Report on the thesis 'Translations, measure and dimension' by Tamás Keleti

Taimis Keleti's dissertation concerns various relations between the additive and measure theoretic structure in euclidean spaces, mostly in terms of sets but in the last part also in terms of functions. The thesis is divided into five chapters which are based on eight journal papers. I shall now comment each chapter separately and give an overview.

The first chapter 'Sets without given patterns' is based on two short papers and it deals with subsets of the reals containing given finite or countable patterns. Asa starting point serves the property of sets of positive Lebesgue measure that they contain a similar copy of any finite set. The results of Keleti show that no similar result is true for sets of Hausdorff dimension 1 thus answering a question asked separately by M.N. Kolountzakis and A. Iosevich. For any finite set A of at, least 3 elements, and somewhat more generally, Keleti constructed a compact subset of the reals of Hausdorff dimension 1 which contains no similar copy of A. This is not surprising but requires a careful and skillful construction and is enligthening also from the point of view of some later topics of the thesis.

The second chapter 'Covering the real line with small sets' has both geometric measure theoretic and set theoretic flavour. It is based on two short papers, one of them with U.B. Darji and one with M. Abért. The starting question is: when is R the union of less than continuum many translates of a given compact set? The question is trivial under the continuum hypothesis, so it is related to the axioms of set theory. A problem presented by Gruenhage, which is open, asks whether R can be covered with less than continuum many translates of a compact set of Lebesgue measure zero. Keleti and Darji give a relevant partial result: R cannot be covered with less than continuum many translates of a compact set of packing dimension less than 1. In the paper with Abért a partition of R with certain type of small sets is established and it is applied to prove that any permutation of the plane can be written as a finite decomposition of horizontal and vertical slides, maps moving points only horizontally or only vertically. This has an interesting group theoretic consequence.

The third chapter 'Density and coverings in R` deals with a classical topic of density and covering theorems. It is based on two papers. The main result of this chapter is related to a question of A. Carbery by proving that if a measurable set on the unit cube in R' intersects a union of axis-parallel rectangular boxes in a small set, then it intersects some of the boxes in a small set (with a quantitative estimate). Several related results are also given. The proof involves interesting application of the 'minimal function', a kind of opposite to the Hardy-Littlewood maximal function. The second paper continues along the same lines but with sets of different geometric shape in place of intervals. Results of this type are relevant in many parts of modern real analysis, for example, differentiation of integrals and Fourier analysis.

The fourth chapter 'The measure of the intersection of two copies of a self-similar or self-affine set' deals with fractal geometry. It is based on a long joint paper with M. Elekes and A. MAth6. There are two main topics of study. Firstly, what can be said about measure properties of the intersection of a self-similar or self-affine set with its translates or more general transformations? When can a measure on a subset of R" be extended to an invariant (for example, under translations or isometrics) measure of the whole space? For the first topic the paper contains many interesting results. One of them says that for self-similar sets in certain cases positive measure of such an intersection is equivalent with non-empty relative interior. Results of this type are very important in dynamical systems. For the second topic the authors first study extension of measures from general Borel sets. A surprising consequence of their investigation is that there is a compact subset of the real line of Hausdorff dimension 1 such that any continuous Borel measure oil it can be extended to a translation invariant Borel measure on the line. They also combine these two topics and obtain nice extension results from self-similar and self-affine sets.

The fifth chapter 'Periodic composition of measurable integer valued functions' studies the question when a measurable integer valued function can be written as a sure of measurable integer valued periodic functions with given periods. This is based on one paper. The first result is a counter-example to the conjecture that a measurable real-valued periodic decomposition would yield an integer valued presentation. Then periods for which this conjecture is true are characterized in terms of certain difference operators. This paper is a valuable addition to a classical topic.

All the papers oil which this dissertation is based are very well written and they are very interesting. The questions studied are difficult, up-to-date and often closely related to main stream mathematical research in other areas such as dynamical systems and Fourier analysis. The results are original, genuinely new (not just modifications of existing results) and significant, and they substantially enrich the discipline and contribute to the development of mathematics. The proofs are often very clever and introduce good new ideas. The work of Keleti and his co-authors is likely to have much influence oil further research on related topics.

I recommend without hesitation that Keleti's dissertation would be accepted for setting a date for the public debate and that the dissertation's work would be accepted.

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