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EXTREMAL PROBLEMS FOR POSITIVE DEFINITE FUNCTIONS AND POLYNOMIALS

Budapest, 2009

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Personal Preface

The present work describes some results of my research done between 2001-2009. Among the three chapters the work of two were done in a great extent abroad, and/or with collaborators whom I met during my stay in Greece and France. Thanks must be given to the European Union for the two Marie Curie fellowships which I obtained and which indeed helped me to revive my interest in mathematics, to turn my attention to new problems, and to open up collaboration with new colleagues. The experiences of these fellowships greatly boosted my research activity in all aspects.

Chapter 3 on idempotent exponential polynomials is a joint work with Aline Bonami – moreover, at a certain point, we even used a suggestion from T. Tao, without whom we might have been blocked at a – rather early! - -stage in our progress. Chapter 2 about the so-called Turán extremal problem for positive definite functions is partially joint work with Mihalis Kolountzakis, and the chapter partially contains my further results. In Chapter 1 on the Turán type converse Markov inequalities in the complex plane, almost everything is exclusively my result – except the main theorem, where also an insightful suggestion of G. Halász is fruitfully employed.

Nevertheless, in all chapters I used many discussions, feedback, references etc., provided by many other colleagues. Mathematics research is not done in a lonely cell, without communication to others - and it is better, nicer and more fair to admit and record the many stimulating interactions than to behave like an outer-worldly creature, doing mathematics in itself, without relying on the stimulating milieu around. I especially enjoyed and benefitted from discussions with V. Totik, G. Halász, J. Kincses, E. Makai, I. Ruzsa and B. Farkas.

Also the Alfréd Rényi Institute of Mathematics in general provided a really outstanding environment for my research. Getting acquainted with other research and academic centers in Europe prompted me to appreciate more and more the place where I have the fortune to work. Hopefully for still some more years!

Each chapters have their own detailed introduction, so there is no need to describe the mathematical content here. Perhaps a few words about the selection of topics and my personal favorites is in order, however.

Putting together the material of a thesis serves several purposes. The candidate must choose a subject which is well-focused and can be explained in itself, while he is to present, in some way, his research work in general. In my case, since relatively independent, different topics frequently occurred in my work, an exhaustive presentation of my research would have required much more space and would not have been really focused. So I had to drop many abstract analysis topics - rendezvous numbers, polarization constants and their relation to general linear potential theory e.g. – which I like and which, on the other hand, do relate, through potential theory, to the harmonic analysis nature of most of the material here. Also, multivariate polynomial inequalities – one of my most cultivated areas – were neglected, too. People knowing my work may argue that this was not the right choice – but I had to make selection in order to keep the size reasonable. Several other issues, like e.g. the recently reviving area of periodic-, or invariant decomposition of functions, also had to be left aside. The current some one and a half hundred pages should be enough for any referee to read – I should not demand more work from anyone.

Still, I feel, that the selected topics more or less exhibit my research spectrum and style, give reasonable samples of my research results, and should be sufficient to give basis for an evaluation. And, after all, that is the main purpose of a thesis.

Most of the material here has been published or is under publication. Nevertheless, writing this summary also prompted me to finish three more papers, pending for long, for their writing was not so simple. So a positive side effect of writing this thesis is perhaps this forced success of finishing what could have been left unpublished otherwise.

Among these, I especially like the otherwise elementary treatise on the Blaschke Rolling Ball theorem. Geometry having been my favorite topics in secondary school, dealing with that brought back the good feelings of doing mathematics so constructively in those old days. A close second is, however, another issue, the very definition of uniform asymptotic upper density, explained and used in §2, which is not a "result", not in the strict sense, but I feel that it is still a very nice and useful mathematical finding which has appealing aesthetic value in itself. And, perhaps, not only aesthetic value, but also use: such an unexpectedly (to me) simple formulation of an extended notion could, and perhaps should, have many good applications in the future. I myself satisfactorily settled the issue I was after (a packing type estimate in the so-called Turán extremal problem), but I am convinced that the notion of u.a.u.d. itself is good for much more things.

Finally I would like to express my sincere gratitude to all those – abroad and in Hungary – who suggested, encouraged, helped, supported my application for the doctor of the academy degree, and my work in putting together all the materials for that. Without their continuous encouragement and support, I would have not accomplish this, not in the current period of my life. Nevertheless, such personal support is perhaps too personal to be recorded by names here. So without naming anyone whom I am really very thankful, let me just record that in the long run surely I will appreciate their support even more than now. Good colleagues and friends form an ever increasing asset of my life, and this aspect of my life is surely enriched by my current experience with this work.

Budapest, April 2009

Szilárd Gy. Révész

CHAPTER 1

Turán-Erőd type converse Markov inequalities for convex domains on the plane

1.1. Introduction

On the complex plane polynomials of degree n admit a Markov inequality[1] $\|p'\|_K \le c_K n^2 \|p\|_K$ on all convex, compact $K \subset \mathbb{C}$. Here the norm $\|\cdot\| := \|\cdot\|_K$ denotes sup norm over values attained on K.

In 1939 Paul Turán studied converse inequalities of the form $\|p'\|_K \ge c_K n^A \|p\|_K$. Clearly such a converse can hold only if further restrictions are imposed on the occurring polynomials p. Turán assumed that all zeroes of the polynomials must belong to K. So denote the set of complex (algebraic) polynomials of degree (exactly) n as $\mathcal{P}_n$, and the subset with all the n (complex) roots in some set $K \subset \mathbb{C}$ by $\mathcal{P}_n(K)$. The (normalized) quantity under our study is thus the "inverse Markov factor"

$$M_n(K) := \inf_{p\in\mathcal{P}_n(K)} M(p) \qquad \text{with} \qquad M := M(p) := \frac{\|p'\|}{\|p\|} . \tag{1.1}$$

Theorem 1.1.1 (**Turán,** [**20**, p. 90]). *If $p \in \mathcal{P}_n(D)$, where D is the unit disk, then we have*

$$\|p'\|_D \ge \frac{n}{2}\|p\|_D . \tag{1.2}$$

Theorem 1.1.2 (**Turán,** [**20**, p. 91]). *If $p \in \mathcal{P}_n(I)$, where $I := [-1, 1]$, then we have*

$$\|p'\|_I \ge \frac{\sqrt{n}}{6}\|p\|_I . \tag{1.3}$$

Theorem 1.1.1 is best possible, as the example of $p(z) = 1+z^n$ shows. This also highlights the fact that, in general, the order of the inverse Markov factor cannot be higher than n. On the other hand, a number of positive results, started with J. Erőd's work, exhibited convex domains having order n inverse Markov factors (like the disk). We come back to this after a moment.

Regarding Theorem 1.1.2, Turán pointed out by the example of $(1 - x^2)^n$ that the $\sqrt{n}$ order is sharp. The slightly improved constant $1/(2e)$ can be found in [**8**], but the value of the constant is computed for all fixed n precisely in [**6**]. In fact, about two-third of the paper [**6**] is occupied by the rather lengthy and difficult calculation of these constants, which partly explains why later authors started to consider this achievement the only content of the paper. Nevertheless, the work of Erőd was much richer, with many important ideas occurring in the various approaches what he had presented.

[1] Namely, to each point z of K there exists another $w \in K$ with $|w - z| \ge \text{diam}(K)/2$, and thus application of Markov's inequality on the segment $[z, w] \subset K$ yields $|p'(z)| \le (4/\text{diam}(K))n^2\|p\|_K$.

In particular, Erőd considered ellipse domains, which form a parametric family E_b naturally connecting the two sets I and D. Note that for the same sets E_b the best form of the Bernstein-Markov inequality was already investigated by Sewell, see [**18**].

THEOREM 1.1.3 (**Erőd,** [**6**, p. 70]). *Let* $0 < b < 1$ *and let* E_b *denote the ellipse domain with major axes* $[-1, 1]$ *and minor axes* $[-ib, ib]$*. Then*

$$\|p'\| \geq \frac{b}{2} n \|p\| \tag{1.4}$$

for all polynomials p *of degree* n *and having all zeroes in* E_b*.*

Erőd himself provided two proofs, the first being a quite elegant one using elementary complex functions, while the second one fitting more in the frame of classical analytic geometry. In 2004 this theorem was rediscovered by J. Szabados, providing a testimony of the natural occurrence of the sets E_b in this context[2].

In fact, the key to Theorem 1.1.1 was the following observation, implicitly already in [**20**] and [**6**] and formulated explicitly in [**8**].

LEMMA 1.1.4 (**Turán, Levenberg-Poletsky**). *Assume that* $z \in \partial K$ *and that there exists a disc* D_R *of radius* R *so that* $z \in \partial D_R$ *and* $K \subset D_R$*. Then for all* $p \in \mathcal{P}_n(K)$ *we have*

$$|p'(z)| \geq \frac{n}{2R} |p(z)| \ . \tag{1.5}$$

So Levenberg and Poletsky [**8**] found it worthwhile to formally introduce the next definition.

DEFINITION 1.1.5. A compact set $K \subset \mathbb{C}$ is called *R-circular*, if for any point $z \in \partial K$ there exists a disc D_R of radius R with $z \in \partial D_R$ and $K \subset D_R$.

With this they formulated various consequences. For our present purposes let us chose the following form, c.f. [**8**, Theorem 2.2].

THEOREM 1.1.6 (**Erőd; Levenberg-Poletsky**). *If* K *is an R-circular set and* $p \in \mathcal{P}_n(K)$*, then*

$$\|p'\| \geq \frac{n}{2R} \|p\| \ . \tag{1.6}$$

Note that here it is not assumed that K be convex; a circular arc, or a union of disjoint circular arcs with proper points of join, satisfy the criteria. However, other curves, like e.g. the interval itself, do not admit such inequalities; as said above, the order of magnitude can be as low as $\sqrt{n}$ in general.

Erőd did not formulate the result that way; however, he was clearly aware of that. This can be concluded from his various argumentations, in particular for the next result.

[2] After learning about the overlap with Erőd's work, the result was not published.

THEOREM 1.1.7 (**Erőd,** [**6**, p. 77]). *If K is a C^2-smooth convex domain with the curvature of the boundary curve staying above a fixed positive constant $\kappa > 0$, and if $p \in \mathcal{P}_n(K)$, then we have*

$$\|p'\| \geq c(K)n\|p\|. \tag{1.7}$$

From Erőd's argument one can not easily conclude that the constant is $c(K) = \kappa/2$; on the other hand, his statement is more general than that. Although the proof is slightly incomplete, let us briefly describe the idea[3].

PROOF. The norm of p is attained at some point of the boundary, so it suffices to prove that $|p'(z)|/|p(z)| \geq cn$ for all $z \in \partial K$. But the usual form of the logarithmic derivative and the information that all the n zeroes $z_1, \ldots, z_n$ of p are located in K allows us to draw this conclusion once we have for a fixed direction $\varphi := \varphi(z)$ the estimate

$$\Re\left(e^{i\varphi}\frac{1}{z - z_k}\right) \geq c > 0 \qquad (k = 1, \ldots, n). \tag{1.8}$$

Choosing φ the (outer) normal direction of the convex curve ∂K at $z \in \partial K$, and taking into consideration that z_k are placed in $K \setminus \{z\}$ arbitrarily, we end up with the requirement that

$$\Re\left(e^{i\varphi}\frac{1}{z - w}\right) = \frac{\cos\alpha}{|z - w|} \geq c \qquad (w \in K \setminus \{z\},\ \ \alpha := \varphi - \arg(z - w)) \ . \tag{1.9}$$

Now if K is strictly convex, then for $z \neq w$ we do not have $\cos\alpha = 0$, a necessary condition for keeping the ratio off zero. It remains to see if $|z - w|/\cos\alpha$ stays bounded when $z \in \partial K$ and $w \in K \setminus \{z\}$, or, as is easy to see, if only $w \in \partial K \setminus \{z\}$. Observe that $F(z, w) := |z - w|/\cos\alpha$ is a two-variate function on ∂K^2, which is not defined for the diagonal $w = z$, but under certain conditions can be extended continuously. Namely, for given z the limit, when $w \to z$, is the well-known geometric quantity $2\rho(z)$, where $\rho(z)$ is the radius of the osculating circle (i.e., the reciprocal of the curvature $\kappa(z)$). (Note here a gap in the argument for not taking into consideration also $(z', w') \to (z, z)$, which can be removed by showing uniformity of the limit.) Hence, for smooth ∂K with strictly positive curvature bounded away from 0, we can define $F(z, z) := 2/\kappa(z) = 2\rho(z)$. This makes F a continuous function all over ∂K^2, hence it stays bounded, and we are done. □

We will return to this theorem and provide a somewhat different, complete proof giving also the value $c(K) = \kappa/2$ of the constant later in §1.8. For an analysis of the slightly incomplete, nevertheless essentially correct and really innovative proof of Erőd see [**15**].

From this argument it can be seen that whenever we have the property (1.9) for all given boundary points $z \in \partial K$, then we also conclude the statement. This explains why Erőd could allow even vertices, relaxing the conditions of the above statement to hold only piecewise on smooth Jordan arcs, joining at vertices. However, to have a fixed bound, either the number of vertices has to be bounded, or some additional condition must be imposed on them. Erőd did not elaborate further on this direction.

[3] For more about the life and work of János Erőd, see [**15**] and [**16**].

Convex domains (or sets) *not* satisfying the R-circularity criteria with any fixed positive value of R are termed to be *flat*. Clearly, the interval is flat, like any polygon or any convex domain which is not strictly convex. From this definition it is not easy to tell if a domain is flat, or if it is circular, and if so, then with what (best) radius R. We will deal with the issue in this work, aiming at finding a large class of domains having cn order of the inverse Markov factor with some information on the arising constant as well.

On the other hand a lower estimate of the inverse Markov factor of the same order as for the interval was obtained in full generality in 2002, see [**8**, Theorem 3.2].

THEOREM 1.1.8 (**Levenberg-Poletsky**). *If $K \subset \mathbb{C}$ is a compact, convex set, $d := \operatorname{diam} K$ is the diameter of K and $p \in \mathcal{P}_n(K)$, then we have*

$$\|p'\| \geq \frac{\sqrt{n}}{20 \operatorname{diam}(K)} \|p\| \ . \tag{1.10}$$

Clearly, we can have no better order, for the case of the interval the $\sqrt{n}$ order is sharp. Nevertheless, already Erőd [**6**, p. 74] addressed the question: "For what kind of domains does the method of Turán apply?" Clearly, by "applies" he meant that it provides cn order of oscillation for the derivative.

The most general domains with $M(K) \gg n$, found by Erőd, were described on p. 77 of [**6**]. Although the description is a bit vague, and the proof shows slightly less, we can safely claim that he has proved the following result.

THEOREM 1.1.9 (**Erőd**). *Let K be any convex domain bounded by finitely many Jordan arcs, joining at vertices with angles $< \pi$, with all the arcs being C^2-smooth and being either straight lines of length $\ell < \Delta(K)/4$, where $\Delta(K)$ stands for the transfinite diameter of K, or having positive curvature bounded away from 0 by a fixed constant. Then there is a constant $c(K)$, such that $M_n(K) \geq c(K)n$ for all $n \in \mathbb{N}$.*

To deal with the flat case of straight line boundary arcs, Erőd involved another approach, cf. [**6**, p. 76], appearing later to be essential for obtaining a general answer. Namely, he quoted Faber [**7**] for the following fundamental result going back to Chebyshev.

LEMMA 1.1.10 (**Chebyshev**). *Let $J = [u, v]$ be any interval on the complex plane with $u \neq v$ and let $J \subset R \subset \mathbb{C}$ be any set containing J. Then for all $k \in \mathbb{N}$ we have*

$$\min_{w_1,\dots,w_k \in R} \max_{z \in J} \left| \prod_{j=1}^{k} (z - w_j) \right| \geq 2 \left(\frac{|J|}{4} \right)^k \ . \tag{1.11}$$

PROOF. This is essentially the classical result of Chebyshev for a real interval, cf. [**2, 9**], and it holds for much more general situations (perhaps with the loss of the factor 2) from the notion of Chebyshev constants and capacity, cf. Theorem 5.5.4. (a) in [**11**]. □

The relevance of Chebyshev's Lemma is that it provides a quantitative way to handle contribution of zero factors at some properly selected set J. One uses this for comparison: if $|p(\zeta)|$ is maximal at $\zeta \in \partial K$, then the maximum on some J can not be larger. Roughly speaking, combining this with geometry we arrive at an effective estimate of the contribution, hence even on the location of the zeroes.

In his recent work [**5**], Erdélyi considered various special domains. Apart from further results for polynomials of some special form (e.g. even or real polynomials), he obtained the following.

THEOREM 1.1.11 (**Erdélyi**). *Let Q denote the square domain with diagonal $[-1,1]$. Then for all polynomials $p \in \mathcal{P}_n(Q)$ we have*

$$\|p'\| \geq C_0 n \|p\| \tag{1.12}$$

with a certain absolute constant C_0.

Note that the regular n-gon K_n is already covered by Erőd's Theorem 1.1.9 if $n \geq 26$, but not the square Q, since the side length h is larger than the quarter of the transfinite diameter Δ: actually, $\Delta(Q) \approx 0.59017\ldots h$, while

$$\Delta(K_n) = \frac{\Gamma(1/n)}{\sqrt{\pi}2^{1+2/n}\Gamma(1/2+1/n)} h > 4h \quad \text{iff} \qquad n \geq 26,$$

see [**11**, p. 135]. Erdélyi's proof is similar to Erőd's argument[4]: sacrificing generality gives the possibility for a better calculation for the particular choice of Q.

Returning to the question of the order in general, let us recall that the term *convex domain* stands for a compact, convex subset of $\mathbb{C}$ *having nonempty interior.* Clearly, assuming boundedness is natural, since all polynomials of positive degree have $\|p\|_K = \infty$ when the set K is unbounded. Also, all convex sets with nonempty interior are *fat*, meaning that $\mathrm{cl}(K) = \mathrm{cl}(\mathrm{int}K)$. Hence taking the closure does not change the sup norm of polynomials under study. The only convex, compact sets, falling out by our restrictions, are the intervals, for what Turán has already shown that his $c\sqrt{n}$ lower estimate is of the right order. Interestingly, it turned out that among all convex compacta only intervals can have an inverse Markov constant of such a small order.

To study (1.1) some geometric parameters of the convex domain K are involved naturally. We write $d := d(K) := \mathrm{diam}\,(K)$ for the *diameter* of K, and $w := w(K) := \mathrm{width}\,(K)$ for the *minimal width* of K. That is,

$$w(K) := \min_{\gamma \in [-\pi,\pi]} \left(\max_{z \in K} \Re(ze^{-i\gamma}) - \min_{z \in K} \Re(ze^{-i\gamma}) \right). \tag{1.13}$$

Note that a (closed) convex domain is a (closed), bounded, convex set $K \subset \mathbb{C}$ with nonempty interior, hence $0 < w(K) \leq d(K) < \infty$. Our main result is the following.

THEOREM 1.1.12 (**Halász and Révész**). *Let $K \subset \mathbb{C}$ be any convex domain having minimal width $w(K)$ and diameter $d(K)$. Then for all $p \in \mathcal{P}_n(K)$ we have*

$$\frac{\|p'\|}{\|p\|} \geq C(K) n \qquad \text{with} \qquad C(K) = 0.0003 \frac{w(K)}{d^2(K)} . \tag{1.14}$$

On the other hand, as regards the order of magnitude, (and in fact apart from an absolute constant factor), this result is sharp for all convex domains $K \subset \mathbb{C}$.

[4] Erdélyi was apparently not aware of the full content of [**6**] when presenting his rather similar argument.

THEOREM 1.1.13. *Let $K \subset \mathbb{C}$ be any compact, connected set with diameter d and minimal width w. Then for all $n > n_0 := n_0(K) := 2(d/16w)^2 \log(d/16w)$ there exists a polynomial $p \in \mathcal{P}_n(K)$ of degree exactly n satisfying*

$$\|p'\| \leq C'(K)\, n\, \|p\| \qquad \text{with} \qquad C'(K) := 600\, \frac{w(K)}{d^2(K)}\ . \tag{1.15}$$

REMARK 1.1.14. Note that here we do not assume that K be convex, but only that it is a connected, closed (compact) subset of $\mathbb{C}$. (Clearly the condition of boundedness is not restrictive, $\|p\|$ being infinite otherwise.)

In the proof of Theorem 1.1.12, due to generality, the precision of constants could not be ascertained e.g. for the special ellipse domains considered in [**6**]. Thus it seems that the general results are not capable to fully cover e.g. Theorem 1.1.3.

However, even that is possible for a quite general class of convex domains with order n inverse Markov factors and a different estimate of the arising constants. This will be achieved working more in the direction of Erőd's first observation, i.e. utilizing information on curvature.

Since these results need some technical explanations, formulation of these will be postponed until §1.8. But let us mention the key ingredient, which clearly connects curvature and the notion of circular domains. In the smooth case, it is well-known as Blaschke's Rolling Ball Theorem, cf. [**1**, p. 116].

LEMMA 1.1.15 (**Blaschke**). *Assume that the convex domain K has C^2 boundary $\Gamma = \partial K$ and that there exists a positive constant $\kappa > 0$ such that the curvature $\kappa(\zeta) \geq \kappa$ at all boundary points $\zeta \in \Gamma$. Then to each boundary points $\zeta \in \Gamma$ there exists a disk D_R of radius $R = 1/\kappa$, such that $\zeta \in \partial D_R$, and $K \subset D_R$.*

Again, geometry plays the crucial role in the investigations of variants when smoothness and conditions on curvature are relaxed. We will strongly extend the classical results of Erőd, showing that conditions on the curvature suffices to hold only almost everywhere (in the sense of arc length measure) on the boundary.

THEOREM 1.1.16. *Assume that the convex domain K has boundary $\Gamma = \partial K$ and that the a.e. existing curvature of Γ exceeds κ almost everywhere, or, equivalently, assume the subdifferential condition* (1.60) *(or any of the equivalent formulations in* (1.55)-(1.60)*) with $\lambda = \kappa$. Then for all $p \in \mathcal{P}_n(K)$ we have*

$$\|p'\| \geq \frac{\kappa}{2} n \|p\|\ . \tag{1.16}$$

This also hinges upon geometry, and we will have two proofs. One is essentially an application of a recent, quite far-reaching extension of the Blaschke Theorem by Strantzen. The other involves even more geometry: it hinges upon a new, discrete version of the Blaschke Rolling Ball Theorem, (which easily implies also Strantzen's Theorem), but which is suitable, at least in principle, to provide also some degree-dependent estimate of $M_n(K)$ by means of the *minimal oscillation* or *change* of the outer unit normal vector(s) along the boundary curve.

For applications to various domains, where yields of the different estimates can also be compared, see the later sections. Before that, in the next section we prove the most general result, Theorem 1.1.12, and we follow by proving sharpness of the result, i.e. proving Theorem 1.1.13.

In §1.4 we start with describing the underlying geometry, and in §1.7 we will describe variants and extensions on the theme of the Blaschke Roling Ball Theorem. Finally, in §1.8 we will formulate the resulting theorems and analyze the yields of them on various parametric classes of domains.

1.2. Proof of the main theorem

IDEA OF PROOF. Throughout we will assume, as we may, that K is also closed, hence a compact convex set with nonempty interior. Our proof will follow the argument of [**13**], with one key alteration, suggested to us by Gábor Halász. Let us first describe the original idea and then the additional suggestion of Halász, even if the reader may understand the proof below without these notes as well.

We start with picking up a boundary point $\zeta \in \partial K$ of maximality of $|p|$, and consider a supporting line at ζ to K. Our original argument of [**13**] then used a normal direction and compared values of p at ζ and on the intersection of K and this normal line. Essential use were made of the fact that in case the length h of this intersection is small (relative to w), then, due to convexity, the normal line cuts K into half unevenly: one part has to be small (of the order of h). That was explicitly formulated in [**13**], and is used implicitly even here through various calculations with the angles.

However, here we compare the values of p at ζ and *on a line slightly slanted off from the normal*. Comparing the calculations here and in [**13**] one can observe how this change led to a further, essential improvement of the result through improving the contribution of the factors belonging to zeroes close to the supporting line. In [**13**] we could get a square term (in h there) only, due to orthogonality and the consequent use of the Pythagorean Theorem in calculating the distances. However, here we obtain *linear dependence* in δ via the general cosine theorem for the slanted segment J. (That insightful observation was provided by G. Halász.)

One of the major geometric features still at our help is the fact, that when h is small, then one portion of K, cut into half by our slightly tilted line, is also small. This is the key feature which allows us to bend the direction of the normal a bit *towards the smaller portion* of K[5].

As a result of the improved estimates squeezed out this way, we do not need to employ the second usual technique, also going back to Turán, i.e. integration of $(p'/p)'$ over a suitably chosen interval. As pointed out already in [**13**], this part of the proof yields

[5]If we try tilting the other way we would fail badly, even if the reader may find it difficult to distill from the proof where, and how. But if there were zeroes close to (or on) the supporting line and far from ζ *in the direction of the tilting*, then these zeroes were farther off from ζ, than from the other end of the intersecting segment. That would spoil the whole argument. However, since K is small in one direction of the supporting line, tilting towards this smaller portion does work.

weaker estimates than cn, so avoiding it is not only a matter of convenience, but is an essential necessity.

PROOF. We list the zeroes of a polynomial $p \in \mathcal{P}_n(K)$ according to multiplicities as $z_1, \ldots, z_n$, and the set of these zero points is denoted as $\mathcal{Z} := \mathcal{Z}(p) := \{z_j \ : \ j = 1, \ldots, n\} \subset K$. (It suffices to assume that all z_j are distinct, so we do not bother with repeatedly explaining multiplicities, etc.) Assume, as we may, $p(z) = \prod_{j=1}^n (z - z_j)$.

We start with picking up a point ζ of K, where p attains its norm. By the maximum principle, $\zeta \in \partial K$, and by convexity there exists a supporting line to K at ζ with inward normal vector $\boldsymbol{\nu}$, say. Without loss of generality we can take $\zeta = 0$ and $\boldsymbol{\nu} = i$. Now by definition of the minimal width $w = w(K)$, there exists a point $A \in K$ with $\Im A \geq w$; by symmetry, we may assume $\Re A \leq 0$, say.

Sometimes we write the zeroes in their polar form

$$z_j = r_j e^{i\varphi_j} \qquad (r_j := |z_j|, \quad \varphi_j := \arg z_j \quad (j = 1, \ldots, n)) \ . \tag{1.17}$$

Throughout the proofs with $[(\varphi, \psi)]$ being any open, closed, half open-half closed or half closed-half open interval we use the notations

$$S[(\varphi, \psi)] := \{z \in \mathbb{C} \ : \ \arg(z) \in [(\varphi, \psi)]\} \tag{1.18}$$

and

$$\mathcal{Z}[(\varphi, \psi)] := \mathcal{Z} \cap S[(\varphi, \psi)] \,, \qquad n[(\varphi, \psi)] := \# \mathcal{Z}[(\varphi, \psi)] \,, \tag{1.19}$$

for the sectors, the zeroes in the sectors, and the number of zeroes in the sectors determined by the angles φ and ψ.

In all our proof we fix the angles

$$\psi := \arctan\left(\frac{w}{d}\right) \in (0, \pi/4] \qquad \text{and} \qquad \theta := \psi/20 \in (0, \pi/80]. \tag{1.20}$$

Since $|p(0)| = \|p\|$, $M \geq |p'(0)/p(0)|$. Observe that for any subset $\mathcal{W} \subset \mathcal{Z}$ we then have

$$M \geq \left|\frac{p'}{p}(0)\right| \geq \Im \frac{p'}{p}(0) = \sum_{j=1}^n \Im \frac{-1}{z_j} \geq \sum_{z_j \in \mathcal{W}} \Im \frac{-1}{z_j} = \sum_{z_j \in \mathcal{W}} \frac{\sin \varphi_j}{r_j} \,, \tag{1.21}$$

since all terms in the full sum are nonnegative.

Let us consider now the ray (straight half-line) emanating from $\zeta = 0$ in the direction of $e^{i(\pi/2 - 2\theta)}$. This ray intersects K in a line segment $[0, D]$, and if $D = 0$, then $K \subset S[\pi/2 - 2\theta, \pi]$ and a standard argument using e.g. Turán's Lemma 1.1.4 yields $M \geq n/(2d)$. Hence we may assume $D \neq 0$.

Consider now any point $B \in K$ with maximal real part, and take $B' := \Re B = \max\{\Re z \ : \ z \in K\}$. Since $D \neq 0$, $B' > 0$, and as $\Re A \leq 0$ and $\Re B$ is maximal, $[A, B']$ intersects $[0, D]$ in a point $D' \in [0, D]$, i.e. $[0, D'] \subset [0, D] \subset K$. Moreover, the angle at B' between the real line and AB' is $-\arg(B' - A) = -\arg(B' - D') \in [\psi, \pi/2)$. Indeed, $\Im(A - B') \geq w$ and $\Re(B' - A) = \Re(B - A) \leq d$ (resulting from $A, B \in K$) imply $-\arg(B' - A) \geq \arctan(w/d) = \psi$.

In the following let us write $\delta := |D'| > 0$; it can not vanish, as $B' \neq 0$ and the line segment $[B', A]$ intersects the real line only in B'. Consider the point $B" \in \mathbb{R}$ with

$B" \geq B' > 0$ and $-\arg(B" - D') = \psi$. We can say now that K lies both in the upper half of the disk with radius d around 0 (which we denote by U), and the halfplane $\Re z \leq B"$ (which we denote by H); moreover, $[0, D'] \subset K \subset (U \cap H)$.

Now we put $D" := 3D'/4$ and take

$$J := \left[D", D'\right] \subset K \quad \text{i.e. } J := \{\tau := te^{i(\pi/2-2\theta)}\delta \ : \ 3/4 \leq t \leq 1\} \ . \tag{1.22}$$

Denoting $D_r(0) := \{z \, : \, |z| \leq r\}$ we split the set $\mathcal{Z}$ into the following parts.

$$\begin{aligned}
\mathcal{Z}_1 : &= \mathcal{Z}[0, \theta] \,, & \mu := \#\mathcal{Z}_1 = n[0, \theta] \\
\mathcal{Z}_2 : &= \mathcal{Z}(\theta, \pi - \theta) \cap \left\{ \Im(e^{i2\theta} z) < \frac{3}{8}\delta \right\} \,, & \nu := \#\mathcal{Z}_2 \\
\mathcal{Z}_3 : &= \mathcal{Z}(\theta, \pi - \theta) \cap \left\{ \Im(e^{i2\theta} z) \geq \frac{3}{8}\delta \right\} \cap D_{2\delta}(0) \,, & \kappa := \#\mathcal{Z}_3 \\
\mathcal{Z}_4 : &= \mathcal{Z}(\theta, \pi - \theta) \cap \left\{ \Im(e^{i2\theta} z) \geq \frac{3}{8}\delta \right\} \setminus D_{2\delta}(0) = & \\
&= \mathcal{Z}(\theta, \pi - \theta) \setminus (\mathcal{Z}_2 \cup \mathcal{Z}_3) \ , & k := \#\mathcal{Z}_4 \\
\mathcal{Z}_5 : &= \mathcal{Z}[\pi - \theta, \pi] \,, & m := \#\mathcal{Z}_5 = n[\pi - \theta, \pi] \ .
\end{aligned} \tag{1.23}$$

In the following we establish an inequality from condition of maximality of $|p(0)|$. First we estimate the distance of any $z_j \in \mathcal{Z}_1$ from J. In fact, taking any point $z = re^{i\varphi} \in H \cap S[0, \theta]$ the sine theorem yields $r \cos\varphi = \Re z \leq |B"| = \delta \sin(\pi/2 + 2\theta - \psi)/\sin\psi = \delta \cos(\psi - 2\theta)/\sin\psi < \delta \cot(18\theta)$, and so

$$r \sin\theta < \frac{\sin\theta}{\cos\varphi} \frac{\delta}{\tan(18\theta)} \leq \delta \frac{\tan\theta}{\tan(18\theta)} < \frac{\delta}{18} \ . \tag{1.24}$$

Now $\text{dist}\,(z, J) = \min_{3/4 \leq t \leq 1} |z - \tau|$, (where $\tau := te^{i(\pi/2-2\theta)}\delta$) and by the cosine theorem $|z-\tau|^2 = t^2\delta^2 + r^2 - 2\cos(\pi/2 - \varphi - 2\theta)\ rt\delta$. Because of $\cos(\pi/2 - \varphi - 2\theta) = \sin(\varphi + 2\theta) \leq \sin(3\theta) \leq 3\sin\theta$, (1.24) implies $|z - \tau|^2 \geq t^2\delta^2 + r^2 - 6t\delta \sin\theta\ r \geq t^2\delta^2 + r^2 - (1/3)t\delta^2$, and thus $\min_{3/4 \leq t \leq 1} |z - \tau|^2 \geq \min_{3/4 \leq t \leq 1} t^2\delta^2 + r^2 - (1/3)t\delta^2 = r^2 + (5/16)\delta^2$. It follows that we have

$$\frac{|z - \tau|^2}{|z|^2} \geq \frac{r^2 + (5/16)\delta^2}{r^2} > 1 + \frac{(90/16)\sin\theta\ \delta}{r} > 1 + \frac{5\sin\theta\ \delta}{d} \quad (\tau \in J) \ ,$$

applying also (1.24) to estimate δ/r in the last but one step. Now $\delta/d \leq 1$ and $5\sin\theta < 0.2$, hence we can apply $\log(1 + x) \geq x - x^2/2 \geq 0.9x$ for $0 < x < 0.2$ to get

$$\frac{|z - \tau|^2}{|z|^2} \geq \exp\left(0.9 \frac{5\sin\theta\ \delta}{d}\right) > \exp\left(\frac{4\sin\theta\ \delta}{d}\right) \quad (\tau \in J) \ .$$

Applying this estimate for all the μ zeroes $z_j \in \mathcal{Z}_1$ we finally find

$$\prod_{z_j \in \mathcal{Z}_1} \left| \frac{z_j - \tau}{z_j} \right| \geq \exp\left(\frac{2\sin\theta\ \delta\mu}{d}\right) \qquad \left(\tau = t\delta e^{i(\pi/2-2\theta)} \in J\right). \tag{1.25}$$

The estimate of the contribution of zeroes from $\mathcal{Z}_5$ is somewhat easier, as now the angle between z_j and τ exceeds $\pi/2$. By the cosine theorem again, we obtain for any $z = re^{i\varphi} \in$

$S[\pi-\theta,\pi]\cap U$ the estimate

$$\begin{aligned} |z-\tau|^2 =& r^2+t^2\delta^2-2\cos(\varphi-(\pi/2-2\theta))\, rt\delta \\ \ge& r^2+t^2\delta^2+2\sin\theta\ rt\delta > r^2\left(1+\frac{3\sin\theta\ \delta}{2d}\right) \ (\tau\in J)\ , \end{aligned} \tag{1.26}$$

as $t \ge 3/4$ and $r \le d$. Hence using again $\delta/d \le 1$ and $1.5\sin\theta < 0.06$ we can apply $\log(1+x) \ge x - x^2/2 \ge 0.97x$ for $0 < x < 0.06$ to get

$$\frac{|z-\tau|}{|z|} \ge \exp\left(\frac{0.97}{2}\,\frac{3\sin\theta\ \delta}{2d}\right) \ge \exp\left(\frac{18\sin\theta\,\delta}{25d}\right) \ (\tau\in J)\ ,$$

whence

$$\prod_{z_j\in\mathcal{Z}_5}\left|\frac{z_j-\tau}{z_j}\right| \ge \exp\left(\frac{18\sin\theta\ \delta m}{25d}\right) \qquad \left(\tau = t\delta e^{i(\pi/2-2\theta)}\in J\right). \tag{1.27}$$

Observe that zeroes belonging to $\mathcal{Z}_2$ have the property that they fall to the opposite side of the line $\Im(e^{i2\theta}z) = 3\delta/8$ than J, hence they are closer to 0 than to any point of J. It follows that

$$\prod_{z_j\in\mathcal{Z}_2}\left|\frac{z_j-\tau}{z_j}\right| \ge 1 \qquad \left(\tau = t\delta e^{i(\pi/2-2\theta)}\in J\right). \tag{1.28}$$

Next we use Lemma 1.1.10 to estimate the contribution of zero factors belonging to $\mathcal{Z}_3$. We find

$$\max_{\tau\in J}\prod_{z_j\in\mathcal{Z}_3}\left|\frac{z_j-\tau}{z_j}\right| \ge 2\left(\frac{|J|}{4}\right)^{\kappa}\prod_{z_j\in\mathcal{Z}_3}\frac{1}{r_j} > \left(\frac{1}{32}\right)^{\kappa} > \exp(-3.5\kappa)\ , \tag{1.29}$$

in view of $|J| = \delta/4$ and $r_j \le 2\delta$.

Note that for any point $z = re^{i\varphi} \in D_{2\delta}(0)\cap\{\Im(e^{i2\theta}z) \ge 3\delta/8\}$ we must have

$$\frac{3\delta}{8} \le \Im(e^{i2\theta}re^{i\varphi}) = r\sin(\varphi+2\theta)\ ,$$

hence by $r \le 2\delta$ also

$$\sin(\varphi+2\theta) \ge \frac{3\delta}{8r} \ge \frac{3}{16}$$

and $\sin\varphi \ge \sin(\varphi+2\theta) - 2\theta \ge 3/16 - \pi/40 > 1/10$. Applying this for all the zeroes $z_j \in \mathcal{Z}_3$ we are led to

$$1 \le \frac{2\delta}{r_j} \le 20\delta\frac{\sin\varphi_j}{r_j} \qquad (z_j\in\mathcal{Z}_3)\ . \tag{1.30}$$

On combining (1.29) with (1.30) we are led to

$$\max_{\tau\in J}\prod_{z_j\in\mathcal{Z}_3}\left|\frac{z_j-\tau}{z_j}\right| \ge \exp\left(-70\delta\sum_{z_j\in\mathcal{Z}_3}\frac{\sin\varphi_j}{r_j}\right)\ . \tag{1.31}$$

Finally we consider the contribution of the zeroes from $\mathcal{Z}_4$, i.e. the "far" zeroes for which we have $\Im(z_je^{2i\theta}) \ge 3\delta/8$, $\varphi_j \in (\theta,\pi-\theta)$ and $|r_j| \ge 2\delta$. Put now $Z := z_je^{2i\theta} = u + iv =$

$re^{i\alpha}$, and $s := |\tau| = t\delta$, say. We then have

$$\left|\frac{z_j-\tau}{z_j}\right|^2 = \frac{|Z-t\delta i|^2}{r^2} = \frac{u^2+(v-s)^2}{r^2} = 1-\frac{2vs}{r^2}+\frac{s^2}{r^2} \tag{1.32}$$
$$> 1-\frac{2vs}{r^2}+\frac{s^2}{r^2}\frac{v^2}{r^2} = \left(1-\frac{vs}{r^2}\right)^2 \geq \left(1-\frac{|v|\delta}{r^2}\right)^2 = \left(1-\frac{\delta|\sin\alpha|}{r}\right)^2 .$$

Recall that $\log(1-x) > -x-\frac{x^2}{2}\frac{1}{1-x} \geq -x(1+1/2)$ whenever $0 \leq x \leq 1/2$. We can apply this for $x := \delta|\sin\alpha|/r_j \leq \delta/r_j \leq 1/2$ using $r = r_j = |z_j| \geq 2\delta$. As a result, (1.32) leads to

$$\left|\frac{z_j-\tau}{z_j}\right| \geq \exp\left(-\frac{3}{2}\delta\frac{|\sin(\varphi_j+2\theta)|}{r_j}\right) , \tag{1.33}$$

and using $|\sin(\varphi_j+2\theta)| \leq \sin(\varphi_j)+\sin(2\theta) \leq 3\sin\varphi_j$ (in view of $\varphi_j \in (\theta, \pi-\theta)$), finally we get

$$\prod_{z_j\in\mathcal{Z}_4}\left|\frac{z_j-\tau}{z_j}\right| \geq \exp\left(-\frac{9\delta}{2}\sum_{z_j\in\mathcal{Z}_4}\frac{\sin\varphi_j}{r_j}\right) \qquad \left(\tau = t\delta e^{i(\pi/2-2\theta)} \in J\right) . \tag{1.34}$$

If we collect the estimates (1.25) (1.27) (1.28) (1.31) and (1.34), we find for a certain point of maxima $\tau_0 \in J$ in (1.31) the inequality

$$1 \geq \frac{|p(\tau_0)|}{|p(0)|} = \prod_{z_j\in\mathcal{Z}}\left|\frac{z_j-\tau_0}{z_j}\right| > \tag{1.35}$$
$$\exp\left\{\frac{18}{25}\sin\theta\,\delta\frac{\mu+m}{d} - 70\delta\sum_{z_j\in\mathcal{Z}_2\cup\mathcal{Z}_3\cup\mathcal{Z}_4}\frac{\sin\varphi_j}{r_j}\right\} ,$$

or, after taking logarithms and cancelling by $18\delta/25$

$$\sin\theta\frac{\mu+m}{d} < \frac{875}{9}\sum_{z_j\in\mathcal{Z}_2\cup\mathcal{Z}_3\cup\mathcal{Z}_4}\frac{\sin\varphi_j}{r_j} \tag{1.36}$$

Observe that for the zeroes in $\mathcal{Z}_2\cup\mathcal{Z}_3\cup\mathcal{Z}_4$ we have $\sin\varphi_j > \sin\theta$, whence also

$$(\nu+\kappa+k)\frac{\sin\theta}{d} \leq \sum_{z_j\in\mathcal{Z}_2\cup\mathcal{Z}_3\cup\mathcal{Z}_4}\frac{\sin\varphi_j}{r_j} . \tag{1.37}$$

Adding (1.36) and (1.37) and taking into account $\#\mathcal{Z} = \sum_{j=1}^5 \#Z_j$, we obtain

$$\sin\theta\frac{n}{d} = \sin\theta\frac{\mu+m+\nu+\kappa+k}{d} < \frac{884}{9}\sum_{z_j\in\mathcal{Z}_2\cup\mathcal{Z}_3\cup\mathcal{Z}_4}\frac{\sin\varphi_j}{r_j} . \tag{1.38}$$

Making use of (1.21) with the choice of $\mathcal{W} := \mathcal{Z}_2\cup\mathcal{Z}_3\cup\mathcal{Z}_4$ we arrive at

$$\sin\theta\frac{n}{d} < \frac{884}{9}M ,$$

that is,

$$M > \frac{9\sin\theta}{884d}n . \tag{1.39}$$

It remains to recall (1.20) and to estimate

$$\sin\theta = \sin\left(\frac{\arctan(w/d)}{20}\right) .$$

As $\theta \in (0,\pi/80]$, $\sin\theta > \theta(1-\theta^2/6) \geq \theta(1-\pi/240) > 0.98\theta$ and as $0 < w/d \leq 1$, $\arctan(w/d) \geq (w/d)(\pi/4)$, whence

$$\sin\theta \geq 0.98\frac{\arctan(w/d)}{20} \geq \frac{0.98\pi}{80}\frac{w}{d} \ .$$

If we substitute this last estimate into (1.39) we get

$$M > \frac{9}{884}\cdot\frac{0.98\pi}{80}\cdot\frac{w}{d^2}\cdot n > 0.0003\frac{w}{d^2}n \ ,$$

concluding the proof. □

1.3. On sharpness of the order n lower estimate of $M_n(K)$

Proof. Take $a, b \in K$ with $|a-b| = d$ and $m \in \mathbb{N}$ with $m > m_0$ to be determined later. Consider the polynomials $q(z) := (z-a)(z-b)$, $p(z) = (z-a)^m(z-b)^m = q^m(z)$ and $P(z) = (z-a)^m(z-b)^{m+1} = (z-b)q^m(z)$. Clearly, $p, P \in \mathcal{P}_n(K)$ with $n = \deg p = 2m$ and $n = \deg P = 2m+1$, respectively. We claim that for appropriate choice of m_0 these polynomials satisfy inequality (1.15) for all $n > 2m_0$.

Without loss of generality we may assume $a = -1$, $b = 1$ and thus $d = 2$, as substitution by the linear function $\Phi(z) := \frac{2}{b-a}z - \frac{a+b}{b-a}$ shows. Indeed, if we prove the assertion for $\widetilde{K} := \Phi(K)$ and for $\widetilde{p}(z) = (z+1)^m(z-1)^m$, $\widetilde{P}(z) = (z+1)^m(z-1)^{m+1}$ defined on $\widetilde{K}$, we also obtain estimates for $p = \widetilde{p}\circ\Phi$ and $P = \widetilde{P}\circ\Phi$ on K. The homothetic factor of the inverse substitution Φ^{-1} is $\Lambda := \left|\frac{b-a}{2}\right| = d(K)/2$, and width changes according to $w(\widetilde{K}) = 2w(K)/d(K)$. Note also that under the linear substitution Φ the norms are unchanged but for the derivatives $\|p'\| = \Lambda^{-1}\|\widetilde{p}'\|$ and $\|P'\| = \Lambda^{-1}\|\widetilde{P}'\|$. So now we restrict to $a = -1$, $b = 1$, $d = 2$ and $q(z) := z^2 - 1$ etc.

First we make a few general observations. One obvious fact is that the imaginary axes separates $a = -1$ and $b = 1$, and as K is connected, it also contains some point $c = it$ of K. Therefore, $\|q\| \geq |q(c)| = 1 + t^2 \geq 1$. Also, it is clear that $q'(z) = 2z = (z-1)+(z+1)$: thus, by definition of the diameter

$$\|q'\| \leq \||z-1| + |z+1|\| \leq 4 \ . \tag{1.40}$$

Let us put $w^+ := \sup_{z\in K}\Im z$ and $w^- := -\inf_{z\in K}\Im z$. We can estimate $w' := \max(w^+, w^-)$ from above by a constant times w. That is, we claim that for any point $\omega = \alpha + i\beta \in K$ we necessarily have $|\beta| \leq \sqrt{2}w$ and so the domain K lies in the rectangle $R := \mathrm{con}\{-1 - i\sqrt{2}w, 1 - i\sqrt{2}w, 1 + i\sqrt{2}w, -1 + i\sqrt{2}w\}$.

To see this first note that $\beta \leq \sqrt{3}$, since $d(K) = 2$ by assumption. Recalling (1.13), take $e^{i\gamma}$ be the direction of the minimal width of K: by symmetry, we may take $0 \leq \gamma < \pi$. Then there is a strip of width w and direction $ie^{i\gamma}$ containing K, hence also the segments $[-1,1]$ and $[\alpha, \alpha+i\beta]$. It follows that $2|\cos\gamma| \leq w$ and $\beta\sin\gamma \leq w$. The second inequality immediately leads to $\beta \leq \sqrt{2}w$ if $\gamma \in [\pi/4, 3\pi/4]$. So let now $\gamma \in [0,\pi/4) \cup [3\pi/4, \pi)$, i.e. $|\cos\gamma| \geq 1/\sqrt{2}$. Applying also $\beta \leq \sqrt{3}$ now we deduce $\beta \leq \sqrt{3} \leq \sqrt{3/2}\,2|\cos\gamma| \leq \sqrt{3/2}w$, whence the asserted $w^\pm \leq \sqrt{2}w$ is proved.

Consider now the norms of the derivatives. As for p, we have $p' = mq'q^{m-1}$, hence

$$\|p'\| \le m\|q'\|\|q\|^{m-1} \le m4\frac{\|p\|}{\|q\|} \le 4m\|p\| \ . \tag{1.41}$$

Concerning P we can write using also (1.41) above

$$\|P'\| \le \|p\| + \|p'\|\|z-1\| \le \|p\| + 2\|p'\| \le (8m+1)\|p\| \ . \tag{1.42}$$

Consider any point $z \in K$ where $\|q\|$, and thus also $\|p\|$ is attained. We clearly have $\|P\| \ge |P(z)| = |z-1|\|p\|$. But here $|z-1| \ge 2/5$: for in case $|z-1| \le 2/5$ we also have $|z+1| \le 12/5$ and thus $|q(z)| \le 24/25 < \|q\|$, as $\|q\| \ge 1$ was shown above. We conclude $\|P\| \ge (2/5)\|p\|$ and (1.42) leads to

$$\|P'\| \le \frac{5(8m+1)}{2}\|P\| < 10n\|P\| \qquad (\ n := 2m+1 = \deg P\ ) \ . \tag{1.43}$$

Now consider first the case $w \ge 2/25$. Using $(25w/2) \ge 1$ we obtain both for p and for P the estimate

$$M(p), M(P) \le 10n \le 125wn \qquad (n := \deg p \text{ or} \deg P, \text{ respectively}). \tag{1.44}$$

Note that here we have these estimates for any $n \in \mathbb{N}$, without bounds on n.

Let now $w < 2/25$. For the central part $Q := \{\alpha + i\beta \in R \ : \ |\alpha| \le 10w\}$ of R we have

$$\|q'\|_{K\cap Q} = \|2z\|_{K\cap Q} \le 2\sqrt{(10w)^2 + (\sqrt{2}w)^2} \le 2\sqrt{102w^2} < 21w, \tag{1.45}$$

while for the remaining part (1.40) remains valid as above.

Next we estimate q in $K \setminus Q$. It is easy to see that here we have $\|q\|_{K\setminus Q} \le \|q\|_{R\setminus Q} = \left|q\left(10w + i\sqrt{2}w\right)\right|$, hence using also $w \le 2/25$ we are led to

$$\begin{aligned}\|q\|^2_{K\setminus Q} &\le \left[(1+10w)^2 + (\sqrt{2}w)^2\right]\left[(1-10w)^2 + (\sqrt{2}w)^2\right] \\ &= 1 - 196w^2 + 10404w^4 \le 1 - 196w^2 + 10000\left(\frac{2}{25}\right)^2 w^2 + 404w^4 \\ &= 1 - 132w^2 + 404w^4 \le 1 - 128w^2 + 4096w^4 = \left[1-(8w)^2\right]^2 \ .\end{aligned} \tag{1.46}$$

Now for $z \in K \cap Q$ we have in view of (1.45) and $\|q\|_K \ge 1$

$$|p'(z)| = m \cdot |q'(z)| \cdot |q^{m-1}(z)| \le m21w\|q\|^m = \frac{21}{2}wn\|p\| \ , \tag{1.47}$$

and for $z \in K \setminus Q$ using $\|p\|_K = \|q\|^m_K \ge 1$, (1.40) and (1.46) we get

$$|p'(z)| \le m \cdot 4 \cdot \|q\|^{m-1}_{K\setminus Q} \le 4m\|p\|\left[1-(8w)^2\right]^{m-1} \ . \tag{1.48}$$

In view of $w < 2/25$, a standard calculation shows that

$$\left[1-(8w)^2\right]^{m-1} \le \frac{25}{2}w \qquad \text{if } m \ge m_0 := \left(\frac{1}{8w}\right)^2 \log\left(\frac{1}{8w}\right) \ . \tag{1.49}$$

Indeed, as $\log(1-x) < -x$ for all $0 < x < 1$, using $w < 2/25$ we find

$$(m-1)\log\left[1-(8w)^2\right] < -(m-1)(8w)^2 < -m(8w)^2 + 0.41,$$

which entails for $m \geq m_0$ that

$$\Big[1-(8w)^2\Big]^{m-1} < e^{-m_0(8w)^2+0.41} = e^{-\log\left(\frac{1}{8w}\right)+0.41} < \frac{25w}{2}\ .$$

It follows from (1.48) and (1.49) that

$$\|p'\|_{K\setminus Q} \leq 25wn\|p\|\ . \tag{1.50}$$

Collecting (1.47) and (1.50) we get also in this case of $w < 2/25$ the estimate

$$\|p'\| \leq 25wn\|p\| \qquad (n = 2m = \deg p,\ m \geq m_0)\,. \tag{1.51}$$

It remains to consider the odd degree case of $n = 2m+1$, i.e. P. Now write

$$|P'(z)| \leq |p(z)| + |p'(z)|\cdot|z-1| \leq \|p\| + 2\|p'\| \leq (1+100wm)\|p\| \quad (m \geq m_0), \tag{1.52}$$

in view of (1.51). As shown above, we have $\|P\| \geq \|p\|/(2/5)$, while $m \geq m_0$ entails $1 \leq m/m_0 < m(8w)(16/25)(1/\log(25/16)) < 12mw$, hence (1.52) yields

$$\|P'\| \leq 112mw\|p\| \leq 280mw\|P\|\ .$$

Since now $n = 2m+1 > 2m$, we finally find

$$\|P'\| < 140wn\|P\| \qquad (n = 2m+1 = \deg P,\ m > m_0)\,. \tag{1.53}$$

Collecting (1.44), (1.51) and (1.53), in view of $\max(125, 25, 140) < 150$ we always get

$$M(p), M(P) < 150wn \quad (n := \deg p \text{ or} \deg P,\ \text{respectively})\ . \tag{1.54}$$

As remarked at the outset, for the general case the homothetic substitution Φ can be considered. That yields $< 600w/d^2$ on the right hand side of (1.54). □

1.4. Some geometrical notions

Let $\mathbb{R}^d$ be the usual Euclidean space of dimension d, equipped with the Euclidean distance $|\cdot|$. Our starting point is the following classical result of Blaschke [**1**, p. 116].

THEOREM 1.4.1 (**Blaschke**). *Assume that the convex domain $K \subset \mathbb{R}^2$ has C^2 boundary $\Gamma = \partial K$ and that with the positive constant $\kappa_0 > 0$ the curvature satisfies $\kappa(\mathbf{z}) \leq \kappa_0$ at all boundary points $\mathbf{z} \in \Gamma$. Then to each boundary points $\mathbf{z} \in \Gamma$ there exists a disk D_R of radius $R = 1/\kappa_0$, such that $\mathbf{z} \in \partial D_R$, and $D_R \subset K$.*

Note that the result, although seemingly local, does not allow for extensions to non-convex curves Γ. One can draw pictures of leg-bone like shapes of arbitrarily small upper bound of (positive) curvature, while at some points of touching containing arbitrarily small disks only. The reason is that the curve, after starting off from a certain boundary point $\mathbf{x}$, and then leaning back a bit, can eventually return arbitrarily close to the point from where it started: hence a prescribed size of disk cannot be inscribed.

On the other hand the Blaschke Theorem extends to any dimension $d \in \mathbb{N}$.

Also, the result has a similar, dual version, too, see [**1**, p. 116]. This was formulated already in Lemma 1.1.15 above.

Now we start with introducing a few notions and recalling auxiliary facts. In §1.5 we formulate and prove the two basic results – the discrete forms of the Blaschke Theorems.

Then we show how our discrete approach yields a new, straightforward proof for a more involved sharpening of Theorem 1.4.1, originally due to Strantzen.

Recall that the term planar *convex body* stands for a compact, convex subset of $\mathbb{C} \cong \mathbb{R}^2$ having nonempty interior. For a (planar) convex body K any interior point z defines a parametrization $\gamma(\varphi)$ – the usual polar coordinate representation of the boundary ∂K, – taking the unique point $\{z + te^{i\varphi} : t \in (0, \infty)\} \cap \partial K$ for the definition of $\gamma(\varphi)$. This defines the closed Jordan curve $\Gamma = \partial K$ and its parametrization $\gamma : [0, 2\pi] \to \mathbb{C}$. By convexity, from any boundary point $\zeta = \gamma(\theta) \in \partial K$, locally the chords to boundary points with parameter $< \theta$ or with $> \theta$ have arguments below and above the argument of the direction of any supporting line at ζ. Thus the tangent direction or argument function $\alpha_-(\theta)$ can be defined as e.g. the supremum of arguments of chords from the left; similarly, $\alpha_+(\theta) := \inf\{\arg(z-\zeta) \ : \ z = \gamma(\varphi), \ \varphi > \theta\}$, and any line $\zeta + e^{i\beta}\mathbb{R}$ with $\alpha_-(\theta) \le \beta \le \alpha_+(\theta)$ is a supporting line to K at $\zeta = \gamma(\theta) \in \partial K$. In particular the curve γ is differentiable at $\zeta = \gamma(\theta)$ if and only if $\alpha_-(\theta) = \alpha_+(\theta)$; in this case the tangent of γ at ζ is $\zeta + e^{i\alpha}\mathbb{R}$ with the unique value of $\alpha = \alpha_-(\theta) = \alpha_+(\theta)$. It is clear that interpreting $\alpha_\pm$ as functions on the boundary points $\zeta \in \partial K$, we obtain a parametrization-independent function. In other words, we are allowed to change parameterizations to arc length, say, when in case of $|\Gamma| = \ell$ ($|\Gamma|$ meaning the length of $\Gamma := \partial K$) the functions $\alpha_\pm$ map $[0, \ell]$ to $[0, 2\pi]$.

Observe that $\alpha_\pm$ are nondecreasing functions with total variation $\mathrm{Var}\,[\alpha_\pm] = 2\pi$, and that they have a common value precisely at continuity points, which occur exactly at points where the supporting line to K is unique. At points of discontinuity $\alpha_\pm$ is the left-, resp. right continuous extension of the same function. For convenience, and for better matching with [**3**], we may even define the function $\alpha := (\alpha_+ + \alpha_-)/2$ all over the parameter interval.

For obvious geometric reasons we call the jump function $\beta := \alpha_+ - \alpha_-$ the *supplementary angle* function. In fact, β and the usual Lebesgue decomposition of the nondecreasing function α_+ to $\alpha_+ = \sigma + \alpha_* + \alpha_0$, consisting of the pure jump function σ, the nondecreasing singular component α_*, and the absolute continuous part α_0, are closely related. By monotonicity there are at most countable many points where $\beta(x) > 0$, and in view of bounded variation we even have $\sum_x \beta(x) \le 2\pi$, hence the definition $\mu := \sum_x \beta(x)\delta_x$ defines a bounded, non-negative Borel measure on $[0, 2\pi)$. Now it is clear that $\sigma(x) = \mu([0, x])$, while $\alpha_*' = 0$ a.e., and α_0 is absolutely continuous. In particular, α or α_+ is differentiable at x provided that $\beta(x) = 0$ and x is not in the exceptional set of non-differentiable points with respect to α_* or α_0. That is, we have differentiability almost everywhere, and

$$\begin{aligned}
\int_x^y \alpha' &= \alpha_0(y) - \alpha_0(x) = \lim_{z \to x-0} \alpha_0(y) - \alpha_0(z) \\
&= \lim_{z \to x-0} \{[\alpha_+(y) - \sigma(y) - \alpha_*(y))] - [\alpha_+(z) - \sigma(z) - \alpha_*(z)]\} \\
&= \alpha_+(y) - \beta(y) - \mu([x, y)) - \lim_{z \to x-0} \alpha_+(z) - \lim_{z \to x-0} [\alpha_*(y) - \alpha_*(z)] \le \alpha_-(y) - \alpha_+(x) \ .
\end{aligned} \tag{1.55}$$

It follows that

$$\alpha'(t) \geq \lambda \qquad \text{a.e.} \quad t \in [0,a] \tag{1.56}$$

holds true if and only if we have

$$\alpha_\pm(y) - \alpha_\pm(x) \geq \lambda(y-x) \qquad \forall x, y \in [0,a] \ . \tag{1.57}$$

Here we restricted ourselves to the arc length parametrization taken in positive orientation. Recall that one of the most important geometric quantities, curvature, is just $\kappa(s) := \alpha'(s)$, whenever parametrization is by arc length s.

Thus we can rewrite (1.56) as

$$\kappa(t) \geq \lambda \qquad \text{a.e.} \quad t \in [0,a] \ , \tag{1.58}$$

or, with radius of curvature $\rho(t) := 1/\kappa(t)$ introduced (writing $1/0 = \infty$),

$$\rho(t) \leq \frac{1}{\lambda} \qquad \text{a.e.} \quad t \in [0,a] \ . \tag{1.59}$$

Again, ρ is a parametrization-invariant quantity (describing the radius of the osculating circle). Actually, it is easy to translate all these conditions to arbitrary parametrization of the tangent angle function α. Since also curvature and radius of curvature are parametrization-invariant quantities, all the above hold for any parametrization.

Moreover, with a general parametrization let $|\Gamma(\eta,\zeta)|$ stand for the length of the counterclockwise arc $\Gamma(\eta,\zeta)$ of the rectifiable Jordan curve Γ between the two points $\zeta, \eta \in \Gamma = \partial K$. We can then say that the curve satisfies a Lipschitz-type increase or *subdifferential condition* whenever

$$|\alpha_\pm(\eta) - \alpha_\pm(\zeta)| \geq \lambda |\Gamma(\eta,\zeta)| \qquad (\forall \zeta, \eta \in \Gamma) \ , \tag{1.60}$$

here meaning by $\alpha_\pm(\xi)$, for $\xi \in \Gamma$, not values in $[0, 2\pi)$, but a locally monotonously increasing branch of $\alpha_\pm$, with jumps in $(0,\pi)$, along the counterclockwise arc $\Gamma(\eta,\zeta)$ of Γ. Clearly, the above considerations show that all the above are equivalent.

In the paper we use the notation α (and also $\alpha_\pm$) for the tangent angle, κ for the curvature, and ρ for the radius of curvature. The counterclockwise taken right hand side tangent unit vector(s) will be denoted by $\mathbf{t}$, and the outer unit normal vectors by $\mathbf{n}$. These notations we will use basically in function of the arc length parametrization s, but with a slight abuse of notation also $\alpha_-(\varphi)$, $\mathbf{t}(\mathbf{x})$, $\mathbf{n}(\mathbf{x})$ etc. may occur with the obvious meaning.

Note that $\mathbf{t}(\mathbf{x}) = i\mathbf{n}(\mathbf{x}))$ and also $\mathbf{t}(\mathbf{x}) = \dot{\gamma}(s)$ when $\mathbf{x} = \mathbf{x}(s) \in \gamma$ and the parametrization/differentiation, symbolized by the dot, is with respect to arc length; moreover, with $\nu(s) : \arg(\mathbf{n}(\mathbf{x}(s))$ we obviously have $\alpha \equiv \nu + \pi/2 \mod 2\pi$ at least at points of continuity of α and ν. To avoid mod 2π equality, we can shift to the universal covering spaces and maps and consider $\widetilde{\alpha}, \widetilde{\nu}$, i.e. $\widetilde{\mathbf{t}}, \widetilde{\mathbf{n}}$ – e.g. in case of $\widetilde{\mathbf{n}}$ we will somewhat detail this right below. However, note a slight difference in handling α and $\widetilde{\mathbf{n}}$: the first is taken as a singlevalued function, with values $\alpha(s) := \frac{1}{2}\{\alpha_-(s) + \alpha_+(s)\}$ at points of discontinuity, while $\widetilde{\mathbf{n}}$ is a multivalued function attaining a full closed interval $[\widetilde{\mathbf{n}}_-(s), \widetilde{\mathbf{n}}_+(s)]$ whenever s is a point of discontinuity. Also recall that curvature, whenever it exists, is $|\ddot{\gamma}(s)| = \alpha'(s) = \widetilde{\mathbf{n}}'(s)$.

In this work we mean by a multi-valued function Φ from X to Y a (non-empty-valued) mapping $\Phi : X \to 2^Y \setminus \{\emptyset\}$, i.e. we assume that the domain of Φ is always the whole of X and that $\emptyset \neq \Phi(x) \subset Y$ for all $x \in X$. Recall the notions of modulus of continuity and minimal oscillation in the full generality of multi-valued functions between metric spaces.

DEFINITION 1.4.2 (**modulus of continuity and minimal oscillation**). Let (X, d_X) and (Y, d_Y) be metric spaces. We call the *modulus of continuity* of the multivalued function Φ from X to Y the quantity

$$\omega(\Phi, \tau) := \sup\{d_Y(y, y') \ : \ x, x' \in X,\ d_X(x, x') \leq \tau,\ y \in \Phi(x),\ y' \in \Phi(x')\}.$$

Similarly, we call *minimal oscillation* of Φ the quantity

$$\Omega(\Phi, \tau) := \inf\{d_Y(y, y') \ : \ x, x' \in X,\ d_X(x, x') \geq \tau,\ y \in \Phi(x),\ y' \in \Phi(x')\}.$$

If we are given a multi-valued *unit vector function* $\mathbf{v}(\mathbf{x}) : H \to 2^{S^{d-1}} \setminus \{\emptyset\}$, where $H \subset \mathbb{R}^d$ and S^{d-1} is the unit ball of $\mathbb{R}^d$, then the derived formulae become:

(1.61)

$$\omega(\tau) := \omega(\mathbf{v}, \tau) := \sup\{\arccos\langle \mathbf{u}, \mathbf{w}\rangle \ : \ \mathbf{x}, \mathbf{y} \in H,\ |\mathbf{x} - \mathbf{y}| \leq \tau,\ \mathbf{u} \in \mathbf{v}(\mathbf{x}), \mathbf{w} \in \mathbf{v}(\mathbf{y})\},$$

and

(1.62)

$$\Omega(\tau) := \Omega(\mathbf{v}, \tau) := \inf\{\arccos\langle \mathbf{u}, \mathbf{w}\rangle \ : \ \mathbf{x}, \mathbf{y} \in H,\ |\mathbf{x} - \mathbf{y}| \geq \tau,\ \mathbf{u} \in \mathbf{v}(\mathbf{x}),\ \mathbf{w} \in \mathbf{v}(\mathbf{y})\}.$$

For a *planar* multi-valued unit vector function $\mathbf{v} : H \to 2^{S^1} \setminus \{\emptyset\}$, where $H \subset \mathbb{R}^2 \simeq \mathbb{C}$ and S^1 is the unit circle in $\mathbb{R}^2$, we can parameterize the unit circle S^1 by the corresponding angle φ and thus write $\mathbf{v}(\mathbf{x}) = e^{i\Phi(\mathbf{x})}$ with $\Phi(\mathbf{x}) := \arg(\mathbf{v}(\mathbf{x}))$ being the corresponding angle. We will somewhat elaborate on this observation in the case when our multi-valued vector function is the outward normal vector(s) function $\mathbf{n}(\mathbf{x})$ of a closed convex curve.

Let γ be the boundary curve of a convex body in $\mathbb{R}^2$, which will be considered as oriented counterclockwise, and let the multivalued function $\mathbf{n}(\mathbf{x}) : \gamma \to 2^{S^1} \setminus \{\emptyset\}$ be defined as the set of all outward unit normal vectors of γ at the point $\mathbf{x} \in \gamma$. Observe that the set $\mathbf{n}(\mathbf{x})$ of the set of values of $\mathbf{n}$ at any $\mathbf{x} \in \gamma$ is either a point, or a closed segment of length less than π. Then there exists a unique lifting $\tilde{\mathbf{n}}$ of $\mathbf{n}$ from the universal covering space $\tilde{\gamma}(\simeq \mathbb{R}$, see below) of γ to the universal covering space $\mathbb{R} = \tilde{S^1}$ of S^1, with the respective universal covering maps $\pi_\gamma : \tilde{\gamma} \to \gamma$ and $\pi_{S^1} : \tilde{S^1} \to S^1$, with properties to be described below. Here we do not want to recall the concept of the universal covering spaces from algebraic topology in its generality, but restrict ourselves to give it in the situation described above. As already said, $\tilde{S^1} = \mathbb{R}$ and the corresponding universal covering map is $\pi_{S^1} : x \to (\cos x, \sin x)$ (We consider, as usual, S^1 as $\mathbb{R} \mod 2\pi$.) Similarly, for γ we have $\tilde{\gamma} = \mathbb{R}$, with universal covering map $\pi_\gamma : \mathbb{R} \to \gamma$ given in the following way. Let us fix some arbitrary point $\mathbf{x_0} \in \gamma$, (the following considerations will be independent of $\mathbf{x_0}$, in the natural sense). Let us denote by ℓ the length of γ. Then for $\lambda \in \mathbb{R} = \tilde{\gamma}$ we have that $\pi_\gamma(\lambda) \in \gamma$ is that unique point $\mathbf{x}$ of γ, for which the counterclockwise measured arc $\mathbf{x_0 x}$ has a length $\lambda \mod \ell$.

Now we describe the postulates for the multivalued function $\tilde{\mathbf{n}} : \mathbb{R} = \tilde{\gamma} \to \tilde{S^1} = \mathbb{R}$, which determine it uniquely. First of all, we must have the equality $\pi_{S^1} \circ \tilde{\mathbf{n}} = \mathbf{n} \circ \pi_\gamma$, where $\circ$ denotes the composition of two multivalued functions. (In algebraic topology this is called *commutativity of a certain square of mappings.*) Second, the values of $\tilde{\mathbf{n}}$ must be either points or non-degenerate closed intervals (of length less than π; however this last property follows from the other ones). Third, $\tilde{\mathbf{n}}$ must be non-decreasing in the following sense: for $\lambda_1, \lambda_2 \in \mathbb{R},\ \lambda_1 < \lambda_2$ we have $r_1 \in \tilde{\mathbf{n}}(\lambda_1), r_2 \in \tilde{\mathbf{n}}(\lambda_2) \Longrightarrow r_1 \le r_2$. Further, $\tilde{\mathbf{n}}$ must be a non-decreasing multivalued function, continuous from the left, i.e., for any $\lambda \in \mathbb{R}$ we have that for any $\varepsilon > 0$ there exists a $\delta > 0$, such that $\cup_{\mu \in (\lambda-\delta,\lambda)} \tilde{\mathbf{n}}(\mu) \subset (\min \tilde{\mathbf{n}}(\lambda) - \varepsilon, \min \tilde{\mathbf{n}}(\lambda))$. Analogously, $\tilde{\mathbf{n}}$ must be a non-decreasing multi-valued function continuous from the right, i.e., for any $\lambda \in \mathbb{R}$ we have that for any $\varepsilon > 0$ there exists a $\delta > 0$, such that $\cup_{\mu \in (\lambda,\lambda+\delta)} \tilde{\mathbf{n}}(\mu) \subset (\max \tilde{\mathbf{n}}(\lambda), \max \tilde{\mathbf{n}}(\lambda) + \varepsilon)$. These are all the postulates for the multi-valued function $\tilde{\mathbf{n}}$. It is clear, that $\tilde{\mathbf{n}}$ exists and is uniquely determined, for fixed $\mathbf{x_0}$ (and, for $\mathbf{x_0}$ arbitrary, only the parametrization of $\mathbb{R} = \tilde{\gamma}$ changes, by a translation.)

The above listed properties imply still one important property of the multi-valued function $\tilde{\mathbf{n}}$: we have for any $\lambda \in \mathbb{R}$ that $\tilde{\mathbf{n}}(\lambda + \ell) = \tilde{\mathbf{n}}(\lambda) + 2\pi$.

Definition 1.4.3. We define the modulus of continuity of the multi-valued normal vector function $\mathbf{n}(\mathbf{x})$ *with respect to arc length* as the (ordinary) modulus of continuity of the multi-valued lift-up function $\tilde{\mathbf{n}} : \mathbb{R} \to \mathbb{R} \setminus \{\emptyset\}$, i.e. as

$$\tilde{\omega}(\tau) := \tilde{\omega}(\mathbf{n}, \tau) := \omega(\tilde{\mathbf{n}}, \tau)$$
$$:= \sup\{|r_1 - r_2| \mid r_1 \in \tilde{\mathbf{n}}(\lambda_1), r_2 \in \tilde{\mathbf{n}}(\lambda_2),\ \lambda_1, \lambda_2 \in \mathbb{R}, |\lambda_1 - \lambda_2| \le \tau\}. \tag{1.63}$$

Similarly, we define the minimal oscillation of the multi-valued normal vector function $\mathbf{n}(\mathbf{x})$ *with respect to arc length* as the (ordinary) minimal oscillation function of $\tilde{\mathbf{n}}$, i.e. as

$$\tilde{\Omega}(\tau) := \tilde{\Omega}(\mathbf{n}, \tau) := \Omega(\tilde{\mathbf{n}}, \tau)$$
$$:= \inf\{|r_1 - r_2| \mid r_1 \in \tilde{\mathbf{n}}(\lambda_1), r_2 \in \tilde{\mathbf{n}}(\lambda_2),\ \lambda_1, \lambda_2 \in \mathbb{R}, |\lambda_1 - \lambda_2| \ge \tau\}. \tag{1.64}$$

By writing "modulus of continuity" we do not mean to say anything like continuity of $\tilde{\mathbf{n}}$. In fact, if for some $\lambda \in \mathbb{R}$ $\tilde{\mathbf{n}}(\lambda)$ is a non-degenerate closed segment, then the left-hand side and right-hand side limits of $\tilde{\mathbf{n}}$ at λ - in the sense of the definition of continuity from the left or right, respectively - are surely different.

We evidently have that the modulus of continuity of $\tilde{\mathbf{n}}$ is subadditive, meaning $\tilde{\omega}(\tau_1 + \tau_2) \le \tilde{\omega}(\tau_1) + \tilde{\omega}(\tau_2)$, and similarly, that the minimal oscillation of $\tilde{\mathbf{n}}$ is superadditive, meaning $\tilde{\Omega}(\tau_1 + \tau_2) \ge \tilde{\Omega}(\tau_1) + \tilde{\Omega}(\tau_2)$. In fact, a standard property of the modulus of continuity of *any (non-empty valued) multivalued function from* $\mathbb{R}$ *(or from any convex set, in the sense of metric intervals) to* $\mathbb{R}$ is subadditivity, and similarly, minimal oscillation of such a function is superadditive. These properties with non-negativity and non-decreasing property also imply that $\tilde{\omega}(\tau)/\tau$ and $\tilde{\Omega}(\tau)/\tau$ have limits when $\tau \to 0$; moreover, $\lim_{\tau\to 0} \tilde{\omega}(\tau)/\tau = \sup \tilde{\omega}(\tau)/\tau$ and $\lim_{\tau\to 0} \tilde{\Omega}(\tau)/\tau = \inf \tilde{\Omega}(\tau)/\tau$. Note that metric convexity is essential here, so e.g. it is not clear if in $\mathbb{R}^d$ any proper analogy could be established.

Observe that if the curvature of γ exists at $\mathbf{x}_0$, then for the non-empty valued multi-valued function $\mathbf{n}(\mathbf{x})$:="set of values of all outer unit normal vectors of γ at $\mathbf{x}$", we necessarily have $\#\mathbf{n}(\mathbf{x}_0) = 1$ and the curvature can be written as

$$\kappa(\mathbf{x}_0) = \lim_{\mathbf{y}\to\mathbf{x}_0\ \mathbf{v}\in\mathbf{n}(\mathbf{y})} \frac{\arccos\langle \mathbf{n}(\mathbf{x}_0), \mathbf{v}\rangle}{|\mathbf{x}_0 - \mathbf{y}|}, \tag{1.65}$$

where the limit in (1.65) exists with arbitrary choice of $\mathbf{v} \in \mathbf{y}$ and is independent of this choice.

The next two propositions are well-known.

PROPOSITION 1.4.4. *Let γ be a planar convex curve. Recall that* (1.61) *and* (1.62) *is the modulus of continuity and the minimal oscillation of the multi-valued normal vector function* $\mathbf{n}(\mathbf{x})$ *with respect to chord length, and that* (1.63) *and* (1.64) *stand for the modulus of continuity and the minimal oscillation of* $\mathbf{n}(\mathbf{x})$ *with respect to arc length. Then for all* $\mathbf{x} \in \gamma$ *with curvature* $\kappa(\mathbf{x}) \in [0, \infty]$ *we have*

$$\lim_{\tau\to 0} \frac{\Omega(\tau)}{\tau} = \lim_{\tau\to 0} \frac{\tilde{\Omega}(\tau)}{\tau} \le \kappa(\mathbf{x}) \le \lim_{\tau\to 0} \frac{\tilde{\omega}(\tau)}{\tau} = \lim_{\tau\to 0} \frac{\omega(\tau)}{\tau}. \tag{1.66}$$

PROOF. First of all, by definition and the obvious fact that chord length does not exceed arc length, it follows that $\Omega(\tau) \le \tilde{\Omega}(\tau) \le \tilde{\omega}(\tau) \le \omega(\tau)$. We have already remarked, that the limits $\lim_{\tau\to 0} \tilde{\Omega}(\tau)/\tau$ and $\lim_{\tau\to 0} \tilde{\omega}(\tau)/\tau$ exist; moreover, the limit $\lim_{\tau\to 0} \tilde{\Omega}(\tau)/\tau = \inf \tilde{\Omega}(\tau)/\tau \le 2\pi/\ell(\gamma)$ is necessarily finite.

On the other hand, let τ be any fixed value, chosen sufficiently small, and choose $0 \le s < t < \ell(\gamma)$, $\gamma(s) = \mathbf{x}$ and $\gamma(t) = \mathbf{y}$ with $|\mathbf{x} - \mathbf{y}| = \tau$ such that $\Omega(\tau) = \arccos\langle \mathbf{u}, \mathbf{v}\rangle$ with some $\mathbf{u} \in \mathbf{n}(\mathbf{x})$, $\mathbf{v} \in \mathbf{n}(\mathbf{y})$. Then clearly $\arg \mathbf{u} = \tilde{\mathbf{n}}_+(s)$, $\arg \mathbf{v} = \tilde{\mathbf{n}}_-(t)$, also $\Omega(\tau) = \tilde{\mathbf{n}}_-(t) - \tilde{\mathbf{n}}_+(s)$, and for all $s < \sigma < t$ we have $\tilde{\mathbf{n}}(\sigma) \subset [\tilde{\mathbf{n}}_+(s), \tilde{\mathbf{n}}_-(t)]$. Moreover, putting $\boldsymbol{\nu}$ for the normal vector of the chord $\mathbf{y} - \mathbf{x}$, having right angle with it in the clockwise direction, we also have $\arg(\boldsymbol{\nu}) \in [\tilde{\mathbf{n}}_+(s), \tilde{\mathbf{n}}_-(t)]$ because $\mathbf{y} - \mathbf{x} = \int_s^t \mathbf{t}(\sigma)d\sigma = \int_s^t i\mathbf{n}(\sigma)d\sigma$, and thus $\arg(\mathbf{y} - \mathbf{x}) \in [\tilde{\mathbf{n}}_+(s) + \pi/2, \tilde{\mathbf{n}}_-(t) + \pi/2] \mod 2\pi$.

Now we compare arc length and chord length. We find $\tau = |\mathbf{y} - \mathbf{x}| = \int_s^t \langle \mathbf{n}(\sigma), \boldsymbol{\nu}\rangle d\sigma \ge (s - t)\cos(\tilde{\mathbf{n}}_-(t) - \tilde{\mathbf{n}}_+(s)) = (s-t)\cos\Omega(\tau)$, and, as $\Omega(\tau) = O(\tau)$, we surely have $\cos(\Omega(\tau)) \to 1$ when $\tau \to 0$. It is also clear that $t - s \to 0$ together with $\tau \to 0$, so for τ chosen sufficiently small,

$$\frac{\Omega(\tau)}{\tau} \ge \frac{\tilde{\Omega}(t-s)}{\tau} = \frac{t-s}{\tau}\frac{\tilde{\Omega}(t-s)}{t-s} \ge \frac{t-s}{\tau}(1-\varepsilon)\lim_{\xi\to 0}\frac{\tilde{\Omega}(\xi)}{\xi} \ge (1-\varepsilon)^2 \lim_{\xi\to 0}\frac{\tilde{\Omega}(\xi)}{\xi}$$

and it follows that the two limits of the oscillation functions coincide.

For the modulus of continuity type quantities note that if $\mathbf{n}$ is really multivalued, i.e. there exists some point $\mathbf{x} \in \gamma$ where $\mathbf{n}(\mathbf{x})$ consists of more than one vector, then $\tilde{\mathbf{n}}$ attains some closed interval and $\tilde{\omega}(\tau)$ does not go to 0 with τ: whence the arising limits must be $+\infty$. Therefore, it suffices to consider the case when $\mathbf{n}$, i.e. $\tilde{\mathbf{n}}$, are single-valued (and thus $\tilde{\mathbf{n}}$ is monotonous and continuous) functions.

Again, consider a given value $\tau > 0$, sufficiently small, and a pair of extremal points $\mathbf{x} = \gamma(s)$ and $\mathbf{y} = \gamma(t)$ with $0 \le s < t < \ell(\gamma)$ such that $\tau = |\mathbf{y} - \mathbf{x}|$ and $\omega(\tau) = \arccos\langle \mathbf{n}(\mathbf{x}), \mathbf{n}(\mathbf{y})\rangle = \tilde{\mathbf{n}}(t) - \tilde{\mathbf{n}}(s)$. As above, for all $s < \sigma < t$ we have $\tilde{\mathbf{n}}(\sigma) \subset [\tilde{\mathbf{n}}(s), \tilde{\mathbf{n}}(t)]$.

Moreover, $\tau = |\mathbf{y}-\mathbf{x}| = \int_s^t \langle \mathbf{n}(\sigma), \boldsymbol{\nu}\rangle d\sigma \geq (t-s)\cos(\tilde{\mathbf{n}}(t)-\tilde{\mathbf{n}}(s)) = (t-s)\cos\omega(\tau)$, and, as $\omega(\tau)\to 0$, we surely have $\cos(\omega(\tau))\to 1$ when $\tau\to 0$. (Observe that we already have $t-s\to 0$ together with $\tau\to 0$ – however, we do not need it here.) At last, we find for τ chosen sufficiently small,

$$\frac{\omega(\tau)}{\tau} \leq \frac{t-s}{\tau}\frac{\tilde{\omega}(t-s)}{t-s} \leq \frac{1}{\cos\omega(\tau)}\sup_{\xi}\frac{\tilde{\omega}(\xi)}{\xi} \leq (1+\varepsilon)\lim_{\xi\to 0}\frac{\tilde{\omega}(\xi)}{\xi}.$$

It follows that the leftmost and rightmost limits in (1.66) exist and are equal to the corresponding limits with respect to arc length. Therefore, it suffices to prove the inequalities involving $\kappa(\mathbf{x})$ for the quantities ω and Ω only.

Clearly, $\Omega(\mathbf{n}, |\mathbf{x}-\mathbf{y}|) \leq \arccos\langle \mathbf{u},\mathbf{v}\rangle \leq \omega(\mathbf{n}, |\mathbf{x}-\mathbf{y}|)$ for all $\mathbf{u}\in\mathbf{n}(\mathbf{x})$, $\mathbf{v}\in\mathbf{n}(\mathbf{y})$. Putting $\tau = |\mathbf{y}-\mathbf{x}|$, and recalling that $\mathbf{n}(\mathbf{x})$ is unique by condition of existence of $\kappa(\mathbf{x})$, we obtain

$$\lim_{\tau\to 0}\frac{\Omega(\mathbf{n},\tau)}{\tau} \leq \left(\kappa(\mathbf{x}) = \right) \lim_{\mathbf{y}\to\mathbf{x}\ \mathbf{v}\in\mathbf{n}(\mathbf{y})} \frac{\arccos\langle \mathbf{n}(\mathbf{x}),\mathbf{v}\rangle}{|\mathbf{x}-\mathbf{y}|} \leq \lim_{\tau\to 0}\frac{\omega(\mathbf{n},\tau)}{\tau}.$$

□

In the following proposition arccos will denote the branch with values in $[0,\pi]$.

PROPOSITION 1.4.5. *Let γ be a closed convex curve, and* (1.61) *and* (1.62) *be the modulus of continuity and the minimal oscillation of the (in general, multi-valued) unit normal vector function* $\mathbf{n}(\mathbf{x})$.

(*i*) *If the curvature exists and is bounded from above by κ_0 all over γ, then there exists a bound $\tau_0 > 0$ so that for any two points $\mathbf{x},\mathbf{y}\in\gamma$ with $|\mathbf{x}-\mathbf{y}|\leq\tau\leq\tau_0$ we must have $\omega(\mathbf{n},\tau) < \pi/2$ and $\arccos\langle\mathbf{n}(\mathbf{x}),\mathbf{n}(\mathbf{y})\rangle \leq \kappa_0\tau/\cos(\omega(\mathbf{n},\tau))$. Thus we also have $\omega(\mathbf{n},\tau)\leq\kappa_0\tau/\cos(\omega(\mathbf{n},\tau))$ for $\tau\leq\tau_0$.*

(*ii*) *If the curvature $\kappa(\mathbf{x})$ exists (linearly, that is, according to arc length parametrization) almost everywhere, and is bounded from below by κ_0 (linearly) almost everywhere on γ, then for any two points $\mathbf{x},\mathbf{y}\in\gamma$ with $|\mathbf{x}-\mathbf{y}|\geq\tau$ and for all $\mathbf{u}\in\mathbf{n}(\mathbf{x})$,$\mathbf{v}\in\mathbf{n}(\mathbf{y})$ we have $\arccos\langle\mathbf{u},\mathbf{v}\rangle\geq\kappa_0\tau$ and hence $\Omega(\mathbf{n},\tau)\geq\kappa_0\tau$.*

PROOF. Consider first (*i*). In this case $\mathbf{n}$ is a single-valued function. Recall that α stands for the *tangent angle* function, and therefore with $\mathbf{x} = \gamma(s_0)$ and $\mathbf{y} = \gamma(t)$ $\arccos\langle\mathbf{n}(\mathbf{x}),\mathbf{n}(\mathbf{y})\rangle = \alpha(t)-\alpha(s_0)$, supposing that on the counterclockwise closed arc $\mathbf{xy}$ of γ the rotation of the outer unit normal vector is at most π. (In case of the rotation exceeding π, the complementary arc must have rotation below π, and considering the negatively oriented curve, i.e. a reflection of γ, we can conclude the same way.) Since the curvature is just $\kappa = \alpha'$ (α written in arc length parametrization), by condition α is an *everywhere differentiable function* (with respect to arc length). Thus we can apply the Lagrange mean value theorem to find some parameter $u\in(s_0,t)$ satisfying

$$\alpha(t)-\alpha(s_0) = \alpha'(u)(t-s_0).$$

Now we can apply the condition $\alpha' = \kappa\leq\kappa_0$ to get

$$\arccos\langle\mathbf{n}(\mathbf{x}),\mathbf{n}(\mathbf{y})\rangle \leq \kappa_0(t-s_0). \tag{1.67}$$

It remains to estimate the arc length $t - s_0$ in function of τ.

Let now $\mathbf{x}, \mathbf{y}$ be two arbitrary points of γ and consider the counterclockwise arc of γ between these points. Let us suppose that this arc has total curvature less than $\pi/2$. Since κ exists and is bounded everywhere by κ_0, a standard compactness argument yields $\omega(\mathbf{n}, \tau) < \pi/2$ for $\tau := |\mathbf{y} - \mathbf{x}| \le \tau_0$. As now $\mathbf{n}(\mathbf{x})$ is single-valued, we have $\arg \mathbf{t}(\mathbf{x}) = \alpha(s_0)$ with the unique tangent vector at $\mathbf{x}$, and we can write

$$t - s_0 = \int_{s_0}^{t} 1 ds \le \int_{s_0}^{t} \frac{\cos(\alpha(s) - \alpha(s_0))}{\cos(\alpha(t) - \alpha(s_0))} ds = \frac{1}{\cos(\alpha(t) - \alpha(s_0))} \int_{s_0}^{t} \langle \dot\gamma(s); \mathbf{t}(\mathbf{x}) \rangle ds$$
$$\le \frac{1}{\cos \omega(\mathbf{n}, |\mathbf{y} - \mathbf{x}|)} \left\langle \int_{s_0}^{t} \dot\gamma(s) ds; \mathbf{t}(\mathbf{x}) \right\rangle = \frac{\langle \mathbf{y} - \mathbf{x}; \mathbf{t}(\mathbf{x}) \rangle}{\cos \omega(\mathbf{n}, \tau)} \le \frac{\tau}{\cos \omega(\mathbf{n}, \tau)}.$$

On combining this with (1.67), the assertion (i) follows.

To prove (ii) we still can use that α is a monotonic function, hence is almost everywhere differentiable and, as detailed above, for any $\mathbf{u} \in \mathbf{n}(\mathbf{x})$, $\mathbf{v} \in \mathbf{n}(\mathbf{y})$ we have

$$\arccos\langle \mathbf{u}, \mathbf{v} \rangle = \arg \mathbf{u} - \arg \mathbf{v} \ge \tilde{\mathbf{n}}_-(t) - \tilde{\mathbf{n}}_+(s_0) = \alpha_-(t) - \alpha_+(s_0) \ge \int_{s_0}^{t} \alpha'(s) ds \ge \kappa_0 (t - s_0)$$

by condition of $\kappa = \alpha' \ge \kappa_0$ (linearly) a.e. on γ. (As above, we may assume that $\arg \mathbf{u} - \arg \mathbf{v}$ does not exceed π, as otherwise we may consider the complementary arc, i.e. the reflected curve with respect to the line of $\mathbf{x}$ and $\mathbf{y}$, e.g.) It is obvious that the arc length of γ between $\mathbf{x}$ and $\mathbf{y}$ is at least the distance of $\mathbf{x}$ and $\mathbf{y}$, hence the assertion follows. □

Rotations of $\mathbb{C} = \mathbb{R}^2$ about the origin O by the counterclockwise measured (positive) angle φ will be denoted by U_φ, that is,

$$U_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}. \tag{1.68}$$

We denote T the reflection to the y-axis, i.e. the linear mapping defined by $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

Definition 1.4.6 (**Mangled n-gons**). Let $2 \le k \in \mathbb{N}$ and put $n = 4k - 4$, $\varphi^* := \frac{\pi}{2k}$. We define the *standard mangled n-gon* as the convex n-gon

$$M_k := \mathrm{con}\,\{A_1, \dots, A_{k-1}, A_{k+1}, \dots, A_{2k-1}, A_{2k+1}, \dots, A_{3k-1}, A_{3k+1}, \dots, A_{4k-1}\}, \tag{1.69}$$

of $n = 4k - 4$ vertices with

$$A_m := \left(\sum_{j=1}^{m} \cos(j\varphi^*) - \sum_{\ell=1}^{\lfloor m/k \rfloor} \cos(\ell k \varphi^*), \sum_{j=1}^{m} \sin(j\varphi^*) - \sum_{\ell=1}^{\lfloor m/k \rfloor} \sin(\ell k \varphi^*) \right), \tag{1.70}$$

where $m \in \{1, \dots, 4k\} \setminus \{k, 2k, 3k, 4k\}$. That is, we consider a regular $4k$-gon of unit sides, but cut out the middle "cross-shape" (i.e., the union of two rectangles which are the convex hulls of two opposite sides of the regular $4k$-gon, these pairs of opposite sides being perpendicular to each other) and push together the left over four quadrants (i.e., shift the vertices $A_{\ell k}$ to the position of $A_{\ell k - 1}$ consecutively to join the remaining sides of the polygon. Observe that taking $A_0 := O$, the same formula (1.70) is valid also for

$A_0 := O = A_{4k} = A_{4k-1}$ and $A_{\ell k} = A_{\ell k-1}$, $\ell = 1, 2, 3, 4$, showing how the vertices of the regular $4k$-gon were moved into their new positions.)

Now let $\tau > 0$, $\alpha \in \mathbb{R}$, $\mathbf{x} \in \mathbb{R}^2$ and $\varphi \in (0, \pi/4]$ be arbitrary. Take $k := \left\lfloor \frac{\pi}{2\varphi} \right\rfloor$, so that $\varphi^* := \frac{\pi}{2k} \geq \varphi$.

Then we write $M(\varphi) := M_k$, and, moreover, we also define

$$M(\mathbf{x}, \alpha, \varphi, \tau) := M(\mathbf{x}, \alpha, \varphi^*, \tau) := U_\alpha(\tau M_k) + \mathbf{x}, \tag{1.71}$$

that is, the copy shifted by $\mathbf{x}$ of the $4k-4$-gon obtained by dilating $M(\varphi) = M_k$ from $O = A_0 = A_{4k-1}$ with τ and rotating it counterclockwise about O by the angle α.

E.g. if $\varphi \in (\pi/6, \pi/4]$, then $k = 2$, $\varphi^* = \pi/4$, $n = 4$, and M_2 is just a unit square, its side lines having direction tangents ± 1 and having its lowest vertex at O. It is the left over part, pushed together, of a regular octagon of unit side length, when the middle cross-shape is removed from its middle.

It is easy to see that the inradius $\rho(\varphi)$ and the circumradius $R(\varphi)$ of $M(\varphi) = M(\varphi^*) = M_k$ are

$$\begin{cases} r(\varphi) &= \frac{1}{2}\left\{\cot\frac{\pi}{4k} - \sqrt{2}\cos\left(\frac{1-(-1)^k}{8k}\pi\right)\right\}, \\ R(\varphi) &= \frac{1}{2}\left\{\cot\frac{\pi}{4k} - 1\right\}, \end{cases} \qquad \left(k := \left\lfloor \frac{\pi}{2\varphi} \right\rfloor\right), \tag{1.72}$$

respectively.

Similarly to the *mangled n-gons* M_k, we also define the *fattened n-gons* F_k.

Definition 1.4.7 (**Fattened n-gons**). Let $k \in \mathbb{N}$ and put $n = 4k$, $\varphi^* := \frac{\pi}{2k}$. We first define the *standard fattened n-gon* as the convex n-gon

$$F_k := \operatorname{con}\{A_1, \dots, A_{k-1}, A_k, A_{k+1}, \dots, A_{4k-1}, A_{4k}\}, \tag{1.73}$$

of $n = 4k$ vertices with

$$A_m := \left(\sum_{j=1}^{m}\cos(j\varphi^*) + \sum_{\ell=0}^{\lfloor m/k \rfloor}\cos(\ell k\varphi^*), \sum_{j=1}^{m}\sin(j\varphi^*) + \sum_{\ell=0}^{\lfloor m/k \rfloor}\sin(\ell k\varphi^*)\right). \tag{1.74}$$

That is, we consider a regular $4k$-gon , but fatten the middle "cross-shape" to twice as wide, and move the four quadrants to the corners formed by this width-doubled cross (i.e., shift the vertices $A_{\ell k}$ to the position of $A_{\ell k-1} + 2(A_{\ell k} - A_{\ell k-1})$ consecutively to join the remaining sides of the polygon). Observe that $A_{4k-1} = (-1, 0)$ and $A_{4k} = (1, 0)$.

Let $\tau > 0$, $\alpha \in \mathbb{R}$, $\mathbf{x} \in \mathbb{R}^2$ and $\varphi \in (0, \pi)$ be arbitrary. Now we take $k := \left\lceil \frac{\pi}{2\varphi} \right\rceil$, whence $\varphi^* := \frac{\pi}{2k} \leq \varphi$.

Then we write $F(\varphi) := F_k$, and, moreover, we also define

$$F(\mathbf{x}, \alpha, \varphi, \tau) := F(\mathbf{x}, \alpha, \varphi^*, \tau) := U_\alpha(\tau F_k) + \mathbf{x}, \tag{1.75}$$

that is, the copy shifted by $\mathbf{x}$ of the $4k$-gon obtained by dilating $F(\varphi) = F_k$ from O with τ and rotating it counterclockwise about O by the angle α.

E.g. if $\varphi \geq \pi/2$, then $k = 1$, $\varphi^* = \pi/2$, $n = 4$, and F_4 is just the square spanned by the vertices (1,0), (1,2), (-1,2), (-1,0) and having sides of length 2.

Observe that using the usual Minkowski addition, we can represent the connections of these deformed n-gons and the regular n-gon easily. Write Q_n for the regular n-gon placed symmetrically to the y-axis but above the x-axis with $O \in \partial Q_n$ a midpoint (hence not a vertex) of a side of Q_n. (This position is uniquely determined.) Also, denote the standard square as $S := Q_4 := \text{con}\ \{(1/2, 0); (1/2, 1); (-1/2, 1); (-1/2, 0)\}$. Then we have $M_k + S = Q_{4k}$ and $Q_{4k} + S = F_k$.

It is also easy to see that the inradius $\mathfrak{r}(\varphi)$ and the circumradius $\mathfrak{R}(\varphi)$ of $F(\varphi) = F(\varphi^*)$ are

$$\mathfrak{r}(\varphi) = \frac{1}{2}\cot\frac{\pi}{4k} + \frac{1}{2} \qquad \left(k := \left\lceil \frac{\pi}{2\varphi} \right\rceil\right), \tag{1.76}$$

and

$$\mathfrak{R}(\varphi) = \begin{cases} \frac{1}{2\sin\frac{\pi}{4k}} + \frac{1}{\sqrt{2}} & \text{if } 2 \nmid k \\ \sqrt{\frac{1}{2} + \frac{1}{4\sin^2\frac{\pi}{4k}} + \frac{1}{\sqrt{2}}\cot\frac{\pi}{4k}} & \text{if } 2 \mid k \end{cases} \qquad \left(k := \left\lceil \frac{\pi}{2\varphi} \right\rceil\right), \tag{1.77}$$

respectively.

The actual values of the above in- and circumradii in (1.72), (1.76), (1.77) are not important, but observe that for $\varphi \to 0$, or, equivalently, for $k \to \infty$, we have the asymptotic relation $r(\varphi) \sim R(\varphi) \sim \mathfrak{r}(\varphi) \sim \mathfrak{R}(\varphi) \sim \frac{1}{\varphi}$.

1.5. The discrete Blaschke theorems

1.6. Discrete versions of the Blaschke Rolling Ball Theorems

THEOREM 1.6.1. *Let $K \subset \mathbb{C}$ be a convex body and $0 < \varphi < \pi/4$. Denote $\mathbf{n}$ the (multi-valued) function of outer unit normal(s) to the closed convex curve $\gamma := \partial K$ and assume that $\omega(\mathbf{n}, \tau) \le \varphi < \pi/4$. Put $k := \left\lfloor \frac{\pi}{2\varphi} \right\rfloor$. If $\mathbf{x} \in \partial K = \gamma$, and $\mathbf{n}_0 = (\sin\alpha, -\cos\alpha) \in \mathbf{n}(\mathbf{x})$ is outer unit normal to γ at $\mathbf{x}$, then $M(\mathbf{x}, \alpha, \varphi, \tau) \subset K$.*

PROOF. Because $\varphi \le \varphi^* := \pi/(2k)$ and $M(\mathbf{x}, \alpha, \varphi, \tau) = M(\mathbf{x}, \alpha, \varphi^*, \tau)$, it suffices to present a proof for the case when $\varphi = \varphi^* = \frac{\pi}{2k}$.

Applying simple transformations we may reduce to the case $\mathbf{x} = O$ and $\alpha = 0$, $\tau = 1$. With these restrictions we are to prove $M_k \subset K$, where $O \in K = \partial\gamma$, $(0, -1)$ is an outer normal to K at O, and $\omega(\mathbf{n}, 1) \le \varphi$. Denote $P = (a, b)$ the first point, along γ following O counterclockwise, satisfying that $(1, 0)$ is outer normal to K at P. Clearly, then γ can be parameterized with the x-values along the x-axis so that $\gamma(x) = (x, g(x))$ for values $x \in [0, a]$, and g is a convex function on $[0, a]$.

Consider $A_m = (a_m, b_m)$ defined in (1.70) for $m = 0, \dots, k-1$, putting here $A_0 := A_{4k-1} = O$, and consider the function

$$f(x) := \begin{cases} \dots \\ (x - a_{m-1})\tan\frac{m\pi}{2k} + a_{m-1} \quad (a_{m-1} \le x \le a_m) \\ \dots \end{cases} \qquad (m = 1, \dots, k-1). \tag{1.78}$$

Moreover, denote the broken line joining $O = A_0, A_1, \dots, A_{k-1}$ as L, that is,

$$L := \{(x, f(x)) \ : \ 0 \le x \le a_{k-1}\}. \tag{1.79}$$

LEMMA 1.6.2. *Let $O \in \gamma = \partial K$, $(0,-1)$ is outer normal to K at O, and $\omega(\mathbf{n}, 1) \le \varphi = \frac{\pi}{2k}$. With the notations above, we have*

(i) $a \ge a_{k-1} = \frac{1}{2}(\cot \frac{\pi}{4k} - 1)(= R(\varphi))$.
(ii) $0 \le g(x) \le f(x)$ *for all* $x \in [0, a_{k-1}]$.
(iii) $g'_\pm(x) \le f'_\pm(x)$ *for all* $x \in (0, a_{k-1})$ *and* $g'_+(0) \le f'_+(0)$, $g'_-(a_{k-1}) \le f'_-(a_{k-1})$.
(iv) $b := g(a) \ge a_{k-1}$.
(v) $L \subset K$.

PROOF. Let $a^* := \min(a, a_{k-1})$. We argue by induction on m, where $m = 1, \dots k-1$, and the inductive assertions will comprise

(i') $a^* \ge a_m$;
(ii') $0 \le g(x) \le f(x)$ for all $x \in [a_{m-1}, a_m]$;
(iii') $g'_\pm(x) \le f'_\pm(x)$ for all $x \in (a_{m-1}, a_m)$ and $g'_+(a_{m-1}) \le f'_+(a_{m-1})$, $g'_-(a_m) \le f'_-(a_m)$.

Clearly, if we show this for all $m = 1, \dots, k-1$ then (i)-(iii) of the Lemma will be proved.

Let us start with $m = 1$. Since $O = (0,0) \in \gamma$, we have $g(a_{m-1}) = g(0) = 0 \le f(a_{m-1}) = f(0) = 0$. Let $S := \{x \in [0, a_1] \ : \ g|_{[0,x]} \le f|_{[0,x]}\}$. Clearly, S is a (possibly degenerate) closed interval with left end point 0, say $[0, X]$. Our aim is to prove that $S = [0, a_1]$. Clearly, if $X = a_1 \in S$, then a relative neighborhood of X belongs to S, too. We prove the same thing for any other $X \in S$. Observe that the distance of $O = A_0$ and A_1 is 1, and all other points of the triangle $\Delta := \Delta(O, (a_1, 0), A_1)$ are closer than 1 to $O = A_0$. In particular, in case $X < a_1$, both $\{(x, g(x)) \ : \ 0 \le x \le X\}$ and also a small neighborhood of $(X, g(X)) \in \Delta$ is also closer to O than 1. It follows that the continuous curve γ runs in the 1-neighborhood of O even in an appropriately small neighborhood of $(X, g(X)) \in \gamma$. Therefore, by assumption on the change of the normal to γ, the vector $(0,1)$ in the counterclockwise taken angular region between the left and right hand side half-tangents (oriented according to the positive orientation of γ) to γ at O, cannot rotate over $(\cos\varphi, \sin\varphi)$ along $\{(x, g(x)) \ : \ 0 \le x \le X + \eta\}$ for some positive value of η. That is, $a^* \ge X + \eta$ and the representation $\gamma(x) = (x, g(x))$ is valid for $x \in [0, X+\eta]$; moreover, $g'_\pm(x) \le \tan\varphi$ for all $x \in [0, X+\eta]$. In conclusion, $g(x) = \int_0^x g'(\xi) d\xi \le x \cdot \tan\varphi = f(x)$ for all $x \in [0, X+\eta]$. As a result, we find that S is relatively open. As it is also closed and nonempty, it is the whole interval $[0, a_1]$. This proves (i') and (ii') for $m = 1$, and (iii') follows from the fact that $\{(x, g(x)) \ : \ 0 \le x \le a_1\} \subset \Delta$ and thus the distance of any point of $\{(x, g(x)) \ : \ 0 \le x \le a_1\}$ from O is at most 1.

We proceed by induction. Let $1 < m < k$ and assume the assertion for all $m' < m$. Then from the inductive hypothesis $a^* \ge a_{m-1}$, $\mu := \mu_{m-1} := g'_-(a_{m-1}) \le f'_-(a_{m-1}) = \tan((m-1)\varphi)$ and $g(a_{m-1}) := y_{m-1} \le f(a_{m-1}) = b_{m-1}$. Consider now the function $h(x) := y_{m-1} + (x - a_{m-1})\tan(m\varphi)$ (defined for $x \in I_m := [a_{m-1}, a_m]$), denote the points $P_{m-1} := (a_{m-1}, y_{m-1})$ and $P_m := (a_m, h(a_m))$, and define the triangle $\Delta := \Delta_m := \Delta(P_{m-1}, (a_m, y_{m-1} + \mu\cos(m\varphi)), P_m)$. Then $h = f|_{I_m} - (b_{m-1} - y_{m-1}) \le f|_{I_m}$, and $h'_\pm = f'_\pm$ on I_m.

Our aim now is to show that γ proceeds inside $\Delta = \Delta_m$. Observe that for points Q inside Δ we have $|Q - P_{m-1}| \leq 1$, with equality holding only if $Q = P_m$. Therefore, for $Q \in \gamma \cap \Delta$ the right half-tangent direction to γ cannot exceed $\arctan \mu + \varphi \leq m\varphi$, and, moreover, the same properties hold even for a relative neighborhood of Q on γ if $Q \neq P_m$.

So we proceed similarly to the case $m = 1$. It is obvious that $a^* > a_{m-1}$ as γ proceeds between slopes μ and $\tan(m\varphi)$ in the 1-neighborhood of P_{m-1}. Take $S := S_m := \{x \in I_m \ : \ g|_{[a_{m-1},x]} \leq h|_{[a_{m-1},x]}\}$. Again, by continuity of g and linearity of h S is a closed interval $[a_{m-1}, X]$, say. Also, if $X = a_m$, then $S = I_m$ and so S is relatively open in I_m, and if $a_m \neq X \in S$, then $(X, g(X)) \in \gamma \cap \Delta$ has a small neighborhood where γ stays within the 1-neighborhood of P_{m-1}, therefore its slope is below $\tan(\arctan \mu + \varphi)$ and $\gamma(x) = (x, g(x))$ extends even until some $X + \eta$; moreover, $a^* \geq X + \eta$ and $\mu \leq g'_\pm \leq \tan(\arctan \mu + \varphi) \leq \tan(m\varphi)$ holds all over $[a_{m-1}, X + \eta]$ (where for a_{m-1} and $X + \eta$ we claim only the inequalities for g_+ and g_-, resp.), proving

(1.80)
$$\mu(x - a_{m-1}) + y_{m-1} \leq g(x) = \int_{a_{m-1}}^{x} g'(\xi) d\xi + y_{m-1} \leq \tan(m\varphi)(x - a_{m-1}) + y_{m-1} = h(x)$$

for all $a_{m-1} \leq x \leq X + \eta$. That is, γ stays inside Δ and S contains a small neighborhood of X, too. It follows that $S \neq \emptyset$ is open and closed, while I_m is connected, thus $S_m = I_m$ and (1.80) holds true even for the whole of I_m. This proves (i')-(iii'), hence (i)-(iii) of the Lemma.

Applying the above we find $a > a_{k-1}$. However, a simple argument immediately gives also $b > a_{k-1}$, too. Indeed, it suffices to consider the new curve $\widehat{\gamma} := T(U_{-\pi/2}(\gamma - (a, b))$, obtained from γ first shifting it by $-P = -(a, b)$, then rotating it by $-\pi/2$ about O, and finally reflecting it at the y-axis. This shows (iv).

Also, applying the Lemma for the reflected curve $\widetilde{\gamma}$ of γ with respect to the y-axis gives a similar result for the part of γ towards the "negative x-direction". That is, we find that γ joins the points $\widetilde{P} = (\widetilde{a}, \widetilde{b})$ and $P = (a, b)$ with (some of their) outer unit normals $(-1, 0)$ and $(1, 0)$, respectively, so that the part strictly between these points (and containing O) does never have horizontal normals, and we have a parametrization $\gamma(x) = (x, g(x))$ for all $\widetilde{a} \leq x \leq a$ with $\widetilde{a} \leq -a_{k-1}$, $a \geq a_{k-1}$, and $0 \leq g(x) \leq f(|x|)$, $|g'_\pm(x)| \leq f'_\pm(|x|)$ for all $x \in [-a_{k-1}, a_{k-1}]$. Note that we also have $\widetilde{b} \geq a_{k-1}$, as above.

Finally let us show (v). Consider any point $(x, f(x))$ of L, where $x \in [0, a_{k-1}]$. There is a vertical line ℓ through it that intersects K in a vertical chord C of K. The lower endpoint of C is $(x, g(x)$. The upper endpoint of C has second coordinate at least $\min\{b, b\tilde{b}\} \geq a_{k-1}$. Hence the point $(x, f(x))$ lies on the chord C of K, whence in K. This proves (v).

□

Continuation of the proof of Theorem 1.6.1. From the above argument – or just reflecting L to the y-axis – it is immediate that also the broken line $\widetilde{L}$ joining $A_{3k+1}, \ldots, A_{4k-1} = O$ in this order that lies on the boundary of M_k belongs to K, too. We are left with the upper part joining $A_{k-1}, A_{k+1}, \ldots, A_{2k-1}, A_{2k+1}, \ldots, A_{3k-1}$. Let

(1.81)
$$L^+ := [A_{k-1}, A_{k+1}] \cup \cdots \cup [A_{2k-2}, A_{2k-1}], \quad L^- := [A_{2k-1}, A_{2k+1}] \cup \cdots \cup [A_{3k-2}, A_{3k-1}].$$

Next, let us apply the Lemma to the curve from P onwards in the counterclockwise sense. That is, take $K_+ := U_{-\pi/2}(K-P)$ (with U_α as defined in (1.68)) and $\gamma_+ := U_{-\pi/2}(\gamma - P)$ and check that $O \in \gamma_+$ and also γ_+ has an outer normal $(0,-1)$ at O; moreover, the same estimate on the modulus of continuity of the normal holds for γ_+. Thus we obtain that $L \subset K_+$, that is, $U_{\pi/2}(L) + P \subset K$. Observe $U_{\pi/2}(L) = L^+ - (a_{k-1}, a_{k-1})$, which entails $L^+ + (a - a_{k-1}, b - a_{k-1}) \subset K$. It suffices to say that $L^+ + (u,v) \subset K$ with $u, v \geq 0$.

Very similarly (or from this and using reflection) we also obtain $L^- + (p,q) \subset K$ with $p \geq 0$ and $q \leq 0$.

We claim that $A_{2k-1} = (0, 2a_{k-1}) \in K$. Indeed, $A_{2k-1} + (0, -2a_{k-1}) = O \in K$ and $A_{2k-1} + (u,v) \in L^+ + (u,v) \subset K$, $A_{2k-1} + (p,q) \in L^- + (p,q) \subset K$, and the convex hull of the vectors $(0, -2a_{k-1})$, (u,v) and (p,q) contains $(0,0)$, hence by convexity $A_{2k-1} \in K$.

Now, for showing $L^+ \subset K$, recall that $A_{k-1} \in L \subset K$, $A_{2k-1} \in K$, and $L^+ + (u,v) \subset K$. So it remains to see that L^+ is in the convex hull of its two endpoints and the set $L^+ + (u,v)$ whenever $u, v \geq 0$. Similarly one obtains $L^- \subset K$. That concludes the proof. □

An even stronger version can be proved considering the modulus of continuity $\tilde{\omega}$ with respect to arc length. We thank this sharpening to Endre Makai, who kindly called our attention to this possibility and suggested the crucial Lemma 1.6.4 for the proof.

THEOREM 1.6.3. *Let $K \subset \mathbb{C}$ be a planar convex body and $0 < \varphi < \pi/4$. Denote $\mathbf{n}$ the (multivalued) function of outer unit normal(s) to the closed convex curve $\gamma := \partial K$ and assume that $\tilde{\omega}(\tau) \leq \varphi < \pi/4$. Put $k := \left\lfloor \frac{\pi}{2\varphi} \right\rfloor$. If $\mathbf{x} \in \partial K = \gamma$, and $\mathbf{n}_0 = (\sin\alpha, -\cos\alpha) \in \mathbf{n}(\mathbf{x})$ is outer unit normal to γ at $\mathbf{x}$, then $M(\mathbf{x}, \alpha, \varphi, \tau) \subset K$.*

PROOF. We can repeat the argument yielding Theorem 1.6.1 with the only change that in the inductive argument for proving Lemma 1.6.2, we have to use twice (once for the case $m = 1$ to start the inductive argument, and once for general m) a slightly sharper geometric assertion to ensure that even in this setting the boundary curve γ of K will again proceed in the triangles $\Delta := \Delta(O, (a_1, 0), A_1)$ and $\Delta := \Delta_m := \Delta(P_{m-1}, (a_m, y_{m-1} + \mu\cos(m\varphi)), P_m)$.

The general situation will be covered by the following lemma.

LEMMA 1.6.4. *Let $\Delta = \Delta(P, Q, R)$ be the right- or obtuse triangle spanned by the points $P = (a,p)$, $Q = (b,q)$ with $b > a$ and $q \geq p$, and $R = (b,r)$ with $r > q$. Denote $\rho := \sqrt{(b-a)^2 + (r-p)^2}$ the length of the longest side of Δ, and let $\mu := (q-p)/(b-a)$, resp. $\nu := (r-p)/(b-a)$ be the slopes of sides PQ and PR, respectively, with corresponding angles $\psi := \arctan\mu$ and $\lambda := \arctan\nu$. Denote $\varphi := \lambda - \psi$ the angle of Δ at P.*

Let Γ be a convex curve of arc length ρ, connecting the points P and $N = (n,s)$ and having all its tangent vectors at all points of Γ (including the right half tangent at P and the left half tangent at N) with angles between ψ and $\psi + \varphi = \lambda$. Then $n \geq b$, the only possibility for equality is when $N = R$, otherwise $n > b$ and Γ intersects the vertical side of Δ at a mesh point $M = (b,m)$ with $q \leq m < r$. Moreover, $s \in [p + \mu(n-a), r]$.

PROOF. By convexity, the non-empty valued, multivalued tangent vector function $\mathbf{t}$ along Γ is continuous (in the weak sense) and nondecreasing, and also we have for the multi-valued tangent angle function $\widehat{\alpha}(\sigma) = \arg \mathbf{t}(\sigma) \in [\psi, \lambda]$ for all $0 \le \sigma \le \rho$, i.e. all along Γ. Therefore,

$$(n-a, s-p) = N - P = \int_0^\rho \mathbf{t}(\sigma) d\sigma = \int_0^\rho (\cos(\alpha(\sigma)), \sin(\alpha(\sigma)) d\sigma,$$

now neglecting the linearly 0-measure set of points where $\mathbf{t}$ or $\widehat{\alpha}$ is indeed multi-valued. By condition we find $n - a \ge \rho \cos \lambda = b - a$ and equality would imply $\cos \alpha(\sigma) = \cos \lambda$ a.e., that is $\Gamma = [P, R]$. Otherwise by $|\Gamma| = \rho$ and $n > b$ we surely have $s < r$. Finally, for the directional tangent of the chord $[P, N]$ we see

$$\frac{s-p}{n-a} = \frac{\int_0^\rho \sin \alpha(\sigma) d\sigma}{\int_0^\rho \cos \alpha(\sigma) d\sigma} \ge \frac{\int_0^\rho \sin \psi d\sigma}{\int_0^\rho \cos \psi d\sigma} = \tan \psi = \mu.$$

The lemma follows. □

In applying the above lemma we start with the observation that by condition $\tilde{\omega}(1) \le \varphi$, the tangent angle of γ can increase at most φ along the part of γ which is closer than 1 to the point O (in case $m = 1$) or to P_{m-1} (in case of the inductive step with general m). Therefore, proceeding along γ with arc length 1 and denoting this arc of γ as Γ, we will have a convex curve, with tangent angles between ψ and $\psi + \varphi$, as in the above lemma. Therefore, Lemma 1.6.4 will ensure that the argument goes through for proving the corresponding version of Lemma 1.6.2 with $\omega(\mathbf{n}, 1)$ replaced by $\tilde{\omega}(1)$. Otherwise the argument is the same. □

THEOREM 1.6.5. *Let $K \subset \mathbb{C}$ be a (planar) convex body and $\tau > 0$. Denote $\mathbf{n}$ the (multivalued) function of outer unit normal(s) to the closed convex curve $\gamma := \partial K$ and assume that $\Omega(\mathbf{n}, \tau) \ge \varphi$. Take $k := \left\lceil \frac{\pi}{2\varphi} \right\rceil$. If $\mathbf{x} \in \partial K = \gamma$, and $\mathbf{n}_0 = (\sin \alpha, -\cos \alpha) \in \mathbf{n}(\mathbf{x})$ is normal to γ at $\mathbf{x}$, then $F(\mathbf{x}, \alpha, \varphi, \tau) \supset K$.*

PROOF. Because $\varphi \ge \varphi^* := \frac{\pi}{2k}$ and $F(\mathbf{x}, \alpha, \varphi, \tau) = F(\mathbf{x}, \alpha, \varphi^*, \tau)$, it suffices to present a proof for the case when $\varphi = \varphi^* = \frac{\pi}{2k}$.

Applying simple transformations we may reduce to the case $\mathbf{x} = O$ and $\alpha = 0$, $\tau = 1$. With these restrictions we are to prove $F_k \supset K$, where $O \in K = \partial\gamma$, $(0, -1)$ is an outer normal to K at O, and $\Omega(\mathbf{n}, 1) \ge \varphi$.

Denote $P = (a, b)$ the first point counterclokwise after O, along γ, satisfying that $(1, 0)$ is an outer unit normal to K at P. Clearly, then γ can be parameterized with the x-values along the x-axis so that $\gamma(x) = (x, g(x))$ for values $x \in [0, a]$, and g is a convex function on $[0, a]$.

Note that in case $a = 0$ we necessarily have $K \subset \{(x, y) \ : \ x \le 0\}$, and so the degenerate case becomes trivial as regards proving $K \cap \{(x, y) \ : \ x \ge 0\} \subset F_k \cap \{(x, y) \ : \ x \ge 0\}$. Therefore, we can assume that we have the non-degenerate case.

Similarly to (1.74), we define $A_m = (a_m, b_m)$ for $m = 0, \dots, k$ (here $A_0 := (0, 0) = O$), and consider the function

$$f(x) := \begin{cases} \dots \\ (x - a_m) \tan \frac{m\pi}{2k} + a_m & (a_m \le x \le a_{m+1}) \qquad (m = 0, \dots, k-1). \\ \dots \end{cases} \tag{1.82}$$

Moreover, now L will denote the broken line joining $O = A_0, A_1, \dots, A_k, \frac{1}{2}(A_k + A_{k+1}) = (a_k, a_k)$ in this order, that is,

$$L := \{(x, f(x)) \ : \ 0 \le x \le a_k\} \cup [A_k, (a_k, a_k)]. \tag{1.83}$$

We write

$$L_1 := L \cup \{(x, 0) \ : \ x \le 0\} \cup \{(a_k, y) \ : \ y \ge a_k\}. \tag{1.84}$$

Then $\mathbb{R}^2 \setminus L_1$ will have two connected components; the convex one will be denoted by K_1.

LEMMA 1.6.6. *Let $O \in K = \partial\gamma$, $(0, -1)$ be an outer normal to K at O, and $\Omega(\mathbf{n}, 1) \ge \varphi = \frac{\pi}{2k}$. With the notations above, we have*

(i) $a \le a_k = \frac{1}{2}(\cot \frac{\pi}{4k} + 1)(= \mathfrak{r}(\varphi))$.
(ii) $0 \le f(x) \le g(x)$ *for all for all* $x \in [0, a]$.
(iii) $f'_\pm(x) \le g'_\pm(x)$ $x \in (0, a)$ *and* $f'_+(0) \le g'_+(0)$, $f'_-(a) \le g'_-(a)$.
(iv) $b := g(a) \le a_k$.
(v) $K \subset K_1$.

PROOF. Since the degenerate case $a = 0$ is trivial (observe that (ii) is then undefinied, but cf. the paragraph before (1.82)), we assume $a > 0$.

Let $a^* := \min(a, a_k)$. We argue by induction on m, where $m = 0, \dots k - 1$, and the inductive assertions will comprise

(i') Either $a \le a_m$ or both(ii') and (iii') hold, where
(ii') $0 \le f(x) \le g(x)$ for all $x \in [a_m, \min(a, a_{m+1})]$;
(iii') $g'_\pm(x) \ge f'_\pm(x)$ for all $x \in [a_m, \min(a, a_{m+1})]$ (except for $g'_-(0)$ and also for $g'_+(a)$ if the second occurs).

Clearly, if we show this for all $m = 0, \dots, k-1$ then (i)-(iii) of the Lemma will be proved.

Let us start with $m = 0$. Since $O = (0, 0) \in \gamma$, we have $g(a_m) = g(0) = 0 \ge f(a_m) = f(0) = 0$. Let $S := \{x \in [0, a_1] \ : \ g|_{[0,x]} \ge f|_{[0,x]}\}$. Clearly, by continuity of f and g, S is a closed interval with left endpoint 0. Our aim is to prove that $S = [0, \min(a, 1)]$. Indeed, since $f|_{[0,1]} \equiv 0$, and as $(0, -1)$ is normal to K at O, we must have $g(x) \ge 0$ for all $0 \le x \le a$, as stated in (ii'). Moreover, since g is a convex curve, $g'_\pm(x) \ge 0 = f'_\pm(x)$ for all $x \in (0, \min(a, 1))$, and also $g'_+(0) \ge f'_+(0) = 0$, furthermore, $g'_-(\min(a, 1)) \ge f'_-(\min(a, 1))$. It remains to show $g'_+(1) \ge \tan\varphi = f'_+(1)$ in case $\min(a, 1) = 1$. But in this case either $a = 1$, and then $g'_+(1)$ does not exist (and the case is listed as exceptional in (iii')), or in view of $|O - (1, g(1))| \ge 1$ any point $(x, g(x))$ along γ in the counterclockwise sense after $(1, g(1))$ but before P (that is, with $1 < x < a$) is of distance > 1 from O, hence by condition its any outer normal direction is at least φ larger than that of the outer normal $(0, -1)$ of O: it follows that $g'_\pm(x) \ge \tan\varphi$ and thus $g'_+(1) \ge \tan\varphi = f'_+(1)$.

We proceed by induction. Let $1 \le m < k$ and assume the assertion for all $0 \le m' < m$. If $\min(a, a_m) = a$, then (i') holds and we have nothing to prove. Let now $a^*_{m+1} := \min(a, a_{m+1})$. If $\min(a, a_m) = a_m < a$, then by the inductive assumption we must have $g(a_m) \ge f(a_m)$ and $g'_-(a_m) \ge f'_-(a_m)$, $g'_+(a_m) \ge f'_+(a_m) = \tan(m\varphi) \equiv f'_\pm|_{(a_m, a_{m+1})}$. In view of convexity we thus obtain $g'_\pm|_{(a_m, a^*_{m+1})} \ge g'_+(a_m) \ge f'_+(a_m) = \tan(m\varphi) \equiv$

$f'_{\pm}|_{(a_m,a^*_{m+1})}$ and by left continuity of the left hand derivative this extends to $g'_{-}(a^*_{m+1}) \geq f'_{-}(a^*_{m+1})$, too. Furthermore, if $a^*_{m+1} = a_{m+1}$, i.e. $a < a_{m+1}$, then $g'_{+}(a^*_{m+1}) = g_{+}(a)$ does not exist (and is listed in (iii') as exceptional). The only case remaining is when $a^*_{m+1} = a_{m+1}$, i.e. $a \geq a_{m+1}$. Let first $a = a_{m+1}$. As before, in this case $g'_{+}(a^*_{m+1}) = g'_{+}(a)$ does not exist and is excepted in (iii'). Let now $a > a_{m+1}$, and consider a small right neighborhood $[a_{m+1}, a_{m+1} + \epsilon]$ of a_{m+1} which is contained fully in $[0, a)$. Then in this neighborhood the parametrization $\gamma(x) = (x, g(x))$ extends for a small arc of γ in the counterclockwise sense from $P_{m+1} := (a_{m+1}, g(a_{m+1}))$, hence for this arc the condition on Ω can be applied. (We will use also the notations $P_0, P_1, \ldots, P_m$ defined analogously as $P_k := (a_k, g(a_k))$, $k = 0, 1, \ldots, m$).

First we prove that $|P_m - P_{m+1}| \geq 1$, which will also imply $|P_m - (x, g(x))| > 1$ for all $x \in [a_{m+1}, a_{m+1}+\epsilon]$, too. For this purpose consider the line $\ell(x) := P_m + (\tan(m\varphi))(x - a_m)$ and let $Q := Q_m := (a_{m+1}, \ell(a_{m+1}))$. Note that between a_m and a_{m+1} the line ℓ runs below the curve of γ, since for any point x between the endpoints $g'_{\pm}(x) \geq f'_{\pm}(x) = \tan(m\varphi) = \ell'(x)$, and $g(a_m) = \ell(a_m)$. It follows that $g(a_{m+1}) \geq \ell(a_{m+1})$ and thus $|P_{m+1} - P_m|^2 \geq (a_{m+1} - a_m)^2 + (\tan(m\varphi) \cdot (a_{m+1} - a_m))^2 = 1$, as stated.

Hence $|P_m - (x, g(x))| > 1$ for all $x \in [a_{m+1}, a_{m+1} + \epsilon]$ holds and the Ω-condition can be applied to get $\arctan g'_{\pm}(x) \geq g'_{+}(a_m) + \varphi \geq f'_{+}(a_m) + \varphi = (m+1)\varphi = \arctan f'_{\pm}(x)$. In view of the right continuity of the right hand derivative, we thus obtain $g'_{+}(a_{m+1}) \geq \tan((m+1)\varphi) = f'_{+}(a_{m+1})$, too.

Therefore, in case (i') does not hold, we conclude (iii'). Since in this case we have $f(a_m) \leq g(a_m)$ by the inductive hypothesis, a simple integration using (iii') proves also (ii').

Therefore, the inductive argument for (i')-(iii') concludes and we obtain (i)-(iii) of the Lemma. It remains to show (iv) and (v). To prove (iv), it suffices to consider the curve $\widehat{\gamma} := T(U_{-\pi/2}(\gamma - (a, b))\ \gamma_1$ from the proof of Lemma 1.6.2, which will have $\widehat{P} = (b, a)$ while satisfying all our requirements.

Finally, let us prove (v): clearly it suffices to prove $\operatorname{int} K \subset K_1$. Because at O K has an outer normal $(0, -1)$, we have $\operatorname{int} K \subset \{(x, y)\ :\ y > 0\}$. Similarly, as at $P = (a, b)$ K has an outer normal $(1, 0)$, in view of Lemma 1.6.6 (i) we also have $\operatorname{int} K \subset \{(x, y)\ :\ x < a\} \subset \{(x, y)\ :\ x < a_k\}$.

In view of $\operatorname{int} K \subset \{(x, y)\ :\ x < a\}$, it remains to show that $(x, y) \in \operatorname{int} K$, $0 < x < a$ imply $y > f(x)$. However, the part of ∂K above the open segment $(O, (a, 0))$ consists of two open arcs, the lower one being $\{(x, g(x))\ :\ 0 < x < a\}$. Thus, for $0 < x < a$, $(x, y) \in \operatorname{int} K$ we necessarily have $y > g(x) \geq f(x)$, as was to be shown. □

LEMMA 1.6.7. *Let K, L, L_1, K_1 as above. Let $L_1 + (u, v)$ a translate of L_1 such that $\operatorname{int} K \subset K_1 + (u, v)$. Further, let $u' \geq u$ and $v' \leq v$. Then also $\operatorname{int} K \subset K_1 + (u', v')$ holds.*

PROOF. In fact, we are to prove that $K_1 \subset K_1 + (w, z)$, with arbitrary $w \geq 0 \geq z$. (Then this can be applied with $(w, z) = (u' - u, v' - v)$ to get $K_1 + (u, v) \subset (K_1 + (w, z)) + (u, v) = K_1 + (u', v')$.) Observe that the special cases with one coordinate of the translation vector being zero already suffice, for $K_1 \subset K_1 + (w, 0) \subset (K_1 + (0, z)) + (0, w) = K_1 + (w, z)$

gives the general case, too. Also observe that by symmetry of K_1 to the line $y = -x$, it suffices to prove one such case, e.g. $K_1 \subset K_1 + (0, z)$. However, as K_1 can be defined as the set of points above a function graph, this last inclusion with$z \le 0$ is evident. □

Continuation of the proof of Theorem 1.6.5. Recall that T is the reflection on the y-axis; let us introduce also S as the reflection on the line $y = a_k$.

From the above argument – or just reflecting L to the y-axis – it is immediate that we have also $K \subset TK_1$.

We are left with the upper part joining $\frac{1}{2}(A_k + A_{k+1}), A_{k+1}, \dots, A_{3k}, \frac{1}{2}(A_{3k} + A_{3k+1})$. Let

$$L^+ := \left[\frac{A_k + A_{k+1}}{2}, A_{k+1}\right] \cup \bigcup_{m=k+1}^{2k-1} [A_m, A_{m+1}] \cup \left[A_{2k}, \frac{A_{2k} + A_{2k+1}}{2}\right], \tag{1.85}$$

$$L^- := \left[\frac{A_{2k} + A_{2k+1}}{2}, A_{2k+1}\right] \cup \bigcup_{m=2k+1}^{3k-1} [A_m, A_{m+1}] \cup \left[A_{3k}, \frac{A_{3k} + A_{3k+1}}{2}\right]. \tag{1.86}$$

Next, let us apply Lemma 1.6.6 to the curve from $P = (a, b)$ onwards to the counterclockwise sense. That is, take $K_+ := U_{-\pi/2}(K - P)$ and $\gamma_+ := U_{-\pi/2}(\gamma - P)$ and check that $O \in \gamma_+$ and also γ_+ has normal $(0, -1)$ at O; moreover, the same estimate on the minimal oscillation of the normal holds for γ_+. Thus we obtain that $K \subset SK_1 + (a - a_k, b - a_k) \subset SK_1$, where the last inclusion follows from $a, b \le a_k$ and Lemma 1.6.7.

Very similarly (or from this and using reflection) we also obtain $K \subset TSK_1$. So putting together the four inclusions, we obtain $K \subset K_1 \cap TK_1 \cap SK_1 \cap TSK_1 = F_k$, i.e. $K \subset F_k$, and the proof concludes.

□

1.7. Extensions of the Blaschke Rolling Ball Theorem

As the first corollaries, we can immediately deduce the classical Blaschke theorems. We denote by $D(\mathbf{x}, r)$ the closed disc of centre $\mathbf{x}$ and radius r.

Proof of Theorem 1.4.1. Let τ_0 be the bound provided by (i) of Proposition 1.4.5. Under the condition, we find (with $\omega(\mathbf{n}, \tau) < \pi/2$)

$$\omega(\mathbf{n}, \tau) \le \frac{\kappa_0 \tau}{\cos(\omega(\mathbf{n}, \tau))} =: \varphi(\tau) \qquad (\tau \le \tau_0). \tag{1.87}$$

Let us apply Theorem 1.6.1 for the boundary point $\mathbf{x} \in \gamma$ with normal vector $\mathbf{n}(\mathbf{x}) = (\sin\alpha, -\cos\alpha)$. If necessary, we have to reduce τ so that the hypothesis $\varphi(\tau) \le \pi/4$ should hold. We obtain that the congruent copy $U_\alpha(\tau M_k) + \mathbf{x}$ of τM_k is contained in K, where $k = \lfloor \pi/2\varphi(\tau) \rfloor$. Note that $U_\alpha(\tau M_k) + \mathbf{x} \supset D(\mathbf{z}, \tau r(\varphi(\tau)))$, where $\mathbf{z} = \mathbf{x} - \tau R(\varphi(\tau))\mathbf{n}(\mathbf{x})$. When $\tau \to 0$, also $\varphi(\tau) \to 0$, therefore also $\omega(\mathbf{n}, \tau) \to 0$ in view of (1.87), and we see

$$\lim_{\tau\to 0} (\tau R(\varphi(\tau))) = \lim_{\tau\to 0} (\tau r(\varphi(\tau))) = \lim_{\tau\to 0} \frac{\tau}{\varphi(\tau)} = \lim_{\tau\to 0} \frac{\cos(\omega(\mathbf{n}, \tau))}{\kappa_0} = \frac{1}{\kappa_0}.$$

Note that we have made use of $\omega(\mathbf{n}, \tau) \to 0$ in the form $\cos(\omega(\mathbf{n}, \tau)) \to 1$. It follows that $D(\mathbf{x} - \frac{1}{\kappa_0}\mathbf{n}(\mathbf{x}), \frac{1}{\kappa_0}) \subset K$, whence the assertion. □

Note that in the above proof of Theorem A we did not assume C^2-boundary, as is usual, but only the existence of curvature and the estimate $\kappa(\mathbf{x}) \le \kappa_0$. So we found the following stronger corollary (still surely well-known).

COROLLARY 1.7.1. *Assume that $K \subset \mathbb{R}^2$ is a convex domain with boundary curve γ, that the curvature κ exists all over γ, and that there exists a positive constant $\kappa_0 > 0$ so that $\kappa \le \kappa_0$ everywhere on γ. Then to all boundary point $\mathbf{x} \in \gamma$ there exists a disk D_R of radius $R = 1/\kappa_0$, such that $\mathbf{x} \in \partial D_R$, and $D_R \subset K$.*

Similarly, one can deduce also the "dual" Blaschke theorem, i.e. Lemma1.1.15, in a similarly strengthened form. In fact, the conditions can be relaxed even further, as was shown by Strantzen, see [**3**, Lemma 9.11]. Our discrete approach easily implies Strantzen's strengthened version, originally obtained along different lines.

COROLLARY 1.7.2 (**Strantzen**). *Let $K \subset \mathbb{R}^2$ be a convex body with boundary curve γ. Assume that the (linearly) a.e. existing curvature κ of γ satisfies $\kappa \ge \kappa_0$ (linearly) a.e. on γ. Then to all boundary point $\mathbf{x} \in \gamma$ there exists a disk D_R of radius $R = 1/\kappa_0$, such that $\mathbf{x} \in \partial D_R$, and $K \subset D_R$.*

PROOF. Now we start with (ii) of Proposition 1.4.5 to obtain $\Omega(\tau) \ge \kappa_0\tau$ for all τ. Put $\varphi := \varphi(\tau) := \kappa_0\tau$. Clearly, when $\tau \to 0$, then also $\varphi(\tau) \to 0$ and $k := \lceil \pi/(2\varphi(\tau)) \rceil \to \infty$. Take $\mathbf{n}(\mathbf{x}) = (\cos\alpha, \sin\alpha)$ and apply Theorem 1.6.5 to obtain $U_\alpha(\tau F_k) + \mathbf{x} \supset K$ for all $\tau > 0$. Observe that $D_\varphi := D((0, \mathfrak{r}(\varphi)), \mathfrak{R}(\varphi)) \supset F_k$, hence $U_\alpha(\tau D_\varphi) + \mathbf{x} \supset K$. In the limit, since $\mathfrak{r}(\varphi(\tau)) \sim \mathfrak{R}(\varphi(\tau)) \sim 1/(\varphi(\tau)) = 1/(\kappa_0\tau)$, we find $D(\mathbf{x} - (1/\kappa_0)\mathbf{n}, 1/\kappa_0) \supset K$, for any $\mathbf{n} \in \mathbf{n}(\mathbf{x})$, that implies the statement. □

1.8. Further results for non-flat convex domains

The above Theorem 1.1.3 was formulated with very precise constants. In particular, it gives a good description of the "inverse Markov factor"

$$M(E_b) := \inf_{p \in \mathcal{P}_n(E_b)} M(p),$$

when n is fixed and $b \to 0$. In this section we aim at a precise generalization of Theorem 1.1.3 using appropriate geometric notions. Our argument stems out of the notion of "circular sets", used in [**8**] and going back to Turán's work. This approach can indeed cover the full content of Theorem 1.1.3. Moreover, the geometric observation and criteria we present will cover a good deal of different, not necessarily smooth domains. First let us have a recourse to Theorem 1.1.7.

THEOREM 1.8.1. *Let $K \subset \mathbb{C}$ be any convex domain with C^2-smooth boundary curve $\partial K = \Gamma$ having curvature $\kappa(\zeta) \ge \kappa$ with a certain constant $\kappa > 0$ and for all points $\zeta \in \Gamma$. Then $M(K) \ge (\kappa/2)n$.*

PROOF. The proof hinges upon geometry in a large extent. For this smooth case we use Blaschke's Rolling Ball Theorem, i.e. Lemma 1.1.15. This means, with our definition above, that if the curvature of the boundary curve of a twice differentiable convex body exceeds $1/R$, then the convex body is R-circular. From this an application of Theorem 1.1.6 yields the assertion. □

So now it is worthy to calculate the curvature of ∂E_b.

LEMMA 1.8.2. *Let E_b be the ellipse with major axes $[-1, 1]$ and minor axes $[-ib, ib]$. Consider its boundary curve Γ_b. Then at any point of the curve the curvature is between b and $1/b^2$.*

PROOF. Now we depart from arc length parameterization and use for $\Gamma_b := \partial E_b$ the parameterization $\gamma(\varphi) := (\cos(\varphi), b\sin(\varphi))$. Then we have

$$\kappa(\gamma(\varphi)) = \frac{|\dot\gamma(\varphi) \times \ddot\gamma(\varphi)|}{|\dot\gamma(\varphi)|^3} ,$$

that is,

$$\begin{aligned}\kappa(\gamma(\varphi)) &= \frac{|(-\sin\varphi, b\cos\varphi) \times (-\cos\varphi, -b\sin\varphi)|}{|(-\sin\varphi, b\cos\varphi)|^3} \\ &= \frac{b\sin^2\varphi + b\cos^2\varphi}{(\sin^2\varphi + b^2\cos^2\varphi)^{3/2}} \\ &= \frac{b}{(\sin^2\varphi + b^2\cos^2\varphi)^{3/2}} .\end{aligned}$$

Clearly, the denominator falls between $(b^2\sin^2\varphi + b^2\cos^2\varphi)^{3/2} = b^3$ and $(\sin^2\varphi + \cos^2\varphi)^{3/2} = 1$, and these bounds are attained, hence $\kappa(\gamma(\varphi)) \in [b, 1/b^2]$ whenever $b \leq 1$. □

PROOF OF THEOREM 1.1.3. The curvature of Γ_b at any of its points is at least b according to Lemma 1.8.2. Hence $M(E_b) \geq (b/2)n$ in view of Theorem 1.8.1, and Theorem 1.1.3 follows. □

However, not only smooth convex domains can be proved to be circular. Eg. it is easy to see that if a domain is the intersection of finitely many R-circular domains, then it is also R-circular. The next generalization is not that simple, but is still true.

LEMMA 1.8.3 (**Strantzen**). *Let the convex domain K have boundary $\Gamma = \partial K$ with angle function $\alpha_\pm$ and let $\kappa > 0$ be a fixed constant. Assume that $\alpha_\pm$ satisfies the curvature condition $\kappa(s) = \alpha'(s) \geq \kappa$ almost everywhere. Then K is $R = 1/\kappa$-circular.*

PROOF. This result is essentially the far-reaching, relatively recent generalization of Blaschke's Rolling Ball Theorem by Strantzen, i.e. Corollary 1.7.2 above. The only slight alteration from the standard formulation in [**3**], suppressed in the above quotations, is that Strantzen's version assumes $\kappa(t) \geq \kappa$ wherever the curvature $\kappa(t) = \alpha'(t)$ exists (so almost everywhere for sure), while above we stated the same thing for almost everywhere, but not necessarily at every points of existence. This can be overcome by reference to the subdifferential version, too. □

Now we are in an easy position to prove Theorem 1.1.16.

PROOF OF THEOREM 1.1.16. The proof follows from a combination of Theorem 1.1.6 and Lemma 1.8.3. □

Let us illustrate the strengths and weaknesses of the above results on the following instructive examples, suggested to us by J. Szabados (personal communication). Consider for any $1 < p < \infty$ the ℓ_p unit ball

$$B^p := \{(x,y) \,:\, |x|^p + |y|^p \le 1\}, \qquad \Gamma^p := \partial B^p = \{(x,y) \,:\, |x|^p + |y|^p = 1\}. \tag{1.88}$$

Also, let us consider for any parameter $0 < b \le 1$ the affine image ("ℓ_p-ellipse")

$$B^p_b := \{(x,y) \,:\, |x|^p + |y/b|^p \le 1\}, \qquad \Gamma^p_b := \partial B^p_b = \{(x,y) \,:\, |x|^p + |y/b|^p = 1\}. \tag{1.89}$$

By symmetry, it suffices to analyze the boundary curve $\Gamma := \Gamma^p_b$ in the positive quadrant. Here it has a parametrization $\Gamma(x) := (x, y(x))$, where $y(x) = b\,(1-x^p)^{1/p}$. As above, the curvature of the general point of the arc in the positive quadrant can be calculated and we get

$$\kappa(x) = \frac{(p-1)bx^{p-2}(1-x^p)^{1/p-2}}{\left(1+b^2x^{2p-2}(1-x^p)^{2/p-2}\right)^{3/2}} \tag{1.90}$$

For $p > 2$, the curvature is continuous, but it does not stay off 0: e.g. at the upper point $x = 0$ it vanishes. Therefore, neither Theorem 1.8.1 nor Theorem 1.1.16 can provide any bound, while Theorem 1.1.12 provides an estimate, even if with a small constant: here $d(B) = 2$, $w(B) = 2b$, and we get $M(B) \ge 0.00015bn$.

When $p = 2$, we get back the disk and the ellipses: the curvature is minimal at $\pm ib$, and its value is b there, hence $M(B) \ge (b/2)n$, as already seen in Theorem 1.1.3. On the other hand Theorem 1.1.12 yields only $M(B) \ge 0.00015bn$ also here.

For $1 < p < 2$ the situation changes: the curvature becomes infinite at the "vertices" at $\pm ib$ and ± 1, and the curvature has a positive minimum over the curve Γ. When $b = 1$, it is possible to explicitly calculate it, since the role of x and y is symmetric in this case and it is natural to conjecture that minimal curvature occurs at $y = x$; using geometric-arithmetic mean and also the inequality between power means (i.e. Cauchy-Schwartz), it is not hard to compute $\min \kappa(x,y) = (p-1)2^{1/p-1/2}$, (which is the value attained at $y = x$). Hence Theorem 1.1.16 (but not Theorem 1.8.1, which assumes C^2-smoothness, violated here at the vertices!) provides $M(B^p) \ge (p-1)2^{1/p-3/2}n$, while Theorem 1.1.12 provides, in view of $w(B^p) = 2^{3/2-1/p}$, something like $M(B^p) \ge 0.0003\,2^{-1/2-1/p}n \ge 0.0001n$, which is much smaller until p comes down very close to 1.

For general $0 < b < 1$ we obviously have $d(B) = 2$, $(\sqrt{2}b <)2b/\sqrt{1+b^2} < w(B) < 2b$, and Theorem 1.1.12 yields $M(B) \ge 0.0001bn$ independently of the value of p.

Now $\min \kappa$ can be estimated within a constant factor (actually, when $b \to 0$, even asymptotically precisely) the following way. On the one hand, taking $x_0 := 2^{-1/p}$ leads to $\kappa(x_0) = (p-1)b2^{1+1/p}/(1+b^2)^{3/2} < b(p-1)2^{1+1/p}$, hence $\min \kappa(x < b(p-1)2^{1+/p}$. Note that when $b \to 0$, we have asymptotically $\kappa(x-0) \sim b(p-1)2^{1+/p}$. On the other hand denoting $\xi := x^p$ and $\beta := 2/p - 1 \in (0,1)$, from (1.90) we get

$$\frac{(p-1)b}{\kappa(x)} = [\xi(1-\xi)]^{\beta}\left[\xi^{1-\beta} + b^2(1-\xi)^{1-\beta}\right]^{3/2} \le 2^{-2\beta}\left[(\xi+(1-\xi))^{1-\beta}(1+(b^2)^{1/\beta})^{\beta}\right]^{3/2},$$

with an application of geometric-arithmetic mean inequality in the first and Hölder inequality in the second factor. In general we can just use $b < 1$ and get

$$\kappa(x) \geq (p-1)b2^{2\beta}\left[1+b^{2/\beta}\right]^{-3\beta/2} \geq (p-1)b2^{\beta/2} = (p-1)b2^{1/p-1/2},$$

within a factor $2^{3/2}$ of the upper estimate for $\min \kappa$.
Thus, inserting this into Theorem 1.1.16 as above, we derive $M(B_b^p) \geq (p-1)b2^{1/p-3/2}n$. In all, we see that Theorems 1.8.1 (essentially due to Erőd) and 1.1.16 usually (but not always, c.f. the case $p \approx 1$ above !)) give better constants, when they apply. However, in cases the curvature is not bounded away from 0, we can retreat to application to the fully general Theorem 1.1.12, which, even if with a small absolute constant factor, but still gives a precise estimate even regarding dependence of the constant on geometric features of the convex domain. According to Theorem 1.1.13, this latter phenomenon is not just an observation on some particular examples, but is a general fact, valid even for not necessarily convex domains.

1.9. Further remarks and problems

In the case of the unit interval also Turán type L^p estimates were studied, see [**22**] and the references therein. It would be interesting to consider the analogous question for convex domains on the plane. Note that already Turán remarked, see the footnote in [**20**, p.141], that on D an L^p version holds, too. Also note that for domains there are two possibilities for taking integral norms, one being on the boundary curve and another one of integrating with respect to area. It seems that the latter is less appropriate and convenient here.

In the above we described a more or less satisfactory answer of the problem of inverse Markov factors for convex domains. However, Levenberg and Poletsky showed that star-shaped domains already do not admit similar inverse Markov factors. A question, posed by V. Totik, is to determine exact order of the inverse Markov factor for the "cross" $C := [-1,1] \cup [-i,i]$; clearly, the point is not in the answer for the cross itself, but in the description of the inverse Markov factor for some more general classes of sets.

Another question, still open, stems from the Szegő extension of the Markov inequality, see [**19**], to domains with sector condition on their boundary. More precisely, at $z \in \partial K$ K satisfies the *outer sector condition* with $0 < \beta < 2$, if there exists a small neighborhood of z where some sector $\{\zeta \ : \ \arg(\zeta - z) \in (\theta, \beta\pi + \theta)\}$ is disjoint from K. Szegő proved, that if for a domain K, bounded by finitely many smooth (analytic) Jordan arcs, the supremum of β-values satisfying outer sector conditions at some boundary point is $\alpha < 2$, then $\|P'\| \ll n^{\alpha}\|P\|$ on K. Then Turán writes: "Es ist sehr wahrscheinlich, daß auch den Szegőschen Bereichen $M(p) \geq cn^{1/\alpha}$...", that is, he finds it rather likely that the natural converse inequality, suggested by the known cases of the disk and the interval (and now also by any other convex domain) holds also for general domains with outer sector conditions.

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CHAPTER 2

Turán type extremal problems for positive definite functions

2.1. Introduction

2.1.1. The Turán problem. We study the following problem, generally investigated under the name of *"Turán's Problem"*, following Stechkin [**86**], who recalls a question posed to him in personal discussion.

PROBLEM 2.1.1. Given an open set Ω, symmetric about 0, and a continuous, positive definite, integrable function f, with $\operatorname{supp} f \subseteq \Omega$ and with $f(0) = 1$, how large can $\int f$ be?

Although this name for the problem is quite widespread, one has to note that all the important versions of the problem were investigated well before the beginning of the seventies, when the discussion of Turán and Stechkin took place.

About the same time when Turán discussed the question with Stechkin, American researchers already investigated in detail the *square integral version of the problem*, see [**28, 67, 20**]. Their reason for searching the extremal function and value came from radar engineering problems at the Jet Propulsion Laboratory.

more importantly, Problem 2.1.1 appears as early as in the thirties [**84**], when Siegel considered the question for Ω being a ball, or even an ellipsoid in Euclidean space $\mathbb{R}^d$, and established the right extremal value $|\Omega|/2^d$. The question occurred to Siegel as a theoretical possibility to sharpen the Minkowski Latice Point Theorem. Although Siegel concluded that, due to the extremal value being just as large as the Minkowski Lattice Point Theorem would require, this geometric statement can not be further sharpened through improvement on the extremal problem, nevertheless he works out the extremal problem fully and exhibits some nice applications in the theory of entire functions.

Furthermore, the same Problem 2.1.1 appeared in a paper of Boas and Kac [**13**] already in the forties, even if the main direction of the study there was a different version, what is nowadays generally called the *pointwise Turán problem.* However, as is realized partially in [**13**] and fully only later in [**55**], the pointwise Turán problem – formulated in the classical setting of Fourier series, but nevertheless equivalent to the Euclidean space settings of [**13**] – goes back already to Caratheodory [**16**] and Fejér [**24**].

The Turán problem was considered by Stechkin on an interval in the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ [**86**] and in $\mathbb{R}$ by Boas and Kac [**13**], but extensions were to follow in several directions.

Such a question is interesting in the study of sphere packings [**30, 17, 18**], in additive number theory [**79, 46, 64, 34**] and in the theory of Dirichlet characters and exponential sums [**57**], among other things.

2.1.2. One dimensional case of the Turán problem. Already the symmetric interval case in one dimension presents nontrivial complications, which were resolved satisfactorily only recently. We discuss the development of the problem from the outset to date.

Actually, Turán's interest might have come from another area in number theory, namely Diophantine approximation. (Let us point out that [**3**] starts with the sentence: "With regard to applications in number theory, P. Turán stated the following problem:", while at the end of the paper there is special expression of gratitude to Professor Stechkin for his interest in this work. Also, Gorbachev writes in [**29**, p. 314]): "Studying applications in number theory, P. Turán posed the problem ...")

One can hypothesise that Turán thought of the elegant proof of the well-known Dirichlet approximation theorem, stating that for any given $\alpha \in \mathbb{R}$ at least one multiple $n\alpha$ in the range $n = 1, \dots, N$ have to approach some integer as close as $1/(N+1)$. The proof, which uses Fourier analysis and Fejér kernels in particular, is presented in [**64**, p. 99], and in a generalized framework it is explained in [**12**], but it is remarked in [**64**, p. 105] that the idea comes from Siegel [**84**], so Turán could have been well aware of it. Let us briefly present the argument right here.

If we wish to detect multiples $n\alpha$ of $\alpha \in \mathbb{R}$ which fall in the δ-neighborhood of an integer, that is which have $\|n\alpha\| < \delta$ (where, as usual in this field, $\|x\| := \operatorname{dist}(x, \mathbb{Z})$), then we can use that for the triangle function $F(x) := F_\delta(x) := \max(1 - \|x\|/\delta)_+$, we have $F(n\alpha) > 0$ iff $\|n\alpha\| < \delta$. So if with an arbitrary $\delta > 1/(N+1)$ we can work through a proof of $F(n\alpha) > 0$ for some $n \in [1, N]$, then the proof yields the sharp form of the Dirichlet approximation theorem. (It is indeed sharp, because for no $N \in \mathbb{N}$ can any better statement hold true, as the easy example of $\alpha := 1/(N+1)$ shows.)

So we take now $S := \sum_{n=1}^{N} (1 - \frac{|n|}{N+1}) F(n\alpha)$, or, since F is even and $F(0) = 1$, consider the more symmetric sum $2S+1 = \sum_{n=-N}^{N} (1 - \frac{|n|}{N+1}) F(n\alpha)$. Note that $\widehat{F_\delta}(t) = \delta \cdot \left(\frac{\sin(\pi\delta t)}{\pi\delta t}\right)^2$, so in particular with the nonnegative coefficients $\widehat{F}(k) = c_k$ we can write (with $e(t) := e^{2\pi i t}$)

$$F_\delta(x) = \sum_{k=-\infty}^{\infty} c_k e(kx) \qquad c_0 = \delta,\ c_k = \delta \cdot \left(\frac{\sin(\pi k\delta)}{\pi k\delta}\right)^2 \ (k = \pm 1, \pm 2, \dots). \tag{2.1}$$

It suffices to show $S > 0$. With the Fejér kernels $\sigma_N(x) := \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) e(nx) = \frac{1}{N+1} \cdot \left(\frac{\sin(\pi(N+1)x)}{\pi x}\right)^2 \geq 0$, after a change of the order of summation we are led to

$$\begin{aligned} 2S + 1 &= \sum_{k=-\infty}^{\infty} c_k \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) e(nk\alpha) \\ &= c_0\sigma_N(0) + 2\sum_{k=1}^{\infty} c_k\sigma_N(k\alpha) \geq c_0\sigma_N(0) = \delta(N+1) > 1, \end{aligned}$$

which concludes the argument.

Now if in place of the triangle function with $\delta = 1/(N+1)$ another positive definite (i.e. $\widehat{f} \geq 0$) function f could be put with $\operatorname{supp} f \subset [-\delta, \delta]$ and $f(0) = 1$ but with

$\widehat{f}(0) > \delta$ then the above argument with f in place of F would give $S > 0$ even for $\delta = 1/(N+1)$, clearly a contradiction since the Dirichlet approximation theorem cannot be further sharpened. That round-about argument already gives that for h a reciprocal of an integer, the triangle function F_h is extremal in the Turán problem for $[-h, h]$. In other words, we obtain Stechkin's result [**86**], (see also below) already from considerations of Diophantine approximation.

So Turán asked Stechkin if for any $h > 0$ the triangle function provides the largest possible integral among all positive definite functions vanishing outside $[-h, h]$ and normalized by attaining the value 1 at 0. Stechkin derived that this is the case for h being the reciprocal of a natural number: by monotonicity in h for other values he could conclude an estimate. Anticipating and slightly abusing the general notations below, denote the extremal value by $T(h)$: then Stechkin obtained $T(h) = h + O(h^2)$. This was sharpened later by Gorbachev [**29**] and Popov [**69**] (cited in [**31**, p. 77]) to $h + O(h^3)$.

The corresponding Turán extremal value $T_{\mathbb{R}}(h)$ on the real line is, by simple dilation, depends linearly on the interval length and is just $hT_{\mathbb{R}}(1)$ for any interval $I = [-h, h]$. On the other hand it follows already from $\lim_{h\to 0+} T(h)/h = 1$ that e.g. for the unit interval $[-1, 1]$ the extremal function must be the triangle function and $T_{\mathbb{R}}(1) = 1$, hence $T_{\mathbb{R}}(h) = h$. In fact, this case was already settled earlier by Boas and Katz in [**13**] as a byproduct of their investigation of the pointwise question.

But there is another observation, seemingly well-known although no written source can be found. Namely, it is also known for some time that for h *not being a reciprocal of an integer number*, the triangle function *can indeed be improved upon* a little. Indeed, the triangle function F_h has Fourier transform which vanishes precisely at integer multiples of $1/h$, and in case $1/h \notin \mathbb{N}$, some multiples fall outside $\mathbb{Z}$. And then the otherwise double zeroes of $\widehat{F_h}$ can even be substituted by a product of two close-by zero factors, allowing a small interval in between, where the Fourier transform can be negative. This negativity spoils positive definiteness regarding the function on $\mathbb{R}$: but on $\mathbb{T}$ it does not, for only the values at integer increments must be nonnegative in order that a function be positive definite on $\mathbb{T}$. With a detailed calculus (using also the symmetric pair of zeroes) such an improvement upon the triangle function is indeed possible. (Note that here $\widehat{F}$, so also $\int \widehat{F} = F(0)$ is perturbed while $\widehat{F}(0) = \int F$ is unchanged.) I have heard this construction explained in lectures during my university studies [**37**]; in Russia, a similar observation was communicated by A. Yu Popov [**69**] and later recorded in writing in [**33, 35, 31**].

As said above, the computation of exact values of $T(h)$ started with Stechkin for $h = 1/q$, $q \in \mathbb{N}$: these are the only cases when $T(h) = h$. Further values, already deviating from this simple formula, were computed for *some rational* h in [**59, 33, 35**] and finally *for all rational* h in [**31, 44**]. Knowing the value for rational h led Ivanov to further investigations which established continuity of the extremal value in function of h, and thus gave the complete solution of Turán's problem on the torus [**43**]. In fact, the above works also established that for $[-h, h] \subset \mathbb{T}$ the Turán extremal problem and the Delsarte extremal problem (see §2.1.4) has the same extremal value (and extremal functions). Note that this coincidence does not hold true in general.

However, it seems that almost nothing is known about Turán extremal values of other, one would say "dispersed" sets not being intervals. A natural conjecture is that e.g. on $\mathbb{R}$ (or perhaps even on $\mathbb{T}$?) a set $\Omega \subset \mathbb{R}$ of fixed measure $|\Omega| = m$ can have maximal Turán constant value if only it is a zero-symmetric interval $[-m/2, m/2]$.

What we know at present from Theorem 2.6.16, (that is from [**56**, Theorem 6]) is that we certainly have $T(\Omega) \leq m/2$, that is, in $\mathbb{R}$ no "better sets", than zero-symmetric intervals, can exist. However, uniqueness is not known, not even for $\mathbb{R}$. The result is in fact a more general estimate in function of the prescribed measure m, but for higher dimensions it is far less precise. Also, regarding the discrete group $\mathbb{Z}$ one must observe that zero-symmetric intervals $[-N, N] \subset \mathbb{Z}$ have the same Turán extremal values as their homothetic copies $k[-N, N]$ ($k \in \mathbb{N}$) which already destroys the hope for "uniqueness only for intervals". In higher dimensions not even the right class of the corresponding "condensed sets", like intervals in dimension one, has been identified.

2.1.3. Turán's problem in the multivariate setting. Already as early as in the 1930's, Siegel [**84**] proved that for an ellipsoid in $\mathbb{R}^d$ the extremal value in Problem 2.1.1 is $|\Omega|/2^d$.

In the 1940's, Boas and Katz [**13**] mentioned that Poisson summation may be used to treat similar questions in higher dimensions. Besides mentioning the group settings, Garcia & al. [**28**] and Domar [**20**] also touches upon the question without going into further details. The packing problem by balls in Euclidean space has already been treated by many authors via multivariate extremal problems of the type, but there the optimal approach is to pose a closely related, still different variant, named Delsarte- (and also as Logan- and Levenshtein-) problem. See e.g. [**30, 17**] and the references therein.

As a direct generalization of Stechkin's work, Andreev [**2**] calculated the Turán constants of cubes Q_h^d in $\mathbb{T}^d$ obtaining $h^d + O(h^{d+1})$. Moreover, he estimated the Turán constant of the cross-politope (ℓ_1-ball) O_h^d in $\mathbb{T}^d$: his estimates are asymptotically sharp when $d = 2$. Gorbachev [**29**] simultaneously sharpened and extended these results proving that for any centrally symmetric body $D \subset [-1, 1]^d$ and for all $0 < h < 1/2$ we always have $\mathcal{T}_{\mathbb{T}^d}(hD) = \mathcal{T}_{\mathbb{R}^d}(D) \cdot h^d + O(h^{d+2})$.

Arestov and Berdysheva [**6**] offers a systematic investigation of the multivariate Turán problem collecting several natural properties. They also prove that the hexagon has Turán constant exactly one fourth of the area of itself. Gorbachov [**29**] proved that the unit ball $B_d \subset \mathbb{R}^d$ has Turán constant $2^{-d}|B_d|$, where $|B_d|$ is the volume (d-dimensional Lebesgue measure) of the ball. Another proof of this fact can be found in [**54**], but we have already noted that the result goes back to Siegel [**84**].

There is a special interest in the case which concerns Ω being a (centrally symmetric) convex subset of $\mathbb{R}^d$ [**6, 7, 29, 54**], since in this case the natural analog of the triangle function, the self-convolution (convolution square) of the characteristic function $\chi_{\frac{1}{2}\Omega}$ of the half-body $\frac{1}{2}\Omega$ is available showing that $\mathcal{T}_{\mathbb{R}^d}(\Omega) \geq |\Omega|/2^d$. The natural conjecture is that for a symmetric convex body this convolution square is extremal, and $\mathcal{T}_{\mathbb{R}^d}(\Omega) = |\Omega|/2^d$. (Note that this fails in $\mathbb{T}^d$, already for $d = 1$, for some sets Ω.) Convex bodies with this property may be called Turán type, or Stechkin-regular, or, perhaps, *Stechkin-Turán domains*, while symmetric convex bodies in $\mathbb{R}^d$ with $\mathcal{T}_{\mathbb{R}^d}(\Omega) > |\Omega|/2^d$ as anti-Turán or *non-Stechkin-Turán* domains. Thus the above mentioned result about the ball can be reworded saying that the ball is of Stechkin-Turán type.

To date, no non-Stechkin-Turán domains are known, although the family of known Stechkin-Turán domains is also quite meager (apart from $d = 1$ when everything is clear for the intervals).

In [**6, 7**] Arestov and Berdysheva prove that if $\Omega \subseteq \mathbb{R}^d$ is a convex polytope which can tile space when translated by the lattice $\Lambda \subseteq \mathbb{R}^d$ (this means that the copies $\Omega + \lambda$, $\lambda \in \Lambda$, are non-overlapping and almost every point in space is covered) then $\mathcal{T}_{\mathbb{R}^d}(\Omega) = |\Omega|/2^d$. Whence the class of Stechkin-Turán domains includes, by the result of Arestov and Berdysheva, convex lattice tiles.

Kolountzakis and Révész [**54**] showed the same formula for all convex domains in $\mathbb{R}^d$ which are *spectral*. This rsult is presented here in Corollary 2.7.3. For the definition and some context see §2.4.2, where it will be explained that all convex tiles are spectral, and so the result of Arestov and Berdysheva is also a consequence of Corollary 2.7.3.

For not necessarily convex sets, further results are contained in our work, e.g. Theorem 2.7.2 for $\mathbb{R}^d$ and Theorem 2.7.1 for finite groups. These appeared first in [**54**]

2.1.4. Variants and relatives of the Turán problem. In the same class of functions $\mathcal{F}$ various similar quantities may be maximized. The two most natural versions concern the *square-integral* of $f \in \mathcal{F}$, henceforth called the *square-integral Turán problem*, and the *function value* at some arbitrarily prescribed point $z \in \Omega$, called the *pointwise Turán problem*. About the latter see §2.1.5.

The square-integral Turán problem occurred for applied scientists in connection with radar design (radar ambiguity and overall signal strength maximizing), see [**67, 28**]. Further interesting results were obtained in [**20**]. Nevertheless, already on the torus $\mathbb{T}$ the exact answer is not known, even if Page [**67**] provides convincing computational evidence for certain conjectures in case $h = \pi/n$, and the existence of *some* extremal function is known.

Further ramifications are obtained with considering different variations of the above definitions. E.g. Belov and Konyagin [**9, 10**] considers functions with integer coefficients, and periodic even functions $f \sim \sum_k a_k \cos(kx)$ with $\sum_k |a_k| = 1$ but with not necessarily $a_k \geq 0$, i.e. not necessarily positive definite. Berdysheva and Berens considers the multivariate question restricted to the class of ℓ_1-radial functions.

A very natural version of the same problem is the Delsarte problem [**19**] (also known under the name of Logan and Levenshtein): here the only change in the conditioning of the extremal problem is that we assume, instead of vanishing of f outside a given set Ω, only the less restrictive condition that f be nonnegative outside the given set. Both extremal problems are suitable in deriving estimates of packing densities through Poisson summation: this is exploited in particular for balls in Euclidean space, see e.g. [**19, 45, 58, 4, 18, 5, 30, 17**].

There are several other rather similar, yet different extremal problems around. E.g. one related intriguing question [**83**], dealt with by several authors, is the maximization of $\int f$ for real functions f supported in $[-1,1]$, admitting $\|f\|_\infty = 1$, but instead of being positive definite, (which in $\mathbb{R}$ is equivalent to being represented as $g * \widetilde{g}$), having only a representation $f = g * g$ with some $g \geq 0$ supported in the half-interval $[-1/2, 1/2]$.

Here we do not consider these relatives of the Turán problems, apart from the so-called *pointwise Turán problem*, to be introduced in more detail next.

2.1.5. The pointwise Turán problem. The natural pointwise analogue of Problem 2.1.1 is the maximization of the function value $f(z)$, for given, fixed $z \in \Omega$, in place of the integral, over functions from the same class than in Problem 2.1.1. (Actually, the question can as well be posed in any LCA group.) For intervals in $\mathbb{T}$ or $\mathbb{R}$ this was studied in [**8**] under the name of "the pointwise Turán problem", although the same problem was already settled in the relatively easy case of an interval $(-h,h) \subset \mathbb{R}$ by Boas and Kac in [**13**]. For general domains in arbitrary dimension the problem was further studied in [**55**]: we present the results of this paper below in §2.9. Here we introduce the problem and make some preparations, too.

Let us denote $\mathbb{T}^d := \left[-\frac{1}{2}, \frac{1}{2}\right)^d \subset \mathbb{R}^d$ with the usual modified topology of periodicity, that is, take the topology of $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$. For $\Omega \subseteq \mathbb{T}^d$ any open domain[1], we put

$$\mathcal{F}^*(\Omega) := \{f : \mathbb{T}^d \to \mathbb{R} \,:\, \operatorname{supp} f \subseteq \Omega, f(0) = 1, f \text{ positive definite}\}, \tag{2.2}$$

and, analogously, when $\Omega \subseteq \mathbb{R}^d$ is any open set,

$$\mathcal{F}(\Omega) := \{f : \mathbb{R}^d \to \mathbb{R} \,:\, \operatorname{supp} f \subseteq \Omega, f(0) = 1, f \text{ positive definite}\}. \tag{2.3}$$

Recall that positive definiteness of functions (and even measures and tempered distributions) can be defined or equivalently characterized by nonnegativity of Fourier transform. In case (2.2) positive definiteness means $\widehat{f}(n) \geq 0 \, (\forall n \in \mathbb{Z}^d)$, while in case (2.3) it means $\widehat{f}(x) \geq 0 \, (\forall x \in \mathbb{R}^d)$.

For general domains in arbitrary dimension the problem can be formulated as follows.

PROBLEM 2.1.2 (Boas-Kac - type pointwise extremal problem for the space). Let $\Omega \subseteq \mathbb{R}^d$ be an open set, and let $f : \mathbb{R}^d \to \mathbb{R}$ be a positive definite function with $\operatorname{supp} f \subseteq \Omega$ and $f(0) = 1$. Let also $z \in \Omega$. What is the largest possible value of $f(z)$? In other words, determine

$$\mathcal{M}(\Omega, z) := \sup_{f \in \mathcal{F}(\Omega)} f(z). \tag{2.4}$$

REMARK 2.1.3. Obviously, $\mathcal{M}(\Omega, z) \leq 1$, as $1 \pm f(z) = \int_{\mathbb{R}} (1 \pm \exp(2\pi i z t)) \widehat{f}(t) dt = \int_{\mathbb{R}} (1 \pm \cos(2\pi z t)) \widehat{f}(t) dt \geq 0$.

[1] Note that $0 \notin \Omega$ entails $f(0) = 0$, hence the function classes $\mathcal{F}^*(\Omega)$ and $\mathcal{F}(\Omega)$ defined in (2.2) and (2.3) are empty; therefore, it suffices to restrict attention to the case $0 \in \Omega$.

One might miss a more precise specification of the function class $f : \mathbb{R}^d \to \mathbb{R}$ here and similarly in the problems listed below. The fact that considering L^1, C or C^∞ leads to the same answer i.e. same extremal values, will be discussed at the beginning of §2.9.1.

PROBLEM 2.1.4 (Turán - type pointwise extremal problem for the torus). Let $\Omega \subseteq \mathbb{T}^d$ be any open set, and let $f : \mathbb{T}^d \to \mathbb{R}$ be a positive definite function with $\operatorname{supp} f \subseteq \Omega$ and $f(0) = 1$. Let also $z \in \Omega$. What is the largest possible value of $f(z)$? In other words, determine

$$\mathcal{M}^*(\Omega, z) := \sup_{f \in \mathcal{F}^*(\Omega)} f(z). \tag{2.5}$$

REMARK 2.1.5. Let $\Omega \subseteq (-\frac{1}{2}, \frac{1}{2})^d$ and $f : \Omega \to \mathbb{R}$. For the function f to be positive definite *on the torus* means a nonnegativity condition for the Fourier Transform

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{2\pi i \langle \xi, x \rangle} f(x)\, dx$$

only for a discrete set of values of ξ, namely $\xi \in \mathbb{Z}^d$, while positive definiteness of f as a function on $\mathbb{R}^d$ is equivalent to nonnegativity of the Fourier transform $\widehat{f}$ for all occurring values. From this it follows that we always have

$$\mathcal{M}^*(\Omega, z) \geq \mathcal{M}(\Omega, z). \tag{2.6}$$

The extremal value in the above Problem 2.1.2 was estimated together with its periodic analogue Problem 2.1.4 in the work [**8**] for dimension $d = 1$. However, Boas and Kac have already solved the $d = 1$ case of Problem 2.1.2, a fact which seems to have been unnoticed in [**8**].

These problems are not only analogous, but also related to each other, and, in fact, Problem 2.1.2 is only a special, limiting case of the more complex Problem 2.1.4 (see Theorem 2.9.18 below). On the other hand, Boas and Kac have already observed, that Problem 2.1.2 (dealt with for $\mathbb{R}$ in [**13**]) is connected to trigonometric polynomial extremal problems. In particular, from the solution to the interval case they deduced the value (2.107) below of the extremal problem due to Carathéodory [**16**] and Fejér [**24**]. They also established a connection (see [**13**, Theorem 6]) what corresponds to the one-dimensional case of the first part of our Theorem 2.9.1.

It is appropriate at this point to consider also the following type of trigonometric polynomial extremal problems. Let us define for any $H \subseteq \mathbb{N}_2 := \mathbb{N} \cap [2, \infty)$

$$\begin{aligned} \Phi(H) := \{\varphi : \mathbb{T} \to \mathbb{R}_+ \ : \ \lambda \in \mathbb{R},\ \varphi \geq 0,& \\ \varphi(t) \sim 1 + \lambda \cos 2\pi t + \sum_{k \in H} c_k \cos 2\pi k t\}& \end{aligned} \tag{2.7}$$

and with a given $m \in \mathbb{N}_2$ and $H \subseteq \mathbb{N}_2$ also

$$\begin{aligned} \Phi_m(H) := \{\varphi : \mathbb{T} \to \mathbb{R} \ : \ \lambda \in \mathbb{R}, \varphi(\frac{j}{m}) \geq 0\, (j \in \mathbb{Z}),\ \varphi(t) =& \\ = 1 + \lambda \cos 2\pi t + \sum_{k \in H} c_k \cos 2\pi k t\}.& \end{aligned} \tag{2.8}$$

PROBLEM 2.1.6 (Carathéodory-Fejér type trigonometric polynomial problem). Determine the extremal quantity

$$M(H) := \sup\{\lambda = 2\widehat{\varphi}(1) \,:\, \varphi \in \Phi(H)\}. \tag{2.9}$$

REMARK 2.1.7. Observe that $M(H) \leq 2$, always, as

$$|\lambda/2| = |\widehat{\varphi}(1)| \leq \|\varphi\|_1 = \int \varphi = \widehat{\varphi}(0) = 1.$$

PROBLEM 2.1.8 (Discretized Carathéodory-Fejér type extremal problem). Determine

$$M_m(H) := \sup\{\lambda = 2\widehat{\varphi}(1) \,:\, \varphi \in \Phi_m(H)\}. \tag{2.10}$$

REMARK 2.1.9. It should be remarked here that obviously we have $\Phi(H) \subseteq \Phi_m(H)$. So we always have $M_m(H) \geq M(H)$.

Here we will present the exact solution of Problem 2.1.2 that is in line with what the paper [**13**] suggests. Actually, we have to acknowledge that Boas and Kac mentioned the possibility of extending one of their methods – Poisson summation – to higher dimensions, so some parts of what follows can be interpreted as implicitly present already in their work [**13**]. But here we obtain some results also for the more complex periodic version.

However, the main result of the present investigation is perhaps the understanding that the above point-value extremal problems are in fact equivalent to the above trigonometric polynomial extremal problems, thus transferring information on one problem to the equivalent other problem in several cases. Until now the equivalence formulated below remained unclear in spite of the fact that, e.g., Boas and Kac found ways to deduce the solution of the trigonometric extremal problems in Problem 2.1.6 from their results on Problem 2.1.2. We also obtain a clear picture of the limiting relation between torus problems and space problems, and, parallel to this, between the finitely conditioned trigonometric polynomial extremal problems of Problem 2.1.8 and the positive definite trigonometric polynomial extremal problems of Problem 2.1.6.

2.1.6. Extension of the problem to LCA groups. Some authors have already extended the investigations, although not that systematically as in case of the multivariate setting, to locally compact abelian groups (LCA groups henceforth). This is the natural settings for a general investigation, since the basic notions used in the formulation of the question – positive definiteness, neighborhood of zero, support in and integral over a 0-symmetric set Ω – can be considered whenever we have the algebraic and topological structure of an LCA group. Note that we always have the Haar measure, which makes the consideration of the integral over a compact set (hence over the support of a compactly supported positive definite function) well defined.

Note that the Turán problem, posed for a sequence $H \subset \mathbb{Z}$, has a key relevance in additive number theory, namely in studying van der Corput sets, see [**46, 79, 64, 36**].

We find the first mention of the group case in [**28**], and a more systematic use of the settings (for the square-integral Turán problem) in [**20**]. Utilizing also the work in [**6**] on extensions to the several dimensional case, the framework below was set up in [**56**], see

§2.1.7 and §2.2.1. In [**56**] there we obtained some fairly general results for compact LCA groups as well as for the most classical non-compact groups: $\mathbb{R}^d$, $\mathbb{T}^d$ and $\mathbb{Z}^d$. These will mostly be presented in §'s 2.6 and 2.7.

In this work we study the problem in the generality of LCA groups. This simplifies and unifies many of the existing results and gives several new estimates and examples. If G is a LCA group a continuous function $f \in L^1(G)$ is positive definite if its Fourier transform $\widehat{f} : \widehat{G} \to \mathbb{C}$ is everywhere nonnegative on the dual group $\widehat{G}$. For the relevant definitions of the Fourier transform we refer to [**47**, Chapter VII] or [**77**].

The set Ω will always be taken in this paper work to be a 0-symmetric, open set in G. This is not serious restriction, since the support of any positive definite function is necessarily symmetric, see §2.2.1, hence Ω could always be substituted by $\Omega \cap (-\Omega)$ in case of lack of symmetry.

We say that f belongs to the class $\mathcal{F}(\Omega)$ of functions if $f \in L^1(G)$ is continuous, positive definite and is supported on a closed subset of Ω. For any positive definite function f it follows that $f(0) \geq f(x)$ for any $x \in G$. This leads to the estimate $\int_G f \leq |\Omega| f(0)$ for all $f \in \mathcal{F}$, which is called (following Andreev [**2**]) the *trivial estimate* from now on.

Definition 2.1.10. The *Turán constant* $\mathcal{T}_G(\Omega)$ of a 0-symmetric, open subset Ω of a LCA group G is the supremum of the quantity $\int_G f / f(0)$, where $f \in \mathcal{F}(\Omega)$, i.e. $f \in L^1(G)$ is continuous and positive definite, and $\operatorname{supp} f$ is a closed set contained in Ω.

In fact, depending on the precise requirements on the functions considered, here we have certain variants of the problem: an account of these is presented below in §2.1.7.

Remark 2.1.11. The quantity $\mathcal{T}_G(\Omega)$ depends on which normalization we use for the Haar measure on G. If G is discrete we use the counting measure and if G is compact and non-discrete we normalize the measure of G to be 1.

The *trivial upper estimate* or *trivial bound* for the Turán constant is thus $\mathcal{T}_G(\Omega) \leq |\Omega|$.

2.1.7. Various equivalent formulations of the Turán problem. In fact, it is worth noting that Turán type problems can be, and have been considered with various settings, although the relation of these has not always been fully clarified. Thus in extending the investigation to LCA groups or to domains in Euclidean groups which are not convex, the issue of equivalence has to be dealt with. One may consider the following function classes.

$$\mathcal{F}_1(\Omega) := \left\{ f \in L^1(G) \; : \; \operatorname{supp} f \subset \Omega, \; f \, \text{positive definite} \right\}, \tag{2.11}$$

$$\mathcal{F}_\&(\Omega) := \left\{ f \in L^1(G) \cap C(G) \; : \; \operatorname{supp} f \subset \Omega, \; f \, \text{positive definite} \right\}, \tag{2.12}$$

$$\mathcal{F}_c(\Omega) := \left\{ f \in L^1(G) \; : \; \operatorname{supp} f \subset\subset \Omega, \; f \, \text{positive definite} \right\}, \tag{2.13}$$

$$\mathcal{F}(\Omega) := \left\{ f \in C(G) \; : \; \operatorname{supp} f \subset\subset \Omega, \; f \, \text{positive definite} \right\}. \tag{2.14}$$

In $\mathcal{F}_1, \mathcal{F}_\&$ supp f is assumed to be merely closed ad not necessarily compact, and in $\mathcal{F}_1, \mathcal{F}_c$ the function f may be discontinuous.

The respective Turán constants are

$$\text{(2.15)}\quad \mathcal{T}_G^{(1)}(\Omega) \text{ or } \mathcal{T}_G^{\&}(\Omega) \text{ or } \mathcal{T}_G^{c}(\Omega) \text{ or } \mathcal{T}_G(\Omega) := \sup\left\{ \frac{\int_G f}{f(0)} \ :\ f \in \mathcal{F}_1(\Omega) \text{ or } \mathcal{F}_\&(\Omega) \text{ or } \mathcal{F}_c(\Omega) \text{ or } \mathcal{F}(\Omega), \text{ resp.} \right\}.$$

In general we should consider functions $f : G \to \mathbb{C}$. But according to (2.17) also $\overline{f}$ and thus even $\varphi := \Re f$ is positive definite, while belonging to the same function class. As we also have $f(0) = \varphi(0)$ and $\int f = \int \varphi$, restriction to real valued functions does not change the values of the Turán constants.

To start with, we prove in §2.2.2.

Theorem 2.1.12. *We have for any LCA group the equivalence of the above defined versions of the Turán constants:*

$$\mathcal{T}_G^{(1)}(\Omega) = \mathcal{T}_G^{\&}(\Omega) = \mathcal{T}_G^{c}(\Omega) = \mathcal{T}_G(\Omega)\,. \tag{2.16}$$

Note that the original formulation, presented also above in Definition 2.1.10, corresponds to $\mathcal{T}_G^{\&}(\Omega)$. Also note that with this setup, e.g. the interval case $\Omega = [-h, h] \subset \mathbb{T}$ or $\mathbb{R}$ admits no extremal function, because the support of Δ_h is the full $\overline{\Omega}$, not a closed subset of the open set $(-h, h)$ In this case an obvious limiting process is neglected in the formulation of the results above.

Remark 2.1.13. It is not fully clarified what happens for functions vanishing only outside of Ω, but having nonzero values up to the boundary $\partial\Omega$.

2.2. Positive definite functions and equivalent formulations of the Turán problem

2.2.1. Positive definite functions on LCA groups. Positive definite functions were introduced by Maximilian Matthias [**60**]. By the analogous definition these can be defined also on locally compact Abelian groups (LCA groups).

In this section we explore a few facts on positive definite, not necessarily continuous functions. We could not decide if anything is new here, as we have found it very hard to locate these facts in the literature without assuming continuity of the positive definite function at the outset. So we collected these facts here. Everything in this section is taken from our joint work with Mihalis Kolountzakis [**56**].

Recall that on a LCA group G a function f is called positive definite if the inequality

$$\sum_{n,m=1}^{N} c_n \overline{c_m} f(x_n - x_m) \geq 0 \qquad (\forall x_1, \dots, x_N \in G, \forall c_1, \dots, c_N \in \mathbb{C}) \tag{2.17}$$

holds true. Note that positive definite functions are not assumed to be continuous. Still, all such functions f are necessarily bounded by $f(0)$ [**77**, p. 18, Eqn (3)]. Moreover, $f(x) = \widetilde{f}(x) := \overline{f(-x)}$ for all $x \in G$ [**77**, p. 18, Eqn (2)], hence the support of f is necessarily symmetric, and the condition supp $f \subset \Omega$ implies also supp $f \subset \Omega \cap (-\Omega)$. The latter set being symmetric, without loss of generality we can assume at the outset that Ω is symmetric itself.

It is immediate from (2.17) that for any subgroup K of G, the restriction $f|_K$ of a positive definite function f is also positive definite on K.

The Fourier transform $\widehat{f}$ of an $f \in L^1(G)$ belongs to $A(\widehat{G}) \subset C_0(\widehat{G})$, and the Fourier transform of the convolution $f * g$ of $f, g \in L^1(G)$, defined almost everywhere, satisfies $\widehat{f * g} = \widehat{f}\widehat{g}$ [**77**, Theorem 1.2.4]. Similarly, for $\nu, \mu \in M(G)$ and their convolution $\mu * \nu \in M(G)$ the Fourier transforms are bounded and uniformly continuous and $\widehat{\mu * \nu} = \widehat{\mu}\widehat{\nu}$ [**77**, Theorem 1.3.3].

In case $f, g \in L^2(G)$, the convolution $h := f * g$ is defined even in the pointwise sense and $h \in C_0(\widehat{G})$ [**77**, Theorem 1.1.6(d)]. For $f \in L^2(G)$ arbitrary (denoting as above, $\widetilde{f}(x) := \overline{f(-x)}$), $f * \widetilde{f}$ is continuous and positive definite with Fourier transform $|\widehat{f}|^2$ [**77**, §1.4.2(a)].

Note that for any given $\gamma \in \widehat{G}$ f is positive definite if and only if $f(x)\gamma(x)$ is positive definite; this can be checked by modifying the coefficients in (2.17) accordingly.

LEMMA 2.2.1. *Suppose that f is (measurable and) positive definite and $g \in L^2(G)$ is arbitrary. Then the product $f \cdot (g * \widetilde{g})$ is positive definite.*

PROOF. As written above, $h := g * \widetilde{g} \in C_0(G)$, while f, being positive definite, is also bounded. Take now $x_n \in G$ and $c_n \in \mathbb{C}$ for $n = 1, \dots, N$ arbitrarily. Then

$$\begin{aligned}
&\sum_{n,m=1}^{N} c_n \overline{c_m} f(x_n - x_m) h(x_n - x_m) \\
&= \sum_{n,m=1}^{N} c_n \overline{c_m} f(x_n - x_m) \int_G g(x_n - y) \overline{g}(x_m - y) dy \\
&= \int_G \sum_{n,m=1}^{N} a_n(y) \overline{a_m(y)} f(x_n - x_m)\, dy\,,
\end{aligned}$$

where $a_n(y) := c_n g(x_n - y) \in L^2(G)$ $(n = 1, \dots, N)$. Since the expression under the integral sign is nonnegative by (2.17) for each given y, also the integral is nonnegative and the assertion follows. $\square$

Observe that Lemma 2.2.1 ensures positive definiteness of fh only if h has a "convolution root" $g \in L^2(G)$. We do not know if the same holds for any positive definite $h \in C(G)$, say, as in general positive definite functions may have no convolution roots, see e.g. [**21**].

Note that we did not assume f to be integrable, and neither the product fh is supposed to belong to any subspace. By positive definiteness, f is bounded; but if G is not compact, $\widehat{f}$ is not necessarily defined. However, as $h \in C_0(G)$, in any case we must have $fh \in L^\infty(G)$. This follows from positive definiteness of fh, too.

The next Lemma is obvious for compact groups as we can take $k = 1$.

LEMMA 2.2.2. *Suppose C is a compact set in a LCA group G and $\delta > 0$ is given. Then there exists a compactly supported, positive definite and continuous "kernel function"*

$k(x) \in C_c(G)$ satisfying $k(0) = 1$, $0 \le k \le 1$, and $k|_C \ge 1 - \delta$. Moreover, we can take $k = h * \widetilde{h}$, where h is the L^2-normalized indicator function of a suitable Borel measurable set $V \supset C$ with compact closure $\overline{V}$.

PROOF. We may clearly assume that G is not compact.

The deduction will follow the proof of 2.6.7 Theorem on page 52 of [**77**] with a slight modification towards the end of the argument. In this proof the compact set C is given, and then another Borel set E and an increasing sequence of Borel sets V_N $(N \in \mathbb{N})$ are found, so that $C \subset E = V_0$ and $|V_N| = (2N+1)^n|E|$ (with n a fixed nonnegative integer constant); moreover, all the V_N have compact closure and $V_N + E \subset V_{N+s}$ is ensured for some fixed s and for all $N \in \mathbb{N}$. Hence for every $c \in C \subset E$ we have $V_{N+s} - c \supset V_N$. Denoting the indicator function of V_{N+s} by χ we are led to $\int_G \chi(x+c)\chi(x)dx \ge |V_N|$. Putting $h := |V_{N+s}|^{-1/2}\chi$ yields $h*\widetilde{h}(c) \ge |V_N|/|V_{N+s}| > 1-\delta$, if N is chosen large enough (depending on the constants n, s and the given δ). With this choice of h and $V := V_{N+s}$ all assertions of the Lemma are true. □

REMARK 2.2.3. As Rudin points out, this argument essentially depends on structure theorems of LCA groups.

LEMMA 2.2.4. *Suppose that $f \in L^1(G)$ is positive definite. Then the Fourier transform $\widehat{f}$ is nonnegative.*

PROOF. Since for any character $\gamma \in \widehat{G}$ we have $\widehat{\gamma f} = \widehat{f}(\cdot - \gamma)$, and f is positive definite precisely when γf is such, it suffices to prove that $\widehat{f}(0) \ge 0$.

For technical reasons, we need to modify f to have compact support. Let δ be any positive parameter. Since $d\nu(x) := f(x)dx$ is a *regular* Borel measure, for some compact set C we have $\|f\|_{L(G\setminus C)} < \delta$. Take the function k provided by Lemma 2.2.2 for the compact set C and the chosen parameter $\delta > 0$. If $g := kf$, Lemma 2.2.1 shows that g is positive definite, while $|\widehat{g}(0) - \widehat{f}(0)| \le |\int_C f - \int_C g| + \|f\|_{L^1(G\setminus C)} < \delta \int_C |f| + \delta \le \delta(1 + \|f\|_{L^1})$. Choosing δ small enough, it follows that there exists a compactly supported positive definite $g \in L^1(G)$ with $\widehat{g}(0) < 0$ provided that $\widehat{f}(0) < 0$. Hence it suffices to prove the assertion for compactly supported positive definite functions g.

Applying definition (2.17) with all c_n chosen as 1 yields

$$0 \le \sum_{n=1}^{N} \sum_{m=1}^{N} g(x_n - x_m)\,.$$

Integrating over C^N (where $C := \operatorname{supp} g$) we obtain

$$0 \le N|C|^N g(0) + (N^2 - N)|C|^{N-1} \int_C g\,,$$

which implies

$$-\frac{|C|g(0)}{N-1} \le \widehat{g}(0).$$

Letting $N \to \infty$ concludes the proof. □

LEMMA 2.2.5. *Suppose that $f, g \in L^1(G)$ are two positive definite functions. Then the convolution $f * g \in L^1(G)$ is uniformly continuous and positive definite.*

PROOF. Since a positive definite function is bounded, we have also $f \in L^\infty(G)$, hence $f * g$ is uniformly continuous c.f. [**77**, Theorem 1.1.6(b)]. For the Fourier transform $\widehat{f * g} = \widehat{f}\widehat{g}$ of the continuous function $f * g$ positive definiteness is equivalent to $\widehat{f * g} \geq 0$. Now Lemma 2.2.4 gives $\widehat{f} \geq 0$ and $\widehat{g} \geq 0$, hence $\widehat{f * g} \geq 0$ and $f * g$ is positive definite. □

LEMMA 2.2.6. *Suppose U is a given neighborhood of 0 in a LCA group G. Then there exists a compactly supported, continuous, positive definite and nonnegative "kernel function" $k(x) \in C_c(G)$ satisfying* $\operatorname{supp} k \subset\subset U$ *and $\int k = 1$. Moreover, we can take $k = h * \widetilde{h}$, with $h = |W|^{-1}\chi_W$, where χ_W is the indicator function of a compact set W satisfying $W - W \subset U$.*

PROOF. By continuity of the operation of subtraction, there exists a compact neighborhood W of 0 satisfying $W - W \subset U$. With the above definitions of k and h we clearly have $\operatorname{supp} k \subset\subset W - W \subset U$ (c.f. [**77**, Theorem 1.1.6(c)] and also $\int k = |W|^{-2}(\int \chi_W)^2 = 1$. As $h, \widetilde{h} \in L^2(G)$, $k \in C_0(G)$, and in view of $\operatorname{supp} k$ being compact, $k \in C_c(G)$. Because h is nonnegative, so is k. Finally, [**77**, 1.4.2(a)] gives positive definiteness of k. □

LEMMA 2.2.7. *Let f be positive definite and integrable. Then for any $\epsilon > 0$ and open set U containing 0, there exists a nonnegative, positive definite function of the form $k = h * \widetilde{h}$ (with $h \in L^2(G)$), so that* $\operatorname{supp} k \Subset U$*, $\int_U k = 1$, and $\|f - f * k\|_1 < \epsilon$.*

PROOF. For the given function f there exists a neighborhood V of 0 with the property that $\|f - f * u\| < \epsilon$ whenever $\int_G u = 1$ and $u \geq 0$ is Borel measurable and vanishing outside V [**77**, Theorem 1.1.8]. Now we can construct for the open set $U_0 := V \cap U$ the kernel function k as in Lemma 2.2.6. Clearly, k satisfies all conditions for u, hence $\|f - f * k\|_1 < \epsilon$ follows. By construction, $\operatorname{supp} k \subset\subset U_0 \subset U$ and $\int_U k = 1$. □

LEMMA 2.2.8. *For any pair of sets $K \Subset U$ with K compact and U open, there exists a neighborhood V of 0 satisfying $K + V \subset U$.*

PROOF. Since addition is continuous, for any open neighborhood U_0 of 0 there exists a neighborhood W so that $W + W \subset U_0$. Take now to each point $x \in K$ an open neighborhood W_x of 0 such that $x + W_x + W_x \subset U$, ie. $W_x + W_x \subset U - x$. Clearly the family of open sets $\{x + W_x \ : \ x \in K\}$ form an open covering of K, so in view of compactness of K there exists a finite subcovering $\{W_{x_k} + x_k \ : \ k = 1, \dots, n\}$. Take now $V := \bigcap_{k=1}^n W_{x_k}$. We claim that $K + V \subset U$. Indeed, if $y \in K$ and $z \in V$ then considering any index k with $y \in x_k + W_{x_k}$, we find $y + z \in (x_k + W_{x_k}) + V \subset x_k + W_{x_k} + W_{x_k} \subset U$. □

LEMMA 2.2.9. *Let $\epsilon > 0$ be arbitrary. Assume that f is measurable and positive definite and compactly supported in the open set Ω. Then there exists another positive definite, but also continuous function g with $f(0) \geq g(0)$ and $\int_G g \geq \int_G f - \epsilon$, also supported compactly in Ω.*

PROOF. Observe that f, being positive definite, is also bounded, and since it is compactly supported, it also belongs to $L^1(G)$. Thus we can use the Fourier transform $\widehat{f}$. Let $K := \operatorname{supp} f \subset\subset \Omega$ and consider a neighborhood U of 0 with $K + U \subset \Omega$. Such a U is provided by Lemma 2.2.8. Lemma 2.2.7 provides a positive definite, continuous kernel $k \in C_c(G)$, compactly supported in U and satisfying $\int_G f * k \geq \int_G f - \epsilon$. In view of $k = h * \widetilde{h}$ and Lemma 2.2.5 also $g := f * k$ is positive definite while obviously $g \in C_c(G)$ is supported compactly in $K + U \subset \Omega$. It remains to note that by $k \geq 0$, $\int k = 1$ and $|f| \leq f(0)$ we also have $g(0) = \int k(x)f(-x)dx \leq f(0) \int k = f(0)$. □

2.2.2. Proof of Theorem 2.1.12 on the equivalence of various definitions of $\mathcal{T}_G(\Omega)$ on LCA groups.

PROPOSITION 2.2.10 (**Kolountzakis-Révész**). *With the definitions above we have that* $\mathcal{T}_G^{(1)}(\Omega) = \mathcal{T}_G^c(\Omega)$.

PROOF. Let $\epsilon > 0$ and $\delta > 0$ be arbitrary and $f \in \mathcal{F}_1(\Omega)$ be chosen so that $\int_G f > \mathcal{T}_G^{(1)}(\Omega) - \delta$. As $f \in L^1(G)$, the measure $|f(x)|dx$ is absolutely continuous with respect to the Haar measure, hence it is also a regular Borel measure and there exists a compact subset $C \subset\subset \operatorname{supp} f$ so that $\int_{G \setminus C} |f| < \delta$. Now an application of Lemma 2.2.2 with C and δ provides us the positive definite, compactly supported kernel function k satisfying $k(0) = 1$, and $k|_C > (1 - \delta)$. Let $g := fk$. Then $\operatorname{supp} g \subset (\operatorname{supp} k \cap \operatorname{supp} f) \subset\subset \operatorname{supp} f$, hence g is compactly supported within Ω. Moreover, $g(0) = 1$ and g is positive definite in view of Lemma 2.2.1. Hence $g \in \mathcal{F}_c(\Omega)$. We now have

$$\begin{aligned} \int g = \int_\Omega kf &= \int_\Omega f - \int_\Omega (1-k) f \\ &\geq \int_\Omega f - \delta \int_C |f| - \int_{\Omega \setminus C} |f| \\ &\geq \int_\Omega f - \delta \int_\Omega |f| - \delta \qquad \geq (1-\delta)\big(\mathcal{T}_G^{(1)}(\Omega) - \delta\big) - \delta. \end{aligned}$$

Clearly, if δ was chosen small enough, we obtain $\int g > \mathcal{T}_G^{(1)}(\Omega) - \epsilon$. Now taking sup over $g \in \mathcal{F}_c(\Omega)$ concludes the proof, since $\epsilon > 0$ was arbitrary. □

PROPOSITION 2.2.11 (**Kolountzakis-Révész**). *With the definitions above we have that* $\mathcal{T}_G(\Omega) = \mathcal{T}_G^c(\Omega)$.

PROOF. Since $\mathcal{F}_c(\Omega) \supset \mathcal{F}(\Omega)$, it suffices to prove $\mathcal{T}_G^c(\Omega) \leq \mathcal{T}_G(\Omega)$.
Let $\epsilon > 0$ and $f \in \mathcal{F}_c(\Omega)$ be chosen so that $\int f > \mathcal{T}_G^c(\Omega) - \epsilon$, while $\operatorname{supp} f$ is a compact subset of the open set Ω. Hence an application of Lemma 2.2.9 provides a $g \in \mathcal{F}(\Omega)$ with $\mathcal{T}_G(\Omega) \geq \int g > \int f - \epsilon > \mathcal{T}_G^c(\Omega) - 2\epsilon$. Now $\epsilon \to 0$ yields the Proposition. □

PROOF OF THEOREM 2.1.12. We have the obvious inclusions $\mathcal{F}_1(\Omega) \supset \mathcal{F}_\&(\Omega) \supset \mathcal{F}(\Omega)$ and $\mathcal{F}_1(\Omega) \supset \mathcal{F}_c(\Omega) \supset \mathcal{F}(\Omega)$, hence $\mathcal{T}_G^{(1)}(\Omega) \geq \mathcal{T}_G^\&(\Omega) \geq \mathcal{T}_G(\Omega)$ and $\mathcal{T}_G^{(1)}(\Omega) \geq \mathcal{T}_G^c(\Omega) \geq \mathcal{T}_G(\Omega)$. On combining these inequalities with Propositions 2.2.10 and 2.2.11 the assertion follows. □

If we consider a *continuous* positive definite function f, then it must also be uniformly continuous [**77**, p. 18, Eqns (3), (4)]. When $\operatorname{supp} f$ has bounded Haar measure (and, in particular, when $\operatorname{supp} f$ is compact) then f belongs to $L^1(G)$, too. For an integrable, continuous and positive definite function f the Fourier transform $\widehat{f}$ of f exists, and the Fourier inversion formula holds, cf. [**77**, §1.5.1]. The well-known Bochner-Weil characterization says that $f \in C(G)$ being positive definite is equivalent to the existence of a non-negative measure μ on the dual group $\widehat{G}$ so that

$$f(x) = \int_{\widehat{G}} \overline{\gamma(x)}\, d\mu(\gamma);$$

moreover, this representation is unique cf. [**77**, §1.4.3], Comparing the Fourier inversion formula and the unique representation above leads to the further characterization that *for a continuous and integrable* f being positive definite is equivalent to $\widehat{f} \geq 0$, compare [**77**, §1.7.3(e)]. Thus it is really advantageous to restrict the function class considered from $\mathcal{F}_1(\Omega)$ to $\mathcal{F}(\Omega)$, say.

Our setting is that Ω is an open (symmetric) set, and we require that f can be nonzero only in Ω. This is an essential condition. In this respect approximation has its limitations: eg. we cannot relax the conditions to require $\operatorname{supp} f \subset \overline{\Omega}$ only.

Indeed, if Ω is not *fat*, meaning that $\Omega = \operatorname{int} \overline{\Omega}$, this can lead to essential changes of the Turán constants. Eg. if $G = \mathbb{R}$ and $\Omega = (-a, a) \setminus \{\pm b\}$, then $\operatorname{int}\overline{\Omega} = (-a, a)$ and $\mathcal{T}_{\mathbb{R}}((-a, a)) = a$, while $\mathcal{T}_{\mathbb{R}}(\Omega) = b$ if $a/2 \leq b \leq a$, see Theorem 2.6.20 below. Similarly, if $G = \mathbb{T}$ and $\Omega = \mathbb{T} \setminus \{\pi\}$, then $\mathcal{T}_{\mathbb{T}}(\Omega) = 1/2$, but obviously $\mathcal{T}_{\mathbb{T}}(\overline{\Omega}) = 1$. That is, forcing the function f to vanish at one single point can, through positive definiteness, bring down the values essentially in general.

In this respect, original formulations of the Turán problem in [**7**] and [**54**] may be misleading, since for a convex body Ω in $\mathbb{R}^d$ or $\mathbb{T}^d$ the allegedly extremal function $\chi_{\Omega/2} * \chi_{\Omega/2}$ does *not* belong to the function class $\mathcal{F}_{\&}(\Omega)$ considered there. Instead, a corresponding limiting argument should provide the same extremal value. In convex or star bodies in Euclidean spaces one can easily obtain a positive definite function supported properly in the body from one that may be "non-zero up to the boundary", by a slight dilation of space, without losing much integral. It is unclear how to do this in general, even for domains in $\mathbb{R}^d$.

2.3. Uniform asymptotic upper density on LCA groups

2.3.1. Measuring large, but not necessarily dense infinite sequences and sets in groups. Although only definitions are constructed here, we feel that in the long run this part may prove to be the most interesting part of the whole analysis we present.

Our aim here is to extend the notion of *uniform asymptotic upper density*, used in case of $\mathbb{R}$ already by Beurling and Pólya in the analysis of entire functions. The same notion is frequently called by others as *Banach density*, c.f. e.g. [**27**, p. 72].

The notion of uniform asymptotic upper density – u.a.u.d. for short – is a way to grab the idea of a set being relatively considerable, even if not necessarily dense or large in some other more easily accessible sense. In many theorems, in particular in Fourier analysis and in additive problems where difference sets or sumsets are considered, the u.a.u.d. is the right notion to express that a set becomes relevant in the question considered. However, to date the notions was only extended to sequences and subsets of the real line, and some immediate relatives like $\mathbb{Z}^d$, $\mathbb{R}^d$, as well as to finite, or at least finitely constructed (e.g. σ-finite) cases.

A framework where the notion might be needed is the generality of LCA groups. In recent decades it is more and more realized that many questions e.g. in additive number theory can be investigated, even sometimes structurally better understood/described, if we leave e.g. $\mathbb{Z}$, and consider the analogous questions in Abelian groups. In fact, when some analysis, i.e. topology also has a role – like in questions of Fourier analysis e.g. – then the setting of LCA groups seems to be the natural framework. And inded several notions and questions, where in classical results u.a.u.d. played a role, have already been defined, even in some extent discussed in LCA groups. Nevertheless, it seems that no attempt has been made to extend the very notion of u.a.u.d. to this setup.

One of the more explicit attempts to really "measure sets in infinite groups" is perhaps the work of Borovik at al. [**15**], [**14**]. Other papers, where some ideas close to ours can be seen, are [**65**] – considering measures, not sets, although the investigation there is focused on local structure at small neighborhoods of points – and in [**66**], where at least the setup of LCA groups is apparent (although the interest is quite different).

For cases of σ-finite groups G it is easy to design the u.a.u.d., compare [**38**]. In the more general framework of discrete groups, I.Z. Ruzsa [**82**] had two constructions to define u.a.u.d..

However, neither of these constructions were the same as ours. Below we will explain, how one may construct notions of u.a.u.d., which finely extend the classical notion.

2.3.2. Some additive number theory flavored results for difference sets. Let us denote the upper density of $A \subset \mathbb{N}$ as $\overline{d}(A) := \limsup_{n\to\infty} A(n)/n > 0$ with $A(n) := \#(A \cap [1, n])$. Erdős and Sárközy (seemingly unpublished, but quoted in [**38**] and in [**81**]) observed the following.

PROPOSITION 2.3.1 (Erdős-Sárközi). *If the upper density $\overline{d}(A)$ of a sequence $A \subset \mathbb{N}$ is positive, then writing the positive elements of the sequence $D(A) := D_1(A) := A - A$ as $D(A) \cap \mathbb{N} = \{(0 <)d_1 < d_2 < \dots\}$ we have $d_{n+1} - d_n = O(1)$.*

This is analogous, but not contained in the following result of Hegyvári, obtained for σ-finite groups. An abelian group is called σ-finite (with respect to H_n), if there exists an increasing sequence of *finite* subgroups H_n so that $G = \cup_{n=1}^{\infty} H_n$. For such a group Hegyvári defines asymptotic upper density (with respect to H_n) of a subset $A \subset G$ as

$$\overline{d}_{H_n}(A) := \limsup_{n\to\infty} \frac{\#(A \cap H_n)}{\#H_n} \ . \tag{2.18}$$

Note that for finite groups this is just $\#(A \cap G)/\#G$. Then Hegyvári proves the following result, see [**38**, Proposition 1].

PROPOSITION 2.3.2 (Hegyvári). *Let G be a σ-finite abelian group with respect to the increasing, exhausting sequence H_n of finite subgroups and let $A \subset G$ have positive upper density with respect to H_n. Then there exists a finite subset $B \subset G$ so that $A - A + B = G$. Moreover, we have $\#B \le 1/\overline{d}_{H_n}(A)$.*

Fürstenberg calls a subset $S \subset G$ in a topological Abelian (semi)group a *syndetic* set, if there exists a compact set $K \subset G$ such that for each element $g \in G$ there exists a $k \in K$ with $gk \in S$; in other words, in topological groups $\cup_{k \in K} S k^{-1} = G$. Then he presents as Proposition 3.19 (a) of [**27**] the following.

PROPOSITION 2.3.3 (Fürstenberg). *Let $S \subset \mathbb{Z}$ with positive upper Banach density. Then $S - S$ is a syndetic set.*

In the following we extend the notion of uniform asymptotic upper density, (also called as Banach density) to arbitrary LCA groups, and present various generalized versions of the above results, which cover all of them.

2.3.3. Various forms of the asymptotic density. We start with the frequently used definition of asymptotic upper density in $\mathbb{R}^d$. Let $K \subset \mathbb{R}^d$ be a *fat body*, i.e. a set with $0 \in \mathrm{int}K$, $K = \overline{\mathrm{intK}}$ and K compact. Then asymptotic upper density with respect to K is defined as

$$\overline{d}_K(A) := \limsup_{r \to \infty} \frac{|A \cap rK|}{|rK|} \ . \tag{2.19}$$

The definition (2.18) is clearly analogous to (2.19). As is easy to see, both (2.19) and (2.18) depends on the choice of the fundamental set K or sequence H_n, even if *positivity* of (2.19) is invariant for a large class of underlying sets including all convex, but also many other bodies. The similar notion of density applies and has the same properties also for the discrete group $\mathbb{Z}^d$. On the other hand, for a given subset A in a σ-finite group G, (2.18) can easily be zero for some fundamental sequence H_n, while being maximal (i.e., 1) for some other choice H'_n of fundamental sequence.

EXAMPLE 2.3.4. Let $G := \mathbb{Q}/\mathbb{Z}$, which is a σ-finite additive abelian group. Let $H_n := \{r \in G \ : \ r = \frac{p}{q}, \ q \leq n\}$; then H_n is an increasing and exhausting sequence of finite subgroups of G. Note that $\#H_n = \sum_{j \leq n} \varphi(j) \sim \frac{6}{\pi^2} n^2$. Let then $A_k := \{r \in G \ : \ r = \frac{p}{q}, \ (p,q) = 1, \ (k^2+k)! < q \leq (k+1)^2!\}$ and $A := \cup_{k=1}^{\infty} A_k$. Then it is not hard to prove that $\liminf_{n \to \infty} \frac{\#A \cap H_n}{\#H_n} = 0$ but $\limsup_{n \to \infty} \frac{\#A \cap H_n}{\#H_n} = 1$. Then it is clear that the value of the upper density can be either 0 or 1 depending on the choice of an appropriate subsequence of H_n as fundamental sequence. With a little modification an example with arbitrary numbers as possible upper densities can be derived.

However, results corresponding to the above ones of Erdős, Sárközi and Hegyvári are easily sharpened by using only a weaker notion, that of *asymptotic uniform upper density*. It could be defined as

$$\overline{D}_{H_n}(A) := \limsup_{n \to \infty} \frac{\sup_{x \in G} \#(A \cap (H_n + x))}{\#H_n} \tag{2.20}$$

for σ-finite abelian groups and is defined as

$$\overline{D}_K(A) := \limsup_{r \to \infty} \frac{\sup_{x \in \mathbb{R}^d} |A \cap (rK + x)|}{|rK|} \tag{2.21}$$

in $\mathbb{R}^d$. It is obvious that these notions are translation invariant, and $\overline{D}_{H_n}(A) \geq \overline{d}_{H_n}(A)$, $\overline{D}_K(A) \geq \overline{d}_K(A)$. It is also well-known, that $\overline{D}_K(A)$ gives the same value for all nice - e.g. for all convex - bodies $K \subset \mathbb{R}^d$, although this fact does not seem immediate from the formulation. Actually, we will obtain this as a side result, being an immediate corollary of Theorem 2.3.6, see Remark 2.3.7.

Similar definitions can be used for $\mathbb{Z}^d$. However, dependence on the fundamental sequence H_n makes the σ-finite case less appealing, and we lack a successful notion for abelian groups in general. In particular, a natural requirement is to find a common generalization of asymptotic upper density, which works both for $\mathbb{R}^d$ and $\mathbb{Z}^d$, and also for a larger class of (say, abelian) groups, including, but not restricted to σ-finite ones.

Note also the following ambiguity in the use of densities in literature. Sometimes even in continuous groups a discrete set Λ is considered in place of A, and then the definition of the asymptotic upper density is

$$\overline{D}^{\#}_K(A) := \limsup_{r\to\infty} \frac{\sup_{x\in\mathbb{R}^d} \#(\Lambda \cap (rK + x))}{|rK|} \quad . \tag{2.22}$$

That motivates our further extension: we are aiming at asymptotic uniform upper densities of *measures*, say measure ν with respect to measure μ, (whether related by ν being the trace of μ on a set or not). E.g. in (2.22) $\nu := \#$ is the cardinality or counting measure of a set Λ, while $\mu := |\cdot|$ is just the volume. The general formulation in $\mathbb{R}^d$ is thus

$$\overline{D}_K(\nu) := \limsup_{r\to\infty} \frac{\sup_{x\in\mathbb{R}^d} \nu(rK + x)}{|rK|} \quad . \tag{2.23}$$

Of course, to extend these notions some natural hypotheses should apply. We are considering abelian groups (although non-abelian groups come to mind naturally, here we do not consider this extension), and in accordance to the group settings only densities with respect to translation-invariant measures μ are suitable. Otherwise we want ν to be a measure, possibly infinite, and μ be another, translation-invariant, nonnegative (outer) measure with strictly positive, but finite values when applied to sets considered.

We will consider two generalizations here. The first applies for the class of abelian groups G, equipped with a topological structure which makes G a LCA (locally compact abelian) group. Considering such groups are natural for they have an essentially unique translation invariant Haar measure μ_G (see e.g. [**77**]), what we fix to be our μ. By construction, μ is a Borel measure, and the sigma algebra of μ-measurable sets is just the sigma algebra of Borel mesurable sets, denoted by $\mathcal{B}$ throughout. Furthermore, we will take $\mathcal{B}_0$ to be the members of $\mathcal{B}$ with compact closure: note that such Borel measurable sets necessarily have finite Haar measure. This will be important for not allowing a certain degeneration of the notion: e.g. if we consider $G = \mathbb{R}$, ν is the counting measure $\#$ and A is some sequence $A = \{a_k \ : \ k \in \mathbb{N}\}$, say tending to infinity, then it is easy to define a (non-compact, but still measurable) union V of decreasingly small neighborhoods of the points a_k such that the Haar measure of V does not exceed 1, but all of A stays in V, hence the relative density of A, with respect to the counting measure, is infinite. (Another way to deal with this phenomenon would have been to fix that $\infty/\infty = 0$, but we prefer not to go into such questions.)

Note if we consider the discrete topological structure on any abelian group G, it makes G a LCA group with Haar measure $\mu_G = \#$, the counting measure. Therefore, our notions below certainly cover all discrete groups. This is the natural structure for $\mathbb{Z}^d$, e.g. On the other hand all σ-finite groups admit the same structure as well, unifying considerations. (Note that $\mathbb{Z}^d$ is not a σ-finite group since it is *torsion-free*, i.e. has no finite subgroups.)

The other measure ν can be defined, e.g., as the *trace* of μ on the given set A, that is, $\nu(H) := \nu_A(H) := \mu_G(H \cap A)$, or can be taken as the counting measure of the points included in some set Λ derived from the cardinality measure similarly: $\gamma(H) := \gamma_\Lambda(H) := \#(H \cap \Lambda)$.

DEFINITION 2.3.5. Let G be a LCA group and $\mu := \mu_G$ be its Haar measure. If ν is another measure on G with the sigma algebra of measurable sets being $\mathcal{S}$, then we define

$$\overline{D}(\nu;\mu) := \inf_{C \Subset G} \sup_{V \in \mathcal{S} \cap \mathcal{B}_0} \frac{\nu(V)}{\mu(C+V)} \ . \tag{2.24}$$

In particular, if $A \subset G$ is Borel measurable and $\nu = \mu_A$ is the trace of the Haar measure on the set A, then we get

$$\overline{D}(A) := \overline{D}(\nu_A;\mu) := \inf_{C \Subset G} \sup_{V \in \mathcal{B}_0} \frac{\mu(A \cap V)}{\mu(C+V)} \ . \tag{2.25}$$

If $\Lambda \subset G$ is any (e.g. discrete) set and $\gamma := \gamma_\Lambda := \sum_{\lambda \in \Lambda} \delta_\lambda$ is the counting measure of Λ, then we get

$$\overline{D}^{\#}(\Lambda) := \overline{D}(\gamma_\Lambda;\mu) := \inf_{C \Subset G} \sup_{V \in \mathcal{B}_0} \frac{\#(\Lambda \cap V)}{\mu(C+V)} \ . \tag{2.26}$$

THEOREM 2.3.6. *Let K be any convex body in $\mathbb{R}^d$ and normalize the Haar measure of $\mathbb{R}^d$ to be equal to the volume $|\cdot|$. Let ν be any measure with sigma algebra of measurable sets $\mathcal{S}$. Then we have*

$$\overline{D}(\nu; |\cdot|) = \overline{D}_K(\nu) \ . \tag{2.27}$$

The same statement applies also to $\mathbb{Z}^d$.

REMARK 2.3.7. In particular, we find that the asymptotic uniform upper density $\overline{D}_K(\nu)$ does not depend on the choice of K. For a direct proof of this one has to cover the boundary of a large homothetic copy of K by standard (unit) cubes, say, and after a tedious ϵ-calculus a limiting process yields the result. However, Theorem 1 elegantly overcomes these technical difficulties.

Furthermore, we also introduce a second notion of density as follows.

DEFINITION 2.3.8. Let G be a LCA group and $\mu := \mu_G$ be its Haar measure. If ν is another measure on G with the sigma algebra of measurable sets being $\mathcal{S}$, then we define

$$\overline{\Delta}(\nu;\mu) := \inf_{F \subset G,\, \#F < \infty} \sup_{V \in \mathcal{S} \cap \mathcal{B}_0} \frac{\nu(V)}{\mu(F+V)} \ . \tag{2.28}$$

In particular, if $A \subset G$ is Borel measurable and $\nu = \mu_A$ is the trace of the Haar measure on the set A, then we get

$$\overline{\Delta}(A) := \overline{\Delta}(\nu_A;\mu) := \inf_{F\subset G,\,\#F<\infty} \sup_{V\in\mathcal{B}_0} \frac{\mu(A\cap V)}{\mu(F+V)} \ . \tag{2.29}$$

If $\Lambda \subset G$ is any (e.g. discrete) set and $\gamma := \gamma_\Lambda := \sum_{\lambda\in\Lambda} \delta_\lambda$ is the counting measure of Λ, then we get

$$\overline{\Delta}^{\#}(\Lambda) := \overline{\Delta}(\gamma_\Lambda;\mu) := \inf_{F\subset G,\,\#F<\infty} \sup_{V\in\mathcal{B}_0} \frac{\#(\Lambda\cap V)}{\mu(F+V)} \ . \tag{2.30}$$

The two definitions are rather similar, except that the requirements for $\overline{\Delta}$ refer to finite sets only. Because all finite sets are necessarily compact in an LCA group, (2.24) of Definition 2.3.5 extends the same infimum over a wider family of sets than (2.28) of Definition 2.3.8; therefore we get

PROPOSITION 2.3.9. *Let G be any LCA group, with normalized Haar measure μ. Let ν be any measure with sigma algebra of measurable sets $\mathcal{S}$. Then we have*

$$\overline{\Delta}(\nu;\mu) \geq \overline{D}(\nu;\mu) \ . \tag{2.31}$$

This specializes to $\mathbb{R}^d$ as follows.

PROPOSITION 2.3.10. *Let us normalize the Haar measure of $\mathbb{R}^d$ to be equal to the volume $|\cdot|$. Let ν be any measure with sigma algebra of measurable sets $\mathcal{S}$. Then we have*

$$\overline{\Delta}(\nu;|\cdot|) \geq \overline{D}(\nu;|\cdot|) \ . \tag{2.32}$$

Moreover, the following is obvious, since in discrete groups the Haar measure is the counting measure and the compact sets are exactly the finite sets.

PROPOSITION 2.3.11. *Let ν be any measure on the sigma algebra $\mathcal{S}$ of measurable sets in a discrete Abelian group G. Take $\mu := \#$ the counting measure, which is the normalized Haar measure of G as a LCA group. Then*

$$\overline{\Delta}(\nu;\#) = \overline{D}(\nu;\#) \ . \tag{2.33}$$

So there is no difference for $\mathbb{Z}$, e.g. In general, however, the two densities, defined above, may well be different: in fact, we would bet for that, but we have no construction to show this.

2.3.4. Proof of Theorem 2.3.6. Proof of $\overline{D}(\nu;|\cdot|) \geq \overline{D}_K(\nu)$.

Let now $\tau < \tau' < \overline{D}_K(\nu)$ and $C \Subset G$ be arbitrary. Since C is compact, for some sufficiently large $r' > 0$ we have $C \Subset r'K$, hence by convexity also $C + rK \subset (r+r')K$ for any $r > 0$. On the other hand by $\tau' < \overline{D}_K(\nu)$ there exist $r_n \to \infty$ and $x_n \in \mathbb{R}^d$ with $|\nu(r_nK + x_n)| > \tau'|r_nK|$. With large enough n, we also have $|(r_n + r')K|/|r_nK| = (1 + r'/r_n)^d < \tau'/\tau$, hence with $V := r_nK + x_n$ we find $\nu(V) > \tau'|r_nK| > \tau|(r_n + r')K| = \tau|x_n + r_nK + r'K| \geq \tau|V + C|$. This proves that $\overline{D}(\nu;|\cdot|) \geq \tau$, whence the assertion.

Proof of $\overline{D}_K(\nu) \geq \overline{D}(\nu;|\cdot|)$.

Take now $\tau < \overline{D}(\nu; |\cdot|)$, put $C := rK$ with some $r > 0$ given, and pick up a measurable set V satisfying $\nu(V) > \tau|V + C|$. We can then write

$$\int \chi_V(t) d\nu(t) > \tau|V + C| \ . \tag{2.34}$$

If $t \in V$, $u \in C(= rK)$, then $t + u \in V + C$, hence $\chi_{V+C}(t + u) = 1$, and we get

$$\chi_V(t) \le \frac{1}{|C|} \int \chi_{V+C}(t + u)\chi_C(u) du \qquad (\forall t \in V) \ .$$

If $t \notin V$, this is obvious, as the left hand side vanishes: hence (2.34) implies
(2.35)

$$\tau|V + C| < \int \frac{1}{|C|} \int \chi_{V+C}(t + u)\chi_C(u) du d\nu(t) = \int \chi_{V+C}(y) \frac{1}{|C|} \int \chi_C(y - t) d\nu(t) dy \ .$$

Sinve $C = -C$, the inner function is

$$f(y) := \frac{1}{|C|} \int \chi_C(y - t) d\nu(t) = \frac{\nu(C + y)}{|C|} \ ,$$

and according to (2.35) we have $\tau|V + C| < \int \chi_{V+C}(y) f(y) dy = \int_{V+C} f$, hence for some appropriate point $z \in V + C$ we must have $\tau < f(z)$. That is, $\nu(C + z) > \tau|C|$, and we get by $C := rK$ the estimate

$$\nu(rK + z) > \tau|rK|. \tag{2.36}$$

Since r was arbitrary, it follows that $\overline{D}_K(\nu) \ge \tau$, and applying this to all $\tau < \overline{D}(\nu; |\cdot|)$ the statement follows.

2.3.5. Extension of the propositions of Erdős-Sárközy, of Hegyvári and of Fürstenberg.

THEOREM 2.3.12. *If G is a LCA group and $A \subset G$ has $\overline{\Delta}(A) > 0$, then there exists a finite subset $B \subset G$ so that $A - A + B = G$. Moreover, we can find B with $\#B \le [1/\overline{\Delta}(A)]$.*

REMARK 2.3.13. We need a translation-invariant (Haar) measure, but not the topology or compactness.

PROOF. Assume that $H \subset G$ satisfies $(A - A) \cap (H - H) = \{0\}$ and let $L = \{b_1, b_2, \ldots, b_k\}$ be any finite subset of H. By condition, we have $(A + b_i) \cap (A + b_j) = \emptyset$ for all $1 \le i < j \le k$. Take now $C := L$ in the definition of density (2.30) and take $0 < \tau < \rho := \overline{\Delta}(A)$. By Definition 2.3.8 of the density $\overline{\Delta}(A)$, there are $x \in G$ and $V \subset G$ open with compact closure – or, a $V \in \mathcal{S}$ with $0 < |V| < \infty$ – satisfying

$$|A \cap (V + x)| > \tau|V + L| \ . \tag{2.37}$$

On the other hand

$$V + L = \bigcup_{j=1}^{k} (V + x + (b_j - x)) \supset \bigcup_{j=1}^{k} (((V + x) \cap A) + b_j) - x \tag{2.38}$$

and as $A + b_j$ (thus also $((V + x) \cap A) + b_j$) are disjoint, and the Haar measure is translation invariant, we are led to

$$|V + L| \ge k|(V + x) \cap A| \ . \tag{2.39}$$

Comparing (2.37) and (2.39) we obtain

$$|A \cap (V + x)| > \tau k |(V + x) \cap A| \qquad \text{and also} \qquad |V + L| > k\tau |V + L| \,, \tag{2.40}$$

hence after cancellation by $|V + L| > 0$ we get $k < 1/\tau$ and so in the limit $k \le K := [1/\rho]$. It follows that H is necessarily finite and $\#H \le K$.

So let now $B = \{b_1, b_2, \ldots, b_k\}$ be any set with the property $(A - A) \cap (B - B) = \{0\}$ (which implies $\#B \le K$) and maximal in the sense that for no $b' \in G \setminus B$ can this property be kept for $B' := B \cup \{b'\}$. In other words, for any $b' \in G \setminus B$ it holds that $(A - A) \cap (B' - B') \ne \{0\}$.

Clearly, if $A - A = G$ then any one point set $B := \{b\}$ is such a maximal set; and if $A - A \ne G$, then a greedy algorithm leads to one in $\le K$ steps.

Now we can prove $A - A + B = G$. Indeed, if there exists $y \in G \setminus (A - A + B)$, then $(y - b_j) \notin A - A$ for $j = 1, \ldots, k$, hence $B' := B \cup \{y\}$ would be a set satisfying $(B' - B') \cap (A - A) = \{0\}$, contradicting maximality of B. $\square$

Corollary 2.3.14. *Let $A \subset \mathbb{R}^d$ be a (measurable) set with $\overline{\Delta}(A) > 0$. Then there exists $b_1, \ldots, b_k$ with $k \le K := [1/\overline{\Delta}(A)]$ so that $\cup_{j=1}^{k} (A - A + b_j) = \mathbb{R}^d$.*

This is interesting as it shows that the difference set of a set of positive Banach density $\overline{\Delta}$ is necessarily rather large: just a few translated copies cover the whole space.

Observe that we have Proposition 2.3.3 as an immediate consequence, since $\mathbb{Z}$ is discrete, and thus the two notions $\overline{\Delta}$ and $\overline{D}$ of Banach densities coincide; moreover, the finite set $B := \{b_1, \ldots, b_K\}$ is a compact set in the discrete topology of $\mathbb{Z}$. But in fact we can as well formulate the following extension.

Corollary 2.3.15. *Let G be a LCA group and $S \subset G$ a set with positive upper Banach density, i.e. $\overline{D}(S) > 0$, where here $\overline{D}(S) = \overline{D}(\mu|_S; \mu)$. Then the difference set $S - S$ is a syndetic set: moreover, the set of translations K, for which we have $G = KS$, can be chosen not only compact, but even to be a finite set with $\#K \le [1/\overline{D}(S)]$ elements.*

This corollary is immediate, because $\overline{\Delta}(S) \ge \overline{D}(S)$ according to Proposition 2.3.9.

This indeed generalizes the proposition of Fürstenberg. Also this result contains the result of Hegyvári: for on σ-finite groups the natural topology is the discrete topology, whence the natural Haar measure is the counting measure, and so on σ-finite groups Corollary 2.3.15 and Theorem 2.3.12 coincides. Finally, this also generalizes and sharpens the Proposition of Erdős and Sárközy. Indeed, on $\mathbb{Z}$ or $\mathbb{N}$ we naturally have $\overline{\Delta}(A) = \overline{D}(A) \ge \overline{d}(A)$, so if the latter is positive, then so is $\overline{D}(A)$; and then the difference set is syndetic, with finitely many translates belonging to a translation set K, say, covering the whole $\mathbb{Z}$. Hence $d_{n+1} - d_n$ is necessarily smaller than the maximal element of the finite set K of translations.

Theorem 2.3.16. *Let G be a LCA group and $S \subset G$ a set with a positive, (but finite) uniform asymptotic upper density, regarding now the counting measure of elements of S in the definition of Banach density, i.e. $\overline{D}(S) = \overline{D}(\#|_S; \mu) > 0$. Then the difference set $S - S$ is a syndetic set.*

REMARK 2.3.17. One would like to say that a density $+\infty$ is "even the better". However, in non-discrete groups this is not the case: such a density can in fact be disastrous. Consider e.g. the set of points $S := \{1/n \ : \ n \in \mathbb{N}\}$ as a subset of $\mathbb{R}$. Clearly for any compact C of positive Haar /i.e. Lebesgue/ measure $|C| > 0$, and for any $V \in \mathcal{B}_0$ of finite measure and compact closure, $|V + C|$ is positive but finite: whence whenever $0 \in \mathrm{int}V$, we automatically have $\#(S \cap V) = \infty$ and also $\#(S \cap V)/|C + V| = \infty$, therefore $\overline{D}(\#|_S; |\cdot|) = \infty$; but $S - S \subset [-2, 2]$ and thus with a compact B it is not possible that $B + S - S$ covers $G = \mathbb{R}$, whence $S - S$ is not syndetic.

PROBLEM 2.3.18. The implicitly occurring set of translations K, for which we have $G = K + (S - S)$, is not controlled in size by the proof below. However, one would like to say that there must be some bound, hopefully even $\mu(K) \leq [1/\overline{D}(S)]$, for an appropriately chosen compact set of translates K. This we cannot prove yet.

PROOF. We are not certain that our argument is the simplest possible: also, it does not give a good estimate for the measure of the required compact set exhibiting the syndetic property of $S - S$. Nevertheless, we consider it worthwhile to present it in full detail, since the various steps, eventually leading to the result, seem to be rather general and useful auxiliary statements, having their own independent interest. Correspondingly, we break the argument in a series of lemmas.

LEMMA 2.3.19. *Let $S \subset G$ and assume $\overline{D}(\#|_S; \mu) = \rho \in (0, \infty)$. Consider any compact set $H \subset G$ satisfying the "packing type condition" $H - H \cap S - S = \{0\}$ with S. Then we necessarily have $\mu(H) \leq 1/\overline{D}(S)$.*

PROOF. Let $0 < \tau < \rho$ be arbitrary. By definition of $\overline{D}(S)$, (using H in place of C) there must exist a measurable set $V \in \mathcal{S} \cap \mathcal{B}_0$, with compact closure so that $\infty > \#(S \cap V) > \tau\mu(V + H)$, therefore also $\#(S \cap V) > \mu((S \cap V) + H)$. However, for any two elements $s \neq s' \in (S \cap V) \subset S$, $(s + H) \cap (s' + H) = \emptyset$, since in case $g \in (s + H) \cap (s' + H)$ we have $g = s + h = s' + h'$, i.e. $s - s' = h - h'$, which is impossible for $s \neq s'$ and $(H - H) \cap (S - S) = \{0\}$. Therefore for each $s \in (S \cap V)$ there is a translate of H, totally disjoint from all the others: i.e. the union $(S \cap V) + H = \cup_{s \in (S \cap V)}(s + H)$ is a disjoint union. By the properties of the Haar measure, we thus have $\mu((V \cap S) + H) = \sum_{s \in (S \cap V)} \mu(s + H) = \#(V \cap S)\mu(H)$.
Whence we find $\#(S \cap V) \geq \tau\#(S \cap V)\mu(H)$, and, since $\#(S \cap V) > \tau\mu(V + H)$ was positive, we can cancel with it and infer $\mu(H) < 1/\tau$. This holding for all $\tau < \rho = \overline{D}(S)$, we obtained that any compact set H, satisfying the packing type condition with S, is necessarily bounded in measure by $1/\overline{D}(S)$. □

LEMMA 2.3.20. *Suppose that $S - S \cap H - H = \{0\}$ with $\overline{D}(\#|_S; \mu) = \rho \in (0, \infty)$ and $H \Subset G$ with $0 < \mu(H - H)$. Then the set $A := S + (H - H)$ has the uniform asymptotic upper density $\overline{D}(\mu|_A; \mu)$, with respect to the Haar measure (restricted to A), not less than $\rho \cdot \mu(H - H)$.*

PROOF. Let $C \Subset G$ be arbitrary and denote $Q := H - H$. We want to estimate from below the ratio $\mu(A \cap V)/\mu(C + V)$ for an appropriately chosen $V \in \mathcal{B}_0$. Let us fix that we will take for V some set of the form $U + Q$ with $U \in \mathcal{B}_0$. Clearly $A \cap V = (S+Q) \cap (U+Q) \supset (S \cap U) + Q$. Now for any two elements $s \neq t \in S$, thus even more for $s, t \in (S \cap V)$, the sets $s + Q$ and $t + Q$ are disjoint, this being an easy consequence of the packing property because $s + q = t + q' \Leftrightarrow s - t = q - r$, which is impossible for $s - t \neq 0$ by condition. Therefore by the properties of the Haar measure we get $\mu((S \cap U) + Q) = \sum_{s \in (S \cap U)} \mu(s + Q) = \#(S \cap U) \cdot \mu(Q)$. In all, we found $\mu(A \cap V) \geq \#(S \cap U) \cdot \mu(Q)$.

It remains to choose V, that is, U, appropriately. For the compact set $C + Q \Subset G$ and for any given small $\varepsilon > 0$, by definition of $\overline{D}(\#|_S; \mu) = \rho$ there exists some $U \in \mathcal{B}_0$ such that $\#(S \cap U) > (\rho - \varepsilon)\mu((C + Q) + U)$. Choosing this particular U and combining the two inequalities we are led to $\mu(A \cap V) \geq (\rho - \varepsilon)\mu(C + Q + U)\mu(Q)$, that is, for $V := U + Q$ written in $\mu(A \cap V)/\mu(C + V) \geq (\rho - \varepsilon)\mu(H - H)$.

As we find such a V for every positive ε, the sup over $V \in \mathcal{B}_0$ is at least $\rho\mu(H - H)$, and because $C \Subset G$ was arbitrary, we infer the assertion. □

LEMMA 2.3.21. *Suppose that* $S - S \cap H - H = \{0\}$ *with* $\overline{D}(\#|_S; \mu) = \rho \in (0, \infty)$ *and* $H \Subset G$ *with* $0 < \mu(H - H)$. *Then there exists a finite set* $B = \{b_1, \ldots, b_k\} \subset G$ *of at most* $k \leq [1/(\rho\mu(H - H))]$ *elements so that* $B + (H - H) - (H - H) + (S - S) = G$. *In particular, the set* $S - S$ *is syndetic with the compact set of translates* $B + (H - H + H - H)$.

PROOF. By the above Lemma 2.3.20 we have an estimate on the density of $A := S + (H - H)$ with respect to Haar measure. But then we may apply Corollary 2.3.15 to see that the difference set $S + (H - H) - (S + (H - H))$ is a syndetic set with the set of translates B admitting $\#B \leq [1/\overline{D}(\mu|_A; \mu)] \leq [1/(\rho\mu(H - H))]$. Because also the set H is compact, this yields that S is syndetic as well, with set of translations being $B + (H - H) + (H - H)$. □

One may think that it is not difficult, for a discrete set S of finite density with respect to counting measure, to find a compact neighborhood R of 0, so that $R \cap (S - S)$ be almost empty with 0 being its only element. If so, then by continuity of subtraction, also for some compact neighborhood H of zero with $(H - H) \subset R$ (and, being a neighborhood, with $\mu(H) > 0$, too) we would have $(H - H) \cap (S - S) = \{0\}$, the packing type condition, whence concluding the proof of Theorem 2.3.16.

Unfortunately this idea turns to be naive. Consider the sequence $S = \{n + 1/n \ : \ n \in \mathbb{N}\} \cup \mathbb{N}$ (in $\mathbb{R}$), which has uniform asymptotic upper density 2 with the cardinality measure, whilst $S - S$ is accumulating at 0.

Nevertheless, this example is instructive. What we will find, is that sets of *finite* positive uniform asymptotic upper density cannot have a too dense difference set: it always splits into a fixed, bounded number of disjoint subsets so that the difference set of each subset already leaves out a fixed compact neighborhood of 0. This will be the substitute for the above naive approach to finish our proof of Theorem 2.3.16 through proving also some kind of subadditivity of the uniform asymptotic upper density – another auxiliary statement interesting for its own right.

LEMMA 2.3.22. *Let* $Q \Subset G$ *be any symmetric compact neighborhood of 0 and let* S *have positive but finite uniform asymptotic upper density with respect to cardinality measure,* *i.e.* $\overline{D}(\#|_S;\mu) = \rho \in (0,\infty)$. *Then there exists a finite disjoint partition* $S = \bigcup_{j=1}^n S_j$ *of* S *such that* $(S_j - S_j) \cap Q = \{0\}$. *Moreover, choosing an appropriate symmetric compact neighborhood* Q *of 0, depending on* $\varepsilon > 0$, *we can even guarantee that the number of subsets in the partition is not more than* $k \le [(1+\varepsilon)\rho\mu(Q)]$.

Proof. Let $s \in S$ be arbitrary, consider $R := s + Q$, and let us try to estimate the number of other elements of S falling in R. Clearly for any $C \Subset G$ we have $\#(S \cap R)/\mu(C + R) \le \sup_{V \in \mathcal{B}_0} \#(S \cap V)/\mu(C+V)$ so for any $\varepsilon > 0$ and with some appropriate $C \Subset G$ this is bounded by $\rho + \varepsilon$ according to the density condition. Note that the choice of C depends only on ε, but not on R. That is, we already have a bound $k := \#(S \cap R) \le (\rho+\varepsilon)\mu(C+R)$ with the given $C = C(\varepsilon)$, independently of R, i.e. of Q.

Next we show how to obtain the bound $k \le [\rho\mu(Q)] + 1$ for some appropriate choice of Q. This hinges upon a lemma of Rudin, stating that for any given compact set $C \Subset G$ and $\varepsilon > 0$ there exists another Borel set V, also with compact closure, so that $\mu(C+V) < (1+\varepsilon)\mu(V)$, c.f. 2.6.7 Theorem on page 52 of [**77**]; moreover, Rudin remarks that this can even be proved (actually, read out from the proof) with open sets V having compact closure. It is a matter of invariance of Haar measure with respect to translations to ascertain that (some) of the interior points of V be 0, so that V is a neighborhood of 0: also, by regularity of the Borel measure, and by compactness of the closure, we can as well take V to be its own closure. Furthermore, the same proof also shows that V can even be taken symmetric. In all, *for an appropriate choice of* V *for* Q, we even have $k := \#(S \cap R) \le (\rho+\varepsilon)\mu(C+R) < (\rho+\varepsilon)(1+\varepsilon)\mu(Q)$. Note that here the dependence on C disappears from the end formula, but there is a dependence of Q on ε. This is equivalent to the estimate in the Lemma.

It remains to construct the partition once we have a compact neighborhood Q of 0 and a finite number $k \in \mathbb{N}$ such that $\#(S \cap (Q+s) \le k$ for all $s \in S$. this is standard argument. Consider a graph on the points of S defined by connecting two points s and t exactly when $t \in s + Q$. Since Q is symmetric, this is indeed a good definition for a graph (and not for a directed graph only).

In this graph by condition the degree of any point of $s \in S$ is at most $k-1$: there are at most $k-1$ further points of S in $s+Q$. But it is well-known that such a graph can be partitioned into k subgraphs with no edges within any of the induced subgraphs.[2] That is, the set of points split into the disjoint union of some S_j with no two points $s,t \in Q$ being in the relation $t \in s+Q$, defining an edge between them.

It is easy to see that now we constructed the required partition: the S_j are disjoint, and so are $(S_j - S_j)$ and $Q \setminus \{0\}$, for any $j = 1,\ldots,k$, too. This concludes the proof. □

Lemma 2.3.23 (subadditivity). *Let* $\nu_0 = \sum_{j=1}^n \nu_j$ *be a sum of measures, all on the common set algebra* $\mathcal{S}$ *of measurable sets. Then we have* $\overline{D}(\nu_0,\mu) \le \sum_{j=1}^n \overline{D}(\nu_j,\mu)$. *In*

[2] The proof of this is very easy for finite or countable graphs: just start to put the points, one by one, inductively into k preassigned sets S_j so that each point is put in a set where no neighbor of it stays; since each point has less than k neighbors, this simple greedy algorithm can not be blocked and the points all find a place. Same for countable many points, while for larger cardinalities transfinite induction is needed to carry out the same reasoning.

particular, this holds for one given measure ν *and a disjoint union of sets* $A_0 = \cup_{j=1}^{n} A_j$*, with* $\nu_j := \nu|_{A_j}$*, for* $j = 0, 1, \dots, k$*.*

Proof. Uniform asymptotic upper density is clearly monotone in the sets considered, therefore all S_j have a density $0 \leq \rho_j \leq \rho < \infty$ Let $\varepsilon > 0$ be arbitrary, and take $C_j \Subset G$ so that for all $V \in \mathcal{B}_0$ in the definition of $\overline{D}(\nu|_A; \mu)$ we have $\nu_j(V) \leq (\rho_j + \varepsilon)\mu(C_j + V)$. Such C_j exists in view of the infinum on $C \Subset G$ in the definition of u.a.u.d.

Consider the (still) compact set $C := C_1 + \cdots + C_n$. By definition of u.a.u.d. there is $V \in \mathcal{B}_0$ such that $\nu(V) \geq (\rho - \varepsilon)\mu(C + V)$. Obviously, $\mu(C_j + V) \leq \mu(C + V)$, so on combining these we obtain

$$\rho - \varepsilon \leq \frac{\nu(V)}{\mu(C+V)} = \frac{\sum_{j=1}^{k} \nu_j(V)}{\mu(C+V)} \leq \sum_{j=1}^{k} \frac{\nu_j(V)}{\mu(C_j+V)} \leq \sum_{j=1}^{k} (\rho_j + \varepsilon),$$

that is, $\rho - \varepsilon \leq \sum_j (\rho_j + \varepsilon)$ holding for all ε, we find $\rho \leq \sum_j \rho_j$, as was to be proved. □

Continuation of the proof of Theorem 2.3.16. We take now an *arbitrary* compact neighborhood $H \Subset G$ of 0, with of course $\mu(H) > 0$, and also $Q := H - H$ again with $0 < \mu(Q) < \infty$ and Q a symmetric neighborhood of 0. By Lemma 2.3.22 there exists a finite disjoint partition $S = \cup_{j=1}^{n} S_j$ with $(S_j - S_j) \cap (H - H) = \{0\}$. By subadditivity of u.a.u.d. (that is, Lemma 2.3.23 above), at least one of these S_j must have positive u.a.u.d. ρ_j (with respect to the counting measure), namely of density $0 < \rho/n \leq \rho_j \leq \rho < \infty$, with $\rho := \overline{D}(\#|_S, \mu)$.

Selecting such an S_j, we can apply Lemma 2.3.21 to infer that already S_j – hence also $S \supset S_j$ – is syndetic. □

2.4. Structural properties of sets – tiling, packing and spectrality

2.4.1. Tiling and packing.

Suppose G is a LCA group.

Definition 2.4.1. We say that a nonnegative function $f \in L^1(G)$ tiles G by translation with a set $\Lambda \subseteq G$ at level $c \in \mathbb{C}$ if

$$\sum_{\lambda \in \Lambda} f(x - \lambda) = c$$

for a.a. $x \in G$, with the sum converging absolutely. We then write "$f + \Lambda = cG$".

We say that f *packs* G with the translation set Λ at level $c \in \mathbb{R}$, and write $f + \Lambda \leq cG$, if

$$\sum_{\lambda \in \Lambda} f(x - \lambda) \leq c,$$

for a.a. $x \in G$. When the same properties hold with constant $c = 1$ for a characteristic function χ_Ω of some $\Omega \in \mathcal{B}_0$, then we simply say that Ω tiles or packs G, and write $\Omega + \Lambda = G$, $\Omega + \Lambda \leq G$, respectively.

Neglecting some measure zero sets, packing occurs when for any point $x \in G$ $x - \lambda \in \Omega$ for at most one point of $\lambda \in \Lambda$, which in turn is equivalent to $\lambda + \Omega$ being disjoint for different $\lambda \in \Lambda$. This explains the term "packing". On the other hand this latter statement is equivalent to saying that $\lambda + x = \lambda' + x'$ with $\lambda, \lambda' \in \Lambda$ and $x, x' \in \Omega$ can occur only if $\lambda = \lambda'$ and hence also $x = x'$. Writing this in the form of differences, $\lambda - \lambda' = x' - x$ only for both sides being 0, that is, $(\Lambda - \Lambda) \cap (\Omega - \Omega) = \{0\}$. This is an equivalent condition to Ω packing with Λ. More generally, we will say that the set S satisfies a "packing type condition" with L, if $(L - L) \cap S \subset \{0\}$, irrespectively of the situation whether S can be represented as a difference set of some other Ω or not.

So in an Euclidean space about a nonnegative $f \in L^1(\mathbb{R}^d)$ we say that f tiles with Λ at level ℓ if

$$\sum_{\lambda \in \Lambda} f(x - \lambda) = \ell, \quad \text{a.e. } x.$$

We denote this latter condition by $f + \Lambda = \ell \mathbb{R}^d$.

In particular, a measurable set $\Omega \subseteq \mathbb{R}^d$ is a *translational tile* if there exists a set $\Lambda \subseteq \mathbb{R}^d$ such that almost all (Lebesgue) points in $\mathbb{R}^d$ belong to exactly one of the translates

$$\Omega + \lambda, \quad \lambda \in \Lambda.$$

We denote this condition by $\Omega + \Lambda = \mathbb{R}^d$.

In any tiling the translation set has some properties of density, which hold uniformly in space.

Definition 2.4.2. A set $\Lambda \subseteq \mathbb{R}^d$ has (uniform) density ρ if

$$\lim_{R \to \infty} \frac{\#(\Lambda \cap B_R(x))}{|B_R(x)|} \to \rho$$

uniformly in $x \in \mathbb{R}^d$. We write $\rho = \operatorname{dens} \Lambda$.

We say that Λ has density uniformly bounded by ρ, if the fraction above is bounded by the constant ρ uniformly for $x \in \mathbb{R}$ and $R > 1$.

Remark 2.4.3. It is not hard to prove (see for example [**51**], Lemma 2.3, where it is proved in dimension one – the proof extends verbatim to higher dimension) that in any tiling $f + \Lambda = \ell \mathbb{R}^d$ the set Λ has density $\ell / \int f$.

When the group is finite (and we do not, therefore, have to worry about the set Λ being finite or not) the tiling condition $f + \Lambda = cG$ means precisely $f * \chi_\Lambda = c$. Taking Fourier transform, this is the same as $\widehat{f}\widehat{\chi_\Lambda} = c|G|\chi_{\{0\}}$, which is in turn equivalent to the condition

$$\operatorname{supp} \widehat{\chi_\Lambda} \subseteq \{0\} \cup \left\{ \widehat{f} = 0 \right\} \quad \text{and} \quad c = \frac{|\Lambda|}{|G|} \sum_{x \in G} f(x). \tag{2.41}$$

The packing type condition $\Omega \cap (\Lambda - \Lambda) = \{0\}$ will be used in Theorem 2.6.4 below. This result will be an essential extension of the earlier result of Arestov and Berdysheva, stating that in $\mathbb{R}^d$ a convex lattice tile is necessarily of the Stechkin-Turán type. Another generalization of this result appears in Corollary 2.7.3, through another structural property of sets, namely spectrality.

2.4.2. Spectra.

DEFINITION 2.4.4. Let G be a LCA group and $\widehat{G}$ be its dual group, that is the group of all continuous group homomorphisms (characters) $G \to \mathbb{C}$. We say that the set $T \subseteq \widehat{G}$ is a *spectrum* of $H \subseteq G$ if and only if T forms an orthogonal basis for $L^2(H)$.

In particular, let Ω be a measurable subset of $\mathbb{R}^d$ and Λ be a discrete subset of $\mathbb{R}^d$. We write

$$\begin{aligned} e_\lambda(x) &= \exp(2\pi i\langle \lambda, x\rangle), \quad (x \in \mathbb{R}^d), \\ E_\Lambda &= \{e_\lambda : \lambda \in \Lambda\} \subset L^2(\Omega). \end{aligned}$$

The inner product and norm on $L^2(\Omega)$ are

$$\langle f, g\rangle_\Omega = \int_\Omega f\overline{g}, \quad \text{and} \quad \|f\|_\Omega^2 = \int_\Omega |f|^2.$$

The pair (Ω, Λ) is called a *spectral pair* if E_Λ is an orthogonal basis for $L^2(\Omega)$. A set Ω will be called *spectral* if there is $\Lambda \subset \mathbb{R}^d$ such that (Ω, Λ) is a spectral pair. The set Λ is then called a *spectrum* of Ω.

EXAMPLE 2.4.5. If $Q_d = (-1/2, 1/2)^d$ is the cube of unit volume in $\mathbb{R}^d$ then $(Q_d, \mathbb{Z}^d)$ is a spectral pair, as is well known by the ordinary L^2 theory of multiple Fourier series.

Bent Fuglede formulated the following famous conjecture in 1974.

CONJECTURE 2.4.6. (**Fuglede** [**26**]) Let $\Omega \subset \mathbb{R}^d$ be a bounded open set. Then Ω is spectral if and only if there exists $L \subset \mathbb{R}^d$ such that $\Omega + L = \mathbb{R}^d$ is a tiling.

One basis for the conjecture was that the lattice case of this conjecture is easy to show, (see for example [**26, 49**]). In the following result the dual lattice Λ^* of a lattice Λ is defined as usual by

$$\Lambda^* = \Big\{x \in \mathbb{R}^d : \ \forall \lambda \in \Lambda \ \ \langle x, \lambda\rangle \in \mathbb{Z}\Big\}.$$

THEOREM 2.4.7 (Fuglede [**26**]). *The bounded, open domain Ω admits translational tilings by a lattice Λ if and only if E_{Λ^*} is an orthogonal basis for $L^2(\Omega)$.*

Note that in Fuglede's Conjecture no relation is claimed between the translation set L and the spectrum Λ.

The Conjecture in its full generality was recently disproved. First, T. Tao showed [**88**] that in $\mathbb{R}^5$ there exists a spectral set, which however fails to tile space. The method, roughly speaking, is to construct counterexamples on finite groups, and then lift them up first to $\mathbb{Z}^d$ and finally to $\mathbb{R}^d$. Soon after that breakthrough, Tao's construction was further sharpened to provide non-tiling spectral sets in $\mathbb{R}^4$ [**61**] and finally even in dimension 3 [**53**].

More importantly, the converse implication was also disproved, first in dimension 5 by Kolountzakis and Matolcsi [**52**]. Subsequently, examples of tiling, but non-spectral sets were constructed in $\mathbb{R}^4$ by Farkas and Révész [**23**], and then even in $\mathbb{R}^3$ by Farkas, Matolcsi and Móra [**22**].

Positive results are far more meager, and basically restrict to special sets on the real line. However, for *planar convex domains*, it also holds true [**42**].

In these results – in particular in the construction of counterexamples – finite groups played a decisive role. Therefore it indeed makes sense to investigate the analogous problem in groups.

Suppose from now on that G is finite.

It follows that $|T| = |H|$, the dimension of $\ell^2(H)$, and with a little more work it follows that T is a spectrum of H if and only if we have the tiling condition

$$|\widehat{\chi_H}|^2 + T = |H|^2 \widehat{G}. \tag{2.42}$$

Indeed, for $t_1, t_2 \in \widehat{G}$ we have by definition of the Fourier transform that

$$\langle t_1, t_2 \rangle_H = \sum_{x \in H} t_1(x)\overline{t_2}(x) = \sum_{x \in H} (t_1 - t_2)(x) = \widehat{\chi_H}(t_1 - t_2).$$

Suppose now that T is a spectrum of H. If $t \in \widehat{G}$ we have (Parseval)

$$\begin{aligned} |H| &= \|t\|^2_{\ell^2(H)} \\ &= \sum_{s \in T} \left| \left\langle t, \frac{s}{\|s\|} \right\rangle \right|^2 \\ &= \frac{1}{|H|} \sum_{s \in T} |\langle t, s \rangle|^2 \\ &= \frac{1}{|H|} \sum_{s \in T} |\widehat{\chi_H}(t - s)|^2, \end{aligned}$$

which is precisely the statement that $|\widehat{\chi_H}|^2 + T = |H|^2 \widehat{G}$. That this tiling condition is also sufficient to imply that T is a spectrum of H follows similarly (we are not using this direction in this work).

By the analysis of tiling shown in §2.4.1 it follows that this happens if and only if

$$\operatorname{supp} \widehat{\chi_T} \subseteq \{0\} \cup (H - H)^c \quad \text{and} \quad |T| = |H|. \tag{2.43}$$

2.4.3. Packing, covering, tiling and uniform asymptotic upper density.

Proposition 2.4.8. *Assume that $H \in \mathcal{B}$ and that $H + \Lambda \leq G$ (H packs G with $\Lambda \subset G$), i.e. $(H - H) \cap (\Lambda - \Lambda) \subseteq \{0\}$. Then Λ must satisfy $\overline{D}^{\#}(\Lambda) \leq 1/\mu(H)$.*

Proof. Let $B \Subset H$ and $V \in \mathcal{B}_0$ be arbitrary. Denote $L := \Lambda \cap V$. Then $B + V \supset B + L = \cup_{\lambda \in L}(B + \lambda)$, and this union being disjoint (as $(B+\lambda) \cap (B+\lambda') \subset (H+\lambda) \cap (H+\lambda') = \emptyset$ unless $\lambda = \lambda'$), from additivity and translation invariance of the Haar measure we obtain $\mu(B + V) \geq \mu(B + L) = \#L\mu(B)$. This yields $\#L/\mu(B + V) \leq 1/\mu(B)$, therefore $\sup_{V \in \mathcal{B}_0} \#(\Lambda \cap V)/\mu(B+V) \leq 1/\mu(B)$. Approximating $\mu(H)$ by $\mu(B)$ of $B \Subset H$ arbitrarily closely, we thus obtain $\inf_{B \Subset H} \sup_{V \in \mathcal{B}_0} \#(\Lambda \cap V)/\mu(B + V) \leq 1/\mu(H)$. However, $\overline{D}^{\#}(\Lambda)$ is a similar infimum extended to a larger family of compact sets, so it can not be larger, and the assertion follows. ☐

Proposition 2.4.9. *Assume that $H \in \mathcal{B}_0$ and that it covers G with $\Lambda \subset G$ ("$H + \Lambda \geq G$"), i.e. $H + \Lambda$ contains μ-almost all points of G. Then we necessarily have $\overline{D}^{\#}(\Lambda) \geq 1/\mu(H)$.*

Proof. Let $C \Subset G$ be arbitrary, and take $W := \overline{H} - C$, which is again a compact set of G by assumption on H and in view of the continuity of the group operation on G. So the Theorem in §2.6.7. on p. 52 of [**77**] applies to the compact set W and to any given $\varepsilon > 0$, and we find some Borel measurable set $U = U_{\varepsilon,C} \in \mathcal{B}_0$ satisfying $\mu(U - W) < (1+\varepsilon)\mu(U)$.

Consider now $V := V_{\varepsilon,C} := U - H \in \mathcal{B}_0$. Then $\mu(C+V) = \mu(C+U-H) \leq \mu(U-(\overline{H}-C)) = \mu(U-W) < (1+\varepsilon)\mu(U)$. Denote $L := \Lambda \cap V$. Then $L = \{\lambda \in \Lambda \ : \ \exists h \in H,\ \lambda + h \in U\} = \{\lambda \in \Lambda \ : \ (\lambda + H) \cap U \neq \emptyset\}$, and so clearly $U \cap (\Lambda + H) = \cup_{\lambda \in L}(\lambda + H)$, while $U_0 := U \setminus (U \cap (\Lambda + H))$ is of measure zero by assumption on the covering property of H with Λ. So in all $\mu(U) \leq \mu(U_0) + \sum_{\lambda \in L} \mu(\lambda + H) = 0 + \# L\mu(H)$ and $\mu(C+V) < (1+\varepsilon)\# L\mu(H)$.

It follows that with the arbitrarily chosen $C \Subset G$ and for all $\varepsilon > 0$ we have with a certain $V_{\varepsilon,C} \in \mathcal{B}_0$

$$\frac{\#(\Lambda \cap V_{\varepsilon,C})}{\mu(C + V_{\varepsilon,C})} \geq \frac{1}{(1+\varepsilon)\mu(H)},$$

so taking supremum over all $V \in \mathcal{B}_0$ we even get $\sup_{V \in \mathcal{B}_0} \#(\Lambda \cap V)/\mu(C+V) \geq 1/\mu(H)$. This holding for all $C \Subset G$, taking infimum over C does not change the lower estimation, so finally we arrive at $\overline{D}^{\#}(\Lambda) \geq 1/\mu(H)$, whence the proposition. □

Tiling means simultaneously packing and covering. Therefore, from the above two propositions the following corollary obtains immediately.

Corollary 2.4.10. *Assume that $H \in \mathcal{B}_0$ tiles with the set of translations $\Lambda \subset G$: $H + \Lambda = G$. Then we also have $\overline{D}^{\#}(\Lambda) = 1/\mu(H)$.*

2.5. Generalities about Turán constants on groups

Again this section is taken entirely from our joint work with Mihalis Kolountzakis [**56**].

2.5.1. Homomorphic images and the Turán constant. Let G and H be two LCA groups, and $\varphi : G \to H$ a continuous group homomorphism which maps G onto H. Denote $K := \mathrm{Ker}(\varphi) \leq G$. By continuity of φ, K is a closed subgroup, hence a LCA group itself. We consider G/K as fixed together with the canonical or natural projection $\pi : G \to G/K$ defined as $\pi(g) := [g] := g + K \in G/K$. By definition of the topology of G/K, π is an open and continuous mapping. Compare §B.2, B.6 in Appendix B of [**77**]. Moreover, $\varphi \circ \pi^{-1} : G/K \to H$ is an isomorphism of the LCA groups G/K and H.

For the determination of the Turán constants, the choice of the Haar measure is relevant. Haar measures are unique up to a constant factor: we can always choose the Haar measures μ_K and $\mu_{G/K}$ so that $d\mu_G = d\mu_K d\mu_{G/K}$, in the sense of (2) in [**77**, §2.7.3]. On the other hand fixing a particular Haar measure μ_H of H always leaves open the question of compatibility with the fixed measure $\mu_{G/K}$ and the mapping φ. Let $A \subset H$ be an arbitrary Borel set. Then one can define $\nu(A) := \mu_{G/K}(\pi(\varphi^{-1}(A)))$; since φ is onto, clearly this defines another Haar measure on H. Since Haar measures are constant multiples of each other, we necessarily have $C := d\mu_H/d\nu$ a constant. Once H and μ_H are given, various homomorphisms φ may generate different measures, but the constant $C = C(\varphi)$ can always be read from this relation.

PROPOSITION 2.5.1 (**Kolountzakis-Révész**). *Let G and H be LCA groups, and $\varphi : G \to H$ be a continuous group homomorphism onto H. Suppose an open subset $\Omega \subset G$ is given, and let $\Theta := \varphi(\Omega) \subset H$. Consider the closed subgroup $K := \mathrm{Ker}(\varphi) \leq G$, and the quotient group G/K together with their Haar measures $\mu_{G/K}$ and μ_K, normalized as above. We then have*

$$\mathcal{T}_G(\Omega) \leq \frac{1}{C}\mathcal{T}_H(\Theta)\mathcal{T}_K(\Omega \cap K) \qquad (C := \frac{d\mu_H}{d\nu}) \ . \tag{2.44}$$

Here $\nu := \mu_{G/K} \circ \pi \circ \varphi^{-1}$ is defined as above.

PROOF. As K is the kernel of the continuous homomorphism φ, K is a closed subgroup of G. Therefore, the factor group G/K is a LCA group, which is continuously isomorphic to H.

The image Θ of the open set Ω is open, since φ is also an open mapping. Indeed, π is open by its definition, and thus $\pi(\Omega)$ is open in G/K for any open Ω in G. However, the isomorphism $\psi : G/K \to H$, defined by $\psi := \varphi \circ \pi^{-1}$, brings over the open set $\pi(\Omega)$ to Θ, which is then open by the isomorphism itself.

Observe that $\Omega_g := \Omega \cap (K + g)$ is relatively open for any $g \in G$, while the coset $K + g$ is closed. Let us choose arbitrarily a representative $g(h) \in G$ of each coset $\varphi^{-1}(h)$ of K to all $h \in H$. Now for any uniformly continuous function $f : G \to \mathbb{C}$ we can define

$$F(h) := \int_K f(g(h) + k)\ d\mu_K(k) = \int_K f(x)\ d\mu_K(x - g(h)). \tag{2.45}$$

Since f is uniformly continuous, the function $F : H \to \mathbb{C}$ is continuous, $F(0) = \int_K f d\mu_K$, and by Fubini's Theorem

$$\begin{aligned}\int_H F(h)d\mu_H(h) &= \int_H \int_{\varphi^{-1}(h)} f(g(h) + k)\ d\mu_K(k)C d\nu(h)\\ &= C\int_H \int_K f(g(h) + k)\ d\mu_K(k)d\mu_{G/K}(\pi\varphi^{-1}(h))\\ &= C\int_{H\times K} f(g(h) + k)\ d\mu_K(k)d\mu_{G/K}([g(h)]) = C\int_G f\ d\mu_G\,,\end{aligned} \tag{2.46}$$

taking into account the choice of normalization of the Haar measures for K and G/K. Next we prove that F is positive definite on H in case f is positive definite on G. Indeed, for any character χ on H there is a character $\gamma := \chi \circ \varphi$ on G, and applying (2.46) to $f\gamma$ yields

$$\int_H F(h)\chi(h)d\mu_H(h) = C\int_G f(g)\gamma(g) \geq 0\ .$$

Thus we have $\int_H F d\mu_H \leq \mathcal{T}_H(\Theta)F(0)$. Moreover, $f|_K$ is positive definite on K, hence we also have $F(0) = \int_{K\cap\Omega} f\ d\mu_K \leq \mathcal{T}_K(K \cap \Omega)f(0)$. Comparing these inequalities with (2.46) yields $C\int_G f\ d\mu_G \leq \mathcal{T}_H(\Theta)\mathcal{T}_K(K \cap \Omega)f(0)$, and taking supremum of $\int_G f d\mu_G/f(0)$ (2.44) obtains.

□

2.5.2. Automorphic invariance of the Turán constant. One of the reasons to work out Proposition 2.5.1 is its corollary to the case when we deal with an automorphism of the group G.

COROLLARY 2.5.2 (**Kolountzakis-Révész**). *Let G be a LCA group and let $\varphi : G \to G$ be an automorphism. Then we have for any open set $\Omega \subset G$ the identity*

$$\mathcal{T}_G(\varphi(\Omega)) = \frac{|\varphi(\Omega)|}{|\Omega|}\mathcal{T}_G(\Omega) \ . \tag{2.47}$$

PROOF. In our case $H = G$ and φ is an automorphism. Clearly then $K = \{0\}$ is the trivial group, $\mu_K = \delta_0$ is the trivial measure, $K \cap \Omega = \{0\}$, $\mathcal{T}_K(K \cap \Omega) = 1$, $\mu_K(K \cap \Omega) = 1$ and $G/K \cong G$, $\mu_{G/K} \cong \mu_G$. Thus we find $\nu = \mu_G \circ \varphi^{-1}$, and $C := d\mu_H/d\nu$ being constant, it can be computed on $\Omega^* := \varphi(\Omega)$ as $C = |\Omega^*|/|\varphi^{-1}(\Omega^*)| = |\Omega^*|/|\Omega|$. Applying Proposition 2.5.1 yields (2.47) with $\geq$ first. However, φ^{-1} is also an automorphism, and that implies the reverse inequality, too. Whence Corollary 2.5.2 follows. □

The important special case when $G = \mathbb{R}^d$ and φ is any linear mapping $A : \mathbb{R}^d \to \mathbb{R}^d$ was already noted in [**6**]. There the computation of the constant C is equivalent to the calculation of the volume element, ie. the determinant, of the linear mapping A.

The next assertion was also observed in [**6**] for $\mathbb{R}^d$.

COROLLARY 2.5.3 (**Kolountzakis-Révész**). *Let $G = G_1 \times \cdots \times G_n$ and $\Omega_j \subset G_j$ ($j = 1, \ldots, n$), $\Omega = \Omega_1 \times \cdots \times \Omega_n$. Then we have*

$$\mathcal{T}_G(\Omega) = \mathcal{T}_{G_1}(\Omega_1) \cdots \mathcal{T}_{G_n}(\Omega_n) \ . \tag{2.48}$$

PROOF. The $\leq$ direction easily follows from iteration of Proposition 2.5.1. On the other hand take any continuous positive definite functions f_j on G_j with $\operatorname{supp} f_j \Subset \Omega_j$ for ($j = 1, \ldots, n$). It is easy to see that then the product $f := f_1 \cdots f_n$ is a positive definite function on G, with $\operatorname{supp} f \Subset \Omega$, hence also the $\geq$ part of (2.48) follows. □

2.5.3. Turán constants on quotient groups.

COROLLARY 2.5.4 (**Kolountzakis-Révész**). *Let G be a LCA group, K a closed subgroup of G, and suppose that the Haar measures μ_K and $\mu_{G/K}$ of G and G/K, respectively, are normalized (as always) so that $d\mu_G = d\mu_K d\mu_{G/K}$. Let Ω be any open set in G and Θ be its projection on G/K, ie. $\Theta := \{g + K \, : \, g \in \Omega\}$. Then we have*

$$\mathcal{T}_G(\Omega) \leq \mathcal{T}_{G/K}(\Theta)\mathcal{T}_K(\Omega \cap K) \ . \tag{2.49}$$

In particular, if $\Omega \cap K = \{0\}$, then $\mathcal{T}_G(\Omega) \leq \mathcal{T}_{G/K}(\Theta)$.

PROOF. Consider $H := G/K$ and the natural projection $\pi : G \to G/K$. It is a continuous group homomorphism and thus Proposition 2.5.1 can be applied with $\varphi := \pi$. In this case $\Theta = \pi(\Omega)$ comprises the class of cosets $K + g$ so that $K + g \cap \Omega \neq \emptyset$, and the arising measure ν is identical to $\mu_{G/K}$. Hence $C = 1$ and we are led to (2.49). The special case is obvious. □

2.5.4. Restrictions to subgroups and the Turán constants. We show now that there is some sort of monotonicity in the first argument of $\mathcal{T}_G(\Omega)$ as well.

Corollary 2.5.5 (**Kolountzakis-Révész**). *Let G be a compact abelian group, and K a closed subgroup of G. Let the Haar measures μ_K and μ_G be normalized arbitrarily, and let Ω be any open set in G. Then we have*

$$\mathcal{T}_G(\Omega) \le \frac{|G|}{|K|} \mathcal{T}_K(\Omega \cap K) \ . \tag{2.50}$$

Here $|G| = \mu_G(G)$ and $|K| = \mu_K(K)$.

Proof. With μ_G and μ_K already given, we can define the Haar measure $\mu_{G/K}$ so that the condition $\mu_G = \mu_K \mu_{G/K}$ still holds. Let $\varphi := \pi$ and $H := G/K$ as in the previous Corollary. Since we always have $\Theta \subset G/K$, and thus $\mathcal{T}_{G/K}(\Theta) \le \mathcal{T}_{G/K}(G/K) = \mu_{G/K}(G/K)$, an application of Corollary 2.5.4 yields $\mathcal{T}_G(\Omega) \le \mu_{G/K}(G/K)\mathcal{T}_K(\Omega \cap K)$. It remains to see that for a *compact* group G and (closed, hence compact) subgroup K also the quotient is compact, and according to our choice of normalization we have $\mu_{G/K}(G/K) = \mu_G(G)/\mu_K(K)$. The assertion follows. $\square$

Example 2.5.6. Let us remark here that Lemma 1 of [**33**] can be proved via Corollary 2.5.5 by taking $G = \mathbb{T}$, Ω to be the interval $(-p/q, p/q) \subseteq \mathbb{T}$ (for some co-prime integers p and q with $p/q \le 1/2$) and K to be the (finite) subgroup of $\mathbb{T}$ generated by $1/q$. The results in [**33**] first show that the Turán problem in this case can be reduced to a finite problem of linear programming (this is obviously the case for any Turán problem on a finite group) and Corollary 2.5.5 shows half the reduction. The reverse inequality is also true in this particular case (this can be shown by “convolving” a positive definite function on the subgroup with a Fejér kernel of half-base $1/q$) but it cannot be expected to hold in general.

2.6. Upper bound from packing

2.6.1. Bounds from packing in some special cases. In the type of results we now present, some kind of “packing” condition is assumed on Ω which leads to an upper bound for $\mathcal{T}_G(\Omega)$. The first result we present here is taken from [**56**]: we repeat it here for sake of a simpler situation which nevertheless may shed light on the general case. The second result, valid in some non-compact cases, will be detailed, too, for a part of its proof will be directly referred to later in the general version.

Theorem 2.6.1 (**Kolountzakis-Révész**). *Suppose that G is a compact abelian group, $\Lambda \subseteq G$, $\Omega \subseteq G$ is a 0-symmetric open set and $(\Lambda - \Lambda) \cap \Omega \subseteq \{0\}$. Suppose also that $f \in L^1(G)$ is a continuous positive definite function supported on Ω. Then*

$$\int_G f(x)\, dx \le \frac{\mu(G)}{\#\Lambda} f(0). \tag{2.51}$$

In other words $\mathcal{T}_G(\Omega) \le \mu(G)/\#\Lambda$.

(Observe that the conditions imply that Λ is finite.)

PROOF. Define $F : G \to \mathbb{C}$ by

$$F(x) = \sum_{\lambda,\mu\in\Lambda} f(x + \lambda - \mu).$$

In other words $F = f * \delta_\Lambda * \delta_{-\Lambda}$, where δ_A denotes the finite measure on G that assigns a unit mass to each point of the finite set A. It follows that $\widehat{F} = \widehat{f}\left|\widehat{\delta_\Lambda}\right|^2 \geq 0$ so that F is continuous and positive definite. Moreover, we also have

$$\operatorname{supp} F \subseteq \operatorname{supp} f + (\Lambda - \Lambda) \subseteq \Omega + (\Lambda - \Lambda) \tag{2.52}$$

and

$$F(0) = \#\Lambda f(0), \tag{2.53}$$

since $\Omega \cap (\Lambda - \Lambda) \subseteq \{0\}$. Finally

$$\int_G F = \#\Lambda^2 \int_G f. \tag{2.54}$$

Applying the trivial upper bound $\int_G F \leq F(0)\mu(\Omega + (\Lambda - \Lambda))$ to the positive definite function F and using (2.53) and (2.54) we get

$$\int_G f \leq \frac{\mu(\Omega + (\Lambda - \Lambda))}{\#\Lambda} f(0). \tag{2.55}$$

Estimating trivially $\mu(\Omega + (\Lambda - \Lambda))$ from above by $\mu(G)$ we obtain the required $\mathcal{T}_G(\Omega) \leq \mu(G)/\#\Lambda$. □

COROLLARY 2.6.2 (**Kolountzakis-Révész**). *Let G be a compact abelian group and suppose $\Omega, H, \Lambda \subseteq G$, $H + \Lambda \leq G$ is a packing at level 1, that $\Omega \subseteq H - H$ and that $f \in \mathcal{F}(\Omega)$. Then (2.51) holds.*

In particular, if $H + \Lambda = G$ is a tiling, we have

$$\mathcal{T}_G(\Omega) \leq \mu(H). \tag{2.56}$$

PROOF. Since $H + \Lambda \leq G$ it follows that $(H - H) \cap (\Lambda - \Lambda) = \{0\}$. Since $\Omega \subseteq H - H$ by assumption it follows that Ω and $\Lambda - \Lambda$ have at most 0 in common. Theorem 2.6.1 therefore applies and gives the result. If $H + \Lambda = G$ then $\mu(G)/\#\Lambda = |H|$ and this proves (2.56). □

A partial extension of the result to the non-compact case was also worked out in [**56**]. However, it used the notion of u.a.u.d. which then restricted considerations to classical groups only. Nevertheless note that some parts of the proof for this theorem will be used even in the proof for our more general result, see the end of Lemma 2.6.5.

THEOREM 2.6.3 (**Kolountzakis-Révész**). *Suppose that G is one of the groups $\mathbb{R}^d$ or $\mathbb{Z}^d$, that $\Lambda \subseteq G$ is a set of uniform asymptotic upper density $\rho > 0$, and $\Omega \subseteq G$ is a 0-symmetric open set such that $\Omega \cap (\Lambda - \Lambda) \subseteq \{0\}$. Let also $f \in L^1(G)$ be a continuous positive definite function on G whose support is a compact set contained in Ω. Then*

$$\int_G f(x)\, dx \leq \frac{1}{\rho} f(0). \tag{2.57}$$

In other words $\mathcal{T}_G(\Omega) \leq 1/\rho$.

Proof. Let $\epsilon > 0$ and choose $R > 0$ and $x \in G$ such that

$$|\Lambda \cap Q_R(x)| \geq (\rho - \epsilon)|Q_R(x)| \geq (\rho - \epsilon)(R-1)^d,$$

where $Q_R(x)$ is the cube of side R and center at x. Assume also that $\operatorname{supp} f \subseteq Q_r(0)$.

Let $\Lambda' = \Lambda \cap Q_R(x)$ and construct the function F as in the proof of Theorem 2.6.1, with Λ' in place of Λ. We now have that

$$\operatorname{supp} F \subseteq \operatorname{supp} f + (\Lambda' - \Lambda') \subseteq Q_{2R+r}(0).$$

This time we do not apply the trivial upper estimate to F as we did in Theorem 2.6.1 (then, we had no detailed information on the support). Instead we use that for $L \in 2\mathbb{N}$

$$\mathcal{T}_G(Q_L(0)) \leq (L/2+1)^d \ . \tag{2.58}$$

The validity of $\mathcal{T}_{\mathbb{R}^d}(Q_L(0)) \leq 2^{-d}L^d$ ($\forall L > 0$) and hence (2.58) in the case of $G = \mathbb{R}^d$ has been proved, for example, in [**6, 7, 54**]. For $G = \mathbb{Z}^d$ we give a proof here.

Notice first that for any finite $\Omega \subseteq \mathbb{Z}^d$ and any large enough positive integer M we have

$$\mathcal{T}_{\mathbb{Z}^d}(\Omega) \leq \mathcal{T}_{\mathbb{Z}^d_M}(\Omega). \tag{2.59}$$

Indeed, if M is large enough (e.g. $M > \operatorname{diam}(\Omega)/2$) then the closed subgroup $K := M\mathbb{Z}^d$ only intersects Ω in 0, while the factor group $\mathbb{Z}^d_M$ will have an injective image Θ of Ω: hence Corollary 2.5.4 applies.

If $\Omega = Q^d_L(0) = \{-L/2, \ldots, L/2\}^d$ define H to be the set $\{0, \ldots, L/2\}^d$ such that $\Omega = H - H$. Take now $M = 10(L/2+1)$, for example, so that (a) H tiles $\mathbb{Z}^d_M$ by translation, and, (b) M is large enough to have all elements of Ω distinct mod $\mathbb{Z}^d_M$. Using Corollary 2.6.2 we obtain (2.58) from (2.56) in the group $\mathbb{Z}^d_M$, and hence also in $\mathbb{Z}^d$ because of (2.59).

Hence taking $L := L(R, r)$ in (2.58) as the least even integer not less than $2R + r$, we obtain both for $G = \mathbb{R}^d$ and $G = \mathbb{Z}^d$ the estimate $\int_G F \leq \mathcal{T}_G(Q_L(0))F(0) \leq (R + r/2 + 2)^d F(0)$. Comparing this with (2.53) and (2.54) (with Λ' in place of Λ) we are led to

$$\int_G f \leq f(0)\frac{(R + r/2 + 2)^d}{|\Lambda'|} \leq f(0)\frac{(R + r/2 + 2)^d}{(\rho - \epsilon)(R-1)^d} \ .$$

Since $\epsilon > 0$ can be taken arbitrarily small and R arbitrarily large, we get $\int_G f \leq \frac{1}{\rho} f(0)$. $\square$

2.6.2. Bounds from packing in general LCA groups. Now we have ready a notion of u.a.u.d. as defined in §2.3. With this notion, we have the following general version of the above particular results.

Theorem 2.6.4. *Let $\Omega \subset G$ be a 0-symmetric open neighborhood of 0 and $\Lambda \subset G$ be a subset satisfying the "packing-type condition" $\Omega \cap (\Lambda - \Lambda) = \{0\}$. If $\rho := \overline{D}^{\#}(\Lambda) > 0$, then we have $\mathcal{T}_G(\Omega) \leq 1/\rho$.*

Proof. Let $\varepsilon > 0$ be fixed small, but arbitrary. By Theorem 2.1.12, there exists $f \in \mathcal{F}(\Omega)$ with $\int_G f > \mathcal{T}_G(\Omega) - \varepsilon$. Denote $S := \operatorname{supp} f$, which is a compact subset of Ω in view of $f \in \mathcal{F}(\Omega)$.

In the following we consider a compact, 0-symmetric neighborhood of 0 which we denote by W. We require W to be the closure of a 0-symmetric open subset O containing $S-S$ in it. (Such a compact set exists: by continuity of the group operation, the compact subset $S\times S$ is mapped to a compact set, i.e. $S-S$ is compact, and then for any symmetric, open neighborhood Q of 0 with compact closure $\overline{Q}$ choosing $O:=(S-S)+Q$, $W:=(S-S)+\overline{Q}$ suffices.)

Let us consider the subgroup G_0 of G, generated by W. Here we repeat the construction on [**77**, p. 52]. First, by [**77**, Lemma 2.4.2], $\langle W\rangle = G_0$ implies that there exists a closed subgroup $K\leq G_0$ which is isomorphic to $\mathbb{Z}^k$ with some natural number k and satisfies $W\cap K=\{0\}$, so that $H:=G_0/K$ is then compact. Let ϕ be the natural homomorphism (projection) of G_0 onto H.

Because $S-S\subset \text{int}W$, there exists an open neighborhood X_1 of S such that $X_1-X_1\subset W$, whence $\phi(x)-\phi(y)=0\in H$ with $x,y\in X_1$ would imply $x-y\in \ker\phi = K$, i.e. $x-y\in K\cap W=\{0\}$ and thus $x=y$. In other words, ϕ is a homeomorphism on X_1, and $Y_1:=\phi(X_1)\subset H$ is open. By compactness of H, finitely many translates of Y_1, say $Y_1,Y_2,\dots,Y_r$ will cover H, and there are open subsets X_i of G_0 with compact closure such that ϕ maps X_i onto Y_i homeomorphically for each $i=1,\dots,r$. If $Y_1':=Y_1$, $Y_i':=Y_i\setminus(\cup_{j=1}^{i-1}Y_j)$ $(i=2,\dots,r)$ and $X_i':=X_i\cap\phi^{-1}(Y_i')$ $(i=1,\dots,r)$, then $E:=\cup_{i=1}^{r}X_i'$ is a Borel set in G_0 with compact closure, ϕ is one-to-one on E, and $\phi(E)=H$, i.e., each $x\in G_0$ can be uniquely represented as $x=e+n$, with $e\in E$ and $n\in K$.

In the following we put $\|n\|:=\max_{1\leq j\leq k}|n_j|$, where $(n_1,\dots,n_k)\in\mathbb{Z}^k$ is the element corresponding to $n\in K$ under the fixed isomorphism from K to $\mathbb{Z}^k$. Note also that $S\subset X_1=X_1'\subset E$ and that $\overline{E}$ is compact. Hence also $E+E-E-E$ has compact closure, and the discrete set K can intersect it only in finitely many points. So we put $s:=\max\{\|n\| \ : \ n\in(E+E-E-E)\cap K\}$, which is finite. Next we define

$$V_N:=\cup\{E+n \ : \ n\in K, \|n\|\leq N\} \qquad (N\in\mathbb{N}). \tag{2.60}$$

Note that $|V_N|=(2N+1)^k|E|$ for all $N\in\mathbb{N}$, and the V_N are Borel sets with compact closure. Let $N,M\in\mathbb{N}$, and $x=e+n$, $y=f+m$ be the decomposition of two elements $x\in V_N$ and $y\in V_M$ in terms of $E+K$, i.e. $e,f\in E$ and $n,m\in K$. Then $x+y=e+f+n+m=g+p+n+m$, where $e+f$ has the standard decomposition $g+p$, and so $p=e+f-g\in(E+E-E)$, therefore in $(E+E-E)\cap K$, and we find $\|p\|\leq s$. In all, we find $x+y\in E+q$, where $q:=p+n+m$ satisfies $q\leq N+M+s$, and so $x+y\in V_{N+M+s}$. It follows that $V_N+V_M\subset V_{N+M+s}$.

LEMMA 2.6.5. *With the above notations we have* $\mathcal{T}_{G_0}(V_N)\leq(N+s+1)^k|E|$ *for arbitrary* $N\in\mathbb{N}$.

PROOF. Consider again the natural homeomorphism (projection) $\phi: G_0\to G_0/K=: H$. Proposition 2.5.1 gives

$$\mathcal{T}_{G_0}(V_N)\leq C\mathcal{T}_H(\phi(V_N))\mathcal{T}_K(V_N\cap K) \qquad (C:=\frac{d\nu}{d\mu_H}) \tag{2.61}$$

with $\nu := \mu_{G_0/K} \circ \pi \circ \phi^{-1} = \mu_{G_0/K}$, as $\pi = \phi$ in our case. Note that now $G_0/K := H$, but the Haar measures are normalized differently: H, as a compact group, has $\mu_H(H) = 1$, $K \cong \mathbb{Z}^k$ has the counting measure as its natural Haar measure, but G_0 has the restriction measure μ_{G_0} inherited from $|\cdot| = \mu_G$. Therefore, following the standard convention (as explained e.g. in [**77**, §2.7.3]), under what convention the above quoted Proposition 2.5.1 holds, we must take care of $d\mu_{G_0} = d\mu_{G_0/K} d\mu_K$, which determines $d\mu_{G_0/K}$ and hence C. It suffices to consider one test function, which we chose to be χ_E, the characteristic function of E. We obtain

$$|E| = \mu_{G_0}(E) = \int_{G_0} \chi_E d\mu_{G_0} = \int_{G_0/K} \int_K \chi_E(x+y) d\mu_K(y) d\mu_{G_0/K}([x])$$

$$= \int_{G_0/K} 1 \; d\mu_{G_0/K}([x]) = \mu_{G_0/K}(G_0/K) \tag{2.62}$$

in view of $\#\{y \in K \ : \ x+y \in E\} = 1$ by the above unique representation of G_0 as $E+K$. It follows that

$$C\left(:= \frac{d\nu}{d\mu_H}\right) = \frac{\mu_{G_0/K}(G_0/K)}{\mu_H(H)} = |E| \tag{2.63}$$

and we are led to

$$\mathcal{T}_{G_0}(V_N) \leq |E| \mathcal{T}_H(\phi(V_N)) \mathcal{T}_K(V_N \cap K). \tag{2.64}$$

Since $E \subset V_N$ and $\phi(E) = H$, $\mathcal{T}_H(\phi(V_N) = \mathcal{T}_H(H) = 1$. Let us write from now on $Q_M := Q_{2M}(0) = \{m \ : \ \|m\| \leq M\}$. On the other hand $(V_N \cap K) \subset Q_{N+s}$, because for any $e \in E \cap K$ we necessarily have $\|e\| \leq s$. These observations yield

$$\mathcal{T}_{G_0}(V_N) \leq |E| \cdot 1 \cdot \mathcal{T}_K\left(\{m \in K \ : \ \|m\| \leq N+s\}\right) = |E| \mathcal{T}_{\mathbb{Z}^k}(Q_{2N+2s}(0)),$$

by the isomorphism of K and $\mathbb{Z}^k$. It remains to see that $\mathcal{T}_{\mathbb{Z}^k}(Q_{2L}(0)) \leq (L+1)^k$, which follows from formula (2.58) from the proof of Theorem 2.6.3. □

LEMMA 2.6.6. *Let V be any Borel measurable subset of G with compact closure and let ν be a Borel measure on G with $\overline{D}_G(\nu;\mu) = \rho > 0$. If $\varepsilon > 0$ is given, then there exists $z \in G$ such that*

$$\nu(V+z) \geq (\rho - \varepsilon)|V|. \tag{2.65}$$

PROOF. Let $D := -V$. V is a Borel set with compact closure $\overline{D} \Subset G$. So by Definition 2.3.5 we can find, according to the assumption on $\overline{D}_G(\nu;\mu) = \rho$, some $Z \in \mathcal{B}_0$ which satisfy

$$\nu(Z) \geq (\rho - \varepsilon)|Z + \overline{D}| \geq (\rho - \varepsilon)|Z + D|. \tag{2.66}$$

We can then write

$$\int \chi_Z(t) d\nu(t) \geq (\rho - \varepsilon)|Z + D|. \tag{2.67}$$

For $t \in Z$ $u \in D(= -V)$ also $t + u \in Z + D$, hence $\chi_{Z+D}(t+u) = 1$, and we get

$$\chi_Z(t) \leq \frac{1}{D} \int \chi_{Z+D}(t+u) \chi_D(u) d\mu(u) \tag{2.68}$$

for all $t \in Z$. But for $t \notin Z$ $\chi_Z(t) = 0$ and the right hand side being nonnegative, inequality (2.68) holds for all $t \in G$, hence (2.67) implies

$$
\begin{aligned}
(\rho - \varepsilon)|Z + D| &\le \frac{1}{|D|} \int\int \chi_{Z+D}(t + u)\chi_D(u)d\mu(u)d\nu(t) \\
&= \int \chi_{Z+D}(y) \left(\frac{1}{|D|} \int \chi_D(y - t)d\nu(t) \right) d\mu(y) \\
&= \int \chi_{Z+D}(y)f(y)d\mu(y) \qquad \left(\text{with} \quad f(y) := \frac{\nu(y - D)}{|D|} \right) \\
&= \int_{Z+D} f d\mu.
\end{aligned}
\tag{2.69}
$$

It follows that there exists $z \in Z + D \subset G$ satisfying $f(z) \ge (\rho - \varepsilon)$. That is, we find $\nu(z - D) \ge (\rho - \varepsilon)|D|$ or $\nu(z + V) = \nu(z - D) \ge (\rho - \varepsilon)|D| = (\rho - \varepsilon)|V|$. □

LEMMA 2.6.7. *If $\overline{D}_G(\nu; \mu) = \rho > 0$ with $\mu = \mu_G$ and ν any given Borel measure on the LCA group G, then for any open subgroup G' of G, compact $D \Subset G'$ and $\varepsilon > 0$ there exist $x \in G$ and $Z \subset G'$, $Z \in \mathcal{B}_0$ so that $\nu(Z + x) \ge (\rho - \varepsilon)\mu(Z + D)$.*

REMARK 2.6.8. One would be tempted to assert that on some coset $G' + x$ of G' the relative density of ν must be at least $\rho - \varepsilon$, i.e. $\overline{D}_{G'}(\nu_x; \mu|_{G'}) = \rho - \varepsilon$ with $\nu_x(Z) := \nu(Z + x)$ for $Z \subset G'$ Borel and $x \in G$. However, this stronger statement does not hold true. Consider e.g. $G = \mathbb{Z}^2$, $G' := \mathbb{Z} \times \{0\}$, $A := \{(k, l) \ : \ k \in \mathbb{N}, l \ge k\}$, and $\nu := \mu_A$ the trace of the counting measure μ of $\mathbb{Z}^2$ on A. Since A contains arbitrarily large squares, $\overline{D}(\nu; \mu) = 1$. (In fact, ν has a positive asymptotic density $\delta(\nu; \mu) = 1/8$, too.) However, for each coset $G' + x = \mathbb{Z} \times \{m\}$ of G' the intersection $A \cap G'$ is only finite and $\overline{D}_{G'}(\nu_x; \mu|_{G'}) = 0$.

PROOF. By condition, for $D \Subset G' \le G$ there exists $V \Subset G$ such that

$$\nu(V) \ge (\rho - \varepsilon)\mu(V + D). \tag{2.70}$$

Let now U be an open set containing $V + D$ and with compact closure $\overline{U} \Subset G$. Because the cosets of G' cover G, we have

$$V + D = \bigcup_{x \in G} \left((V + D) \cap (G' + x) \right) \subset \bigcup_{x \in G} \left(U \cap (G' + x) \right).$$

Since both U and G' are open, and $V + D$ is compact, the covering on the right hand side has a finite subcovering; moreover, we can select all covering cosets only once, hence arrive at a disjoint covering

$$V + D \subset \bigcup_{j=1}^{m} U_j \qquad \left(\, U_j := U \cap (G' + x_j), \quad j = 1, \dots, m \, \right).$$

Take now $V_j := U_j \cap (V + D)$. As the U_j are disjoint, so are the V_j; and as the U_j together cover $V + D$, so do the V_j. So we have the disjoint covering $V + D = \cup_{j=1}^m V_j$. Furthermore, if $x \in (V + D) \cap (G' + x_j) \subset V + D$, it must belong to V_j, for all V_i with $i \ne j$ are disjoint from $G' + x_j$ and hence $x \notin V_i$ for $i \ne j$. Therefore all V_j are compact, in view of $V_j = U_j \cap (V + D) = (V + D) \cap U \cap (G' + x_j) = (V + D) \cap (G' + x_j)$ because

$V + D$ is compact and $G' + x_j$ is also closed (as an open subgroup, hence its cosets, are always closed, too.)

Next we define $W_j := V \cap V_j$. Plainly, $W_j \Subset G$ and disjoint, and $V = \cup_{j=1}^m W_j$. Moreover, $W_j + D = V_j$; indeed, $W_j + D = (V \cap (G' + x_j)) + D = (V + D) \cap (G' + x_j)$ since $D \subset G'$ and $G' \le G$. So we find

$$\nu(V) = \sum_{j=1}^m \nu(W_j) \tag{2.71}$$

and also

$$\mu(V + D) = \sum_{j=1}^m \mu(V_j) = \sum_{j=1}^m \mu(W_j + D) = \sum_{j=1}^m \mu(W_j - x_j + D) \tag{2.72}$$

Collecting (2.71), (2.70) and (2.72) we conclude

$$\sum_{j=1}^m \nu(W_j) \ge (\rho - \varepsilon) \sum_{j=1}^m \mu(W_j - x_j + D), \tag{2.73}$$

hence for some appropriate $j \in [1, m]$ we also have $\nu(W_j) \ge (\rho - \varepsilon)\mu(W_j - x_j + D)$. Taking $Z := W_j - x_j$ and $x = x_j$ concludes the proof. $\square$

End of the proof of Theorem 2.6.4. Let now $\nu := \delta_\Lambda$ be the counting measure of the (discrete) set $\Lambda \subset G$. Then $\overline{D}_G(\nu; \mu) = \overline{D}_G^{\#}(\Lambda) = \rho > 0$ and Lemma 2.6.6 applies providing some $z := z_N \in G$ with

$$M := \#\left(\Lambda \cap (V_N + z)\right) \ge (\rho - \varepsilon)|V_N|. \tag{2.74}$$

Take now $\Lambda' := \Lambda \cap (V_N + z) = \{\lambda_m \ : \ m = 1, \ldots, M\}$. Put $F := f \star \delta_{\Lambda'} \star \delta_{-\Lambda'}$, i.e.

$$F(x) := \sum_{m=1}^M \sum_{n=1}^M f(x + \lambda_m - \lambda_n),$$

which is a positive definite continuous function supported in $S + (V_N + z) - (V_N + z) = S + V_N - V_N = S + E - E + Q_{2N} \subset E + E - E + Q_{2N} \subset V_{2N+s}$. (Recall $Q_L := Q_{2L}(0) = \{m \in \mathbb{Z}^d \ : \ \|m\| \le L\}$.) Furthermore, as $S \subset G_0$,

$$\int_{G_0} F = M^2 \int_{G_0} f \ge M^2(\mathcal{T}_G(\Omega) - \varepsilon) \tag{2.75}$$

and

$$F(0) = \sum_{m=1}^M \sum_{n=1}^M f(\lambda_m - \lambda_n) = M f(0) = M, \tag{2.76}$$

because if $\lambda_m - \lambda_n \in S$ then $\lambda_m - \lambda_n \in S \cap (\Lambda - \Lambda) \subset \Omega \cap (\Lambda - \Lambda) = \{0\}$ and $\lambda_m = \lambda_n$, i.e. $n = m$. By this construction we derive that

$$\begin{aligned} \mathcal{T}_{G_0}(V_{2N+s}) &\ge \frac{1}{F(0)} \int_{G_0} F \ge M(\mathcal{T}_G(\Omega) - \varepsilon) \\ &\ge (\rho - \varepsilon)(\mathcal{T}_G(\Omega) - \varepsilon)|V_N| = (\rho - \varepsilon)(\mathcal{T}_G(\Omega) - \varepsilon)(2N + 1)^k |E|. \end{aligned} \tag{2.77}$$

On the other hand Lemma 2.6.5 provides us

$$\mathcal{T}_{G_0}(V_{2N+s}) \le (2N + s + 1)^k |E|. \tag{2.78}$$

On comparing (2.77) and (2.78) we conclude $(\rho-\varepsilon)(\mathcal{T}_G(\Omega)-\varepsilon)(2N+1)^k|E| \le (2N+s+1)^k|E|$, that is

$$\mathcal{T}_G(\Omega)-\varepsilon \le \frac{1}{\rho-\varepsilon}\left(\frac{2N+s+1}{2N+1}\right)^k.$$

Letting $N\to\infty$ and $\varepsilon\to 0$ gives the assertion. □

COROLLARY 2.6.9. *Suppose that $\Omega\subset G$ is an open and symmetric set and $\Omega = H - H$, where H tiles space with $\Lambda\subset G$. Moreover, assume that H has compact closure $\overline{H}\Subset G$ and is measurable, i.e. $H\in\mathcal{B}_0$. Then $\mathcal{T}_G(\Omega)=\mu(H)$.*

PROOF. First, observe that for any $A\Subset H$ we have $f:=\chi_A*\chi_{-A}\in\mathcal{F}_\&(\Omega)$. Indeed, $\widetilde{\chi_A}=\chi_{-A}$ because χ_A is real valued, also $\chi_A\in L^2(G)$, and such a convolution representation guarantees that $f\in C(G)\cap L^1(G)$ is positive definite; furthermore, if $f(x)\neq 0$, then necessarily $x=a-a'$ with some $a,a'\in A\subset H$, hence $\operatorname{supp} f\subset\Omega$.

Therefore, calculating with the admissible function f, we find $\mathcal{T}_G(\Omega)\ge\int_G f/f(0) = \mu(A)^2/\mu(A) = \mu(A)$. Since H is Borel measurable, its measure can be approximated arbitrarily closely by measures of inscribed compact sets A: therefore, taking supremum over compact sets $A\Subset H$, we obtain the lower estimate $\mathcal{T}_G(\Omega)\ge\mu(H)$.

On the other hand, $H+\Lambda=G$ entails that H packs with Λ, and so an application of Theorem 2.6.4 gives $\mathcal{T}_G(\Omega)\le 1/\overline{D}^{\#}(\Lambda)$. Now we can apply that H also covers G with Λ, so that Proposition 2.4.9 also applies, giving $\overline{D}^{\#}(\Lambda)\ge 1/\mu(H)$. On combining the last two inequalities, $\mathcal{T}_G(\Omega)\le\mu(H)$, whence the assertion, follows. □

2.6.3. Sharpness and further examples. Again, everything in this section is from our joint work with Mihalis Kolountzakis [**56**].

First let us point out that the bound (2.51) can be sharp. Take, for example, Ω to be a subgroup of G of finite index and $H=\Omega$. Take also Λ to a complete set of coset representatives of G/Ω, so that $|\Lambda|<\infty$. Then $H+\Lambda=G$ and Corollary 2.6.2 applies and gives

$$\sum_{x\in G} f(x)\le|\Omega|f(0) \tag{2.79}$$

for every positive definite function $f:G\to\mathbb{C}$ supported in Ω, which is also the trivial bound. Taking $f=\chi_\Omega$, which is positive definite because Ω is a group, gives equality in (2.79).

More generally (and as in the next example) the inequality (2.51) is sharp whenever $H+\Lambda=G$ and $\Omega=H-H$. In such a case the function $f=\chi_H*\chi_{-H}$ achieves equality in (2.51).

EXAMPLE 2.6.10. Take $G=\mathbb{Z}_8=\{0,1,\dots,7\}$, $H=\{0,1,4,5\}$, $\Omega=H-H=\{0,1,3,4,5,7\}$ and $\Lambda=\{0,2\}$, so that $\Lambda-\Lambda=\{0,2,6\}$ and $H+\Lambda=G$. It follows that

$$\sum_{x\in G} f(x)\le 4f(0)$$

for any positive definite function on $\mathbb{Z}_8$ which vanishes on ± 2, instead of the trivial $\sum_{x\in G} f(x)\le 6f(0)$. The equality can be achieved by the function $f=\chi_H*\chi_{-H}$.

EXAMPLE 2.6.11. Let $G := \mathbb{Z}$ and $\Omega := \Omega_N := \{-N, -1, 0, 1, N\}$; then the trivial estimate is $A(N) := \mathcal{T}_{\mathbb{Z}}(\Omega_N) \leq 5$. Let $f \in \mathcal{F}(\Omega)$ be a positive definite and real valued function: then $f(k) = f(-k)$, that is, f is even. The dual group is $\mathbb{T}$, and positive definiteness of f means $p(x) := 1 + 2f(1)\cos x + 2f(N)\cos Nx \geq 0$ (as $f(0) = 1$ by normalization). In the Turán problem we are to maximize $\int_{\mathbb{Z}} f = 1+2f(1)+2f(N) = p(0)$; we have $A(N) = \max p(0)$.

To find $A(N)$ in case when $N = 2n+1$ is odd we may look at the value $p(\pi) = 1 - 2f(1) - 2f(2n+1) \geq 0$ to see that $p(0) = 2 - p(\pi) \leq 2$. Clearly, any function with $f(1) + f(2n+1) = 1/2$ achieves this bound while $p \geq 0$ if additionally we require $0 \leq f(1), f(2n+1)$. Hence $A(2n+1) = 2$.

If $N = 2n$ is even, the solution is less simple, see [**72**]. We claim that $A(N) = 1 + 1/\cos\frac{\pi}{2n+1} =: C(N)$, say, and the extremal function is

$$p_0(x) := 1 + \frac{2n}{(2n+1)\cos\frac{\pi}{2n+1}}\cos x + \frac{1}{(2n+1)\cos\frac{\pi}{2n+1}}\cos 2nx\,.$$

Clearly $p_0(0) = C(N)$, and standard calculus proves nonnegativity of p_0, hence it is an admissible trigonometric polynomial and $A(N) \geq C(N)$.

To show its extremality we consider a general $p(x) = 1 + a\cos x + b\cos 2nx$ (where $a := 2f(1)$, $b := 2f(N)$) at the point $z_0 := \pi + \pi/(2n+1)$, which yields $0 \leq p(z_0) = 1 - a\cos\frac{\pi}{2n+1} - b\cos\frac{\pi}{2n+1}$. Thus $p(0) = 1+a+b = 1+(1-p(z_0))/\cos\frac{\pi}{2n+1} \leq C(N)$, and the calculation is concluded.

Now let us consider the estimates obtainable from the use of Theorem 2.6.3. In case N is odd, taking $\Lambda := 2\mathbb{Z}$ is optimal. Indeed, since Λ is a subgroup, $\Lambda - \Lambda = \Lambda$, and it does not intersect Ω_N (apart from 0), hence an application of Theorem 2.6.3 gives the right value $A(N) \leq 1/\mathrm{dens}(\Lambda) = 2$. Hence in this case Theorem 2.6.3 is sharp.

Let us see that it is *not* in the case when $N = 2n$ is even. To this, first we have to find the best upper density, that is,

$$L(N) := \sup_{\Omega_N \cap (\Lambda - \Lambda) = \{0\}} \overline{\mathrm{dens}}(\Lambda)\,.$$

Let us consider the set $\Lambda^* := \{0, 2, \ldots, 2n-2\} \cup \{2n+1, 2n+3, \ldots, 4n-1\} + (4n+2)\mathbb{Z}$, which contains $2n$ elements in each interval $\big[k(4n+2), (k+1)(4n+2)\big)$ of $4n+2$ numbers and hence has density $n/(2n+1)$. A direct calculation shows that $\Omega_N \cap (\Lambda^* - \Lambda^*) = \{0\}$, hence $L(N) \geq n/(2n+1)$. On the other hand we assert that for no Λ satisfying $\Omega_N \cap (\Lambda - \Lambda) = \{0\}$ can any interval $I = [k, k+2n]$ of $2n+1$ consecutive numbers contain more than n elements of Λ. Indeed, no pair of neighboring numbers belong to Λ, because $1 \in \Omega_N$, and (at least) $n+1$ non-neighboring numbers can be placed into I only if all $m \in I$ with the same parity as k is contained. However, then both k and $k+2n$ is contained, having difference $2n \in \Omega_N$, a contradiction. Hence for a Λ satisfying our condition, the upper density can not exceed $n/(2n+1)$, which proves $L(N) = n/(2n+1)$.

Now we can compare the best estimate $\mathcal{T}_{\mathbb{Z}}(\Omega_N) \leq 1/L(N) = 2 + 1/n$ arising from Theorem 2.6.3 to the exact value $2 + 1/\cos\frac{\pi}{2n+1}$ found above. It shows that application of Theorem 2.6.3 – although much better than the trivial estimate, but still – is not optimal in this case. This example highlights also the fact that number theoretical, intrinsic structural properties – like e.g. N being even or odd – essentially influence the values of the Turán constants and sharpness of the estimates we have.

EXAMPLE 2.6.12. Another example of a nice set with nontrivial, but not sharp estimate arising from Theorem 2.6.3 is the unit disk D in $\mathbb{R}^2$ (with Lebesgue measure). The area of D is π and the right value of the Turán constant, first computed by Siegel [**84**] and then again by Gorbachev [**29**], is $|D|/2^d = \pi/4$ in this case. Now D is the difference set of $H := D/2$, and the best density we can have is, in fact, the *sphere packing constant* of $\mathbb{R}^2$. It is well-known [**1**] that the best packing is the regular hexagon lattice packing, hence $L(D) = 2/\sqrt{3}$ and the arising estimate is $\sqrt{3}/2$. In comparison, note that the estimate of §2.6.5 gives $|D|/2 = \pi/2$, while the estimate of Theorem 2.7.2 from the spectral approach does not apply, since the ball is *not* spectral. The above values compare as $\pi/4 = 0.785\cdots < \sqrt{3}/2 = 0.866\cdots < \pi/2 = 1.57\ldots$.

EXAMPLE 2.6.13. We see that for a general $\Omega \subset H - H$ or even $\Omega = H - H$ the "best translational set", (i.e. the maximal number of elements or the highest possible upper density), does not always achieve an exact bound of $\mathcal{T}_G(\Omega)$. In this respect it is worth mentioning that, on the other hand, results of Herz [**39**], [**40**] show that each subgroup Λ of G provides the theoretically best possible, sharp estimate for *some* open set Ω. E.g. if G is compact, and Λ is a finite subgroup having n elements, there exists a Borel set H with the properties $|H| = 1/n$, $\Omega := H - H$ is open, and $\Omega \cap \Lambda = \{0\}$. See also [**77**, §7.4.1]. Clearly for this Ω and H we have that $H + \Lambda = G$ is a tiling, and $\mathcal{T}_G(\Omega) = 1/n$, achieved by $\chi_H * \chi_{-H}$.

EXAMPLE 2.6.14. The size of the Turán constant of a set Ω may be extremely small. Take for example in the group $G = \mathbb{Z}_{2n}$ the set $\Omega = \{0\} \cup K^c$, where K is the subgroup generated by 2. Let then $\Lambda = K$ and apply Theorem 2.6.1. It follows that $\mathcal{T}_G(\Omega) \leq 2$ while $|\Omega| = n + 1$.

The same way we have $\mathcal{T}_{\mathbb{Z}}(\Omega) \leq 2$ for any subset $\Omega \subset (\{0\} \cup (2\mathbb{Z}+1))$ in view of Theorem 2.6.3 and considering the set $\Lambda := 2\mathbb{Z}$. (This covers the N odd case of Example 2.6.11, too.)

The generality of this example should be obvious.

2.6.4. The Turán constant of difference sets of tiles in $\mathbb{R}^d$ or $\mathbb{Z}^d$. Here we show how to generalize the results in [**7**] (see also [**54**]). In [**7**] the Turán constant of convex polytopes which tile $\mathbb{R}^d$ by lattice translation was determined.

Actually being a polytope and *lattice* translation need not be assumed as it is a fact (see e.g. the references in [**54**]) that any convex body that tiles space by translation is a polytope and can also tile by lattice translation.

From Theorem 2.6.3 it follows that if H is any measurable set of finite measure that tiles $\mathbb{R}^d$ or $\mathbb{Z}^d$ by translation with Λ then the Turán constant of $H - H$ is equal to $1/\operatorname{dens}\Lambda = |H|$.

Whenever Ω is a convex body in $\mathbb{R}^d$ one can take $H = \frac{1}{2}\Omega$, so Theorem 2.6.3 is indeed a generalization of the result in [**7**].

However, Theorem 2.6.3 can determine the Turán constant of many more sets than those dealt with in [**7**].

EXAMPLE 2.6.15. Let $H \subset \mathbb{Z}^2$ be the three-element set $\{(0,0), (0,1), (1,0)\}$ and Ω be the difference set $H - H = \{(-1,0), (-1,1), (0,-1), (0,0), (0,1), (1,-1), (1,0)\}$. Then $|\Omega| = 7$, but H tiles $\mathbb{Z}^2$, hence Theorem 2.6.3 applies and yields $\mathcal{T}_{\mathbb{Z}^2}(\Omega) = 3$. Observe that the set $\Lambda := \mathbb{Z}(1,1) + \mathbb{Z}(2,-1)$ provides a translational set. Indeed, any points $(n+2m, n-m)$ of Λ, and thus also of $\Lambda - \Lambda$, has the property that the first coordinate is congruent to the second $\mod 3$, hence $\Omega \cap \Lambda - \Lambda = \{(0,0)\}$. On the other hand all points of $\mathbb{Z}^2$ with the above congruence property belong to Λ, i.e. Λ is a subgroup of index 3. It follows that the density of Λ is $1/3$, and Theorem 2.6.3 gives the assertion.

2.6.5. The Turán constant of dispersed sets. As an application of Theorem 2.6.3 we show that, in $\mathbb{R}$, the Turán constant of a set of given length is the largest if the set is an interval. The construction extends to $\mathbb{Z}$, and even to $\mathbb{R}^d$ and $\mathbb{Z}^d$ giving a generally valid improvement of the trivial bound by about a factor of 2.

THEOREM 2.6.16 (**Kolountzakis-Révész**). *Let $\Omega \subseteq \mathbb{R}^d$ be an open set of finite measure m. Then we have*

$$\mathcal{T}_{\mathbb{R}^d}(\Omega) \le \frac{m}{2} \ . \tag{2.80}$$

Let $\Omega \subseteq \mathbb{Z}^d$ be a set of size m containing the origin and denote by m^+ the number of lattice points in the "nonnegative half of Ω", i.e. in $\Omega \cap \big([0,\infty) \times \mathbb{Z}^{d-1}\big)$. Then we have

$$\mathcal{T}_{\mathbb{Z}^d}(\Omega) \le m^+ \ . \tag{2.81}$$

PROOF. Let us denote $P := [0,\infty) \times \mathbb{R}^{d-1}$ or $[0,\infty) \times \mathbb{Z}^{d-1}$, respectively, and put $\Omega^+ := \Omega \cap P$. Note that in $\mathbb{R}^d$ we simply have $m^+ := |\Omega^+| = m/2$. It is easy to see that Theorem 2.1.12 (on the equivalent formulations of the Turán constant), allows us to assume that Ω is bounded: so let $\Omega \subset B(0,r)$ with some fixed ball of radius r. Take a large parameter $L_0 > \max\{2, r\}$, define $L_k = L_0^{2^k} = L_{k-1}^2$ $(\forall k \in \mathbb{N})$, say, and put

$$Q_k := Q_{L_k}((L_k, 0, \ldots, 0)) = [0, 2L_k] \times [-L_k, L_k]^{d-1} \quad (k \in \mathbb{N}), \quad Q_0 := \emptyset \ . \tag{2.82}$$

Note that $|Q_k| = (2L_k)^d$ in $\mathbb{R}^d$ and $(2L_k+1)^d$ in $\mathbb{Z}^d$. Define

$$S_k := Q_k \setminus (Q_{k-1} + \Omega) \quad (k \in \mathbb{N}) \ . \tag{2.83}$$

Obviously, S_k are closed sets of measure

$$|S_k| \ge |Q_k| - |Q_{k-1} + \Omega| \ge (2L_k)^d - ((2L_{k-1}+1) + 2r)^d \ge 2^d L_k^d \left(1 - (\frac{2+r}{L_{k-1}})^d\right) \quad (k \in \mathbb{N}), \tag{2.84}$$

satisfying $(S_k - S_n) \cap \Omega = \emptyset$ for $k \ne n$. We aim at constructing the discrete set

$$\Lambda := \bigcup_{k=1}^{\infty} \Lambda_k, \qquad \Lambda_k \subset S_k \quad (k \in \mathbb{N}) \tag{2.85}$$

with as many as possible elements but satisfying $(\Lambda_k - \Lambda_k) \cap \Omega = \{0\}$. Note that if the latter condition is satisfied, then we will also have $(\Lambda - \Lambda) \cap \Omega = \{0\}$ in view of the respective property of $S_k \supset \Lambda_k$. So now we define the elements of Λ_k inductively by a "greedy algorithm" as follows. Let $\lambda_0^{(k)}$ be any element of the nonempty set S_k with first coordinate 0. Such an element clearly exists. Then for $n \geq 1$ take any

$$\lambda_n^{(k)} := (x_{1,n}, \ldots, x_{d,n}) \in \big(S_k \setminus \bigcup_{j=1}^{n-1} (\lambda_j^{(k)} + \Omega^+)\big)$$

(2.86) with

$$x_{1,n} = \min\left\{x_1 \ : \ \exists\ x = (x_1, \ldots, x_d) \in \big(S_k \setminus \bigcup_{j=1}^{n-1} (\lambda_j^{(k)} + \Omega^+)\big)\right\} .$$

Defining new elements $\lambda_n^{(k)}$ of Λ_k terminates in a finite number of steps, but not before $\cup_{j=1}^{n-1} (\lambda_j^{(k)} + \Omega^+)$ covers S_k, so with $m^+ := |\Omega^+|$ we must have

$$\#\Lambda_k \geq \frac{|S_k|}{|\Omega^+|} \geq \frac{2^d L_k^d \left(1 - (\frac{2+r}{L_{k-1}})^d\right)}{m^+} \quad (k \in \mathbb{N}) . \tag{2.87}$$

By construction, for any $n > j$ $\lambda_n^{(k)} - \lambda_j^{(k)} \in \Omega$ is not possible, hence $\Lambda - \Lambda \cap \Omega = \{0\}$. Moreover, in view of (2.87) we have

$$\overline{\mathrm{dens}}\,\Lambda \geq \limsup_{k\to\infty} \frac{\#\Lambda_k}{|Q_k|} \geq \limsup_{k\to\infty} \frac{\left(1 - (\frac{2+r}{L_{k-1}})^d\right)}{m^+} = \frac{1}{m^+} . \tag{2.88}$$

Now an application of Theorem 2.6.3 with Λ concludes the proof. □

REMARK 2.6.17. For $d = 1$ (2.80) is sharp for intervals in $\mathbb{R}$. It is plausible, but we do not know if intervals are the only cases of equality.

REMARK 2.6.18. As Ω is always symmetric, in $\mathbb{Z}$ we always have $m^+ = (m+1)/2$. The estimate (2.81) can also be sharp at least for $d = 1$. Take e.g. $\Omega = \Omega_0$ or Ω_1 from Example 2.6.11, or, more generally, take $\Omega := [-N, N]$. Then $m = 2N+1$, $m^+ = (m+1)/2 = N+1$, and the Fejér kernel shows that this value can be achieved. Thus $\mathcal{T}_{\mathbb{Z}}([-N, N]) = N+1$, and intervals have maximal Turán constants once again. However, here the sets $k[-N, N] := \{kn \ : \ |n| \leq N\}$ of similar size have equally large Turán constants, hence intervals are not the only extremal examples in $\mathbb{Z}$.

REMARK 2.6.19. It can be proved that the asymptotic uniform upper density of all sets remain the same both in $\mathbb{R}^d$ and in $\mathbb{Z}^d$ if we define it replacing Q_R by RK with any other convex body K. Thus in the above proof one can consider the slightly modified basic sets $RQ(T)$, where $RQ(T)$ is the R-dilated copy of the unit box rotated by the isometry $T \in SO(d)$. If we choose T to be "irrational" in the sense that no lattice point (apart from the origin) moves to the hyperplane $\{x_1 = 0\}$, then with these sets a similar argument leads the same estimate but now with $m^+ = \#\Omega^+ = (m+1)/2$. We leave the details to the reader.

2.6.6. The Turán constant of an interval missing two points. Our next result shows the effect of forcing a positive definite function to vanish at a neighborhood of one point in an interval.

THEOREM 2.6.20 (**Kolountzakis-Révész**). *Suppose* $0 < b < a \le 2b$ *and let*

$$\Omega = (-a, -b) \cup (-b, b) \cup (b, a).$$

Then $\mathcal{T}_{\mathbb{R}}(\Omega) = \mathcal{T}_{\mathbb{R}}(-b, b) = b$.

PROOF. Simply take $\Lambda = b\mathbb{Z}$ and apply Theorem 2.6.3 to obtain that $\mathcal{T}_{\mathbb{R}}(\Omega) \le b$. The other direction is obvious by the monotonicity of $\mathcal{T}_G(\cdot)$. □

The condition $a < 2b$ is necessary in Theorem 2.6.20. Indeed, if $a > 2b$ then, with $c = \min\{b, (a-b)/2\} > b/2$ and $d := (a+b)/2$ the function $f := \chi_{(0,c)} * \chi_{(-c,0)} * (\delta_0 + \delta_d) * (\delta_0 + \delta_{-d})$, whose graph consists of three triangles centered at 0 and $\pm d$ of width $2c$ and heights 1 (for the central triangle) and 1/2 (for the other two) is positive definite and supported in Ω, yet has $f(0) = 2c$ and $\int_{\mathbb{R}} f = 4c^2$. Hence $\mathcal{T}_{\mathbb{R}}(\Omega) \ge 2c > b$.

2.7. Upper bound from spectral sets

2.7.1. Some easy cases of using spectrality for estimating the Turán constant. The second type of result we give is analogous to that proved in [**54**]. Here we suppose that Ω can be embedded in the difference set of a spectral set (see definition in §2.4.2) and we derive an upper bound for $\mathcal{T}_G(\Omega)$ from that.

THEOREM 2.7.1 (**Kolountzakis-Révész**). *Suppose G is a finite abelian group, $\Omega, H \subseteq G$, $\Omega \subseteq H - H$, and that H is a spectral set with spectrum $T \subseteq \widehat{G}$. Then for any positive definite function on G with support in Ω we have*

$$\sum_{x \in G} f(x) \le |H| f(0). \tag{2.89}$$

In other words $\mathcal{T}_G(\Omega) \le |H|$.

PROOF OF THEOREM 2.7.1. Since T is a spectrum of H we have (see §2.4.2)

$$\begin{aligned} \operatorname{supp} \widehat{\chi_T} &\subseteq \{0\} \cup (H - H)^c \\ &\subseteq \{0\} \cup \Omega^c \\ &\subseteq \{0\} \cup \{f = 0\}. \end{aligned}$$

Hence $\widehat{f} + T = c\widehat{G}$ is a tiling and $c = |T| f(0)$, as $\int_{\widehat{G}} \widehat{f} = \left|\widehat{G}\right| f(0)$.

Since $\widehat{f} \ge 0$ in $\widehat{G}$ it follows that $\widehat{f}(0) \le c$ or

$$\sum_{x \in G} f(x) \le |T| f(0) = |H| f(0).$$

2.7.2. Estimates of the Turán constant of spectral sets in $\mathbb{R}^d$. What was essentially proved in [**54**] was a "continuous" version of Theorem 2.7.1.

THEOREM 2.7.2 (**Kolountzakis-Révész**). *If H is a bounded open set in $\mathbb{R}^d$ which is spectral, then for the difference set $\Omega = H - H$ we have $\mathcal{T}_{\mathbb{R}^d}(\Omega) = |H|$.*

Originally, we formulated in [**54**] only the following special case of the above result. The possibility of deriving even Theorem 2.7.2 essentially from the same proof, was noted only later in [**56**]; however, no detailed proof appeared in writing yet. In §2.7.3 we will give the full proof.

COROLLARY 2.7.3 (**Kolountzakis-Révész**). *Let $\Omega \subseteq \mathbb{R}^d$ be a convex domain. If Ω is spectral, then it has to be a Stechkin-Turán domain as well.*

PROOF. First let us note that convex spectral domains are necessarily symmetric according to the result in [**49**]. Let now Ω be a symmetric convex domain. Then taking $H := \frac{1}{2}\Omega$, we have $H - H = \Omega$. Moreover, if Ω is spectral, say with spectrum Λ, then also H is clearly spectral with the dilated spectrum 2Λ. So Theorem 2.7.2 applies and we are done, in view of $|H| = |\frac{1}{2}\Omega| = |\Omega|/2^d$. □

COROLLARY 2.7.4 (Arestov-Berdysheva). *Suppose the symmetric convex domain $\Omega \subseteq \mathbb{R}^d$ is a translational tile. Then it is a Stechkin-Turán domain.*

PROOF OF COROLLARY 2.7.4. We start with the following result which claims that every convex tile is also a lattice tile.

THEOREM 2.7.5 (Venkov [**89**] and McMullen [**63**]). *Suppose that a convex body K tiles space by translation. Then it is necessarily a symmetric polytope and there is a lattice L such that*

$$K + L = \mathbb{R}^d.$$

A complete characterization of the tiling polytopes is also among the conclusions of the Venkov-McMullen Theorem but we do not need it here and choose not to give the full statement as it would require some more definitions.

So, if a convex domain is a tile, it is also a lattice tile, hence spectral by Theorem 2.4.7, and as such it is Stechkin-Turán, by Corollary 2.7.3. □

REMARK 2.7.6. If one wants to avoid using the Venkov-McMullen theorem in the proof of Corollary 2.7.4 one should enhance the assumption of Corollary 2.7.4 to state that Ω is a lattice tile. Arestov and Berdysheva in [**7**] prove Corollary 2.7.4 without going through spectral domains.

The result of [**6**] about the hexagon being a Stechkin-Turán domain is thus a special case of our Corollary 2.7.4, but not the result in [**84**] and [**29**] about the ball being Stechkin-Turán type. The ball, and essentially every smooth convex body [**41**], is known not to be spectral, in accordance with the Fuglede Conjecture.

Fuglede's Conjecture for convex domains is still open except for dimension $d = 2$, in which case it was answered in the affirmative recently [**42**]. Thus our Theorem 2.7.3

conceivably (though not very likely) applies to a wider class of convex domains than just convex tiles, dealt with in Corollary 2.7.4.

2.7.3. Proof of Theorem 2.7.2.

Proof of Theorem 2.7.2. The proof of the theorem relies on Fourier theoretic characterizations of translational tiling [**49**].

First, let $C \Subset H$ be any compact set: then $\Phi := \chi_C + \chi_{-C}$ is supported in the compact set $C - C \Subset \Omega$, is in $L^1(G) \cap C(G)$, and is positive definite with Fourier transform $|\widehat{\chi_C}|^2$, that is, it is in $\mathcal{F}_{\&}$, and it provides the lower estimation $\mathcal{T}_{\mathbb{R}^d}(\Omega) \geq \int \chi_C = |C|$, which can be arbitrarily close to $|H|$, so $\mathcal{T}_{\mathbb{R}^d}(\Omega) \geq |H|$ as well.

Without loss of generality let us assume from now on that H has measure 1.

Let H have spectrum $\Lambda \subseteq \mathbb{R}^d$. This is equivalent to the following (see [**49**])

$$\sum_{\lambda \in \Lambda} |\widehat{\chi_H}|^2(x - \lambda) = 1, \quad \text{for a.e. } x \in \mathbb{R}^d. \tag{2.90}$$

That is, $|\widehat{\chi_H}|^2$ *tiles* $\mathbb{R}^d$ with translation set Λ at level 1, i.e. $|\widehat{\chi_H}|^2 + \Lambda = \mathbb{R}^d$. According to Remark 2.4.3 we then have $\operatorname{dens} \Lambda = 1/|H| = 1$.

For any given $\Lambda \subset \mathbb{R}^d$ with bounded density (see Definition 2.4.2) we denote by δ_Λ the (infinite) measure $\sum_{\lambda \in \Lambda} \delta_\lambda$. This is a tempered distribution, as the total mass in a ball of radius R grows polynomially with R, and therefore we can speak of its Fourier transform.

We shall use the following result from [**49**].

Lemma 2.7.7 (**Kolountzakis** [**49**]). *Suppose that $f \geq 0$ is not identically 0, that $f \in L^1(\mathbb{R}^d)$, $\widehat{f} \geq 0$ has compact support and $\Lambda \subset \mathbb{R}^d$. If $f + \Lambda$ is a tiling then*

$$\operatorname{supp} \widehat{\delta_\Lambda} \subseteq \left\{ x \in \mathbb{R}^d : \ \widehat{f}(x) = 0 \right\} \cup \{0\}. \tag{2.91}$$

When applied to our case, $f = |\widehat{\chi_H}|^2$. Note that H being bounded and open, the function $\widehat{f} = \chi_H * \chi_{-H}$ is nonzero exactly at points of $\Omega = H - H$. It follows that

$$\operatorname{supp} \widehat{\delta_\Lambda} \subseteq \{0\} \cup (H - H)^c = \{0\} \cup \Omega^c. \tag{2.92}$$

The necessary support condition (2.91) in Lemma 2.7.7 cannot by itself guarantee that f tiles with Λ. The reason is that a tempered distribution, such as $\widehat{\delta_\Lambda}$, which is supported in the zero-set of a function, is not necessarily killed when multiplied by that function. One has to know some extra information about the order of the distribution ("what order derivatives it involves") versus the degree of vanishing of the function on the support of the distribution[3].

In the following partial converse to Lemma 2.7.7 (see [**49**]), this problem is solved as the separation of the supports guarantees infinite order of vanishing of f.

[3] An important special case is when one knows the distribution to be a measure, as is the case when Λ is either a lattice or fully periodic. In that case any vanishing of the function will do and the implication in Lemma 2.7.7 can essentially be reversed.

LEMMA 2.7.8 (**Kolountzakis** [**49**]). *Suppose that $g \in L^1(\mathbb{R}^d)$, and that $\Lambda \subset \mathbb{R}^d$ has uniformly bounded density. Suppose also that $O \subset \mathbb{R}^d$ is open, that*

$$\widehat{g}(0) = \int g \neq 0, \tag{2.93}$$

and that for some $\delta > 0$

$$\operatorname{supp} \widehat{\delta_\Lambda} \setminus \{0\} \ \subseteq \ O \quad \text{and} \quad O + B_\delta(0) \subseteq \ \{\widehat{g} = 0\}. \tag{2.94}$$

Then $g + \Lambda$ is a tiling at level $\widehat{g}(0) \cdot \widehat{\delta_\Lambda}(\{0\})$.

The conclusion of Lemma 2.7.8 demands some explanation. Conditions (2.93) and (2.94) imply that in a neighborhood of 0 the tempered distribution $\widehat{\delta_\Lambda}$ is supported at 0 only. That's because $\widehat{g}$ is continuous and, since $\widehat{g}(0) \neq 0$, it does not vanish in some neighborhood of 0. It then follows that, near 0, $\widehat{\delta_\Lambda}$ is not only a tempered distribution but a measure, that is, it is just a point mass at 0 (see [**51**], Theorem 5.1, Step 1, for the proof in dimension 1, which works in any dimension). For this reason it makes sense to write $\widehat{\delta_\Lambda}(\{0\})$ for that point mass.

From Lemma 2.7.9 below, it follows that the value of this constant is precisely the density of Λ, if such a density exists.

LEMMA 2.7.9 (**Kolountzakis** [**50**]). *Suppose that $\Lambda \in \mathbb{R}^d$ is a multiset with density ρ, $\delta_\Lambda = \sum_{\lambda \in \Lambda} \delta_\lambda$, and that $\widehat{\delta_\Lambda}$ is a measure in a neighborhood of* 0. *Then $\widehat{\delta_\Lambda}(\{0\}) = \rho$.*

Conclusion of the proof of Theorem 2.7.2. If Ω is of non-Stechkin-Turán type. Then there exists a positive definite function F supported in $S := \operatorname{supp} F \Subset \Omega$ with $F(0) = 1$ and $\int F > |\Omega|/2^d = |H| = 1$.

Now define

$$\widehat{G}(x) = F\left((1+\epsilon)x\right),$$

where $\epsilon > 0$ is to be taken so small that we still have $G(0) = \int \widehat{G} > 1$ and that we also have $S' := \operatorname{supp} \widehat{G} = (1+\varepsilon)^{-1} S \Subset \Omega$. The function $\widehat{G}$ is also positive definite, and $\widehat{G}(0) = F(0) = 1$.

Because of (2.92) we can now write $\operatorname{supp} \widehat{\delta_\Lambda} \ \subseteq \ \{0\} \cup \left\{\widehat{G} = 0\right\}$, but instead of this immediate fact, we would like to have the two inclusions

$$\operatorname{supp} \widehat{\delta_\Lambda} \subseteq \{0\} \cup O \qquad \text{and} \qquad O + B_\delta(0) \subset \left\{\widehat{G} = 0\right\},$$

with some proper open set O and a positive δ, in order to apply Lemma 2.7.8 with $g = G$.

As $S' \Subset \Omega$, the distance of S' from the the complement Ω^c is positive, so taking $\delta < d(S', \Omega^c)/2$, we find that for $O := \Omega^c + B_\delta(0)$ we still have $d(O, S') > \delta$. Note that according to (2.92), $\operatorname{supp} \widehat{\delta_\Lambda} \subseteq \{0\} \cup \Omega^c$, so indeed $\operatorname{supp} \widehat{\delta_\Lambda} \subseteq \{0\} \cup O$ with this open set O. But then we are done, since $\operatorname{supp} \widehat{G} = S'$ means that $\left\{\widehat{G} = 0\right\}$ certainly contains S'^c, so even $O + B_\delta(0)$ in view of $d(O, S') > \delta$. In all, we can indeed apply Lemma 2.7.8 with $g = G$. Note that (2.93) is also satisfied here as $\widehat{g}(0) = \widehat{G}(0) = F(0) = 1$ by the definition of $\widehat{G}$.

Thus Lemma 2.7.8 can be applied and we find that G tiles $\mathbb{R}^d$ with translation set Λ at level $\widehat{g}(0)\widehat{\delta_\Lambda}\{(0)\} = \widehat{G}(0)\text{dens}\,\Lambda$ by Lemma 2.7.9. Here $\widehat{G}(0) = 1$ and also $\text{dens}\,\Lambda = 1$ in view of the considerations following formula (2.90). Thus the level of tiling by G and Λ is found to be 1.

However, $G(0) = \int \widehat{G} > 1$, hence the continuous nonnegative function G can not tile $\mathbb{R}^d$ at level 1. This contradiction proves that there is no function F with the given properties as supposed at the outset. That is, Ω is a Stechkin-Turán domain. $\square$

2.8. Some further results on the Turán constant

2.8.1. Another proof that the ball is a Turán domain. Here we give a new proof, rather different from that in [**29**], that the ball is a Stechkin-Turán domain. Recall that the result was already proved by Siegel in 1935, see [**84**]. Let B denote the unit ball in $\mathbb{R}^d$. To say that the ball is Turán is to prove the inequality

$$\int f \le 2^{-d}|B|f(0) \tag{2.95}$$

for every positive definite f supported in B. By an easy approximation argument it is enough to prove (2.95) under the extra assumption that f is smooth. Noticing further that both sides of the inequality are linear functionals of f invariant under rotation, we can assume that f is radial by examining the spherical average of f

$$\int f(Tx)\, dT$$

the integral being over $T \in O_n$, the group of all orthogonal transformations equipped with Haar measure.

The key ingredient in the proof is the following result.

THEOREM 2.8.1 (Rudin [**78**]). *Suppose f is a radial smooth positive definite function with support in the ball B. Then f can be written as a uniformly convergent series*

$$f = \sum_{k=1}^{\infty} f_k * \widetilde{f_k}, \quad (\widetilde{f_k}(x) = \overline{f_k(-x)}), \tag{2.96}$$

with f_k being smooth and supported in the half ball $(1/2)B$.

Notice that for any integrable function g with support in the compact set K we have for $f = g * \widetilde{g}$

$$\begin{aligned}
\int f &= \left|\int g\right|^2 \\
&\le \int |g|^2 |K| \quad \text{(Cauchy-Schwartz)} \\
&\le \int |g|^2 \cdot 2^{-d}|K-K| \quad \text{(Brunn-Minkowski)} \\
&= f(0)\cdot 2^{-d}|K-K|.
\end{aligned}$$

Taking $K = \overline{(1/2)B}$ we obtain for every f_k in (2.96):

$$\int f_k * \widetilde{f_k} \le 2^{-d}|B|(f_k * \widetilde{f_k})(0).$$

Summing the series we have (2.95) for f.

2.8.2. Comparison of Theorems 2.6.1, 2.6.3 and 2.7.1, 2.7.2. Now we show that there are cases when Theorems 2.7.1 and 2.7.2 give provably better results than any application of Theorems 2.6.1 and 2.6.3, respectively. For this we use one of Tao's [**88**] recent examples which show one direction of Fuglede's conjecture to be false.

First we give an example when Theorem 2.7.1 gives a better bound than any possible application of Theorem 2.6.1. Let $G = \mathbb{Z}_2^{12}$ and $H = \{e_1, e_2, \dots, e_{12}\}$, where e_i is the vector in G with all zeros except at the i-th position where we have 1. The set H was recently shown by Tao [**88**] to have a spectrum, and it is clear that H cannot tile G since $|H| = 12$ does not divide $|G| = 2^{12}$.

Let $\Omega = H - H$. This means that Ω consists of the all-zero vector plus all vectors in G with precisely two 1's, hence $|\Omega| = \binom{12}{2} + 1 = 67$.

By Theorem 2.7.1 we have that if $f : G \to \mathbb{C}$ is a positive definite function supported on Ω then

$$\sum_{x\in G} f(x) \le 12 f(0).$$

Suppose now that Theorem 2.6.1 applies with some $\Lambda \subseteq G$, such that $\Omega \cap (\Lambda - \Lambda) = \{0\}$. Since $\Omega = H - H$ this implies that $H + \Lambda \le G$ is a packing at level 1, hence $|\Lambda| \le \frac{1}{12}|G|$. In fact $|\Lambda| < \frac{1}{12}|G|$ as $|\Lambda|$ is an integer but $\frac{1}{12}|G|$ is not. Clearly then (2.51) is inferior than $\sum_{x\in G} f(x) \le 12f(0)$ given by Theorem 2.7.1.

Tao [**88**] also shows how to construct a domain (in fact, a finite union of unit cubes) in $\mathbb{R}^d$, $d \ge 5$, which is spectral but not a translational tile. Suppose H is such a domain. Theorem 2.7.2 shows that $\mathcal{T}_{\mathbb{R}^d}(H - H) \le |H|$. We claim that Theorem 2.6.3 gives a worse upper bound for the set $\Omega = H - H$. Indeed, suppose that $\Lambda \subseteq \mathbb{R}^d$ is a set for which

$$\Omega \cap (\Lambda - \Lambda) = \{0\},$$

as required by Theorem 2.6.3, and that ρ is the upper density of Λ. Condition (2.97) means that $H + \Lambda$ is a packing, hence $|H| \operatorname{dens} \Lambda \le 1$. The fact that H is not a tile implies (this requires a proof, an easy diagonal argument) that the inequality above is strict, so that $1/\rho > |H|$, which shows that any application of Theorem 2.6.3 gives a worse result than Theorem 2.7.2 for $H - H$.

2.9. The pointwise Turán problem

2.9.1. Preliminaries. Formulation of the Equivalence Results. Note that in the above definitions (2.2), (2.3) or (2.7), (2.8) it is left a bit unclear, what function classes are considered as $\mathbb{R}^d \to \mathbb{R}$, $\mathbb{T}^d \to \mathbb{R}$ or $\mathbb{T} \to \mathbb{R}$. However, this causes no ambiguity, since it is not hard to see that the extremal problems (2.4), (2.5), (2.9) or (2.10) yield the same extremal values when e.g., integrable functions (with continuity of f supposed only at z in case of (2.4) or (2.5)) are considered, and when e.g., compactly supported C^∞ functions are taken into account. Indeed, on $\mathbb{T}$ or $\mathbb{T}^d$ this follows after a convolution by e.g. the Fejér kernels. The same way we can restrict ourselves even to trigonometric polynomials in $\Phi(H)$ or $\Phi_m(H)$ as well.

Passing on to the case of the real space $\mathbb{R}^d$, first we show that it suffices to consider *bounded* open sets only. To this end let us consider the auxiliary positive definite function

$$\Delta_R(x) := \frac{1}{|B_{R/2}|}\chi_{B_{R/2}} * \chi_{B_{R/2}} \tag{2.97}$$

with $B_r := \{x \in \mathbb{R}^d \,:\, |x| \le r\}$, and take $f_N := f\Delta_N$ to obtain

$$\mathcal{M}(\Omega, z) = \lim_{N\to\infty} \mathcal{M}(\Omega_N, z) = \lim_{N\to\infty} \mathcal{M}(\operatorname{int}\Omega_N, z),$$

where $\Omega_N := \{x \in \Omega \,:\, |x| \le N\} = \Omega \cap B_N$, and thus $\Omega_N \subseteq \operatorname{int}\Omega_{N+1}$.

Next observe that for any bounded open Ω, the condition $\operatorname{supp} f \subseteq \Omega$ entails that $\operatorname{supp} f$ is compact and of a fixed positive distance η from the boundary of Ω. Thus convolution of f with the (convolution) square of some approximate identity k_δ with $\operatorname{supp} k_\delta \subseteq B_\delta$ leads to a function $f_\delta := f * k_\delta * k_\delta$ satisfying $\operatorname{supp} f_\delta \subseteq \operatorname{supp} f + B_{2\delta} \subseteq \Omega$ if $\delta < \frac{1}{2}\eta$. Hence with a smooth k_δ we have $f_\delta \in \mathcal{F}(\Omega) \cap C^\infty(\Omega)$, while for arbitrary fixed $\epsilon > 0$ and with δ correspondingly small enough $f_\delta(z) \ge f(z) - \epsilon$ in view of the continuity of f at z.

Now let us define for $z \in \Omega$ the derived set

$$H(\Omega, z) := \{k \in \mathbb{N}_2 \,:\, kz \in \Omega,\ -kz \in \Omega\} \tag{2.98}$$

Our first goal is to show that in fact the Boas-Kac type Problem 2.1.2 is a one-dimensional problem. This is contained in the following result.

THEOREM 2.9.1 (**Kolountzakis-Révész**). *Let $0 \in \Omega \subseteq \mathbb{R}^d$ be any open set and $z \in \Omega \cap (-\Omega)$. With the above notations we have*

$$\mathcal{M}(\Omega, z) = \frac{1}{2} M(H(\Omega, z)).$$

REMARK 2.9.2. Note that in case $z \in \Omega$, $z \notin -\Omega$, we trivially conclude that $\mathcal{M}(\Omega, z) = 0$ since for all $f \in \mathcal{F}(\Omega)$, $\operatorname{supp} f \subseteq \Omega \cap (-\Omega)$ follows from (2.103) below. Also $0 \in \Omega$ is necessary, for a positive definite function f must vanish a.e. if $0 \notin \operatorname{supp} f$.

To tackle the Turán-type Problem 2.1.4, one may consider $f \in L^1(\mathbb{T}^d)$ with continuity supposed at z, or even $f \in C^\infty(\mathbb{T}^d)$.

Here positive definiteness of f is equivalent to $\widehat{f}(n) \ge 0 \quad (\forall n \in \mathbb{Z}^d)$, and similarly to (2.103), one gets $f(x) = f(-x) \quad (\forall x \in \mathbb{T}^d)$. Thus $\operatorname{supp} f$ is symmetric, hence $\operatorname{supp} f \subseteq \Omega \cap (-\Omega)$.

Once again we see that (2.5) vanishes unless $z \in \Omega \cap (-\Omega)$ and that it suffices to restrict ourselves to sets symmetric about the origin. In other words, if $z \notin \Omega$ or if $z \notin (-\Omega)$, then $\mathcal{M}^*(\Omega, z) = 0$, while for $z = 0$ obviously $\mathcal{M}^*(\Omega, 0) = 1$. These are the trivial cases, and for the remaining cases we introduce a further notation. Put

$$\mathcal{Z} := \mathcal{Z}(z) := \{nz \,(\operatorname{mod} \mathbb{T}^d) \mid n \in \mathbb{Z}\}. \tag{2.99}$$

The set $\mathcal{Z}$ is finite if and only if we have $z \in \mathbb{Q}^d$, that is, $z = (\frac{p_1}{q_1}, \dots, \frac{p_d}{q_d})$ with p_j, $q_j \in \mathbb{Z}$, $(p_j, q_j) = 1$ $(j = 1, \dots, d)$. In this case we have with $m = [q_1, \dots, q_d]$, the least common multiple of the denominators, that $mz = 0 \,(\mathrm{mod}\, \mathbb{T}^d)$, and for arbitrary $n, n' \in \mathbb{Z}$ $nz = n'z \,(\mathrm{mod}\, \mathbb{T}^d)$ if and only if $n \equiv n' \,(\mathrm{mod}\, m)$.

Let us keep the definition (2.98) with an interpretation $(\mathrm{mod}\ \mathbb{T}^d)$ for infinite $\mathcal{Z}$. On the other hand, in case $\#\mathcal{Z} = m$ we put

$$H_m(\Omega, z) := \{k \in [2, m/2] \,:\, kz \in \Omega,\ -kz \in \Omega\} = H(\Omega, z) \cap [2, m/2]. \tag{2.100}$$

Moreover, for any set $H \subset \mathbb{Z}$ we define

$$H(m) := \{k \in [2, m/2] \,:\, \exists h \in H \text{such that} \pm k \equiv h \pmod{m}\}.$$

REMARK 2.9.3. Note the following relations for an arbitrary $H \subseteq \mathbb{N}_2$. First, if there exists any index $k \in H$ with $k \equiv 1 \pmod{m}$, then we obtain $M_m(H) = \infty$, because $1 + a\cos 2\pi t - a\cos 2k\pi t$ is nonnegative at j/m for all $j = 1, \dots, m$ and for any $a \in \mathbb{R}$. Similarly, for $k \equiv \ell \pmod{m}$ $\cos 2k\pi t - \cos 2\ell\pi t$ vanishes at all points of the form j/m, hence the frequencies can be changed mod m to reduce φ to a trigonometric polynomial of degree at most m. Moreover, since this can be used even for negative indices, and as $\cos(-k2\pi t) = \cos k2\pi t$, we can reduce the support of $\widehat{\varphi}$ to $[0, m/2]$. That is, either $M_m(H) = \infty$ (in case there is a $k \in H$ with $k \equiv \pm 1 \pmod{m}$), or $M_m(H) = M_m(H(m))$.

Now we can formulate

THEOREM 2.9.4 (**Kolountzakis-Révész**). *Let $0 \in \Omega \subseteq \mathbb{T}^d$ be any open set and $z \in \Omega \cap (-\Omega)$. Then the extremal quantity* (2.5) *depends only on the set $\mathcal{Z}$. In case $\mathcal{Z}$ is infinite, we have*

$$\mathcal{M}^*(\Omega, z) = \frac{1}{2} M(H(\Omega, z)). \tag{2.101}$$

In case $\#\mathcal{Z} = m$ is finite, we have

$$\mathcal{M}^*(\Omega, z) = \frac{1}{2} M_m(H_m(\Omega, z)). \tag{2.102}$$

2.9.2. Proof of Theorem 2.9.1. First note that it suffices to consider symmetric sets $\Omega' = \Omega \cap (-\Omega)$ only. Indeed, if Ω is arbitrary, and $f \in \mathcal{F}(\Omega)$, $f \in C_0^\infty(\mathbb{R}^d)$, then by $\widehat{f} \geq 0$ Fourier inversion yields

$$f(x) = \overline{f(x)} = \overline{\int \widehat{f}(y) e^{2\pi i \langle x, y \rangle} dy} = \int \widehat{f}(y) e^{-2\pi i \langle x, y \rangle} dy = f(-x). \tag{2.103}$$

Thus for all $f \in \mathcal{F}(\Omega)$ $\mathrm{supp}\, f$ is necessarily symmetric. On the other hand, $H(\Omega, z)$ is symmetrized by definition (2.98) with respect to Ω. Hence we can restrict ourselves to symmetric sets. Without loss of generality we can assume that Ω is also bounded.

Now given a bounded symmetric open set Ω the proof consists of proving the two inequalities below.

$\mathcal{M}(\Omega, z) \leq M(H(\Omega, z))/2$

Let f have $f(0) = 1$, be positive definite and have support in Ω. Define also the positive definite Radon measure

$$\mu_z := \sum_{k \in \mathbb{Z}} \delta_{kz}.$$

The function f being continuous, the measure

$$\nu_z = f \cdot \mu_z = \sum_{k \in \mathbb{Z}} f(kz) \delta_{kz} \tag{2.104}$$

is well defined and positive definite as well.

Notice now, because of the boundedness of Ω, that the sum in (2.104) is actually a finite one. More precisely, if we have e.g., $\Omega \subseteq B_n$, then we find

$$\nu_z := \sum_{k=-(n-1)}^{n-1} f(kz)\delta_{kz} = \delta_0 + f(z)(\delta_z + \delta_{-z}) + \sum_{k \in H(\Omega,z)} f(kz)(\delta_{kz} + \delta_{-kz}),$$

and that

$$0 \leq \widehat{\nu_z}(x) = 1 + 2f(z) \cos 2\pi \langle z, x \rangle + \sum_{k \in H(\Omega,z)} 2f(kz) \cos 2\pi k \langle z, x \rangle, \quad (x \in \mathbb{R}^d).$$

Setting $t = \langle z, x \rangle$ and observing that the trigonometric polynomial

$$1 + 2f(z) \cos 2\pi t + \sum_{k \in H(\Omega,z)} 2f(kz) \cos 2\pi kt$$

is nonnegative, we obtain $2f(z) \leq M(H(\Omega, z))$.

$\mathcal{M}(\Omega, z) \geq M(H(\Omega, z))/2$

For a function $\varphi : \mathbb{T} \to \mathbb{R}$ let us call the *(restricted) spectrum* of φ the set $S := S(\varphi) := \operatorname{supp} \widehat{\varphi} \cap \mathbb{N}_2 \subseteq \mathbb{N}_2$. Also, we will use the term *full spectrum* and the notation $S' := S'(\varphi)$ for the set $S' := \{-1, 0, 1\} \cup S \cup (-S)$, whether the exponential Fourier coefficients at $-1, 0$ or 1 happen to vanish or not.

Take any trigonometric polynomial $\varphi \in \Phi(H)$ with spectrum $S \subseteq H := H(\Omega, z)$. Recall that taking the supremum in (2.9) over the function class (2.7) yields the same result as considering such trigonometric polynomials only. Consider the measure

$$\alpha_z := \delta_0 + (\lambda/2)(\delta_z + \delta_{-z}) + \sum_{k \in S} (c_k/2)(\delta_{kz} + \delta_{-kz}),$$

whose Fourier transform is essentially equal to the polynomial $\varphi(t)$ in (2.7). Hence α_z is a positive definite measure.

Take now the "triangle function" Δ_ϵ defined as in (2.97), but here with a subscript ϵ small enough to guarantee that

(1) The sets $kz + B_\epsilon$, $k \in S'$, are disjoint, i.e., $\epsilon < \frac{|z|}{2}$, and
(2) These sets are all contained in Ω, i.e., $\epsilon < \operatorname{dist}\{\partial\Omega, S'z\}$.

Finally define

$$f := \alpha_z * \Delta_\epsilon,$$

which is a positive definite function supported in Ω with value 1 at the origin and with $f(z) = \lambda/2$. This proves that $\mathcal{M}(\Omega, z) \geq M(H(\Omega, z))/2$, as desired.

2.9.3. Applications of Theorem 2.9.1. The first application concerns the original convex case of the pointwise Boas-Kac type problem formulated in Problem 1. A symmetric, bounded convex domain with nonempty interior – that is, a *convex body* – defines a norm. So for a vector x let $||x||$ denote the norm of x defined by Ω, that is

$$||x|| := \inf\left\{\lambda > 0 : \ \frac{1}{\lambda}x \in \Omega\right\}.$$

In other words, Ω is the unit ball of the norm $|| \cdot ||$.

COROLLARY 2.9.5. **(Boas – Kac [13]).** *Let $\Omega \subseteq \mathbb{R}^d$ be a convex open domain, symmetric about 0. Suppose that*

$$\frac{1}{n+1} \leq ||z|| < \frac{1}{n}, \tag{2.105}$$

for some $n \geq 1$. Then

$$\mathcal{M}(\Omega, z) = \cos\frac{\pi}{n+2}.$$

PROOF OF COROLLARY 2.9.5. First observe that for the symmetric, convex, bounded, open set Ω the norm of z satisfies (2.105) if and only if $H(\Omega, z) = [2, n]$. Thus by Theorem 2.9.1 the problem reduces to the extremal problem

$$M_n := \sup\{\lambda \ : \ \exists\varphi(t) \geq 0, \varphi(t) = 1 + \lambda\cos 2\pi t + \sum_{k=2}^{n} c_k \cos 2\pi kt\}. \tag{2.106}$$

This problem was settled by Fejér, see e.g., [**24**] or [**25**, p. 869-870]. To finish the proof, we quote from these or from [**68**, Problem VI. 52, p. 79] the formula

$$M_n = 2\cos\frac{\pi}{n+2}. \tag{2.107}$$

□

Note that [**8**, Theorem 2] gave the estimate $\frac{n}{n+1} \leq \mathcal{M}(\Omega, z) \leq \frac{1}{2}(1 + \cos(\frac{\pi}{n+1}))$ for the one-dimensional case. The above exact solution and some calculation shows that both of these estimates are sharp for $n = 1$, but none of them is for $n > 1$. However, this is covered (at least for $d = 1$) by [**13**, Theorem 2].

Now the $n \to \infty$ limiting case easily leads to

COROLLARY 2.9.6. **(Boas – Kac [13]).** *Suppose that the open set $\Omega \subseteq \mathbb{R}^d$ contains all integer multiples of the point $z \in \mathbb{R}^d$. Then $\mathcal{M}(\Omega, z) = 1$.*

Moreover, we also derive easily the d-dimensional extension of [**13**, Theorem 3].

COROLLARY 2.9.7. **(Boas – Kac).** *Suppose that for some $n \in \mathbb{N}$ the open set $\Omega \subset \mathbb{R}^d$ contains no integer multiples kz of the point $z \in \mathbb{R}^d$ with $k > n$. Then we have again $\mathcal{M}(\Omega, z) \le M_n = 2\cos\frac{\pi}{n+2}$.*

Apart from the convex case there are several cases of (2.4) when through the trigonometric extremal problem (2.9) either the precise value, or at least some estimate can be found.

THEOREM 2.9.8 (**Kolountzakis-Révész**). *Let Ω be a symmetric open set and $z \in \Omega$. Then the value of the extremal quantity* (2.4) *satisfies the following relations.*

(i) *If $H(\Omega, z) = \{n\}$, then $\mathcal{M}(\Omega, z) = \frac{1}{2\cos\frac{\pi}{2n}}$.*
(ii) *If $H(\Omega, z) = \mathbb{N}_2 \setminus \{n\}$, then $\mathcal{M}(\Omega, z) = \cos\frac{\pi}{2n}$.*
(iii) *If $H(\Omega, z) = (n, \infty) \cap \mathbb{N}_2$, then $\mathcal{M}(\Omega, z) = \frac{1}{2\cos\frac{\pi}{n+2}}$.*
(iv) *If $H(\Omega, z) = 2\mathbb{N} + 1$, then $\mathcal{M}(\Omega, z) = \frac{2}{\pi}$.*
(v) *If $H(\Omega, z) = 2\mathbb{N}$, then $\mathcal{M}(\Omega, z) = \frac{\pi}{4}$.*

REMARK 2.9.9. The extremal quantities $\mathcal{M}$ and M are monotonic in the sets Ω and H, respectively, hence the above relations imply the corresponding inequalities when we know only that e.g., $nz \in \Omega$, etc. We skip the formulation.

PROOF OF THEOREM 2.9.8. In view of Theorem 2.9.1, the calculation of $\mathcal{M}(\Omega, z)$ hinges on finding the value of $M(H(\Omega, z))$. The solutions of the corresponding trigonometric polynomial extremal problems, relevant to the above list (i)-(v), can be looked up from the literature as follows.

(i) An easy calculation, see e.g., [**70**].
(ii) See [**70**], Proposition 1.
(iii) See [**73**].
(iv) See the end of [**85**].
(v) See [**71**, p. 492-493].

□

When $\mathcal{M}(\Omega, z)$ is known for a certain $H(\Omega, z)$, then further cases can be obtained via the following duality result.

LEMMA 2.9.10. **(see [70]).** *Let $H \subseteq \mathbb{N}_2$ be arbitrary. Then we have*

$$M(H)M(\mathbb{N}_2 \setminus H) = 2.$$

In fact, this gives (ii) once (i) is known; (iii) and Corollary 2.9.7 and also (iv) and (v) are similarly related, although they were obtained differently in the works mentioned above.

To formulate the corresponding relation in Problem 2.1.2 we can record

COROLLARY 2.9.11 (**Kolountzakis-Révész**). *For any open set $\Omega \subseteq \mathbb{R}^d$ and $z \in \Omega$ we have*

$$\mathcal{M}(\Omega, z)\mathcal{M}(\Omega^*, z) = \frac{1}{2},$$

where Ω^* is any open, symmetric set containing 0, z and $(\mathbb{N}_2 \setminus H(\Omega, z))z$, but disjoint from $H(\Omega, z)z$.

Ending this section, let us recall that investigation of Turán-type problems started with keeping an eye on number theoretic applications and connected problems. The interesting papers of Gorbachev and Manoshina [**33, 35**] mention [**57**]; applications to van der Corput sets were mentioned in the introduction. Here we mention another question of a number theoretic relevance.

Problem 2.9.12. Determine

$$\Delta(n) := \sup\{M(H)/2 \ : \ H \subseteq \mathbb{N}_2, |H| = n\}.$$

We only know (cf [**70**])

$$1 - \frac{5}{(n+1)^2} \leq \Delta(n) \leq 1 - \frac{0.5}{(n+1)^2}.$$

The question is relevant to the Beurling theory of generalized primes, see [**74**].

2.9.4. Proof of Theorem 2.9.4. As above, without loss of generality we can restrict ourselves to sets Ω symmetric about the origin. Similarly to the proof of Theorem 2.9.1, we are to prove two inequalities for both cases.

Case $\#\mathcal{Z} = \infty \ : \ \mathcal{M}^*(\Omega, z) \leq M(H(\Omega, z))/2$

Let $f \in \mathcal{F}^*(\Omega) \cap \ C^\infty(\mathbb{T}^d)$. We consider the measure

$$\sigma_z^{(N)} := \sum_{k=-N}^{N} (1 - \frac{|k|}{N}) \delta_{kz}.$$

This measure is positive definite since for all $n \in \mathbb{Z}^d$ we have

$$\widehat{\sigma_z^{(N)}}(n) = \int_{\mathbb{T}^d} e^{-2\pi i \langle n, x\rangle} d\sigma_z^{(N)}(x) = \sum_{k=-N}^{N} (1 - \frac{|k|}{N}) e^{2\pi i \langle n, z\rangle} =: K^{(N)}(2\pi\langle n, z\rangle),$$

where $K^{(N)}$ is the usual Fejér kernel, which is nonnegative. Let us denote $H(N) := H(\Omega, z) \cap [2, N]$.

The function f being continuous and even, the measure

$$\rho_z := f \cdot \sigma_z^{(N)} = f(0)\delta_0 + \sum_{k \in \{1\} \cup H(N)} (1 - \frac{k}{N}) f(kz)(\delta_{kz} + \delta_{-kz}) \tag{2.108}$$

is well defined and, by $\widehat{\rho_z} = \widehat{f} * \widehat{\sigma_z^{(N)}}$, is positive definite as well. In view of $f(0) = 1$ we now find for arbitrary $n \in \mathbb{Z}^d$ that

$$0 \leq \widehat{\rho_z}(n) = 1 + (2 - \frac{2}{N}) f(z) \cos 2\pi\langle z, n\rangle + \sum_{k \in H(N)} (2 - \frac{2k}{N}) f(kz) \cos 2\pi k \langle z, n\rangle.$$

Setting $t := \langle z, n\rangle$ yields

$$0 \leq \varphi_N(t) := 1 + 2(1 - \frac{1}{N}) f(z) \cos 2\pi t + \sum_{k \in H(N)} 2(1 - \frac{k}{N}) f(kz) \cos 2\pi k t.$$

Since $\#\mathcal{Z} = \infty$, here for the various values of $n \in \mathbb{Z}^d$ the derived variable t will be dense in $\mathbb{T}$.

Hence we can conclude that in the infinite case $\varphi_N(t) \in \Phi(H(\Omega, z))$. This gives $2(1 - \frac{1}{N})f(z) \leq M(H(\Omega, z))$ for all $N \in \mathbb{N}$. Whence the stated inequality.

Case $\#\mathcal{Z} = m < \infty \; : \; \mathcal{M}^*(\Omega, z) \leq M_m(H_m(\Omega, z))/2$

Let again $f \in \mathcal{F}^*(\Omega) \cap \; C^\infty(\mathbb{T}^d)$. Now we consider the measure

$$\sigma_{z,m} := \frac{1}{2} \sum_{k=-[\frac{m-1}{2}]}^{[\frac{m-1}{2}]} \delta_{kz} + \frac{1}{2} \sum_{k=-[\frac{m}{2}]}^{[\frac{m}{2}]} \delta_{kz}.$$

For all $n \in \mathbb{Z}^d$ we have

$$\widehat{\sigma_{z,m}}(n) = \int_{\mathbb{T}^d} e^{-2\pi i \langle n,x \rangle} d\sigma_{z,m}(x) = 1 + \sum_{k=1}^{[\frac{m-1}{2}]} \cos 2\pi k \langle n, z \rangle + \sum_{k=1}^{[\frac{m}{2}]} \cos 2\pi k \langle n, z \rangle.$$

Since $\#\mathcal{Z} = m < \infty$, where $m = [q_1, \ldots, q_d]$ with $z = (\frac{p_1}{q_1}, \ldots, \frac{p_d}{q_d})$, $(p_j, q_j) = 1$ $(j = 1, \ldots, d)$, for the various values of $n \in \mathbb{Z}^d$ the derived variable $t := \langle n, z \rangle$ will cover exactly the values of $j/m \, (\operatorname{mod} \mathbb{T})$. For these values, however, direct calculation shows that the above sum is either exactly m (in case $j \equiv 0 \, (\operatorname{mod} m)$), or vanishes. Thus, again, the measure $\sigma_{z,m}$ will be positive definite.

The function f being continuous and symmetric, the measure

$$\rho_{z,m} := f \cdot \sigma_{z,m} = f(0)\delta_0 + \sum_{k=1}^{[\frac{m-1}{2}]} f(kz)(\delta_{kz} + \delta_{-kz}) + \sum_{k=1}^{[\frac{m}{2}]} f(kz)(\delta_{kz} + \delta_{-kz}) \tag{2.109}$$

is well defined and, by $\widehat{\rho_{z,m}} = \widehat{f} * \widehat{\sigma_{z,m}}$, is positive definite as well. In view of $f(0) = 1$ we now find for all $n \in \mathbb{Z}^d$

$$0 \leq \widehat{\rho_z}(n) = 1 + 2f(z) \cos 2\pi t + \sum_{k=2}^{[\frac{m-1}{2}]} f(kz) \cos 2\pi kt + \sum_{k=2}^{[\frac{m}{2}]} f(kz) \cos 2\pi kt, \tag{2.110}$$

where $t = \langle z, n \rangle$ as above. So let us write now

$$\varphi_{z,m}(t) := 1 + 2f(z) \cos 2\pi t + \sum_{k=2}^{[\frac{m-1}{2}]} f(kz) \cos 2\pi kt + \sum_{k=2}^{[\frac{m}{2}]} f(kz) \cos 2\pi kt.$$

It follows that

$$\varphi_{z,m}(t) = 1 + 2f(z) \cos 2\pi t + \sum_{k \in H_m(\Omega,z)} c_k^* \cos 2\pi kt,$$

for some $c_k^* \in \mathbb{R}$. Similarly as above, (2.110) implies $\varphi_{z,m}(j/m) \geq 0 \, (j = 0, \ldots, m-1)$. That is, we conclude $\varphi_{z,m} \in \Phi_m(H_m(\Omega, z))$ and thus $2f(z) \leq M_m(H_m(\Omega, z))$. Hence the statement.

Case $\#\mathcal{Z} = \infty \; : \; \mathcal{M}^*(\Omega, z) \geq M(H(\Omega, z))/2$

Let φ be any trigonometric polynomial from the class (2.7). Then φ has (restricted) spectral set S and full spectrum $S' := \{-1, 0, 1\} \cup \pm S$ with $S \subseteq H := H(\Omega, z)$ necessarily finite. Note that the supremum in the definition (2.9) of $M(H(\Omega, z))$ can be restricted to the trigonometric polynomials of (2.7).

Consider the measure

$$\alpha_z = \delta_0 + (\lambda/2)(\delta_z + \delta_{-z}) + \sum_{k\in S}(c_k/2)(\delta_{kz} + \delta_{-kz}),$$

whose Fourier transform $\widehat{\alpha_z}(n) = \varphi(\langle z, n\rangle)\,(n \in \mathbb{Z}^d)$ is essentially the polynomial $\varphi(t)$ itself. Hence α_z is a positive definite measure.

Take now the "triangle function" Δ_ϵ, defined in (2.97), with a parameter ϵ small enough to guarantee that

(1) The sets $kz + B_\epsilon$, $(k \in S')$, are disjoint, and
(2) These sets are all contained in Ω, i.e., $\epsilon < \operatorname{dist}\{\partial\Omega, S'z\}$.

Since we consider only a finite subset S of H, and $S' = \{-1, 0, 1\} \cup \pm S)$, these conditions are met with some positive ϵ as no two different multiples of z are equal in $\mathbb{T}^d$. Finally define

$$f := \alpha_z * \Delta_\epsilon,$$

which is a positive definite function supported in Ω with value 1 at the origin and with $f(z) = \lambda/2$. This proves that $\mathcal{M}^*(\Omega, z) \geq \lambda/2$, hence taking supremum over all polynomials $\varphi \in \Phi(H)$ concludes the proof.

Case $\#\mathcal{Z} = m < \infty\,:\, \mathcal{M}^*(\Omega, z) \geq M_m(H_m(\Omega, z))/2$

We denote here $H := H_m(\Omega, z)$. Now take any φ in (2.8).

Consider the measure

$$\alpha_z = \delta_0 + (\lambda/2)(\delta_z + \delta_{-z}) + \sum_{k<\frac{m}{2}, k\in H}(c_k/2)(\delta_{kz} + \delta_{-kz}) + c_{m/2}\delta_{mz/2},$$

with the last term appearing only if m is even and $m/2$ belongs to the spectral set (2.100). Observe that for the true spectrum of this measure we have

$$S^* := \operatorname{supp}\widehat{\alpha_z} := S^*(\alpha_z) \subseteq \{-1, 0, 1\} \cup \pm H \setminus \{-m/2\} = S' \setminus \{-m/2\}, \tag{2.111}$$

where the last term $(\setminus\{-m/2\})$ appears only if m is even. Thus it is easy to see that the multiples $kz\,(k \in S^*)$ are different even in $\mathbb{T}^d$.

Now let us prove that α_z is positive definite. Taking $n \in \mathbb{Z}^d$ arbitrarily, consider the Fourier transform

$$\widehat{\alpha_z}(n) = 1 + \lambda \cos 2\pi\langle z, n\rangle + \sum_{k<\frac{m}{2}, k\in H} c_k \cos 2\pi k\langle z, n\rangle + c_{m/2}e^{-im\pi\langle z,n\rangle}.$$

Here, by the condition $\langle z, n\rangle = j/m$ for some integer j, we have in the last term $e^{-m\pi\langle z,n\rangle} = (-1)^j = \cos \pi j = \cos m\pi\langle z, n\rangle$ and we get $\widehat{\alpha_z}(n) = \varphi(\langle z, n\rangle) = \varphi(j/n)$. It follows that $\widehat{\alpha_z}(n) \geq 0$ by definition (2.8).

Take now the "triangle function" Δ_ϵ defined in (2.97) with a parameter ϵ small enough to ensure

(1) The sets $kz + B_\epsilon$, $(k \in S^*)$, are disjoint, and
(2) These sets are all contained in Ω, i.e., $\epsilon < \text{dist}\{\partial\Omega, S^* z\}$.

These conditions are met with some positive ϵ since no two different multiples kz $(k \in S^*)$ are equal in $\mathbb{T}^d$, and by definitions (2.8) and (2.111) we necessarily have $S^* z \subseteq \Omega$.

Finally define

$$f = \alpha_z * \Delta_\epsilon,$$

which is a positive definite function supported in Ω with value 1 at the origin and with $f(z) = \lambda/2$. This proves that $\mathcal{M}^*(\Omega, z) \geq \lambda/2$, hence taking supremum over all polynomials $\varphi \in \Phi_m(H)$ concludes the proof.

2.9.5. Applications of Theorem 2.9.4 and further connections. Arestov, Berdysheva and Berens [**8**] mention the one dimensional symmetric interval special case of the following fact.

Proposition 2.9.13 (**Kolountzakis-Révész**). *Suppose $\Omega \subseteq (-\frac{1}{2}, \frac{1}{2})^d$ is an open set. Then*

$$\mathcal{M}(\Omega, z) \leq \mathcal{M}^*(\Omega, z).$$

Proof. The original proof of [**8**] uses the natural periodization of functions $f \in \mathcal{F}(\Omega)$. Taking $g(x) := \sum_{n \in \mathbb{Z}^d} f(x - n)$ maps $\mathcal{F}(\Omega)$ injectively to $\mathcal{F}^*(\Omega)$, which proves the Proposition. However, we have also an alternative argument here, as Theorems 2.9.1 and 2.9.4 translate the extremal problems in question to extremal problems for trigonometric polynomials. In case $\#\mathcal{Z} = \infty$ the $\mathbb{R}^d$ and $\mathbb{T}^d$ interpretations of (2.98) give $H_{\mathbb{R}^d}(\Omega, z) \subset H_{\mathbb{R}^d}(\Omega + \mathbb{Z}^d, z) = H_{\mathbb{T}^d}(\Omega, z)$. For $\#\mathcal{Z} = m < \infty$ $H_{\mathbb{R}^d}(\Omega, z) \subseteq [2, m-2]$. Indeed, $-z \in \Omega \subseteq (-\frac{1}{2}, \frac{1}{2})^d$, and as $0 \neq mz$ but $mz \equiv 0 \pmod{\mathbb{T}^d}$, we obtain that $(m-1)z \notin \Omega$ in $\mathbb{R}^d$, and similarly for $k \geq m$ $kz \notin [-\frac{1}{2}, \frac{1}{2})^d$ excludes the possibility of $k \in H_{\mathbb{R}^d}(\Omega, z)$. Thus it is easy to see that

$$M_m(H_m(\Omega, z)) = M_m(H(\Omega, z) \cap [2, m-2]) = M_m(H_{\mathbb{R}^d}(\Omega, z)). \tag{2.112}$$

Now it is obvious that $\Phi_m(H) \supseteq \Phi(H)$ and thus $M_m(H) \geq M(H)$ for arbitrary $H \subseteq \mathbb{N}_2$, and we get the assertion even for the finite case. □

Corollary 2.9.14 (**Kolountzakis-Révész**). *Let $\Omega \subseteq (-\frac{1}{2}, \frac{1}{2})^d$ be a convex, symmetric domain. Then we have*

$$\mathcal{M}^*(\Omega, z) \geq w(||z||), \qquad \text{where} \qquad w(t) := \cos \frac{\pi}{\lceil 1/t \rceil + 1}.$$

Proof. Corollary 2.9.5 gives $\mathcal{M}(\Omega, z) \geq w(||z||)$. Thus combining Proposition 2.9.13 and Corollary 2.9.5 proves the assertion. □

REMARK 2.9.15. The above estimate is a sharpening of (14) in [**8**, Theorem 3].

The following assertion is obvious both directly and by Theorem 2.9.1.

PROPOSITION 2.9.16 (**Kolountzakis-Révész**). *For all open sets* $\Omega \subseteq \mathbb{R}^d$ *and* $z \in \mathbb{R}^d$, $\alpha > 0$ *we have*

$$\mathcal{M}(\alpha\Omega, \alpha z) = \mathcal{M}(\Omega, z).$$

PROPOSITION 2.9.17 (**Kolountzakis-Révész**). *For* $\Omega \subseteq (-\frac{1}{2}, \frac{1}{2})^d$ *open,* $z \in \mathbb{T}^d$ *and* $N \in \mathbb{N}$ *we have*

$$\mathcal{M}^*(\frac{1}{N}\Omega, \frac{1}{N}z) \leq \mathcal{M}^*(\Omega, z).$$

PROOF. One can work out the generalization of the proof of [**8**, Lemma 5], which is the one-dimensional interval special case of this assertion. Instead, we note that $k\frac{1}{N}z \in \frac{1}{N}\Omega$ (mod $\mathbb{T}^d$) entails $kz \in \Omega$ (mod $\mathbb{T}^d$), and by Theorem 2.9.4 the $\#\mathcal{Z} = \infty$ case follows.

On the other hand for finite $\#\mathcal{Z}(z) = m < \infty$ we have $\#\mathcal{Z}(\frac{1}{N}z) = Nm$ and $\Phi_m(H) \supseteq \Phi_{mN}(H)$. Thus combining (2.102) and (2.112) yields

$$\begin{aligned} 2\mathcal{M}^*(\Omega, z) = &\ M_m(H_m(\Omega, z)) = M_m(H_{\mathbb{R}^d}(\Omega, z) \\ = &\ M_m(H_{\mathbb{R}^d}(\tfrac{1}{N}\Omega, \tfrac{1}{N}z) \geq M_{mN^*}(H_{\mathbb{R}^d}(\tfrac{1}{N}\Omega, \tfrac{1}{N}z) \\ = &\ M_{mN^*}(H_{mN^*}(\tfrac{1}{N}\Omega, \tfrac{1}{N}z)) = 2\mathcal{M}^*(\tfrac{1}{N}\Omega, \tfrac{1}{N}z). \end{aligned}$$

□

.

The next assertion is the generalization of [**8**, Theorem 4].

THEOREM 2.9.18 (**Kolountzakis-Révész**). *For any bounded open set* $\Omega \subset \mathbb{R}^d$ *and* $z \in \mathbb{R}^d$ *we have*

$$\lim_{\alpha \to +0} \mathcal{M}^*(\alpha\Omega, \alpha z) = \mathcal{M}(\Omega, z).$$

REMARK 2.9.19. Here the condition of boundedness ensures that for α small enough we have $\alpha\Omega \subset (-\frac{1}{2}, \frac{1}{2})^d$ and the expression under the limit on the left hand side is defined by (2.5).

PROOF. Again, extending the original arguments of [**33, 35**] or [**8**] leads to a proof. There the idea is to multiply $f \in \mathcal{F}^*(\alpha\Omega)$ by a fixed positive kernel, say $\Delta_{\frac{1}{4}}$, and exploit that for α small $\Delta_{\frac{1}{4}}|_{\alpha\Omega}$ is approximately 1.

Alternatively, we can argue as follows. Let Ω, be bounded by R, and let $\alpha < \frac{1}{2R}$: then $\alpha\Omega \subseteq (-\frac{1}{2}, \frac{1}{2})^d$. Moreover, using $\mathbb{R}^d$ interpretation of the arising sets we always have

$$H_{\mathbb{R}^d}(\Omega, z) = H_{\mathbb{R}^d}(\alpha\Omega, \alpha z) \subset \left[2, \frac{R}{|z|}\right], \tag{2.113}$$

while $m(\alpha) := \#\mathcal{Z}(\alpha z) \geq \frac{1}{\alpha|z|} \to \infty \quad (\alpha \to 0)$. Note that here for irrational α we can have $m(\alpha) = +\infty$, but defining the index function $m(\alpha)$ in this extended sense does not question the asserted limit relation.

In what follows we unify terminology by writing $H_\infty(\Theta, w) = H(\Theta, w)$ while keeping the notation $H_n(\Theta, w) = H(\Theta, w) \cap [2, n/2]$ for finite n. For the finite case we have $H_{\mathbb{R}^d}(\alpha\Omega, \alpha z) = H_{\mathbb{R}^d}(\Omega, z) \subseteq [2, \frac{m(\alpha)}{2}]$, and in view of (2.98) and (2.113) $H := H_{m(\alpha)}(\alpha\Omega, \alpha z) = H_{\mathbb{T}^d}(\alpha\Omega, \alpha z) \cap [2, \frac{m(\alpha)}{2}] = H_{\mathbb{R}^d}(\alpha\Omega, \alpha z) = H_{\mathbb{R}^d}(\Omega, z)$, too. Now if $m(\alpha) = \infty$, then we are to consider the normalized, nonnegative trigonometric polynomials $\varphi \in \Phi_\infty(H) := \Phi(H)$ defined by (2.7), while for finite $m(\alpha) < \infty$, the function set to be considered is $\Phi_m(H)$ defined by (2.8).

Let now $\alpha_n \to 0$, and φ_n be an extremal polynomial in $\Phi_{m(\alpha_n)}(H)$. In view of the nonnegativity conditions for these sets we get $|c_k| \le 2 \quad (k \in H)$, applying finite Fourier Transform in case $m(\alpha_n) < \infty$. Hence with $K := \left\lceil \frac{2R}{|z|} \right\rceil$ we find $\varphi_n \in \mathcal{F}_K := \{\varphi(t) = 1 + 2\sum_{k=1}^K a_k \cos 2\pi kt \mid |a_k| \le 1,\, k = 1, \dots, K\}$, which is a compact subset of $C(\mathbb{T})$. Thus without loss of generality we can suppose that $\varphi_n \to \phi \in \mathcal{F}_K$ uniformly as $n \to \infty$. Since $m(\alpha_n) \to \infty$, we must have $\phi \ge 0$. Moreover, if we write $\phi(t) = 1 + 2\sum_{k=1}^K a_k \cos 2\pi kt$ and $\varphi_n(t) = 1 + 2\sum_{k=1}^K a_k^{(n)} \cos 2\pi kt$, then $\lim_{n\to\infty} a_k^{(n)} = a_k$, so $\phi \in \Phi(H)$ and

$$\lim_{n\to\infty} \mathcal{M}^*(\alpha_n\Omega, \alpha_n z) = \lim_{n\to\infty} a_1^{(n)} = a_1 \le \mathcal{M}(\Omega, z).$$

On the other hand Proposition 2.9.13 gives the converse inequality. □

2.9.6. Calculations of extremal values for some special cases. Now we formulate a periodic case analogue of the Boas-Kac result Corollary 2.9.6.

PROPOSITION 2.9.20 (**Kolountzakis-Révész**). *Suppose that the open set $\Omega \subseteq \mathbb{T}^d$ contains all integer multiples of the point $z \in \mathbb{T}^d$, i.e., $\mathcal{Z} \subset \Omega$ with $\mathcal{Z}$ defined in* (2.99). *Then $\mathcal{M}^*(\Omega, z) = 1$.*

PROOF. In case $\#\mathcal{Z} = \infty$, Theorem 2.9.4 immediately gives the equality $\mathcal{M}^*(\Omega, z) = M(H(\Omega, z))/2 = M(\mathbb{N}_2)/2 = 1$. Let now $\#\mathcal{Z} = m < \infty$. Then Theorem 2.9.4 yields the equality $\mathcal{M}^*(\Omega, z) = M_m(H_m(\Omega, z))/2 = M_m([2, m/2])/2$. To see that this quantity achieves 1, it suffices to consider the cosine polynomial

$$\varphi_m(t) := 1 + \sum_{k=1}^{[\frac{m-1}{2}]} \cos 2\pi kt + \sum_{k=1}^{[\frac{m}{2}]} \cos 2\pi kt.$$

Direct calculation proves again $\varphi_m(j/m) \ge 0 \quad (j \in \mathbb{N})$, thus $\varphi_m \in \Phi_m([2, m/2])$ and now we find $M_m([2, m/2])/2 = 1$. □

With the following applications in mind we first prove

LEMMA 2.9.21 (**Kolountzakis-Révész**). *For $m \in 2\mathbb{N}$ even we have $M_m([2, m/2)) = 1 + \cos\frac{2\pi}{m}$.*

PROOF. Let $m = 2n$ and

$$\varphi(t) = 1 + \sum_{k=1}^{n-1} c_k \cos 2\pi kt \in \Phi_m(M_m([2, n))).$$

Using the finite Fourier Transform coefficient formula and $\varphi(j/m) \geq 0 \quad (j \in \mathbb{N})$ we obtain

$$\begin{aligned} c_1 &= \frac{2}{m}\sum_{j=0}^{m-1}\varphi(\frac{j}{m})\cos\frac{2\pi j}{m} \\ &= \frac{1}{n}\sum_{l=0}^{n-1}\varphi(\frac{l}{n})\cos\frac{2\pi l}{n} + \frac{1}{n}\sum_{l=0}^{n-1}\varphi(\frac{2l+1}{m})\cos(\frac{2\pi l}{n}+\frac{\pi}{n}) \\ &\leq \frac{1}{n}\sum_{l=0}^{n-1}\varphi(\frac{l}{n}) + \frac{1}{n}\sum_{l=0}^{n-1}\varphi(\frac{l}{n}+\frac{1}{m})\cos(\frac{\pi}{n}) = 1+\cos(\frac{\pi}{n}). \end{aligned}$$

On the other hand take the cosine polynomial

$$\phi_m(t) := 1 + \sum_{k=1}^{n-1}(1+\cos\frac{\pi k}{n})\cos 2\pi kt.$$

Direct calculation gives

$$\phi_m(\frac{j}{m}) = \begin{cases} m & j \equiv 0 \pmod{m} \\ m/2 & j \equiv \pm 1 \pmod{m} \\ 0 & \text{otherwise,} \end{cases}$$

whence $\phi_m(\frac{j}{m}) \geq 0 \quad (j \in \mathbb{N})$ and $\phi_m \in \Phi_m(M_m([2,n)))$. ☐

Corollary 2.9.22. **(Arestov – Berdysheva – Berens [8])** *For dimension one we have*

(i) *For* $(p,q)=1$, q *even we have* $\mathcal{M}^*((-\frac{1}{2},\frac{1}{2}),\frac{p}{q}) = \frac{1}{2}(1+\cos\frac{2\pi}{q})$.
(ii) *For* $(p,q)=1$, q *odd we have* $\mathcal{M}^*((-\frac{1}{2},\frac{1}{2}),\frac{p}{q}) = 1$.
(iii) *For* $z \notin \mathbb{Q}$ *we have* $\mathcal{M}^*((-\frac{1}{2},\frac{1}{2}),z) = 1$.

Proof. In case (i) $\#\mathcal{Z} = q = 2r$, and $H(\Omega,z) = \mathbb{N}_2 \setminus r\mathbb{N}$, $H_q^*(\Omega,z) = [2,r-1]$. Hence in view of Theorem 2.9.4 it suffices to show that $M_q^*([2,r)) = 1+\cos(2\pi/q)$, which follows from Lemma 2.9.21. For the cases (ii) and (iii) we clearly have $\mathcal{Z} \subseteq \Omega$, hence Proposition 2.9.20 applies. ☐

Similarly to the above result of Arestov et al, we can also answer the pointwise Turán extremal problem for $\Omega = (-\frac{1}{2},\frac{1}{2})^d$.

Theorem 2.9.23 (**Kolountzakis-Révész**). *Let* $\Omega = (-\frac{1}{2},\frac{1}{2})^d \in \mathbb{T}^d$. *Then we have*

(i) $\mathcal{M}^*((-\frac{1}{2},\frac{1}{2})^d,z) = 1$ *if* $z \notin \mathbb{Q}^d$.

Moreover, if $z \in \mathbb{Q}^d$, $z = (\frac{p_1}{q_1},\ldots,\frac{p_d}{q_d})$ *with* $(p_j,q_j) = 1$, $q_j = 2^{s_j}t_j \quad (s_j \in \mathbb{N})$, $t_j \in 2\mathbb{N}+1 \quad (j=1,\ldots,d)$ *and* $m := [q_1,\ldots,q_d] = 2^s t \quad t \in 2\mathbb{N}+1$, *then we have either*

(ii) $1 \leq s = s_1 = \cdots = s_d$, *and then* $\mathcal{M}^*((-\frac{1}{2},\frac{1}{2})^d,z) = \frac{1}{2}(1+\cos\frac{2\pi}{m})$, *or*
(iii) $s=0$ *or* $\exists j$, $1 \leq j \leq d$ *with* $s_j < s$ *and then* $\mathcal{M}^*((-\frac{1}{2},\frac{1}{2})^d,z) = 1$.

Proof. Case (i) is covered by Proposition 2.9.20 above. If $z \in \mathbb{Q}^d$, then the set defined in (2.99) is finite and we have $\#\mathcal{Z} = m = [q_1,\ldots,q_d]$. Let us determine the set $H(\Omega,z)$ first. For $k \in \mathbb{N}$ we have $kz \notin \Omega$ iff $kp_j/q_j \equiv 1/2 \pmod 1 \quad (j=1,\ldots,d)$, i.e., $2kp_j/q_j \equiv 1 \pmod 2 \quad (j = 1, \ldots, d)$. It follows that $q_j | 2k \quad (j = 1, \ldots, d)$, and we can not have a solution $k \in \mathbb{N}$ if $\exists j$ so that q_j is odd, since then $2k/q_j$ must be even. Hence we can consider the case when all $s_j \geq 1$ and, by $(p_j, q_j) = 1$, all p_j is odd. Then using $p_j \in 2\mathbb{Z} + 1$ the condition becomes $2k/q_j \equiv 1 \pmod 2 \quad (j = 1, \ldots, d)$. Hence $m = [q_1, \ldots, q_d] | 2k$ and $s = s_j \quad (j = 1, \ldots, d)$ since otherwise for any $s_j < s$ we get $2k/q_j = nm/q_j = n2^{s-s_j}t/t_j \equiv 0 \pmod 2$. In all, $kz \notin \Omega$ occurs only in case (ii), while case (iii) will again be covered by Proposition 2.9.20. In case (ii), when $kz \notin \Omega$ happens, it occurs precisely for multiples of $m/2 \in \mathbb{N}$. That is, case (ii) now reduces to the determination of $\mathcal{M}^*(\Omega, z) = M_m([2, m/2))/2 = (1 + \cos 2\pi/m)/2$ in view of Theorem 2.9.4 and Lemma 2.9.21. $\square$

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CHAPTER 3

Integral concentration of idempotent trigonometric polynomials with gaps

3.1. Introduction

Let us first record here that the results in this chapter are all belong to our joint work with Aline Bonami. That is, theorems, lemmas, propositions etc. the authorship of which are not given explicitly, are all joint results of Bonami & Révész.

Put $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ for the circle, and denote $e(t) := e^{2\pi it}$ the usual exponential function adjusted to interval length 1. We will denote $e_h(t)$ the function $e(ht)$. For obvious reasons of being convolution idempotents, the set

$$\mathcal{P} := \left\{ \sum_{h\in H} e_h \ : \ H \subset \mathbb{N}, \ \#H < \infty \right\} \tag{3.1}$$

is called the set of *(convolution-)idempotent exponential (or trigonometric) polynomials*, or just *idempotents* for short.

Observe that we assume all frequencies of idempotents under consideration to be non-negative. This we can do without loss of generality since we will only be interested in the modulus of idempotents, which is not modified by multiplication by some exponential e_N. We will denote as well

$$\mathcal{T} := \left\{ \sum_{h\in H} a_h e_h \ : \ H \subset \mathbb{N}, \ \#H < \infty \, ; \ a_h \in \mathbb{C}, \ h \in H \right\} \tag{3.2}$$

the space of all trigonometric polynomials.

The starting point of our work was a conjecture in [**3**] regarding the impossibility of the concentration of the integral norm of idempotents.

Before recording the main result of the paper [**3**], let us give some notations and definitions. We first start by the notion of concentration on symmetric open sets, for which results are more complete, and proofs are more elementary.

A set E is symmetric if $x \in E$ implies $-x \in E$.

Definition 3.1.1. Let $p > 0$ and $a \in \mathbb{T}$. We say that there is p-concentration at a if there exists a constant $c > 0$ so that for any symmetric open set E that contains a, one can find an idempotent $f \in \mathcal{P}$ with

$$\int_E |f|^p \geq c \int_{\mathbb{T}} |f|^p. \tag{3.3}$$

Moreover, the supremum of all such constants c will be denoted as $c_p(a)$: it is called the *level of the p-concentration at a*. Such an idempotent f will be called a *p-concentrating polynomial.*

DEFINITION 3.1.2. Let $p > 0$. We say that there is p-concentration if there exists a constant $c > 0$ so that for any symmetric non empty open set E one can find an idempotent $f \in \mathcal{P}$ with

$$\int_E |f|^p \geq c \int_{\mathbb{T}} |f|^p. \tag{3.4}$$

Moreover, c_p will denote the supremum of all such constants c. Correspondingly, c_p is called the *level of p-concentration.* If $c_p = 1$, we say that there is *full p-concentration.*

Clearly, as remarked in [**14**], the local constant $c_p(a)$ is an upper semi-continuous function on $\mathbb{T}$, and $c_p = \inf_{a \in \mathbb{T}} c_p(a)$.

REMARK 3.1.3. We have taken symmetric open sets because the function $|f|$ is even for $f \in \mathcal{P}$. Without the assumption of symmetry, the constant $c_p(a)$ would be at most $1/2$ for a different from 0 and $1/2$. With this definition, as we will see, $c_p(a)$ and even c_p can achieve the maximal value 1. Nevertheless, using the alternative definition with arbitrary open sets (or just intervals) would only mean taking half of our constants $c_p(a)$ for $a \neq 0, 1/2$ and of c_p.

The question of p-concentration, and the computation or at least estimation of the best constant c_p, originated from the work of Cowling [**13**], and of Ash [**4**] on comparison of restricted type and strong type for convolution operators. This is described recently in the survey [**5**]. It has since then been the object of considerable interest, with improving lower bounds obtained by Pichorides, Montgomery, Kahane and Ash, Jones and Saffari, see [**1, 2, 3**] for details. In 1983 Déchamps-Gondim, Piquard-Lust and Queffélec [**14, 15**] answered a question from [**1**], proving the precise value

$$c_2 = \sup_{0 \leq x} \frac{2 \sin^2 x}{\pi x} = 0.46 \cdots . \tag{3.5}$$

Moreover, they obtained $c_p \geq 2^{1-\frac{p}{2}} c_2^{p/2}$ for all $p > 2$.

As in [**14, 15, 2, 3**], we will consider the same notion of p-concentration of (convolution-)idempotents for measurable sets, too.

DEFINITION 3.1.4. Let $p > 0$ and $a \in \mathbb{T}$. We say that there is p-concentration *for measurable sets at a*, if there exists a constant $\gamma > 0$ so that for any symmetric measurable set E, with a being a density point of E, there exists some idempotent $f \in \mathcal{P}$ with

$$\int_E |f|^p \geq \gamma \int_{\mathbb{T}} |f|^p. \tag{3.6}$$

The supremum of all such constants γ will be denoted as $\gamma_p(a)$. Furthermore, we say that there is p-concentration *for measurable sets* if such an inequality holds for any symmetric measurable set E of positive measure. The supremum of all such constants is denoted by γ_p.

It is clear that p-concentration for measurable sets implies p-concentration. On the other hand it is not clear, if $\gamma_p(a)$ is upper semicontinuous, too. If we knew this, by our methods that would easily imply the same strength of the results for measurable sets, as we will obtain for open sets.

The main theorem of [**3**] can be stated as:

THEOREM 3.1.5 (**Anderson, Ash, Jones, Rider, Saffari**). *There is p-concentration for measurable sets for all $p > 1$.*

We also refer to them for the fact that $\gamma_2 = c_2$ is given by (3.5). The proof of [**2, 3**] is based on the properties of the function

$$D_n(x)D_n(qx), \tag{3.7}$$

where D_n stands for the Dirichlet kernel. We will use the same notation as in [**3**] and define the Dirichlet kernel as

$$D_n(x) := \sum_{\nu=0}^{n-1} e(\nu x) = e^{\pi i(n-1)x}\frac{\sin(\pi n x)}{\sin(\pi x)}. \tag{3.8}$$

The idea is that the first Dirichlet kernel in (3.7) will have sufficiently peaky behavior (regarding $|\cdot|^p$), while the second one simulates a Dirac delta, so that the p-th integral outside very close neighborhoods of the points k/q is small. They use the multiplicative group structure of $\mathbb{Z}/q\mathbb{Z}$, when q is prime, to prove that concentration at k/q and concentration at $1/q$ may be compared.

Their proof yields p-concentration only with $c_p \to 0$ when $p \to 1$. Based on these and some other heuristical arguments and calculations the authors conjectured that for p-concentration the value 1 should be a natural limit. We will disprove this conjecture, even for measurable sets and we will even prove more: all concentrating idempotents can be taken with arbitrarily large gaps. Recall that the trigonometric polynomial

$$f(x) := \sum_{k=1}^{K} a_k e(n_k x) \tag{3.9}$$

has gaps larger than N if it satisfies the gap condition $n_{k+1} - n_k > N$ $(k = 1, \ldots, K-1)$. Before describing our results more precisely, we need other definitions.

DEFINITION 3.1.6. We say that there is p-concentration with gap (resp. p- concentration with gap for measurable sets) at a if for all $N > 0$ the p-concentrating polynomial in (3.3) (resp. in (3.6)) can be chosen with gap larger than N. If this holds for every a, we say that there is p-concentration with gap (resp. p-concentration with gap for measurable sets). If, moreover, the constant c can be taken arbitrarily close to 1, we say that there is full p-concentration with gap (resp. p-concentration with gap for measurable sets).

With these definitions, we can give our main theorems.

THEOREM 3.1.7. *For all $0 < p < \infty$, we have p-concentration. Moreover, if p is not an even integer, then we have full concentration, i.e. $c_p = 1$. When considering even integers, we have c_2 given by* (3.5)*, then $0.495 < c_4 \le 1/2$, and for all other even integers*

$0.483 < c_{2k} \leq 1/2$. *Moreover, unless* $p = 2$, *we have concentration with gap at the same level of concentration. On the other hand for* $p = 2$ *requiring arbitrarily large gaps would decrease the level of concentration to* 0.

For measurable sets, our results are just as good for $p > 1$. Arriving at the limits of our current methods, we leave it as an open problem what happens for $p \leq 1/2$, and whether there is full concentration for $1/2 < p \leq 1$.

THEOREM 3.1.8. *For all* $1/2 < p < \infty$ *we have* p*-concentration for measurable sets. If* p *is not an even integer, then we have full concentration for measurable sets when* $p > 1$. *If* $p = 2$, *the level of the concentration is given by* (3.5), *and for* $p = 4$ *we have* $0.495 < \gamma_4 \leq 1/2$. *For other even integers we have uniformly* $0.483 < \gamma_{2k} \leq 1/2$. *Moreover, unless* $p = 2$, *the same level of concentration can be achieved with arbitrarily large gaps.*

This improves considerably the constants given in [**2, 3**], which tend to zero when $p \to \infty$ or when $p \to 1^+$ (however, to compare constants, be aware of the notational difference between us and [**3, 2**]).

We postpone to section 3.10 what concerns measurable sets. The proofs will follow from an adaptation of the methods that we develop for open sets, and also from the use of diophantine approximation. As in [**3**], we do not know whether constants γ_p and c_p differ when $p \neq 2$, except when we know that both of them are 1, which is the case of all $p > 1$ not an even integer.

Let us hint some of the key ideas in our proofs, which may be of independent interest. The first one is an explicit construction of concentrating idempotents for the points 0 and 1/2 at a level of concentration arbitrarily close to 1 and with arbitrarily large gaps. To emphasize their role in our construction, we will term such concentrating idempotents as ‘‘peaking idempotents”, or, when referring to the large gaps required, as “gap-peaking idempotents” – for a more precise meaning see the beginning of §3.3.

PROPOSITION 3.1.9. *For all* $p > 0$, *except for* $p = 2$, *one has full* p*-concentration with gap at* 0. *For* $p = 2$, *positive concentration with arbitrarily large gaps is possible at neither points* $a \in \mathbb{T}$.

Note that, using the Dirichlet kernel that peaks at 0, we find full p-concentration at 0 for $p > 1$. For $p \leq 1$, the Dirichlet kernel cannot be used. For a given concentration, our examples will be obtained using idempotents of much higher degree. So as for the behavior at point 0 and $p > 1$ different from 2, the novelty is the fact that the peaking polynomial may have arbitrarily large gaps.

This is what cannot occur in L^2, in view of Ingham's inequalities [**23, 45**]. The somewhat surprising new fact here is that it does occur for all other values of p.

Zygmund [**45**, Chapter V §9, page 380] pointed out concerning Ingham's results on essentially uniform distribution of square integrals (norms) for Fourier series with large gaps: “Nothing seems to be known about possible extensions to classes L^p, $p \neq 2$”. To the best of our knowledge the problem has not been addressed thus far. But now we find that an Ingham type inequality is characteristic to the Hilbertian case, and for no $p \neq 2$ one can have similar inequalities, not even when restricting to idempotent polynomials.

The next proposition is even more surprising. It is the key to full concentration at other points than 0.

PROPOSITION 3.1.10. *Full p-concentration with gap at $1/2$ holds whenever $p > 0$ is not an even integer. On the other hand, for $p = 2k \in 2\mathbb{N}$, $c_{2k}(1/2) = 1/2$.*

The assertion for p an even integer will follow directly from the work of Déchamps-Gondim, Lust-Piquard and Queffélec [**14, 15**].

For $0 < p < 2$ we base our argument on the properties of the bivariate idempotent $1 + e(y) + e(x + 2y)$.

For $p > 2$, we will rely on a construction of Mockenhaupt and Schlag, see [**30**], given in their work on the Hardy-Littlewood majorant problem, which we describe now in its original formulation. Following Hardy and Litlewood, f is said to be a majorant to g if $|\widehat{g}| \leq \widehat{f}$. Obviously, then f is necessarily a positive definite function. The (upper) majorization property (with constant 1) is the statement that whenever $f \in L^p(\mathbb{T})$ is a majorant of $g \in L^p(\mathbb{T})$, then $\|g\|_p \leq \|f\|_p$. Hardy and Littlewood proved this for all $p \in 2\mathbb{N}$. On the other hand, already Hardy and Littlewood observed that this fails for $p = 3$: they took $f = 1 + e_1 + e_3$ and $g = 1 - e_1 + e_3$ (where $e_k(x) := e(kx)$) and calculated that $\|f\|_3 < \|g\|_3$.

The failure of the majorization property for $p \notin 2\mathbb{N}$ was shown by Boas [**8**] (see also [**7**] for arbitrarily large constants, and also [**18, 29**] for further comments and similar results in other groups.) Montgomery conjectured that it fails also if we restrict to majorants belonging to $\mathcal{P}$, see [**31**, p. 144]. This has been recently proved by Mockenhaupt and Schlag in [**30**].

THEOREM 3.1.11 (**Mockenhaupt & Schlag**). *Let $p > 2$ and $p \notin 2\mathbb{N}$, and let $k > p/2$ be arbitrary. Then for the trigonometric polynomials $g := (1 + e_k)(1 - e_{k+1})$ and $f := (1 + e_k)(1 + e_{k+1})$ we have $\|g\|_p > \|f\|_p$.*

Our proof of Proposition 3.1.10 for $p > 2$ and $p \notin 2\mathbb{N}$, will be based on the construction of Mockenhaupt and Schlag.

Once we have our peaking polynomials at 1/2, we conclude in proving the following assertion.

PROPOSITION 3.1.12. *Let $p > 0$ and assume that we have full p-concentration with gap at $1/2$ for this value of p. Then we also have p-concentration. Moreover, $c_p = 1$ and we have full p-concentration with gap.*

The proof of Proposition 3.1.12 consists of considering products like

$$D_r(s_1 x) \cdots D_r(s_n x) T(qx), \tag{3.10}$$

where the similarity to (3.7) may be misleading in regard of the role of the Dirichlet kernels here: the role of the "approximate Dirac delta" is fully placed on T, which is a peaking function at 1/2 with large gaps that insure that the product is still an idempotent. The first factors will be chosen in such a way that they coincide with a power of a Dirichlet kernel on some grid $\frac{1}{2q}+\mathbb{Z}/q\mathbb{Z}$. For measurable sets, the use of diophantine approximation forces us to take at most two factors, resulting in the restriction $p > 1/2$.

When there is not full p-concentration at $1/2$, i.e. for $p = 2k$, we could not determine c_{2k} precisely. Still, we can use a peaking function at 0, provided by Proposition 3.1.9, thus obtaining reasonable uniform bounds.

Our final results for $p > 1$ derive from the consideration of the class of positive definite trigonometric polynomials

$$\mathcal{P}^+ := \left\{ \sum_{h\in H} a_h e_h \ : \ H \subset \mathbb{N}, \ \#H < \infty\,; \ a_h > 0 \text{ for } h \in H \right\}, \tag{3.11}$$

for which full p-concentration for measurable sets can be proved for $p > 0$ not an even integer. We then use a randomization process to transfer this result to the class $\mathcal{P}$ for $p > 2$, and then using that even to $p > 1$.

As seen above, the conjecture of Ash, Anderson, Jones, Rider and Saffari on nonexistence of L^1-concentration, described after Theorem 3.1.5, fails. Moreover, we have full concentration (for open sets), and for measurable sets the level of concentration is also considerably large. In fact, pushing our methods somewhat further, we show in Theorem 3.14.1 that the L^1 concentration constant for measurable sets satisfy even $\gamma_1 > 0.96$, quite close to 1. Moreover, once again we can ascertain this level of concentration even with arbitrarily large gaps.

Nevertheless, in a sense this all is due to a "cheating" in the extent that we can simulate powers of Dirichlet kernels by products of their scaled versions. In Theorem 3.13.3 we show, however, that on the finite groups $\mathbb{Z}/q\mathbb{Z}$ uniform in q L^1 concentration does really fail.

In summary, for open sets or positive definite polynomials on measurable sets we proved full concentration unless $p \in 2\mathbb{N}$. Furthermore, we proved that the constant γ_p is equal to 1 when $p > 1$ and p is not an even integer. As for the exceptional situation for $p \in 2\mathbb{N}$, this is in line with the fact that L^p norms behave differently depending on whether p is an even integer or not in a certain number of problems, such as the Hardy-Littlewood majorant problem (*does an inequality on absolute values of Fourier coefficients imply an inequality on L^p norms?*), as well as Zygmund's question (*does a Wiener-Ingham type essentially uniform distribution of the p-norm holds on intervals longer than $2\pi/N$, when $f \in L^p(\mathbb{T})$ has gaps exceeding N in its Fourier series?*) or the Wiener property for periodic positive definite functions (*does a positive definite function belong to L^p when it is the case on a small interval around 0?*). These we will have a closer look in Section 3.15.

Indeed, our results are built on partial results already stronger than a negative answer to Zygmund's question, posed by A. Zygmund in his classical book, see Notes to Chapter V §9, page 380 in [**45**]. The answer to this question was only partially known, in the extent that constructions were given by Erdős and Rényi [**17**] for $p \notin 2\mathbb{N}$ and $p > 2$ with an existential (probabilistic) proof, and, for $p > 6$, by Turán [**41**] with a concrete construction, (based on primes). In these examples they provided lacunarity, i.e. large gaps, in the Fourier series, while local boundedness in L^p was ascertained somewhere in a small interval not containing 0. From our constructions the negative answer to Zygmund's question for all $p \neq 2$ can easily be seen e.g. already from the combination of Propositions 3.3.1 and 3.3.4 below.

Our results are also well adapted to give counter-examples for the Wiener property. But to exploit our methods better, here in the last section we will provide joint counterexamples to these famous problems in very strong forms. To explain the results more precisely, let us discuss the Wiener problem in a bit more detail.

Let f be a periodic integrable function which is positive definite, that is, has non negative Fourier coefficients. Assume that it is bounded (in $\|\cdot\|_\infty$) in a neighborhood of 0, then it necessarily belongs to $L_\infty(\mathbb{T})$, too. In fact, its maximum is obtained at 0 and, as $f(0) = \sum_k \widehat{f}(k)$, f has an absolutely convergent Fourier series.

The same question can be formulated in any L^p space. Actually, the following question was posed by Wiener in a lecture, after he proved the L^2 case.

Wiener's Problem. *Let $1 \leq p < \infty$. Is it true, that if for some $\varepsilon > 0$ a positive definite function $f \in L^p(-\varepsilon, \varepsilon)$, then we necessarily have $f \in L^p(\mathbb{T})$, too?*

We refer to [**36**] for the story of this conjecture, see also [**26**] and [**43**]. The observation that the answer is positive if $p \in 2\mathbb{N}$ has been given by Wainger [**42**], as well as by Erdős and Fuchs [**16**]. The key of the proof is Wiener's Inequality stating that all 1-periodic positive definite trigonometric polynomials satisfy

$$\frac{1}{2a}\int_{-a}^{+a} |f|^p \geq \frac{1}{2}\int_{-1/2}^{+1/2} |f|^p \tag{3.12}$$

for $p = 2$, and hence for all $p \in 2\mathbb{N}$, see [**36**]. For optimality of constants, see [**26, 27**]. Generalizations in higher dimension may be found in [**22**] for instance.

It was shown by Shapiro [**36**] and Wainger [**42**] that the answer to Wiener's problem is to the negative for all other values of p. Negative results were obtained for other groups in e.g. [**18**] and [**26**].

There is even more evidence that the Wiener property must hold when $p = 2$ and we prescribe large gaps in the Fourier series of f. Indeed, in this case by well-known results of Wiener and Ingham, see e.g. [**43, 45**], we necessarily have an essentially uniform distribution of the L^2 norm on intervals longer than the reciprocal of the gap, even without the assumption that f be positive definite. To clarify the notions, for f with Fourier series $\sum_k a_k e^{2i\pi n_k x}$, where n_k is increasing, we define $\mathrm{Gap}(f)$ by

$$\mathrm{Gap}(f) := \min_k (n_{k+1} - n_k)\,. \tag{3.13}$$

Then we say that f has gaps tending to ∞ when $\mathrm{Gap}(f - S_N(f))$ tends to ∞, where $S_N(f)$ denote the partial sums of the Fourier series of f.

As Zygmund pointed out, see the Notes to Chapter V §9, page 380 in [**45**], Ingham type theorems were not known for $p \neq 2$, nevertheless, one would feel that prescribing large gaps in the Fourier series should lead to better control of the global behavior by means of having control on some subset like e.g. $(-\varepsilon, \varepsilon)$. So the analogous Wiener question can be posed restricting to positive definite functions having gaps tending to ∞. That is, we combine conditions of the Wiener problem (positive definiteness) and that of the Wiener-Ingham inequalities, i.e. the problem of Zygmund.

However, we answer negatively as well.

THEOREM 3.1.13. *For all $0 < p < \infty$, p not an even integer, whenever a 0-symmetric measurable set E of positive measure $|E| > 0$ is given, then for all $\varepsilon > 0$ there exists $f \in \mathcal{T}^+$ so that*

$$\int_{^cE} |f|^p \leq \varepsilon \int_{\mathbb{T}} |f|^p. \tag{3.14}$$

Moreover, f can be taken such that $\mathrm{Gap}(f)$ *is arbitrarily large. When E is an open set or when $p > 1$, then f can be chosen an idempotent for all p.*

Theorem 3.1.13 allows us to see immediately that there is no inequality like (3.12) for p not an even integer. What is new, compared to the results of Shapiro and Wainger, is the fact that this is also the case if f has arbitrarily large gaps, and that we can replace intervals $(-a, +a)$ by arbitrary measurable sets of measure less than 1.

In this strong form the question, to the best of our knowledge, has not been dealt with yet. Neither extension can be obtained by a straightforward use of the methods of Shapiro and Wainger. In fact, our construction gives a simultaneous, combined negative answer to the Wiener problem and to Zygmund's question of L^p versions of the Ingham-Wiener theorems for functions with large gaps in the Fourier series. Nevertheless, we will obtain a few further sharpening and pose some open questions, too.

Let us finally fix some notations that will be used all over. We denote

$$\mathcal{T}_q := \left\{ \sum_{h=0}^{q-1} a_h e_h \,;\; a_h \in \mathbb{C} \text{ for } h = 0, \cdots, q-1 \right\} \tag{3.15}$$

the space of trigonometric polynomials of degree smaller than q and

$$\mathcal{P}_q := \left\{ \sum_{h \in H} e_h \;:\; H \subset \{0, 1, \cdots q-1\} \right\} \tag{3.16}$$

the set of idempotents of degree smaller than q.

3.2. Negative results regarding concentration when $p \in 2\mathbb{N}$

Let us first start with proving that in case $p = 2$, requiring arbitrarily large gaps decreases the level of concentration to 0, as said in Theorem 3.1.7 and Proposition 3.1.9 (and, consequently, in Theorem 3.1.8, too).

For this there is a well known argument. We take an interval E centered at 0 and a triangular function Δ supported by $2E$ and equal to 1 at zero. Let N be an integer and f an idempotent with gap N. Then

$$\int_E |f|^2 dt \leq 2 \int \Delta |f|^2 dt = 2 \sum_m \sum_n \widehat{\Delta}(m) \widehat{f}(n) \overline{\widehat{f}(n-m)}.$$

If we write separately the term with $m = 0$ and insert $\widehat{\Delta}(0) = |E|$, then the right hand side becomes

$$2|E| \sum_n |\widehat{f}(n)|^2 + 2 \sum_{|m|>N} \widehat{\Delta}(m) \sum_n \widehat{f}(n) \overline{\widehat{f}(n-m)}.$$

Finally, by an application of the Cauchy-Schwarz inequality,

$$\int_E |f|^2 dt \leq 2|E| \sum_n |\widehat{f}(n)|^2 + 2 \sum_{|m|>N} |\widehat{\Delta}(m)| \sum_n |\widehat{f}(n)|^2.$$

According to Parseval's identity $\int_{\mathbb{T}} |f|^2 dt = \sum_n |\hat{f}(n)|^2$, hence

$$\frac{\int_E |f|^2 dt}{\int_{\mathbb{T}} |f|^2 dt} \leq 2|E| + 2 \sum_{|m|>N} |\widehat{\Delta}(m)|.$$

The last estimate can be taken arbitrarily small by taking the interval E small enough, and then the gap N large enough, using the fact that the Fourier series of Δ is absolutely convergent. This contradicts the peaking property with gap.

Remark 3.2.1. The same proof, using for Δ a triangular function supported by E, gives the reverse inequality

$$\frac{\int_E |f|^2 dt}{\int_{\mathbb{T}} |f|^2 dt} \geq \frac{|E|}{2+\epsilon},$$

valid for functions with sufficiently large gaps, depending on E and $\epsilon > 0$. These type of estimates are known as Ingham type inequalities, and various generalizations have many applications e.g. in control theory, see [**25**], [**39**], [**40**]. The fact that one can have full p-concentration with gap at 0 may be interpreted as the impossibility of an Ingham type inequality for $p \neq 2$. This settles to the negative a problem posed by Zygmund.

Next, we explain how to obtain the necessary condition $c_{2k} \leq 1/2$. In fact one knows more, since this is also valid for the problem of concentration on the class $\mathcal{P}^+$ of positive definite exponential polynomials (see (3.11)). Let us denote by c_p^+ and $c_p(a)^+$, as well as γ_p^+ and $\gamma_p^+(a)$, the corresponding concentration constants, with the class $\mathcal{P}$ of idempotents replaced by the class $\mathcal{P}^+$. One has the inequalities

$$c_p(a) \leq c_p^+(a), \quad c_p \leq c_p^+, \qquad \gamma_p(a) \leq \gamma_p^+(a) \quad \gamma_p \leq \gamma_p^+.$$

It was proved in [**14, 15**] that $c_2^+(1/2) = 1/2$. From this we obtain that for $p = 2k$ an even integer, $c_{2k}(1/2) \leq c_{2k}^+(1/2) \leq 1/2$. Indeed, if $f \in \mathcal{P}^+$, so is f^k, and using the already known value $c_2^+(1/2) = 1/2$ we infer $c_{2k}(1/2) \leq c_2^+(1/2) = 1/2$. In fact we have equality,

$$c_{2k}(1/2) = c_{2k}^+(1/2) = 1/2,$$

taking the Dirichlet kernel $D_N(2x)$ as concentrating polynomial.

While [**14, 15**] gives also $c_2^+ = 1/2$, we do not know the exact values of c_{2k} and c_{2k}^+ for $k > 1$.

We do not have any other negative result than the ones in this §.

3.3. Full concentration with gap and peaking functions

In this section, we will prove Proposition 3.1.9 and Proposition 3.1.10. For $a = 0$ or $1/2$, we are interested in the construction of gap-peaking idempotents, that is, for all ε, δ and $N > 0$, idempotent exponential polynomials

$$T(x) := \sum_{k=1}^{K} e(n_k x), \tag{3.17}$$

with gap condition $n_{k+1} - n_k > N$ $(k = 1, \dots, K)$, so that

$$\int_{a-\delta}^{a+\delta} |T|^p > (1-\varepsilon) \int_{\mathbb{T}} |T|^p. \tag{3.18}$$

The first step is to prove the following.

Proposition 3.3.1. *Let f be an idempotent exponential polynomial in two variables and of the form*

$$f(x,y) = \sum_{k=1}^{K} e(n_k x + m_k y), \tag{3.19}$$

where $K \in \mathbb{N}$ and $n_k, m_k \in \mathbb{N}$ are two sequences of nonnegative integers, with m_k strictly increasing. Assume that f has the property that its "marginal p-integral", given by

$$F(x) := \int_0^1 |f(x,y)|^p dy, \tag{3.20}$$

has a strict maximum at a, for $a = 0$ or $a = 1/2$. Then one has full p-concentration with gap at the point a.

Proof. Choose M with $0 \le m_k, n_k < M$ for all k and consider the Riesz product

$$g(x) := g_{R,J}(x) := \prod_{j=1}^{J} f(x, R^j x) \tag{3.21}$$

where R is a very large integer, f is given by (3.19) satisfying the assumption, and J will be chosen later on. If we take $R > M(J+1)$, then $g \in \mathcal{P}$; moreover, g will obey a gap condition of size N if R is large enough depending on J, M and N. Recall that the marginal p–integral (3.20) has a strict maximum at a. For any fixed interval I, the integral of $|g|^p$ on I will approach the integral of F^J on I as $R \to \infty$. Indeed,

$$\int_I |g|^p = \int_I \prod_{j=1}^{J} |f(x, R^j x)|^p dx\,,$$

and as the function $|f|^p \in C(\mathbb{T}^2)$, we can apply Lemma 3.3.2 below.

Lemma 3.3.2. *Assume that $\varphi \in C(\mathbb{T} \times \mathbb{T}^J)$. Denote the marginal integrals by $\Phi(x) := \int_{\mathbb{T}^J} \varphi(x, \mathbf{y}) d\mathbf{y}$. Then, for E a measurable set of positive measure, we have*

$$\lim_{n_1, n_2, \dots, n_J \to \infty} \int_E \varphi\left(x, n_1 x, n_1 n_2 x, \dots, n_1 n_2 \cdots n_J x\right) dx = \int_E \Phi(x) dx. \tag{3.22}$$

Here by $n_1, \ldots, n_J \to \infty$ we naturally mean $\min(n_1, \ldots, n_J) \to \infty$. For the sake of remaining self-contained, we give a proof below, even if this one is standard, mentioned also e.g. in [**29, 31, 7**] (for $J = 1$).

PROOF. By density, it is sufficient to prove this for φ an exponential polynomial on $\mathbb{T}\times\mathbb{T}^J$. By linearity, it is sufficient to consider a monomial. When it does not depend on the second variable there is nothing to prove. Assume that $\varphi(x, \mathbf{y}) = e(kx + l_1 y_1 + \cdots + l_J y_J)$, with at least one of the l_j's being nonzero. We want to prove that

$$\int_E \varphi\left(x, n_1 x, n_1 n_2 x, \ldots, n_1 n_2 \cdots n_J x\right) dx \longrightarrow 0 \qquad (n_1, \ldots, n_J \to \infty).$$

This integral is the Fourier coefficient of the characteristic function of E at the frequency $k + n_1 l_1 + n_1 n_2 l_2 + \cdots + n_1 n_2 \cdots n_J l_J$, which tends to infinity for $n_1, \ldots, n_J \to \infty$. We conclude using the Riemann-Lebesgue Lemma. □

Let us go back to our Riesz product g in (3.21). Let us first choose J large enough. Then F^J will be arbitrarily concentrated on $I := [a - \delta, a + \delta]$ in integral because F has a strict global maximum at a. More precisely, we fix J large enough so that

$$\int_I F^J > (1 - \varepsilon) \int_{\mathbb{T}} F^J.$$

Once J is fixed, we use Lemma 3.3.2 for the function

$$\varphi(x, \mathbf{y}) := \prod_{j=1}^{J} |f(x, y_j)|^p.$$

We know that

$$\lim_{R\to\infty} \int_I |g_{R,J}|^p = \int_I F^J,$$

and the same for the integral over the whole torus. The proposition is proved. □

This concludes the proof of Proposition 3.1.9, assuming that the condition of Proposition 3.3.1 holds. Next we will focus on this point.

REMARK 3.3.3. The function $|f|$ is even in the sense that $|f(-x, -y)| = |f(x, y)|$, since the quantities inside the absolute value sign are just complex conjugates. Therefore, F is even. Moreover it can have a *unique* maximum in $\mathbb{T}$ if only this maximum is either at 0 or at 1/2.

PROPOSITION 3.3.4. *Let $f(x, y) := 1 + e(y) + e(x + 2y)$. Then the marginal integral function $F_p(x) := \int_0^1 |f(x, y)|^p dy$ is a continuous function, which has a unique, strict maximum at 0 for $p > 2$, while it has a strict maximum at 1/2 for $p < 2$.*

PROOF. Since F_p is even, it suffices to prove that it is monotonic on $[0, \frac{1}{2}]$, with the required monotonicity. Note that

$$|f(x, y)| = \left|2e(x/2) \cos\left(\pi(x + 2y)\right) + 1\right|.$$

So

$$\begin{aligned}F_p(x) &= \int_{-1/2}^{1/2} |2e(x/2)\cos(2\pi y) + 1|^p \, dy \\ &= \int_{-1/4}^{1/4} \left(|2e(x/2)\cos(2\pi y) + 1|^p + |2e(x/2)\cos(2\pi y) - 1|^p\right) dy.\end{aligned}$$

It is sufficient to show that for fixed $y \in (-\frac{1}{4}, \frac{1}{4})$ the quantity

$$\Phi(x, y) := |2e(x/2)\cos(2\pi y) + 1|^p + |2e(x/2)\cos(2\pi y) - 1|^p$$

is monotonic in x for $0 < x < \frac{1}{2}$. Considering its derivative

$$\begin{aligned}\frac{\partial \Phi}{\partial x}(x, y) = &- 2p\pi \sin(\pi x)\cos(2\pi y) \\ &\times \left\{ |2e(\frac{x}{2})\cos(2\pi y) + 1|^{p-2} - |2e(\frac{x}{2})\cos(2\pi y) - 1|^{p-2} \right\}\end{aligned}$$

we find that its signum is the opposite of the signum of the difference in the second line. It follows that Φ, hence F_p has a strict global maximum at zero when $p > 2$ and a strict global maximum at $1/2$ when $p < 2$. □

This concludes for the existence of a peaking function at 0 for $p > 2$, and for a peaking function at $1/2$ for $p < 2$.

We will need the following lemma later on.

Lemma 3.3.5. *The function F_p is a $\mathcal{C}^2$ function for $p > 2$ and its second derivative at* 0 *is strictly negative. For all values of p it is a $\mathcal{C}^\infty$ function outside* 0*. Its second derivative at* $1/2$ *is strictly negative for $p < 2$.*

Proof. For $p > 2$ the smoothness of the composite function follows from smoothness of $|\cdot|^p$. We already know from monotonicity of $\Phi(x, y)$ for fixed y that $\Phi''_{xx}(0, y)$ is non positive. Since it is clearly not identically 0, it is somewhere strictly negative, hence $F''_p(0) < 0$. To prove that F_p is a $\mathcal{C}^\infty$ function outside 0, it is sufficient to remark that $f(x, y)$ does not vanish for $x \neq 0$. The same reasoning as above gives the sign of the second derivative at $1/2$. □

Proof of Proposition 3.1.10. Let us now concentrate on peaking functions at $1/2$ for $p > 2$ not an even integer and prove Proposition 3.1.10. We will prove the following, which relies entirely on the methods of Mockenhaupt and Schlag [**30**], but tailored to our needs with introducing also a second variable and slightly changing the occurring idempotents, too.

Proposition 3.3.6. *Let $p > 2$ not an even integer. For k an odd number that is larger than $p/2$, the bivariate idempotent function*

$$g(x, y) := (1 + e_1(x)e_k(y))(1 + e_1(x)e_{k+1}(y)) \tag{3.23}$$

is such that its marginal integral $G_p(x) := \int_{\mathbb{T}} |g(x, y)|^p dy$ has a strict maximum at $1/2$*. Moreover, it is a $\mathcal{C}^4$ function, whose second derivative at* $1/2$ *is strictly negative.*

PROOF. After a change of variables, we see that

$$G_p(x) = 4^p \int_0^1 |\cos(\pi k y)|^p \left|\cos\left(\pi(k+1)(y - \frac{x}{k(k+1)})\right)\right|^p dy.$$

The smoothness of G_p follows from the fact that it is the convolution of two functions of class $\mathcal{C}^2$. Mockenhaupt and Schlag have computed that

$$2^p |\cos(\pi y)|^p = \sum_n (-1)^n c_n e^{2i\pi n y}$$

with real coefficients $c_n = c_{-n}$, such that, for non negative n,

$$c_{n+1} = \frac{n - \frac{p}{2}}{n + \frac{p}{2} + 1}\, c_n.$$

In the convolution, only frequencies that are multiples of both k and $k+1$ are present, so that

$$G_p(x) = \sum_n (-1)^n c_{kn} c_{(k+1)n} e^{2i\pi n x}.$$

Indeed, the Fourier coefficient $\widehat{G_p}(n)$ is equal to $c_m c_{m'}$, where $km = (k+1)m'$, and $n = m'/k$, which gives also $m = (k+1)n$.

Now, looking at the inductive formula for the coefficients, and using the fact that all $c_{kn}c_{(k+1)n}$ are positive for $k > p/2$, we find that G_p is maximum when $e^{2i\pi nx} = (-1)^n$ for all n, that is, for $x = 1/2$. The computation of the Fourier series of its second derivative implies that it is strictly negative at this point. □

It remains to prove that we have the gap peaking property at 0 for $0 < p < 2$. It could be deduced from the theorems below, but we can also build on the construction of Mockhenhaupt and Schlag. Indeed, consider for $0 < p < 2$, the bivariate idempotent

$$h(x, y) := (1 + e_1(y))(1 + e_1(x)e_3(y)).$$

Using the computations of Mockenhaupt and Schlag, similarly to the above it is again straightforward to see that the p-th marginal integral $H_p(x) := \int_{\mathbb{T}} |h(x, y)|^p dy$ has a strict maximum at 0.

This concludes the proof of Proposition 3.1.10. □

REMARK 3.3.7. Note that $1 + re \pm r^{k+1} e_{k+1}$, with $r > 0$ very small, already occurred in the work of Boas [**8**] as a counterexample to the Hardy-Littlewood majorant property for $2k - 2 < p < 2k$, while already Hardy and Littlewood [**21**] has shown that $1 + e \pm e_3$ is a counterexample for $p = 3$. This seems to be the motivation to Montgomery for formulating his conjecture on existence of idempotent counterexamples for all $p \neq 2\mathbb{N}$. Mockenhaupt [**29**] discussed that $1 + e \pm e_3$ is a counterexample for all $2 < p < 4$, but his argument is not complete, with the quoted "numerical estimates" not existing where he had referred to them. Nevertheless, it seems that he thought that even $1 + e \pm e_{k+1}$ is a counterexample for all $2k - 2 < p < 2k$. At the end the answer to Montgomery's Conjecture came from giving this up and considering instead a 4-term idempotent, which on the other hand can be calculated more easily due to its product structure. Thus the original idea of $1 + e \pm e_{k+1}$ being a three-term idempotent counterexample for all $2k - 2 < p < 2k$ remains an open question. In this respect, however, note that the $k = 1$ case of this is implied by the above Proposition 3.3.4, by comparing $x = 0$ and $x = 1/2$, where $e(x) = -1$.

3.4. Restriction to a discrete problem of concentration

The second step of our proof consists of restricting the problem of p-concentration of an idempotent polynomial on a small interval into the one of concentration of an idempotent polynomial at one point of either of the two discrete grids

$$\mathbb{G}_q := \frac{1}{q}\mathbb{Z}/q\mathbb{Z} \qquad \mathbb{G}_q^\star := \frac{1}{2q} + \frac{1}{q}\mathbb{Z}/q\mathbb{Z}. \tag{3.24}$$

The idea is that if we take a gap-peaking polynomial T, then multiplication by $T(qx)$ will concentrate integrals on a neighborhood of the grid: for the first grid we need T to be peaking at 0, and for the second one we need T to do so at $1/2$.

Definition 3.4.1. For $f \in \mathcal{T}$ we denote by $\mathbf{\Pi}_q(f)$ the polynomial in $\mathcal{T}_q$ which coincides with f on the grid $\mathbb{G}_q$, that is, the polynomial having Fourier coefficients

$$\widehat{\mathbf{\Pi}_q(f)}(k) := \sum_{j\in\mathbb{N}} \widehat{f}(k + jq), \qquad k = 0, 1, \cdots, q - 1.$$

In particular, if f is positive definite, so is $\mathbf{\Pi}_q(f)$. However, in general the class of idempotent polynomials is not preserved by this projection.

Let us first define concentration on $\mathbb{G}_q$.

Definition 3.4.2. We shall say that there is p-concentration at a/q on $\mathbb{G}_q$ with constant $c > 0$ if there exists an idempotent polynomial R such that

$$\left|R\left(\frac{a}{q}\right)\right|^p > c\sum_{k=0}^{q-1}\left|R\left(\frac{k}{q}\right)\right|^p. \tag{3.25}$$

The next well-known lemma (see [**14, 3**] etc.) allows to restrict to $a = 1$.

Lemma 3.4.3. *Assume that there is p-concentration at $1/q$ on $\mathbb{G}_q$ with constant c, that is, with some appropriate idempotent R we have*

$$\left|R\left(\frac{1}{q}\right)\right|^p > c\sum_{k=0}^{q-1}\left|R\left(\frac{k}{q}\right)\right|^p. \tag{3.26}$$

Let now $a \in \mathbb{N}$, $0 < a < q$ be a natural number so that a and q are relatively prime. Then there is also p-concentration at a/q on $\mathbb{G}_q$ with constant c: that is, (3.26) *implies* (3.25) *with some appropriately chosen (possibly different) idempotent R.*

Proof. Let Q be the idempotent that satisfies (3.26). Let now $a \not\equiv 0, 1$ (mod q, of course) be another value, coprime to q. We then have a multiplicative inverse b of a mod q so that $1 \le b < q$ and $ab \equiv 1$ mod q. With this particular b we can consider

$$R(x) := Q(bx). \tag{3.27}$$

Clearly we have $R(0) = Q(0)$, $R(a/q) = Q(ab/q) = Q(1/q)$, and the values of $R(j/q) = Q(jb/q)$ with $j = 0, \ldots, q-1$ will cover all values of $Q(k/q)$ with $k = 0, 1, \ldots, q-1$, exactly once each. Therefore, we conclude that (3.25) holds with a and R. $\square$

REMARK 3.4.4. If Q is in $\mathcal{P}_q$, then instead of $Q(bx)$ we can take for R the polynomial in $\mathcal{T}_q$ which coincides with $Q(bx)$ on the grid $\mathbb{G}_q$, that is, the polynomial $\mathbf{\Pi}_q(Q(b\,\cdot))$ of Definition 3.4.1. Indeed, it is also an idempotent polynomial since b and q are coprime.

So now it makes sense to formally define the following concentration coefficient.

DEFINITION 3.4.5. We define, for $q \in \mathbb{N}$,

$$c_p^\sharp(q) := \sup_{R\in\mathcal{P}} \frac{\left|R\left(\frac{1}{q}\right)\right|^p}{\sum_{k=0}^{q-1}\left|R\left(\frac{k}{q}\right)\right|^p}, \tag{3.28}$$

and

$$c_p^\sharp := \liminf_{q\to\infty} c_p^\sharp(q). \tag{3.29}$$

We want to extend concentration results on discrete point grids to the whole of $\mathbb{T}$, and keep track of constants. We state this as a proposition.

PROPOSITION 3.4.6. *Let $p > 0$ be such that there is full p-concentration with gap at 0. If $c_p^\sharp > 0$, then p-concentration holds for the whole of $\mathbb{T}$, and we have the inequality*

$$c_p \geq 2c_p^\sharp. \tag{3.30}$$

Moreover, the same level of concentration holds with gap.

PROOF. Let us fix a symmetric open set E and construct a related peaking idempotent. First, there exists some interval $J := \left[\frac{a}{q} - \frac{1}{2q}, \frac{a}{q} + \frac{1}{2q}\right]$ with $(a,q)=1$, such that J and $-J$ are contained in E. We fix R that gives the p-concentration at a/q on $\mathbb{G}_q$ with a constant C: this can be done with C arbitrarily close to $c_p^\sharp(q)$ in view of Lemma 3.4.3.

Now, let ε be given. By uniform continuity we may choose $0 < \delta < 1/2$ so that we have the inequalities

$$|R(t+a/q)|^p \geq |R(a/q)|^p - \varepsilon\,|R(a/q)|^p\ , \qquad (|t| \leq \delta/q) \tag{3.31}$$

and, for $k = 0, 1, \cdots, q-1$,

$$|R(t+k/q)|^p \leq |R(k/q)|^p + \varepsilon|R(0)|^p, \qquad (|t| \leq \delta/q)$$

which implies immediately

$$\sum_{k=0}^{q-1} |R(t+k/q)|^p \leq (1+q\varepsilon)\sum_{k=0}^{q-1}|R(k/q)|^p. \qquad (|t| \leq \delta/q) \tag{3.32}$$

Once δ is chosen, we will take T a gap-peaking idempotent at 0, provided by Proposition 3.1.9 – compare also (3.17)-(3.18) – with the given ε, δ as above, and N larger than the degree of R, so that

$$S(x) := R(x)T(qx) \tag{3.33}$$

is an idempotent, too. It remains to show

$$2C\int_{\mathbb{T}} |S|^p \leq \kappa(\varepsilon)\int_E |S|^p, \tag{3.34}$$

with $\kappa(\varepsilon)$ getting arbitrarily close to 1 when ε is chosen appropriately small.

Denoting $\tau^p := \int_{\mathbb{T}} |T|^p$ and $I := [\frac{a}{q} - \frac{\delta}{2q}, \frac{a}{q} + \frac{\delta}{2q}]$, we find

$$
\begin{aligned}
\frac{1}{2}\int_E |S|^p \geq \int_J |S|^p &\geq (1-\varepsilon)\,|R(a/q)|^p \int_I |T(qx)|^p dx \\
&\geq (1-\varepsilon)\,|R(a/q)|^p \;\frac{1}{q}\int_{-\delta}^{\delta} |T|^p \\
&\geq \frac{(1-\varepsilon)^2\tau^p}{q}\,|R(a/q)|^p\,. \qquad (3.35)
\end{aligned}
$$

We now estimate the whole integral of $|S|^p$. We define the intervals

$$J_k := \left[\frac{k}{q} - \frac{1}{2q}, \frac{k}{q} + \frac{1}{2q}\right], \quad I_k := \left[\frac{k}{q} - \frac{\delta}{q}, \frac{k}{q} + \frac{\delta}{q}\right] \quad (k = 0, \dots, q-1).$$

Then, if we proceed as in (3.35), using (3.32) this time, we find that

$$\sum_{k=0}^{q-1}\int_{I_k} |S|^p = \int_{I_0}\sum_{k=0}^{q-1} |R(t+k/q)|^p |T(qt)|^p \leq \frac{\tau^p}{q}(1+q\varepsilon)\sum_{k=0}^{q-1} |R(k/q)|^p,$$

while

$$
\begin{aligned}
\int_{J_k\setminus I_k} |S|^p \leq 2\,|R(0)|^p \int_{\frac{k}{q}+\frac{\delta}{q}}^{\frac{k}{q}+\frac{1}{2q}} |T(qx)|^p dx &= \frac{2}{q}\,|R(0)|^p \int_{\frac{\delta}{q}}^{\frac{1}{2}} |T(x)|^p dx \\
\leq \frac{\varepsilon\tau^p}{q}\,|R(0)|^p \leq \frac{\varepsilon\tau^p}{q}\sum_{k=0}^{q-1} |R(k/q)|^p.
\end{aligned}
$$

Taking the sum over k for the last integrals and adding the above sum for integrals over the I_k's, we obtain the estimate

$$\int_{\mathbb{T}} |S|^p \leq \frac{\tau^p}{q}(1+2q\varepsilon)\sum_k |R(k/q)|^p. \qquad (3.36)$$

Combining (3.35) and (3.36), (3.34) obtains with $\kappa(\varepsilon) := (1-\varepsilon)^{-2}(1+2q\varepsilon)$.

Let us finally prove p-concentration with gap. It is sufficient to remark that instead of taking the polynomial R in (3.33) we could have as well taken the polynomial $R((Mq{+}1)x)$, with M arbitrarily large. From this point, the proof is identical, since the two polynomials take the same values on the grid. If the gaps of the peaking idempotent T are taken large enough, then S will have gaps larger than M. $\square$

We can modify slightly the previous proof of Proposition 3.4.6 to prove concentration results on the corresponding second grid, using the peaking property with gap at 1/2 instead of 0.

Definition 3.4.7. We shall say that there is p-concentration at $\frac{2a+1}{2q}$ on the grid $\mathbb{G}_q^\star$ with constant c if there exists an idempotent polynomial R such that

$$\left|R\left(\frac{2a+1}{2q}\right)\right|^p > c\sum_{k=0}^{q-1}\left|R\left(\frac{2k+1}{2q}\right)\right|^p. \qquad (3.37)$$

Remark that in particular we restrict to idempotents R that do not vanish identically on the grid under consideration, which we assume in the following definition.

DEFINITION 3.4.8. If $q \in \mathbb{N}$, then we define

$$c_p^\star(q) := \sup_{R\in\mathcal{P}} \frac{\left|R\left(\frac{1}{2q}\right)\right|^p}{\sum_{k=0}^{q-1}\left|R\left(\frac{2k+1}{2q}\right)\right|^p} \tag{3.38}$$

and

$$c_p^\star := \liminf_{q\to\infty} c_p^\star(q). \tag{3.39}$$

Again, the first step is to restrict to $1/(2q)$.

LEMMA 3.4.9. *Assume that there is p-concentration at $1/(2q)$ on $\mathbb{G}_q^\star$ with constant c. Let now $a \in \mathbb{N}$, $0 \le a < q$ be so that $2a+1$ and q are relatively prime. Then there is also p-concentration at $(2a+1)/(2q)$ on the grid $\mathbb{G}_q^\star$ with the same constant c.*

PROOF. Let Q be the idempotent that satisfies (3.37) with $a = 0$. We then have a multiplicative inverse b of $2a+1 \mod 2q$ so that $1 \le b < 2q$ and $(2a+1)b \equiv 1 \mod 2q$; hence, in particular, also b is odd. Now with this particular b we can consider $R(x) := Q(bx)$ exactly as before in (3.27).

Clearly we have $R(0) = Q(0)$, $R((2a+1)/(2q)) = Q((2a+1)b/(2q)) = Q(1/(2q))$, and the values of $R(j/(2q)) = Q(jb/(2q))$ with $j = 0, \dots, 2q-1$ will cover all values of $Q(k/(2q))$ with $k = 0, 1, \dots, 2q-1$, exactly once each, and such a way, that odd j's correspond to odd k's. Therefore, we conclude that (3.37) holds with $2a+1$ and R. □

REMARK 3.4.10. As in Remark 3.4.4, if Q is in $\mathcal{P}_{2q}$ then instead of $Q(bx)$ we can take for R the polynomial $\mathbf{\Pi}_{2q}(Q(b\,\cdot))$, which coincides with $Q(bx)$ at each point of the grid $\mathbb{G}_{2q}$, hence a priori on $\mathbb{G}_q^\star$.

The corresponding proposition goes as follows:

PROPOSITION 3.4.11. *Let $p > 0$ be such that there is full p-concentration with gap at 1/2. If $c_p^\star > 0$, then p-concentration holds for the whole of $\mathbb{T}$ and we have the inequality*

$$c_p \ge 2c_p^\star. \tag{3.40}$$

Moreover, the same property holds with arbitrarily large gaps.

PROOF. Similarly to the above, it suffices to derive the concentration phenomenon for the symmetrized of an interval $J := [\frac{a}{q}, \frac{(a+1)}{q}]$ for q a sufficiently large number, $2a+1$ coprime to $2q$.

In this setup for any $c < c_p^\star(q)$ Lemma 3.4.9 leads to the inequality

$$\left|R\left(\frac{2a+1}{2q}\right)\right|^p > c\sum_{k=0}^{q-1}\left|R\left(\frac{2k+1}{2q}\right)\right|^p. \tag{3.41}$$

with an appropriate $R \in \mathcal{P}$.

At this point, the proof is exactly the same as the one of the previous proposition, considering intervals I_k centered at $(2k+1)/(2q)$ with radius δ/q, with δ small enough so that R is nearly constant on I_k, and then considering $S(x) := R(x)\cdot T(qx)$ again, where T is now a gap-peaking idempotent at $1/2$, with gaps sufficiently large, so that S is still an idempotent. Using the fact that outside I_k but within $(k/q,(k+1)/q)$, the integral of T is arbitrarily small in view of the peaking property at $1/2$, we obtain the assertion as before. The only difference is the fact that 0 is no more in the grid, so that the quotient of $R(0)^p$ with $\sum_{k=0}^{q-1}|R((2k+1)/(2q))|^p$ appears in the rests, but does not change the limit since it remains fixed while ε tends to 0.

The p-concentration with gap at the same level of concentration is obtained also in a similar way. □

3.5. p-concentration by means of peaking at $1/2$

We now prove the part of Theorem 3.1.7 concerning p not an even integer, which we state separately for the reader's convenience. The following proof contains also the one of Proposition 3.1.12, which we gave in the introduction as a hint for the methods.

PROPOSITION 3.5.1. *Let $p>0$ be a given value for which there is full p-concentration with gap at $1/2$. Then for each nonempty symmetric open set $E\subset\mathbb{T}$ and each constant $c<1$ we can find an idempotent $S\in\mathcal{P}$ with the property that*

$$\int_E |S|^p > c\int_{\mathbb{T}}|S|^p. \tag{3.42}$$

Moreover, S may be chosen with arbitrarily large gaps.

PROOF. By Proposition 3.4.11, it is sufficient to prove that $c_p^\star = 1/2$, that is,

$$\liminf_{q\to\infty}\sup_{R\in\mathcal{P}}\frac{\left|R\left(\frac{1}{2q}\right)\right|^p}{\sum_{k=0}^{q-1}\left|R\left(\frac{2k+1}{2q}\right)\right|^p}=\frac{1}{2}. \tag{3.43}$$

We will restrict to a sub-family of polynomials in $\mathcal{P}$, obtained by products of Dirichlet kernels. Observe first that for $r<q$, the product

$$D_r(x)\prod_{l=1}^{L-1}D_r\Big(((2q)^l+1)x\Big)$$

is also an idempotent polynomial, the modulus of which coincides with the L-th power of $|D_r|$ on the grid under consideration. So we are to prove also the last inequality in

$$\frac{1}{2c_p^\star}=\limsup_{q\to\infty}\frac{1}{2c_p^\star(q)}\le\inf_L\limsup_{q\to\infty}\min_{r<q}\frac{1}{2}\cdot\frac{\sum_{k=0}^{q-1}\left|D_r\left(\frac{2k+1}{2q}\right)\right|^{Lp}}{\left|D_r\left(\frac{1}{2q}\right)\right|^{Lp}}\le 1. \tag{3.44}$$

Let us define

$$A(\lambda,r,q):=\left|\frac{\sin\left(\frac{\pi}{2q}\right)}{\sin\left(\frac{r\pi}{2q}\right)}\right|^\lambda\sum_{k\in\mathbb{N}\,;k<q/2}\left|\frac{\sin\left(\frac{(2k+1)r\pi}{2q}\right)}{\sin\left(\frac{(2k+1)\pi}{2q}\right)}\right|^\lambda. \tag{3.45}$$

Substituting the explicit value of D_r and using parity, the quantity appearing inside the $\min_{r<q}$ in (3.44) can be written as $A(Lp,r,q)$ for q even. When q is odd, we have to subtract half of the term obtained for $k=(q-1)/2$, which gives only a 0 contribution to the limit below. In any case, we have the inequality

$$\left(\frac{1}{2c_p^\star(q)}\le\right)\frac{1}{2}\frac{\sum_{k=0}^{q-1}\left|D_r\left(\frac{2k+1}{2q}\right)\right|^{Lp}}{\left|D_r\left(\frac{1}{2q}\right)\right|^{Lp}}\le A(Lp,r,q). \tag{3.46}$$

We then have the following lemma.

LEMMA 3.5.2. *For fixed $\lambda>1$, we have the inequality*

$$\limsup_{q\to\infty}\min_{r<q}A(\lambda,r,q)\le\inf_{0<t<1/2}A(\lambda,t), \tag{3.47}$$

where

$$A(\lambda,t):=\frac{1}{(\sin(\pi t))^\lambda}\sum_{k=0}^{\infty}\left|\frac{\sin\left((2k+1)\pi t\right)}{2k+1}\right|^\lambda. \tag{3.48}$$

PROOF. Let us fix $t\in(0,1/2)$, and consider the limit of $A(\lambda,2[qt],q)$ when q tends to ∞. It has the same limit as

$$\left|\frac{\frac{\pi}{2}}{\sin(\pi t)}\right|^\lambda\sum_{k=0}^{(q-1)/2}\left|\frac{\sin\left(\frac{(2k+1)[qt]\pi}{q}\right)}{q\sin\left(\frac{(2k+1)\pi}{2q}\right)}\right|^\lambda.$$

As $q\sin(\frac{(2k+1)\pi}{2q})\ge(2k+1)$, Lebesgue's theorem for series justifies taking the limit termwise. This concludes the proof of the lemma. ☐

So in view of Lemma 3.5.2 $1/(2c_p^\star(q))\le\inf_L\inf_t A(Lp,t)$. If we take $t=1/4$, all the absolute values of the occurring sines in $A(\lambda,t)$ are equal, hence cancel out. It remains

$$A(\lambda,1/4)=\sum_k(2k+1)^{-\lambda}=(1-2^{-\lambda})\zeta(\lambda).$$

Now we can take L, or $\lambda=Lp$, arbitrarily large. Therefore, the infimum in (3.44) is just 1. ☐

Note that we found that $1/(2c_p^\star)\le\inf_L\inf_t A(Lp,t)$ holds always.

Let us conclude this section by a remark that will be used later on for measurable sets, where we will not be able to consider large products of Dirichlet kernels for $p\le1$, and will have to restrict to two factors, that is, take $L=2$. Observe that each term $\left|\frac{\sin((2k+1)\pi t)}{(2k+1)\sin(\pi t)}\right|$ is below 1, so that $A(\lambda,t)$ and $\inf_t A(\lambda,t)$ are strictly decreasing functions of λ.

Moreover, $\inf_{0<t<1/2}A(2,t)$ can be computed explicitly. To compute the summation, we can use Plancherel Formula once we have recognized the Fourier coefficients (at k and $-k$) of the function

$$\frac{\pi}{2}\left(\chi_{[-t/2,t/2]}(x)-\chi_{[-t/2,t/2]}(x-1/2)\right).$$

It follows that

$$A(2,t)=\frac{\pi^2t}{4\sin^2(\pi t)}. \tag{3.49}$$

Substituting $x = \pi t$ and recalling (3.5) we find $1/\min_t A(2,t) = 2c_2 \approx 0.92...$, which is already much larger than $1/2$, and close to 1.

3.6. Uniform lower bounds for p-concentration

We now prove the lower estimation in the $p \in 2\mathbb{N}$ part of Theorem 3.1.7. We proceed as in the last section, using Proposition 3.4.6 instead of Proposition 3.4.11, since we have now gap-peaking idempotents at 0 only. Similarly to the above, we consider a product of Dirichlet kernels:

$$R(x) := D_r(x) \prod_{\ell=1}^{L-1} D_r((q^\ell + 1)x). \tag{3.50}$$

We have to consider the quantities (3.28) and (3.29), i.e. we are to calculate

$$\frac{2}{c_p} \le \frac{1}{c_p^\sharp} = \limsup_{q\to\infty} \frac{1}{c_p^\sharp(q)} \le \inf_L \limsup_{q\to\infty} \min_{r<q} \frac{\sum_{k=0}^{q-1} \left|D_r\left(\frac{k}{q}\right)\right|^{Lp}}{\left|D_r\left(\frac{1}{q}\right)\right|^{Lp}}. \tag{3.51}$$

As before, in order to estimate the quotient in (3.51) we have to consider the equivalent quantity $B(Lp, r, q)$ defined by

$$B(\lambda, r, q) := \left|\frac{r\sin\left(\frac{\pi}{q}\right)}{\sin\left(\frac{r\pi}{q}\right)}\right|^\lambda + 2\left|\frac{\sin\left(\frac{\pi}{q}\right)}{\sin\left(\frac{r\pi}{q}\right)}\right|^\lambda \sum_{k=1}^{q/2} \left|\frac{\sin\left(\frac{kr\pi}{q}\right)}{\sin\left(\frac{k\pi}{q}\right)}\right|^\lambda. \tag{3.52}$$

We then have the following lemma.

Lemma 3.6.1. *For fixed $\lambda > 1$, we have the inequality*

$$\limsup_{q\to\infty} \min_{r<q} B(\lambda, r, q) \le \inf_{0<t<1/2} B(\lambda, t), \tag{3.53}$$

where

$$B(\lambda, t) := \left(\frac{\pi t}{\sin \pi t}\right)^\lambda \left(1 + 2\sum_{k=1}^\infty \left|\frac{\sin(k\pi t)}{k\pi t}\right|^\lambda\right). \tag{3.54}$$

Proof. For fixed $t \in (0, 1/2)$, the left hand side of (3.53) is bounded by the value that we obtain when letting $q \to \infty$ with r/q tending to t at the same time. We conclude as in Lemma 3.5.2. □

Let us define for any fixed value of $\kappa > 0$, the quantity

$$\beta(\kappa) := \limsup_{\lambda\mapsto\infty} B\left(\lambda, \kappa\sqrt{6/\lambda}\right), \tag{3.55}$$

which will be useful later on, since $2/c_p \le \inf_L \inf_t B(Lp, t) \le \beta(\kappa)$. For fixed s, the quantity $\left(\sqrt{\lambda}/s \cdot \sin(s/\sqrt{\lambda})\right)^\lambda$ tends to $\exp(-s^2/6)$. We use this for the computation of $\beta(\kappa)$ and see that the first factor of (3.54) tends to $\exp(\kappa^2\pi^2)$.

Applying the well-known Weierstrass product for sin we get

$$\log\left(\frac{\sin x}{x}\right) = \sum_{n=1}^\infty \log\left(1 - \frac{x^2}{n^2\pi^2}\right) \le -\left(\sum_{n=1}^\infty \frac{1}{n^2}\right)\frac{x^2}{\pi^2} = -\frac{x^2}{6}.$$

For the log function here we must restrict to $0 < x < \pi$: that provides us the useful inequality

$$\frac{\sin x}{x} \le \exp\left(-\frac{x^2}{6}\right) \qquad (0 < x < \pi),$$

what we apply in the second factor of (3.54) for the range $1 \le k < 1/t$. Thus (at the end extending the sum up to ∞) we are led to

$$\sum_{k<1/t} \left|\frac{\sin(k\pi t)}{k\pi t}\right|^\lambda \le \sum_{k<1/t} \exp\left(-\lambda \frac{k^2\pi^2 t^2}{6}\right) \le \sum_{k=1}^{\infty} e^{-\kappa^2 k^2 \pi^2}. \tag{3.56}$$

Using the trivial bound $|\sin u| \le 1$, the tail sum can be estimated as

$$\sum_{k\ge 1/t} \left|\frac{\sin(k\pi t)}{k\pi t}\right|^\lambda \le (\pi t)^{-\lambda}\left(t^\lambda + \int_{1/t}^{\infty} \frac{du}{u^\lambda}\right) = \pi^{-\lambda}\left(1 + \frac{1/t}{\lambda - 1}\right), \tag{3.57}$$

which tends to 0 with $t = \kappa\sqrt{6/\lambda}$ and $\lambda \to \infty$.

Collecting the above estimates for $\beta := \inf_{\kappa>0} \beta(\kappa)$, we are led to

$$\beta \le \inf_{\kappa>0} e^{\pi^2\kappa^2}\left\{1 + 2\sum_{k=1}^{\infty} e^{-\kappa^2 k^2 \pi^2}\right\}. \tag{3.58}$$

Note that the sum in the last curly brackets is well-known as Jacobi's theta function. Choosing here $\kappa = 0.225$, we can compute $\beta \le 4.13273$, which leads to $c_p \ge 2/\beta \ge 0.48394$, surprisingly close to the theoretical upper bound of $1/2$.

The computation of $\inf_{0<t<1/2} B(\lambda, t)$ can be executed explicitly for $\lambda = 4$. We recognize the Fourier coefficients of the convolution product $\chi_{[-t/2,t/2]} * \chi_{[-t/2,t/2]}$, whose L^2 norm is equal to $(2t^3/3)^{1/2}$. Then we use the Plancherel Formula and obtain that

$$c_4 \ge \max_{0<t<1/2} \frac{3\left(\sin^4(\pi t)\right)}{\pi^4 t^3} > 0.495, \tag{3.59}$$

the concrete numerical value having been obtained for the choice of $t = 0.267$.

Comparing the results of the last two sections, it should become clear why gap-peaking at $1/2$ is even more useful for us, than gap-peaking at 0. Indeed, once we can apply gap-peaking at $1/2$, we are able to consider $\mathbb{G}_q^\star$ in place of $\mathbb{G}_q$: and that means that instead of the *second largest term* $|D_r(1/q)|$, we can consider *the very largest term* $|D_r(1/2q)|$ in comparison to the whole grid sum. Thus in the translated grid case we can take advantage of considering arbitrarily large powers L, eventually killing all other terms compared to our $|D_r(1/2q)|^L$, while in the original grid $\mathbb{G}_q$ this is subject to a fine balance, restricted by the necessity of keeping control of the dominance of the very largest term $D_r(0)^L$.

Part III : Concentration for measurable sets

We will go back to all steps of the previous proofs in order to partly generalize the results to measurable sets. We start by using the theorem of Khintchine on diophantine approximation, see [**24**]. We prove that a symmetric measurable set of positive measure contains large parts of intervals which are centered at a point of one of the two grids, $\mathbb{G}_q$

or $\mathbb{G}_q^\star$. This is done in Section 3.7. Then in Section 3.8 we prove the gap-peaking property at 0 or 1/2 in the even stronger form that some measurable set of measure $2\eta\delta$ can be deleted from the interval $[-\delta, +\delta]$. In Section 3.9 we prove that values of an idempotent, concentrating on the grid, does not take too different values on the intervals of length 2δ. Here we may consider additional assumptions on the degree of the polynomials. Based on the results of these sections, we will prove p-concentration for measurable sets when $p > 1/2$, with some estimates on constants. We conclude the proof of Theorem 3.1.8 finally in §3.10.

3.7. The use of Diophantine Approximation

We will state two propositions, used respectively on $\mathbb{G}_q$ and $\mathbb{G}_q^\star$. The first one is a direct corollary of Khintchine's Theorem, while the second one is its inhomogeneous extension, first proved by Szüsz [**37**] and later generalized by Schmidt [**35**].

PROPOSITION 3.7.1. *Let E be a measurable set of positive measure in $\mathbb{T}$. For all $\theta > 0$, $\eta > 0$ and $Q \in \mathbb{N}$, there exists an irreducible fraction k/q such that $q > Q$ and*

$$\left| \left[\frac{k}{q} - \frac{\theta}{q^2}, \frac{k}{q} + \frac{\theta}{q^2} \right] \cap E \right| \geq (1-\eta)\frac{2\theta}{q^2}. \tag{3.60}$$

Moreover, given a positive integer ν, it is possible to choose q such that $(\nu, q) = 1$.

PROPOSITION 3.7.2. *Let E be a measurable set of positive measure in $\mathbb{T}$. For all $\theta > 0$, $\eta > 0$ and $Q \in \mathbb{N}$, there exists an irreducible fraction $(2k+1)/(2q)$ such that $q > Q$ and*

$$\left| \left[\frac{2k+1}{2q} - \frac{\theta}{q^2}, \frac{2k+1}{2q} + \frac{\theta}{q^2} \right] \cap E \right| \geq (1-\eta)\frac{2\theta}{q^2}. \tag{3.61}$$

Moreover, given a positive integer ν, it is possible to choose q such that $(\nu, q) = 1$.

PROOF OF PROPOSITIONS 3.7.1 AND 3.7.2. Let α be 0 or 1/2. Then according to Szüsz' Theorem [**37**] for ξ belonging to a set of full measure,

$$\|q\xi - \alpha\| \leq \frac{\theta}{q} \tag{3.62}$$

has an infinite number of solutions. For $\alpha = 1/2$, for instance, it means that with a certain $k \in \mathbb{N}$ $(0 \leq k < q)$ we have

$$|q\xi - 1/2 - k| < \frac{\theta}{q}, \qquad \text{i.e.} \qquad \left|\xi - \frac{2k+1}{2q}\right| < \frac{\theta}{q^2}. \tag{3.63}$$

We may assume, and we will do it, that the denominator and numerator are coprime: if not, we cancel out the common factors, and the error, compared to the new denominator q', is even better. Note that for irrational ξ we have infinitely many different such denominators q': indeed, if not we get a contradiction with the fact that the error tends to zero with q.

Let us choose for ξ an irrational density point of E having infinitely many solutions of (3.62). This we can do, since almost every point of E is such. For η fixed and q sufficiently large we then have

$$\left| \mathbb{T} \setminus E \cap \left[\xi - \frac{2\theta}{q^2}, \xi + \frac{2\theta}{q^2} \right] \right| \leq \frac{2\eta\theta}{q^2}.$$

So, if q and k are such that (3.63) holds and if q is large enough, then (3.61) is satisfied by the triangle inequality.

It remains to prove that the denominators q can be taken so that $(\nu, q) = 1$. Schmidt proves in [**35**] that, for each polynomial P with integer coefficients and each $\alpha \in \mathbb{T}$, for almost every ξ one can find an infinite number of integers r such that

$$\|P(r)\xi - \alpha\| \leq \frac{\theta}{r}. \tag{3.64}$$

Both for $\alpha = 0$ or $1/2$, it suffices to consider $P(r) = \nu r + 1$. Schmidt's Theorem then allows (3.64) for a.e. ξ by infinitely many r. So we can approach ξ for $\alpha = 0$ by fractions $k/(\nu r+1)$, and for $\alpha = 1/2$ by fractions $\frac{2k+1}{2(\nu r+1)}$, eventually simplified. So the denominator and ν will always remain coprime. The rest of the proof is identical. □

3.8. Peaking idempotents at 0 and 1/2

We will prove the following, which is a more accurate statement than those of Section 3.3.

PROPOSITION 3.8.1. *Let $p > 2$. For $\varepsilon > 0$ there exists $\delta_0 > 0$ and $\eta > 0$ such that, for all $\delta < \delta_0$ and $N \in \mathbb{N}$, if E is a measurable set that satisfies $|E \cap [-\delta, \delta]| > 2(1-\eta)\delta$, then there exists an idempotent T with gaps larger than N such that*

$$\int_{E\cap[-\delta,\delta]} |T|^p > (1-\varepsilon) \int_0^1 |T|^p.$$

Let $p > 0$ not an even integer. Then for $\varepsilon > 0$ there exists $\delta_0 > 0$ and $\eta > 0$ such that, for all $\delta < \delta_0$ and $N \in \mathbb{N}$, if E is a measurable set that satisfies $|E \cap [\frac{1}{2}-\delta, \frac{1}{2}+\delta]| > 2(1-\eta)\delta$, then there exists an idempotent T with gaps larger than N such that

$$\int_{E\cap[\frac{1}{2}-\delta,\frac{1}{2}+\delta]} |T|^p > (1-\varepsilon) \int_0^1 |T|^p.$$

PROOF. We will proceed as in Section 3.3. The main point is, for our peaking bivariate functions f, to find an appropriate power L of the marginal function F for which the same kind of estimate is valid: we will then take a Riesz product with L factors. The proposition will be a consequence of the following lemma, with F the associated marginal function.

LEMMA 3.8.2. *Let $F : [0, 1/2] \to [0, \infty)$ be a nonnegative, continuous function, having a strict global maximum at 0. Moreover, assume that there exist $0 < a < A$ and $\Delta > 0$ with F admitting the estimates*

$$F(0)\exp(-Ax^2) < F(x) < F(0)\exp(-ax^2) \qquad (x \in [0, \Delta]). \tag{3.65}$$

Then for all $\varepsilon > 0$ there exists an $\eta > 0$ so that for any $0 < \delta < \Delta$ there is an $L = L(\varepsilon, \delta) \in \mathbb{N}$ with the property that whenever $E \subset [0, 1/2)$ is a measurable set satisfying $|E \cap [0, \delta]| > (1-\eta)\delta$, then we have the inequality

$$\int_{E\cap[0,\delta]} F^L > (1-\varepsilon) \int_0^{1/2} F^L.$$

REMARK 3.8.3. Observe that (3.65) certainly holds true in case F has a nonvanishing second derivative (from the right) at 0. Also note the validity of the obvious modification for even functions on $[-1/2, 1/2]$ assuming the analogous two-sided conditions.

PROOF. We can assume $F(0) = 1$. By condition, $\max_{[\Delta,1/2]} F < 1$, hence – perhaps with a different value of a, which still depends only on F – we have $F(x) < \exp(-ax^2)$ on the whole of $[0, 1/2]$. Extending F to the halfline $[0, \infty)$ as 0 outside $[0, 1/2]$, we thus still have this estimate.

Let now $H := [0, \infty) \setminus ([0, \delta] \cap E)$. Then we have

$$\int_H F^L < \int_{[0,\delta]\setminus E} 1 + \int_\delta^\infty e^{-Lax^2} dx < |[0, \delta] \setminus E| + \int_\delta^\infty \frac{dx}{Lax^2} < \eta\delta + \frac{1}{\delta a L}. \tag{3.66}$$

On the other hand with a very similar calculation we obtain

$$\begin{aligned} \int_{[0,\delta]\cap E} F^L &> \int_{[0,\delta]\cap E} e^{-LAx^2} dx = \left(\int_0^\infty - \int_{[0,\delta]\setminus E} - \int_\delta^\infty \right) e^{-LAx^2} dx \\ &\geq \frac{1}{2}\sqrt{\frac{\pi}{LA}} - |[0, \delta] \setminus E| - \int_\delta^\infty \frac{dx}{Lax^2} > \frac{1}{2}\sqrt{\frac{\pi}{LA}} - \eta\delta - \frac{1}{\delta a L}. \end{aligned} \tag{3.67}$$

A combination of (3.66) and (3.67) reveals that it suffices to ascertain

$$\eta\delta < \frac{\varepsilon}{8}\sqrt{\frac{\pi}{LA}} \qquad \text{and} \qquad \frac{1}{\delta a L} < \frac{\varepsilon}{8}\sqrt{\frac{\pi}{LA}},$$

that is, with a constant $C = C(a, A) = C_F$,

$$\eta\delta\sqrt{L} \leq \varepsilon' \qquad \text{and} \qquad \frac{1}{\delta\sqrt{L}} \leq \varepsilon' \qquad \text{with} \quad \varepsilon' := \varepsilon/C.$$

Thus we conclude the proof choosing $L := \lceil \varepsilon'^{-2}\delta^{-2} \rceil$ and $\eta = \varepsilon'^2/2$. □

To prove both cases of the proposition, note that we can also translate F so that the maximum point falls to $1/2$ instead of 0.

At this point the proof of the proposition is identical to the proofs of Section 3.3, using Lemma 3.3.2. For the given E, we find an idempotent T such that integrals of $|T|^p$, respectively on $E \cap [-\delta, +\delta]$ and on the whole torus, satisfy the same inequality as the corresponding integrals for the function F^L. □

3.9. Bernstein-type inequalities

In order to adapt our proof of Proposition 3.4.6, we need to control the error done when replacing values of idempotents in a neighborhood of one of the grids by its values on the grid.

We introduce the following notation, which will simplify the proofs. For f a periodic function, we will use the sums of its values on the two grids, which we denote by

$$\mathbf{\Sigma}_q(f) := \sum_{k=0}^{q-1} f\left(\frac{k}{q}\right), \qquad \mathbf{\Sigma}_q^\star(f) := \sum_{k=0}^{q-1} f\left(\frac{2k+1}{2q}\right). \tag{3.68}$$

The aim of this paragraph is to recall classical inequalities, and modify them according to our purposes. Let us prove the following lemma.

LEMMA 3.9.1. *For $1 < p < \infty$ there exists a constant C_p such that, for $P \in \mathcal{T}_q$ and for $|t| < 1/2$, we have the two inequalities*

$$\sum_{k=0}^{q-1} |P(t + k/q)|^p \leq C_p \sum_{k=0}^{q-1} |P(k/q)|^p, \tag{3.69}$$

$$\sum_{k=0}^{q-1} ||P(t + k/q)|^p - |P(k/q)|^p| \leq C_p |qt| \sum_{k=0}^{q-1} |P(k/q)|^p. \tag{3.70}$$

PROOF. For $1 < p < \infty$, the L^p norm of a trigonometric polynomial in $\mathcal{T}_q$ is equivalent to the ℓ^p norm of its values on the grid $\mathbb{G}_q$. This is known as the Marcinkiewicz-Zygmund Theorem: the implied constants depend only on p but tend to ∞ for p tending to 1 or ∞. For the exact form fitting to our Taylor polynomials see Theorem (7.10), p. 30 chapter X in [**45**]; see also [**32**] for recent extensions. Inequality (3.69) then follows using the Marcinkiewicz-Zygmund Theorem twice, and invariance by translation of the L^p norm.

To obtain (3.70), we use a variant of Bernstein's Inequality, which may be stated, for $P \in \mathcal{T}_q$, as

$$\int_0^1 |P(x + t) - P(x)|^p dx \leq (2\pi q|t|)^p \int_0^1 |P(x)|^p dx. \tag{3.71}$$

Since this is not the usual form of Bernstein's Inequality, we indicate how to obtain it. We write, for positive t,

$$|P(x + t) - P(x)|^p \leq t^{p-1} \int_x^{x+t} |P'(u)|^p du,$$

apply this estimate on the left hand side of (3.71) and then change the order of integration. We then conclude by using Bernstein's Inequality as stated in Theorem (3.16), chapter X in [**45**], that is,

$$\int_0^1 |P'(x)|^p dx \leq (2\pi q)^p \int_0^1 |P(x)|^p dx.$$

Let us proceed with the proof of (3.70). By using the Marcinkiewicz-Zygmund Theorem for both sides of (3.71), we find that, for $1 < p < \infty$, there exists some constant C_p, (independent of $P \in \mathcal{T}_q$), such that

$$\sum_{k=0}^{q-1} |P(t + k/q) - P(k/q)|^p \leq C_p |qt|^p \sum_{k=0}^{q-1} |P(k/q)|^p. \tag{3.72}$$

Let us use the elementary inequality

$$||a|^p - |b|^p| \leq p|a - b| \left(|a|^{p-1} + |b|^{p-1}\right) \tag{3.73}$$

and the Hölder Inequality together with (3.72), as well as our notation given in (3.68). We obtain the estimate

$$\sum_{k=0}^{q-1} ||P(t + k/q)|^p - |P(k/q)|^p| \leq$$

$$pC_p^{\frac{1}{p}} |qt| \left(\mathbf{\Sigma}_q |P|^p\right)^{\frac{1}{p}} \cdot \left(\mathbf{\Sigma}_q \left(\left(|P|^{p-1} + |P(t + \cdot)|^{p-1}\right)^{\frac{p}{p-1}}\right)\right)^{\frac{p-1}{p}}.$$

After having used Minkowski's inequality and the estimate (3.69), i.e. $\mathbf{\Sigma}_q(|P(t+\cdot)|^p) \le C_p\mathbf{\Sigma}_q(|P|^p)$, the last factor on the right hand side becomes $C'_p\left(\mathbf{\Sigma}_q(|P|^p)\right)^{\frac{p-1}{p}}$, which concludes the proof of (3.70). ☐

The following is an easy consequence of Lemma 3.9.1.

LEMMA 3.9.2. *For $1 < p < \infty$ and with the same constant C_p as in Lemma 3.9.1 we have the following property. Whenever $P \in \mathcal{T}_{2q}$ satisfies*

$$\sum_{k=0}^{q-1} |P(k/q)|^p \le K\mathbf{\Sigma}_q^{\star}(|P|^p), \tag{3.74}$$

then, for any $|t| < 1/2$, we have the two inequalities

$$\sum_{k=0}^{2q-1} \left|P\left(\frac{k}{2q}+t\right)\right|^p \le C_p(K+1)\mathbf{\Sigma}_q^{\star}(|P|^p), \tag{3.75}$$

$$\sum_{k=0}^{2q-1} \left|\left|P\left(\frac{k}{2q}+t\right)\right|^p - \left|P\left(\frac{k}{2q}\right)\right|^p\right| \le 2C_p(K+1)|qt|\mathbf{\Sigma}_q^{\star}(|P|^p). \tag{3.76}$$

This lemma explains why we introduce the next definition.

DEFINITION 3.9.3. Let $0 < p < \infty$ and $q \in \mathbb{N}$. We say that a polynomial f satisfies the *grid-condition with constant K*, if we have

$$\sum_{k=0}^{q-1} \left|f\left(\frac{k}{q}\right)\right|^p \le K \sum_{k=0}^{q-1} \left|f\left(\frac{2k+1}{2q}\right)\right|^p, \tag{3.77}$$

that is, with the notation (3.68), $\mathbf{\Sigma}_q(|f|^p) \le K\mathbf{\Sigma}_q^{\star}(|f|^p)$.

REMARK 3.9.4. When $P \in \mathcal{T}_q$, (3.74) – i.e., the grid condition (3.77) for P – holds with $K = C_p$ depending only on $p > 1$: just use (3.69) for the translated by $1/2q$ polynomial.

We will use these considerations for products of such polynomials as well.

LEMMA 3.9.5. *For $1/2 < p < \infty$ there exists a constant A_p such that, whenever $Q \in \mathcal{T}_{2q}$ satisfies the grid-condition (3.77) with exponent $2p$, i.e.*

$$\mathbf{\Sigma}_q(|Q|^{2p}) := \sum_{k=0}^{q-1} |Q(k/q)|^{2p} \le K\mathbf{\Sigma}_q^{\star}(|Q|^{2p}), \tag{3.78}$$

then for the product polynomial $R(x) := Q(x)Q((2q+1)x)$ we have for all $|t| < 1/2$ and for all $a := 0, 1, \ldots, q-1$ the two inequalities

$$\sum_{k=0}^{q-1} \left|R\left(t+\frac{2k+1}{2q}\right)\right|^p \le (1 + A_p(K+1)q^2|t|)\mathbf{\Sigma}_q^{\star}(|Q|^{2p}), \tag{3.79}$$

$$\left|R\left(t+\frac{2a+1}{2q}\right)\right|^p \ge \left|R\left(\frac{2a+1}{2q}\right)\right|^p - A_p(K+1)q^2|t|\mathbf{\Sigma}_q^{\star}(|Q|^{2p}). \tag{3.80}$$

PROOF. Let us put, for $k = 0, 1, \ldots 2q-1$,

$$X_k(t) := \left| Q\left(t + \frac{k}{2q}\right)\right|^{2p}. \tag{3.81}$$

Note that the two factors of R take the same values on the grid $\mathbb{G}_q^\star$. Moreover, since $Q \in \mathcal{T}_{2q}$ and $2p > 1$, it follows from Lemma 3.9.2, formula (3.76) that

$$\sum_{k=0}^{2q-1} |X_k(t) - X_k(0)| \leq 2C_{2p}(K+1)q|t|\mathbf{\Sigma}_q^\star(|Q|^{2p}).$$

Let us pass to

$$Y_k(t) := \left| R\left(t + \frac{k}{2q}\right)\right|^{p} = \sqrt{X_k(t)X_k((2q+1)t)}.$$

Using the Cauchy-Schwarz Inequality and the previous inequality, we find for all $|t| \leq 1/2$

$$\begin{aligned}
\sum_{k=0}^{q-1} Y_{2k+1}(t) &\leq \left(\sum_{k=0}^{q-1} X_{2k+1}(t)\right)^{\frac{1}{2}} \left(\sum_{k=0}^{q-1} X_{2k+1}((2q+1)t)\right)^{\frac{1}{2}} \\
&\leq \left(\mathbf{\Sigma}_q^\star(|Q|^{2p}) + \sum_{k=0}^{2q-1} |X_k(t) - X_k(0)|\right)^{\frac{1}{2}} \\
&\qquad \times \left(\mathbf{\Sigma}_q^\star(|Q|^{2p}) + \sum_{k=0}^{2q-1} |X_k((2q+1)t) - X_k(0)|\right)^{\frac{1}{2}} \\
&\leq (1 + 2C_{2p}(2q+1)q(K+1)|t|)\mathbf{\Sigma}_q^\star(|Q|^{2p}).
\end{aligned}$$

We have proved (3.79). We can write in the same way that

$$\begin{aligned}
Y_{2a+1}(t)^2 \geq &\left(X_{2a+1}(0) - \sum_{k=0}^{2q-1} |X_k(t) - X_k(0)|\right) \\
&\times \left(X_{2a+1}(0) - \sum_{k=0}^{2q-1} |X_k((2q+1)t) - X_k(0)|\right),
\end{aligned}$$

so that

$$Y_{2a+1}(t) \geq Y_{2a+1}(0) - 2C_{2p}(K+1)(2q+1)q|t|\mathbf{\Sigma}_q^\star(|Q|^{2p}).$$

□

3.10. From discrete concentration to concentration for measurable sets

DEFINITION 3.10.1. We define

$$\gamma_p^\sharp := \liminf_{q\to\infty} \gamma_p^\sharp(q), \qquad \gamma_p^\sharp(q) := \sup_{R\in\mathcal{P}_q} \frac{\left|R\left(\frac{1}{q}\right)\right|^p}{\sum_{k=0}^{q-1}\left|R\left(\frac{k}{q}\right)\right|^p}. \tag{3.82}$$

Using the notation (3.54), the results of Section 3.6 give immediately

$$(\gamma_p^\sharp)^{-1} \leq \inf_{0<t<1/2} B(p,t), \tag{3.83}$$

valid for any $p > 1$.

Let us give the corresponding definition for the grid $\mathbb{G}_q^\star$.

DEFINITION 3.10.2. We define

$$\gamma_p^\star := \liminf_{q\to\infty} \gamma_p^\star(q), \qquad \gamma_p^\star(q) := \sup_{R\in\mathcal{P}_q} \frac{\left|R\left(\frac{1}{2q}\right)\right|^p}{\sum_{k=0}^{q-1}\left|R\left(\frac{2k+1}{2q}\right)\right|^p}. \tag{3.84}$$

We have seen in Section 3.5 that, for $p>1$, with the notation (3.48),

$$(2\gamma_p^\star)^{-1} \le \inf_{0<t<1/2} A(p,t). \tag{3.85}$$

PROPOSITION 3.10.3. *Let $p>1/2$ not an even integer. Then there is p-concentration for measurable sets, and $\gamma_p \ge 2\gamma_{2p}^\star$; furthermore, there is p concentration for measurable sets with gap at the same level.*

PROOF. We postpone to the end of the proof the fact that peaking idempotents can be taken with arbitrary large gaps.

The proof is organized as the one of Proposition 3.4.6. At the outset we have a measurable and symmetric set $E\subset\mathbb{T}$ with $|E|>0$. Let us first take $C<\gamma_{2p}^\star$ arbitrarily close to $\gamma_{2p}^\star$, then fix ε a small constant. Let η and δ_0 be given by Proposition 3.8.1 (second case), depending on ε. Let $\theta>0$ be a small constant which will be fixed later on, and q_0 large enough so that, for $q>q_0$ one has $C<\gamma_{2p}^\star(q)$ and $\theta/q\le\delta_0$. With this data we consider some interval centered at $(2a+1)/(2q)$ given by Proposition 3.7.2. Let $P\in\mathcal{P}_q$ be such that

$$\left|P\left(\frac{1}{2q}\right)\right|^{2p} > C\sum_{k=0}^{q-1}\left|P\left(\frac{2k+1}{2q}\right)\right|^{2p} = C\mathbf{\Sigma}_q^\star(|P|^{2p}). \tag{3.86}$$

By Lemma 3.4.9, and Remark 3.4.10, we can find an idempotent $Q\in\mathcal{P}_{2q}$ such that we have

$$\left|Q\left(\frac{2a+1}{2q}\right)\right|^{2p} = \left|P\left(\frac{1}{2q}\right)\right|^{2p} \tag{3.87}$$

and

$$\sum_{k=0}^{q-1}\left|Q\left(\frac{2k+1}{2q}\right)\right|^{2p} = \sum_{k=0}^{q-1}\left|P\left(\frac{2k+1}{2q}\right)\right|^{2p}, \tag{3.88}$$

and also

$$\sum_{k=0}^{q-1}\left|Q\left(\frac{k}{q}\right)\right|^{2p} = \sum_{k=0}^{q-1}\left|P\left(\frac{k}{q}\right)\right|^{2p}. \tag{3.89}$$

Recall that $P\in\mathcal{P}_q$, so for $2p>1$ according to Remark 3.9.4 it satisfies the grid condition (3.77) with a constant C_{2p} depending only on p. Since P and Q attain exactly the same set of values both on the two grids $\mathbb{G}_q$ and $\mathbb{G}_q^\star$, the idempotent Q also satisfies the grid-condition (3.77) for $2p$ with the constant C_{2p}. So the idempotent

$$R(x) := Q(x)Q((2q+1)x), \tag{3.90}$$

matching with Q^2 on both grids, also satisfies

$$|R(0)|^p \le \mathbf{\Sigma}_q(|R|^p) \le C_{2p}\mathbf{\Sigma}_q^\star(|R|^p), \tag{3.91}$$

i.e. the grid condition (3.77) holds for R, too (with $K = C_{2p}$). Whence Lemma 3.9.5 applies to R, so choosing θ satisfying $A_p(C_{2p}+1)C^{-1}\theta \le \varepsilon$ and in view of (3.87), (3.88) and (3.89) for all $|t| < \theta/q^2$ we obtain the estimates

$$\sum_{k=0}^{q-1} \left| R\left(t + \frac{2k+1}{2q}\right)\right|^p \le (1+\varepsilon)\mathbf{\Sigma}_q^\star(|P|^{2p}) = (1+\varepsilon)\mathbf{\Sigma}_q^\star(|R|^p), \tag{3.92}$$

$$\left| R\left(t + \frac{2a+1}{2q}\right)\right|^p \ge (1-\varepsilon)\left| R\left(\frac{2a+1}{2q}\right)\right|^p, \tag{3.93}$$

using also that, on comparing (3.86), (3.87), (3.88) and (3.90) we are led to

$$C\mathbf{\Sigma}_q^\star(|R|^p) \le \left| R\left(\frac{2a+1}{2q}\right)\right|^p. \tag{3.94}$$

Next, we will need a peaking idempotent at 1/2, as obtained by Proposition 3.8.1. This one will depend on our given constants ε, η, $\delta = \theta/q$ and N larger than the degree of R, and also on a measurable set of finite measure E_ε that we define now. The mapping $x \mapsto qx$ is bijective from $J := (k/q, (k+1)/q)$ onto $(0,1)$, and we take for E_ε the image of $E \cap J$. It is clear that the condition

$$|E_\varepsilon \cap [\frac{1}{2} - \delta, \frac{1}{2} + \delta]| > 2(1-\eta)\delta$$

has been satisfied. We take the idempotent T provided by Proposition 3.8.1 for this data, satisfying

$$\int_{E_\varepsilon \cap [\frac{1}{2}-\delta, \frac{1}{2}+\delta]} |T|^p > (1-\varepsilon)\int_0^1 |T|^p. \tag{3.95}$$

We finally consider the product

$$S(x) := T(qx)R(x), \tag{3.96}$$

which is also an idempotent. We will prove as in Section 3.4 that

$$2C\int_{\mathbb{T}} |S|^p \le \kappa(\varepsilon)\int_E |S|^p, \tag{3.97}$$

with $\kappa(\varepsilon)$ being arbitrarily close to 1 when ε is sufficiently small. In order to do this, we put

$$J_k := \left[\frac{k}{q}, \frac{k+1}{q}\right], \qquad I_k := \left[\frac{2k+1}{2q} - \frac{\theta}{q^2}, \frac{2k+1}{2q} + \frac{\theta}{q^2}\right] \tag{3.98}$$

for $k = 0, \dots, q-1$. From now on the proof of the proposition is similar to the one of Proposition 3.4.6. We repeat briefly the steps for the reader's convenience. Denoting $\tau^p := \int_{\mathbb{T}} |T|^p$, we find, using the property (3.95), that

$$\begin{aligned} \frac{1}{2}\int_E |S|^p \ge \int_{I_a \cap E} |S|^p &\ge (1-\varepsilon)\left| R\left(\frac{2a+1}{2q}\right)\right|^p \int_{I_a \cap E} |T(qx)|^p dx \\ &\ge (1-\varepsilon)\left| R\left(\frac{2a+1}{2q}\right)\right|^p \frac{1}{q}\int_{E_\varepsilon \cap [\frac{1}{2}-\delta, \frac{1}{2}+\delta]} |T|^p \\ &\ge \frac{(1-\varepsilon)^2\tau^p}{q}\left| R\left(\frac{2a+1}{2q}\right)\right|^p. \end{aligned} \tag{3.99}$$

Then we give an upper bound for the integral on the whole torus:

$$\begin{aligned}\sum_{k=0}^{q-1}\int_{I_k}|S|^p &= \int_{-\theta/q^2}^{\theta/q^2}\sum_{k=0}^{q-1}\left|R\left(\frac{2k+1}{2q}+t\right)\right|^p |T(qt)|^p\,dt\\ &\leq (1+\varepsilon)\mathbf{\Sigma}_q^\star(|R|^p)\frac{\tau^p}{q},\end{aligned}$$

while

$$\begin{aligned}\int_{J_k\setminus I_k}|S|^p &\leq 2\|R\|_\infty^p\int_{\frac{k}{q}+\frac{\delta}{q}}^{\frac{k}{q}+\frac{1}{2q}}|T(qx)|^p dx = 2\,|R(0)|^p\,\frac{1}{q}\int_{\frac{\delta}{q}}^{\frac{1}{2}}|T(x)|^p dx\\ &\leq \frac{\varepsilon\tau^p}{q}\,|R(0)|^p \leq \frac{C_{2p}\varepsilon\tau^p}{q}\mathbf{\Sigma}_q^\star(|R|^p),\end{aligned}$$

making use of (3.91), too. Summing the last integrals over k, we obtain

$$\int_{\mathbb{T}}|S|^p \leq \frac{\tau^p}{q}(1+\varepsilon+C_{2p}\varepsilon)\mathbf{\Sigma}_q^\star(|R|^p). \tag{3.100}$$

Now (3.94), (3.99) and (3.100) give (3.97) with $\kappa(\varepsilon) :=(1+\varepsilon+C_{2p}\varepsilon)(1-\varepsilon)^{-2}$, concluding the proof, except for assuring arbitrarily large gaps.

It remains to indicate how to modify the proof to get peaking idempotents with arbitrarily large gaps. So we fix ν as a large odd integer, and we will prove that we can replace the polynomial $Q(x)$ by some polynomial $\tilde{Q}(\nu x)$, with gaps at least ν. Recall first that we can take arbitrarily large q satisfying $(\nu, q) = 1$. So we now choose $\tilde{Q}$ similarly as before, to be the polynomial of degree $2q$ that coincides with $P(bx)$ on the grid $\mathbb{G}_q$, but now with b chosen so that $\nu b(2a+1) \equiv 1 \mod 2q$. Such a b exists, as $\nu(2a+1)$ and $2q$ are coprime. We then fix

$$R(x) := \tilde{Q}(\nu x)\tilde{Q}((2q+1)\nu x).$$

There is an additional factor ν, which modifies the value of θ, but otherwise the proof is identical. We know that $\tilde{Q}(\nu x)$ and $P(bx)$, and thus $P(x)$, take globally the same values on both grids $\mathbb{G}_q$ and $\mathbb{G}_q^\star$, because in each case we multiply by an odd integer that is coprime with $2q$. So in particular the grid condition (3.78) is satisfied with C_{2p} once again. □

Similarly, but with the grid $\mathbb{G}_q$ instead of $\mathbb{G}_q^\star$, we obtain the following.

PROPOSITION 3.10.4. *Let $p > 2$ an even integer. Then there is p-concentration for measurable sets, and $\gamma_p \geq 2\max\left(\gamma_p^\sharp, \gamma_{2p}^\sharp\right)$. Moreover, we can choose the concentrating trigonometric polynomials with arbitrarily large gaps.*

PROOF. We do not give the proof, since most modifications are straightforward, and even simpler. Now if $\gamma_p^\sharp \geq \gamma_{2p}^\sharp$, we consider $C < \gamma_p^\sharp$ and P satisfying

$$\left|P\left(\frac{1}{q}\right)\right|^p > \ C\sum_{k=0}^{q-1}\left|P\left(\frac{k}{q}\right)\right|^p. \tag{3.101}$$

We build $R := Q := \mathbf{\Pi}_q P(b\ \cdot)$ of degree lower than q, using Lemma 3.4.3 and Remark 3.4.4, with b chosen such that $b \cdot a \equiv 1 \mod q$, and thus a/q is mapped on $1/q$. Thus we obtain the required concentration as above.

If $\gamma_p^\sharp < \gamma_{2p}^\sharp$, we take $C < \gamma_{2p}^\sharp$ and an idempotent $P \in \mathcal{P}_q$ satisfying

$$\left|P\left(\frac{1}{q}\right)\right|^{2p} > C\sum_{k=0}^{q-1}\left|P\left(\frac{k}{q}\right)\right|^{2p}. \tag{3.102}$$

In this case we consider $R := R(x) := Q(x)Q((q+1)x)$ with $Q := \mathbf{\Pi}_q P(b\ \cdot) \in \mathcal{P}_q$, and the proof is even more like the above argument.

□

3.11. Positive definite trigonometric polynomials

The proof of Proposition 3.10.3 generalizes directly to the class $\mathcal{P}^+$, with the main difference that, when considering the values of a polynomial P on some grid $\mathbb{G}_q$ or $\mathbb{G}_q^\star$, we can always consider the projected polynomial $\mathbf{\Pi}_{2q}(P)$, taking the same values on $\mathbb{G}_{2q}$ and hence both on $\mathbb{G}_q$ and on $\mathbb{G}_q^\star$: here we need not be concerned for occasional coincidences of projected terms in the sum, as the projection $\mathbf{\Pi}_{2q}$ leaves $\mathcal{P}^+$ invariant anyway. Therefore, the concentration constants γ_p^+, that we will obtain for positive definite functions and measurable sets, will be the same as the ones for open sets (i.e. c_p). In particular, we have the following.

Theorem 3.11.1. *Let $p > 0$ not an even integer. Then there is full p-concentration for the class $\mathcal{P}^+$ for measurable sets. Moreover, we can choose the concentrating positive definite trigonometric polynomials with arbitrarily large gaps.*

Proof. The proof follows the same lines as the one of Proposition 3.10.3, but is simpler. We know that for $p \notin 2\mathbb{N}$ there is full p-concentration at 1/2, and also from Section 3.5 that this implies $c_p^\star = 1/2$, c.f. the proof of Proposition 3.5.1. So it is sufficient to prove the following lemma, which is very similar to Proposition 3.10.3. □

Lemma 3.11.2. *Let $p > 0$. Then there is p-concentration for the class $\mathcal{P}^+$ for measurable sets, and if $p \notin 2\mathbb{N}$, then the level of concentration satisfies $\gamma_p^+ \geq 2c_{Lp}^\star$ for any L such that $Lp > 1$. Moreover, unless $p = 2$, we can choose the concentrating trigonometric polynomials with arbitrarily large gaps.*

Proof. We only sketch the modifications to accomplish in the proof of Proposition 3.10.3. Now $C < c_{Lp}^\star$. Naturally, we choose $P \in \mathcal{P}_q$ such that,

$$\left|P\left(\frac{1}{2q}\right)\right|^{Lp} > C\sum_{k=0}^{q-1}\left|P\left(\frac{2k+1}{2q}\right)\right|^{Lp}. \tag{3.103}$$

Then, as before, we choose $Q := \mathbf{\Pi}_{2q}(P(b\ \cdot))$. Now we can take $R := Q^L$, as clearly $R \in \mathcal{P}^+$, and its degree is less than $2Lq$ (instead of $2q(2q+1)$ previously). So the Bernstein type inequalities can be applied more easily, with better estimates than previously, not restricting the value of L in this case. (In fact, we could as well consider $\mathbf{\Pi}_{2q}R \in \mathcal{T}_{2q} \cap \mathcal{P}^+$, too.)

Note that here there is no need to $L \to \infty$, but only to take some $L > 1/p$, as we already have $c_p^\star = 1/2$ for $p \notin 2\mathbb{N}$. On the other hand $L > 1/p$ we really do need, as we apply Marcinkievicz-Zygmund inequalities in the proof.

Otherwise the proof for $Lp > 1$ can be adapted from Proposition 3.10.3, with all other modifications being straightforward. $\square$

When $p \in 2\mathbb{N}$, we do not have gap-peaking at 1/2, but, unless $p = 2$, we have that at 0. With a completely analogous argument, we obtain the corresponding result as follows.

THEOREM 3.11.3. *Let $p \neq 2$ be an even integer. Then there is p-concentration for the class $\mathcal{P}^+$ for measurable sets at the level $\gamma_p^+ \geq 2\sup_{L\in\mathbb{N}} c_{Lp}^\sharp$. Moreover, we can choose the concentrating positive definite trigonometric polynomials with arbitrarily large gaps.*

3.12. Concentration of random idempotents

We will see that part of the estimates proved for $\mathcal{P}^+$ in Section 3.11 extend to $\mathcal{P}$. This will be shown by certain random constructions of idempotents.

We have seen in Section 3.6 that $\inf_t B(\lambda, t)$ appears naturally when proving lower bounds for c_p when $p > 2$ is an even integer: for c_p (and thus for γ_p^+) we obtained the lower bound $\sup_L 2/\inf_t B(Lp, t)$. We will now prove the same lower bound for γ_p.

PROPOSITION 3.12.1. *Let $p > 2$ an even integer. Then, for $L \geq 1$ an integer, $\gamma_p \geq 2/\inf_t B(Lp, t)$.*

PROOF. Let $C < 1/\min_t B(Lp, t) = 1/B(Lp, t_0)$, say, and let us chose some $c := c(L, p) < t_0$. Then let q be large enough, and $P \in \mathcal{P}_q$ such that

$$\left|P\left(\frac{1}{q}\right)\right|^{Lp} > C\sum_{k=0}^{q-1}\left|P\left(\frac{k}{q}\right)\right|^{Lp}. \tag{3.104}$$

Reflecting back to Section 3.6, we know that P may be taken as some Dirichlet kernel D_r, with $r = [t_0 q] > cq$. (This is the only specific property of D_r that we will use.) Let us take $R := M^{-1}\mathbf{\Pi}_q(P^L)$, which coincides with $M^{-1}P^L$ on the grid $\mathbb{G}_q$. Choosing $M := Lr^{L-1}$, which is a majorant of the Fourier coefficients of $\mathbf{\Pi}_q(P^L)$, the polynomial R may be written as

$$R = \sum_{k=0}^{q-1} \alpha_k e_k,$$

with all $\alpha_k \in [0, 1]$ and $\sum_k \alpha_k = R(0) = r/L$. By construction, we also have

$$\left|R\left(\frac{1}{q}\right)\right|^{p} > C\sum_{k=0}^{q-1}\left|R\left(\frac{k}{q}\right)\right|^{p}. \tag{3.105}$$

We now define a random idempotent R_ω by

$$R_\omega = \sum_{k=0}^{q-1} X_k(\omega) e_k,$$

where X_k are independent Bernoulli random variables, with X_k of parameter α_k, that is, $\mathbb{P}(X_k = 1) = \alpha_k$. We want to prove that for any $\varepsilon > 0$ and for $q > q_0(\varepsilon)$, with positive probability the random idempotent R_ω satisfies the inequality

$$\left|R_\omega\left(\frac{1}{q}\right)\right|^p > K(\varepsilon)\sum_{k=0}^{q-1}\left|R_\omega\left(\frac{k}{q}\right)\right|^p, \tag{3.106}$$

with $K(\varepsilon) := K_p(\varepsilon)$ arbitrarily close to C with ε sufficiently small.

Observe that our random idempotents R_ω are such that $\mathbb{E}(R_\omega(x)) = R(x)$, so in view of (3.105), in order to prove (3.106) we have to measure the error done when replacing R_ω by its expectation. Let us center our Bernoulli variables X_k by considering $\widetilde{X}_k := X_k - \alpha_k$. Clearly, $\widetilde{X}_k$ has variance $\mathbb{V}(\widetilde{X}_k) = \alpha_k(1-\alpha_k) \le \alpha_k$, so $R_\omega(k/q)$ has expectation $R(k/q)$ and variance bounded by r/L. Also, by assumption, $|R(1/q)| > C^{1/p}R(0) > \frac{cC^{1/p}}{L}q$, so after an application of Markov's Inequality we find

$$\mathbb{P}\left(\left|\frac{R_\omega(1/q)}{R(1/q)}\right| \le 1-\varepsilon\right) \le A\varepsilon^{-2}q^{-1},$$

where A depends on p, c, L, but is independent of q and ε. Whence for q large enough, the inequality

$$\left|\frac{R_\omega(1/q)}{R(1/q)}\right| > 1-\varepsilon \tag{3.107}$$

holds with probability say at least 2/3.

Let us now consider the sums

$$S(\omega) := \sum_{k=0}^{q-1}\left|R_\omega\left(\frac{k}{q}\right)\right|^p \qquad S := \sum_{k=0}^{q-1}\left|R\left(\frac{k}{q}\right)\right|^p,$$

which we want to compare. So we also put

$$\widetilde{R}_\omega(k/q) := R_\omega(k/q) - R(k/q), \qquad \widetilde{S}(\omega) := \sum_{k=0}^{q-1}\left|\widetilde{R}_\omega(k/q)\right|^p.$$

We claim that

$$\mathbb{E}(\widetilde{S}(\omega)) \le qC_p\left(1+\sum\alpha_k\right)^{\frac{p}{2}} = qC_p\left(1+\frac{r}{L}\right)^{\frac{p}{2}}. \tag{3.108}$$

Let us first assume this inequality and conclude the proof of the proposition. So, using (3.108), $S \ge R(0)^p = (r/L)^p$ and $\widetilde{S}(\omega) \ge 0$ we are led to

$$\mathbb{P}\left(C(\varepsilon)\widetilde{S}(\omega) \ge \varepsilon S\right) \le \frac{C(\varepsilon)}{\varepsilon S}\cdot qC_p\left(1+\frac{r}{L}\right)^{\frac{p}{2}} \le A\varepsilon^{-1}q^{1-p/2}.$$

Therefore the inequality

$$C(\varepsilon)\widetilde{S}(\omega) < \varepsilon S \tag{3.109}$$

also holds with probability at least 2/3 for q large enough.

Next we will need the elementary inequality

$$|a|^p \le (1+\varepsilon)|b|^p + C(\varepsilon)|a-b|^p, \tag{3.110}$$

valid for arbitrary $\varepsilon > 0$ with some corresponding constant $C(\varepsilon)$. This is indeed obvious in case we have $|a| \le \mu|b|$ with $\mu := (1+\varepsilon)^{1/p} > 1$, while otherwise we can write $|a-b| \ge |a| - |b| \ge |a|(1-1/\mu))$, therefore $|a| \le \mu/(\mu-1)|a-b|$ and we obtain the inequality again. So applying this inequality with $a = R_\omega(k/q)$ and $b = R(k/q)$ we can estimate $|R_\omega(k/q)|^p$ by $(1+\varepsilon)|R(k/q)|^p + C(\varepsilon)|\widetilde{R}_\omega(k/q)|^p$, yielding

$$S(\omega) \le (1+\varepsilon)S + C(\varepsilon)\widetilde{S}(\omega).$$

Therefore, taking into account (3.109), (3.105) and (3.107), we find that

$$CS(\omega) \le C(1+2\varepsilon)S < (1+2\varepsilon)|R(1/q)|^p \le \frac{1+2\varepsilon}{(1-\varepsilon)^p}|R_\omega(1/q)|^p$$

holds with probability at least 1/3 for $q > q_0 = q_0(\varepsilon, p, c, L)$.

So we find that (3.106) does indeed hold with $K(\varepsilon) := C(1-\varepsilon)^p/(1+2\varepsilon)$ and for some appropriate idempotent R_ω, once we have (3.108), which we prove now. This is a consequence of the following lemma, which is certainly classical, but which we give here for the reader's convenience.

LEMMA 3.12.2. *For $p > 1$ there exists some constant C_p with the following property. Let $\alpha_k \in [0,1]$ and $a_k \in \mathbb{C}$ be arbitrary for $k = 0, 1, \ldots, N$. Let X_k be a sequence of independent Bernoulli random variables with parameter α_k, and let $\widetilde{X}_k := X_k - \alpha_k$ be their centered version, again for $k = 0, 1, \ldots, N$. Then we have*

$$\mathbb{E}\left(\left|\sum_{k=0}^N a_k \widetilde{X}_k\right|^{2p}\right) \le C_p \cdot \max_{k=1,\ldots,N} |a_k|^{2p} \cdot (1 + \sum_{k=0}^N \alpha_k)^p.$$

PROOF. We can normalize by taking $\max_{k=1,\ldots,N} |a_k| = 1$. It follows from classical martingale inequalities (see [**12**]) that

$$\mathbb{E}\left(\left|\sum_{k=0}^N a_k \widetilde{X}_k\right|^{2p}\right) \le A_p \mathbb{E}\left(\left|\sum_{k=0}^N \widetilde{X}_k^2\right|^p\right).$$

So we are left with proving the inequality

$$\mathbb{E}\left(\left|\sum_{k=0}^N \widetilde{X}_k^2\right|^p\right) \le A'_p \left(1 + \sum_{k=0}^N \alpha_k\right)^p. \tag{3.111}$$

If $0 \le \alpha \le 1$ and Y is a centered Bernoulli variable with parameter α, then

$$\mathbb{E}\left(e^{Y^2}\right) = \alpha e^{(1-\alpha)^2} + (1-\alpha)e^{\alpha^2} \le \alpha(1 + e(1-\alpha)^2) + (1-\alpha)(1 + e\alpha^2) \le e^{e\alpha},$$

because $e^x \le 1 + ex$ for $0 \le x \le 1$ and $1 + e\alpha(1-\alpha) \le 1 + e\alpha \le e^{e\alpha}$. So

$$\mathbb{E}\left(e^{\sum_{k=0}^N \widetilde{X}_k^2}\right) \le e^{e\sum_{k=0}^N \alpha_k}. \tag{3.112}$$

Finally, we use the fact that, whenever Z is a nonnegative random variable such that $\mathbb{E}(e^Z) \le e^\kappa$, then

$$\mathbb{E}(Z^p) = p\int_0^\infty \mathbb{P}(Z>\lambda)\lambda^{p-1}d\lambda \le (2\kappa)^p + p\int_{2\kappa}^\infty e^{\kappa-\lambda}\lambda^{p-1}d\lambda$$
$$\le 2^p\kappa^p + p\int_0^\infty e^{-\lambda/2}\lambda^{p-1}d\lambda = 2^p\kappa^p + A_p'' \le (2^p + A_p'')(1+\kappa)^p.$$

Putting $Z := \sum_k \widetilde{X}_k^2$ and $\kappa := e\sum_k \alpha_k$, (3.112) leads to (3.111). □

So there exists $R_\omega \in \mathcal{P}_q$ with (3.106), whence $\liminf_{q\to\infty} \gamma_p^\sharp(q) \ge C$, even $\gamma^\sharp := \liminf_{q\to\infty} \gamma_p^\sharp(q) \ge 1/\inf_t B(Lp,t)$, and referring to Proposition 3.10.4 concludes the proof of Proposition 3.12.1. □

Note that the result implies $\gamma_4 \ge 2/\inf_t B(4,t) = 0.495\ldots$, as computed in (3.59) at the end of Section 3.6 for the sake of c_4, and similarly $\gamma_{2k} \ge 0.483\ldots$ for general $k>2$ according to the calculations of (3.58).

REMARK 3.12.3. These results could also have been obtained by applying the direct estimates of Salem and Zygmund [**33**], which allow here to have estimates of the maximum value of $|\widetilde{R}_\omega|$ on the grid $\mathbb{G}_q$. The same remark holds for the next case, using the grid $\mathbb{G}_{2Lq}$.

The use of the same methods for $p>2$ not an even integer is somewhat more delicate: nevertheless, we will prove full p-concentration with gap for measurable sets. According to Proposition 3.10.3, it would suffice to show $\gamma_p^\star = 1/2$ for $p>2$. Essentially, we will do this, but with some necessary modifications. On the other hand we do know $c_p^\star = 1/2$ e.g. from the proof of Proposition 3.5.1: this proof also provides us a concrete construction, with the product of certain Dirichlet kernels in the proof, which we will make use in some extent. We start with

LEMMA 3.12.4. *Let $p>2$. Then for all $C<1/2$, there exists a constant $K := K_p(C)$ with the property that for q large there exists an idempotent $P \in \mathcal{P}_{2q}$ which satisfies the two inequalities*

$$\left|P\left(\frac{1}{2q}\right)\right|^p > C\sum_{k=0}^{q-1}\left|P\left(\frac{2k+1}{2q}\right)\right|^p, \tag{3.113}$$

$$\left|P\left(\frac{1}{2q}\right)\right|^p > K\sum_{k=0}^{q-1}\left|P\left(\frac{k}{q}\right)\right|^p. \tag{3.114}$$

PROOF. We use now from Section 3.5 that for L large enough and q large enough there exists an idempotent in $\mathcal{P}_q$, which actually can be taken some Dirichlet kernel D_r, with say $r := [q/4] > cq$ (for some fixed value of $c = c(L,p) < 1/4$), such that

$$\left|D_r\left(\frac{1}{2q}\right)\right|^{Lp} > C\sum_{k=0}^{q-1}\left|D_r\left(\frac{2k+1}{2q}\right)\right|^{Lp}.$$

From now on we fix L, so that constants may as well depend on L.

Next, we wish to ensure, with some constant $K = K(C, p, L)$, that

$$\left| D_r\left(\frac{1}{2q}\right) \right|^{Lp} > K \sum_{k=0}^{q-1} \left| D_r\left(\frac{k}{q}\right) \right|^{Lp}. \tag{3.115}$$

In view of the concrete form of the Dirichlet kernel, it is obvious, that $|D_r(1/2q)| \geq |D_r(1/q)|$. Consider now, recalling the estimation of the concentration constants $c_p^\sharp(q) \to c_p^\sharp$ in Section 3.6, and in particular reflecting back to (3.50) – (3.52), the lower estimates

$$\left| D_r\left(\frac{1}{q}\right) \right|^{Lp} > \frac{1}{B(Lp, [q/4], q)} \sum_{k=0}^{q-1} \left| D_r\left(\frac{k}{q}\right) \right|^{Lp}.$$

As $B(Lp, [q/4], q) \to B(Lp, 1/4) > 0$ $(q \to \infty)$, this clearly implies (3.115).

At this point we proceed as above. First we consider the L^{th} power of D_r and take for P the projected polynomial $M^{-1}\mathbf{\Pi}_{2q}(D_r^L)$, with $M := Lr^{L-1}$ a majorant of the Fourier coefficients of D_r^L. The polynomial P may be written as

$$P = \sum_{k=0}^{2q-1} \alpha_k e_k,$$

with all $\alpha_k \in [0, 1]$ and $\sum \alpha_k = P(0) = r/L$ $(\simeq c(L)q)$. So we have

$$\left| P\left(\frac{1}{2q}\right) \right|^{p} > C \sum_{k=0}^{q-1} \left| P\left(\frac{2k+1}{2q}\right) \right|^{p}. \tag{3.116}$$

Moreover, by construction we also have the grid condition

$$\left| P\left(\frac{1}{2q}\right) \right|^{p} > K \sum_{k=0}^{q-1} \left| P\left(\frac{k}{q}\right) \right|^{p} \tag{3.117}$$

with a certain constant $K = K(C, p, L)$.

Observe that the only required property what P does not have is being an idempotent: here $P \in \mathcal{T}_{2q} \cap \mathcal{P}^+$, while we need some polynomial in $\mathcal{P}_{2q}$. So we define, as before, a random idempotent P_ω by

$$P_\omega := \sum_{k=0}^{2q-1} X_k(\omega) e_k,$$

where X_k are independent Bernoulli random variables, with X_k of parameter α_k, that is, $\mathbb{P}(X_k = 1) = \alpha_k$. Then again $P(x) = \mathbb{E}P_\omega(x)$, and we measure the error done when replacing P_ω by its expectation.

Let us write $X_k = \alpha_k + \widetilde{X}_k$, where $\widetilde{X}_k$ is centered and has variance $\alpha_k(1-\alpha_k) \leq \alpha_k$. So $P_\omega(k/(2q))$ has expectation $P(k/(2q))$ and variance bounded by r/L.

By construction $|P(1/(2q))| > K^{1/p}P(0) > \frac{cK^{1/p}}{L}q$. So, by Markov Inequality, as before, we find that for q large enough, the inequalities

$$\left| \frac{P_\omega(1/(2q))}{P(1/(2q))} \right| > 1 - \varepsilon, \qquad \left| \frac{P_\omega(1/q)}{P(1/q)} \right| > 1 - \varepsilon \tag{3.118}$$

hold with probability 2/3.

Denoting again $\widetilde{P}_\omega(x) := P_\omega(x) - P(x)$, let us now consider the sums

$$S(\omega) := \sum_{k=0}^{q-1} \left| P_\omega \left(\frac{2k+1}{2q} \right) \right|^p, \qquad S'(\omega) := \sum_{k=0}^{q-1} \left| P_\omega \left(\frac{k}{q} \right) \right|^p,$$
$$S := \sum_{k=0}^{q-1} \left| P \left(\frac{2k+1}{2q} \right) \right|^p, \qquad S' := \sum_{k=0}^{q-1} \left| P \left(\frac{k}{q} \right) \right|^p,$$
$$\widetilde{S}(\omega) := \sum_{k=0}^{q-1} \left| \widetilde{P}_\omega \left(\frac{2k+1}{2q} \right) \right|^p, \qquad \widetilde{S}'(\omega) := \sum_{k=0}^{q-1} \left| \widetilde{P}_\omega \left(\frac{k}{q} \right) \right|^p.$$

To compare these again we use the elementary inequality (3.110) to get $|P_\omega(k/(2q))|^p \leq (1+\varepsilon)|P(k/(2q))|^p + C(\varepsilon)|\widetilde{P}_\omega(k/(2q))|^p$ and thus

$$S(\omega) \leq (1+\varepsilon)S + C(\varepsilon)\widetilde{S}(\omega), \qquad S'(\omega) \leq (1+\varepsilon)S' + C(\varepsilon)\widetilde{S}'(\omega).$$

Applying Lemma 3.12.2 as before, analogously to (3.108) we now obtain

$$\mathbb{E}|\widetilde{P}_\omega(k/(2q))|^p \leq qC_p \left(1 + \sum \alpha_k\right)^{\frac{p}{2}} \leq c'(p, L) q^{1+p/2}.$$

So for q large enough, similarly to (3.109), we prove as before that the inequalities

$$C(\varepsilon)\widetilde{S}(\omega) < \varepsilon S, \qquad C(\varepsilon)\widetilde{S}'(\omega) < \varepsilon S'$$

hold with probability 2/3, thus combining with the above, we even have

$$S(\omega) < (1+2\varepsilon)S, \qquad S'(\omega) < (1+2\varepsilon)S'$$

with probability at least 1/3. Taking into account also (3.116), (3.117) and (3.118), we can summarize our estimates so that with positive probability

$$CS(\omega) < \frac{1+2\varepsilon}{(1-\varepsilon)^p} \left| P_\omega \left(\frac{1}{2q} \right) \right|^p,$$
$$KS'(\omega) < \frac{1+2\varepsilon}{(1-\varepsilon)^p} \left| P_\omega \left(\frac{1}{2q} \right) \right|^p.$$

Since ε is arbitrary, we conclude that some $P_\omega \in \mathcal{P}_{2q}$ satisfies the requirements of the Lemma. □

At this point, we have all the elements to have the best constant for all $p > 1$ not even.

PROPOSITION 3.12.5. *Let $p > 1$ not an even integer. Then there is full p-concentration with gap for measurable sets.*

PROOF. The proof follows the same lines as the proof of Proposition 3.10.3. We take now $C < 1/2$ and, instead of choosing $P \in \mathcal{P}_q$ satisfying (3.86) and starting the consruction of Q with that, we start with choosing $P \in \mathcal{P}_{2q}$ given by Lemma 3.12.4, with exponent $2p > 2$.

Note that the only point of the proof of Proposition 3.10.3 using the fact that P is in $\mathcal{P}_q$ is the grid condition (3.78), which is given now by (3.114). Thus Lemma 3.9.5 applies even in this case, while otherwise the proof is exactly as for Proposition 3.10.3. □

3.13. p-concentration on $\mathbb{Z}_q$

In all the above proofs, the same kind of estimates as (3.4), but with finite sums on a grid of points replacing integrals, plays a central role. So it was natural to get interested in best constants on these finite structures. This led us to the same problem, but taken on finite groups, which we describe now.

Consider $\mathbb{Z}_q := \mathbb{Z}/q\mathbb{Z}$, which identifies with the grid (or subgroup) $\mathbb{G}_q \subset \mathbb{T}$, defined in (3.24). We still denote by $e(x) := e^{2\pi i x/q}$ the exponential function adapted to the group $\mathbb{Z}_q$ and by e_h the function $e(hx)$. Note that the dual group of $\mathbb{Z}_q$ is again $\mathbb{Z}_q$, i.e. characters of the group $\mathbb{Z}_q$ are the exponential functions $\{e_h \ : \ h \in \mathbb{Z}_q\}$. Again the set

$$\mathcal{P}_q := \left\{ \sum_{h\in H} e_h \ : \ H \subset \mathbb{Z}_q \simeq \{0, \cdots, q-1\} \right\} \tag{3.119}$$

is called the set of *idempotents* on $\mathbb{Z}_q$. In this context, the set of idempotents has 2^q elements.

We then adapt the definition of p-concentration to the setting of $\mathbb{Z}_q$. This strongly corresponds to Definition 3.10.1, the reason for the repetitious explanation being only the slight change of context (from $\mathbb{G}_q$ to $\mathbb{Z}_q$). Nevertheless, we keep the same notations for the various concentration constants which strictly correspond to each other.

DEFINITION 3.13.1. Let $p > 0$. We then define

$$\gamma_p^\sharp(q) := \max_{f\in\mathcal{P}_q} \frac{|f(1)|^p}{\sum_{k=0}^{q-1} |f(k)|^p} \tag{3.120}$$

and also

$$\gamma_p^\sharp := \liminf_{q\to\infty} \gamma_p^\sharp(q). \tag{3.121}$$

Note the slight alteration of notions between the continuous group $\mathbb{T}$ and the discrete group $\mathbb{Z}_q$. Since the latter occurred as a technical tool in the analysis of $\mathbb{T}$, in the definition of $\gamma_p^\sharp(q)$ we did not keep considering only symmetric sets. Also the reader may note that in order to define p-concentration in the group $\mathbb{Z}_q$, one should also look for f that satisfies (3.120), but with $f(a)$, for some arbitrary $a \in \mathbb{Z}_q$, in the left hand side.

Again for technical convenience, as it was sufficient in the analysis of $\mathbb{T}$, above we mainly restricted considerations to cases when q was a prime. Then it was easy, for $a = 0$ the Dirac mass at 0, which is an idempotent, has the required concentration property with constant 1, and for $a \neq 0$ and f chosen to satisfy (3.120), the function $g(x) := f(a^{-1}x)$ – with a^{-1} being the unique inverse of a for the multiplication in $\mathbb{Z}_q$ – satisfies the same inequality, but with $g(a)$ in the numerator. Indeed, $g(a) = f(1)$, and all other values taken by f are taken by g since multiplication is one-to-one in $\mathbb{Z}_q$ for q prime, so that the denominator is the same sum, but in different order, for f and g.

We can also replace 1 by a in the numerator of (3.120) when q is any integer, but a and q are coprime. For composite q, and with a consideration of symmetry again for allowing full concentration with constant 1, we give the definition, corresponding to definitions 3.1.1, 3.1.2 and 3.1.4, too.

Note that distinction of constants according to openness or measurability of concentrating subsets does not occur here. On the other hand, it seems to be reasonable to make a new distinction regarding the restriction if q is assumed to be prime, or arbitrary. As in the case of open an measurable sets, at the end it may turn out that the constants coincide: but at the outset we don't have it for granted. That is, a precise definition of p-concentration on $\mathbb{Z}_q$ would be the following.

DEFINITION 3.13.2. The concentration constants of $\mathbb{Z}_q$ is defined as

$$\gamma_p(\mathbb{Z}_q) := \min_{\substack{\emptyset\neq E\subset\mathbb{Z}_q\\ E=-E}} \max_{f\in\mathcal{P}_q} \frac{\sum_{a\in E}|f(a)|^p}{\sum_{k=0}^{q-1}|f(k)|^p}$$

$$= \min\left(\min_{\substack{a\in\mathbb{Z}_q\\ a\equiv -a \mod q}} \gamma_p(a;\mathbb{Z}_q), \quad \min_{\substack{a\in\mathbb{Z}_q\\ a\not\equiv -a \mod q}} 2\gamma_p(a;\mathbb{Z}_q) \right), \tag{3.122}$$

where

$$\gamma_p(a;\mathbb{Z}_q) := \max_{f\in\mathcal{P}_q} \frac{|f(a)|^p}{\sum_{k=0}^{q-1}|f(k)|^p} \tag{3.123}$$

Then the *uniform level of p-concentration on prime* $\mathbb{Z}_q$ is defined as

$$C_p := \liminf_{q\to\infty,\ q \text{ prime}} \gamma_p(\mathbb{Z}_q)$$

and the *uniform level of p-concentration on integer* $\mathbb{Z}_q$ is defined as

$$\Gamma_p := \liminf_{q\to\infty} \gamma_p(\mathbb{Z}_q).$$

For q prime – and anyway, for q odd – the first part in the minimum of the second line of (3.122) can be neglected, as $\gamma_p(0,\mathbb{Z}_a) = 1$ anyway. Moreover, if $q \in 2\mathbb{N}$, i.e. $q = 2r$, then the first minimum is the minimum of 1 (arising from concentration at 0) and $1/2$, as concentration at $q/2$ is always $1/2$: it can not be larger, since $|f(r)| \le f(0)$, but also it achieves $1/2$, since the function $f := (1/2)(\delta_0 + \delta_r) = \delta_0(2\,\cdot)/2 = \sum_{h=1}^{q} e_{2h}$ exhibits $f(a) = 0$ for $a \not\equiv 0, r \mod q$, while $f(0) = f(r) = r$. So we have

$$\gamma_p(\mathbb{Z}_q) = \begin{cases} \min\left(\frac{1}{2}, \quad 2 \min\limits_{\substack{a\in\mathbb{Z}_q\\ a\not\equiv -a \mod q}} \gamma_p(a;\mathbb{Z}_q)\right) & q\in 2\mathbb{N} \\ 2 \min\limits_{\substack{a\in\mathbb{Z}_q\\ a\not\equiv -a \mod q}} \gamma_p(a;\mathbb{Z}_q) & q \in 2\mathbb{N}+1 \end{cases}, \tag{3.124}$$

and in particular

$$\gamma_p(\mathbb{Z}_q) = 2\gamma_p^\sharp(q) \qquad q \text{ prime}, \tag{3.125}$$

for $a \not\equiv 0 \mod q$ has a multiplicative inverse $\mod q$, giving $\gamma_p(a;\mathbb{Z}_q) = \gamma_p(1;\mathbb{Z}_q)$, while $\gamma_p(0;\mathbb{Z}_q) = 1$, always.

Here we can formulate a discrete analogue of the problem in [**2, 3**]: *Does q-uniform concentration fail for* $p = 1$*?* Also, for its own sake, we can ask for the determination of $\gamma_p(\mathbb{Z}_q)$ and of $C_p = 2\gamma_p^\sharp$ or Γ_p. The interest in the latter is purely a matter of curiosity, because in the transference to the torus it suffices to consider prime q's: henceforth also our interest will be restricted to this case here, leaving the somewhat more number theoretical question of Γ_p for future investigations.

As we said, p-concentration on $\mathbb{Z}_q$ plays a role in proofs for p-concentration on the torus. In order to solve the 2-concentration problem on the torus, Déchamps-Gondim, Piquard-Lust and Queffélec [**14, 15**] have considered the concentration problem on $\mathbb{Z}_q$, proving the precise value that we already mentioned,

$$C_2 = \lim_{q\to\infty,\ q \text{ prime}} \gamma_2(\mathbb{Z}_q) = 2\gamma_2^\sharp = \sup_{0\le x} \frac{2\sin^2 x}{\pi x} = 0.4613\cdots . \tag{3.126}$$

Moreover, they obtained $\gamma_p^\sharp \ge (\gamma_2^\sharp)^{p/2}$ for all $p > 2$. The last assertion is an easy consequence of the decrease of ℓ^p norms with p, and we have, in general,

$$\gamma_p^\sharp \ge (\gamma_{p'}^\sharp)^{p/p'} \tag{3.127}$$

for $p > p'$.

Let us also mention that they considered the same problem for the class of positive definite polynomials (3.11) (with the interpretation of e_h changing from $e^{2\pi iht}$ to $e^{2\pi iht/q}$ corresponding to the shift to $\mathbb{Z}_q$). The corresponding concentration constants in Definition 3.13.1, but with $\mathcal{P}_q$ replaced by $\mathcal{P}_q^+ (\simeq \mathcal{P}^+$ on $\mathbb{Z}_q)$, are denoted by $c_p^+(q)$ and c_p^+.

With these notations, it has been proved in [**14**] that $c_2^+ = 1/4$. Since the class of positive definite polynomials is stable by taking products, it follows that, for all even integers $2k$,

$$\gamma_{2k}^\sharp \le c_{2k}^+ \le 1/4.$$

It is easy to see that there is uniform p-concentration on $\mathbb{Z}_q$ for all $p > 1$, using Dirichlet kernels. This has been extensively used above, where the discrete problem under consideration here has been largely studied, at least for p an even integer.

On the other hand, coming back to our main point, i.e. to the case of $p = 1$, and using the recent results of B. Green and S. Konyagin [**19**], we answer negatively in this case, which gives an affirmative answer to the conjecture of [**3**] for finite groups $\mathbb{Z}_q$.

All the results on $\mathbb{Z}_q$ summarize in the following theorem, which gives an almost complete answer to the p-concentration problem under consideration, except for the best constants, which are not known for $p \ne 2$.

THEOREM 3.13.3. *For all* $1 < p < \infty$ *we have uniform* p*-concentration on* $\mathbb{Z}_q$*. We have* $2\gamma_2^\sharp$ *given by* (3.126)*, then* $0.495 < 2\gamma_4^\sharp \le 1/2$*. For all* $p > 2$*, we have* $2\gamma_p^\sharp > 0.483$*. On the other hand for* $p \le 1$ *we do not have uniform* p*-concentration, moreover, already* $C_1 = 0$*.*

Positiveresults are already contained in the above, where they were used as tools for the problem of concentration on the torus. As for upper bounds for $\gamma_p^\sharp$, since the polynomials f with positive coefficients have their maximum at 0, we have the trivial upper bound $\gamma_p^\sharp \le 1/3$. Moreover, for p an even integer, we have seen that $2\gamma_p^\sharp \le 1/2$. Observe that (3.127) provides an improvement on the bound $2/3$ between two even integers. Indeed, for $p \le 2k$, using $2\gamma_{2k}^\sharp \le 1/2$, we obtain

$$\gamma_p^\sharp \le 2^{-p/k}.$$

Recall the situation on the group $\mathbb{Z}_q$ by transferring the results that have been obtained for the grid $\mathbb{G}_q \subset \mathbb{T}$, defined in (3.24). In spite of a slight abuse of notation, let us still keep the notation $\mathcal{P}_q$ from (3.16) for the set of trigonometrical idempotents of degree less than q on $\mathbb{T}$, with e_h denoting the exponential $e_h(x) := e^{2\pi i h x}$ adapted to $\mathbb{T}$. When restricted to $\mathbb{G}_q$ identified with $\frac{1}{q}\mathbb{Z}_q$, it coincides with the corresponding idempotent (the coefficients are the same, but the exponential is now adapted to $\mathbb{Z}_q$) on $\mathbb{Z}_q$. This is a one-to-one correspondence between idempotents of $\mathbb{Z}_q$ and idempotents of degree less than q, since these last ones are determined by their values on q points, and, in particular, on $\mathbb{G}_q$. We will prefer to deal with ordinary trigonometrical polynomials, and see $\mathbb{Z}_q$ as the grid $\mathbb{G}_q$.

Unless explicitly mentioned, we will only consider Taylor polynomials, that is, trigonometrical polynomials with only non negative frequencies.

Again with the same slight lack of precision, we identify the quantities $\gamma_p^\sharp$, $\gamma_p^\sharp(q)$ in Definition 3.13.1 with the ones in Definition 3.10.1.

One can obtain a lower bound of $\gamma_p^\sharp$, with $p > 1$, by the only consideration of the Dirichlet kernels (3.8). Here the constraint on the degree restricts us to $n < q$ and $L = 1$, but essentially formula (3.51) provides the required estimate of $1/\gamma_p^\sharp$. Having n and q tend to infinity with n/q tending to t, we can refer to Lemma 3.6.1 for the fact that for $p > 1$ the inequality

$$(\gamma_p^\sharp)^{-1} \leq \inf_{0<t<1/2} B(p,t), \tag{3.128}$$

holds with $B(\lambda, t)$ (defined in (3.54) for all $\lambda > 1$).

It is clear that $B(\lambda, t)$ is bounded for $\lambda > 1$, so that $\gamma_p^\sharp > 0$ and there is uniform p-concentration: just take as a bound the value for $t = 1/4$. Let us try to get more precise estimates. The computation of $\inf_{0<t<1/2} B(\lambda, t)$ can be executed explicitly for $\lambda = 2$ and $\lambda = 4$. In the first case we recognize in the sum the Fourier coefficients of $\chi_{[-t/2,t/2]}$, whose L^2 norm is $\sqrt{t}$. So (3.128) leads to the minimization of the function $\frac{2\sin^2 t}{\pi t}$, and to the estimate $\gamma_2^\sharp \geq \sup_{0\leq t} \frac{2\sin^2 t}{\pi t} = 0.4613\cdots$. This is the formula given by Déchamps-Gondim, Lust-Piquard and Queffélec in [**14**]. We refer to them for the necessity of the condition, for which they give a smart proof. For $\lambda = 4$, repeating the argument leading to (3.59), we recognize in the sum of (3.54) the Fourier coefficients of the convolution product $\chi_{[-t/2,t/2]} * \chi_{[-t/2,t/2]}$, whose L^2 norm is equal to $(2t^3/3)^{1/2}$, and then using Plancherel Formula we obtain that

$$\gamma_4^\sharp \geq \max_{0<t<1/2} \frac{3\left(\sin^4(\pi t)\right)}{\pi^4 t^3} > 0.495. \tag{3.129}$$

For larger integer values of λ, the computations do not seem to be easily handled. But we can prove that there exists a uniform lower bound for $\gamma_p^\sharp$ when $p > 2$. To see this, we will use a version of Lemma 3.12.2 above. Let us first give new definitions, relative to positive definite polynomials.

Similarly as for idempotents, by the same slight abuse of notation, let us denote

$$\mathcal{P}_q^+ := \left\{ \sum_{h\in H} a_h e_h \;:\; a_h \geq 0,\, h \in \{0, \cdots, q-1\} \right\}. \tag{3.130}$$

the set of trigonometrical polynomials with non negative coefficients of degree less than q on $\mathbb{T}$, with e_h denoting the exponential adapted to $\mathbb{T}$. Again, when restricted to $\mathbb{G}_q$, it coincides with the corresponding positive definite polynomial with non negative coefficients on $\mathbb{Z}_q$, and this defines a one-to-one correspondence between positive definite polynomials of $\mathbb{Z}_q$ and positive definite polynomials on $\mathbb{T}$ of degree less than q. The constant C_p^+ can then be defined by

$$C_p^+ := \liminf_{q\to\infty} C_p^+(q), \qquad C_p^+(q) := \sup_{R\in\mathcal{P}_q^+} \frac{\left|R\left(\frac{1}{q}\right)\right|^p}{\sum_{k=0}^{q-1}\left|R\left(\frac{k}{q}\right)\right|^p}. \tag{3.131}$$

It is much easier to find positive definite polynomials in $\mathcal{P}_q^+$ than idempotents. The reason for that is that here the restriction on the degree is not essential: any polynomial $P \in \mathcal{P}^+$ can be projected (see the notation of Definition 3.4.1) to $\widetilde{P} := \mathbf{\Pi}_q P \in \mathcal{P}_q^+$ while keeping its values on the grid i.e. on $\mathbb{Z}_q$. The closedness of $\mathcal{P}_+$ under projections explains why the concentration constants $C_p^+(q)$ and C_p^+ are in fact equal to the constants $c_p^+(q)$, c_p^+, (i.e. the same constants lacking any restrictions on the degree), which can be defined analogously to Definition 3.4.8 of $c_p^\sharp$, $c_p^\sharp(q)$ with $\mathcal{P}$ changed to $\mathcal{P}^+$.

In particular, whenever P is in $\mathcal{P}_q$, then, for each positive integer L the polynomial $Q := \mathbf{\Pi}_q P^L$ is also in $\mathcal{P}_q^+$. So we can take as well powers of Dirichlet kernels as polynomials R in the right hand side of (3.131). This leads to the following bounds, via (3.51) and Lemma 3.6.1. The first estimate gives a non explicit bound for a fixed p:

$$C_p^+ \geq \sup_{L\geq 1} \sup_{0<t<1/2} B(Lp,t)^{-1}. \tag{3.132}$$

The next two estimates

$$\inf_{L\geq 1} \inf_{0<t<1/2} B(Lp,t) \leq \inf_{\kappa>0} \limsup_{\lambda\mapsto\infty} B\left(\lambda, \kappa\sqrt{6/\lambda}\right) \leq 4.13273 \tag{3.133}$$

may be found in and right after (3.58). These lead to

$$2C_p^+ > 0.483. \tag{3.134}$$

We prove now that we have the same estimates for $\gamma_p^\sharp$ when $p > 2$.

THEOREM 3.13.4. *We have* $2\gamma_p^\sharp > 0.483$ *uniformly for all* $p > 2$.

This is a consequence of the following proposition, which is more general than the corresponding Lemma 3.12.2 above.

PROPOSITION 3.13.5. *Let* $p > 2$ *and* $c > 0$, $\varepsilon > 0$. *Then there exists* $q_0 := q_0(c, \varepsilon)$ *such that, if* $q > q_0$ *and* $P := \sum_0^{q-1} a_h e_h$ *is a polynomial of degree less than* q *that satisfies the two conditions*

$$cq \max_h |a_h| \leq \sum |a_h| \leq c^{-1}|P(1/q)|, \tag{3.135}$$

$$|P(1/q)| \geq c\left(\sum_{k=0}^{q-1} |P(k/q)|^p\right)^{1/p}, \tag{3.136}$$

then there exists a polynomial Q of degree less than q, whose coefficients are either $a_h/|a_h|$ or 0, such that

$$|Q(1/q)| \geq (1-\varepsilon)|P(1/q)|, \tag{3.137}$$

$$\left(\sum_{k=0}^{q-1} |Q(k/q) - P(k/q)|^p\right)^{1/p} \leq \varepsilon |P(1/q)|. \tag{3.138}$$

Observe that, for P positive definite, Q is an idempotent. In this case, the first condition can be reduced to $P(0) \geq cq \max_h |a_h|$. Indeed, the fact that $|P(1/q)| \geq cP(0)$ follows from the second one.

Let us take the proposition for granted, and use it in our context.

Proof of Theorem 3.13.4. Let us take a positive definite polynomial P of degree less than q for which

$$\frac{2\left|P\left(\frac{1}{q}\right)\right|^p}{\sum_{k=0}^{q-1}\left|P\left(\frac{k}{q}\right)\right|^p} \geq c_0 > 0.483.$$

Such $P \in \mathcal{P}_q^+$ exists in view of (3.134). We claim that there exists an idempotent Q for which the same ratio is bounded from below by $c_0 C(\varepsilon)$, with $C(\varepsilon)$ being arbitrarily close to 1 when $\varepsilon > 0$ is chosen sufficiently small. Indeed, we can apply the proposition as soon as we have proved that P satisfies the condition (3.135) (uniformly for q large). We have seen that P can be taken as $\mathbf{\Pi}_q D_n^L$, i.e. the polynomial of degree less than q, which coincides with D_n^L on the grid $\mathbb{G}_q$, for n chosen in such a way that $n/q \approx t = \kappa\sqrt{6/\lambda}$ is small enough so that we approach the extremum in (3.133). Next, it is easy to see that $P(0) = n^L$, while $|\hat{P}(k)| \leq Ln^{L-1}$. So we have (3.135) with a very small constant c, but what is important that it does not depend on q tending to ∞ (for fixed ε). To conclude the proof, we use the fact that, by Minkowski's inequality, and using the assumption on P, we have

$$\begin{aligned}
\left(\sum_{k=0}^{q-1}\left|Q\left(\frac{k}{q}\right)\right|^p\right)^{1/p} &\leq \left(\sum_{k=0}^{q-1}\left|P\left(\frac{k}{q}\right)\right|^p\right)^{1/p} + \varepsilon|P(1/q)| \\
&\leq ((2/c_0)^{1/p} + \varepsilon)|P(1/q)| \\
&\leq (1-\varepsilon)((2/c_0)^{1/p} + \varepsilon)|Q(1/q)|.
\end{aligned}$$

The constant tends to $(2/c_0)^{1/p}$ when ε tends to 0, which concludes the proof. □

The same method leads to

$$\gamma_p^\sharp \geq \sup_{L\geq 1} \sup_{0<t<1/2} B(Lp,t)^{-1}. \tag{3.139}$$

This finishes the proof of the part of Theorem 3.13.3 concerning $p > 1$, except for the proof of Proposition 3.13.5, which we do now. It relies on the construction of random polynomials, which may have an independent interest.

Proof of Proposition 3.13.5. Without loss of generality we may assume $\max_h |a_h| = 1$. We put $\alpha_k := |a_k|$ and $\sigma; = \sum \alpha_k$, so that $0 \leq \alpha_k \leq 1$ and $cq \leq \sigma \leq c^{-1}|P(1/q)|$. We take a sequence of independent random variables $X_0, X_1, \ldots, X_{q-1}$ that follow the Bernoulli law with parameters $\alpha_0, \alpha_1, \ldots, \alpha_{q-1}$ on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and set

$$P_\omega := \sum_0^{q-1} b_h X_h(\omega) e_h$$

with $b_h := a_h/|a_h|$ for $a_h \neq 0$, otherwise $b_h = 0$. Then the expectation of P_ω is equal to P. We will prove that $Q = P_\omega$ satisfies (3.137) and (3.138) with positive probability. Let us first consider (3.137), and prove that the converse inequality holds with probability less than 1/3 for q large enough. Indeed, one has the inclusions

$$\{\omega; |P_\omega(1/q)| \leq (1-\varepsilon)|P(1/q)|\} \subset \{\omega; |P_\omega(1/q) - P(1/q)| > \varepsilon|P(1/q)|\},$$

so that, by Markov inequality, using the fact that the variance of $P_\omega(1/q)$ is $\sum \alpha_k(1-\alpha_k) \leq \sigma$, we have

$$\mathbb{P}\left(\left|\frac{P_\omega(1/q)}{P(1/q)}\right| \leq 1-\varepsilon\right) \leq c^{-2}\varepsilon^{-2}\sigma^{-1}.$$

By (3.135) we know that this quantity is small for q large.

Next, to show (3.138), in view of (3.135) it is sufficient to prove that with probability 2/3,

$$\sum_{k=0}^{q-1} |P_\omega(k/q) - P(k/q)|^p \leq c^p \varepsilon^p \sigma^p.$$

We claim that there exists some uniform constant C_p, for $p > 2$, such that, for each k,

$$\mathbb{E}(|P_\omega(k/q) - P(k/q)|^p) \leq C_p \sigma^{p/2}. \tag{3.140}$$

Let us take this for granted and finish the proof. By simple estimation

$$\mathbb{P}\left(\sum |P_\omega(k/q) - P(k/q)|^p \geq (c\varepsilon\sigma)^p\right) \leq c^{-p}\varepsilon^{-p} C_p\, q\, \sigma^{-p/2}.$$

From this we conclude easily, using the fact that $\sigma \geq cq$, so that the right hand side tends to 0 when q tends to infinity.

Finally, (3.140) is a well-known property of independent sums of Bernoulli variables, see Lemma 3.12.2 above. □

Of course one would like to know whether constants are the same for classes $\mathcal{P}_q$ and $\mathcal{P}_q^+$. Thanks to the work of Déchamps-Gondim, Lust-Piquard and Queffélec, we know that it is not the case for $p = 2$, but the last proposition induces to conjecture that they are the same for $p > 2$.

Note that Proposition 3.13.5 holds when (3.135) is replaced by the weaker assumption $\sigma \geq \delta(q) q^{2/p} \max |a_h|$, with δ tending to infinity with q. However, for sparse coefficients this cannot go through, take e.g. $P(x) = frac12e(x)$. Therefore, one would like to ascertain somehow that the extremal P in $C^+(q)$ has a somewhat dense coefficient sequence, an issue to be studied more closely.

Finally, we prove here the negative result of Theorem 3.13.3. It will be more convenient, in this part, to work directly on $\mathbb{Z}_q$, and not on the grid $\mathbb{G}_q$. We now restrict to q prime, which is sufficient to conclude negatively.

Assume that there exists some constant c and some idempotent $f = \sum_{h\in H} e_h$ such that

$$|f(1)| \geq c \sum_{k=0}^{q-1} |f(k)|. \tag{3.141}$$

We claim that H may be assumed having cardinality $\leq q/2$. Indeed, H is certainly not the whole set $\{0, \cdots, q-1\}$, since the corresponding idempotent is q times the Dirac mass at 0. Moreover, the idempotent $\widetilde{f}$, having spectrum ${}^cH$, takes the same absolute values as f outside 0, while its value at 0 is $q - \#H$. So, if $\#H > q/2$, then $\widetilde{f}$ satisfies also (3.141).

From now on, let $r := \#H \leq q/2$. We have by assumption (3.141) $\sum_{k=0}^{q-1} |f(k)| \leq |f(1)|/c \leq f(0)/c = r/c$. So the function

$$g := r^{-1} (f - r\delta_0)$$

is 0 at 0, has ℓ^1 norm bounded by $\frac{1}{c} + 1$, while its Fourier coefficients are equal to $1/r - 1/q$ (r of them), or $-1/q$, since the delta function has all Fourier coefficients equal to $1/q$. But, according to Theorem 1.3 of [**19**], we should have $q \min_k |\hat{g}(k)|$ tending to 0 when q tends to ∞ (note that the Fourier transform here is replaced by the inverse Fourier transform in [**19**], which is the reason for multiplication by q compared to the statement given there). This gives a contradiction, and allows to conclude that there is no uniform 1-concentration. This finishes the proof.

We leave the following as an open question.

PROBLEM 3.13.6. With the notation of Definition 3.13.1, we obtained that $\gamma_1^\sharp(q)) \to 0$ as $q \to \infty$. Determine $\beta := \liminf_{q\to\infty} \log(1/\gamma_1^\sharp(q))/\log\log q$.

Using the full strength of the result of [**19**], the constant c in the proof of Theorem 3.13.3 may be chosen uniformly bounded from below in q by $\log^{-\alpha} q$, with α less than $1/3$ (that is, the proof by contradiction shows that $c > \log^{-\alpha} q$ is not possible, hence $\beta \geq 1/3$). On the other hand the Dirichlet kernel exhibits $\gamma_1^\sharp(q) \geq C/\log q$, i.e. $\beta \leq 1$. This leaves open the question if β achieves 1, i.e. $\log(1/\gamma_1^\sharp(q))/\log\log q$ can be taken anything less than 1. The problem is in relation with the Littlewood conjecture on groups $\mathbb{Z}_q$, for which there has been new improvements by Sanders [**34**].

3.14. More on L^1 concentration

Recall the definition of p-concentration for measurable sets given in Definition 3.1.4. The main theorem of [**3**], described in Theorem 3.1.5, formulated that there is p-concentration for all $p > 1$. We have proved above that there is p-concentration even for all $p > 1/2$, while the same authors conjectured that idempotent concentration fails already for $p = 1$.

Furthermore, we proved that the constant γ_p is equal to 1 when $p > 1$ and p is not an even integer. As for the exceptional situation for $p \in 2\mathbb{N}$, this is in line with the fact that L^p norms behave differently depending on whether p is an even integer or not in a certain number of problems, such as the Hardy-Littlewood majorant problem (*does an inequality on absolute values of Fourier coefficients imply an inequality on* L^p *norms?*), Zygmund's question (*does a Wiener-Ingham type essentially uniform distribution of the* p*-norm holds, at least on intervals longer than* $2\pi/N$*, when* $f \in L^p(\mathbb{T})$ *has gaps exceeding* N *in its Fourier series?*) or the Wiener property for periodic positive definite functions (*does a positive definite function belong to* L^p *when it is the case on a small interval around 0?*). To these we shall return in Section 3.15.

The other open question is what happens for $0 < p < 1$, where there is full concentration (for open sets), but we could not achieve the same strength of results for measurable sets as well. In particular, even if we disproved the conjecture of [**3**] for $p = 1$, the situation is not yet entirely clear. Indeed, when we restrict the class of symmetric measurable sets to symmetric open sets or enlarge the class of idempotent polynomials to all positive definite ones, that is, allow all non negative coefficients and not only 0 or 1, then we obtain full concentration, i.e. $c_1 = 1$ and $c_1^+ = \gamma_1^+ = 1$. So one may conjecture that even $\gamma_1 = 1$ (even if we understand that one should be cautious with such conjectures).

We can not achieve this ultimate result, neither we can determine the exact value of γ_1. Nevertheless, by pushing forward our techniques, we can somewhat further improve our previous estimate on the concentration constant γ_1 for measurable sets, and prove a lower estimation quite close to 1. Our goal here is to prove the following numerically improved concentration estimate in the critical case of $p = 1$.

THEOREM 3.14.1. *For* $p = 1$ *there is concentration for measurable sets at the level* $\gamma_1 > 0.96$*. Moreover, for arbitrarily large given* N *the corresponding concentrating idempotent can be chosen with gaps at least* N *between consecutive frequencies.*

Our argument will be a refined version of the proof of Proposition 3.10.3, where the grid condition of Definition 3.9.3 played a crucial role. In fact, now it is better to make this role more explicit by introducing the following modified discrete concentration constants. We start with some further notations for the grid condition itself.

DEFINITION 3.14.2. Let $0 < p < \infty$ and $q \in \mathbb{N}$. The set of polynomials satisfying a grid-condition (3.77) of Definition 3.9.3 with the constant K is denoted as $\mathcal{T}(K)$. Furthermore, for arbitrary degree $m \in \mathbb{N}$ we write $\mathcal{T}_m(K) := \mathcal{T}(K) \cap \mathcal{T}_m$, and we also set $\mathcal{P}(K) := \mathcal{P} \cap \mathcal{T}(K)$ and $\mathcal{P}_m(K) := \mathcal{P}_m \cap \mathcal{T}(K)$.

Then we can define the modified concentration constants as follows.

DEFINITION 3.14.3. With $\mathcal{P}_{2q}(K)$ defined in Definition 3.14.2 we define

$$\Gamma_p^\star := \sup_{K<\infty} \liminf_{q\to\infty} \Gamma_p^\star(q,K), \quad \Gamma_p^\star(q,K) := \sup_{R\in\mathcal{P}_{2q}(K)} \frac{\left|R\left(\frac{1}{2q}\right)\right|^p}{\sum_{k=0}^{q-1}\left|R\left(\frac{2k+1}{2q}\right)\right|^p}. \tag{3.142}$$

In other words, $\Gamma_p^\star$ is positive when there is uniform concentration at $1/2q$, (which is the case for $p > 1$), but the grids $\mathbb{G}_q$ and $\mathbb{G}_q^\star$ do not play the same role; the constant $\Gamma_p^\star$ is only the relative concentration on $\mathbb{G}_q^\star$ (the coset $\frac{1}{2q} + \mathbb{G}_q$ of $\mathbb{G}_q$ in $\mathbb{G}_{2q}$), which we try to maximize.

Taking the supremum in K means that we do not mind and do not take into account the actual value of the bound, until it stays bounded uniformly.

In order to carry over the proof of Theorem 3.14.1, we need to control the error done when replacing idempotents in a neighborhood of points of a grid by its values on the grid. So as above, we will make use of the Bernstein type Lemma 3.9.1.

This also explains why we have introduced Definitions 3.14.2 and 3.14.3. When $P \in \mathcal{T}_q$, the grid condition (3.77) for P holds with $K = C_p$ depending only on $p > 1$, c.f. Remark 3.9.4. Therefore, we immediately obtain $\mathcal{P}_{2q}(K) \supset \mathcal{P}_q$ for $p > 1$ and $K \geq C_p$, resulting in $\Gamma_p^\star(q, K) \geq \gamma_p^\star(q)$ for $K \geq C_p$ and $p > 1$. Whence

$$\Gamma_p^\star \geq \gamma_p^\star \qquad (p > 1). \tag{3.143}$$

It is then clear that proving inequalities with $\Gamma_p^\star$ can yield sharper results than proven before.

Remark 3.14.4. We can also replace 1 by $2a + 1$ in the numerator of (3.142) when q is any integer, but $2a + 1$ and $2q$ - i.e, $2a + 1$ and q – are coprime.

This is the same as Lemma 3.4.9. Note that if $R(x) \in \mathcal{P}_{2q}(K)$ and b is the multiplicative inverse of $2a + 1$ mod q, then $Q(x) := \mathbf{\Pi}_{2q} R(bx) \in \mathcal{P}_{2q}(K)$ attaining the same set of values both on $\mathbb{G}_q$ and on $\mathbb{G}_q^\star$, while $Q(\frac{1}{2q}) = R(\frac{2a+1}{2q})$. Here $\mathbf{\Pi}_{2q}$ does not spoil being an idempotent, since for $(b, 2q) = 1$ $bk \equiv bj \mod 2q$ occurs only for $k \equiv j \mod 2q$, so the projection of nonzero coefficient frequencies is one to one.

A lower bound for $\Gamma_p^\star$ is provided by the estimate below.

Lemma 3.14.5. *For $p > 1$, we have the inequality*

$$\frac{1}{2\Gamma_p^\star} \leq \inf_{0<t<1/2} A(p, t), \tag{3.144}$$

where for $\lambda > 1$ $A(\lambda, t)$ is the quantity defined in (3.48).

Proof. This is a combination of (3.46) with $L = 1$ – when it is easy to see that $D_r \in \mathcal{P}_{2q}(K)$ – and Lemma 3.5.2.

Obviously, taking Dirichlet kernels D_r, with $r/2q$ tending to $t \in (0, 1/2)$, leads to $D_r \in \mathcal{P}_q$. The boundedness condition (3.77), to ensure also $D_r \in \mathcal{P}_{2q}(K)$, is provided either by a concrete calculation, or referring to Remark 3.9.4, where in the full generality of $P \in \mathcal{P}_q$ and $p > 1$ we have noted the validity of a grid condition with $K = C_p$ independently also of q, hence uniformly in q.

Then an application of Lemma 3.5.2 concludes the argument. □

Observe that $A(\lambda, t)$ tends to ∞ when t tends to 0, so that the infimum in the right hand side of (3.144) is obtained away from 0. Also note that (for fixed t) $A(\lambda, t)$, and hence also $\inf_{0<t<1/2} A(\lambda, t)$ are decreasing functions of λ. Above in (3.49) we have calculated $A(2, t)$. Substituting $x = \pi t$ and recalling (3.5), (3.49), (3.85) and (3.143), we find that

$$\Gamma_2^\star \geq \gamma_2 \geq \frac{1}{2 \inf_{0<t<1/2} A(2, t)} = c_2 \approx 0.4613.$$

Moreover, it is easy to see that $\inf_{0<t<1/2} A(\lambda, t)$ is left continuous in λ at 2, so that

$$\liminf_{p\to 2-0} \Gamma_p^\star \geq 0.4613. \tag{3.145}$$

Let us next give a slight reformulation of Lemma 3.9.2.

LEMMA 3.14.6. *Let $1 < p < \infty$ and C_p be the same constant as in Lemma 3.9.1. Then for all $P \in \mathcal{T}_{2q}(K)$ and for any $|t| < 1/2$ we have the two inequalities*

$$\sum_{k=0}^{2q-1} \left| P\left(\frac{k}{2q} + t\right)\right|^p \leq C_p(K+1)\mathbf{\Sigma}_q^\star(|P|^p), \tag{3.146}$$

$$\sum_{k=0}^{2q-1} \left| \left| P\left(\frac{k}{2q} + t\right)\right|^p - \left| P\left(\frac{k}{2q}\right)\right|^p \right| \leq 2C_p(K+1)|qt|\mathbf{\Sigma}_q^\star(|P|^p). \tag{3.147}$$

Let us revisit now the above Lemma 3.12.4! This Lemma means that if $p > 2$, then for all $C < 1/2$ we have $\Gamma_p^\star(q, K) \geq C$ for $K = K(C)$ and $q > q_0(K, C, p)$. An immediate corollary is

COROLLARY 3.14.7. *For $p > 2$ we have $\Gamma_p^\star = 1/2$.*

Actually, we will prove the following result.

THEOREM 3.14.8. *Let $p > 1/2$ not an even integer, and $1 < r, s$ such that $p/r + p/s = 1$. Then there is p-concentration for measurable sets, and $\gamma_p \geq 2 \cdot \Gamma_r^{\star p/r} \cdot \Gamma_s^{\star p/s}$; furthermore, there is p-concentration for measurable sets with gap at the same level.*

This implies Theorem 3.14.1 because for $p = 1$ we can choose $r < 2$, but converging to 2, and $s := r/(r-1) > 2$. Then $\Gamma_s^\star = 1/2$ in view of Corollary 3.14.7. Thus using also (3.145) we obtain

$$\begin{aligned}\gamma_1 &\geq \liminf_{r\to 2-0} 2\Gamma_r^{\star 1/r}\Gamma_s^{\star 1/s} \\ &= 2\liminf_{r\to 2-0} \Gamma_r^{\star 1/r} 2^{1/r-1} = \sqrt{2\liminf_{r\to 2-0} \Gamma_r^\star} \geq \sqrt{0.9226} > 0.96.\end{aligned}$$

This is the assertion stated in Theorem 3.14.1.

PROOF OF THEOREM 3.14.8. The proof is organized essentially as the proof of Proposition 3.10.3. However, a number of technical changes are necessary, so we give a complete proof.

In the following we denote $p_1 := r$ and $p_2 := s$: the index j will always cover the two values $j = 1$ and $j = 2$.

Let us first take $C_j < \Gamma_{p_j}^\star$ arbitrarily close to $\Gamma_{p_j}^\star$, and q_0 and $K_j := K(C_j)$ be constants so that for all $q > q_0$ one has $C_j < \Gamma_{p_j}^\star(q, K_j)$.

As the combined constants

$$C := C_1^{p/p_1} C_2^{p/p_2}, \qquad K := K_1^{p/p_1} K_2^{p/p_2} \tag{3.148}$$

appear frequently, we will use the short notations given above.

We fix $\varepsilon > 0$ a small constant. Let η and δ_0 be given by Proposition 3.8.1, depending on ε, and let $\theta > 0$ be a small constant which will be fixed later on satisfying $\theta/q_0 \le \delta_0$.

Let now $q > q_0$ be chosen with this data so that there exists some interval centered at $(2a+1)/(2q)$ satisfying (3.61) (with a in place of k) of Proposition 3.7.2.

By definition, we have idempotents $P_j \in \mathcal{P}_{2q}(K_j)$ such that

$$\left|P_j\left(\frac{1}{2q}\right)\right|^{p_j} > C_j \sum_{k=0}^{q-1}\left|P_j\left(\frac{2k+1}{2q}\right)\right|^{p_j} = C_j \mathbf{\Sigma}_q^\star(|P_j|^{p_j}), \tag{3.149}$$

and

$$\sum_{k=0}^{q-1}\left|P_j\left(\frac{k}{q}\right)\right|^{p_j} < K_j \sum_{k=0}^{q-1}\left|P_j\left(\frac{2k+1}{2q}\right)\right|^{p_j} = K_j \mathbf{\Sigma}_q^\star(|P_j|^{p_j}). \tag{3.150}$$

Next we consider a multiplication by $b \in \mathbb{N}$, where b is the multiplicative inverse of $2a+1$ mod $2q$. With this unique b the multiplication $x \to bx$ maps the points of both grids $\mathbb{G}_q$ and $\mathbb{G}_q^\star$ onto themselves bijectively.

So the idempotents $Q_j(x) := \mathbf{\Pi}_{2q} P_j(bx) \in \mathcal{P}_{2q}$ attain exactly the same set of values both on the two grids $\mathbb{G}_q$ and $\mathbb{G}_q^\star$ as P_j do. Consequently, we have

$$\left|Q_j\left(\frac{2a+1}{2q}\right)\right|^{p_j} = \left|P_j\left(\frac{1}{2q}\right)\right|^{p_j} \tag{3.151}$$

and

$$\sum_{k=0}^{q-1}\left|Q_j\left(\frac{2k+1}{2q}\right)\right|^{p_j} = \sum_{k=0}^{q-1}\left|P_j\left(\frac{2k+1}{2q}\right)\right|^{p_j}, \tag{3.152}$$

and also

$$\sum_{k=0}^{q-1}\left|Q_j\left(\frac{k}{q}\right)\right|^{p_j} = \sum_{k=0}^{q-1}\left|P_j\left(\frac{k}{q}\right)\right|^{p_j}. \tag{3.153}$$

As a result, we now have

$$\left|Q_j\left(\frac{2a+1}{2q}\right)\right|^{p_j} > C_j \mathbf{\Sigma}_q^\star(|Q_j|^{p_j}) \tag{3.154}$$

and the idempotents Q_j also satisfy the grid-conditions (3.150) for p_j with the constant K_j:

$$\mathbf{\Sigma}_q(|Q_j|^{p_j}) \le K_j \mathbf{\Sigma}_q^\star(|Q_j|^{p_j}). \tag{3.155}$$

Our example will be the idempotent

$$R(x) := Q_1(x)Q_2((2q+1)x), \tag{3.156}$$

matching with Q_1Q_2 on both grids. By an application of Hölder's inequality and (3.155) we obtain

$$\begin{aligned} |R(0)|^p = Q_1(0)^p Q_2(0)^p &\le \mathbf{\Sigma}_q(|Q_1|^{p_1})^{p/p_1} \mathbf{\Sigma}_q(|Q_2|^{p_2})^{p/p_2} \\ &\le K \mathbf{\Sigma}_q^\star(|Q_1|^{p_1})^{p/p_1} \mathbf{\Sigma}_q^\star(|Q_2|^{p_2})^{p/p_2}, \end{aligned} \tag{3.157}$$

with K in (3.148). Now we assume that θ satisfies

$$\theta < \frac{\varepsilon'}{\max\left(2C_{p_1}(K_1+1), 2C_{p_2}(K_2+1)\right)}$$

with some ε', to be chosen in function of ε. Lemma 3.14.6 applies to both Q_j, so with this choice of θ in view of (3.151), (3.152), (3.153) and (3.155) for all $|t| < \theta/q^2$ we obtain the estimates

$$\sum_{k=0}^{2q-1} \left| Q_j \left(\frac{k}{2q} + t \right) \right|^{p_j} \leq C_{p_j}(K_j+1)\mathbf{\Sigma}_q^\star(|Q_j|^{p_j}), \tag{3.158}$$

$$\sum_{k=0}^{q-1} \left| \left| Q_1 \left(\frac{2k+1}{2q} + t \right) \right|^{p_1} - \left| Q_1 \left(\frac{2k+1}{2q} \right) \right|^{p_1} \right| \leq \varepsilon' \mathbf{\Sigma}_q^\star(|Q_1|^{p_1}). \tag{3.159}$$

and

$$\sum_{k=0}^{q-1} \left| \left| Q_2 \left((2q+1) \left(\frac{2k+1}{2q} + t \right) \right) \right|^{p_2} - \left| Q_2 \left(\frac{2k+1}{2q} \right) \right|^{p_2} \right| \leq \varepsilon' \mathbf{\Sigma}_q^\star(|Q_2|^{p_2}). \tag{3.160}$$

In particular, we get

$$\begin{aligned} \left| Q_1 \left(\frac{2a+1}{2q} + t \right) \right|^{p_1} &\geq \left| Q_1 \left(\frac{2a+1}{2q} \right) \right|^{p_1} - \varepsilon' \mathbf{\Sigma}_q^\star(|Q_1|^{p_1}) \\ &\geq \left(1 - \varepsilon'/C_1\right) \left| Q_1 \left(\frac{2a+1}{2q} \right) \right|^{p_1}, \end{aligned} \tag{3.161}$$

and similarly

$$\left| Q_2 \left(\frac{2a+1}{2q} + (2q+1)t \right) \right|^{p_2} \geq \left(1 - \varepsilon'/C_2\right) \left| Q_2 \left(\frac{2a+1}{2q} \right) \right|^{p_2}. \tag{3.162}$$

As a result, we find

$$\begin{aligned} \left| R \left(t + \frac{2a+1}{2q} \right) \right|^{p} &\geq \left(1 - \varepsilon'/C_1\right)^{p/p_1} \left(1 - \varepsilon'/C_2\right)^{p/p_2} \left| R \left(\frac{2a+1}{2q} \right) \right|^{p} \\ &> (1-\varepsilon) \left| R \left(\frac{2a+1}{2q} \right) \right|^{p} \qquad \left(|t| \leq \frac{\theta}{q^2} \right), \end{aligned} \tag{3.163}$$

if ε' is chosen appropriately small in function of C_j, p_j, p and ε.

We estimate the shifted grid sums of $|R|^p$ now. By Hölder's inequality and using (3.159), (3.160) for arbitrary $|t| \leq \theta/q^2$ we are led to

$$\begin{aligned} &\sum_{k=0}^{q-1} \left| R \left(t + \frac{2k+1}{2q} \right) \right|^{p} \\ &\leq \left(\sum_{k=0}^{q-1} \left| Q_1 \left(\frac{2k+1}{2q} + t \right) \right|^{p_1} \right)^{\frac{p}{p_1}} \left(\sum_{k=0}^{q-1} \left| Q_2 \left(\frac{2k+1}{2q} + (2q+1)t \right) \right|^{p_2} \right)^{\frac{p}{p_2}} \\ &\leq (1+\varepsilon')^{p/p_1} \mathbf{\Sigma}_q^\star(|Q_1|^{p_1})^{p/p_1} (1+\varepsilon')^{p/p_2} \mathbf{\Sigma}_q^\star(|Q_2|^{p_2})^{p/p_2} \\ &\leq (1+\varepsilon) \mathbf{\Sigma}_q^\star(|Q_1|^{p_1})^{p/p_1} \mathbf{\Sigma}_q^\star(|Q_2|^{p_2})^{p/p_2}, \end{aligned} \tag{3.164}$$

if again ε' is chosen small enough in function of p, the p_j and ε.

Next, we will need a peaking idempotent at 1/2, as obtained by Proposition 3.8.1. This one will depend on our given constants ε, η, $\delta = \theta/q$ and N larger than the degree of R, and also on a measurable set of finite measure E_ε that we define now. The mapping $x \mapsto qx$ is bijective from $J := (a/q, (a+1)/q)$ onto $(0,1)$, and we take for E_ε the image of $E \cap J$. It is clear that the condition

$$|E_\varepsilon \cap [\frac{1}{2} - \delta, \frac{1}{2} + \delta]| > 2(1-\eta)\delta$$

has been satisfied. We take the idempotent T provided by Proposition 3.8.1 for this data, satisfying

$$\int_{E_\varepsilon \cap [\frac{1}{2}-\delta, \frac{1}{2}+\delta]} |T|^p > (1-\varepsilon) \int_0^1 |T|^p. \tag{3.165}$$

We finally consider the product

$$S(x) := T(qx)R(x), \tag{3.166}$$

which is also an idempotent. We will now prove that

$$2C \int_{\mathbb{T}} |S|^p \leq \kappa(\varepsilon) \int_E |S|^p, \tag{3.167}$$

with $\kappa(\varepsilon)$ being arbitrarily close to 1 when ε is sufficiently small. In order to do this, we put

$$J_k := \left[\frac{k}{q}, \frac{k+1}{q}\right], \qquad I_k := \left[\frac{2k+1}{2q} - \frac{\theta}{q^2}, \frac{2k+1}{2q} + \frac{\theta}{q^2}\right] \tag{3.168}$$

for $k = 0, \dots, q-1$.

Denoting $\tau^p := \int_{\mathbb{T}} |T|^p$, we find, using the property (3.165), that

$$\begin{aligned}
\frac{1}{2}\int_E |S|^p \geq \int_{I_a \cap E} |S|^p &\geq (1-\varepsilon)\left|R\left(\frac{2a+1}{2q}\right)\right|^p \int_{I_a \cap E} |T(qx)|^p dx \\
&\geq (1-\varepsilon)\left|R\left(\frac{2a+1}{2q}\right)\right|^p \frac{1}{q}\int_{E_\varepsilon \cap [\frac{1}{2}-\delta, \frac{1}{2}+\delta]} |T|^p \\
&\geq \frac{(1-\varepsilon)^2\tau^p}{q}\left|R\left(\frac{2a+1}{2q}\right)\right|^p \\
&= \frac{(1-\varepsilon)^2\tau^p}{q}\left|Q_1\left(\frac{2a+1}{2q}\right)\right|^p \left|Q_2\left(\frac{2a+1}{2q}\right)\right|^p \\
&\geq \frac{(1-\varepsilon)^2\tau^p}{q}\; C\; \mathbf{\Sigma}_q^\star(|Q_1|^{p_1})^{p/p_1} \mathbf{\Sigma}_q^\star(|Q_1|^{p_2})^{p/p_2},
\end{aligned} \tag{3.169}$$

at the end applying the definition (3.148) of the constants C_j and (3.155), too.

Then we give an upper bound for the integral on the whole torus. First of all, (3.164) yields

$$\begin{aligned}
\sum_{k=0}^{q-1}\int_{I_k} |S|^p &= \int_{-\theta/q^2}^{\theta/q^2} \sum_{k=0}^{q-1}\left|R\left(\frac{2k+1}{2q} + t\right)\right|^p |T(qt)|^p\, dt \\
&\leq (1+\varepsilon)\mathbf{\Sigma}_q^\star(|Q_1|^{p_1})^{p/p_1} \mathbf{\Sigma}_q^\star(|Q_2|^{p_2})^{p/p_2} \frac{\tau^p}{q}.
\end{aligned}$$

For the integration on the remaining parts of the intervals (3.157) can be used to get

$$\int_{J_k\setminus I_k} |S|^p \le 2\|R\|_\infty^p \int_{\frac{k}{q}+\frac{\delta}{q}}^{\frac{k}{q}+\frac{1}{2q}} |T(qx)|^p dx = 2\,|R(0)|^p\,\frac{1}{q}\int_{\frac{\delta}{q}}^{\frac{1}{2}} |T(x)|^p dx$$
$$\le \frac{\varepsilon\tau^p}{q}\,|R(0)|^p \le \frac{\varepsilon\tau^p}{q}\;K\;\mathbf{\Sigma}_q^\star(|Q_1|^{p_1})^{p/p_1}\mathbf{\Sigma}_q^\star(|Q_2|^{p_2})^{p/p_2}.$$

Summing the last integrals over k, we obtain

$$\int_{\mathbb{T}} |S|^p \le \frac{\tau^p}{q}\,(1+\varepsilon+\varepsilon K)\,\mathbf{\Sigma}_q^\star(|Q_1|^{p_1})^{p/p_1}\mathbf{\Sigma}_q^\star(|Q_2|^{p_2})^{p/p_2}. \tag{3.170}$$

Now (3.169) and (3.170) give (3.167) with

$$\kappa(\varepsilon) := (1+\varepsilon+\varepsilon K)\,(1-\varepsilon)^{-2}$$

concluding the proof, except for assuring arbitrarily large gaps.

It remains to indicate how to modify the proof to get peaking idempotents with arbitrarily large gaps. So we fix ν as a large odd integer, and we will prove that we can replace the polynomial $Q(x)$ by some polynomial $\tilde{Q}(\nu x)$, with gaps at least ν. Recall first that we can take arbitrarily large q satisfying $(\nu, q) = 1$. So we now choose $\tilde{Q}$ similarly as before, to be the polynomial of degree $2q$ that coincides with $P(bx)$ on the grid $\mathbb{G}_q$, but now with b chosen so that $\nu b(2a+1) \equiv 1 \mod 2q$. Such a b exists, as $\nu(2a+1)$ and $2q$ are coprime. We then fix

$$R(x) := \tilde{Q}(\nu x)\tilde{Q}((2q+1)\nu x).$$

There is an additional factor ν, which modifies the value of θ, but otherwise the proof is identical. We know that $\tilde{Q}(\nu x)$ and $P(bx)$, and thus $P(x)$, take globally the same values on both grids $\mathbb{G}_q$ and $\mathbb{G}_q^\star$, because in each case we multiply by an odd integer that is coprime with $2q$. So in particular the grid condition (3.155) is satisfied with K_j once again. □

3.15. Counterexamples in the problems of Wiener and Zygmund

Our above results enable us to draw strong conclusions in the Wiener-Zygmund circle of questions. We formulated in the Introduction Theorem 3.1.13 as a first example. This can now be seen simply by referring to Theorems 3.1.7, 3.1.8 and 3.11.1, above.

Theorem 3.1.13 can be strengthened for open sets, using an improvement of the methods of Shapiro in [**36**]. The construction is closely related to the failure of the Hardy Littlewood majorant property.

THEOREM 3.15.1. *For all $0 < q \le p < 2$, whenever a 0-symmetric open set E of positive measure $|E| > 0$ is given, then for all $\varepsilon > 0$ there exists $f \in \mathcal{T}^+$ which satisfies*

$$\int_{{}^cE} |f|^p \le \varepsilon \left(\int_{-1/2}^{+1/2} |f|^q\right)^{p/q}. \tag{3.171}$$

The same is valid for $q < p$ with p not an even integer, provided that q is sufficiently close to p, that is $q > q(p)$, where $q(p) < p$.

Proof. Let us first assume that $p < 2$. Then, for D_n the Dirichlet kernel defined by (3.8), with n sufficiently large depending on ε, there exists a choice of $\eta_k = \pm 1$ such that

$$\|D_n\|_p \leq \varepsilon \| \sum_{k=0}^{n} \eta_k e_k \|_q.$$

Indeed, if it was not the case, taking the q-th power, integrating on all possible signs and using Khintchine's Inequality, we would find that $c\varepsilon\sqrt{n} \leq \|D_n\|_p \leq Cn^{1-\frac{1}{p}}$ $(p > 1)$, $c\varepsilon\sqrt{n} \leq \|D_n\|_1 \leq C \log n$ and $c\varepsilon\sqrt{n} \leq \|D_n\|_p \leq C$ $(0 < p < 1)$ which leads to a contradiction. We note $g(t) := \sum_{k=0}^{n} \eta_k e_k(t)$ and $G(t) := D_n(t)$.

We assume that E contains $I \cup (-I)$, where $I := (\frac{k}{N}, \frac{k+1}{N})$. Let Δ be a triangular function based on the interval $(-\frac{1}{2N}, +\frac{1}{2N})$, that is, $\Delta(t) := (1 - 2N|t|)_+$. We finally consider the function

$$f(t) := \Delta(t-a)g(2Nt) + \Delta(t+a)g(2Nt) + 2\Delta(t)G(2Nt),$$

where a is the center of the interval I. Then an elementary computation of Fourier coefficients, using the fact that Δ has positive Fourier coefficients while the modulus of those of g and G are equal, allows to see that f is positive definite. Let us prove that one has (3.171). The left hand side is bounded by $\frac{2}{N}\|G\|_p^p$, while $\int_{\mathbb{T}} |f|^q$ is bounded below by $\frac{1}{2N}\|g\|_q^q - \frac{2}{N}\|G\|_q^q$. We conclude the proof choosing n, N sufficiently large.

Let us now consider $p > 2$ not an even integer. Mockenhaupt and Schlag in [**30**] have given counter-examples to the Hardy Littlewood majorant conjecture, which are based on the following property: for $j > p/2$ an odd integer, the two trigonometric polynomials $g_0 := (1+e_j)(1-e_{j+1})$ and $G_0 := (1+e_j)(1+e_{j+1})$ satisfy the inequality $\|G_0\|_p < \|g_0\|_p$. By continuity, this inequality remains valid when p is replaced by q in the right hand side, with $q > q(p)$, for some $q(p) < p$. By a standard Riesz product argument, for K large enough, as well as $N_1, N_2, \cdots N_K$, depending on ε, the functions $g(t) := g_0(t)g_0(N_1 t) \cdots g_0(N_K t)$ and $G(t) := G_0(t)G_10(N_1 t) \cdots G_0(N_K t)$ satisfy the inequality $\|G\|_p \leq \varepsilon \|g\|_q$. From this point the proof is identical. □

We do not know whether, for $p > 2$ not an even integer, that is $2k < p < 2k+2$, we can take $q(p) = 2k$. Due to (3.12), we cannot take $q(p) < 2k$. We do not know either whether the statement is valid for functions with arbitrarily large gaps.

In the next theorems, $H^q(\mathbb{T})$ denotes the space of periodic distributions f whose negative coefficients are zero, and such that the function f_r are uniformly in $L^q(\mathbb{T})$ for $0 < r < 1$, where $f_r(t) := \sum_n \hat{f}(n) r^n e^{2i\pi nt}$.

Moreover, the norm (or quasi-norm) of f is given by $\|f\|_{H^q(\mathbb{T})}^q := \sup_{0<r<1} \int_0^1 |f_r|^q$. It is well known that, for $f \in H^q(\mathbb{T})$, the functions f_r have an a. e. limit f^* for r tending to 1. The function f^*, which we call the pointwise boundary value, belongs to $L^q(\mathbb{T})$. When $q \geq 1$, then f is the distribution defined by f^*, and $H^q(\mathbb{T})$ coincides with the subspace of functions in $L^q(\mathbb{T})$ whose negative coefficients are zero. In all cases the space $H^q(\mathbb{T})$ identifies with the classical Hardy space when identifying the distribution f with the holomorphic function $\sum_{n\geq 0} \hat{f}(n) z^n$ on the unit disc. This explains the use of the term of boundary value.

THEOREM 3.15.2. *Let $0 < p < \infty$, and $p \notin 2\mathbb{N}$. Then for any symmetric, measurable set $E \subset \mathbb{T}$ with $|E| > 0$ and any $q < p$, there exists a function f in the Hardy space $H^q(\mathbb{T})$ with positive Fourier coefficients, so that its pointwise boundary value f^* is in $L^p({}^cE)$ while $f^* \notin L^p(\mathbb{T})$. Moreover, f can be chosen with gaps tending to ∞.*

The key of the proof is Theorem 3.1.13. Observe that we can assume that $p > q > 1$. Indeed, f^ℓ is a positive definite function when f is, and counter-examples for some $p > 1$ will lead to counter-examples for p/ℓ. We do not give the details of the proof, which is analogous to the one of the next theorem.

Using Theorem 3.15.1 instead of Theorem 3.1.13, we have the following.

THEOREM 3.15.3.

(i) *Let $p > 2$, with $p \notin 2\mathbb{N}$, and let $\ell \in \mathbb{N}$ such that $2\ell < p < 2(\ell + 1)$. Then, for any symmetric open set $U \subset \mathbb{T}$ with $|U| > 0$ and $q > q(p)$, there exists a positive definite function $f \in L^{2\ell}(\mathbb{T})$, whose negative coefficients are zero, such that $f \notin L^q(\mathbb{T})$ while f is in $L^p({}^cU)$.*

(ii) *Let $0 < p < 2$. Then for any symmetric open set $U \subset \mathbb{T}$ with $|U| > 0$ and any $s < q < p$, there exists a function f in the Hardy space $H^s(\mathbb{T})$ with non negative Fourier coefficients, so that $f \notin H^q(\mathbb{T})$ while f^* is in $L^p({}^cU)$.*

PROOF. Let us first prove (i). We can assume that ${}^cU$ contains a neighborhood of 0. So, by Wiener's property, if f is integrable and belongs to $L^p({}^cU)$, then f is in $L^{2\ell}(\mathbb{T})$. Let us prove that there exists such a function, whose Fourier coefficients satisfy the required properties, and which does not belong to $L^q(\mathbb{T})$. By using Theorem 3.15.1, we can find positive definite polynomials f_k such that $\|f_k\|_q = 2^{k/2} \to \infty$, while $\|f_k\|_{L^p({}^cU_k)} \leq 2^{-k/2}$ with $U_k \subset U$, so that $\sum \|f_k\|_{L^p({}^cU)} < \infty$. Moreover, we may choose the U_k disjoint and such that $|U_k| < 2^{-\alpha_k}$, with $\alpha(1 - 1/q) = 1$. Then, using Hölder's Inequality, we obtain

$$\int_{\mathbb{T}} |f_k| \leq 2^{-\alpha(1-1/q)k} \left(\int_{U_k} |f_k|^q\right)^{1/q} + \left(\int_{\mathbb{T}\setminus U_k} |f_k|^p\right)^{1/p} \leq 2\; 2^{-k} \left(\int_{\mathbb{T}} |f_k|^q\right)^{1/q},$$

so that $\sum \|f_k\|_1 < \infty$. The function $f := \sum_{k\geq 1} e_{m_k} f_k$ has the required properties. Indeed,

$$\|f\|_{H^q(\mathbb{T})} \geq \|f^*\|_{L^q(U_k)} \geq \|f_k\|_q - \sum_j \|f_j\|_{L^q({}^cU_j)} \geq 2^{\frac{k}{2}} - \sum_{j>0} 2^{-\frac{j}{2}},$$

from which we infer that f is not in $H^q(\mathbb{T})$.

Let us now consider $1 < p < 2$ and $q < 1$, from which we conclude for (ii). We proceed as before, with f_k's given by Theorem 3.15.1, such that $\|f_k\|_q = 2^{k/2}$ and $\|f_k\|_{L^p({}^cU_k)} \leq 2^{-k/2}$. The U_k's are assumed to be disjoint and of small measure, so that $\sum_k \|f_k\|_{H^s}^s < \infty$. It follows that $f \in H^s(\mathbb{T})$. The proof follows the same lines, even if f is not a function, in general, but a distribution.

REMARK 3.15.4. As Wainger in [**42**], we can prove a little more: the function f may be chosen such that $\sup_{r<1} |f_r|$ is in $L^p({}^cU)$. Let us give the proof in the case (i). We can assume that U may be written as $I \cup (-I)$ for some interval I. Let J be the interval of same center and length half, and take f constructed as wished, but for the open set $J \cup (-J)$. Finally, write $f = \phi + \psi$, with $\phi := f\chi_{{}^c(J\cup(-J))}$. Then using the maximal theorem we know that $\sup_{r<1}|\phi_r| \in L^p(\mathbb{T})$, while the Poisson kernel $P_t(x-y)$ is uniformly bounded for $x \notin U$ and $y \in J \cup (-J)$, so that $\sup_{r<1}|\psi_r|$ is uniformly bounded outside U.

In the case (ii), the proof is more technical, f being only a distribution. We use the fact that derivatives of the Poisson kernel $P_t(x-y)$ are also uniformly bounded for $x \notin U$ and $y \in J \cup (-J)$.

□

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We thank Terence Tao, who suggested the construction of peaking functions through bivariate idempotents and Riesz products [**38**]. Although Riesz products form a well-known technique, see e.g. [**7, 20, 29**], and bivariate idempotents have already been occurred in the subject, too, see [**2, 3**], combining these for the particular construction did not occur to us, so our goal could not have been achieved without this contribution.

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