Fixed points and choices: stable marriages and beyond

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Contents

Introduction

In their famous paper, Gale and Shapley proposed the following problem [35]. Imagine that each of n men and n women ranks the members of the opposite gender as a possible spouse. If these people form n married couples then the hence determined matching is unstable if there is a man and a woman who mutually prefer one another to their eventual partners. A natural goal is to match these people such that this kind of instability does not occur. This idea includes many possible generalizations. The players may seek more than one partners. A practice-motivated example for this is the college admission problem where two the sets of players that rank each other are colleges and applicants instead of men and women, and each college has an individual quota that bounds the number of acceptable applicants from above. In an admission scheme, each applicant is admitted by at most one college and no college exceeds is quota. Such a scheme is unstable if there is an applicant who is not admitted to a certain college but both the college would be happy to accept the applicant (and possibly to fire another less preferred applicant to comply with the quota) and the applicant would improve by being admitted to this college. It is worth mentioning that the admission scheme determined by the education office of the Hungarian government avoids exactly this kind of instability.

It is also possible that not every man-woman (or college-applicant) pair can be realized, that is, the bipartite graph representing possible marriages is not a complete one. In fact this graph does not even have to be bipartite. The roommates problem leads to this generalization. In this problem, the goal is to allocate a set of students into two persons dormitory rooms, where each student ranks his or her possible roommates. Instability of a room assignment here means that no two students would be happier to share a room rather that living with their roommates given by the scheme.

A further direction of possible generalizations with a clear practical motivation is dropping the requirement on strict preference orders of the players, and hence allowing ties. For example, in the college admission problem, each applicant is expected to formulate a strict preference order on the colleges, while for colleges it is allowed to rank two different applicant equally. As we know, the ranging of each college is based solely on the entrance exam score that might well be equal for two different applicants.

On the original problem the following results are well-known. With the help of the deferred acceptance algorithm, Gale and Shapely proved that there exists a stable solution for both the marriage and the college admission problems. Knuth attributes the observation to Conway that stable matchings form a lattice [52]. Later, this observation turned out to be crucial for several further structural results about stable matchings. Actually, the study of stable matchings employs tools of several disciplines: Knuth intended his above mentioned book as an introduction to the theory of algorithms through demonstrating methods and tools in the design and analysis of algorithms [52]. In connection with further algorithmic complexity aspects, we should mention the book of Gusfield and Irving [37]. Optimization over stable matchings is another natural task, where polyhedral methods play an important role here [68, 61, 8, 58, 1, 66, 21]. The book of Roth and Sotomayor leans on notions and methods from Game Theory [60].

Based on our present knowledge, it is fair to say that by the introduction of the notion of stable matchings, Gale and Shapley achieved much more than their direct goal articulated in their paper, namely the popularization of the mathematical and game theoretic approach. It becomes clear from the research built upon their work that their notion is exceptionally successful both for practical applications and for theoretical results. For the former fact, we do not need a better argument than the 2012 Nobel Memorial Prize in Economic Sciences having been awarded to Roth and Shapley for their work on mechanism design and on the theory of stable matchings. A possible example of the latter successes is Galvin's proof for the Dinitz' conjecture that essentially leans on stable matchings [36] or the unexpected breakthrough by Király's approximation algorithm [49].

It is worth mentioning some results connected to generalizations. The generalized notion of stability may manifest in a common antichain of posets or in a common independent set of matroids [25], but it is also possible to define stability of network flows that can serve as some model of supply chains known in Economics [23]. Our stable flow model is closely related to the supply chain model by Ostrovsky which is more general in some sense and more restricted in some other sense than flows [55]. As a matter of fact, in the Economics literature, Ostrovsky's result is considered a clear breakthrough that became the origin of further important results.

These days, stable matchings have a wide literature. Regarding only the algorithmic aspects, the state of the art around 2013 is collected in Manlove's imposing book [54]. The present work aims for a much more modest goal: it intends to introduce the reader to an unorthodox method and to point out certain interesting aspects. Incidentally, it tries to change our understanding of the history of stable matchings. As we know, Gale and Shapley published their seminal paper in 1962. Later Roth pointed out that a variant of the deferred acceptance algorithm is used in the USA from 1952 [57], hence this should be the origin of the known history of stable matchings. We attempt to date the beginning of the story to as early as 1928 when Knaster and Tarski published (back then without a proof) a fixed point theorem on monotone set functions [51]. Later, in 1955, Tarski proved a lattice theoretical generalization and illustrated its applicability by deducing various mean value theorems known from Analysis [65]. It turned out that this fixed point theorem has nontrivial consequences already in the finite case. It clearly explains the correctness of the deferred acceptance algorithm of Gale and Shapley and the lattice property of stable matchings. As we shall explain, the link between stable matchings and monotone mappings are so-called choice functions introduced by Kelso and Crawford [46].

Without going into the details, we mention some further significant results on generalizations of stable matchings. For some time, it was unclear whether there is a polynomial time algorithm that decides the existence of a stable matching in a nonbipartite graph. The positive answer is due to Irving whose algorithm has a first phase based on the steps of the deferred acceptance algorithm and in the second phase it eliminates so-called rotations until it either finds a stable matching or concludes that no stable matching exists in the input graph [45]. Later, Tan extended Irving's algorithm and hence he proved the existence of stable half-matchings (that may contain edges of weight $\frac{1}{2}$ as well) [64]. It also turned out that a graph contains a stable matching if and only if no stable half-matching contains an odd cycle of $\frac{1}{2}$-weight edges. Aharoni and Fleiner have pointed out that the existence of stable half-matchings is a consequence of Scarf's lemma, and Scarf's lemma can be regarded as a close relative of Brouwer's fixed point theorem [6]. Hence, stable matchings can be viewed as fixed points: in case of a bipartite graph, stable matchings are fixed points of a monotone mapping, while in case of nonbipartite graphs, stable matchings are fixed points of a more complicated function. Cechlárová and Fleiner proposed a generalization of Irving's algorithm to find a stable b-matching [13], while Biró, Cechlárová and Fleiner studied the change of stable matchings when a new player emerges on the market. A consequence of their result is that also in case of a nonbipartite graph, it is possible to define a certain friend and an enemy relation between the players by observing how the situation of one player changes when the other player leaves the market [10].

Our present work is structured as follows.

- In Chapter 1, we introduce stable matchings in graphs, build our framework on (lattice-based) choice functions, define choice-function related kernels, and point out that stable matchings are examples of such kernels. Then we introduce Tarski's fixed point theorem and indicate the connection between fixed points of a monotone mapping and the previously defined kernels. In particular, we show that the deferred acceptance algorithm of Gale and Shapley can be regarded as an iteration of a monotone mapping. Our approach described here is an improved version of the one in [20].

- In Chapter 2, we deduce two independent results on kernels (i.e. on generalized stable matchings): one about fractional kernels in posets and another one on matroid-kernels.

- Chapter 3 is devoted to consequences of the lattice structure of kernels.

- Chapter 4 illustrates three applications: we connect Pym's linking theorem to kernels, then we prove two extensions of Galvin's result on list-colorings of graphs and we exhibit an interesting application of the matroid-kernel result on a specific college admission problem with lower quotas.

- Chapter 5 discusses the characterization of kernel-related polyhedra. It turns out that in the general case, a blocking-antiblocking type characterization follows from earlier results of Hoffman and Schwartz on lattice polyhedra and of Fulkerson on blocking polyhedra. In the special case of stable b-matchings, a structural result from Chapter 3 (namely the splitting property) allows us to prove a less implicit characterization based on a polynomial number of constraints.

- Next, Chapter 6 is devoted to the recent concept of stability in network flows. We prove that stable flows always exists and exhibit some observations on the structur of stable flows. Our results are closely related to a recent topic in Economics on stability of supply chains that started with the celebrated paper by Ostrovsky [55].

- At last, in Chapter 7, we study stable matchings and generalizations in graphs that are not necessarily bipartite. We point out another fixed-point connection and show the existence of fractional stable matchings. Then we reduce the problem of stable b-matchings to stable matchings and describe several generalizations of Irving's algorithm. Our last topic is an extension of Tan's characterization on stable matchings to the nonbipartite case with choice functions.

- Finally, we conclude with a subjective review on our contribution.

As the goal of the present dissertation is to explain the author's scientific contribution, we apply the following convention. Results of the author are highlighted by underlining or boxing. Such an expression indicates that the result belongs (at least partly) to the author, in the latter case the result has not been used to obtain a scientific degree so far.

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Chapter 1

Foundations

In this chapter we introduce our terminology that turns out to be useful in generalizing and extending stable matching related results. First we give a nonstandard proof on the existence of ordinary stable matchings and then we build up our choice function based framework and present the connection to Tarski's fixed point theorem.

1.1 Stable matchings

Let $G = (V, E)$ be a graph and let $\preceq_v$ be a linear (preference) order on the set $E(v)$ of edges incident with v for each vertex v of G. We say that edge e is *better* for v than edge f if $e \preceq_v f$ holds. Subset $M \subseteq E$ of edges is a *matching*, if no two edges of M have a common vertex, i.e. if $d_M(v) \leq 1$ holds or each vertex $v \in V$. For given bounds $b : V \to \mathbb{N}$ set $M \subseteq E$ of edges is a *b-matching* if $d_M(v) \leq b(v)$ holds for each vertex v of G. Clearly a matching is a special case of a b-matching for $b \equiv 1$. Matching *M dominates edge $e = uv$ at u* if M contains some edge m with $m \preceq_u f$. Similarly, b-matching *M b-dominates edge $e = uv$ at u* if there are edges $m_1, \ldots, m_{b(u)}$ of M such that $m_i \preceq_u f$ holds for each $1 \leq i \leq b(u)$. Matching M *(b-)dominates* edge $e = uv$ if M (b-)dominates e at u or at v. Edge e *blocks* (b-)matching M if M does not (b-)dominate e. At last, (b-)matching M is a *stable* (b-)matching if there is no blocking edge, that is, if M dominates each edge of G in $E \setminus M$. If, for example G is an odd cycle such that each vertex prefers the former vertex to the latter vertex in some orientation of the cycle then it can be seen easily that no stable matching exists in G. This is not the case for bipartite graphs as the following theorem shows.

Theorem 1.1 (Gale and Shapley [35]). *If graph G is bipartite then for any preferences there exists a stable matching.*

Note that Gale and Shapley proved Theorem 1.1 only for complete bipartite graph $K_{n,n}$ and they extended this result by showing that there must exist a stable b-matching if $b \equiv 1$ on one side of G. Gale and Shapley actually showed that their deferred acceptance algorithm finds a stable matching for any input instance. This algorithm can be described in man-woman terminology as follows. In the beginning, each man proposes to his first choice. If men propose to different women then proposals become marriages and this is a stable scheme. Otherwise, there is a woman that received more than one proposal. All these women refuse all but their best proposer. If a refusal took place then each man proposes again to his first choice that did not refuse the particular man. Sooner or later no refusal takes place. The last proposals become marriages and this is output by the algorithm.

In their paper, Gale and Shapley remark that their result is an excellent counterexample for the stereotypical beliefs that any reasonable mathematical deduction must contain difficult calculations or obscure formulas. For example, though the description and the proof of correctness of the deferred acceptance algorithm is free of all these, it clearly is a decent mathematical proof. Without disputing this statement, we point out an unusual proof for the correctness of the deferred acceptance algorithm. This is based on the following easy observation that is valid also for nonbipartite graphs.

Lemma 1.2 (Fleiner). *Assume that for each vertex v of not necessarily bipartite graph G, linear preference order $\preceq_v$ is given on the set of edges $E(v)$ incident to v. Assume that $e \prec_v f$ holds for edge $e = uv$ that is best according to preference order $\preceq_u$. Then the set of stable matchings in G coincides with the set of stable matchings in $G - f$.*

Proof. Let M be a stable matching of G. If $e \in M$ then $f \notin M$ as M is a matching and if $e \notin M$ then M dominates e at v and hence $f \notin M$ holds again. Consequently, M is a stable matching of $G - f$.

Assume now that M is a stable matching of $G - f$. If $e \in M$ then M dominates f, and hence M is stable in G, as well. If $e \notin M$ then M must dominate e at v. Therefore M also dominates f at v, and M is a stable matching of M again. □

According to Lemma 1.2, we may remove certain edges from G "for free" without creating or destroying a single stable matching. If we keep on applying this operation on a bipartite graph, sooner or later we reach a state where no more edges can be removed, and hence both men and women have different people on the top of their preference lists. It is easy to see that both the first choices of the men and the first choices of the women represent a stable matching in the graph resulted after all the edge-deletions. Hence, due to Lemma 1.2, these will be stable matchings also in the original graph G. Moreover, it also follows immediately that the stable matching output by the deferred acceptance algorithm is *man-optimal*, meaning that each man receives a wife that is best for him among those women that are achievable for him in some stable matching of G. Furthermore, this also implies that this procedure provides the worst husband for each woman out of those men that can be the partner of the particular woman in some stable matching of G.

The above proof with obvious changes justifies the correctness of the appropriate extension of the deferred acceptance algorithm for b-matchings and also shows that the output stable b-matching is optimal in the above sense for the proposing side.

Using standard graph-theoretical tools, it is not difficult to prove the following lattice property of stable matchings. If M_1 and M_2 are stable (b-)matchings and each man picks his favorite (b-)matching edges out of $M_1 \cup M_2$ then the hence chosen edges from a stable (b-)matching. Later on, we shall see far reaching generalizations of this fact.

1.2 Choice functions

A most useful tool to study stable matchings and generalizations is the notion of choice functions that helps us to describe the preferences of the players. Our unusual way to

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introduce choice functions is based on so-called determinants. For ground set E, mapping $\mathcal{F} : 2^E \to 2^E$

- is a *choice function* if there exists a mapping $\mathcal{D} : 2^E \to 2^E$ such that $\mathcal{F}(X) = X \cap \mathcal{D}(X)$ holds for each subset X of E. (Such mapping $\mathcal{D}$ is called a *determinant* of $\mathcal{F}$);
- is *monotone* if $\mathcal{F}(X) \subseteq \mathcal{F}(Y)$ holds for $X \subseteq Y \subseteq E$ and
- is *antitone*, if $\mathcal{F}(X) \subseteq \mathcal{F}(Y)$ holds whenever $Y \subseteq X \subseteq E$ and at last
- is *substitutable* if $\mathcal{F}$ is a choice function that has an antitone determinant[1].

Obviously, $\mathcal{F}$ is a choice function iff $\mathcal{F}(X) \subseteq X$ holds for any subset X of E, moreover the same choice function may have several different determinants. The textbook example of a substitutable choice function is the one of men in the Gale-Shapley model.

Example 1.3. *Let $G = (V, E)$ be a finite bipartite graph where sets A of men and B of women are the parts and let $\preceq_v$ be a linear order on the set $E(v)$ of edges incident to v for each vertex v of G. For each subset X of E, define set $\mathcal{F}_A(X)$ as those edges $e = mw$ in X that are $\preceq_m$-best for man m in X.*

It is easy to see that an antitone determinant of the above mapping $\mathcal{F}_A$ is $\mathcal{D}_A(X) := \bigcup_{v\in A} \bigcap_{e\in X\cap E(v)} \mathcal{D}_A(v, e)$ where $\mathcal{D}_A(v, e) := \{e' \in E(v) : e' \preceq_v e\}$ is the set of those edges that are not $\preceq_v$-worse for v than e. Hence $\mathcal{D}_A(X)$ consists of all edges of E that are not dominated by another edge of X at some vertex of A. Similarly, as in Example 1.3 above, we may define choice function $\mathcal{F}_B$ of women and the corresponding antitone determinant $\mathcal{D}_B$.

Observation 1.4 (Fleiner). *Subset M of E is a stable matching in bipartite graph $G = (V, E)$ iff there exist subsets X and Y of E with $M = X \cap Y$ and $\mathcal{D}_A(X) = Y$ and $\mathcal{D}_B(Y) = X$ hold.*

Proof. Assume that M is a stable matching and define $X := \mathcal{D}_B(M)$ and $Y := \mathcal{D}_A(M)$. As matching M is stable and no edge $e \in X \setminus M$ is dominated by M at a vertex of B, each of these edges are dominated by M at some vertex of A. Consequently, $\mathcal{D}_A(X) = \mathcal{D}_A(M) = Y$. A similar proof shows that $\mathcal{D}_B(Y) = X$.

Assume now that $M = X \cap Y$, $\mathcal{D}_A(X) = Y$ and $\mathcal{D}_B(Y) = X$ holds for subsets X and Y of E. If $e \prec_v f$ holds for edge e and f of M for some $v \in A$ then $f \notin \mathcal{D}_A(M) \supseteq \mathcal{D}_A(X) = Y$, hence $f \notin X \cap Y = M$, a contradiction would follow. The same holds in case of $v \in B$, so M is a matching.

If $f \notin M = X \cap Y$ then $f \notin X = \mathcal{D}_B(Y)$ or $f \notin Y = \mathcal{D}_A(X)$. In the first case, f is dominated by some other edge of Y at some vertex v of B. If e denotes the $\preceq_v$-best such edge of Y then $e \in X = \mathcal{D}_B(Y)$ and hence $e \in Y \cap X \in M$ holds, showing that M dominates f at vertex v of B. A similarly proof shows that, in the second case when $f \notin Y = \mathcal{D}_A(X)$ then f is dominated by M at some vertex v of A. Consequently, M is a stable matching. □

[1] The standard definition of substitutability requires that $X \cap F(X + e) \subseteq F(X)$ holds for any $X \subseteq E$ and $e \in E$. This means that if we are not interested in a certain choice then this choice does not become more interesting if the choices set grows. Our definition is somewhat stronger, but only for an infinite ground set E.

The following useful lemma shows an important property of substitutable choice functions.

Lemma 1.5. *If choice function $\mathcal{F} : 2^E \to 2^E$ is substitutable and $\mathcal{F}(X) \subseteq Y \subseteq X$ then $\mathcal{F}(X) \subseteq \mathcal{F}(Y)$ holds.*

Proof. For antitone determinant $\mathcal{D}$ of $\mathcal{F}$ we get $\mathcal{F}(Y) = Y \cap \mathcal{D}(Y) \supseteq Y \cap \mathcal{D}(X) \supseteq \mathcal{F}(X) \cap \mathcal{D}(X) = (X \cap \mathcal{D}(X)) \cap \mathcal{D}(X) = X \cap \mathcal{D}(X) = \mathcal{F}(X)$. □

We define the following crucial properties. Assume that mapping $w : 2^E \to \mathbb{R}_+$ is *strictly monotone* (that is, $w(\emptyset) = 0$ and $w(a) < w(b)$ holds whenever $a \prec b$). (For a finite set E, $w(X) := |X|$ is an example of such a function, but any positive weight function on E induces such a function.) Choice function $\mathcal{F} : 2^E \to 2^E$

- has the *IRC* property if $\mathcal{F}(X) = \mathcal{F}(Y)$ holds for $\mathcal{F}(X) \subseteq Y \subseteq X$,
- $\mathcal{F}$ is *path-independent* if $X, Y \subseteq E \Rightarrow \mathcal{F}(X \cup Y) = \mathcal{F}(X \cup \mathcal{F}(Y))$,
- and $\mathcal{F}$ is *increasing*, if $|\mathcal{F}(X)| \leq |\mathcal{F}(Y)|$ holds whenever $X \subseteq Y$.
- and $\mathcal{F}$ is *w-increasing*, if $w(\mathcal{F}(X)) \leq w(\mathcal{F}(Y))$ holds whenever $X \subseteq Y$.

Clearly, the increasing property is a special case of the w-increasing one for function w defined by $w(X) = |X|$.

Observation 1.6 (Fleiner). *Let E be a finite set and assume that $\mathcal{F} : 2^E \to 2^E$ is a substitutable choice function. Then $\mathcal{F}$ has the IRC property if and only if $\mathcal{F}$ is path-independent. Moreover, if $\mathcal{F}$ is w-increasing for some strictly monotone mapping w on 2^E then $\mathcal{F}$ has the IRC property (and hence $\mathcal{F}$ is path-independent) as well.*

Proof. Let $\mathcal{D}$ be an antitone determinant of $\mathcal{F}$. Assume $\mathcal{F}$ has the IRC property and let $X, Y \subseteq E$. Clearly,

$$\mathcal{F}(X \cup Y) = (X \cup Y) \cap \mathcal{D}(X \cup Y) = (X \cap \mathcal{D}(X \cup Y)) \cup (Y \cap \mathcal{D}(X \cup Y) \subseteq \\ \subseteq X \cup (Y \cap D(X \cup Y)) \subseteq X \cup (Y \cap \mathcal{D}(Y)) = X \cup \mathcal{F}(Y) \subseteq X \cup Y$$

yielding $\mathcal{F}(X \cup Y) \subseteq X \cup \mathcal{F}(Y) \subseteq X \cup Y$, hence by the IRC property $\mathcal{F}(X \cup Y) = \mathcal{F}(X \cup \mathcal{F}(Y))$ follows. This means that $\mathcal{F}$ is path-independent.

Assume now that $\mathcal{F}$ is path-independent and $\mathcal{F}(X) \subseteq Y \subseteq X$. Path-independence of $\mathcal{F}$ implies $\mathcal{F}(X) = \mathcal{F}(X \cup Y) = \mathcal{F}(\mathcal{F}(X) \cup Y) = \mathcal{F}(Y)$, hence $\mathcal{F}$ has the IRC property.

Assume at last that $\mathcal{F}$ is w-increasing and $\mathcal{F}(X) \subseteq Y \subseteq X$. As $\mathcal{F}(X) \subseteq \mathcal{F}(Y)$ by Lemma 1.5, the w-increasing property of $\mathcal{F}$ implies $w(\mathcal{F}(X)) \geq w(\mathcal{F}(Y))$ and $\mathcal{F}(X) = \mathcal{F}(Y)$ follows. Hence $\mathcal{F}$ has the IRC property, indeed. □

Lemma 1.7 (Fleiner, Jankó [30]). *Let E be a finite set and assume that $\mathcal{F} : 2^E \to 2^E$ is a substitutable choice function. Then $\mathcal{F}$ is path-independent if and only if there exists an antitone determinant $\mathcal{D}_\mathcal{F}$ of $\mathcal{F}$ such that*

$$\mathcal{D}_\mathcal{F}(X) = \mathcal{D}_\mathcal{F}(\mathcal{F}(X)) \text{ holds for any subset } X \text{ of } E\ . \quad (1.1)$$

Proof. Assume that $\mathcal{D}_\mathcal{F}$ is an antitone determinant with property (1.1). Let $\mathcal{F}(X) \subseteq Y \subseteq X$. By the antitone property of $\mathcal{D}_\mathcal{F}$ we get $\mathcal{D}_\mathcal{F}(X) \subseteq \mathcal{D}_\mathcal{F}(Y) \subseteq \mathcal{D}_\mathcal{F}(\mathcal{F}(X)) = \mathcal{D}_\mathcal{F}(X)$ hence $\mathcal{D}_\mathcal{F}(X) = \mathcal{D}_\mathcal{F}(Y)$ holds. Consequently, $\mathcal{F}(Y) = Y \cap \mathcal{D}_\mathcal{F}(Y) = Y \cap \mathcal{D}_\mathcal{F}(X) \subseteq X \cap \mathcal{D}_\mathcal{F}(X) = \mathcal{F}(X)$ and Lemma 1.5 implies $\mathcal{F}(X) = \mathcal{F}(Y)$. This means that $\mathcal{F}$ has the IRC property, hence $\mathcal{F}$ is path-independent by Observation 1.6.

Assume now that $\mathcal{F}$ is path-independent with antitone determinant $\mathcal{D}$. Define

$$\mathcal{D}_\mathcal{F}(X) := \bigcap \{\mathcal{D}(X') : \mathcal{F}(X') = \mathcal{F}(X)\} \tag{1.2}$$

for each subset X of E. As $\mathcal{F}(X) \subseteq \mathcal{D}_\mathcal{F}(X) \subseteq \mathcal{D}(X)$, mapping $\mathcal{D}_\mathcal{F}$ is also a determinant of $\mathcal{F}$. By (1.2), $\mathcal{D}_\mathcal{F}(X) = \mathcal{D}_\mathcal{F}(Y)$ holds whenever $\mathcal{F}(X) = \mathcal{F}(Y)$. So $\mathcal{D}_\mathcal{F}(X) = \mathcal{D}_\mathcal{F}(\mathcal{F}(X))$ follows from path-independence by $\mathcal{F}(X) = \mathcal{F}(\emptyset \cup X) = \mathcal{F}(\emptyset \cup \mathcal{F}(X)) = \mathcal{F}(\mathcal{F}(X))$. To finish the proof, we have to show that $\mathcal{D}_\mathcal{F}$ is antitone, so assume that $X \subseteq Y \subseteq E$. If $\mathcal{F}(X') = \mathcal{F}(X)$ then path-independence of $\mathcal{F}$ implies that $\mathcal{F}(X' \cup Y) = \mathcal{F}(\mathcal{F}(X') \cup Y) = \mathcal{F}(\mathcal{F}(X) \cup Y) = \mathcal{F}(X \cup Y) = \mathcal{F}(Y)$, hence

$$\mathcal{D}_\mathcal{F}(X) := \bigcap \{\mathcal{D}(X') : \mathcal{F}(X') = \mathcal{F}(X)\} \supseteq \bigcap \{\mathcal{D}(X' \cup Y) : \mathcal{F}(X') = \mathcal{F}(X)\} \supseteq$$
$$\bigcap \{\mathcal{D}(X' \cup Y) : \mathcal{F}(X' \cup Y) = \mathcal{F}(Y)\} \supseteq \bigcap \{\mathcal{D}(Y') : \mathcal{F}(Y') = \mathcal{F}(Y)\} = \mathcal{D}_\mathcal{F}(Y)$$

holds, i.e. $\mathcal{D}_\mathcal{F}$ is indeed an antitone determinant of $\mathcal{F}$ with property (1.1). □

1.3 Tarski's fixed point theorem and the deferred acceptance algorithm

Poset $(L, \preceq)$ is a *lattice* if there is a greatest lower bound (denoted by $x \wedge y$) and a least upper bound (denoted by $x \vee y$) for any elements $x, y \in L$. Ha a partial order $\preceq$ is clear from the context then we may talk about lattice L. Lattice L is *complete*, if for any subset X of L there exists a greatest lower bound (denoted by $\bigwedge X$) and a least upper bound (denoted by $\bigvee X$). In case of a complete lattice, 0 and 1 denotes its least and greatest elements, that is, $0 = \bigwedge L$ and $1 = \bigvee L$. Clearly, any finite lattice is complete, but not vice verse: finite subsets of $\mathbb{N}$ form a lattice on ordinary set inclusion but this lattice has no greatest element, hence it is not complete.

Remark 1.8. *Notions defined in section 1.2 can also be defined in the more general setting of lattices by replacing order relation* $\subseteq$ *by* $\preceq$ *and lattice operations* $\cap$ *and* $\cup$ *by* $\wedge$ *and* $\vee$*. In section 1.2, we worked out the framework carefully, and we used only operations* $\cap$ *and* $\cup$ *and avoided set-difference. For this reason, our results and proofs in section 1.2 are valid also for choice functions defined on complete lattices.*

Our main tool in this work is Tarski's fixed point theorem about complete lattices. Note that although we claim corollaries of Tarski's theorem on ordinary choice functions, these results (with the exception of the second part of Corollary 1.17) are valid in the more general lattice choice function setting, as well.

Theorem 1.9 (Tarski [65]). *If* $\mathcal{F} : L \to L$ *is a monotone mapping on complete lattice* $(L, \preceq)$*, then fixed points of* $\mathcal{F}$ *form a nonempty complete lattice for order* $\preceq$*.* □

Note that in the special case of $(L, \preceq) = (2^E, \subseteq)$ Theorem 1.9 was proved by Knaster and Tarksi [51]. It is worth observing that in case of a finite lattice, the least fixed point can be constructed as the greatest element of chain $0 \preceq \mathcal{F}(0) \preceq \mathcal{F}(\mathcal{F}(0)) \preceq \dots$. (Similarly, the greatest fixed point is the least element of chain $1 \succeq \mathcal{F}(1) \succeq \mathcal{F}(F(1)) \succeq \dots$.) Tarski illustrated the application of Theorem 1.9 by deducing various mean value theorems. Another well-known application, the proof of the Cantor-Schröder-Bernstein theorem also involves infinite lattices. However, Theorem 1.9 has interesting implications already on finite lattices.

Theorem 1.9 shows that fixed points of a monotone mapping on a complete lattice L form a nonempty lattice subset of L. Our next goal is to prove a strengthening of this result by showing that if a further condition holds then fixed points of a monotone mapping form a sublattice, that is, the lattice operations restricted to fixed points will be the lattice operations on the set of fixed points.

Recall that $w : L \to \mathbb{R}$ is a *strictly monotone* function on lattice $(L, \preceq)$, that is, $w(0) = 0$ and $w(a) < w(b)$ holds whenever $a \prec b$. Mapping $\mathcal{F} : L \to L$ is a *w-contraction* if

$$|w(a) - w(b)| \geq |w(\mathcal{F}(a)) - w(\mathcal{F}(b))| \text{ holds for any comparable elements } a, b \in L\ .$$

Theorem 1.10 (Fleiner [20]). *If $(L, \preceq)$ is a complete lattice, $w : L \to \mathbb{R}_+$ is strictly monotone function on L and $\mathcal{F} : L \to L$ is a monotone w-contraction then fixed points of $\mathcal{F}$ form a nonempty sublattice of L.*

Proof. As the set of fixed points is nonempty by Theorem 1.9, it suffices to prove that fixed points are closed on lattice operations $\wedge$ and $\vee$ of L. So assume $\mathcal{F}(a) = a$ and $\mathcal{F}(b) = b$ are fixed points. From $a \wedge b \preceq a \preceq a \vee b$ we get that $\mathcal{F}(a \wedge b) \preceq \mathcal{F}(a) = a \preceq \mathcal{F}(a \vee b)$ by monotonicity of $\mathcal{F}$. So $\mathcal{F}(a \wedge b) \preceq a \wedge b$ and $a \vee b \preceq \mathcal{F}(a \vee b)$ and consequently

$$\begin{aligned} 0 &\leq |w(a \vee b) - w(a)| - |w(\mathcal{F}(a \vee b)) - w(\mathcal{F}(a))| \\ &= w(a \vee b) - w(a) - (w(\mathcal{F}(a \vee b)) - w(\mathcal{F}(a))) = w(a \vee b) - w(\mathcal{F}(a \vee b)) \leq 0 \end{aligned}$$

follows as $\mathcal{F}$ is a w-contraction and w is strict monotone. So we have equality throughout, in particular $w(\mathcal{F}(a \vee b)) = w(a \vee b)$. Hence $\mathcal{F}(a \vee b) = a \vee b$ follows from strict monotonicity of w and $a \vee b \preceq \mathcal{F}(a \vee b)$.

By a similar argument, $\mathcal{F}(a \wedge b) = a \wedge b$ follows from

$$\begin{aligned} 0 &\leq |w(a) - w(a \wedge b)| - |w(\mathcal{F}(a)) - w(\mathcal{F}(a \wedge b))| \\ &= w(a) - w(a \wedge b) - (w(\mathcal{F}(a)) - w(\mathcal{F}(a \wedge b))) = w(a \wedge b) - w(\mathcal{F}(a \wedge b)) \leq 0\ . \end{aligned}$$ □

We return to the original lattice version of Tarski's theorem.

Corollary 1.11 (Fleiner [20]). *If $\mathcal{F}, \mathcal{G} : 2^E \to 2^E$ are substitutable choice functions then there exist subsets X, Y of E such that $Y = \mathcal{D}_\mathcal{F}(X)$ és $X = \mathcal{D}_\mathcal{G}(Y)$ holds where $\mathcal{D}_\mathcal{F}$ and $\mathcal{D}_\mathcal{G}$ are antitone determinants of $\mathcal{F}$ and $\mathcal{G}$, respectively. Moreover, pairs (X, Y) with this property form a lattice with partial order $\preceq$ where $(X_1, Y_1) \preceq (X_2, Y_2)$ holds whenever $X_1 \subseteq X_2$ és $Y_2 \subseteq Y_1$.*

Proof. Observe that $L = (2^E \times 2^E, \preceq)$ is a complete lattice, mapping $\mathcal{H}(X, Y) := (\mathcal{D}_\mathcal{G}(Y), \mathcal{D}_\mathcal{F}(X))$ is monotone on L and element $(X, Y) \in L$ is a fixed point of $\mathcal{H}$ iff $Y = \mathcal{D}_\mathcal{F}(X)$ és $X = \mathcal{D}_\mathcal{G}(Y)$ holds. Fixed points of $\mathcal{H}$ form a lattice by Theorem 1.9. □

Corollary 1.11 motivates the following definition.

Definition 1.12. *Let $\mathcal{F}, \mathcal{G} : 2^E \to 2^E$ be substitutable choice functions. Subset K of ground set E is called an $\mathcal{F}\mathcal{G}$-*kernel *if there exist subsets X and Y of E and antitone determinants $\mathcal{D}_\mathcal{F}$ and $\mathcal{D}_\mathcal{G}$ of $\mathcal{F}$ and $\mathcal{G}$ such that*

$$K = X \cap Y, \; Y = \mathcal{D}_\mathcal{F}(X) \text{ and } X = \mathcal{D}_\mathcal{G}(Y) \tag{1.3}$$

holds.

Definition 1.12 seems to be the right way to generalize the notion of bipartite stable matchings as the example below shows.

Example 1.13. *If $\mathcal{F}_A$ and $\mathcal{F}_B$ denotes the preferences of men and women as in Example 1.3 then subset M of $E(G)$ is a stable matching if and only if M is an $\mathcal{F}_A\mathcal{F}_B$-kernel. It is easy to check that if $\mathcal{D}_A$ and $\mathcal{D}_B$ are determinants of $\mathcal{F}_A$ and $\mathcal{F}_B$, respectively then iteration of $\mathcal{D}_B \circ \mathcal{D}_A$ on $E(G)$ is equivalent with the deferred acceptance algorithm of Gale and Shapley.*

According to Example 1.13, stable matchings are $\mathcal{F}_A\mathcal{F}_B$-kernels for particular choice functions $\mathcal{F}_A$ and $\mathcal{F}_B$. As our framework for $\mathcal{F}\mathcal{G}$-kernels requires only fairly general properties of choice functions $\mathcal{F}$ and $\mathcal{G}$ (like substituability, path-independence or eventually increasingness), we will be able to generalize many results on stable matchings in our framework. We continue the study of general $\mathcal{F}\mathcal{G}$-kernels.

Observation 1.14 (Fleiner). *If (1.3) holds for substitutable choice functions $\mathcal{F}$ and $\mathcal{G}$ then $\mathcal{F}(X) = \mathcal{G}(Y) = K$ holds. If $\mathcal{F}$ and $\mathcal{G}$ are path-independent as well then $\mathcal{F}(K) = \mathcal{G}(K) = K$ is also true.*

Proof. By definition, we have $\mathcal{F}(X) = X \cap \mathcal{D}_\mathcal{F}(X) = X \cap Y = K$ and $\mathcal{G}(Y) = Y \cap \mathcal{D}_\mathcal{G}(Y) = Y \cap X = K$. If $\mathcal{F}$ is path-independent then $K = \mathcal{F}(X) = \mathcal{F}(\mathcal{F}(X)) = \mathcal{F}(K)$ and similar holds for $\mathcal{G}$. □

Antitone determinants in Definition 1.12 are not unique in general. However, for path-independent choice functions, this does not matter according to the following lemma.

Lemma 1.15 (Fleiner [20]). *Let $\mathcal{F}, \mathcal{G} : 2^E \to 2^E$ be path-independent and substitutable choice functions, let K be an $\mathcal{F}\mathcal{G}$-kernel and let $\mathcal{D}_\mathcal{F}$ and $\mathcal{D}_\mathcal{G}$ be antitone determinants of $\mathcal{F}$ and $\mathcal{G}$ respectively with property (1.1). Then there exist subsets X, Y of E that satisfy (1.3).*

Proof. We show that (1.3) holds for $X := \mathcal{D}_\mathcal{G}(K)$ and $Y := \mathcal{D}_\mathcal{F}(K)$. As K is an $\mathcal{F}\mathcal{G}$-kernel, we may assume that $K = X' \cap Y'$ and $Y' = \mathcal{D}'_\mathcal{F}(X')$, $X' = \mathcal{D}'_\mathcal{G}(Y')$ where $\mathcal{D}'_\mathcal{F}$ and $\mathcal{D}'_\mathcal{G}$ are antitone determinants of $\mathcal{F}$ and $\mathcal{G}$, respectively. By Observation 1.14, as $\mathcal{D}_\mathcal{F}$ is an antitone determinants of $\mathcal{F}$ with property (1.1), we get

$$K = \mathcal{F}(X') = X' \cap \mathcal{D}_\mathcal{F}(X') = X' \cap \mathcal{D}_\mathcal{F}(\mathcal{F}(X')) = X' \cap \mathcal{D}_\mathcal{F}(K) = X' \cap Y \; .$$

Consequently,

$$\begin{aligned} \mathcal{G}(Y' \cup Y) = (Y' \cup Y) \cap \mathcal{D}'_\mathcal{G}(Y' \cup Y) \subseteq (Y' \cup Y) \cap \mathcal{D}'_\mathcal{G}(Y') = \\ = (Y' \cup Y) \cap X' = (Y' \cap X') \cup (Y \cap X') = K \cup K = K \; , \end{aligned}$$

hence $\mathcal{G}(Y' \cup Y) \subseteq K \subseteq Y \subseteq Y' \cup Y$ and $\mathcal{G}(Y' \cup Y) \subseteq K \subseteq Y' \subseteq Y' \cup Y$. So by the IRC property of $\mathcal{G}$ we have $\mathcal{G}(Y) = \mathcal{G}(Y' \cup Y) = \mathcal{G}(Y') = K$. Property (1.1) of $\mathcal{D}_\mathcal{G}$ implies $\mathcal{D}_\mathcal{G}(Y) = \mathcal{D}_\mathcal{G}(\mathcal{G}(Y)) = \mathcal{D}_\mathcal{G}(K) = X$ and a similar proof shows that $\mathcal{D}_\mathcal{F}(X) = Y$. To finish the proof, we observe that $K = \mathcal{G}(Y) = Y \cap \mathcal{D}_\mathcal{G}(Y) = Y \cap X$. □

The proof of Lemma 1.15 shows that in case of path-independent substitutable choice functions, $\mathcal{FG}$-kernels are essentially fixed points of a monotone mapping. This is formulated in the following lemma.

Lemma 1.16 (Fleiner). *If $\mathcal{F}, \mathcal{G} : 2^E \to 2^E$ are path-independent and substitutable choice functions and $\mathcal{D}_\mathcal{F}$ and $\mathcal{D}_\mathcal{G}$ are their antitone determinants with property (1.1) then $\mathcal{D}_\mathcal{G}$ defines a bijection between $\mathcal{FG}$-kernels and fixed points of mapping $\mathcal{D}_\mathcal{G} \circ \mathcal{D}_\mathcal{F}$.*

Proof. We have seen in the proof of Theorem 1.15 that if K is an $\mathcal{FG}$-kernel then $X = \mathcal{D}_\mathcal{G}(K)$ and $Y = \mathcal{D}_\mathcal{F}(K)$ satisfy (1.3). As $\mathcal{F}(X) = X \cap \mathcal{D}_\mathcal{F}(X) = X \cap Y = K$, determinant $\mathcal{D}_\mathcal{G}$ is injective on $\mathcal{FG}$-kernels. If (1.3) holds for X, Y and K then $\mathcal{D}_\mathcal{G}(\mathcal{D}_\mathcal{F}(X)) = \mathcal{D}_\mathcal{G}(Y) = X$, hence $\mathcal{FG}$-kernels are mapped into different fixed points of $\mathcal{D}_\mathcal{G} \circ \mathcal{D}_\mathcal{F}$. At last, if X is a fixed point of $\mathcal{D}_\mathcal{G} \circ \mathcal{D}_\mathcal{F}$ then (1.3) holds for for $Y = \mathcal{D}_\mathcal{F}(X)$ and $K = X \cap Y$, meaning that $\mathcal{D}_\mathcal{G}$ is indeed a bijection between $\mathcal{FG}$-kernels and fixed points of $\mathcal{D}_\mathcal{G} \circ \mathcal{D}_\mathcal{F}$. □

Due to Lemmata 1.7 and 1.15, Corollary 1.11 can be formulated for path-independent choice functions as follows. Function $w : 2^E \to \mathbb{R}_+$ is *modular* if $w(A) + w(B) = w(A \cap B) + w(A \cup B)$ holds for any subsets A, B of E. It is easy to see that $w(A) := |A|$ defines a strictly monotone and modular function on 2^E.

Corollary 1.17 (Fleiner [20]). *If $\mathcal{F}, \mathcal{G} : 2^E \to 2^E$ are path-independent and substitutable choice functions then $\mathcal{FG}$-kernels form a nonempty lattice for partial order $\preceq_\mathcal{F}$ defined by $K_1 \preceq_\mathcal{F} K_2$ whenever $\mathcal{F}(K_1 \cup K_2) = K_1$. Moreover, partial order $\preceq_\mathcal{G}$ is the opposite of $\preceq_\mathcal{F}$ on $\mathcal{FG}$-kernels.*

Furthermore, if the above choice functions $\mathcal{F}$ and $\mathcal{G}$ are also w-increasing for some strictly monotone modular mapping $w : 2^E \to \mathbb{R}_+$ then for any $\mathcal{FG}$-kernels K_1, K_2 sets $K_1 \wedge_\mathcal{F} K_2 := \mathcal{F}(K_1 \cup K_2)$ and $K_1 \vee_\mathcal{F} K_2 := \mathcal{G}(K_1 \cup K_2)$ are also $\mathcal{FG}$-kernels such that

$$\chi(K_1) + \chi(K_2) = \chi(K_1 \vee_\mathcal{F} K_2) + \chi(K_1 \wedge_\mathcal{F} K_2) \tag{1.4}$$

holds for the characteristic functions of these kernels. Moreover, $w(K_1) = w(K_2)$ holds for any $\mathcal{FG}$-kernels K_1 and K_2.

Proof. Determinant $\mathcal{D}_\mathcal{G}$ defines a bijection between $\mathcal{FG}$-kernels and fixed points of $\mathcal{D}_\mathcal{G} \circ \mathcal{D}_\mathcal{F}$ by Lemma 1.16. As both $\mathcal{D}_\mathcal{G}$ and $\mathcal{D}_\mathcal{F}$ are antitone, their composition is monotone, hence by Theorem 1.9 of Tarski, fixed points of $\mathcal{D}_\mathcal{G} \circ \mathcal{D}_\mathcal{F}$ form a lattice for set-inclusion. So for the first part of Corollary 1.17, we only have to show that by this bijection, set inclusion corresponds to the partial order given in Corollary 1.17. For this reason, assume that $\mathcal{FG}$-kernels K_1 and K_2 correspond to $X_1 = \mathcal{D}_\mathcal{G}(K_1)$, $Y_1 = \mathcal{D}_\mathcal{F}(K_1)$ and $X_2 = \mathcal{D}_\mathcal{G}(K_2)$, $Y_2 = \mathcal{D}_\mathcal{F}(K_2)$, respectively. As we have seen before, $\mathcal{F}(X_1) = X_1 \cap \mathcal{D}_\mathcal{F}(X_1) = X_1 \cap Y_1 = K_1$ and $\mathcal{F}(X_2) = K_2$.

Assume first that $\mathcal{F}(K_1 \cup K_2) = K_1$. By antitonicity and property (1.1) of $\mathcal{D}_\mathcal{F}$,

$$Y_2 = \mathcal{D}_\mathcal{F}(K_2) \supseteq \mathcal{D}_\mathcal{F}(K_1 \cup K_2) = \mathcal{D}_\mathcal{F}(\mathcal{F}(K_1 \cup K_2)) = \mathcal{D}_\mathcal{F}(K_1) = Y_1$$

and hence $X_2 = \mathcal{D}_{\mathcal{F}}(Y_2) \subseteq \mathcal{D}_{\mathcal{F}}(Y_1) = X_1$ follows by the antitone property of $\mathcal{D}_{\mathcal{F}}$. A similar proof shows that $\mathcal{G}(K_1 \cup K_2) = K_1$ implies $Y_2 \subseteq Y_1$.

Suppose now that $X_2 \subseteq X_1$ holds. Property (1.1) of $\mathcal{D}_{\mathcal{G}}$ implies

$$Y_1 = \mathcal{D}_F(K_1) = \mathcal{D}_{\mathcal{F}}(\mathcal{F}(X_1)) = \mathcal{D}_F(X_1) \subseteq \mathcal{D}_F(X_2) = \mathcal{D}_{\mathcal{F}}(\mathcal{F}(X_2) = \mathcal{D}_{\mathcal{F}}(K_2) = Y_2 \ .$$

Now

$$\mathcal{F}(K_1 \cup K_2) = \mathcal{F}(\mathcal{F}(X_1) \cup \mathcal{F}(X_2)) = \mathcal{F}(X_1 \cup \mathcal{F}(X_2)) = \mathcal{F}(X_1 \cup X_2) = \mathcal{F}(X_1) = K_1$$

and

$$\mathcal{G}(K_1 \cup K_2) = \mathcal{G}(\mathcal{G}(Y_1) \cup \mathcal{G}(Y_2)) = \mathcal{G}(Y_1 \cup \mathcal{G}(Y_2)) = \mathcal{G}(Y_1 \cup Y_2) = \mathcal{G}(Y_2) = K_2$$

hold by path-independence of $\mathcal{F}$ and $\mathcal{G}$. This justifies the first part of the corollary.

For the second part of the theorem, assume that $\mathcal{F}$ and $\mathcal{G}$ are also w-increasing and define $\mathcal{D}'_{\mathcal{F}}$ and $\mathcal{D}'_{\mathcal{G}}$ by

$$\mathcal{D}'_{\mathcal{F}}(X) = \mathcal{D}_{\mathcal{F}}(X) \cup (E \setminus X) \text{ and } \mathcal{D}'_{\mathcal{G}}(Y) = \mathcal{D}_{\mathcal{G}}(Y) \cup (E \setminus Y) \ .$$

Clearly $\mathcal{D}'_{\mathcal{F}}$ and $\mathcal{D}'_{\mathcal{G}}$ are determinants of $\mathcal{F}$ and of $\mathcal{G}$, respectively, and being the union of two antitone mappings, both of them are antitone. We show that both these mappings are w-contractions. Assume that $X \subseteq Y$ holds and observe that

$$\begin{aligned} w(X) + \mathcal{D}'_{\mathcal{F}}(X) &= w(X \cap \mathcal{D}'_{\mathcal{F}}(X)) + w(X \cup \mathcal{D}'_{\mathcal{F}}(X)) = w(\mathcal{F}(X)) + w(E) \\ &\leq w(\mathcal{F}(Y)) + w(E) = w(Y \cap \mathcal{D}'_{\mathcal{F}}(Y)) + w(Y \cup \mathcal{D}'_{\mathcal{F}}(Y)) = w(Y) + \mathcal{D}'_{\mathcal{F}}(Y) \end{aligned}$$

holds by the modular and strict monotone properties of w. This shows that $w(Y) - w(X) \geq \mathcal{D}'_{\mathcal{F}}(X) - \mathcal{D}'_{\mathcal{F}}(Y)$, that is, $\mathcal{D}'_{\mathcal{F}}$ is a w-contraction. A similar proof shows that $\mathcal{D}'_{\mathcal{G}}$ is also a w-contraction, hence their composition $\mathcal{D}'_{\mathcal{G}} \circ \mathcal{D}'_{\mathcal{F}}$ is a monotone w-contraction.

Assume now that $\mathcal{FG}$-kernels K_1 and K_2 are determined by sets $K_1 = X_1 \cap Y_1$ (with $Y_1 = \mathcal{D}'_{\mathcal{F}}(X_1)$ and $X_1 = \mathcal{D}'_{\mathcal{G}}(Y_1)$) and $K_2 = X_2 \cap Y_2$ (with $Y_2 = \mathcal{D}'_{\mathcal{F}}(X_2)$ and $X_2 = \mathcal{D}'_{\mathcal{G}}(Y_2)$). As X_1 and X_2 are fixed points of monotone w-contraction $\mathcal{D}'_{\mathcal{G}} \circ \mathcal{D}'_{\mathcal{F}}$, both $X_1 \cap X_2$ and $X_1 \cup X_2$ are fixed points by Theorem 1.10. Similarly, as Y_1 and Y_2 are fixed points of monotone w-contraction $\mathcal{D}'_{\mathcal{F}} \circ \mathcal{D}'_{\mathcal{G}}$, both $Y_1 \cap Y_2$ and $Y_1 \cup Y_2$ are fixed points of this latter mapping. Consequently, $\mathcal{F}(X_1 \cup X_2) = \mathcal{G}(Y_1 \cap Y_2)$ and $\mathcal{F}(X_1 \cup X_2) = \mathcal{G}(Y_1 \cup Y_2)$ coincide with $\mathcal{FG}$-kernels $K_1 \wedge_{\mathcal{F}} K_2$ and $K_1 \vee_{\mathcal{F}} K_2$, moreover $\mathcal{F}(X_1 \cup X_2) = \mathcal{F}(\mathcal{F}(X_1) \cup \mathcal{F}(X_2)) = \mathcal{F}(K_1 \cup K_2)$ and $\mathcal{G}(Y_1 \cup Y_2) = \mathcal{G}(\mathcal{G}(Y_1) \cup \mathcal{G}(Y_2)) = \mathcal{G}(K_1 \cup K_2)$ holds by path-independence of $\mathcal{F}$ and $\mathcal{G}$. To finish the proof of the second part, it is left to show (1.4).

The above proof shows that if $\mathcal{FG}$-kernels K_1 and K_2 are determined by pairs (X_1, Y_1) and (X_2, Y_2) then $K_1 \wedge_{\mathcal{F}} K_2$ and $K_1 \vee_{\mathcal{F}} K_2$ are determined by pairs $(X_1 \cup X_2, Y_1 \cap Y_2)$ and $(X_1 \cap X_2, Y_1 \cup Y_2)$. Consequently, $K_1 = X_1 \cap Y_1$, $K_2 = X_2 \cap Y_2$, $K_1 \vee_{\mathcal{F}} K_2 = (X_1 \cup X_2) \cap Y_1 \cap Y_2$ and $K_1 \wedge_{\mathcal{F}} K_2 = X_1 \cap X_2 \cap (Y_1 \cup Y_2)$. From this latter observation, it is straightforward to check that each element of E contributes the same to both sides of (1.4).

For the last statement, let K_1 and K_2 be $\mathcal{FG}$-kernels and assume indirectly that $w(K_1) > w(K_2)$. From modularity of w, (1.4) and the w-increasing property of $\mathcal{F}$ and $\mathcal{G}$ we get

$$\begin{aligned} 2 \cdot w(K_1) > w(K_1) + w(K_2) = w(K_1 \wedge_{\mathcal{F}} K_2) + w(K_1 \vee_{\mathcal{F}} K_2) = w(\mathcal{F}(K_1 \cup K_2)) \\ + w(\mathcal{G}(K_1 \cup K_2)) \geq w(\mathcal{F}(K_1)) + w(\mathcal{G}(K_1)) = w(K_1) + w(K_1) = 2 \cdot w(K_1), \end{aligned}$$

a contradiction, proving $w(K_1) = w(K_2)$ must hold. This finishes the proof of the theorem. □

By Corollary 1.17, for path-independent substitutable choice functions $\mathcal{F}$ and $\mathcal{G}$ on E, there is an $\mathcal{F}$*-optimal* and a $\mathcal{G}$*-optimal* $\mathcal{FG}$-kernel $K_{\mathcal{F}}$ and $K_{\mathcal{G}}$ such that $\mathcal{F}(K \cup K_{\mathcal{F}}) = K_{\mathcal{F}}$ and $\mathcal{G}(K \cup K_{\mathcal{G}}) = K_{\mathcal{G}}$ holds for any $\mathcal{FG}$-kernel K. These $\mathcal{FG}$-kernels $K_{\mathcal{F}}$ and $K_{\mathcal{G}}$ are the $\preceq_{\mathcal{F}}$-maximal and $\preceq_{F}$-minimal elements of the lattice of $\mathcal{FG}$-kernels.

Let us illustrate Corollary 1.17 with the following application.

Example 1.18 (2007 Kürschák Competition, problem 3).
Prove that any finite subset H of gridpoints on the plane has a subset K with the property that

1. *any line parallel with one of the axes (i.e. vertical or horizontal) intersects K in at most 2 points,*
2. *any point of $H \setminus K$ is on a segment with end points in K and parallel with one of the axes.*

Proof. Define choice functions $\mathcal{F}, \mathcal{G} : 2^H \to 2^H$ such that for any subset L of H, $\mathcal{F}(L)$ denotes as the set of extreme elements of L on the horizontal gridlines and $\mathcal{G}(L)$ denotes the set of extreme elements of L on the vertical gridlines. It is fairly straightforward to check that $\mathcal{F}$ and $\mathcal{G}$ are substitutable and path-independent, moreover K satisfies the required properties if K is an $\mathcal{FG}$-kernel. Hence the claim follows from Corollary 1.17.□

Theorem 1.1 follows immediately from Observation 1.4 and Corollary 1.11. More is true, however. Due to Observation 1.4 stable matchings coincide with $\mathcal{F}_A\mathcal{F}_B$-kernels that form fixed points of a monotone mapping. These fixed points form a complete lattice by Theorem 1.9 hence there is a least and a greatest fixed point. This immediately implies the existence of a man-optimal stable matching (in which each man is assigned to his best partner he can receive in a stable matching and at the same time each women is matched to her worst stable partner) and the existence of a woman-optimal stable matching (that can be defined by exchanging the role of men and women). It also turns out that the deferred acceptance algorithm of Gale and Shapley can be regarded as the iteration of the monotone mapping $\mathcal{D}_{\mathcal{G}} \circ \mathcal{D}_{\mathcal{F}}$ in the proof of Corollary 1.17. (We have seen that this iteration finds the least and the greatest fixed point.) It is also not difficult to prove Blair's theorem on the lattice property from Corollary 1.17.

Theorem 1.19 (Blair [12]). *Let $G = (V, E)$ be a bipartite graph with parts A and B and for each vertex $v \in V$ let $\mathcal{F}_v : 2^{E(v)} \to 2^{E(v)}$ a substitutable choice function having the IRC property. Let $\mathcal{F}_A(X) := \bigcup\{\mathcal{F}_v(X \cap E(v) : v \in A\}$ and $\mathcal{F}_B(X) := \bigcup\{\mathcal{F}_v(X \cap E(v) : v \in B\}$. Then $\mathcal{F}_A\mathcal{F}_B$-kernels form a nonempty (complete) lattice for partial order $\preceq_B$ where $X \preceq_B Y$ holds whenever $\mathcal{F}_B(X \cup Y) = X$.*

Proof. As $\mathcal{F}_A$ and $\mathcal{F}_B$ are substitutable choice functions with the IRC property on 2^E, these choice functions are path-independent as well by Observation 1.6. Theorem 1.19 directly follows from Corollary 1.17. □

It is worth mentioning that the celebrated ground breaking work of Hatfield and Milgrom [42] is based on the rediscovery of the connection between Corollary 1.17 and Theorem 1.9 of Tarski.

Chapter 2

Kernel type results

We illustrate the choice-function approach by proving a result of Baïou and Balinski [9] that generalize the theorem on stable b-matchings. Just like several earlier stable matching related results, the proof of Baïou and Balinski introduces an appropriate generalization of the Gale-Shapley algorithm. Our approach is that we reduce the problem to Corollary 1.17. The choice function in this case works on a complete lattice that is not a subsetlattice. To claim the theorem, we need to fix some terminology.

The *stable allocation problem* is defined by finite disjoint sets W and F of workers and firms, a map $q : W \cup F \to \mathbb{R}$, a set E of edges between W and F along with a map $p : E \to \mathbb{R}$ and for each worker or firm $v \in W \cup F$ a linear order $<_v$ on those pairs of E that contain v. We shall refer to pairs of E as "edges" and hopefully it will not cause ambiguity. Quota $q(v)$ denotes the maximum of total assignment that worker or firm v can accept and capacity $p(wf)$ of edge $e = wf$ means the maximum allocation that worker w can be assigned to firm f along e. An *allocation* is a nonnegative map $g : E \to \mathbb{R}$ such that $g(e) \le p(e)$ holds for each $e \in E$ and for any $v \in W \cup F$ we have

$$g(v) := \sum_{x:vx\in E} g(vx) \le q(v) \ , \tag{2.1}$$

that is, the total assignment $g(v)$ of player v cannot exceed quota $q(v)$ of v. If (2.1) holds with equality then we say that player v is *g-saturated*. An allocation is *stable* if for any edge wf of E at least one of the following properties hold:

$$\begin{gathered}\text{either } g(wf) = p(wf)\\ \text{(the particular employment is realized with full capacity)}\end{gathered} \tag{2.2}$$

$$\begin{gathered}\text{or } \textstyle\sum_{wf'\le_w wf} g(wf') = q(w)\text{, that is worker } w \text{ is } g\text{-saturated and } w \text{ does not}\\ \text{prefer } f \text{ to any of his employers (we say that } wf \text{ is } g\textit{-dominated at } w\text{)}\end{gathered} \tag{2.3}$$

$$\begin{gathered}\text{or } \textstyle\sum_{w'f\le_f wf} g(w'f) = q(f)\text{, that is firm } f \text{ is } g\text{-saturated and } f \text{ does not}\\ \text{prefer } w \text{ to any of its employees (we say that } wf \text{ is } g\textit{-dominated at } f\text{).}\end{gathered} \tag{2.4}$$

Note that (2.4) and (2.3) imply that if g is a stable allocation, then for each firm f and each worker w

$$\text{there is at most one edge } e \text{ dominated at } f \text{ with } g(e) > 0 \text{ and} \tag{2.5}$$

$$\text{there is at most one edge } e \text{ dominated at } w \text{ with } g(e) > 0\ . \tag{2.6}$$

If g_1 and g_2 are allocations and $w \in W$ is a worker then we say that *allocation g_1 dominates allocation g_2 for worker w* (in notation $g_1 \leq_w g_2$) if one of the following properties is true:

$$\text{either } g_1(wf) = g_2(wf) \text{ for each } f \in F \text{ or} \tag{2.7}$$

$$\begin{aligned} \textstyle\sum_{f' \in F} g_1(wf') = \sum_{f' \in F} g_2(wf') = q(w), \text{ and} \\ g_1(wf) < g_2(wf) \text{ and } g_1(wf') > 0 \text{ implies that } wf' <_w wf. \end{aligned} \tag{2.8}$$

That is, if w can freely choose his allocation from $\max(g_1, g_2)$ then w would choose g_1 either because g_1 and g_2 are identical for w or because w is saturated in both allocations and g_1 represents w's choice out of $\max(g_1, g_2)$. By exchanging the roles of workers and firms, one can define domination relation $\leq_f$ for any firm f, as well.

The stable allocation problem was introduced by Baïou and Balinski as a certain "continuous" version of the stable marriage problem in [9]. Below we state and prove a nondiscrete generalization of the Baïou-Balinski result.

Theorem 2.1 (See Baïou and Balinski [9]). *1. In any stable allocation problem instance described by W, F, E, p and q, there exists a stable allocation g. Moreover, if p and q are integral, then there exists an integral stable allocation g.*

2. If g_1 and g_2 are stable allocations and $v \in W \cup F$ then $g_1 \leq_v g_2$ or $g_2 \leq_v g_1$ holds.

3. Stable allocations have a natural lattice structure. Namely, if g_1 and g_2 are stable allocations then $g_1 \vee g_2$ and $g_1 \wedge g_2$ are stable allocations, where

$$(g_1 \vee g_2)(wf) = \begin{cases} g_1(wf) & \text{if } g_1 \leq_w g_2 \\ g_2(wf) & \text{if } g_2 \leq_w g_1 \end{cases} \tag{2.9}$$

and

$$(g_1 \wedge g_2)(wf) = \begin{cases} g_1(wf) & \text{if } g_1 \leq_f g_2 \\ g_2(wf) & \text{if } g_2 \leq_f g_1 \end{cases} \tag{2.10}$$

In other words, if workers choose from two stable allocations then we get another stable allocation, and this is also true for the firms' choices. Moreover, it is true that

$$(g_1 \vee g_2)(wf) = \begin{cases} g_1(wf) & \text{if } g_1 \geq_f g_2 \\ g_2(wf) & \text{if } g_2 \geq_f g_1 \end{cases} \tag{2.11}$$

and

$$(g_1 \wedge g_2)(wf) = \begin{cases} g_1(wf) & \text{if } g_1 \geq_w g_2 \\ g_2(wf) & \text{if } g_2 \geq_w g_1 \end{cases} \tag{2.12}$$

That is, in stable allocation $g_1 \vee g_2$ where each worker picks his better assignment, each firm receives the worse out of the two. Similarly, in $g_1 \wedge g_2$ the choice of the firms means the less preferred situation to the workers.

Sketch of the proof of Theorem 2.1. Define lattice $L^p = (\{l : E \to \mathbb{R}_+, l \leq p\}, \leq)$ and observe that L^p is complete. For $l \in L^p$ let $\mathcal{F}(l)$ denote the choice of workers: if $e_1 <_v e_2 < \ldots$ is the preference order of the edges of $E(w)$ then

$$\mathcal{F}(l)(e_i) = \min\left(l(e_i), q(w) - \sum_{j=1}^{i-1} \mathcal{F}(l)(e_j) \right).$$

That is, workers choose their best possible assignment that does not exceed l. In this choices, workers do not care about the firm quotas. By definition, $\mathcal{F}(l) \leq l$, hence $\mathcal{F}$ is a choice function on L^p. It is easy to check that $\mathcal{F}(l) = \min(l, \mathcal{D}_{\mathcal{F}}(l))$ where

$$\mathcal{D}_{\mathcal{F}}(l)(e_i) = \max\left(0, q(w) - \sum_{j=1}^{i-1} l(e_j)\right)$$

for the above edge e_i. Clearly, $\mathcal{D}_{\mathcal{F}}$ is antitone, hence $\mathcal{F}$ is substitutable, and it is also straightforward to see that $\mathcal{F}$ is path-independent as well. We can define choice function $\mathcal{G}$ and its determinant $\mathcal{D}_{\mathcal{G}}$ similarly where the role of workers is played by the firms.

To finish the proof, we observe that stable allocations are exactly $\mathcal{FG}$-kernels and Theorem 2.1 follows from Corollary 1.17. The proof of the integrality property in the first part of the theorem is exactly the same, we only have to replace L^p by complete lattice $L'^p = (\{l : E \to \mathbb{N}, l \leq p\}, \leq)$. □

2.1 Poset-kernels

The fixed-point based approach in Chapter 1 allows us to generalize or extend several earlier results. A common antichain K of finite posets $P_1 = (E, \leq_1)$ and $P_2 = (E, \leq_2)$ is called a P_1P_2-*kernel* if for any $e \in E$ there is some $k \in K$ such that $e \leq_1 k$ or $e \leq_2 k$ holds. The first part of the following generalization of Theorem 1.1 of Gale and Shapley can be easily deduced from the result by Sands, Sauer, and Woodrow [62]. (Moreover, it is also true that the Sands-Sauer-Woodrow theorem is a consequence of the first part of Theorem 2.2.)

Theorem 2.2 (Fleiner [20]). *For any finite posets $P_1 = (E, \leq_1)$ and $P_2 = (E, \leq_2)$ there exists a P_1P_2-kernel. Moreover, P_1P_2-kernels form a lattice for partial order $\prec_1$ where $A \prec_1 A'$ holds for two antichains of P_1 if each element of A has an $\leq_1$-upper bound in A'.*

Theorem 2.2 follows from Corollary 1.4 and the observation that mapping each subset X of the poset to the set of maxima of X defines a path-independent choice function. We omit the formal proof as we show a generalization of Theorem 2.2 later in this chapter. The following application illustrates the first part of Theorem 2.2.

Example 2.3 (2016 Kürschák Competition, problem 2).
Prove that any finite subset A of the positive integers has a subset B with the properties below.

- *If b_1 and b_2 are different elements of B then neither b_1 and b_2, nor b_1+1 and b_2+1 are multiples of one another, and*
- *for any element a of set A there exists some element b of B such that a divides b or $(b+1)$ divides $(a+1)$.*

Proof. Define two partial orders $P_1 = (A, \leq_1)$ and $P_2 = (A, \leq_2)$ by $a \leq_1 b$ holds if $b \mid a$ and $a \leq_2 b$ holds if $a+1 \mid b+1$. Then subset B of A satisfies the requirements in Example 2.3 if and only if B is a P_1P_2-kernel that does exist by Theorem 2.2. □

Aharoni, Berger, and Gorelik proved a weighted version of Theorem 2.2. We need a couple of definitions to state it. Let $P = (V, \leq)$ be a finite poset, let $w : V \to \mathbb{N}$ be a demand function and let $f : V \to \mathbb{N}$ be a weight function. For any $v \in V$ let

$$f^{\uparrow}_{\leq}(v) = \max\{f(c_1) + f(c_2) + \ldots : v = c_1 < c_2 < \ldots\}$$

denote the maximum weight of chains starting from v. This weight function f is $(\leq, w)$-*independent* if

- for any chain $c_1 < c_2 < \ldots < c_k$ we have $\sum_{i=1}^{k} f(c_i) \leq \max\{w(c_i) : 1 \leq i \leq k\}$ and
- $f(v) \cdot f^{\uparrow}_{\leq}(v) \leq f(v) \cdot w(v)$ for any $v \in V$.

(The first condition means that the total weight of no chain exceeds the maximal demand of its elements while due to the second condition the total weight of a chain starting at element v of positive weight does not exceed the demand of v.) It is immediate that $(\leq, 1)$-independent weight functions coincide with the characteristic vectors of antichains.

The above weight function f *w-dominates* element c_1 of poset P if there is some chain $c_1 < c_2 < \ldots < c_k$ such that $w(c_1) \leq \sum_{i=1}^{k} f(c_i)$ holds, that is, if there is a chain starting at c_1 with a total weight not less than the demand of c_1. Now let $P_1 = (V, \leq_1)$ and $P_2 = (V, \leq_2)$ be finite posets on common ground set V and let $w_1, w_2 : V \to \mathbb{N}$ be two demand functions. By a (w_1, w_2)-*kernel* of these posets, we mean a weight function $f : V \to \mathbb{N}$ that is both $(\leq_1, w_1)$-independent and $(\leq_2, w_2)$-independent and moreover f dominates each element of ground set V, more precisely each element of V is w_1-dominated by f in P_1 or w_2-dominated in P_2. It is worth observing that a $(1,1)$-kernel coincides with the previously defined kernel. Now we can claim the theorem on weighted kernels.

Theorem 2.4 (Aharoni, Berger, Gorelik [5]). *Let $P_1 = (V, \leq_1)$ and $P_2 = (V, \leq_2)$ be finite posets and let $w : V \to \mathbb{N}$ be a demand function. Then these posets have a (w, w)-kernel.*

An extension of Corollary 1.11 of Theorem 1.9 to lattices (according to Remark 1.8) allows us to generalize Theorem 2.4 as follows.

Theorem 2.5 (Fleiner, Jankó [30]). *Let $P_1 = (V, \leq_1)$ and $P_2 = (V, \leq_2)$ be finite posets, and let $w_1 : V \to \mathbb{N}$ and $w_2 : V \to \mathbb{N}$ two demand functions. Then there exists a (w_1, w_2)-kernel of these posets. The set of (w_1, w_2)-kernels form a lattice for the partial order $\preceq_1$ of weight functions where $f \preceq_1 g$ holds for weight functions f and g if $f^{\uparrow}_{\leq_1} \leq g^{\uparrow}_{\leq_1}$.*

Note that Theorem 2.2 is a special case of Theorem 2.5 for $w_1 = w_2 = 1$. Here, we give a sketch of the proof, the interested reader finds the details in [30].

Proof. Let $w = \max(w_1, w_2)$ and define complete lattice $L^w := (\{f : V \to \mathbb{N}, f \leq w\}, \leq)$ and refer to the elements of L^w as *weight functions*. We shall define two choice functions on L^w.

Let $V = \{v_1, v_2, \ldots, v_n\}$ be a linear extension of $\leq_1$, that is, if $v_i \leq_1 v_j$ then $j \leq i$ holds. (So v_1 is $\leq_1$-maximal element of V and v_{i+1} is a $\leq_1$-maximal element of $V \setminus \{v_1, v_2, \ldots, v_i\}$ for $i = 1, 2, \ldots$.) For weight function $f \in L^w$, define $\mathcal{F}(f)$ for values of

$v_1, v_2, \ldots, v_n$ in this order in the following certain greedy manner. After we calculated the values of $[\mathcal{F}(f)](v_1), \ldots, [\mathcal{F}(f)](v_{i-1})$, we determine value $[\mathcal{F}(f)](v_i) = \alpha$ such that $\alpha \le f(v_i)$ and α is maximal with the property that $\mathcal{F}(f)$ is $(\le_1, w_1)$-independent on any chain $v_i \le_1 l_1 \le_1 l_2 \le_1 \ldots$ starting at v_i. More precisely,

$$[\mathcal{F}(f)](v_i) = \min\{f(v_i), \max\{0, w_1(v_i) - [\mathcal{F}(f)]^*(v_i)\}\} \tag{2.13}$$

where $[\mathcal{F}(f)]^*(v) = 0$ if v is a $\le_1$-maximal element of V, otherwise

$$[\mathcal{F}(f)]^*(v) = \max\{[\mathcal{F}(f)](c_1) + [\mathcal{F}(f)](c_2) + \ldots : v <_1 c_1 <_1 c_2 <_1 \ldots\} \tag{2.14}$$

By definition, $\mathcal{F}(f)(v_i) \le f(v_i)$ holds for each element v_i of V, hence mapping $\mathcal{F}$ is a choice function on L^w. Moreover, $\mathcal{F}(f)$ is $(\le_1, w_1)$-independent for any weight $f \in L^w$ as we have chosen each value $\mathcal{F}(f)(v)$ such that every chain $v \preceq c_1 \preceq c_2 \preceq \ldots$ satisfies the property that independence requires.

We define choice function $\mathcal{G}$ similarly as $\mathcal{F}$, using $\le_2$ and w_2 instead of $\le_1$ and w_1 with linear extension $V = \{u_1, u_2, \ldots, u_n\}$ of $\le_2$.

We prove that $\mathcal{F}$ and $\mathcal{G}$ are substitutable by the following lemma the proof of which is omitted.

Lemma 2.6 (Fleiner, Jankó [30]). *For any $f \in L^w$ and $v \in V$ define $[\mathcal{M}_\mathcal{F}(f)](v) = 0$ if v is $\le_1$-maximal otherwise let*

$$\begin{aligned}[\mathcal{M}_\mathcal{F}(f)](v) := \max\{&f'(c_1) + f'(c_2) + \ldots + f'(c_k) : \\ &f' \le f \text{ and } f' \text{ is } (\le_1, w_1)\text{-independent and } v <_1 c_1 <_1 c_2 <_1 \ldots <_1 c_k\}\end{aligned} \tag{2.15}$$

as the maximum total f'-weight of a chain above v where f' is a $(\le_1, w_1)$-independent lower bound of f. Then $\mathcal{D}_\mathcal{F} := \max\{0, w - \mathcal{M}_\mathcal{F}\}$ is a determinant of $\mathcal{F}$, that is

$$[\mathcal{F}(f)](v) = \min\{f(v), \max\{0, w(v) - [\mathcal{M}_\mathcal{F}(f)](v)\}\}\ . \tag{2.16}$$

Moreover, $\mathcal{D}_\mathcal{F}$ has property (1.1). □

Mapping $\mathcal{M}$ is monotone by (2.15), hence determinant $\mathcal{D}_\mathcal{F}$ is antitone. Consequently choice function $\mathcal{F}$ is substitutable and path-independent by Lemma 1.7. A similar proof shows that $\mathcal{G}$ is also substitutable and path-independent. We finish the proof by showing that (w_1, w_2)-kernels are exactly the $\mathcal{FG}$-kernels. Once we do so, Theorem 2.5 directly follows from the first part of Corollary 1.17.

Assume first that f is a (w_1, w_2)-kernel. To show that f is an $\mathcal{FG}$-kernel, it is enough to prove that

$$f = \min\{\mathcal{D}_\mathcal{F}(f), \mathcal{D}_\mathcal{G}(f)\}\ . \tag{2.17}$$

As f is $(\le_1, w_1)$-independent and $(\le_2, w_2)$-independent, $f = \mathcal{F}(f) = \mathcal{G}(f)$, so $f \le \min\{\mathcal{D}_\mathcal{F}(f), \mathcal{D}_\mathcal{G}(f)\}$ by the definition of the determinant. Now pick any $v \in V$. As f is a (w_1, w_2)-kernel, v is either w_1-dominated or w_2-dominated by f (or both). In the first case, $[\mathcal{D}_\mathcal{F}(f)](v) = f(v)$ and in the second one $[\mathcal{D}_G(f)](v) = f(v)$ holds, that is

$$f \ge \min\{\mathcal{D}_\mathcal{F}(f), \mathcal{D}_\mathcal{G}(f)\} = \min\left\{\mathcal{D}_\mathcal{F}(f), \mathcal{D}^w_\mathcal{G}(f)\right\}\ ,$$

by Lemma 2.6. This proves (2.17) hence f is an $\mathcal{FG}$-kernel.

Now suppose that f is an $\mathcal{FG}$-kernel. As $f = \mathcal{F}(f) = \mathcal{G}(f)$, f is both $(\leq_1, w_1)$-independent and $(\leq_2, w_2)$-independent. Moreover, (2.17) holds by (1.3). Pick any $v \in V$. Now Lemma 2.6 implies

$$f(v) = \min\left\{\left[\mathcal{D}_{\mathcal{F}}(f)\right](v), \left[\mathcal{D}_{\mathcal{G}}(f)\right](v)\right\}$$

So either $f(v) = [\mathcal{D}_{\mathcal{F}}(f)](v)$ or $f(v) = [\mathcal{D}_{\mathcal{G}}(f)](v)$ holds (or both). In the first case v is $(\leq_1, w_1)$-dominated by f and in the second case v is $(\leq_2, w_2)$-dominated by f. This proves that f is indeed a (w_1, w_2)-kernel. □

There are other poset-related kernel type results follow from our framework. Below, we describe a formal generalization of the Hatfield-Milgrom result [42] after introducing some Economics motivated terminology. The generalization of the stable marriage theorem (i.e. Theorem 1.1) by Hatfield and Milgrom can be formulated as follows. We have a two-sided market: agents on one side are hospitals and agents on the other side are doctors. (In our terminology, hospitals and doctors are the vertices of the underlying graph.) Set X represents a set of bilateral contracts, each involving a hospital and a doctor. There may be several different contracts possible involving the same two agents. (That is, X stands for the edges of bipartite graph G and parallel edges are possible.) Each agent v has some preference on her contracts and this is described by a choice function C_v that from any set of possible contracts selects those that v would choose if she is allowed to pick freely regardless of any other agents. These individual choice functions can be combined to define two choice functions on 2^X: one for the hospitals ($\mathcal{C}_h$) and one for the doctors ($\mathcal{C}_D$). Hatfield and Milgrom proved that if both $\mathcal{C}_H$ and $\mathcal{C}_D$ are substitutable and path-independent then there always exist a stable allocation of doctors to hospitals, that is, a subset X' of X such that $\mathcal{C}_D(X') = \mathcal{C}_H(X') = X'$ and no contract x exists such that $x \in \mathcal{C}_D(X' \cup \{x\}) \cap \mathcal{C}_H(X' \cup \{x\})$ holds.

Note that in certain practical applications hospitals (or doctors) have preferences that do not correspond to a substitutable choice function. This happens in particular, if there is a partial order $\preceq$ on the set X of contracts such that if $x' \preceq x$ and x is available in a choice set then x' must also be available. This condition might mean that it is aways possible for both parties to "downgrade" an offered contract. The following example shows such a situation.

Example 2.7. *Assume we have two hospitals h and $\bar{h}$ and two doctors d and d'. Contract x'_i represents an i-days job of d' at h, and $\bar{x}_j$ stands for a j-days occupation for d at $\bar{h}$, etc. That is, $X := \{z_i : 1 \leq i \leq 5,\ z \in \{x, x', \bar{x}, \bar{x}'\}\}$ is the set of possible contracts. Partial order $\preceq$ is defined by relations of type $z_i \preceq z_j$ for $i \leq j$ and $z \in \{x, x', \bar{x}, \bar{x}'\}$.*

Assume that d is a famous doctor with a high salary expectation, so each hospital wants to employ her but for a minimum amount of time. Doctor d' can do the same job equally well but she is young and hence costs less to the employer. Assume that each hospital needs 5 days of work and from a given set of options it selects 1 day of work of doctor d, the maximum amount of work for doctor d' up to 5 days altogether and for the missing days it selects d if she is still available. For example, $\mathcal{C}_h(x_1, x_2, x_3, x'_1, x'_2, x'_3) = \{x_2, x'_3\}$, $\mathcal{C}_h(x_1, x_2, x_3, x'_1, x'_2, x'_3, x'_4) = \{x_1, x'_4\}$ and $\mathcal{C}_{\bar{h}}(\bar{x}_1, \bar{x}_2, \bar{x}'_1, \bar{x}'_2) = \{\bar{x}_2, \bar{x}'_2\}$. Assume moreover that both doctors d and d' look for 5 days of work, and both of them prefer hospital h to h': $\mathcal{C}_d(x_1, x_2, x_3, \bar{x}_1, \bar{x}_2, \bar{x}_3) = \{x_3, \bar{x}_2\}$.

It is easy to see that none of choice functions $\mathcal{C}_H$ and $\mathcal{C}_D$ on 2^X are substitutable.

It turns out that our framework can be applied for Example 2.7 as well, and the key is that we restrict the domain of the choice functions so as to achieve substitutability. If $P = (X, \preceq)$ is a poset then $Y \subset X$ is a *lower ideal* if it is downward closed, that is, $a \preceq b \in Y$ implies $a \in Y$. It is easy to see that lower ideals are closed on arbitrary intersection and union, hence lower ideals form a complete lattice that we shall denote by $\mathcal{L}_P$. As we argued above, the domain of our choice functions $\mathcal{C}_H$ and $\mathcal{C}_D$ will be $\mathcal{L}_P$. The range of these choice functions however is the set of antichains of P, as neither a doctor nor a hospital can pick two comparable contracts. It is easy to see that (at least for finite posets P) there is a bijection between lower ideals and antichains as each antichain A determines a lower ideal $\mathrm{Li}(A) = \{x \in X : x \preceq a \in A\}$ and each lower ideal I determines an antichain $\max I$. Consequently, (according to Remark 1.8) we may think about choice functions $\mathcal{C}_H$ and $\mathcal{C}_D$ as a choice function on $\mathcal{L}_P$. Notice that if $\mathcal{C}_H$ and $\mathcal{C}_D$ come from Example 2.7 then both these choice functions are substitutable and path-independent. (Moreover, these choice functions are even w-monotone if $w(x_i) = w(x'_i) = w(\bar{x}_i) = w(\bar{x}'_i) = i$ holds for each $i = 1, 2, \ldots, 5$.)

As we saw before, choice function $\mathcal{C} : \mathcal{L}_P \to \mathcal{L}_P$ is substitutable if there is an antitone function $\mathcal{D} : \mathcal{L}_P \to \mathcal{L}_P$ such that $\mathcal{C}(l) = l \cap \mathcal{D}(l)$ holds for any element l of $\mathcal{L}_P$. According Definition 1.12, subset Y of X is a $\mathcal{C}_H\mathcal{C}_D$*-kernel* if there exist lower ideals $l_1, l_2 \in \mathcal{L}_P$ such that $Y = \max(l_1 \cap l_2)$ and $\mathcal{D}_D(l_1) = l_2$ and $\mathcal{D}_H(l_2) = l_1$. Hence the application of Corollary 1.17 gives the following result

Theorem 2.8 (Farooq, Fleiner and Tamura [18]). *Let X be a set of possible contracts between set D of doctors and H of hospitals and $= (P)$ be the lattice of lower ideals of some partial order $P = (X, \preceq)$. If both joint choice functions $\mathcal{C}_H$ and $\mathcal{C}_D$ are substitutable and path independent then there exist a $\mathcal{C}_H\mathcal{C}_D$-kernel and $\preceq_{\mathcal{C}_D}$ and $\preceq_{\mathcal{C}_H}$ are opposite partial orders on $\mathcal{C}_D\mathcal{C}_H$-kernels and both of them define a lattice.*

2.2 Matroid-kernels

Kernel type results can be proved on other structures than posets. Let $\mathcal{M}_1 = (E, \mathcal{I}_1)$ and $\mathcal{M}_2 = (E, \mathcal{I}_2)$ be matroids and let $\leq_1$ and $\leq_2$ be a linear order on their common ground set E. Common independent set K of these matroids are called an $\mathcal{M}_1\mathcal{M}_2$*-kernel* if for any element $e \in E \setminus K$ there exist a cycle C of matroid $\mathcal{M}_i$ for some $i \in \{1, 2\}$ such that $C \subseteq K \cup \{e\}$ and $c \leq_i e$ holds for each $c \in C - e$.

Theorem 2.9 (Fleiner [20]). *For any matroids $\mathcal{M}_1 = (E, \mathcal{I}_1)$ and $\mathcal{M}_2 = (E, \mathcal{I}_2)$ and for any linear orders $\leq_1$ and $\leq_2$ on E, there exists an $\mathcal{M}_1\mathcal{M}_2$-kernel. If $K_1, K_2 \subseteq E$ are $\mathcal{M}_1\mathcal{M}_2$-kernels then the greedy algorithm on $\mathcal{M}_i$ for processing order $\leq_i$ selects an $\mathcal{M}_1\mathcal{M}_2$-kernel from $K_1 \cup K_2$ for any $i = 1, 2$. These two operations determine a lattice on $\mathcal{M}_1\mathcal{M}_2$-kernels with property 1.4. Moreover, $\mathrm{span}_{\mathcal{M}_1} K_1 = \mathrm{span}_{\mathcal{M}_1} K_2$ and $\mathrm{span}_{\mathcal{M}_2} K_1 = \mathrm{span}_{\mathcal{M}_2} K_2$ holds for any $\mathcal{M}_1\mathcal{M}_2$-kernels K_1 and K_2.*

The keys to Theorem 2.9 are Corollary 1.17 and the fact that the choice function defined by the greedy algorithm is substitutable and increasing.

Proof. Define choice functions $\mathcal{F}$ and $\mathcal{G}$ on E as follows. For subset X of E let $x_1 <_1 x_2 <_1 x_3 <_1 \ldots$ be the $\leq_1$ order of the elements of X and let $\mathcal{F}_0(X) = \emptyset$ and for

$i = 1, 2, \ldots$ define

$$\mathcal{F}_i(X) = \begin{cases} \mathcal{F}_{i-1}(X) \cup \{x_i\} & \text{if } \mathcal{F}_{i-1}(X) \cup \{x_i\} \in \mathcal{I}_1 \\ \mathcal{F}_{i-1}(X) & \text{if } \mathcal{F}_{i-1}(X) \cup \{x_i\} \notin \mathcal{I}_1 \end{cases},$$

and let $\mathcal{F}(X) := \mathcal{F}_{|X|}(X)$. That is, $\mathcal{F}_i(X)$ is the basis of X that the greedy algorithm selects from X in matroid $\mathcal{M}_1$ for order $\leq_1$. Define choice function $\mathcal{G}$ similarly for matroid $\mathcal{M}_2$ and order $\leq_2$. Clearly, $\mathcal{F}$ and $\mathcal{G}$ are choice functions and $|\mathcal{F}(X)| = rk_1(X)$ and $|\mathcal{G}(X)| = rk_2(X)$, so by the monotone property of the rank function, both $\mathcal{F}$ and $\mathcal{G}$ are increasing. A standard property of matroids is that choice function $\overline{\mathcal{F}}$ defined by $\overline{\mathcal{F}}(X) = X \setminus \mathcal{F}(X)$ as the nonselected elements of $\mathcal{F}$ is monotone. Hence $\mathcal{D}_{\mathcal{F}}(X) := E \setminus \overline{\mathcal{F}}(X) = \mathcal{F}(X) \cup (E \setminus X)$ is an antitone determinant of $\mathcal{F}$, showing that $\mathcal{F}$ (and $\mathcal{G}$ for a similar reason) is an increasing substitutable choice function.

To show that $\mathcal{M}_1\mathcal{M}_2$-kernels are exactly $\mathcal{FG}$-kernels, assume first that K is an $\mathcal{FG}$-kernel, i.e. $K = X \cap Y$ where $X = \mathcal{D}_{\mathcal{G}}(Y)$ and $Y = \mathcal{D}_{\mathcal{F}}(X)$. Now K is a common independent set of $\mathcal{M}_1$ and $\mathcal{M}_2$ as $\mathcal{F}(X) = X \cap \mathcal{D}_{\mathcal{F}}(X) = X \cap Y = \mathcal{D}_{\mathcal{G}}(Y) \cap Y = \mathcal{G}(Y)$. Let $e \in E \setminus K$ and assume no cycle $C \subseteq K \cup \{e\}$ of $\mathcal{M}_1$ exist such that $c \leq_1 e$ holds for each element c of $C - e$. This means $e \notin X$ as otherwise $e \in \mathcal{F}(X) = K$ would hold. However, then $e \in \mathcal{D}_{\mathcal{F}}(X) = Y$ and $e \notin K = \mathcal{G}(Y)$, hence the greedy algorithm does not pick e from Y, i.e. there must exist some cycle C' if $\mathcal{M}_2$ in Y such that $C' \subset K \cup \{e\}$ and $c \leq_2 e$ holds for each element c of $C' - e$. Consequently, K is an $\mathcal{M}_1\mathcal{M}_2$-kernel.

Assume now that K is an $\mathcal{M}_1\mathcal{M}_2$-kernel and define

$$X := K \cup \{e \in E : e \notin \mathcal{F}(K \cup \{e\}\} \qquad \text{and} \qquad Y := \mathcal{D}_{\mathcal{F}}(X)\,.$$

Set K is independent in $\mathcal{M}_1$ hence $K = \mathcal{F}(X) = X \cap Y$ holds. So $Y = \mathcal{D}_{\mathcal{F}}(X) = K \cup (E \setminus X) = \{e \in E : e \in \mathcal{F}(K \cup \{e\}\}$. As K is an $\mathcal{M}_1\mathcal{M}_2$-kernel, for any element e of $Y \setminus K$ there exists a cycle $C \subset K \cup \{e\}$ of $\mathcal{M}_2$ such that $c \leq_2 e$ holds for any $c \in C$. Consequently, $\mathcal{G}(Y) = K$ and hence $\mathcal{D}_{\mathcal{G}}(Y) = E \setminus (X) \cup K = X$, that is, K is an $\mathcal{FG}$-kernel.

Hence the existence of $\mathcal{M}_1\mathcal{M}_2$-kernels and the structural properties of $\mathcal{M}_1\mathcal{M}_2$-kernels directly follow from Corollary 1.17 and the definition of choice functions $\mathcal{F}$ and $\mathcal{G}$. □

It is interesting to see that Theorem 2.9 generalizes the following well-known result.

Theorem 2.10. *If graph G is bipartite then $d_M \equiv d_{M'}$ holds for any two stable b-matching M and M'. Moreover, $M(v) = M'(v)$ whenever $|M(v)| < b(v)$ holds.*

Note that the famous Rural Hospitals Theorem of Roth [57] is the special case of the above Theorem 2.10 where G has bipartiton (A, B) and $b(a) = 1$ holds for each vertex $a \in A$.

Proof. Stable b-matchings of G are exactly $\mathcal{M}_1\mathcal{M}_2$-kernels where $\mathcal{M}_1$ and $\mathcal{M}_2$ are partition matroids on E (the partitions are given by the stars of the two color classes), linear orders $\leq_1$ and $\leq_2$ are compatible with the preferences of the vertices, and b determines the rank of the parts in the partiton. If M and M' are stable b-matchings of G then $span_{\mathcal{M}_1}(M) = span_{\mathcal{M}_1}(M')$ and $span_{\mathcal{M}_2}(M) = span_{\mathcal{M}_2}(M')$ holds by Theorem 2.9 and this directly implies Theorem 2.10. □

Chapter 3

The structure of kernels

In this chapter, we study the structure of various $\mathcal{FG}$-kernels. We shall see that the lattice structure of kernels (due to Corollary 1.17) allows us to prove various interesting structural results.

3.1 Uncrossing of kernels and median kernels

First, we deduce as a corollary of Corollary 1.17 that $\mathcal{FG}$-kernels can be efficiently uncrossed.

Theorem 3.1 (Fleiner [20]). *Let $\mathcal{F}, \mathcal{G} : 2^E \to 2^E$ be w-increasing substitutable choice functions for some modular and strictly monotone function $w : E \to \mathbb{R}_+$ and let $K_1, K_2, \ldots, K_m$ be arbitrary $\mathcal{FG}$-kernels. Then there exist a chain $K^1 \preceq_{\mathcal{F}} K^2 \preceq_{\mathcal{F}} \ldots \preceq_{\mathcal{F}} K^m$ of $\mathcal{FG}$-kernels such that $\sum_{i=1}^m \chi(K_i) = \sum_{i=1}^m \chi(K^i)$ and $K^j = \mathcal{F}\left(\text{supp}\left(\sum_{i=1}^m \chi(K_i) - \sum_{i=1}^{j-1} \chi(K^i)\right)\right)$ hold for $1 \leq j \leq m$.*

Proof. If $\mathcal{FG}$-kernels K_i and K_j are not comparable then $K_i \wedge_{\mathcal{F}} K_j$ and $K_i \vee_{\mathcal{F}} K_j$ are both $\mathcal{FG}$-kernels with the property that $\chi(K_i) + \chi(K_j) = \chi(K_i \wedge_{\mathcal{F}} K_j) + \chi(K_i \vee_{\mathcal{F}} K_j)$ by the second part of Corollary 1.17. This means that if we replace K_i and K_j by $\mathcal{FG}$-kernels $K_i \wedge_{\mathcal{F}} K_j$ and $K_i \vee_{\mathcal{F}} K_j$, then this does not change $\sum_{i=1}^m \chi(K_i)$, while it increases $\sum_{i=1}^m |K_i|^2$. So at some point no such replacement is possible any more and hence we have a chain of $\mathcal{FG}$-kernels as required by the theorem. It follows from $\sum_{i=1}^m \chi(K_i) = \sum_{i=1}^m \chi(K^i)$ that

$$K^j = K^j \wedge_{\mathcal{F}} K^{j+1} \wedge_{\mathcal{F}} \ldots \wedge_{\mathcal{F}} K^m = \mathcal{F}(K^j \cup K^{j+1} \cup \ldots \cup K^m) = F\left(\text{supp}\left(\sum_{i=j}^m \chi(K^i)\right)\right)$$

$$= \mathcal{F}\left(\text{supp}\left(\sum_{i=1}^m \chi(K_i) - \sum_{i=1}^m \chi(K^i) + \sum_{i=j}^m \chi(K^i)\right)\right) = \mathcal{F}\left(\text{supp}\left(\sum_{i=1}^m \chi(K_i) - \sum_{i=1}^{j-1} \chi(K^i)\right)\right)$$

holds, and this finishes the proof. □

In case of stable b-matchings, Theorem 3.1 has an especially simple form that relies on the following generalization by Baïou and Balinski of the Comparability Theorem [59] by Roth and Sotomayor.

Theorem 3.2 (Baïou and Balinski [7]). *Let $G = (V, E)$ be a bipartite graph and let $\preceq_v$ be a linear order on the set $E(v)$ of edges incident to vertex v and let $b : V \to \mathbb{N}$. If S_1 and S_2 are stable matchings and $v \in V$ then one of the two properties below must hold.*

- *$S_1(v) = S_2(v)$ or*
- *$|S_1(v)| = |S_2(v)| = b(v)$ and the $b(v)$ $\preceq_v$-best element of set $S_1(v) \cup S_2(v)$ is either $S_1(v)$ or $S_2(v)$.*

It is not difficult to prove Theorem 3.2 with elementary tools, i.e. by studying the domination in the symmetric difference of two stable b-matchings. A corollary of Theorem 3.2 is that for any vertex v of G there is a linear order on those subsets of edges of $E(v)$ that a stable b-matching can contain. As it (hopefully) does not cause ambiguity's, this linear order is denoted also by $\preceq_v$. The theorem below in particular implies that if each agent in one part of the bipartite graph select the ith best assignment out of k given stable b-matchings then these choices result in another stable b-matching.

Theorem 3.3 (Fleiner [19]). *Let $G = (V, E)$ be a bipartite graph with parts A and B and let $\preceq_v$ be a linear order on the set $E(v)$ of edges incident with v for each vertex v and let $b : V \to \mathbb{N}$. Then for any stable b-matchings $S_1, S_2, \ldots, S_k$, and for any $1 \le i, j \le k$, sets $S_A^i := \bigcup\{S^i(v) : v \in A\}$ and $S_B^j := \bigcup\{S^i(v) : v \in B\}$ are stable b-matchings with the property that $S_A^i = S_B^{k+1-i}$ holds for $1 \le i \le k$ where $S^1(v), S^2(v), \ldots, S^k(v)$ denotes edge sets $S_1(v), S_2(v), \ldots, S_k(v)$ in $\preceq_v$-order.*

Proof. For vertex v of G and stable matchings S and S' let $S \preceq_v S'$ denote that v prefers S to S'. According to Theorem 3.2, we can list $S_1, S_2, \ldots, S_n$ as $S_v^1 \preceq_v S_v^2 \preceq_v \ldots \preceq_v S_v^n$ for each vertex v. Observe that $S_A^i = \bigvee_{a\in A} \bigwedge_{\ell=1}^{i} S_a^\ell$ and $S_B^j = \bigwedge_{b\in B} \bigvee_{\ell=1}^{j} S_b^\ell$ where we use the choice function of A for operation $\wedge$ and operation $\vee$ is calculated with the choice function of B. Chains $S_A^1, S_A^2, \ldots, S_A^k$ and $S_B^1, S_B^2, \ldots, S_B^k$ are opposite, hence $S_A^i = S_B^{k+1-i}$. □

Theorem 3.3 above has been proved for stable matchings by Teo and Sethuraman with the help of linear programming tools [66]. Later, not being aware of Theorem 3.3, Klaus and Klijn gave a very similar short proof for a special case [50].

3.2 The splitting property of kernels

The following less known splitting property of stable b-matchings turns out to be especially useful when proving a linear description of stable b-matching polyhedra.

Theorem 3.4 (Fleiner [21]). *Let $\preceq_v$ be a linear order on the set $E(v)$ of edges incident to v for each vertex v of bipartite graph $G = (V, E)$ and let $b : V \to \mathbb{N}$. Then there exists a partition $E(v) = E_1(v) \cup \ldots \cup E_{b(v)}(v)$ for each vertex $v \in V$ such that $|E_i(v) \cap S| \le 1$ holds for any stable b-matching S, vertex v and index $1 \le i \le b(v)$.*

Proof. Let $E'(v)$ be the set of those edges of $E(v)$ that can appear in some stable b-matching of G. Note that it is enough to partition $E'(z)$ into $b(z)$ parts with the required property, as we can assign edges of $E(z) \setminus E'(z)$ into any of the parts without violating the condition required by Theorem 3.4.

If M and M' are stable b-matchings and the $\preceq_v$ minimal edge of $M(v)$ is the same as the $\preceq_v$-minimal edge of $M'(v)$ then Theorem 3.2 yields that $M(v) = M'(v)$. Hence there is a linear order $M^1(v) \prec_v M^2(v) \prec_v \ldots \prec_v M^\ell(v)$ on possible sets $M(v)$ so that for $i < j$ the $\preceq_v$-worst $|M^i(v) \setminus M^j(v)|$ edges of $M^i(z) \cup M^j(z)$ are the edges of $M^j(v)$. By induction on k, we show how to partition $\bigcup_{i=1}^k M^i(v)$ into $b(v)$ parts $E_1^k(v), E_2^k(v), \ldots, E_{b(v)}^k(v)$ so that any $M^j(v)$ contains at most one edge of each part $E_i^k(v)$ for $j \geq k$.

As $M^1(v)$ comes from some b-matching, we can partition $M^1(v)$ into $b(v)$ (possibly empty) parts $E_1^1(v), E_2^1(v), \ldots, E_{b(v)}^l(v)$, of size at most one. Thus $M^1(v)$ intersects each $E_i^1(v)$ in at most one edge. Assume that we have a partition $E_1^k(v), E_2^k(v), \ldots, E_{b(v)}^k(v)$ of $\bigcup_{i=1}^k M^i(z)$ with the above property. To construct partition $E_1^{k+1}(v), E_2^{k+1}(v), \ldots, E_{b(v)}^{k+1}(v)$ of $\bigcup_{i=1}^{k+1} M^i(v)$, we extend the old parts $E_i^k(v)$ by assigning the new edges of $M^{k+1}(v) \setminus \bigcup_{i=1}^k M^i(v)$ to certain parts so that the required property is preserved.

By Theorem 3.2, $|M^{k+1}(v) \setminus M^k(v)| = |M^k(v) \setminus M^{k+1}(v)|$, moreover no edge of $M^{k+1}(v) \setminus M^k(v)$ is present in $M^i(z)$ for $i \leq k$ (those are the worst edges that we did not see so far). So we can distribute the unassigned edges of $M^{k+1}(v) \setminus M^k(v)$ into the parts of the edges of $M^k(v) \setminus M^{k+1}(v)$ in such a way that we put exactly one edge to each of the parts. By this, we partition $\bigcup_{i=1}^{k+1} M^i(v)$ into $b(v)$ parts $E_1^{k+1}(v), E_2^{k+1}(v), \ldots, E_{b(v)}^{k+1}(v)$ so that any $M^i(v)$ intersects any part $E_j^{k+1}(v)$ in at most one edge for $i \leq k+1$ and $1 \leq j \leq b(z)$. Partition $E_1(v), E_2(v), \ldots, E_{b(v)}(v)$ required by Theorem 3.4 can be constructed from $E_1^\ell(v), E_2^\ell(v), \ldots, E_{b(v)}^\ell(v)$ by putting unassigned edges to arbitrary parts. □

It follows from Theorem 3.4 that for any graph G with arbitrary preference orders on its vertices and for arbitrary vertex capacities there exists a graph G' such that G can be obtained from G' by merging certain vertex subsets of G', moreover any stable b-matching of G corresponds to a stable matching of G'. Although graph G' can be constructed efficiently from graph G, preferences $\preceq_v$ and quotas q, this observation does not allow us to reduce the search of a stable b-matching to a search of stable matchings. The reason is that it is not true that any stable matching of G' is coming from a stable b-matching of G as the following Example shows.

Example 3.5. *Let G have two vertices u and v and 4 parallel uv edges and assume $b(u) = b(v) = 2$ and $e_1 <_u e_2 <_u e_3 <_u e_4$ and $e_4 <_v e_3 <_v e_2 <_v e_1$. Then stable b-matchings are $\{e_1, e_2\}, \{e_2, e_3\}$ and $\{e_3, e_4\}$, so the only possible star-partitions provided by Theorem 3.4 (up to permutation of parts) is $E_1(u) = E_1(v) = \{e_1, e_3\}$ and $E_2(u) = E_2(v) = \{e_2, e_4\}$. Now $\{e_1, e_4\}$ is a stable matching of graph G' while it is not a stable b-matching in G.*

Observe that if in Theorem 3.4, we pick some vertex v and a set $E_i(v)$ such that $E_i^1(v)$ in the proof is nonempty then then $|E_i(v) \cap S| = 1$ holds for any stable b-matching S. Our last result in this section is a certain extension of Theorem 3.4 to a more general setting. First, we need a handy lemma stating that $\mathcal{FG}$-kernels form a clutter (to be defined in Chapter 5).

Lemma 3.6 (Fleiner [20]). *If $K_1 \preceq_\mathcal{F} K_2 \preceq_\mathcal{F} K_3$ is a chain of of $\mathcal{FG}$-kernels for path-independent substitutable choice functions $\mathcal{F}, \mathcal{G} : 2^E \to 2^E$ then $K_1 \cap K_3 \subseteq K_2$ must hold.*

Proof. Let $\mathcal{D}_\mathcal{F}$ be a determinant of $\mathcal{F}$ with property (1.1). From $K_1 \preceq_\mathcal{F} K_2 \preceq_\mathcal{F} K_3$ we get $\mathcal{F}(K_1 \cup K_2 \cup K_3) = K_1$ by Corollary 1.17, hence $\mathcal{D}_\mathcal{F}(K_1 \cup K_2 \cup K_3) = \mathcal{D}_\mathcal{F}(K_1)$ holds. Using Observation 1.14, the antitone property of $\mathcal{D}_F$ and Corollary 1.17,

$$\begin{aligned} K_2 = \mathcal{F}(K_2) = \mathcal{F}(K_2 \cup K_3) = (K_2 \cup K_3) \cap \mathcal{D}_\mathcal{F}(K_2 \cup K_3) \supseteq K_3 \cap \mathcal{D}_\mathcal{F}(K_1 \cup K_2 \cup K_3) \\ = K_3 \cap \mathcal{D}_\mathcal{F}(K_1) \supseteq K_3 \cap (\mathcal{D}_\mathcal{F}(K_1) \cap K_1) = K_3 \cap \mathcal{F}(K_1) = K_3 \cap K_1 \end{aligned}$$

follows. □

3.3 Transversals of kernels

In this section, our main result is we prove that there exists a transversal of $\mathcal{FG}$-kernels that will will be crucial in Chapter 5. For convenience, let $\mathcal{K}_{\mathcal{FG}}$ denote the set of $\mathcal{FG}$-kernels for choice functions $\mathcal{F}, \mathcal{G} : 2^E \to 2^E$.

Theorem 3.7 (Fleiner [20]). *For any modular and strictly monotone function $w : E \to \mathbb{R}_+$ and for any w-increasing substitutable choice functions $\mathcal{F}, \mathcal{G} : 2^E \to 2^E$, there is a subset X of E such that*

$$|X \cap K| = 1 \text{ holds for any } \mathcal{FG}\text{-kernel } K. \tag{3.1}$$

Proof. For element x of E and subset Y of E define

$$\begin{aligned} K_x &:= \bigwedge\nolimits_\mathcal{F} \{K \in \mathcal{K}_{\mathcal{FG}} : x \in K\}, \\ K^x &:= \bigvee\nolimits_\mathcal{F} \{K \in \mathcal{K}_{\mathcal{FG}} : x \in K\}, \text{ and} \\ K_{\overline{Y}} &:= \bigwedge\nolimits_\mathcal{F} \{K \in \mathcal{K}_{\mathcal{FG}} : K \cap Y = \emptyset\} \end{aligned}$$

and observe from Lemma 3.6 and (1.4) that

$$\{K \in \mathcal{K}_{\mathcal{FG}} : x \in K\} = \{K \in \mathcal{K}_{\mathcal{FG}} : K_x \preceq_\mathcal{F} K \preceq_\mathcal{F} K^x\} = [K_x, K^x], \tag{3.2}$$

that is, the set of $\mathcal{FG}$-kernels that contain element x of E are exactly the ones in the interval between K_x and K^x. Consequently, our task is to partition the lattice of $\mathcal{FG}$-kernels into disjoint intervals $I_x = [K_x, K^x]$.

Pick an arbitrary element x_1 of $\mathcal{F}$-optimal $\mathcal{FG}$-kernel $K^1 = K_\mathcal{F}$ and let I_{x_1} be the first interval in our partition. We create the intervals one by one, so assume that for some n we have elements $x_1, x_2, \ldots, x_n$ of E such that for any $i \leq n$ we have the following properties:

$$\text{Intervals } I_{x_1}, I_{x_2}, \ldots, I_{x_n} \text{ are pairwise disjoint and} \tag{3.3}$$

$$K^{x_1} \prec_\mathcal{F} K^{x_2} \prec_F \ldots \prec_F K^{x_n} \text{ and} \tag{3.4}$$

$$\begin{gathered} \textstyle\bigcup_{j=1}^{i} I_{x_j} = [K_\mathcal{F}, K^{x_i}], \text{ that is, intervals } I_{x_j} \text{ cover} \\ \text{the part of the lattice of } \mathcal{FG}\text{-kernels below } K^{x_i}. \end{gathered} \tag{3.5}$$

Case 1. If $x_n \in K_\mathcal{G}$, i.e., if $K^{x_n} = K_\mathcal{G}$ then $X = \{x_1, x_2, \ldots, x_n\}$ has the property that Theorem 3.7 requires, and the proof is done.

Case 2. Otherwise, if $x_n \notin K_{\mathcal{G}}$ then define $K := K_{\overline{\{x_1,x_2,\ldots,x_n\}}}$. As $K \cap \{x_1, x_2, \ldots, x_n\} = \emptyset$ by (3.2) and the definition of K, (3.5) for $i = n$ implies $K \not\preceq_{\mathcal{F}} K^{x_n}$. Clearly,

$$K = \bigvee_{\mathcal{F}} \{K_x : x \in K\}, \tag{3.6}$$

hence there must exist some $x_{n+1} \in K$ such that

$$K_{x_{n+1}} \not\preceq_{\mathcal{F}} K^{x_n} \ . \tag{3.7}$$

Thus $K_{x_{n+1}} \cap \{x_1, x_2, \ldots, x_n\} = \emptyset$ holds by (3.5) and $K \preceq_{\mathcal{F}} K_{x_{n+1}} \preceq_{\mathcal{F}} K$ follows from (3.6) and the definition of K. This proves that

$$K = K_{\overline{\{x_1,x_2,\ldots,x_n\}}} = K_{x_{n+1}} \tag{3.8}$$

Our next goal is to prove that (3.3) and (3.4) hold for $n+1$ and (3.5) holds for $i = n+1$. To show (3.3), assume indirectly $K \in I_{x_{n+1}} \cap I_{x_j}$, for some $j \leq n$. This means that $K_{x_{n+1}} \preceq_{\mathcal{F}} K \preceq_{\mathcal{F}} K^{x_j} \preceq_{\mathcal{F}} K^{x_n}$, contradicting (3.7). This contradiction proves (3.3) for $n+1$.

For (3.4), it is enough to justify $K^{x_n} \preceq_{\mathcal{F}} K^{x_{n+1}}$. As

$$\chi(K_{x_{n+1}}) + \chi(K^{x_n}) = \chi(K_{x_{n+1}} \wedge_{\mathcal{F}} K^{x_n}) + \chi(K_{x_{n+1}} \wedge_{\mathcal{F}} K^{x_n})$$

holds by (1.4), and $x_{n+1} \notin K_{x_{n+1}} \wedge_{\mathcal{F}} K^{x_n}$ by the definition of K_{x_n} and by (3.7). Consequently, $x_{n+1} \in K_{x_{n+1}} \vee_{\mathcal{F}} K^{x_n}$ and hence

$$K^{x_n} \preceq_{\mathcal{F}} K_{x_{n+1}} \vee_{\mathcal{F}} K^{x_n} \preceq_{\mathcal{F}} K^{x_{n+1}},$$

proving (3.4) for $n+1$.

To show (3.5) for $i = n+1$, assume that $K' \preceq K^{x_{n+1}}$ holds for some $\mathcal{FG}$-kernel K. If $x_{n+1} \notin K'$ then $K' \not\succeq_{\mathcal{F}} K_{x_{n+1}}$, hence $K' \not\succeq_{\mathcal{F}} K = K_{\overline{\{x_1,x_2,\ldots,x_n\}}}$ due to (3.8). Hence $K' \cap \{x_1, x_2, \ldots, x_n\} \neq \emptyset$ and (3.5) for $i = n+1$ follows.

To finish the proof, we construct elements $x_1, x_2, \ldots$ one after another as described in Case 2. above. As $\mathcal{K}_{\mathcal{FG}}$ is finite, at some point we end up in Case 1, and this completes the proof of Theorem 3.7. □

Chapter 4

Applications of kernels

This chapter illustrates some applications of well-known theorems on stable matchings and less known results on kernels.

Below, we briefly sketch such an application. It is relatively easy to see that the Cantor-Bernstein theorem on cardinalities (stating that $|A| \leq |B|$ and $|B| \leq |A|$ implies $|A| = |B|$) can be deduced from an infinite version of the stable matching theorem that we did not state in this work. Note that the Cantor-Bernstein theorem is a standard application of Theorem 1.9 of Tarski. Note also that the infinite stable matching theorem can be proved from Tarski's fixed point theorem similarly as we proved the finite version due to Gale and Shapley. It is well-known (and easy to see) that the Cantor-Bernstein theorem can be regarded as an infinite version of the Mendelsohn-Dulmage theorem stating that if some matching covers subset A' of part A of bipartite graph G and some other matching of G covers B' of part B then some matching of G covers $A \cup B$. From this perspective, it is not surprising that the Mendelsohn-Dulmage result can also be proved with stable matchings. Furthermore, the matroid generalization of the Mendelsohn-Dulmage theorem, namely the Kundu-Lawler theorem follows easily from Theorem 2.9, the matroid-generalization of Theorem 1.1 of Gale and Shapley.

In what follows, we survey kernel-related results about graph paths, graph colorings and college admissions.

4.1 Applications on paths

As we mentioned earlier, the graph-kernel result below can be proved from Tarski's fixed point theorem (and also from Theorem 2.2).

Theorem 4.1 (Sands, Saurer, Woodrow [62]). *If E_1 and E_2 are two loopless arc sets on vertex set V then there is a subset U of V with the two properties below.*

- *There is no directed path in E_1 or in E_2 connecting two different vertices of U and*
- *from any vertex $v \in V \setminus U$ there exists a directed path of E_1 or of E_2 that terminates in U.*

A perhaps less self-explanatory neat application of the theorem of Gale and Shapley is the proof of Pym's theorem below.

Theorem 4.2 (Pym [56]). *Let each of $\mathcal{P}$ and $\mathcal{Q}$ be a set of vertex-disjoint directed paths in digraph $D = (V, E)$. Then there exists a set $\mathcal{R}$ of vertex-disjoint directed paths of D such that*

- *there is a path of $\mathcal{R}$ starting at each starting vertex of a path of $\mathcal{P}$ and each path of $\mathcal{R}$ starts in a starting point of a path of $\mathcal{P}$ or of $\mathcal{Q}$ and*
- *any terminal of any path of $\mathcal{Q}$ is a terminal of some path of $\mathcal{R}$ and each terminal of each path of $\mathcal{R}$ is a terminal of a path of $\mathcal{P}$ or of $\mathcal{Q}$, furthermore*
- *each path of $\mathcal{R}$ is a concatenation of a (possibly empty) starting segment of a path of $\mathcal{P}$ and a (possibly empty) end segment of a path of $\mathcal{Q}$.*

Proof. Define bipartite graph G on vertex set $\mathcal{P} \cup \mathcal{Q}$ such that edges correspond to common vertices of a path $P \in \mathcal{P}$ and $Q \in \mathcal{Q}$. For vertex $P \in \mathcal{P}$ define linear order $\leq_P$ on $E(P)$ according to the order on P of the intersection vertices of P. Similarly, linear order $\leq_Q$ for path $Q \in \mathcal{Q}$ comes from the opposite order of the intersection vertices of Q. Let S be the intersection vertices of a stable matching of G. Define $\mathcal{R}$ as those paths of $\mathcal{P} \cup \mathcal{Q}$ that do not contain any vertex of S and for each vertex v of S construct path R_v of $\mathcal{R}$ by concatenating the starting segment of a $\mathcal{P}$ path terminating at v with the terminal segment of a $\mathcal{Q}$-path starting at v. Clearly, all three requirements for $\mathcal{R}$ hold, the only thing we have to check is that paths of $\mathcal{R}$ are vertex-disjoint.

Let v be an arbitrary vertex of D. As edge e_v of G that corresponds to v does not block stable matching S, e_v must be dominated by some edge e_u of S such that u and v are on the same path $Z \in \mathcal{P} \cup \mathcal{Q}$. If Z belongs to $\mathcal{P}$ then the starting part of Z that belongs to $\mathcal{R}$ does not contain v and if $Z \in \mathcal{Q}$ then the terminal segment of Z that belongs to $\mathcal{R}$ does not contain v. Hence no vertex v can belong to two paths of $\mathcal{R}$, that is, $\mathcal{R}$ consists of vertex-disjoint paths, as we claimed. This observation finishes the proof. □

4.2 List-edge-colorings

In this subsection, we study edge-colorings of graphs. Let $G = (V, E)$ be a finite loopless graph. For each edge $e \in E$, let $L(e) \subset \mathbb{N}$ be a set of available colors for e. We say that G is *L-edge-choosable* if G has an *L-edge-coloring*, that is, a proper edge-coloring $c : E \to \mathbb{N}$ such that $c(e) \in L(e)$ holds for each edge e of E. Graph G is called *k-edge-choosable* if G is L-edge-choosable for any $L : E \to \binom{\mathbb{N}}{k}$. The famous list coloring conjecture states that any finite loopless graph G is $\chi'(G)$-edge-choosable, where chromatic index $\chi'(G)$ denotes the minimum number of colors needed to properly color the edges of G. By generalizing the Dinitz conjecture in [36], Galvin justified the list coloring conjecture for bipartite multigraphs.

Theorem 4.3 (Galvin [36]). *Every bipartite multigraph G is $\Delta(G)$-edge-choosable.*

Galvin's method can be extended to nonbipartite graphs as follows.

Theorem 4.4 (Fleiner [24]). *Let $G = (V, E)$ be a graph and $c : E \to \{1, 2, \ldots, k\}$ be a proper edge-coloring of G. Let $L(e)$ be a list of k colors for each edge $e \in E$. If* *the color lists of the edges of no odd cycle of G contain a common element then G has a proper edge coloring l that colors each edge from its list, i.e. $l(e) \in L(e)$ holds for each edge e of G.*

Proof. For $i = 1, 2, \ldots$ define $E_i := \{e \in E : 2i-1 \le c(e) \le 2i\}$. Clearly, $E = E_1 \cup E_2 \cup \ldots \cup E_{\lceil k/2 \rceil}$. As the maximum degree in $G_i = (V, E_i)$ is not more than 2, each component of G_i is a path or a cycle. Orient the edges of G such that each component of each G_i becomes a directed path or a directed cycle. For edge $e = uv \in E_i$ define

$$r_v(e) = \begin{cases} i & \text{if } v \text{ is the head of the arc that corresponds to } e \\ k+1-i & \text{if } v \text{ is the tail of the arc that corresponds to } e. \end{cases}$$

Assume that $r_v(e) = r_v(f) = j$. If $j < \frac{k+1}{2}$ then $e, f \in E_j$ and v is the head of both e and f. If $j > \frac{k+1}{2}$ then $e, f \in E_{k+1-j}$ and v is the tail of both e and f. At last, if $j = \frac{k+1}{2}$ then $c(e) = c(f) = k$. In all three cases, $e = f$ must hold. Consequently, rank function r_v determines linear order $\preceq_v$ on $E(v)$ where $e \preceq_v f$ means that $r_v(e) \le r_v(f)$. Let uv be the oriented version of edge $e \in E_i$. From $r_u(e) = i$ and $r_v(e) = k+1-i$ we see that

$$|\{f \in E(u) : f \prec_u e\}| + |\{f \in E(v) : f \prec_v e\}| \le i - 1 + (k+1-i) - 1 = k - 1\ . \quad (4.1)$$

Observation (4.1) enables us to employ Galvin's method to finish the proof. Define $E^i := \{e \in E : i \in L(e)\}$ as the set of i-colorable edges and let $G^i := (V, E^i)$. As none of the G^is contain an odd cycle by the assumption, each G^i is bipartite. For $i = 0, 1, 2, \ldots$ define M^i as a stable matching of graph $G^i \setminus (M^0 \cup \ldots \cup M^{i-1})$ with restricted linear orders $\preceq_v$. Such matchings exist by Theorem 1.1.

To show that G is L-edge-choosable, give color i to edges of M^i. Clearly, no two edges of the same color share a vertex and each colored edge receives its color from its list. The only thing left is to show that each edge of G receives some color.

Observe that if edge $e = uv$ of G^i does not receive color i, (i.e. if $e \notin M^i$) then either $e \in M^j$ for some $j < i$ (hence e received color j before M^i was defined) or M^i contains an edge f such that $f \prec_u e$ or $f \prec_v e$. So if e does not receive any color, that is, if $e \notin \bigcup\{M^j : j \in L(e)\}$ then there is an $f^j \in M^j$ for each $j \in L(e)$ with $f^j \prec_u e$ or $f^j \prec_v e$. As $|L(e)| \ge k$, this is impossible by (4.1) and this contradiction proves that the above algorithm finds a proper L-edge-coloring of G. □

There also exists a common generalization of Galvin's theorem and the theorem on balanced coloring of bipartite graphs. To formulate this extension, we introduce a partial order on (not necessarily proper) edge-colorings of graphs.

To define this partial order, we start from a little afar. For a nonnegative integer n, a *(number theoretic) partition of n* is a way to decompose n as a sum of positive integers where the order of the terms does not matter. That is, if two such sums only differ in the order of the terms then those determine the same partition. For a number theoretic partition π let $\pi(i)$ denote the ith greatest term in π, where we count each addend with its multiplicity. That is, if π is partition $2+3+2+5+1+1$ of 14 then $\pi(3) = 2$, $\pi(5) = 1$ and (slightly abusing notation) $\pi(8) = 0$. We say that partition π of n is *better* than partition π' of n' (denoted by $\pi \preceq \pi'$) if $\sum_{i=1}^k \pi(i) \le \sum_{i=1}^k \pi'(i)$ holds for all positive integers k. It follows immediately from the definition that among partitions of n, $n = 1 + 1 + \ldots + 1$ is the best one and the one-term partition $n = n$ is the worst one.

Let us turn to edge-colorings now. Each k-edge-coloring $c : E(G) \to \{1, 2, \ldots, k\}$ and each vertex v of G induce a partition $\pi(c, v)$ of degree $d(v)$ of v into (at most k) terms that describe how many edges of each color of c are incident with v. In particular, edge-coloring c is a proper one if and only if $\pi(c, v)$ is the best partition of $d(v)$ for each vertex v of G.

If c and c' are two edge-colorings of G then edge-coloring c is *better* than c' if $\pi(c, v) \preceq \pi(c', v)$ holds for each vertex v of G, that is, if c induces a better partition on each degree than c' does. This definition yields in particular that the best edge-colorings are the proper ones. Now we can claim our theorem.

Theorem 4.5 (Fleiner, Frank [26]). *Let $G = (V, E)$ be a finite bipartite graph and π_v be a partition of $d(v)$ into at most k terms. If $L(e)$ is a list of at least k possible colors for each edge e of G then we can pick a color $c(e)$ of $L(e)$ for each edge e of G such that the partition c induces at v is better than π_v at each vertex v of G.*

Note that Theorem 4.5 implies both the edge-coloring theorem of Kőnig and Theorem 4.3 of Galvin if we apply it to the finest partitions of each degree $d(v)$. Another immediate corollary of Theorem 4.5 is the following.

Corollary 4.6 (Fleiner, Frank [26]). *If c is a (not necessarily proper) k-coloring of the edges of bipartite graph G and $|L(e)| \geq k$ for each edge e of G then there exists a list edge coloring l of G such that $l \preceq c$.*

Proof. Apply Theorem 4.5 to partitions $\pi(v) = \pi(c, v)$. □

Here is yet another consequence of Theorem 4.5 that has to do with balanced k-edge-colorings.

Corollary 4.7 (Fleiner, Frank [26]). *Assume that G is a bipartite graph and for each edge e of G, list $L(e)$ contains at least k colors. Then it is possible to pick a color $c(e) \in L(e)$ for each edge e of G such that no vertex v is incident with more than $\left\lceil \frac{d(v)}{k} \right\rceil$ edges of the same color.*

Proof. Applying Theorem 4.5 to G where π_v denotes the partition of $d(v)$ into k terms each of which is either $\left\lceil \frac{d(v)}{k} \right\rceil$ or $\left\lfloor \frac{d(v)}{k} \right\rfloor$ gives a list edge-coloring c such that $\pi(c, v)(1) \leq \pi_v(1) = \left\lceil \frac{d(v)}{k} \right\rceil$ for all vertices v of G. This is exactly what Corollary 4.7 requires. □

Before justifying Theorem 4.5, we recall some definitions. If G is a graph and S is a set of vertices of G then by *merging the vertices of S* we mean the operation that we delete S from G, introduce a new vertex (say v_S) and in each edge e of G incident with some vertex of S we replace vertices of S by v_S. Note that we may create parallel edges and loops by merging. Clearly, if G' is obtained from G by merging the vertices of S then G and G' has the same number of edges and the degree of v_S in G' is the sum of the degrees of the vertices of S in G. If S contains k vertices then we say that we can get graph G from G' by *detaching v_S into k parts*. Note that merging vertices is a unique operation unlike detaching a vertex into k parts that can be done several ways.

We need some basics also on partitions. We say that partition π of n is the *conjugate* of partition σ of n if $\pi(i) = \max\{j : \sigma(j) \geq i\}$. It is well-known that turning the Ferrers diagram of a partition by 90 degrees (and taking mirror image) we get the Ferrers diagram of the conjugate partition hence if σ is the conjugate of π then π is the conjugate of σ, as well.

Proof of Theorem 4.5. Construct graph G' by detaching each vertex v of G into vertices $v_1, v_2, \ldots, v_{\pi_v(1)}$ in such a way that $d_{G'}(v_1) + d_{G'}(v_2) + \ldots + d_{G'}(v_k)$ is the conjugate partition of π_v. Clearly $k \geq d_{G'}(v_1) \geq d_{G'}(v_2) \geq d_{G'}(v_3) \geq \ldots$ holds by our choice, so $\Delta(G') \leq k$. For each edge e' of G' define $L(e') := L(e)$ where e' corresponds to edge e' of G. By Theorem 4.3 of Galvin, there exists a list edge-coloring of G', that is, we can pick a color $c'(e) \in L'(e)$ for each edge e' of G such that c' is a proper edge-coloring of G'. For each edge e of G define $c(e) := c'(e')$ where e' corresponds to e in G'. By definition, $c(e) \in L(e)$ holds.

The only thing left is to show that $\pi(c, v) \preceq \pi_v$ for each vertex v of G. To this end, it is enough to prove that for any positive integer i and any set C of i colors, no more than $\pi_v(1) + \pi_v(2) + \ldots + \pi_v(i)$ edges incident with v have been colored to a color of C. So fix set C of i colors and let $E(C, v) := \{e \in E(v) : c(e) \in C\}$ be the set of edges incident with v with a color of C. (Here $E(v)$ stands for the set of edges incident with v.) Let $E'(C, v)$ be the set of edges of G' that correspond to edges $E(C, v)$. Clearly, each vertex v_j of G' is incident with at most $\min(d_{G'}(v_j), i)$ edges of $E'(C, v)$. This means that

$$|E(C, v)| = |E'(C, v)| \leq \sum_{j=1}^{\pi_v(1)} \min(d_{G'}(v_j), i) = \pi_v(1) + \pi_v(2) + \ldots + \pi_v(i) \ ,$$

where the last equality follows from the fact that partitions $\pi_v(1)+\pi_v(2)+\ldots+\pi_v(i)$ and $\min(d_{G'}(v_1), i) + \min(d_{G'}(v_2), i) + \ldots + \min(d_{G'}(v_{\pi_v(1)}), i)$ are conjugates of one another.

We got that for each vertex v of G there cannot be more edges incident with v colored with at most i colors than the sum of the i greatest term of prescribed vertex-partition π_v. This means that list edge-coloring c induces a better partition on $d(v)$ than π_v, and this is exactly what we wanted to prove. □

4.3 Applications on college admissions

In this section, we study practice motivated applications, namely different variants of the college admission problem. In this problem, we have given a set of colleges, a set of applicants and a set of applications where each application is defined by an applicant and a college. Each applicants has a linear preference order on her applications and we assume that each college c also has a linear order on its applications. (Note that in practice, colleges' preference orders are usually determined by entrance exam scores hence ties may be present on colleges' preferences.) In addition, each college c has an upper quota $q(c)$ of admissible applicants. An *admission scheme* is an assignment of applicants to colleges such that each applicant is assigned to at most one college c and each college c admits at most $q(c)$ applicants. An admission scheme is *stable* if for each application (a, c) either a is admitted to a college that is not worse for a than c or c has admitted $q(c)$ applicants such that each of the admitted applicants are better for c than a. A standard extension of Theorem 1.1 shows that a stable admission scheme always exists.

In practice however there are further requirements that the admission scheme must satisfy. One such requirement can be a set of common quotas, that is, various sets of colleges may have a common upper bound on the number of their admitted applicants, meaning that a feasible admission scheme must satisfy these further upper bounds given by the common quotas. In this model, we require that if two colleges are in the same set with a common quota then the preference orders of these colleges on their common applicants coincide. A feasible admission scheme is *common-quota stable* if for each application (a, c), either a is admitted to a college that is not worse for a than c or there exists a common quota set containing college c that admitted quota-many applicants, each of them is better than a. In this college admission model, we have two results: a positive and a negative one. By a *laminar* family of sets we mean a family with the property that any two members are either disjoint or one contains the other.

Theorem 4.8 (Biró, Fleiner, Irving, Manlove [11]). *The decision problem on the existence of common-quota stable admission scheme is NP-complete.*

However, if common quota sets form a laminar family on colleges, then there always exists a common-quota stable admission scheme.

We refer the reader to [11] for the details of the proof of the first part of Theorem 4.8. The second part can be deduced from Theorem 2.9, but we omit the proof as we shall prove a generalization, namely Theorem 4.10.

Another practice-motivated restriction on college admission schemes is that colleges (or sets of colleges) may have lower quotas as well. The task is then to find a stable admission scheme that obeys the lower quotas as well (or to conclude that no such scheme exists). Note that in presence of lower quotas, there are at least two different reasonable notions of stability that we may want to expect. One has to do with blocking coalitions: an admission scheme is *group-stable* if for each application (a, c), either a is admitted to a college that is not worse for a than c or c admitted quota-many applicants, each of them is better than a or college c did not admit any applicants. We require moreover, that if c did not admit any applicant then there are less than k applications (a, c) such that a is not admitted to a better college than c where k is the lower quota for c. In this model again, it turns out that group-stability is intractable.

Theorem 4.9 (Biró, Fleiner, Irving, Manlove [11]). *In presence of lower quotas, deciding the existence of a group-stable admission scheme is NP-complete.*

The proof of Theorem 4.9 can be found in [11]. Note however, that if we look for an admission scheme that is stable in the ordinary sense then the problem is tractable. The main result in this section is to show a generalization for models where both common quota sets and lower quotas are present.

We define the **2LCSM** problem (that stands for "2-sided laminar classified stable matching") as follows. Let $G = (V, E)$ be a bipartite graph with a linear order $\preceq_v$ on $E(v)$ for each vertex v and let $\mathcal{C}_v$ be a laminar system of sets of $E(v)$ with lower and upper quotas $l(C) \leq u(C)$ on each member C of $\mathcal{C}_v$. Subset M of $E(G)$ is an *lu-matching*, if $l(C) \leq |M \cap C| \leq u(C)$ holds for each vertex v and each set $C \in \mathcal{C}_v$. lu-matching M *lu-dominates* edge $e \in E \setminus M$ if e has some vertex v and set $C \in \mathcal{C}_v$ such that $|M \cap C| = u(C)$ and $m \leq_v e$ holds for each $m \in M \cap C$. An lu-matching M is called *lu-stable* if it lu-dominates each edge in $E \setminus M$. The **2LCSM** problem is the decision of the existence of an

lu-stable l-matching. This problem is a generalization of the so-called LCSM problem but Huang [44]. Huang's **LCSM** problem is motivated by a practical variant of the college admission problem as unlike in case of the stable b-matching problem, the present model can handle conditions that require that certain colleges have to admit a certain number of students to be able to operate or certain colleges must obey a common quota on the total number of admitted students. A possible solution for the **2LCSM** problem can be obtained from the following result.

Theorem 4.10 (Fleiner, Kamiyama [31]). *For any 2LCSM problem on graph $G = (V, E)$, it is possible to construct matroids $\mathcal{M}_P$ and $\mathcal{M}_Q$ in polynomial time such that if there exists an lu-stable matching then the set of lu-stable matchings coincide with the set of $\mathcal{M}_P\mathcal{M}_Q$-kernels. Furthermore, an $\mathcal{M}_P\mathcal{M}_Q$-kernel M is an lu-stable matching if and only if M is an lu-matching.*

According to Theorem 4.10, it is enough to find a single $\mathcal{M}_P\mathcal{M}_Q$-kernel M: if M is an lu-matching then we are done as M is lu-stable as well, otherwise, if M is not an lu-matching then no lu-stable matching exists whatsoever.

Before proving Theorem 4.10, we introduce some further terminology. In the **2LCSM** problem, we are given a finite bipartite graph $G = (V, E)$ with color classes P and Q. For each vertex v of V, there is a laminar family $\mathcal{C}_v$ of subsets of $E(v)$. Define

$$\mathcal{C}_P := \bigcup_{v\in P} \mathcal{C}_v, \quad \mathcal{C}_Q := \bigcup_{v\in Q} \mathcal{C}_v \quad \text{and} \quad \mathcal{C} := \mathcal{C}_P \cup \mathcal{C}_Q.$$

We are given lower and upper quota functions $l\colon \mathcal{C} \to \mathbb{Z}_+$ and $u\colon \mathcal{C} \to \mathbb{Z}_+$. In the sequel, we call a member C of $\mathcal{C}$ a *class*.

Let M be a subset of E. We say that M *obeys l* (resp., u) *for a class C of $\mathcal{C}$* if

$$l(C) \le |M \cap C| \qquad (\text{resp., } |M \cap C| \le u(C)).$$

We call M *feasible for a vertex v of V* if M obeys l and u for any class of $\mathcal{C}_v$, i.e.,

$$l(C) \le |M \cap C| \le u(C)$$

for any class C of $\mathcal{C}_v$. If M is feasible for any vertex of V, then M is an *lu-matching*.

Let M be an lu-matching. In our model, each vertex v has a strict linear order $<_v$ on $E(v)$. We regard this linear order as the preference order of v on its edges, the most preferred one is the $<_v$-smallest edge. An edge e of $E \setminus M$ is called *free for an endpoint v of e* if

$$M + e \text{ is feasible for } v, \text{ or}$$
$$\text{there is an edge } f \text{ of } M(v) \text{ such that } e <_v f \text{ and } M + e - f \text{ is feasible for } v.$$

An edge e of $E \setminus M$ *blocks M* if e is free for both endpoints of e. An lu-matching M of E is *stable* if no edge of $E \setminus M$ blocks M. Then, the **2LCSM** problem is to find an lu-stable matching if exists.

A class C of $\mathcal{C}$ is a *child of a class C' of $\mathcal{C}$* if C is a proper subset of C' and there is no class C° of $\mathcal{C}$ such that $C \subsetneq C^\circ \subsetneq C'$. Without loss of generality, we can make the following assumptions.

Assumption 4.11. *For any vertex v of V and any edge e of $E(v)$, $\{e\} \in \mathcal{C}_v$.*

(We can define $l(\{e\}) = 0$ and $u(\{e\}) = 1$.) By Assumption 4.11, for any class C of $\mathcal{C}$, either C has no child or children $C_1, \ldots, C_k$ of C form a partition of C, i.e. $C_1 \cup \cdots \cup C_k = C$.

Assumption 4.12. *For any vertex v of V, $E(v) \in \mathcal{C}_v$.*

(If $E(v) \notin \mathcal{C}_v$ then we can add $E(v)$ to $\mathcal{C}$ with $l(E(v)) = l(C_1) + l(C_2) + \ldots + l(C_k)$ and $u(E(v)) = |E(v)|$, where $C_1, \ldots, C_k$ are the inclusionwise maximal members of $\mathcal{C}_v$.)

Assumption 4.13. *If a class C of $\mathcal{C}$ has children $C_1, \ldots, C_k$ then*

$$l(C_1) + \cdots + l(C_k) \leq l(C) \leq u(C). \tag{4.2}$$

(We can do so because if the second relation does not hold then clearly there exists no lu-matching. If the first relations fails then we do not change the problem if we change $l(C)$ to $l(C_1) + \cdots + l(C_k)$.)

For a class C of $\mathcal{C}$, we denote by $\mathcal{C}_C$ the set of classes C' of $\mathcal{C}$ such that $C' \subseteq C$. The *level of a class C of $\mathcal{C}$* is the maximum integer k for which there are classes $C_1, \ldots, C_k$ of $\mathcal{C}$ such that $C_1 = C$, C_{i+1} is a child of C_i for any $i \in [k-1]$, and C_k has no child. For a class C of $\mathcal{C}$, we define a function $d_C \colon 2^C \to \mathbb{Z}_+$ as follows. If C has no child, then

$$d_C(F) := \max(|F|,\ l(C))$$

for a subset F of C. If C has children $C_1, \ldots, C_k$, then

$$d_C(F) := \max(d_{C_1}(F \cap C_1) + \cdots + d_{C_k}(F \cap C_k),\ l(C))$$

for a subset F of C. A subset F of a class C of $\mathcal{C}$ is *deficient on C* if the following conditions hold. If C has no child, then F does not obey l for C. If C has children $C_1, \ldots, C_k$, then $d_{C_1}(F \cap C_1) + \cdots + d_{C_k}(F \cap C_k) < l(C)$.

Lemma 4.14. *Let C be a class of $\mathcal{C}$.*

(a) $d_C(F + e) \leq d_C(F) + 1$ for any subset F of C and any edge e of C.
(b) $d_C(F_1) \leq d_C(F_2)$ for any subsets F_1, F_2 of C such that $F_1 \subseteq F_2$.
(c) If a subset F of C obeys l for any class of $\mathcal{C}_C$, then $d_C(F) = |F|$.
(d) If a subset F of C is deficient on C, then $d_C(F + e) = d_C(F)$ for any edge e of C.

Proof. Statements (a) to (c) can be easily proved by induction on the level of C. Statement (d) follows from Statement (a). □

For a class C of $\mathcal{C}$, we define a family $\mathcal{I}_C$ of subsets I of C by

$$\mathcal{I}_C := \{I \subseteq C \mid d_{C'}(I \cap C') \leq u(C') \text{ for any } C' \in \mathcal{C}_C\}.$$

Our next goal is to prove that $\mathcal{M}_C = (C, \mathcal{I}_C)$ is a matroid for any class C of $\mathcal{C}$.

Lemma 4.15. *If C is a class of $\mathcal{C}$, $I, J \in \mathcal{I}_C$ and $|I \cap C'| \geq |J \cap C'|$ for any class C' of $\mathcal{C}_C$ on which $I \cap C'$ is deficient, then $d_C(J) - d_C(I) \geq |J| - |I|$.*

Proof. We prove the lemma by induction on the level of C. If the level of C is one, that is, if C is a singleton then the lemma is straightforward. Assume that the lemma holds for any class with level at most r for some $r \geq 1$ and take a class C of level $r+1$. If I is deficient on C, then $|I| \geq |J|$ by the condition in the lemma. So,

$$d_C(J) - d_C(I) = d_C(J) - l(C) \geq l(C) - l(C) = 0 \geq |J| - |I|,$$

where the first equality is due to that I is deficient on C.

Suppose that I is not deficient on C. Let $C_1, \ldots, C_k$ be the children of C, $I_i = I \cap C_i$ and $J_i = J \cap C_i$. For any class C' of $\mathcal{C}_{C_i}$, $I \cap C' = I_i \cap C'$ and $J \cap C' = J_i \cap C'$. So, $I_i, J_i \in \mathcal{I}_{C_i}$ by $I, J \in \mathcal{I}_C$. Moreover, by assumption, $|I_i \cap C'| \geq |J_i \cap C'|$ holds for any class C' of $\mathcal{C}_{C_i}$ on which $I_i \cap C'$ is deficient. So, by the induction hypothesis, $d_{C_i}(J_i) - d_{C_i}(I_i) \geq |J_i| - |I_i|$. Thus,

$$\begin{aligned} d_C(J) - d_C(I) = d_C(J) - \sum_{i \in [k]} d_{C_i}(I_i) &\geq \sum_{i \in [k]} d_{C_i}(J_i) - \sum_{i \in [k]} d_{C_i}(I_i) \\ &\geq \sum_{i \in [k]} |J_i| - \sum_{i \in [k]} |I_i| = |J| - |I|, \end{aligned}$$

where the first equality follows from the fact that I is not deficient on C. ☐

Lemma 4.16. *If C is a class of $\mathcal{C}$, $I, J \in \mathcal{I}_C$ and $|I| < |J|$, then $I + e \in \mathcal{I}_C$ for some edge e of $J \setminus I$.*

Proof. We prove the lemma by induction on the level of C. If the level of C is one, then the lemma is straightforward as C is a singleton. Assume that the lemma holds if the level of C is at most r for some $r \geq 1$, and take a class C of level $r+1$.

Case 1. $|I \cap C^*| < |J \cap C^*|$ for some class C^* of $\mathcal{C}_C$ on which $I \cap C^*$ is deficient. Let $I^* = I \cap C^*$ and $J^* = J \cap C^*$. By $I, J \in \mathcal{I}_C$, we have $I^*, J^* \in \mathcal{I}_{C^*}$. So, by the induction hypothesis, $I^* + e^* \in \mathcal{I}_{C^*}$ for some edge e^* of $J^* \setminus I^*$. Since I^* is deficient on C^*, $d_{C^*}(I^* + e^*) = d_{C^*}(I^*)$ by Lemma 4.14(d). From this, we shall prove that $I + e^* \in \mathcal{I}_C$.

Let $L = I + e^*$ and $L' = L \cap C'$ for a class C' of $\mathcal{C}_C$. It suffices to prove that $d_{C'}(L') \leq u(C')$ for any class C' of $\mathcal{C}_C$. If $C' \cap C^* = \emptyset$, then this holds by $e^* \notin C'$ and $I \in \mathcal{I}_C$. If $C' \in \mathcal{C}_{C^*}$, then this holds by $I^* + e^* \in \mathcal{I}_{C^*}$. If $C^* \subseteq C'$, then this follows from $d_{C^*}(I^* + e^*) = d_{C^*}(I^*)$ and the fact that $d_{C'}(L')$ does not change if $d_{C^\circ}(L' \cap C^\circ)$ does not change for any child C° of C'.

Case 2. Assume that $|I \cap C'| \geq |J \cap C'|$ for any class C' of $\mathcal{C}_C$ on which $I \cap C'$ is deficient. By Lemma 4.15, $d_C(J) - d_C(I) \geq |J| - |I| > 0$. This implies that

$$d_C(I + e) \leq d_C(I) + 1 \leq d_C(J) \leq u(C) \tag{4.3}$$

for any edge e of $J \setminus I$, where the first inequality follows from Lemma 4.14(a) and the third from $J \in \mathcal{I}_C$. Let $C_1, \ldots, C_k$ be the children of C, $I_i = I \cap C_i$ and $J_i = J \cap C_i$. By $I, J \in \mathcal{I}_C$, we have $I_i, J_i \in \mathcal{I}_{C_i}$. Let N be the set of $i \in [k]$ such that $|I_i| < |J_i|$. Notice that $N \neq \emptyset$ by $|I| < |J|$. By the induction hypothesis, for any $i \in N$ there is an edge e_i of $J_i \setminus I_i$ such that $I_i + e_i \in \mathcal{I}_{C_i}$. So, by (4.3), $I + e_i \in \mathcal{I}_C$ for any $i \in N$. This completes the proof. ☐

We are now ready to prove a key observation for Theorem 4.10

Lemma 4.17. *For any class C of $\mathcal{C}$, $\mathcal{M}_C = (C, \mathcal{I}_C)$ is a matroid.*

Proof. By the first inequality of (4.2), $d_{C'}(\emptyset) = l(C')$ for any class C' of $\mathcal{C}_C$. So, by the second inequality of (4.2), $\emptyset \in \mathcal{I}_C$, i.e., $\mathcal{I}_C \neq \emptyset$. Furthermore, the independence axioms follow from Lemmata 4.14(b) and 4.16. □

In what follows, we describe our algorithm for the **2LCSM** problem. By Lemma 4.17, $\mathcal{M}_{E(v)}$ is a matroid for any vertex v of V. Let $\mathcal{M}_P = (E, \mathcal{I}_P, <_P)$ be an ordered matroid such that $(E, \mathcal{I}_P)$ is the direct sum of matroids $\mathcal{M}_{E(v)}$ for all vertices v of P and $<_P$ is a strict linear order defined in such a way that $e <_P f$ whenever $e <_v f$ for some vertex v of P. For the vertex class Q, we similarly define an ordered matroid $\mathcal{M}_Q = (E, \mathcal{I}_Q, <_Q)$. Then, our algorithm, called Algorithm_2LCSM, can be described as follows. Note that Step 1 of the algorithm is a natural generalization of the proposal algorithm of Gale and Shapley (with the choice function represented by the greedy algorithm), described in [20].

Algorithm_2LCSM

Step 1: Find an $\mathcal{M}_P\mathcal{M}_Q$-kernel K.

Step 2: If K obeys l for any class of $\mathcal{C}$, then we output K, i.e., K is a stable assignment. Otherwise, there is no stable assignment.

It is easy to see that Algorithm_2LCSM runs in polynomial time.
Our next goal is to prove the correctness of Algorithm_2LCSM. By Lemma 4.14(c),

$$\text{a subset } M \text{ of } E \text{ is feasible for a vertex } v \text{ of } V \text{ if and only if } M(v) \in \mathcal{I}_{E(v)} \text{ and } M \text{ obeys } l \text{ for any class of } \mathcal{C}_v. \tag{4.4}$$

The following lemma gives connects lu-stable matchings and $\mathcal{M}_P\mathcal{M}_Q$-kernels.

Lemma 4.18. *A subset M of E is a lu-stable matching if and only if M is an $\mathcal{M}_P\mathcal{M}_Q$-kernel and M obeys l for any class of $\mathcal{C}$.*

Proof. We first prove sufficiency. Let M be an $\mathcal{M}_P\mathcal{M}_Q$-kernel obeying l for any class of $\mathcal{C}$. By (4.4), M is an lu-matching. Let e be an edge of $E \setminus M$. Since M is an $\mathcal{M}_P\mathcal{M}_Q$-kernel, without loss of generality, we can assume that $e \in \mathcal{D}_{\mathcal{M}_P}(M)$. Let v be the endpoint of e in P. By the definition of $\mathcal{M}_P$, $M(v) + e \notin \mathcal{I}_{E(v)}$. So, by (4.4), $M + e$ is not feasible for v. Let F be the set of arcs f of $M(v)$ such that $M + e - f$ is feasible for v. Now we prove that $f <_v e$ for any edge f of F. By (4.4), $M(v) + e - f \in \mathcal{I}_{E(v)}$, i.e., f is an edge of the basic circuit of e with respect to $M(v)$ in $\mathcal{M}_{E(v)}$ (also, M in $\mathcal{M}_P$). Since M is an $\mathcal{M}_P\mathcal{M}_Q$-kernel, we have $f <_P e$. So, by the definition of $<_P$, we have $f <_v e$.

For the necessity, let M be a lu-stable matching. By (4.4), $M \in \mathcal{I}_P \cap \mathcal{I}_Q$ and M obeys l for any class of $\mathcal{C}$. Let e be an edge of $E \setminus M$. Since M is a lu-stable matching, e is not free for at least one endpoint v of e. Without loss of generality, we can assume that $v \in P$. Now we prove that $e \in \mathcal{D}_{\mathcal{M}_P}(M)$. Since $M + e$ is not feasible for v, $M(v) + e \notin \mathcal{I}_{E(v)}$. Let D be the basic circuit of e with respect to $M(v)$ in $\mathcal{M}_{E(v)}$ (also, M in $\mathcal{M}_P$). Now we prove that $f <_P e$ for any edge f of $D - e$. For this, we need the following claim.

Claim 4.19. *$M+e-f$ obeys l for any class of $\mathcal{C}_v$.*

Proof. Let $M_1 = M+e$ and $M_2 = M+e-f$. Since $M(v) \in \mathcal{I}_{E(v)}$ and $M_1(v) \notin \mathcal{I}_{E(v)}$, there is a class C of $\mathcal{C}_v$ such that $e \in C$ and $d_C(M_1 \cap C) > u(C)$. By $M_2 \in \mathcal{I}_{E(v)}$, we have $f \in C$. So, $|M_2 \cap C'| = |M \cap C'| \geq l(C')$ for any class C' of $\mathcal{C}_v$ such that $C \subsetneq C'$. Thus, it suffices to prove that M_2 obeys l for any class of $\mathcal{C}_C - C$. Assume that M_2 does not obey l for some class C^* of $\mathcal{C}_C - C$. Let $M_1^* = M_1 \cap C^*$ and $M_2^* = M_2 \cap C^*$. Since M_1 obeys l for C^*, we have $f \in C^*$. So, if we can prove that $d_{C^*}(M_2^*) = d_{C^*}(M_1^*)$, then $d_C(M_2 \cap C) = d_C(M_1 \cap C) > u(C)$, which contradicts the fact that $M_2(v) \in \mathcal{I}_{E(v)}$. Since M_2 does not obey l for C^* but M obeys l for C^*, we have $|M_1^*| = l(C^*)$. Moreover, since M_1 obeys l for any class of $\mathcal{C}_{C^*}$, we have $d_{C^*}(M_1^*) = |M_1^*|$ by Lemma 4.14(c). So,

$$l(C^*) \leq d_{C^*}(M_2^*) \leq d_{C^*}(M_1^*) = |M_1^*| = l(C^*),$$

where the second inequality follows from Lemma 4.14(b) and the fact that $M_2 \subseteq M_1$. This implies that $d_{C^*}(M_2^*) = d_{C^*}(M_1^*)$, which completes the proof. □

By Claim 4.19 and (4.4), $M+e-f$ is feasible for v. Since M is a lu-stable matching, we have $f <_v e$. So, by the definition of $<_P$, we have $f <_P e$. □

Lemma 4.20. *If an $\mathcal{M}_P\mathcal{M}_Q$-kernel K does not obey l for a class C of $\mathcal{C}_P$ but K obeys l for any class of $\mathcal{C}_C - C$, then $\mathrm{span}_{\mathcal{M}_P}(K) \cap C = \mathrm{span}_{\mathcal{M}(C)}(K \cap C)$.*

Proof. Obviously,

$$\mathrm{span}_{\mathcal{M}(C)}(K \cap C) \subseteq \mathrm{span}_{\mathcal{M}_P}(K) \cap C.$$

To prove the opposite direction, let e be an edge of $(\mathrm{span}_{\mathcal{M}_P}(K) \cap C) \setminus K$ and $L = K+e$. By the definition of e, $d_{C^*}(L \cap C^*) > u(C^*)$ for some class C^* of $\mathcal{C}_P$. Recall that K obeys l for any class of $\mathcal{C}_C - C$. So, if C has children $C_1, \ldots, C_k$, then

$$d_{C_1}(K \cap C_1) + \cdots + d_{C_k}(K \cap C_k) = |K \cap C|$$

by Lemma 4.14(c). Thus, since K does not obey l for C, $K \cap C$ is deficient on C. So, $d_C(L \cap C) = d_C(K \cap C)$ by Lemma 4.14(d). This implies that $d_{C'}(L \cap C') = d_{C'}(K \cap C')$ for any class of C' of $\mathcal{C}_P$ such that $C \subseteq C'$. So, by $K \in \mathcal{I}_P$, we have $C^* \in \mathcal{C}_C$, i.e., $e \in \mathrm{span}_{\mathcal{M}(C)}(K \cap C)$. □

Lemma 4.21. *If an $\mathcal{M}_P\mathcal{M}_Q$-kernel K does not obey l for some class C of $\mathcal{C}$ but K obeys l for each class of $\mathcal{C}_C - C$, then no $\mathcal{M}_P\mathcal{M}_Q$-kernel obeys l for C.*

Proof. Without loss of generality, we can assume that $C \in \mathcal{C}_P$. For any $\mathcal{M}_P\mathcal{M}_Q$-kernel L,

$$L \cap C \subseteq \mathrm{span}_{\mathcal{M}_P}(L) \cap C = \mathrm{span}_{\mathcal{M}_P}(K) \cap C = \mathrm{span}_{\mathcal{M}(C)}(K \cap C),$$

where the first equality follows from Theorem 2.9 and the second from Lemma 4.20. Since $K \cap C$ and $L \cap C$ are independent sets of $\mathcal{I}_C$, $|L \cap C| \leq |K \cap C| < l(C)$ holds. □

The above proof of Lemma 4.21 implies the correctness of Algorithm_2LCSM and concludes the proof of Theorem 4.10.

Note that Theorem 4.10 can be extended to generalized (poly)matroids, see Yokoi [69].

Chapter 5

Stable matching polyhedra

For a given weight function on the edges, it is a natural question how to find a maximum weight stable matching, that is, a stable matching with maximum sum of weights of its edges. A standard approach is to optimize over the polyhedron spanned by the characteristic vectors of stable matchings. This can be done if a linear characterization of this polyhedron is available. The first step towards this direction was made by Vande Vate who described such a linear description in case of complete bipartite graphs [68]. This was followed by Rothblum who gave a linear characterization for arbitrary bipartite graphs [61], and then Baïou and Balinski came up with a linear description of the stable b-matching polytope (with an exponential number of constraints) for the special case when bound b is constant 1 on one part of the graph. In this section, we give a linear charcterization of the stable b-matching polyhedron for general b. Furthermore, as an extension, we describe various $\mathcal{FG}$-kernel polyhedra for increasing substitutable choice functions. In particular, we provide a linear description of matroid-kernel polyhedra.

5.1 Stable b-matching polyhedra

Our first focus is the stable b-matching problem related polyhedron. Recall that the stable b-matching problem is the generalization of the stable admissions problem in which agents on both sides of the market may have a quota greater than one. In what follows, we fix some terminology. A *bipartite preference system* is a pair $(G, \mathcal{O})$ where $G = (U \cup V, E)$ is a finite bipartite graph with bipartition (U, V), and $\mathcal{O} = \{\leq_z : z \in U \cup V\}$ is a family of linear orders, $\leq_z$ being an order on the set $E(z)$ of edges incident with the vertex z. We denote by $P^b(G, \mathcal{O})$ the convex hull in $\mathbb{R}^E$ of characteristic vectors of stable b-matchings of bipartite preference system $(G, \mathcal{O})$. As usual in linear programming, we denote by $\widetilde{x}(S)$ the sum $\sum\{x(e) : e \in S\}$ for a vector $x \in \mathbb{R}^E$ and subset S of E.

Theorem 5.1 (Vande Vate '89 [68]). *Let $(G, \mathcal{O})$ be a bipartite preference system with $|U| = |V|$ and $E = U \times V$. Then*

$$P^{\mathbf{1}}(G, \mathcal{O}) = \{x \in \mathbb{R}^E : x \geq \mathbf{0},\ \widetilde{x}(E(z)) = 1\ \forall\ z \in U \cup V,\ \widetilde{x}(\psi(e)) \leq 1\ \forall\ e \in E\}$$

where $\psi(uv) := \{f \in E : f \geq_u uv \text{ or } f \geq_v uv\}$. □

Clearly, the right hand side of the description is a convex polytope that contains the left hand side, that is any characteristic vector χ^M of a stable matching M: a characteristic vector is obviously nonnegative; for each vertex v of G, there is exactly one edge of M that is incident with v; and at last, if for an edge e of E there are at least 2 different edges of M that are less preferred than e in some of the preferences, then e is a blocking edge, hence M is not stable. The more difficult part of Vande Vate's result is to prove that any vector of the right hand side is a convex combination of characteristic vectors of stable matchings.

Rothblum gave a shorter proof of a modified description for a more general problem in [61], and his proof was further simplified by Roth *et al.* in [58].

Theorem 5.2 (Rothblum '92 [61]). *Let $(G, \mathcal{O})$ be a bipartite preference system. Then*

$$P^{\mathbf{1}}(G, \mathcal{O}) = \{x \in \mathbb{R}^E : x \geq \mathbf{0},\ \widetilde{x}(E(z)) \leq 1\ \forall\ z \in U \cup V,\ \widetilde{x}(\phi(e)) \geq 1\ \forall\ e \in E\}$$

where $\phi(uv) := \{f \in E : f \leq_u uv \text{ or } f \leq_v uv\}$. □

The third type condition in Rothblum's characterization is the linear relaxation of the condition that for any edge e of G, either e belongs to stable matching M or there is an edge of M that dominates e.

Based on the above results, standard tools of linear programming allow us to find a maximum weight stable matching in polynomial time. Eventually, a linear programming approach has been developed to the theory of stable matchings by Abeledo, Blum, Roth, Rothblum, Sethuraman, Teo and others (see [3, 4, 1, 2, 58, 66]). However, these results handle only the stable matching problem and do not say much about stable b-matchings. The following theorem of Baïou and Balinski [8] is an exception as it gives a linear description of the stable admissions polytope and generalizes Theorem 5.2.

Theorem 5.3 (Baïou and Balinski 2000 [8]). *Let $(G, \mathcal{O})$ be a bipartite preference system and $b : U \cup V \to \mathbb{N}$ be a quota function so that $b(u) = 1$ for all vertices u of U. Then*

$$\begin{aligned} P^b(G, \mathcal{O}) = \{x \in \mathbb{R}^E : & x \geq \mathbf{0}, \\ & \widetilde{x}(E(u)) \leq 1\ \forall\ u \in U,\ \widetilde{x}(E(v)) \leq b(v)\ \forall\ v \in V, \\ & \widetilde{x}(C(v, u_1, u_2, \ldots, u_{b(v)})) \geq b(v) \\ & \text{for all combs } C(v, u_1, u_2, \ldots, u_{b(v)})\}\ , \end{aligned}$$

where a comb is defined for $v \in V$ and $vu_1 >_v vu_2 >_v \ldots >_v vu_{b(v)}$ as

$$\begin{aligned} C(v, u_1, u_2, \ldots, u_{b(v)}) = & \{uv \in E : uv \leq_v u_1 v\} \cup \\ & \cup \{u_i v' \in E : u_i v' \leq_{u_i} u_i v \text{ for some } i = 1, 2, \ldots, b(v)\}\ . \end{aligned}$$ □

By Theorem 5.3, if a nonnegative vector x on the edges of G satisfies certain conditions, then it is in the stable admissions polytope, hence it can be decomposed as a convex combination of characteristic vectors of stable b-matchings. These conditions are that the total sum of its coordinates on edges incident with agent u of U is at most 1, the coordinate sum along agent v of V is at most $b(v)$. At last, no matter how we pick an agent v of V and possible partners $u_1, u_2, \ldots u_{b(v)}$ of v such that u_1 is the worst of these partners, if we sum up the coordinates of x on those edges of v that are not $<_v$-worse than vu_1 and we add the coordinate sum of those edges of u_i that are $<_{u_i}$-better than vu_i (for $i = 1, 2, \ldots, b(v)$) then the sum is at least $b(v)$. This latter condition corresponds to the fact that for any stable b-matching M, edge u_1v is either dominated at vertex v, and hence M contains $b(v)$ edges adjacent to v that are preferred by v to u_1v or u_1v is dominated at v and hence all u_iv's are dominated at v by M, so we find one edge m_i of M for each u_i such that agent u_i prefers partnership along m_i than with v.

The following theorem gives a linear description for the stable b-matching polyhedron.

Theorem 5.4 (Fleiner [21]). *Let $(G, \mathcal{O})$ be a bipartite preference system and $b : U \cup V \to \mathbb{N}$ be a quota function. Then*

$$P^b(G,\mathcal{O}) \quad = \{x \in \mathbb{R}^E : \; x \geq 0, \qquad (5.1)$$
$$\widetilde{x}(E_i(v)) \leq 1 \quad \forall v \in V, \quad \forall 1 \leq i \leq b(v),$$
$$\widetilde{x}(\Phi_{i,j}(uv)) \geq 1 \quad \forall uv \in E \quad \forall 1 \leq i \leq b(u), \quad \forall 1 \leq j \leq b(v)\} \; ,$$

where $E_1(v), E_2(v), \ldots, E_{b(v)}(v)$ denotes subsets of $E(v)$ according to Theorem 3.4 and

$$\phi_{i,j}(uv) := \{uv\} \cup \{uv' : uv' >_u uv, uv' \in E_i(u)\} \cup \{u'v : u'v >_v uv, u'v \in E_j(v)\} \; .$$

That is, if a nonnegative vector x on the edges of G satisfies certain conditions then it is in the convex hull of characteristic vectors of stable b-matchings. These conditions are that the coordinate sum of x on the edges of any part of a star-partition is at most one, moreover, for any edge uv and any i, j, the coordinate sum on edge uv, together with the coordinate sum on the edges that are preferred to uv in i-th part of the star of u plus the coordinate sum on the edges that are preferred to uv in the jth part of v is at least one. This latter condition is the linear relaxation of the property that for any stable b-matching M and for any partnership uv either uv is realized in M or either u or v has the property that in each part of its partition contains an edge of M that is preferred to uv. Note that Theorem 5.4 is a genuine generalization of Theorem 5.2, as if $b \equiv \mathbf{1}$ then $\phi(u, v) = \phi_{1,1}(u, v)$ for any edge uv.

Proof of Theorem 5.4. For partitions in Theorem 3.4, the characteristic vector χ^M of any stable b-matching M will satisfy the right hand side of (5.1): $\chi^M \geq \mathbf{0}$; M contains at most one edge of $E_i(z)$; and for any edge e, either e belongs to M or it has an end vertex z so that for $1 \leq k \leq b(z)$ each $E_k(z)$ will contain an edge m of M with $e >_z m$. Hence the polyhedron described on the right hand side of (5.1) contains $P^b(G, \mathcal{O})$.

To justify the opposite containment, we shall decompose a vector x satisfying the right hand side of (5.1) as a convex combination of characteristic vectors of stable b-matchings. To do this, we need the following lemma.

Lemma 5.5. *Let x be a vector satisfying the right hand side of (5.1) and $uv \in E_i(u) \cap E_j(v)$. Then edge uv is the most preferred edge in $E_i(u) \cap \mathrm{supp}(x)$ if and only if uv is the least preferred edge of $E_j(v) \cap \mathrm{supp}(x)$.*

Proof. From $x(\phi_{i.j}(uv)) \geq 1$ and $x(E_j(v)) \leq 1$ it follows that if uv is the most preferred edge of $E_i(u) \cap \mathrm{supp}(x)$ then uv is the least preferred edge of $E_j(v) \cap \mathrm{supp}(x)$. This means that $\mathrm{supp}(x)$ intersects at least as many $E_j(v)$'s for $v \in V$ as many $E_i(u)$'s for $u \in U$. The same argument holds if we exchange the role of U and V, thus $\mathrm{supp}(x)$ intersects exactly as many $E_j(v)$'s as many $E_i(u)$'s. So the set of most preferred edges of $E_i(u) \cap \mathrm{supp}(x)$ for $u \in U$ and $1 \leq i \leq b(u)$ is the same as the set of least preferred edges of $E_j(v) \cap \mathrm{supp}(x)$ for $v \in V$ and $1 \leq j \leq b(v)$. ∎

Let x be a vector satisfying the right hand side of (5.1) and let M consist of the most preferred edges of sets $E_i(u) \cap \mathrm{supp}(x)$ for $u \in U$ and $1 \leq i \leq b(u)$. Let δ denote amount $\min\{x(m) : m \in M\}$. As $x - \delta\chi^M$ has a strictly smaller support than x has, to finish the proof by induction on $|\mathrm{supp}(x)|$, it is enough to show that M is a stable b-matching and that $x' := \frac{1}{1-\delta}(x - \delta\chi^M)$ satisfies the constraints in the right hand side of (5.1).

First we prove that M is a stable b-matching. By Lemma 5.5, M contains at most one edge from each $E_k(z)$ for $z \in U \cup V$ and $1 \leq k \leq b(z)$, hence M is indeed a b-matching. For domination, fix edge uv. If property uv is not dominated then there must be an integer i so that $1 \leq i \leq b(u)$ and there is no edge m of $M \cap E_i(u)$ with $m \leq_u uv$.

If $x(uv) > 0$ then choose j so that $uv \in E_j(v)$. From $x(\phi_{i,j}(uv)) \geq 1$ and $x(E_j(v)) \leq 1$, it follows that uv is the least preferred edge by v from $E_j(v) \cap \mathrm{supp}(x)$. Then by Lemma 5.5, uv is selected into M, so uv is not undominated by M.

Otherwise $x(uv) = 0$. For any $1 \leq j \leq b(v)$, we have $x(\phi_{i,j}(uv)) \geq 1$ and $x(E_j(v)) \leq 1$. This implies that $x(\{e \in E_j(v) : e <_v uv\}) = 1$ so set $E_j(v) \cap \mathrm{supp}(x)$ is not empty. Let m_j be the least preferred edge by v of $E_j(v) \cap \mathrm{supp}(x)$. As $m_j <_v uv$ for all j, edge uv is dominated by M.

It remains to check that x' satisfies the constraints of (5.1). By our choice of δ, vector x is nonnegative. As we have chosen one edge from each nonempty $E_k(z) \cap \mathrm{supp}(x)$ for all vertices z of G, condition $x'(E_i(z)) \leq 1$ holds for all vertices z. For the third type constraint, pick an edge uv of G and indices i, j with $1 \leq i \leq b(u)$ and $1 \leq j \leq b(v)$. If $u'v >_v uv$ for the $<_v$-worst edge $u'v$ of $\mathrm{supp}(x) \cap E_j(v)$ then

$$x'(\phi_{i,j}(uv)) \geq \frac{1}{1-\delta}(x(\phi_{i,j}(uv)) - \delta) \geq \frac{1-\delta}{1-\delta} = 1$$

holds. Otherwise let $u'v \in E_k(u')$. By Lemma 5.5, $u'v$ is the $<_{u'}$-best edge of $\mathrm{supp}(x) \cap E_k(u')$, so

$$\begin{aligned} x'(\phi_{i,j}(uv)) \geq & \frac{1}{1-\delta}(x(E_j(v) \cap \{e \in E : e \leq_v uv\}) - \delta) = \\ = & \frac{1}{1-\delta}(x(\phi_{i,k}(m)) - \delta) \geq \frac{1-\delta}{1-\delta} = 1 \ . \end{aligned}$$

□

Note that Theorem 5.2 is a special case of Theorem 5.4 and that Király and Pap proved that the linear description in Theorem 5.2 has total dual integrality property [47].

5.2 General kernel-polyhedra

It turns out that $\mathcal{FG}$-kernel polyhedra other than the stable b-matching polyhedron have a linear description that allows one to optimize in polynomial time. To formulate the results, we need some notation. Recall that for substitutable choice functions $\mathcal{F}, \mathcal{G} : 2^E \to 2^E$, $\mathcal{K}_{\mathcal{FG}}$ denotes the set of $\mathcal{FG}$-kernels, and let

$$\begin{aligned} \mathcal{A}_{\mathcal{FG}} &:= \{A \subseteq E : |A \cap K| \leq 1 \quad \forall K \in \mathcal{K}_{\mathcal{FG}}\} \text{ and} \\ \mathcal{B}_{\mathcal{FG}} &:= \{B \subseteq E : |B \cap K| \geq 1 \quad \forall K \in \mathcal{K}_{\mathcal{FG}}\} \end{aligned}$$

stand for the *blocker* and *antiblocker* of $\mathcal{FG}$-kernels. If $\mathcal{Z} \subseteq 2^E$ is a family of sets then $\mathcal{P}_{\mathcal{Z}} := \text{conv}\ \{\chi(Z) : Z \in \mathcal{Z}\}$ is the convex hull of the characteristic vectors of members of $\mathcal{Z}$. If $\mathcal{P} \subseteq \mathbb{R}^E_+$ is a polyhedron then

$$\begin{aligned} \mathcal{P}^{\uparrow} &:= \mathcal{P} + \mathbb{R}^E_+ \\ \mathcal{P}^{\downarrow} &:= (\mathcal{P} + \mathbb{R}^E_-) \cap \mathbb{R}^E_+ \end{aligned}$$

are the *dominant* and *submissive* polyhedra of $\mathcal{P}$. Polyhedron $P \subseteq \mathbb{R}^d_+$ is a *blocking type polyhedron* if $P = P^{\uparrow}$, and it is an *antiblocking type polyhedron* if $P = P^{\downarrow}$. We have the following result.

Theorem 5.6 (Fleiner [20]). *If $\mathcal{F}, \mathcal{G} : 2^E \to 2^E$ are w-increasing substitutable choice functions for strictly monotone and modular mapping $w : 2^E \to \mathbb{R}_+$ then*

$$\begin{aligned} \mathcal{P_{FG}}^{\uparrow} &= \{x \in \mathbb{R}^E : x \geq \mathbf{0},\ \widetilde{x}(B) \geq 1 \ \ \textit{for } B \in \mathcal{B_{FG}}\}\ , \qquad (5.2) \\ \mathcal{P_{FG}}^{\downarrow} &= \{x \in \mathbb{R}^E : x \geq \mathbf{0},\ \widetilde{x}(Z_{\mathcal{FG}}) = 0 \textit{ and } \widetilde{x}(A) \leq 1 \textit{ for any } A \in \mathcal{A_{FG}}\}\ , \qquad (5.3) \\ \mathcal{P_{FG}} &= \{x \in \mathbb{R}^E : x \geq 0, \quad \widetilde{x}(B) \geq 1 \quad \forall B \in \mathcal{B_{FG}}, \quad \widetilde{x}(A) \leq 1 \quad \forall A \in \mathcal{A_{FG}}\} \qquad (5.4) \end{aligned}$$

where $Z_{\mathcal{FG}} = E \setminus \bigcup \mathcal{K_{FG}}$ denotes the set of those elements of E that are not contained in any $\mathcal{FG}$-kernel.

It is interesting to observe that in the linear programming problems describing the polyhedra in Theorem 5.6 all the coefficients and constants are 0 or 1. It is also worth pointing out that each linear constraint in (5.1) is a special case of a constraint in (5.4). Our main tool to prove Theorem 5.6 is the well-known result of Hoffman and Schwartz.

To state the Hoffman-Schwartz theorem, a basic result on lattice polyhedra, we need to introduce the notion of a clutter. Fix a ground set X and a family $\mathcal{L}$ of subsets of X. A partial order $<$ on $\mathcal{L}$ is called *consistent* if $A \cap C \subseteq B$ holds for any members A, B, C of $\mathcal{L}$ with $A < B < C$. Family $\mathcal{L}$ is a *clutter* if there is a consistent lattice order $<$ on $\mathcal{L}$ with lattice operations $\wedge$ and $\vee$ such that $\chi^A + \chi^B = \chi^{A \wedge B} + \chi^{A \vee B}$ holds for any members A, B of $\mathcal{L}$.

Theorem 5.7 (Hoffman-Schwartz [43]). *Let $\mathcal{L} \subseteq 2^X$ be a clutter for consistent lattice order $<$ and lattice operations $\wedge, \vee$ and let $d : X \to \mathbb{N} \cup \{\infty\}$ be and arbitrary function. If $r : \mathcal{L} \to \mathbb{N}$ is submodular then system (5.5) below is TDI.*

$$\{x \in \mathbb{R}^X : 0 \leq x \leq d,\ x(A) \leq r(A) \textit{ for any } A \in \mathcal{L}\} \qquad (5.5)$$

If $r : \mathcal{L} \to \mathbb{N}$ is supermodular then system (5.6) below is TDI.

$$\{x \in \mathbb{R}^X : 0 \leq x \leq d,\ x(A) \geq r(A) \textit{ for any } A \in \mathcal{L}\}\square \qquad (5.6)$$

In Theorem 5.7, $r : \mathcal{L} \to \mathbb{N}$ is *submodular* if $r(A) + r(B) \geq r(A \wedge B) + r(A \vee B)$ holds for any $A, B \in \mathcal{L}$; r is *supermodular* if the reverse inequality is valid. We say that system $Ax \leq b$ of linear inequalities is *totally dual integral* (or *TDI*) if the dual of linear programming problem $\max\{cx : Ax \leq b\}$ has an integral optimum whenever c is an integral vector and a fractional optimum for the dual problem exists. From this property, it follows that system $Ax \leq b$ describes an integral polyhedron. By Lemma 3.6, $\mathcal{K_{FG}}$ is a clutter for $\preceq_{\mathcal{F}}$. Then application of Theorem 5.7 yields the following.

Theorem 5.8 (Fleiner [20]). *If $\mathcal{F}$ and $\mathcal{G}$ are w-increasing substitutable functions for a strictly monotone and modular mapping $w : 2^E \to \mathbb{R}_+$ then*

$$\mathcal{P}^{\uparrow}_{\mathcal{B}_{\mathcal{FG}}} = \{x \in \mathbb{R}^E : x \geq \mathbf{0} \text{ and } x(K) \geq 1 \text{ for any } K \in \mathcal{K}_{\mathcal{FG}}\} \text{ and} \quad (5.7)$$
$$\mathcal{P}^{\downarrow}_{\mathcal{A}_{\mathcal{FG}}} + \mathcal{C}_{Z_{\mathcal{FG}}} = \{x \in \mathbb{R}^E : x \geq \mathbf{0} \text{ and } x(K) \leq 1 \text{ for any } K \in \mathcal{K}_{\mathcal{FG}}\}, \quad (5.8)$$

holds where $\mathcal{C}_{Z_{\mathcal{FG}}} = \{x \in \mathbb{R}^E : x \geq 0, x(e) = 0 \quad \forall e \notin Z_{\mathcal{FG}}\}$ denotes the cone of $Z_{\mathcal{FG}}$.

Proof. Obviously, the polyhedra on the left hand side of (5.7-5.8) are the integer hulls of the polyhedra described by right hand sides.

By our Lemma 3.6, $\mathcal{K}_{\mathcal{FG}}$ is a clutter. Let $d(v) := \infty$ and $r(K) := 1$ for all $v \in X$ and $K \in \mathcal{K}_{\mathcal{FG}}$. Clearly, r is both sub- and supermodular. By Theorem 5.7, linear systems in (5.7,5.8) are TDI, hence the polyhedra on the right hand sides are integer. As the right hand side of (5.7) is a blocking type polyhedron, it is the dominant of the convex hull of its lowest integer points. Clearly, the support of any gridpoint x on the right hand side of (5.7) is a member of $\mathcal{B}_{\mathcal{FG}}$, hence the right hand side of (5.7) is contained $\mathcal{P}^{\uparrow}_{\mathcal{B}_{\mathcal{FG}}}$, and (5.7) follows.

Similarly, the right hand side of (5.8) is an antiblocking type polyhedron that contains $\mathcal{C}_{Z_{\mathcal{FG}}}$ and $\text{supp}(x) \setminus Z_{\mathcal{FG}} \in \mathcal{A}_{\mathcal{FG}}$ holds for each gridpoint of the right hand side of (5.8). This observation implies (5.8). □

In order to prove Theorem 5.6, we introduce some basic notions from the theory of blocking and antiblocking polyhedra. For a polyhedron $P \subseteq \mathbb{R}^d_+$

$$B(P) := \{x \in \mathbb{R}^d_+ : x^T y \geq 1 \text{ for all } y \in P\} \text{ and}$$
$$A(P) := \{x \in \mathbb{R}^d_+ : x^T y \leq 1 \text{ for all } y \in P\}$$

are the *blocking* and *antiblocking polyhedron* of P, respectively. As suggested by the name, if P is a polyhedron then both $A(P)$ and $B(P)$ are polyhedra. An important tool in handling blocking and antiblocking polyhedra is the following result of Fulkerson.

Theorem 5.9 (Fulkerson [32, 33, 34]). *If P is a blocking type polyhedron then $B(P)$ is a blocking type polyhedron and $P = B(B(P))$. If P is an antiblocking type polyhedron then $A(P)$ is an antiblocking type polyhedron and $P = A(A(P))$. Furthermore,*

$$B(\text{conv}\,\{x_1, x_2, \ldots, x_n\}^{\uparrow}) = \{y \in \mathbb{R}^d_+ : \quad y^T x_i \geq 1 \text{ for } i \in [n]\} \quad (5.9)$$
$$A(\text{conv}\,\{x_1, x_2, \ldots, x_n\}^{\downarrow} + \mathcal{C}_Z) = \{y \in \mathbb{R}^d_+ : \quad y^T x_i \leq 1 \text{ for } i \in [n] \text{ and } \widetilde{x}(Z) = 0\} \quad (5.10)$$

for any $n \in \mathbb{N}$, elements x_i ($i \in [n]$) of $\mathbb{R}^E_+$ and subset Z of E, where $Z = \{e \in E : x_i(e) = 0 \forall 1 \leq i \leq n\}$ and $\mathcal{C}_Z = \{x \in \mathbb{R}^E : x \geq 0, x(e) = 0 \quad \forall e \notin Z\}$ denotes the cone of Z. □

Proof of Theorem 5.6. By (5.7) and (5.9), $\mathcal{P}^{\uparrow}_{\mathcal{B}_{\mathcal{FG}}} = B(\mathcal{P}_{\mathcal{FG}}{}^{\uparrow})$. From Theorem 5.9, we get that $\mathcal{P}_{\mathcal{FG}}{}^{\uparrow} = B(\mathcal{P}^{\uparrow}_{\mathcal{B}_{\mathcal{FG}}})$, and (5.2) follows from (5.9). Similarly, $\mathcal{P}^{\downarrow}_{\mathcal{A}_{\mathcal{FG}}} = A(\mathcal{P}_{\mathcal{FG}}{}^{\downarrow})$ from (5.8) and (5.10). Theorem 5.9 implies that $\mathcal{P}_{\mathcal{FG}}{}^{\downarrow} = A(\mathcal{P}^{\downarrow}_{\mathcal{A}_{\mathcal{FG}}})$, so (5.3) follows from (5.10).

By Corollary 1.17, w is constant on $\mathcal{FG}$-kernels, say $w(K) = \alpha$ holds for each $\mathcal{FG}$-kernel K. Hence $\mathcal{P}_{\mathcal{FG}} = \{x \in \mathcal{P}_{\mathcal{FG}}{}^{\uparrow} : w(x) = \alpha\} = \{x \in \mathcal{P}_{\mathcal{FG}}{}^{\downarrow} : w(x) = \alpha\}$ follows from

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strict monotonicity of w. Consequently, $w(y) > \alpha$ holds for each $y \in \mathcal{P}_{\mathcal{FG}}{}^{\uparrow} \setminus \mathcal{P}_{\mathcal{FG}}$. As $w(x) \leq \alpha$ holds for each $x \in \mathcal{P}_{\mathcal{FG}}{}^{\downarrow}$, this follows that

$$\mathcal{P}_{\mathcal{FG}} = \mathcal{P}_{\mathcal{FG}}{}^{\uparrow} \cap \mathcal{P}_{\mathcal{FG}}{}^{\downarrow}$$
$$= \{x \in \mathbb{R}^X : x \geq \mathbf{0},\ \widetilde{x}(B) \geq 1\ \forall B \in \mathcal{B}_{\mathcal{FG}}, \widetilde{x}(Z_{\mathcal{FG}}) = 0 \text{ and } \widetilde{x}(A) \leq 1\ \forall A \in \mathcal{A}_{\mathcal{FG}}\}\ .$$

Let

$$P = \{x \in \mathbb{R}^X : x \geq \mathbf{0},\ \widetilde{x}(B) \geq 1\ \forall B \in \mathcal{B}_{\mathcal{FG}} \text{ and } \widetilde{x}(A) \leq 1\ \forall A \in \mathcal{A}_{\mathcal{FG}}\}$$

and let $X \in \mathcal{A}_{\mathcal{FG}} \cap \mathcal{B}_{\mathcal{FG}}$ be a transversal of $\mathcal{FG}$-kernels. Such an X exists by Theorem 3.7. Let $x \in P$ be arbitrary. As both $Y := X \setminus Z_{\mathcal{FG}}$ and $Y' := X \cup Z_{\mathcal{FG}}$ are transversals of $\mathcal{FG}$-kernels (i.e. $Y, Y' \in \mathcal{A}_{\mathcal{FG}} \cap \mathcal{B}_{\mathcal{FG}}$), we get $\widetilde{x}(Y) = \widetilde{x}(Y') = 1$, hence $\widetilde{x}(Z_{\mathcal{FG}}) = \widetilde{x}(Y') - \widetilde{x}(Y) = 1 - 1 = 0$ follows. Consequently, $P = \mathcal{P}_{\mathcal{FG}}$ and this proves the correctness of (5.4). □

An interesting question is, whether linear descriptions (5.2-5.4) are good characterizations, that is, whether the separation problem over these polyhedra can be solved efficiently. The answer is yes, and a possible way for $\mathcal{P}_{\mathcal{FG}}{}^{\uparrow}$ is explained in [25].

Finally, to contrast Theorem 5.6, we prove that it is NP-complete to decide whether in case of certain posets P and P', a particular element of their ground set can belong to some PP'-kernel or not. It means that unless P=NP, it is necessary to have some extra assumption (like the increasing property) beyond substitutability and path-independence to hope for a good characterization of the corresponding $\mathcal{FG}$-kernel polytope, $\mathcal{P}_{\mathcal{FG}}$.

Theorem 5.10 (Fleiner [25]). *If undirected graph $G = (V, E)$ and $k \in \mathbb{N}$ are given then it is possible to construct posets P and P' and an element s of their common ground-set X in time polynomial in $|V|$, such that s belongs to a PP'-kernel if and only if G contains an independent set of size k.*

Proof. We may assume $k \leq |V|$, otherwise the theorem is trivial. Let

$$X := \{s\} \cup \{a_j, a'_j : 1 \leq j \leq k\} \cup \{v_j, v'_j : v \in V, 1 \leq j \leq k\}.$$

Partial orders $<$ and $<'$ are determined by

$$a_j < s, \qquad u_j < v'_j, \qquad u_j < u'_l, \qquad w_l < v'_j, \qquad v_j < a'_j$$
$$a'_j <' s, \qquad u'_j <' v_j, \qquad u'_j <' u_l, \qquad w'_l <' v_j, \qquad v'_j <' a_j$$

for $1 \leq j \leq k, 1 \leq l \leq k$, $j \neq l$, $u, v, w \in V$, $u \neq v$ and $vw \in E$ or $v = w$.

If G has an independent set $I = \{i^1, i^2, \ldots, i^k\} \subseteq V$ of size k, then $S := \{s\} \cup \{i^j_j, i^{j\prime}_j : 1 \leq j \leq k\}$ is a PP'-kernel. On the other hand, if s belongs to a PP'-kernel S then neither a_j, nor a'_j can belong to S. Thus for every j there must exist elements i^j and e^j of V such that $i^j_j, e^{j\prime}_j \in S$. By the kernel property, $i^j = e^j \neq i^l$ and $i^j i^l \notin E$ for $j \neq l$, in other words $I := \{i^1, i^2, \ldots, i^k\}$ is an independent set of G of size k. □

It is worth mentioning that the above results motivated further work on stable matching related polyhedra [48, 17, 38].

Chapter 6

Stability of network flows

As well-known, the problem of finding a maximum size matching in a bipartite graph G can be formulated as a network flow problem: introduce terminals s and t, draw an arc from s to each vertex of one part (say A) of G while sending an arc from each vertex of the other part (say B) to t and orient each edge of G from A to B. If we assign capacity 1 to each arc of the digraph then matchings of G bijectively correspond to integral flows of the just constructed network in such a way that the size of a matching coincides with the value of the corresponding flow. Hence the search for a maximum size matching reduces to the well-known problem of finding an integral flow of maximum value.

It is a natural question whether the network flow model has an extension that contains the stable matching problem exactly as the maximum flow problem contains the maximum size matching problem. The answer to this question is positive and we work out the details below.

Let (D, s, t, c) be a network, i.e. D is a digraph, s and t are different vertices (*terminals*) in D and $c : A(D) \to \mathbb{R}_+$ is a nonnegative capacity function on the arcs of D. As usual, a *feasible flow* is a function $f : A(D) \to \mathbb{R}_+$ that satisfies the capacity condition (i.e. $f \leq c$) and obeys the Kirchhoff rule, that is, for any nonterminal vertex v the net inflow into v is the same as the net outflow from v. Assume further that $\preceq_v$ is a linear order on set $A(v)$ of arcs incident with v, such that v prefers e to f if $e \preceq_v f$ holds. (In fact, we only need a linear order on the arcs entering v and another one on the arcs leaving v as we never have to compare an arc entering v to another one leaving v.) We say that walk $v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_{k+1}$ *blocks* flow f if

$$e_i = v_i v_{i+1} \text{ holds for each } 1 \leq i \leq k \text{ and} \tag{6.1}$$

$$\text{arcs } e_1, e_2, \ldots, e_k \text{ are pairwise different and} \tag{6.2}$$

$$f(e_i) < c(e_i) \text{ holds for } 1 \leq i \leq k \text{ (i.e. } e_i \text{ unsaturated) and} \tag{6.3}$$

$$v_1 \in \{s, t\} \text{ or there is an arc } e = v_1 x \text{ such that } f(e) > 0 \text{ and } e_1 \prec_{v_1} e, \text{ and at last} \tag{6.4}$$

$$v_{k+1} \in \{s, t\} \text{ or there is some arc } e = x v_{k+1} \text{ such that } f(e) > 0 \text{ and } e_k \prec_{v_{k+1}} e. \tag{6.5}$$

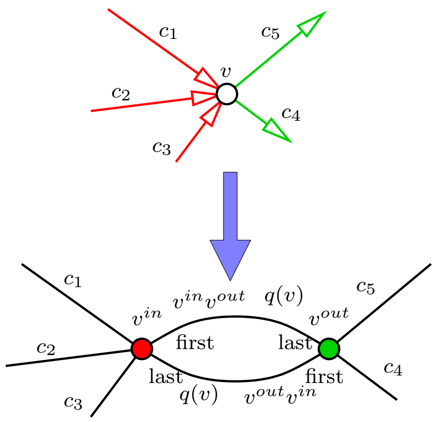
Feasible flow f is *stable* if no walk blocks f. The above model can be motivated as follows. Nonterminal vertices of D represents tradesmen who buy and sell a certain kind of product. Arcs represent possible trades and capacities give an upper bound on the amount of the product traded in between the two players. A feasible trading scheme on the above market can be described by a feasible flow as each tradesman sells the same amount of product as he buys. If tradesman u picks arc uv rather than arc uw (that is, if $uv \prec_u uw$) then this means that u prefers to sell to v rather than to w, hence he is eager to decrease the amount he sells to w in order to increase the amount sold to v if this is possible. Of course, this is possible only if v can sell the extra amount further to some other player, and so on. In this story, a walk blocking a feasible flow f means that the two terminals of the walk would be eager to reroute some of their trades onto the path while intermediate vertices are better off by trading more. This way, a blocking walk causes some kind of instability on the trading scheme described by f as the players involved in the walk have a joint interest to divert from the trading scheme along the walk. Hence a stable flow describes a situation where there is no coalition of players have a common will to divert.

If digraph D consists of a source s, a set A of sellers, a set B of buyers and a sink t and there is an arc from s to each vertex of A and from each vertex of B to t and certain further arcs from A to B and all capacities are 1 then stable flows correspond bijectively to stable matchings in the graph we get from the unoriented version of D by removing s and t (and keeping the preferences, of course).

Theorem 6.1 (Fleiner [23]). *For any network (D, s, t, c) and for any linear preferences $\preceq_v$ on $A(v)$ for each vertex v of D there exists a stable flow. If capacity c takes integral values on the arcs then there exists an integral stable flow.*

With the help of the given instance of the stable flow problem, we shall define a particular instance of the stable allocation problem. For each vertex v of D calculate

$$M(v) := \max\left(\sum_{x:xv\in A(D)} c(xv), \sum_{x:vx\in A(D)} c(vx)\right),$$



that is, $M(v)$ is the maximum of total capacity of those arcs of D that enter and leave v. So $M(v)$ is an upper bound on the amount of flow that can flow through vertex v. Choose $q(v) := M(v) + 1$. Construct graph G_D as follows. Split each vertex v of D into two distinct vertices v^{in} and v^{out}, and for each arc uv of D add edge $u^{out}v^{in}$ to G_D.

For each nonterminal vertex v of D, add two parallel edges between v^{in} and v^{out}: to distinguish between them, we will refer them as $v^{in}v^{out}$ and $v^{out}v^{in}$. Let $p(v^{in}v^{out}) = p(v^{out}v^{in}) := q(v)$, $p(u^{out}v^{in}) := c(uv)$ and $q(v^{in}) = q(v^{out}) := q(v)$. To finish the construction of the stable allocation problem, we need to fix a linear preference order for each vertex of G_D. For vertex v^{in}, let $v^{in}v^{out}$ be the most preferred and $v^{out}v^{in}$ be the least preferred edge (if these edges are present, that is, if v is nonterminal), and the order of the other edges incident to v^{in} are coming from the preference order of v on the corresponding arcs. For vertex v^{out}, the most preferred edge is $v^{out}v^{in}$ and the least preferred one is $v^{in}v^{out}$ (if these edges are present), and the other preferences are coming from $<_v$.

The proof of Theorem 6.1 is a consequence of Theorem 2.1 of Baïou and Balinski and the following Lemma 6.2 that describes a close relationship between stable flows and stable allocations.

Lemma 6.2 (Fleiner [23]). *If network (D, s, t, c) and preference orders $<_v$ describe a stable flow problem then $f : A(D) \to \mathbb{R}$ is a stable flow if and only if there is a stable allocation g of G_D such that $f(uv) = g(u^{out}v^{in})$ holds for each arc uv of D.*

Proof. Assume first that g is a stable allocation in G_D. This means that none of the $v^{in}v^{out}$ edges are blocking, so either $g(v^{in}v^{out}) = p(v^{in}v^{out}) = q(v)$ or $v^{in}v^{out}$ must be g-dominated at v^{out}, hence v^{out} is assigned to $q(v^{out}) = q(v)$ amount of allocation. As $q(v)$ is more than the total capacity of arcs leaving v, $g(v^{in}v^{out}) > 0$ or $g(v^{out}v^{in}) > 0$ must hold. So v^{out} must have exactly $q(v)$ amount of allocation whenever $v^{in}v^{out}$ is present. An exchange of "*in*" and "*out*" shows that the presence of $v^{out}v^{in}$ implies that v^{in} has exactly $q(v^{in}) = q(v)$ allocation. These observations directly imply that the Kirchhoff law holds for f at each vertex different from s and t. The capacity condition is also trivial for f, hence f is indeed a flow of D. Observe that by the choice of q, neither s nor t is g-saturated hence no edge is g-dominated at s or at t.

Assume that walk $P = (v_1, v_2, \ldots, v_k)$ blocks flow f. As P is f-unsaturated, each edge $v_i^{out}v_{i+1}^{in}$ of G_D must be g-dominated at v_i^{out} or at v_{i+1}^{in}. Walk P is blocking, hence either v_1 is terminal, and hence $v_1^{out}v_2^{in}$ cannot be dominated at v_1 or there is a v_1u arc with positive flow value such that $v_1u >_{v_1} v_1v_2$. In both cases, edge $v_1^{out}v_2^{in}$ has to be g-dominated at v_2^{in}. It means that $g(v_2^{in}v_2^{out}) > 0$. As arc v_2v_3 is f-unsaturated, it follows that edge $v_2^{out}v_3^{in}$ must be g-dominated at v_3^{in}. This yields that $g(v_3^{in}v_3^{out}) > 0$. Again, arc v_3v_4 is f-unsaturated, hence edge $v_3^{out}v_4^{in}$ has to be g-dominated at v_4^{in}, and so on. At the end we get that $v_{k-1}^{out}v_k^{in}$ is g-dominated at v_k^{in}. As terminal vertices s and t are g-unsaturated, v_k cannot be a terminal vertex. So by the blocking property of P, there is an arc wv_k with positive flow and $v_{k-1}v_k <_{v_k} wv_k$, hence again, $v_{k-1}^{out}v_k^{in}$ cannot be g-dominated at v_k^{in}. The contradiction shows that no walk P can block f.

Assume now that f is a stable flow of D. We have to exhibit a stable allocation g of G_D such that f is the "restriction" of g. Define $g(u^{out}v^{in}) := f(uv)$, so we only need to determine the $g(v^{in}v^{out})$ and $g(v^{out}v^{in})$ values for all nonterminal vertices v. Actually, the stable allocation we look for might not be unique. In what follows, we shall construct a particular one, the so-called *canonical representation* g_f of f.

Let S be the set of those vertices u of D such that there exists an f-unsaturated directed walk $P = (v_1, v_2, \ldots, v_k = u)$ that is not f-dominated at v_1. As no walk can block f, S is disjoint from terminal vertices s, t. To determine g_f, for each nonterminal vertex v allocate the remaining quota of v to $v^{in}v^{out}$ or to $v^{out}v^{in}$ depending on whether $v \in S$ or $v \notin S$ holds. More precisely, define

$$g_f(v^{in}v^{out}) = \begin{cases} q(v) - \sum_{x \in V(D)} f(vx) & \text{if } v \in S \\ 0 & \text{if } v \notin S \end{cases} \quad \text{and} \tag{6.6}$$

$$g_f(v^{out}v^{in}) = \begin{cases} q(v) - \sum_{x \in V(D)} f(xv) & \text{if } v \notin S \\ 0 & \text{if } v \in S\ . \end{cases} \tag{6.7}$$

By the definition of q, both $g_f(v^{in}v^{out})$ and $g_f(v^{out}v^{in})$ are nonnegative. If $v \in S$ then the amount of total allocation of v^{out} is $q(v) = q(v^{out})$ by (6.6), and for $v \notin S$ the amount of total allocation of v^{in} is $q(v) = q(v^{in})$ by (6.7). So if $v \neq s, t$ then the total allocation of v^{in} and v^{out} is $q(v)$ by the Kirchhoff law. The total allocations of s^{in}, s^{out} and t^{in}, t^{out} is less than $q(s)$ and $q(t)$ respectively, by the choice of q. That is, g_f is an allocation on G_D.

To justify the stability of g_f, we have to show that no blocking edge exists. To see that neither $v^{in}v^{out}$ nor $v^{out}v^{in}$ is blocking, observe that both v^{in} and v^{out} are saturated in g_f. So $v^{in}v^{out}$ is dominated at v^{out} and $v^{out}v^{in}$ is dominated at v^{in}. Assume now that $g_f(v^{out}u^{in}) < p(v^{out}u^{in}) = c(vu)$ holds. Our goal is to prove that $v^{out}u^{in}$ is not blocking.

If there is an f-unsaturated walk P ending with arc vu that is not f-dominated at its starting vertex then $u \in S$ by the definition of S, hence $g_f(u^{out}u^{in}) = 0$. Moreover, if some edge $w^{out}u^{in}$ with $v^{out}u^{in} <_{u^{in}} w^{out}u^{in}$ would have positive allocation then walk P would block f, a contradiction. As u^{in} has $q(u^{in})$ amount of total allocation, edge $v^{out}u^{in}$ is g_f-dominated at u^{in}.

The last case is when any f-unsaturated walk that ends with arc vu is f-dominated at its starting vertex. In particular, $v \notin S$, so $g_f(v^{in}v^{out}) = 0$. Moreover, f-unsaturated walk (v, u) must be f-dominated at v, hence $v \notin \{s, t\}$ and $v^{out}u^{in}$ is g_f-dominated at v^{out} as v^{out} has $q(v) = q(v^{out})$ amount of allocation. The conclusion is that $g := g_f$ is a stable allocation, just as we claimed. □

At this point, we are ready to prove our result on stable flows.

Proof of Theorem 6.1. There is a stable allocation for G_D by Theorem 2.1, hence there is a stable flow for D due to the first part of Lemma 6.2. If c is integral then $q(v)$ is an integer for each vertex v of D hence p is integral for G_D. The integrality property of stable allocations in the first part of Theorem 5.3 shows that there is an integral stable allocation g of G_D that describes an integral stable flow f of D. □

A stable flow can also be constructed by an appropriate generalization of the Gale-Shapley algorithm and there can be several different stable flows on the same network with preferences. However, unlike stable matchings, these stable flows do not form a lattice. As we will see, a certain lattice property can be defined as soon as each vertex is declared either to be a seller or a buyer. If f is a stable flow then some vertices can be clearly classified into one class, but for some others we are free to declare either possibility. However, the "Rural Hospitals Theorem" does extend as it has the following generalization for this model.

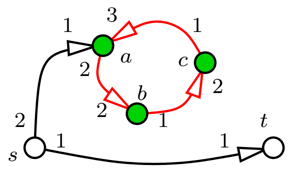
Theorem 6.3 (Fleiner [23]). *If f_1 and f_2 are stable flows in network (D, s, t, c) with linear vertex preferences $\preceq_v$ then $f_1(a) = f_2(a)$ holds for each arc a incident to terminal s or t. Consequently, $m_{f_1} = m_{f_2}$ holds, that is, any two stable flow has the same value.*

Proof. As the value of a flow is the net amount that leaves s, one can calculate it in G_D as the difference of total allocation of s^{out} and s^{in}. This means that the first part of the theorem directly implies the second one. So below we prove only the first part of Theorem 6.3.

Let g_1 and g_2 be the canonical representations of flows f_1 and f_2 defined in Lemma 6.2. As there is no edge between s^{out} and s^{in}, the definition of $q(s)$ implies that both s^{out} and s^{in} are g_1-unsaturated. Hence property (2.8) can hold neither for s^{in} nor for s^{out}. Moreover, Theorem 2.1 implies that g_1 and g_2 are $\leq_{s^{out}}$ and $\leq_{s^{in}}$-comparable. So property (2.7) must be true for flows g_1 and g_2 for both vertices $v = s^{out}$ and $v = s^{in}$. In particular, $g_1(a) = g_2(a)$ holds for each arc a incident to s. This shows the second part of the theorem for s. The argument for t is analogous to the above one. □

Let us point out a weakness of our stability concept. The motivation behind the notion is that we look for a flow that corresponds to an equilibrium situation where the players represented by the vertices of the network act in a selfish way. This equilibrium situation occurs if no coalition of the players can block the underlying flow f, and this blocking is defined by an f-unsaturated walk with a certain property. Along such a walk the players are capable and prefer to increase the flow. However, any closed f-unsaturated walk C per se causes some kind of instability because the players along C mutually agree to send some extra flow along C, even if properties (6.4) and (6.5) does not hold for vertex $v_1 = v_k$ of C. This motivates us to define flow f of network (D, s, t, c) with preferences to be *completely stable* if f is stable and there exists no f-unsaturated cycle in D whatsoever. If f is a stable flow then we can "augment" along f-unsaturated cycles, and hence we can construct a flow $f' \geq f$ such that there no longer exists an f'-unsaturated cycle. Unfortunately flow f' might not be stable any more because we might have created a blocking walk by the cycle-augmentations.

In fact, there exist networks with preferences that do not have a completely stable flow. One example is on the figure: each arc has unit capacity, preferences are indicated around the vertices: lower rank is preferred to the higher.
As no arc leaves subset $U := \{a, b, c\}$ of the vertices, no flow can leave U, hence no flow enters U. In particular, arc sa has zero flow. If we assume indirectly that f is a completely stable flow then cycle abc cannot block, hence there must be a unit flow along it. Consequently, walk sa is blocking, a contradiction. Let us mention the following related result.



Stable flows have a blocking cycle

Theorem 6.4 (Fleiner, Jankó, Schlotter, Teytelboym [28]). *The problem of deciding the existence of a completely stable flow is NP-complete.*

Next, we turn back to the lattice structure of stable flows. Assume that f is a stable flow in network $(D, s, t, c,)$ with preferences, and let stable allocation g_f of G_D be the canonical representation of f, as in the proof of Lemma 6.2.

Observe that any nonterminal vertex, v, of D, exactly one of $g_f(v^{in}v^{out})$ and $g_f(v^{out}v^{in})$ is positive by the choice of q and g_f. For stable flow f, we can classify the vertices of D different from s and t as follows: v is an f-*vendor* if $g_f(v^{in}v^{out}) > 0$, and v is an f-*customer* if $g_f(v^{out}v^{in}) > 0$. If v is an f-vendor, then no edge $v^{out}u^{in}$ can be g_f-dominated at v^{out} (as $g_f(v^{in}v^{out}) > 0$); hence, player v sends as much flow to other vertices as they accept. Similarly, if v is an f-customer, then no edge $u^{out}v^{in}$ can be g_f-dominated at v^{out}; that is, player v receives as much flow as the others can supply her.

To explore the promised lattice structure of stable flows, let f_1 and f_2 be two stable flows with canonical representations g_{f_1} and g_{f_2}, respectively. From Theorem 2.1, we know that stable allocations form a lattice; so, $g_{f_1} \vee g_{f_2}$ and $g_{f_1} \wedge g_{f_2}$ are also stable allocations of G_D, and by Theorem 6.1, these stable allocations define stable flows $f_1 \vee f_2$ and $f_1 \wedge f_2$, respectively. How can we determine these latter flows directly, without the canonical representations? To answer this, we translate the lattice property of stable allocations on G_D to stable flows of D.

Theorem 6.3 shows that stable flows cannot differ on arcs incident to terminal vertex s or t, so on these arcs, $f_1 \vee f_2$ and $f_1 \wedge f_2$ are determined. However, vertices different from s and t may have completely different situations in stable flows f_1 and f_2. The two color classes of graph G_D are formed by the v^{in}- and v^{out}-type vertices, respectively.

Therefore, by Theorem 2.1, $g_{f_1} \vee g_{f_2}$ can be determined, such that (say) each vertex, v^{out}, selects the better allocation and each vertex, v^{in}, receives the worst allocation out of the ones that g_{f_1} and g_{f_2} provide them. Similarly, for stable allocation $g_{f_1} \wedge g_{f_2}$, the "in"-type vertices choose according to their preferences, and the "out"-type ones are left with the less preferred allocations. This means the following in the language of flows. If we want to construct $f_1 \vee f_2$ and v is a vertex different from s and t, then either all arcs entering v will have the same flow in $f_1 \vee f_2$ as in f_1 or for all arcs a entering v, equation $(f_1 \vee f_2)(a) = f_2(a)$ holds. A similar statement is true for the arcs leaving v. To determine which of the two alternatives is the right one, the following rules apply:

- If v is an f_1-vendor and an f_2-customer, then v chooses f_2. If v is an f_2-vendor and an f_1-customer, then v chooses f_1. That is, each vertex strives to be a customer.
- If v is an f_1-vendor and an f_2-vendor and v transmits more flow in f_1 than in f_2 (*i.e.*, $0 < g_{f_1}(v^{in}v^{out}) < g_{f_2}(v^{in}v^{out})$) then v chooses f_1. That is, vendors prefer to sell more.
- If v is an f_1-customer and an f_2-customer and v transmits more flow in f_1 than in f_2 (*i.e.*, $0 < g_{f_1}(v^{out}v^{in}) < g_{f_2}(v^{out}v^{in})$), then v chooses f_2. That is, customers prefer to buy less.
- Otherwise, v is a customer in both f_1 and f_2 or v is a vendor in both flows, and v transmits the same amount in both flows (*i.e.*, $g_{f_1}(v^{out}v^{in}) = g_{f_2}(v^{out}v^{in})$ and $g_{f_1}(v^{in}v^{out}) = g_{f_2}(v^{in}v^{out})$). In this situation, v chooses the better "selling position" and gets the worse "buying position" out of stable flows f_1 and f_2.

Clearly, for the construction of $f_1 \wedge f_2$, one always has to choose the "other" options rather than the one that the above rules describe.

We finish the discussion of stable flows by mentioning that the notion of a stable flow has motivated further work [16, 15, 53].

The stable flow model is closely related to the stability of supply chain networks defined by Ostrovsky [55]. In Ostrovsky's model, vertices of an acyclic digraph represent individual players and arcs indicate possible trade between the players. Each vertex v is equipped with a choice function C_v that maps any subset X of arcs incident with v to subset $C_v(X)$ of X that v would like to pick for trade if she can decide freely. All these choice functions C_v must obey properties *SSS* és *CSC* (that stand for "same side substitutability" and "cross side complementarity") meaning that if e enters and f leaves v then

$$C_v^+(X \cup \{e\}) \subseteq C_v^+(X) \cup \{e\} \text{ és } C_v^-(X) \subseteq C_v^-(X \cup \{e\})$$

and

$$C_v^-(X \cup \{f\}) \subseteq C_v^-(X) \cup \{f\} \text{ és } C_v^+(X) \subseteq C_v^+(X \cup \{f\})$$

must hold where upperscipts $+$ and $-$ stand for incoming and outgoing arcs, respectively. So the above properties imply that if a new purchase opportunity pops up then the underlying player will sell along all arcs where she sold before and does not buy along all arcs where she did not buy before (but had the possibility to buy). A new selling opportunity means that all buying trades are realized along which v were buying before and no nonrealized selling opportunity gets realized. In a supply chain network, in case of preferences described by such choice functions an equilibrium is a subset S of arcs such that $C_v(S(v)) = S(v)$ holds for each player v and does not exists a *blocking chain*, that is a a directed path P that is arc-disjoint from S such that $P(v) \subseteq C_v(S(v) \cup P(v))$ holds for each vertex v of P. In his celebrated paper, Ostrovsky proved the following theorem.

Theorem 6.5 (Ostrovsky [55]). *For any acyclic digraph D with arbitrary choice functions C_v of the vertices with the SSS and CSC properties, there always exists an equilibrium.*

Ostrovsky's work motivated several new results (e.g. [39, 41, 40]). Note that Ostrovsky's model allows far more general choice functions than those that used in the stable flow model and obey the Kirchhoff law. However, in case of stable flows we do not require the acyclicity of the underlying graph and choice functions do not have to be discrete. A common generalization of the two results is a yet unpublished joint work of Fleiner, Jankó, Teytelboym and Tamura that also compares various stability notions on networks [29].

Chapter 7

Stable matchings and kernels in nonbipartite graphs

We have seen that in case of a bipartite graph, there exists a stable (b-)matching for any linear preferences. It is easy to see however that if in a 3-cycle each vertex prefers its right neighbour to the left one then no stable matching exists. It is natural to ask whether there is an efficient algorithm that finds a stable matching for a given input graph and linear vertex preferences or concludes that no stable matching exists. As Lemma 1.2 is valid also for nonbipartite graphs, it is a natural idea that our algorithm applies it as long as it is possible to remove an edges from the graph according to the lemma. As we saw in case of bipartite graphs, if we cannot remove any further edge then both the boys first choices and the girls first choices form a stable matching. This is not the case for nonbipartite graphs, however. From an instance where we cannot remove any further edge due to Lemma 1.2 it may be pretty unclear whether a stable matching exists or not. The first efficient algorithm that achieved this goal is due to Irving [45]. Irving's algorithm starts indeed with removing edges according to Lemma 1.2, but it needs a further operation, namely the so-called rotation elimination. In this transformation we also remove edges from the graph in such a way that no new stable matching is created (just like in case of Lemma 1.2). However, a rotation elimination may kill a stable matching. The key property of rotation elimination is that it cannot kill *all* stable matchings in the graph, hence even if we loose one (or more) there certainly remains at least one stable matching. Irving observed that if none of the above operations can be executed on a graph then the graph itself is either a matching (which is obviously a stable matching of the original input graph) or the graph has an odd cycle component that can be oriented such that each vertex of the cycle prefers the outgoing arc to the incoming one. As no such odd cycle has a stable matching, the original input graph has no stable matching either.

7.1 Fractional stable matchings

Using Irving's algorithm, Tan pointed out that in case of nonbipartite graphs, there always exists a so-called stable partition that we shall call a stable half-matching and define below [64]. This stable partition of Tan is either a matching (and hence a stable matching) or it is a succinct proof for the nonexistence of a stable matching. To be able to generalize this result, we need the following terminology.

Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph on vertex set V, that is, $E \subseteq V$ holds for each hyperedge $E \in \mathcal{E}$. Let $\preceq_v$ be a linear order in the set $\mathcal{E}(v)$ of hyperedges incident to vertex v. Set $\mathcal{E}'$ of hyperedges is a *matching* if hyperedges of $\mathcal{E}'$ are disjoint, i.e. $|\mathcal{E}'(v)| \leq 1$ holds for each vertex v of V. Nonnegative vector $x \in \mathbb{R}_+^{\mathcal{E}}$ is said to be a *fractional matching* if $\widetilde{x}(\mathcal{E}(v)) \leq 1$ hols for each vertex $v \in V$. If x is a fractional matching and $x(E) \in \{0, \frac{1}{2}, 1\}$ holds for each hyperedge $E \in \mathcal{E}$ then x is called a *half-matching* and if we have $x(E) \in \{0, 1\}$ for the hyperedges then x is an *integral matching*. (It is clear that integral matchings are exactly the characteristic vectors of matchings.) Fractional matching x is *stable* if each hyperede $E \in \mathcal{E}$ has a vertex v such that $\sum\{x(F) : F \preceq_v E\} \geq 1$ holds, that is, if hyperedges on v that are not worse than E fully cover v. It is easy to see that if x is a stable half-matching of G then $x^{-1}(1)$ is a matching in G and the components of $x^{-1}(\frac{1}{2})$ are cycles with cyclic preferences. In the above terminology, Theorem 1.1 can be stated such that each bipartite graph for any vertex preferences has a stable integral matching. Tan proved the following theorem.

Theorem 7.1 (Tan, [64]). *If $G = (V, E)$ is a finite graph and $\preceq_v$ is a linear preference order on $E(v)$ for each vertex v of V then G has a stable half-matching. Moreover, if x and y are stable half-matchings then $x^{-1}(\frac{1}{2})$ and $y^{-1}(\frac{1}{2})$ contain the same odd cycles. Consequently, G has a stable matching if and only if G has a stable half-matching such that $x^{-1}(\frac{1}{2})$ contains no odd cycle.*

Scarf's lemma known from Game Theory [63] allows us to prove the following generalization of Tan's theorem.

Theorem 7.2 (Aharoni, Fleiner [6]). *If $\mathcal{H} = (V, \mathcal{E})$ is a hypergraph and $\preceq_v$ is a linear order on $\mathcal{E}(v)$ for each vertex $v \in V$ then there exists a stable fractional matching of $\mathcal{H}$.*

If $\mathcal{H}$ is a graph then there exists a stable half-matching.

Before the proof of Theorem 7.2, we state our main tool.

Lemma 7.3 (Scarf [63]). *Let $n < m$ be positive integers, b a vector in $\mathbb{R}_+^n$. Also let $B = (b_{i,j}), C = (c_{i,j})$ be matrices of dimensions $n \times m$, satisfying the following three properties: the first n columns of B form an $n \times n$ identity matrix (i.e. $b_{i,j} = \delta_{i,j}$ for $i, j \in [n]$), the set $\{x \in \mathbb{R}_+^n : Bx = b\}$ is bounded, and $c_{i,i} < c_{i,k} < c_{i,j}$ for any $i \in [n]$, $i \neq j \in [n]$ and $k \in [m] \setminus [n]$.*

Then there is a nonnegative vector x of $\mathbb{R}_+^m$ such that $Bx = b$ and the columns of C that correspond to $\mathrm{supp}(x)$ form a dominating set, that is, for any column $i \in [m]$ there is a row $k \in [n]$ of C such that $c_{k,i} \leq c_{k,j}$ for any $j \in \mathrm{supp}(x)$.

Proof of Theorem 7.2. Let B be the incidence matrix of H, with the identity matrix adjoined to it at its left. Let C' be a $V \times E$ matrix satisfying the following two conditions:

(1) $c'_{v,e} < c'_{v,f}$ whenever $v \in e \cap f$ and $e <_v f$

(2) $c'_{v,f} < c'_{v,e}$ whenever $v \in f \setminus e$.

Let C be obtained from C' by adjoining to it on its left a matrix so that C satisfies the conditions of Lemma 7.3. Let x be a vector as in Lemma 7.3 for B and C, for $b = \mathbf{1}$. Define $x' = x|_E$, namely the restriction of x to E. Clearly, x' is a fractional matching. To see that it is dominating, let e be an edge of H. By the conditions on x, there exists a vertex v such that $c_{v,e} \leq c_{v,j}$ for all $j \in \mathrm{supp}(x)$. Since $c_{v,v} < c_{v,e}$ it follows that

$v \notin \text{supp}(x)$. Since $Bx = \mathbf{1}$ it follows that supp(x) contains an edge f containing v (otherwise $(Bx)_v = 0$). Since $c_{v,f} \geq c_{v,e}$ it follows by condition (2) above that $v \in e$. The condition $(Bx)_v = 1$ now implies that e is dominated by x at v.

If $\mathcal{H} = G = (V, E)$ is a graph then x is in the fractional matching matching polytope P of G, hence x is a convex combination of vertices of P. It is well-known that vertices of P are half-integral and it is straightforward to check that the vertices in the convex combination must also be stable fractional matchings. Consequently, if $\mathcal{H}$ is a graph then there exists a stable half-matching. □

7.2 Reduction of stable matching problems

It is worth mentioning that the reduction to Lemma 7.3 does not provide us with a polynomial time algorithm as the related problem is PPAD-complete. Hence the next natural question is whether there is an efficient algorithm to find a stable solution for various generalizations of the stable roommates problem, e.g. finding a stable b-matching if exists. Due to Cechlárová and Fleiner, there is an elementary transformation that can be used to reduce the problem of finding a stable b-matching to the problem of finding a stable matching [13]. Moreover, this work also extends Irving's algorithm to the stable b-matching problem and proves a generalization of Theorem 3.4 to nonbipartite graphs.

Theorem 7.4 (Ceclárová, Fleiner [13]). *For any finite graph $G = (V, E)$, for arbitrary linear orders $\preceq_v$ on $E(v)$ for each vertex $v \in V$ and for arbitrary quota function $b : V \to \mathbb{N}_+$ on the vertices, it is possible to construct finite graph G^b and preference orders on the stars in polynomial time such that any stable matching of G^b corresponds to a stable b-matching of G and for any stable b-matching M of G there is at least one stable matching M^b of G^b that M^b corresponds to M.*

Moreover, if G is bipartite then G^b is also bipartite.

Theorem 7.4 shows that as soon as we have an efficient algorithm that finds a stable matching (like Irving's) then we can apply it to find a stable b-matching, as well.

Proof. The construction of G^b is done in two steps. First, we achieve the so-called many-to-one property, that is, $b(u) = 1$ or $b(v) = 1$ for every edge uv. If G happens to have this property then we do nothing in the first step. Otherwise as long as there is an edge $e = uv \in E$ such that $b(u) > 1 < b(v)$ then we replace edge uv by a little graph shown in figure 7.1. That is, we introduce new vertices u_0^e, u_1^e, u_2^e and v_0^e, v_1^e, v_2^e all with quotas $b = 1$ and we introduce a cycle on them with cyclic preferences such that edges uu_0^e and vv_0^e are in between the cycle edges according to u_0^e and v_0^e and edges uu_0^e and vv_0^e take place of edge uv in the preference orders of u and v.

Observe that graph G' after the construction has one less edge than G with the property that both endvertices have quota more than one. Observe moreover that this construction preserves bipartiteness, that is G' is bipartite whenever G is bipartite. Observe at last that if M is a stable b-matching of G' and (say) $uu_0^e \in M$ then by stability $u_1^e v_2^e \in M$ and hence (again by stability) $v_0^e v \in M$. That is, either both of uu_0^e and vv_0^e belong to M or none of them.

This follows that if M' is a stable b-matching of G' then M' corresponds to a b-matching M of G: edge e will belong to M if $uu_0^e, vv_0^e \in M$. Moreover, it is straightfor-

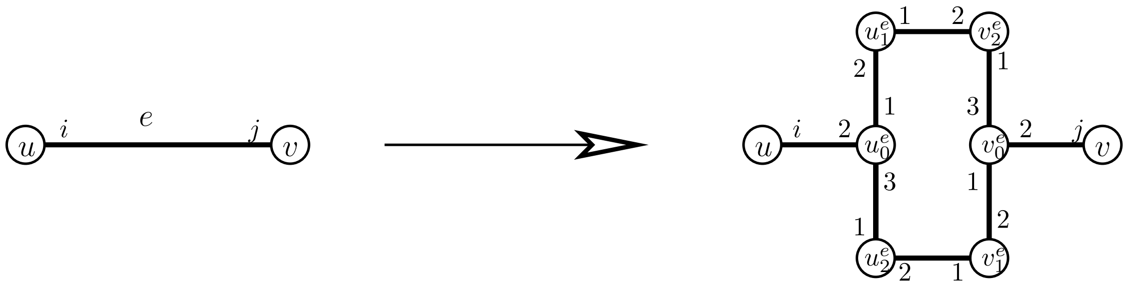


Figure 7.1: Achieving the many-to-one property

ward to check that if there is an edge blocking M then M' is not a stable b-matching of G'.

To finish the proof of correctness of the above transformation, we show that if M is a stable b-matching of G then there is another stable b-matching M' of G' such that M' corresponds to M in the above sense. To construct M', the only thing we should care of is what to do on the graph inserted instead of edge e. Three cases are possible.

- If $e \in M$ then $M' := M \setminus \{e\} \cup \{uu_0^e, u_1^e v_2^e, vv_0^e, v_1^e u_2^e\}$.
- If $e \notin M$ and e is dominated at u then $M' := M \cup \{u_0^e u_2^e, v_0^e v_1^e, u_1^e v_2^e\}$.
- If $e \notin M$ and e is dominated at v then $M' := M \cup \{v_0^e v_2^e, u_0^e u_1^e, v_1^e u_2^e\}$.

It is straightforward to check that M' is a stable b-matching of G'. After transforming G at most $|E|$ times, we may assume that our graph (that we still call G) has the many-to-one property. From now on, we assume this property and describe the second transformation.

We may assume that $b \leq |V|$, as if $b(v) > |V|$ would hold then we may reduce $b(v)$ to $|V|$ without changing the set of stable b-matchings. We construct graph G^b the following way. For each vertex u, we create $b(u)$ clones of u: we remove u from G and add new vertices $u_1, u_2, \ldots, u_{b(u)}$ and add edges $\{u_i v : 1 \leq i \leq b(u), uv \in E\}$. Define $b(u_i) = 1$ for $1 \leq i \leq b(u)$ and each u_i inherits the preference order of u. If $vu \in E$ then the preference order of v changes such that we keep the preference order on all edges different from uv, new edges $u_i v$ compare to old edges just like uv did and $u_1 v \prec_v u_2 v \prec_v \ldots \prec_v u_{b(u)} v$. Clearly, this cloning operation preserves bipartiteness, that is, G^b is bipartite whenever G is bipartite.

It is straightforward (and a bit laborious) to check that if M^b is a stable matching of G^b then

$$M := \{uv \in E : u_i v_j \in M \text{ for some } i, j\}$$

is a stable b-matching of G. Moreover it is also not difficult to see that if M is a stable b-matching of G then

$$M^b := \{u_i v_j : uv \in M \text{ and } uv \text{ is the } i\text{th edge in } M \text{ according to } u \text{ and the } j\text{th edge in } M \text{ according to } v.\}$$

is a stable matching of G^b. □

The key to Theorem 7.4 were two graph-transformations. Below we show a third one that proves the splitting property for nonbipartite graphs.

Theorem 7.5 (Cechlárová, Fleiner [13]). *Let $\preceq_v$ be a linear order on the set $E(v)$ of edges incident to v for each vertex v of finite graph $G = (V, E)$ and let $b : V \to \mathbb{N}$. Then there exists a partition $E(v) = E_1(v) \cup \ldots \cup E_{b(v)}(v)$ for each vertex $v \in V$ such that $|E_i(v) \cap S| \leq 1$ holds for any stable b-matching S, vertex v and index $1 \leq i \leq b(v)$.*

Proof. Define bipartite graph $G' = (V_1 \cup V_2, E')$ where $V_1 = \{v_1 : v \in V\}$, $V_2 = \{v_2 : v \in V\}$ and $E' = \{u_1v_2 : uv \in E\}$ by duplicating the vertex set of G and introducing two edges for each edge e of G. Define $b' : V_1 \cup V_2 \to \mathbb{N}$ by $b'(v_1) = b'(v_2) = b(v)$ for each vertex $v \in V$. Observe that if M is a stable b-matching of G then $M' = \{u_1v_2 : uv \in M\}$ is a stable b'-matching of G'. Hence Theorem 7.5 follows directly by applying Theorem 3.4 on G' and b'. □

Note that the transformation in the above proof can also be used to generalize the Rural Hospitals Theorem to nonbipartite graphs.

7.3 Extensions of Irving's algorithm

We have seen that if we are looking for a stable matching in graph G then a stable half-matching contains all the information we need: either it contains an odd cycle of half-weight edges (and then no stable matching exists in G) or, if its half-weight edges form only even cycles then removing every second edges of these even cycles we get a stable matching of G. One may wonder whether there is a similar characterizing structure for stable b-matchings. The answer is yes and it can be proved by a fairly natural extension of the proof of Theorem 7.2. However, as this proof leans on Scarf's lemma, we do not get an efficient algorithm that finds a stable half-b-matching. Moreover, the transformations in Theorem 7.4 does not provide us with a completely satisfactory result either: although it is true that one can reduce finding a stable b-matching of G to finding a stable matching in some other graph G^b, it is quite laborious to prove the existence of a stable half-b-matching of G from the existence of a stable half-matching of G^b. Instead of this, we use a different approach. It turns out that Irving's algorithm can be generalized and this generalization is an efficient algorithm that either finds a stable b-matching or concludes that no such matching exists. (In the latter case, it is not difficult to prove that Irving's algorithm finds a stable half-b-matching, but we do not pursue this line here.)

7.3.1 Stable b-matchings

The input of the extension of Irving's algorithm (the *EI algorithm*, for short) is an instance $(G, \mathcal{O}, b) = (G_0, \mathcal{O}_0, b)$ (where $\mathcal{O} = \{\preceq_v : v \in V(G)\}$ stand for the system of preferences), and its output is either a stable b-matching M of $(G, \mathcal{O}, b)$or a conclusion that no stable b-matching of $(G, \mathcal{O}, b)$ exists. The EI algorithm has two phases. In both phases, it transforms an instance $(G_i, \mathcal{O}_i, b)$ to another instance $(G_{i+1}, \mathcal{O}_{i+1}, b)$ in such a way that

$$G_{i+1} \text{ is a proper subgraph of } G_i, \tag{7.1}$$

$$\text{if } (G_i, \mathcal{O}_i, b) \text{ has a stable } b\text{-matching then } (G_{i+1}, \mathcal{O}_{i+1}, b) \text{ has one,} \tag{7.2}$$

$$\text{any stable } b\text{-matching of } (G_{i+1}, \mathcal{O}_{i+1}, b) \text{ is a stable } b\text{-matching of } (G_i, \mathcal{O}_i, b). \tag{7.3}$$

Define $B(u, G_i)$ as the set of the $b(u)$ best edges of G_i in $\prec_u$ and let $D(u, G_i)$ denote those edges f of G_i that can only be b-dominated at u:

$$\begin{aligned} B(u, G_i) &:= \{f = ux \in E(u, G_i) : |\{g \in E(u, G_i) : g \prec_u f\}| < b(u)\} \\ D(u, G_i) &:= \{f = ux \in E(u, G_i) : f \in B(x, G_i)\} \ . \end{aligned}$$

That is, $B(u, G_i)$ contains the first choices of u and $D(u, G_i)$ are those edges of u that are first choices of the other vertex. We say that instance $(G_i, \mathcal{O}_i, b)$ has the *first-last property* if for each vertex u of G_i and for each edge $e \in E(G_i)$ incident with u

$$|\{f \in D(u, G_i) : f \prec_u e\}| < b(u) \tag{7.4}$$

holds. That is, no edge e is incident with u that is preceded in $\preceq_u$ by $b(u)$ edges along which a proposal comes to u.

The name of the first-last property comes from the stable roommates terminology where it means that each agent is the last choice of his/her first choice. Consequently, each agent is the first choice of his/her last choice. This is generalized in the following lemma.

Lemma 7.6. *If instance $(G_i, \mathcal{O}_i, b)$ has the first-last property then $|B(u, G_i)| = |D(u, G_i)|$ for each vertex u of G_i.*

Proof. Observe that $|D(u, G_i)| \leq |B(u, G_i)|$ for each vertex u of G_i. This is because if $|B(u, G_i)| < b(u)$ then $D(u, G_i) \subseteq E(u, G_i) = B(u, G_i)$. Otherwise $|B(u, G_i)| = b(u)$, and $|D(u, G_i)| \leq b(u)$ by the first-last property. By double counting those edges that cannot be dominated at one endvertex we get that $\sum\{|B(u, G_i)| : u \in V(G_i)\} = \sum\{|D(u, G_i)| : u \in V(G_i)\}$, so Lemma 7.6 follows. □

If instance $(G_i, \mathcal{O}_i, b)$ does not have the first-last property then the algorithm makes a first phase step. That is, it finds an edge $e = uv$ that violates (7.4) and deletes e from G_i to get G_{i+1}. To construct $\mathcal{O}_{i+1}$, we restrict each order of $\mathcal{O}_i$ to the remaining edges. The motivation is that each agent selects his/her best possible partners according to his/her quota and proposes to them. If agent v receives at least $b(v)$ proposals than he/she will never be a partner of another agent who is worse than the $b(v)$th proposer of v. The next lemma is the b-matching counterpart of Lemma 1.2.

Lemma 7.7. *If $(G_{i+1}, \mathcal{O}_{i+1}, b)$ is constructed from $(G_i, \mathcal{O}_i, b)$ by deleting e in a first phase step then properties (7.1–7.3) hold.*

Proof. Property (7.1) holds trivially for $(G_{i+1}, \mathcal{O}_{i+1}, b)$. Assume that M is a stable b-matching of $(G_i, \mathcal{O}_i, b)$. By the definition of the first phase step, there are different edges $f_1, f_2, \ldots, f_{b(u)} \in D(u, G_i)$ such that $f_j \prec_u e$ for $j = 1, 2, \ldots, b(u)$. The definition of $D(u, G_i)$ implies that either all f_j's belong to M or M must b-dominate some f_j at u. In both cases, e is b-dominated by M at u, hence $e \notin M$, so M is a stable b-matching of $(G_{i+1}, \mathcal{O}_{i+1}, b)$. This proves (7.2).

Observe that after the deletion of e, we still have $f_j \in D(u, G_{i+1})$ for $j = 1, 2, \ldots, b(u)$. So the above argument applies to any stable b-matching M' of $(G_{i+1}, \mathcal{O}_{i+1}, b)$, and shows that M' b-dominates e. Hence M' is a stable b-matching of $(G_i, \mathcal{O}_i, b)$, justifying (7.3).□

The above proof justifies the following observation as well.

Observation 7.8. *If edge e is deleted in the first phase of the EI algorithm then e does not belong to any stable b-matching of $(G_0, \mathcal{O}_0, b)$.*

If the first phase step cannot be executed, i.e. $(G_i, \mathcal{O}_i, b)$ has the first-last property then either the edges of graph G_i form a b-matching M which (by property (7.3)) is a stable b-matching of $(G_0, \mathcal{O}_0, b)$, or the algorithm makes a second phase step, that is, it finds and eliminates a so called rotation to get $(G_{i+1}, \mathcal{O}_{i+1}, b)$. A *rotation* of $(G_i, \mathcal{O}_i, b)$ is a pair of edge sets $R = (\{e_1, e_2, \dots, e_k\}, \{f_1, f_2, \dots, f_k\})$ such that $e_j = u_j v_j$, $f_j = u_j v_{j+1}$ (here and further on, addition in the indices is modulo k), e_j is maximal (i.e. worst) in $\prec_{v_j}$ and f_j is the $(b(u_j) + 1)$st least (i.e. $(b(u_j) + 1)$st best) element of $\prec_{u_j}$. The above rotation R *covers* vertices $u_1, v_1, u_2, v_2 \dots u_k, v_k$. We denote the degree function of graph G_i by d_{G_i}.

Lemma 7.9. *If $(G_i, \mathcal{O}_i, b)$ has the first-last property and $E(G_i)$ is not a b-matching then there exists a rotation R of $(G_i, \mathcal{O}_i, b)$ such that $d_{G_i}(v) > b(v)$ for each vertex v covered by R.*

Proof. Define arc set A on $V(G_i)$ by introducing an arc $\vec{a} = \vec{uv}$ if $e = uw$ is the $\prec_u$-maximal edge and $f = wv$ is the $(b(w) + 1)$st least edge of $\prec_w$. Observe that if $d_{G_i}(u) > b(u)$ then for the $\prec_u$-maximal edge $e = uw$ we have $e \notin B(u, G_i)$, hence $e \notin D(w, G_i)$. This yields that $E(w) \neq D(w, G_i)$, so $E(w) \neq B(w, G_i)$, that is, $d_{G_i}(w) > b(w)$ by Lemma 7.6. This means that whenever $d_{G_i}(u) > b(u)$ then there is an arc $\vec{a}$ of A going from u to some vertex v with $d_{G_i}(v) > b(v)$. As $E(G_i)$ is not a b-matching, A is nonempty and contains a cycle. This A-cycle defines a rotation in a natural way. □

If the EI algorithm finds a rotation $R = (\{e_1, e_2, \dots, e_k\}, \{f_1, f_2, \dots, f_k\})$ as in Lemma 7.9 such that $\{e_1, e_2, \dots, e_k\} = \{f_1, f_2, \dots, f_k\}$ then it concludes that no stable matching of instance $(G, \mathcal{O})$ exists. Otherwise the algorithm *eliminates* rotation R, i.e. it constructs G_{i+1} and $\mathcal{O}_{i+1}$ by deleting edges $\{e_1, e_2, \dots, e_k\}$ from G_i and by restricting orders of $\mathcal{O}_i$ to the remaining edges. The following lemma justifies the correctness of the second phase step.

Lemma 7.10. *Let instance $(G_i, \mathcal{O}_i, b)$ have the first-last property and let $R = (\{e_1, e_2, \dots, e_k\}, \{f_1, f_2, \dots, f_k\})$ be a rotation of $(G_i, \mathcal{O}_i, b)$ as in Lemma 7.9.*

A *Sets $\{e_1, e_2, \dots, e_k\}$ and $\{f_1, f_2, \dots, f_k\}$ are disjoint or identical. In the latter case, $(G_i, \mathcal{O}_i, b)$ has no stable b-matching.*

B *If $(G_{i+1}, \mathcal{O}_{i+1}, b)$ is the SMA instance after the elimination of rotation R then properties (7.1–7.3) hold.*

Note that if $\{e_1, e_2, \dots, e_k\} = \{f_1, f_2, \dots, f_k\})$ for rotation R in Lemma 7.10 then $\{e_1, e_2, \dots, e_k\} \subseteq x^{-1}(\frac{1}{2})$ holds for any stable half-b-matching x.

Proof of part A. Assume that $e_j = f_l$, for some j, l. As $d_{G_i}(v_j) > b(v_j)$ and e_j is the $\prec_{v_j}$-maximal edge, $e_j \in B(u_j, G_i)$ by the first-last property of $(G_i, \mathcal{O}_i, b)$. Clearly, $f_l \notin B(u_l, G_i)$, so $u_j \neq u_l$ hence $u_j = v_{l+1}$ and $u_l = v_j$ must hold. So $d_{G_i}(u_l) = b(u_l) + 1$ as e_j is maximal and f_l is the $(b(u_l) + 1)$st least element of $\prec_{u_l}$. In particular, $b(u_l) = |B(u_l, G_i)| = |D(u_l, G_i)|$ by Lemma 7.6. In other words, $|B(u_l, G_i) \cap D(u_l, G_i)| = b(u_l) - 1$,

$D(u_l, G_i) \setminus B(u_l, G_i) = \{e_j\}$ and $B(u_l, G_i) \setminus D(u_l, G_i) = \{e\}$ for a unique edge e in $E(u_l, G_i)$.

By the definition of R, both edges e_l and f_{j-1} are incident with u_l and by the degree condition of Lemma 7.9, $e_l, f_{j-1} \notin D(u_l, G_i)$. This yields that $e_l = e = f_{j-1}$. That is, $e_j = f_l$ implies $e_l = f_{j-1}$ that in turn implies $e_{j-1} = f_{l-1}$. By induction, $\{e_1, e_2, \ldots, e_k\} = \{f_1, f_2, \ldots, f_k\}$ follows.

By the definition of a rotation, $e_1, f_1, e_2, f_2, \ldots, e_k, f_k$ is a closed walk. As all edges e_i are different and $\{e_1, e_2, \ldots, e_k\} = \{f_1, f_2, \ldots, f_k\}$, it follows that each edge is used exactly twice in the walk. So edges e_i (in an appropriate order) form a cycle C of length k. We have seen that $e_j = f_l$ and $e_l = f_{j-1}$, so $j - l \equiv l - (j-1) (\mathrm{mod}\ k)$, that is, $2(j - l) \equiv 1 (\mathrm{mod}\ k)$. This means that k is odd, and C is an odd cycle.

Now let M be a stable b-matching of $(G_i, \mathcal{O}_i, b)$. Pick some vertex (say $u_l = v_j$) of C. If $f \in E(u_l, G_i)$ and $e_j \neq f \neq e_l$ then $f \in B(u_l, G_i) \cap D(u_l, G_i)$, so $f \in M$. At most one of e_j and e_l can belong to M because $d_{G_i}(u_l) > b(u_l)$ and M is a b-matching. As $e_l \in D(u_l, G_i)$, if $e_l \notin M$ then M must b-dominate e_l at u_l, so $e_j \in M$ follows. We got that from two neighbouring edges of C exactly one belongs to M. As C is an odd cycle, this is impossible. That is, no stable b-matching of $(G_i, \mathcal{O}_i, b)$ exists. □

Proof of part B. Property (7.1) holds trivially. Let N be a stable b-matching of $(G_i, \mathcal{O}_i, b)$. Clearly, if N contains none of the e_j's then N is a stable b-matching of $(G_{i+1}, \mathcal{O}_{i+1}, b)$. Assume that $e_j \in N$. Then by Lemma 7.9, $d_{G_i}(v_j) > b(v_j)$, hence $|D(v_j, G_i)| = b(v_j)$. Moreover, $D(v_j, G_i) \subseteq N$ because e_j being maximal in $\prec_{v_j}$, no edge of G_i is b-dominated by N at v_j. By definition, $f_{j-1} \notin D(v_j, G_i)$, so $f_{j-1} \notin N$, implying that f_{j-1} is b-dominated by N at u_{j-1}. As f_{j-1} is preceded by exactly $b(u_{j-1})$ edges in $\prec_{u_{j-1}}$, all of those edges (in particular e_{j-1}) must belong to N. We got that $e_j \in N$ implies $e_{j-1} \in N \not\ni f_{j-1}$, hence $\{e_1, e_2, \ldots e_k\} \subseteq N$ and $\{f_1, f_2, \ldots f_k\}$ is disjoint from N.

Define $M := N \cup \{f_1, f_2, \ldots f_k\} \setminus \{e_1, e_2, \ldots e_k\}$. As the e_j's and the f_j's cover the same set of vertices, M is a b-matching. The above argument also shows that M contains the best $b(u_j)$ edges of $\prec_{u_j}$ in G_{i+1}. So if some edge e of G_{i+1} is b-dominated by N at u_j then e is still b-dominated by M at u_j. As N b-dominates no edge of G_i at v_j (because $e_j \in N$ is $\prec_{v_j}$-maximal), M is a stable matching of $(G_{i+1}, \mathcal{O}_{i+1}, b)$. This proves property (7.2).

Let M be a stable b-matching of $(G_{i+1}, \mathcal{O}_{i+1}, b)$ and $e_j = u_j v_j$ be an edge that has been deleted in the elimination of R. Assume that e_j is not b-dominated by M at v_j. By Lemma 7.9, $d_{G_i}(v_j) > b(v_j)$, so $|D(v_j, G_i)| = b(v_j)$ and $e_j \in D(v_j, G_i)$ by property (7.4) and Lemma 7.6. If $e \notin M$ for some edge $e \in D(v_j, G_i)$ other than e_j then e has to be b-dominated by M at v_j, and this means that e_j is also b-dominated by M at v_j, a contradiction. So $D(v_j, G_i) \setminus \{e_j\} \subseteq M$. This yields that $f_{j-1} \notin M$ because otherwise e_j would be b-dominated by M at v_j, as $f_{j-1} \notin D(v_j, G_i)$ and $f_{j-1} <_{v_j} e_j$. Hence f_{j-1} has to be b-dominated by M at its other vertex u_{j-1}. This is impossible as after the deletion of e_{j-1} in the elimination of R, there are only $b(u_j) - 1$ edges left in G_{i+1} that preceded f_{j-1} in $\prec_{u_j}$. The contradiction shows that no e_j can block M, so M is a stable b-matching of $(G_i, \mathcal{O}_i, b)$. This justifies property (7.3). □

After each rotation elimination, the algorithm returns to the first phase. Table 7.1 contains a pseudocode summarizing the algorithm.

Lemmata 7.7, 7.9 and 7.10 imply the following theorem.

The EI algorithm
Input: SMA instance $(G_0, \mathcal{O}_0, b)$
Output: stable b-matching of $(G_0, \mathcal{O}_0, b)$, if one exists
begin
i:=0
while $E(G_i)$ is not a b-matching **do**
begin
if $(G_i, \mathcal{O}_i, b)$ does not have the first-last property
then find $e \in E(G_i)$ that violates (7.4)
delete e to get $(G_{i+1}, \mathcal{O}_{i+1}, b)$
else find rotation $(\{e_1, e_2, \dots, e_k\}, \{f_1, f_2, \dots, f_k\})$
if $\{e_1, e_2, \dots, e_k\} = \{f_1, f_2, \dots, f_k\}$
then STOP: No stable b-matching of $(G_0, \mathcal{O}_0, b)$ exists
else delete $\{e_1, e_2, \dots, e_k\}$ to get $(G_{i+1}, \mathcal{O}_{i+1}, b)$
end if
end if
increase i by 1
end
STOP: Output stable b-matching $E(G_i)$ of $(G_0, \mathcal{O}_0, b)$
end

Table 7.1: Pseudocode of the extension of Irving's algorithm

Theorem 7.11 (Cechlárová, Fleiner [13]). *Let $G_0 = (V, E)$ be a finite graph, $\mathcal{O}_0 = \{\preceq_v : v \in V\}$ where $\preceq_v$ is a linear order on $E(v)$ and let $b : V \to \mathbb{N}$. Then the EI algorithm finds a stable b-matching of $(G_0, \mathcal{O}_0, b)$ or concludes that no stable matching of $(G_0, \mathcal{O}_0, b)$ exists in $O((n+m)^2)$ time where $n = |V|$ and $m = |E|$.*

Proof. In step i, the algorithm deletes at least one edge of G_i. So there are at most $n+m$ steps, each taking constant time. Finding all sets $B(u, G_i)$ takes $O(m)$ time. Each edge deletion costs constant time if edges are stored in doubly linked lists along orders $\prec_v$. We can update sets $B(u, G_i)$ after the deletion of k edges in $O(k)$ time. We can recognize the edge to be deleted in the first phase in $O(m)$ time. Finding a rotation according to Lemma 7.9 takes $O(\min(m, n))$ time, so altogether we get $O(m^2)$ for the complexity of the EI algorithm. □

Note that the complexity of the EI algorithm is worse than that of Irving's which has complexity $O(m)$. In case of the SR problem, the two algorithms are slightly different: in Irving's algorithm there is a so called "first phase" in which only first phase steps of the EI algorithm are executed. This is followed by a "second phase", where rotations are eliminated, and Irving's algorithm never returns to the first phase. For Irving, a rotation elimination consists of an EI second phase step, but the algorithm also executes some strictly specified first phase steps as well, so that the first-last property is preserved. The EI algorithm returns to the first phase, but still, it does the same, although it does not tell exactly which first phase steps to take. This is one reason that the complexity grows.

An advantage of the EI algorithm is that its correctness is somewhat easier to prove. Our main motivation here was to give a generalization of Irving's algorithm and we did not care much about complexity. Note, that by a better organization of the EI algorithm it is possible to reduce the complexity to $O(n+m)$, and this is done in Cechlárová and Vaľová [14] (see also [67]).

7.3.2 Weak preferences and forbidden edges

Irving's algorithm can be generalized in a further direction. Assume that we have given graph $G = (V, E)$ and a poset $(P_v, \leq_v)$ together with a mapping $f_v : E_v \to P_v$ for each vertex $v \in V$. We say that $e \preceq_v e'$ if $f_v(e) \leq_v f_v(e')$ and we call relation $\preceq_v$ a *weak preference order*. If $e, e' \in E(v)$ then there are exactly four possibilities: either e is better than e' or e' is better than e or e and e' are equally good or e and e' are incomparable. Assume further that we have given a set F of *forbidden edges* of G. Edges of $E \setminus F$ are called *free*. Matching M of G is *super-stable* if $M \subseteq E \setminus F$ and for every edge $e \in E$ there exists a vertex v and an edge $m \in M$ such that $m \preceq_v e$ holds. Note that using the tools described in part 7.3.1, our treatment below can also handle super-stable b-matchings, as well.

Theorem 7.12 (Fleiner, Irving, Manlove [27]). *There is a polynomial-time algorithm that finds a super-stable matching or concludes that no such matching exists for any finite graph $G = (V, E)$, set of forbidden edges $F \subseteq E$ and weak preferences $\preceq_v$.*

To prove Theorem 7.12, we introduce some notation. Let us fix a preference model $(G_0, F_0, \mathcal{O}_0)$, as the input of our algorithm. Our goal is to design an algorithm that finds a super-stable matching, if it exists. The algorithm shall handle so-called 1-arcs and 2-arcs that are oriented versions of certain edges of the underlying model. The sets of these arcs after the ith step of the algorithm are denoted by A_i^1 and A_i^2, respectively. In the beginning, $A_0^1 = A_0^2 = \emptyset$. The algorithm works step by step. In the $(i+1)$st step, it changes the current instance $(G_i, F_i, A_i^1, A_i^2, \mathcal{O}_i)$ into a "simpler" model $(G_{i+1}, F_{i+1}, A_{i+1}^1, A_{i+1}^2, \mathcal{O}_{i+1})$ in such a way that the answer to the latter problem is also a valid answer to the former one. That is,

$$\begin{gathered}\text{any super-stable matching in } (G_{i+1}, F_{i+1}, \mathcal{O}_{i+1}) \\ \text{is a super-stable matching in } (G_i, F_i, \mathcal{O}_i) \text{ and}\end{gathered} \tag{7.5}$$

and

$$\begin{gathered}\text{if there is a super-stable matching in } (G_i, F_i, \mathcal{O}_i) \text{ then} \\ \text{there has to be a super-stable matching in } (G_{i+1}, F_{i+1}, \mathcal{O}_{i+1}) \text{ as well.}\end{gathered} \tag{7.6}$$

We employ four different kinds of transformations to achieve this goal: we find 1-arcs and 2-arcs, we forbid edges and we delete forbidden edges.

To describe these transformations, we need a couple of definitions. We say that edge $e \in E_i(v)$ of G_i (forbidden or not) is a *first choice edge of* v, if there is no edge $f \in E_i(v) \setminus F_i$ with $f <_v e$ (i.e., if no free edge dominates e at vertex v). Note that there can exist more than one first choice of v. Moreover, an edge e can be a first choice of both of its vertices. Further, if v is not an isolated vertex then there is at least one first choice of v. If $e = vu$ is a first choice of v then we may change our current instance

$(G_i, F_i, A_i^1, A_i^2, \mathcal{O}_i)$ into $(G_{i+1}, F_{i+1}, A_{i+1}^1, A_{i+1}^2, \mathcal{O}_{i+1})$ where $G_{i+1} = G_i, E_{i+1} = E_i, A_{i+1}^1 = A_i^1 \cup \{(vu)\}, A_{i+1}^2 = A_i^2, \mathcal{O}_{i+1} = \mathcal{O}_i$ and we say that (vu) is a *1-arc*. This 1-arc finding transformation clearly satisfies conditions (7.5) and (7.6).

An edge $e \in E_i(v)$ is a *second choice of* v if e is not a first choice of v and $e >_v f \notin F_i$ implies that f is a first choice of v. In other words, e is a second choice if any free edge that dominates e at v is a first choice of v and there is at least one such free edge. Note that there can be several second choices of v present in an instance. Moreover, the set of second choices of v is nonempty if and only if there exist two free edges incident to v such that one dominates the other at v. If $e = vu$ is a second choice of v then we may change our current instance $(G_i, F_i, A_i^1, A_i^2, \mathcal{O}_i)$ into $(G_{i+1}, F_{i+1}, A_{i+1}^1, A_{i+1}^2, \mathcal{O}_{i+1})$ where $G_{i+1} = G_i, E_{i+1} = E_i, A_{i+1}^1 = A_i^1, A_{i+1}^2 = A_i^2 \cup \{(uv)\}, \mathcal{O}_{i+1} = \mathcal{O}_i$ and we say that (uv) is a *2-arc*. This 2-arc finding transformation clearly satisfies conditions (7.5) and (7.6). Note that the definition of a 2-arc is somewhat counterintuitive: unlike in case of a 1-arc, a 2-arc points to that vertex whose second choice it represents. Later we shall see the reason for this. For each j, we require the following property after the jth step of our algorithm.

$$\text{Each arc of } A_j^1 \text{ is a 1-arc of } (G_j, F_j, \mathcal{O}_j) \text{ and} \tag{7.7}$$

$$\text{each arc of } A_j^2 \text{ is a 2-arc of } (G_j, F_j, \mathcal{O}_j). \tag{7.8}$$

Clearly, 1-arc finding and 2-arc finding steps do not violate conditions (7.7) and (7.8).

If e is a free edge of G_i, then *forbidding* e means that $G_{i+1} := G_i$, $F_{i+1} := F_i \cup \{e\}$, and $\mathcal{O}_{i+1} := \mathcal{O}_i$. After forbidding, $A_{i+1}^1 = A_i^1$ and $A_{i+1}^2 = A_i^2$, unless we explicitly state otherwise. The algorithm may forbid e if either no super-stable matching contains e or if e is not contained in all super-stable matchings of $(G_i, F_i, \mathcal{O}_i)$. (Note that neither of these conditions implies the other.) Such a forbidding transformation clearly satisfies (7.5) and (7.6). Forbidding a subset C of E means that we simultaneously forbid all edges of C. We shall do so if properties (7.5) and (7.6) hold for $j = i + 1$.

If e is a forbidden edge of G_i then *deleting* e means that we delete e from G_i and F_i to get G_{i+1}: $E_{i+1} := E_i \setminus \{e\}$, $F_{i+1} := F_i \setminus \{e\}$, $A_{i+1}^1 := A_i^1 \setminus \{a \in A_i^1 : a = \vec{e}\}$, $A_{i+1}^2 := A_i^2 \setminus \{a \in A_i^2 : a = \vec{e}\}$ (where $a = \vec{e}$ means that 1-arc or 2-arc a is coming from first or second choice e) and the partial orders in $\mathcal{O}_{i+1}$ are the restrictions of the corresponding partial orders of $\mathcal{O}_i$, to the corresponding stars of G_{i+1}. The algorithm may delete forbidden edge e if there exists no matching in $(G_i, F_i, \mathcal{O}_i)$ that is blocked exclusively by e. This implies that the set of super-stable matchings in $(G_i, F_i, \mathcal{O}_i)$ and in $(G_{i+1}, F_{i+1}, \mathcal{O}_{i+1})$ is the same, so (7.5) and (7.6) clearly hold for $j = i + 1$. As first and second choices do not change after deleting a forbidden edge, properties (7.7) and (7.8) are true for $j = i + 1$.

As we mentioned already, our algorithm works in steps and in each step it changes the instance according to some of the above transformations. There is a certain hierarchy between these steps: the current move of the algorithm is always chosen to have the highest priority among the executable steps. Our description of the step types is in the order of this hierarchy.

0th priority (proposal) step If edge $e = vw$ is a first choice of v and does not belong to A_i^1 then find 1-arc vw, that is $A_{i+1}^1 = A_i^1 \cup \{(vw)\}$.

We have seen that conditions (7.5) and (7.6) are satisfied after a proposal step and by definition, (7.7) and (7.8) also hold for $j = i + 1$. As soon as the algorithm has found all 1-arcs, it looks for a

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1st priority (mild rejection) step If $\vec{e} = uv$ is a 1-arc of A_i^1, $A_i^2 = \emptyset$ and $E_i(v) \ni f \not<_v e$ (that is, f is not strictly better than e according to v in G_i) then forbid f.

Obviously, if f belongs to some matching M then $e \notin M$, and hence e (being a first choice at u) blocks M. So f does not belong to any super-stable matching, hence we can safely forbid it. Clearly, any first choice remains a first choice after forbidding edge e, hence (7.7) remains true for $i+1$. Moreover, after forbidding e, a second choice either remains a second choice or it becomes a first choice. Consequently, for $j = i+1$, properties (7.7) and (7.8) remain true with the default choice $A_{i+1}^1 = A_i^1$ and $A_{i+1}^2 = \emptyset$.

Eventually, the algorithm deletes certain forbidden edges in the following way.

2nd priority (firm rejection) step If $e = uv$ is a free 1-arc of A_i^1 and $e <_v f \in E_i(v)$ (e is strictly better than f according to v in G_i) then we delete f.

Note that the above f is already forbidden by a 1st priority mild rejection step. Assume that f blocks some matching M, hence, in particular, $e \notin M$. However, e, being a first choice of u, also blocks M. So deleting f does not change the set of super-stable matchings of the preference model.

Note that the so called 1st phase steps in Irving's algorithm [45] for the super-stable roommates problem are special cases of our proposal and firm rejection steps. It is true for the super-stable roommates problem that as soon as no more 1st phase steps can be executed, the preference model has the so called first-last property: if some edge $e = uv$ is a first choice of u, then e is the last choice of v. The next lemma shows that generalization of this property holds also in our setting. Assume that the algorithm cannot execute a 0th, 1st or 2nd priority step for $(G_i, F_i, A_i^1, A_i^2, \mathcal{O}_i)$. Let V_i^0 denote the set of those vertices of G_i that are not incident with any free edges, V_i^1 stand for the set of those vertices of G_i that are incident with a bioriented free 1-arc and V_i^2 refer to the set of the remaining vertices of G_i. The following properties are true.

Lemma 7.13. *Assume that no proposal or rejection step is possible in instance* $(G_i, F_i, A_i^1, A_i^2, \mathcal{O}_i)$ *and let* V_i^0, V_i^1 *and* V_i^2 *be defined as above.*

If $v \in V_i^1 \cup V_i^2$ *then there is a unique 1-arc entering* v *and there is a unique 1-arc leaving* v *and both of these 1-arcs are free. There is no edge of* G_i *that leaves* V_i^0*. Bioriented free 1-arcs form a matching* M_1 *that covers* V_i^1 *and no more edges are incident with* V_i^1 *in* G_i*.*

M *is a super-stable matching of* $(G_i, F_i, \mathcal{O}_i)$ *if and only if the following properties hold:*
(1) each vertex of V_i^0 *is isolated and* *(2)* $M_1 \subseteq M$ *and*
(3) $M \setminus M_1$ *is a super-stable matching of the model restricted to* V_i^2*.*

Proof. Let $v \in V_i^1 \cup V_i^2$. By definition, there is at least one free edge incident with v, hence there is at least one free 1-arc leaving v. On the other hand, no proposal or rejection step (mild or firm) can be made in G_i, hence at most one free 1-arc enters v. By definition, no free 1-arc enters any vertex of V_i^0, and this means that 1-arcs leaving vertices of $V_i^1 \cup V_i^2$ enter this very same vertex set. Consequently, there is a unique free 1-arc leaving and entering each vertex of $V_i^1 \cup V_i^2$. Can there be a forbidden 1-arc e incident with a vertex v of $V_i^1 \cup V_i^2$? The answer is no and we prove it indirectly. Assume that $\vec{e}$ is such a 1-arc. If $\vec{e}$ enters v then v would be able to reject, a contradiction. So $\vec{e} = (vw)$ is a 1-arc of A_i^1 from $V_i^1 \cup V_i^2$ to V_i^0. However, w is not incident with any free arcs by definition, thus (vu) is also a 1-arc of A_i^1 that enters vertex u of $V_i^1 \cup V_i^2$, contradiction again. Hence each 1-arc of A_i^1 incident with $V_i^1 \cup V_i^2$ is free.

Let $u \in V_i^0$ and $e = uv$ be an edge of G_i. Clearly $\vec{e} = (uv)$ is a 1-arc and $\vec{e} \in F_i$ by the definition of V_i^0, so $v \in V_i^0$ holds. This means that each edge of G_i incident with a vertex of V_i^0 is completely inside V_i^0.

If v is in V_i^1 then there is a unique 1-arc a that leaves v, so a must be bioriented by the definition of V_i^1. If $e = uv$ is an edge of G_i then either e is the unoriented version of a or e is not a first choice of v, hence $a <_v e$ holds. In this latter case, v should delete e in a firm rejection step as a is a 1-arc entering v. This argument shows that edges of G_i that are incident with V_i^1 are all bioriented and form a matching M_1 covering V_i^1.

Assume now that M is a super-stable matching of G_i. No edge of G_i incident with a vertex of V_i^0 can block M, hence V_i^0 must consist of isolated vertices. As M is not blocked by an edge of M_1, edges of M_1 all belong to M. As there is no edge of G_i that leaves V_i^2, edges of M in V_i^2 form a super-stable matching of the restricted model to V_i^2.

Let now M_2 be a super-stable matching of the model restricted to V_i^2 and assume that V_i^0 consists of isolated vertices. Let $M := M_2 \cup M_1$. Clearly, M is a matching. If some edge e blocks M then e cannot be incident with V_i^0, as these vertices are isolated, and e cannot have a vertex in V_i^1 either, as vertices of V_i^1 are only incident with edges of M_1. Hence e is an edge within V_i^2, contradicting to the fact that M_2 is a super-stable matching of the model restricted to V_i^2. □

Lemma 7.13 shows that as soon as we have a (forbidden) edge incident with some vertex of V_i^0 for an instance where no proposal or rejection step is possible then there exists no super-stable matching in our instance, so the algorithm can terminate with the conclusion that in the original instance there is no super-stable matching whatsoever. Another possible conclusion of the algorithm is that eventually no proposal or rejection step can be made and $V_i^2 = \emptyset$ holds. In this case, if V_i^0 consists of isolated vertices then graph G_i is just matching M_1 and this is a super-stable matching for the instance after the ith step, hence it is also a super-stable matching for the original instance. So our goal from now on is to get rid off the V^2 part and to achieve this, the algorithm will work only on V_i^2.

Assume that in instance $(G_i, F_i, A_i^1, A_i^2, \mathcal{O}_i)$, the algorithm can execute no 0th, 1st or 2nd priority step. By Lemma 7.13, every vertex v of V_i^2 is incident with at least one free second choice edge: in the "worst case" it is the unique 1-arc pointing to v.

3rd priority (2-arc finding) step If $e = vw \notin A_i^2$ is a second choice of v then find 2-arc wv.

What is the meaning of a 2-arc? Let vv' and uu' be 1-arcs and $u'v$ be a 2-arc. As vv' is the only free edge dominating $u'v$ at v, we get that if uu' is present in a super-stable matching M then uu' does not dominate uv', hence $vv' \in M$ follows. In other words, 2-arcs represent implications on 1-arcs. This allows us to build an implication structure on the set of 1-arcs.

In this structure, two 1-arcs e and f are called *sm-equivalent*, if there is a directed cycle D formed by 1-arcs and 2-arcs in an alternating manner such that D contains both e and f. (Note that D may use the same vertex more than once.) Sm-equivalence is clearly an equivalence relation and if C is an sm-class and M is a super-stable matching then either C is disjoint from M or C is contained in M.

Beyond determining sm-equivalence classes, 2-arcs yield further implications between sm-classes: if uu' is a 1-arc of sm-class C and vv' is a 1-arc of sm-class C' and $u'v$ is a 2-arc, then sm-class C "implies" sm-class C' in such a way that if C is not disjoint from super-stable matching M then M contains both classes C and C'. Assume that sm-class C is on the top of this implication structure, i.e. C is not implied by any other sm-class (but C may imply certain other classes). Formally, we have that

$$\begin{array}{c}\text{if } vv' \text{ is a 1-arc of } C \text{ and } w'v \text{ is a 2-arc} \\ \text{then (the unique) 1-arc } ww' \text{ is sm-equivalent to } vv'.\end{array} \tag{7.9}$$

To find a top sm-class C, introduce an auxiliary digraph on the vertices of G_i, such that if uu' is a 1-arc and $u'v$ is a 2-arc, then we introduce an arc uv of the auxiliary graph. It is well known that by depth first search, we can find a source strong component of the auxiliary graph in linear time. If it contains vertices $u_1, u_2, \dots, u_k$ then it determines a top sm-class $C = \{u_1u_1', u_2u_2', \dots, u_ku_k'\}$ formed by 1-arcs. Note that it is possible here that $u_l = u_t'$ for different l and t. After we have found all 2-arcs (and there are no proposal or rejection steps) in instance $(G_i, F_i, A_i^1, A_i^2, \mathcal{O}_i)$ then the algorithm looks for a

4th priority (2-arc elimination) step If for 1-arcs $u_lu_l', u_tu_t' \in C$ there are 2-arcs vu_l and vu_t with $vu_l \not<_v vu_t$ then forbid vu_l and keep 1-arcs and 2-arcs: $A_{i+1}^1 = A_i^1$ and $A_{i+1}^2 = A_i^2$.

To justify this step, assume that $vu_l \in M$ for some super-stable matching M of G_i. As vu_l does not dominate vu_t, vu_t has to be dominated at u_t by $u_tu_t' \in M$. As u_lu_l' and u_tu_t' are sm-equivalent, this means that u_tu_t' also belongs to M, a contradiction. So vu_l does not belong to any super-stable matching and after forbidding it, the set of super-stable matchings does not change. This proves (7.5) and (7.6). As the forbidden edge vu_l is a second choice of u_l and not $\le_v$-better than vu_t, vu_l is a first choice of neither v nor u_l. Consequently, after forbidding vu_l, first and second choices remain first and second choices, respectively. It follows that a 4th priority step preserves conditions (7.7) and (7.8). Note that though a 4th priority step does not change first choices, it may create new second choices hence the algorithm might continue with a 3rd priority step after executing a 4th priority one. Note also that if preferences are linear (rather than partial) orders then no 4th priority step is possible.

If none of the above steps is possible any more then the top sm-equivalence class C can be forbidden. This is the step that corresponds to the 'rotation elimination' step in Irving's algorithm. Note that by the impossibility of a 4th priority step, any top sm-equivalence class $C = \{(u_lu_l') : 1 \le l \le k\}$ has the property that there is exactly one 2-arc entering each u_l, that is, there is a unique second choice of each vertex u_l.

5th priority (top class elimination) step Forbid all edges of C in $(G_i, F_i, \mathcal{O}_i)$ and set $A_{i+1}^2 = \emptyset$.

As we forbid 1-arcs, first and second choices along the vertices of C change after a 5th priority step. In particular, the unique second choices of the u_l vertices of C become first choices. For this reason we change $A_{i+1}^1 = A_i^1 \cup S^{-1}$, where S denotes the set of those 2-arcs that enter some vertex u_l of C and S^{-1} is the set of oppositely oriented arcs of S. After these changes, all arcs in A_{i+1}^1 are clearly first choices of their initial vertices, hence (7.7) and (7.8) hold for $j = i + 1$. To justify properties (7.5) and (7.6) for the 5th priority step, we distinguish two cases.

Case 1: C is not a matching. This means that $u_l = u_t'$ for some $l \ne t$. As a subset of a matching is a matching, no matching (hence no super-stable matching) can contain C. So by sm-equivalence, C is disjoint from any super-stable matching of G_i, and forbidding C does not change the set of super-stable matchings.

Case 2: C is a matching. Each u_l is adjacent to at least two free edges: the incoming and the outgoing 1-arcs. So each u_l receives at least one free 2-arc. This free 2-arc must come from some u'_t by property (7.9). Let C' denote the set of free 2-arcs of the form $u'_t u_l$. As we have seen, each u_l receives at least one arc of C', hence $|C'| \geq k$. As we cannot execute any more 4th priority steps in $(G_i, F_i, \mathcal{O}_i)$, from each u'_t there is at most one arc of C' leaving, implying $|C'| \leq k$. This means that $|C'| = k$ and each u_l receives exactly one arc of C' and each u'_l sends exactly one arc of C'. As sets $\{u_1, u_2, \ldots, u_k\}$ and $\{u'_1, u'_2, \ldots, u'_k\}$ are disjoint, this means that set C' forms a perfect matching on vertices $u_1, u'_1, u_2, u'_2, \ldots, u_k, u'_k$.

Let M be a super-stable matching of $(G_i, F_i, \mathcal{O}_i)$. If M is disjoint from C then M is super-stable in $(G_{i+1}, F_{i+1}, \mathcal{O}_{i+1})$ as well. Otherwise, by sm-equivalence, M contains all edges of C and disjoint from C'. We claim that $M' := M \setminus C \cup C'$ is another super-stable matching of $(G_i, F_i, \mathcal{O}_i)$ and hence it is a super-stable matching of $(G_{i+1}, F_{i+1}, \mathcal{O}_{i+1})$, as well.

Indeed: M' is a matching, as C and C' cover the same set of vertices. Each edge $u_l u'_l$ is dominated at u'_l by M' by Lemma 7.13. Each forbidden 2-arc of type $u'_t u_l$ is dominated at u'_t by the 4th priority step. For the remaining edges, if some edge e does not have a vertex u_l then e is dominated the same way in M' as in M. Otherwise, if u_l is a vertex of e then e is neither a first nor a second choice of u_l as we have already checked these edges. This means that the free 2-arc pointing to u_l is dominating e, so C' and thus M' also dominates e at u_l.

The pseudocode on Table 7.2 summarizes how our algorithm works. The organization of the steps is justified by the fact that a firm rejection step always deletes a forbidden edge, hence no new first choice is created. Similarly, 2-arc finding and 2-arc elimination steps do not change the set of first choices and preserve properties described in Lemma 7.13.

The following theorem justifies the correctness of our algorithm.

Lemma 7.14. *Assume that the algorithm cannot execute any more step at some instance $(G_i, F_i, A_i^1, A_i^2, \mathcal{O}_i)$. Then $V_i^2 = \emptyset$.*

Proof. Assume indirectly that v is a vertex of V_i^2, so by Lemma 7.13, v sends a free 1-arc, and also receives a free 1-arc different from the opposite of the previous one. It follows that there is a 2-arc pointing to v. This implies that a 4th or a 5th priority step can be executed, a contradiction. □

To finish the description of the algorithm, we should recall our earlier remark. By Theorem 7.14, when the algorithm terminates then we have $V_i^2 = \emptyset$, so by Lemma 7.13, if V_i^0 spans some edge then the conclusion is that there is no super-stable matching, otherwise, if each vertex of V_i^0 is isolated then there is a super-stable matching of the original instance, and the edge set E_i of G_i forms such a matching. The following lemma estimates the complexity of our algorithm and finishes the proof of Theorem 7.12.

Lemma 7.15. *Assume that preference model $(G, F, \mathcal{O})$ is such that G has n vertices and m edges we can decide for each edge $e = uv$ of G whether e is a first or second choice of u in constant time for any preference model created from $(G, F, \mathcal{O})$ after forbidding and deleting edges. The algorithm we described above finds a super-stable matching or concludes that no super-stable matching exists in $O(m \cdot (n + m))$ time.*

Input: $(G_0, F_0, \mathcal{O}_0)$ **Output**: Super-stable matching, if exists

$A_0^1 := A_0^2 := \emptyset,\ i := 0$

1 **IF** there is a first choice uv of u that is not a 1-arc
  THEN $(G_{i+1}, F_{i+1}, \mathcal{O}_{i+1}, A_{i+1}^1, A_{i+1}^2) := (G_i, F_i, \mathcal{O}_i, A_i^1 \cup \{uv\}, A_i^2)$,
    $i := i+1$, **GO TO 1**
  ELSE

2 **IF** mild rejection is possible for some edge uv of G_i
  THEN $(G_{i+1}, F_{i+1}, \mathcal{O}_{i+1}, A_{i+1}^1, A_{i+1}^2) := (G_i, F_i \cup \{uv\}, \mathcal{O}_i, A_i^1, A_i^2)$,
    $i := i+1$, **GO TO 1**
  ELSE

3 **IF** firm rejection is possible for some edge uv of G_i
  THEN $(G_{i+1}, F_{i+1}, \mathcal{O}_{i+1}, A_{i+1}^1, A_{i+1}^2) :=$
    $:= (G_i - \{uv\}, F_i \setminus \{uv\}, \mathcal{O}_i|_{G_{i+1}}, A_i^1 \setminus \{uv\}, A_i^2) \setminus \{uv\}$,
    $i := i+1$, **GO TO 3**
  ELSE

4 **IF** there is a second choice uv of u that is not a 2-arc
  THEN $(G_{i+1}, F_{i+1}, \mathcal{O}_{i+1}, A_{i+1}^1, A_{i+1}^2) := (G_i, F_i, \mathcal{O}_i, A_i^1, A_i^2 \cup \{vu\})$,
    $i := i+1$, **GO TO 4**
  ELSE

5 **IF** some 2-arc $uv \in A_i^2$ can be eliminated
  THEN $(G_{i+1}, F_{i+1}, \mathcal{O}_{i+1}, A_{i+1}^1, A_{i+1}^2) := (G_i, F_i \cup \{uv\}, \mathcal{O}_i, A_i^1, A_i^2)$,
    $i := i+1$, **GO TO 4**
  ELSE

6 **IF** some sm-equivalence class C can be eliminated
  THEN $(G_{i+1}, F_{i+1}, \mathcal{O}_{i+1}, A_{i+1}^1, A_{i+1}^2) :=$
    $(G_i - C, F_i \setminus C, \mathcal{O}_i, A_i^1 \cup S^{-1}, \emptyset)$, $i := i+1$, **GO TO 1**
  ELSE

7 **IF** each vertex of V_i^0 is isolated
  THEN **OUTPUT** super-stable matching E_i
  ELSE **OUTPUT** "No super-stable matching exists"
  END IF
  END IF
  END IF
  END IF
  END IF
  END IF
STOP

Table 7.2: Pseudocode of the super-stable matching algorithm

Proof. We have seen that if the algorithm terminates then it has the right answer, so we only need to prove that the running time is $O(m \cdot (n+m))$. As we have seen, in each step, the algorithm changes the current instance by changing the set of 1-arcs or 2-arcs or by forbidding or deleting certain edges. Let us call the latter two transformations *major events*. Clearly, during the course of the algorithm there can be at most $2m$ major events as there are m edges that can be forbidden or eventually deleted. We show that between two consecutive major events the algorithm needs $O(n+m)$ time.

If a major event is a 1st priority (mild rejection) step, then previously we had to find all 1-arcs (in $O(n+m)$ time) and finding the forbidden edge after this can be done in $O(n+m)$ time again. If the major event is a 2nd priority (firm rejection) step then it is preceded by 0th priority proposal steps (taking $O(n+m)$ time again) and checks for 1st priority (mild rejection) steps taking $O(n+m)$ time. We need again $O(n+m)$ time to find the edge to be deleted in the 2nd priority (firm rejection) step.

The next major event type is a 4th priority (2-arc eliminating) step. It is preceded by executing all 0th priority (proposal) steps and checking for 1st and 2nd priority steps that take altogether $O(n+m)$ time. Then we find all 2-arcs in $O(n+m)$ time, find top sm-class C by depth first search in $O(n+m)$ time and find the deleted edge $O(n+m)$ time again.

The remaining major event happens in a 5th priority step. So after the previous major event we had at most $O(n)$ 0th priority proposal steps that take $O(n+m)$ time, checks for 1st and 2nd priority rejection steps taking $O(n+m)$ time, we had to find all 2-arcs in $O(n+m)$ time, we find top sm-class C in $O(n+m)$ time and check for 2-arc elimination in $O(n+m)$ time again.

The above estimates prove that there is $O(n+m)$ time between consecutive major events. We have seen that there are $O(m)$ major events, so our algorithm terminates in $O(m \cdot (n+m))$ time, just as we claimed in the theorem. □

The time complexity in Theorem 7.15 is pretty rough. This is partly due to the fact that in 5th priority (top class elimination) steps we throw away all 2-arcs in spite of the fact that most of them can be reused. Probably by paying more attention to the changes of second choices and by using more appropriate data structures one can streamline the algorithm to approach the complexity of Irving's original algorithm described in [45]. As we mentioned, our goal was not a competitive algorithm but a description of a polynomial-time method with a compact proof of correctness that gives hope to find further structural results on super-stable matchings. This goal is definitely achieved.

7.4 Roommates with generalized choice functions

It turns out that Irving's algorithm is more powerful than what we have seen in the previous sections. In this section, we point out that its appropriate extension can also be used in a far reaching generalization of the stable roommates problem. We have seen that in case of bipartite graphs, the stable marriage theorem of Gale and Shapley can be extended to choice function based general model. For this reason, it is a natural problem to find a generalized stable matching (a kernel, in our terminology) in case of nonbipartite graphs and preferences determined by choice functions.

Assume finite graph $G = (V, E)$ is given and for each vertex $v \in V$ we have a substitutable choice function $C_v : 2^{E(v)} \to 2^{E(v)}$. Subset S of the edges is a C-*kernel* if

- $C_v(S(v)) = S(v)$ holds for all $v \in V$ and
- each $e \in E \setminus S$ has a vertex v such that $e \notin C_v(S(v) \cup \{e\})$ holds.

The first condition is often called as individual rationality, while the second requires that no blocking edge exists.

For choice function C_v, we say that subset X of E *v-dominates* element x of $E(v)$ if $x \in \overline{C}_v(X(v) \cup \{x\})$. Less formally, x is dominated, if v is not interested in x even if beside the set X of possible options we make option x available for v. If it causes no ambiguity, then instead of v-domination we may speak about *domination*. For a choice function C_v, let $D_v(X)$ denote the set of elements (v-)dominated by X. It is easy to see that $\mathcal{D}_{C_v}(X) = E(v) \setminus D_v(X)$ is a determinant of C_v, in fact it is the one defined in (1.2). Hence $X \setminus C_v(X) \subseteq D_v(X)$ and $C_v(X) = X(v) \setminus D_v(X)$ holds for any subset X of $E(v)$.

Lemma 7.16. *If choice function C_v is substitutable then dominance function D_v is monotone.*

If choice function C_v is substitutable and increasing and $Y \subseteq D_v(X)$ then $C_v(X \cup Y) = C_v(X)$, thus $D_v(X \cup Y) = D_v(X)$.

Proof. The first part directly follows from the fact that $\mathcal{D}_{C_v}(X) = E(v) \setminus D_v(X)$ is a determinant of C_v.

For the second part, let $y \in Y \cup \overline{C}_v(X)$ be an arbitrary element dominated by X. So $y \in \overline{C}_v(X \cup \{y\}) \subseteq \overline{C}_v(X \cup Y)$, where the second relation follows from the monotonicity of $\overline{C}_v$. Hence $Y \cup \overline{C}_v(X) \subseteq \overline{C}_v(X \cup Y)$, that is, $C_v(X \cup Y) = (X \cup Y) \setminus \overline{C}_v(X \cup Y) \subseteq (X \cup Y) \setminus (\overline{C}_v(X) \cup Y) \subseteq X \setminus \overline{C}_v(X) = C_v(X)$. As $X \subseteq X \cup Y$, the increasing property of C_v implies that $|C_v(X)| \leq |C_v(X \cup Y)|$, so $C_v(X) = C_v(X \cup Y)$ follows. □

The notion of dominance allows us to reformulate the notion of a C-kernel.

Theorem 7.17. *If $G = (V, E)$ is a finite graph and for each vertex v, C_v is a choice function on $E(v)$ then S is a C-kernel if and only if $E \setminus S = \bigcup_{v \in V} D_v(S(v))$, that is, if S dominates exactly $E \setminus S$.*

Proof. If S is a C-kernel then by individual rationality, no edge of S is dominated by S. As no blocking edge exists, each edge outside S is dominated by S. On the other hand, if $E \setminus S = \bigcup_{v \in V} D_v(S \cap E(v))$ then no edge of S is dominated by S, thus S is individually rational. As each edge outside S is dominated, no blocking edge exists. □

Let C_v be a choice function and let $X \subseteq E(v)$. For an edge x in $C_v(X)$ the *X-replacement* of x according to C_v is the set $R = C_v(X \setminus \{x\}) \setminus C_v(X)$. Roughly speaking, if option x is not available for v any more, then v selects from X options of R instead of x.

Lemma 7.18. *If C_v is an increasing and substitutable choice function on $E(v)$, $X \subseteq E(v)$ and $x \in C_v(X)$, then the X-replacement R of x contains at most one element.*

Proof. We have $C_v(X) \setminus \{x\} \subseteq C_v(X \setminus \{x\})$ by substitutability, so $C_v(X \setminus \{x\}) = C_v(X) \cup R \setminus \{x\}$. From the increasing property of C_v, we get $|C_v(X)| \geq |C_v(X \setminus \{x\})| = |C_v(X) \cup R \setminus \{x\}| = |C_v(X)| + |R| - 1$, and the lemma follows. □

To define the notion of a half-integral C-kernel, we introduce the nonstandard notion of an *oriented edge* of an undirected graph $G = (V, E)$ as an edge $e = uv$ that is labelled with at least one of the two possible orientations uv and vu. So an unoriented edge is an ordinary edge, there can be *bidireced edges* that are labelled with both possible orientations and there may exists simple oriented edges that are similar to arcs. The reason that we do so is that these objects are edges for the choice function, but behave like arcs in other situations we deal with.

Let $G = (V, E)$ be a graph and for each vertex v, let C_v be a choice function on $E(v)$. Let S be a subset of E and fix disjoint subsets $S_1, S_2, \ldots, S_l$ of S such that each S_i is an odd cycle (i.e. a closed walk in G) and fix an orientation for each S_i. (Note that cycle S_i is not necessarily a circuit, as S_i can traverse the same vertex several times.) Let $S_i^+(v)$ and $S_i^-(v)$ denote the set of oriented edges of S_i that leave and enter vertex v, respectively. Define edge set $S^+(v) := S(v) \setminus \bigcup_{i=1}^{l} S_i^-(v)$ as the unoriented edges of $S(v)$ together with the oriented edges of the S_i's that leave v. Similarly let $S^-(v) := S(v) \setminus \bigcup_{i=1}^{l} S_i^+(v)$, denote the set unoriented edges of $S(v)$ and all oriented edges of the S_i's that enter v. We say that $(S, S_1, \ldots, S_l)$ is a *half-integral C-kernel* if

1. for each $v \in V$, $C_v(S(v)) = S^+(v)$. Moreover,
2. If oriented edges $e \in S_i^-(v)$ and $f \in S_i^+(v)$ are consecutive on S_i then $\{e\}$ is the $S(v)$-replacement of f according to C_v.
3. $E \setminus S = \bigcup_{v \in V} D_v(S^-(v))$.

A consequence of the definition is that if $(S, S_1, \ldots, S_l)$ is a half-integral C-kernel and $e = uv$ is an edge then exactly one of the following three possibilities holds. Either e is an unoriented edge of S (that does not belong to any of the S_i's), or e is an edge of some S_i, and hence if $e = S_i^-(v)$ then e is dominated by $S_i^+(v)$, or $e \notin S$ and hence e is dominated by $S^-(u)$ at some vertex u of e. If $(S, S_1, \ldots, S_l)$ has properties 1. and 2. and edge $e = uv \notin S$ is not dominated (i.e. $e = uv$ and $e \notin D_u(S^-(u)) \cup D_v(S^-(v))$) then we say that e is *blocking* $(S, S_1, \ldots, S_l)$. Observe that (S) is a half-integral C-kernel (that is, no oriented odd cycles are present) if and only if S is a C-kernel. Now we can state our main result.

Theorem 7.19 (Fleiner [22]). *If $G = (V, E)$ is a finite graph and for each vertex v, choice function C_v on $E(v)$ is increasing and substitutable then there exists a half-integral C-kernel. Moreover, if $(S, S_1, \ldots, S_l)$ and $(S', S'_1, \ldots, S'_m)$ are half-integral C-kernels, then $l = m$ and sets of oriented cycles $\{S_1, \ldots, S_l\}$ and $\{S'_1, \ldots, S'_m\}$ are identical.*

Corollary 7.20. *If $(S, S_1, \ldots, S_l)$ is a half-integral C-kernel then either $l = 0$ and S is a C-kernel, or no C-kernel exists whatsoever.*

Corollary 7.20 shows that to solve the C-kernel problem in case of increasing substitutable choice functions, it is enough to find a half-integral C-kernel. Note that Theorem 7.19 is a generalization of Tan's result [64] on stable half-matchings (or on "stable partitions" in Tan's terminology).

To prove Theorem 7.19, we follow Tan's method. Tan extended Irving's algorithm in such a way that it finds a stable half-matching, and, with the help of the algorithm, he proved the unicity of the oriented odd cycles. Here, instead of linear orders, we work with increasing substitutable choice functions. To handle this situation, we shall generalize Irving's algorithm to our setting. Irving's algorithm works in such a way that it keeps on deleting edges so that no new stable matching is created after a deletion, and, if there was a stable matching before a deletion, there should be one after it, as well. Irving's algorithm terminates if the current graph is a matching, which, by the deletion rules is a stable matching for the original instance. Similarly, our algorithm will delete edges in such a way that after a deletion no new half-integral C-kernel can be created. Moreover, if there was a half-integral C-kernel $(S, S_1, \ldots, S_l)$ before some deletion, then we cannot delete an edge of any of the S_i's and at least one half-integral C-kernel has to survive the deletion. If our algorithm terminates then we are left with a graph G' such that edge set $E(G')$ of G' is a half-integral C-kernel of G', hence it is a half-integral C-kernel of G, as well. Our algorithm has different deletion rules, and there is a priority of them. The algorithm always takes a highest priority step that can be made. In this section we describe and justify our algorithm. Note that our definitions and theorems always refer to the "current" graph, so if we check how the algorithm works on a given instance then this instance is changing after each step. In particular, the set E of edges is changing, as it is the edge set of the current graph.

To start the algorithm we need some definitions. We say that the *first choices* of v are the edges of $C_v(E(v))$. These are the best possible options for agent v. If edge $e = vw$ is a first choice of v then we call oriented edge $e = vw$ a 1-*arc*. Note that if vw is a 1-arc then it is possible that wv is also a 1-arc. Let A denote the set of 1-arcs. For a vertex v let $A^+(v)$ and $A^-(v)$ stand for the set of 1-arcs that are directed away from v and towards v, respectively.

The **1st priority (proposal) step** is that we find and orient all 1-arcs.

As the instance did not change (we did not delete anything), the set of half-integral C-kernels is the same as it was before the orientation. After we found all 1-arcs, we execute the

2nd priority (rejection) step: If $D_v(A^-(v)) \neq \emptyset$ for some vertex v then we delete $D_v(A^-(v))$.

Lemma 7.21. *The set of half-integral C-kernels does not change by a 2nd priority step.*

Proof. Assume that $(S, S_1, \ldots, S_l)$ is a half-integral C-kernel after the deletion. This means that for each 1-arc $f = uv$ of $A^-(v)$ either f belongs to $S^-(v)$, or, as f cannot be dominated by $S^-(u)$ at u, f is dominated by $S^-(v)$ at v. Lemma 7.16 implies that $D_v(A^-(v)) = D_v(C_v(A^-(v))) \subseteq D_v(S^-(v) \cup C_v(A^-(v))) = D_v(S^-(v))$, hence no deleted edge can block $(S, S_1, \ldots, S_l)$. This means that no new half-integral C-kernel can emerge after a 2nd priority deletion.

Assume now that $(S, S_1, \ldots, S_l)$ is a half-integral C-kernel before the deletion and some edge $e \in D_v(A^-(v))$ belongs to S. Similarly to the previous argument, this means that for each 1-arc $f = uv$ of $C_v(A^-(v))$ either f belongs to $S^-(v)$, or (as f cannot be dominated by $S^-(u)$ at u) f is dominated by $S^-(v)$ at v. Lemma 7.16 implies that $e \in D_v(A^-(v)) = D_v(C_v(A^-(v))) \subseteq D_v((S^-(v)) \cup C_v(A^-(v)) = D_v(S^-(v))$, so e cannot belong to $S(v)$, a contradiction. □

Later we need the following lemma.

Lemma 7.22. *If no 1st and 2nd priority steps can be made then $|A^+(v)| = |A^-(v)|$ for each vertex v.*

Proof. By the increasing property of C_v, we have $|A^-(v)| = |C_v(A^-(v))| \leq |C_v(E(v))| = |A^+(v)|$. So each vertex v has at least as many outgoing 1-arcs as the number of 1-arcs entering v. As both the total number of outgoing 1-arcs and the total number of ingoing 1-arcs is exactly $|A|$, the previous inequality must be an equality for each vertex v. □

If no more 1st and 2nd priority steps can be made then the following two cases are possible.

Case 1. All 1-arcs are bidirected, that is, if $e = uv$ is a 1-arc, then its reverse vu is also a 1-arc. In other words, $A^+(v) = A^-(v) = A(v) = E(v)$ for each vertex v. This means that the edge set of our graph is a C-kernel, so the algorithm terminates and outputs (S).

Case 2. There exists at least one 1-arc $e = uv$ that is not bidirected. In this situation, the algorithm looks for replacements. For a non bidirected 1-arc $e = uv$, let $e^r = uw$ denote the $E(u)$-replacement of e according to C_u. (It might happen that e^r does not exist.)

3rd priority (replacement) step: For any non bidirected 1-arc $e = uv$, find $E(u)$-replacement e^r of e.

As we do not delete anything in a 3rd priority step, the set of half-integral C-kernels does not change by this step. Next we study replacements of 1-arcs.

Lemma 7.23. *Assume that no 1st and 2nd priority steps can be made and that 1-arc $e = uv$ is not bidirected, that is, vu is not a 1-arc. Then there exists an $E(u)$-replacement e^r of e.*

Proof. By Lemma 7.22 and the increasing property of C_u, we have

$$|A^+(u)| = |A^-(u)| = |C_u(A^-(u))| \leq |C_u(E(u) \setminus \{e\})| \leq |C_u(E(u))| = |A^+(u)| \ , \quad (7.10)$$

where the first equality is from Lemma 7.22, the second is follows from the impossibility of a rejection step, the inequalities are due to the increasing property of C_v and the last equality comes from the definition of a 1-arc. Hence there is equality throughout (7.10). In particular, we see that $|C_u(E(u) \setminus \{e\})| = |A^+(u)|$. By substitutability of C_u, we have $A^+(u) \setminus \{e\} = C_u(E(u)) \setminus \{e\} \subseteq C_u(E(u) \setminus \{e\})$. This means that the $E(u)$-replacement of e (that is, $C_u(E(u) \setminus \{e\}) \setminus A^+(u)$) is a unique edge of $E(u)$. □

Lemma 7.24. *Assume that no 1st and 2nd priority step can be executed and $e = uv$ is a 1-arc such that vu is not a 1-arc. If $e^r = uw$ is the $E(u)$-replacement of e, then $D_w(\{e^r\} \cup A^-(w))$ contains exactly one edge, say $e^r_r = xw$. Moreover, $e^r_r = xw$ is a 1-arc and its reverse wx is not a 1-arc.*

Proof. By the 2nd priority step $e^r \notin D_w(A^-(w))$, hence $e^r \in C_w(\{e^r\} \cup A^-(w))$. The increasing property of C_w gives that $|A^-(w)| = |C_w(A^-(w))| \leq |C_w(\{e^r\} \cup A^-(w))| \leq |C_w(E(w))| = |A^+(w)| = |A^-(w)|$, where the last equality is due to Lemma 7.22. So we have equality throughout, i.e. $|A^-(w)| = |C_w(\{e^r\} \cup A^-(w))|$, so e^r has a unique $(\{e^r\} \cup A^-(w))$-replacement $e^r_r = xw$. Clearly, if wx was a 1-arc then $e^r_r \in C_w(\{e^r\} \cup A^-(w))$ holds, a contradiction. □

Assume now that no 1st and 2nd priority steps are possible and the 3rd priority steps are also finished. As there exists a 1-arc $e = uv$ that is not bidirected, it has an $E(u)$-replacement e^r. Edge e^r_r in Lemma 7.24 is another 1-arc that is not bidirected. Following the alternating sequence of nonbidirected 1-arcs and their replacements (and possibly by changing the starting 1-arc to the first repetition in the sequence $e, e^r_r, (e^r_r)^r_r \dots$), we shall find a sequence $(e_1, (e_1)^r, e_2, (e_2)^r, \dots, e_m, (e_m)^r, e_{m+1} = e_1)$ in such a way that $(e_i)^r_r = e_{i+1}$ for $i = 1, 2, \dots, m$ and edges $e_1, e_2, \dots, e_m$ are different 1-arcs, none of them is bidirected. After Irving, we call such an alternating sequence $(e_1, (e_1)^r, e_2, (e_2)^r, \dots, e_m, (e_m)^r)$ of 1-arcs and edges a *rotation.*

Lemma 7.25. *Assume that $(S, S_1, \dots, S_l)$ is a half-integral C-kernel in graph G and no 1st, 2nd and 3rd priority step is possible. If $(e_1, (e_1)^r, e_2, (e_2)^r, \dots, e_m, (e_m)^r)$ is a rotation and $e_i = xv \in S(v)$ then $e_{i-1} = uw \in S^+(u)$ follows, where addition is meant modulo m.*

In particular, if $e_i \in S$ then $\{e_1, e_2, \dots, e_m\} \subseteq S$.

Proof. First we show that $e^r_{i-1} = uv \notin S^-(v) \cup D_v(S^-(v))$. Indirectly, assume $e^r_{i-1} = uv \in S^-(v) \cup D_v(S^-(v))$. If $f = zv \in A^-(v)$ is a 1-arc then f (being a first choice) cannot be dominated at z, so $f \in S^-(v) \cup D_v(S^-(v))$ follows, that is,

$$A^-(v) \subseteq S^-(v) \cup D_v(S^-(v)) \ . \tag{7.11}$$

By Lemma 7.16,

$$e_i = (e_{i-1})^r_r \in D_v(A^-(v) \cup \{(e_{i-1})^r\}) \subseteq D_v(S^-(v) \cup D_v(S^-(v)) \cup \{(e_{i-1})^r\}) = D_v(S^-(v))$$

so $e_i \notin S(v)$, a contradiction. Thus $e^r_{i-1} = uv \notin S^-(v)$, hence $e^r_{i-1} = uv \notin S^+(u)$ and $e^r_{i-1} \notin D_v(S^-(v))$, hence $e^r_{i-1} \in D_u(S^+(u))$. As e^r_{i-1} is the $E(u)$-replacement of first choice e_{i-1}, it follows that $e_{i-1} \in S^+(u)$, as Lemma 7.25 claims. □

Lemma 7.26. *Assume that no 1st, 2nd and 3rd priority step is possible for the current graph and that $(S, S_1, \dots, S_l)$ is a half-integral C-kernel. If $(e_1, (e_1)^r, e_2, (e_2)^r, \dots, e_m, (e_m)^r)$ is a rotation then sets $\{e_1, e_2, \dots, e_m\}$ and $\{e^r_1, e^r_2, \dots, e^r_m\}$ are either disjoint or identical.*

If $\{e_1, e_2, \dots, e_m\} = \{e^r_1, e^r_2, \dots, e^r_m\}$ then m is odd and $\{e_1, e_2, \dots, e_m\}$ is one of the S_j's.

If sets $\{e_1, e_2, \dots, e_m\}$ and $\{e^r_1, e^r_2, \dots, e^r_m\}$ are disjoint and $e_i \in S$ then $\{e_1, \dots, e_m\} \subseteq S \setminus (S_1 \cup \dots \cup S_l)$. Moreover, $(S', S_1, \dots, S_l)$ is a half-integral C-kernel for $S' = S \setminus \{e_1, e_2, \dots, e_m\} \cup \{e^r_1, e^r_2, \dots, e^r_m\}$.

Proof. Assume first that sets $\{e_1, e_2, \dots, e_m\}$ and $\{e^r_1, e^r_2, \dots, e^r_m\}$ are not disjoint, so $e_i = uv = (e_{i+k})^r$ for some $i \in \{1, 2, \dots, m\}$ and $1 \leq k < m$, where addition is meant modulo m. If 1-arc e_{i+k} is a first choice of x then $(e_{i+k})^r$ cannot be the first choice of x by the definition of a replacement. This means that 1-arc $e_{i+k} = vw$ is a first choice of v (and not of u). As $(e_{i+k})^r$ is an $E(v)$-replacement of first choice e_{i+k} of v, it follows that $(e_{i+k})^r \in D_v(F)$ implies $e_{i+k} \in F$. Choose $F = A^-(v) \cup \{(e_{i-1})^r\}$. We know that $(e_{i+k})^r = e_i = (e_{i-1})^r_r \in D_v(F)$ by the definition of $(e_{i-1})^r_r$, hence $e_{i+k} \in A^-(v) \cup \{(e_{i-1})^r\}$. As $e_{i+k} = vw$ is a 1-arc in the rotation, $e_{i+k} \notin A^-(v)$, so $e_{i+k} = (e_{i-1})^r$.

We proved that $e_i = (e_{i+k})^r$ implies $e_{i+k} = (e_{i-1})^r$. The above argument for the latter equality means that $e_{i-1} = (e_{i+k-1})^r$. That is, $e_i = (e_{i+k})^r$ yields $e_{i-1} = (e_{i+k-1})^r$, $e_{i-2} = (e_{i+k-2})^r$, $e_{i-3} = (e_{i+k-3})^r$, and so on. So for any j, we have $e_j = (e_{j+k})^r$. In particular, we see that $\{e_1, e_2, \dots, e_m\} = \{e_1^r, e_2^r, \dots, e_m^r\}$.

Another consequence is that $e_{i+k} = (e_{i-1})^r = e_{i-1-k}$, so $i + k \equiv i - 1 - k (mod\ m)$, that is, $2k + 1 \equiv 0(mod\ m)$. As $1 \le k < m$, we get that $m = 2k + 1$, and edges $e_1, e_2, \dots, e_m$ form a cycle in order $e_1, e_{k+1}, e_m, e_k, e_{m-1}, e_{k-1}, \dots, e_2, e_{k+2}$. We shall prove that $\{e_1, e_2, \dots, e_m\} \subseteq S$. If $e_i \in S$ then this follows by Lemma 7.25. Otherwise, $(e_{i+k})^r = e_i \notin S$. This means that $(e_{i+k})^r \in D_v(S^-(v))$, so $e_{i+k} \in S^-(v) \subseteq S(v)$, and $\{e_1, e_2, \dots, e_m\} \subseteq S$ by Lemma 7.25 again.

By property (7.11), $A^-(v) \subseteq S^-(v) \cup D_v(S^-(v))$. Hence, by Lemma 7.16, we have

$$\begin{aligned} e_i = (e_{i-1})_r^r \in D_v(A^-(v) \cup \{(e_{i-1})^r\}) \\ = D_v(A^-(v) \cup \{e_{i+k}\}) \subseteq D_v(S^-(v) \cup D_v(S^-(v)) \cup \{e_{i+k}\}) \\ \subseteq D_v(S^-(v) \cup D_v(S^-(v)) \cup S^+(v)) = D_v(S^+(v)) \ , \end{aligned}$$

thus, $e_i \in S(v) \setminus S^+(v)$. This means that e_i belongs to one of the cycles S_j of half-integral C-kernel $(S, S_1, S_2, \dots, S_l)$, and e_i is the replacement of e_{i+k} for all i. Thus $\{e_1, e_2, \dots, e_m\}$ is indeed one of the S_j's.

To finish the proof, we settle the remaining case when set $\{e_1, e_2, \dots, e_m\}$ is disjoint from $\{(e_1)^r, (e_2)^r, \dots, (e_m)^r\}$. If $e_i \in S$ then $\{e_1, e_2, \dots, e_m\} \subseteq S$ and 1-arc $e_i = uv$ is in $S^+(u)$ by Lemma 7.25. If, indirectly $e_i \in S_j$, so $e_i \notin S^-(u)$ then $(e_i)^r = uz \notin D_u(S^-(u))$ as $(e_i)^r$ is the $E(u)$-replacement of e_i. So either $(e_i)^r \in D_z(S^-(z))$ or $(e_i)^r \in S^-(u)$. In the former case, $e_{i+1} = (e_i)_r^r \in D_z(A^-(z) \cup \{(e_i)^r\})$. By property (7.11), $A^-(z) \subseteq S^-(z) \cup D_z(S^-(z))$, so by Lemma 7.16 we have $e_{i+1} \in D_z(S^-(z) \cup D_z(S^-(z)) \cup \{(e_i)^r\}) = D_z(S^-(z))$, that contradicts to $e_{i+1} \in S$. So $(e_i)^r \in S^-(u)$ holds. As $e_{i+1} \in S$ is the $E(v)$-replacement of $(e_i)^r$, this can only happen if $e_i, (e_i)^r$ and e_{i+1} all belong to the same odd cycle S_j. The argument shows that S_j is exactly the cycle $(e_1, (e_1)^r, e_2, (e_2)^r, \dots, e_m, (e_m)^r)$, which is a contradiction, as $|S_j|$ is not odd. So $\{e_1, \dots, e_m\} \subseteq S \setminus (S_1 \cup \dots \cup S_l)$, as we claimed.

Consider $(S', S_1, \dots, S_l)$. (Recall that $S' = S \setminus \{e_1, e_2, \dots, e_m\} \cup \{e_1^r, e_2^r, \dots, e_m^r\}$.) To see that it is a half-integral C-kernel we check the three properties of the definition. Fix a vertex v and let R^+ and R^- be the set of 1-arcs of rotation $(e_1, (e_1)^r, e_2, (e_2)^r, \dots, e_m, (e_m)^r)$ that leave and enter v, respectively. Let moreover $T^+ := \{(e_i)^r : e_i \in R^+\}$ and $T^- := \{(e_{i-1})^r : (e_{i-1})_r^r = e_i \in R^-\}$. Let S^* be one of the sets $S(v)$, $S^-(v)$ or $S(v) \setminus \{e\}$ for some edge e of some $S_j^+(v)$. To prove properties 1. and 2., we show that $C_v(S^* \setminus (R^+ \cup R^-) \cup T^+ \cup T^-) = C_v(S^*) \setminus (R^+ \cup R^-) \cup T^+ \cup T^-$.

The definition of replacement, property (7.11) and the monotonicity of D_v (Lemma 7.16) implies that $R^- \subseteq D_v(A^-(v) \cup T^-) \subseteq D_v(S^-(v) \cup D_v(S^-(v)) \cup T^-) \subseteq D_v(S^-(v) \cup T^-) \subseteq D_v(S^* \cup T^-)$, hence R^- is disjoint from $C_v(S^* \cup T^-)$ and hence $C_v(S^* \cup T^-) = C_v(S^* \cup T^- \setminus R^-)$. Substitutability of C_v gives $C_v(S^* \cup T^-) \cap S^* \subseteq C_v(S^*)$, thus $C_v(S^* \cup T^- \setminus R^-) = C_v(S^* \cup T^-) \subseteq C_v(S^*) \setminus R^- \cup T^-$. The increasing property of C_v implies that $|(C_v(S^*)| \le |C_v(S^* \cup T^-)| \le |C_v(S^*) \setminus R^- \cup T^-| = |C_v(S^*)| - |R^-| + |T^-|$, so from $|R^-| = |T^-|$ we get that $C_v(S^* \cup T^-) = C_v(S^* \cup T^- \setminus R^-) = C_v(S^*) \cup T^- \setminus R^-$.

Assume that edge $(e_i)^r = uv \in T^+$ is in S. Property (7.11) shows that $A^-(u) \subseteq S^-(u) \cup D_u(S^-(u))$, so $e_{i+1} = (e_i)_r^r \in D_u(A^-(u) \cup \{(e_i)^r\}) \subseteq D_u(S^-(u) \cup D_u(S^-(u)) \cup \{(e_i)^r\}) = D_u(S^-(u) \cup \{(e_i)^r\})$. This contradicts $e_{i+1} \in S \setminus (S_1 \cup \dots \cup S_l)$. This argument proves that $(e_i)^r \notin S$ and that $(e_i)^r \notin D_u(S(u))$. So $(e_i)^r$ has to be dominated at v: $(e_i)^r \in D_v(S^-(v))$. As $(e_i)^r$ was an arbitrary edge of T^+, we proved that T^+ is disjoint from S, moreover $T^+ \subseteq D_v(S^-(v)) \subseteq D_v(S^* \cup S^-) = D_v(S^*) \subseteq D_v(S^* \cup T^-) = D_v(S^* \cup T^- \setminus R^-)$ (we used the monotonicity of D_v), thus $C_v(S^* \cup T^- \setminus R^-) = C_v(S^* \cup T^- \setminus R^- \cup T^+)$. We use the substitutability of C_v for $S^* \cup T^- \cup T^+ \setminus (R^- \cup R^+) \subseteq S^* \cup T^- \cup T^+ \setminus R^-$:

$$\begin{aligned} C_v(S^*) \cup T^- \setminus (R^- \cup R^+) &= \left(C_v(S^*) \cup T^- \setminus R^-\right) \cap \left(S^* \cup T^- \cup T^+ \setminus (R^- \cup R^+)\right) \qquad (7.12)\\ &= C_v(S^* \cup T^- \setminus R^-) \cap \left(S^* \cup T^- \cup T^+ \setminus (R^- \cup R^+)\right)\\ &= C_v(S^* \cup T^- \cup T^+ \setminus R^-) \cap \left(S^* \cup T^- \cup T^+ \setminus (R^- \cup R^+)\right)\\ &\subseteq C_v(S^* \cup T^- \cup T^+ \setminus (R^- \cup R^+)) \ . \end{aligned}$$

As edges of T^+ are $E(v)$-replacements, it follows that $T^+ \subseteq C_v(S^* \cup T^- \cup T^+ \setminus (R^- \cup R^+))$, so with (7.12) we have $C_v(S^*) \cup T^- \cup T^+ \setminus (R^- \cup R^+) \subseteq C_v(S^* \cup T^- \cup T^+ \setminus (R^- \cup R^+))$. The increasing property of C_v gives that

$$\begin{aligned} |C_v(S^*)| + |T^-| + |T^+| - (|R^-| + |R^+|) &= |C_v(S^*) \cup T^- \cup T^+ \setminus (R^- \cup R^+)|\\ &\le |C_v(S^* \cup T^- \cup T^+ \setminus (R^- \cup R^+))| \le |C_v(S^* \cup T^- \cup T^+ \setminus R^-)|\\ &= |C_v(S^*) \cup T^- \setminus R^-| = |C_v(S^*)| + |T^-| - |R^-| \ , \end{aligned}$$

hence $C_v(S^*) \cup T^- \cup T^+ \setminus (R^- \cup R^+) = C_v(S^* \cup T^- \cup T^+ \setminus (R^- \cup R^+))$, as we claimed. So we justified properties 1. and 2. for $(S', S_1, \dots, S_l)$.

For property 3., we have already seen that $C_v(S'(v))$ is disjoint from S', so it remains to check that any edge $e \in E \setminus S'$ is dominated at some vertex. There are two cases for e: either $e = e_i \in R^-$ is a 1-arc of our rotation. The above argument for $S^* = S^-(v)$ shows that $R^- \subseteq D_v((S')^-(v))$, so we may assume that $e \notin S$, hence $e \in D_v(S^-(v))$ for some vertex v. Again the above proof shows that everything that $S^-(v)$ is dominating according to C_v is also dominated by $(S')^-(v)$, except for R^+. This proves property 3.□

4th priority (rotation elimination) step: Find a rotation $(e_1, (e_1)^r, e_2, (e_2)^r, \dots, e_m, (e_m)^r)$ with disjoint sets $\{e_1, e_2, \dots, e_m\}$ and $\{e_1^r, e_2^r, \dots, e_m^r\}$ and delete $\{e_1, e_2, \dots, e_m\}$.

To justify the rotation elimination step, we only have to check that it does not create a new half-integral C-kernel.

Lemma 7.27. *Any half-integral C-kernel after a 4th priority step is also a half-integral C-kernel before this step.*

Proof. Observe that after the elimination, each edge $(e_i)^r$ becomes a 1-arc. Assume that $(S, S_1, \dots, S_l)$ is a half-integral C-kernel after the step. As in the proof of Lemma 7.26, let R^- denote the set of deleted 1-arcs of our rotation that enter a fixed vertex v, let $T^- := \{(e_{i-1})^r : (e_{i-1})^r_r = e_i \in R^-\}$ be the new 1-arcs entering v and let A^- be the set of those 1-arcs that enter v and have not been deleted during the step.

By the definition of the rotation, from property (7.11) and the monotonicity of D_v we get $R^- \subseteq D_v(A^- \cup R^- \cup T^-) = D_v(A^- \cup T^-) \subseteq D_v(S^-(v) \cup D_v(S^-(v))) = D_v(S^-(v))$, and this is exactly what we wanted to prove. □

The following theorem finishes the proof of Theorem 7.19.

Theorem 7.28. *If no 1st, 2nd, 3rd and 4th priority step can be made on graph G then $(E(G), S_1, S_2, \ldots, S_l)$ is a half-integral C-kernel, where cycles S_i are given by the rotations.*

Proof. We have already seen that if all 1-arcs are bidirected then we have a C-kernel, which is a special case of a half-integral C-kernel. So assume that no further step can be executed but we still have a 1-arc e that is not bidirected. We have also seen that if we follow the alternating sequence of non bidirected 1-arcs and their replacements $e, e^r, e^r_r, (e^r_r)^r, (e^r_r)^r_r, \ldots$ then we find a rotation $(e_1, (e_1)^r, e_2, (e_2)^r, \ldots, e_m, (e_m)^r)$, that must be an odd cycle S_i that cannot be eliminated. This means that $m = 2k+1$, and $e_i = (e_{i+k})^r = (e_{i-1})^r_r$ for $1 \le i \le m$, where addition is modulo m. We shall prove that our starting point, 1-arc e is an edge of this rotation. Hence, if our algorithm cannot make a step then each 1-arc is either bidirected or belongs to exactly one odd rotation.

To this end, we may assume that e^r_r is an edge of the rotation, namely $e^r_r = uv = e_i = (e_{i+k})^r$. That is, e_i is the $E(v)$-replacement of first choice e_{i+k}, that is $C_v(E(v) \setminus \{e_{i+k}\}) \cup D_v(E(v) \setminus \{e_{i+k}\}) = E(v) \setminus \{e_{i+k}\}$, in other words, if $e_i \in D_v(X)$ for some subset X of $E(v)$ then $e_{i+k} \in X$ must hold. From the definition of e^r_r it follows that $e_i \in D_v(A^-(v) \cup \{e_r\})$, so $e_{i+k} \in A^-(v) \cup \{e_r\}$. 1-arc $e_{i+k} \in A^+(v)$ is not bidirected, hence $e_{i+k} \notin A^-(v)$, thus $e_{i+k} = e^r$ is an edge of the rotation, as well.

Similarly as above, $e_{i+k} = vw$ is the $E(w)$-replacement of first choice e_{i-1} of w, hence $C_w(E(w) \setminus \{e_{i-1}\}) \cup D_w(E(w) \setminus \{e_{i-1}\}) = E(w) \setminus \{e_{i-1}\}$. This means that if e_{i+k} is the $E(w)$-replacement of edge e then $e = e_{i-1}$ must hold. So e is an edge of our rotation, and all non bidirected 1-arcs of G belong to odd rotations.

Next we prove that all edges of G are 1-arcs. If, indirectly, $e = uv$ is not a 1-arc, then $e \in D_u(A^+(u))$ by the 1st priority step and $e \notin D_u(A^-(u))$ by the 2nd priority step. So u is incident with certain nonbidirected 1-arcs such that these 1-arcs all belong to odd rotations, and each 1-arc of $A^-(u)$ is the $E(u)$-replacement of pairwise different 1-arcs of $A^+(u)$. As $e \notin D_u(A^-(u))$, we have $e \in C_u(A^-(u) \cup \{e\})$, and $|C_u(A^-(u) \cup \{e\})| \le |C_u(E(u))| = |C_u(A^+(u))| = |C_u(A^-(u))|$ implies that there is a unique 1-arc $f \in A^-(u)$ that is dominated: $f \in D_u(A^-(u) \cup \{e\})$. However, f is the $E(u)$-replacement of some other 1-arc $g \in A^+(u)$, and we have already seen twice in this proof that this means that g is a member of each edge set that C_v-dominates f: $g \in A^-(u) \cup \{e\}$. Clearly, this is a contradiction as 1-arc g is not bidirected and leaves u and e is not a 1-arc.

So if the algorithm cannot make any further step then our graph consists of bidirected 1-arcs and odd rotations $S_1, S_2, \ldots, S_k$. It is trivial from the definition that $(E(G), S_1, \ldots, S_k)$ is a half-integral C-kernel. □

7.4.1 Complexity issues and structural results

Irving's original algorithm [45] is very efficient: it runs in linear time. However, this algorithm is different from the one that we get if we apply our algorithm to an ordinary stable roommates problem. The difference is that Irving's algorithm has two phases: in the first phase it makes only 1st and 2nd priority steps, and after the 1st phase is over, it keeps on eliminating rotations, and never gets back to the 1st phase. The explanation is that Irving's rotation elimination deletes not just first choices but removes some other edges as well.

Actually, it is rather straightforward to modify our algorithm to work similarly, and this improves even its time-complexity. The reason that we did not do this in the previous section is that the proof is more transparent this way. So how can we speed up the algorithm?

Observe that after a rotation elimination (4th priority) step, if 1-arc $e_i = uv$ is deleted, then e_i ceases to be a first choice of u. The new first choice instead of e_i will be its replacement $(e_i)^r$. So we can (with no extra cost) orient each replacement edge. Of course, refusal (2nd priority) steps may still be possible, but only at those vertices that the newly created 1-arcs enter. The definition of a rotation implies that if we apply a refusal step at such a vertex then no 1-arc gets deleted, but we might delete some undirected edges. So if we modify the rotation elimination step in such a way that we also include these extra 1st and 2nd priority steps within the rotation elimination step, then once we start to eliminate rotations, we never go back to the first phase. That is, we shall never have to make a proposal or a rejection step again.

To analyze the above (modified) algorithm, we have to say something about the calculation of the choice function and the dominance function. Assume that functions C_v and D_v are given by an oracle for all vertices v of G, such that for an arbitrary subset X of $E(v)$ these oracles output $C_v(X)$ and $D_v(X)$ in unit time. Note that if we have only the oracle for D_v then we can easily construct one for C_v from the identity $C_v(X) = X \setminus D_v(X)$. Similarly, if we have an oracle for C_v then $D_v(X)$ can be calculated by $O(n)$ calls of the C_v-oracle, according to the definition of D_v. (As usual, n and m denotes the number of vertices and edges of G, respectively.)

The algorithm starts with n C-calls, and continues with n D-calls. After this, each deletion in a 2nd priority step involves one C-call at vertex u where we deleted, and, if this C-call generates a new 1-arc $e = uv$, then we also have to make one D-call at the other end v of e. So the first phase (the 1st and 2nd priority steps) uses at most $O(n + m)$ C-calls and $O(n + m)$ D-calls.

In the second phase, the algorithm makes 3rd priority steps and modified rotation elimination steps. We do it in such a way that we start from a nonbidirected 1-arc e and follow the sequence $e, e^r, e^r_r, (e^r_r)^r, \ldots$, until we find a rotation. The rotation will be a suffix of this sequence, and after eliminating this rotation, we reuse the prefix of this sequence, and continue the rotation search from there. This means that for the 3rd type steps we need altogether $O(m)$ C-calls. The modified rotation elimination steps consist of deleting each 1-arc $e_i = uv$ of the rotation, orienting edges $(e_i)^r = uw$ and applying refusal steps at vertices w. As we delete at most m edges in all rotation eliminations, this will add at most $O(m)$ D-calls. All additional work of the algorithm can be allocated to the oracle calls, so we got the following.

Theorem 7.29 (Fleiner [22]). *If we modify the rotation elimination step as described above, then our algorithm uses $O(n + m)$ C-calls and D-calls to find a half-integral C-kernel and runs in linear time.*

We have seen, in case of bipartite graphs a C-kernel always exists for path independent substitutable choice functions (see [20]). That is, we do not have to require the increasing property of functions C_v if we want to solve the C-kernel problem on a bipartite graph. A natural question is if it is possible to generalize our result on C-kernels to substitutable, but not necessarily increasing choice functions on nonbipartite graphs. In our proof, we heavily used the fact that if no proposal and rejection steps can be made then each 1-arc has a replacement and these replacements improve some other 1-arc at their other vertices. This property is not valid in the more general setting. Below we show that the C-kernel problem for substitutable choice functions is NP-complete by reducing the 3-SAT problem to it.

Theorem 7.30 (Fleiner [22]). *For any 3-CNF boolean expression ϕ, we can construct a graph G_ϕ and path independent substitutable choice functions C_v on the stars of G_ϕ in polynomial time in such a way that ϕ is satisfiable if and only if there exists a C-kernel in G_ϕ for the choice functions C_v.*

Proof. Define directed graph $\vec{G}_\phi$ such that $\vec{G}_\phi$ has three vertices a_C, b_C and v_C for each clause C of ϕ and two vertices t_x and f_x for each variable x of ϕ. The arc set of $\vec{G}_\phi$ consists of arcs t_xf_x and f_xt_x for each variable x of ϕ, arcs of type v_Ct_x (and v_Cf_x) if literal x (literal $\bar{x}$) is present in clause C of ϕ. Moreover, we have arcs a_Cb_C, b_Cv_C and v_Ca_C for each clause C of ϕ. If A is a set of arcs incident with some vertex v of $\vec{G}_\phi$ then $C'_v(A) = A$ if no arc of A leaves v, otherwise $C'_v(A)$ is the set of arc of A that leave v. It is easy to check that choice function C'_v is path independent and substitutable. Let G_ϕ be the undirected graph that corresponds to $\vec{G}_\phi$ and let C_v denote the choice function induced by $C'(v)$ on the undirected edges of G_ϕ. We shall show that ϕ is satisfiable if and only if there is a C-kernel of G_ϕ for choice functions C_v, that is, if and only if there is a subset S of arcs of $\vec{G}_\phi$ such that S does not contain two consecutive arcs and for any arc uv outside S there is an arc vw of S.

Assume now that ϕ is satisfiable, and consider an assignment of logical values to the variables of ϕ that determine a truth evaluation of ϕ. If the value of variable x is true then add arc f_xt_x, if it is false, then add arc t_xf_x to S. Do this for all variables of ϕ. Furthermore, add all arcs a_Cb_C to S. If variable x is true then add all arcs v_ct_x to S for all clauses that contain variable x. If variable y is false then add all arcs v_cf_x to S for all clauses that contain negated variable $\bar{y}$. Clearly, the just defined S does not contain two consecutive arcs. If some arc of type t_xf_x or f_xt_x is not in S then it is dominated by the other, which is in S. Each arc of type v_Ca_C is dominated by arc a_Cb_C of S and each arc of type b_Cv_C is dominated by some arc of type xt_x or yf_y as C has a variable that makes C true.

To finish the proof, assume that S is a C-kernel of $\vec{G}_\phi$. Observe that for each variable x either t_xf_x or f_xt_x belongs to S, as no other arc dominates these arcs. If $t_xf_x \in S$ then set variable x to be false, else assign logical value true to x. We have to show that for this assignment the evaluation of each clause C is true, that is, there is an arc of S from v_C to some t_x or f_x. Indirectly, if there is no such arc then the corresponding edges of S should form a C-kernel on directed circuit $v_Ca_Cb_C$, which is impossible.

So the decision problem of the existence of a C-kernel is NP-complete. □

Note that though Theorem 7.30 shows that the C-kernel problem is difficult for nonincreasing choice functions, it does not imply that Theorem 7.19 fails for substitutable choice functions. Actually, the increasing property is encoded into the definition of a half-integral C-kernel, as replacements of a single element cannot contain more than one element. So, in this sense Theorem 7.19 is not true for a directed cycle of length three if we add a parallel copy to each edge and use choice functions from the proof of Theorem 7.30. However, there is a natural way to extend the definition of a half-integral C-kernel so that a generalization of Theorem 7.19 makes more sense. As we cannot state here any nontrivial fact, we do not go into the details.

The following theorem is an extension of the well-known Rural Hospital Theorem that states that if a hospital cannot fill up its quota with residents in some stable outcome, then no matter which stable outcome is selected, it always receives the same applicants.

Theorem 7.31 (Fleiner [22]). *Assume that S and S' are C-kernels on (not necessarily bipartite) graph $G = (V, E)$. Then $|S(u)| = |S'(u)|$ for each vertex u of G. Moreover, if C_v is a linear choice function with quota q then $|S(v)| < q$ implies $S(v) = S'(v)$.*

Proof. No edge xy of $S \setminus S'$ is blocking S', so S' must dominate xy at x or at y. Similarly, each edge of $S' \setminus S$ is dominated by S at one end vertex. So we can orient each edge of the symmetric difference $S \Delta S'$ to that end vertex where the particular edge is dominated.

We prove that for any vertex u of G the number of oriented edges pointing to u is not more than the number of oriented edges leaving u. Let S_+, S_-, S'_+ and S'_- denote those oriented edges of S and S' that leave and enter vertex u and let $T := S(u) \cap S'(u)$. By the definition of the orientation we have that $S(u) = C_u(S(u) \cup S'(u))$ and $S'(u) = C_u(S'(u) \cup S_-)$. Let $X := S_- \cup S'_- \cup T$. As $X \subseteq S(u)$, we get by substitutability of C_u that $S_- \cup T \subseteq C(X)$. From $X \subseteq S'(u)$ it follows that $S'_- \cup T \subseteq C(X)$, hence $C(X) = X$ follows. The increasing property of C_u implies that

$$|S_-| + |S'_-| + |T| = |C_u(X)| \leq |C_u(S(u))| = |S(u)| = |S_-| + |T| + |S_+| \ ,$$

hence $|S'_-| \leq |S_+|$. A similar proof shows (with exchanging the role of S and S' that $|S_-| \leq |S'_+|$. So indeed: $|S_- \cup S'_-| \leq |S_+ \cup S'_+|$, that is, for each vertex u at least as many oriented edges leave u as enter it. As each oriented edge is leaving and entering exactly one vertex, the latter inequality can hold only if there is equality, that is our oriented edges form an Eulerian graph, and hence $|S(u)| = |S'(u)|$. This proves the first part of the theorem.

Consider now our vertex v that could not fill up its quota with C-kernel S. If $S(v) \neq S'(v)$ then $S(v) \Delta S'(v)$ is nonempty, and half of its edges have to be oriented only towards v. So there is at least one edge of the symmetric difference that is dominated at u by $S(v)$ or $S'(v)$. This means that v could fill up its quota in S or in S', a contradiction.□

The first part of Theorem 7.31 is an implication of the following result that generalizes a result by Cechlárová and Fleiner [13] on the splitting property of stable b-matchings.

Theorem 7.32 (Fleiner [22]). *Let S be a C-kernel for graph $G = (V, E)$ and increasing substitutable choice functions C_v. For each vertex v it is possible to partition $E(v)$ into (possibly empty) parts $E_0(v), E_1(v), E_2(v), \dots, E_{|S(v)|}(v)$ in such a way that for any C-kernel S' we have $S' \cap E_0(v) = \emptyset$ and $|S' \cap E_i(v)| = 1$ holds for $i = 1, 2, \dots, |S(v)|$.*

Proof. Let us find some C-kernel S by the algorithm in the previous section. Fix a vertex v and determine the partition of $E(v)$ in the following manner. Each element of $S(v)$ will belong to a different part. Follow the algorithm backwards, that is, we start from S and we build up the original G by adding edges according to the deletions of the algorithm. If we add an edge that is not incident with v, then we do not do anything. If we add an edge e of $E(v)$ that was deleted by a 2nd priority step, then we put e into part $E_0(v)$. This is a good choice, since e is contained in no C-kernel. If e was deleted in a 4th priority step along a rotation then this rotation contains another (replacement) edge f incident with v. Lemma 7.26 shows that if we assign e to that part $E_i(v)$ that contains

f, then still no C-kernel can contain two edges of the same part $E_i(v)$. Let us build up the graph by backtracking the algorithm. This way, we find a part for each edge of $E(v)$, and this partition clearly has the property we need. □

There is an aesthetic problem with Theorem 7.32, namely, that part $E_0(v)$ of the star of v is redundant in the following sense. If we remove all edges from $E_0(v)$ and independently from one another we assign each of them to an arbitrary part $E_i(v)$ (for $1 \leq i \leq |S(v)|$) then the resulted partition also satisfies the requirements of Theorem 7.32 and $E_0(v) = \emptyset$ for all vertices v. In what follows, by proving a strengthening of Theorem 7.32, we exhibit an interesting connection between the C-kernel problem and the stable roommates problem.

If $G = (V, E)$ is a graph and v is a vertex of it then *detaching v into k parts* is the inverse operation of merging k vertices into one vertex. That is, we delete vertex v, introduce new vertices $v^1, v^2, \ldots, v^k$ and each edge that was originally incident with v will be incident with one of $v^1, v^2, \ldots, v^k$. If $k : V \to \{1, 2, 3, \ldots\}$ is a function then a *k-detachment of G* is a graph G^k that we get by detaching each vertex v of G into $k(v)$ parts. Clearly, there is a natural correspondence between the edges of G and those of G^k. With this notation, there is an equivalent formulation of Theorem 7.32: if G is a graph, and increasing substitutable choice function C_v is given for each vertex v of G and S is a C-kernel then there exists a k-detachment G^k of G in such a way that any C-kernel of G corresponds to a matching of G^k, where $k(v) := |S(v)|$ for each vertex v of G.

Theorem 7.33 (Fleiner [22]). *Let S be a C-kernel for graph $G = (V, E)$ and increasing substitutable choice functions C_v. Let $k(v) := \max\{|S(v)|, 1\}$. There is a k-detachment G^k of G and there are linear orders $<_{v_i}$ on the stars of G^k such that any C-kernel of G corresponds to a stable matching of G^k.*

Proof. Just like in the proof of Theorem 7.32, we start from a C-kernel S', produced by our algorithm and we build up G^k and construct orders $<_{v_i}$ by following the algorithm backwards.

Let $G_i = (V, E_i)$ denote the underlying graph after the ith step of our algorithm, that is, $G_0 = G$ and $G_t = (V, S')$ for some t. Assume that we have a k-detachment G_i^k of G_i and suitable linear orders such that any C-kernel of G_i (for the restricted choice functions $C_v|_{E_i}$) corresponds to a stable matching of G_i^k. We show how to find a k-detachment G_{i-1}^k of G_{i-1} and extensions of the linear orders such that any C-kernel of G_{i-1} corresponds to a matching of G_{i-1}^k. If we do this, then G_0^k with the constructed linear orders is a k-detachment we look for.

First we construct G_t^k by detaching G_t into a matching. This means that each vertex v incident with S' is detached into $|S'(v)| = |S(v)| = k(v)$ parts (and we do not detach isolated vertices of G_t). As each degree of G_t^k is 0 or 1, the linear orders are trivial. Clearly the only C-kernel S' of G corresponds to the unique stable matching of G_t^k. So assume we have have already constructed G_i^k and the linear orders. If the ith step of the algorithm was 1st or 3rd type then $G_{i-1} = G_i$, hence we can choose $G_{i-1}^k = G_i^k$ and the same linear orders on the stars.

Assume that the ith step is a 1st type rejection step, that is, we delete some edges incident with some vertex v, say $vu_1, vu_2, \ldots, vu_p$. Define k-detachment G_{i-1}^k by adding p edges to the G_i^k in such a way that the edge corresponding to vu_i will be edge (say) $v^1u_j^1$. The extended linear orders on the stars of G_{i-1}^k will be the same as those of G_i^k, except for we append the new edges $v^1u_j^1$ to the end of these orders, that is, these new edges will be the least preferred ones of the vertices. Lemma 7.21 implies that the set of C-kernels of G_i and of G_{i-1} is the same, so it is enough to check that no stale matching S^k of G_i^k that corresponds to a C-kernel S of G_i is blocked by some edge $v^1u_j^1$.

Edge vu_j is not blocking S, hence at least one of v and u_j is covered by S. Corollary 7.31 implies that $|S(v)| = |S'(v)|$ and $S(u_j)| = |S'(u_j)|$, so this means that S^k has an edge e that covers v^1 or u_j^1. The definition of the linear orders on the stars of G_{i-1} implies $v^1u_j^1$ is dominated by e, so $v^1u_j^1$ cannot block stable matching S^k.

The remaining case is that the ith step of the algorithm is a 4th type rotation elimination. Assume the eliminated rotation is $(e_1, (e_1)^r, e_2, (e_2)^r, \dots, e_m, (e_m)^r)$, so we delete edges $e_1, e_2, \dots, e_m$ where 1-arc $e_j = u_jv_j$ is a first choice of u_j. After the elimination, each edge $(e_j)^r = u_jv_{j+1}$ becomes a first choice of u_j for $j = 1, 2, \dots, m$.

To construct G_{i-1}^k, we add an edge e_j^k to G_i^k that correspond to e_j for $j = 1, 2, \dots, m$. Assume that edges $(e_{j-1})^r = u_{j-1}v_j$ and $(e_j)^r = u_jv_{j+1}$ of G_i correspond to edges $u_{j-1}^t v_j^{t'}$ and $u_j^s v_{j+1}^{s'}$, of G_i^k, respectively. Then the edge of G_{i-1}^k that corresponds to e_j will be $e_j^k := u_j^s v_j^t$. In other words, we choose k-detachment G_{i-1}^k in such a way that edges of the rotation correspond to a cycle. We insert e_j^k into the linear order of u_j^s in such a way that e_j^k and $((e_j)^r)^r$ are consecutive and e_j^k is preceding $((e_j)^r)^r$. We insert e_j^k into the linear order of v_j^t in such a way that and e_j^k and $((e_{j-1})^r)^k$ will also be consecutive according to the order of v_j^t, but e_j^k succeeds $((e_{j-1})^r)^k$. We do this for all $j = 1, 2, \dots, m$, hence determining k-detachment G_{i-1}^k and linear orders on its stars.

First we prove that for any eliminated edge e_j of the rotation, edge $((e_j)^r)^k$ is the first edge in the linear order of u_j^s in G_i^k. By the definition of the replacement and rotation elimination, $(e_j)^r$ is a first choice of u_j in G_i. So after the $(i-1)$st step of the algorithm, $(e_j)^r$ never could be an u_j-replacement of another edge. This means, that from the ith step on, we never inserted an edge right before $(e_j)^r$ in the linear order of u_j^s. So if $(e_j)^r$ is an edge of C-kernel S' produced by our algorithm then $(e_j)^r$ is still the most preferred edge of u_j^s. If $(e_j)^r$ is deleted after the $(i-1)$st step then we had to delete it in the lth step, in a rotation elimination, as a first choice of u_j. This means on one hand that $((e_j)^r)^k$ is first in the linear order of u_j^s, in G_l^k. As we did not insert any edge before $((e_j)^r)^k$ during the construction of $G_{l-1}^k, G_{l-2}^k, \dots, G_i^k$, we see that $((e_j)^r)^k$ is first in the linear order of u_j^s in G_i^k.

We prove that any C-kernel of G_{i-1} corresponds to a stable matching of G_{i-1}^k. If S is a C-kernel of G_{i-1} then either $e_1, e_2, \dots, e_m \in S$ or S is a C-kernel of G_i by Lemma 7.25. In the first case, Lemma 7.26 implies that

$$S \setminus \{e_1, e_2, \dots, e_m\} \cup \{(e_1)^r, (e_2)^r, \dots, (e_m)^r\}$$

is a C-kernel, hence it corresponds to a stable matching of G_i^k by the induction hypothesis. As we have chosen edges e_j^k and $((e_j)^r)^k$ and e_j^k and $((e_{j-1})^r)^k$ consecutive in the linear orders of u_j^s and of v_j^t, we see that no edge can block the matching that corresponds to S in G_{i-1}^k.

In the second case, when S is a C-kernel of G_i, we have to show that the stable matching of G_i^k that corresponds to S (by the induction hypothesis) is not blocked by edge e_j^k. If $((e_j)^r)^k$ is in the stable matching then it dominates e_j^k at v_j^t. If $((e_j)^r)^k$ does not belong to the stable matching then it cannot block it, hence, as $((e_j)^r)^k$ is the best edge of $u_{j+1}^{t'}$, the stable matching dominates it at v_j^t. So this matching that corresponds to S in G_{i-1}^k also dominates e_j^k. This completes the proof. □

Note that Theorem 7.33 is not an equivalence: it is not true that for any C-kernel problem there exists a detachment with appropriate linear orders in such a way that C-kernels correspond bijectively to stable matchings. A counterexample is a graph on two vertices, four parallel edges with opposite linear orders on the vertices. The choice function of both vertices is the best two edges of the offered set, that is the C-kernel problem is a stable 2-matching problem. It is easy to see that there are exactly 3 different C-kernels, but any 2-detachment has 1, 2 or 4 stable matching.

Conclusion

The aim of the present dissertation is to illustrate a fairly recent approach and some of its consequences to the interdisciplinary topic of stable matchings and its generalizations. The story below aims to explain this sentence in more details.

Some twenty years ago, this topic was interesting for roughly three more or less disjoint communities: Economists (including Game Theorists), Computer Scientists (in particular Algorithm Design people) and Mathematicians (especially Graph Theorists), with the latter group publishing significantly less on the topic than the first two. It seems that communication between these groups were limited in those days and perhaps due to this fact, these groups were not really aware of each other's achievements. Examples are choice functions ("invented" by Economists) that describe nonlinear preferences for practical applications or the graph terminology and graph theoretic methods that were not standard tools for the first two groups. With a bit of exaggeration one may say that a standard result back then was the description of an appropriate generalization of the deferred acceptance algorithm with a proof of correctness from scratch.

The present work is the account of the miracle that at a certain point powerful tools of the different groups started to work together giving new insights and producing interesting results. These tools were the choice function based approach of Kelso and Crawford, the translation of the problems to combinatorial (especially graph theoretical) language, the application of combinatorial methods (like reduction of problems to one another) and most importantly, finding the connection of various problems to fixed points of certain mappings. This new framework provides us with a novel approach that sometimes outperforms the "traditional" one. Namely, it allows us to prove earlier results simply by describing the problem with choice functions and checking that these choice functions have certain properties.

Though the new results were appealing, the communication barrier between the groups did not vanish. Independently of our work, Economists explored the fixed point machinery again and turned it into a horn of plenty for new results. Then, something unexpected occurred again. In 2012, Roth and Shapley received the Nobel Memorial prize in Economic Sciences for the theory of stable allocations and the practice of market design. This extraordinary success of the field attracted many young and mathematically capable colleagues to the topic. Hence perhaps it becomes possible at last that the different groups can join their forces and start to explore together. This is the main motivation behind our ongoing work.

The author of this dissertation hopes that if we look back after many years from now then the content of the present dissertation will be the introduction of a success story.

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