábor Hofer-Szabó, "Separate common causal explanation and the Bell inequalities," *International Journal of Theoretical Physics*, **51** 110-123 (2012).

Chapter 10. Gábor Hofer-Szabó, "EPR correlations, Bell inequalities and common cause systems," in D. Aerts, S. Aerts and C. de Ronde (eds.), *Probing the Meaning of Quantum Mechanics: Physical, Philosophical and Logical Perspectives*, 263-277 (2014).

The results in the above papers have been presented at more than 60 international workshops and department seminars. I thank the audience of these workshop and seminars for their valuable comments. The papers benefited a lot from these discussions.

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Introduction and overview

The philosophical understanding of the foundations of quantum theory is one of the most thrilling questions in today's philosophy of science. What is the correct conceptual basis of quantum mechanics? How can our most fundamental philosophical concepts such as 'causality', 'probability' or 'locality' be accommodated in this theory?

There is a very influential approach to the foundational problems of quantum theory which intends to accommodate quantum phenomena in a so-called *classical, locally causal* world picture. This world picture is *classical* since it adopts a classical ontology of events represented by a Boolean mathematical structure in a classical spacetime; it is *local*, since the events in question are localized in a well-defined region of the spacetime; and finally it is *causal* in the sense that the relation between these events meets the relativistic requirement of 'no superluminal propagation'. The first advocate of such a theory was John Bell. In a number of seminal papers Bell carefully studied the philosophical intuitions lying behind our concept of locality and causality. His major contribution, however, consisted in translating these intricate notions into a simple probabilistic framework which made these notions tractable both for mathematical treatment and later for experimental testability. Since the central question was as to whether quantum theory can be accommodated in a classical framework, therefore both Bell and the subsequent authors used a *classical* probabilistic language in their analysis. Events were understood as classical events represented by a commutative mathematical structure and all the assumptions representing locality and causality were formulated in the classical probability theory.

This classical, local and causal framework, however, turned out soon to be inappropriate to account for quantum theory. Bell showed that these classical probabilistic assumptions lead to some mathematical constraints—the so-called Bell inequalities—which were shown to be violated in some quantum scenarios, thereby inhibiting a classical, locally causal interpretation of quantum mechanics. Bell's work has been followed by an extensive research to locate the assumptions responsible for the violation of the Bell inequalities, and many authors analyzed the philosophical consequences of giving up either the one or the other of these assumptions. Since these assumptions represented our natural intuitions concerning locality and causality, abandoning any of them resulted in acknowledging the limits of a locally causal interpretation of quantum mechanics.

Many of the papers contained in this anthology can be considered as an attempt to make a completely new start in the locally causal approach to quantum theory. The core idea in brief is this: let us give up the *classical ontology* in order to save *locality* and *causality*. In other words, contrary to the standard strategy, we should not stick to a classical ontology at the price of making our explanation either nonlocal, non-causal or introducing other undesirable features, but we should straightly abandon the classical—that is, commutative—character of causality, and investigate what we may gain and what philosophical price we must pay for such a change in our conceptual framework. Noncommutativity has a well-established place in the formalism of quantum theory, but its role in causal explanation is completely unexplored. Exploring the causal explanatory role of noncommutativity in local causality, introducing noncommutative causal concepts into our explanatory framework can both broaden our formal strategies to causally account for quantum phenomena, and also deepen our understanding of the nonclassical nature of causality in quantum theory.

There is, however, another more conservative research line pursued in this volume. This follows the down-to-Earth Humean tradition and asks how far we get by adhering to the standard ontology of physics which is both local and classical. How can quantum theory be reconstructed from this ontology and how quantum probabilities can be accounted for in terms of classical relative frequencies. What kind of causal and probabilistic independencies one should assume between the elements of reality of this classical ontology on the one hand and measurement choices of the experimenter on the other hand?

These are the main questions and topics of this volume.

The first three chapters of anthology lie on the conservative side. The topic of Chapter 1 is to analyze the reconstructability of quantum mechanics from classical conditional probabilities representing measurement outcomes conditioned on measurement choices. It will be investigated how the quantum mechanical representation of classical conditional probabilities is situated within the broader frame of noncommutative representations. To this goal, I adopted some parts of the quantum formalism and asked whether empirical data can constrain the rest of the representation to conform to quantum mechanics. I will show that as the set of empirical data grows conventional elements in the representation gradually shrink and the noncommutative representations narrow down to the unique quantum mechanical representation.

Chapter 2 sheds light on the broader landscape of the relation among the most notorious principles in the foundations of quantum mechanics. I compare here three principles accounting for correlations, namely Reichenbach's Common Cause Principle, Bell's Local Causality Principle, and Einstein's Reality Criterion and relate them to the Bell inequalities. I show that there are two routes connecting the principles to the Bell inequalities. In case of Reichenbach's Common Cause Principle and Bell's Local Causality Principle one assumes a non-conspiratorial joint common cause for a set of correlations. In case of Einstein's Reality Criterion one assumes strongly non-conspiratorial separate common causes for a set of perfect correlations. Strongly non-conspiratorial separate common causes for perfect correlations, however, form a non-conspiratorial joint common cause. Hence the two routes leading the Bell inequalities meet.

Chapter 3 addresses the problem of the so-called no-conspiracy. No-conspiracy is the requirement that measurement settings should be probabilistically independent of the elements of reality responsible for the measurement outcomes. In this chapter I investigate what role no-conspiracy generally plays in a physical theory; how it influences the semantical role of the event types of the theory; and how it relates to such other concepts as separability, compatibility, causality, locality and contextuality.

In Chapters 4-6 I turn towards the definition of Bell's notion of local causality in local physical theories. The questions asked here are how local causality is related to Causal Markov Condition, d-separation and whether complete specification is in contradiction with no-conspiracy.

The aim of Chapter 4 is to relate Bell's notion of local causality to the Causal Markov Condition. To this end, first a framework, called local physical theory, will be introduced integrating spatiotemporal and probabilistic entities and the notions of local causality and Markovity will be defined. Then, illustrated in a simple stochastic model, it will be shown how a discrete local physical theory transforms into a Bayesian network and how the Causal Markov Condition arises as a special case of Bell's local causality and Markovity.

Chapter 5 aims to motivate Bell's notion of local causality by means of Bayesian networks. In a locally causal theory any superluminal correlation should be screened off by atomic events localized in any so-called *shielder-off region* in the past of one of the correlating events. In a Bayesian network any correlation between non-descendant random variables are screened off by any so-called *d-separating set* of variables. I will argue that the shielder-off regions in the definition of local causality conform in a well defined sense to the d-separating sets in Bayesian networks.

A physical theory is called locally causal if any correlation between space-like separated events is screened-off by local beables *completely specifying* an appropriately chosen region in the past of the events. In Chapter 6 I will define local causality in a clear-cut framework, called local physical theory which integrates both probabilistic and spatiotemporal entities. Then I will argue that, contrary to the claim of Seevinck and Uffink (2011), complete specification does not stand in contradiction to the free variable (no-conspiracy) assumption.

In Chapter 7 it will be argued that embracing noncommuting common causes in the causal explanation of quantum correlations in algebraic quantum field theory has the following two beneficial consequences: it helps (i) to maintain the validity of Reichenbach's Common Causal Principle and (ii) to provide a local common causal explanation for a set of correlations violating the Bell inequality.

In Chapter 8 the relation between the standard probabilistic characterization of the common cause (used for the derivation of the Bell inequalities) and Bell's notion of local causality will be investigated in the isotone net framework borrowed from algebraic quantum field theory. The logical role of two components in Bell's definition will be scrutinized; namely that the common cause is localized in the *intersection* of the past of the correlated events; and that it provides a *complete specification* of the 'beables' of this intersection.

In Chapter 9 I ask how the following two facts are related: (i) a set of correlations has a local, non-conspiratorial *separate* common causal explanation; (ii) the set satisfies the Bell inequalities. My answer will be partial: we show that no set of correlations violating the Clauser–Horne inequalities can be given a local, non-conspiratorial separate common causal model *if* the model is *deterministic*.

Chapter 10 is again devoted to separate common cause systems. Namely, standard common causal explanations of the EPR situation assume a so-called joint common cause system that is a common cause for all correlations. However, the assumption of a joint common cause system together with some other physically motivated assumptions concerning locality and no-conspiracy results in various Bell inequalities. Since Bell inequalities are violated for appropriate measurement settings, a local, non-conspiratorial joint common causal explanation of the EPR situation is ruled out. But why do we assume that a common causal explanation of a set of correlation consists in finding a joint common cause system for *all* correlations and not just in finding *separate* common cause systems for the different correlations? What are the perspectives of a local, non-conspiratorial *separate* common causal explanation for the EPR scenario? And finally, how do Bell inequalities relate to the weaker assumption of *separate* common cause systems?

Chapter 1

Quantum mechanics as a noncommutative representation of classical conditional probabilities

The aim of this paper is to analyze the reconstructability of quantum mechanics from classical conditional probabilities representing measurement outcomes conditioned on measurement choices. We will investigate how the quantum mechanical representation of classical conditional probabilities is situated within the broader frame of noncommutative representations. To this goal, we adopt some parts of the quantum formalism and ask whether empirical data can constrain the rest of the representation to conform to quantum mechanics. We will show that as the set of empirical data grows conventional elements in the representation gradually shrink and the noncommutative representations narrow down to the unique quantum mechanical representation.

1.1 Introduction

In the quantum information theoretical paradigm one is usually looking for the reconstruction of quantum mechanics from information-theoretic first principles (Hardy, 2008; Chiribella, D'Ariano and Perinotti, 2015). This approach has produced many fascinating mathematical results and greatly contributed to a better understanding of the complex formal structure of quantum mechanics. As a top-down approach, however, its prime aim was to clarify the relation of the theory to higher-order (rationality, information-theoretic, etc.) principles and payed less attention to the "legs" of the theory connecting it to experience.

In this paper we take an opposite, bottom-up route and ask—in the spirit of the good old empiricist tradition—as to how the theory can be reconstructed not from first principles but from experience. More precisely, we will ask whether we can reconstruct the formalism of quantum mechanics from using simply classical conditional probabilities.

Why classical conditional probabilities?

Quantum mechanics as a probabilistic theory provides us quantum probabilities for certain observables. The question is how to connect these quantum probabilities to experience. The correct answer is that the probabilities provided by the Born rule should be interpreted as *classical conditional probabilities*. They are *classical* since they are nothing but the long-run relative frequency of certain measurement outcomes explicitly testable in the lab; and they are *conditional* on the fact that a certain measurement had been chosen and performed (E. Szabó, 2008). For example, the quantum probability of the outcome "spin-up" in direction z is the relative frequency of the outcomes "up"—but not in the statistical ensemble of *all* measurement outcomes (which may also comprise spin measurements in *other* directions) but only in the *sub*ensemble when spin was measured in direction z.

What does it mean to reconstruct quantum mechanics from classical conditional probabilities?

First note that all we are empirically given are classical conditional probabilities. The question is how to represent these empirical data. As it was shown in (Bana and Durt 1997), (E. Szabó 2001) and (Rédei 2010) classical conditional probabilities conforming to the probabilistic predictions of quantum mechanics need *not* necessarily be represented in the formalism of quantum mechanics. The so-called "Kolmogorovian Censorship Hypothesis" (or better, Proposition) states that there is always a Kolmogorovian representation of the quantum probabilities *if* the measurement conditions also make part of the representation. Thus, a stubborn classicist will always find a way to represent the empirical content of quantum mechanics in a purely classical framework.

On the other hand, quantum mechanics has proved to be an extremely elegant and economic representation of these empirical data. It provides a principled representation of an enormous collection of conditional probabilities together with their dynamical evolution.

Our paper is a kind of interpolation between the two sides. Our strategy will be to accept *some* parts of the quantum mechanical representation of classical conditional probabilities and ask whether the *rest* follows. More precisely, we accept the noncommutative probability theory which in our case will boil down to representing observables and states by linear operators. We also adopt the Born rule connecting the quantum probabilities to real-world classical conditional probabilities; and the quantum mechanical representation of measurement settings and measurement outcomes. The only "free variable" will be the representation of the *state* of the system. Our main question will then be as to what empirical data ensure that the state of a system is represented by a density operator.

By this strategy we are going to analyze how quantum mechanics is situated within a noncommutative probability theory and to study whether the specific quantum mechanical representation of classical conditional probabilities within this broader frame can be traced back to purely empirical facts or is partly of conventional nature.

In the paper we will proceed as follows. In Section 2 we introduce the general scheme of a noncommutative representation of classical conditional probabilities. In the subsequent three sections we gradually enhance the set of empirical data that is the set of classical conditional probability of measurement outcomes. We ask whether by increasing the set of empirical data the noncommutative representation of these data necessarily narrows down to the quantum mechanical representation or some extra conventional elements are also needed. The empirical situation we are going to represent will be three yes-no measurements in Section 3, k measurements each with n outcomes in Section 4, and finally a continuum set of measurements with n outcomes in Section 5. We will see how the conventional part gradually shrinks as experience grows until the representation finally zooms in on the quantum mechanical representation. We discuss our results in Section 6.

1.2 Quantum mechanical and noncommutative representation

Suppose there is a physical system in state s and we perform a set $\{a_i\}$ $(i \in I)$ of measurements on the system. Denote the outcomes of measurement a_i by $\{A_i^j\}$ $(j \in J)$. Suppose that by repeating the measurements many times we obtain a probability $p_s(A_i^j|a_i)$ that is a stable long-run relative frequency for each outcome A_i^j *given* measurement a_i is performed. Now, quantum mechanics represents these conditional probabilities as it is summarized in the following table:

Quantum mechanical representation:

Operator assignment:			Born rule:
System	$\longrightarrow$	$\mathcal{H}$: Hilbert space	
Measurements: a_i	$\longrightarrow$	O_i: self-adjoint operators	
Outcomes: A_i^j	$\longrightarrow$	P_i^j: spectral projections of O_i	$p_s(A_i^j\|a_i) = \mathrm{Tr}(W_s P_i^j)$
States: s	$\longrightarrow$	W_s: density operators	

In the table the different concepts are presented. On the left hand side of the arrow/equation sign stand the empirical concepts to be represented; on the right hand side stand the mathematical representation of the empirical concepts. The two are not to be mixed. Although we do not use "hat" to denote operators, throughout the paper we carefully distinguish empirical concepts (measurements, outcomes, states) from their representation (self-adjoint operators, projections, density operators). Thus, the physical system under investigation is associated to a Hilbert space $\mathcal{H}$; each measurement a_i is represented by a self-adjoint operator O_i; the outcomes A_i^j of a_i are represented by the orthogonal spectral projections of O_i; and the state s of the system is represented by a density operator W_s, a self-adjoint, positive semidefinite operator with trace equal to 1. In the second column the mathematical representation is connected to experience by the Born rule: the representation is correct only if the quantum mechanical trace formula $\mathrm{Tr}(W_s P_i^j)$ correctly yields the empirical conditional probability $p_s(A_i^j|a_i)$ for any outcome A_i^j of measurement a_i and any state s.

Note the following two facts. First, the trace formula is associated to a *conditional* probability, not to a probability *simpliciter*. This means, among others, that in joint measurements one always needs to combine different measurement conditions. Second, the trace formula is "holistic" in the sense that the empirically testable conditional probabilities are associated to the trace of the *product* of two operators, one representing the state and the other representing the measurement. This leaves a lot of freedom to account for the same empirical content in terms of operators.

The main question of our paper is whether the above quantum mechanical representation of classical conditional probabilities is constrained upon us if the set of empirical data is large enough or whether we need some extra theoretical, aesthetic etc. considerations to arrive at it. In order to decide on this question, we consider first a wider class of representations which we will call *noncommutative representations*. We will then ask whether a noncommutative representation of a set of large enough data is necessarily a quantum mechanical representation.

What is a noncommutative representation?

Generally, a noncommutative representation is simply an association of measurements and states to linear operators acting on a Hilbert space such that some functional of the representants provides the correct empirical conditional probabilities. Obviously this association can be done in many different ways. In our paper we pick a special noncommutative representation which is very close to the quantum mechanical representation: We retain all the assignments (denoted by $\longrightarrow$) of the above table except the last one. That is we will represent the system by a Hilbert space, the measurements by self-adjoint operators, and the outcomes by the orthogonal spectral projections. We also retain the Born rule connecting the formalism to experience. The only part of the representation which we let vary will be the association of the state of the system to linear operators. That is we do not demand that states should necessarily be represented by density operators. We summarize this scheme in the following table:

Noncommutative representation:

Operator assignment:			Born rule:
System	$\longrightarrow$	$\mathcal{H}$: Hilbert space	
Measurements: a_i	$\longrightarrow$	O_i: self-adjoint operators	$p_s(A_i^j \vert a_i) = \mathrm{Tr}(W_s P_i^j)$
Outcomes: A_i^j	$\longrightarrow$	P_i^j: spectral projections of O_i	
States: s	$\longrightarrow$	W_s: linear operators	

Obviously, our noncommutative representation is only one special choice among many. One could well take different routes. For example one could demand that the state should be represented by density operators but abandon that the projections representing the outcomes should be orthogonal. Or one could replace the Born rule by another expression connecting the formalism to the world. As said above, the connection of the formalism of quantum mechanics and experience is of holistic nature; one can fix one part of the formalism and see how the rest may vary such that the resulting probabilities are in tune with experience. With respect to our aim which is to see how we are compelled to adopt the quantum mechanical representation by increasing the number of conditional probabilities to be represented, our above choice is just as good as any other.

What we will test in the subsequent sections is whether our noncommutative representation is necessarily a quantum mechanical representation. In other words, we will test whether for any choice of operators representing a certain set of measurements and the outcomes such that the Born rule yields the correct conditional probabilities, the state will necessarily be represented by a density operator. In Section 3 we start off as a warm-up with three measurements; in Section 4 we continue with k measurements; and in Section 5 we end up by uncountably many measurements. It will turn out that the gap between noncommutative and quantum mechanical representation gradually shrinks as the set of empirical data grows.

1.3 Case 1: Three yes-no measurements

Consider a box filled with balls. Denote the preparation of the box by s. Suppose you can perform three different measurements on the system; you can measure the *color*, the *size* or the *shape* of the balls. Denote the three measurements as follows:

a:	Color measurement
b:	Size measurement
c:	Shape measurement

Suppose that each measurement can have only two outcomes:

A^+:	Black	A^-:	White
B^+:	Large	B^-:	Small
C^+:	Round	C^-:	Oval

Suppose you pick a measurement, perform it many times (putting the balls always back into the box), and count the probability, that is the long-run relative frequency, of the outcomes. What you obtain is the *conditional* probability of the outcomes *given* the measurement you picked is performed on the system prepared in state s:

$$p_a^{\pm} := p_s(A^{\pm}|a) \tag{1.1}$$
$$p_b^{\pm} := p_s(B^{\pm}|b) \tag{1.2}$$
$$p_c^{\pm} := p_s(C^{\pm}|c) \tag{1.3}$$

Now, suppose you are going to represent the above empirical facts not in the standard classical probability theory but in a quantum fashion. Since our model contains only two-valued (yes-no) measurements, it suffices to use only a minor fragment of quantum mechanics. Again, we summarize it in a table:

Quantum mechanical representation:

Operator assignment:			Born rule:
System:	$\longrightarrow$	$\mathbf{C}_2$	
Color: a	$\longrightarrow$	$O_a = \mathbf{a}\boldsymbol{\sigma}$	
Size: b	$\longrightarrow$	$O_b = \mathbf{b}\boldsymbol{\sigma}$	
Shape: c	$\longrightarrow$	$O_c = \mathbf{c}\boldsymbol{\sigma}$	$p_a^{\pm} = \mathrm{Tr}(W_s P_a^{\pm}) = \frac{1}{2}(1 \pm \mathbf{sa})$
Black/White: $A^{\pm}$	$\longrightarrow$	$P_a^{\pm} = \frac{1}{2}(\mathbf{1} \pm \mathbf{a}\boldsymbol{\sigma})$	$p_b^{\pm} = \mathrm{Tr}(W_s P_b^{\pm}) = \frac{1}{2}(1 \pm \mathbf{sb})$
Large/Small: $B^{\pm}$	$\longrightarrow$	$P_b^{\pm} = \frac{1}{2}(\mathbf{1} \pm \mathbf{b}\boldsymbol{\sigma})$	$p_c^{\pm} = \mathrm{Tr}(W_s P_c^{\pm}) = \frac{1}{2}(1 \pm \mathbf{sc})$
Round/Oval: $C^{\pm}$	$\longrightarrow$	$P_c^{\pm} = \frac{1}{2}(\mathbf{1} \pm \mathbf{c}\boldsymbol{\sigma})$	
State: s	$\longrightarrow$	$W_s = \frac{1}{2}(\mathbf{1} + \mathbf{s}\boldsymbol{\sigma})$	

Here, the Hilbert space associated to the system is the two-dimensional complex space $\mathbf{C}_2$; and the operators associated to the measurements, outcomes and the state are all self-adjoint operators acting on $\mathbf{C}_2$. According to this representation, called the Bloch sphere representation, a self-adjoint operator O_a associated to measurement a can be represented by the inner product of a unit vector $\mathbf{a} = (a_x, a_y, a_z)$ in $\mathbf{R}^3$ and the Pauli vector $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$. The two outcomes $A^{\pm}$ of measurement a are associated to the spectral projections $P_a^{\pm} = \frac{1}{2}(\mathbf{1} \pm \mathbf{a}\boldsymbol{\sigma})$ of O_a, where $\mathbf{1}$ is the two-dimensional identity operator. Finally, the density operator W_s associated to the state s of the system is of the form $W = \frac{1}{2}(\mathbf{1}+\mathbf{s}\boldsymbol{\sigma})$, where $\mathbf{s} = (s_x, s_y, s_z)$ is in the unit ball $B = \{\mathbf{r} \in \mathbf{R}^3 : |\mathbf{r}| \leqslant 1\}$ of $\mathbf{R}^3$. If $|\mathbf{s}| = 1$, then s is said to be a pure state, otherwise a mixed state. Again, the empirical content of the representation is ensured by the Born rule which in this two-dimensional case boils down to the inner product: $p_a^{\pm} = \frac{1}{2}(1 \pm \mathbf{sa})$. (Similarly for b and c.)

Now, to give a quantum mechanical representation for the above situation we need to associate the three measurements to three Bloch vectors and the state of the system to a fourth Bloch vectors (either unit or smaller) such that the Born rule (the trace formula) yields the pre-given conditional probabilities (1.1)-(1.3). Thus, assign to each measurement a unit vector in $\mathbf{R}^3$:

$$\{a, b, c\} \mapsto \{\mathbf{a}, \mathbf{b}, \mathbf{c}\} \tag{1.4}$$

Suppose that the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are linearly independent. First, we show that given three pairs of empirical conditional probabilities $p_a^{\pm}$, $p_b^{\pm}$ and $p_c^{\pm}$ and also the assignment (1.4), the operator W_s associated to the state s gets uniquely fixed. Schematically,

$$p_a^{\pm}, p_b^{\pm}, p_c^{\pm} \quad \& \quad \mathbf{a}, \mathbf{b}, \mathbf{c} \quad \Longrightarrow \quad W_s$$

To see this, observe that any linear operator acting on $\mathbf{C}_2$ can be written as

$$W_s = s_0\mathbf{1} + \mathbf{s}\boldsymbol{\sigma} + i(s_0'\mathbf{1} + \mathbf{s}'\boldsymbol{\sigma})$$

where $s_0, s_0' \in \mathbf{R}$ and $\mathbf{s}, \mathbf{s}' \in \mathbf{R}^3$. Now, applying the Born rule to the three measurements we get:

$$\begin{aligned} p_a^{\pm} &= \mathrm{Tr}(W_s P_a^{\pm}) = s_0 \pm \mathbf{sa} + i(s_0' \pm \mathbf{s}'\mathbf{a}) \\ p_b^{\pm} &= \mathrm{Tr}(W_s P_b^{\pm}) = s_0 \pm \mathbf{sb} + i(s_0' \pm \mathbf{s}'\mathbf{b}) \\ p_c^{\pm} &= \mathrm{Tr}(W_s P_c^{\pm}) = s_0 \pm \mathbf{sc} + i(s_0' \pm \mathbf{s}'\mathbf{c}) \end{aligned}$$

which, assuming that $p_a^{\pm}$, $p_b^{\pm}$ and $p_c^{\pm}$ are real and $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are linearly independent, yield

$$s_0 = \frac{1}{2} \qquad s_0' = 0 \qquad \mathbf{s}' = \mathbf{0}$$

and hence

$$\begin{aligned} p_a^{\pm} &= \frac{1}{2} \pm \mathbf{sa} \\ p_b^{\pm} &= \frac{1}{2} \pm \mathbf{sb} \\ p_c^{\pm} &= \frac{1}{2} \pm \mathbf{sc} \end{aligned}$$

the solution of which is $W_s = \frac{1}{2}(\mathbf{1} + \mathbf{s}\boldsymbol{\sigma})$ with

$$\mathbf{s} = \frac{(p_a^+ - \frac{1}{2})(\mathbf{b} \times \mathbf{c}) + (p_b^+ - \frac{1}{2})(\mathbf{c} \times \mathbf{a}) + (p_c^+ - \frac{1}{2})(\mathbf{a} \times \mathbf{b})}{\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})}$$

where $\times$ is the cross product. (The linear independence of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is needed for the triple product in the denominator not to be zero.)

This is a well-known result. Since the late 60s and early 70s there has begun an intensive research for the *empirical* determination of the state of a quantum system. In a series of papers Band and Park (1970, 1971) have extensively investigated how the expectation value of certain observables determine the state of a system. They investigated the minimal number of observables, called the *quorum*, needed for such state determinations; the structure and geometry of this set; and many other important features. The study of the *quorum* has become an eminent research project also in the new quantum informational paradigm. Quantum tomography, quantum state reconstruction, quantum state estimation etc. all follow the same path: they start from a set of observables and aim to end up with a more-or-less fixed state using empirical input (see for example (D'Ariano, Maccone and Paris, 2001)).

However, all these endeavors have a common pre-assumption, namely that *the association of measurements to operators is already settled*. They all start from a set of operators and (by means of a set of empirical probabilities) aim to reconstruct the quantum state of a system. But an operator is not a measurement but only a representation of a measurement. Calling operators *observables* overshadows the fact that the operators are already on the mathematical side of the project and without providing an association of measurements to operators the state determination cannot rightly be called "empirical". This measurement-operator assignment is that which we are going to make explicit in what comes.

Consider the following measurement-operator assignment in the context of our above model: we associate the following three Bloch vectors to the measurements a, b and c:

$$\mathbf{a} = \mathbf{x} = (1,0,0) \tag{1.5}$$

$$\mathbf{b} = (0, \cos\varphi, -\sin\varphi) \tag{1.6}$$

$$\mathbf{c} = \mathbf{z} = (0,0,1) \tag{1.7}$$

and for the sake of simplicity we set the conditional probabilities as follow:

$$p_a^+ = p_b^+ = p_c^+ =: p \tag{1.8}$$

The Bloch vector $\mathbf{s}$ for these special directions and empirical probabilities will then be the following:

$$\mathbf{s} = (p - \frac{1}{2}) \left(1, \frac{1+\cos\varphi+\sin\varphi}{1+\cos\varphi-\sin\varphi}, 1\right) \tag{1.9}$$

But the operator W_s associated to the Bloch vector $\mathbf{s}$ will not necessarily be a density operator. For example for any

$$p \in [0.76, 1] \quad \text{and} \quad \varphi \in [\pi/3, \pi/2) \tag{1.10}$$

the vector $\mathbf{s}$ will be longer than 1 and hence W_s will not be positive semidefinite, that is, a density operator.

Thus, we have provided a noncommutative but not quantum mechanical representation of the above scenario. All the assignments of the table at the beginning of this section hold except the last one: the state of the system is represented by a linear operator but not a density operator.

This toy-example is, however, special in two senses: (i) the number of measurements is finite and (ii) the number of outcomes is two, that is, the scenario is represented in the two-dimensional Hilbert space which is always a special case. We tackle point (ii) in the next section and point (i) in the one after the next.

1.4 Case 2: k measurements with n outcomes

Let us then see whether a larger set of probabilities can also be given a noncommutative but not quantum mechanical representation. Suppose we perform k measurements on a system such that each measurement can have n outcomes. Suppose we obtain the following empirical conditional probabilities:

$$p_i^j := p(A_i^j|a_i) \geqslant 0 \quad \text{with} \quad \sum_i p_i^j = 1 \quad \text{for all} \quad i = 1 \ldots k;\, j = 1 \ldots n$$

Just as above we represent each measurement a_i by a self-adjoint operator O_i in the Hilbert space $\mathcal{H}_n$ and the measurement outcomes $\{A_i^j\}$ of a_i by the orthogonal spectral projections $\{P_i^j\}$. The representation is connected to experience by the Born rule:

$$p_i^j := p(A_i^j|a_i) = \mathrm{Tr}(W_s P_i^j)$$

where W_s is a linear operator representing the state s of the system. Again, we do not assume that W_s is a density operator; our task is just to see whether it follows that W_s is always a density operator.

Now, the empirically given probability distributions together with the conventionally chosen sets of minimal orthogonal projections provide constraints on W_s via the Born rule. For a certain number of measurements W_s gets completely fixed. Schematically,

$$\{p_1^j\},\, \{p_2^j\} \ldots \{p_k^j\} \quad \& \quad \{P_1^j\},\, \{P_2^j\} \ldots \{P_k^j\} \quad \Longrightarrow \quad W_s$$

How many measurements are needed to uniquely fix W_s?

W_s gets uniquely fixed if $\mathrm{Tr}(W_s A)$ is given for n^2 linearly independent operators A. Our operators are minimal projections. The first set of minimal orthogonal projections provides n linearly independent equations. Any further linearly independent set of orthogonal projection provides $n-1$ extra equations since in each set the projections sum up to the unity. That is k linearly independent sets of minimal orthogonal projections provide $k(n-1)+1$ linearly independent equations which is equal to n^2 if $k = n+1$. Thus, performing $k = n+1$ measurements on our system (resulting in $k = n+1$ probability distributions) and representing all the outcomes by orthogonal projections in $\mathcal{H}_n$, the linear operator W_s gets uniquely fixed.

But it will not necessarily be a density operator!

Our question is then: Do $k = n+1$ measurements constrain W_s to be a density operator for *all* linearly independent sets of orthogonal projections representing the outcomes and *all* probability distributions generated from the projections by the Born rule? Again, what we test here is whether a noncommutative representation is necessarily a quantum mechanical representation.

Now, we show that the answer is: *no*.

As said above, a density operator is a self-adjoint, positive semidefinite operator with trace equal to 1. Self-adjoint operators in $\mathcal{H}_n$ form a vector space V over the field of real numbers. This vector space can also be endowed with an inner product induced by the trace: $(A, B) := \mathrm{Tr}(AB)$. The operators with trace equal to 1 form an affin subspace E in V and the positive semidefinite operators form a convex cone C_+. (A subset C of a real vector space V that linearly spans V is a *convex cone* if for any $A_1, A_2 \in C$ and $r_1, r_2 \in \mathbf{R}_+$, $r_1A_1 + r_2A_2 \in C$ and $A, -A \in C \Rightarrow A = 0$). The intersection of the two, $\mathcal{C}_+ \cap E$, is a convex set in the affin subspace. The extremal elements of this set are the minimal projections in $\mathcal{H}_n$. Denote this set of minimal projections in $\mathcal{H}_n$ by $\mathcal{P}_n$.

Now, for any cone C in V, the *dual cone* C^* is defined as

$$C^* := \{A \in V \,|\, \mathrm{Tr}(AB) \geqslant 0 \quad \text{for all } B \in C\}$$

According to Fejér's Trace Theorem the cone of the positive semidefinite operators is self-dual that is $C_+^* = C_+$.

Now, let us return to our example. Consider the $k = n+1$ linearly independent sets of orthogonal projections representing the measurement outcomes in $\mathcal{H}_n$. Let D be the convex cone expanded by these projections in $\mathcal{P}_n$ as extremal elements. Obviously, $D \subset C_+$ and consequently $D^* \supset C_+^* = C_+$. Pick an element from $(D^* \setminus C_+) \cap E$ and call it W_s. Lying outside C_+, W_s will *not* be positive semidefinite but, lying in E, W_s will be of trace 1. Hence for any set of orthogonal projections it generates a probability distribution by the Born rule.

Thus, we have found a counter-example (actually, continuously many counter-examples): $k = n+1$ linearly independent sets of orthogonal projections representing measurement outcomes and $k = n + 1$ probability distributions such that the latter is generated from the former by the Born rule with an operator W_s which is *not* a density operator (since is not positive semidefinite). Hence, we have provided a noncommutative but not quantum mechanical representation for a situation in which $k = n + 1$ measurements with n outcomes are performed on a system. This shows that our previous result is not a consequence of the fact that the Hilbert space is the special $\mathcal{H}_2$. Conditional probabilities of finitely many measurements with finitely many outcomes can always be given a noncommutative but *not* quantum mechanical representation.

But what is the situation if we are going to the continuum limit? Does our counter-example survive if the cardinality of the set of conditional probabilities to be represented is uncountable? To this we turn in the next section.

1.5 Case 3: A continuum set of measurements with n outcomes

There is a theorem which immediately comes to one's mind when going to the continuum limit, namely Gleason's theorem.

Suppose we are given a continuum set of probability distributions of measurements with, say, n outcomes. We are to represent this set in an n-dimensional Hilbert space $\mathcal{H}_n$. Now, suppose that we assign self-adjoint operators to the measurements such that the spectral projections of the various operators together cover the *full* set $\mathcal{P}_n$ of minimal projections in $\mathcal{H}_n$. In other words, there is no minimal projection in $\mathcal{P}_n$ which does not represent a measurement outcome. In this case we can invoke Gleason's theorem to decide on the question as to whether there exist noncommutative representations which are not the quantum mechanical representation. Gleason's theorem answers this question in the negative.

Gleason's theorem namely claims that for every state ϕ in a Hilbert space with dimension greater than 2 there is a density operator W (and vica versa) such that the Born rule $\phi(P) = \mathrm{Tr}(PW)$ holds for *all* projections. In other words, if *all* projections are considered, then the state will uniquely be represented by a density operator. Translating it into our case, the theorem claims that *if* one represents the continuum set of measurement outcomes by the *full* set $\mathcal{P}_n$ of projections of a given Hilbert space, then one has no other choice to account for the whole set of conditional probabilities, than to represent the state by a density operator.

Note, however, that the previous sentence is a conditional: *if* we represent the measurement outcomes by the full set $\mathcal{P}_n$ then Gleason's theorem tells us that the only representation is the quantum mechanical. This raises the following question: Are we compelled to represent a continuum set of measurement outcomes necessarily by the full set of minimal projections? Can we not "compress" somehow the set of projections representing the measurement outcomes such that (i) the outcome-projection assignment is injective (no two outcomes of different measurements are represented by the same projection), still (ii) the set of projections is only a proper subset of $\mathcal{P}_n$? As we saw in the previous section, in this case we can always represent the state of the system by a linear operator which is not a density operator. Or to put it briefly, can we avoid Gleason's theorem by not making use of all minimal projections of $\mathcal{P}_n$?

As stressed in Section 2, it is of crucial importance to discern physical measurements from operators mathematically representing them. When we use Gleason's theorem we intuitively assume that *all* projections in a Hilbert space represent a measurement outcome for a real-world physical measurement. The case of spin enforces this intuition since the Bloch sphere representation of spin-half particles nicely pairs the spatial orientations of the Stern-Gerlach apparatus with the projections of $\mathcal{P}_2$. In general, however, we have no *a priori* knowledge of the measurement-operator assignment. Particularly, we cannot assume that a set of measurements *just because it is an uncountable set* has be represented by the *full* set of projections of a given Hilbert space. *A priori* it is perfectly conceivable that a set of real-world measurements, even if its cardinality is uncountable, can be represented by a proper subset of $\mathcal{P}_n$.

The question of how the *cardinality* of the empirical data influences the possible representations should be discerned from another question, namely the *content* of the empirical data. What is the empirical data that we are going to represent? It is the empirical content of quantum mechanics itself — one may respond. But what is that?

Suppose that for a given Hilbert space $\mathcal{H}_n$ *all* the self-adjoint operators on $\mathcal{H}_n$ represent a real-world empirical measurement with n outcomes and *all* states on $\mathcal{H}_n$ represent a real-world preparation of the system to be measured. In other words, take it at face value that the *full formalism* of an n-dimensional quantum mechanics has empirical meaning. Again, this assumption is legitimate for $n = 2$ where one can see how self-adjoint operators in $\mathcal{H}_2$ nicely align with real-world spin measurements of electrons in different spatial directions. This matching for, say, $n = 13$, however, is not so obvious. Be as it may, suppose we coin the term "empirical content of the n-dimensional quantum mechanics" for the (continuum) set of conditional probabilities provided by the Born rule that is gained by taking the trace of the *all* the different spectral projections multiplied by the *all* the different density operators on $\mathcal{H}_n$. Then our question is this: can the empirical content of the n-dimensional quantum mechanics be represented in $\mathcal{H}_n$ in a noncommutative but *not* quantum mechanical way?

Thus, we have two different questions. 1. Is a noncommutative representation of a set of empirical probabilities necessarily a quantum mechanical representation if the cardinality of the set is continuum? 2. Is a noncommutative re-representation of the empirical content of quantum mechanics is necessarily a quantum mechanical representation? In what comes we will show that the answer to the first question is *no* and the answer to the second question is *yes*.

We start with the first question. Our task is to represent a continuum set of empirical probabilities in a noncommutative but not quantum mechanical way. The set we pick will be the set of probabilities of spin measurements in all the different spatial directions performed on an electron prepared in *one* given state. This set is obviously a continuum set but not yet the full empirical content of the two-dimensional quantum mechanics since we consider only one state. The continuum set of empirical conditional probabilities is the following:

$$\left\{p_a^{\pm} := p_s(A^{\pm}|a); \quad s \text{ is fixed}\right\} \tag{1.11}$$

Here a denotes the spin measurement in direction $\mathbf{a}$ and $A^{\pm}$ are the two spin outcomes. Now, in the Bloch sphere representation one associates two unit vectors

$$\begin{aligned} \mathbf{a} &= (1, \vartheta, \varphi) \\ \mathbf{s} &= (1, 0, 0) \end{aligned}$$

to the spin measurement a and state s of the system, respectively, such that the Born rule yields the conditional probabilities (1.11):

Operator assignment:				Born rule:
Outcomes: $A^{\pm}$	$\longrightarrow$	$P_a^{\pm} = \frac{1}{2}(\mathbf{1} \pm \mathbf{a}\boldsymbol{\sigma})$	$\mathbf{a} \in \mathbf{R}^3$, $\lvert\mathbf{a}\rvert = 1$	$p_a^{\pm} = \mathrm{Tr}(W_s P_a^{\pm})$
Pure state: s	$\longrightarrow$	$W_s = \frac{1}{2}(\mathbf{1} + \mathbf{s}\boldsymbol{\sigma})$	$\mathbf{s} \in \mathbf{R}^3$, $\lvert\mathbf{s}\rvert = 1$	

As is well-known, the measurement outcomes in the Bloch sphere representation are associated to the *full* set of minimal projections $\mathcal{P}_2$, and hence W_s must be represented by a density operator due to Gleason's theorem.

However, the Bloch sphere representation is not the only possible noncommutative representation of (1.11). Here is an alternative.

Consider the following two functions:

$$f : S^2 \to S^2;\ \mathbf{a} \mapsto f(\mathbf{a})$$
$$g : S^2 \to \mathbf{R}^3;\ \mathbf{s} \mapsto g(\mathbf{s})$$

and suppose that instead of $\mathbf{a}$ and $\mathbf{s}$ we associate

$$\begin{aligned} f(\mathbf{a}) &= (1, \vartheta', \varphi') \\ g(\mathbf{s}) &= (r, 0, 0) \end{aligned}$$

to a and s, respectively, where

$$\vartheta' = \arccos\left(\frac{\cos(\vartheta)}{r}\right) \quad \text{for } \varphi \in [0, 2\pi] \tag{1.12}$$

$$\varphi' = \begin{cases} 0 & \text{for } \vartheta = 0 \\ \varphi & \text{for } \vartheta \in (0, \pi) \\ \pi & \text{for } \vartheta = \pi \end{cases} \tag{1.13}$$

and $r > 1$. Observe that f is injective but not surjective: a spherical cap around the "North Pole" and "South Pole" is not in the image of f. It is easy to check that by these associations we obtain a noncommutative representation for the conditional probabilities (1.11):

		Operator assignment:		Born rule:
Outcomes: $A^\pm$	$\longrightarrow$	$P_a^\pm = \frac{1}{2}(\mathbf{1} \pm f(\mathbf{a})\boldsymbol{\sigma})$	$f(\mathbf{a}) \in \mathbf{R}^3$, $\lvert f(\mathbf{a})\rvert = 1$	$p_a^\pm = \mathrm{Tr}(W_s P_a^\pm)$
Pure state: s	$\longrightarrow$	$W_s = \frac{1}{2}(\mathbf{1} + g(\mathbf{s})\boldsymbol{\sigma})$	$g(\mathbf{s}) \in \mathbf{R}^3$, $\lvert g(\mathbf{s})\rvert > 1$	

The representation is a noncommutative but not a quantum mechanical representation since W_s is not positive semidefinite and hence not a density operator. Note again that we have avoided Gleason's theorem because we did not use the full Bloch sphere to represent measurement outcomes but only a "belt" defined by the angles (1.12)-(1.13). To sum up, even though the set of measurements is uncountable, the noncommutative representation is not necessarily quantum mechanical since the set of projections representing the outcomes is not the full set of projections $\mathcal{P}_2$ of the Hilbert space $\mathcal{H}_2$.

However, (1.11) contains only the conditional probabilities of the spin measurement for *one* state. Can we apply the above technique of "pecking a hole" in the surface of the Bloch sphere and "pushing out" s such that W_s will not be a density operator in the case when we take into consideration *all states*? In other words, can we provide a noncommutative but not quantum mechanical representation for the full empirical content of the two-dimensional quantum mechanics? This was our second question above.

This is point where the representation of the set of conditional probabilities gets rigid. It will turn out that if one is to represent the conditional probability of *all* measurement outcomes of *all* spin measurement in *all* states, then there is no other noncommutative representation but the quantum mechanical. We prove it by the following lemma.

Lemma 1. Consider the Bloch sphere representation of spin. That is let $\mathbf{a}$ and $\mathbf{s}$ two unit vectors associated to the spin measurement a and state s of the system, respectively, such that the Born rule yields the conditional probabilities:

$$\mathrm{Tr}(W_s P_a^{\pm}) = \mathrm{Tr}\left(\frac{1}{2}(\mathbf{1}+\mathbf{s}\boldsymbol{\sigma})\frac{1}{2}(\mathbf{1}\pm\mathbf{a}\boldsymbol{\sigma})\right) = \frac{1}{2}(1\pm\mathbf{sa}) \tag{1.14}$$

Then, if there are two functions

$$f : S^2 \to S^2;\ \mathbf{a} \mapsto f(\mathbf{a})$$
$$g : S^2 \to \mathbf{R}^3;\ \mathbf{s} \mapsto g(\mathbf{s})$$

such that all the conditional probabilities (1.14) are preserved that is

$$\mathbf{as} = f(\mathbf{a})g(\mathbf{s}) \tag{1.15}$$

for all $\mathbf{a}, \mathbf{s} \in S^2$, then

(i) f and g are the restrictions of the bijective linear maps

$$\hat{f}\colon \mathbf{R}^3 \to \mathbf{R}^3$$
$$\hat{g}\colon \mathbf{R}^3 \to \mathbf{R}^3$$

to S^2, respectively;

(ii) $\hat{f}$ is the orthogonal transformation;

(iii) $\hat{g} = \hat{f}$.

For the proof of Lemma 1 see the Appendix.

Lemma 1 shows that there is no other transformation of the Bloch vectors which preserve all the empirical conditional probabilities encoded in the inner product but the orthogonal transformation. Consequently, one cannot avoid Gleason's theorem and provide a counter-example of the above type in which the state is represented by a linear but not density operator.

In the rest of the section we prove that this result holds not only in $\mathcal{H}_2$ but in any n-dimensional Hilbert space. We show that one cannot preserve all the empirical conditional probabilities encoded in the inner product of the Hilbert space by other transformation than the unitary transformation. Thus, "compressing" the empirical content in a proper subset of $\mathcal{P}_n$ of a given Hilbert space is not a viable route to follow. If all the inner products of minimal projections have an empirical meaning then the only way to represent them is via quantum mechanics.

Lemma 2. Let $\mathcal{H}$ be an n-dimensional Hilbert space and let $\mathcal{P}_n$ be the set of minimal projections in $\mathcal{B}(\mathcal{H}) \simeq M_n(\mathbf{C})$. If there are two functions

$$f : \mathcal{P}_n \to \mathcal{P}_n$$
$$g : \mathcal{P}_n \to M_n(\mathbf{C})$$

such that

$$\mathrm{Tr}(PQ) = \mathrm{Tr}(f(P)g(Q)) \tag{1.16}$$

for all $P, Q \in \mathcal{P}_n$ then

(i) f and g are the restrictions of the bijective linear maps

$$\hat{f}\colon M_n(\mathbf{C}) \to M_n(\mathbf{C})$$
$$\hat{g}\colon M_n(\mathbf{C}) \to M_n(\mathbf{C})$$

to $\mathcal{P}_n$, respectively;

(ii) $\hat{f}$ is unitary with respect to the inner product on $M_n(\mathbf{C})$ provided by the trace;

(iii) $\hat{g} = \hat{f}$.

For the proof of Lemma 2 see again the Appendix.[1]

[1] I thank Péter Vecsernyés for his help in proving both Lemma 1 and 2.

1.6 Discussion

Is quantum mechanics the only possible way to represent an empirically given set of classical conditional probabilities in a noncommutative way; or is this representation picked out from a broader set of representations by convention? Ultimately, this was the question we posed in this paper. To make this question precise, we specified a set of representations, called noncommutative representations, in which measurement choices and measurement outcomes were represented in the quantum fashion and the Born rule connecting the quantum probabilities to classical conditional probabilities was respected. We asked whether experience can ensure that this representation becomes not just partly but fully quantum mechanical, that is, the state will be represented by a density operator. Our answer was the following:

1. In case of finitely many measurements with finitely many outcomes the probability distribution of outcomes can always be given a noncommutative but not quantum mechanical representation.

2. In case of infinitely many measurements the probability distributions can be given a noncommutative but not quantum mechanical representation only if one can avoid Gleason's theorem by not using all the projections of the Hilbert space in representing measurement outcomes.

3. If the physical situation is so complex that the inner product of any pair of minimal projections is of empirical meaning, then there exists no noncommutative representation which is not quantum mechanical.

The relation between point 2 and 3 is very subtle. It shows that simply the cardinality of the set of measurements does not decide on whether the situation can be given a noncommutative but not quantum mechanical representation. By "compressing" the projections representing measurement outcomes into a real subset of the full set of minimal projections of the given Hilbert space one can go beyond the quantum mechanical representation. The representation becomes rigid only if the inner product of any pair of minimal projections in a Hilbert space can be given an empirical content. This is case for spin-half particles where projections can directly be associated to preparation and measurement directions. Whether one can provide a similar empirical account for the inner product of any pair of minimal projections in a Hilbert space of higher dimension, is a question which cannot be decided *a priori*.

Appendix

Proof of Lemma 1. (i) Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \subset S^2$ be an orthonormal basis in $\mathbf{R}^3$. Then due to (1.15) the sets $\{f(\mathbf{e}_1), f(\mathbf{e}_2), f(\mathbf{e}_3)\}$ and $\{g(\mathbf{e}_1), g(\mathbf{e}_2), g(\mathbf{e}_3)\}$ are biorthogonal:

$$(f(\mathbf{e}_i), g(\mathbf{e}_j)) = \delta_{i,j} \quad i, j = 1, 2, 3$$

Biorthogonal sets with cardinality d in $\mathbf{R}^d$ form (in general two different) linear bases of $\mathbf{R}^d$. Hence, if $\mathbf{a} = \sum_i \alpha_i \mathbf{e}_i \in S^2$ and $f(\mathbf{a}) = \sum_i \alpha_i^f f(\mathbf{e}_i) \in \mathbf{R}^3$ with $\alpha_i, \alpha_i^f \in \mathbf{R}$, then

$$\alpha_i = (\mathbf{a}, \mathbf{e}_i) = (f(\mathbf{a}), g(\mathbf{e}_i)) = \sum_j \alpha_j^f (f(\mathbf{e}_j), g(\mathbf{e}_i)) = \alpha_i^f, \quad i = 1, 2, 3 \quad (1.17)$$

Hence, $f(\sum_i \alpha_i \mathbf{e}_i) = \sum_i \alpha_i f(\mathbf{e}_i)$, that is f is the restriction of the bijective linear map $\hat{f}$ characterized by the image linear basis $\{f(\mathbf{e}_1), f(\mathbf{e}_2, f(\mathbf{e}_3)\}$ of the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. A similar argument shows that g is the restriction of the bijective linear map $\hat{g}$ to S^2.

(ii) Using polarization identity

$$(\mathbf{a}, \mathbf{b}) = \frac{1}{4}\big[(\mathbf{a}+\mathbf{b}, \mathbf{a}+\mathbf{b}) - (\mathbf{a}+\mathbf{b}, \mathbf{a}+\mathbf{b})\big], \quad \mathbf{a}, \mathbf{b} \in \mathbf{R}^3$$

it is enough to show that

$$(\mathbf{a}, \mathbf{a}) = (\hat{f}(\mathbf{a}), \hat{f}(\mathbf{a})), \quad \mathbf{a} \in \mathbf{R}^3$$

which, however, holds since

$$1 = (\mathbf{a}, \mathbf{a}) = (f(\mathbf{a}), f(\mathbf{a})) = (\hat{f}(\mathbf{a}), \hat{f}(\mathbf{a})), \quad \mathbf{a} \in S^2$$

and $\hat{f}$ is linear.

(iii) Using (1.15) and the orthogonality of $\hat{f}$ one has

$$(\mathbf{a}, \mathbf{b}) = (\hat{f}(\mathbf{a}), \hat{g}(\mathbf{b})) = (\mathbf{a}, \hat{f}^{-1}(\hat{g}(\mathbf{b}))), \ \mathbf{a}, \mathbf{b} \in \mathbf{R}^3.$$

Hence, $\hat{g} = \hat{f}$ due to the uniqueness of the inverse map. ■

Proof of Lemma 2. (i) Since the trace is a faithful positive linear functional on $M_n(\mathbf{C})$,

$$(A, B) := \mathrm{Tr}(A^* B), \quad A, B \in M_n(\mathbf{C})$$

defines an inner product on the n^2-dimensional complex linear space $M_n(\mathbf{C})$. The real linear combinations of the projections in $\mathcal{P}_n$ span the real vector space of self-adjoint elements in $M_n(\mathbf{C})$, and the complex linear combinations span the complex vector space $M_n(\mathbf{C})$. Let $\{P_i, i = 1, \dots, n^2\} \subset \mathcal{P}_n$ be a linear basis in $M_n(\mathbf{C})$. Then the inner product matrix $g \in M_{n^2}(\mathbf{C})$ given by matrix elements $g_{ij} := (P_i, P_j) \geq 0$ is an invertible matrix. Since $\mathrm{Tr}(f(P_i)g(P_j)) = g_{ij}$ due to (1.16) $\{f(P_i), i = 1, \dots, n^2\} \subset \mathcal{P}_n$ and $\{g(P_i), i = 1, \dots, n^2\} \subset M_n(\mathbf{C})$ are linear bases in $M_n(\mathbf{C})$ due to invertibility of g. Defining the bijective linear maps $\hat{f}, \hat{g} \colon M_n(\mathbf{C}) \to M_n(\mathbf{C})$ by the linear extension of these image bases for $P = \sum_i \alpha_i P_i \in \mathcal{P}_n$ one has

$$(f(P), g(P_j)) = (P, P_j) = (\sum_i \alpha_i P_i, P_j) = \sum_i \alpha_i (P_i, P_j) = \sum_i \alpha_i (f(P_i), g(P_j))$$
$$= (\sum_i \alpha_i f(P_i), g(P_j)) =: (\hat{f}(\sum_i \alpha_i P_i), g(P_j)) = (\hat{f}(P), g(P_j)), \quad j = 1, \dots, n^2.$$

Hence, f is the restriction of the bijective linear map $\hat{f}$ to $\mathcal{P}_n$, indeed. A similar argument shows that g is the restriction of the bijective linear map $\hat{g}$ to $\mathcal{P}_n$.

(ii) Using polarization identity

$$(A, B) = \frac{1}{4}\big[(A + B, A + B) - (A - B, A - B)\big], \quad A, B \in M_n(\mathbf{C})$$

it is enough to show unitarity on 'diagonal' inner products:

$$(A, A) = (\hat{f}(A), \hat{f}(A)), \quad A \in M_n(\mathbf{C})$$

Since $f(\mathcal{P}_n) \subset \mathcal{P}_n$ by assumption, using the normalization $\mathrm{Tr}(P) = 1, P \in \mathcal{P}_n$ of the trace it follows that

$$(P, P) = 1 = (f(P), f(P)) = (\hat{f}(P), \hat{f}(P)),\ P \in \mathcal{P}_n$$

i.e. $\hat{f}$ is unitary on diagonals from $\mathcal{P}_n$. Using a spectral decomposition of self-adjoint elements by orthogonal minimal projections one concludes that $\hat{f}$ maps the real vector space of self-adjoint elements in $M_n(\mathbf{C})$ into itself, moreover, it is unitary on diagonals from the space of self-adjoint elements. Since $A \in M_n(\mathbf{C})$ can be written uniquely as a sum of self-adjoint elements: $A = R + iI$ with $R := (A + A^*)/2$ and $I := (A - A^*)/2i$ it follows that

$$\begin{aligned}(A, A) &= (R + iI, R + iI) = (R, R) + (I, I) = (\hat{f}(R), \hat{f}(R)) + (\hat{f}(I), \hat{f}(I)) \\ &= (\hat{f}(R) + i\hat{f}(I), \hat{f}(R) + i\hat{f}(I)) = (\hat{f}(A), \hat{f}(A)),\end{aligned}$$

that is $\hat{f}$ is unitary on diagonals from $M_n(\mathbf{C})$, which provides unitarity of $\hat{f}$.

(iii) Using (1.16) and unitarity of $\hat{f}$ one has

$$(A, B) = (\hat{f}(A), \hat{g}(B)) = (A, \hat{f}^{-1}(\hat{g}(B))),\ \ A, B \in M_n(\mathbf{C}).$$

Hence, $\hat{g} = \hat{f}$ due to the uniqueness of the inverse map. ■

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Chapter 2

Three principles leading to the Bell inequalities

In the paper we compare three principles accounting for correlations, namely Reichenbach's Common Cause Principle, Bell's Local Causality Principle, and Einstein's Reality Criterion and relate them to the Bell inequalities. We show that there are two routes connecting the principles to the Bell inequalities. In case of Reichenbach's Common Cause Principle and Bell's Local Causality Principle one assumes a non-conspiratorial joint common cause for a set of correlations. In case of Einstein's Reality Criterion one assumes strongly non-conspiratorial separate common causes for a set of perfect correlations. Strongly non-conspiratorial separate common causes for perfect correlations, however, form a non-conspiratorial joint common cause. Hence the two routes leading the Bell inequalities meet.

2.1 Introduction

Many were pondering on the historical reasons of why it took thirty years to get from the EPR argument to the Bell inequalities. (See for example (Bell 1964/2004); (Howard 1985); (Redhead 1987); (Hájek and Bub 1992); (Fine 1996); (Norton 2004); (Szabó 2008); (Goldstein et al. 2011); (Maudlin 2014) and (Lewis 2015).) This paper has nothing to say about these historical and conceptual reasons. It rather intends to show that the route leading from Einstein's Reality Criterion to the Bell inequalities is no longer than the route starting off from two other principles standardly used to causally account for correlations, namely Reichenbach's Common Cause Principle and Bell's Local Causality Principle.

In the paper we will handle the three principles side by side and show how they relate to one another and to the Bell inequalities. In Section 2 we show how the principles are used to causally account for correlations; in Section 3 we use them to explain conditional correlations; and in Section 4 we trace the routes leading from the principles to the Bell inequalities. In the paper we deliberately keep the philosophical analysis short so that the formal parallelism will not be lost sight of.

2.2 Explaining correlations

Let A and B be two correlated but causally separated events represented in a classical probability space (Σ, p):

$$p(A \wedge B) \neq p(A)\, p(B) \tag{2.1}$$

One can invoke three principles to causally account for this correlation. If one is concerned only with the probabilistic aspects, one can apply

Reichenbach's Common Cause Principle: If there is a correlation between two events and there is no direct causal (or logical) connection between the correlated events, then there always exists a common cause of the correlation.

Formally, a common cause of the correlation (10.1) is a partition $\{C_k\}$ $(k \in K)$ in (Σ, p)—or in an extension of (Σ, p); see (Hofer-Szabó, Rédei and Szabó 2013)—such that for any $k \in K$:

$$p(A \wedge B|C_k) = p(A|C_k)\, p(B|C_k) \tag{2.2}$$

If one furthermore assumes that the events A and B also have spatiotemporal localization, for example they are located in spatially separated regions, V_A and V_B, respectively, then to causally account for them, one can invoke a further principle:

Bell's Local Causality Principle: "A theory will be said to be locally causal if the probabilities attached to values of local beables in a space-time region V_A are unaltered by specification of values of local beables in a spatially separated region V_B, when what happens in the backward light cone of V_A is already sufficiently specified, for example by a full specification of local beables in a space-time region V_C." (Bell 1990/2004, 239-240)

The figure Bell is attaching to this formulation is reproduced in Fig. 6.1 with the original caption. In a locally causal theory for any correlation between events A and B localized in spatially separated regions V_A and V_B, respectively, the *atomic partition* $\{C_k\}$ $(k \in K)$ in the probability space (Σ, p) associated to any region V_C causally shielding-off V_A from the common past of V_A and V_B as depicted Fig. 6.1 should satisfy (2.2).

Finally, suppose we interpret the correlation (10.1) epistemologically as a *prediction*. That is we interpret A as a *predicting event* and B as a *predicted event* and the prediction as a correlation between the two. After all, a prediction is ontologically nothing but an (ideally strong) correlation between two event types. Weather forecast is simply a correlation between the today announcement and the tomorrow weather. Moreover, in a prediction the predicted event cannot causally influence the predicting events. One can predict the tomorrow weather but not the yesterday weather.

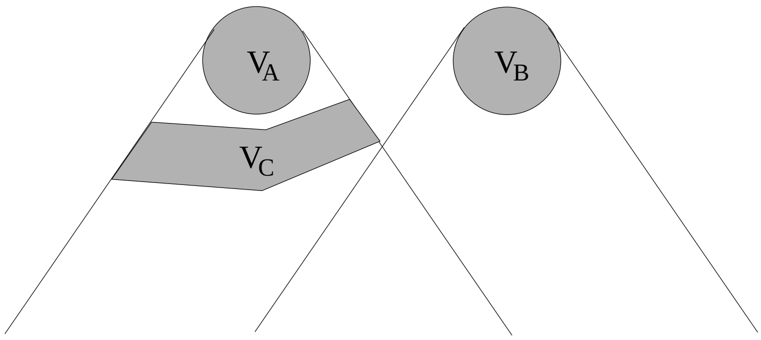

Figure 2.1: Full specification of what happens in V_C makes events in V_B irrelevant for predictions about V_A in a locally causal theory.

Suppose furthermore that the following two requirements also hold: (i) The predicting event is also causally irrelevant for the predicted event. This can happen for example when the two events are spatially separated. (ii) The correlation between A and B is *perfect*:

$$p(A \wedge B) = p(A) = p(B) \tag{2.3}$$

If all these hold, then we have a third principle to account for the correlation (2.3):

Einstein's Reality Criterion: "If, without in any way disturbing a system, we can predict with certainty (i.e. with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity." (EPR 1935, 777-778)

Observe, that the term "without in any way disturbing a system" is just condition (i) above, and the term "predict with certainty" is just condition (ii). What Einstein's Reality Criterion requires is that in case of a perfect prediction, that is perfect correlation between causally separated events, an element of reality should account for the correlation.

What is an element of reality?

The distinctive feature of an element of reality (see (Gömöri and Hofer-Szabó 2017) for the details) is that it determines the predicted event with certainty. Formally, an element of reality is a partition $\{C^+, C^-\}$ in (Σ, p) such that the following holds:

$$p(A \wedge B|C^+) = 1 \tag{2.4}$$

$$p(A \wedge B|C^-) = 0 \tag{2.5}$$

Now, let us go back to the Reichenbach's Common Cause Principle. It is well known that for perfect correlations a common cause that is a partition $\{C_k\}$ $(k \in K)$

satisfying (2.2) is *deterministic*: for any $k \in K$

$$p(A \wedge B|C_k) \in \{0, 1\} \tag{2.6}$$

Hence, the indices $k \in K$ can be grouped into two groups K^+ and K^- with $K^+ \vee K^- = K$ such that

$$\begin{aligned} C^+ &= \vee_{k \in K^+} C_k & (2.7) \\ C^- &= \vee_{k \in K^-} C_k & (2.8) \end{aligned}$$

and $\{C^+, C^-\}$ satisfies (2.4)-(2.5). Common causes for perfect correlations understood as predictions are just elements of reality.

To sum up, a correlation between two events depending on whether we understand it purely probabilistically or spatiotemporally or in the context of predictions can be explained by three different principles: by Reichenbach's Common Cause Principle, by Bell's Local Causality Principle or by Einstein's Reality Criterion.

2.3 Explaining conditional correlations

Now, let us apply the above reasoning to measurements. Let a_i and b_j ($i \in I, j \in J$) be *measurement choices* and let $\{A_i, A'_i\}$ and $\{B_j, B'_j\}$ be binary *measurement outcomes* on two spatially separated systems. We will represent the measurement choices as two partitions $\{a_i\}$ ($i \in I$) and $\{b_j\}$ ($j \in J$) in a classical probability space (Σ, p), and the measurement outcomes by further partitioning the appropriate measurement choices a_i and b_j, respectively:

$$\begin{aligned} A_i \wedge A'_i = 0 \qquad A_i \vee A'_i = a_i && (2.9) \\ B_j \wedge B'_j = 0 \qquad B_j \vee B'_j = b_j && (2.10) \end{aligned}$$

Suppose that for a given $i \in I$ and $j \in J$ the measurement outcomes A_i and B_j are *conditionally* correlated in the following sense:

$$p(A_i \wedge B_j|a_i \wedge b_j) \neq p(A_i|a_i)\, p(B_j|b_j) \tag{2.11}$$

What is the causal explanation of this conditional correlation?

Before we turn to the above principles, we make the following stipulation: Whatever explains the above correlations, it has to be causally and hence probabilistically independent of the measurement choices. In other words, in applying the above principles we will always require:

No-conspiracy: If a partition $\{C_k\}$ ($k \in K$) represents a set of events explaining the correlation (10.13), then for any $k \in K$ the following relation is required:

$$p(a_i \wedge b_j \wedge C_k) = p(a_i \wedge b_j)\, p(C_k) \tag{2.12}$$

Next, we formulate the three principles causally accounting for the conditional correlation between the measurement outcomes given certain measurement choices:

Reichenbach's Common Cause Principle: The common cause of the conditional correlation (10.13) is a partition $\{C_k\}$ in (Σ, p) such that for any $k \in K$:

$$p(A_i \wedge B_j | a_i \wedge b_j \wedge C_k) = p(A_i | a_i \wedge C_k)\, p(B_j | b_j \wedge C_k) \tag{2.13}$$

$$p(a_i \wedge b_j \wedge C_k) = p(a_i \wedge b_j)\, p(C_k) \tag{2.14}$$

Bell's Local Causality Principle: Suppose there is a conditional correlation (10.13) between measurement outcomes A_i and B_j given measurement choices a_i and b_j. Suppose further that A_i and a_i are localized in regions V_A and B_j and b_j are localized in regions V_B spatially separated from V_A. Then, if the theory accounting for this correlation is locally causal, then the atomic partition $\{C_k\}$ $(k \in K)$ in (Σ, p) associated to the region V_C (see Fig. 6.1) should satisfy (2.13)-(3.3).

Einstein's Reality Criterion: Suppose that the conditional correlation (10.13) represents now a prediction. That is let A_i denote the outcome of a predicting event a_i and let B_j denote the outcome of the predicted event b_j. Suppose furthermore that A_i, a_i and B_j, b_j are causally separated. Also suppose that we can predict the outcome B_j of the measurement b_j by obtaining outcome A_i for the prediction a_i for sure. In other words, suppose that the conditional correlation is perfect:

$$p(A_i \wedge B_j | a_i \wedge b_j) = p(A_i | a_i) = p(B_j | b_j) \tag{2.15}$$

Then Einstein's Reality Criterion claims that there are elements of reality that is a partition $\{C^+, C^-\}$ in (Σ, p) explaining correlation (2.15) in the following sense:

$$p(A_i \wedge B_j | a_i \wedge b_j \wedge C^+) = 1 \tag{2.16}$$

$$p(A_i \wedge B_j | a_i \wedge b_j \wedge C^-) = 0 \tag{2.17}$$

$$p(a_i \wedge b_j \wedge C^+) = p(a_i \wedge b_j)\, p(C^+) \tag{2.18}$$

$$p(a_i \wedge b_j \wedge C^-) = p(a_i \wedge b_j)\, p(C^-) \tag{2.19}$$

Just as above, in case of a perfect correlation a common cause $\{C_k\}$ $(k \in K)$ satisfying (2.13)-(3.3) is deterministic, hence a suitable grouping of the C_k-s *via* (2.7)-(2.8) will yield the elements of reality C^+ and C^-. In short, Einstein's Reality Criterion is a special case of Reichenbach's Common Cause Principle when the correlation is perfect. (For the details see Gömöri and Hofer-Szabó 2017.)

To sum up, the core of all three principles is to account for correlations in terms of a non-conspiratorial common cause. In case of Reichenbach's Common Cause Principle only the probabilistic aspects (2.13)-(3.3) of the common cause are taken into consideration. In case of Bell's Local Causality Principle both the correlated events and also the common cause have a spatiotemporal localization. In case of Einstein's Reality Criterion the whole correlation scenario is interpreted in the framework of a prediction and the correlation is taken to be perfect.

Before we move on to the relation of the principles to the Bell inequalities, let us see how the conditional and unconditional correlations and their explanations relate to one another.

First, observe that if the measurement choices are causally and therefore probabilistically independent, that is if for any $i \in I$ and $j \in J$:

$$p(a_i \wedge b_j) \quad = \quad p(a_i)\, p(b_j) \tag{2.20}$$

and the algebraic inclusions (2.9)-(2.10) hold, then the outcomes A_i and B_j are correlated in the *conditional* sense

$$p(A_i \wedge B_j | a_i \wedge b_j) \neq p(A_i | a_i)\, p(B_j | b_j) \tag{2.21}$$

if and only if they are correlated in the *unconditional* sense

$$p(A_i \wedge B_j) \neq p(A_i)\, p(B_j) \tag{2.22}$$

Second, given (2.9)-(2.10) and (2.20), $\{C_k\}$ is a non-conspiratorial common cause of the *conditional* correlation (2.21):

$$p(A_i \wedge B_j | a_i \wedge b_j \wedge C_k) \quad = \quad p(A_i | a_i \wedge C_k)\, p(B_j | b_j \wedge C_k) \tag{2.23}$$

$$p(a_i \wedge b_j \wedge C_k) \quad = \quad p(a_i \wedge b_j)\, p(C_k) \tag{2.24}$$

if and only if $\{C_k\}$ is a non-conspiratorial common cause of the *unconditional* correlation (2.22):

$$p(A_i \wedge B_j | C_k) \quad = \quad p(A_i | C_k)\, p(B_j | C_k) \tag{2.25}$$

$$p(a_i \wedge b_j \wedge C_k) \quad = \quad p(a_i \wedge b_j)\, p(C_k) \tag{2.26}$$

(For the proof see (Hofer-Szabó, Rédei and Szabó 2013, Lemma 9.8).) Therefore, on the assumptions (2.9)-(2.10) and (2.20), the common causal explanations (2.23)-(3.5) and (2.25)-(3.19) are interchangeable.

2.4 From the principles to the Bell inequalities

How the above three principles serving for a causal explanation of correlations relate to the Bell inequalities? The crucial point is to see how the different principles relate to the common causal explanation of *more correlations*. Principally, there are two possible ways: either the different correlations are explained by a *joint common cause* or each correlation is explained by a *separate common cause*. The standard derivation of the Bell inequalities from Reichenbach's Common Cause Principle and Bell's Local Causality Principle assumes a joint common cause; whereas the derivation of the Bell inequalities from Einstein's Reality Criterion assumes only separate common causes. Since the assumption of separate common causes is weaker than that of a joint common cause, the derivation of the Bell inequalities from Einstein's Reality Criterion needs a stronger version of no-conspiracy.

Let us see the derivations in turn:

Reichenbach's Common Cause Principle. Suppose that $I = J = \{1, 2\}$ and the events A_i and B_j are all conditionally correlated that is for any $i, j \in I$:

$$p(A_i \wedge B_j | a_i \wedge b_j) \neq p(A_i | a_i) p(B_j | b_j) \tag{2.27}$$

The four correlations are said to have a *non-conspiratorial joint common cause* if there is a *single* partition $\{C_k\}$ $(k \in K)$ in (Σ, p) (or in an extension of (Σ, p)) such that for all $i, j \in I$ and $k \in K$ the following hold:

$$p(A_i \wedge B_j | a_i \wedge b_j \wedge C_k) = p(A_i | a_i \wedge C_k)\, p(B_j | b_j \wedge C_k) \tag{2.28}$$

$$p(a_i \wedge b_j \wedge C_k) = p(a_i \wedge b_j)\, p(C_k) \tag{2.29}$$

We claim that the events A_i, B_j, a_i and b_j with a non-conspiratorial joint common causal explanation satisfy the *Clauser–Horne inequalities* that is for any $i, i', j, j' \in I$ and $i \neq i', j \neq j'$:

$$\begin{aligned} &-1 \leqslant p(A_i \wedge B_j | a_i \wedge b_j) + p(A_i \wedge B_{j'} | a_i \wedge b_{j'}) \\ &+p(A_{i'} \wedge B_j | a_{i'} \wedge b_j) - p(A_{i'} \wedge B_{j'} | a_{i'} \wedge b_{j'}) \\ &-p(A_i | a_i) - p(B_j | b_j) \leqslant 0 \end{aligned} \tag{2.30}$$

For the proof see the Appendix.

Bell's Local Causality Principle. Again, let $I = J = \{1, 2\}$. Suppose that the events A_i and B_j localized in spatially separated regions V_A and V_B respectively, are all conditionally correlated in the sense of (2.27). In a *locally causal theory* the atomic partition of the local algebra associated to V_C (see again Fig. 6.1) is a non-conspiratorial joint common cause in the sense of (2.28)-(2.29). Hence the Clauser–Horne inequalities (9.24) follow, just as in the case of Reichenbach's Common Cause Principle.

Einstein's Reality Criterion. Suppose now that $I = J = \{1, 2, 3, 4\}$ and there is a perfect conditional correlation between (the predicting events) A_i and (the predicted events) B_j for any $i = j \in I$:

$$p(A_i \wedge B_i | a_i \wedge b_i) = p(A_i | a_i) = p(B_i | b_i) \tag{2.31}$$

First, observe that the four correlations in (2.31) are not the same as the correlations (2.27) above. In (2.27) $I = J = \{1, 2\}$ and the four correlations were not necessarily perfect; in (2.31) $I = J = \{1, 2, 3, 4\}$ and the four correlations are the $i = j$ perfect correlations.

Now, Einstein's Reality Criterion does *not* assume that all four correlations in (2.31) have a *joint* common cause. All it assumes is that there are *separate* elements of reality to each correlation, that is for any $i \in I$ there is a partition $\{C_i^+, C_i^-\}$ satisfying

$$p(A_i \wedge B_i | a_i \wedge b_i \wedge C_i^+) = 1 \tag{2.32}$$

$$p(A_i \wedge B_i | a_i \wedge b_i \wedge C_i^-) = 0 \tag{2.33}$$

However, instead of simply requiring no-conspiracy:

$$p(a_i \wedge b_j \wedge C_k^+) = p(a_i \wedge b_j)\, p(C_k^+) \tag{2.34}$$

$$p(a_i \wedge b_j \wedge C_k^-) = p(a_i \wedge b_j)\, p(C_k^-) \tag{2.35}$$

$(i, j, k \in I)$ one requires *strong no-conspiracy*, namely that *any* element C in the Boolean algebra generated by the four pairs of elements of reality $\{C_k^{\pm}\}$ should be independent of *any* combination of the measurement choices:

$$p(a_i \wedge b_j \wedge C) = p(a_i \wedge b_j)\, p(C) \tag{2.36}$$

In short, in case of more correlations Einstein's Reality Criterion requires *less* than the other two principles since it requires only separate elements of reality for the different correlations, but also requires *more* since it requires all Boolean combinations of the elements of reality to be independent of the measurement choices.

The derivation of the Clauser–Horne inequalities (9.24) from a strongly non-conspiratorial separate common causal explanation is straightforward. From (2.31), (2.32)-(2.33) and (2.36) it follows that for any $i, j \in I$:

$$p(A_i|a_i) = p(B_i|b_i) = p(C_i^+) \tag{2.37}$$

$$p(A_i \wedge B_j|a_i \wedge b_j) = p(C_i^+ \wedge C_j^+) \tag{2.38}$$

Now, it is an elementary fact of classical probability theory that for any four events $C_i^+, C_{i'}^+, C_j^+$ and $C_{j'}^+$ in (Σ, p) we have:

$$\begin{aligned} -1 \leqslant p(C_i^+ \wedge C_j^+) + p(C_i^+ \wedge C_{j'}^+) + p(C_{i'}^+ \wedge C_j^+) \\ -p(C_{i'}^+ \wedge C_{j'}^+) - p(C_i^+) - p(C_j^+) \leqslant 0 \end{aligned} \tag{2.39}$$

Substituting (2.37)-(2.38) into (2.39) one arrives at (9.24).

What one proves here is that the atomic partition composed of the intersections of strongly non-conspiratorial separate common causes *for perfect correlations* form a non-conspiratorial joint common cause for all correlations. Note that in the general case that is *for non-perfect correlations* the relation between separate and joint common causes is not so straightforward and the relation of strongly non-conspiratorial separate common causes to the Bell inequalities is not known. (See (Hofer-Szabó, Rédei and Szabó 2013, Conjecture 9.11.))

To sum up, one can arrive at the Bell inequalities from the three principles on two different routes. In the standard derivation based on Reichenbach's Common Cause Principle or Bell's Local Causality Principle one takes four correlations and assumes that they have a non-conspiratorial joint common cause. In case of Einstein's Reality Criterion one takes four perfect correlations and assumes that each has a separate common cause which together are strongly non-conspiratorial. Both routes lead directly to the Clauser-Horne inequalities.

2.5 Conclusions

In this paper we compared three principles accounting for correlations and related them to the Bell inequalities. Reichenbach's Common Cause Principle, in the original sense at least, refers only to one correlation: it demands a common cause for a given correlation if the direct causal link between the correlata can be excluded.

In the derivation of the Bell inequalities, however, the principle had to be used in a stronger sense, namely demanding one and the same cause for a set of correlations. Bell's Local Causality Principle has already been formulated originally in this strong sense: all correlations localized in spatially separated regions were to be screened-off by the "full specification" of an appropriately localized third spacetime region. In this sense Bell's Local Causality Principle is a stronger principle than Reichenbach's Common Cause Principle. Finally, Einstein's Reality Criterion again assumes elements of reality to each correlation separately, similarly to Reichenbach's Common Cause Principle. Moreover, it does so only in case of perfect correlations. In this sense Einstein's Reality Criterion seems to be even weaker than Reichenbach's Common Cause Principle.

Note, however, that not even the strongest of the three principles, namely Bell's Local Causality Principle implies the Bell inequalities on its own. Even this principle needs to assume that the common causes or elements of reality causally responsible for the correlations are causally and hence probabilistically independent from the measurement choices. To be sure, no-conspiracy seems to be a natural requirement for an element of reality to deserve its name. No-conspiracy, however, can be defined in different strength. And this is the point where the principles faring worse at the beginning can catch up. Even though Einstein's Reality Criterion provides only separate elements of reality for the correlations, if these elements of reality are strongly non-conspiratorial, then they suffice to derive the Bell inequalities. In short, no-conspiracy together with joint elements of reality and strong no-conspiracy together with separate elements of reality fare equally well in the derivation of the Bell inequalities.

Appendix

Proof. It is an elementary fact of arithmetic that for any $\alpha, \alpha', \beta, \beta' \in [0,1]$ we have

$$-1 \leqslant \alpha\beta + \alpha\beta' + \alpha'\beta - \alpha'\beta' - \alpha - \beta \leqslant 0 \tag{2.40}$$

Now, let $\alpha, \alpha', \beta, \beta'$ be

$$\alpha = p(A_i|a_i \wedge C_k) \tag{2.41}$$

$$\alpha' = p(A_{i'}|a_{i'} \wedge C_k) \tag{2.42}$$

$$\beta = p(B_j|b_j \wedge C_k) \tag{2.43}$$

$$\beta' = p(B_{j'}|b_{j'} \wedge C_k) \tag{2.44}$$

Substituting (9.26)–(9.29) into (9.25) we get

$$\begin{aligned} -1 \leqslant\ & p(A_i|a_i \wedge C_k)p(B_j|b_j \wedge C_k) + p(A_i|a_i \wedge C_k)p(B_{j'}|b_{j'} \wedge C_k) \\ & + p(A_{i'}|a_{i'} \wedge C_k)p(B_j|b_j \wedge C_k) - p(A_{i'}|a_{i'} \wedge C_k)p(B_{j'}|b_{j'} \wedge C_k) \\ & - p(A_i|a_i \wedge C_k) - p(B_j|b_j \wedge C_k) \leqslant 0 \end{aligned} \tag{2.45}$$

Using the screener-off condition (2.28) we obtain

$$\begin{aligned} -1 \leqslant p(A_i \wedge B_j | a_i \wedge b_j \wedge C_k) + p(A_i \wedge B_{j'} | a_i \wedge b_{j'} \wedge C_k) \\ +p(A_{i'} \wedge B_j | a_{i'} \wedge b_j \wedge C_k) - p(A_{i'} \wedge B_{j'} | a_{i'} \wedge b_{j'} \wedge C_k) \\ -p(A_i | a_i \wedge C_k) - p(B_j | b_j \wedge C_k) \leqslant 0 \end{aligned} \tag{2.46}$$

Multiplying by $p(C_k)$, using no-conspiracy (2.29) and summing up for k one arrives at (9.24). ■

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Chapter 3

How human and nature shake hands: the role of no-conspiracy in physical theories

No-conspiracy is the requirement that measurement settings should be probabilistically independent of the elements of reality responsible for the measurement outcomes. In this paper we investigate what role no-conspiracy generally plays in a physical theory; how it influences the semantical role of the event types of the theory; and how it relates to such other concepts as separability, compatibility, causality, locality and contextuality.

3.1 Introduction

As the old *bon mot* has it, in experiment human and nature shake hands. This portrayal of the experiment as the celebration of a good business pact between two parties highlights two features of experimentation, namely that both human and nature are equally contributing to its success and that both parties are being independent. This independence is the topic of the present paper.

In the foundations of quantum mechanics probably the most significant research project has been for decades to precisely identify and conceptually analyze those assumptions that go into the derivation of the Bell inequalities and can be made responsible for their violation in the EPR scenario. Locality, factorization, Common Cause Principle, determinism—these were the main concepts and principles on the table. There was, however, one additional premise which, though being indispensable in the derivation of the Bell inequalities, remained much more obscure concerning its status, meaning and relation to the other premises.

The palpable evidence for this embarrassment around this assumption is that there has not even been coined a name for it. It has been referred to by many names such as (no) "conspiratorial entanglement" (Bell, 1981), "hidden autonomy" (Van Fraassen, 1982), "independence assumption" (Price 1996), "free will assumption" (Tumulka, 2007), "measurement independence" (Sanpedro, 2013), (no) "superdeterminism" (Price and Wharton, 2015), and—probably in its most well-known form—"no-conspiracy" (Hofer-Szabó, Rédei and Szabó, 1999; Placek and Wroński, 2009). This latter is the phrase we are going to use in this paper.

The fact that no-conspiracy has been used by so many names attests that there is a wide range of topics which it can be related to. It has been explicitly addressed by Bell in his 1981 paper and its rejection has been qualified as "even more mind boggling than one in which causal chains go faster than light" (Bell, 1981, p. 57). No-conspiracy made its way into the philosophy of physics via Van Fraassen's 1982 careful analysis of the assumptions leading to the Bell inequalities. Ever since these two influential papers no-conspiracy has been given some attention in the philosophy of science. A topic gaining probably the greatest philosophical interest was that how no-conspiracy is related to free will. The first to identify conspiracy as a lack of free will was Bell (1977, 1981) himself and has been followed by many others (Price 1996; Conway and Kochen, 2006; Tumulka, 2007, Price and Wharton, 2015).

The present paper does not concern any of the topics mentioned above: neither free will, nor EPR, nor Bell inequalities. It does not investigate no-conspiracy at the level of the specific scientific theories such as quantum mechanics, quantum field theory, etc. (For this see (Bell, 1977, 1981), (Butterfield, 1995), (Sanpedro, 2013, 2014), (Hofer-Szabó, Rédei and Szabó, 2013), (Price and Wharton, 2015)). Our aim is more general: to investigate what role no-conspiracy plays in a physical theory. To this aim in Section 2 we will first unfold a general scheme of the ontology of a physical theory. We will discern two event types making the ontology: measurement event types and elements of reality. Measurement event types can be of two types: measurement settings and measurement outcomes. We will clarify how measurement settings and measurement outcomes provide semantics for a physical theory. To illustrate the general scheme we introduce a toy model in Section 3 which will then be used throughout the paper. No-conspiracy enters in Section 4. Here we show how the presence of no-conspiracy can deprive measurement settings and measurement outcomes of their semantical role and directs them into pragmatics. In Section 5 some examples will be given for situations when no-conspiracy is violated. In Sections 6 to 10 we will investigate in turn the relationship of no-conspiracy to such concepts as separability, compatibility, causality, locality and contextuality. We conclude with a discussion in Section 11.

This paper is written in the down-to-earth physicalist philosophy of László E. Szabó to whom I dedicate it.

3.2 The ontology of experiment

In this Section we expose the main philosophical ideas lying behind our approach in a concise manner. In the following Section all these general considerations will be made concrete on a simple toy model. The approach we are following here is a strict actualist approach where the key concepts such as causality, probability, etc. all supervene on particulars instantiating certain event types in a Humean manner. This framework is certainly not necessary to address the question of no-conspiracy; I presume that most claims of the paper also hold in other metaphysical frameworks. I follow this approach simply because the present paper is part of a larger research project aiming to explore how far one can get in understanding physical theories and especially quantum mechanics within a Humean framework.

A *physical theory* can be reconstructed as a formal system plus a semantics connecting the formal system to the world. The *formal system* consists of a formal language with some logical axioms and derivation rules, some mathematical and physical axioms. The *semantics* provides an interpretation for the formalism; it connects the formal system to reality. Note that here 'semantics' does not mean a connection between the formal system and some models of the system as in model theory; here semantics means a down-to-earth physical interpretation of the formal system. We stress again that the semantics is an indispensable part of a physical theory. A formal system in itself is *not* yet a physical theory (Szabó, 2011).

The *semantics* settles the ontology of the theory. This can be done in many ways but typically the semantics fixes the ontological *types* or *categories* out there in the world and provides some means to decide when a certain *token* falls in the category of a given type making a certain sentence of the theory true. The types and tokens which we will be interested in here are *event types* and *token events*. The ontology of a physical theory is an *event algebra* constructed from these event types. Note that concerning the ontology of the types our approach is not committed metaphysically either to the realist nor to the nominalist camp.

Physical theories are verified by *experiments*. The rough picture of an experiment is the following. An experimenter performs a procedure by setting a measurement apparatus in a certain way, obtaining a measurement outcome and repeating this procedure many times. The two essential ontological categories of an experiment are the *measurement settings* and the *measurement outcomes*. These categories are event types just as the other ontological types of the theory. The token events are the *instances* of these event types in the different runs of the experiment. Sometimes I will simply refer to these token events as the *runs* of the experiment.

Measurement settings and measurement outcomes do not appear directly in the textbook form of a theory but they are indispensable part of the semantics (not of pragmatics!): without them the theory cannot be linked to reality. More than that, these two types are the only types an experimenter has direct empirical access to. Everything else posited by the theory has to ultimately boil down to some relations between these observable categories. To be more specific, any deductive or inductive relation between the ontological types of the theory has to be accounted for in terms of *correlations* between the token events falling in the category of measurement settings and measurement outcomes. As the empiricist thesis teaches, one has no other access to physical reality than via observation.

Correlations between measurement settings and measurement outcomes can be accounted for in terms of *probabilities*. In our actualist framework the probability of an outcome type is understood as the *long-run relative frequency* of those runs of the experiment which fall in that type if the experiment is repeated appropriately many times. Specifically, the probability of an outcome *given* a certain measurement setting is simply the number of those runs which fall in both the type of the outcome and the setting divided by the number of those runs which fall in the type of the setting. More importantly, *any* probability assignment to any ontological type to which we have no direct empirical access must be based on *type assignments* to the individual runs of the experiment in the long-run frequency sense: the probability of a given type is p only if the relative frequency of the individual runs (instances) falling in the type in question is p. Probability supervenes on the Humean mosaic of token events.

In order to account for the observable measurement outcomes physical theories typically introduce a further, not directly accessible event type, which we will call *elements of reality*. In this sense our approach is scientifically realist. Elements of reality come in two sorts: they can either determine the measurement outcomes for a given measurement setting *for sure*, or they can fix only the *probability* of the measurement outcomes. We will call the first event type *property* and the second event type *propensity*. Whereas measurement outcomes are clearly causally influenced by and therefore probabilistically dependent on the elements of reality, it is not a priori clear what the relation between the measurement settings and the elements of reality should be. This is what we are going to analyze in what comes.

3.3 A toy model

Let us make these abstract considerations more concrete on a simple model. (For a general scheme of a physical theory see the Appendix.) Consider a box containing colored dice (Szabó, 2008). Let us try to develop a physical theory of this system. Whatever theory we develop, the semantics of the theory has to minimally specify the measurement settings and measurement outcomes. These are the categories which are directly accessible for an experimenter. Suppose that the measurement settings are the following:

a_1: drawing a die from the box and checking its *color*

a_2: drawing a die from the box, throwing it and checking the *number on its upper face*

Suppose furthermore that the measurement outcomes are

A_1^i: the color of the die is *black* (A_1^1) or *white* (A_1^2)

A_2^j: the number on the *upper face* of the die is j ($j = 1 \ldots 6$)

So the semantics of the theory posits the following event types: the measurement settings a with two subcategories a_1 and a_2, and the measurement outcomes A with two plus six sub-subcategories A_1^i and A_2^j.

As the experimenter is repeating the experiment, the token events, that is the runs falling in the different event types, are accumulating giving rise to a probabilistic description of the experiment. She can calculate for example the conditional probability of obtaining a black die on the condition that she had performed the color measurement:

$$p(A_1^1|a_1) = \frac{\#(A_1^1 \wedge a_1)}{\#(a_1)}$$

This probability is empirically accessible: one just reads off from the relative frequency of the measurement outcomes and measurement settings. (Here we set aside problems concerning the convergence of the relative frequencies.)

The experimenter can of course try to enrich her theory and introduce a new ontological category into her theory. The motivation behind this move is to obtain an answer to the question: "Why was the outcome of the color measurement black in a certain run of the experiment?" A natural answer to this question is to say: "Because the die itself was black." This answer amounts to introducing a third event type into our ontology, which we will call *property*. What is a property?

The defining feature of the property black is the following: whenever a die with the property black is subjected to a color measurement, the outcome will always be black. Denote the property black by α_1^1 and the property white by α_1^2. (So our notation is the following: we use lower case Latin letter for the measurement settings (a); capital Latin letters for the measurement outcomes (A); and Greek letters for the elements of reality (α).) The property black is an event type and each token event that is each run of the experiment can be characterized by either falling into this event type or not. Therefore, one can also meaningfully speak about the probability of the property black, $p(\alpha_1^1)$, as the long-run relative frequency of those runs of the experiment which fall into the event type α_1^1. Consequently, one can also express the defining feature of the property black and white in terms of probabilities as follows:

$$p(A_1^i|a_1 \wedge \alpha_1^k) = \delta_{ik} \qquad\qquad i,k = 1,2 \tag{3.1}$$

That is in each run of the experiment when the die was black and the color has been measured, the outcome was black and never white; and in each run of the experiment when the die was white and the color has been measured, the outcome was white and never black. A property is nothing but an event type which, if instantiated and measured in a certain run of experiment, brings with it a definite outcome.

Let us now go over to the case of throwing the dice and ask a similar question to that of the color measurement: "Why does the outcome six come up with a certain probability in the experiment?" Here the natural answer is this: "Because the die has a certain mass distribution." This leads us to introducing another event type which we will call *propensity*.

Suppose that the box is containing dice with two different mass distributions. Denote them by α_2^1 and α_2^2. Here the lower index 2 indicates that the measurement setting is of the second type, namely checking the upper face of the die (and not the color), and the upper index discerns the two mass distributions. The mass distribution α_2^1 is again an event type just as α_1^1, the property black was. In every single run of the experiment it is either instantiated or not that is each die has either the mass distribution α_2^1 or not. Hence one can speak about the probability $p(\alpha_2^1)$ as the relative frequency of those runs which fall into the event type α_2^1. If a die with mass distribution α_2^1 is drawn from the box and thrown, then let the probability of its coming up j be denoted by q^{j1}. Similarly, if a die with mass distribution α_2^2 is drawn from the box and thrown, then the probability of coming up j is q^{j2}. This means that the mass distribution of a given die fixes the *probability* of the die coming up with a certain face upon throwing. In terms of probabilities this can be expressed as follows:

$$p(A_2^j|a_2 \wedge \alpha_2^l) = q^{jl} \qquad\qquad j = 1\dots 6,\, l = 1,2 \tag{3.2}$$

where $\sum_j q^{jl} = 1$ for $l = 1, 2$.

Metaphysically, the new event type α_2 is the *propensity* of the die to come up with a certain face in the second type of measurement setting. Note that the propensity here is not something which the notion of probability should be reduced to as in the literature on the interpretations of probability. Here propensity is an event type and probability is simply long-run relative frequency. Moreover, one can meaningfully speak about the "probability of a given propensity" as the long-run frequency of those token events which instantiate the event type of the propensity in question.

Also observe that a property mathematically differs from a propensity only in that the q^{jl}-s fixing the conditional probabilities are all either 0 or 1 for the properties, whereas they can be any number between 0 and 1 for the propensities. Being black fixes the measurement outcomes for the color measurement, whereas having mass distribution α_2^1 fixes only the probability of obtaining a six. The defining equation (3.1) of properties is a special case of the defining equation (3.2) of propensities. Still, it is worth discerning these two event types. If in a given theory the probabilities, correlations, etc. of the measurement outcomes can all be accounted for by postulating purely properties then the theory can rightly be called deterministic, whereas if propensities are also needed then the theory is indeterministic.

To sum up, in our "theory of dice" we have two measurement event types, the event type of measurement settings and the event type of measurement outcomes. Beyond these we can introduce into our ontology two elements of reality for explanatory purposes, the event type of properties, α_1, with two subcategories α_1^1 (black) and α_1^2 (white); and the event type of propensities, α_2, with two subcategories α_2^1 (first mass distribution) and α_2^2 (second mass distribution). From now on we will coin the term *measurement event type* for measurement settings and measurement outcomes and *element of reality* for properties and propensities. The event algebra of the theory will be composed as the Boolean combination of the measurement event types and elements of reality. This algebra will be built up from $2 \cdot (2 \cdot 6) \cdot (2 \cdot 2)$ atomic events associated to the different combinations of measurement settings, measurement outcomes, properties and propensities. Each run of the experiment will instantiate an element of this algebra. Probabilities enter the theory by simply counting how many runs are instantiating certain elements of the algebra.

3.4 No-conspiracy

So far, so good. But physics is a procedure to move from the observable to the unobservable. Do we have any means to infer from the first two event types to the second two? Can we say something about properties and propensities based on measurement settings and measurement outcomes?

Here is a sufficient condition which entitles us to such an inference. Suppose that the elements of reality are probabilistically independent of the measurement settings. In case of the properties this means that

$$p(a_1 \wedge \alpha_1^k) = p(a_1)\, p(\alpha_1^k) \qquad k = 1, 2 \tag{3.3}$$

in case of the propensities:

$$p(a_2 \wedge \alpha_2^l) = p(a_2)\, p(\alpha_2^l) \qquad l = 1, 2 \tag{3.4}$$

Taking the conjunctions we obtain:

$$p(a_1 \wedge a_2 \wedge \alpha_1^k \wedge \alpha_2^l) = p(a_1 \wedge a_2)\, p(\alpha_1^k \wedge \alpha_2^l) \qquad k, l = 1, 2 \tag{3.5}$$

Now, consider all those other equations which arise from (3.5) by substituting one or more event types by their complements; for example:

$$p(\sim a_1 \wedge a_2 \wedge \alpha_1^k \wedge \sim \alpha_2^l) = p(\sim a_1 \wedge a_2)\, p(\alpha_1^k \wedge \sim \alpha_2^l) \qquad k, l = 1, 2 \tag{3.6}$$

Including (3.5) one obtains thus altogether $2 \cdot 2 \cdot 4 \cdot 4 = 64$ equations. Let us refer to this set of 64 equations as *no-conspiracy*. No-conspiracy expresses a probabilistic independence between the various Boolean combinations of measurement settings and the various Boolean combinations of elements of reality. To make reference easier we will sometimes refer solely to (3.5) as no-conspiracy requirement without mentioning the other 63 equations arising from complementation.

No-conspiracy does us a great service: we can reproduce the observable probabilities of the theory in terms of the probabilities of the elements of reality. For example the conditional probability $p(A_1^1|a_1)$ of obtaining a black die upon color measurement turns out to be just the probability $p(\alpha_1^1)$ of the property black:

$$\begin{aligned} p(A_1^1|a_1) &= \frac{p(A_1^1 \wedge a_1)}{p(a_1)} = \frac{\sum_k p(A_1^1 \wedge a_1 \wedge \alpha_1^k)}{p(a_1)} = \frac{\sum_k p(A_1^1|a_1 \wedge \alpha_1^k) p(a_1 \wedge \alpha_1^k)}{p(a_1)} \\ &= \frac{\sum_k p(A_1^1|a_1 \wedge \alpha_1^k) p(a_1) p(\alpha_1^k)}{p(a_1)} = \sum_k p(A_1^1|a_1 \wedge \alpha_1^k) p(\alpha_1^k) \\ &= \sum_k \delta_{1k} p(\alpha_1^k) = p(\alpha_1^1) \end{aligned} \tag{3.7}$$

where we used only the theorem of total probability, the defining feature (3.1) of a property and no-conspiracy (3.3).

By similar reasoning we can reproduce the conditional probability $p(A_2^6|a_2)$ of obtaining the outcome six upon "upper face" measurement in terms of weighted averages of the probability of propensities $p(\alpha_2^l)$:

$$p(A_2^6|a_2) = q^{61}\, p(\alpha_2^1) + q^{62}\, p(\alpha_2^2) \tag{3.8}$$

Equations (3.7) and (3.8) are of central importance. They explain why in the text book form of a physical theory one need not speak about measurement settings and measurement outcomes. If no-conspiracy holds, then the conditional probabilities of the measurement outcomes on measurement settings simply mirror the (unconditional) probabilities of the elements of reality (properties and propensities). Consequently, the deductive and inductive relations between the measurement event types simply reveal deductive and inductive relations between the elements of reality. For example, observing the relation that the probability of a die coming up six is higher than that of being black

$$p(A_2^6|a_2) > p(A_1^1|a_1) \tag{3.9}$$

reveals the unobservable fact that

$$q^{61}\, p(\alpha_2^1) + q^{62}\, p(\alpha_2^2) > p(\alpha_1^1) \tag{3.10}$$

More than that, the relations between measurement settings and measurement outcomes do not just reveal the hidden relations between the unobservable categories but by the same move they also *seem* to make measurement event types superfluous. If the role of these "surface" relations is simply to reflect the deep structural relationships of the unobservable categories with which real physics is concerned—then why one would care about them? Why one would care about measurement settings and measurement outcomes if one can also speak about the "real stuff" directly? In short, no-conspiracy can contribute to delegating measurement settings and measurement outcomes from semantics to mere pragmatics.

May this rationale be as fruitful in displaying textbook theories as it is, in a philosophical reflection, I think, one should not concede that no-conspiracy blurs the general semantical role of measurement settings and measurement outcomes. Just recall the general frame: a physical theory is a formal system *plus* a semantics connecting the formal system to the world. The very two categories which lend empirical meaning to a physical theory are the measurement settings and the measurement outcomes. They are the only event types which an observer have direct access to. Consequently, they cannot be eliminated from a physical theory—neither by appealing to no-conspiracy, nor by appealing to anything else. Otherwise the whole theory would lose its empirical content. It would turn into an uninterpreted formalism. No consideration can deprive a physical theory of those constituents which make up its semantics.

But let us return now to no-conspiracy. What if no-conspiracy does not hold? In this case the inference from the measurement event types to the elements of reality via (3.7) and (3.8) is not possible. But does it make the knowledge of the unobservable categories impossible? Is no-conspiracy a kind of Kantian “condition of the possibility of experience”?

Some seem to think so. In his famous 'cat' paper Schrödinger (1935) likens the free measurement choice of the EPR experiment to a situation when a class of students are asked a set of question such that each student may be asked any of questions. If the answer to the questions are all correct, then one can conclude that *all* students know *all* answers. Analyzing Schrödinger's example Maudlin (2014) writes the following:

> “Recall Schrödinger's class of identically prepared students. We are told they can all answer any of a set of questions correctly, but each can only answer one, and then forgets the answers to the rest. It's an odd idea, but we can still test it: we ask the questions at random, and find that we always get the right answer. Of course it is possible that each student only knows the answer to one question, which always happens to be the very one we ask! But that would require a massive coincidence, on a scale that would undercut the whole scientific method.” (Maudlin, 2014 p. 23)

In short, the independence of the measurement choices and the elements of reality is a precondition of pursuing science *per se*. But is it really so?

3.5 When no-conspiracy does not hold

Consider the following examples:

Example 1. Suppose that the black painting on the dice is not durable enough: if you just touch the dice, the black color is wearing off it and it turns white.

Example 2. Suppose that each die is filled with a high viscosity fluid which can stream and swirl inside the die. By every throw the fluid is put in motion which changes the mass distribution of the die and hence the propensity of the outcome at that very throw.

Example 3 is special case of Example 2. Suppose again that the dice are filled with a fluid which can stream inside them before tossing. But by tossing the dice (due to the heavy shaking, say) the fluid “freezes out” in such a biased way that the dice can come up with only one definite face.

The above three examples are all illustrating a situation when no-conspiracy is violated. In the first example the property α_1^1 (black) has turned into another property α_1^2 (white) as a result of the measurement setting a_1 (drawing a die from the box). In the second example the propensity α_2^1 (first mass distribution) has turned into another propensity α_2^2 (second mass distribution) as a result of the measurement setting a_2 (tossing a die). Finally, in the third example we find a change of category. Recall that properties and propensities differed only in whether they determined the outcome for sure or only up to a certain probability. In the third example there was some non-trivial probability for the different faces of the dice to come up before the throw. After the tossing, however, the die could come up only with a given face. That means that here a propensity (one sort of mass distribution) has been turned into a property (a special mass distribution exactly fixing the outcome) as a result of the measurement setting a_2 (tossing a die). In each case no-conspiracy is violated. (For the relevance of these examples to the interpretations of quantum mechanics see (Gömöri and Hofer-Szabó, 2016).)

In the above three examples no-conspiracy was violated due to the causal influence of the measurement settings on the elements of reality. But it can also fail due to an opposite causal connection when the elements of reality have causal influence on the measurement settings:

Example 4. Suppose that touching the dice of the second mass distribution is unpleasant for your hand; so you toss them hastily rather then keep them in hand and check the color.

Yet another example for the violation of no-conspiracy is a common causal connection between the elements of reality and the measurement settings. It is a combination of example 1 and 4.

Example 5. Suppose that the dice of the second mass distribution are too heavy to be tossed; so you rather perform a color measurement on them. Suppose furthermore that being heavy and having a second mass distribution have a common cause—say, these dice are being made in the same factory.

In all the above examples no-conspiracy was violated due to a causal connection between the measurement settings and the elements of reality. But is causal connection the only way to violate no-conspiracy? We come back to this question in Section 8.

Now, we go over to our central question: Under what circumstances can we adopt no-conspiracy in our physical theory, and when are we forced to abandon it? In the upcoming five Sections we investigate five concepts in turn which can qualify the decision. They are separability, compatibility, causality, locality and contextuality.

3.6 Separability

Niels Bohr's notorious insistence on the use of classical concepts in the description of quantum phenomena is one of the hallmarks of his philosophy. In his contribution to the 1949 Einstein *Festschrift* Bohr writes:

> It is decisive to recognize that, however far the phenomena transcend the scope of classical physical explanation, the account of all evidence must be expressed in classical terms. The argument is simply that by the word "experiment" we refer to a situation where we can tell others

> what we have done and what we have learned and that, therefore, the account of the experimental arrangement and of the results of the observations must be expressed in unambiguous language with suitable application of the terminology of classical physics. (Bohr 1949, p. 209).

Many Bohr scholars have made significant efforts to understand the meaning and role of Bohr's doctrine on the primacy of classical concepts. Camilleri and Schlosshauer (2015) argue that Bohr's doctrine is primarily a general epistemological thesis articulating the epistemology of *experiment* rather than a special interpretation of quantum mechanics (for this see also (Zinkernagel, 2015)). The epistemological problem according to Bohr is that whereas the very notion of experiment presupposes that the measured objects possess a definite state which is independent from the state of the measurement apparatus, quantum mechanics makes this distinction between object and apparatus ambiguous by treating the two as a single, composite, entangled system:

> ...the impossibility of subdividing the individual quantum effects and of separating the behaviour of the objects from their interaction with the measuring instruments serving to define the conditions under which the phenomena appear implies an ambiguity in assigning conventional attributes to atomic objects which calls for a reconsideration of our attitude towards the problem of physical explanation. (Bohr 1948, p. 317).

If entanglement between object and apparatus is *the* obstacle to an unambiguous description of quantum phenomena, then such a description in classical terms can be realized when the subsystems are not entangled, that is when they are *separable.* This is exactly Don Howard's (1994) suggestion for the reconstruction of Bohr's doctrine on classical concepts:

> ...for Bohr, classical concepts are necessary because they embody the assumption of instrument-object separability, and that such separability must be assumed, in spite of its denial by quantum mechanics, in order to secure an unambiguous and thus objective description of quantum phenomena. (Howard 1994, p. 209).

Howard's suggestion to analyze classical description in terms of separability boils down to the requirement to reproduce the statistical predictions of a given quantum phenomenon in terms of an "appropriate mixture." The state of a composite system is called *separable*, if it is a *mixture* that is a convex sum of product states of the components. Since product states represent probabilistically independent components, a mixture is simply a convex combination of these states which expresses a classical probabilistic correlation between the components. Mixtures give rise to a classical, ignorance interpretation of the statistics of the phenomenon under investigation. This analysis via the notion of an "appropriate mixture" has been picked up for example by Halvorson and Clifton (2002) who provide an elegant analysis of the EPR experiment from Bohr's perspective along the lines suggested by Howard.

But how separability as a reconstruction of Bohr's demand on classicality relates to no-conspiracy as a kind of independence principle between measurement settings and the elements of reality attributed to the system? Clearly, separability is a broader concept than no-conspiracy: separability simply requires that the relation between the measurement settings and elements of reality should be expressed as a mixture of probabilistic independences; whereas no-conspiracy requires that the two should be probabilistically independent. In our toy model for example separability requires the probability of the color measuring and the system's possessing the property black to be the following:

$$p(a_1 \wedge \alpha_1^1) = \lambda_1\, p(a_1)\, p(\alpha_1^1) + \lambda_2\, p(a_1)\, p(\sim\alpha_1^1) + \lambda_3\, p(\sim a_1)\, p(\alpha_1^1) + \lambda_4\, p(\sim a_1)\, p(\sim\alpha_1^1) \quad (3.11)$$

with any $\lambda_i \in [0,1]$ and $\sum_{i=1}^4 \lambda_i = 1$; whereas no-conspiracy requires that

$$p(a_1 \wedge \alpha_1^1) \;=\; p(a_1)\, p(\alpha_1^1) \quad (3.12)$$

Observe that separability (3.11) does not give any restriction in our case; it simply means that p is a classical probability which we already knew since we took probabilities to be relative frequencies.

All the five examples in the previous Section, though violating no-conspiracy, are completely classical; they provide an unambiguous description of how the unobservable properties or propensities change upon throwing the dice. They even provide a mechanism for the causal dependence. In Example 1 for instance when upon drawing the black color is wearing off the dice, obviously

$$p(a_2 \wedge \alpha_1^1) \;\neq\; p(a_2)\, p(\alpha_1^1) \quad (3.13)$$

Throwing the dice and being black will not be probabilistically independent due to the causal relation between the two event types.

Thus, the "unambiguous language" requires only to attribute *some* properties to the system which stand in *some* classical probabilistic relation to the measurement settings but it does not require them to be probabilistically independent of one another. Hence, separability as a weaker requirement than no-conspiracy cannot be used to back the latter. (In addition, according to Howard even the demand on classicality as separability is too restrictive from perspective of a general epistemology of experiment.)

3.7 Compatibility

Now, let us go over to our second concept which is *compatibility* of the measurement settings. Up to now we have considered measurement settings only separately. Let us see now what happens when we perform a joint measurement.

Again, consider our toy model and suppose that we perform the measurement $a_1 \wedge a_2$ that is we are drawing a die from the box, throwing it and checking its color *and* also the number on its upper face. Suppose that after performing both measurements we disregard the upper face and consider only the color. Suppose that we observe that the probability of the outcome black in this joint measurement is not the same as in the measurement a_1. That is we find that

$$p(A_1^1|a_1 \wedge a_2) \neq p(A_1^1|a_1) \tag{3.14}$$

Let us call (3.14) *incompatibility* of the two measurements. Note that incompatibility does not mean here that $a_1 \wedge a_2$ cannot be performed; it means that a_1 and a_2 are disturbing one another.

What is incompatibility a sign of?

First, observe that the condition a_1 on the right hand side of (3.14) does not mean that we performed only a_1—this would be $a_1 \wedge \sim a_2$. The condition a_1 means that we consider all the runs in which a_1 has been performed, irrespectively whether a_2 has been performed or not—that is $a_1 = (a_1 \wedge a_2) \vee (a_1 \wedge \sim a_2)$. So what (3.14) expresses is that whether we perform a_2 or not *does* count in measuring a_1 and producing outcome A_1^1.

Generally, one can take two positions towards incompatibility. I will call the first the *purist* or *Bridgmanian* strategy and the second the *stubborn* strategy.

According to the purist strategy if the probability of the outcome of a given measurement can vary depending whether another measurement is performed or not, then this measurement is *not yet well defined*. Consider the following example. In a regiment two tests are performed: it is tested how good shots are the soldiers (a_1) and how much alcohol they can drink (a_2). Obviously, whether the second test is performed or not, crucially influences the outcome of the first. So the two tests are incompatible in the above sense (and not in the sense that they cannot be performed at the same time: they can—although it is not recommended). So the correct definition of the first test is this: let the soldiers shoot but do not give them alcohol ($a_1 \wedge \sim a_2$).

So the purist attitude towards (3.14) is that a_1 in itself is not yet a well defined measurement procedure since the probability of the outcomes depends on whether a_2 is performed or not. So instead of taking *two* measurement settings, a_1 and a_2, we should rather take *four*, $a_1 \wedge a_2$, $a_1 \wedge \sim a_2, \sim a_1 \wedge a_2$ and $\sim a_1 \wedge \sim a_2$ (in this latter case we do nothing). By this move we can eliminate incompatibility since the four new measurements are logically mutually exclusive. They cannot be co-performed and hence cannot disturb one another. Generally, the purist strategy is to take the conjunctions of incompatible measurements until they become either compatible or logically exclusive.

We call this strategy Bridgmanian since it is in tune with Bridgman's ideas on the correct definition of measurement unfolded for example in *The Logic of Modern Physics*:

> Implied in this recognition of the possibility of new experience beyond our present range, is the recognition that no element of a physical situation, no matter how apparently irrelevant or trivial, may be dismissed as without effect on the final result until proved to be without effect by actual experiment. (Bridgman 1958, p. 3)

Returning to no-conspiracy, the Bridgmanian strategy renders all co-measurable measurements compatible with one another. Therefore, the problem of incompatibility disappears and we are back to our single case measurement scenario. The purist strategy teaches nothing new about no-conspiracy.

Let us go over to the stubborn strategy. I call it stubborn since it takes a_1 and a_2 to be correct measurement settings in spite of their incompatibility (3.14)? What does then (3.14) say about no-conspiracy?

This is a point where we need to go one step further concerning the relation between measurement event types and elements of reality. We need to specify how the elements of reality behave when jointly measured. Therefore suppose that the following relation also holds (in addition to (3.1) and (3.2)):

$$p(A_1^i \wedge A_2^j | a_1 \wedge a_2 \wedge \alpha_1^k \wedge \alpha_2^l) = \delta_{ik}\, q^{jl} \qquad i,k,l = 1,2;\ j = 1\ldots 6 \tag{3.15}$$

Requirement (3.15) expresses a kind of non-disturbance relation between the measurements which can be better seen if we sum up first for i then for j:

$$p(A_1^i | a_1 \wedge a_2 \wedge \alpha_1^k \wedge \alpha_2^l) = \delta_{ik} = p(A_1^i | a_1 \wedge \alpha_1^k) \tag{3.16}$$
$$p(A_2^j | a_1 \wedge a_2 \wedge \alpha_1^k \wedge \alpha_2^l) = q^{jl} = p(A_2^j | a_2 \wedge \alpha_2^l) \tag{3.17}$$

(Here the second equation in both rows are due to the defining equation (3.1) of the property and (3.2) of the propensity, respectively.) (3.16) and (3.17) express that the probability of an outcome conditioned on an element of reality and a measurement setting does not change by further conditioning it on other elements of reality or measurement settings. From (3.16) (where the element of reality is a property) it also follows that

$$p(A_1^i | a_1 \wedge a_2 \wedge \alpha_1^k) = p(A_1^i | a_1 \wedge \alpha_1^k \wedge \alpha_2^l) = p(A_1^i | a_1 \wedge \alpha_1^k) \tag{3.18}$$

Now, suppose that no-conspiracy also holds that is

$$p(a_1 \wedge a_2 \wedge \alpha_1^k \wedge \alpha_2^l) = p(a_1 \wedge a_2)\, p(\alpha_1^k \wedge \alpha_2^l) \qquad k,l = 1,2 \tag{3.19}$$

From (3.15) and (3.19) it is easy to show (via a derivation similar to (3.7)) that

$$p(A_1^1 | a_1 \wedge a_2) = p(A_1^1 | a_1) \tag{3.20}$$

in contradiction to incompatibility (3.14). This means that incompatibility between the measurements implies that we have to abandon either the non-disturbance of the measurement procedures (3.15) or no-conspiracy (3.19).

Thus, in case of the stubborn strategy compatibility of the measurement settings is a good sign of that both non-disturbance and no-conspiracy hold; and incompatibility is a good sign of that either the one or the other is violated. Whether to blame the one or the other is a question for further investigation.

3.8 Causality

Our third concept in the row is *causality.* In Section 5 we saw several examples for causal connections between the measurement settings and the elements of reality. In Example 1 for instance we supposed that the black painting on the dice is not durable enough and if one touches the dice, the color black is wearing off. Causal connection between elements of reality and measurement settings is a prime source of no-conspiracy.

Causal connection comes in two sorts. It can be either a direct causal connection as in Examples 1 to 4; or it can be a common causal connection as in Example 5. Reichenbach's Common Cause Principle states that all correlations should be accounted for by one of the two causal connections. On the other hand, probabilistic independence between the measurement settings and the elements of reality is a sign of causal independence (assuming that causal effects do not cancel one another). Hence, no-conspiracy can be ensured if any causal connection between the measurement settings and the elements of reality can be excluded.

Before turning to this point, first we need to clarify what we mean by a causal connection between two event types, say, the color measurement, a_1, and the property black, α_1^1. By that we mean that the color measurement and the property black are causally related in a *tokenwise* manner. In other words, there is a *pairing* of token events instantiating these two types such that for each pair the token events of the pair stand in either a direct or a common causal connection to one another. But how to create pairs?

Consider a certain run of the experiment which instantiates $a_1 \wedge \alpha_1^1$. Up to now we treated this run of the experiment as *one single* run in which one performed a color measurement *and* the property of the dice which has been drawn was black. How can the color measurement cause the property black in this single run? If this run of the experiment is taken as one single token event, then there can be no tokenwise causal connection; simply because we have only one token. In order to have a causal connection, one needs to decompose this one single run of the experiment instantiating $a_1 \wedge \alpha_1^1$ into a *pair* of token events such that the one token event instantiates a_1 and the other token event instantiates α_1^1. In order to speak about a tokenwise causal relation, one token event is not enough. One possibility to perform this decomposition is to say that the first token event occurred here and the other token event occurred over there. Localization is a typical method for individuation. We come back to the question of localization in the next Section.

Now, suppose that we can separately individuate the token events of the color measurement and the token events of the property black. Then a causal connection between a_1 and α_1^1 means that for each pair either the token instantiating a_1 is the cause of the token instantiating α_1^1; or *vice versa*; or there is a third token instantiating a third event type which is the common cause of both. Is there a way to exclude both a direct and also a common causal connection between the token events and by this to ensure no-conspiracy? What might come to mind first is to rely on some locality consideration. This is the topic of the next Section.

3.9 Locality

Is there a spatiotemporal arrangement of the event types a_1 and α_1^1 such that one can safely say that all possible causal connections between the measurement settings and the elements of reality are shielded off? Suppose that we take a snapshot of the world and it turns out that the pairs of token events instantiating the color measurement and the property black are localized in spacelike separated regions. Thus, in the first run of the experiment the token event instantiating a_1 is spacelike separated from the token event instantiating α_1^1; and similarly for the second, third, etc. run. This is the best scenario a spacetime localization can provide for causal independence. Does it guarantee that there is no causal and hence probabilistic dependence between a_1 and α_1^1? As one expects, the answer to this question is *no*.

Even if the token events of each pair are spacelike separated, they can still be causally related to one another both in a direct and also in a common causal way. As for direct causal connection, just note that in order to produce a measurement outcome these two token events need to interact somewhere in spacetime. Hence even if they are spacelike separated at a certain moment, they will not be so at the moment of bringing about the outcome black. Therefore their direct causal effect on one another at the time of their interaction cannot be excluded based on the fact that at a previous time they were localized in a spacelike separated way. The situation is similar or even worse in case of a common cause. Even if the two token events are spacelike separated, there well can be a common cause in their common past causally influencing both.

To sum up, locality considerations do not help us in excluding causal connections and hence to ensure no-conspiracy. Thus, we have fallen back to the situation in the previous Section: to guarantee no-conspiracy we need to exclude causal connection in some way without making use of spatiotemporal considerations.

3.10 Contextuality

Up to now it may have appeared that the only source for the violation of no-conspiracy is a causal connection between the elements of reality and the measurement settings. However, there is a further way to violate no-conspiracy which is not related to causality. Two events can be correlated even if they are not causally related; namely if they *logically* depend on one another. This leads us to the problem of *contextuality*.

A little reflection on the definition of property and propensity can convince us that (3.1) and (3.2) say nothing about whether the elements of reality and the measurement settings are logically independent or not. It can well be the case that by specifying the measurement setting we partly specify also the elements of reality. Consider the following example:

Example 6. Let $\alpha_2^1(x)$ denote the following property of the dice: the mass distribution of the dice is of the first type *and* the initial conditions (position plus momentum) of its toss is x. $\alpha_2^1(x)$ is obviously a property since together with the toss a_2 it determines the upper face for sure; that is

$$p(A_2^j|a_2 \wedge \alpha_2^1(x))$$

is either 0 or 1 for any j and x.

However, a_2 and $\alpha_2^1(x)$ are *not* logically independent. If you tossed the die, then the initial velocity is surely not zero and the die must have been located somewhere around the table. That is the measurement setting partly specifies the initial conditions. This logical dependence between the element of reality and the measurement setting is called *contextuality.*

How contextuality leads to the failure of no-conspiracy? First, consider an initial condition x which can reasonably be regarded as "tossing the dice" (that is for the tossing of the die with x, it will land on the table and after a couple of rolls it will stop on the table, etc.). For such an x, $\alpha_2^1(x)$ is algebraically contained in a_2, therefore

$$p(a_2 \wedge \alpha_2^1(x)) = p(\alpha_2^1(x)) \neq p(a_2)\, p(\alpha_2^1(x)) \tag{3.21}$$

if $p(a_2) \neq 1$ and hence no-conspiracy is violated. Second, suppose that x does not count as "tossing the dice" (the die flies over the table, say). Then a_2 and $\alpha_2^1(x)$ are algebraically disjoint and hence

$$p(a_2 \wedge \alpha_2^1(x)) = 0 \neq p(a_2)\, p(\alpha_2^1(x)) \tag{3.22}$$

if $p(a_2) \neq 0$ and no-conspiracy is again violated. In short, the logical dependence between the measurement settings and the elements of reality directly implies (for non-extremal probabilities, that is typically) a probabilistic dependence between them; that is a violation of no-conspiracy.

To sum up, even if the elements of reality and the measurement settings are causally detached, they can still violate no-conspiracy if the measurement settings wholly or partially contribute to the definition of the elements of reality. Such a situation cannot be excluded a priori; at least the definitions of the property and the propensity do not exclude it. The logical dependence between elements of reality and measurement settings suffices to establish conspiracy. Contextuality is the other main source for the violation of no-conspiracy.

3.11 Discussion

In this paper we have adopted the following empiricist philosophical position. A physical theory was reconstructed as a formal system plus a semantics connecting the formal system to the world. The semantics has to minimally specify what event types inhabit the world. Event types can be of two sorts: measurement event types and elements of reality. Typically we have direct access to the former but not to the latter. There are two measurement event types: measurement settings and measurement outcomes and there are also two types of elements of reality:

properties and propensities. The probability of an event type is understood as simply the long-run relative frequency of the token events instantiating the event type in question. In an experiment the token events are the runs of the experiment.

Adopting the above philosophical position we have argued for the following. No-conspiracy is the requirement that elements of reality should be probabilistically independent of the measurement settings. There is no a priori guarantee that no-conspiracy does hold. If it does, probabilistic relations between the measurement event types will mirror probabilistic relations between the elements of reality. This licenses physics to forget about measurement settings and measurement outcomes and to talk directly about elements of reality. The temptation to delete measurement event types from the semantics of the theory, however, should be resisted.

No-conspiracy is a concept situated within a web of related concepts such as separability, compatibility, causality, locality and contextuality. In the paper I concentrated only on those threads of the web which connected these notions to no-conspiracy. But certainly there are many other interconnections. Causality and contextuality are complementary terms: the more the measurement settings and elements of reality are logically depend on one another, the less room there is for causal connection between them. Separability and spacetime localization do not orient us about causal connections between measurement settings and elements of reality; whereas incompatibility is often due to a direct causal link between them; as in case of the soldiers' shooting and drinking.

Going back to no-conspiracy, the following can be said. Three of the five concepts, namely separability, compatibility and locality do not bring us closer to no-conspiracy. Separability is a weaker concept than no-conspiracy, so one cannot back the latter by the former. Compatibility of measurement settings is empty in case of a purist strategy and only a partial motivation in case of the stubborn strategy. Finally, locality cannot be used to support no-conspiracy at all. However, the remaining two concepts, namely causality and contextuality, are closely linked to no-conspiracy. No-conspiracy can be guaranteed if both causal and logical dependence between the measurement settings and the elements of reality can be excluded. In the first case one needs to ensure that there is no direct or common causal connection between the individual runs of the experiment. In the second case that measurement settings should not contribute to the very definition of the elements of reality.

Whether this can be done and hence a non-conspiratorial physical theory can be provided for a given phenomena is a question that can be answered only by a thorough scrutiny of the phenomena in question. Whether any conspiratorial description of a physical scenario can be replaced by a "better" non-conspiratorial one; whether adopting no-conspiracy can be in conflict, as in the EPR-Bell scenario, with other principles such as local causality, Common Cause Principle, etc.—well, these questions cannot be decided at a general metaphysical level. No-conspiracy is neither an analytic nor a transcendental truth; it is an extra constraint on theory construction the success of which can be decided only on a case-by-case basis.

Appendix

Throughout the paper we used a simple toy model for a physical theory. Here we provide a general mathematical picture of a physical theory.

Let $a_i\,(i = 1\ldots I)$ be the measurement settings in a given theory and let $A_i^{j_i}\,(j_i = 1\ldots J_i)$ denote the jth outcome of the ith measurement. Suppose furthermore that there is an element of reality $\alpha_i^{k_i}$ $(k_i = 1\ldots K_i)$ (either a property or a propensity) associated to each measurement setting a_i such that

$$p(A_i^{j_i}|a_i \wedge \alpha_i^{k_i}) = q_i^{j_i k_i} \tag{3.23}$$

where $\sum_{j_i=1}^{J_i} q_i^{j_i k_i} = 1$ for any $i = 1\ldots I$ and $k_i = 1\ldots K_i$. For a given $i \in I$ the element of reality $\alpha_i^{k_i}$ is a property iff $J_i = K_i$ and $q_i^{j_i k_i} = \delta_{j_i k_i}$. Otherwise $\alpha_i^{k_i}$ is a propensity.

Suppose that the elements of reality are related nicely to the measurement event types not only in case of a single measurement but also in case of a joint measurement. (Note the word "single" does not mean that the other measurements are not performed; it means rather that it is not taken into consideration whether they are performed or not.) Therefore, select I' measurement settings out of the possible I and let now the index i run from 1 to I'. What we require is that for any such selection (among them the no-selection) the following should hold:

$$p(A_1^{j_1} \wedge \ldots \wedge A_{I'}^{j_{I'}}|a_1 \wedge \ldots \wedge a_{I'} \wedge \alpha_1^{k_1} \wedge \ldots \wedge \alpha_{I'}^{k_{I'}}) = q_1^{j_1 k_1} \times \cdots \times q_{I'}^{j_{I'} k_{I'}} \tag{3.24}$$

Now, the elements of reality $\{\alpha_i^{k_i}\}$ are said to satisfy *no-conspiracy* iff

$$p(a_1 \wedge \ldots \wedge a_I \wedge \alpha_1^{k_1} \wedge \ldots \wedge \alpha_I^{k_I}) = p(a_1 \wedge \ldots \wedge a_I)\, p(\alpha_1^{k_1} \wedge \ldots \wedge \alpha_I^{k_I}) \tag{3.25}$$

holds *together* with those "complemented" variants of (3.25) where one or more event types are substituted by their complements. From no-conspiracy it follows that they also satisfy no-conspiracy for all selections, among them

$$p(a_i \wedge \alpha_i^{k_i}) = p(a_i)\, p(\alpha_i^{k_i}) \tag{3.26}$$

By means of (3.24) and no-conspiracy (3.25) one can transform for any selection the probabilistic relations among the measurement event types into probabilistic relations among elements of reality as follows:

$$p(A_1^{j_1} \wedge \ldots \wedge A_{I'}^{j_{I'}}|a_1 \wedge \ldots \wedge a_{I'}) = \sum_{k_1 \ldots k_{I'}} q_1^{j_1 k_1} \times \cdots \times q_{I'}^{j_{I'} k_{I'}}\, p(\alpha_1^{k_1} \wedge \ldots \wedge \alpha_{I'}^{k_{I'}}) \tag{3.27}$$

Specifically, if all the event types $\{\alpha_i^{k_i}\}$ are properties, then (3.27) reads as

$$p(A_1^{j_1} \wedge \ldots \wedge A_{I'}^{j_{I'}}|a_1 \wedge \ldots \wedge a_{I'}) = p(\alpha_1^{j_1} \wedge \ldots \wedge \alpha_{I'}^{j_{I'}}) \tag{3.28}$$

and in the special case of a single measurement as

$$p(A_i^{j_i}|a_i) = p(\alpha_i^{j_i}) \tag{3.29}$$

for all $i = 1 \ldots I$. Equation (3.27) shows that the probability of the outcomes conditioned on the measurement settings is mirrored in the probability of the properties.

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Chapter 4

Relating Bell's local causality to the Causal Markov Condition

The aim of the paper is to relate Bell's notion of local causality to the Causal Markov Condition. To this end, first a framework, called local physical theory, will be introduced integrating spatiotemporal and probabilistic entities and the notions of local causality and Markovity will be defined. Then, illustrated in a simple stochastic model, it will be shown how a discrete local physical theory transforms into a Bayesian network and how the Causal Markov Condition arises as a special case of Bell's local causality and Markovity.

4.1 Introduction

Local causality is a concept introduced into the foundations of quantum theory by John Stewart Bell. A physical theory is said to be locally causal if, fixing its past, any event happening in a given spacetime region will be probabilistically independent of any other event localized in a spatially separated region.

Causal Markov Condition is the central notion of the theory of Bayesian networks. Here events are represented both as random variables in a probability space and also as vertices in a causal graph. A set of events is said to satisfy the Causal Markov Condition relative to the graph, if, conditioned on its causal parents, any event will be probabilistically independent of any of its causal non-descendants.

The similarity between the logical schema of both principles is conspicuous even at first blush: if events are localized in the spacetime/causal graph in a certain way, then they are to satisfy certain probabilistic independencies. In this paper I will argue that this intuition is correct: Bell's local causality, read in an appropriate way, is a Causal Markov Condition. Causal Markov Condition relates random variables to causal structures, local causality relates them to a net of spacetime regions. We will show that the causal graph generated by the net structure of a local physical theory transforms the theory into a Bayesian network and yields the Causal Markov Condition as a kind of composition of Bell's local causality plus a similar screening-off condition, called Markovity.

To treat physical events both as probabilistic and also as spatiotemporal/causal entities in a unified framework and to be able to infer from spatiotemporal/causal relations to probabilistic independencies one needs to have a common conceptual schema integrating both spatiotemporal/causal and probabilistic concepts. This formalism is thoroughly worked out in the theory of Bayesian networks. Here Causal Markov Condition is functioning as a 'bridge law' connecting the causal and the probabilistic side of the theory. In the foundations of quantum physics, however, local causality is used in a much more intuitive way. Here one simply "reads off" probabilistic independencies from the spatiotemporal localization of the events in question. Hence our first task is to introduce a mathematically well-defined and physically well-motivated framework which treats probabilistic and spatiotemporal entities in a common mathematical formalism. We will call such a theory a *local physical theory*. We will borrow a lot from the most elaborate physical theory offering such a general framework, namely algebraic quantum field theory (AQFT). Having such a framework integrating spatiotemporal and probabilistic aspects, we will be able to provide a clear-cut formulation of Bell's notion of local causality.

To relate Bell's local causality to the Causal Markov Condition, we will introduce a simple stochastic local classical theory on a discretized two dimensional spacetime. This toy theory will display all the features previously defined in an abstract way, and provide us a useful tool to study the properties of local causality in a more manageable way, and to trace its connections to the Causal Markov Condition.

In the paper we will proceed as follows. In Section 2 we make a historical detour and take a closer look at Bell's different definitions of local causality. In Section 3 we introduce the concept of a local physical theory and give a precise mathematical definition of Bell's notion of local causality together with Markovity within this framework. In Section 4 our stochastic local classical theory will be introduced. In Section 5 we define the Causal Markov Condition and show how a local physical theory gives rise to a Bayesian network and how local causality plus Markovity go over to the Causal Markov Condition. We will conclude in Section 6.

There is a huge literature available relating the Causal Markov Condition to the EPR scenario and to the Bell inequalities. The standard way to derive the Bell inequalities is to start with Reichenbach's Common Cause Principle together with some locality conditions. Since Reichenbach's Common Cause Principle is a special case of the Causal Markov Condition, many authors start the derivation directly from this latter. Glymour (2006) shows that the EPR case has no causal explanation compatible with the Causal Markov Condition. Suárez and Iniaki (2011) systematically apply the Causal Markov Condition to the EPR scenario and make a connection to the robustness condition, a probabilistic causality condition thoroughly discussed in the early 1990's. On the other hand, Hausman and Woodward

(1999) argue that the Causal Markov Condition is inapplicable to the EPR scenario since the non-separability of the quantum state renders interventions, a necessary criterion for applicability, unavailable. As a reply to their claim see Suárez (2013). Hofer-Szabó, Rédei and Szabó (2013) connect the Causal Markov Condition both to the so-called *common*-common-causal and also to the separate-common-causal explanation of the EPR case. They show that hidden locality, an assumption of the standard derivation of the Bell inequalities, can be justified by the Causal Markov Condition only in case of common common causes but not in case of separate common causes.

Despite the rich literature on the topic I am unaware of any work relating the Causal Markov Condition directly to Bell's notion of local causality. This paper intends to fill this gap.

4.2 Bell's three definitions of local causality

Local causality is the idea that causal processes propagate though space continuously and with velocity less than the speed of light. John Stewart Bell formulates this intuition in a 1988 interview as follows:

> "[Local causality] is the idea that what you do has consequences only nearby, and that any consequences at a distant place will be weaker and will arrive there only after the time permitted by the velocity of light. Locality is the idea that consequences propagate continuously, that they don't leap over distances." (Mann and Crease, 1988)

Bell has returned to this intuitive idea of local causality from time to time and provided a more and more elaborate formulation of it. First he addressed the notion of local causality in his "The theory of local beables" delivered at the Sixth GIFT Seminar in 1975; later in a footnote added to his 1986 paper "EPR correlations and EPW distributions" intending to clean up the first version; and finally in the most elaborate form in his "La nouvelle cuisine" posthumously published in 1990. Below I will overview the different versions briefly commenting on each of them.

Version 1. Bell's first definition of local causality reads as follows:

> "Consider a theory in which the assignment of values to some beables Λ implies, not necessarily a particular value, but a probability distribution, for another beable A. Let $p(A|\Lambda)$ denote[1] the probability of a particular value A given particular values Λ. Let A be localized in a space-time region A. Let B be a second beable localized in a second region B separated from A in a spacelike way. (Fig. 4.1.) Now my intuitive notion of local causality is that events in B should not be 'causes' of events in A, and vice versa. But this does not mean that

[1] For the sake of uniformity throughout the paper I slightly changed Bell's denotation and figures.

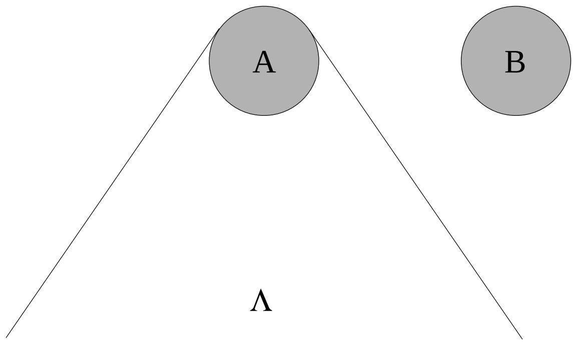

Figure 4.1: Bell's first figure illustrating local causality (1975).

the two sets of events should be uncorrelated, for they could have common causes in the overlap of their backward light cones. It is perfectly intelligible then that if Λ in (8.6) does not contain a complete record of events in that overlap, it can be usefully supplemented by information from region B. So in general it is expected that

$$p(A|\Lambda, B) \neq p(A|\Lambda) \tag{4.1}$$

However, in the particular case that Λ contains already a *complete* specification of beables in the overlap of the light cones, supplementary information from region B could reasonably be expected to be redundant.

Let C_2 denote a specification of *all* beables, of some theory, belonging to the overlap of the backward light cones of spacelike regions A and B. Let C_1 be a specification of some beables from the remainder of the

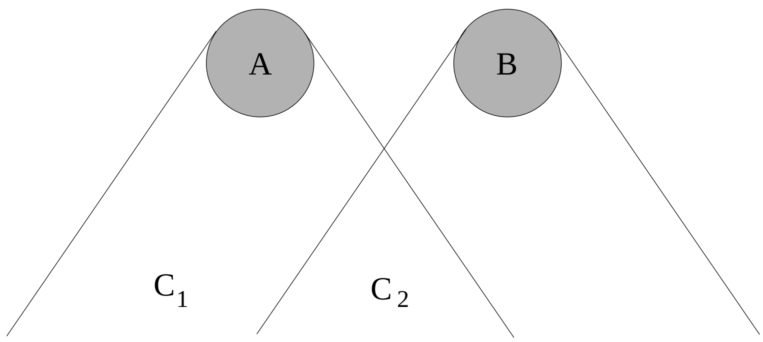

Figure 4.2: Bell's second figure illustrating local causality (1975).

backward light cone of A, and B of some beables in the region B. (See

> Fig. 4.2.) Then in a *locally causal theory*
>
> $$p(A|C_1, C_2, B) = p(A|C_1, C_2) \tag{4.2}$$
>
> whenever both probabilities are given by the theory." (Bell, 1975/2004, p. 54)

First, let us comment briefly on the terminology Bell is using in his first version of local causality.

The term "beable" has been introduced into the literature by Bell himself. It is intended to be opposed to the term "observable" used in quantum theory and to refer to something that "really" exists. "The word 'beable' will also be used to carry another distinction already in classical theory between 'physical' and 'non-physical' quantities. In Maxwell's electromagnetic theory, for example, the fields **E** and **H** are physical (beables, we will say) but potentials **A** and ϕ are non-physical." (Bell, 1975/2004, p. 52) Without the clarification of what the "beables" of a given theory really are, one cannot even formulate local theory.

"Beables" are to be local. "We will be particularly concerned with local beables, those which (unlike for example the total energy) can be assigned to some bounded space-time region. For example, in Maxwell's theory the beables local to a given region are just the fields **E** and **H**, in that region, and all functionals thereof." (Bell, 1975/2004, p. 53)

Finally, the beables localized in the region C_1 are to provide a "completely specification" of the region in question. We will come back to this point later on.

Although the beables are to be local, in his screening-off condition (8.7) Bell takes into account the *whole causal past* of the events in question. He does not assume some kind of Markovity rendering superfluous the remote past regions below a certain Cauchy surface. The second version of his formulation of local causality can be regarded as a step towards this Markovian direction.

Version 2.

> "The notion of local causality presented in this reference [namely in (Bell, 1975/2004)] involves complete specification of the beables in an infinite space-time region. The following conception is more attractive in this respect: In a locally-causal theory, probabilities attached to values of local beables in one space-time region, when values are specified for all local beables in a second space-time region fully obstructing the backward light cone of the first, are unaltered by specification of values of local beables in a third region with spacelike separation from the first two." (Bell, 1986/2004, p. 200)

Bell's second version is in a footnote; it is very succinct and contains no figure. The new element is the phrasing "space-time region fully obstructing the backward light cone of the first". This idea gets a more precise exposition in Bell's third, final version of local causality.

Version 3.

> "A theory will be said to be locally causal if the probabilities attached to values of local beables in a space-time region A are unaltered by specification of values of local beables in a space-like separated region B, when what happens in the backward light cone of A is already sufficiently specified, for example by a full specification of local beables in a space-time region C (Fig. 4.3)." (Bell, 1990/2004, p. 239-240)

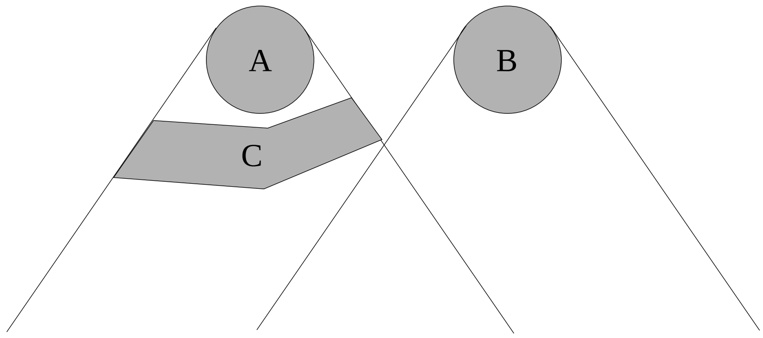

Figure 4.3: Bell's figure illustrating local causality (1990).

The localization of region C is of crucial importance. It is not enough that C completely cuts across the causal past of region A; it also has to "obstruct the backward lightcone of the first". Bell explicitly stresses this point: "It is important that region C completely shields off from A the overlap of the backward light cones of A and B." (Bell, 1990/2004, p. 240) This requirement will play a central role in our investigation on the relation of local causality to the Causal Markov Condition. We will come back to that having defined local causality in the next Section.

4.3 Local causality in local physical theories

The framework integrating probabilistic and spatiotemporal entities can be defined as follows. (For the details and motivations of the definition see (Hofer-Szabó and Vecsernyés, 2015a,b).)

Definition 1. A $\mathcal{P}_{\mathcal{K}}$-*covariant local physical theory* is a net $\{\mathcal{A}(V), V \in \mathcal{K}\}$ associating algebras of events to spacetime regions which satisfies *isotony*, *microcausality* and *covariance* defined as follows (Haag, 1992):

Isotony. Let $\mathcal{M}$ be a globally hyperbolic spacetime and let $\mathcal{K}$ be a covering collection of bounded, globally hyperbolic subspacetime regions of $\mathcal{M}$ such that $(\mathcal{K}, \subseteq)$ is a directed poset under inclusion $\subseteq$. The net of local observables is given by the isotone map $\mathcal{K} \ni V \mapsto \mathcal{A}(V)$ to unital C^*-algebras, that is $V_1 \subseteq V_2$ implies that $\mathcal{A}(V_1)$ is a unital C^*-subalgebra of $\mathcal{A}(V_2)$. The *quasilocal algebra* $\mathcal{A}$ is defined to be the inductive limit C^*-algebra of the net $\{\mathcal{A}(V), V \in \mathcal{K}\}$ of local C^*-algebras.

Microcausality (also called as *Einstein causality*) is the requirement that $\mathcal{A}(V')' \cap \mathcal{A} \supseteq \mathcal{A}(V), V \in \mathcal{K}$, where primes denote spacelike complement and algebra commutant, respectively.

Spacetime covariance. Let $\mathcal{P}_{\mathcal{K}}$ be the subgroup of the group $\mathcal{P}$ of geometric symmetries of $\mathcal{M}$ leaving the collection $\mathcal{K}$ invariant. A group homomorphism $\alpha\colon \mathcal{P}_{\mathcal{K}} \to \operatorname{Aut} \mathcal{A}$ is given such that the automorphisms $\alpha_g, g \in \mathcal{P}_{\mathcal{K}}$ of $\mathcal{A}$ act covariantly on the observable net: $\alpha_g(\mathcal{A}(V)) = \mathcal{A}(g \cdot V), V \in \mathcal{K}$.

If the quasilocal algebra $\mathcal{A}$ of the local physical theory is commutative, we speak about a *local classical theory*; if it is noncommutative, we speak about a *local quantum theory.* For local classical theories microcausality fulfills trivially.

A *state* ϕ in a local physical theory is defined as a normalized positive linear functional on the quasilocal observable algebra $\mathcal{A}$. The corresponding GNS representation $\pi_\phi\colon \mathcal{A} \to \mathcal{B}(\mathcal{H}_\phi)$ converts the net of C^*-algebras into a net of C^*-subalgebras of $\mathcal{B}(\mathcal{H}_\phi)$. Closing these subalgebras in the weak topology one arrives at a net of local von Neumann observable algebras: $\mathcal{N}(V) := \pi_\phi(\mathcal{A}(V))'', V \in \mathcal{K}$. Von Neumann algebras are generated by their projections representing *quantum events.* The net $\{\mathcal{N}(V), V \in \mathcal{K}\}$ of local von Neumann algebras also obeys isotony, microcausality, and $\mathcal{P}_{\mathcal{K}}$-covariance, hence one can also refer to a net $\{\mathcal{N}(V), V \in \mathcal{K}\}$ of local *von Neumann algebras* as a local physical theory.

Why von Neumann algebras?

Classical field theories are characterized by their sets of field configurations. Taking the equivalence classes of those field configurations which have the same field values on a given spacetime region one can generate local (cylindrical) σ-algebras. One can translate σ-algebras into the language of abelian von Neumann algebras and then generalize this framework also for non-abelian von Neumann algebras. We come back to the details of this procedure in the next section when we introduce our stochastic local classical theory. Thus, we translate Bell's term "local beables" into the language of local physical theories simply as "elements of a local von Neumann algebra". Now, how to translate the term "a complete specification of beables"? We are of the opinion that the natural translation of this term is simply "an atomic event of a local von Neumann algebra" (Henson, 2013). Here it is assumed that the local algebras of the net are atomic, which is not the case, for example, in Poincaré covariant algebraic quantum field theory. (For a more general definition of local causality see (Hofer-Szabó and Vecsernyés, 2015a).) With these notions in hand now one can formulate Bell's notion of local causality in a local physical theory as follows:

Definition 2. A local physical theory represented by a net $\{\mathcal{N}(V), V \in \mathcal{K}\}$ of von Neumann algebras is called *locally causal*, if for any pair $A \in \mathcal{N}(V_A)$ and $B \in \mathcal{N}(V_B)$ of projections supported in spacelike separated regions $V_A, V_B \in \mathcal{K}$ and for every locally normal and faithful state ϕ establishing a correlation $\phi(AB) \neq \phi(A)\phi(B)$ between A and B, and for any spacetime region V_C such that

(i) $V_C \subset J_-(V_A)$,

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(ii) $V_A \subset V_C''$,

(iii) $J_-(V_A) \cap J_-(V_B) \cap \big(J_+(V_C) \setminus V_C\big) = \emptyset$,

(see Fig. 6.2) and for any atomic event C_k of $\mathcal{A}(V_C)$ $(k \in K)$, the following holds:

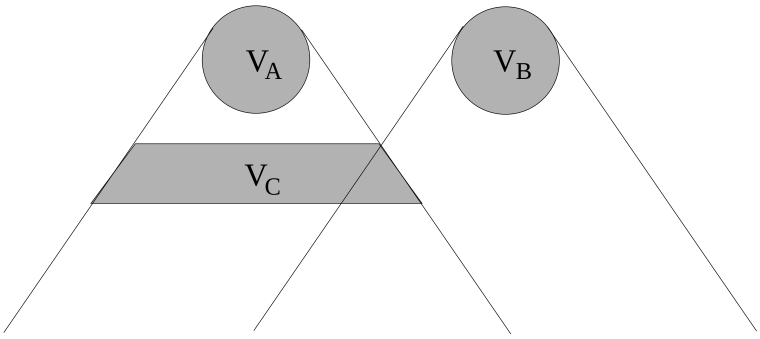

Figure 4.4: A region V_C satisfying Requirements (i)-(iii).

$$\frac{\phi(C_k ABC_k)}{\phi(C_k)} = \frac{\phi(C_k AC_k)}{\phi(C_k)} \frac{\phi(C_k BC_k)}{\phi(C_k)} \tag{4.3}$$

In case of local *classical* theories a locally faithful state ϕ determines uniquely a locally nonzero probability measure p by $p(A) := \phi(A), A \in \mathcal{P}(\mathcal{N}(V))$. By means of this (6.1) can be written both in the symmetric form

$$p(AB|C_k) = p(A|C_k)p(B|C_k) \tag{4.4}$$

and also in the equivalent asymmetric form

$$p(A|BC_k) = p(A|C_k) \tag{4.5}$$

featuring in Bell's first version of local causality.

Now, the localization of region V_C by Requirements (i)-(iii) is a bit more liberal than that required in Bell's second version. Although V_C "completely shields off" region V_A from the common past of V_A and V_B, it is not spacelike separated from V_B (as is, for example, region V_C in Fig. 4.3). But why not to be more liberal? Why Requirement (iii) is needed at all? Why does a region V_C such as the one depicted in Fig. 4.5 not suffice? The brief answer to this question is that the region *above* V_C (lighter shaded in Fig. 4.5) can contain stochastic events which, though completely specified by the region V_C, still, being stochastic, could establish a correlation between A and B in a classical stochastic theory (Norsen, 2011; Seevinck and Uffink 2011; Hofer-Szabó 2015c). Indeed, exactly this will be the case in our model introduced in the next section.

In order to relate Bell's local causality to the Causal Markov Condition we need to introduce a screening-off condition similar to local causality, namely Markovity:

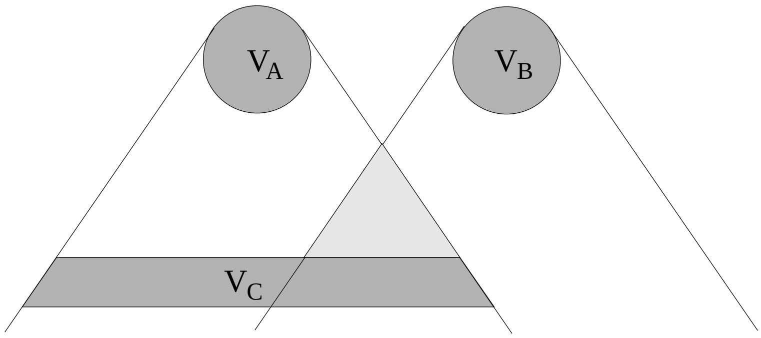

Figure 4.5: A region V_C for which Requirement (iii) does not hold.

Definition 3. A local physical theory represented by a net $\{\mathcal{N}(V), V \in \mathcal{K}\}$ of von Neumann algebras is called *Markov*, if for any pair $A \in \mathcal{N}(V_A)$ and $B \in \mathcal{N}(V_B)$ of projections supported in regions $V_A, V_B \in \mathcal{K}$ with $V_B \subset I_-(V_A)$ and for every locally normal and locally faithful state ϕ establishing a correlation $\phi(AB) \neq \phi(A)\phi(B)$ between A and B, and for any spacetime region V_C such that

(i) $V_C \subset J_-(V_A)$,

(ii) $V_A \subset V_C''$,

(iii') $V_B \subset J_-(V_C)$,

(see Fig. 4.6) and for any atomic event C_k of $\mathcal{A}(V_C)$ $(k \in K)$ (6.1) holds.

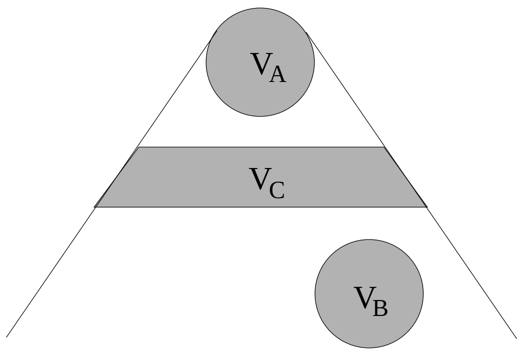

Figure 4.6: A region V_C satisfying Requirements (i)-(iii') of Markovity.

The relation between local causality and Markovity is straightforward. In both cases events localized in region V_A and V_B, respectively are screened-off by the atomic events in region V_C. If V_A and V_B are spacelike separated and V_C is localized according to Requirements (i)-(iii), then (6.1) expresses local causality. If V_A and

V_B are timelike separated and V_C is localized according to Requirements (i)-(iii'), then (6.1) expresses Markovity. As we will see later Causal Markov Condition will be a special case of the composition of local causality and Markovity.

4.4 A simple stochastic local classical theory

In this section we will develop a simple stochastic local classical theory. Before introducing it in a full-fledged form, let us sketch it in brief. The *spacetime* of the theory will be a 1+1 dimensional discretized Minkowski spacetime covered by minimal double cones. (See Fig. 4.7.) The *field configurations* of the theory are

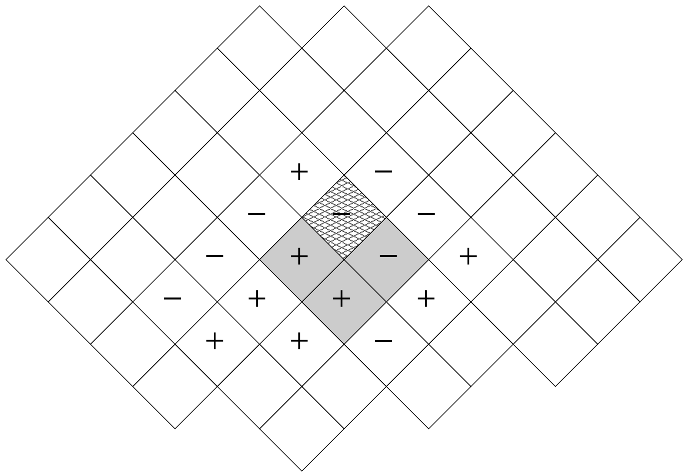

Figure 4.7: A simple stochastic local classical theory.

given by mappings assigning a $+$ or a $-$ to each minimal double cone. The *dynamics* of the theory is generated by the following *transition probabilities*: The value $+$ or $-$ in a given minimal double cone is probabilistically fixed by the product of the values in the three minimal double cones adjacent to it *from below*, irrespectively of the value in other minimal double cones, like earlier or spatially separated ones. The probabilistic dependence is this: If the product of the values in the three adjacent minimal double cones is $+$, then the value in the upper minimal double cone will be $+$ with probability p and $-$ with probability $1-p$; if the product is $-$, the value will be $-$ with probability p and $+$ with probability $1-p$. The process is *deterministic*, if $p \in \{0,1\}$ and *stochastic*, if $p \in (0,1)$. Now, let us see the theory in a more detailed way.

Consider a discretized version of the two dimensional Minkowski spacetime $\mathcal{M}^2$ which is composed of minimal double cones $V^m(t,i)$ of unit diameter with their center in (t,i) for $t,i \in \mathbb{Z}$ or $t,i \in \mathbb{Z}+1/2$. The set $\{V^m(t,i), i \in \frac{1}{2}\mathbb{Z}\}$ of such minimal double cones with $t = 0, -1/2$ defines a 'thickened' Cauchy surface in this spacetime, denoted by $\mathcal{S}_0$. For double cones sitting on $\mathcal{S}_0$ we will drop the time coordinate and simply write V_i^m. (See Fig. 8.14.)

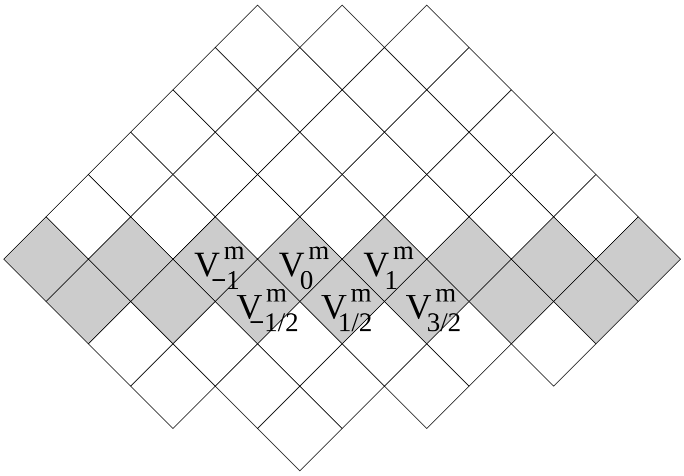

Figure 4.8: Two dimensional discrete Minkowski spacetime with a 'thickened' Cauchy surface.

A double cone $V(t,i;s,j)$ is defined to be the smallest double cone containing both $V^m(t,i)$ and $V^m(s,j)$, that is generated by them: $V(t,i;s,j) := V^m(t,i) \vee V^m(s,j)$. The directed poset of such double cones is denoted by $\mathcal{K}^m$ and the directed poset of double cones generated by minimal double cones sticked to the Cauchy surface $\mathcal{S}_0$ is denoted by $\mathcal{K}_0^m$. Obviously, $\mathcal{K}_0^m$ will be left invariant by integer space translations and $\mathcal{K}^m$ will be left invariant by integer space and time translations. By shifting the time coordinates of the minimal double cones by t one can similarly define the Cauchy surface $\mathcal{S}_t$ and the net $\mathcal{K}_t^m$.

Let S^m denote the set of minimal double cones of $\mathcal{M}^2$ and let $\mathbf{Z}_2$ be the multiplicative group of the integers $\{1,-1\}$. Define the set $\mathcal{C}$ of *configurations* of the theory as: $\mathcal{C} := \{c\colon S^m \to \mathbf{Z}_2\}$. The maximal σ-algebra of classical events $(\mathcal{C},\mathcal{P}(\mathcal{C}))$ is given by the power set $\mathcal{P}(\mathcal{C})$ of the set of configurations. But one can also obtain a narrower σ-algebra in tune with the net structure $\mathcal{K}^m$. This is done by taking the equivalence classes of those configurations which have the same field values on a given region in $\mathcal{K}^m$. The sets $\mathcal{C}_V$ of local equivalence classes (the 'cylindrical subsets' of $\mathcal{C}$ concentrated on V) are obtained by the equivalence relation: $c \sim_V c'$ if $c_{|V} = c'_{|V}$. Clearly, $\mathcal{C}_V$ contains $2^{|V|}$ elements, where $|V|$ is the number of minimal double cones in V. One can get the power set $\mathcal{P}(\mathcal{C}_V)$ of $\mathcal{C}_V$ by defining the following map Z_V for $V \in \mathcal{K}^m$:

$$Z_V\colon \mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathcal{C}),\ C \mapsto \{c' \in \mathcal{C}\,|\,\exists c \in C : c_{|V} = c'_{|V}\} \tag{4.6}$$

For a given $V \in \mathcal{K}^m$ the image sets of Z_V define a unital σ-subalgebra $\Sigma(V)$ of $\mathcal{P}(\mathcal{C})$, which is isomorphic to the power set $\mathcal{P}(\mathcal{C}_V)$ of $\mathcal{C}_V$. By ranging over $V \in \mathcal{K}^m$

one obtains an isotone net structure $\{(\mathcal{C}, \Sigma(V)), V \in \mathcal{K}^m\}$. The $2^{|V|}$ dimensional abelian local von Neumann algebra $\mathcal{N}(V)$ corresponding to the local σ-algebra $\Sigma(V)$ is spanned by the orthogonal set of minimal projections $P_V^c, c \in \mathcal{C}_V$ corresponding to characteristic functions $\chi_V^c : \mathcal{C} \to \mathbf{C}$ which are 1 on the cylindrical subset $c \in \mathcal{C}_V$ of $\mathcal{C}$ and 0 otherwise. Clearly, $\{\mathcal{N}(V), V \in \mathcal{K}^m\}$ is an isotone net of finite dimensional abelian von Neumann algebras, hence it defines a local classical theory.

The quasilocal C^*-algebra $\mathcal{A}$ is given by the inductive limit of the local von Neumann algebras $\mathcal{N}(V), V \in \mathcal{K}^m$, and similarly the unital C^*-subalgebras $\mathcal{A}_0$ of $\mathcal{A}$ is given by the inductive limit of the local von Neumann algebras $\mathcal{N}(V), V \in \mathcal{K}_0^m$. Now, a stochastic theory can be regarded as a state extension procedure from the subalgebra $\mathcal{A}_0$ (or from any $\mathcal{A}_t$) to the quasilocal algebra $\mathcal{A}$ by means of so-called transition probabilities. This is done in the following way.

Let $V(t+\frac{1}{2})$ be a finite set of minimal double cones on the time slice $t+\frac{1}{2}$. Define the nearest past of $V(t+\frac{1}{2})$ as follows: $\mathcal{P}_t(V(t+\frac{1}{2})) \equiv \mathcal{S}_t \cap (\mathcal{S}_t \setminus J_-(V(t+\frac{1}{2})))'$. Specifically, the nearest past $\mathcal{P}_t(V^m(t+\frac{1}{2}, i))$ of the minimal double cone $V^m(t+\frac{1}{2}, i)$ contains the three minimal double cones adjacent to $V^m(t+\frac{1}{2}, i)$ from below, namely $V^m(t, i-\frac{1}{2})$, $V^m(t-\frac{1}{2}, i)$ and $V^m(t, i+\frac{1}{2})$. For a given configuration $c \in \mathcal{C}$ define the *generating transition probabilities* from the equivalence class $c_{\mathcal{P}_t(V^m(t+\frac{1}{2}, i))}$ to the equivalence class $c_{V^m(t+\frac{1}{2}, i)}$ as follows:

$$p(c_{V^m(t+\frac{1}{2},i)} | c_{\mathcal{P}_t(V^m(t+\frac{1}{2},i))}) := \begin{cases} p, & \text{if } c(t+\frac{1}{2}, i) = c(t, i-\frac{1}{2})c(t-\frac{1}{2}, i)c(t, i+\frac{1}{2}) \\ 1-p, & \text{if } c(t+\frac{1}{2}, i) = -c(t, i-\frac{1}{2})c(t-\frac{1}{2}, i)c(t, i+\frac{1}{2}) \end{cases} \tag{4.7}$$

where $c(t, i)$ is short for $c(V^m(t, i))$, the value of the configuration c at the minimal double cone $V^m(t, i)$. Assuming that the generating transition probabilities are independent with respect to spacelike separation, one can define the *transition probabilities* from the Cauchy surface $\mathcal{S}_t$ to the time slice $t+\frac{1}{2}$ as:

$$p(c_{V(t+\frac{1}{2})} | c_{\mathcal{P}_t(V(t+\frac{1}{2}))}) := \prod_{V^m(t+\frac{1}{2}, i) \in V(t+\frac{1}{2})} p(c_{V^m(t+\frac{1}{2},i)} | c_{\mathcal{P}_t(V^m(t+\frac{1}{2},i))}) \tag{4.8}$$

Intuitively, these transition probabilities do the following: The value $+$ or $-$ in a given minimal double cone is probabilistically fixed purely by the product of the values in the three minimal double cones adjacent to it from below. (See again Fig. 4.7.) Negatively speaking, they do not depend on the value of other minimal double cones, like *earlier* or *spatially separated* ones. As we will see, these two independencies are closely connected to Markovity and local causality, respectively. If the product is $+$, then the value is $+$ with probability p and $-$ with probability $1-p$; if the product is $-$, the value is $-$ with probability p and $+$ with probability $1-p$.

Finally, let $U(t)$ be a finite set of minimal double cones on the Cauchy surface $\mathcal{S}_t$. We define the state on the equivalence class $c_{V(t+\frac{1}{2})} \cap c_{U(t)}$ as follows:

$$\phi(c_{V(t+\frac{1}{2})} \cap c_{U(t)}) := p(c_{V(t+\frac{1}{2})} | c_{\mathcal{P}_t(V(t+\frac{1}{2}))}) \phi(c_{\mathcal{P}_t(V(t+\frac{1}{2}))} \cap c_{U(t)}) \tag{4.9}$$

Thus, starting from ϕ_0 on $\mathcal{A}_0$ one can recursively define the state ϕ on the whole $\mathcal{A}$. (For the Cauchy surfaces below $\mathcal{S}_0$ we use Bayes theorem for the extension.)

To simplify things, introduce the following denotation. Let i^+ and i^- denote three different things at the same time: the two cylindrical subsets of $\mathcal{C}_{V_i^m}$ concentrated on the minimal double cone V_i^m on the Cauchy surface $\mathcal{S}_0$; the two corresponding characteristic functions; and also the two corresponding orthogonal projections in $\mathcal{N}(V_i^m)$. If we are not specifying which of the two sets/characteristic functions/projections we are speaking about, we simply write i. The nth forward and backward *space* translates of i will be denoted by $(i+n)$ and $(i-n)$, respectively ($n \in \frac{1}{2}\mathbb{N}$); the tth forward and backward *time* translates will be denoted by i_t and i_{-t}, respectively ($t \in \mathbb{N}$).

Let, furthermore,

$$i \cdot (i+\frac{1}{2}) \dots (j-\frac{1}{2}) \cdot j$$

denote the product of a sequence of projections localized on the Cauchy surface $\mathcal{S}_0$ between minimal double cones V_i^m and V_j^m, and let $p_{i\dots j}$ denote the probability thereof in state ϕ. Since we will deal only with *projections* of abelian von Neumann algebras, from now on instead of ϕ we simply write p. Finally, we will express the condition

$$c(t+\frac{1}{2},i) = c(t,i-\frac{1}{2})c(t-\frac{1}{2},i)c(t,i+\frac{1}{2})$$

in (4.7) by the Dirac delta symbol

$$\delta_{c(t+\frac{1}{2},i),c(t,i-\frac{1}{2})c(t-\frac{1}{2},i)c(t,i+\frac{1}{2})}$$

or in the short form

$$\delta_{i_1,(i-\frac{1}{2})i(i+\frac{1}{2})}$$

Now, let $A = i_t$ and $B = j_s$ be two projections localized in the minimal double cones $V^m(t,i)$ and $V^m(s,j)$, respectively, with $i < j$. Suppose that $V^m(t,i)$ and $V^m(s,j)$ are spatially separated, that is $|j-i| > |s-t|$. To calculate the probability of A, B and AB, we need a little geometry. (See Fig. 4.9.) Consider the minimal double cone $V^m(u,k)$ (striped horizontally) at the 'top of the common past' of regions $V^m(t,i)$ and $V^m(s,j)$. The coordinates of $V^m(u,k)$ are the following:

$$u = \frac{1}{2}(t+s+i-j) \qquad k = \frac{1}{2}(i+j+t-s) \tag{4.10}$$

Consider now the Cauchy surface $\mathcal{S}_{\lceil u \rceil}$ fitting $V^m(u,k)$, where the ceiling function $\lceil \cdot \rceil$ in the subscript is just to round up the u coordinates if half integers. Let the number of minimal double cones in the causal past of $V^m(t,i)$ *above* $\mathcal{S}_0$ (including $V^m(t,i)$ but not including double cones on $\mathcal{S}_0$) be denoted by n, and the number of minimal double cones in the causal past of $V^m(t,i)$ *above* $\mathcal{S}_{\lceil u \rceil}$ (again including $V^m(t,i)$ but not including double cones on $\mathcal{S}_{\lceil u \rceil}$) by n'. Similarly, the number of minimal double cones in the causal past of $V^m(s,j)$ *above* $\mathcal{S}_0$ and $\mathcal{S}_{\lceil u \rceil}$ be denoted by m and m', respectively. Finally, denote the number of minimal double cones in the causal past of $V^m(u,k)$ *above* $\mathcal{S}_0$ by l. The numbers n, n', m', m and l are the

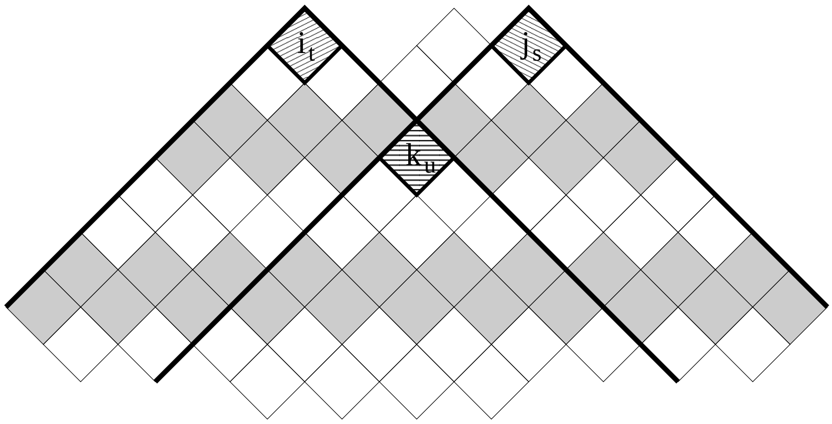

Figure 4.9: A little geometry.

following functions of i, j, t and s:

$$n = \begin{cases} -t + 4\sum_{x=1}^{t} x, & \text{if } i \in \mathbb{N} \\ t + 4\sum_{x=1}^{t}(x-1), & \text{if } i \in \frac{1}{2}\mathbb{N} \end{cases} \tag{4.11}$$

$$n' = \begin{cases} -t + 4\sum_{x=\lceil u \rceil}^{t} x, & \text{if } i \in \mathbb{N} \\ t + 4\sum_{x=\lceil u \rceil}^{t}(x-1), & \text{if } i \in \frac{1}{2}\mathbb{N} \end{cases} \tag{4.12}$$

$$m = \begin{cases} -s + 4\sum_{x=1}^{s} x, & \text{if } j \in \mathbb{N} \\ s + 4\sum_{x=1}^{s}(x-1), & \text{if } j \in \frac{1}{2}\mathbb{N} \end{cases} \tag{4.13}$$

$$m' = \begin{cases} -s + 4\sum_{x=\lceil u \rceil}^{s} x, & \text{if } j \in \mathbb{N} \\ s + 4\sum_{x=\lceil u \rceil}^{s}(x-1), & \text{if } j \in \frac{1}{2}\mathbb{N} \end{cases} \tag{4.14}$$

$$l = \begin{cases} -\lceil u \rceil + 4\sum_{x=1}^{\lceil u \rceil} x, & \text{if } k \in \mathbb{N} \\ \lceil u \rceil + 4\sum_{x=1}^{\lceil u \rceil}(x-1), & \text{if } k \in \frac{1}{2}\mathbb{N} \end{cases} \tag{4.15}$$

In Fig. **??**, for example, $n = m = 3$, $n' = m' = 21$ and $l = 6$. With these numbers one can also calculate the number r of minimal double cones *between* $\mathcal{S}_{\lceil u \rceil}$ and $\mathcal{S}_0$ (including double cones on $\mathcal{S}_{\lceil u \rceil}$ but not on $\mathcal{S}_0$):

$$r = n - n' + m - m' - l \tag{4.16}$$

which is 30 in Fig. 4.9. Now, using the above numbers (4.11)-(4.16) the probability of A, B and AB will be the following:

$$
\begin{aligned}
p(A) &= \sum_{(i-t-\{i+\frac{1}{2}\}),\ldots,(i+t+\{i+\frac{1}{2}\})} \Big[q_n \delta_{i_t,(i-t+\{i\})\ldots(i+t-\{i\})} \\
&\quad +(1-q_n)\delta_{-i_t,(i-t+\{i\})\ldots(i+t-\{i\})} \Big] p_{(i-t-\{i+\frac{1}{2}\})\ldots(i+t+\{i+\frac{1}{2}\})} \qquad (4.17)\\
p(B) &= \sum_{(j-s-\{j+\frac{1}{2}\}),\ldots,(j+s+\{j+\frac{1}{2}\})} \Big[q_m \delta_{j_s,(j-s+\{j\})\ldots(j+s-\{j\})} \\
&\quad +(1-q_m)\delta_{-j_s,(j-s+\{j\})\ldots(j+s-\{j\})} \Big] p_{(j-s-\{j+\frac{1}{2}\})\ldots(j+s+\{j+\frac{1}{2}\})} \qquad (4.18)\\
p(AB) &= \sum_{(i-t+\{i\}),\ldots,(j+s-\{j\})} \Big[q_{n'} q_{m'} q_r\, \delta_{i_t,(i-t+\{i\})\ldots(i+t-\{i\})} \delta_{j_s,(j-s+\{j\})\ldots(j+s-\{j\})} \\
&\quad +q_{n'}(1-q_{m'})q_r\, \delta_{i_t,(i-t+\{i\})\ldots(i+t-\{i\})} \delta_{-j_s,(j-s+\{j\})\ldots(j+s-\{j\})} \\
&\quad +(1-q_{n'})q_{m'} q_r\, \delta_{-i_t,(i-t+\{i\})\ldots(i+t-\{i\})} \delta_{j_s,(j-s+\{j\})\ldots(j+s-\{j\})} \\
&\quad +(1-q_{n'})(1-q_{m'})q_r\, \delta_{-i_t,(i-t+\{i\})\ldots(i+t-\{i\})} \delta_{-j_s,(j-s+\{j\})\ldots(j+s-\{j\})} \Big] \\
&\quad \times p_{(i-t-\{i+\frac{1}{2}\})\ldots(j+s+\{j+\frac{1}{2}\})} \qquad (4.19)
\end{aligned}
$$

where the fractional part function $\{\cdot\}$ in the subscript is again to treat integer and half integer coordinates together, and q_x $(x = n, n', m, m', r)$ is the even part of the binomial expression:

$$q_x \quad := \quad p^x + \binom{x}{2} p^{x-2}(1-p)^2 + \binom{x}{4} p^{x-4}(1-p)^4 + \ldots \qquad (4.20)$$

Obviously, in the general case:

$$p(AB) \neq p(A)p(B) \qquad (4.21)$$

so there is a superluminal correlation between A and B.

Example 1. As an example, let $A = i_1^+$ and $B = j_1^+$, where $j = i + 2 \in \mathbb{N} + \frac{1}{2}$. (See Fig. 8.16.) Let the 'prior' probabilities $p_{(i-1)\ldots(j+1)}$ on $\mathcal{S}_0$ be fixed as follows:

$$
\begin{aligned}
p_{+++++++++} &= \frac{1}{2} \qquad (4.22)\\
p_{+++++++-+} &= \frac{1}{4} \qquad (4.23)\\
p_{+-+++++++} &= \frac{1}{4} \qquad (4.24)
\end{aligned}
$$

and all the other combinations be 0. Then the probability of A, B and AB is the

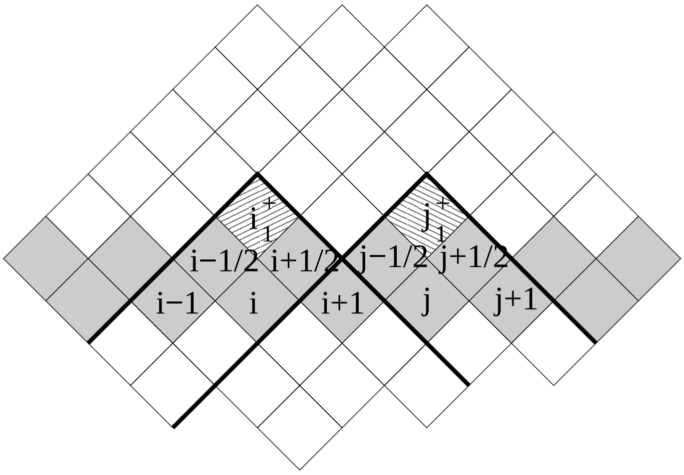

Figure 4.10: Superluminally correlating events i_1^+ and j_1^+.

following:

$$\begin{aligned}
p(A) &= \sum_{(i-1),\ldots,(i+1)} \left[p\,\delta_{i_1^+,(i-\frac{1}{2})i(i+\frac{1}{2})} + (1-p)\delta_{-i_1^+,(i-\frac{1}{2})i(i+\frac{1}{2})}\right] p_{(i-1)\ldots(i+1)} = \frac{1}{2}\left(\frac{1}{2}+p\right)\\
p(B) &= \sum_{(j-1),\ldots,(j+1)} \left[p\,\delta_{j_1^+,(j-\frac{1}{2})j(j+\frac{1}{2})} + (1-p)\delta_{-j_1^+,(j-\frac{1}{2})j(j+\frac{1}{2})}\right] p_{(j-1)\ldots(j+1)} = \frac{1}{2}\left(\frac{1}{2}+p\right)\\
p(AB) &= \sum_{(i-1),\ldots,(j+1)} \Big[p^2\,\delta_{i_1^+,(i-\frac{1}{2})i(i+\frac{1}{2})}\delta_{j_1^+,(j-\frac{1}{2})j(j+\frac{1}{2})} + p(1-p)\delta_{i_1^+,(i-\frac{1}{2})i(i+\frac{1}{2})}\delta_{-j_1^+,(j-\frac{1}{2})j(j+\frac{1}{2})}\\
&\quad +(1-p)p\,\delta_{-i_1^+,(i-\frac{1}{2})i(i+\frac{1}{2})}\delta_{j_1^+,(j-\frac{1}{2})j(j+\frac{1}{2})} + (1-p)^2\delta_{-i_1^+,(i-\frac{1}{2})i(i+\frac{1}{2})}\delta_{-j_1^+,(j-\frac{1}{2})j(j+\frac{1}{2})}\Big] p_{(i-1)}\\
&\quad = \frac{1}{2}p
\end{aligned}$$

thus A and B are correlating whenever $p \neq \frac{1}{2}$.

Example 2. In the second example, let $A = i_2^+$ and $B = j_2^+$, where again $j = i+2 \in \mathbb{N}+\frac{1}{2}$. (See Fig. 8.17.) With the 'prior' probabilities $p_{(i-2)\ldots(j+2)}$:

$$p_{+++++++++++++} = \frac{1}{2} \tag{4.28}$$

$$p_{+++++++++++-+} = \frac{1}{4} \tag{4.29}$$

$$p_{+-+++++++++++} = \frac{1}{4} \tag{4.30}$$

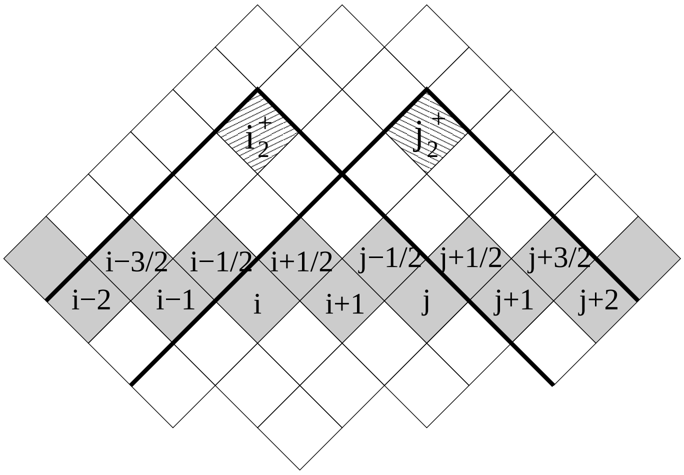

Figure 4.11: Superluminally correlating events i_2^+ and j_2^+.

(and the rest is 0) one obtains the probability of A, B and AB as:

$$\begin{aligned}
p(A) &= \sum_{(i-2),\dots,(i+2)} \left[q_6\, \delta_{i_2^+,(i-\frac{3}{2})\dots(i+\frac{3}{2})} + (1-q_6)\delta_{-i_2^+,(i-\frac{3}{2})\dots(i+\frac{3}{2})} \right] p_{(i-2)\dots(i+2)} \\
&= \frac{1}{2}\left(\frac{1}{2}+q_6\right) \qquad (4.31) \\
p(B) &= \sum_{(j-2),\dots,(j+2)} \left[q_6\, \delta_{j_2^+,(j-\frac{3}{2})\dots(j+\frac{3}{2})} + (1-q_6)\delta_{-j_2^+,(j-\frac{3}{2})\dots(j+\frac{3}{2})} \right] p_{(j-2)\dots(j+2)} \\
&= \frac{1}{2}\left(\frac{1}{2}+q_6\right) \qquad (4.32) \\
p(AB) &= \sum_{(i-2),\dots,(j+2)} \Big[p^2 q_9\, \delta_{i_2^+,(i-\frac{3}{2})\dots(i+\frac{3}{2})} \delta_{j_2^+,(j-\frac{3}{2})\dots(j+\frac{3}{2})} \\
&\quad + p\,(1-p)\,q_9\, \delta_{i_2^+,(i-\frac{3}{2})\dots(i+\frac{3}{2})} \delta_{-j_2^+,(j-\frac{3}{2})\dots(j+\frac{3}{2})} \\
&\quad + (1-p)\,p\,q_9\, \delta_{-i_2^+,(i-\frac{3}{2})\dots(i+\frac{3}{2})} \delta_{j_2^+,(j-\frac{3}{2})\dots(j+\frac{3}{2})} \\
&\quad + (1-p)^2 q_9\, \delta_{-i_2^+,(i-\frac{3}{2})\dots(i+\frac{3}{2})} \delta_{-j_2^+,(j-\frac{3}{2})\dots(j+\frac{3}{2})} \Big] p_{(i-2)\dots(j+2)} = \frac{1}{2} p q_9 \qquad (4.33)
\end{aligned}$$

thus A and B are correlating whenever $\frac{1}{4}(\frac{1}{2}+q_6)^2 \neq \frac{1}{2}pq_9$ which is the typical case.

The difference between Example 1 and 2 is that in Example 1 there is *no* minimal double cone above $\mathcal{S}_0$ in the common past of A and B, whereas in Example 2 there *is* such a minimal double cone, namely $V^m(1, i+1)$.[2] This difference will have crucial consequences concerning local causality to which we turn now.

First, we prove that the above local classical theory is locally causal. Actually, we prove a little less: local causality for a specific choice of V_A, V_B and V_C. (For a general proof see (Hofer-Szabó and Vecsernyés 2015a).) Let $V_A = V^m(t, i)$ and $V_B = V^m(s, j)$ be two spatially separated minimal double cones with $i < j$, and let V_C be generated by the intersection of the causal past of V_A and a Cauchy surface "shielding off" V_A from the common past of V_A and V_B. Any Cauchy surface $\mathcal{S}_v$ with $\lceil u \rceil \leqslant v \leqslant t$ will be such a "shielder-off" Cauchy surface, where u is defined in (4.10). (For a "shielder-off" Cauchy surface see Fig. 4.9.) The region V_C generated by this intersection will obviously satisfy Requirements (i)-(iii) in Definition 6 of local causality.

Now, we prove local causality with respect to these regions.

Proposition 1. The stochastic local classical theory $\{\mathcal{N}(V), V \in \mathcal{K}^m\}$ is locally causal for any three regions V_A, V_B and V_C specified above.

Proof. Let $A = i_t$ and $B = j_s$ be two projections localized in V_A and V_B, respectively, and correlating in the probability measure p. We are to show that for any atomic event

$$C = \left(i - t + v - \{i + \frac{1}{2}\}\right)_v \ldots \left(i + t - v + \{i + \frac{1}{2}\}\right)_v$$

of V_C the following holds:

$$p(AB|C) = p(A|C)p(B|C) \tag{4.34}$$

First, for the sake of convenience, shift the Cauchy surface $\mathcal{S}_0$ up to $\mathcal{S}_v$ and denote the new time coordinates by a prime: $t' := t - v$ and $s' := s - v$. Similarly let q'_n and q'_m denote the appropriate number of minimal double cones with respect to the shifted Cauchy surface. With this notation the conditional probabilities are the

[2]See also our remark in the last paragraph of Section 3.

following:

$$
\begin{aligned}
p(A|C) &= \left[q'_n \delta_{i_{t'},(i-t'+\{i\})\dots(i+t'-\{i\})} + (1-q'_n)\delta_{-i_{t'},(i-t'+\{i\})\dots(i+t'-\{i\})}\right] && (4.35)\\
p(B|C) &= \sum_{(j-s'-\{j+\frac{1}{2}\}),\dots,(j+s'+\{j+\frac{1}{2}\})} \Big[q'_m \delta_{j_{s'},(j-s'+\{j\})\dots(j+s'-\{j\})} \\
&\quad +(1-q'_m)\delta_{-j_{s'},(j-s'+\{j\})\dots(j+s'-\{j\})}\Big] p_{C(j-s'-\{j+\frac{1}{2}\})\dots(j+s'+\{j+\frac{1}{2}\})} && (4.36)\\
p(AB|C) &= \sum_{(j-s'-\{j+\frac{1}{2}\}),\dots,(j+s'+\{j+\frac{1}{2}\})} \Big[q'_n q'_m\, \delta_{i_{t'},(i-t'+\{i\})\dots(i+t'-\{i\})}\delta_{j_{s'},(j-s'+\{j\})\dots(j+s'-\{j\})} \\
&\quad +q'_n(1-q'_m)\, \delta_{i_{t'},(i-t'+\{i\})\dots(i+t'-\{i\})}\delta_{-j_{s'},(j-s'+\{j\})\dots(j+s'-\{j\})} \\
&\quad +(1-q'_n)q'_m\, \delta_{-i_{t'},(i-t'+\{i\})\dots(i+t'-\{i\})}\delta_{j_{s'},(j-s'+\{j\})\dots(j+s'-\{j\})} \\
&\quad +(1-q'_n)(1-q'_m)\, \delta_{-i_{t'},(i-t'+\{i\})\dots(i+t'-\{i\})}\delta_{-j_{s'},(j-s'+\{j\})\dots(j+s'-\{j\})}\Big] \\
&\quad \times p_{C(j-s'-\{j+\frac{1}{2}\})\dots(j+s'+\{j+\frac{1}{2}\})} && (4.37)
\end{aligned}
$$

where $p_{C(j-s'-\{j+\frac{1}{2}\})\dots(j+s'+\{j+\frac{1}{2}\})}$ is a short for

$$p_{(i-t'-\{i+\frac{1}{2}\}\dots(i+t'+\{i+\frac{1}{2}\})(j-s'-\{j+\frac{1}{2}\})\dots(j+s'+\{j+\frac{1}{2}\})}$$

From (4.35)-(4.37) the screening-off (6.2) follows immediately. ■

One can see from the proof that if V_C is a segment of Cauchy surface satisfying Requirements (i)-(iii) in Definition 6, that is a segment of Cauchy surface located *at or above the top* of the common causal past of the correlating events A and B, then from (4.19) the q_r terms will drop out leaving no correlation between the conditional probabilities. Note that V_C need not necessarily be *above* the common past of A and B, it can also intersect *with the top* of it (see again Fig. 4.5). All is needed is that there is no region *above* V_C in the common past. Such a region, namely, can contain stochastic events which could establish a correlation between A and B. Mathematically this means that from (4.19) the q_r terms would *not* drop out and hence the correlation would not be screened off by the atomic events of V_C. Requirement (iii) in the definition of local causality is just to exclude this case. The next proposition shows that Requirement (iii) also is a necessary condition in the localization of V_C.

Proposition 2. The local classical theory $\{\mathcal{N}(V), V \in \mathcal{K}^m\}$ would *not* be locally causal if Requirement (iii) was dropped from Definition 6.

Proof. Consider Example 2 of the previous Section that is let $A = i^+_2$ and $B = (i+2)^+_2$ and the prior probabilities those fixed in (4.28)-(4.30). Let C be the minimal projection

$$(i-2)^+(i-\frac{3}{2})^+(i-1)^+(i-\frac{1}{2})^+i^+(i+\frac{1}{2})^+(i+1)^+(i+\frac{3}{2})^+(i+2)^+$$

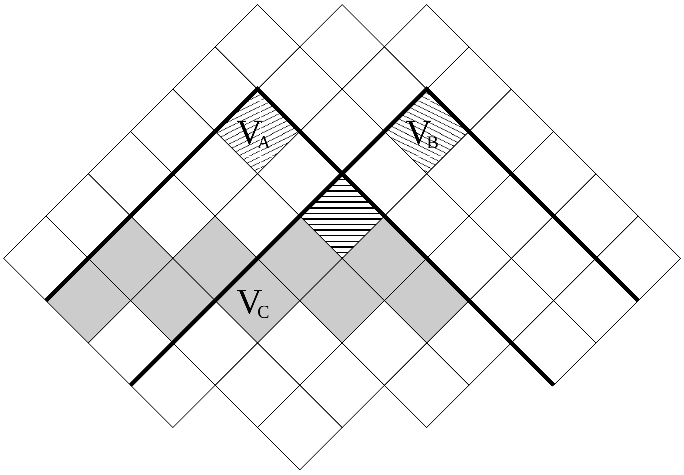

Figure 4.12: A region V_C for which Requirement (iii) does not hold.

localized in region V_C. (See Fig. 4.12.) For the region V_C Requirement (iii) does not hold since there is a minimal double cone, $V^m(1, i+1)$ (the one with horizontal stripes) above region V_C in the common past of V_A and V_B.

Using the identity

$$\sum_{(i+\frac{5}{2}),(i+3),(i+\frac{7}{2}),(i+4)} (i+\frac{5}{2})(i+3)(i+\frac{7}{2})(i+4) = \mathbf{1} \tag{4.38}$$

it is easy to see that C does *not* screen off the correlation between A and B since

$$p(A|C) = q_6 \tag{4.39}$$

$$p(B|C) = \frac{\sum_{(i+\frac{5}{2}),(i+3),(i+\frac{7}{2}),(i+4)} p\big(B\big|C(i+\frac{5}{2}),(i+3),(i+\frac{7}{2}),(i+4)\big)p_{C(i+\frac{5}{2}),(i+3),(i+\frac{7}{2}),(i+4)}}{p(C)}$$

$$= \frac{1}{3}(1+q_6) \tag{4.40}$$

$$p(AB|C) = \frac{\sum_{(i+\frac{5}{2}),(i+3),(i+\frac{7}{2}),(i+4)} p\big(AB\big|C(i+\frac{5}{2}),(i+3),(i+\frac{7}{2}),(i+4)\big)p_{C(i+\frac{5}{2}),(i+3),(i+\frac{7}{2}),(i+4)}}{p(C)}$$

$$= \frac{1}{3}(1+p)pq_9 \tag{4.41}$$

for any C of non-zero measure. But typically

$$\frac{1}{3}q_6(1+q_6) \neq \frac{1}{3}(1+p)pq_9 \tag{4.42}$$

since the left and right hand side are of different ordo in p. ∎

Next we prove that the above local classical theory is also *Markov*. Again, we prove a little less: local causality for a minimal double cone $V_A = V^m(t,i)$, another minimal double cone $V_B = V^m(s,j)$ lying in the causal past of V_A, and a third region V_C generated by the intersection of the causal past of V_A and a Cauchy surface "shielding off" V_A from V_B. (See Fig. 4.13.) V_C will obviously satisfy

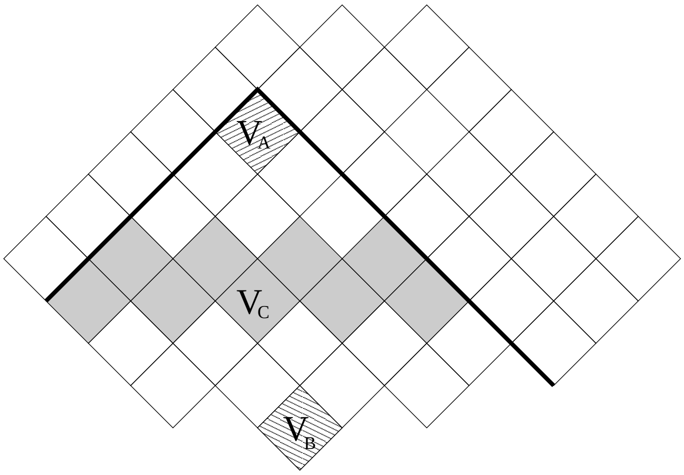

Figure 4.13: The regions V_A, V_B and V_C for which Markovity holds.

Requirements (i)-(iii') in Definition 3 of Markovity.

Proposition 3. The stochastic local classical theory $\{\mathcal{N}(V), V \in \mathcal{K}^m\}$ is Markov for any three regions V_A, V_B and V_C specified above.

Proof. Let $A = i_t$ and $B = j_s$ be two projections localized in V_A and V_B, respectively, and correlating in the probability measure p. We are to show that for any atomic event

$$C = \left(i - t + v - \{i + \frac{1}{2}\}\right)_v \dots \left(i + t - v + \{i + \frac{1}{2}\}\right)_v$$

of V_C with $s < v < t$ the following holds:

$$p(A|C) = p(A|CB) \tag{4.43}$$

But it does, since both sides of (6.3) are simply

$$q'_n \delta_{i_{t'},(i-t'+\{i\})\dots(i+t'-\{i\})} + (1 - q'_n)\delta_{-i_{t'},(i-t'+\{i\})\dots(i+t'-\{i\})}$$

where again $t' := t - v$ and q'_n denotes the appropriate number of minimal double cones with respect to the shifted Cauchy surface. ■

4.5 Local Causality, Causal Markov Condition and d-separation

Now, I connect local causality and Markovity to the Causal Markov Condition used in the theory of Bayesian networks (see (Pearl, 2000) and (Spirtes, Glymour and Scheines, 2000)). Consider a directed acyclic graph $\mathcal{G}$ and a set of random variables $\mathcal{V}$ on a classical probability space (Σ, p) such that the elements $X, Y \ldots$ of $\mathcal{V}$ are represented by the vertices of $\mathcal{G}$ and the arrows $X \to Y$ on the graph represent that X is *causally relevant* for Y. For any $X \in \mathcal{V}$ let $Par(X)$, the *parents* of X, be the set of vertices that have directed edges in X; let $Anc(X)$, the *ancestors* of X, be the set of vertices from which a directed paths is leading to X; and finally let $Des(X)$, the *descendants* of X, be the set of vertices that are endpoints of a directed paths from X. The set $\mathcal{V}$ is said to satisfy the *Causal Markov Condition* relative to the graph $\mathcal{G}$ if for any $X \in \mathcal{V}$ and any $Y \notin Des(X)$ the following is true:

$$p(X|Par(X) \wedge Y) \quad = \quad p(X|Par(X)) \tag{4.44}$$

In other words, conditioning on its parents the random variable X will be probabilistically independent from any of its non-descendant. Non-descendants of X can be of two types: either ancestors or *non-relatives* (non-descendants and non-ancestors). As we will see, being independent of *ancestors* is related to the Markovity, whereas being independent of *non-relatives* is related to local causality.

We say that the set $\mathcal{V}$ is *faithful* relative to the graph $\mathcal{G}$ if all probabilistic independencies between the random variables of $\mathcal{V}$ are implied by the Causal Markov Condition. This implication can neatly be depicted graphically by the so-called *d-separation* criterion. Let $\mathcal{P}$ be a path in $\mathcal{G}$. A variable C on $\mathcal{P}$ is a *collider* if there are arrows to C from both its neighbors on $\mathcal{P}$. Now, let $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ be three disjoint sets of vertices in $\mathcal{G}$. $\mathcal{X}$ and $\mathcal{Y}$ are said to be *d-connected* by $\mathcal{Z}$ in $\mathcal{G}$ iff there exists a path $\mathcal{P}$ between some vertex in $\mathcal{X}$ and some vertex in $\mathcal{Y}$ such that for every collider C on $\mathcal{P}$, either C or a descendant of C is in $\mathcal{Z}$, and no non-collider on $\mathcal{P}$ is in $\mathcal{Z}$. $\mathcal{X}$ and $\mathcal{Y}$ are said to be *d-separated* by $\mathcal{Z}$ in $\mathcal{G}$ iff they are not d-connected by $\mathcal{Z}$ in $\mathcal{G}$. Specifically, the Causal Markov Condition entails that the variables X and Y are probabilistically independent conditional upon the subset $\mathcal{Z}$ just in case $\mathcal{Z}$ d-separates X and Y in $\mathcal{G}$.

Now, consider the stochastic local classical theory $\{\mathcal{N}(V), V \in \mathcal{K}^m\}$ introduced in the previous Section. A local von Neumann algebra $\mathcal{N}(V)$ of the theory gives rise to a graph $\mathcal{G}(V)$ and a set of random variables $\mathcal{V}(V)$ on a classical probability space (Σ, p) in the following way. Consider a region V in $\mathcal{K}^m$ with the set $\{V^m\}$ of minimal double cones contained in V. Let the minimal double cones be the *vertices* of a causal graph and draw an *arrow* to every minimal double cone $V^m(t,i)$ from the three minimal double cones adjacent to it *from below*, that is from $V^m(t-\frac{1}{2}, i-\frac{1}{2})$, $V^m(t-1, i)$ and $V^m(t-\frac{1}{2}, i+\frac{1}{2})$, if all contained in V. (See Fig. 4.14.) The set of vertices and arrows will uniquely determine a *causal graph* $\mathcal{G}(V)$ associated to V.

As for the set of *random variables* $\mathcal{V}(V)$, to each minimal double cone $V^m(t,i)$ in V assign simply the two cylindrical subsets of $\mathcal{C}_{V(t,i)}$, denoted by $c^+_{V^m(t,i)}$ and

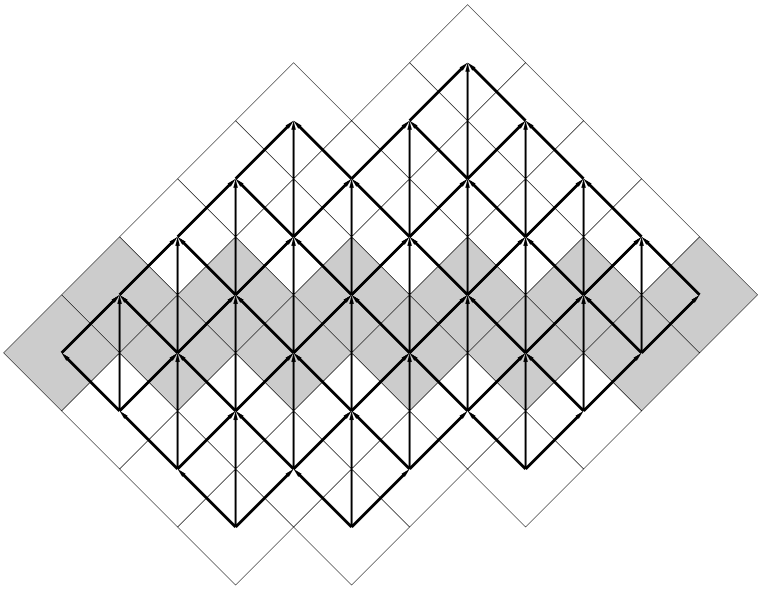

Figure 4.14: The causal graph $\mathcal{G}(V)$ associated to V.

$c^{-}_{V^m(t,i)}$, or equivalently the projections i_t^+ and i_t^+, respectively. Thus, the parents of a given random variable will be the projections in the three past timelike related adjacent minimal double cones, the descendants of a random variable will be the projections in the future timelike related minimal double cones, etc. The pair $\big(\mathcal{G}(V), \mathcal{V}(V)\big)$ will form a *Bayesian network*.

The translation manual between the vocabulary of the theory of Bayesian networks and that of the stochastic local classical theory $\{\mathcal{N}(V), V \in \mathcal{K}^m\}$ is shown in the following table:

Theory of Bayesian networks	**Stochastic local classical theory**
Bayesian network $\big(\mathcal{G}(V), \mathcal{V}(V)\big)$	Associated to every $V \in \mathcal{K}^m$
Causal graph $\mathcal{G}(V)$	Local von Neumann algebra $\mathcal{N}(V)$ with $V \in \mathcal{K}^m$
Vertices	Minimal double cones in V
Arrows	Pointing to future timelike related adjacent minimal double cones
Random variables $\mathcal{V}(V)$	Projections localized in the minimal double cones contained in V
Parents	Projections in past timelike related adjacent minimal double cones
Ancestors	Projections in past timelike related minimal double cones
Descendants	Projections in future timelike related minimal double cones
Causal Markov Condition	Bell's local causality plus Markovity

The last line of the table contains the central point of our discussion, namely:

1. The Causal Markov Condition is a consequence of Bell's local causality and Markovity when applied to the parents of a random variable.
2. Bell's local causality/Markovity are consequences of the Causal Markov Condition, since the set of random variables localized in a region satisfying Requirements (i)-(iii)/(iii') is d-separating.

We prove the first claim in the following proposition and illustrate the second in the subsequent examples.

Proposition 4. Let $\{\mathcal{N}(V), V \in \mathcal{K}^m\}$ be the stochastic local classical theory introduced above satisfying local causality and Markovity. Then for any pair $\big(\mathcal{G}(V), \mathcal{V}(V)\big)$ associated to any $V \in \mathcal{K}^m$ the Causal Markov Condition holds.

Proof. First we prove Causal Markov Condition for *non-relatives* which follows from the theory being locally causal. Let $V \in \mathcal{K}^m$ and let $V^m(t,i)$ and $V^m(s,j)$ be two minimal double cones in V such that $i < j$. Suppose that $V^m(t,i)$ and $V^m(s,j)$ are spatially separated (non-relatives), that is $|j-i| > |s-t|$. Without loss of generality we also can assume that $t = \frac{1}{2}$ and $s \geqslant t$, as depicted in Fig. 4.15. We are to show that the Causal Markov Condition (5.1) holds for $X = i_1$ and $Y = j_s$ in the Bayesian network $\big(\mathcal{G}(V), \mathcal{V}(V)\big)$ associated to V.

First, observe the parents of the variable i_1 are $(i-\frac{1}{2})$, i and $(i+\frac{1}{2})$. Thus, the Causal Markov Condition (5.1) reads as follows:

$$p\left(i_1 \middle| (i-\frac{1}{2}) i (i+\frac{1}{2})\, j_s\right) = p\left(i_1 \middle| (i-\frac{1}{2}) i (i+\frac{1}{2})\right) \tag{4.45}$$

or equivalently

$$p\left(i_1 j_s \middle| (i-\frac{1}{2}) i (i+\frac{1}{2})\right) = p\left(i_1 \middle| (i-\frac{1}{2}) i (i+\frac{1}{2})\right) p\left(j_s \middle| (i-\frac{1}{2}) i (i+\frac{1}{2})\right) \tag{4.46}$$

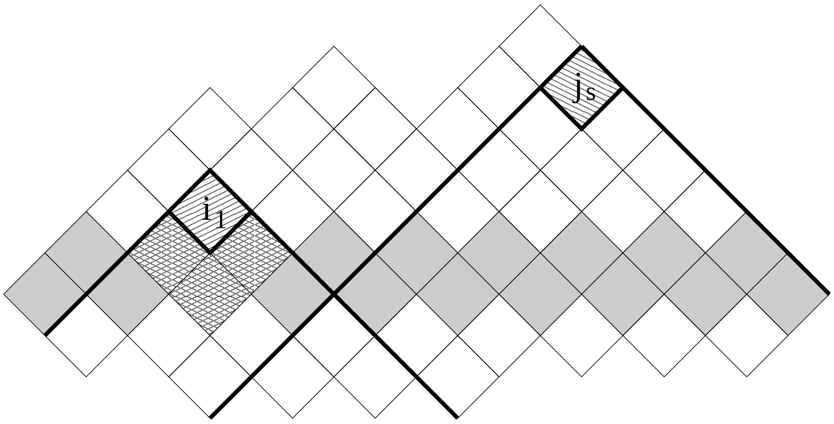

Figure 4.15: Causal Markov Condition follows from Bell's local causality relative to the parents.

Or in other words, the atomic events $(i-\frac{1}{2})i(i+\frac{1}{2})$ screen off the correlation between i_1 and j_s. But (4.46) does hold, since from (4.35)-(4.37) it follows that

$$
\begin{aligned}
p\left(i_1 \middle| (i-\frac{1}{2})i(i+\frac{1}{2})\right) &= \left[p\delta_{i_1,(i-\frac{1}{2})i(i+\frac{1}{2})} + (1-p)\delta_{-i_1,(i-\frac{1}{2})i(i+\frac{1}{2})}\right] \qquad (4.47)\\
p\left(j_s \middle| (i-\frac{1}{2})i(i+\frac{1}{2})\right) &= \sum_{(i-\frac{1}{2}),\ldots,(j+s+\{j+\frac{1}{2}\})} \Big[q_m\delta_{j_s,(j-s+\{j\})\ldots(j+s-\{j\})}\\
&\quad + (1-q_m)\delta_{-j_s,(j-s+\{j\})\ldots(j+s-\{j\})}\Big] p_{(i-\frac{1}{2})i(i+\frac{1}{2})(j-s+\{j\})\ldots(j+s-\{j\})} \qquad (4.48)\\
p\left(i_1 j_s \middle| (i-\frac{1}{2})i(i+\frac{1}{2})\right) &= \left[p\delta_{i_1,(i-\frac{1}{2})i(i+\frac{1}{2})} + (1-p)\delta_{-i_1,(i-\frac{1}{2})i(i+\frac{1}{2})}\right] \times\\
&\quad \sum_{(i-\frac{1}{2}),\ldots,(j+s+\{j+\frac{1}{2}\})} \Big[q_m\delta_{j_s,(j-s+\{j\})\ldots(j+s-\{j\})}\\
&\quad + (1-q_m)\delta_{-j_s,(j-s+\{j\})\ldots(j+s-\{j\})}\Big] p_{(i-\frac{1}{2})i(i+\frac{1}{2})(j-s+\{j\})\ldots(j+s-\{j\})} \qquad (4.49)
\end{aligned}
$$

Next we prove Causal Markov Condition for *ancestors* which follows from the theory being Markov. Let again $V \in \mathcal{K}^m$ and let $V^m(t,i)$ and $V^m(s,j)$ be two minimal double cones in V such that $V^m(s,j)$ is in the causal past (is an ancestor) of $V^m(t,i)$, that is $|j-i| \leqslant |s-t|$. Again, we can assume that $t = \frac{1}{2}$ and $s \geqslant t$, as depicted in Fig. 4.16. To prove (4.45) just observe that both sides equal to

$$p\delta_{i_1,(i-\frac{1}{2})i(i+\frac{1}{2})} + (1-p)\delta_{-i_1,(i-\frac{1}{2})i(i+\frac{1}{2})}$$

This completes the proof. ∎

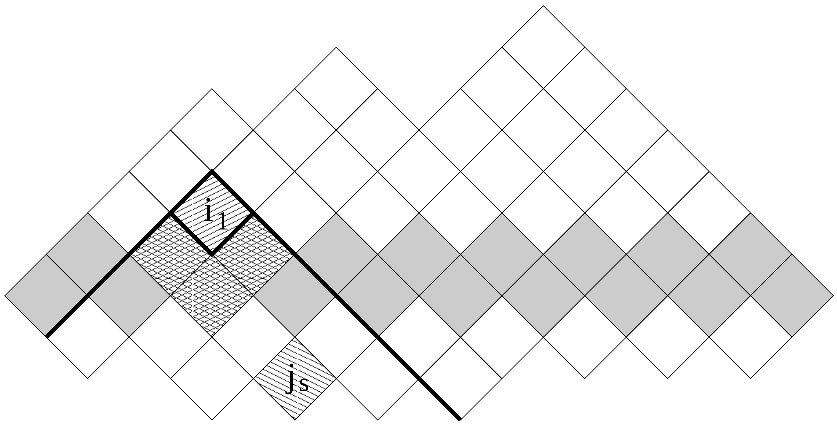

Figure 4.16: Causal Markov Condition follows from Markovity relative to the parents.

Thus, the Causal Markov Condition is a special case of Bell's local causality and Markovity in the stochastic local classical theory $\{\mathcal{N}(V), V \in \mathcal{K}^m\}$, namely when V_C is a special spacetime region: the union of the three parental minimal double cones, that is minimal double cones adjacent to a given minimal double cone from below. We stress again that Causal Markov Condition is a composition of two screening-off conditions: one for the ancestors and the other for the non-relatives. The first is the consequence of Markovity, the second is the consequence of local causality.

Now, we go over to our inverse claim, namely that Bell's local causality/Markovity are consequences of the Causal Markov Condition, since the set of random variables localized in a region V_C satisfying Requirements (i)-(iii)/(iii') is d-separating. Here we do not prove this claim generally, but only illustrate the connection of Requirements (i)-(iii) in the definition of local causality to d-separation on our previous two examples.

Example 1. Consider the smallest region $V \in \mathcal{K}^m$ in our Example 1 (in Section 4) containing the superluminally correlating events i_1^+ and j_1^+ with $j = i + 2 \in \mathbb{N} + \frac{1}{2}$ and a region V_C satisfying Requirements (i)-(iii) in the definition of local causality. (See Fig. 4.17.)

Now, consider the Bayesian network $\big(\mathcal{G}(V), \mathcal{V}(V)\big)$ associated to this V. The causal graph of the network is illustrated in Fig. 4.18. Let the variables be $X = i_1$, $Y = j_1$ and the subset $\mathcal{Z}$ be defined as:

$$\mathcal{Z} := \left\{(i-1), (i-\frac{1}{2}), i, (i+\frac{1}{2}), (i+1)\right\}$$

In other words, $\mathcal{Z}$ contains the random variables associated to the minimal double cones of V_C.

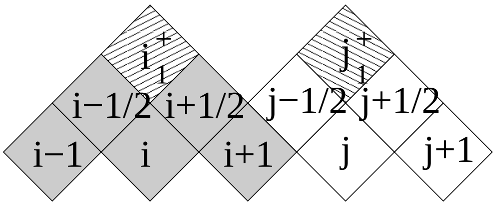

Figure 4.17: The smallest region containing the scenario of Example 1.

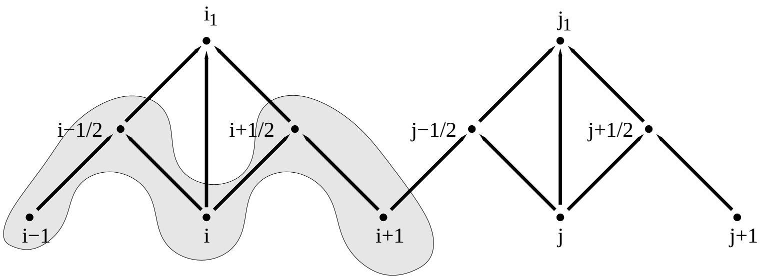

Figure 4.18: A d-separating scenario.

Now, $\mathcal{Z}$ d-separates i_1 and j_1 in $\mathcal{G}(V)$, since for every path $\mathcal{P}$ connecting i_1 and j_1 in $\mathcal{G}(V)$ there is a non-collider in $\mathcal{Z}$, namely, $(i+1)$. Therefore, i_1 and j_1 are probabilistically independent conditional upon any atomic event

$$(i-1)^{\pm}(i-\frac{1}{2})^{\pm}i^{\pm}(i+\frac{1}{2})^{\pm}(i+1)^{\pm}$$

This fact is the Bayesian network analogon of the situation illustrated in Fig. 8.16 where V_C is such that there is *no* minimal double cone above V_C in the intersection of the causal past of the correlating events. As said before, this is due to the fact that V_C satisfies Requirement (iii) in the definition of local causality. If Requirement (iii) does not fulfil, region V_C turns into d-connecting, as is shown in the next example.

Example 2. Consider the smallest region $V \in \mathcal{K}^m$ in our Example 2 containing the superluminally correlating events i_2^+ and j_2^+ with $j = i + 2 \in \mathbb{N} + \frac{1}{2}$ and a region V_C still in the causal past of i_2^+ but *not* satisfying Requirement (iii). (See Fig. 4.19.)

The causal graph $\mathcal{G}$ of the network is illustrated in Fig. 4.20. Let the variables be $X = i_2$, $Y = j_2$ and let

$$\mathcal{Z} := \left\{(i-\frac{3}{2}), (i-1), (i-\frac{1}{2}), i, (i+\frac{1}{2}), (i+1), (i+\frac{3}{2}) = (j-\frac{1}{2})\right\}$$

again a subset containing the random variables associated to the minimal double cones within V_C.

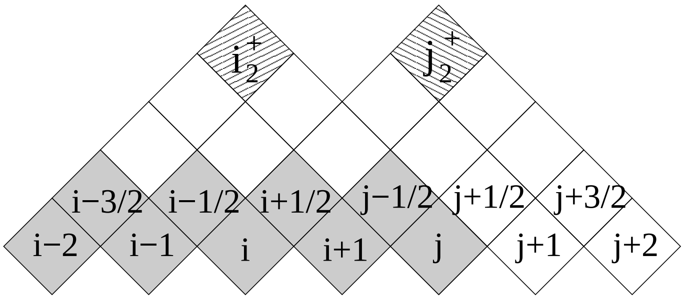

Figure 4.19: The smallest region containing the scenario of Example 2.

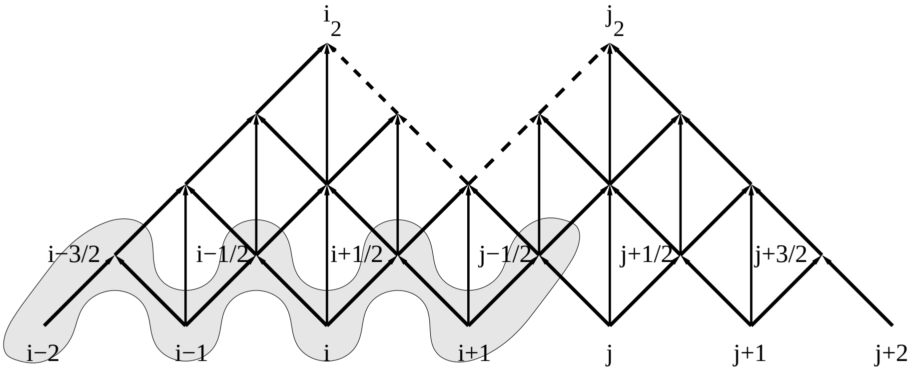

Figure 4.20: A d-connecting scenario.

Now, $\mathcal{Z}$ does *not* d-separate i_2 and j_2 in $\mathcal{G}$, since the path

$$\mathcal{P} := \left\{ i_2, (i+\frac{1}{2})_1, (i+1)_1, (j-\frac{1}{2})_1, j_2 \right\}$$

(denoted by a broken line in Fig. 4.20) connecting i_2 and j_2 in $\mathcal{G}(V)$ contains only non-colliders which are outside $\mathcal{Z}$. Therefore, the probabilistic independence of i_1 and j_1 conditional upon the atomic events

$$(i-\frac{3}{2})^{\pm}(i-1)^{\pm}(i-\frac{1}{2})^{\pm}i^{\pm}(i+\frac{1}{2})^{\pm}(i+1)^{\pm}(i+\frac{3}{2})^{\pm}$$

is not ensured by the Causal Markov Condition (and if the graph is faithful, it is even excluded). This fact is the Bayesian network analogon of the situation illustrated in Fig. 8.17 where V_C *does not* satisfy Requirement (iii) in the definition of local causality.

These examples point in the same direction: the Causal Markov Condition and the d-separation together ensure that Bell's local causality will hold for the atomic projections localized in a region satisfying Requirements (i)-(iii). Moreover, they also show that Requirements (iii) is a necessary condition.

4.6 Conclusions

In the paper I was arguing, based on a simple stochastic local classical model, that Bell's local causality, read in an appropriate way, is a Causal Markov Condition. I have not though provided a general proof. This would amount to solve the following

Open problem. Let $\{\mathcal{N}(V), V \in \mathcal{K}\}$ be a discrete local physical theory, discrete in the sense that every $V \in \mathcal{K}$ contains only a finite number of elements of $\mathcal{K}$ and the local von Neumann algebras $\mathcal{N}(V)$ are finite. Construct the Bayesian network $\big(\mathcal{G}(V), \mathcal{V}(V)\big)$ associated to a region V in $\mathcal{K}$. Prove (or falsify) that $\{\mathcal{N}(V), V \in \mathcal{K}\}$ is Markov and locally causal in Bell's sense *iff* $\big(\mathcal{G}(V), \mathcal{V}(V)\big)$ fulfils the Causal Markov Condition for every $V \in \mathcal{K}$.

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Chapter 5

Bell's local causality is a d-separation criterion

This paper aims to motivate Bell's notion of local causality by means of Bayesian networks. In a locally causal theory any superluminal correlation should be screened off by atomic events localized in any so-called *shielder-off region* in the past of one of the correlating events. In a Bayesian network any correlation between non-descendant random variables are screened off by any so-called *d-separating set* of variables. We will argue that the shielder-off regions in the definition of local causality conform in a well defined sense to the d-separating sets in Bayesian networks.

5.1 Introduction

John Bell's notion of local causality is one of the central notions in the foundations of relativistic quantum physics. Bell himself has returned to the notion of local causality from time to time providing a more and more refined formulation for it. The final formulation stems from Bell's posthumously published paper "La nouvelle cuisine." It reads as follows:[1]

> A theory will be said to be locally causal if the probabilities attached to values of local beables in a space-time region V_A are unaltered by specification of values of local beables in a space-like separated region V_B, when what happens in the backward light cone of V_A is already sufficiently specified, for example by a full specification of local beables in a space-time region V_C. (Bell, 1990/2004, p. 239-240)

The figure Bell is attaching to his formulation of local causality is reproduced in Fig. 5.1 with Bell's original caption. In a rough translation, a theory is locally causal if any superluminal correlation can be screened-off by a "full specification of local beables in a space-time region" in the past of one of the correlating events.

[1] For the sake of uniformity we slightly changed Bell's notation and figure.

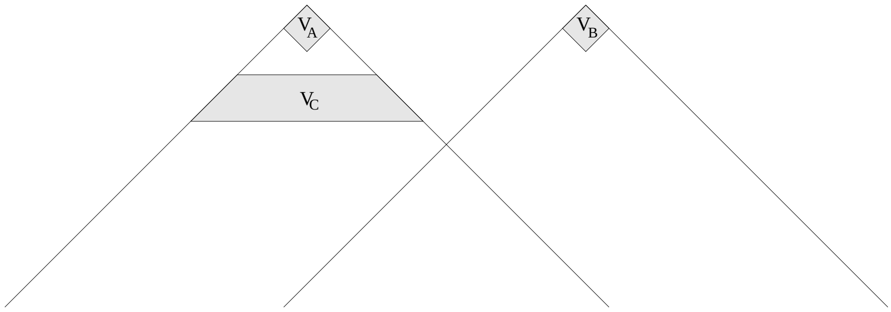

Figure 5.1: Full specification of what happens in V_C makes events in V_B irrelevant for predictions about V_A in a locally causal theory.

The terms in quotation marks, however, need clarification. What are "local beables"? What is "full specification" and why is it important? Which are those regions in spacetime which, if fully specified, render superluminally correlating events probabilistically independent? The first two questions have attracted much interest among philosophers of science. As Bell puts it, "*be*ables of the theory are those entities in it which are, at least tentatively, to be taken seriously, as corresponding to something real" (Bell, 1990/2004, p. 234). Furthermore, "it is important that events in V_C be specified completely. Otherwise the traces in region V_B of causes of events in V_A could well supplement whatever else was being used for calculating probabilities about V_A" (Bell, 1990/2004, p. 240).

The third question, however, concerning the localization of the screener-off regions has gained much less attention in the literature. How to characterize the regions which region V_C in Fig. 5.1 is an example of? Bell's answer is instructive but brief: "It is important that region V_C completely shields off from V_A the overlap of the backward light cones of V_A and V_B." (Bell, 1990/2004, p. 240) But why to shield off the common past of the correlating events? Why the region V_C cannot be in the remote past of V_A as for example in Figure 5.2? Well, intuition dictates

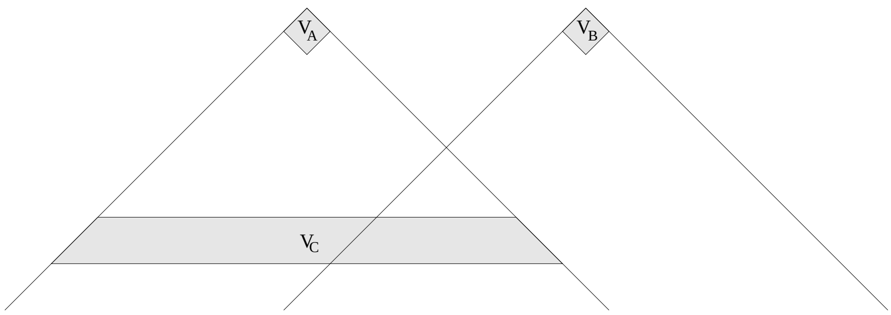

Figure 5.2: A *not* completely shielding-off region V_C.

that in this latter case some event might occur *above* the shielder-off region but still *within* the common past establishing a correlation between events in V_A and V_B. This intuition is correct. The aim of this paper, however, is to provide a more precise explanation for the localization of the shielder-off regions in spacetime. This explanation will consists in drawing a parallel between local physical theories and Bayesian networks. It will turn out that *the shielder-off regions in the definition of local causality play an analogous role to the so-called d-separating sets of random variables in Bayesian networks.*

There is a renewed interest in Bell's notion of local causality (Norsen, 2009, 2011; Maudlin 2014), its relation to separability (Henson, 2013b); the role of full specification in local causality (Seevinck and Uffink, 2011; Hofer-Szabó 2015a); its role in relativistic causality (Butterfield 2007; Earman and Valente, 2014; Rédei 2014); its status as a local causality principle (Henson, 2005; Rédei and San Pedro, 2012; Henson 2013a). A similar closely related topic, the Common Cause Principle is also given much attention (Rédei 1997; Rédei and Summers 2002; Hofer-Szabó and Vecsernyés 2012a, 2013a). On the other hand, there is also an intensive discussion on the applicability of the Causal Markov Condition in the EPR scenario (Glymour, 2006; Suárez and Iniaki, 2011; Hausman and Woodward, 1999; Suárez, 2013; Hofer-Szabó, Rédei and Szabó, 2013). Despite the rich and growing literature on the topic I am unaware of any work relating Bayesian networks and especially d-separation directly to local causality. This paper intends to fill this gap. For a precursor of this paper investigating Causal Markov Condition in a specific local physical theory see (Hofer-Szabó, 2015b). For a comprehensive formally rigorous investigation of the relation of Bell's local causality to the Common Cause Principle and other relativistic locality concepts see (Hofer-Szabó and Vecsernyés, 2015); for a more philosopher-friendly version see (Hofer-Szabó and Vecsernyés, 2016).

In the paper we will proceed as follows. In Section 2 we introduce the basics of the theory of Bayesian networks and the notion of d-separation and m-separation. In Section 3 we define the notion of a local physical theory and formulate Bell's notion of local causality within this framework. We prove our main claim in Section 4 and conclude in Section 5.

5.2 Bayesian networks and d-separation

A *Bayesian network* (Pearl, 2000; Glymour, Scheines and Spirtes, 2000) is a pair $(\mathcal{G}, \mathcal{V})$ where $\mathcal{G}$ is a directed acyclic graph and $\mathcal{V}$ is a set of random variables on a classical probability space (X, Σ, p) such that the elements $A, B \ldots$ of $\mathcal{V}$ are represented by the vertices of $\mathcal{G}$ and the arrows (directed edges) $A \to B$ on the graph represent that A is *causally relevant* for B. Two vertices are called *adjacent* if they are connected by an arrow. For a given $A \in \mathcal{V}$, the set of vertices that have directed edges in A is called the *parents* of A, denoted by $Par(A)$; the set of vertices from which a directed paths is leading to A is called the *ancestors* of A, denoted by $Anc(A)$; and finally the set of vertices that are endpoints of a directed paths from A is called the *descendants* of A, denoted by $Des(A)$. For a set $\mathcal{C}$ of vertices $Par(\mathcal{C})$, $Anc(\mathcal{C})$ and $Des(\mathcal{C})$ are defined similarly.

The set $\mathcal{V}$ is said to satisfy the *Causal Markov Condition* relative to the graph $\mathcal{G}$ if for any $A \in \mathcal{V}$ and any $B \notin Des(A)$ the following is true:

$$p(A \,|\, Par(A) \wedge B) \quad = \quad p(A \,|\, Par(A)) \tag{5.1}$$

or equivalently

$$p(A \wedge B \,|\, Par(A)) \quad = \quad p(A \,|\, Par(A))\, p(B \,|\, Par(A)) \tag{5.2}$$

That is conditioning on its parents any random variable will be probabilistically independent from any of its non-descendant. Non-descendants can be of two types: either ancestors or *collaterals* (non-descendants and non-ancestors). As we will see, being independent of collaterals is what relates the Causal Markov Condition to Bell's local causality.

Causal Markov Condition establishes a special conditional independence relation between some random variables of $\mathcal{V}$. But there are many other conditional independences. In a faithful Bayesian network these other conditional independences are all implied by the Causal Markov Condition by means of the so-called *d-separation* criterion. Let $\mathcal{P}$ be a *path* in $\mathcal{G}$, that is a sequence of adjacent vertices. A variable E on $\mathcal{P}$ is a *collider* if there are arrows to E from both its neighbors on $\mathcal{P}$ ($D \to E \leftarrow F$). Now, let $\mathcal{C}$ be a set of vertices and let A and B two different vertices not in $\mathcal{C}$. The vertices A and B are said to be *d-connected* by $\mathcal{C}$ in $\mathcal{G}$ iff there exists a path $\mathcal{P}$ between A and B such that every non-collider on $\mathcal{P}$ is not in $\mathcal{C}$ and every collider is in $Anc(\mathcal{C})$. A and B are said to be *d-separated* by $\mathcal{C}$ in $\mathcal{G}$, iff they are not d-connected by $\mathcal{C}$ in $\mathcal{G}$.

The intuition behind d-separation is the following. A vertex E on a path (not at the endpoints) can be either a *collider* ($D \to E \leftarrow F$), an *intermediary cause* ($D \to E \to F$) or a *common cause* ($D \leftarrow E \to F$). The idea here is that only intermediary and common causes (together called *non-colliders*) can transmit causal dependence and hence establish probabilistic dependence. This dependence can be *blocked* by conditioning on the non-collider. Colliders behave just the opposite way. They represent two events causing a common effect. These two causes are causally and probabilistically independent, but become dependent upon conditioning on their common effect. Moreover, they also become dependent upon conditioning on any of the descendants of the effect. Putting these together, the causal dependence on a path $\mathcal{P}$ connecting two vertices is blocked by a set $\mathcal{C}$ if either there is at least one non-collider on $\mathcal{P}$ which is in $\mathcal{C}$ or there is at least one collider E on $\mathcal{P}$ such that either E or a descendant of E is not in $\mathcal{C}$. The two vertices are d-separated by $\mathcal{C}$ if causal dependence is blocked on every path connecting them.

As an example for d-connection and d-separation consider the causal graph in Fig. 5.3. (The arrows are directed to up, left up and right up.) Let A be the left "peak" and B the right "peak" in the graph and let $\mathcal{C}$, $\mathcal{C}'$ and $\mathcal{C}''$ be the sets shown in the figure containing 3, 5 and 7 vertices, respectively. Then A and B are d-separated by $\mathcal{C}$ since the parents are always d-separating due to the Causal Markov Condition. A and B are d-separated also by $\mathcal{C}'$ since for every path connecting the

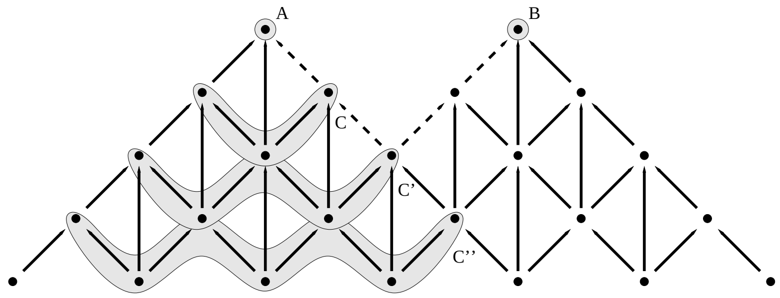

Figure 5.3: A and B are d-separated by $\mathcal{C}$ and $\mathcal{C}'$ but d-connected by $\mathcal{C}''$.

peaks there is a non-collider in $\mathcal{C}'$. However, A and B are d-connected by $\mathcal{C}''$ since there is a path (denoted by a broken line in Fig. 5.3) connecting the peaks which contains only non-colliders outside $\mathcal{C}''$. Consequently, the following probabilistic relations hold:

$$p(A \wedge B \,|\, \mathcal{C}) \;=\; p(A \,|\, \mathcal{C})\, p(B \,|\, \mathcal{C}) \tag{5.3}$$

$$p(A \wedge B \,|\, \mathcal{C}') \;=\; p(A \,|\, \mathcal{C}')\, p(B \,|\, \mathcal{C}') \tag{5.4}$$

$$p(A \wedge B \,|\, \mathcal{C}'') \;\neq\; p(A \,|\, \mathcal{C}'')\, p(B \,|\, \mathcal{C}'') \tag{5.5}$$

Looking at in Fig. 5.3, what stands out immediately is that a set which is too far in the causal past of A cannot d-separate A from a collateral event since there might be paths connecting them “above” the set. As we will see, a similar moral will be valid in case of local causality: regions with are too far in the causal past of an event cannot screen it off from a spacelike separated event since there might be events “above” the region which can establish correlation between them.

In analyzing local causality sometimes we need to go beyond directed acyclic graphs. A graph which may contain both directed ($A \to B$) and bi-directed ($A \leftrightarrow B$) edges is called *mixed*. The d-separation criterion extended to mixed acyclic graphs is called *m-separation*. (Richardson and Spirtes, 2002; Sadeghi and Lauritzen, 2014) Two vertices A and B are said to be *m-connected* by $\mathcal{C}$ in a mixed acyclic graph $\mathcal{G}$ iff there exists a path $\mathcal{P}$ between A and B such that every non-collider on $\mathcal{P}$ is not in $\mathcal{C}$ and every collider is in $Anc(\mathcal{C})$. A and B are said to be *m-separated* by $\mathcal{C}$ in $\mathcal{G}$, iff they are not m-connected by $\mathcal{C}$ in $\mathcal{G}$. In a directed acyclic graph m-separation reduces to d-separation.

An example for a mixed acyclic graph is depicted in Fig. 5.4. Here the bi-directed edges are represented by dotted lines. Again, let A be the left “peak” and B the right “peak” in the graph and let $\mathcal{C}$, $\mathcal{C}'$ and $\mathcal{C}''$ be the sets shown in the figure containing 3, 5 and 7 vertices, respectively. Then A and B are m-separated by $\mathcal{C}$ but m-connected by both $\mathcal{C}'$ and $\mathcal{C}''$. The connecting path is the shortest path

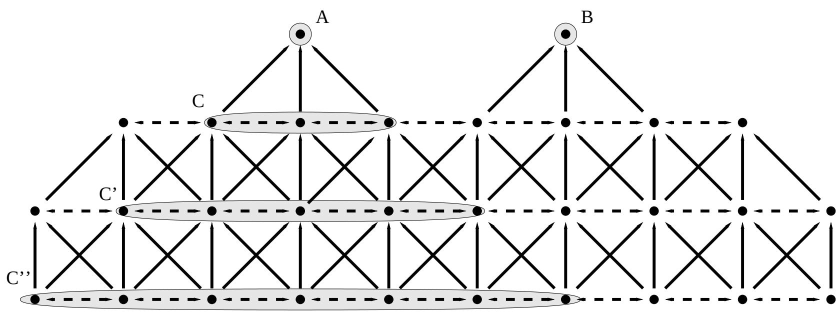

Figure 5.4: A and B are m-separated by $\mathcal{C}$ but m-connected by both $\mathcal{C}'$ and $\mathcal{C}''$.

connecting A and B.

Now, let us connect the terminology of Bayesian networks to that of standard physics. Before doing that note that probability is commonly interpreted in Bayesianism subjectively as partial belief and in physics objectively as long-run relative frequency. This interpretative difference, however, does not undermine the analogy between local causality and d-separation, since Bayesian networks are well open to statistical interpretation and, conversely, there is a growing tendency to understand quantum physics in a subjectivist way.

Let us start with random variables. A random variable is a real-valued Borel-measurable function on X. Each random variable $A \in \mathcal{V}$ generates a sub-σ-algebra of Σ by the inverse image of the Borel sets:

$$\sigma(A) := \left\{A^{-1}(b) \,|\, b \in \mathcal{B}(\mathbf{R})\right\} \tag{5.6}$$

Similarly, each set $\mathcal{C}$ of n random variables generates a sub-σ-algebra of Σ by the inverse image of the n-dimensional Borel sets:

$$\sigma(\mathcal{C}) := \left\{(C_1, C_2 \dots C_n)^{-1}(b) \,|\, C_i \in \mathcal{C},\ b \in \mathcal{B}(\mathbf{R}^n)\right\} \tag{5.7}$$

From this perspective d-separation tells us which sub-σ-algebras are probabilistically independent conditioned on which other sub-σ-algebras of Σ.

Now, instead of using σ-algebras it is more instructive to use a richer structure in physics, namely *von Neumann algebras.* Consider the characteristic functions on X projecting on the elements of Σ, called *events.* The set $\{\chi_S \,|\, S \in \Sigma\}$ of characteristic functions generates an abelian von Neumann algebra, namely $\mathcal{L}^\infty(X, \Sigma, p)$, the space of essentially bounded complex-valued functions on X. Starting from the characteristic functions of the sub-σ-algebra $\sigma(A)$, one arrives at a subalgebra of $\mathcal{L}^\infty(X, \Sigma, p)$. Denote this abelian von Neumann algebra determined by the random variable A by $\mathcal{N}_A$. Similarly, denote by $\mathcal{N}_\mathcal{C}$ the von Neumann algebra determined by a set $\mathcal{C}$ of random variables.

Instead of using a probability measure on Σ or on a sub-σ-algebra $\sigma(A)$, one can also use a state on the corresponding von Neumann algebra $\mathcal{N}_A$. A *state* ϕ is a positive linear functional of norm 1 on a von Neumann algebra. States on $\mathcal{N}_A$ and probability measures on $\sigma(A)$ mutually determine one another: a state restricted to the characteristic functions in $\mathcal{N}_A$ is a probability measure on $\sigma(A)$; and vice versa, integrating elements of $\mathcal{N}_A$ according to a probability measure on $\sigma(A)$ yields a state on $\mathcal{N}_A$.

Therefore, a conditional independence between random variables A and B given the set $\mathcal{C}$

$$p(A \wedge B \,|\, \mathcal{C}) = p(A \,|\, \mathcal{C}) \, p(B \,|\, \mathcal{C}) \tag{5.8}$$

can be rewritten as follows: for any projection $A \in \mathcal{N}_A$, $B \in \mathcal{N}_B$ and $C \in \mathcal{N}_\mathcal{C}$:

$$\frac{\phi(A \wedge B \wedge C)}{\phi(C)} = \frac{\phi(A \wedge C)}{\phi(C)} \frac{\phi(B \wedge C)}{\phi(C)} \tag{5.9}$$

Although in this paper we stay at the classical level, the theory of von Neumann algebras is wide enough to incorporate also quantum physics. In this case the von Neumann algebras are nonabelian. The events, just like in the classical case, are represented by projections of the von Neumann algebras. In the quantum case conditional independence between the projection $A \in \mathcal{N}_A$ and $B \in \mathcal{N}_B$ given $C \in \mathcal{N}_\mathcal{C}$ reads as follows:

$$\frac{\phi(CABC)}{\phi(C)} = \frac{\phi(CAC)}{\phi(C)} \frac{\phi(CBC)}{\phi(C)} \tag{5.10}$$

which in the classical case reduces to (5.9).

The last point in converting the formalism of Bayesian networks into physics, is to swap the causal graph for spacetime. We can then replace the causal relations embodied in the causal graph by spatiotemporal relations of a given spacetime. Instead of saying that a random variable is the *ancestor* of another variable we will then say that an event is in the *past* of the other. But to do so first we need to *localize* events in spacetime that is we need to have an association of algebras of events to spacetime regions. Such a principled association is offered by the formalism of algebraic quantum field theory. Hence, in the next section we will introduce some elements of algebraic quantum field theory which is indispensable for our purpose which is to come up with a mathematically precise definition of Bell's notion of local causality.

5.3 Bell's local causality in a local physical theory

Let $\mathcal{M}$ be a globally hyperbolic spacetime and let $\mathcal{K}$ be a covering collection of bounded, globally hyperbolic subspacetime regions of $\mathcal{M}$ such that $(\mathcal{K}, \subseteq)$ is a directed poset under inclusion $\subseteq$. A *local physical theory* is a net $\{\mathcal{A}(V), V \in \mathcal{K}\}$ associating algebras of events to spacetime regions which satisfies *isotony* and *microcausality* defined as follows (Haag, 1992; Halvorson 2007; Hofer-Szabó and Vecsernyés 2015, 2016):

Isotony. The net of local observables is given by the isotone map $\mathcal{K} \ni V \mapsto \mathcal{A}(V)$ to unital C^*-algebras, that is $V_1 \subseteq V_2$ implies that $\mathcal{A}(V_1)$ is a unital C^*-subalgebra of $\mathcal{A}(V_2)$. The *quasilocal algebra* $\mathcal{A}$ is defined to be the inductive limit C^*-algebra of the net $\{\mathcal{A}(V), V \in \mathcal{K}\}$ of local C^*-algebras.

Microcausality: $\mathcal{A}(V')' \cap \mathcal{A} \supseteq \mathcal{A}(V), V \in \mathcal{K}$, where primes denote spacelike complement and algebra commutant, respectively.

If the quasilocal algebra $\mathcal{A}$ of the local physical theory is commutative, we speak about a *local classical theory*; if $\mathcal{A}$ is noncommutative, we speak about a *local quantum theory.* For local classical theories microcausality fulfills trivially.

Given a state ϕ on the quasilocal algebra $\mathcal{A}$, the corresponding GNS representation $\pi_\phi \colon \mathcal{A} \to \mathcal{B}(\mathcal{H}_\phi)$ converts the net of C^*-algebras into a net of C^*-subalgebras of $B(\mathcal{H}_\phi)$. Closing these subalgebras in the weak topology one arrives at a net of local von Neumann observable algebras: $\mathcal{N}(V) := \pi_\phi(\mathcal{A}(V))'', V \in \mathcal{K}$. The *net* $\{\mathcal{N}(V), V \in \mathcal{K}\}$ *of local von Neumann algebras* also obeys isotony and microcausality, hence we can also refer to it as a local physical theory.

Given a local physical theory, we can turn now to the definition of Bell's notion of local causality. Recall that according to Bell a theory is locally causal if any superluminal correlation is screened-off by a "full specification of local beables in a space-time region V_C" as shown in Fig. 5.1. As indicated in the Introduction we need to address three questions. What are "local beables"? What is "full specification"? Which are the shielder-off regions? The brief answer to the first two questions is the following. In a local physical theory a "local beable" in a region V is an *element* of the local von Neumann algebra $\mathcal{N}(V)$. A "full specification" of local beables in region V is an *atomic element* of the local von Neumann algebra $\mathcal{N}(V)$. In this paper we do not comment on these two answers. For a more thoroughgoing discussion on why we think this to be the correct translation of Bell's intuition into our framework see (Hofer-Szabó and Vecsernyés, 2015, 2016).

To the third question, which is the topic of our paper, the answer is this: a shielder-off region V_C is a region in the causal past of V_A which can block any causal influence on V_A arriving from the common past of V_A and V_B. But there is an ambiguity in this answer. Bell's Fig. 5.1 suggests that a shielder-off region should not intersect with the common past. Whereas the requirement of simply blocking causal influences from the past allows for also regions depicted in Fig. 5.5 intersecting with the common past. This means that one can define a *shielder-off region of V_A relative to V_B* either as a region V_C satisfying:

$\mathbf{L}_1 : V_C \subset J_-(V_A)$ (V_C is in the causal past of V_A),

$\mathbf{L}_2 : V_A \subset V_C''$ (V_C is wide enough such that its causal shadow contains V_A),

$\mathbf{L}_3^Q : V_C \subset V_B'$ (V_C is spacelike separated from V_B)

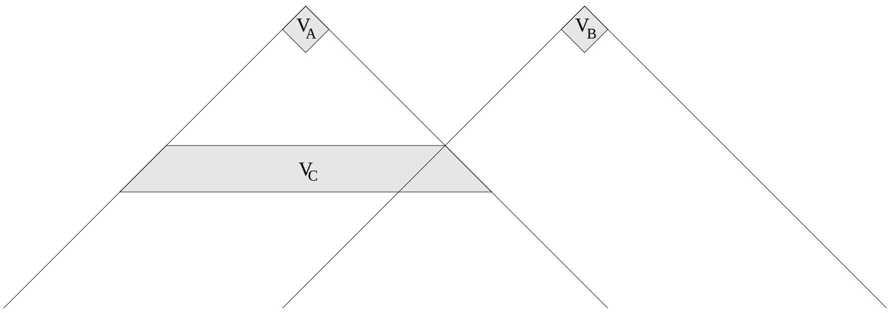

Figure 5.5: A completely shielding-off region V_C intersecting with the common past of V_A and V_B.

in tune with Bell's Fig. 5.1; or one can replace L_3^Q by the weaker requirement

> $\mathbf{L}_3^C$: $J_-(V_C) \supset J_-(V_A) \cap J_-(V_B)$ (The causal past of V_C contains the common past of V_A and V_B)

allowing for regions such as in Fig. 5.2. It turns out that (with respect to the Bell inequalities, see (Hofer-Szabó and Vecsernyés, 2012b, 2013b)) it is more appropriate to demand L_3^Q in case of a local *quantum* theory and L_3^C in case of a local *classical* theory (hence the superscripts). But note that as the covering regions become infinitely thin shrinking down to a Cauchy surface, requirement L_3^C coincides with requirement L_3^Q.

With all these considerations in mind Bell's notion of local causality in the framework of a local physical theory will be the following:

Definition 4. A local physical theory represented by a net $\{\mathcal{N}(V), V \in \mathcal{K}\}$ of von Neumann algebras is called *locally causal* (in Bell's sense), if

1. for any pair $A \in \mathcal{N}(V_A)$ and $B \in \mathcal{N}(V_B)$ of events represented by projections in spacelike separated regions $V_A, V_B \in \mathcal{K}$;
2. for every locally normal and faithful state ϕ establishing a correlation $\phi(AB) \neq \phi(A)\phi(B)$ between A and B;
3. for any spacetime shielder-off region V_C defined by requirements L_1, L_2 and L_3^Q/L_3^C;
4. for any event C in the set $\mathcal{C}$ of atomic events in $\mathcal{A}(V_C)$

the following screening-off condition holds:

$$\frac{\phi(CABC)}{\phi(C)} = \frac{\phi(CAC)}{\phi(C)} \frac{\phi(CBC)}{\phi(C)} \tag{5.11}$$

which for a local *classical* theory is equivalent to

$$p(A \wedge B \,|\, \mathcal{C}) = p(A \,|\, \mathcal{C})\, p(B \,|\, \mathcal{C}) \tag{5.12}$$

In short, a local physical theory is locally causal in Bell's sense if every superluminal correlation is screened off by all atomic events in all shielder-off region. (For many delicate questions such as what if the algebras are non-atomic, how this definition of local causality relates to the Common Cause Principle and the Bell inequalities see again (Hofer-Szabó and Vecsernyés, 2015, 2016).)

The question left is, however: *why* shielder-off regions are characterized by requirements L_1, L_2 and L_3^Q/L_3^C? To this we turn in the next Section.

5.4 Shielder-off regions are d-separating

The point we are going to make in this Section is that shielder-off regions in the definition of local causality conform to d-separating sets in directed acyclic graphs and to m-separating sets in mixed acyclic graphs.

First we show how a local physical theory gives rise to a causal graph. Consider a local *classical* theory $\{\mathcal{N}(V), V \in \mathcal{K}\}$ where the covering collection is induced by a partition $\mathcal{T}$ of a spacetime $\mathcal{M}$. By *partition* we mean a countable set of disjoint, bounded spacetime regions such that their union is $\mathcal{M}$. Whether we demand global hyperbolicity from the elements of the partition will turn out to play an important role in the type of the graph we can construct. For some specific globally hyperbolic coverings we will get directed acyclic graphs, otherwise only a mixed graph.

Let the vertices of the $\mathcal{G}$ be the regions in the partition, $\{V \in \mathcal{T}\}$. Denote the vertex corresponding to the region $V \in \mathcal{T}$ by A_V and the region corresponding to a vertex A by V_A. Similarly, denote the *set* of vertices corresponding to the region $V \in \mathcal{K}$ by $\mathcal{C}_V$ and the region corresponding to a set of vertices $\mathcal{C}$ by $V_\mathcal{C}$. Define the *ancestors* of a vertex B as:

$$Anc(B) := \{A \in \mathcal{V} \,|\, A \neq B,\ V_A \cap J_-(V_B) \neq \emptyset\}$$

and the *parents* of B, $Par(B)$, as those elements in $Anc(B)$ for which there is a causal curve connecting V_A and V_B directly (that is without entering a third region between them). Now, let there be an arrow $A \to B$ between vertex A and B in $\mathcal{T}$ if and only if $A \in Par(B)$. It will turn out that the type of the graph we obtain is crucially depending on the partition $\mathcal{T}$ of the spacetime. Let us see the different cases.

If $\mathcal{M}$ is the 1+1 dimensional Minkowski spacetime, then it can be covered by double cones of equal size. (See Fig. 5.6.) Double cones are globally hyperbolic. (For the details of this example see (Hofer-Szabó, 2015b).) The causal graph corresponding to this covering emerges simply by connecting the midpoints of those adjacent double cones which lie in the causal past of one another. What we get is just the directed acyclic graph depicted in Fig. 5.3 in Section 2.

Fig. 5.6 is a kind of "superposition" of a spacetime diagram and a Bayesian network. Consider for example region $V_{\mathcal{C}'}$. Reading Fig. 5.6 as a spacetime diagram, one sees that $V_{\mathcal{C}'}$ is a *shielder-off region* (similar to the one depicted in Fig. 5.5). Reading Fig. 5.6 as a causal graph, one observes that the set $\mathcal{C}'$ corresponding to $V_{\mathcal{C}'}$ (depicted in Fig. 5.3) is a *d-separating set*. Similarly, one can check that the

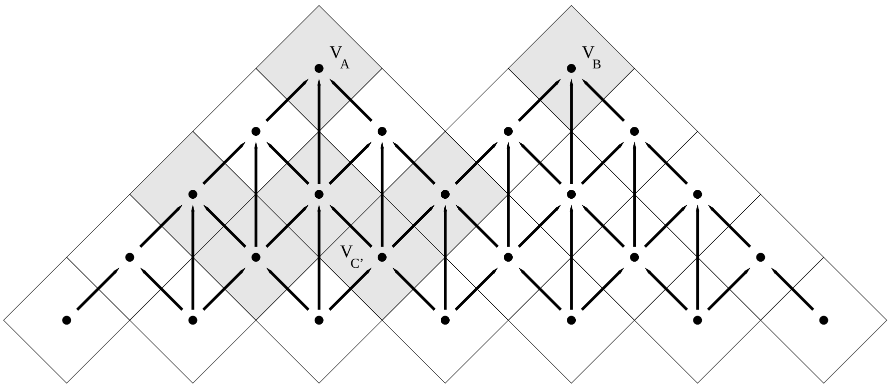

Figure 5.6: The directed acyclic graph generated by double cones of equal size covering the 1+1 dimensional Minkowski spacetime.

region associated to the d-separating set $\mathcal{C}$ in Fig. 5.3 is a shielder-off region and the region associated to the d-connecting set $\mathcal{C}''$ is not a shielder-off region.

A general spacetime $\mathcal{M}$ cannot be partitioned to globally hyperbolic regions, let alone to double cones. Still one can construct the causal graph corresponding to a partition $\mathcal{T}$. In Fig. 5.7 we illustrate such a construction where a 1+1 dimensional

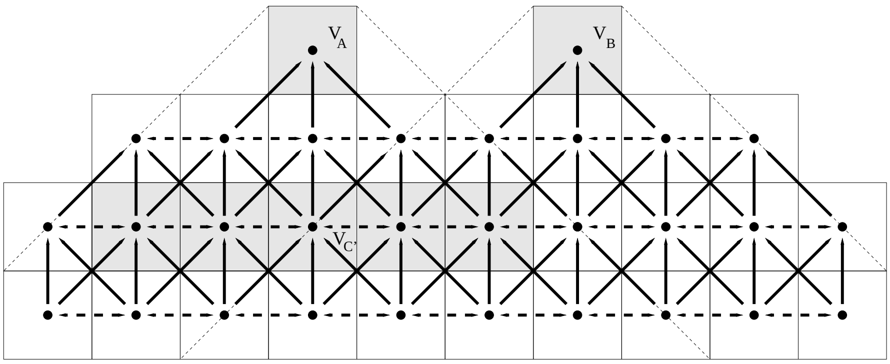

Figure 5.7: The mixed acyclic graph generated by boxes of equals size covering of the 1+1 dimensional Minkowski spacetime.

Minkowski spacetime is covered by boxes of equals size. (This example, in contrast to the previous one, can be generalized for a 3+1-dimensional Minkowski spacetime covered by 3+1-dimensional boxes of equals size.) The causal graph emerging from this construction is not a directed acyclic graph since it contains bi-directed edges: spacelike neighboring boxes will be *spouses*. What we get is a mixed acyclic graph depicted in Fig. 5.4. Again, confronting Fig. 5.4 and Fig. 5.7 one can see that the set $\mathcal{C}'$ is *not* an m-separating set and at the same time the corresponding region

$V_{C'}$ is *not* a shielder-off region of V_A relative to V_B.

The exact characterization of the graphs emerging from a different coverings of a given spacetime is a subtle question which we do not go into here. Instead we turn now to the construction of random variables. Let $\mathcal{N}(V)$ be the local von Neumann algebra associated to the spacetime region $V \in \mathcal{T}$. Denote by $\sigma(V)$ the sigma-algebra of the projections of $\mathcal{N}(V)$. Let the random variable (also denoted by) A_V associated to V be any Borel-measurable function from $\sigma(V)$ to $\mathcal{B}(\mathbf{R})$. Any state ϕ will then define a probability measure p on $\sigma(V)$ for any $V \in \mathcal{T}$ and, due to isotony of the net, also for any V which is a *finite* union of regions in $\mathcal{T}$. (Note that $\sigma(\mathcal{M})$ may not be a sigma-algebra since the quasilocal algebra $\mathcal{A}$ is not necessarily a von Neumann algebra, so it may not contain projections.)

In sum, any finite set of regions of a local classical theory $\{\mathcal{N}(V), V \in \mathcal{K}\}$ generated by a globally hyperbolic partition of $\mathcal{M}$ defines a Bayesian network $(\mathcal{G}, \mathcal{V})$. If global hyperbolicity is not required, then $\mathcal{G}$ is not a directed acyclic but only a mixed graph.

Now, we state and prove the main claim of the paper.

Proposition 5. Let G be a directed/mixed acyclic graph constructed from a local classical theory $\{\mathcal{N}(V), V \in \mathcal{K}\}$ where $\mathcal{K}$ is generated by a partition $\mathcal{T}$ of $\mathcal{M}$. Suppose that $\{\mathcal{N}(V), V \in \mathcal{K}\}$ is locally causal in the sense of Definition 4. Then for any shielder-off region V defined by L_1, L_2 and L_3^C, the corresponding set $\mathcal{C}_V$ is d-separating/m-separating.

Proof. Let A and B two collateral vertices in $\mathcal{G}$ corresponding to two spacelike separated regions V_A and V_B, respectively ($V_A, V_B \in \mathcal{T}$). Call a set $\mathcal{C}$ of random variables a *shielder-off set* (for A relative to B), if $V_{\mathcal{C}}$ is a shielder-off region (for V_A relative to V_B). Shielder-off sets block every directed path from $Anc(A) \wedge Anc(B)$, the set of common ancestors of A and B, to A (that is every directed path has to pass through $\mathcal{C}$).

We show that shielder-off sets are d-separating/m-separating. Let $\mathcal{C}$ be a shielder-off set for A relative to B. We have to show that $\mathcal{C}$ blocks every path connecting A and B. First consider those paths that contain no colliders. These paths need to pass through the set of common ancestors of A and B, $Anc(A) \wedge Anc(B)$. Hence, the shielder-off set $\mathcal{C}$ blocks them. So there remain only those paths to be blocked which contain at least one collider. It is easy to see that these latter paths need to contain at least one collider E such that $E \notin Anc(A)$. But then neither E nor any descendant of E is in $\mathcal{C}$, hence $\mathcal{C}$ blocks also these paths. ■■

The converse of Proposition 13 is not true: d-separating sets are not necessarily shielder-off sets. Tian, Paz, and Pearl (1998) list algorithms to find the so-called *minimal d-separating sets* for two random variables A and B, that is sets that are d-separating but taking away any vertex from the set they will cease to be d-separating. It turns out that any minimal d-separating set is sitting in the *union* of the ancestors of A and B (including also A and B), $Anc(A) \vee Anc(B) \vee A \vee B$. However, a minimal d-separating set need not satisfy relations L_1, L_2 and L_3^C. For example the sets $\mathcal{D}$, $\mathcal{D}'$ and $\mathcal{D}''$ in Fig. 5.8 are all minimal d-separating sets but not shielder-off regions for A relative to B.

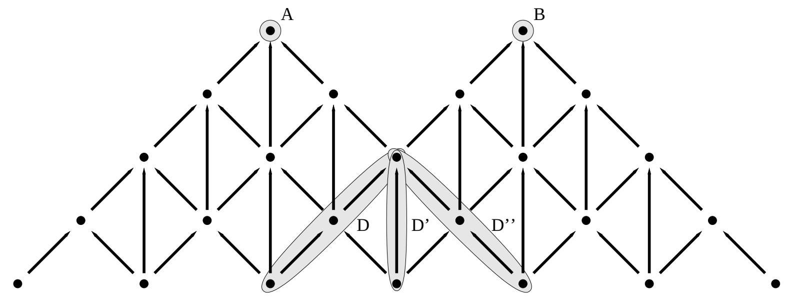

Figure 5.8: Minimal d-separating but not shielder-off regions.

At any event, shielder-off regions are d-separating, and this was to be shown in this paper.

5.5 Conclusions

The aim of the paper was to motivate Bell's definition of local causality by means of Bayesian networks. To this aim, first we constructed a causal graph from the covering collection of a spacetime. In certain cases the graph was a directed acyclic graph, in other cases only a mixed acyclic graph. Similarly, we have associated random variables to the local algebras of a local physical theory. By this move shielder-off regions turned out be specific d-separation (m-separating) sets on the causal graph. Hence, Bell's definition of local causality requiring that spacelike separated events should be screened-off by events in a shielder-off region turned out to be a d-separation criterion.

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Chapter 6

Local causality and complete specification: a reply to Seevinck and Uffink

A physical theory is called locally causal if any correlation between spacelike separated events is screened-off by local beables *completely specifying* an appropriately chosen region in the past of the events. In this paper I will define local causality in a clear-cut framework, called local physical theory which integrates both probabilistic and spatiotemporal entities. Then I will argue that, contrary to the claim of Seevinck and Uffink (2011), complete specification does not stand in contradiction to the free variable (no-conspiracy) assumption.

6.1 Introduction

Local causality is the idea that causal processes propagate though space continuously and with velocity less than the speed of light. John Stewart Bell formulates this intuition in a 1988 interview as follows:

> "[Local causality] is the idea that what you do has consequences only nearby, and that any consequences at a distant place will be weaker and will arrive there only after the time permitted by the velocity of light. Locality [= local causality] is the idea that consequences propagate continuously, that they don't leap over distances." (Mann and Crease, 1988)

Bell has returned to this intuitive idea of local causality from time to time and provided a more and more elaborate formulation of it. First he addressed the notion of local causality in his "The theory of local beables" delivered at the Sixth GIFT Seminar in 1975; later in a footnote added to his 1986 paper "EPR correlations and EPW distributions" intending to clean up the first version; and finally in the most elaborate form in his "La nouvelle cuisine" posthumously published in 1990. In this latter paper local causality obtains the following formulation:[1]

> "A theory will be said to be locally causal if the probabilities attached to values of local beables in a space-time region V_A are unaltered by specification of values of local beables in a space-like separated region V_B, when what happens in the backward light cone of V_A is already sufficiently specified, for example by a full specification of local beables in a space-time region V_C." (Bell, 1990/2004, p. 239-240)

We reproduce the figure Bell is attaching to his formulation in Fig. 6.1. (The captation is Bell's original.)

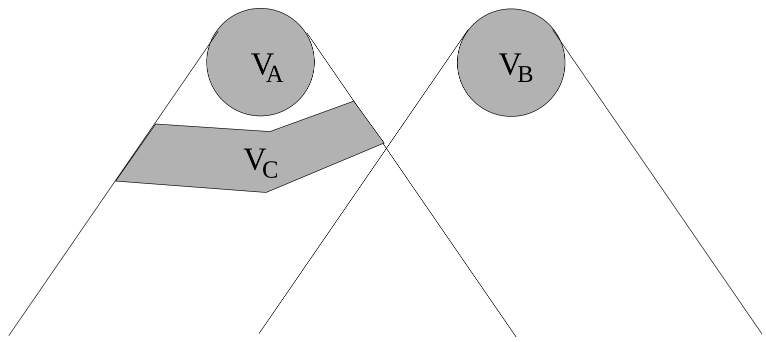

Figure 6.1: Full specification of what happens in V_C makes events in V_B irrelevant for predictions about V_A in a locally causal theory.

Some brief remarks concerning Bell's terminology are in place here (for a detailed analysis see (Norsen 2009, 2011)):

(i) The term "beable" in the quote is Bell's own neologism and is contrasted to the term "observable" used in quantum theory. "The *be*ables of the theory are those entities in it which are, at least tentatively, to be taken seriously, as corresponding to something real" (Bell, 1990/2004, p. 234).

(ii) Beables are to be local: "*Local* beables are those which are definitely associated with particular space-time regions. The electric and magnetic fields of classical electromagnetism, $\mathbf{E}(t,x)$ and $\mathbf{B}(t,x)$ are again examples." (p. 234).

(iii) Local beables in region V_C are to be "fully specified" in order to block causal influences arriving at V_A from the common past of V_A and V_B.

This latter point is of central importance and is also stressed by Bell:[2]

[1] For the sake of conformity with the rest of the paper I slightly changed Bell's notation and figure.

[2] But, to be fair, see (Bell 1980/2004, p. 106), (Bell 1980/2004, p. 152) and the above (Bell 1990/2004, p. 234) for Bell's hesitation on the issue.

> "It is important that region V_C completely shields off from V_A the overlap of the backward light cones of V_A and V_B. And it is important that events in V_C be specified completely. Otherwise the traces in region V_B of causes of events in V_A could well supplement whatever else was being used for calculating probabilities about V_A. The hypothesis is that any such information about V_B becomes redundant when V_C is specified completely." (Bell, 1990/2004, p. 240)

In a recent paper Michael Seevinck and Jos Uffink (2011) have questioned the necessary role of *complete* specification in the definition of local causality and recommended *sufficient* specification instead. They argue that complete specification is too strong: it contradicts to the so-called no-conspiracy (free variable) condition which requires that the common cause of the correlation be probabilistically independent of the choice of the measurement settings.

I do not see this contradiction and my aim in this paper is to articulate my point. I will proceed as follows. The logical schema of Bell's definition of local causality is the following: if events are localized in the spacetime in such-and-such a way, then these events are to satisfy such-and-such probabilistic independencies. This schema is highly intuitive and easily applicable in the physical praxis, however, in order to account for these inferences from spatiotemporal to probabilistic relations in a mathematically transparent way, one needs to have a *framework* integrating both spatiotemporal and also probabilistic entities. Only after having such a common framework can one define Bell's notion of local causality in a clear-cut way. Thus, in Section 2 first this framework, called *local physical theory*, will be introduced and then Bell's notion of local causality will be formulated within this framework. In Section 3 the relation of local causality to the Bell inequalities will be explicated. The main section is Section 4; here it will be argued that there is no tension between complete specification and no-conspiracy. I conclude in Section 5.

6.2 Bell's local causality in a local physical theory

In developing the notion of a local physical theory one is lead by the following intuitions. A local physical theory is to describe "beables," let them be classical or nonclassical; it is to account for the logical combination of these events; these events should be capable of bearing a probabilistic interpretation; the theory is to provide some way to localize these event in the spacetime, and is also to provide some physically well-motivated principles guiding the association of spacetime regions to physical events; the theory is to guarantee that the symmetries of the spacetime are in tune with the symmetries of the events. (For the details see (Hofer-Szabó and Vecsernyés, 2015 a,b).) All these preliminary intuitions are captured in the following definition (Haag, 1992):

Definition 5. A $\mathcal{P}_{\mathcal{K}}$-*covariant local physical theory* is a net $\{\mathcal{A}(V), V \in \mathcal{K}\}$ associating algebras of events to spacetime regions which satisfies *isotony*, *microcausality* and *covariance* defined as follows:

1. *Isotony.* Let $\mathcal{M}$ be a globally hyperbolic spacetime and let $\mathcal{K}$ be a covering collection of bounded, globally hyperbolic subspacetime regions of $\mathcal{M}$ such that $(\mathcal{K}, \subseteq)$ is a directed poset under inclusion $\subseteq$. The net of local observables is given by the isotone map $\mathcal{K} \ni V \mapsto \mathcal{A}(V)$ to unital C^*-algebras, that is $V_1 \subseteq V_2$ implies that $\mathcal{A}(V_1)$ is a unital C^*-subalgebra of $\mathcal{A}(V_2)$. The *quasilocal algebra* $\mathcal{A}$ is defined to be the inductive limit C^*-algebra of the net $\{\mathcal{A}(V), V \in \mathcal{K}\}$ of local C^*-algebras.

2. *Microcausality* (also called as *Einstein causality*) is the requirement that $\mathcal{A}(V')' \cap \mathcal{A} \supseteq \mathcal{A}(V), V \in \mathcal{K}$, where primes denote spacelike complement and algebra commutant, respectively.

3. *Spacetime covariance.* Let $\mathcal{P}_{\mathcal{K}}$ be the subgroup of the group $\mathcal{P}$ of geometric symmetries of $\mathcal{M}$ leaving the collection $\mathcal{K}$ invariant. A group homomorphism $\alpha\colon \mathcal{P}_{\mathcal{K}} \to \operatorname{Aut}\mathcal{A}$ is given such that the automorphisms $\alpha_g, g \in \mathcal{P}_{\mathcal{K}}$ of $\mathcal{A}$ act covariantly on the observable net: $\alpha_g(\mathcal{A}(V)) = \mathcal{A}(g \cdot V), V \in \mathcal{K}$.

If the quasilocal algebra $\mathcal{A}$ of the local physical theory is commutative, we speak about a *local classical theory*, if it is noncommutative, we speak about a *local quantum theory.* For local classical theories microcausality fulfills trivially.

A *state* ϕ in a local physical theory is defined as a normalized positive linear functional on the quasilocal observable algebra $\mathcal{A}$. The corresponding GNS representation $\pi_\phi\colon \mathcal{A} \to \mathcal{B}(\mathcal{H}_\phi)$ converts the net of C^*-algebras into a net of C^*-subalgebras of $\mathcal{B}(\mathcal{H}_\phi)$. Closing these subalgebras in the weak topology one arrives at a net of local von Neumann observable algebras: $\mathcal{N}(V) := \pi_\phi(\mathcal{A}(V))'', V \in \mathcal{K}$. Von Neumann algebras are generated by their projections, which are called *quantum events* since they can be interpreted as 0-1–valued observables. The net $\{\mathcal{N}(V), V \in \mathcal{K}\}$ of local von Neumann algebras given above also obeys isotony, microcausality, and $\mathcal{P}_{\mathcal{K}}$-covariance, hence we can also refer to a net $\{\mathcal{N}(V), V \in \mathcal{K}\}$ of local von Neumann algebras as a local physical theory.

Now, a local physical theory is locally causal in Bell's sense if any correlation between spatially separated events is screened off by "local beables" which are localized in a "shielding-off" region and which "completely specify" that region. How to translate Bell's terms of "local beable" and "complete specification" into the language of a local physical theory?

In a classical field theory beables are characterized by sets of field configurations. Taking the equivalence classes of those field configurations which have the same field values on a given spacetime region one can generate local (cylindrical) σ-algebras. Translating σ-algebras into the language of abelian von Neumann algebras one can represent Bell's notion of "local beables" in the framework of local physical theories. In a more general way, one can also use the term "local beables" both for abelian and non-abelian local von Neumann algebras, hence treating local classical and quantum theories on an equal footing. Translating "local beables" simply as "elements of a local algebra" naturally brings with it the translation of the term "a complete specification of beables" as "an atomic event of a local algebra" (Henson, 2013). To be sure, here it is assumed that the local algebras of the net are atomic, which is typically not the case, for example, in Poincaré covariant algebraic quantum field theory. (For the relation between σ-algebras and von Neumann algebras and for a more general definition of local causality see (Hofer-Szabó and Vecsernyés, 2015 a,b).) With these notions in hand now one can formulate Bell's notion of local causality in a local physical theory as follows:

Definition 6. A local physical theory represented by a net $\{\mathcal{N}(V), V \in \mathcal{K}\}$ of von Neumann algebras is called *locally causal* (in Bell's sense), if for any pair $A \in \mathcal{N}(V_A)$ and $B \in \mathcal{N}(V_B)$ of projections supported in spacelike separated regions $V_A, V_B \in \mathcal{K}$ and for every locally normal and faithful state ϕ establishing a correlation $\phi(AB) \neq \phi(A)\phi(B)$ between A and B, and for any spacetime region V_C such that

(i) $V_C \subset J_-(V_A)$,

(ii) $V_A \subset V_C''$,

(iii) $J_-(V_A) \cap J_-(V_B) \cap \big(J_+(V_C) \setminus V_C\big) = \emptyset$,

(see Fig. 6.2) and for any atomic event C_k of $\mathcal{A}(V_C)$ $(k \in K)$, the following holds:

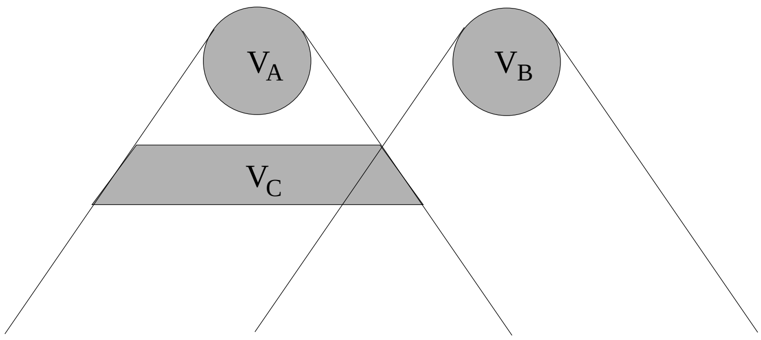

Figure 6.2: A region V_C satisfying Requirements (i)-(iii).

$$\frac{\phi(C_k ABC_k)}{\phi(C_k)} = \frac{\phi(C_k AC_k)}{\phi(C_k)} \frac{\phi(C_k BC_k)}{\phi(C_k)} \tag{6.1}$$

Remarks:

1. A *locally normal* state is a normal state on the local von Neumann algebras. A *locally faithful* state ϕ means that any projection $A \in \mathcal{P}(\mathcal{N}(V))$ in the local von Neumann algebra $\mathcal{N}(V)$ has nonzero expectation value. In case of local classical theories a locally faithful state ϕ determines uniquely a locally nonzero probability measure p by $p(A) := \phi(A), A \in \mathcal{P}(\mathcal{N}(V))$. By means of this (6.1) can be written in the following 'symmetric' form:

$$p(AB|C_k) = p(A|C_k)p(B|C_k) \tag{6.2}$$

which further is equivalent to the 'asymmetric' screening-off condition:

$$p(A|BC_k) = p(A|C_k) \tag{6.3}$$

sometimes used in the literature (for example in (Bell, 1975/2004 , p. 54)).

2. The role of Requirement (iii) in the definition is to ensure that "V_C shields off from V_A the overlap of the backward light cones of V_A and V_B". A spacetime region *above* V_C in the common past of the correlating events (see Fig. 6.3) namely may contain stochastic events which could establish a correlation

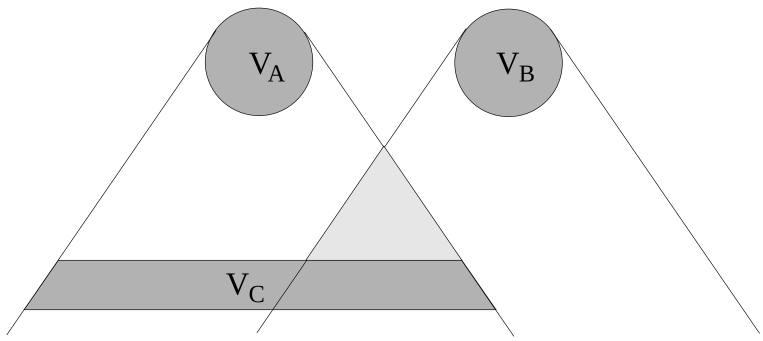

Figure 6.3: A region V_C for which Requirement (iii) does not hold.

between A and B in a classical stochastic theory (Norsen, 2011; Seevinck and Uffink 2011). Requirement (iii) is somewhat weaker than Bell's original localization (see Fig. 6.1) which can be formulated as:

(iv) $J_-(V_A) \cap J_-(V_B) \cap V_C = \emptyset$

The difference is that Requirement (iii) does, but Requirement (iv) does not allow for region V_C to penetrate into the 'top part' of the common past. However, both requirements coincide, if V_C 'shrinks down' to a Cauchy surface. In local classical theories it suffices to use Requirement (iii).

Finally, note that the question whether a given local classical or quantum theory is locally causal is a highly nontrivial question depending on such factors as the atomicity of the local algebras, the fulfilment of the so-called local primitive causality,[3] or whether there exists a causal dynamics in the theory, etc. (For the details see again (Hofer-Szabó and Vecsernyés, 2015 a,b).)

Next I turn to the relation of Bell's local causality to the Bell inequalities.

6.3 Local causality and the Bell inequalities

From this section on we restrict ourselves to local *classical* theories since beables are standardly taken to be classical entities. Consider a local classical theory rep-

[3]For any globally hyperbolic bounded subspacetime regions $V \in \mathcal{K}$, $\mathcal{A}(V'') = \mathcal{A}(V)$.

resented by a net $\{\mathcal{N}(V), V \in \mathcal{K}\}$ of local abelian von Neumann algebras. Suppose that Bell's local causality holds in this theory. Let V_A and V_B be two spatially separated regions in $\mathcal{M}$, and V_C a third region (see Fig. 6.4) such that

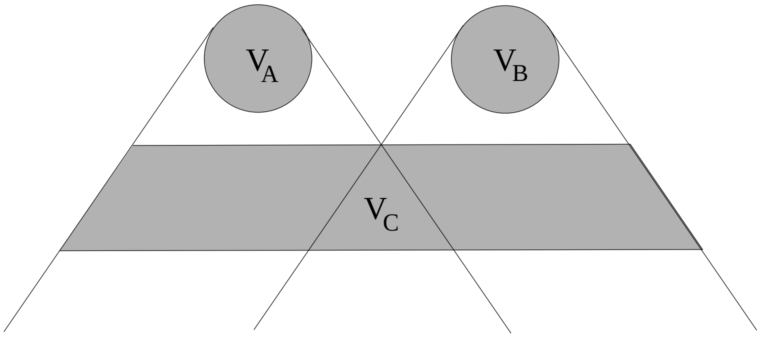

Figure 6.4: Localization of regions V_A, V_B and V_C.

$$V_C \subset J_-(V_A \cup V_B) \tag{6.4}$$
$$(V_A \cup V_B) \subset V_C'' \tag{6.5}$$
$$J_-(V_A) \cap J_-(V_B) \cap \big(J_+(V_C) \setminus V_C\big) = \emptyset \tag{6.6}$$

Divide V_C into six regions V_C^L, $V_{C'}^L$, V_C^M, $V_{C'}^M$, V_C^R and $V_{C'}^R$, for example as depicted in Fig. 6.5. Here the superscripts L, M and R stand for 'left', 'middle' and 'right', representing those parts of V_C which fall into region $J_-(V_A) \setminus J_-(V_B)$, $J_-(V_A) \cap J_-(V_B)$ and $J_-(V_B) \setminus J_-(V_A)$, respectively. Now, let the various events be

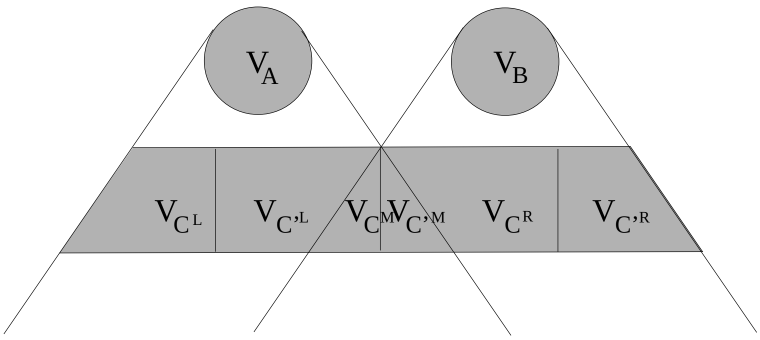

Figure 6.5: Dividing up region V_C.

localized in these regions as follows. Let A_i and B_j be *measurement outcomes* and a_i, b_j *measurement choices* localized in the appropriate regions: $A_i, a_i \in \mathcal{A}(V_A)$, $B_j, b_j \in \mathcal{A}(V_B)$. (See Fig. 6.6.) Let, furthermore, C_k^L, $C_l'^L$, C_m^M, $C_n'^M$, C_p^R, $C_q'^R$ be *atomic* events (minimal projections) in $\mathcal{A}(V_C^L)$, $\mathcal{A}(V_{C'}^L)$, $\mathcal{A}(V_C^M)$, $\mathcal{A}(V_{C'}^M)$, $\mathcal{A}(V_C^R)$

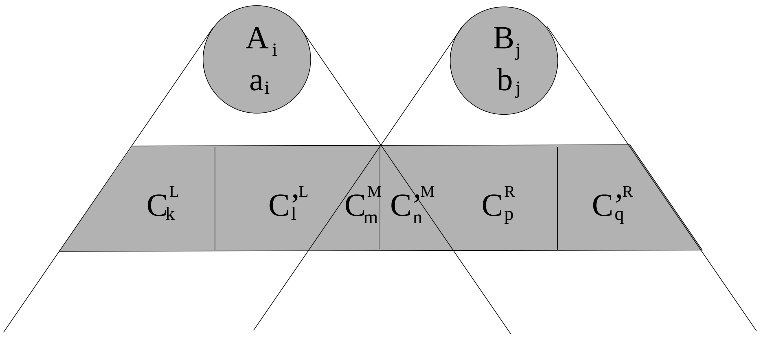

Figure 6.6: Localization of the various events.

and $\mathcal{A}(V^R_{C'})$, respectively, where the indices $i, j, k \ldots$ are taken from appropriate index sets. Now, the difference between the primed and the unprimed events in V_C is that the *primed* events probabilistically *depend* on the the measurement choices a_i and b_j, whereas the *unprimed* events are probabilistically *completely independent* of them:

$$
\begin{aligned}
p(a_i b_j C^L_l C^M_m C^R_p) &= p(a_i)p(b_j)p(C^L_l)p(C^M_m)p(C^R_p) && (6.7)\\
p(a_i b_j C^L_l C^M_m) &= p(a_i)p(b_j)p(C^L_l)p(C^M_m) && (6.8)\\
&\ldots && (6.9)\\
p(a_i b_j C^R_p) &= p(a_i)p(b_j)p(C^R_p) && (6.10)
\end{aligned}
$$

Let us call these conditions *no-conspiracy conditions*.

To sum up, here we assume that *any* of the left, middle and right region of V_C, respectively can be decomposed into two subregions such that each of these subregions contains exclusively *either* events 'influencing' the measurement choices *or* events being independent of them. Obviously, only this latter class of events can be regarded as the *common cause* of the correlation between the measurement outcomes; the former events are playing a role in fixing the measurement settings. As we will see later, this assumption of the decomposability of V_C into *six* regions is too tolerant if our aim is to derive the Bell inequalities. It will turn out that there are only *five* regions, the middle region can contain only *unprimed* events.

Now, *local causality* of local physical theory represented by a net $\{\mathcal{N}(V), V \in \mathcal{K}\}$ implies (among others) the following conditional independencies:

$$
\begin{aligned}
p(A_i a_i | B_j b_j C^L_k C'^L_l C^M_m C'^M_n C^R_p C'^R_q) &= p(A_i a_i | C^L_k C'^L_l C^M_m C'^M_n) && (6.11)\\
p(B_j b_j | C^L_k C'^L_l C^M_m C'^M_n C^R_p C'^R_q) &= p(B_j b_j | C^M_m C'^M_n C^R_p C'^R_q) && (6.12)\\
p(a_i | b_j C^L_k C'^L_l C^M_m C'^M_n C^R_p C'^R_q) &= p(a_i | C^L_k C'^L_l C^M_m C'^M_n) && (6.13)\\
p(b_j | C^L_k C'^L_l C^M_m C'^M_n C^R_p C'^R_q) &= p(b_j | C^M_m C'^M_n C^R_p C'^R_q) && (6.14)
\end{aligned}
$$

which together with the *complete independence* of the events C^L_k, C'^L_l, C^M_m, C'^M_n,

C_p^R and $C_q'^R$:

$$p(C_k^L C_l'^L C_m^M C_n'^M C_p^R C_q'^R) = p(C_k^L)p(C_l'^L)p(C_m^M)p(C_n'^M)p(C_p^R)p(C_q'^R) \tag{6.15}$$

$$p(C_k^L C_l'^L C_m^M C_n'^M C_p^R) = p(C_k^L)p(C_l'^L)p(C_m^M)p(C_n'^M)p(C_p^R) \tag{6.16}$$

$$\ldots \tag{6.17}$$

$$p(C_p^R C_q'^R) = p(C_p^R)p(C_q'^R) \tag{6.18}$$

yield the following *screening-off* or *factorization conditions*:

$$p(A_iB_j|a_ib_jC_k^L C_l'^L C_m^M C_n'^M C_p^R C_q'^R) = p(A_i|a_iC_k^L C_l'^L C_m^M C_n'^M)p(B_j|b_jC_m^M C_n'^M C_p^R C_q'^R) \tag{6.19}$$

$$p(A_iB_j|a_ib_jC_k^L C_m^M C_n'^M C_p^R) = p(A_i|a_iC_k^L C_m^M C_n'^M)p(B_j|b_jC_m^M C_n'^M C_p^R) \tag{6.20}$$

$$p(A_iB_j|a_ib_jC_l'^L C_m^M C_n'^M C_q'^R) = p(A_i|a_iC_l'^L C_m^M C_n'^M)p(B_j|b_jC_m^M C_n'^M C_q'^R) \tag{6.21}$$

$$p(A_iB_j|a_ib_jC_m^M C_n'^M) = p(A_i|a_iC_m^M C_n'^M)p(B_j|b_jC_m^M C_n'^M) \tag{6.22}$$

(For the proof see Appendix A.) These equations show that not only the *atomic* events $C_k^L C_l'^L C_m^M C_n'^M C_p^R C_q'^R$ localized in the entire V_C screen off the conditional correlation

$$p(A_iB_j|a_ib_j) \neq p(A_i|a_i)p(B_j|b_j) \tag{6.23}$$

but one can freely sum up for any of the *primed* and *unprimed* events both in the *left* and the *right* region without vitiating the screening-off. In other words, the non-atomic (coarse-grained) events $C_k^L C_m^M C_n'^M C_p^R$, $C_l'^L C_m^M C_n'^M C_q'^R$ and $C_m^M C_n'^M$, respectively localized in appropriate subregions of V_C will all be screener-offs for the correlation (10.1).[4] That one can freely sum up for both the primed and the unprimed events is a consequence of the fact that in the derivation of (6.19)-(6.22) no-conspiracy (6.7)-(6.10) does *not* play a role.

However, for events localized in the *middle* region one *cannot* sum up! As a consequence, one cannot get rid of the *primed* terms $C_n'^M$ in equations (6.19)-(6.22). So for example it will *not* be generally true that

$$p(A_iB_j|a_ib_jC_m^M) = p(A_i|a_iC_m^M)p(B_j|b_jC_m^M) \tag{6.24}$$

(See Appendix B.) However, if we cannot get rid of the *primed* terms $C_n'^M$, we will not be able to derive the Bell inequalities since in the derivation we need to use no-conspiracy (6.7)-(6.10) which holds only for the *unprimed* terms. (See Appendix C.)

This shows that our decomposition of region V_C into *six* regions was too liberal. We have to make one step back and restrict our previous schema to the one depicted in Fig. 6.7. *Outside* the common past of the correlating events one can have both primed and unprimed events that is events influencing the measurement choices and events being independent of them. However, *within* the common past there can be

[4]Note again that the term 'common cause' is used only for those screener-offs which are composed of *unprimed* events.

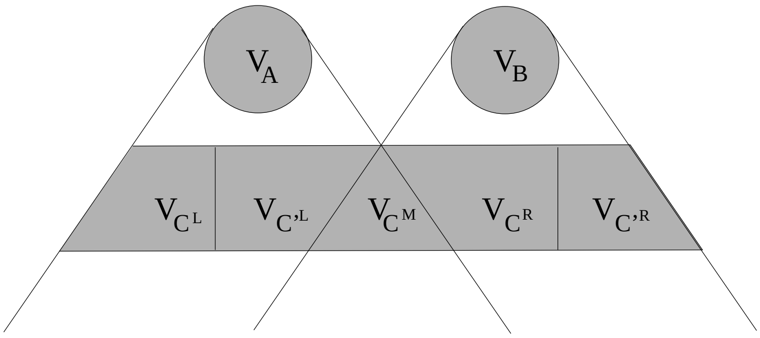

Figure 6.7: The most general scenario from which the Bell inequalities can be derived.

only events which are probabilistically independent of the measurement choices. Within this schema the Bell inequalities can be derived.

To sum up, given a locally causal local classical theory represented by a net $\{\mathcal{N}(V), V \in \mathcal{K}\}$ with regions localized as in Fig. 6.7 and elements in the appropriate regions, complete independence (6.15)-(6.18) and no-conspiracy (6.7)-(6.10) together imply the Bell inequalities.

6.4 Complete *versus* sufficient specification

Now I turn to the question of 'complete *versus* sufficient specification' raised by Norsen (2009) and unfolded by Seevinck and Uffink (2011). In his illuminating paper, comparing the notion of 'completeness' used in Bell's *vs.* Jarrett's writings, Norsen (2009) raised the following concern:[5] Since "the past light cones of [the measurement choices] a and b overlap with the region containing C – and C by definition is supposed to contain a *complete* specification of beables in this region ...one wonders how a and b could possibly *not* be causally influenced by C (in a locally causal theory)" (Norsen 2009, p. 283.) Seevinck and Uffink take Norsen's point and argue that complete specification is too strong "when formalising the notion of local causality. It is only needed that the specification is *sufficiently* specified, in the relevant sense" (p. 5); and then they go on to develop this relevant sense in terms of Fisher's statistical concept of sufficiency.

The argument of Seevinck and Uffink against complete specification is put in the form of a dilemma:

> "C cannot be expected to be a complete specification of region V_C because one must allow for the possibility of traces in region V_C of the causal past of both the settings [measurement choices], and given the

[5]Again for the sake of consistency I changed the notation of both Norsen (2009) and Seevinck and Uffink (2011).

> independence of C and the settings, these traces cannot be included in C.
>
> An alternative understanding of this point is that one is here faced with a dilemma. That is, the following two assumptions cannot both hold: (i) the free variables [no-conspiracy] assumption, and (ii) the assumption that C is completely specified, i.e., contains the description of all and every beable in region V_C." (Seevinck and Uffink, 2011, p. 5)

In brief, the complete specification of region V_C contradicts to the no-conspiracy condition since if C completely specifies region V_C, then it also specifies the measurement choices a and b, and hence C and a, b cannot be probabilistically independent.

I see, however, no contradiction between complete specification and no-conspiracy. I have a weaker and a stronger claim supporting my point. I start with the weaker one. The upshot of this weaker claim is that the events which satisfy complete specification *need not be the same* as the events which satisfy no-conspiracy.

Complete specification of a spacetime region, as said before, is simply an atomic event in that region. If our "candidate theory" represented by a net of local algebras is given, then to every bounded region V_C there is an algebra $\mathcal{A}(V_C)$ associated; and if the algebra is atomic, the complete specifications that is the atomic events of the region are also given. Consider region V_C in Fig. 6.7. The event $C_k^L C_l'^L C_m^M C_p^R C_q'^R$ is a complete specification in V_C, but the unprimed event $C_k C_m C_p$ and the primed event $C_l' C_q'^R$ separately are not. These latter two play different theoretical roles: No-conspiracy holds for $C_k C_m C_p$, therefore it is interpreted as a (possible) common cause of the conditional correlation (10.1). For $C_l' C_q'^R$ no-conspiracy does not hold (and a fortiori neither does for the complete specification $C_k^L C_l'^L C_m^M C_p^R C_q'^R$). Thus $C_l' C_q'^R$ has another interpretation: it allows "for the possibility of traces in region V_C of the causal past of both the settings." This 'division of labor' between the unprimed $C_k C_m C_p$ and the primed $C_l' C_q'^R$, however, is no worry: together they provide a complete specification of region V_C and enable the derivation of the Bell inequalities as long as the middle region, $V_C \cap V_A \cap V_B$ contains no primed term violating no-conspiracy. In short, in order to derive the Bell inequalities from local causality, those events which completely specify region V_C *need not be the same events* as those satisfying no-conspiracy.

But here is my stronger claim: *they can.* Namely, there is no contradiction between complete specification and no-conspiracy even if we require them to hold *for the same events.* To see this, simply consider the case when the subregions $V_{C'}^L$ and $V_{C'}^R$ are empty, that is when V_C contains exclusively unprimed elements (see Fig. 6.8). In this case the event $C_k^L C_m^M C_p^R$ will *both* completely specify region V_C *and* satisfy no-conspiracy. Consequently, the Bell inequalities will follow. More importantly, this independence between the common causes and the measurement choices does not trivialize the theory, for example by dissolving the conditional correlation (10.1) between the measurement outcomes.

The next proposition illustrates this latter point.

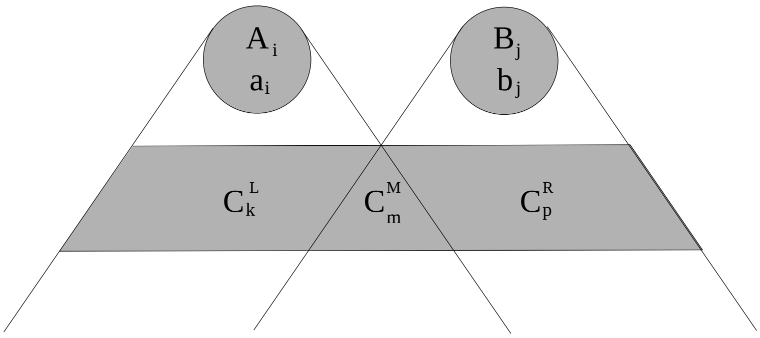

Figure 6.8: No contradiction between complete specification and no-conspiracy.

Proposition 6. There exists a locally causal local classical theory with events $A_i, a_i \in \mathcal{A}(V_A)$, $B_j, b_j \in \mathcal{A}(V_B)$ in spatially separated regions V_A and V_B conditionally correlating in the sense of (10.1), and atomic events $C_k^L \in \mathcal{A}(V_C^L)$, $C_m^M \in \mathcal{A}(V_C^M)$ and $C_p^R \in \mathcal{A}(V_C^R)$, where $V_C = V_C^L \cup V_C^M \cup V_C^R$ satisfies requirements (6.4)-(6.6), such that no-conspiracy (6.7)-(6.10), moreover complete independence (6.15)-(6.18) hold.

Proof. Let A_i, a_i, B_j, b_j, C_k^L, C_m^M and C_p^R be events localized as in Fig. 6.8. Suppose that for the atomic events C_k^L, C_m^M and C_p^R completely specifying region V_C both complete independence

$$p(C_k^L C_m^M C_p^R) = p(C_k^L C_m^M) p(C_p^R) = p(C_k^L) p(C_m^M C_p^R) = p(C_m^M) p(C_k^L C_p^R) = p(C_k^L) p(C_m^M) p(C_p^R) \tag{6.25}$$

and also no-conspiracy

$$p(a_i b_j C_k^L C_m^M C_p^R) = p(a_i b_j) p(C_k^L C_m^M C_p^R) = \cdots = p(a_i) p(b_j) p(C_k^L) p(C_m^M) p(C_p^R) \tag{6.26}$$

hold for any combination of the indices. Let the net containing the events be locally causal; for example let

$$p(A_i B_j | a_i b_j C_k^L C_m^M C_p^R) = p(A_i | a_i C_k^L C_m^M) p(B_j | b_j C_m^M C_p^R) = (p_i^L \delta_{1k} \delta_{1m})(p_j^R \delta_{1m} \delta_{1p}) \tag{6.27}$$

where $\sum_i p_i^L = \sum_j p_j^R = 1$. Now, the conditional probabilities are given as follows:

$$p(A_i|a_i) = \sum_{k,m} p(A_i|a_i C_k^L C_m^M) p(C_k^L C_m^M) = p_i^L p(C_1^L) p(C_1^M) \tag{6.28}$$

$$p(B_j|b_j) = \sum_{m,p} p(B_j|b_j C_m^M C_p^R) p(C_m^M C_p^R) = p_j^R p(C_1^M) p(C_1^R) \tag{6.29}$$

$$\begin{aligned} p(A_i B_j|a_i b_j) &= \sum_{k,m,p} p(A_i B_j|a_i b_j C_k^L C_m^M C_p^R) p(C_k^L C_m^M C_p^R) \\ &= \sum_{k,m,p} p(A_i|a_i C_k^L C_m^M) p(B_j|b_j C_m^M C_p^R) p(C_k^L) p(C_m^M) p(C_p^R) \\ &= p_i^L p_j^R p(C_1^L) p(C_1^M) p(C_1^R) \end{aligned} \tag{6.30}$$

Thus, there is a conditional correlation (10.1) between A_i and B_j whenever $p(C_1^M) \neq 0$ or 1. ■

Consequently, there is no contradiction between complete specification and no-conspiracy even if both are applied to the same events, namely the atomic events of the entire V_C. The measurement choices can be free of the common causes even if the causal past of the region containing them is completely specified. This independence does not abolish the conditional correlation between the measurement outcomes: atomic events can be probabilistically irrelevant to the measurement *choices* and at the same time relevant to the measurement *outcomes*. Moreover, the independence of the measurement choices of the atomic events does not mean that the former are not 'determined' (probabilistically) by the latter. They are: the conditional probabilities $p(a_i b_j|C_k^L C_m^M C_p^R)$ are set in a local physical theory, even if they are equal to $p(a_i b_j)$.

Thus, based on these two claims, I think, there is no need to replace 'complete specification' in Bell's definition of local causality by 'sufficient specification'.

6.5 Conclusions

The main claims of this paper were the following:

(i) The definition of Bell's notion of local causality presupposes a clear-cut framework in which probabilistic and spatiotemporal entities can be related. This goal can be met by introducing the notion of a local physical theory represented by an isotone net of algebras.

(ii) In a local classical theory the measurement outcomes, measurement choices and common cause can be localized in the spacetime such that one can derive the Bell inequalities from local causality, no-conspiracy and independence.

(iii) Contrary to the claim of Seevinck and Uffink, there is no need to weaken the requirement of complete specification in the definition of local causality on the ground that it contradicts to no-conspiracy.

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Appendix A

First we prove equation (6.22) from local causality (6.11)-(6.14) and the complete independence condition (6.15)-(6.18):

$$
\begin{aligned}
& p(A_iB_j|a_ib_jC_m^MC_n'^M) = \frac{p(A_iB_ja_ib_jC_m^MC_n'^M)}{p(a_ib_jC_m^MC_n'^M)} \\
= \quad & \frac{\sum_{klpq} p(A_iB_ja_ib_jC_k^LC_l'^LC_m^MC_n'^MC_p^RC_q'^R)}{\sum_{klpq} p(a_ib_jC_k^LC_l'^LC_m^MC_n'^MC_p^RC_q'^R)} \\
= \quad & \frac{\sum_{klpq} p(A_iB_ja_ib_j|C_k^LC_l'^LC_m^MC_n'^MC_p^RC_q'^R)p(C_k^LC_l'^LC_m^MC_n'^MC_p^RC_q'^R)}{\sum_{klpq} p(a_ib_j|C_k^LC_l'^LC_m^MC_n'^MC_p^RC_q'^R)p(C_k^LC_l'^LC_m^MC_n'^MC_p^RC_q'^R)} \\
= \quad & \frac{\sum_{klpq} p(A_ia_i|B_jb_jC_k^LC_l'^LC_m^MC_n'^MC_p^RC_q'^R)p(B_jb_j|C_k^LC_l'^LC_m^MC_n'^MC_p^RC_q'^R)p(C_k^LC_l'^LC_m^MC_n'^MC_p^RC_q'^R)}{\sum_{klpq} p(a_i|b_jC_k^LC_l'^LC_m^MC_n'^MC_p^RC_q'^R)p(b_j|C_k^LC_l'^LC_m^MC_n'^MC_p^RC_q'^R)p(C_k^LC_l'^LC_m^MC_n'^MC_p^RC_q'^R)} \\
\overset{(6.11)-(6.14)}{=} \quad & \frac{\sum_{klpq} p(A_ia_i|C_k^LC_l'^LC_m^MC_n'^M)p(B_jb_j|C_m^MC_n'^MC_p^RC_q'^R)p(C_k^LC_l'^LC_m^MC_n'^MC_p^RC_q'^R)}{\sum_{klpq} p(a_i|C_k^LC_l'^LC_m^MC_n'^M)p(b_j|C_m^MC_n'^MC_p^RC_q'^R)p(C_k^LC_l'^LC_m^MC_n'^MC_p^RC_q'^R)} \\
\overset{(6.15)-(6.18)}{=} \quad & \frac{\sum_{klpq} p(A_ia_i|C_k^LC_l'^LC_m^MC_n'^M)p(B_jb_j|C_m^MC_n'^MC_p^RC_q'^R)p(C_k^LC_l'^LC_m^MC_n'^M)p(C_p^RC_q'^R)}{\sum_{klpq} p(a_i|C_k^LC_l'^LC_m^MC_n'^M)p(b_j|C_m^MC_n'^MC_p^RC_q'^R)p(C_k^LC_l'^LC_m^MC_n'^M)p(C_p^RC_q'^R)} \\
= \quad & \left(\frac{\sum_{kl} p(A_ia_i|C_k^LC_l'^LC_m^MC_n'^M)p(C_k^LC_l'^LC_m^MC_n'^M)}{\sum_{kl} p(a_i|C_k^LC_l'^LC_m^MC_n'^M)p(C_k^LC_l'^LC_m^MC_n'^M)}\right)\left(\frac{\sum_{pq} p(B_jb_j|C_m^MC_n'^MC_p^RC_q'^R)p(C_p^RC_q'^R)}{\sum_{pq} p(b_j|C_m^MC_n'^MC_p^RC_q'^RC_p^R)p(C_p^RC_q'^R)}\right) \\
= \quad & \left(\frac{\sum_{kl} p(A_ia_i|C_k^LC_l'^LC_m^MC_n'^M)p(C_k^LC_l'^LC_m^MC_n'^M)}{\sum_{kl} p(a_i|C_k^LC_l'^LC_m^MC_n'^M)p(C_k^LC_l'^LC_m^MC_n'^M)}\right)\left(\frac{\sum_{pq} p(B_jb_j|C_m^MC_n'^MC_p^RC_q'^R)p(C_p^RC_q'^R)}{\sum_{pq} p(b_j|C_m^MC_n'^MC_p^RC_q'^R)p(C_p^RC_q'^R)}\right)\left(\frac{p(C \cdots}{p(C \cdots} \right. \\
\overset{(6.15)-(6.18)}{=} \quad & \left(\frac{\sum_{kl} p(A_ia_i|C_k^LC_l'^LC_m^MC_n'^M)p(C_k^LC_l'^LC_m^MC_n'^M)}{\sum_{kl} p(a_i|C_k^LC_l'^LC_m^MC_n'^M)p(C_k^LC_l'^LC_m^MC_n'^M)}\right)\left(\frac{\sum_{pq} p(B_jb_j|C_m^MC_n'^MC_p^RC_q'^R)p(C_m^MC_n'^MC_p^RC_q'^R)}{\sum_{pq} p(b_j|C_m^MC_n'^MC_p^RC_q'^R)p(C_m^MC_n'^MC_p^RC_q'^R)} \right. \\
= \quad & \left(\frac{\sum_{kl} p(A_ia_iC_k^LC_l'^LC_m^MC_n'^M)}{\sum_{kl} p(a_iC_k^LC_l'^LC_m^MC_n'^M)}\right)\left(\frac{\sum_{pq} p(B_jb_jC_m^MC_n'^MC_p^RC_q'^R)}{\sum_{pq} p(b_jC_m^MC_n'^MC_p^RC_q'^R)}\right) \\
= \quad & \left(\frac{p(A_ia_iC_m^MC_n'^M)}{p(a_iC_m^MC_n'^M)}\right)\left(\frac{p(B_jb_jC_m^MC_n'^M)}{p(b_jC_m^MC_n'^M)}\right) = p(A_i|a_iC_m^MC_n'^M)p(B_j|b_jC_m^MC_n'^M)
\end{aligned}
$$

where the numbers over the equation signs refer to the equation used at that step.

The proof of (6.21), (6.20) and (6.19), respectively can be obtained from the above proof by simply omitting certain summations. For (6.21) just omit summation for l and r; for (6.20) omit summation for k and q; and for (6.19) omit all four.

Appendix B

Here we prove that (6.24) does not generally hold. The proof follows that in Appendix A, except that here there is an extra summation also for n, which causes the trouble in the row below starting with a $\neq$ sign:

$$
\begin{aligned}
& p(A_iB_j|a_ib_jC_m^M) = \frac{p(A_iB_ja_ib_jC_m^M)}{p(a_ib_jC_m^M)} \\
= \quad & \frac{\sum_{klnpq} p(A_iB_ja_ib_jC_k^LC_l^{\prime L}C_m^MC_n^{\prime M}C_p^RC_q^{\prime R})}{\sum_{klnpq} p(a_ib_jC_k^LC_l^{\prime L}C_m^MC_n^{\prime M}C_p^RC_q^{\prime R})} \\
= \quad & \frac{\sum_{klnpq} p(A_iB_ja_ib_j|C_k^LC_l^{\prime L}C_m^MC_n^{\prime M}C_p^RC_q^{\prime R})p(C_k^LC_l^{\prime L}C_m^MC_n^{\prime M}C_p^RC_q^{\prime R})}{\sum_{klnpq} p(a_ib_j|C_k^LC_l^{\prime L}C_m^MC_n^{\prime M}C_p^RC_q^{\prime R})p(C_k^LC_l^{\prime L}C_m^MC_n^{\prime M}C_p^RC_q^{\prime R})} \\
= \quad & \frac{\sum_{klnpq} p(A_ia_i|B_jb_jC_k^LC_l^{\prime L}C_m^MC_n^{\prime M}C_p^RC_q^{\prime R})p(B_jb_j|C_k^LC_l^{\prime L}C_m^MC_n^{\prime M}C_p^RC_q^{\prime R})p(C_k^LC_l^{\prime L}C_m^M \ldots}{\sum_{klnpq} p(a_i|b_jC_k^LC_l^{\prime L}C_m^MC_n^{\prime M}C_p^RC_q^{\prime R})p(b_j|C_k^LC_l^{\prime L}C_m^MC_n^{\prime M}C_p^RC_q^{\prime R})p(C_k^LC_l^{\prime L}C_m^MC_n^{\prime M} \ldots} \\
\overset{(6.11)-(6.14)}{=} \quad & \frac{\sum_{klnpq} p(A_ia_i|C_k^LC_l^{\prime L}C_m^MC_n^{\prime M})p(B_jb_j|C_m^MC_n^{\prime M}C_p^RC_q^{\prime R})p(C_k^LC_l^{\prime L}C_m^MC_n^{\prime M}C_p^RC_q^{\prime R})}{\sum_{klnpq} p(a_i|C_k^LC_l^{\prime L}C_m^MC_n^{\prime M})p(b_j|C_m^MC_n^{\prime M}C_p^RC_q^{\prime R})p(C_k^LC_l^{\prime L}C_m^MC_n^{\prime M}C_p^RC_q^{\prime R})} \\
\overset{(6.15)-(6.18)}{=} \quad & \frac{\sum_{klnpq} p(A_ia_i|C_k^LC_l^{\prime L}C_m^MC_n^{\prime M})p(B_jb_j|C_m^MC_n^{\prime M}C_p^RC_q^{\prime R})p(C_k^LC_l^{\prime L}C_m^MC_n^{\prime M})p(C_p^RC_q^{\prime R})}{\sum_{klnpq} p(a_i|C_k^LC_l^{\prime L}C_m^MC_n^{\prime M})p(b_j|C_m^MC_n^{\prime M}C_p^RC_q^{\prime R})p(C_k^LC_l^{\prime L}C_m^MC_n^{\prime M})p(C_p^RC_q^{\prime R})} \\
= \quad & \frac{\sum_n \left(\sum_{kl} p(A_ia_i|C_k^LC_l^{\prime L}C_m^MC_n^{\prime M})p(C_k^LC_l^{\prime L}C_m^MC_n^{\prime M}) \sum_{pq} p(B_jb_j|C_m^MC_n^{\prime M}C_p^RC_q^{\prime R})p(C_p^RC \ldots\right.}{\sum_n \left(\sum_{kl} p(a_i|C_k^LC_l^{\prime L}C_m^MC_n^{\prime M})p(C_k^LC_l^{\prime L}C_m^MC_n^{\prime M}) \sum_{pq} p(b_j|C_m^MC_n^{\prime M}C_p^RC_q^{\prime R})p(C_p^RC_q^{\prime R} \ldots\right.} \\
= \quad & \frac{\sum_n \left(\sum_{kl} p(A_ia_i|C_k^LC_l^{\prime L}C_m^MC_n^{\prime M})p(C_k^LC_l^{\prime L}C_m^MC_n^{\prime M}) \sum_{pq} p(B_jb_j|C_m^MC_n^{\prime M}C_p^RC_q^{\prime R})p(C_p^RC \ldots\right.}{\sum_n \left(\sum_{kl} p(a_i|C_k^LC_l^{\prime L}C_m^MC_n^{\prime M})p(C_k^LC_l^{\prime L}C_m^MC_n^{\prime M}) \sum_{pq} p(b_j|C_m^MC_n^{\prime M}C_p^RC_q^{\prime R})p(C_p^RC_q^{\prime R} \ldots\right.} \\
\neq \quad & \left(\frac{\sum_{kln} p(A_ia_i|C_k^LC_l^{\prime L}C_m^MC_n^{\prime M})p(C_k^LC_l^{\prime L}C_m^MC_n^{\prime M})}{\sum_{kln} p(a_i|C_k^LC_l^{\prime L}C_m^MC_n^{\prime M})p(C_k^LC_l^{\prime L}C_m^MC_n^{\prime M})}\right)\left(\frac{\sum_{npq} p(B_jb_j|C_m^MC_n^{\prime M}C_p^RC_q^{\prime R})p(C \ldots}{\sum_{npq} p(b_j|C_m^MC_n^{\prime M}C_p^RC_q^{\prime R})p(C_m^M \ldots}\right. \\
\overset{(6.15)-(6.18)}{=} \quad & \left(\frac{\sum_{kln} p(A_ia_i|C_k^LC_l^{\prime L}C_m^MC_n^{\prime M})p(C_k^LC_l^{\prime L}C_m^MC_n^{\prime M})}{\sum_{kln} p(a_i|C_k^LC_l^{\prime L}C_m^MC_n^{\prime M})p(C_k^LC_l^{\prime L}C_m^MC_n^{\prime M})}\right)\left(\frac{\sum_{npq} p(B_jb_j|C_m^MC_n^{\prime M}C_p^RC_q^{\prime R})p(C_k \ldots}{\sum_{npq} p(b_j|C_m^MC_n^{\prime M}C_p^RC_q^{\prime R})p(C_k^L \ldots}\right. \\
= \quad & \left(\frac{\sum_{kln} p(A_ia_iC_k^LC_l^{\prime L}C_m^MC_n^{\prime M})}{\sum_{kln} p(a_iC_k^LC_l^{\prime L}C_m^MC_n^{\prime M})}\right)\left(\frac{\sum_{npq} p(B_jb_jC_m^MC_n^{\prime M}C_p^RC_q^{\prime R})}{\sum_{npq} p(b_jC_m^MC_n^{\prime M}C_p^RC_q^{\prime R})}\right) \\
= \quad & \left(\frac{p(A_ia_iC_m^M)}{p(a_iC_m^M)}\right)\left(\frac{p(B_jb_jC_m^M)}{p(b_jC_m^M)}\right) = p(A_i|a_iC_m^M)p(B_j|b_jC_m^M)
\end{aligned}
$$

where again the numbers over the equation signs refer to the equation used at that step.

Appendix C

Here we prove why in the derivation of the Clauser-Horne inequality

$$-1 \leqslant p(A_iB_j|a_ib_j) + p(A_iB_{j'}|a_ib_{j'}) + p(A_{i'}B_j|a_{i'}b_j) - p(A_{i'}B_{j'}|a_{i'}b_{j'}) - p(A_i|a_ib_j) - p(B_j|a_ib_j) \leqslant 0 \quad (6.3$$

one should use (6.24) instead of (6.22). The standard derivation goes as follows:

It is a simple arithmetic fact that for any $\alpha, \alpha', \beta, \beta' \in [0,1]$:

$$-1 \leqslant \alpha\beta + \alpha\beta' + \alpha'\beta - \alpha'\beta' - \alpha - \beta \leqslant 0 \tag{6.34}$$

Now let $\alpha, \alpha', \beta, \beta'$ first be the conditional probabilities taken from (6.22):

$$\begin{aligned}
\alpha &\equiv p(A_i|a_i C_m^M C_n'^M) && (6.35)\\
\alpha' &\equiv p(A_{i'}|a_{i'} C_m^M C_n'^M) && (6.36)\\
\beta &\equiv p(B_j|b_j C_m^M C_n'^M) && (6.37)\\
\beta' &\equiv p(B_{j'}|b_{j'} C_m^M C_n'^M) && (6.38)
\end{aligned}$$

Plugging (9.26)-(9.29) into (9.25) one obtains

$$\begin{aligned}
-1 \leqslant p(A_i|a_i C_m^M C_n'^M)p(B_j|b_j C_m^M C_n'^M) + p(A_i|a_i C_m^M C_n'^M)p(B_{j'}|b_{j'} C_m^M C_n'^M)\\
+p(A_{i'}|a_{i'} C_m^M C_n'^M)p(B_j|b_j C_m^M C_n'^M) - p(A_{i'}|a_{i'} C_m^M C_n'^M)p(B_{j'}|b_{j'} C_m^M C_n'^M)\\
-p(A_i|a_i C_m^M C_n'^M) - p(B_j|b_j C_m^M C_n'^M) \leqslant 0 \quad (6.39)
\end{aligned}$$

which using (6.22) transforms into

$$\begin{aligned}
-1 \leqslant p(A_i B_j|a_i b_j C_m^M C_n'^M) + p(A_i B_{j'}|a_i b_{j'} C_m^M C_n'^M)\\
+p(A_{i'} B_j|a_{i'} b_j C_m^M C_n'^M) - p(A_{i'} B_{j'}|a_{i'} b_{j'} C_m^M C_n'^M)\\
-p(A_i|a_i C_m^M C_n'^M) - p(B_j|b_j C_m^M C_n'^M) \leqslant 0 \quad (6.40)
\end{aligned}$$

Finally, multiplying the above inequality by $p(C_m^M C_n'^M)$ and summing up for the indices m, n one obtains

$$\begin{aligned}
-1 \leqslant \sum_{mn} \Big[p(A_i B_j|a_i b_j C_m^M C_n'^M) + p(A_i B_{j'}|a_i b_{j'} C_m^M C_n'^M)\\
+p(A_{i'} B_j|a_{i'} b_j C_m^M C_n'^M) - p(A_{i'} B_{j'}|a_{i'} b_{j'} C_m^M C_n'^M)\\
-p(A_i|a_i C_m^M C_n'^M) - p(B_j|b_j C_m^M C_n'^M) \Big] p(C_m^M C_n'^M) \leqslant 0 \quad (6.41)
\end{aligned}$$

which is equivalent to (9.24) only if

$$p(a_i b_j C_m^M C_n'^M) \quad = \quad p(a_i b_j) p(C_m^M C_n'^M) \tag{6.42}$$

were the case, which is not, since $C_n'^M$ is not independent of a_i and b_j.

Now, starting the whole reasoning again with conditional probabilities taken from (6.24):

$$\begin{aligned}
\alpha &\equiv p(A_i|a_i C_m^M) && (6.43)\\
\alpha' &\equiv p(A_{i'}|a_{i'} C_m^M) && (6.44)\\
\beta &\equiv p(B_j|b_j C_m^M) && (6.45)\\
\beta' &\equiv p(B_{j'}|b_{j'} C_m^M) && (6.46)
\end{aligned}$$

the derivation goes through since instead of (6.42) one is to use

$$p(a_i b_j C_m^M) \quad = \quad p(a_i b_j) p(C_m^M) \tag{6.47}$$

which is one of the no-conspiracy conditions (6.7)-(6.10). Thus one can use (6.24) in the derivation of the Clauser-Horne inequality but not (6.22).

Chapter 7

Noncommutative causality in algebraic quantum field theory

In the paper it will be argued that embracing noncommuting common causes in the causal explanation of quantum correlations in algebraic quantum field theory has the following two beneficial consequences: it helps (i) to maintain the validity of Reichenbach's Common Causal Principle and (ii) to provide a local common causal explanation for a set of correlations violating the Bell inequality.

7.1 Introduction

Algebraic quantum field theory (AQFT) is a mathematically transparent quantum theory with clear conceptions of locality and causality (see (Haag, 1992) and (Halvorson, 2007)). In this theory observables are represened by a net of local C^*-algebras associated to bounded regions of a given spacetime. This correspondence is established due to the axioms of the theory such as isotony, microcausality and covariance. A state ϕ in this theory is defined as a normalized positive linear functional on the quasilocal observable algebra $\mathcal{A}$ which is the inductive limit of local observable algebras. The representation $\pi_\phi \colon \mathcal{A} \to \mathcal{B}(\mathcal{H})$ corresponding to the state ϕ transforms the net of C^*-algebras into a net of von Neumann observable algebras by closures in the weak topology.

In AQFT *events* are typically represented by projections of a von Neumann algebra. Although due to the axiom of microcausality two projections A and B commute if they are contained in local algebras supported in spacelike separated regions, they can still *be correlating* in a state ϕ, that is

$$\phi(AB) \neq \phi(A)\phi(B) \tag{7.1}$$

in general. In this case the correlation between these events is said to be *superluminal*. A remarkable characteristics of Poincaré covariant theories is that there exist "many" normal states establishing superluminal correlations (for the precise meaning of "many" see (Summers, Werner 1988) and (Halvorson, Clifton 2000)). Since spacelike separation excludes direct causal influence, one may look for a causal explanation of these superluminal correlations in terms of *common causes.*

The first probabilistic definition of the common cause is due to Hans Reichenbach (1956). Reichenbach characterizes the notion of the common cause in the following probabilistic way. Let (Σ, p) be a classical probability measure space and let A and B be two positively correlating events in Σ that is let

$$p(A \wedge B) > p(A)\, p(B). \tag{7.2}$$

Definition 7. An event $C \in \Sigma$ is said to be the *common cause* of the correlation (A, B) if the following conditions hold:

$$\begin{aligned} p(A \wedge B|C) &= p(A|C)p(B|C) & (7.3)\\ p(A \wedge B|C^{\perp}) &= p(A|C^{\perp})p(B|C^{\perp}) & (7.4)\\ p(A|C) &> p(A|C^{\perp}) & (7.5)\\ p(B|C) &> p(B|C^{\perp}) & (7.6) \end{aligned}$$

where $C^{\perp}$ denotes the orthocomplement of C and $p(\,\cdot\,|\,\cdot\,)$ is the conditional probability.

The above definition, however, is too specific to be applied in AQFT since (i) it allows only for causes with a *positive* impact on their effects, (ii) it excludes the possibility of a *set* of cooperating common causes, (iii) it is silent about the spatiotemporal *localization* of the events and (iv) most importantly, it is *classical.* Therefore we need to generalize Reichenbach's original definition of the common cause. For the sake of brevity, we do not repeat here all the intermediate steps of the entire definitional process (for this see (Hofer-Szabó and Vecsernyés, 2012a)), but jump directly to the most general definition of the common cause in AQFT.

Let $\mathcal{P}(\mathcal{N})$ be the non-distributive lattice of projections (events) in a von Neumann algebra $\mathcal{N}$ and let $\phi\colon \mathcal{N} \to \mathbf{C}$ be a state on it. A set of mutually orthogonal projections $\{C_k\}_{k\in K} \subset \mathcal{P}(\mathcal{N})$ is called a *partition of the unit* $\mathbf{1} \in \mathcal{N}$ if $\sum_k C_k = \mathbf{1}$. Such a partition defines a *conditional expectation*

$$E\colon \mathcal{N} \to \mathcal{C},\; A \mapsto E(A) := \sum_{k\in K} C_k A C_k, \tag{7.7}$$

that is a unit preserving positive surjection onto the unital C^*-subalgebra $\mathcal{C} \subseteq \mathcal{N}$ obeying the bimodule property $E(B_1AB_2) = B_1E(A)B_2; A \in \mathcal{N}, B_1, B_2 \in \mathcal{C}$. We note that $\mathcal{C}$ contains exactly those elements of $\mathcal{N}$ that commute with $C_k, k \in K$. Recall that $\phi \circ E$ is also a state on $\mathcal{N}$.

Now, let $A, B \in \mathcal{P}(\mathcal{N})$ be two commuting events correlating in state ϕ in the sense of (7.1). (We note that in case of projection lattices we will use only algebra operations (products, linear combinations) instead of lattice operations $(\vee, \wedge)$. In case of commuting projections $A, B \in \mathcal{P}(\mathcal{N})$ we have $A \wedge B = AB$ and $A \vee B = A + B - AB$.)

Definition 8. A partition of the unit $\{C_k\}_{k\in K} \subset \mathcal{P}(\mathcal{N})$ is said to be a *common cause system* of the correlation (7.1) if

$$\frac{(\phi\circ E)(ABC_k)}{\phi(C_k)} = \frac{(\phi\circ E)(AC_k)}{\phi(C_k)}\frac{(\phi\circ E)(BC_k)}{\phi(C_k)} \tag{7.8}$$

for $k \in K$ with $\phi(C_k) \neq 0$. If C_k commutes with both A and B for all $k \in K$ we call $\{C_k\}_{k\in K}$ a *commuting* common cause system, otherwise a *noncommuting* one. A common cause system of size $|K| = 2$ is called a *common cause.* Reichenbach's definition (without the inequalities (7.5)-(7.6)) is a commuting common cause in the sense of (7.8).

Some remarks are in place here. First, in case of a commuting common cause system $\phi\circ E$ can be replaced by ϕ in (7.8) since $(\phi\circ E)(ABC_k) = \phi(ABC_k), k \in K$. Second, using the decompositions of the unit, $\mathbf{1} = A + A^\perp = B + B^\perp$, (7.8) can be rewritten in an equivalent form:

$$(\phi\circ E)(ABC_k))(\phi\circ E)(A^\perp B^\perp C_k) = (\phi\circ E)(AB^\perp C_k)(\phi\circ E)(A^\perp BC_k),\ k \in K. \tag{7.9}$$

One can even allow here the case $\phi(C_k) = 0$ since then both sides of (7.9) are zero. Third, it is obvious from (7.9) that if $C_k \leq X$ with $X = A, A^\perp, B$ or $B^\perp$ for all $k \in K$, then $\{C_k\}_{k\in K}$ serves as a (commuting) common cause system of the given correlation independently of the chosen state ϕ. Hence, these solutions are called *trivial common cause systems.* If $|K| = 2$, triviality means that $\{C_k\} = \{A, A^\perp\}$ or $\{C_k\} = \{B, B^\perp\}$. Obviously, for superluminal correlation one looks for nontrival common causal explanations.

In AQFT one also has to specify the spacetime localization of the common causes. They have to be in the past of the correlating events. But in which past? One can define different pasts of the bounded regions V_A and V_B in a given spacetime as:

$$\begin{aligned}
\textit{weak past:} \quad & wpast(V_A, V_B) := I_-(V_A) \cup I_-(V_B)\\
\textit{common past:} \quad & cpast(V_A, V_B) := I_-(V_A) \cap I_-(V_B)\\
\textit{strong past:} \quad & spast(V_A, V_B) := \cap_{x\in V_A\cup V_B} I_-(x)
\end{aligned}$$

where $I_-(V)$ denotes the union of the backward light cones $I_-(x)$ of every point x in V (Rédei, Summers 2007). Clearly, $wpast \supset cpast \supset spast$.

With all these definitions in hand we can now define six different common cause systems in local quantum theories according to (i) whether *commutativity* is required and (ii) whether the common cause system is localized in the *weak*, *common* or *strong* past. Thus we can speak about *commuting/noncommuting (weak/strong) common cause systems.*

To address the EPR-Bell problem we will need one more concept. In the EPR scenario the real challenge is to provide a common causal explanation *not* for one *single* correlating pair but for a *set* of correlations (typically three or four correlations). Therefore, we also need to introduce the notion of the so-called joint[1] common cause system:

[1] In (Hofer-Szabó and Vecsernyés, 2012a, 2013a) called *common* common cause system.

Definition 9. Let $\{A_m; m = 1, \ldots M\}$ and $\{B_n; n = 1, \ldots N\}$ be finite sets of projections in the algebras $\mathcal{A}(V_A)$ and $\mathcal{A}(V_B)$, respectively, supported in spacelike separated regions V_A and V_B. Suppose that all pair of spacelike separated projections (A_m, B_n) correlate in a state ϕ of $\mathcal{A}$ in the sense of (7.1). Then the set $\{(A_m, B_n); m = 1, \ldots M; n = 1, \ldots N\}$ of correlations is said to possess a commuting/noncommuting (weak/strong) *joint* common cause system if there exists *a single* commuting/noncommuting (weak/strong) common cause system for *all* correlations (A_m, B_n).

Since providing a *joint* common cause system for a set of correlations is much more demanding than simply providing a common cause system for a *single* correlation, therefore we keep the question of the common causal explanation separated from that of the *joint* common causal explanation. In Section 2 we will investigate the possibility of a common causal explanation for a *single* correlation—or in the philosophers' jargon, the status of Reichenbach's famous Common Cause Principle in AQFT. In Section 3 we will address the more intricate question as to whether EPR correlations can be given a *joint* common causal explanation. The crucial common element in both sections will be *noncommutativity.* We will argue that embracing *noncommuting* common causes in our causal explanation helps us in both cases: (i) in the case of common causal explanation it helps to maintain the validity of Reichenbach's Common Causal Principle in AQFT; (ii) in the case of *joint* common causal explanation it helps to provide a local, *joint* common causal explanation for a set of correlations violating the Bell inequalities. We conclude the paper in Section 4.

7.2 Noncommutative Common Cause Principles in AQFT

Reichenbach's Common Cause Principle (CCP) is the following metaphysical claim: If there is a correlation between two events and there is no direct causal (or logical) connection between the correlating events, then there exists a common cause of the correlation. The precise definition of this informal statement that fits to AQFT is the following:

Definition 10. A local quantum theory is said to satisfy the Commutative/Noncommutative (Weak/Strong) CCP if for any pair $A \in \mathcal{A}(V_A)$ and $B \in \mathcal{A}(V_B)$ of projections supported in spacelike separated regions V_A, V_B and for every locally faithful state $\phi: \mathcal{A} \to \mathbf{C}$ establishing a correlation between A and B in the sense of (7.1), there exists a *nontrivial* commuting/noncommuting common cause system $\{C_k\}_{k \in K} \subset \mathcal{A}(V)$ such that the localization region V is in the (weak/strong) common past of V_A and V_B.

What is the status of these six different CCPs in AQFT?

The question as to whether the Commutative CCPs are valid in a Poincaré covariant local quantum theory in the von Neumann algebraic setting was first raised by Rédei (1997, 1998). As a positive answer to this question, Rédei and Summers (2002, 2007) have shown that the Commutative Weak CCP holds in algebraic quantum field theory with locally infinite degrees of freedom in the following sense: for every locally normal and faithful state and for every superluminally correlating pair of projections there exists a weak common cause, that is a common cause system of size 2 in the weak past of the correlating projections. They have also shown that the localization of a common cause cannot be restricted to $wpast(V_A, V_B) \setminus I_-(V_A)$ or $wpast(V_A, V_B) \setminus I_-(V_B)$ due to logical independence of spacelike separated algebras.

Concerning the Commutative (Strong) CCP less is known. If one also admits projections localized only in *un*bounded regions, then the Strong CCP is known to be false: von Neumann algebras pertaining to complementary wedges contain correlated projections but the strong past of such wedges is empty (see (Summers and Werner, 1988) and (Summers, 1990)). In spacetimes having horizons, e.g. those with Robertson–Walker metric, there exist states which provide correlations among local algebras corresponding to spacelike separated bounded regions such that the common past of these regions is again empty (Wald 1992). Hence, CCP is not valid there. Restricting ourselves to *local* algebras in Minkowski spaces the situation is not clear. We are of the opinion that one cannot decide on the validity of the (Strong) CCP without an explicit reference to the dynamics.

Coming back to the proof of Rédei and Summers, the proof had a crucial premise, namely that the algebras in question are *von Neumann algebras of type III.* Although these algebras are the typical building blocks of Poincaré covariant theories, other local quantum theories apply von Neumann algebras of other type. For example, theories with locally finite degrees of freedom are based on von Neumann algebras of type I. This raised the question as to whether the Commutative Weak CCP is generally valid in AQFT. To address the problem Hofer-Szabó and Vecsernyés (2012a) have chosen a specific local quantum field theory, the local quantum Ising model having locally finite degrees of freedom. It turned out that the Commutative Weak CCP does *not* hold in the local quantum Ising model and it cannot hold either in theories with locally finite degrees of freedom in general.

But why should we require commutativity between the common cause and its effects at all?

Commutativity has a well-defined role in any quantum theories. In standard quantum mechanics observables should commute to be simultaneously measurable. In AQFT the axiom of microcausality ensures that observables with spacelike separated supports—roughly, events happening 'simultaneously'—commute. But cause and effect are typically *not* such simultaneous events! If one considers ordinary QM, one well sees that observables do not commute even with their own time translates in general. For example, the time translate $x(t) := U(t)^{-1}xU(t)$ of the position operator x of the harmonic oscillator in QM does *not* commute with $x \equiv x(0)$ for generic t, since in the ground state vector ψ_0 we have

$$\left[x, x(t)\right] \psi_0 = \frac{-i\hbar \sin(\hbar\omega t)}{m\omega}\psi_0 \not\equiv 0. \tag{7.10}$$

Thus, if an observable A is not a conserved quantity, then the commutator $[A, A(t)] \neq$

0 in general. So why should the commutators $[A, C]$ and $[B, C]$ vanish for the events A, B and for their common cause C supported in their (weak/common/strong) past? We think that commuting common causes are only unnecessary reminiscense of their classical formulation. Due to their relative spacetime localization, that is due to the time delay between the correlating events and the common cause, it is also an unreasonable assumption.

Abandoning commutativity in the definition of the common cause is therefore a desirable move. The first benefit of allowing noncommuting common causes is that the noncommutative version of the result of Rédei and Summers can be regained. This result has been formulated in (Hofer-Szabó and Vecsernyés 2013a) in the following:

Proposition 7. The *Noncommutative* Weak CCP holds in local UHF-type quantum theories. Namely, if $A \in \mathcal{A}(V_A)$ and $B \in \mathcal{A}(V_B)$ are projections with spacelike separated supports V_A and V_B correlating in a locally faithful state ϕ on $\mathcal{A}$, then there exists a common cause $\{C, C^\perp\}$ localized in the weak past of V_A and V_B.

Now, let us turn to the more complicated question as to whether a *set* of correlations violating the Bell inequality can have a *joint* common causal explanation in AQFT. Since our answer requires some knowledge of the main concepts of the Bell scenario in AQFT and some acquaintance with the model in which our results were formulated, we start the next section with a short tutorial on these issues (for more details see (Hofer-Szabó, Vecsernyés, 2012b, 2013b).

7.3 Noncommutative joint common causal explanation for correlations violating the Bell inequality

The Bell problem is treated in AQFT in a subtle mathematical way (Summers and Werner, 1987a,b, Summers 1990); here we introduce, however, only those concepts which are related to the problem of common causal explanation (for more on that see (Hofer-Szabó, Vecsernyés, 2013b)).

Let $A_1, A_2 \in \mathcal{A}(V_A)$ and $B_1, B_2 \in \mathcal{A}(V_B)$ be projections with spacelike separated supports V_A and V_B, respectively. We say that in a locally faithful state ϕ the Clauser–Horne-type *Bell inequality is satisfied* for A_1, A_2, B_1 and B_2 if the following inequality holds:

$$-1 \leqslant \phi(A_1B_1 + A_1B_2 + A_2B_1 - A_2B_2 - A_1 - B_1) \leqslant 0 \tag{7.11}$$

otherwise we say that the *Bell inequality is violated*. (Sometimes in the EPR-Bell literature another inequality, the so-called Clauser–Horne–Shimony–Holte-type Bell inequality is used as a constraint on the expectation of (not *projections* but) self-adjoint *contractions*. Since these two inequalities are equivalent, in what follows we will simply use (9.24) as *the* definition of the Bell inequality.)

In the literature it is a received view that if a set of correlations violates the Bell inequality, then the set cannot be given a joint common causal explanation. The following proposition proven in (Hofer-Szabó and Vecsernyés 2013b) shows that this view is correct *only if* joint common causal explanation is meant as a *commutative* joint common causal explanation:

Proposition 8. Let $A_1, A_2 \in \mathcal{A}(V_A)$ and $B_1, B_2 \in \mathcal{A}(V_B)$ be four projections localized in spacelike separated spacetime regions V_A and V_B, respectively, which correlate in the locally faithful state ϕ. Suppose that $\{(A_m, B_n); m, n = 1, 2\}$ has a joint common causal explanation in the sense of Definition 9. Then the following Bell inequality

$$-1 \leqslant (\phi \circ E_c)(A_1B_1 + A_1B_2 + A_2B_1 - A_2B_2 - A_1 - B_1) \leqslant 0. \qquad (7.12)$$

holds for the state $\phi \circ E_c$. If the joint common cause is a *commuting* one, then the original Bell inequality (9.24) holds for the original state ϕ.

Proposition 8 states that in order to yield a *commuting* joint common causal explanation for the set $\{(A_m, B_n); m, n = 1, 2\}$ the Bell inequality (9.24) has to be satisfied. This result is in complete agreement with the usual approaches to Bell inequalities (see e.g. (Butterfield 1989, 1995, 2007)). But what is the situation with *noncommuting* common cause systems? Since—apart from (7.12)—Proposition 8 is silent about the relation between a *noncommuting* joint common causal explanation and the Bell inequality (9.24), the question arises: Can a *set* of correlations violating the Bell inequality (9.24) have a *noncommuting* joint common causal explanation?

In (Hofer-Szabó, Vecsernyés, 2012b, 2013b) it has been shown that the answer to the above question is positive: the violation of the Bell inequality does *not* exclude a joint common causal explanation *if* common causes can be noncommuting. Moreover, these common causes turned out to be localizable just in the 'right' spacetime region (see below). For this result, we applied a simple AQFT with locally finite degrees of freedom, the so-called local quantum Ising model (for more details see (Hofer-Szabó, Vecsernyés, 2012b, 2013b); for a Hopf algebraic introduction of the model see (Szlachányi, Vecsernyés, 1993), (Nill, Szlachányi, 1997), (Müller, Vecsernyés)).

Consider a 'discretized' version of the two dimensional Minkowski spacetime $\mathcal{M}^2$ covered by minimal double cones $V^m_{t,i}$ of unit diameter with their center in (t, i) for $t, i \in \mathbb{Z}$ or $t, i \in \mathbb{Z} + 1/2$ (see Fig. 7.1). A non-minimal double cone $V_{t,i;s,j}$ in this covering can be generated by two minimal double cones in the sense that $V_{t,i;s,j}$ is the smallest double cone containing both $V^m_{t,i}$ and $V^m_{s,j}$. The set of double cones forms a directed poset which is left invariant by integer space and time translations.

The 'one-point' observable algebras associated to the minimal double cones $V^m_{t,i}$ are defined to be $\mathcal{A}(V^m_{t,i}) \simeq M_1(\mathbf{C}) \oplus M_1(\mathbf{C})$. By introducing appropriate commutation and anticommutation relations between the unitary selfadjoint generators of the 'one-point' observable algebras (which relations respect microcausality) one can generate the net of local algebras. Since there is an increasing sequence of double

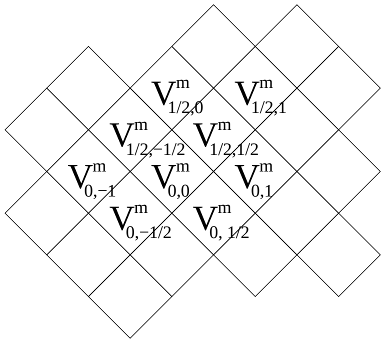

Figure 7.1: The two dimensional discrete Minkowski spacetime covered by minimal double cones.

cones covering $\mathcal{M}^2$ such that the corresponding local algebras are isomorphic to full matrix algebras $M_{2^n}(\mathbb{C})$, the quasilocal observable algebra $\mathcal{A}$ is a uniformly hyperfinite (UHF) C^*-algebra and consequently there exists a unique (non-degenerate) normalized trace $\mathrm{Tr}\colon \mathcal{A} \to \mathbf{C}$ on it.

Now, consider the double cones $V_A := V^m_{0,-1} \cup V^m_{\frac{1}{2},-\frac{1}{2}}$ and $V_B := V^m_{\frac{1}{2},\frac{1}{2}} \cup V^m_{0,1}$ and the 'two-point' algebras $\mathcal{A}(V_A)$ and $\mathcal{A}(V_B)$ pertaining to them (see Fig. 7.2). It turns out that all the minimal projections in $A(\mathbf{a}) \in \mathcal{A}(V_A)$ and $B(\mathbf{b}) \in \mathcal{A}(V_B)$

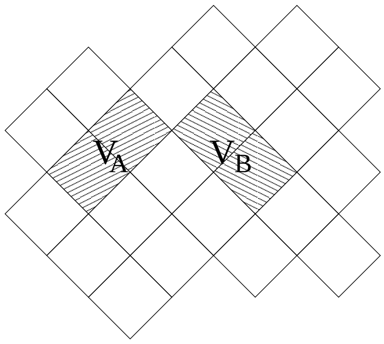

Figure 7.2: Correlations between events in V_A and V_B.

can be parametrized by unit vectors $\mathbf{a}$ and $\mathbf{b}$, respectively in $\mathbf{R}^3$. Now, consider two projections $A_m := A(\mathbf{a^m}); m = 1, 2$ localized in V_A, and two other projections $B_n := B(\mathbf{b^n}); n = 1, 2$ localized in the spacelike separated double cone V_B.

Let the state of the system be the singlet state ϕ^s defined in an appropriate way (by a density operator composed of specific combinations of generators taken from various 'one-point' algebras). It turns out that in state ϕ^s the correlation between

A_m and B_n will the one familiar from the EPR situation:

$$corr(A_m, B_n) := \phi^s(A_m B_n) - \phi^s(A_m)\,\phi^s(B_n) = -\frac{1}{4}\,\langle \mathbf{a^m}, \mathbf{b^n} \rangle \qquad (7.13)$$

where $\langle\ ,\ \rangle$ is the scalar product in $\mathbf{R}^3$. In other words A_m and B_n will correlate whenever $\mathbf{a^m}$ and $\mathbf{b^n}$ are not orthogonal. To violate the Bell inequalitity (9.24) set $\mathbf{a^m}$ and $\mathbf{b^n}$ as follows:

$$\mathbf{a^1} = (0,1,0) \qquad (7.14)$$

$$\mathbf{a^2} = (1,0,0) \qquad (7.15)$$

$$\mathbf{b^1} = \frac{1}{\sqrt{2}}(1,1,0) \qquad (7.16)$$

$$\mathbf{b^2} = \frac{1}{\sqrt{2}}(-1,1,0) \qquad (7.17)$$

With this setting (9.24) will be violated at the lower bound since

$$\phi^s(A_1B_1 + A_1B_2 + A_2B_1 - A_2B_2 - A_1 - B_1) =$$

$$-\frac{1}{2} - \frac{1}{4}\left(\langle \mathbf{a^1}, \mathbf{b^1} \rangle + \langle \mathbf{a^1}, \mathbf{b^2} \rangle + \langle \mathbf{a^2}, \mathbf{b^1} \rangle - \langle \mathbf{a^2}, \mathbf{b^2} \rangle\right) = -\frac{1+\sqrt{2}}{2} \qquad (7.18)$$

Now, the question as to whether the four correlations $\{(A_m, B_n); m, n = 1, 2\}$ violating the Bell inequality (9.24) have a joint common causal explanation was answered in (Hofer-Szabó, Vecsernyés, 2012b) by the following

Proposition 9. Let $A_m := A(\mathbf{a^m}) \in \mathcal{A}(V_A), B_n := B(\mathbf{b^n}) \in \mathcal{A}(V_B); m, n = 1, 2$ be four projections parametrized by the unit vectors *via* (7.14)-(7.17) violating the Bell inequality in the sense of (7.18). Then there exist a noncommuting join common cause $\{C, C^\perp\}$ of the correlations $\{(A_m, B_n); m, n = 1, 2\}$ localizable in the common past $V_C := V_{0,-\frac{1}{2};0,\frac{1}{2}}$ of V_A and V_B (see Fig. 7.3).

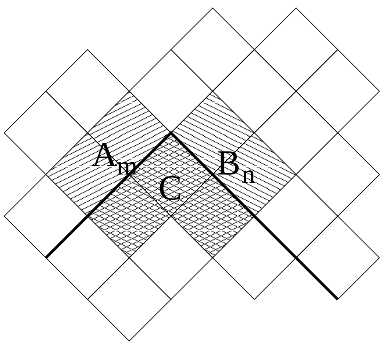

Figure 7.3: Localization of a common cause for the correlations $\{(A_m, B_n)\}$.

Observe that C is localized in the *common past* of the four correlating events that is in the region which seems to be the 'physically most intuitive' localization of the common cause.

Proposition 8 and 9 together show that the relation between the common causal explanation and the Bell inequality in the noncommutative case is different from that in the commutative case. In the latter case the satisfaction of the Bell inquality is a necessary condition for a set of correlations to have a joint common causal explanation. In the noncommutative case, however, the violation of the Bell inequality for a given set of correlations does *not* exclude the possibility of a joint common causal explanation for the set. And indeed, as Proposition 9 shows, one can find a common cause even for a set of correlations violating the Bell inequality. To sum it up, taking seriously the noncommutative character of AQFT where events are represented by not necessarily commuting projections, one can provide a common causal explanation in a much wider range than simply sticking to commutative common causes.

7.4 Conclusions

In the paper we were arguing that embracing noncommuting common causes in our explanatory framework is in line with the spirit of quantum theory and it gives us extra freedom in the search of common causes for correlations. Specifically, it helps to maintain the validity of Reichenbach's Common Causal Principle in the context of AQFT and it also helps to provide a local, *joint* common causal explanation for a set of correlations even if they violate the Bell inequalities.

Using noncommuting common causes naively to address the basic problems of the causal explanation in quantum theory in a formal way is no use whatsoever, if it is not underpinned by a viable ontology on which the causal theory can be based. This is a grandious research project. I conclude here simply by posing the central question of such a project:

Question. What ontology exactly is forced upon us by using noncommuting common causes in our causal explanation?

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Chapter 8

On the relation between the probabilistic characterization of the common cause and Bell's notion of local causality

In this paper the relation between the standard probabilistic characterization of the common cause (used for the derivation of the Bell inequalities) and Bell's notion of local causality will be investigated in the isotone net framework borrowed from algebraic quantum field theory. The logical role of two components in Bell's definition will be scrutinized; namely that the common cause is localized in the *intersection* of the past of the correlated events; and that it provides a *complete specification* of the 'beables' of this intersection.

8.1 Introduction

Standard derivations of the Bell inequalities start from a set of equations representing a probabilistic common causal explanation of correlations. This common causal explanation has three components: a *screening-off condition*, going back to Reichenbach's (1956) original characterization of the common cause, a *locality condition*, expressing probabilistic independences between spacelike separated measurement outcomes and measurement settings, and a *no-conspiracy condition* representing another independency between the common cause and the measurement settings. If one is asked what justifies these probabilistic constraints in representing a proper common causal explanation, the common answer is this: one obtains these equations immediately if one endorses special relativity and looks at the spacetime localization of the events in question. The aim of this paper is to understand more thoroughly this quick answer.

In order to see more clearly how the spatiotemporal and probabilistic characterization of the common cause relate to one another, one has to be clear first of all on three points:

1. To address the problem at all, we need to have a mathematically well-defined and physically well-motivated *framework* connecting events understood as elements of a probability space and regions understood as subsets of a spacetime.

2. Having such a firm framework connecting spatiotemporal and probabilistic entities, we need to *localize* events, among them common causes, in the spacetime.

3. Finally, we have to be clear on what we mean under "*justification* of the probabilistic common causal explanation on spatiotemporal grounds".

Here we briefly comment on the above three points in turn.

Ad 1. Concerning the framework, interestingly enough, there is not a wide choice of mathematical structures representing this highly important connection between probabilistic and spatiotemporal entities. Discounting one approach (Henson, 2005; commented on in the Conclusion and discussion), we are aware of only one such structure, the *isotone net structure* used in algebraic quantum field theory (AQFT). In AQFT observables are represented by (C^*-)algebras associated to bounded regions of a spacetime. This association is called a *net.* A *state* ϕ is defined as a normalized positive linear functional on the quasilocal algebra $\mathcal{A}$ which is the inductive limit of the net. From our perspective, the two important axioms of the net are *isotony* and *local primitive causality. Isotony* requires that if a region V_1 is contained in another region V_2, then the local algebra $\mathcal{A}(V_1)$ associated to V_1 is a (unital C^*-)subalgebra of $\mathcal{A}(V_2)$. *Local primitive causality* is the requirement that for any region V, $\mathcal{A}(V) = \mathcal{A}(V'')$, where V'' is the causal completion (shadow) of V. The framework of isotone nets seems to be flexible enough to be used also for our purposes. The nets which we will use in this paper will be *classical* nets generated by local σ-subalgebras of a Boolean σ-algebra Σ. Thus we borrow a useful mathematical technique from AQFT without endorsing the operational ontology thereof.

Ad 2. Having a neat framework in hand, next we have to localize events. The localization of measurement outcomes and measurement settings is fairly straightforward, but where should we localize common causes? Obviously, the common cause is an event C happening somewhere in the past of two correlated events, say A and B. But in which past? Relativistically two spacelike separated events can have (at least) two different pasts. Let V_A and V_B denote the regions where A and B, respectively are localized. One can then define the *weak past* of A and B as $\mathcal{P}^W(V_A, V_B) := I_-(V_A) \cup I_-(V_B)$ and the *strong past* of A and B as $\mathcal{P}^S(V_A, V_B) := I_-(V_A) \cap I_-(V_B)$ where $I_-(V)$ denotes the union of the causal pasts $I_-(x)$ of every point x in V. Let us call the appropriate common causes *weak* and *strong common causes*, respectively (see Fig. 8.1).

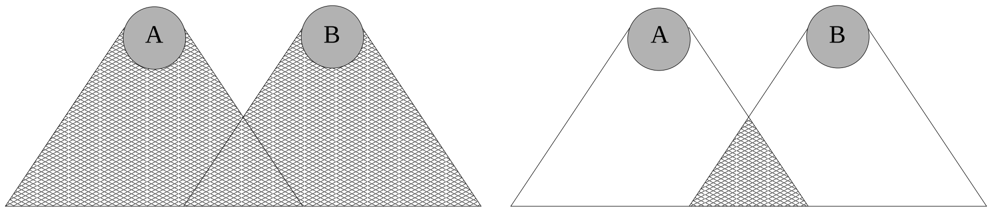

Figure 8.1: Weak and the strong past of the correlated events A and B.

Now, one might consider the *strong* past as a more natural candidate for the localization of the common cause, and indeed plenty of classical examples attest that the strong past is a reasonable choice. The correlation between two fans' shouting at the same time at a football match is explained by the goals scored, that is by events localized in the strong past of the shouts. Curiously enough, however, in AQFT common causes are typically understood as *weak* common causes. It is not difficult to see why.

Consider an isotone net representing a system in AQFT. Suppose that there is a (superluminal) correlation, $\phi(AB) \neq \phi(A)\phi(B)$, between events $A \in \mathcal{A}(V_A)$ and $B \in \mathcal{A}(V_B)$ such that V_A and V_B are spacelike separated. Consider the local algebra $\mathcal{A}((V_A \cup V_B)'')$ associated to the causal completion of $V_A \cup V_B$ and suppose that we find a common cause C of the correlation in $\mathcal{A}((V_A \cup V_B)'')$. In which past of V_A and V_B can C be located? Consider a region V in the *weak* past $P^W(V_A, V_B)$ which is 'wide' enough to ensure that $(V_A \cup V_B) \subset V''$. Due to isotony, $\mathcal{A}(V_A \cup V_B)$ will be a subalgebra of $\mathcal{A}(V'')$ which, due to local primitive causality, is identical to $\mathcal{A}(V)$. Thus, C will be located in V and hence in the weak past of V_A and V_B. To sum up, isotony and local primitive causality together ensures that if a superluminal correlation has a common cause, then it can be localized in the weak past.

Can the common cause be localized also in the strong past? It might, but if so, this will not be simply due to the axioms of AQFT. If V is in $P^S(V_A, V_B)$, then isotony and local primitive causality does not help to relate $\mathcal{A}(V)$ to $\mathcal{A}((V_A \cup V_B)'')$. One also needs to know about the dynamics of the system. The axioms of AQFT are completely silent about whether one can locate the common cause in the strong past. As a consequence, weak common causes *cannot* be excluded *a priori* from our explanatory arsenal. Thus, we had better open leave the question regarding the apt spacetime localization of the common cause.

Ad 3. Finally, we have to pin down the meaning of the term "justification of the probabilistic common causal explanation on spatiotemporal grounds". What we mean here is this: we need to have a *principle* regulating the probabilistic independences of events on the basis of their possible causal connectedness in tune with special relativity. An analogy for such a regulating principle might help. The theory of Bayesian nets involves two parts: a causal graph representing the causal relations among certain events and a probability space with random variables. How are these two parts of the theory related to one another? The bridge relating the two components is called the *Causal Markov Condition.* It says that if the nodes on the graph are related to one another in such-and-such a way, then the variables pertaining to the nodes should satisfy such-and-such probabilistic independences. So the role of the Causal Markov Condition in the theory of Bayesian nets is to coordinate the probabilistic and the graphical description of causal relations.

A principle playing a similar coordinating role in the causal explanation of correlations has been introduced into the literature by John S. Bell (1975/2004) and called *local causality.* Local causality is a relativistic principle tailor-made to study probabilistic relations between events localized in different spacetime regions, among them the relation between the common cause and the correlated events. Thus, we will understand the term "justification of the probabilistic common causal explanation on spatiotemporal grounds" similarly to the Bayesian net theorist: local causality *implies* just those probabilistic independences which characterize the standard common causal explanation.

Putting Points 1-3 together we are faced with the following

Project. Given the isotone net framework connecting events and spacetime regions (Point 1), and given the spatiotemporal localization of the various measurement outcomes, measurement settings and common causes (Point 2), one is to define local causality in the isotone net framework such that the probabilistic independences implied by local causality (Point 3) are just the ones used in the standard probabilistic characterization of the common causal explanation.

In brief, the accommodation of a set of correlations within a locally causal net implies that for any correlations there exist common causes satisfying certain probabilistic constraints.

This, however, is only the coarse-grained story of the paper. Reading Bell's careful formulation of local causality, two requirements will stand out in the definition: one is *atomicity* representing the "complete specification" of the causal past of the correlated events, the other is the *localization* of the common cause in the *strong past.* Our fine-grained story will be to analyze the significance of these ingredients in the definition of local causality. It will turn out that the link between the spatiotemporal and the probabilistic characterization of the common cause is very sensitive to these components of the definition of local causality, as was rightly emphasized by Bell himself. In detail, we would like to address the following questions:

(i) What is the exact role of atomicity in the justification of the probabilistic characterization of the common cause by local causality?

(ii) Do the probabilistic constraints imposed on the notion of common cause restrict the possible spacetime localization of the common cause? Do we need to choose, for example, between weak and strong common causes?

(iii) How do atomicity and localization relate to one another; which of the common causes localized in different pasts need to be atomic?

Our paper follows a research line which has been followed by many. To our knowledge, the first to "survey the ways in which one could associate regions" with events such that it makes "plausible not only completeness and locality, but other assumptions of the Bell inequality" was Butterfield (1989, p. 135). Also, the necessity to introduce spatiotemporal concepts so as to understand the Common Cause Principle was pointed out by Uffink (1999). Common Cause Principle and its role in the EPR-Bell scenario has been thoroughly investigated by The Bern group (Grasshoff, Portmann and Wüthrich, 2005), The Cracow group (Placek and Wronski, 2009), and The Budapest group (Hofer-Szabó, Rédei and Szabó, 2013, especially in Chapter 8 and 9). The status of the Common Cause Principle in AQFT was first investigated by Rédei (1997), and further analyzed in Poincaré covariant AQFT by Rédei and Summers (2002) and in lattice AQFT by Hofer-Szabó and Vecsernyés (2012a, 2013a). Butterfield analysed the assumptions leading to the Bell inequalities in AQFT in (Butterfield, 1995), and the relation of the Common Cause Principe to the Bell inequalities and to various forms of Stochastic Einstein Locality in (Butterfield, 2007). For an earlier discussion on the relation of Stochastic Einstein Locality to the axioms of AQFT, see (Rédei 1991) and (Muller and Butterfield 1994). Hofer-Szabó and Vecsernyés (2012b, 2013b) reassessed the assumptions of the Bell inequalities in AQFT with respect to non-commuting common causes. In a formalism very close or maybe identical to our isotone net formalism, Henson (2013b) treated an important topic, namely that giving up separability does not block the derivation of the Bell inequalities. An interesting debate between Henson, Rédei and San Pedro (Henson, 2005; Rédei and San Pedro, 2012; Henson, 2013a) has been taking place recently in this Journal. We will comment on this debate in the Conclusion and discussions. For a parallel approach to ours, where the assumptions of the Bell inequalities are backed not by spatiotemporal considerations but by the Causal Markov Condition, see (Glymour 2006). For the relation of Causal Markov Condition to EPR correlations see (Suárez, 2013). For a general treatment of Bell's local causality in local physical theories see the more technical (Hofer-Szabó and Vecsernyés 2014a) or its philosopher-friendly version (Hofer-Szabó and Vecsernyés 2014b).

Our paper is structured as follows. In Section 2 the standard requirements of the probabilistic common causal explanation will be recalled. In Section 3 Bell's original idea of local causality will be delineated and redefined in the isotone net formalism. Section 4 will be devoted to the first ingredient of Bell's definition, namely atomicity; Section 5 to the second one, namely localization. In order to proceed in a more picturesque way, both in Section 4 and 5 classical toy models will be introduced helping us to explicate the more abstract results. We conclude the paper in Section 6. Some technicalities are put in the Appendices.

8.2 Common causal explanation

As mentioned above, the first probabilistic characterization of the common cause is due to Reichenbach. There is a long route leading from Reichenbach's original idea of the common cause to the sophisticated probabilistic requirements used today in the philosophy of quantum physics. Here we will not detail the steps of how the notion of common cause evolved and became more and more suitable for causal explanation of the EPR-Bell scenario (for this see (Hofer-Szabó, Rédei and Szabó, 2013), or for a short version (Hofer-Szabó and Vecsernyés, 2012a)). Instead we will jump directly to the full-fledged probabilistic characterization of the common cause and give a brief motivation of the requirements thereafter.

Let $\{a_m\}$ and $\{b_n\}$ $(m \in M, n \in N)$ be two sets of measurement procedures (thought of as happening in two spacelike separated spacetime regions). Suppose that each measurement can have two outcomes and denote the 'positive' outcomes by A_m and B_n and the 'negative' outcomes by $\overline{A}_m$ and $\overline{B}_n$, respectively. Let all these events be accommodated in a classical probability space (Σ, p). Suppose that there is a conditional correlation between the measurement outcomes in the sense that for any $m \in M$ and $n \in N$

$$p(A_m \wedge B_n | a_m \wedge b_n) \neq p(A_m | a_m)\, p(B_n | b_n) \tag{8.1}$$

representing that if we measure the pair a_m and b_n, the appropriate outcomes will be correlated.

The standard probabilistic characterization of a common causal explanation of the correlations (10.1) is the following. A partition $\{C_k\}$ in Σ (that is a set of mutually exclusive events adding up to the unit) is said to be a *local, non-conspiratorial joint common causal explanation* of the correlations (10.1) if for any $m, m' \in M$ and $n, n' \in N$ the following requirements hold:

$$p(A_m \wedge B_n | a_m \wedge b_n \wedge C_k) = p(A_m | a_m \wedge b_n \wedge C_k)\, p(B_n | a_m \wedge b_n \wedge C_k) \qquad \text{(screening-off)} \tag{8.2}$$

$$p(A_m | a_m \wedge b_n \wedge C_k) = p(A_m | a_m \wedge b_{n'} \wedge C_k) \qquad \text{(locality)} \tag{8.3}$$

$$p(B_n | a_m \wedge b_n \wedge C_k) = p(B_n | a_{m'} \wedge b_n \wedge C_k) \qquad \text{(locality)} \tag{8.4}$$

$$p(a_m \wedge b_n \wedge C_k) = p(a_m \wedge b_n)\, p(C_k) \qquad \text{(no-conspiracy)} \tag{8.5}$$

The motivation behind requirements (10.6)-(8.5) is the following. *Screening-off* (10.6) (also called as *outcome independence* (Shimony, 1986), *completeness* (Jarrett, 1984) and *causality* (Van Fraassen, 1982)) is simply the application of Reichenbach's original characterization of the common cause as a screener-off to conditional correlations: although A_m and B_n are correlated when conditioned on a_m and b_n, they will cease to be so, if we further condition on C_k. *Locality* (8.3)-(8.4) (also called as *parameter independence* (Shimony, 1986), *locality* (Jarrett, 1984) and *hidden locality* (Van Fraassen, 1982)) is the constraints that the measurement outcome on the one side can depend only on the measurement choice on the same side and the value of the common cause, but not on the measurement choice on the opposite side (for more on this, see below). Finally, no-conspiracy (8.5) is the requirement that the common cause system and the measurement settings should not influence each other: they should be probabilistically independent.

Now, it is a well known fact that if a set of correlations has a local, non-conspiratorial joint common causal explanation in the above sense, then the set of correlations has to satisfy various Bell inequalities.[1] If quantum correlations are interpreted as classical conditional correlations *á la* (10.1), these Bell inequalities are violated, excluding a local, non-conspiratorial joint common causal explanation of the EPR scenario. Our aim, however, is not to follow the route leading *from* the common causal explanation (10.6)-(8.5) to the Bell inequalities, but rather the route leading *to* the common causal explanation itself. At any rate, in the EPR-Bell literature (10.6)-(8.5) is regarded as the correct probabilistic characterization of the common cause. But observe that the above motivations for the probabilistic independence relations (10.6)-(8.5) are completely meaningless unless we first decide on Points 1 and 2 of the Introduction: that is unless we have a principled way to associate *events* understood as elements of the probability space (Σ, p) to *regions* of a given spacetime (Point 1), and unless we localize the events in question somewhere in the spacetime (Point 2).

So suppose that we do have such an association in form of an isotone net $\mathfrak{N}$ associating bounded regions of the Minkowski spacetime to σ-subalgebras of Σ. Suppose furthermore that we localize common causes in one of the two above mentioned ways, that is common causes are either weak or strong common causes. To address Point 3 of the Introduction, namely the 'bridge law' between the spacetime and probabilistic considerations, we have to introduce one more notion, namely local causality. We do this in Section 3.

8.3 Local causality

As mentioned in the Introduction, there is an influential tradition according to which equations (10.6)-(8.5) are consequences of the requirement that a certain set of correlations are to be accommodated in a *locally causal* theory. The clearest formulation of such a theory is due to Bell himself:

> "Consider a theory in which the assignment of values to some beables Λ implies, not necessarily a particular value, but a probability distribution, for another beable A. Let $p(A|\Lambda)$ denote[2] the probability of a particular value A given particular values Λ. Let A be localized in a space-time region A. Let B be a second beable localized in a second region B separated from A in a spacelike way. (Fig. 8.2.) Now my intuitive notion of local causality is that events in B should not be 'causes' of events in A, and vice versa. But this does not mean that the two sets of events should be uncorrelated, for they could have common causes in the overlap of their backward light cones. It is perfectly intelligible then that if Λ in (8.6) does not contain a complete record of events in that overlap, it can be usefully supplemented by information

[1] For the derivation of one of the simplest Bell inequality, the Clauser–Horne inequality, see Appendix A.

[2] For the sake of uniformity throughout the paper, I slightly changed Bell's notation and figures.

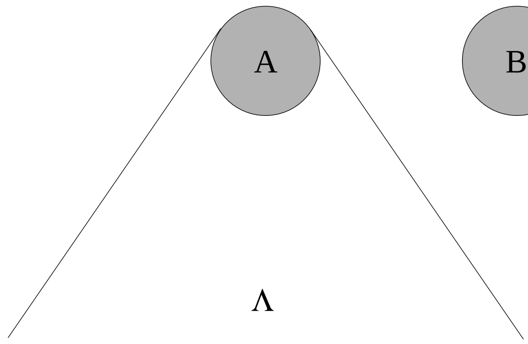

Figure 8.2: Local causality I.

> from region B. So in general it is expected that
>
> $$p(A|\Lambda, B) \neq p(A|\Lambda) \tag{8.6}$$
>
> However, in the particular case that Λ contains already a *complete* specification of beables in the overlap of the light cones, supplementary information from region B could reasonably be expected to be redundant."

And here comes the definition of a locally causal theory.

> "Let C denote a specification of *all* beables, of some theory, belonging to the overlap of the backward light cones of spacelike regions A and B. Let a be a specification of some beables from the remainder of the

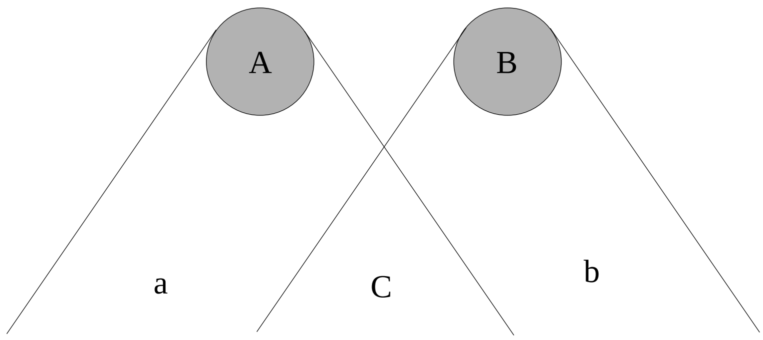

Figure 8.3: Local causality II.

> backward light cone of A, and B of some beables in the region B. (See Fig. 8.3.) Then in a *locally causal theory*
>
> $$p(A|a, C, B) = p(A|a, C) \tag{8.7}$$

whenever both probabilities are given by the theory." (Bell, 1975/2004, p. 54)

Now, let us spell out Bell's characterization of local causality in our isotone net framework. To this end we need to 'translate' a number of terms Bell uses in his formulation into our language.

First, we need to translate Bell's language using random *variables* in (8.7) into a language using *events*. This is straightforward since events are special random variables, namely characteristic functions.

Second, we are to interpret the term 'beable'. 'Beable' is Bell's neologism and is contrasted to the term 'observable' used in quantum theory. "The *be*ables of the theory are those entities in it which are, at least tentatively, to be taken seriously, as corresponding to something real" (Bell, 1990/2004, p. 234). Without the clarification of what the "beables" of a given theory really are, one cannot even formulate local theory since "there *are* things which do go faster than light. British sovereignty is the classical example. When the Queen dies in London (long may it be delayed) the Prince of Wales, lecturing on modern architecture in Australia, becomes instantaneously King" (Bell, 1990/2004, p. 236). In order to vitiate such 'violation' of local causality, the clarification of the "beables" of a theory is indispensable. (Cf. Norsen 2011.) What are the beables in the isotone net structure? Since these nets are classical and hence they represent objective physical events, *any* element of *any* local algebra will be regarded here as a beable.

Third, translating 'beable' simply as 'elements of an algebra' naturally brings with it the translation of the term 'complete specification of beables' as an 'atom of the algebra in question'. Here of course it is assumed that the local algebras of the net are atomic (which is typically not the case in AQFT). (For the translation of 'complete specification' into atomicity see (Henson, 2013a, p. 1015).)

Finally, an important point. Both in his wording and also in his figures Bell seems to take into account the *whole causal past* of the events in question. In the formulation of local causality he does not assume some kind of Markovian condition rendering superfluous the infinite tail of the past regions below a certain Cauchy surface. Other parts of Bell's text, however, speak for a more local interpretation of beable.[3] Moreover, Bell's *La nouvelle cuisine* (Bell, 1990/2004), a posthumous paper on the same subject provides another definition of local causality where the screener-off regions are definitely *finite*. This definition is closer in spirit to the formalism of isotone nets since here only bounded regions are associated to local algebras. Therefore, we will here endorse this "finite" reading of local causality. (We will come back to this point in the Conclusion and discussion.)

With this 'translation manual' in hand, Bell's notion of local causality can be paraphrased as follows.

[3] Cf. "We will be particularly concerned with *local* beables, those which (unlike for example the total energy) can be assigned to some *bounded* [my italics] space-time region." (Bell, 1975/2004, p. 53)

Definition 11. An isotone net $\mathfrak{N}$ associating bounded regions of the Minkowski spacetime to σ-subalgebras of Σ is called *locally causal*, if for any classical probability measure p on Σ[4], and for any two events $A_m \in \mathcal{A}(V_A)$ and $B_n \in \mathcal{A}(V_B)$ localized in the spacelike separated regions V_A and V_B and correlating in the probability measure p, the following holds.

Let V_a, V_b and V_C be three spacetime regions (see Fig. 8.4) such that

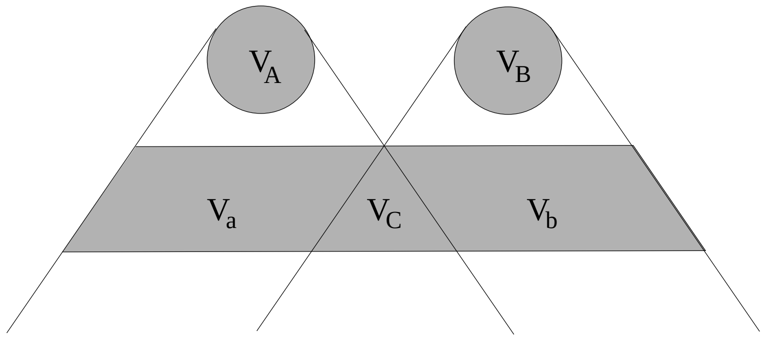

Figure 8.4: Local causality in isotone nets

$$V_a \subset (I_-(V_A) \setminus I_-(V_B)) \tag{8.8}$$
$$V_b \subset (I_-(V_B) \setminus I_-(V_A)) \tag{8.9}$$
$$V_C \subset \mathcal{P}^S(V_A, V_B) \tag{8.10}$$
$$V_C \subset \mathcal{P}^S(V_a, V_b) \tag{8.11}$$
$$V_A \subset (V_a \cup V_C)'' \tag{8.12}$$
$$V_B \subset (V_C \cup V_b)'' \tag{8.13}$$

Let a_m, b_n and C_k be any three atoms of the algebras $\mathcal{A}(V_a)$, $\mathcal{A}(V_b)$ and $\mathcal{A}(V_C)$, respectively, associated to the appropriate regions. Then the following conditional probabilistic independences hold:

$$p(A_m|a_m \wedge C_k \wedge B_n) = p(A_m|a_m \wedge C_k) \tag{8.14}$$
$$p(B_n|A_m \wedge C_k \wedge b_n) = p(B_n|b_n \wedge C_k) \tag{8.15}$$
$$p(A_m|a_m \wedge C_k \wedge b_n) = p(A_m|a_m \wedge C_k) \tag{8.16}$$
$$p(B_n|a_m \wedge C_k \wedge b_n) = p(B_n|b_n \wedge C_k) \tag{8.17}$$

Why four equations instead of Bell's single (8.7)? Observe that (8.15) is just the symmetric version of (8.14) where A_m and a_m are interchanged with B_n and b_n. Equations (8.16)-(8.17), however, are slight extensions of Bell's formulation.

[4]Or, in the more general AQFT case (which we do not need now): for any state ϕ on the quasilocal algebra $\mathcal{A}$. (Cf. Section 1 above.)

Observe that V_A is spacelike separated not only from V_B but also from V_b; moreover, V_C is in the strong past of A and B, $\mathcal{P}^S(V_A, V_b)$. Therefore, conditioned on the complete specification of $V_a \cup V_C$, the same independence should hold between A_m and b_n as between A_m and B_n. Thus (8.16) is the application of Bell's idea to algebras $\mathcal{A}(V_A)$ and $\mathcal{A}(V_b)$, and (8.17) to algebras $\mathcal{A}(V_b)$ and $\mathcal{A}(V_A)$. There are no more spacelike separated regions in Fig. 8.4 to which local causality could be applied.

How do the above considerations relate to the probabilistic characterization (10.6)-(8.5) of the common cause delineated in the previous Section?

First observe that (8.16)-(8.17) are equivalent to *locality* (8.3)-(8.4) and from (8.14)-(8.17) *screening-off* (10.6) follows directly. This proves that the probabilistic characterization of the common cause by the requirements of screening-off and locality can be 'derived' from Bell's notion of local causality imposed on an isotone net associating spacetime regions and local algebras.

There is, however, an important proviso. The third requirement in the definition of a common causal explanation, namely no-conspiracy (8.5) *cannot* be 'derived' from Bell's notion of local causality in a similar way. No-conspiracy is an independent assumption stating that the events $a_m \wedge b_n$ and C_k are probabilistically independent.

Let us come back for a moment to the definition of a locally causal net. In Definition 11 we required (8.14)-(8.17) and hence (10.6)-(8.4) to hold only for the *atoms* a_m and C_k of the algebras $\mathcal{A}(V_a)$ and $\mathcal{A}(V_C)$, respectively. Bell's original definition, however, seems to be more stringent; here (8.7) is required not only for the atoms of $\mathcal{A}(V_a)$ but *for any element.* This might suggest that our definition is weaker than that of Bell. This, however, is not the case. In Proposition 12 at the end of the paper we will show that in a locally causal net (10.6)-(8.4) hold not only for the atomic events a_m, b_n and C_k, but (given some independence condition) also for any Boolean combination $a := \vee_{m\in M'} a_m$, $b := \vee_{n\in N'} a_m$ $(M' \subseteq M, N' \subseteq N)$ of the measurement conditions. Note, however, that the common cause system C_k cannot be 'aggregated' in this way: (10.6)-(8.4) will *not* necessarily hold for the Boolean combination $C := \vee_{k\in K'} C_k$ $(K' \subseteq K)$. This is why it is *necessary* to demand atomicity ("complete specification") in the strong past of the correlated events and *sufficient* to demand it outside it. We will come back to this point later.

An interesting question with respect to AQFT is the following. What is the relation between *local primitive causality* as standardly used in AQFT and our definition of local causality? The answer is given in the following proposition:

Proposition 10. A classical, atomic isotone net which satisfies local primitive causality ($\mathcal{A}(V) = \mathcal{A}(V'')$ for any region V), automatically satisfies also local causality (8.14)-(8.17) for events in regions as shown in Fig. 8.4.

Proof. Consider first (8.14). Due to isotony and local primitive causality $\mathcal{A}(V_A) \subset \mathcal{A}((V_a \cup V_C)'') = \mathcal{A}(V_a \cup V_C)$ and hence for any atom $a_m \wedge C_k$ of $\mathcal{A}(V_a \cup V_C)$: either (i) $A_m \wedge a_m \wedge C_k = 0$ or (ii) $A_m \wedge a_m \wedge C_k = a_m \wedge C_k$. In case (i) both sides of

(8.14) is zero, in case (ii) both sides of (8.14) is one. One obtains (8.15)-(8.17) in a similar fashion. ■

Intuitively, isotony and local primitive causality together ensure that the atoms of $\mathcal{A}(V_a \cup V_C)$ will also be atoms of $\mathcal{A}(V_A)$, hence screening off every correlation. For a more general proposition stating that in *any* atomic classical or quantum isotone net satisfying local primitive causality local causality also holds, see (Hofer-Szabó and Vecsernyés 2014a, Prop. 1) and (Hofer-Szabó and Vecsernyés 2014b, Sec. 3). For relating local causality (Stochastic Einstein Locality) to the axioms of AQFT (treated in the tradition of the so-called syntactical view of scientific theories), see (Rédei 1991) and (Muller and Butterfield 1994).

Reading Bell's formulation of local causality carefully, two ingredients of the definition stand out clearly. The one is that (i) the common cause system provides "a *complete* specification of beables", and (ii) it is located in the "overlap of the light cones". In our terminology, (i) C_k is an *atom* of the appropriate algebra, (ii) it is located in the *strong past* of the correlated events. Bell explicitly stresses both points, and in all the subsequent papers of Van Fraassen (1982), Jarrett (1984), Shimony (1986) etc. trying to turn spacetime considerations into probabilistic independences these two requirements have been (explicitly or implicitly) made.

However, neither requirements are *a priori* concerning the idea of a common cause. One can easily make up common causes which are either *non-atomic* or *not located in the strong past* of the correlated events. How do these common causes relate to Bell's notion of local causality? In the following two Sections the relation between local causality and probabilistic characterization of the common cause will be studied first in the case of non-atomic common causes, then in case of weak common causes. In each Section toy models will be introduced first, then the formal results will be gathered.

8.4 Non-atomic common causes

Example 1. Consider the following toy model. There are five lighthouses on the ocean in a line at equal distances from one another. (See Fig. 8.5.) Let us count

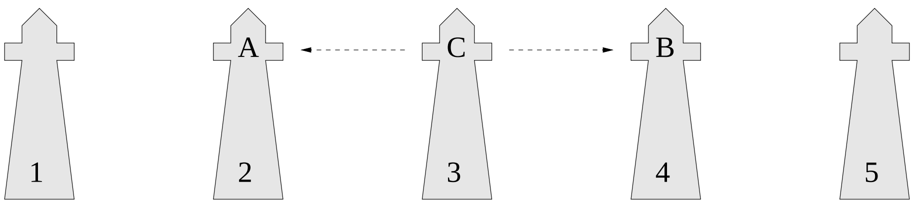

Figure 8.5: Lighthouses I.

them from left to right. In the middle one, that is in lighthouse 3 the lighthouse keeper C has three lamps, C', C'' and C'''. He has the following strategy for turning the lamps on: either he turns on only the lamp C', or only lamp C''', or all three lamps, or none. He never turns on the lamps in any other combination. He chooses between these four options with equal probability (say, by tossing two coins). Let us denote that a given lamp is turned on and off by C and $\overline{C}$, respectively. Using this notation the four possible state of the lamps are the following:

$$C_1 \equiv C' \wedge \overline{C}'' \wedge \overline{C}''' \tag{8.18}$$
$$C_2 \equiv \overline{C}' \wedge \overline{C}'' \wedge C''' \tag{8.19}$$
$$C_3 \equiv C' \wedge C'' \wedge C''' \tag{8.20}$$
$$C_4 \equiv \overline{C}' \wedge \overline{C}'' \wedge \overline{C}''' \tag{8.21}$$

each with probability

$$p(C_k) = \frac{1}{4} \tag{8.22}$$

Now, in the left neighboring lighthouse, that is in lighthouse 2, there is another lighthouse keeper, A; and his role is simply to watch the light signals arriving from either the left or from the right, that is from either lighthouse 1 or lighthouse 3. He does not know that lighthouse 1 is empty, therefore he spends equal time watching both neighboring lighthouses. Suppose furthermore that if he is watching to the left, he will miss the light signals coming from the right. This means that with probability $\frac{1}{2}$ he observes the signals coming from lighthouse 3 and with probability $\frac{1}{2}$ he will miss them. Denoting the event that the lighthouse keeper A is watching to the left and to the right by a_L and a_R, respectively and denoting by A the event that he observes a light signal (disregarding from which lamp it comes), one obtains the following conditional probabilities:

$$p(A|a_m \wedge C_k) = \begin{cases} 1 & \text{if } m = R, k = 1,2,3 \\ 0 & \text{otherwise.} \end{cases} \tag{8.23}$$

In other words, the lighthouse keeper A observes the light signal only if he is watching right and there is a signal sent from C.

Suppose that the same thing happens also in lighthouse 4. The lighthouse keeper B is watching in both directions with equal probability, but since lighthouse 5 is empty, he misses the light signal coming from lighthouse 3 with probability $\frac{1}{2}$. Denoting again the events that the lighthouse keeper B is watching to the left and to the right by b_L and b_R, respectively and denoting by B the event that he observes a signal, one obtains the following conditional probabilities for B's observing a light signal:

$$p(B|b_n \wedge C_k) = \begin{cases} 1 & \text{if } n = L, k = 1,2,3 \\ 0 & \text{otherwise.} \end{cases} \tag{8.24}$$

This situation completely characterizes a probability space. The event algebra is generated by the following events:

$$A,\ \overline{A},\ B,\ \overline{B},\ a_m,\ b_n,\ C_k$$

with $m, n = L, R$ and $k = 1, 2, 3, 4$. The event algebra has 64 atoms, 16 of which have non-zero probability:

$$
\begin{aligned}
p(A \wedge B \wedge a_m \wedge b_n \wedge C_k) &= \frac{1}{16} && \text{if } m = R,\ n = L,\ k = 1, 2, 3 \\
p(A \wedge \overline{B} \wedge a_m \wedge b_n \wedge C_k) &= \frac{1}{16} && \text{if } m, n = R,\ k = 1, 2, 3 \\
p(\overline{A} \wedge B \wedge a_m \wedge b_n \wedge C_k) &= \frac{1}{16} && \text{if } m, n = L,\ k = 1, 2, 3 \\
p(\overline{A} \wedge \overline{B} \wedge a_m \wedge b_n \wedge C_k) &= \frac{1}{16} && \text{if } \begin{cases} m = L,\ n = R,\ k = 1, 2, 3, \\ \text{or } k = 4 \end{cases}
\end{aligned}
$$

and the remaining 48 are of probability zero. By means of the probability of the atoms one can easily calculate the probability of any events of the algebra.

Now, it is easy to see that there is a correlation between events A and B that is between the lighthouse keepers' observing a light signal, both in the non-conditional and conditional sense:

$$
\frac{3}{16} = p(A \wedge B) \quad \neq \quad p(A)\, p(B) = \frac{3}{8} \cdot \frac{3}{8} \tag{8.25}
$$

$$
\frac{3}{4} = p(A \wedge B | a_m \wedge b_n) \quad \neq \quad p(A|a_m)\, p(B|b_n) = \frac{3}{4} \cdot \frac{3}{4} \qquad \text{if } m = R, n = L \tag{8.26}
$$

As one expects, the correlation is due to C's signaling: C_k is a local, (non-conspiratorial) joint common causal explanation of the correlation (8.26) in the sense of (10.6)-(8.5):

$$
\begin{aligned}
p(A \wedge B | a_m \wedge b_n \wedge C_k) = p(A|a_m \wedge b_n \wedge C_k)\, p(B|a_m \wedge b_n \wedge C_k) &= \begin{cases} 1 & \text{if } m = R, n = L, k = 1, 2, 3 \\ 0 & \text{otherwise} \end{cases} \\
p(A|a_m \wedge b_n \wedge C_k) = p(A|a_m \wedge b_{n'} \wedge C_k) &= \begin{cases} 1 & \text{if } m = R, k = 1, 2, 3 \\ 0 & \text{otherwise} \end{cases} \\
p(B|a_m \wedge b_n \wedge C_k) = p(B_n|a_{m'} \wedge b_n \wedge C_k) &= \begin{cases} 1 & \text{if } n = L, k = 1, 2, 3 \\ 0 & \text{otherwise} \end{cases} \\
p(a_m \wedge b_n \wedge C_k) = p(a_m \wedge b_n)\, p(C_k) &= \frac{1}{4} \cdot \frac{1}{4}
\end{aligned}
$$

Example 2. Suppose we take a coarser clustering of the switching of the lamps, say $D_1 \equiv C_1 \vee C_2 \vee C_3$ and $D_2 \equiv C_4$. Physically, D_1 is the event that *any* light is on in lighthouse 3, and D_2 is the event that *no* light is on. As one expects, for this coarser partition the common cause equations (10.6)-(8.5) will hold just as well as for the partition $\{C_k\}$:

$$\begin{aligned}
p(A \wedge B | a_m \wedge b_n \wedge D_k) = p(A|a_m \wedge b_n \wedge D_k)\, p(B|a_m \wedge b_n \wedge D_k) \quad &= \quad \begin{cases} 1 & \text{if } m = R, n = L, k = 1 \\ 0 & \text{otherwise} \end{cases} \\
p(A|a_m \wedge b_n \wedge D_k) = p(A|a_m \wedge b_{n'} \wedge D_k) \quad &= \quad \begin{cases} 1 & \text{if } m = R, k = 1 \\ 0 & \text{otherwise} \end{cases} \\
p(B|a_m \wedge b_n \wedge D_k) = p(B_n|a_{m'} \wedge b_n \wedge D_k) \quad &= \quad \begin{cases} 1 & \text{if } n = L, k = 1 \\ 0 & \text{otherwise} \end{cases} \\
p(a_m \wedge b_n \wedge D_k) = p(a_m \wedge b_n)\, p(D_k) \quad &= \quad \begin{cases} \frac{1}{4} \cdot \frac{3}{4} & \text{if } n = L, k = 1 \\ \frac{1}{4} \cdot \frac{1}{4} & \text{otherwise} \end{cases}
\end{aligned}$$

Thus, $\{D_k\}$ is also a local, (non-conspiratorial) joint common causal explanation of the correlation (8.26).

Example 3. Now, consider a coarser clustering of the switchings 'in the wrong way': $D'_1 \equiv C_1 \vee C_2 \vee C_4$ and $D'_2 \equiv C_3$ mixing together lights being on with lights being off. Contrary to the previous case, for this coarser partition the requirement of screening-off is violated. For example:

$$\frac{2}{3} = p(A \wedge B | a_R \wedge b_L \wedge D'_1) \quad \neq \quad p(A|a_R \wedge b_L \wedge D'_1)\, p(B|a_R \wedge b_L \wedge D'_1) = \frac{2}{3} \cdot \frac{2}{3}$$

(Locality and no-conspiracy will hold even in this case.) Hence $\{D'_k\}$ is *not* a local, (non-conspiratorial) joint common causal explanation of the correlation (8.26).

Now, let us consider the spacetime diagram of the above examples depicted in Fig. 8.6. Let $\mathfrak{N}$ be a locally causal net associating bounded spacetime regions

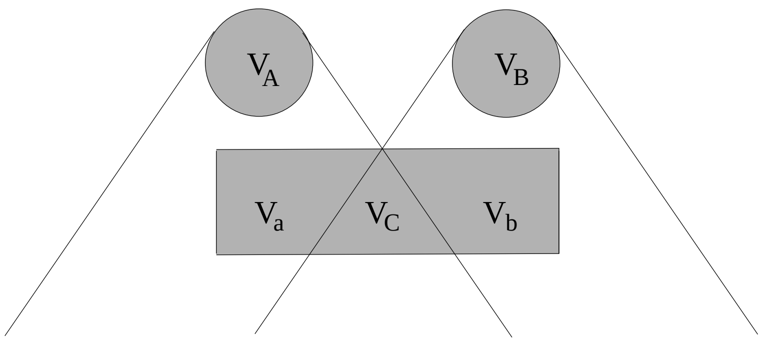

Figure 8.6: Spacetime diagram of Examples 1, 2 and 3.

to local algebras such that $A \in \mathcal{A}(V_A)$, $B \in \mathcal{A}(V_B)$, $a_m \in \mathcal{A}(V_a)$, $b_n \in \mathcal{A}(V_b)$ and $C_k, D_k, D'_k \in \mathcal{A}(V_C)$ for all m, n and k. As shown in Section 2, local causality of the net implies that the set $\{C_k\}$—being an atomic partition localized in the strong past $\mathcal{P}^S(V_A, V_B)$—satisfies (10.6)-(8.4), hence providing a local, joint common causal explanation of the correlation (8.26). (No-conspiracy (8.5), as already stressed in Section 2, is not a consequence of local causality but is assumed in the toy model.) Thus, $\{C_k\}$ is an *atomic*, *strong*, local, non-conspiratorial joint common cause system.

What about *non-atomic* partitions localized in the strong past? Again, both $\{D_k\}$ and $\{D'_k\}$ are localized in $\mathcal{P}^S(V_A, V_B)$, but whereas $\{D_k\}$ is a common cause system of the correlation (8.26), $\{D'_k\}$ is *not*. Thus, local causality is completely silent about whether a *coarse-grained partition* of a local algebra in the strong past is a common cause system of the correlated events or not. This 'non-aggregable' character of the atomic common cause relies heavily on the fact that it is localized in the *strong* past—as will be seen in Proposition 12 in the next Section when contrasted with the opposite character of *weak* common causes. Moreover, the satisfaction of equations (10.6)-(8.5) for a given partition also does not ensure that finer-grained partitions will also do so (this is Simpson's paradox; see e.g. (Uffink 1999)). In this sense the existence of a common cause system characterized by the probabilistic constraints (10.6)-(8.5) for a given correlation is a *weaker requirement* than the accommodation of the same correlation in a locally causal theory. There are many more local, non-conspiratorial joint common cause systems than the atomic ones required by locally causal theories.

Obviously, from the perspective of the EPR-Bell scenario this difference is not of central importance, since the violation of the Bell inequalities derived from (10.6)-(8.5) also excludes atomic common cause systems and hence the possibility of a locally causal theory. But focusing simply on the logical relation between Bell's local causality and the probabilistic equations (10.6)-(8.5), it is fair to say that local causality 'justifies' only one of the multiple common causal explanations, namely the atomic one. The coarse-grained common cause system $\{D_k\}$, however, is an entirely salient physical explanation of the the correlation ("Observers see light signals only if *some* lamps are switched on"), even if the existence of such a common causal explanation is not a consequence of the accommodation of the physical scenario into a locally causal theory.

Now we turn to the role of the other ingredient in Bell's formulation, namely the localization of the common cause in the strong past.

8.5 Weak common causes

Example 4. Now, let us modify the population of the lighthouses. Let A and B remain in their places, that is in lighthouse 2 and 4, respectively: but suppose that lighthouses 1, 3 and 5 are inhabited by three lighthouse keepers C', C'' and C''', respectively, each having the corresponding one of the three lamps introduced in the previous Section. (See Fig. 8.7.) That is suppose that now lighthouse keeper C' in lighthouse 1 operates lamp C', lighthouse keeper C'' in lighthouse 3 operates lamp C'' and lighthouse keeper C''' in lighthouse 5 operates lamp C'''. Suppose furthermore that the ons and offs of the different lamps follow just the same statistics as defined in (8.18)-(8.22), that is $p(C_k) = \frac{1}{4}$ for every $k = 1, 2, 3, 4$ (only lamp C' is on, only lamp C''', all three lamps are on, none is on).

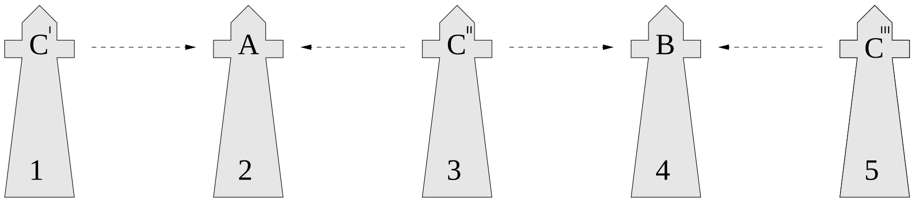

Figure 8.7: Lighthouses II.

Now, the role of lighthouse keepers A and B is just as in Section 4: to watch the light signals arriving at lighthouse 2 and 4, respectively. But now both can obtain a signal from both directions. Suppose that both A and B can only see the light signal sent from a neighboring lighthouse. That is, A cannot see the signal sent from C''' (say, because it is too far or the lighthouses hide each other); and B cannot see the signal sent from C'. Now, again the event algebra has 16 atoms with non-zero probability:

$$
\begin{aligned}
p(A \wedge B \wedge a_m \wedge b_n \wedge C_k) &= \frac{1}{16} \qquad \text{if } k = 3 \\
p(A \wedge \overline{B} \wedge a_m \wedge b_n \wedge C_k) &= \frac{1}{16} \qquad \text{if } m = L,\ k = 1 \\
p(\overline{A} \wedge B \wedge a_m \wedge b_n \wedge C_k) &= \frac{1}{16} \qquad \text{if } n = R,\ k = 2 \\
p(\overline{A} \wedge \overline{B} \wedge a_m \wedge b_n \wedge C_k) &= \frac{1}{16} \qquad \text{if } \begin{cases} m = R,\ k = 1, \\ \text{or } n = L,\ k = 2, \\ \text{or } k = 4 \end{cases}
\end{aligned}
$$

and there is a conditional and non-conditional correlation between event A and B, the detections of light signals in lighthouse 2 and 4, respectively, both in the non-conditional and conditional sense:

$$
\frac{1}{4} = p(A \wedge B) \quad \neq \quad p(A)\, p(B) = \frac{3}{8} \cdot \frac{3}{8} \tag{8.27}
$$

$$
\frac{1}{4} = p(A \wedge B | a_m \wedge b_n) \quad \neq \quad p(A|a_m)\, p(B|b_n) = \begin{cases} \frac{1}{4} \cdot \frac{1}{4} & \text{if } m = R, n = L, \\ \frac{1}{4} \cdot \frac{1}{2} & \text{if } m, n = R, \\ \frac{1}{2} \cdot \frac{1}{4} & \text{if } m, n = L. \end{cases} \tag{8.28}
$$

As one expects, $\{C_k\}$ is a local, (non-conspiratorial) joint common causal explanation of the correlation:

$$\begin{aligned}
p(A \wedge B|a_m \wedge b_n \wedge C_k) = p(A|a_m \wedge b_n \wedge C_k)\, p(B|a_m \wedge b_n \wedge C_k) &= \begin{cases} 1 & \text{if } m = R, n = L, k = 3 \\ 0 & \text{otherwise} \end{cases} \\
p(A|a_m \wedge b_n \wedge C_k) = p(A|a_m \wedge b_{n'} \wedge C_k) &= \begin{cases} 1 & \text{if } m = L, k = 1 \\ 1 & \text{if } k = 3 \\ 0 & \text{otherwise} \end{cases} \\
p(B|a_m \wedge b_n \wedge C_k) = p(B_n|a_{m'} \wedge b_n \wedge C_k) &= \begin{cases} 1 & \text{if } m = R, k = 2 \\ 1 & \text{if } k = 3 \\ 0 & \text{otherwise} \end{cases} \\
p(a_m \wedge b_n \wedge C_k) = p(a_m \wedge b_n)\, p(C_k) &= \frac{1}{4} \cdot \frac{1}{4}
\end{aligned}$$

Now, consider again the spacetime diagram of Example 4 depicted in Fig. 8.8. Here $\{C_k\}$ is localized *not* in the *strong* past but in the *weak* past of the correlated

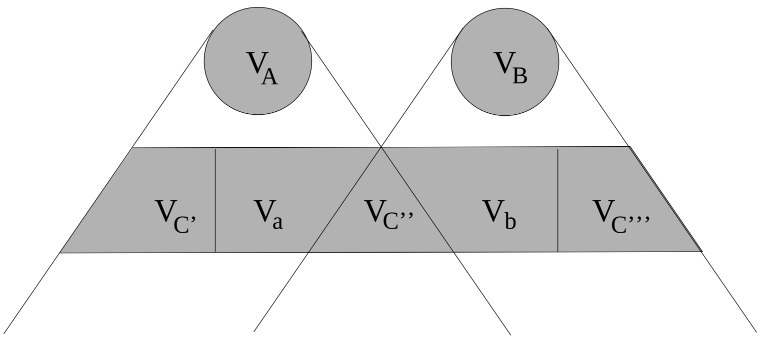

Figure 8.8: Spacetime diagram of Example 4.

events. How do these weak common causes relate to Bell's local causality? This question is answered in the following

Proposition 11. Let $\mathfrak{N}$ be again a locally causal net associating bounded spacetime regions to local algebras and let $A \in \mathcal{A}(V_A)$, $B \in \mathcal{A}(V_B)$, $a_m \in \mathcal{A}(V_a)$, $b_n \in \mathcal{A}(V_b)$, $C'_i \in \mathcal{A}(V_{C'})$, $C''_j \in \mathcal{A}(V_{C''})$ and $C'''_l \in \mathcal{A}(V_{C'''})$ for all m, n, i, j, l be atoms of the appropriate algebras with the regions as shown in Fig. 8.8. (In Example 4 $C'_1 \equiv C'$, $C'_2 \equiv \overline{C}'$ and similarly for C''_j and C'''_l.) Then

$$\{C_{ijl}\} \equiv \{C'_i \wedge C''_j \wedge C'''_l\}$$

is a *weak*, local, joint common cause of the conditional correlations

$$p(A \wedge B|a_m \wedge b_n) \neq p(A|a_m)\, p(B|b_n) \tag{8.29}$$

in the sense that the following equations hold:

$$p(A \wedge B_n | a_m \wedge b_n \wedge C_{ijl}) = p(A|a_m \wedge b_n \wedge C_{ijl})\, p(B|a_m \wedge b_n \wedge C_{ijl}) \quad (8.30)$$
$$p(A|a_m \wedge b_n \wedge C_{ijl}) = p(A|a_m \wedge b_{n'} \wedge C_{ijl}) \quad (8.31)$$
$$p(B|a_m \wedge b_n \wedge C_{ijl}) = p(B|a_{m'} \wedge b_n \wedge C_{ijl}) \quad (8.32)$$

Proof. The proof is straightforward. Local causality of the net implies that for the atoms $a'_{im} \equiv C'_i \wedge a_m \in \mathcal{A}(V_{C'} \cup V_a)$, $b'_{nl} \equiv b_n \wedge C'''_l \in \mathcal{A}(V_b \cup V_{C'''})$ and $C''_j \in \mathcal{A}(V_{C''})$ the following equations hold (being analogous to local causality (8.14)-(8.17)):

$$p(A \wedge B_n | a'_{im} \wedge b'_{nl} \wedge C''_j) = p(A|a'_{im} \wedge b'_{nl} \wedge C''_j)\, p(B|a'_{im} \wedge b'_{nl} \wedge C''_j) \quad (8.33)$$
$$p(A|a'_{im} \wedge b'_{nl} \wedge C''_j) = p(A|a'_{im} \wedge b'_{n'l'} \wedge C''_j) \quad (8.34)$$
$$p(B|a'_{im} \wedge b'_{nl} \wedge C''_j) = p(B|a'_{i'm'} \wedge b'_{nl} \wedge C''_j) \quad (8.35)$$

In other words, $\{C''_j\}$ is a *strong*, local, joint common cause of the conditional correlations

$$p(A \wedge B | a'_{im} \wedge b'_{nl}) \neq p(A|a'_{im})\, p(B|b'_{nl}) \quad (8.36)$$

with the *new* conditions a'_{im} and b'_{nl}. (Again, no-conspiracy

$$p(a'_{im} \wedge b'_{nl} \wedge C''_j) = p(a'_{im} \wedge b'_{nl})\, p(C''_j) \quad (8.37)$$

does *not* follow from local causality of the net.) But (8.33)-(8.35) are just equivalent to (8.30)-(8.32) proving that $\{C_{ijl}\}$ is a *weak*, local, joint common cause of the conditional correlations (10.13). ■

As we saw before, the correlated events $A \in \mathcal{A}(V_A)$, $B \in \mathcal{A}(V_B)$ in a locally causal net always have an atomic, strong common cause system $C''_j \in \mathcal{A}(V_{C''})$. Now, Proposition 13 states that this strong common cause system can always be *spatially extended* into a weak common cause system by simply adding some elements C'_i and C'''_l from the spacelike separated regions $V_{C'}$ and $V_{C'''}$, respectively. These extra terms will not spoil the screening-off: they can be freely added to the strong common cause. Moreover, as will turn out from Proposition 12, these extra terms need not be atomic either: any Boolean combination $C' = \vee_i C'_i$ and $C''' = \vee_l C'''_l$ can also be added without violating the probabilistic constraints (10.6)-(8.4). Thus, local causality does *not* determine the localization of the common cause, it is compatible both with strong and weak common causes.

But what is the exact relation between the weak and the strong common cause systems arising from the local causality of a given net?

In Example 4 one might find it peculiar that even though the common cause $\{C_{ijl}\}$ was non-conspiratorial (it was probabilistically independent of a_m and b_n), still there was a 'conspiracy' *within* the common cause: C'_i, C''_j and C'''_l were *not* probabilistically independent. For example it never happened that only lamp C'' was switched on. This fact does not raise any problem until one asks whether the common cause is localized at one place: for example, as in Example 1, where all the three lamps were localized in lighthouse 3. But in Example 4 the common cause was scattered around in three different locations. It was located in three different lighthouses. The problem with such a common cause that it may well question our whole project to provide a common causal explanation for a correlation. If the *explanans* itself has a built-in correlation, then what is the point in using it for explaining correlations? Can we not come up with a common causal model in which C'_i, C''_j and C'''_l are spacelike separated but still independent, say, regulated by three independent coin tossings in lighthouse 1, 3 and 5, respectively. Can one obtain a *weak* common cause for a given correlation without a *built-in correlation*? In the next proposition we will answer this question in the negative.

Let $\{C_{ijl}\} \equiv \{C'_i \wedge C''_j \wedge C'''_l\}$ be a weak common cause of a given correlation. (Here $\{C'_i\}, \{C''_j\}$ and $\{C'''_l\}$ are general partitions of $\mathcal{A}(V_{C'})$, $\mathcal{A}(V_{C''})$ and $\mathcal{A}(V_{C'''})$, respectively, and not those special ones specified in the above Examples.) Let us call $\{C_{ijl}\}$ a *genuine weak common cause*, iff $\{C''_j\}$—the 'middle part' of $\{C_{ijl}\}$—is *not* a strong common cause. In what follows we will show that the above mentioned 'built-in correlation' is a necessary condition to explain a correlation by a genuine weak common cause. In other words, we will show that if $\{C_{ijl}\} \equiv \{C'_i \wedge C''_j \wedge C'''_l\}$ is a common cause of the correlation (10.13) and C'_i, C''_j and C'''_l are probabilistically independent, then also $\{C''_j\}$ will be a common cause of the correlation.

Proposition 12. Suppose that $\{C'_i \wedge C''_j \wedge C'''_l\}$ is a common cause of the correlation between A_m and B_n in the sense that the following equations hold:

$$\begin{aligned} p(A_m \wedge B_n|a_m \wedge b_n \wedge C'_i \wedge C''_j \wedge C'''_l) &= p(A_m|a_m \wedge b_n \wedge C'_i \wedge C''_j \wedge C'''_l)\, p(B_n|a_m \wedge b_n \wedge C'_i \wedge C''_j \wedge \qquad (8 \\ p(A_m|a_m \wedge b_n \wedge C'_i \wedge C''_j \wedge C'''_l) &= p(A_m|a_m \wedge b_{n'} \wedge C'_i \wedge C''_j \wedge C'''_l) \qquad (8 \\ p(B_n|a_m \wedge b_n \wedge C'_i \wedge C''_j \wedge C'''_l) &= p(B_n|a_{m'} \wedge b_n \wedge C'_i \wedge C''_j \wedge C'''_l) \qquad (8 \\ p(a_m \wedge b_n \wedge C'_i \wedge C''_j \wedge C'''_l) &= p(a_m \wedge b_n)\, p(C'_i \wedge C''_j \wedge C'''_l) \qquad (8 \end{aligned}$$

and suppose that C'_i, C''_j and C'''_l are independent, that is

$$p(C'_i \wedge C''_j \wedge C'''_l) = p(C'_i)\, p(C''_j)\, p(C'''_l) \tag{8.42}$$

then $\{C''_j\}$ is also a common cause of the correlation:

$$\begin{aligned} p(A_m \wedge B_n|a_m \wedge C''_j) &= p(A_m|a_m \wedge b_n \wedge C''_j)\, p(B_n|a_m \wedge b_n \wedge C''_j) && (8.43) \\ p(A_m|a_m \wedge b_n \wedge C''_j) &= p(A_m|a_m \wedge b_{n'} \wedge C''_j) && (8.44) \\ p(B_n|a_m \wedge b_n \wedge C''_j) &= p(B_n|a_{m'} \wedge b_n \wedge C''_j) && (8.45) \\ p(a_m \wedge b_n \wedge C''_j) &= p(a_m \wedge b_n)\, p(C''_j) && (8.46) \end{aligned}$$

For the proof see Appendix B. Since in Example 4 $\{C_{ijl}\} \equiv \{C'_i \wedge C''_j \wedge C'''_l\}$ was localized in the weak past and $\{C''_j\}$ was localized in the strong past, we can interpret Proposition 12 as follows: a weak common cause without a 'built-in correlation' is always 'parasitic' on a strong common cause in the sense that there is no other way to provide a genuine weak common cause for a given correlation than to make the spacelike separated parts of the common cause probabilistically dependent. In brief, there is no genuine weak common cause without 'built-in correlation'.

Proposition 12 nicely explains why we are compelled to use strong common causes in *classical* common causal explanations. If we want to avoid explaining correlations in terms of other correlations, we cannot apply genuine weak common causes. So instead of appealing to non-genuine ('parasitic') weak common causes, it is more informative to use simply strong common causes.

The type of the common cause, however, is not always a matter of what we might want. As was mentioned in the Introduction, the common causes that naturally arise in AQFT are *weak* and *not strong* common causes. Why is that? The mathematical answer, namely that only (the possibility of) weak common causes follows from the axioms of the theory (see (Rédei 1997) and also (Hofer-Szabó and Vecsernyés 2012a, b)), is not very intuitive. In search of a more intuitive explanation, we conclude this paper with a highly speculative question:

Question: Is the fact that common causes in AQFT are weak common causes somehow related to or a consequence of the following two facts? (If these latter are facts at all.)

1. In AQFT quantum states establishing a superluminal correlation between two spacelike separated events also establish (or 'typically' establish) a 'built-in correlation' between the spacelike separated parts of the weak common causes of this correlation.

2. An analogue of Proposition 12 holds in AQFT: stating that, roughly speaking, a 'built-in correlation' is a necessary condition to explain a correlation by a genuine weak common cause.

Were these two facts to hold, one could understand why weak common causes in AQFT are genuine common causes, that is why they do not reduce to strong common causes. (For more on this see (Hofer-Szabó and Vecsernyés 2014a, b).)

8.6 Conclusion and discussion

In this paper, we gave a framework connecting stochastic events and spacetime regions, the isotone net framework of AQFT (Point 1) such that, on a certain specification and localization of the events in question (Point 2), local causality, defined in this framework in an appropriate way, *implies* (up to no-conspiracy) the standard probabilistic characterization of the common causal explanation (Point 3). The subtle roles of the choice of specification (atomic *vs.* non-atomic) and localization (strong *vs.* weak) were analyzed with respect to the relations of the spatiotemporal and probabilistic characterizations of the common cause. Specifically, it was shown that (i) the existence of non-atomic probabilistic common causes does not follow from the accommodation of the correlations in question into a locally causal net; (ii) the probabilistic characterization of the common cause is also compatible with weak common causes; and (iii) genuine weak common causes can be provided for a given correlation only at the cost of introducing a 'built-in correlation' between the spacelike separated parts of the common cause. We also asked whether this latter fact can help us understand how weak common causes arise naturally in AQFT.

Finally, we would like to briefly comment on an ongoing debate between Henson, Rédei and San Pedro on "comparing-distinguishing-confounding causality principles" (Henson, 2005; Rédei and San Pedro, 2012; Henson, 2013a). The debate is about the status of a proposition proved in Henson (2005) claiming that the Strong and Weak Common Cause Principles are equivalent. Here Strong/Weak Common Cause Principles say that any atom of the algebra pertaining to the strong/weak past of a pair of correlated events is a screener-off. The use of atoms (there called "full specifications") in the Common Cause Principles is inspired—just as in this paper—by Bell's work (see also Norsen, 2011), and further motivated as a means to evade Simpson paradoxes (see also Uffink, 1999). The first point to make is that since Henson's framework connecting spacetime regions and probability spaces is *not* the isotone net formalism used in this paper, and his Common Cause Principles are *not* the *non-conspiratorial, local, joint* common causal explanation (10.6)-(8.5) (used to explain *conditional* correlations!), it is not easy to see how Henson's result exactly relates to ours. In the isotone net formalism only bounded regions are associated to local algebras, whereas Henson's "least domains of decidability" formalism is not restricted to such regions. Rédei and San Pedro (2012) challenge Henson's result on the basis of its incompatibility with some propositions in AQFT (Rédei and Summers, 2002, Proposition 3). They claim that Henson's proof crucially depends on the regions being allowed to be infinite; and they question the validity of a similar proof for finite regions.[5] For finite regions, such as the regions in our approach, Henson acknowledges that his proof "cannot be modified so that" the two Common Cause Principles are equivalent; "at least not assuming that there are no correlations between events on spacelike sections of initial hypersurface" (Henson, 2005, 532). In the light of our results and discussion above, we would like to interpret: (i) the first part of this quote as claiming that (provided the two formalisms are equivalent) there is no contradiction between Henson's proof and our sharp distinction between weak and strong common causes; and (ii) the second half of the quote as stating something parallel to Proposition 12. Nonetheless, it would be highly desirable to investigate the relation between the two approaches more thoroughly.

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[5]Their characterization of "finite", however, is defective, since the region they want to have as infinite turns out to be finite; which fact is revealed in Henson's (2013a) reply. Here is a better characterization: V is *finite* iff $\big(I_-(V'') \setminus (V'')\big)'' \supseteq V''$.

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Appendix A

Here we will show that if a set of correlations $\{(A_m, B_n)|m, n = 1, 2\}$ has a local, non-conspiratorial joint common causal explanation in the sense of (10.6)-(8.5), then the following Clauser–Horne inequalities have to hold for any $m, m', n, n' = 1, 2$; $m \neq m', n \neq n'$:

$$\begin{aligned}-1 \leqslant p(A_m \wedge B_n|a_m \wedge b_n) + p(A_m \wedge B_{n'}|a_m \wedge b_{n'}) + p(A_{m'} \wedge B_n|a_{m'} \wedge b_n)\\ -p(A_{m'} \wedge B_{n'}|a_{m'} \wedge b_{n'}) - p(A_m|a_m \wedge b_n) - p(B_n|a_m \wedge b_n) \leqslant 0\end{aligned} \tag{8.47}$$

The derivation of (9.24) from (10.6)-(8.5) is simple. It is an elementary fact of arithmetic that for any $\alpha, \alpha‘, \beta, \beta‘ \in [0, 1]$ the number

$$\alpha\beta + \alpha\beta‘ + \alpha‘\beta - \alpha‘\beta‘ - \alpha - \beta \tag{8.48}$$

lies in the interval $[-1, 0]$. Now let $\alpha, \alpha‘, \beta, \beta‘$ be the following conditional probabilities:

$$\alpha \equiv p(A_m|a_m \wedge b_n \wedge C_k) \tag{8.49}$$
$$\alpha‘ \equiv p(A_{m'}|a_{m'} \wedge b_{n'} \wedge C_k) \tag{8.50}$$
$$\beta \equiv p(B_n|a_m \wedge b_n \wedge C_k) \tag{8.51}$$
$$\beta‘ \equiv p(B_{n'}|a_{m'} \wedge b_{n'} \wedge C_k) \tag{8.52}$$

Plugging (9.26)-(9.29) into (9.25) and using locality (8.3)-(8.4) one obtains

$$\begin{aligned}-1 \leqslant p(A_m|a_m \wedge b_n \wedge C_k)p(B_n|a_m \wedge b_n \wedge C_k) + p(A_m|a_{m'} \wedge b_n \wedge C_k)p(B_{n'}|a_{m'} \wedge b_n \wedge C_k)\\ +p(A_{m'}|a_{m'} \wedge b_n \wedge C_k)p(B_n|a_{m'} \wedge b_n \wedge C_k) - p(A_{m'}|a_{m'} \wedge b_{n'} \wedge C_k)p(B_{n'}|a_{m'} \wedge b_{n'} \wedge C_k)\\ -p(A_m|a_m \wedge b_n \wedge C_k) - p(B_n|a_m \wedge b_n \wedge C_k) \leqslant 0\end{aligned} \tag{8.53}$$

Using screening-off (10.6) one obtains

$$\begin{aligned}-1 \leqslant p(A_m \wedge B_n|a_m \wedge b_n \wedge C_k) + p(A_m \wedge B_{n'}|a_{m'} \wedge b_n \wedge C_k)\\ +p(A_{m'} \wedge B_n|a_{m'} \wedge b_n \wedge C_k) - p(A_{m'} \wedge B_{n'}|a_{m'} \wedge b_{n'} \wedge C_k)\\ -p(A_m|a_m \wedge b_n \wedge C_k) - p(B_n|a_m \wedge b_n \wedge C_k) \leqslant 0\end{aligned} \tag{8.54}$$

Finally, multiplying the above inequality by $p(C_k)$, then summing up for the indices k and using no-conspiracy (8.5) one arrives at (9.24).

Appendix B

Here we prove Proposition 13. Suppose that $\{C'_i \wedge C''_j \wedge C'''_l\}$ is a common cause of the correlation between A_m and B_n in the sense of (8.38)-(8.41) and suppose that

C_i', C_j'' and C_l''' are independent in the sense of (8.42). First, observe that (8.41) and (8.42) together entail that:

$$p(a_m \wedge b_n \wedge C_i' \wedge C_j'' \wedge C_l''') = p(a_m \wedge b_n)\, p(C_i')p(C_j'')p(C_l''') \qquad (8.55)$$

Then C''_j is a *strong* common cause. That is (8.43)-(8.46) hold:

$$
\begin{aligned}
p(A_m \wedge B_n | a_m \wedge b_n \wedge C''_j) &= \frac{p(A_m \wedge B_n \wedge a_m \wedge b_n \wedge C''_j)}{p(a_m \wedge b_n \wedge C''_j)} \\
&\overset{(8.55)}{=} \frac{\sum_{il} p(A_m \wedge B_n | a_m \wedge b_n \wedge C'_i \wedge C''_j \wedge C'''_l) p(a_m \wedge b_n) p(C'_i) p(C''_j) p(C'''_l)}{p(a_m \wedge b_n) p(C''_j)} \\
&\overset{(8.38)}{=} \sum_{il} p(A_m | a_m \wedge b_n \wedge C'_i \wedge C''_j \wedge C'''_l) p(B_n | a_m \wedge b_n \wedge C'_i \wedge C''_j \wedge C'''_l) p(C'_i) p(C \\
&\overset{(8.39)(8.40)}{=} \sum_{il} p(A_m | a_m \wedge b_n \wedge C'_i \wedge C''_j) p(B_n | a_m \wedge b_n \wedge C''_j \wedge C'''_l) p(C'_i) p(C'''_l) \\
&\overset{(8.55)}{=} p(A_m | a_m \wedge b_n \wedge C''_j)\, p(B_n | a_m \wedge b_n \wedge C''_j) \\
p(A_m | a_m \wedge b_n \wedge C''_j) &= \frac{p(A_m \wedge a_m \wedge b_n \wedge C''_j)}{p(a_m \wedge b_n \wedge C''_j)} \\
&\overset{(8.55)}{=} \frac{\sum_{il} p(A_m | a_m \wedge b_n \wedge C'_i \wedge C''_j \wedge C'''_l) p(a_m \wedge b_n) p(C'_i) p(C''_j) p(C'''_l)}{p(a_m \wedge b_n) p(C''_j)} \\
&\overset{(8.39)}{=} \sum_{il} p(A_m | a_m \wedge b_{n'} \wedge C'_i \wedge C''_j \wedge C'''_l) p(C'_i) p(C'''_l) \\
&= \frac{\sum_{il} p(A_m | a_m \wedge b_{n'} \wedge C'_i \wedge C''_j \wedge C'''_l) p(a_m \wedge b_{n'}) p(C'_i) p(C''_j) p(C'''_l)}{p(a_m \wedge b_{n'}) p(C''_j)} \\
&\overset{(8.55)}{=} \frac{p(A_m \wedge a_m \wedge b_{n'} \wedge C''_j)}{p(a_m \wedge b_{n'} \wedge C''_j)} = p(A_m | a_m \wedge b_{n'} \wedge C''_j) \\
p(B_n | a_m \wedge b_n \wedge C''_j) &= \frac{p(B_n \wedge a_m \wedge b_n \wedge C''_j)}{p(a_m \wedge b_n \wedge C''_j)} \\
&\overset{(8.55)}{=} \frac{\sum_{il} p(B_n | a_m \wedge b_n \wedge C'_i \wedge C''_j \wedge C'''_l) p(a_m \wedge b_n) p(C'_i) p(C''_j) p(C'''_l)}{p(a_m \wedge b_n) p(C''_j)} \\
&\overset{(8.40)}{=} \sum_{il} p(B_n | a_{m'} \wedge b_n \wedge C'_i \wedge C''_j \wedge C'''_l) p(C'_i) p(C'''_l) \\
&= \frac{\sum_{il} p(B_n | a_{m'} \wedge b_n \wedge C'_i \wedge C''_j \wedge C'''_l) p(a_{m'} \wedge b_n) p(C'_i) p(C''_j) p(C'''_l)}{p(a_{m'} \wedge b_n) p(C''_j)} \\
&\overset{(8.55)}{=} \frac{p(B_n \wedge a_{m'} \wedge b_n \wedge C''_j)}{p(a_{m'} \wedge b_n \wedge C''_j)} = p(B_n | a_{m'} \wedge b_n \wedge C''_j) \\
p(a_m \wedge b_n \wedge C''_j) &= \sum_{il} p(a_m \wedge b_n \wedge C'_i \wedge C''_j \wedge C'''_l) \\
&\overset{(8.55)}{=} \sum_{il} p(a_m \wedge b_n \wedge C'_i \wedge C'''_l) p(C''_j) = p(a_m \wedge b_n)\, p(C''_j)
\end{aligned}
$$

where the numbers over the equation signs refer to the equation used at that step.

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Chapter 9

Separate common causal explanation and the Bell inequalities

In the paper we ask how the following two facts are related: (i) a set of correlations has a local, non-conspiratorial *separate* common causal explanation; (ii) the set satisfies the Bell inequalities. Our answer will be partial: we show that no set of correlations violating the Clauser–Horne inequalities can be given a local, non-conspiratorial separate common causal model *if* the model is *deterministic*.

9.1 Introduction

According to the standard interpretation a common causal explanation of a set of EPR correlations consists in providing a so-called *common* common cause system that is a common screener-off for *all* correlations of the set such that this common screener-off is local and non-conspiratorial. (For the precise definitions see below.) However, it is well known that the assumption that a set of correlations has a local, non-conspiratorial *common* common cause system results in various Bell inequalities. Since these Bell inequalities are violated for appropriate measurement settings a common causal explanation of the EPR correlations is excluded—at least according to this interpretation of the common causal explanation.

However, in 1996 Belnap and Szabó came up with a weaker interpretation of the common causal explanation (Belnap, Szabó, 1996). The idea was that a set of correlations may not have a *common* common cause system but only a set of *separate* common cause systems explaining the correlations separately. In 2000 Szabó raised the question whether this idea provides a satisfactory common causal explanation for the EPR scenario (Szabó, 2000). To test his idea Szabó took a set of EPR correlations violating the appropriate Bell inequalities and then developed a computer program to generate local, non-conspiratorial separate common cause systems for the given set. The result of the computer simulations was that the chosen set of EPR correlations could be given a local separate common causal explanation, however the common cause systems were conspiratorial in a very tricky way. (See below.) Being unable to remove the unwanted conspiracies Szabó concluded the paper with the conjecture that EPR correlation can not be given a local, non-conspiratorial separate common causal explanation.

Szabó's idea inspired a whole series of papers devoted to the clarification of the possibility of a separate common causal explanation of EPR correlations. In 2005 Grasshoff, Portmann and Wüthrich derived the Wigner-type Bell inequalities from Szabó's premises *plus* the assumption that the set of correlations consisted of only perfect anticorrelations. (Grasshoff et al, 2005). The assumption of perfect anticorrelations, however, had two unpleasant consequences. First, the fate of the separate common causal explanation of the EPR scenario hinged on a precise experimental test of perfect anticorrelations. Second, the assumption of perfect anticorrelations reduced the separate common causal derivation of the Bell inequalities to a standard *common* common causal derivation. This reduction has been shown by Hofer-Szabó in (Hofer-Szabó, 2008). In the same paper Hofer-Szabó has presented a derivation of Bell inequalities from local, non-conspiratorial separate common causes without assuming perfect anticorrelations. Since a common cause is a special common cause system (a common cause system of size 2) the result was not general enough. In 2007 Portmann and Wüthrich have eliminated the restriction to common causes from the derivation and derived the Clauser-Horne inequality from local, non-conspiratorial separate common cause systems in the context of almost perfect anticorrelations (Portmann and Wüthrich, 2007). Hofer-Szabó generalized this derivation for any Bell(δ) inequality that is an inequality differing from some Bell inequality in a term of order of δ (Hofer-Szabó, 2011). In the light of this derivation a $\delta > 0$ threshold could be given for any set of correlations violating the standard Bell inequalities such that if an approriate subset of the original set of correlations differ from perfect anticorrelations less then δ then the set can not be given a local, non-conspiratorial separate common causal explanation. These results have settled the problem concerning the relation between the separate common causal explanations and the EPR scenario. *However, they have not settled the relation between the separate common causal explanations and the Bell inequalities.*

On closer examination the strategies used in the papers of the above authors (including the author of the present paper) had a very baffling structure. The reaction of the authors to Szabó's inability to provide a local, non-conspiratorial separate common causal explanation for a set of EPR correlations was the following. The chosen set of correlations cannot have a separate common causal explanation since it violates a Bell inequality which can be derived from the assumption that the given set has a local, non-conspiratorial separate common causal explanation. Of course, the failure of a separate common causal explanation may result from other reasons as well since separate common cause explanations may bring in other constraints between the probability of the correlating events different from the Bell inequalities; still the idea motivating the explanation of this fact was to derive some Bell inequalities from Szabó's premisses. However, it was *not* that happened. Instead of deriving the appropriate Bell inequality from the assumption that the *original set of the correlations* chosen by Szabó has a local, non-conspiratorial separate common causal explanation, all the mentioned authors have chosen *another set containing only perfect anticorrelations.* Then from the assumption that *this set of perfect anticorrelations* has a local, non-conspiratorial separate common causal explanation they have derived a Bell inequality for the *correlations of the original set.* So the Bell inequality they reached did *not* pertain to the original set but to the newly chosen set of perfect anticorrelations.

The effort of all the subsequent papers (Portmann and Wüthrich, 2007), (Hofer-Szabó, 2008) and (Hofer-Szabó, 2011) was to release the strong requirement of perfect anticorrelations in the derivation and to substitute perfect anticorrelations by almost perfect anticorrelations.

Of course, this strategy is impeccable as long as the aim of the proof is to exclude a local, non-conspiratorial separate common causal explanation of the EPR scenario *in general.* However, it does *not* explain why Szabó could not provide a local, non-conspiratorial separate common causal explanation for *his own* set of correlations. Since Szabó's concern was not to give a separate common causal explanation for the *perfect anticorrelation set*, the violation of Bell inequalities derived from the assumption that the *perfect anticorrelation set* has a separate common causal explanation did *not* explain Szabó's failure of providing a separate common causal explanation for *his own* set. In order to explain this fact one should derive some Bell inequalities from the assumption that Szabó's *original set* has a local, non-conspiratorial separate common causal explanation.

Here we will provide a partial answer to this problem. We will show that no set of correlations violating the Clauser–Horne inequalities can be given a *deterministic*, local, non-conspiratorial separate common causal explanation. Since the elimination of the requirement of determinism from the proof is not straightforward, the general question whether correlations violating the Clauser–Horne inequalities can be given a (not necessary deterministic) local, non-conspiratorial separate common causal explanation remains open.

In Section 2 we summarize the assumptions of a local, non-conspiratorial *common* common causal and separate common causal explanation of a set of EPR correlations respectively. In Section 3 we show in sketch the steps how these assumptions result in the Clauser–Horne inequalities *if* the set for which we are looking for a local, non-conspiratorial separate common causal explanation is a set of perfect or almost perfect anticorrelations. Finally, in Section 4 we drop these extra correlations and derive the Clauser–Horne inequalities from Szabó's original set of correlations for deterministic, local, non-conspiratorial separate common cause systems.

9.2 Common causal explanations of EPR correlations

Consider the Bohm version of the EPR experiment with a pair of spin-$\frac{1}{2}$ particles prepared in the singlet state $|\Psi_s\rangle$. Let a_i denote the event that the measurement apparatus is set to measure the spin in direction $\vec{a}_i$ in the left wing where i is an element of an index set I of spatial directions; and let $p(a_i)$ stand for the probability of a_i. Let b_j and $p(b_j)$ respectively denote the same for direction $\vec{b}_j$ in the right wing where j is again in the index set I. (Note that $i = j$ does not mean that $\vec{a}_i$ and $\vec{b}_j$ are parallel directions.) Furthermore, let $p(A_i)$ stand for the probability that the spin measurement in direction $\vec{a}_i$ in the left wing yields the result "up" and let $p(B_j)$ be defined in a similar way in the right wing for direction $\vec{b}_j$. According to quantum mechanics the quantum probability of getting "up" in direction $\vec{a}_i$ in the left wing; getting "up" in direction $\vec{b}_j$ in the right wing; and getting "up" in both directions $\vec{a}_i$ and $\vec{b}_j$ are given by the following relations

$$
\begin{aligned}
Tr(W_{|\Psi_s\rangle}\,(P_{A_i}\otimes I)) &= \frac{1}{2} && (9.1)\\
Tr(W_{|\Psi_s\rangle}\,(I\otimes P_{B_j})) &= \frac{1}{2} && (9.2)\\
Tr(W_{|\Psi_s\rangle}\,(P_{A_i}\otimes P_{B_j})) &= \frac{1}{2}\sin^2\left(\frac{\theta_{a_ib_j}}{2}\right) && (9.3)
\end{aligned}
$$

where Tr is the trace function; $W_{|\Psi_s\rangle}$ is the density operator pertaining to the pure state $|\Psi_s\rangle$; P_{A_i} and P_{B_j} denote projections on the eigensubspaces with eigenvalue $+1$ of the spin operators associated with directions $\vec{a}_i$ and $\vec{b}_j$ respectively; and $\theta_{a_ib_j}$ denotes the angle between directions $\vec{a}_i$ and $\vec{b}_j$.

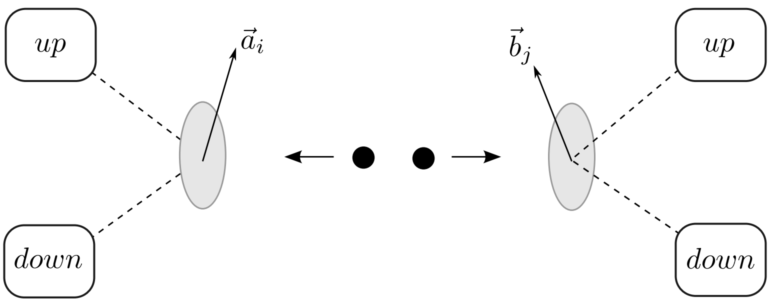

Figure 9.1: EPR–Bohm setup for spin-$\frac{1}{2}$ particles

The standard way to interpret quantum probabilities is to identify them with conditional probabilities as follows:

$$p(A_i|a_ib_j) = Tr(W_{|\Psi_s\rangle}(P_{A_i} \otimes I)) \tag{9.4}$$
$$p(B_j|a_ib_j) = Tr(W_{|\Psi_s\rangle}(I \otimes P_{B_j})) \tag{9.5}$$
$$p(A_iB_j|a_ib_j) = Tr(W_{|\Psi_s\rangle}(P_{A_i} \otimes P_{B_j})) \tag{9.6}$$

where the events A_i, B_j, a_i and b_j $(i, j \in I)$ respectively are elements of a classical probability measure space (X, S, p) and the conditional probabilities are defined in the usual way. With this identification quantum mechanics predicts correlation between classical conditional probabilities for non-perpendicular directions $\vec{a}_i$ and $\vec{b}_j$:

$$p(A_iB_j|a_ib_j) \neq p(A_i|a_ib_j)p(B_j|a_ib_j) \tag{9.7}$$

Specially, if the measurement directions $\vec{a}_i$ and $\vec{b}_j$ are parallel then there is a perfect anticorrelation between the outcomes A_i and B_i:

$$p(A_iB_j|a_ib_j) = 0 \tag{9.8}$$

A further consequence of (9.4)-(9.5) is the so-called surface locality that is for any $i, i', j, j' \in I$ the following relations hold

$$p(A_i|a_ib_j) = p(A_i|a_ib_{j'}) \tag{9.9}$$
$$p(B_j|a_ib_j) = p(B_j|a_{i'}b_j) \tag{9.10}$$

Now, let (A_i, B_j) $(i, j \in I)$ denote a pair correlating conditionally according to (10.1) and let $\{(A_i, B_j)\}_{i,j\in I}$ stand for a set of correlating pairs pertaining to the index set I. What does a common causal explanation of the set $\{(A_i, B_j)\}_{i,j\in I}$ of correlations consist in? In the following we expose the components of such an explanation.

Let us begin with the definition of the common cause. Let (X, S, p) be a classical probability measure space and let A and B be two (positively) correlating events. Then the common cause of the correlation is the following:

Definition 12. An event C in S is said to be the *common cause* of the correlation between events A and B only if the events A, B and C satisfy the following relations:

$$p(AB|C) = p(A|C)p(B|C) \tag{9.11}$$
$$p(AB|C^\perp) = p(A|C^\perp)p(B|C^\perp) \tag{9.12}$$
$$p(A|C) > p(A|C^\perp) \tag{9.13}$$
$$p(B|C) > p(B|C^\perp) \tag{9.14}$$

where $C^\perp$ denotes the orthocomplement of C. Equations (10.2)-(10.3) are called *screening-off* properties since conditioning on C and $C^\perp$ respectively screens off the correlation between A and B. Inequalities (10.4)-(10.5) express positive statistical relevance of the cause C on the two effects A and B respectively.

The above definition of the common cause goes back to Reichenbach (Reichenbach, 1956); (although Reichenbach himself did not regard (10.2)-(10.5) as a sufficient condition for an event to be a common cause). From the time of Reichenbach's first characterization on the common cause concept has been generalized in two important ways. First, it has been generalized for situations where there are more than one causes present that is for a system of cooperating common causes (Hofer-Szabó, Rédei, 2004, 2006). Second, the inequalities expressing positive statistical relevance have gradually been redarded as being too restrictive and hence have been dropped. As a result the common cause has been characterized simply as a screener-off partition of the algebra defined as follows:

Definition 13. Let again (X, S, p) be a classical probability measure space and let A and B be two correlating events in S. Then a partition $\{C_k\}_{k\in K}$ in S is said to be the *common cause system* of the correlation between events A and B if and only if the following factorization holds for all $k \in K$:

$$p(AB|C_k) = p(A|C_k)p(B|C_k) \tag{9.15}$$

where $|K|$, the cardinality of K is said to be the *size* of the common cause system. A common cause system of size 2 is called a common cause.

Definition 13 of the common cause system referred to a *single* correlation. However, generally we are looking for the causal explanation for a *set* of correlations. This explanation can be understood in two different ways. Either we provide a separate common cause system for each separate correlation of the given set; or we are looking for a so-called *common* common cause system that is a partition screening off *all* correlations of the set. This latter option puts extra requirements on the explanation since it demands that the common cause system pertaining to the different correlations be the *same*.

Now, let us apply the concept of common cause systems to EPR correlations. First note that EPR correlations are *conditional* correlations in the sense of (10.1) where the conditions represent the choice of the measurement directions. Looking at the spatiotemporal arrengement of the events representing the measurement choices and the measurement outcomes respectively in the opposite wings and the set of events representing the common cause system at the source we can read off the following two spatial separations. The outcome events A_i in the left wing are spatially separated from the measurement choice events b_j in the right wing; and similarly events B_j are spatially separated from events a_i. The measurement choice events a_i and b_j are spatially separeted from the events of the common cause system $\{C_k\}$. Turning these two spatiotemporal considerations in statistical relationships we get the so-called *locality* and *no-conspiracy* requirements. Applying the above definition of the common cause systems that is the *screening-off* requirement for conditional probabilities we obtain altogether three demands that a common causal explanation should satisfy. If we demand on the top that the common cause sytem be the *same* for all correlations of the given set then we arrive at a local, non-conspiratorial *common* common causal explanation.

Definition 14. Let $\{(A_i, B_j)\}_{i,j\in I}$ be a set of correlating pairs pertaining to the index set I such that A_i, B_j, a_i and b_j are elements of a classical probability measure space (X, S, p). Then a *local, non-conspiratorial common common causal explanation* of the set $\{(A_i, B_j)\}_{i,j\in I}$ consists in providing a partition $\{C_k\}_{k\in K}$ of S such that $\{C_k\}_{k\in K}$ is local, non-conspiratorial and screens off all the correlations of $\{(A_i, B_j)\}_{i,j\in I}$ in the sense that for every $i, i', j, j' \in I$ and $k \in K$ the following relations hold:

$$p(A_i|a_i b_j C_k) = p(A_i|a_i b_{j'} C_k) \qquad \text{(locality)} \qquad (9.16)$$

$$p(B_j|a_i b_j C_k) = p(B_j|a_{j'} b_j C_k) \qquad \text{(locality)} \qquad (9.17)$$

$$p(a_i b_j C_k) = p(a_i b_j) p(C_k) \qquad \text{(no-conspiracy)} \qquad (9.18)$$

$$p(A_i B_j|a_i b_j C_k) = p(A_i|a_i b_j C_k) p(B_j|a_i b_j C_k) \qquad \text{(screening-off)} \qquad (9.19)$$

On the other hand, if we let the common cause sytem be *different* for the different correlations of the set then our explanation will be called a local, non-conspiratorial separate common causal explanation.

Definition 15. Let $\{(A_i, B_j)\}_{i,j\in I}$ be a set of correlating pairs pertaining to the index set I such that A_i, B_j, a_i and b_j are elements of a classical probability measure space (X, S, p). Then a *local, non-conspiratorial separate common causal explanation* of the set $\{(A_i, B_j)\}_{i,j\in I}$ consists in finding a separate partition $\{C_k^{ij}\}_{k(ij)\in K(i,j)}$ of S for each correlation in $\{(A_i, B_j)\}_{i,j\in I}$ such that each $\{C_k^{ij}\}_{k(ij)\in K(i,j)}$ is local, non-conspiratorial and screens off the appropriate correlation in $\{(A_i, B_j)\}_{i,j\in I}$ in the sense that for every $i, i', j, j' \in I$ and $k(ij) \in K(i, j)$ the following relations hold:

$$p(A_i|a_i b_j C_k^{ij}) = p(A_i|a_i b_{j'} C_k^{ij}) \qquad \text{(locality)} \qquad (9.20)$$

$$p(B_j|a_i b_j C_k^{ij}) = p(B_j|a_{j'} b_j C_k^{ij}) \qquad \text{(locality)} \qquad (9.21)$$

$$p(a_i b_j F) = p(a_i b_j) p(F) \qquad \text{(no-conspiracy)} \qquad (9.22)$$

$$p(A_i B_j|a_i b_j C_k^{ij}) = p(A_i|a_i b_j C_k^{ij}) p(B_j|a_i b_j C_k^{ij}) \qquad \text{(screening-off)} \qquad (9.23)$$

where F in equation (10.20) is an element of the algebra $S' \subset S$ generated by all the elements of *every* separate common cause system.

To motivate why it is important to demand no-conspiracy (10.20) in this strong sense namely for any element of the generated algebra and not just for the C_k^{ij} elements, recall the tricky conspiracies in Szabó's separate common causal model. As mentioned in the Introduction Szabó was able to construct a local separate common causal model for a special set of EPR correlations that was non-conspiratorial in the sense the every a_i and b_j were independent of every C_k^{ij}. However, this model was conspiratorial at a deep level—the measurement choices a_i and b_j correlated with some *disjunctions* of elements of separate common cause systems such as $C_k^{ij} \cup C_{k'}^{i'j'}$. To exclude all these type of conspiracies we demand no-conspiracy in the strong form (10.20).

Now, we turn to the relation between the local, non-conspiratorial *common* or separate common causal explanations of the EPR correlations on the one hand and the Bell inequalities on the other.

9.3 Bell inequalities

Now, let us be more specific concerning our set $\{(A_i, B_j)\}_{i,j \in I}$. Let the correlation set consist of four correlating pairs $(A_1, B_3), (A_1, B_4), (A_2, B_3)$ and (A_2, B_4). The standard question is usually whether this set can be given a local, non-conspiratorial *common* common causal explanation in the sense of Definition 14. The answer is well known. $\{(A_i, B_j)\}_{i=1,2;j=3,4}$ can be given a local, non-conspiratorial *common* common causal explanation only if the correlations of the set for any $i, i' = 1, 2$; $j, j' = 3, 4$ and $i \neq i', j \neq j'$ satisfy the Clauser–Horne inequalities

$$-1 \leqslant p(A_i B_j | a_i b_j) + p(A_i B_{j'} | a_i b_{j'}) + p(A_{i'} B_j | a_{i'} b_j) - p(A_{i'} B_{j'} | a_{i'} b_{j'}) - p(A_i | a_i b_j) - p(B_j | a_i b_j) \leqslant 0 \quad (9.24)$$

The proof is simple. It is a trivial fact of arithmetic that for any $\alpha, \alpha', \beta, \beta' \in [0, 1]$ the expression

$$\alpha\beta + \alpha\beta' + \alpha'\beta - \alpha'\beta' - \alpha - \beta \tag{9.25}$$

lies in the bound $[-1, 0]$. Now let $\alpha, \alpha', \beta, \beta'$ be the following conditional probabilities:

$$\begin{aligned} \alpha &\equiv p(A_i | a_i b_j C_k) && (9.26) \\ \alpha' &\equiv p(A_{i'} | a_{i'} b_{j'} C_k) && (9.27) \\ \beta &\equiv p(B_j | a_i b_j C_k) && (9.28) \\ \beta' &\equiv p(B_{j'} | a_{i'} b_{j'} C_k) && (9.29) \end{aligned}$$

Plugging (9.26)-(9.29) into (9.25) and using locality (9.16)-(9.17) one gets that

$$\begin{aligned} -1 \leqslant p(A_i | a_i b_j C_k) p(B_j | a_i b_j C_k) + p(A_i | a_{i'} b_j C_k) p(B_{j'} | a_{i'} b_j C_k) + p(A_{i'} | a_{i'} b_j C_k) p(B_j | a_{i'} b_j C_k) \\ -p(A_{i'} | a_{i'} b_{j'} C_k) p(B_{j'} | a_{i'} b_{j'} C_k) - p(A_i | a_i b_j C_k) - p(B_j | a_i b_j C_k) \leqslant 0 \end{aligned}$$

Using screening-off (10.17) one gets that

$$\begin{aligned} -1 \leqslant p(A_i B_j | a_i b_j C_k) + p(A_i B_{j'} | a_{i'} b_j C_k) + p(A_{i'} B_j | a_{i'} b_j C_k) \\ -p(A_{i'} B_{j'} | a_{i'} b_{j'} C_k) - p(A_i | a_i b_j C_k) - p(B_j | a_i b_j C_k) \leqslant 0 \end{aligned}$$

Finally, multiplying by $p(C_k)$, summing up for the indices k and using no-conspiracy (10.16) one obtains (9.24).

An example for a correlation set which violates (9.24) and hence can not be given a local, non-conspiratorial *common* common causal explanation is the one Szabó used in his paper (2000). Here the angles $\theta_{a_i b_j}$ between the directions $\vec{a}_i$ and $\vec{b}_j$ are set as follows:

$$\theta_{a_1 b_3} = \theta_{a_1 b_4} = \theta_{a_2 b_3} = \frac{2\pi}{3} \quad \text{and} \quad \theta_{a_2 b_4} = 0 \tag{9.30}$$

For this choice of the measurement directions there is a conditional correlation for every (A_i, B_j) pair $(i = 1, 2;\ j = 3, 4)$:

$$\frac{3}{8} = p(A_1B_3|a_1b_3) \quad \neq \quad p(A_1|a_1b_3)\, p(B_3|a_1b_3) = \frac{1}{2}\cdot\frac{1}{2} \tag{9.31}$$

$$\frac{3}{8} = p(A_1B_4|a_1b_4) \quad \neq \quad p(A_1|a_1b_4)\, p(B_4|a_1b_4) = \frac{1}{2}\cdot\frac{1}{2} \tag{9.32}$$

$$\frac{3}{8} = p(A_2B_3|a_2b_3) \quad \neq \quad p(A_2|a_2b_3)\, p(B_3|a_2b_3) = \frac{1}{2}\cdot\frac{1}{2} \tag{9.33}$$

$$0 = p(A_2B_4|a_2b_4) \quad \neq \quad p(A_2|a_2b_4)\, p(B_4|a_2b_4) = \frac{1}{2}\cdot\frac{1}{2} \tag{9.34}$$

Denote this set of correlations by $\{(A_i, B_j)\}_{CH}$. This set violates the Clauser–Horne inequality

$$-1 \leqslant p(A_1B_3|a_1b_3) + p(A_1B_4|a_1b_4) + p(A_2B_4|a_2b_4) - p(A_2B_4|a_2b_4) - p(A_1|a_1b_3) - p(B_3|a_1b_3) \leqslant 0 \tag{9.35}$$

at the upper bound as follows:

$$\frac{3}{8} + \frac{3}{8} + \frac{3}{8} - 0 - \frac{1}{2} - \frac{1}{2} \quad \nleqslant \quad 0 \tag{9.36}$$

Consequently, $\{(A_i, B_j)\}_{CH}$ can not be given a local, non-conspiratorial *common* common causal explanation.

Now, let us go over to the question whether $\{(A_i, B_j)\}_{CH}$ (or any other correlation set violating the Clauser–Horne inequalities) can have a local, non-conspiratorial *separate* common causal explanation. As mentioned in the Introduction Szabó was unable to present a local, non-conspiratorial separate common cause model for $\{(A_i, B_j)\}_{CH}$ because of the unwanted conspiracies. The natural intuition towards this fact was that a local, non-conspiratorial separate common causal explanation of the set $\{(A_i, B_j)\}_{CH}$ results in some Bell inequalities—for example in the above Clauser–Horne inequalities—and the violation of these inequalities for the above setting is responsible for the lack of a separate common causal explanation. Thus, the program has been to show up a derivation of some Bell inequalities from the assumption that $\{(A_i, B_j)\}_{CH}$ has four local, non-conspiratorial separate common cause systems satisfying (9.20)-(10.21).

Curiously enough, none of the authors has taken this route. Instead of taking the above set and then searching for a derivation of some Bell inequality from the assumption that *this set* has a local, non-conspiratorial separate common causal explanation they have chosen *another set*. This set again consisted of the four correlations of $\{(A_i, B_i)\}$ $(i = 1, 2, 3, 4)$ for any of which the angle $\theta_{a_ib_i}$ was set to 0. In other words, this set was composed of *perfect anticorrelations*. Denote this set by $\{(A_i, B_i)\}_{PA}$. For the relation between the sets $\{(A_i, B_j)\}_{CH}$ and $\{(A_i, B_j)\}_{PA}$ see Figure 9.2 where the continuous lines represent the Clauser–Horne correlations and the dotted lines represent the perfect anticorrelations.

Now, the reasoning has run as follows (for the details see (Grasshoff et al. 2005) and (Hofer-Szabó, 2008)). Suppose that $\{(A_i, B_i)\}_{PA}$ has a local, non-conspiratorial separate common causal explanation that is four local, non-conspiratorial

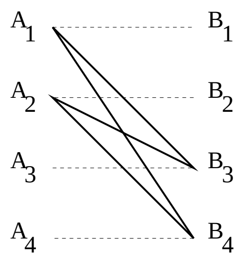

Figure 9.2: The Clauser–Horne correlation set and the perfect anticorrelation set

separate common cause systems $\{C^{ii}_k\}_{k\in K(i)}$ $(i = 1, 2, 3, 4)$ satisfying (9.20)-(10.21). Since $\{(A_i, B_i)\}_{PA}$ consists of only perfect anticorrelations it is easy to show that from assumptions (9.20)-(10.21) it follows that for any $i = 1, 2, 3, 4$ there exist a vector $\varepsilon^{ii} \in \{0, 1\}^{K(i)}$ such that defining C^{ii} and $C^{ii\perp}$ as

$$C^{ii} \equiv \bigcup_{k\in K(i)} \varepsilon^{ii}_k\, C^{ii}_k; \qquad C^{ii\perp} \equiv \bigcup_{k\in K(i)} (1-\varepsilon^{ii}_k)\, C^{ii}_k \tag{9.37}$$

the partitions $\{C^{ii}, C^{ii\perp}\}$ $(i = 1, 2, 3, 4)$ will be local, non-conspiratorial separate common *causes* that is a separate common cause systems of size 2 for the set $\{(A_i, B_i)\}_{PA}$. Moreover, every $\{C^{ii}, C^{ii\perp}\}$ will satisfy (9.20)-(10.21) deterministically that is each term in (9.20)-(10.21) will be either 0 or 1. Finally, the probability of the separate common causes will equal to the probability of the conditional probabilities $p(A_i|a_ib_i)$ and $p(B_i|a_ib_i)$:

$$p(C^{ii}) = p(A_i|a_ib_i) \tag{9.38}$$
$$p(C^{ii\perp}) = p(B_i|a_ib_i) \tag{9.39}$$

Notice that in this reasoning there has been no mention of the original set $\{(A_i, B_j)\}_{CH}$. How do the correlations of $\{(A_i, B_j)\}_{CH}$ come into the picture?

The joint and marginal conditional probabilities of the Clauser–Horne correlations appear simply using locality (9.20)-(9.21) and no-conspiracy (10.20) *for the perfect anticorrelation set.* That is for any $i, j = 1, 2, 3, 4$; $i \neq j$

$$p(C^{ii}) = p(A_i|a_ib_j) \tag{9.40}$$
$$p(C^{jj\perp}) = p(B_j|a_ib_j) \tag{9.41}$$
$$p(C^{ii}C^{jj\perp}) = p(A_iB_j|a_ib_j) \tag{9.42}$$

Now, consider the four events C^{11}, C^{22}, $C^{33\perp}$ and $C^{44\perp}$ in S. For these events the following simple probabilistic constraint applies:

$$-1 \leqslant p(C^{11}C^{33\perp}) + p(C^{11}C^{44\perp}) + p(C^{22}C^{33\perp}) - p(C^{22}C^{44\perp}) - p(C^{11}) - p(C^{33\perp}) \leqslant 0 \tag{9.43}$$

Subtituting the probabilities of (9.43) by the conditional probabilities of (9.40)-(9.42) we get the Clauser–Horne inequality (9.35) *for the correlation set* $\{(A_i, B_j)\}_{CH}$.

Since for the measuring setup (9.30) this inequality is violated there can be given no local, non-conspiratorial separate common causal explanation of the *perfect anticorrelation set* $\{(A_i, B_j)\}_{PA}$!

To put is briefly, the necessary condition for $\{(A_i, B_j)\}_{PA}$ to have a local, non-conspiratorial separate common causal explanation is that $\{(A_i, B_j)\}_{CH}$ satisfies the Clauser–Horne inequality (9.35).

The papers (Portmann and Wüthrich, 2007) and (Hofer-Szabó, 2008, 2011) have repeated the same argumentation for *almost* perfect anticorrelations. Here we sketch the argument of (Hofer-Szabó, 2011). Consider again a set consisting of four correlating pairs $\{(A_i, B_i)\}_{i=1,2,3,4}$ and suppose that for any $i = 1, 2, 3, 4$ the angle $\theta_{a_i b_i}$ between the measurement choices is such that

$$|\pi - \theta_{a_i b_i}| \quad < \quad 2 \arcsin \sqrt{1 - 2\delta} \tag{9.44}$$

or more simply, let the correlations be such that for any $i = 1, 2, 3, 4$

$$p(A_i B_i | a_i b_i) \quad \leqslant \quad \delta \tag{9.45}$$

Denote such a set of correlations by $\{(A_i, B_i)\}_{PA(\delta)}$. Again suppose that $\{(A_i, B_i)\}_{PA(\delta)}$ has a local, non-conspiratorial separate common causal explanation. As above, from this assumption it follows that there exist a vector $\varepsilon^{ii} \in \{0, 1\}^{K_i}$ for any $i = 1, 2, 3, 4$ such that defining C^{ii} and $C^{ii\perp}$ as in (9.37) one get four partitions $\{C^{ii}, C^{ii\perp}\}_{i=1,2,3,4}$ for which—instead of (9.38)-(9.39)—the following inequalities will hold:

$$|p(C^{ii}) - p(A_i | a_i b_i)| \quad \leqslant \quad 4\delta \tag{9.46}$$
$$|p(C^{ii\perp}) - p(B_i | a_i b_i)| \quad \leqslant \quad 4\delta \tag{9.47}$$

Call these partitions *quasi* common causes since although they are constructed out of the elements of the common cause systems $\{C^{ii}_k\}$ they do not satisfy screening-off (10.21) (however they satisfy locality (9.20)-(9.21) and no-conspiracy (10.20)).

Now as above, using locality (9.20)-(9.21) and no-conspiracy (10.20) *for the set* $\{(A_i, B_i)\}_{PA(\delta)}$ we get that for any $i, j = 1, 2, 3, 4$

$$|p(C^{ii}) - p(A_i | a_i b_j)| \quad \leqslant \quad 4\delta \tag{9.48}$$
$$|p(C^{jj\perp}) - p(B_j | a_i b_j)| \quad \leqslant \quad 4\delta \tag{9.49}$$
$$|p(C^{ii} C^{jj\perp}) - p(A_i B_j | a_i b_j)| \quad \leqslant \quad 8\delta \tag{9.50}$$

Consider again inequality (9.43) composed of the *quasi* common causes C^{11}, C^{22}, $C^{33\perp}$ and $C^{44\perp}$ and substitute the probabilities of (9.43) by the conditional probabilities of (9.48)-(9.50). Each substitution will cause an error of order of either 4δ or 8δ. Adding up the errors we obtain the following inequality.

$$-1 \leqslant p(A_1 B_3 | a_1 b_3) + p(A_1 B_4 | a_1 b_4) + p(A_2 B_3 | a_2 b_4) - p(A_2 B_4 | a_2 b_3) - p(A_1 | a_1 b_3) - p(B_3 | a_1 b_3) - 40\delta \leqslant 0 \tag{9.51}$$

We refer to this inequlity as a Clauser–Horne(δ) inequality since (9.51) differs from the original Clauser–Horne inequality (9.43) in a term of order of δ. Again for the measuring setup (9.30) the Clauser–Horne(δ) inequality (9.51) is violated as long as $\delta < \frac{1}{320}$. This excludes a local, non-conspiratorial separate common causal explanation of the *almost perfect anticorrelation set* $\{(A_i, B_j)\}_{PA(\delta)}$.

This strategy can be generalized for arbitrary Bell(δ) inequality. In (Hofer-Szabó, 2011) a recipe has been given for deriving *any* Bell(δ) inequality composed of marginal probabilities $p(A_i|a_ib_j)$, $p(B_j|a_ib_j)$ and joint probabilities $p(A_iB_j|a_ib_j)$. The recipe is roughly this. Consider a Bell inequality resulting from the local, non-consipratorial *common* common causal explanation of a set $\{(A_i, B_j)\}$ of correlations. Consider the set $\{(A_i, B_i)\}_{PA(\delta)}$ of almost perfect anticorrelations pertaining to the events A_i or B_j which appear in either a marginal or a joint probability in the Bell inequality. Suppose that $\{(A_i, B_j)\}_{PA(\delta)}$ has a local, non-conspiratorial separate common causal explanation. This assumption results in a Bell(δ) inequality differing from the original Bell inequality in a term of order of δ where the exact magnitude of this term is the function of the approximation. Choose the setting which violates the Bell inequality maximally. If the δ term is smaller than the violation of the original Bell inequality than the new Bell(δ) inequality will also be violated—excluding a local, non-conspiratorial separate common causal explanation *almost perfect anticorrelation set* $\{(A_i, B_j)\}_{PA(\delta)}$.

9.4 No deterministic, local, non-conspiratorial separate common causal explanation of the Clauser–Horne set

In the last Section we have posed a question and answered another one. The question was whether the set $\{(A_i, B_j)\}_{CH}$ has a local, non-conspiratorial separate common causal explanation. However, the answer was this. The necessary condition for $\{(A_i, B_j)\}_{PA}$ (or $\{(A_i, B_j)\}_{PA(\delta)}$) to have a local, non-conspiratorial separate common causal explanation is that $\{(A_i, B_j)\}_{CH}$ satisfies the Clauser–Horne inequality (9.24). This answer is perfectly adequate if our intention is to exclude the local, non-conspiratorial separate common causal explanation of the EPR scenario *as such*—as it was the aim of the paper (Grasshoff et al. 2005). But it *does not at all explain* the fact why Szabó was not able to give a local, non-conspiratorial separate common causal explanation of his original set $\{(A_i, B_j)\}_{CH}$. This latter question can be answered only if we derive some Bell inequalities from the assumption that the original set $\{(A_i, B_j)\}_{CH}$ has a local, non-conspiratorial separate common causal explanation; or we show up other reasons for the failure.

In this Section we give an answer to the original question—a partial answer confined to the deterministic case. The answer is this. $\{(A_i, B_j)\}_{CH}$ can not have a deterministic, local, non-conspiratorial separate common causal explanation since this separate common causal explanation implies the same Clauser–Horne inequalities as the local, non-conspiratorial *common* common causal explanation.

Proposition 13. Let $\{(A_i, B_j)\}_{i=1,2;j=3,4}$ be a set of correlating pairs such that A_i, B_j, a_i and b_j are elements of a classical probability measure space (X, S, p). Suppose furthermore that $\{(A_i, B_j)\}_{i=1,2;j=3,4}$ has a deterministic, local, non-conspiratorial separate common causal explanation in the sense that there exist a separate partition $\{C_k^{ij}\}_{k(ij)\in K(i,j)}$ of S for each correlation of $\{(A_i, B_j)\}_{i=1,2;j=3,4}$ such that $\{C_k^{ij}\}_{k(ij)\in K(i,j)}$ satisties (9.20)-(10.21) and $p(A_i|a_ib_jC_k^{ij}), p(B_j|a_ib_jC_k^{ij}) \in \{0,1\}$ for any $i = 1,2; j = 3,4$ and $k(ij) \in K(i,j)$. Then for any $i,i' = 1,2; j,j' = 3,4$; $i \neq i', j \neq j'$ the Clauser–Horne inequality (9.24) follows.

Proof. Consider the separate common cause system $\{C_k^{ij'}\}$ $(i = 1,2; j' = 3,4)$ pertaining to the correlation $(A_i, B_{j'})$ and let K' denote the set of those indices $k \in K$ for which

$$p(A_iB_{j'}|a_ib_{j'}C_k^{ij'}) \quad = \quad 1 \tag{9.52}$$

Similarly consider the separate common cause system $\{C_l^{i'j}\}$ $(i' = 1,2; j = 3,4$; $i \neq i', j \neq j')$ pertaining to the correlation $(A_{i'}, B_j)$ and let L' denote the set of those indices $l \in L$ for which

$$p(A_{i'}B_j|a_{i'}b_jC_l^{i'j}) \quad = \quad 1 \tag{9.53}$$

With the index sets K' and L' in hand define the following two elements of the algebra generated by the separate common cause systems $\{C_k^{ij'}\}$ and $\{C_l^{i'j}\}$

$$C^{ij'} \quad \equiv \quad \bigcup_{k\in K'} C_k^{ij'} \tag{9.54}$$

$$C^{i'j} \quad \equiv \quad \bigcup_{l\in L'} C_l^{i'j} \tag{9.55}$$

Now, since due to locality (9.20)-(9.21) for any $k \in K'$ and $l \in L'$

$$\begin{aligned} p(A_i|a_ib_jC_k^{ij'}) &= 1 \\ p(B_j|a_ib_jC_l^{i'j}) &= 1 \end{aligned}$$

and hence for $C^{ij'}$ and $C^{i'j}$

$$\begin{aligned} p(A_i|a_ib_jC^{ij'}) &= 1 \\ p(B_j|a_ib_jC^{i'j}) &= 1 \end{aligned}$$

it follows that

$$a_ib_jC^{ij'} \quad \subseteq \quad A_i \tag{9.56}$$

$$a_ib_jC^{i'j} \quad \subseteq \quad B_j \tag{9.57}$$

except for a set of zero measure. From (9.56)-(9.57) we obtain that

$$a_ib_j\left(C^{ij'} \cup C^{i'j}\right) \quad \subseteq \quad A_i \cup B_j$$

again except for a set of zero measure and hence

$$p(a_i b_j (C^{ij'} \cup C^{i'j})) \quad \leqslant \quad p(A_i \cup B_j)$$

which using no-conspiracy (10.20) results in

$$p(C^{ij'} \cup C^{i'j}) \quad \leqslant \quad p(A_i \cup B_j | a_i b_j) = p(A_i | a_i b_j) + p(B_j | a_i b_j) - p(A_i B_j | a_i b_j) \tag{9.58}$$

Again, due to locality (9.20)-(9.21) from (9.52)-(9.53) for any $k \in K'$ and $l \in L'$ one gets

$$\begin{aligned} p(B_{j'} | a_{i'} b_{j'} C^{ij'}_k) &= 1 \\ p(A_{i'} | a_{i'} b_{j'} C^{i'j}_l) &= 1 \end{aligned}$$

and hence

$$p(B_{j'} | a_{i'} b_{j'} C^{ij'}) = 1 \tag{9.59}$$

$$p(A_{i'} | a_{i'} b_{j'} C^{i'j}) = 1 \tag{9.60}$$

From (9.59)-(9.60) we obtain that

$$\begin{aligned} a_{i'} b_{j'} C^{ij'} &\subseteq B_{j'} \\ a_{i'} b_{j'} C^{i'j} &\subseteq A_{i'} \end{aligned}$$

except for a set of zero measure and hence

$$a_{i'} b_{j'} \, (C^{ij'} C^{i'j}) \quad \subseteq \quad A_{i'} B_{j'} \tag{9.61}$$

again except for a set of zero measure. From (9.61) it follows that

$$p(a_{i'} b_{j'} (C^{ij'} C^{i'j})) \quad \leqslant \quad p(A_{i'} B_{j'})$$

or using no-conspiracy (10.20)

$$p(C^{ij'} C^{i'j}) \quad \leqslant \quad p(A_{i'} B_{j'} | a_{i'} b_{j'}) \tag{9.62}$$

Now, from (9.52)-(9.53) using the theorem of total probability and no-conspriracy (10.20) one obtains that

$$\begin{aligned} p(C^{ij'}) &= p(A_i B_{j'} | a_i b_{j'}) \\ p(C^{i'j}) &= p(A_{i'} B_j | a_{i'} b_j) \end{aligned}$$

which using the fact that

$$p(C^{ij'} \cup C^{i'j}) \quad = \quad p(C^{ij'}) + p(C^{i'j}) - p(C^{ij'} C^{i'j})$$

transforms (9.62) into

$$p(C^{ij'} \cup C^{i'j}) \quad \geqslant \quad p(A_i B_{j'} | a_i b_{j'}) + p(A_{i'} B_j | a_{i'} b_j) - p(A_{i'} B_{j'} | a_{i'} b_{j'}) \tag{9.63}$$

Constrasting (9.58) to (9.63) we get the Clauser–Horne inequality (9.24) at the upper bound. To get the inequality at the lower bound just replace A_i by $A_i^{\perp}$ and follow the steps of the above reasoning. $\Box$

9.5 Conclusions

In the paper we addressed the problem as to why Szabó (2000) was unable to yield a local, non-conspiratorial separate common causal model for the EPR scenario. We have shown that the usual answer claiming that the correlation set used by Szabó violates the Clauser–Horne inequalities if we assume that there is a local, non-conspiratorial separate common causal model of *another set*, is not satisfactory. To explain Szabó's situation one should derive some Bell inequalities from the assumption that there is a local, non-conspiratorial separate common causal model of the *original set*.

Here we provided a partial answer to this problem. We have shown that no set of correlations violating the Clauser–Horne inequalities can be given a *deterministic*, local, non-conspiratorial separate common causal explanation. This result was partial since we could not eliminate the requirement of determinism from the proof. So we conclude the paper with the following

Open question: Is it true that no set of correlations violating the Clauser–Horne inequalities can be given a (not necessarily deterministic) local, non-conspiratorial separate common causal explanation? Or in other words, does Proposition 13 hold generally that is without the assumption that $p(A_i|a_i b_j C_k^{ij})$, $p(B_j|a_i b_j C_k^{ij}) \in \{0, 1\}$ for any $i = 1, 2; j = 3, 4$ and $k(ij) \in K(i, j)$?

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Chapter 10

EPR correlations, Bell inequalities and common cause systems

Standard common causal explanations of the EPR situation assume a so-called joint common cause system that is a common cause for all correlations. However, the assumption of a joint common cause system together with some other physically motivated assumptions concerning locality and no-conspiracy results in various Bell inequalities. Since Bell inequalities are violated for appropriate measurement settings, a local, non-conspiratorial joint common causal explanation of the EPR situation is ruled out. But why do we assume that a common causal explanation of a set of correlation consists in finding a joint common cause system for *all* correlations and not just in finding *separate* common cause systems for the different correlations? What are the perspectives of a local, non-conspiratorial *separate* common causal explanation for the EPR scenario? And finally, how do Bell inequalities relate to the weaker assumption of *separate* common cause systems?

10.1 Introduction

In the history of probabilistic causation Reichenbach's definition (Reichenbach, 1956) was the first formal grasp of the notion of common cause. The conceptual novelty of the Reichenbachian definition has attracted immense interest among philosophers of science from the very beginning (Salmon, 1975; van Fraassen, 1982). From the physical side, the need for a common causal explanation of the EPR situation called attention to the definition of the common cause, even though in standard hidden variable strategies a slightly different common causal concept than the Reichenbachian has been applied (Bell, 1971; Jarrett, 1984; van Fraassen 1989). An important step in the conceptual clarification of the common cause in the EPR-Bell situation was the paper of Belnap and Szabó (1996) in which the difference between the so-called *joint* and *separate* common cause had been first recognized. Belnap and Szabó pointed out that in standard common causal explanations of the EPR correlations common cause is actually meant as a joint common cause accounting for *all* correlations.

Concerning the algebraic-probabilistic features of the Reichenbachian common cause Hofer-Szabó, Rédei and Szabó (1999) proved the following proposition. Classical (and also non-classical) correlations can be given a probabilistic common causal explanation in the sense that any classical probability measure space with correlating pairs of events can be extended such that the extension contains a Reichenbachian *separate* common cause for each correlation. (For the precise definitions see below.) Then in (Hofer-Szabó, Rédei, Szabó, 2002) it was proven that this proposition does not apply if Reichenbachian *separate* common causes are replaced with Reichenbachian *joint* common causes. In other words, classical probability measure spaces containing correlating pairs of events generally cannot be extended such that the extension contains a Reichenbachian *joint* common cause for *all* correlations. Thus, being a joint common cause of a set of correlations turned out to be a much stronger demand than being a common cause of a single correlation.

The first to apply the concept of separate common cause to the EPR situation was Szabó (2000). Since factorizability, locality and no-conspiracy together entail various types of Bell inequalities, EPR correlations cannot be given a local, non-conspiratorial, joint common causal model. Now, Szabó's idea was to replace the joint common causes with separate common causes and thus to give a separate common causal model for the EPR correlations. He constructed a number of separate common causal models which were local and non-conspiratorial in the usual sense that the measurement settings were statistically independent of the different common causes. However, the models were conspiratorial on a deeper level. The measurement settings statistically correlated with various algebraic combinations of the separate common causes. This fact called attention to the subtle but important difference between the so-called *weak* no-conspiracy where statistically independence is required only from the measure settings and the common causes *themselves*, and *strong* no-conspiracy where statistically independence is required from *any Boolean combination* of the measure settings and *any Boolean combination* of the common causes. After numerous computer simulations aiming to remove the unwanted conspiracies Szabó concluded with the conjecture that EPR cannot be given a local, *strongly* non-conspiratorial, separate common causal model.

The conjecture of Szabó has been first proven by Grasshoff, Portmann and Wüthrich (2005). The proof consisted in deriving some Bell inequality from the same assumptions that Szabó intended to apply in his separate common causal models for the EPR correlations. A crucial premise of this derivation was that the (anti)correlation between some events be *perfect*. Assuming perfect anticorrelations, however, turned the *separate* common causal explanations into a *joint* common causal explanation. This fact has been shown in (Hofer-Szabó, 2008). In the same paper Hofer-Szabó eliminated the assumption of perfect anticorrelations and presented a separate common causal derivation of some Bell-like inequalities (Bell(δ)

inequalities). At the same time Portmann and Wüthrich (2007) presented a very similar result for the Clauser-Horne inequality replacing separate common causes with the more general notion of the so-called separate common cause systems (see below). Finally, in Hofer-Szabó (2011, 2012) a general recipe has been given how to derive any type of Bell(δ) inequality provided that the original Bell inequality can be derived from a set of perfect anticorrelations.

Although due to the above proofs the separate common causal explanation of the EPR scenario has been excluded, there is a sense in which Szabó's conjecture is still not decided. Szabó's original conjecture referred to the so-called Clauser–Horne set that is a set of four correlations violating the Clauser–Horne inequality. His question was as to whether the Clauser–Horne set can be given a local, strongly non-conspiratorial, separate common causal model. Interestingly enough—in the face of the above results—this question is still open.

In Section 2 we make explicit the concepts and propositions introduced informally in the Introduction. In Section 3 the standard *joint* common causal explanation of EPR correlations will be recalled. In Section 4 and 5 we explicate what has been and what has *not* been proven in the local, non-conspiratorial, *separate* common causal explanation of the EPR scenario. We conclude the paper in Section 6.

10.2 Joint and separate common cause systems

Let us start the common causal explanation with Reichenbach's (1956) definition of the common cause. Let (Σ, p) be a classical probability measure space and let $A, B \in \Sigma$ be two positively correlating events, i.e.

$$p(A \cap B) > p(A)p(B) \tag{10.1}$$

Reichenbach then defines the common cause of the correlation as follows:

Definition 16. An event $C \in \Sigma$ is said to be the *Reichenbachian common cause* of the correlation between A and B, if the events A, B and C satisfy the following relations:

$$\begin{aligned} p(A \cap B|C) &= p(A|C)p(B|C) && (10.2)\\ p(A \cap B|\overline{C}) &= p(A|\overline{C})p(B|\overline{C}) && (10.3)\\ p(A|C) &> p(A|\overline{C}) && (10.4)\\ p(B|C) &> p(B|\overline{C}) && (10.5) \end{aligned}$$

where $\overline{C}$ denotes the complement of C and the conditional probability is defined in the usual way. Equations (10.2)-(10.3) are referred to as "screening-off" properties and inequalities (10.4)-(10.5) as "positive statistical relevance" conditions. (Here we do not discuss the problem as to whether conditions (10.2)-(10.5) are necessary or sufficient conditions for an event C to be a common cause and simply take them to be the *definition* of the common cause.)

Physicists use the notion of 'common cause' in a different meaning. We obtain this meaning if (i) we drop the positive statistical relevance conditions (10.4)-(10.5) from the definition, and (ii) we do not restrict the screening-off properties (10.2)-(10.3) to the partition $\{C, \overline{C}\}$ of Σ:

Definition 17. Let (Σ, p) be a classical probability measure space and let (A, B) be a correlating pair of events in Σ. A partition $\{C_k\}$ $(k \in K)$ of Σ is said to be the *common cause system* of the pair (A, B) if for all $k \in K$ the following conditions are satisfied:

$$p(A \cap B|C_k) = p(A|C_k)p(B|C_k) \tag{10.6}$$

The cardinality $|K|$ (the number of events in the partition) is called the *size* of the common cause system. We will refer to a common cause system of size 2 (that is of the form $\{C, \overline{C}\}$) as a *common cause.* (Sometimes we will also refer to C as a common cause.)

Now, let (Σ, p) be a classical probability measure space as before and let (A_1, B_1) and (A_2, B_2), respectively be two positively correlating pairs of events in Σ, i.e. for $i = 1, 2$

$$p(A_i \cap B_i) \neq p(A_i)p(B_i) \tag{10.7}$$

In order to give a common causal explanation for *both* correlating pairs we have two options. Either we assume that the two correlations arise from the same causal source or we attribute different causal sources to the correlations. In the first case we explain the correlation by a so-called *joint* common cause system, in the second case we employ two *separate* common cause systems. The definition of joint and separate common cause systems, respectively are the following:

Definition 18. A partition $\{C_k\}$ $(k \in K)$ of Σ is said to be the *joint common cause system* of correlations (A_i, B_i) $(i = 1, 2)$, respectively if for $i = 1, 2$ and $k \in K$ the following relations are satisfied:

$$p(A_i \cap B_i|C_k) = p(A_i|C_k)p(B_i|C_k) \tag{10.8}$$

Definition 19. Two different partitions $\{C^i_k\}$ $(i = 1, 2;\ k(i) \in K(i))$ of Σ are said to be *separate common cause systems* of the correlations (A_i, B_i) $(i = 1, 2)$, respectively if for $i = 1, 2$ and $k(i) \in K(i)$ the following relations hold:

$$p(A_i \cap B_i|C^i_k) = p(A_i|C^i_k)p(B_i|C^i_k) \tag{10.9}$$

Having defined different common causal structures let us turn to the procedure of causal explanation. A common causal explanation of a given correlation is realized mathematically by the extension of the probabilistic measure space in such a way that for the original correlation there exists a common cause system in the extended probabilistic measure space. In the case of two (or more) correlations we can extend the algebra in two different ways according to our causal intuition. In order to model a joint common causal source of the correlations we extend the algebra such that in the extended algebra all correlations have a joint common cause system. On the other hand to account for separate causal mechanisms we extend the algebra such that in the extended algebra different correlations have *separate* common cause systems.

The extendability of the probabilistic measure spaces by joint respectively separate common causal structures crucially depends on the size of the common cause system. In the case of a common cause system of size 2 that is in the case of a common cause there is a great difference between joint and separate common cause extensions as it is shown in the following two propositions:

Proposition 14. (Hofer-Szabó, Rédei, Szabó, 1999) Let (Σ, p) be a classical probability measure space and let (A_1, B_1) and (A_2, B_2), respectively be two correlating pairs of events in Σ. Then there always exists a (Σ', p') extension of (Σ, p) such that for the correlation (A_1, B_1) there exists a common cause C^1 and for the correlation (A_2, B_2) there exists a common cause C^2 in (Σ', p').

Proposition 15. (Hofer-Szabó, Rédei, Szabó, 2002) There exists a (Σ, p) classical probability measure space and two correlating pairs (A_1, B_1) and (A_2, B_2), respectively in Σ such that there is no (Σ', p') extension of (Σ, p) which contains a joint common cause C in (Σ', p') for both correlations.

Proposition 14 claims that for two correlating pairs a separate common causal explanation is always possible by extending the probability measure space in an appropriate way. (Moreover, if Σ contains $n \in \mathbb{N}$ correlating pairs, *each* correlation can be given a *separate* common causal explanation.) However, according to Proposition 15 this strategy does not work generally if we are going to obtain the same common cause for the two (or more) correlating pairs. Thus, being a joint common cause imposes much stronger demand on C than simply being a separate common cause.

However, strangely enough this difference between the common and separate common causal extendability of a probability measure space disappears if the size of the common cause system is *not* specified. In other words, to find a joint common cause system *of arbitrary size* for a set of correlations is *not* a stronger demand than to find separate common cause systems for the same set. To see this, let (A_1, B_1) and (A_2, B_2) be two arbitrary correlating pairs in Σ. Then the partition

$$\{A_1 \cap B_1, A_1 \cap B_2, A_2 \cap B_1, A_2 \cap B_2, \}$$

is always a joint common cause system in Σ for both correlations. Obviously, this partition can be regarded only as a *trivial* joint common cause system of the correlations. This makes it clear that without further specification a joint common causal explanation is not more compelling than a separate common causal explanation. In the following sections we will see how these two types of explanations diverge due to extra requirements.

10.3 No local, non-conspiratorial joint common cause system for the EPR

Consider the standard EPR-Bohm experimental setup with a source emitting pairs of spin-$\frac{1}{2}$ particles prepared in the singlet state $|\Psi_s\rangle$. Let $p(a_i)$ denote the probability that the spin measurement apparatus is set to measure the spin in direction $\vec{a}_i$ ($i \in I$) in the left wing and let $p(b_j)$ denote the same for direction $\vec{b}_j$ ($j \in J$) in the right wing. Furthermore, let $p(A_i)$ stand for the probability that the spin measurement in direction $\vec{a}_i$ in the left wing yields the result $+1$ ('up') and let $p(\overline{A}_i)$ denote the probability of the result -1 ('down'). Let $p(B_j)$ and $p(\overline{B}_j)$ be defined in a similar way in the right wing for direction $\vec{b}_j$. (See Fig. 1) Quantum mechanics then yields the following conditional probabilities for the events in question:

$$p(A_i \cap B_j | a_i \cap b_j) = Tr(W_{|\Psi_s\rangle}\,(P_{A_i} \otimes P_{B_j})) \quad = \quad \frac{1}{2}\sin^2(\frac{\theta_{a_i b_j}}{2}) \qquad (10.10)$$

$$p(A_i | a_i \cap b_j) = Tr(W_{|\Psi_s\rangle}\,(P_{A_i} \otimes I)) \quad = \quad \frac{1}{2} \qquad (10.11)$$

$$p(B_j | a_i \cap b_j) = Tr(W_{|\Psi_s\rangle}\,(I \otimes P_{B_j})) \quad = \quad \frac{1}{2} \qquad (10.12)$$

where $W_{|\Psi_s\rangle}$ is the density operator pertaining to the pure state $|\Psi_s\rangle$; P_{A_i} and P_{B_j} denote projections on the eigensubspaces with eigenvalue $+1$ of the spin operators associated with directions $\vec{a}_i$ and $\vec{b}_j$, respectively; and $\theta_{a_i b_j}$ denotes the angle between directions $\vec{a}_i$ and $\vec{b}_j$.

Thus, for non-perpendicular directions $\vec{a}_i$ and $\vec{b}_j$ there is a conditional correlation

$$p(A_i \cap B_j | a_i \cap b_j) \quad \neq \quad p(A_i | a_i \cap b_j) p(B_j | a_i \cap b_j) \qquad (10.13)$$

and for parallel directions there is a perfect anticorrelation between the outcomes:

$$p(A_i \cap B_j | a_i \cap b_j) \quad = \quad 0 \qquad (10.14)$$

Now, consider a set $\{(A_i, B_j)\}_{(i,j) \in I \times J}$ of EPR correlations in the sense of (10.13). A full-fledged common causal explanation of the set $\{(A_i, B_j)\}_{(i,j) \in I \times J}$ must comply with three demands on the statistical level. Firstly, all the correlations must be screened-off by a joint common cause system. Secondly, statistical relations among the measurement outcomes and the measurement settings must reflect the spacetime location of these events in the sense that spatially separated events have to be statistically independent. Thirdly, the measurement settings and the common cause should not influence each other, they have to be statistically independent. We refer to these requirements in turn as 'joint common cause system', 'locality' and 'no-conspiracy'. In the case of 'no-conspiracy' we will distinguish two types: the 'weak' and the 'strong no-conspiracy'. The precise probabilistic formulation of these demands is the following:

1. *Joint common cause system:* There exists a partition $\{C_k\}$ of Σ such that for every A_i, B_j, a_i and b_j in Σ $(i \in I, j \in J)$ and for any $k \in K$ the following factorization holds:

$$p(A_i \cap B_j | a_i \cap b_j \cap C_k) = p(A_i | a_i \cap b_j \cap C_k) p(B_j | a_i \cap b_j \cap C_k) \quad (10.15)$$

2. *Locality:* For every A_i, B_j, a_i, b_j and C_k in Σ $(i \in I, j \in J, k \in K)$ the following screening-off relations hold:

$$p(A_i | a_i \cap b_j \cap C_k) = p(A_i | a_i \cap C_k) \quad p(B_j | a_i \cap b_j \cap C_k) = p(B_j | b_j \cap C_k) \quad (10.16)$$

3. *a. Weak no-conspiracy:* For every a_i, b_j and C_k in Σ $(i \in I, j \in J, k \in K)$ the following independence holds:

$$p(a_i \cap b_j \cap C_k) = p(a_i \cap b_j) p(C_k) \quad (10.17)$$

   b. Strong no-conspiracy: Consider two Boolean subalgebras $\mathfrak{A}$ and $\mathfrak{C}$ of Σ such that $\mathfrak{A}$ is generated by the partition of the different measurement choices $\{a_i b_j\}$ $(i \in I, j \in J)$ on the opposite wings, and $\mathfrak{C}$ is generated by the partition of the common cause system $\{C_k\}$ $(k \in K)$. Then for any element $E \in \mathfrak{A}$ and $F \in \mathfrak{C}$ the following independence holds:

$$p(E \cap F) = p(E) p(F) \quad (10.18)$$

It is straightforward to see that in the case of *joint* common cause systems (10.17) and (10.18) are equivalent, the probabilistic independence of the *Boolean combinations* of common causes and the measurement settings does not demand more than simply the probabilistic independence of the common causes and the measurement settings *themselves.* Thus, in the case of the joint common cause system type explanations equation (10.17) will suffice as a no-conspiracy requirement.

However, as it is well-known (10.15)-(10.17) result in various Bell inequalities which are violated for special measurement settings in the EPR experiment. For the simplest set of correlations, namely for the Clauser–Horne set $\{(A_i, B_j)\}_{(i,j) \in CH}$ where $CH = I \times J$ with $I = \{1, 2\}$ and $J = \{3, 4\}$ the Bell theorem is the following:

Proposition 16. (Clauser, Horne, 1974) For some measurement directions $\vec{a}_1, \vec{a}_2$ and $\vec{b}_3, \vec{b}_4$ there cannot exist extension of the probability space (Σ, p) such that the extension contains local, (weakly or strongly) non-conspiratorial joint common cause systems for all EPR correlations of $\{(A_i, B_j)\}_{(i,j) \in CH}$.

Consequently, EPR correlations fall short of a local, non-conspiratorial, joint common cause system type explanation. One premise has to be given up.

10.4 Local, weakly non-conspiratorial separate common cause systems do exist for the EPR

Strategies aiming to avoid Bell inequalities and to give a common causal explanation for the EPR correlations can be grouped according the abandoned premise. The first group consists of approaches abandoning locality and preserving the joint common causal background and no-conspiracy. Bohmian mechanics is an eminent representative of this group. The second group consists of less attractive models in which no-conspiracy is given up. Examples of this approach are Brans' and Szabó's models (Brans, 1988; Szabó, 1995). In these models the authors relinquished no-conspiracy and provided a local, deterministic but conspiratorial joint common cause system type explanation for the EPR. (For the problem of free will and no-conspiracy see (SanPedro, 2013.) In this paper, however, we will follow a third strategy which gives up the hypothesis of a joint common cause system. The key idea here is to replace the concept of joint common cause system with that of separate common cause systems and to provide a local, non-conspiratorial, *separate* common cause system type explanation for the EPR. A *separate* common cause system type explanation for a set $\{(A_i, B_j)\}_{(i,j)\in I\times J}$ consists in finding for every $(i,j) \in I \times J$ index pair a *separate* partition $\{C_k^{ij}\}$ $(k(ij) \in K(ij))$ such that screening-off, locality, and (weak or strong) no-conspiracies holds in the following sense:

1. *Separate common cause systems:* For every A_i, B_j, a_i and b_j in Σ $(i \in I, j \in J)$ there exists a separate partition $\left\{C_k^{ij}\right\}$ of Σ such that for any $k(ij) \in K(ij)$ the following factorization holds:
$$p(A_i \cap B_j|a_i \cap b_j \cap C_k^{ij}) = p(A_i|a_i \cap b_j \cap C_k^{ij})p(B_j|a_i \cap b_j \cap C_k^{ij}) \tag{10.19}$$
2. *Locality:* For every $i \in I, j \in J$ and $k(ij) \in K(ij)$ the following screening-off relations hold:
$$p(A_i|a_i \cap b_j \cap C_k^{ij}) = p(A_i|a_i \cap C_k^{ij}), \quad p(B_j|a_i \cap b_j \cap C_k^{ij}) = p(B_j|b_j \cap C_k^{ij}) \tag{10.20}$$
3. *a. Weak no-conspiracy:* For every a_i, b_j and $C_k^{i'j'}$ in Σ $(i, i' \in I;\ j, j' \in J;\ k(i'j') \in K(i'j'))$ the following independence holds:
$$p(a_i \cap b_j \cap C_k^{i'j'}) = p(a_i \cap b_j)p(C_k^{i'j'}) \tag{10.21}$$
b. Strong no-conspiracy: Consider again two Boolean subalgebras $\mathfrak{A}$ and $\mathfrak{C}$ of Σ such that $\mathfrak{A}$ is generated by the partition of the different measurement choices $\{a_i b_j\}$ $(i \in I, j \in J)$ and $\mathfrak{C}$ is generated by the partition of *all the different* common cause systems $\left\{\cap_{ij} C_k^{ij}\right\}$ $(k \in K)$. Then for any element $E \in \mathfrak{A}$ and $F \in \mathfrak{C}$ the following independence holds:
$$p(E \cap F) = p(E)p(F) \tag{10.22}$$

Here, requirement (10.21) does *not* entail (10.22), that is the independence of the separate common cause systems of the choice of the measurement settings does *not* assure that *any Boolean combination* of the common causes will also be independent of *any Boolean combination* of the measurement settings. Thus, in the case of separate common cause system type explanations one has to take into consideration two different versions of no-conspiracy.

The idea to replace the concept of a joint common cause system with that of separate common cause systems and to provide a local, non-conspiratorial separate common cause system type explanation for the EPR was first raised by Szabó (2000). Actually, Szabó replaced the joint common cause system with separate common cause systems of size 2 that is with separate common causes. Szabó provided a number of separate common causal models for the Clauser–Horne set $\{(A_i, B_j)\}_{(i,j)\in CH}$ such that the models were local and non-conspiratorial in the *weak* sense of (10.22). In a precise form, Szabó's proposition was the following:

Proposition 17. (Szabó, 2000) Let $\{(A_i, B_j)\}_{(i,j)\in CH}$ be the Clauser–Horne set of correlations in (Σ, p). Then for any measurement directions $\vec{a}_1, \vec{a}_2$ and $\vec{b}_3, \vec{b}_4$ there exists an extension of the probability space (Σ, p) such that the extension contains local, *weakly* non-conspiratorial separate common causes for the correlations of $\{(A_i, B_j)\}_{(i,j)\in CH}$.

The common causal models provided by Szabó, however, were all conspiratorial in the *strong* sense of (10.22). After numerous computer simulations aiming to remove the unwanted conspiracies Szabó finally concluded with the conjecture that EPR *cannot* be given any local, separate common causal model free from *all* type of conspiracies.

10.5 Local, strongly non-conspiratorial separate common cause systems for the EPR?

Szabó's conjecture is then the following:

Conjecture 1. For some measurement directions $\vec{a}_1, \vec{a}_2$ and $\vec{b}_3, \vec{b}_4$ there cannot exist extension of the probability space (Σ, p) such that the extension contains local, *strongly* non-conspiratorial separate common cause systems for the correlations of $\{(A_i, B_j)\}_{(i,j)\in CH}$.

Although a lot has happened since 2000 in understanding the status of the separate common causal explanation of the EPR scenario, Szabó's conjecture in its original form is *still an open question.* What has actually been excluded, is not a local, strongly non-conspiratorial separate common causal explanation of the the Clauser–Horne set $\{(A_i, B_j)\}_{(i,j)\in CH}$, but that of *another set.* Let $I = J = \{1, 2, 3, 4\}$ and let PA be the following subset of $I \times J$:

$$PA = \{(1,1), (2,2), (3,3), (4,4)\}$$

Then one can prove the following proposition:

Proposition 18. For some measurement directions $\{\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4\}$ and $\{\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4\}$ there cannot exist extension of the probability space (Σ, p) such that the extension contains local, *strongly* non-conspiratorial separate common cause systems for all EPR correlations of $\{(A_i, B_j)\}_{(i,j)\in PA}$.

The above proposition was first proved by Grasshoff, Portmann and Wüthrich (2005). They have shown that no local, *strongly* non-conspiratorial separate common cause systems are possible for all correlations of $\{(A_i, B_j)\}_{(i,j)\in PA}$, *if* for any index pair $(i,j) \in PA$ there is a *perfect anticorrelation* (hence the denotation 'PA') in the sense of (10.14).

The assumption of perfect anticorrelations, however, was unsatisfactory in two respects. The first problem concerns experimental testability. Since perfect anticorrelations cannot be tested experimentally with absolute precision, the proof of Grasshoff, Portmann and Wüthrich did not provide an experimentally verifiable refutation of a separate common causal explanation of the EPR.

The second problem was more conceptual. Standard derivations of the Bell inequalities assume a joint common cause system. The chief virtue of the proof of Grasshoff, Portmann and Wüthrich was that it avoided this strong concept of a *joint* common cause system and used the weaker concept of *separate* common cause systems instead. However, in the perfect anticorrelation case the assumptions of separate common cause systems turned out to be reducible to the assumptions of the standard joint common cause system as it was shown in the following proposition:

Proposition 19. (Hofer-Szabó, 2008) Let $\left\{C_k^{ij}\right\}_{(i,j)\in PA}$ be local, strongly non-conspiratorial *separate* common cause systems for the correlations of $\{(A_i, B_j)\}_{(i,j)\in PA}$. Then the partition $\{D_l\} := \left\{\cap_{ij} C_k^{ij}\right\}$ generated by the intersections of the different separate common cause systems is a local, non-conspiratorial *joint* common cause system of the same correlations of $\{(A_i, B_j)\}_{(i,j)\in PA}$.

The assumption of perfect anticorrelations, however, turned out not to be indispensable in the proof of Proposition 18. Portmann and Wüthrich (2007) and Hofer-Szabó (2008) have shown that Proposition 18 also holds if one only assumes that the correlations to be explained form an *almost perfect anticorrelation* set, $\{(A_i, B_j)\}_{(i,j)\in PA(\delta)}$, in the sense that there exists a δ of some small but not zero value such that

$$p(A_i \cap B_j | a_i \cap b_j) \quad \leqslant \quad \delta \tag{10.23}$$

for any index pair $(i,j) \in PA(\delta)$.

Finally, Hofer-Szabó (2011, 2012) generalized this proof by deriving arbitrary Bell(δ) inequality— that is to say, an inequality differing from the corresponding Bell inequality in a term of order δ. The recipe of this derivation is roughly the following. Consider a Bell inequality resulting from the local, non-conspiratorial *joint* common causal explanation of a given set of correlations $\{(A_i, B_j)\}_{(i,j)\in I\times J}$ (not necessarily $\{(A_i, B_j)\}_{CH}$). Now, define the set PA for $\{(A_i, B_j)\}_{(i,j)\in I\times J}$ as follows: let PA contain all the index pairs (k, k) in $(I \cup J) \times (I \cup J)$ that is all indices appearing either on the left or the right hand side of the correlations in $\{(A_i, B_j)\}_{(i,j) \in I \times J}$.

Now consider the set $\{(A_i, B_j)\}_{PA(\delta)}$ of almost perfect anticorrelations and suppose that it has a local, strongly non-conspiratorial separate common causal explanation. This assumption results in a Bell(δ) inequality differing from the original Bell inequality in a term of order of δ where the exact magnitude of this term is the function of the approximation. Choose the setting which violates the Bell inequality maximally. If the δ term is smaller than the violation of the original Bell inequality, then the Bell(δ) inequality will also be violated, excluding a local, strongly non-conspiratorial separate common causal explanation of the set $\{(A_i, B_j)\}_{PA(\delta)}$.

10.6 Conclusions

In the paper, first, different common causal concepts ranging from Reichenbach's definition to the most general concept of the common cause system have been listed. Then the role of the different causal notions in the common causal explanation of the EPR scenario has been exposed. It was said that a completely satisfactory common causal explanations of the EPR would consist in finding a *joint* common causal source for all correlations which is local and non-conspiratorial. Since these assumptions together entail various Bell inequalities one assumption has to be abandoned. The ambition of the separate common cause system type approach of the EPR was to preserve the latter two physically motivated assumptions of locality and no-conspiracy at the expense of replacing the strong concept of the joint common cause system with the weaker concept of separate common cause systems. It has been shown, however, that the weakening of the common causal concept does not provide a solution to this problem since the weakened assumptions still entail some Bell and Bell(δ) inequalities. Consequently, there exists neither a local, (weakly or strongly) non-conspiratorial *separate* common causal explanation of the EPR.

A weakness of all the above no-go theorems, however, is that they are all based on either *perfect* or *almost perfect* EPR correlations. As it was made clear in Proposition 19 the separate common causal explanation of such correlations is always parasitic on some joint common causal explanation. Therefore it would be highly desirable to derive some Bell inequality form a local, strongly non-conspiratorial separate common causal explanation of a set of *genuine* (not almost perfect) EPR correlations. For example it would be widely wanted to prove or falsify Szabó's original conjecture (Conjecture 1)—that is for the set $\{(A_i, B_j)\}_{(i,j) \in CH}$ violating the Clauser–Horne inequality

(i) either to derive the Clauser–Horne inequality (or some other constraint) from the assumption that $\{(A_i, B_j)\}_{(i,j) \in CH}$ has a local, strongly non-conspiratorial separate common causal explanation;

(ii) or to show up local, strongly non-conspiratorial separate common cause systems for the set $\{(A_i, B_j)\}_{(i,j) \in CH}$.

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Neither option seems to be a trivial task.

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