Extremal Theorems for Matrices

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Tartalomjegyzék

1. fejezet

Bevezetés

Jelen disszertációban összegyűjtött eredmények vezér motívuma a mátrix formában megfogalmazható extremális kombinatorikai, illetve halmazrendszeres problémák. Három fő részből áll. Az első *tiltott részkonfigurációkkal*, vagy más néven *nyomokkal* foglalkozik. A második részben Sperner rendszerek *Vapnik-Chervonenkis dimenzióját* vizsgáljuk. Ezzel kapcsolatban bevezetjük a *rendezett szétzúzás* fogalmát. Az első két részben $0-1$-mátrixokkal foglalkozunk amelyek halmazrendszereket írnak le. A harmadik rész ezzel szemben relációs adatbázis modellek kombinatorikai problémáival foglalkozik. Egy relációs adatbázis legegyszerűbb modellje az a mátrix, melynek sorai az egyedi rekordoknak, oszlopai pedig az egyes tulajdonságoknak, azaz attribútumoknak felelnek meg. A különböző integritási feltételek az adatbázis mátrixokon érdekes extremális kombinatorikai problémákhoz vezetnek.

1.1. Tiltott részkonfigurációk

$0-1$-mátrixok tiltott részkonfigurációinak vizsgálata az extremális gráfelmélet hipergráfokra való kiterjesztésének tekinthető. Egy $m \times n$-es A mátrix *egyszerű*, ha bármely két oszlopa különböző. Ekkor az A mátrix egy n élű az $\{1, 2, \ldots, m\}$ csúcshalmazú hipergráfot ír le, amennyiben a mátrix oszlopait az élek karakterisztikus vektorainak tekintjük. Azt mondjuk, hogy a $k \times l$-es F (nem feltétlenül egyszerű) $0-1$-mátrix az A mátrix *részkonfigurációja*, ha van A-nak olyan részmátrixa, amelyik F-ből sorok és oszlopok permutációjával kapható. Néha a részkonfigurációt *nyomnak* is nevezik és tekinthető a részgráf fogalom általánosításának hipergráfokra.

A 2. Fejezetben tárgyalt probléma a következő. Jelölje $\mathsf{forb}(m, F)$ a legkisebb olyan n értéket (m és F függvényében), amelyre igaz, hogy ha A egy egyszerű $m \times n$-es $0-1$-mátrix amelyik nem tartalmazza F-et rész-konfigurációként, akkor $n \leq \mathsf{forb}(m, F)$. Az, hogy a definíció értelmes, és hogy $\mathsf{forb}(m, F) = O(m^k)$ Füredi egy észrevételéből [Für83] Sauer, Perles és Shelah, illetve Vapnik és Chervonenkis [Sau72, She72, VC71] tétele alapján következik.

A fejezetben szereplő eredmények a [ABS09, AFFS05, AFS01, AGS97, ARS02, AS05, FS09] cikkekből származnak. A fő motiváció az [AS05]-ban leírt sejtés, ami a $\mathsf{forb}(m, F)$ nagyságrendjét adja meg és bizonyos értelemben az Erdős-Stone-Simonovits Tételre hasonlít. A 2.2.3 Sejtés azt mondja ki, hogy $\mathsf{forb}(m, F)$ nagyságrendi meghatározásához elegendő három alap mátrix típusból képzett direkt szozatokat vizsgálni. Ez a három típus az egységmátrix, annak $0-1$-komplementere, valamint az a felső háromszög mátrix, amlynek főátlójában és felette 1-esek vannak. Tegyük fel, hogy az $m_i \times n_i$-es A_i mátrixok egyszerűek $1 \leq i \leq t$ esetén. Ekkor t-szeres direkt szozat $A_1 \times A_2 \times \cdots \times A_t$ azt a $(\sum m_i) \times (\Pi n_i)$-es egyszerű mátrixot jelöli, melynek oszlopait úgy kapjuk, hogy az első m_1 sorra A_1 egy oszlopát tesszük, majd a következő m_2 sorra A_2 egy oszlopát, ... és így tovább, minden lehetséges kombinációban. A 2.2.3 Sejtés sejtés szerint $\mathsf{forb}(m, F) = \Theta(m^\ell)$ arra az ℓ természetes számra, melyre van olyan ℓ tényezős direkt szorzat, úgy hogy minden tényező a három alap mátrix egyike, és nincs F részkonfigurációja, viszont az alap mátrixok *bármely* $\ell + 1$ tényezős szorzata már tartalmazza F-et konfigurációként. A sejtés érdekessége, hogy $\mathsf{forb}(m, F)$ nagyságrendje mindig *m egész kitevős* hatványa.

A 2.3 alfejezetben a 2.2.3 Sejtést igazoljuk $k \times l$-es F-re $k \leq 3$ esetén. $k = 2$-re a 2.3.2. Tétel legérdekesebb esetében irányított gráfot definiálunk az F részkonfigurációt nem tartalmazó egyszerű A mátrix sorain, mint csúcshalmazon. F „hiánya” lefordítható ennek az irányított gráfnak a tulajdonságaira, amelynek segítségével kapjuk a felső becsléseket. Az alsó becslések a direkt szorzat konstrukcióból kaphatóak.

$k = 3$ esetben a 2.3.5. Tétel bizonyításában relációs adatbázisok funkcionális függőségeihez hasonló *implikációkat* vezetünk be. Ezen implikációk halmazából tudunk egy kvadratikus méretű fedő rendszert kiválasztani, ami a kvadratikus felső korlátok bizonyításának alapja.

A 2.2.3 Sejtés alapján két olyan maximális $k \times l$-es részkonfiguráció létezik, melynek tiltása a felső korlátot $\Theta(m^k)$-ról leviszi $O(m^{k-1})$-re. Ezek közül az egyiknek a helyességét bizonyítjuk a 2.3.3 alfejezetben, a 2.3.11. Tételben. A bizonyítás alapja az lemma, aminek segítségével az adott részkonfigurációt nem tartalmazó egyszerű A mátrixból el tudunk hagyni $O(m^{k-1})$ oszlopot úgy, hogy azok után már a Sauer, Perles és Shelah, illetve Vapnik és Chervonenkis tétel alkalmazható legyen rá. A lemma bizonyításának érdekessége, hogy elvezet Lovász egy 3-kritikus hipergráfokról szóló tételének [Lov76] erősítéséhez, illetve általánosításához. Ez a *partíció kritikus* illetve *rendezet-*

ten 3-kritikus hipergráfok fogalmán alapszik, amelyeket a 2.5. alfejezetben vezetünk be.

Pontos eredmények teljes általánosságban a probléma természetéből adódóan nem várhatóak. Azonban, konkrét tiltott részkonfigurációkra teljesen pontos becslések adhatók. Ezeket gyűjtjük össze a 2.4. alfejezetben. Mivel a bizonyítások sokszor hosszadalmasak, ezért csak két 4×2-es konfigurációra vonatkozó eredményt írunk le részletesen. Ezek az [ABS09] cikkben fognak megjelenni. A 2.4.4. Tétel érdekessége a leszámlálási technika és az extremális rendszer karakterizációja. A 2.4.8. Tétel pedig rámutat a téma és a *kombinatorikus design elmélet* kapcsolatára. Azaz, az alsó korlát konstrukcióban a főtag együtthatója m növekedtével egymásba skatulyázott design-okkal javítható.

A 2.5. alfejezetben Toft és Lovász eredményeinek élesítését és általánosítását tárgyaljuk. Egy k-uniform hipergráf $\mathcal{H} = (V, \mathcal{E})$ *ℓ-kritikus*, ha nem $\ell-1$-színezhető, de bármely csúcsát vagy élét elhagyva $\ell-1$-színezhető hipergráfot kapunk. Toft bizonyította [Tof73], hogy rögzített $k, \ell > 3$ és $n \to \infty$, esetén létezik k-uniform ℓ-kritikus $\Omega(n^k)$ elű hipergráf n csúcson. Azonban, minden 3-kritikus k-uniform hipergráf élszáma $o(n^k)$. Toft kérdésére válaszolva Lovász bizonyította, hogy egy 3-kritikus k-uniform hipergráf élszáma legfeljebb $\binom{n}{k-1}$. A 2.5. alfejezet 2.5.5. Tételében rendezetten 3-kritikus hipergráfokra bizonyítjuk ugyanezt a felső korlátot, lineáris algebrai módszerekkel. Ezen kívül partíció kritikus hipergráfokra nagyságrendileg ugyanekkora felső korlátot adunk, valamint egy konstrukciót, melynek élszáma pontosan $\binom{n}{k-1}$. A felső és alsó korlát nagyságrendje $\Theta(n^{k-1})$, a különbségüké $\Theta(n^{k-3})$. A k-uniform $\mathcal{E} \subseteq \binom{[n]}{k}$ hipergráf az X n-elemű alaphalmazon *partíció kritikus* ha a következő feltételeket teljesíti. Az $\mathcal{E}$ élhalmazon adott egy sorbarendezés $E_1, E_2, \ldots E_t$, valamint minden élhez elő van írva egy partíció $A_i \cup B_i = E_i$ ($A_i \cap B_i = \emptyset$), úgy hogy minden $i = 1, 2, \ldots, t$-re létezik az alaphalmaznak egy partíciója $C_i \cup D_i = X$ ($C_i \cap D_i = \emptyset$) úgy, hogy $E_i \cap C_i = A_i$ és $E_i \cap D_i = B_i$, de sem $E_j \cap C_i \neq A_j$ sem $E_j \cap C_i \neq B_j$ $j < i$-re. Azaz, az alaphalmaz i-k partíciója az i-k élet az előírt módon vágja el, de semelyik korábbi élet sem az előírt módon vág szét. A hipergráf *rendezetten 3-kritikus*, ha minden i-re az előírt partíció $A_i = E_i$, $B_i = \emptyset$. Világos, hogy egy 3-kritikus hipergráf az rendezetten 3-kritikus is, és egy rendezetten 3-kritikus hipergráf az partíció kritikus is.

1.2. Antiláncok VC-dimenziója

A 3. Fejezetben amelynek kiinduló pontja Frankl [Fra89] sejtése, amiben összekapcsolja az extremális halmazrendszerek elméletének két klasszikus eredményét, Sauer és Sperner tételeit, a [AS97, ARS02] cikkek eredményeit írjuk le. Mivel halmazrendszerek és egyszerű $0-1$ mátrixok azonosíthatóak, beszélhetünk halmazrendszerek részkonfigurációiról is, melyeket ebben a kontextusban *nyomnak* is szoktak nevezni. Jelölje K_k a $k \times 2^k$-as egyszerű $0-1$ mátrixot. Az $\mathcal{F} \subseteq 2^{[m]}$ halmazrendszer *Vapnik-Chervonenkis-dimenziója* (VC-dimenziója) az a legnagyobb k egész szám, amelyre $\mathcal{F}$-nek van K_k részkonfiurációja, illetve nyoma. Másképpen fogalmazva, az $\mathcal{F}$ halmazrenszer VC-dmenziója a legnagyobb olyan k egész szám, amelyre létezik az alaphalmaznak egy $|S| = k$ részhalmaza, melyre $|\{F \cap S \mid F \in \mathcal{F}\}| = 2^k$. Ekkor azt mondjuk, hogy $\mathcal{F}$ *szétzúzza* S-et. Frankl [Fra89] sejtése szerint ha $\mathcal{F}$ egy *antilánc*, amelyik nem zúz szét k vagy annál nagyobb elemszámú halmazt, akkor $\mathcal{F}| \leq \binom{m}{k-1}$. Az 3.2 alfejezetben, a 3.2.4., 3.2.5. és 3.2.6. Tételekben Frankl sejtését bizonyítjuk be $k \leq 4$-re. A bizonyítás alapja indukció, és az, hogy $k \leq 3$-ra karakterizálni tudjuk az egyenlőség esetét.

A 3. Fejezetben tárgyalt fő fogalom a *rendezett szétzúzás* fogalma. Ez a klasszikus szétzúzás és a Bollobás és Radcliff [BLR89] által „fordított Sauer" egyenlőtlenségekhez bevezetett *strongly traced* fogalom közé esik, az alábbi értelemben. Jelölje $\mathsf{sh}(\mathcal{F})$ az $\mathcal{F}$ halmazrendszer által szétzúzott halmazok családját. (Ekkor $\mathsf{sh}(\mathcal{F})$ leszálló halmazrendszer és $\mathsf{sh}(\mathsf{sh}(\mathcal{F})) = \mathsf{sh}(\mathcal{F})$.) A rendezett szétzúzást S méretére vonatkozó indukcióval definiáljuk. $S = \emptyset$ esetén elegendő, ha $\mathcal{F}$ nem üres. Egyébként pedig azt mondjuk, hogy $\mathcal{F}$ rendezetten szétzúzza az $S = \{s_1, s_2, \dots, s_k\}$ halmazt ($s_1 < s_2 < \cdots < s_k$), ha létezik $\mathcal{F}$-nek $2^{|S|}$ eleme, melyek két halmazrendszerbe sorolhatóak, $\widetilde{\mathcal{F}_0}$-ba és $\widetilde{\mathcal{F}_1}$-be, úgy hogy $T = \{s_k + 1, s_k + 2, \dots, m\}$ esetén (T lehet üres halmaz) igaz az, hogy $T \cap C = T \cap D$ minden $C \in \widetilde{\mathcal{F}_0}$, $D \in \widetilde{\mathcal{F}_1}$, valamint $\{s_k\} \cap C = \emptyset, \{s_k\} \cap D = \{s_k\}$ minden $C \in \widetilde{\mathcal{F}_0}$, $D \in \widetilde{\mathcal{F}_1}$, továbbá $\widetilde{\mathcal{F}_0}$ és $\widetilde{\mathcal{F}_1}$ is külön-külön rendezetten szétzúzza $(S - \{s_k\})$-et. Jelölje $\mathsf{osh}(\mathcal{F})$ az $\mathcal{F}$ által rendezetten szétzúzott halmazok családját.Ekkor, hasonlóan $\mathsf{sh}(\mathcal{F})$-hez, igaz hogy $\mathsf{osh}(\mathcal{F})$ leszálló halmazrendszer és $\mathsf{osh}(\mathsf{osh}(\mathcal{F})) = \mathsf{osh}(\mathcal{F})$. Bollobás és Radcliff következőképpen definiálja a strongly traced fogalmat. $S \subseteq [m]$ strongly traced $\mathcal{F}$ szerint, ha létezik egy olyan $B \subseteq [m]-S$, amelyre $\{E \cap S : E \in \mathcal{F},\, E \cap ([m]-S) = B\} = 2^S$. A definíciók alapján világos, hogy $\mathsf{st}(\mathcal{F}) \subseteq \mathsf{osh}(\mathcal{F}) \subseteq \mathsf{sh}(\mathcal{F})$. Az $\mathsf{osh}(\mathcal{F})$ legfontosabb tulajdonsága, hogy $|\mathsf{osh}(\mathcal{F})| = |\mathcal{F}|$, amiből például Sauer, Perles és Shelah, Vapnik és Chervonenkis tétele azonnal következik.

A 3.3. alfejezetben először indukciót használva bizonyítjuk a 3.1.6. Tételt, ami az $\mathsf{osh}(\mathcal{F})$ előbb említett alap tulajdonságát mondja ki.

A 3.3.1. alfejezetben Frankl és Pach tételének [FP84], amelyik Frankl sejtése uniform halmazrendszerre, egy élesítését bizonyítjuk. A „nincs k méretű szétzúzott halmaz" feltételt helyettesítjük a „nincs k méretű rendezetten szétzúzott halmaz" feltétellel. A bizonyítás lényegi eleme az a karakterizáció, amit uniform halmazrenszerek által rendezetten szétzúzható halmazokra adunk a 3.3.3. Lemmában. Ennek igazi jelentősége nem csupán a tétel bizonyításában van, hanem az algebrai vonatkozásokban [ARS02, HR03b, HR03a, BRR06, HR06, BHR08] található. Egy halmazrendszer elemei természetesen azonosíthatóak monmialokkal $F \subseteq [m]$-hez hozzárendelhető $x_F := \prod_{j \in F} x_j$, és viszont. Ha adott egy $\mathcal{F}$ halmazrendszer, akkor tekinthetjük azon m-változós polinomok $\mathcal{I}$ ideálját, melyek az $\mathcal{F}$-beli halmazok karakterisztikus vektorain 0 értéket vesznek fel. Ezen ideál standard monomjait lehet leírni a rendezett szétzúzás segítségével. A standard monomok kulcsszerepet játszanak a Gröbner bázisok elméletében. Jelölje $\mathsf{Sm}(\mathcal{F}) := \{F \subseteq [m] : x_F \in \mathsf{sm}(\mathcal{I})\}$, ahol $\mathsf{sm}(\mathcal{I})$ az $\mathcal{I}$ ideál standard monomjainak halmaza. Ekkor igaz, hogy $\mathsf{osh}(\mathcal{F}) = \mathsf{Sm}(\mathcal{F})$.

Mivel az uniform halmazrendszerrel rendezetten szétzúzható halmazok karakterizációja lehetőséget adott a Frankl sejtés megfelelő speciális esetének igazolására, ezért a következő lépés azon halmazok leírása, amelyeket antilánccal lehet rendezetten szétzúzni. Egy ilyen, egyszerű numerikus karakterizációt adunk meg a 3.3.2. alfejezet 3.3.5. Tételében.

1.3. Adatbázis mátrixok

A 4. Fejezetben három különböző típusú problémával foglalkozunk, amelyek mindegyike relációs adatbázis modellek vizsgálata során kerül elő. Az új eredmények a [DKS92, DKS95, DKS98, sS98, ADKS00, AS07, SS08a, GOHKSS08, BS] cikkekből valóak.

A 4.3. alfejezet alapkérdése a következő. Tegyük fel, hogy $k \leq n$, $p \leq q < m$ pozitív egész számok és az $m \times n$-es M mátrix teljesíti az alábbi két tulajdonságot:

- Tetszőleges módon kiválasztva k különböző oszlopot, $c_1, c_2, \ldots, c_k$-t, létezik $q+1$ sora M-nek úgy, hogy a különböző értkek száma ezekben a sorokban minden c_i $(1 \leq i \leq k-1)$ oszlopban legfeljebb p, azonban mind a $q+1$ érték a c_k oszlopban ezeken a sorokon különböző;

- Az előbbi feltétel már nem teljesül semmilyen $k+1$ különböző oszlop választása esetén sem.

A célunk m minimalizálása, rögzített n, p, q, k esetén.

A 4.4. alfejezetben egy adatbázisok motiválta kódelméleti problémát vizsgálunk. Egy q elemű ábécé feletti n hosszúságú kód *Armstrong(q, k, n)-kód*, ha a kódszavak minimális távolsága $n-k+1$, valamint tetszőleges $k-1$

koordináta pozícióhoz létezik két olyan kódszó, melyek ott egyeznek meg, azaz a minimális távolság „minden irányban" felvétetik.

A 4.5. alfejezetben egy diszkrepancia típusú eredményt bizonyítunk, amelyet adat olvasás optimalizálás motivál.

A 4.1. alfejezetben áttekintjük azokat a relációs adatbázis modellekhez kapcsolódó matematikai fogalmakat, amelyekre a 4. Fejezetben szükségünk lesz. Relációs adatbázis legegyszerűbb modellje egy mátrix, melynek oszlopai felelnek meg az attribútumoknak, azaz adat típusoknak, míg a sorai az egyes egyedek rekordjainak. Például egy munkahelyi adatbázis attribútumai lehetnek: Név, Anyja neve, Személyi szám, beosztás, Fizetés. Az adatbázis mátrix egy tipikus sora lehet (Nagy Jenő, Kiss Emeralda, 151543QW, portás, 97800). A matematikai modellben feltesszük, az általánosság korlátozása nélkül, hogy a mátrix elemei természetes számok. Egy adatbázishoz hozzátartoznak különböző *integritási feltételek* is. Ezek közül a legtöbbet használt és vizsgált fajta a *funkcionális függőség*. Az Y attribútum halmaz *funkcionális függ* az X attribútum halmaztól, ha egy rekord X-ben felvett értékei *egyértelműen* meghatározzák az Y-ban felvett értékeket. Azaz, ha a mátrix két sora megegyezik az X-beli pozíciókon, akkor megegyeznek az Y-belieken is. Relációs adatbázisok esetében megkülönböztetünk két fajta funkcionális függőséget. Az első az, amit tervezéskor előírnak, hogy teljesüljön, azaz tényleges integritási feltétel, a második fajta pedig az, ami az adatbázis pillanatnyi állapotában, az éppen aktuális adatbázis példányban teljesül, de nem következménye az előírt integritási feltételeknek.

Az $U \to V$ funkcionális függőség *logikai következménye* a Σ függőség halmaznak, jelölésben $\Sigma \models U \to V$, ha minden olyan adatbázis példányban, amiben Σ minden függősége teljesül, teljesül $U \to V$ is. A Σ (funcionális) függőség halmaz *Armstrong* példánya az r példány (adatbázis mátrix), ha $U \to V$ akkor és csak akkor teljesül r-ben, ha $\Sigma \models U \to V$. Funkcionális függőségi rendszerek Armstrong példányainak létezését Armstrong [Arm74] és Demetrovics [Dem79] bizonyították.

Egy függőségi rendszer minimális Armstrong példányának mérete a rendszer bonyolultságának egy mértéke. Adatbányászati szempontból tekintve, funkcionális függőségek keresésée esetén bizonyos függőségi rendszerek kizárhatóak a vizsgált példány mérete alapján. A 4.2. alfejezetben áttekintjük funkcionális függőségi rendszerek minimális Armstrong példányaival (reprezentációival) kapcsolatos eredményeket. Ezek igen bonyolult extremális kombinatorkai problémákhoz vezetnek. A felső becslésekhez használt konstrukciók sokszor design elmélet jellegűek. Az egyik esetben egy teljesen új vizsgálati rányt indítottak el, az *ortogonális kettős fedések* elméletét [BW90, GG87, Che92, GGM94, CD94, GMS95, Gro02].

A 4.3. alfejezetben funcionális függőségek egy általánosítását vezetjük be, és az azzal kapcsolatos kombinatorikai kérdéseket vizsgáljuk. A 4.3.1. Definíció szerint az $a \in \mathbf{R}$ attribútum (p,q)-függ az X attribútum halmaztól (jelölésben $X \xrightarrow{(p,q)} a$), ha az R relációnak (mátrixnak) nincs $q+1$ olyan sora, melyek legfeljebb p különböző értéket tartalmaznak X-beli oszlopokban, azonban az a oszlopban felvett értékeik mind különbözőek. Az $(1,1)$-függőség pontosan a funkcionális függőség. Ellentétben a funkcionális függőségi rendszerekkel, (p,q)-függőségeknek nem feltétlenül létezik Armstrong példányuk. Pontosabban fogalmazva, a következő a helyzet. A 4.1. alfejezetben leírjuk, funkcionális függőségek családjai ekvivalensek az attribútumok halmazán értelmezett lezárási operátorokkal, és ezen lezárások Armstrong példányait tekintjük. A 4.3. alfejezetben belátjuk, hogy a (p,q)-függőségek egy általánosabb fogalomhoz, a 4.3.2. Definícióban leírt *kiterjesztésekhez* vezetnek. A 4.3.1. alfejezetben elégséges feltételeket adunk arra, hogy egy kiterjesztésnek legyen Armstrong példánya, azaz (p,q)-függőséggel reprezentálható legyen, a 4.3.4. Tételben. A $p=q$ esetben a (p,p)-függőség által meghatározott kiterjesztés az lezárás is. Érdekes tehát vizsgálni, hogy milyen lezárásoknak létezik Armstrong példánya (p,p)-függőségek körében. Egy adott $\mathcal{L}$ lezárás *spektruma* $\mathsf{sp}(\mathcal{L})$ azon p természetes számokból áll, amelyekre $\mathcal{L}$-nek létezik Armstrong példánya (p,p)-függőségek körében. A 4.3.9. Tételben pontosan leírjuk az *uniform* lezárások spektrumát. Az eredmény érdekessége, hogy a spektrumhoz tartozó „sporadikus" pontokat is sikerült megadni.

A 4.3.2. alfejezetben kiterjesztések és lezárások minimális Armstrong példányaival foglalkozunk, különféle p,q-függőségek esetében. Mivel a minimális reprezentáció már funkcionális függőségek, azaz $p=q=1$ esetben is nehéz probléma, továbbá általános esetben maga az Armstrong példány létezésének kérdése is nehéz kérdés, ezért általános eredményeket nem várhatunk el. A 4.3.2. alfejezetben egy kiétellével csak uniform lezárásokkal foglalkozunk. A 4.33. Lemmában egy általános alsó korlátot adunk meg, ami a funkcionális függőségekre létező alsó korlát adaptációja. Az alfejezet fő eredményeiben konstrukciókkal bizonyítjuk, hogy a 4.33. Lemma alsó korlátja nagyságrendileg helyes. A 4.3.22. Tételben véges projektív síkokat használunk a konstrukcióban, nem triviális módon. A 4.3.24. Tételben négy pontos eredményt gyűjtünk össze. Ezek közül kettő nagyságrendileg javít a 4.33. Lemma alsó korlátján. A bizonyítások közül csak az érdekesebbik kettőt vettük be a dolgozatba. A (ppn) esetben Lovász egy 1979-es tételét használjuk az alsó korlát bizonyítására, amelyik k-erdő hipergráfok maximális élszámát adja meg. Az (122) esetben a felső korlát érdekes. Ehhez egy n-elemű halmaz q-elemű részhalmazait kell úgy beosztanunk diszjunkt párokba, hogy ezek a párok egymás közt speciális metszet feltételt teljesítsenek (4.3.25. Tétel). Ez utóbbihoz egy Dirac-típusú tételt mondunk ki speciális Hamilton-körök létezéséről (4.3.26. Tétel). A 4.3.25. Tétel érdekessége, hogy lehetővé teszi hogy a diszjunkt k-elemű részhalmazok rendezetlen párjainak „terén” egy távolság megadását, és kódelméleti jellegű kérdesek vizsgálatát [EK01, BK01, BKL, KS04, Qui05, Qui09, DD06].

A 4.4. alfejezetben egy másik típusú kódelméleti kérdést tárgyalunk. Ezt korlátos értékkészletű attribútumok motiválják. Armstrong és Demetrovics eredményében, miszerint minden lezárásnak létezik Armstrong példánya finkcionális függőségek körében, szükséges feltételezés, hogy az egyes attribútumok értékkészlete tetszőlegesen nagy lehet. Azonban a *magasabbrendű adatmodell*, azaz *egymásba skatulyázott attribútumok* [HLS04, Sal04, SS06, SS08b] esetében a *számláló attribútumok* értékkészlete véges, valamint a valós életben is sok olyan helyzet fordul elő, amikor természetesen korlátos az egyes mezőkben felvehető értékek halmaza. Ilyen fordul elő például egy autó kölcsönző adatbázisnál, ahol az autó osztály besorolása csak a {mini, kompakt, alsó-közép, közép, felső, SUV, sport, minibusz} kategóriák egyike lehet.

A 4.4. alfejezet kiinduló pontja az a kérdés, hogy mlyen q, n, k értékekre létezik az n-elemű alaphalmazon k uniform lezárásnak Armstrong példánya, ha az attribútumok értékkészlete q elemű. Egy ilyen adatbázis mátrix sorai n hosszú, q elemű ábécé feletti kódszavaknak tekinthetőek. Ekkor semelyik két kódszó sem egyezhet meg k koordináta pozícióban, viszont bármely $k-1$ koordináta pozícióhoz léteznie kell két kódszónak, amelyek ott megegyeznek. Az ilyen kódokat nevezzük *Armstrong(q, k, n)-kódnak.* $f(q, k)$ jelöli azt a legnagyobb n értéket, amelyre Armstrong(q, k, n)-kód létezik. A 4.4.3. Tételben [GOHKSS08], alsó és felső becsléseket adunk $f(q, k)$-ra. Az egyik fő eredmény, hogy $q = 2$ esetben sikerül egy $c > 1$ konstans létezését bizonyítani melyre $\lfloor ck \rfloor \leq f(2, k)$. A 4.4.4. Állításban egy pontos és egy majdnem pontos értéket határozunk meg. Ez utóbbi érdekessége, hogy a 4.2.8. Tétel, ami speciális típusú ortogonális kettős fedésekről szól és kombinatorikus design elméleti hátterű, ad a felső korlátnál csak eggyel kisebb alsó becslést. Nagy k értékekre 4.4.5. Tételben [SS08a], sikerül a 4.4.3. Tétel alsó és felső korlátait megjavítani. Az alsó korláthoz a véletlen konstrukciót adunk a Lovász Lokális Lemma használatával. A felső korláthoz az Armstrong(q, k, n)-kódot beáagyazzuk az $n' = (q-1)n$-dimenziós euklideszi térbe mint egy szferikus kódot. Ehhez a lehetséges q szimbólumot egy $q-1$-dimenziós szabályos szimplex csúcsainak feleltetjük meg. A kód minimális távolsága meghatározza a szferikus kód minimális szögét. Ez Rankin egy tétele [Ran55] alapján felső becslést ad a szferikus kód pontszámára. Az Armstrong tulajdonság pedig, miszerint a minimális távolság minden irányban felvétetik, ad alsó becslést. A kettő összevetéséből kapjuk n-re a felső korlátot.

A 4.4.1. alfejezetben bináris Armstrong kódok konstrukcióit írjuk le. A 4.4.7. Állítás és a 4.4.8. Tétel [BS], bizonyításának alapja, hogy először egy kellően nagy minimális távolságú „váz-kódot” készítünk, majd az $n-k+1$-elemű koordináta pozíció halmazokat partícionáljuk úgy, hogy egy osztályba eső pozíció halmazok kellően távol legyenek egymástól. Az Armstrong kód a váz-kód szavaiból, valamint azoknak és a megfelelő pozíció halmazok karakterisztikus vektorainak összegeiből áll.

A 4.5. alfejezetben egy diszkrepancia típusú eredményt tárgyalunk. Földrajzi, de egyéb adatbázisok is használják a 2-dimenziós képernyőt adatszervező eszközként. Azaz, a felhasználó kijelöli a képernyő egy területét, és az ahhoz tartozó adatokat kéri le. A modellt, amit használunk Abdel-Gafar és Abbadi [AGA97] vezette be. A feltételezés szerint az adatok párhuzamosan olvasható háttér tárolókon vannak, a minél gyorsabb adatolvasás érdekében a kijelölt képernyő területhez tartozó adatot minél több háttértárolón kell elosztani. A matematikai modellben feltesszük, hogy a felhasználó *téglalap* alakú területet jelöl ki. A képernyőt $n_1 \times n_2$ *csempére* osztjuk, egy csempéhez tartozó adatok egy háttér tárolón helyezkednek el. A felhasználó által kijelölt téglalapot két sarkának koordinátáival írhatjuk le $\mathcal{R} = \mathcal{R}[(i_1, j_1), (i_2, j_2)] = \{(i, j) \colon i_1 \leq i \leq i_2 \text{ és } j_1 \leq j \leq j_2\}$. Minden (i, j) csempéhez egy $f(i, j)$, 1 és m közé eső, egész számot rendelünk ami azt mondja meg, hogy a csempe adata melyik tárolón van. Egy ilyen hozzárendelés akkor jó, ha minden előforduló téglalapra, a benne legtöbbször, illetve legkevesebbszer szereplő tároló szám előfordulásának számai közt a különbség a kicsi.

Definiáljuk egy $f(i, j)$ hozzárendelés diszkrepancáját, majd ezt használva az m szám diszkrepanciáját. Latin négyzeteket használunk optimális hozzárendelés megadásához. A konstrukció indukción alapul, latin négyzetek direkt szozatát használjuk. Továbbá szükségünk van egyfajta „összeadás” lehetőségére is latin négyzetek között. Ehhez definiáljuk egy transzverzális diszkrepanciáját, majd ezt használva tudunk $n \times n$-es latin négyzetről $n+1 \times n+1$-esre áttérni. Az alfejezet két fő tétele a 4.5.4. és 4.5.5. tételek, amelyek logaritmikus diszkrepancájú hozzárendelést adunk meg és bizonyítjuk, hogy latin négyzet típusú hozzárendeléssel ez az lehető legjobb. Az alfejezet anyaga a [ADKS00] konferencia cikken és [AS07] folyóirat cikken alapszik.

1.4. Köszönetnyilvánítás

Sokaknak tartozom köszönettel, mert tanítottak, segítettek pályám során. Szakmai pályafutásom elindítója, évtizedeken át támogatója és mind a mai napig meghatározója Katona Gyula, akinek nemcsak a matematikai támogatásért, hanem barátságáért is köszönettel tartozom. Kandidátusi disszertációm témavezetője Füredi Zoltán volt, akitől azután is sok szakmai segítséget kaptam és hatékony módszereket tanultam. Hálás vagyok Demetrovics Jánosnak, aki adatbázis elméleti cikkeim legtöbbjének társszerzője. Külön köszönet illeti Simonyi Gábort, akivel élmény volt együtt dolgozni gráfelméleti problémákon. Sok-sok figyelmet és segítséget köszönök Recski Andrásnak, Simonovits Miklósnak és T. Sós Verának. Hamburger Péternek hálás vagyok a barátsága és támogatása mellett azért, mert feltárta előttem a Venn-diagrammok matematikai-művészeti világát.Vendégszeretetükért és barátságukért illeti köszönet Czabarka Évát és Székely Lászlót. Végül, de nem utolsó sorban, rendkívüli hálás vagyok Richard Anstee-nek hogy bevezetett a tiltott részkonfigurációk és a rendezett szétzúzás elméletébe.

Természetesen köszönettel és hálával tartozom családomnak is. Feleségem, Kovács Ildi szakadatlan és feltétlen hite bennem segített, hogy a szakmai munkára tudjak koncentrálni. Szüleimnek köszönöm, hogy életem nehéz pillanataiban is mindig mellettem álltak. Fiaim a legtöbbel ajándékoztak meg, amit egy apa kaphat, a barátságukkal. Köszönöm nekik az együtt sportolás élményét.

Chapter 2

Forbidden Configurations

2.1 Basic concepts

The study of forbidden configurations is a problem in extremal set theory. It is convenient to use the language of matrix theory. We define a *simple* matrix as a (0,1)-matrix with no repeated columns. Such an $m \times n$ simple matrix can be thought of a family of n subsets of $\{1, 2, \ldots, m\}$ with the rows indexing the elements and the columns indexing the subsets. Assume we are given a $k \times l$ (0,1)-matrix F. We say that a matrix A has a *configuration* F if a submatrix of A is a row and column permutation of F and so F is referred to as a *configuration* of A (sometimes called *trace*).

We define $\mathsf{forb}(m, F)$ as the smallest value (depending on m and F) so that if A is a simple $m \times n$ matrix and A has no configuration F then $n \leq \mathsf{forb}(m, F)$. Alternatively $\mathsf{forb}(m, F)$ is the smallest value so that if A is an $m \times (\mathsf{forb}(m, F) + 1)$ simple matrix then A must have a configuration F. We are focusing on a single fixed forbidden configuration, although similarly to extremal graph theory families of forbidden configurations could also be considered, as it is done in for example [BB05].

Remark 2.1.1 *Let A^c denote the 0-1-complement of A. Then* $\mathsf{forb}(m, F^c) = \mathsf{forb}(m, F)$.

Remark 2.1.2 *If F' is a row and column permutation of a submatrix of F (i.e. F has a configuration F'), then* $\mathsf{forb}(m, F') \leq \mathsf{forb}(m, F)$.

When giving results it is often convenient to note when do we have $\mathsf{forb}(m, F') = \mathsf{forb}(m, F)$ where F' is a configuration in F. Typically one has a construction working for F' (a simple matrix A with no configuration F') which then necessarily works for F and we have a bound for $\mathsf{forb}(m, F)$

which certainly applies to $\mathsf{forb}(m, F')$. Equality (or asymptotic equality) of the construction and the bound then yields equality (or asymptotic equality) for $\mathsf{forb}(m, F')$ and $\mathsf{forb}(m, F)$ as well as any matrices intermediate between F' and F.

Some notations help us describe the most important matrices. Let K_k denote the $k \times 2^k$ simple matrix of all possible (0,1)-columns on k rows and let K_k^s denote the $k \times \binom{k}{s}$ simple matrix of all possible columns of column sum s. Many results have been obtained about $\mathsf{forb}(m, F)$ but the following is the most fundamental [Sau72, She72, VC71] .

Theorem 2.1.3 *[Sauer, Perles and Shelah, Vapnik and Chervonenkis] We have that*

$$\mathsf{forb}(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0} \tag{2.1}$$

and so $\mathsf{forb}(m, K_k)$ *is* $\Theta(m^{k-1})$.

An easy consequence of Theorem 2.1.3 using Remark 2.1.2 is the following

Corollary 2.1.4 *For a $k \times l$ simple matrix F,* $\mathsf{forb}(m, F) = O(m^{k-1})$ *holds.*
□

It would seem reasonable to consider (0,1)-matrices F which are not simple as well. Füredi [Für83] noted the following general bound that follows from Theorem 2.1.3.

Theorem 2.1.5 ([Für83]) *Let F be a $k \times l$ (0,1)-matrix. Then* $\mathsf{forb}(m, F)$ *is $O(m^k)$.*

2.2 A Conjecture for asymptotic bounds

Our investigations have led us to a conjecture on the asymptotic growth of $\mathsf{forb}(m, F)$ for a fixed F as m goes to infinity. We had noted early on that all our results had $\mathsf{forb}(m, F) = \Theta(m^e)$ for an integer e. Our conjecture involves a cross product construction.

Definition 2.2.1 *Let A_i be an $m_i \times n_i$ simple matrix for $1 \le i \le t$. Denote the t-fold product $A_1 \times A_2 \times \cdots \times A_t$ as the $(\sum m_i) \times (\Pi n_i)$ simple matrix whose columns are formed in all possible ways by putting a column of A_1 in the first m_1 rows and putting a column of A_2 in the next m_2 rows etc. Let*

I_h denote the $h \times h$ identity matrix and I_h^c denotes its (0,1)-complement. Let T_h denote the $h \times h$ triangular *matrix*

$$T_h = \begin{bmatrix} 1 & & & 1's \\ & 1 & & \\ & & \ddots & \\ 0's & & & 1 \end{bmatrix}. \tag{2.2}$$

The three matrices I, I^c, T are our proposed building blocks for the product construction. Note that if each A_i in the t-fold product above is of size $m/t \times m/t$ then the t-fold product has m rows and $\Theta(m^t)$ columns. Let F be a $k \times l$ (0,1)-matrix.

Definition 2.2.2 *Let $X(F)$ be the smallest p so that F is a configuration in $A_1 \times A_2 \times \cdots \times A_p$ for every choice of A_i as either $I_{m/p}$, $I_{m/p}^c$ or $T_{m/p}$. Alternatively, assuming F is not a configuration in at least one of I, I^c, T, then $X(F) - 1$ is the largest choice of p so that F is not a configuration in $A_1 \times A_2 \times \cdots \times A_p$ for some choice of A_i as either $I_{m/p}$, $I_{m/p}^c$ or $T_{m/p}$.*

We are assuming m is large and divisible by p, in particular that $m \geq (k+1)(kl+1)$ so that $m/p \geq kl+1$. Divisibility by p does not affect the asymptotics since we can use a simple submatrix of a simple matrix that avoids F for construction purposes. We are also using the fact that we need only consider p-fold products for $p \leq k+1$, since F is a configuration in $l \cdot K_k$ and we can find $l \cdot K_k$ (and hence F) as a configuration in $A_1 \times A_2 \times \cdots \times A_{k+1}$ by taking 1 row from each of the first k products (each row has [01]) and then, since we are taking zero rows from the final A_{k+1}, we get the configuration $(m/(k+1)) \cdot K_k$ in the product (this also follows from Theorem 2.1.5).

If F is a configuration in the p-fold product $A_1 \times A_2 \times \cdots \times A_p$, assume that a_i rows of A_i are used with $\sum_{i=1}^{p} a_i = k$. If we form the submatrix of A_i of a_i rows, then we would be interested in at most l copies of a given column on these rows (F has l columns) if this is possible. Now for $t \geq k+l$, any a_i rows of K_t^1 contains l columns of 0's as well as a copy of $K_{a_i}^1$. The analogous result is true for K_t^{t-1}. Also for $t \geq kl+l$, the a_i rows of T_t consisting of rows $l+1, 2l+1, 3l+1, \ldots, kl+1$ have l columns of 0's and $l \cdot T_{a_i}$. Thus as long as $m \geq (k+1)(kl+1)$ we are able to use the matrices A_i as if they were arbitrarily large.

Conjecture 2.2.3 *[AS05]*

$$\mathsf{forb}(m, F) = \Theta(m^{X(F)-1}). \tag{2.3}$$

Note that the definition of $X(F)$ ensures $\mathsf{forb}(m,F)$ is $\Omega(m^{X(F)-1})$, via the product construction, although for $X(F)=1$ a little care must be taken. The earliest use of the product construction is in [AGS97] and its non trivial application in Theorem 2.6[AGS97] and Theorem 3.4[AGS97] for cases with $k=2$ and $k=3$. The Conjecture 2.2.3 has been verified for $k=2$ in Theorem 2.3.2, $k=3$ in Theorem 2.3.5, $l=2$ in Theorem 2.3.14, and other cases. Moreover the Conjecture has motivated recent work such as finding the boundary between $O(m^{k-1})$ and $\Theta(m^k)$ configurations.

It is important to note that the constant in front of the leading term $m^{X(F)-1}$ of $\mathsf{forb}(m,F)$ is not predicted by the Conjecture and so the Conjecture is little help concerning exact bounds.

We have yet to make a direct connection between our proofs of asymptotic bounds for $\mathsf{forb}(m,F)$ with the derivation of $X(F)$. We think of this problem as a configuration version of the Erdős-Stone-Simonovits [P. 66] Theorem for the maximum number of edges in a graph avoiding some specified subgraph H where $\chi(H)$ is relevant.

2.3 Complete asymptotic results

In this section we describe the known complete results and give the proofs of some of the interesting cases. Further details can be found in [AGS97], [AFS01], and [AS05].

For completeness we consider $1\times l$ F (Theorem 5.1 and Corollary 5.2 from [AFS01]).

Theorem 2.3.1 *Assume F is a $1\times l$ (0,1)-matrix with p 1's and with $p\geq l-p\geq 0$ and let F' be the $1\times p$ (0,1)-matrix with p 1's. Assume $m\geq p-1\geq 1$. Then*

$$\mathsf{forb}(m,F')=\mathsf{forb}(m,F)=\lfloor\frac{pm}{2}\rfloor+1. \tag{2.4}$$

□

2.3.1 $k=2$

For the case F is $2\times l$, the asymptotic classification of $\mathsf{forb}(m,F)$ is completed in [AGS97]. We need some special matrices

$$F_1=\begin{bmatrix}1\\0\end{bmatrix}\qquad F_2(t)=\begin{bmatrix}0\,\overbrace{1\cdots1}^{t}\,0\,1\\0\,0\cdots0\,1\,1\end{bmatrix}\qquad F_3(t)=\begin{bmatrix}0\,\overbrace{1\cdots1}^{t}\,\overbrace{0\cdots0}^{t}\\0\,0\cdots0\,1\cdots1\end{bmatrix} \tag{2.5}$$

Theorem 2.3.2 *Let F be a $2 \times l$ (0,1)-matrix.*
(Constant Cases) If $F = F_1$, then $\mathsf{forb}(m, F) = \Theta(1)$.
(Linear Cases) If F has at least one configuration from K_2^0, K_2^1, K_2^2, $[2 \cdot F_1]$, and if F is a configuration in $F_2(t)$, $F_3(t)$, $F_3(t)^c$ for some $t \geq 1$, then $\mathsf{forb}(m, F) = \Theta(m)$.
(Quadratic Cases) If F has at least one configuration from $2 \cdot K_2^0$, $[K_2^0 | 2 \cdot K_2^1 | K_2^2]$, or $2 \cdot K_2^2$ then $\mathsf{forb}(m, F) = \Theta(m^2)$.
In addition, any $2 \times l$ (0,1)-matrix F will fall into one of the three Cases.

Proof of Theorem 2.3.2 All the lower bounds follow from the constructions given in Conjecture 2.2.3.

The quadratic bound in general for 2-rowed forbidden configurations follows from Theorem 2.1.5.

The linear bound for $\mathsf{forb}(m, F_2(t))$ is Theorem 2.2[AGS97]. It is proven by easy induction using the following method which we refer to as *standard argument* [Ans95].

If a matrix A does not have multiple columns, then we can decompose it with respect to any row as follows:

$$\begin{array}{|c|c|c|c|}\hline \multicolumn{2}{|c|}{1\,1\,\ldots\,1} & \multicolumn{2}{c|}{0\,0\,\ldots\,0} \\ \hline & & & \\ B_1 & B_2 & B_2 & B_3 \\ & & & \\ \hline \end{array} \tag{2.6}$$

Here B_2 denotes the collection of repeated columns after dropping the chosen row. We shall call this *standard decomposition* and refer to the parts B_i accordingly. Note, that the parts B_1, B_2, B_3 are simple matrices. There is a general method to establish upper bounds for $\mathsf{forb}(m, F)$, based on the standard decomposition [Ans95]. We decompose the forbidden configuration F as follows:

$$\begin{array}{|c|c|}\hline 1\,1\,\ldots\,1 & 0\,0\,\ldots\,0 \\ \hline & \\ F_1 & F_2 \\ & \\ \hline \end{array}$$

Now $[B_1B_2B_3]$ is a simple matrix of $m-1$ rows with no configuration F and B_2 is a simple matrix that does not contain *both* F_1 and F_2 as configurations at the same time. This implies the following inequality:

$$\mathsf{forb}(m,F) \le \mathsf{forb}(m-1,F) + \max\{\mathsf{forb}(m-1,F_1), \mathsf{forb}(m-1,F_2)\}. \quad (2.7)$$

The linear bound for $\mathsf{forb}(m, F_3(t))$ is Theorem 2.3[AGS97]. We give the proof here, since it is a characteristic example of application of graphs.

Let us suppose that A is an $m \times n$ matrix with no configuration $F_3(t)$. Then in each pair of rows either we are missing $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ or we have less than t of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ or less than t of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.We may assume that of the four possible columns on two rows, $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, at most one occurs less than $2t-1$ times on a given pair (i,j) of rows: For if two columns appear less than $2t-1$ times each, then we can apply induction as follows. We have three possibilities according to which two types of columns occur less than $2t-1$ times:

Case 1.

$$\begin{array}{ccccccccc} i & 1 & \ldots & 1 & 0 & \ldots & 0 & \overbrace{* \ldots *}^{\le 4t-4} \\ j & 0 & \ldots & 0 & 1 & \ldots & 1 & * \ldots * \end{array} \quad (2.8)$$

Case 2.

$$\begin{array}{ccccccccc} i & 1 & \ldots & 1 & 1 & \ldots & 1 & \overbrace{* \ldots *}^{\le 4t-4} \\ j & 0 & \ldots & 0 & 1 & \ldots & 1 & * \ldots * \end{array} \quad (2.9)$$

Case 3.

$$\begin{array}{ccccccccc} i & 0 & \ldots & 0 & 0 & \ldots & 0 & \overbrace{* \ldots *}^{\le 4t-4} \\ j & 0 & \ldots & 0 & 1 & \ldots & 1 & * \ldots * \end{array} \quad (2.10)$$

Case 4.

$$\begin{array}{ccccccccc} i & 0 & \ldots & 0 & 1 & \ldots & 1 & \overbrace{* \ldots *}^{\le 4t-4} \\ j & 0 & \ldots & 0 & 1 & \ldots & 1 & * \ldots * \end{array} \quad (2.11)$$

In each of these cases we can drop row i and the at most $4t-4$ columns denoted by *'s to obtain a matrix $\tilde{A}$ of $m-1$ rows and distinct columns with no configuration $F_3(t)$.

We form a digraph with vertex set $[m] = \{1, 2, \ldots, m\}$ the set of rows of A. The edge set E contains pairs (i,j) if less than $2t-1$ columns have 0 in row i and 1 in row j.

Now note that if $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ occurs less than $2t-1$ times on a pair of rows, then it occurs in fact less than t times, otherwise we could find configuration $F_3(t)$. On the other hand, if $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ occurs few times, then it must not be present at all.

Claim 2.3.3 *The graph defined above is* transitive.

Indeed, suppose that $(i,j),(j,k) \in E$ but $(i,k) \notin E$. Thus, on the pair (i,k) of rows we have at least $2t-1$ columns with $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Row j contains at least t 0's or t 1's on these columns, and either case contradicts the assumption. For example, t 1's would mean at least t columns of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ on the pair of rows (i,j), which contradicts to $(i,j) \in E$.

Claim 2.3.4 *If* $(i,j),(i,k) \in E$ *then* (j,k) *or* $(k,j) \in E$.

Indeed, $(i,j) \in E$ implies that at least $2t-1$ of the $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ are in rows i and j. If neither $(j,k) \in E$ nor $(k,j) \in E$, then $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ must be missing in rows j and k. Thus, we must have ones in row k in those columns where $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ occurs in rows i and j. That implies at least $2t-1$ columns with zero in row i and one in row k, a contradiction.

Now let us assume that our directed graph is disconnected regarded as undirected graph. That is, suppose that there is a set S of vertices such that there is no edge between S and $[m] \setminus S$. Thus, $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ cannot occur on a pair (i,j) of rows if $i \in S$ and $j \in [m] \setminus S$. Then A looks like

$$\begin{array}{c} S \\ {[m] \setminus S} \end{array} \left[\begin{array}{c|c} 1 & \geq 1 \text{ 0's per column} \\ \hline * & 1 \end{array} \right] = \left[\begin{array}{c|c} 1 & B \\ \hline C & 1 \end{array} \right]. \tag{2.12}$$

Let the number of rows of S be m'. By induction we have

$$\begin{aligned} &\#\text{of columns of } A = \\ &\quad \#\text{of columns of } B + \#\text{of columns of } C \\ &\leq (4t-4)m' + (4t-4)(m-m') \\ &= (4t-4)m. \end{aligned} \tag{2.13}$$

So we may assume that our directed graph is (weakly) connected. From Claim 2.3.3 and assumptions on A, our digraph is acyclic. Deleting all the edges implied by transitivity we obtain a directed tree T where each node has at most one descendant by Claim 2.3.4. Let us count how many columns have the property that there is no $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ on rows (i, j) if there is no edge between i and j in the graph and furthermore there is no $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ on a pair (i, j) if $i \to j$ is an edge of the tree T. We order the rows so that edges (i, j) in T have $i < j$. Thus, the root of T is row m. Take such a column and assume that the first 0 from top is in row i according to the labelling.

There is a 0 in any row k with $(i, k) \in E$. Indeed, there is a directed path from i to k in T, otherwise (i, k) is not implied by transitivity. Going along that path, we are forced to put 0 on each vertex of it. In the remaining vertices 1's must stand, otherwise there would be a $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ on a pair of rows that are not connected by an edge of the graph. So there are at most $m + 1$ such columns.

Every other column either contains $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ on rows (i, j) if there is no edge between i and j in the graph or $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ on a pair (i, j) where (i, j) is an edge of the tree T. In the first case we obtain an immediate contradiction, since on that pair of rows every other type of column occurs at least $2t - 1$ times, yielding a configuration F. Taking $(t - 1)(m - 1) + 1$ columns of the second type we obtain at least t columns that contain $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ on the *same* edge of the tree, again yielding a configuration. Thus,

$$(m + 1) + (t - 1)(m - 1) + 1 < (4t - 4)m \tag{2.14}$$

columns give us an $F_3(t)$. □

2.3.2 $k = 3$

For the case F is $3 \times l$, the asymptotic classification of $\mathsf{forb}(m, F)$ is begun in [AGS97],[AFS01] and was completed in [AS05]. The following configurations are needed for Theorem 2.3.5

$$F_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad F_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad F_3 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \tag{2.15}$$

$$F_4(t) = \begin{bmatrix} 0 & \overbrace{1\cdots1}^{t} & \overbrace{0\cdots0}^{t} & 0 & \overbrace{1\cdots1}^{t} & 1 & \overbrace{0\cdots0}^{t} & 1 \\ 0 & 0\cdots0 & 1\cdots1 & 0 & 1\cdots1 & 0 & 1\cdots1 & 1 \\ 0 & 0\cdots0 & 0\cdots0 & 1 & 0\cdots0 & 1 & 1\cdots1 & 1 \end{bmatrix} \quad (2.16)$$

$$F_5(t) = \begin{bmatrix} 0 & \overbrace{1\cdots1}^{t} & \overbrace{0\cdots0}^{t} & 0 & 1 & \overbrace{1\cdots1}^{t} & \overbrace{0\cdots0}^{t} & 1 \\ 0 & 0\cdots0 & 1\cdots1 & 0 & 1 & 0\cdots0 & 1\cdots1 & 1 \\ 0 & 0\cdots0 & 0\cdots0 & 1 & 0 & 1\cdots1 & 1\cdots1 & 1 \end{bmatrix} \quad (2.17)$$

$$F_6(t) = \begin{bmatrix} 0 & \overbrace{1\cdots1}^{t} & \overbrace{0\cdots0}^{t} & \overbrace{0\cdots0}^{t} & \overbrace{1\cdots1}^{t} & \overbrace{1\cdots1}^{t} \\ 0 & 0\cdots0 & 1\cdots1 & 0\cdots0 & 1\cdots1 & 0\cdots0 \\ 0 & 0\cdots0 & 0\cdots0 & 1\cdots1 & 0\cdots0 & 1\cdots1 \end{bmatrix} \quad (2.18)$$

Theorem 2.3.5 *Let F be a $3 \times l$ (0,1)-matrix.*
(Linear Cases) If F has at least one column and if F is a configuration in F_2 then $\mathsf{forb}(m, F) = \Theta(m)$.
(Quadratic Cases) If F has at least one configuration from K_3^0, K_3^1, K_3^2, K_3^3, $2 \cdot F_1$, $2 \cdot F_1^c$ or F_3 and if F is a configuration in $F_4(t)$, $F_5(t)$, $F_6(t)$ or $F_6(t)^c$ for some $t \geq 1$, then $\mathsf{forb}(m, F) = \Theta(m^2)$.
(Cubic Cases) If F has at least one configuration from $2 \cdot K_3^0$, $[2 \cdot K_3^1 | K_3^2]$, $[2 \cdot K_3^1 | K_3^3]$, $[K_3^0 | 2 \cdot K_3^2]$, $[K_3^1 | 2 \cdot K_3^2]$ or $2 \cdot K_3^3$ then $\mathsf{forb}(m, F) = \Theta(m^3)$.
In addition, any $3 \times l$ (0,1)-matrix F will fall into one of the three Cases.

Proof of Theorem 2.3.5 All the lower bounds follow from the constructions given in Conjecture 2.2.3, although many of them were already given in [AGS97]. Also, some better multiplicative constants were given in particular cases.

The linear bound for $\mathsf{forb}(m, F_2)$ is Theorem 3.3[AGS97]. The quadratic bound for $\mathsf{forb}(m, F_4(t))$ is Theorem 3.9[AGS97]. The quadratic bound for $\mathsf{forb}(m, F_5(t))$ is Theorem 4.2, while the quadratic bound for $\mathsf{forb}(m, F_6(t))$ is Theorem 4.1 in [AS05]. The cubic bound for all 3-rowed F follows from Theorem 2.1.5. We include the proofs of $\mathsf{forb}(m, F_5(t))$ and $\mathsf{forb}(m, F_6(t))$ because the methods are interesting and recent developments [AF09] show that they are applicable in more general setting.

What is missing if a configuration F is avoided ?

A careful consideration is required to see what is missing from A when either $F_5(t)$ or $F_6(t)$ is not a configuration in A. We wish to use the following terminology. Let $\{i, j, k\}$ be a triple of rows of a matrix $A = (a_{rs})$. We say that we have

$$\text{no}\ \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} d \\ e \\ f \end{bmatrix} \tag{2.19}$$

if in every column q of A we do not have $a_{iq} = d, a_{jq} = e$ and $a_{kq} = f$ all occurring. As well, we say that there are

$$\text{at most } t-1 \text{ of } \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} d \\ e \\ f \end{bmatrix} \tag{2.20}$$

if there are at most $t-1$ columns q of A in which $a_{iq} = d, a_{jq} = e$ and $a_{kq} = f$ all occur.

Let S_3 denote the symmetric group on three symbols.

Proposition 2.3.6 *Let A be a (0,1)-matrix with no configuration $F_6(t)$. Let a, b, c be a triple of rows of A. Then we either have a permutation $\pi_1 \in S_3$ with $\pi_1(a) = i$, $\pi_1(b) = j$, $\pi_1(c) = k$ (note that $\{a, b, c\}$ and $\{i, j, k\}$ are the same as sets) with*

$$\textit{no}\ \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tag{2.21}$$

or if we do not have (2.21), then we have a permutation $\pi_2 \in S_3$ with $\pi_2(a) = i$, $\pi_2(b) = j$, $\pi_2(c) = k$ with

$$\textit{at most } t-1 \textit{ of } \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{2.22}$$

or if we do not have (2.21),(2.22), then we have a permutation $\pi_3 \in S_3$ with $\pi_3(a) = i$, $\pi_3(b) = j$, $\pi_3(c) = k$ with

$$\textit{at most } t-1 \textit{ of } \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \textit{and at most } t-1 \textit{ of } \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}. \tag{2.23}$$

□

Proposition 2.3.7 *Let A be a (0,1)- matrix with no configuration $F_5(t)$. Let a, b, c be a triple of rows of A. Then we either have a permutation $\pi_1 \in S_3$ with $\pi_1(a) = i$, $\pi_1(b) = j$, $\pi_1(c) = k$ with*

$$\textit{no}\ \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \ \textit{or no}\ \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \ \textit{or no}\ \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \ \textit{or no}\ \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{2.24}$$

or if we do not have (2.24), then we have a permutation $\pi_2 \in S_3$ *with* $\pi_2(a) = i$, $\pi_2(b) = j$, $\pi_2(c) = k$ *with*

$$\text{at most } t-1 \text{ of } \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \text{and at most } t-1 \text{ of } \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \tag{2.25}$$

or if we do not have (2.24),(2.25), then we have a permutation $\pi_3 \in S_3$ *with* $\pi_3(a) = i$, $\pi_3(b) = j$, $\pi_3(c) = k$ *with*

$$\text{at most } t-1 \text{ of } \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad \text{and at most } t-1 \text{ of } \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \tag{2.26}$$

or if we do not have (2.24),(2.25),(2.26), then we have a permutation $\pi_4 \in S_3$ *with* $\pi_4(a) = i$, $\pi_4(b) = j$, $\pi_4(c) = k$ *with*

$$\text{at most } t-1 \text{ of } \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \text{and at most } t-1 \text{ of } \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}. \tag{2.27}$$

□

Implications

We will use the notation

$$ij \to_0 k, \tag{2.28}$$

where i, j, k are row indices, and refer to this as a 0-implication. It is not required that i, j, k be distinct. We say that a column q of a matrix $A = (a_{rs})$ *violates* the 0-implication $ij \to_0 k$ if we have $a_{iq} = 0$, $a_{jq} = 0$, and $a_{kq} = 1$. We say that a column q satisfies the 0-implication if the column does not violate the 0-implication, namely when $a_{iq} = 0$, $a_{jq} = 0$, we have $a_{kq} = 0$.

If the matrix A has

$$\text{at most } t-1 \quad \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{2.29}$$

then the 0-implication $ij \to_0 k$ is violated by at most $t-1$ columns of A.

We can also refer by analogy to 1-implications using the notation

$$ij \to_1 k \tag{2.30}$$

by interchanging the roles of 0 and 1. Again, if the matrix A has

$$\text{at most } t-1 \qquad \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}. \tag{2.31}$$

then the 1-implication $ij \to_1 k$ is violated at most $t-1$ times.

Let $\mathsf{Imp}_0(A)$ denote the 0-implications $ij \to_0 k$ which are violated at most $t-1$ times by columns of A. We form a directed graph $D_0(A)$ from $\mathsf{Imp}_0(A)$ with node set equal to all $\binom{m}{2}$ pairs of rows. We have an arc $ij \to kl$ in D_0 if and only if $ij \to_0 k$ and $ij \to_0 l$ are in $\mathsf{Imp}_0(A)$. We can use $ij \to_0 i$ and $ij \to_0 j$ as 0-implications that are never violated and so $ij \to ik$ and $ij \to jk$ follow from $ij \to_0 k$.

Since $D_0(A)$ is a directed graph, we can apply the standard decomposition and topological sort to identify the strongly connected components and also order the nodes of $D_0(A)$ so that if $ij \to kl$ in $D_0(A)$, then either kl appears later in the ordering or ij and kl belong to the same strongly connected component.

Let $C(ij)$ denote the strongly connected component of $D_0(A)$ containing the node ij. We define the support of a strongly connected component C as

$$\mathsf{supp}(C) = \bigcup_{ij \in C} \{i, j\} \tag{2.32}$$

Our goal is to select a small subset of the implications of $\mathsf{Imp}_0(A)$ (preferably of size $O(m^2)$) so that a column that violates one of the implications also violates one of the chosen implications. A pigeonhole principle then ensures that the number of columns with violations is at most $t-1$ times number of chosen implications (and hence $O(m^2)$ if we were lucky). The motivation for implications is partly in Theorem 2.2 in [AFS01] and partly from *functional dependencies* in database theory.

The following classification was very useful. An implication $ij \to_0 k$ of $\mathsf{Imp}_0(A)$ is called an *outside* implication if $k \notin \mathsf{supp}(C(ij))$ and an implication $ij \to_0 k$ of $\mathsf{Imp}_0(A)$ is called an *inside* implication if $k \in \mathsf{supp}(C(ij))$.

Lemma 2.3.8 *We can select $O(m^2)$ inside 0-implications from $\mathsf{Imp}_0(A)$ so that if an inside 0-implication in $\mathsf{Imp}_0(A)$ is violated then one of the selected 0-implications is violated.* □

Proof of Lemma 2.3.8 We define a subset $\mathsf{Imp}_0'(A)$ of the inside implications $\mathsf{Imp}_0(A)$ where $\mathsf{Imp}_0'(A)$ consists of $O(m^2)$ inside implications. Inside each strongly connected component C of $D_0(A)$ on p nodes (p pairs of rows), we can find at most $2p-2$ arcs so that the directed graph consisting of the nodes of C and the up to $2p-2$ arcs results in a strongly connected graph. Thus we can select up to $2\binom{m}{2}-2$ arcs from those within strongly connected components to form a new directed graph $D'_0(A)$ which has the same strongly connected components as $D_0(A)$. If we have $ij \to kl$ in $D'_0(A)$, we can now form the inside implications in $\mathsf{Imp}'_0(A)$ by selecting from $\mathsf{Imp}_0(A)$ the inside implications $ij \to_0 k$ and $ij \to_0 l$. Thus $\mathsf{Imp}'_0(A)$ has at most $4\binom{m}{2}$ inside implications.

Imagine having a column q of A which violates an inside implication $ij \to_0 k$ with $k \in \mathsf{supp}(C(ij))$. Thus we have $a_{iq}=0$, $a_{jq}=0$ and $a_{kq}=1$. Now with $k \in \mathsf{supp}(C(ij))$, there must be some row l with $kl \in C(ij)$ and hence there is a directed path in $D'_0(A)$

$$ij = u_1v_1 \to u_2v_2 \to u_3v_3 \to \cdots \to u_qv_q = kl \tag{2.33}$$

But then $\mathsf{Imp}'_0(A)$ contains the implications

$$ij \to_0 u_2, \quad ij \to_0 v_2, \quad u_2v_2 \to_0 u_3, \quad u_2v_2 \to_0 v_3, \quad \ldots, u_{q-1}v_{q-1} \to_0 k \tag{2.34}$$

With $a_{iq}=0$, $a_{jq}=0$ and $a_{kq}=1$, we deduce that some implication of $\mathsf{Imp}'_0(A)$ is violated by column q. □

Sometimes we have two 0-implications on a triple i,j,k and so more can be deduced.

Lemma 2.3.9 *If a triple i,j,k of rows of A has the property that two of the three possible 0-implications $ij \to_0 k$, $ik \to_0 j$, $jk \to_0 i$ are in $\mathsf{Imp}_0(A)$, then these two 0-implications are inside 0-implications.*

Proof of Lemma 2.3.9 Assume $ij \to_0 k$. Thus on the triple i,j,k we may assume without loss of generality that $ik \to_0 j$. Now $ij \to_0 k$ yields $ij \to ik$ in $D_0(A)$. Also $ik \to_0 j$ yields $ik \to ij$ in $D_0(A)$. Thus $ik \in C(ij)$ and so $k \in \mathsf{supp}(C(ij))$ and so $ij \to_0 k$ is an inside implication. □

We can do certain reductions on outside implications that we summarize in what follows.

Lemma 2.3.10 *We can choose a subset $\mathsf{Imp}''_0(A)$ of $\mathsf{Imp}_0(A)$ so that every violation of a 0-implication in $\mathsf{Imp}_0(A)$ yields a violation of a 0-implication in $\mathsf{Imp}''_0(A)$ with the property that if we have outside 0-implications $ij \to_0 k$, $ij \to_0 l$ in $\mathsf{Imp}''_0(A)$, we do not have $ik \to_0 l \in \mathsf{Imp}_0(A)$ and if we have outside 0-implications $ij \to_0 k$, $ij \to_0 l$, $ij \to_0 h$ in $\mathsf{Imp}''_0(A)$, we do not have $kl \to_0 h \in \mathsf{Imp}_0(A)$.*

Proof of Lemma 2.3.10 We rely on the topological ordering of the nodes of $D_0(A)$ which are the pairs of rows of A. We first do the reduction in Lemma 2.3.8. We start with $\mathsf{Imp}_0''(A)$ consisting of $\mathsf{Imp}_0'(A)$ plus all the outside implications in $\mathsf{Imp}_0(A)$. We successively reduce $\mathsf{Imp}_0''(A)$ by processing pairs of rows in the topological order as follows. For each pair ij, we delete as many outside implications from ij as possible while preserving the property that every violation of an implication in $\mathsf{Imp}_0(A)$ yields a violation of an implication in the new reduced set of implications $\mathsf{Imp}_0''(A)$. If we are processing outside implications from ij and we have $ij \rightarrow_0 k$, $ij \rightarrow_0 l$ in $\mathsf{Imp}_0''(A)$ and $ik \rightarrow_0 l$ in $\mathsf{Imp}_0(A)$, then we can delete $ij \rightarrow_0 l$. This is because a violation of $ij \rightarrow_0 l$ will either violate $ij \rightarrow_0 k$ or $ik \rightarrow_0 l$. Now $ij \rightarrow_0 k$ is in $\mathsf{Imp}_0''(A)$. Also ik is later in the topological ordering than ij (in view of the implication $ij \rightarrow_0 k$ which yields $ij \rightarrow ik$ in $D_0(A)$ and the fact that $k \notin \mathsf{supp}(C(ij))$ so we do not have ik in $C(ij)$). If $ik \rightarrow_0 l$ is currently in $\mathsf{Imp}_0''(A)$ we are done. We note that we have not yet have processed outside implications from ik so if $ik \rightarrow_0 l$ is not currently in $\mathsf{Imp}_0''(A)$ then it must be because $ik \rightarrow_0 l$ is an inside implication that was deleted using Lemma 2.3.8. Now we can use the argument in Lemma 2.3.8 to verify that some remaining inside implication in $\mathsf{Imp}_0'(A)$ is violated. Thus deleting $ij \rightarrow_0 l$ from $\mathsf{Imp}_0''(A)$ will preserve the property that every violation of an implication in $\mathsf{Imp}_0(A)$ yields a violation of an implication in the new reduced set of implications $\mathsf{Imp}_0''(A)$.

We proceed in an inductive way to delete implications while preserving the property that every violation of an implication in $\mathsf{Imp}_0(A)$ yields a violation of an implication in the new reduced set of implications $\mathsf{Imp}_0''(A)$.

In a similar fashion we can assume that if we have $ij \rightarrow_0 k$, $ij \rightarrow_0 l$, $ij \rightarrow_0 h$ in $\mathsf{Imp}_0''(A)$, and $kl \rightarrow_0 h$ in $\mathsf{Imp}_0(A)$, then we can delete $ij \rightarrow_0 h$. □

The ordering was crucial to the formation of $\mathsf{Imp}_0''(A)$. Other reductions are possible. For example we can show that for $k \notin \mathsf{supp}(C(ij))$ we need only keep one implication of the form $uv \rightarrow_0 k$ for $uv \in C(ij)$. We did not need this in our proofs.

Proofs of Quadratic Bounds

Let t be given. Let A be a simple $m \times n$ matrix with no configuration $F_6(t)$. Use Proposition 2.3.6. Each triple of rows i, j, k either has (2.21) or a 0-implication or two 1-implications.

Using Lemma 2.3.9 on the 1-implications, we can select at most $2m^2$ 1-implications so that a column of A which has violations of 1-implications has violations of one of the at most $2m^2$ 1-implications. By the pigeonhole principle, there are at most $2(t-1)m^2$ columns in A which have violations of 1-implications. Delete these columns to form a new $m \times n'$ simple matrix A' with $n \leq n' + 2(t-1)m^2$, where A' has no $F_6(t)$. We can reapply Proposition 2.3.6 to A' to deduce perhaps additional 0-implications (additional 1-implications would not be generated given the precedence in Proposition 2.3.6). But now if we have (2.23), we have

$$\text{no } \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{ and no } \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}. \tag{2.35}$$

We reduce the 0-implications $\mathsf{Imp}_0(A')$ to $\mathsf{Imp}''_0(A')$ as described in Lemmas 2.3.8, 2.3.10 but in view of Lemma 2.3.8 we focus attention on outside 0-implications. Assume we have in $\mathsf{Imp}''_0(A')$ outside implications

$$ij \to_0 k, \quad ij \to_0 l, \quad k, l \notin \mathsf{supp}(C(ij)) \tag{2.36}$$

Consider the triple i, k, l. By Lemma 2.3.10, we do not have $ik \to_0 l$ or $il \to_0 k$. Thus if we have a 0-implication of the triple i, k, l, then it must be $kl \to_0 i$. If the triples i, k, l and j, k, l both have 0-implications then we have $kl \to_0 i$ and $kl \to_0 j$ which yields $kl \to ij$ in $D_0(A')$. Then $ij \to_0 k,$, $ij \to_0 l$ yields $ij \to kl$ in $D_0(A')$ and so $kl \in C(ij)$ contradicting $k, l \notin \mathsf{supp}(C(ij))$. Thus we may assume the triple j, k, l does not have a 0-implication (we can make a similar argument if the triple i, k, l has no 0-implication). Hence it either has no column of 0's (on rows j, k, l) or two 1-implications (on rows j, k, l). If we have two 1-implications, we may assume we have $jk \to_1 l$. Now we have

$$\text{at most } t-1 \text{ of } \begin{matrix} i \\ j \\ k \\ l \end{matrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ \ \end{bmatrix} \text{ and at most } t-1 \text{ of } \begin{matrix} i \\ j \\ k \\ l \end{matrix} \begin{bmatrix} 0 \\ 0 \\ \ \\ 1 \end{bmatrix} \text{ and no } \begin{matrix} i \\ j \\ k \\ l \end{matrix} \begin{bmatrix} \ \\ 1 \\ 1 \\ 0 \end{bmatrix} \tag{2.37}$$

forces

$$\text{at most } t-1 \text{ of } \begin{matrix} i \\ j \\ k \\ l \end{matrix} \begin{bmatrix} 0 \\ \ \\ 1 \\ 0 \end{bmatrix} \tag{2.38}$$

(since for any column q of $A' = (a'_{rs})$, $a'_{lq} = 0$ and $a'_{kq} = 1$ forces $a'_{jq} = 0$) which is the 0-implication $il \to_0 k$. But this must have been discovered while computing $\mathsf{Imp}_0(A')$. Now this again contradicts Lemma 2.3.10. Hence we deduce that neither triple i, k, l or j, k, l contains two 1-implications and so one of the triples, say i, k, l has no column of 0's.

When the outside implications from ij are $ij \rightarrow_0 v_1, ij \rightarrow_0 v_2, \cdots, ij \rightarrow_0 v_q$ we repeat the above argument to deduce that for each pair v_r, v_s (with $1 \leq r < s \leq q$) we have

$$\text{no}\ \begin{matrix} i \\ v_r \\ v_s \end{matrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \ \text{or no}\ \begin{matrix} j \\ v_r \\ v_s \end{matrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{2.39}$$

Thus a column that violates one of the outside implications from ij will violate at least $q-1$ of the q outside implications, and hence at least $\frac{1}{2}$ of the implications. Thus at most $2(t-1)\binom{m}{2}$ columns of A' can violate outside 0-implications. Also at most $O(m^2)$ columns of A' can violate inside 0-implications by Lemma 2.3.8. The number of columns with no violations is at most $\mathsf{forb}(m, K_3)$ which is $O(m^2)$ by Theorem 2.1.3. Thus we have shown n' is $O(m^2)$ and so n is also $O(m^2)$. □

2.3.3 The boundary between $O(m^{k-1})$ and $\Theta(m^k)$

Conjecture 2.2.3 allowed us to predict the following theorem. We need some notations. Let D_{12} denote the simple matrix of all columns of column sum at least 1 with no K_2^2 on rows 1 and 2. Define

$$F_k(t) = [\mathbf{0}|(t+1) \cdot D_{12}]. \tag{2.40}$$

Theorem 2.3.11 *Let $k \geq 2$ be given and let F be a k-rowed (0,1)-matrix. Suppose that the largest column multiplicity in F is $t+1$. If F is a configuration in $F_k(t)$ or $\overline{F_k(t)}$ or F is a configuration in $F_B(t) = [K_k | t \cdot [K_k \setminus B]]$, for some choice of B a $k \times (k+1)$ simple matrix with one column of each column sum, then $\mathsf{forb}(m, F)$ is $O(m^{k-1})$. Otherwise, $\mathsf{forb}(m, F)$ is $\Theta(m^k)$.*

Proof of Theorem 2.3.11 Let us first consider the lower bound. Since F is not a configuration in any $F_B(t)$, F contains $2 \cdot K_k^\ell$ for some choice $\ell \in \{0, 1, \ldots, k\}$.

For $\ell = k$, we can deduce $\mathsf{forb}(m, 2 \cdot K_k^k)$ is $\Omega(m^k)$ using the construction K_m^k. Similarly, for $\ell = 0$, $\mathsf{forb}(m, 2 \cdot K_k^0)$ is $\Omega(m^k)$. For $\ell \neq 0, 1, k-1, k$ we have $\mathsf{forb}(m2 \cdot K_k^\ell) = \Omega(m^k)$ as follows. Consider the k-fold product A consisting of ℓ factors of $I_{m/k}$ and $k-\ell$ factors of $\overline{I}_{m/k}$. If we form a submatrix of k rows of A with 1 row from each of the factors then we cannot obtain a $k \times 2$ submatrix of two identical columns of ℓ 1's and $k-\ell$ 0's. So there is no copy of $2 \cdot K_k^\ell$ on these rows. Nor can we find even K_k^ℓ on a submatrix involving two rows from a single factor, as $\ell \neq 0, 1, k-1, k$ implies that on every pair of rows of K_k^ℓ there is both $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}0\\0\end{bmatrix}$, which are not configurations in $I_{m/k}$ and $\overline{I}_{m/k}$, respectively. Therefore $2 \cdot K_k^\ell$ is not a configuration in A.

From what we have said, then, F contains $2 \cdot K_k^1$ or contains $2 \cdot K_k^{k-1}$, or $\mathsf{forb}(m, F) = \Omega(m^k)$. We consider the case where F contains $2 \cdot K_k^1 = 2 \cdot I_k$, as the case of $2 \cdot K_k^{k-1}$ is the (0,1)-complement. We may assume that F does not contain $2 \cdot \mathbf{0} = 2 \cdot K_k^0$. Therefore, as F is not a configuration in $F_k(t)$, F contains $[2 \cdot I_k|G]$, where G is some matrix with $\begin{bmatrix}1\\1\end{bmatrix}$ in every pair of rows. Similarly to before the k-fold product A consisting of one factor $I_{m/k}$ and $k-1$ factors $\overline{I}_{m/k}$ witnesses that $\mathsf{forb}(m, F)$ is $\Omega(m^k)$. For G (and hence F) has $\begin{bmatrix}1\\1\end{bmatrix}$ in every pair of rows, so that a submatrix of k rows in A taking two rows from the factor $I_{m/k}$ does not have F as a configuration; while similarly I_k for $k \geq 3$ (and hence F) has $\begin{bmatrix}0\\0\end{bmatrix}$ in each pair of rows, so that a submatrix of k rows in A taking two rows from some factor $\overline{I}_{m/k}$ has no configuration F. But when we take any submatrix with one row from each factor there is only one copy of the column with 1 in the factor $I_{m/k}$ and 0's elsewhere. So there is no configuration $2 \cdot I_k$ in such a submatrix. Thus A has no configuration F. We include here a proof of the $\mathsf{forb}(m, [K_k|t \cdot [K_k \setminus B]]) = O(m^{k-1})$ result based on the paper [AFFS05], since it leads to a generalization of Lovász' result on color critical hypergraphs. An induction proof of the same result can be found in [AF08], while the proof of $\mathsf{forb}(m, F_k(t)) = O(m^{k-1})$ is a very recent result of Anstee and Fleming [AF09] using the linear algebra method in a refined way.

Let A be an $m \times n$ simple 0-1 matrix, and B be a $k \times (k+1)$ matrix consisting of one column of each possible column sum. Suppose that A does not have $F_B = [K_k \,|\, t \cdot (K_k - B)]$ as configuration. This implies that on a given k-tuple L of rows either K_k is missing, or if all possible columns of size k occur on L, then $t \cdot (K_k - B)$ must be missing. This latter means, that for some $0 \leq s \leq k$, two columns of column sum s occur at most $t-1$ times on L, respectively. Let $\mathcal{K}$ be the set of k-tuples of rows where the latter happens. Using Lemma 2.3.12 a set of columns of size $O(m^{k-1})$ can be removed from A to obtain A', so that for all $L \in \mathcal{K}$ a column (in fact two) is missing on L in A'. However, this implies that K_k is not a configuration in A', thus by Theorem 2.1.3 A' has at most $O(m^{k-1})$ columns. □

Let $\mathcal{K}$ be a system of k-tuples of rows such that $\forall K \in \mathcal{K}$ there are two $(k \times 1)$ columns, $\alpha_K \neq \beta_K$ specified. We say that a column x of A *violates* (K, α_K), if $x|_K = \alpha_K$, similarly, x *violates* (K, β_K), if $x|_K = \beta_K$.

Lemma 2.3.12 *Assume, that for every* $K \in \mathcal{K}$ *there are at most* $t-1$ *columns of* A *that violate* (K, α_K)*, and at most* $t-1$ *columns of* A *violate* (K, β_K)*. Then there exists a subset* X *of columns of* A*, such that* $|X| = O(m^{k-1})$ *and no column of* $A - X$ *violates any of* (K, α_K) *or* (K, β_K)*.*

Proof of Lemma 2.3.12 It can be assumed without loss of generality that for all $K \in \mathcal{K}$ $\alpha_K = \alpha$ and $\beta_K = \beta$ independent of K. Indeed, there are $2^k \times 2^k$ possible α_K, β_K pairs, that is a constant number of them .Thus, $\mathcal{K}$ can be partitioned into a constant number of parts, so that in each part $\alpha_K = \alpha$ and $\beta_K = \beta$ holds. We apply induction on k using the simplification given above. $k = 1$ is obvious.

Consider now $k \times 1$ columns $\alpha \neq \beta$. Assume first, that $\alpha \neq \overline{\beta}$. That is, there is a coordinate where α and β agree, say both have 1 as their ℓth coordinate. The case of a common 0 coordinate is similar. For the ith row of A we count how many columns have violation so that for some $K \in \mathcal{K}$ the ℓth coordinate in K is exactly row i. Let $\mathcal{K}_{i,\ell}$ be the set of these k-tuples from $\mathcal{K}$. Columns that have violation on k-tuples from $\mathcal{K}_{i,\ell}$ have 1 in the ith row, let $A_{i,1}$ denote matrix formed by the set of columns that have 1 in row i. If row i is removed from $A_{i,1}$, the remaining matrix $A'_{i,1}$ is still simple. Let $\mathcal{K}'_{i,\ell}$ denote the set of $(k-1)$-tuples obtained from k-tuples of $\mathcal{K}_{i,\ell}$ by removing their ℓth coordinate, i, furthermore let α' (β', respectively) denote the $(k-1) \times 1$ column obtained from α (β) by removing the ℓth coordinate, 1. Note, that $\alpha' \neq \beta'$. A column of A has a violation on $K \in \mathcal{K}_{i,\ell}$ iff its counterpart in $A'_{i,1}$ has a violation on the corresponding $K' \in \mathcal{K}'_{i,\ell}$. The number of those columns is at most $c\,m^{k-2}$ by the inductive hypothesis. Since $\mathcal{K} = \cup_{i=1}^{m} \mathcal{K}_{i,\ell}$, we obtain that the number of columns of A having violation on some $K \in \mathcal{K}$ is at most $m \cdot c\,m^{k-2}$.

Let us assume now, that $\alpha = \overline{\beta}$. A subset $\mathcal{J} \subseteq \mathcal{K}$ is called *independent* if there exists an ordering $J_1, J_2, \ldots J_g$ of the elements of $\mathcal{J}$ such that for every $J_i \in \mathcal{J}$ there exists an $m \times 1$ 0-1 column that violates J_i and does not violate any $J_j \in \mathcal{J}$ for $j < i$. Let us call a *maximal* independent subset $\mathcal{B}$ of $\mathcal{K}$ a *basis* of $\mathcal{K}$. If a column of A has a violation on $K \in \mathcal{K}$, then it has a violation on some $B \in \mathcal{B}$, as well. Indeed, either $K \in \mathcal{B}$ holds, or if $K \notin \mathcal{B}$, then by the maximality of $\mathcal{B}$, K cannot be added to it as a $|\mathcal{B}|+1$st element in the order, so the column having violation on K must have a violation on $B \in \mathcal{B}$, for some B. By (2.67) of Theorem 2.5.5 for a basis $\mathcal{B}$ we have

$$|\mathcal{B}| \leq \binom{m-1}{k-1} + \binom{m-1}{k-2} + \ldots + \binom{m-1}{0}, \tag{2.41}$$

since a column violating a k-tuple B_i from $\mathcal{B}$, but none of B_j for $j < i$, gives an appropriate partition of the set of rows. Thus, there could be at most $(2t-2)\binom{m}{k-1}$ columns violating some $K \in \mathcal{K}$. □

2.3.4 $l = 2$

For the case F is $k \times 2$, the asymptotic classification of $\mathsf{forb}(m, F)$ is given by Anstee and Keevash in [AK06].

Definition 2.3.13 *Let F_{abcd} be the $(a+b+c+d) \times 2$ $(0,1)$-matrix which has a rows of* $[11]$*, b rows of* $[10]$*, c rows of* $[01]$*, d rows of* $[00]$*.*

By interchanging columns we see that $\mathsf{forb}(m, F_{abcd}) = \mathsf{forb}(m, F_{acbd})$, and by considering (0,1)-complements we see that $\mathsf{forb}(m, F_{abcd}) = \mathsf{forb}(m, F_{dcba})$. Therefore we may assume that $a \geq d$ and $b \geq c$. Anstee and Keevash's result for the function $\mathsf{forb}(m, F_{abcd})$ is the following.

Theorem 2.3.14 *Suppose $a \geq d$ and $b \geq c$. If either $b > c$ or $a, b \geq 1$, then $\mathsf{forb}(m, F_{abcd})$ is $\Theta(m^{a+b-1})$. Furthermore, $\mathsf{forb}(m, F_{a00d})$ is $\Theta(m^a)$ and $\mathsf{forb}(m, F_{0bb0})$ is $\Theta(m^b)$.*

Their main technique is a strong stability result Theorem 2.3.15 and induction such as Lemma 2.3.16.

The strong stability result used in proving Theorem 2.3.14 considers a k-uniform set system with no $F_{0,r+1,r+1,0}$ which is equivalent to having the set system be $k-r$-intersecting. Let numbers k, r_1, r_2 be given and suppose G and H are given disjoint sets with $|G| = k - r_1 + r_2$. We define $\mathcal{I}^k_{r_1,r_2}$ on the pair (H, G) to be the family consisting of all sets of size k in $G \cup H$ that intersect G in at least $k - r_1 = |G| - r_2$ points. Note that any two sets in $\mathcal{I}^k_{r_1,r_2}$ have at least $|G| - 2r_2 = k - r_1 - r_2$ points in common, i.e. $\mathcal{I}^k_{r_1,r_2}$ is $(k-r)$-intersecting, where $r = r_1 + r_2$. The Complete Intersection Theorem, conjectured by Frankl, and proved by Ahlswede and Khachatrian [AK97], is that any k-uniform, $(k-r)$-intersecting family of maximum size on a given ground set is isomorphic to $\mathcal{I}^k_{r-p,p}$, for some $0 \leq p \leq r$, which depends on the size of the ground set. Note that $|\mathcal{I}^k_{r_1,r_2}|$ is $O(m^r)$ ($\Theta(m^r)$ for $|G|$ and $|H|$ being $\Theta(m)$). Anstee and Keevash prove the following results.

Theorem 2.3.15 *Suppose $\mathcal{A}$ is a k-uniform $(k-r)$-intersecting set system on $[m]$ of size at least $(5r)^{5r} m^{r-1}$. Then $\mathcal{A} \subseteq \mathcal{I}^k_{r-p,p}$ for some $0 \leq p \leq r$.*

Lemma 2.3.16 *Let F be a $k \times l$ $(0,1)$-matrix for which $\mathsf{forb}(m, F)$ is $O(m^t)$. Then with*

$$F' = \begin{bmatrix} 11\cdots1 \\ 00\cdots0 \\ F \end{bmatrix} \tag{2.42}$$

configuration F	construction	bound	reference
$\overbrace{\begin{bmatrix}0\cdots 0\\1\cdots 1\end{bmatrix}}^{q}$	$\left\lfloor\frac{(q+1)m}{2}\right\rfloor+2$	$\left\lfloor\frac{(q+1)m}{2}+\frac{(q-3)m}{2(m-2)}\right\rfloor+2$	[AFS01]
$\begin{bmatrix}0\\1\end{bmatrix}$	2	2	[AFS01]
$\begin{bmatrix}0&0\\1&1\end{bmatrix}$	$m+2$	$m+2$	[AFS01]
$\begin{bmatrix}0&0&0\\1&1&1\end{bmatrix}$	$2m+2$	$2m+2$	[AFS01]
$\begin{bmatrix}1&0&0\\0&1&1\end{bmatrix}$	$\lfloor\frac{3m}{2}\rfloor+1$	$\lfloor\frac{3m}{2}\rfloor+1$	[AGS97]
$\begin{bmatrix}1&0&0&0\\0&1&1&1\end{bmatrix}$	$\lfloor\frac{7m}{3}\rfloor+1$	$\lfloor\frac{7m}{3}\rfloor+1$	[AFS01]
$\begin{bmatrix}1&1&0&0&0\\0&0&1&1&1\end{bmatrix}$	$\lfloor\frac{8m}{3}\rfloor$	$\lfloor\frac{8m}{3}\rfloor$	[AFS01]
$\begin{bmatrix}\overbrace{1\cdots 1}^{p}\overbrace{0\cdots 0}^{q}\\0\cdots 0\,1\cdots 1\end{bmatrix}$	$(\frac{p+q}{2}+O(1))m$	$qm-q+2$	[AK07], [AFS01]
$\begin{bmatrix}\overbrace{1\cdots 1}^{p}\overbrace{0\cdots 0}^{p}\\0\cdots 0\,1\cdots 1\end{bmatrix}$	$pm-p+2$	$pm-p+2$	[AFS01]

Table 2.1: Two-rowed exact cases

we have $\mathsf{forb}(m, F')$ *being* $O(m^{t+1})$.

2.4 Exact results

There are a number of exact results. We collect here those that are found in papers [AGS97, AFS01, AS05, ABS09]. Table 2.1 lists the two-rowed cases.

For three-rowed cases consider the following matrices.

$$F_7 = \begin{bmatrix}1&0\\0&1\\0&1\end{bmatrix}, F_8 = \begin{bmatrix}1&1\\0&1\\0&0\end{bmatrix} \tag{2.43}$$

Theorem 2.4.1 *(Theorem 3.1 [AGS97])* $\mathsf{forb}(m, F_7) = \lfloor 3m/2\rfloor + 1$ □

Theorem 2.4.2 *(Theorem 3.2 [AGS97])* $\mathsf{forb}(m, F_8) = \lfloor m+2\rfloor + 1$ □

configuration	$\mathsf{forb}(m, F)$	configuration	$\mathsf{forb}(m, F)$
$F_{4000} = \begin{bmatrix}1&1\\1&1\\1&1\\1&1\end{bmatrix}$	$\binom{m}{4}+\binom{m}{3}+\binom{m}{2}+\binom{m}{1}+\binom{m}{0}$	$F_{3100} = \begin{bmatrix}1&1\\1&1\\1&1\\1&0\end{bmatrix}$	$\binom{m}{3}+\binom{m}{2}+\binom{m}{1}+\binom{m}{0}$
$F_{3001} = \begin{bmatrix}1&1\\1&1\\1&1\\0&0\end{bmatrix}$	$\binom{m}{3}+\binom{m}{2}+\binom{m}{1}+\binom{m}{0}+\binom{m}{m}$	$F_{2200} = \begin{bmatrix}1&1\\1&1\\1&0\\1&0\end{bmatrix}$	$\binom{m}{3}+\binom{m}{2}+\binom{m}{1}+\binom{m}{0}$
$F_{2110} = \begin{bmatrix}1&1\\1&1\\1&0\\0&1\end{bmatrix}$	$\geq (\frac{29}{21})\binom{m}{2}+\binom{m}{1}+\binom{m}{0}$ $\leq 2\binom{m}{2}+\binom{m}{1}+\binom{m}{0}$	$F_{2101} = \begin{bmatrix}1&1\\1&1\\1&0\\0&0\end{bmatrix}$	$\binom{m}{2}+\binom{m}{1}+\binom{m}{0}+\binom{m}{m}$
$F_{2002} = \begin{bmatrix}1&1\\1&1\\0&0\\0&0\end{bmatrix}$	$\binom{m}{2}+\binom{m}{1}+\binom{m}{0}+\binom{m}{m-1}+\binom{m}{m}$	$F_{1300} = \begin{bmatrix}1&1\\1&0\\1&0\\1&0\end{bmatrix}$	$\binom{m}{3}+\binom{m}{2}+\binom{m}{1}+\binom{m}{0}$
$F_{1210} = \begin{bmatrix}1&1\\1&0\\1&0\\0&1\end{bmatrix}$	$\binom{m}{2}+\binom{m}{1}+\binom{m}{0}+\binom{m}{m}$	$F_{1201} = \begin{bmatrix}1&1\\1&0\\1&0\\0&0\end{bmatrix}$	$\binom{m}{2}+\binom{m}{1}+\binom{m}{0}+\binom{m}{m}$
$F_{1111} = \begin{bmatrix}1&1\\1&0\\0&1\\0&0\end{bmatrix}$	$4m-4$	$F_{0400} = \begin{bmatrix}1&0\\1&0\\1&0\\1&0\end{bmatrix}$	$\binom{m}{3}+\binom{m}{2}+\binom{m}{1}+\binom{m}{0}$
$F_{0310} = \begin{bmatrix}1&0\\1&0\\1&0\\0&1\end{bmatrix}$	$\binom{m}{2}+\binom{m}{1}+\binom{m}{0}+\binom{m}{m}$	$F_{0220} = \begin{bmatrix}1&0\\1&0\\0&1\\0&1\end{bmatrix}$	$\binom{m}{2}+2m-1$

Table 2.2: 4×2 exact bounds

Theorem 2.4.3 *(Theorem 3.3 [AGS97]) Let F_2 be the configuration defined in (2.15). Then* $\mathsf{forb}(m, F_2) = 2m$ □

In the paper [ABS09] we give exact answers for all but one 4×2 configuration. The notation of Definition 2.3.13 is used. In several cases, the extremal matrices are completely determined. Given the nature of the proof of the asymptotic bounds for $k \times 2$ F for larger k of Theorem 2.3.14, we do not expect that many exact bounds will be available for general F (F_{0330} would be the first hurdle). We are unable to obtain an exact bound for F_{2110}. It appears that computing $\mathsf{forb}(F_{2110})$ encodes a difficult design theoretic problem. The results and proofs give a feel for the surprising diversity. Table 2.2 lists our results for 4×2 configurations. We include here the proof of two of them, F_{0220} for its complexity and F_{2110} for the design theoretic connection. If we consider forbidding F_{0220} as a special case of requiring any pair of sets to have symmetric distance at most 2, we can view the following result in relation to the result of Kleitman [Kle66] who considered the maximal size of a set family such that any two sets have symmetric difference at most $2t$ and so in our case $t = 1$.

Theorem 2.4.4

$$\mathsf{forb}(m, F_{0220}) = \binom{m}{2} + 2m - 1. \tag{2.44}$$

Moreover, up to row and column permutations, there are $m-1$ simple $m \times \mathsf{forb}(m, F_{0220})$ matrices with no F_{0220}.

Proof of Theorem 2.4.4 We first prove the bound. Let A be an m-rowed matrix with no F_{0220}. We focus on the columns of each column sum; let c_k denote the number of columns of column sum k in A. We easily deduce that $c_0+c_1+c_{m-1}+c_m \le 2m+2$ (note that columns of column sum $0, 1, m-1, m$ cannot contribute to F_{0220}). If we show

$$\sum_{i=2}^{m-2} c_i \le \sum_{i=2}^{m-2} (m-i+1) = \binom{m}{2} - 3 \tag{2.45}$$

then the bound (2.44) would follow.

Define I_a to be the $a \times a$ identity matrix and I_a^c to be the (0,1)-complement of I_a. Let $J_{a\times b}$ denote the $a \times b$ matrix of 1's and let $0_{a\times b}$ denote the $a \times b$ matrix of 0's. For each $2 \le t \le m-2$ we can divide the rows into three disjoint sets $A_t, B_t, C_t \subseteq \{1, 2, \ldots, m\}$ so that after permuting the rows the columns of column sum t can either be given as

$$\text{type 1:}\quad \begin{matrix} A_t\{ \\ B_t\{ \\ C_t\{ \end{matrix} \begin{bmatrix} I_{|A_t|} \\ J_{|B_t|\times|A_t|} \\ 0_{|C_t|\times|A_t|} \end{bmatrix} \quad \text{or type 2:}\quad \begin{matrix} A_t\{ \\ B_t\{ \\ C_t\{ \end{matrix} \begin{bmatrix} I^c_{|A_t|} \\ J_{|B_t|\times|A_t|} \\ 0_{|C_t|\times|A_t|} \end{bmatrix}. \tag{2.46}$$

We will say t *is of type* i ($i = 1$ or $i = 2$) if the columns of column sum t are of type i. These were introduced and used in [Ans90]. It is straightforward to consider two columns of column sum t and note that because of the forbidden configuration F_{0220} they must have $t-1$ rows where both have 1's. This can be viewed as type 1 with $|B_t| = t-1$, $|A_t| = 2$. If $c_t \le 2$, then we will consider t to be of type 1 (although one could also say they were of type 2). Now considering a third column, we either find ourselves with type 1 or type 2. Moreover adding a fourth or subsequent column of t 1's to either construction of type 1 or type 2 while avoiding F_{0220} will result in a construction of the same type with either one row deleted from C_t and that row added to A_t in type 1 or one row deleted from B_t and that row added to A_t in type 2.

If the columns of column sum t are of type 1 then $|B_t| = t-1$ and so $|A_t| + |C_t| = m - t + 1$. Hence

$$c_t = |A_t| \le m - t + 1. \tag{2.47}$$

If the columns of column sum t are of type 2 then $|A_t| + |B_t| = t + 1$, $|C_t| = m - t - 1$ and so

$$c_t = |A_t| = t + 1 - |B_t| \leq t + 1. \tag{2.48}$$

Let us begin by considering that t is of type 1 for all $t = 2, 3, \ldots, m - 2$ and so $c_t \leq m - t + 1$ by (2.47). Then

$$\sum_{i=2}^{m-2} c_i \leq \sum_{i=2}^{m-2} (m - i + 1) = \binom{m}{2} - 3 \tag{2.49}$$

and we are done. Similarly we are done if t is of type 2 for all $t = 2, 3, \ldots, m-2$. Note, however that these cases are also covered by the arguments below.

Claim 2.4.5 *Assume that k, l satisfy $2 \leq k < l \leq m - 2$ and k is of type 1 and l is of type 2. Assume $k \geq |B_l| + 2$ (i.e. $|B_k| > |B_l|$ since $|B_k| = k - 1$). Then $|A_k| \leq l - k + 2$.*

Given $k < l$ and the forbidden configuration F_{0220}, we deduce that we cannot find a configuration F_{0200}:

$$\begin{matrix} 1 & 0 \\ 1 & 0 \end{matrix} \tag{2.50}$$

where the first column of 2 1's comes from a column of column sum k and the second column of 2 0's comes from a column of column sum l, which corresponds to two rows where the column of l 1's has a 0 while the column of k 1's has a 1. Given that $k < l$, there will be at least 2 rows where the column of l 1's has a 1 while the column of k 1's has a 0. But this yields the configuration F_{0220} on these two columns.

We have two cases to prove the bound $|A_k| \leq l - k + 2$:

Case 1 $B_k \cap C_l \neq \emptyset$.

This forces $|B_k \cap C_l| = 1$, $B_k \backslash C_l \subseteq B_l$ and $A_k \subseteq B_l$ to avoid having (2.50) on two rows with the two 1's coming from a column of column sum k and the two 0's coming from a column of column sum l. Thus

$$|A_k| + |B_k| \leq |B_l| + 1 < l + 1 \tag{2.51}$$

given that $|A_l| + |B_l| = l + 1$, and $|A_l| \geq 3$ (by definition of type 2). Since $|B_k| = k - 1$, we have

$$|A_k| \leq |B_l| - k + 2 < l - k + 2, \tag{2.52}$$

which is the claimed bound.

Case 2 $B_k \cap C_l = \emptyset$.

Given that $|B_k| > |B_l|$, we deduce that $B_k \cap A_l \neq \emptyset$. This forces $A_k \cap C_l = \emptyset$ to avoid creating (2.50) whose first column is from a column of column sum l and whose second column is from a column of column sum k. Thus

$$|A_k| + |B_k| \leq |A_l| + |B_l|. \tag{2.53}$$

Given that $|B_k| = k-1$ and $|A_l|+|B_l| = l+1$ we deduce that $|A_k| \leq l-k+2$. This finishes Claim 1.

We note that given $k < l$ and $|B_l| < |B_k| = k-1$, we have

$$|B_l| + 2 \leq k \leq l - 1. \tag{2.54}$$

Thus for a given value for l, there are $l - |B_l| - 2$ possible choices for k for which Claim 2.4.5 might be applicable.

We now propose bounds for c_k as follows. Let

$$v_k = |\{p : p > k \text{ and } p \text{ is of type 2 and } k \geq |B_p| + 2\}|. \tag{2.55}$$

We define

$$u_k = \begin{cases} m-k+1-v_k & \text{if } k \text{ is of type 1} \\ m-k+1-v_k+k-|B_k|-2 & \text{if } k \text{ is of type 2} \end{cases} \tag{2.56}$$

Claim 2.4.6 $c_k \leq u_k$.

Case 1. k is of type 1.

If $v_k = 0$, then (2.47) yields the result. Assume $v_k > 0$. Let l be the smallest index of type 2 satisfying $l > k$ and $|B_l|+2 \leq k$. Then $v_k \leq m-1-l$ (the possible p would be in $\{l, l+1, \ldots, m-2\}$). By Claim 2.4.5,

$$c_k = |A_k| \leq l-k+2 = m-k+1-(m-l-1) \leq m-k+1-v_k = u_k. \tag{2.57}$$

Case 2. k is of type 2.

By our preliminary observation (2.48), $c_k = |A_k| = k - |B_k| + 1$. As in the case above, $v_k \leq m-k-2$ (possible p in $\{k+1, k+2, \ldots, m-2\}$). Thus

$$c_k = k - |B_k| + 1 = m-k+1-(m-k-2)+(k-|B_k|-2) \tag{2.58}$$

$$\leq m-k+1-v_k+(k-|B_k|-2) = u_k. \tag{2.59}$$

This finishes Claim 2.4.6.

Claim 2.4.7 $\sum_{k=2}^{m-2} u_k = \sum_{k=2}^{m-2}(m-k+1)$.

We first note that

$$\sum_{p:\,p\text{ of type }2}(p-|B_p|-2)=\sum_{k=2}^{m-2}v_k \tag{2.60}$$

by noting that if p is of type 2, it contributes 1 to each $v_k\in\{|B_p|+2,|B_p|+3,\ldots,p-2,p-1\}$ (p contributes to v_k if $|B_p|+2\le k\le p-1$) for a total contribution of $p-|B_p|-2$. We now obtain the claim from (2.56) noting that

$$\sum_{k=2}^{m-2}u_k=\sum_{k=2}^{m-2}(m-k+1)-\sum_{k=2}^{m-2}v_k+\sum_{p:\,p\text{ of type }2}(p-|B_p|-2) \tag{2.61}$$

$$=\sum_{k=2}^{m-2}(m-k+1). \tag{2.62}$$

This completes our proof of (2.45) and hence (2.44) as an inequality. Note that our proof did not need to consider a pair k,l with k of type 1, l of type 2 and $k>l$.

Let A be an $m\times(\binom{m}{2}+2m-1)$ simple matrix with no configuration F_{0220}. We note that to achieve the bound (2.44) we must have $c_k=u_k$ for all k and also $\sum_{i=2}^{m-2}u_i=\sum_{i=2}^{m-2}(m-i+1)$ which implies equality in (2.57) or (2.59) for all k.

In Case 2 of Claim 2.4.6, we have that k is of type 2 forces $v_k=m-k-2$, namely that $k+1,k+2,\ldots,m-2$ are all type 2. Thus either we may assume that either k is of type 1 for all $2\le k\le m-2$ or there is an r, $1\le r\le m-3$ such that $2,3,\ldots,r$ are of type 1 and $r+1,r+2,\ldots,m-2$ are of type 2. The case $r=1$ will handle the case that k is of type 2 for all $2\le k\le m-2$ and then taking complements will handle the case k is of type 1 for all $2\le k\le m-2$ which we can envision as the case $r=m-2$.

In Case 1 of Claim 2.4.6, we have that k is of type 1 forces $|A_k|=(r+1)-k+2$ and also $v_k=m-1-(r+1)$.

We claim that for each r $1\le r\le m-2$ there is, up to row and column permutations, one simple $m\times(\binom{m}{2}+2m-1)$ matrix with no F_{0220} as follows: Column sums $\{2,3,\ldots,r\}$ are of type 1 and column sums $\{r+1,r+2,\ldots,m-2\}$ are of type 2. For the column sums of type 1 define $A_2=\{1,2,\ldots,r+1\}$, $A_3=\{1,2,\ldots,r\}$, $A_r=\{1,2,3\}$, $B_2=\{r+2\}$, $B_3=\{r+1,r+2\}$, ...,$B_r=\{4,5,\ldots,r+2\}$ and $C_2=C_3=\cdots=C_r=\{r+3,r+4,\ldots,m\}$. For the columns of type 2 define $A_{r+1}=\{1,2,\ldots,r+2\}$, $A_{r+2}=\{1,2,\ldots,r+3\}$, ...,$A_{m-2}=\{1,2,\ldots m-1\}$, $B_{r+1}=B_{r+2}=\cdots=B_{m-2}=\emptyset$ and $C_{r+1}=$

$\{r+3, r+4, \ldots, m\}$, $C_{r+2} = \{r+4, r+5, \ldots, m\}$, ..., $C_{m-2} = \{m\}$. We leave checking these examples as an exercise and their existence completes the proof of (2.44).

We deduce that

$$c_q = \begin{cases} r - q + 3 & \text{if } q = 2, 3, \ldots, r \\ q + 1 & \text{if } q = r+1, r+2, \ldots, m-2 \end{cases} \tag{2.63}$$

By (2.48), we have $B_{r+1} = B_{r+2} = \cdots = B_{m-2} = \emptyset$. We have without loss of generality that $A_{m-2} = \{1, 2, \ldots, m-1\}, C_{m-2} = \{m\}$. Our proofs that the remaining sets A_k, B_k, C_k take the stated form will follow by showing that if not there exist $p < q$, two rows r_1, r_2 and two columns α, β where the column sum of α is p and the column sum of β is q where α has 1's in rows r_1, r_2 while β has 0's in rows r_1, r_2, namely (2.50). Given the column sums we deduce that there are (more than) two additional rows with α being 0's and β being 1's which produces a copy of F_{0220}, a contradiction.

For $r+1 \leq p < q \leq m-2$ we can deduce that $C_q \subseteq C_p$. If not, there is an $r_1 \in C_q \backslash C_p = C_q \cap A_p$ for which every column but one of sum p has 1 in row r_1 and all columns of sum q have 0 in row r_1. Given that $3 \leq |A_p|$, there is an $r_2 \in A_p \backslash r_1$ and so all but one column of sum p has 1 in row r_2 and at least one column of sum q has 0 in row r_2 ($r_2 \in A_q \cup C_q$). This yields two columns α, β as described and a contradiction so that for $r+1 \leq p < q \leq m-2$ we deduce that $C_q \subseteq C_p$ and so by reordering rows if necessary we may assume $A_{r+1} = \{1, 2, \ldots, r+2\}$, $A_{r+2} = \{1, 2, \ldots, r+3\}$, ...,$A_{m-2} = \{1, 2, \ldots m-1\}$ and $C_{r+1} = \{r+3, r+4, \ldots, m\}$ $C_{r+2} = \{r+4, r+5, \ldots, m\}$, ..., $C_{m-2} = \{m\}$.

Now consider columns of type 1. We have $|A_p| + |B_p| = r + 2$ and $|B_p| = p - 1 > 0$ for each $2 \leq p \leq r$. We can deduce that $A_p \cup B_p = A_{r+1}$. If not, there is an $r_1 \in (A_p \cup B_p) \backslash A_{r+1}$. If $r_1 \in B_p$ then all columns of sum p have 1 in row r_1 and all columns of sum $r+1$ have 0 in row r_1 ($r_1 \in C_{r+1}$). Choose $r_2 \in (A_p \cup B_p) \backslash r_1$. Then at least one column of sum p has 1 in row r_2 and at least one column of sum $r+1$ has a 0 in row r_2 (note $B_{r+1} = \emptyset$). This yields two columns α, β as described and a contradiction. If $r_1 \in A_p$ then one column of sum r has a 1 in row r_1 and all columns of sum $r+1$ have 0 in row r_1. Choose $r_2 \in B_p$. Then all columns of sum r have 1 in row r_1 and as before at least one column of sum $r+1$ has 1 in row r_2. This yields two columns α, β as described and a contradiction. We may assume $A_p \cup B_p = A_{r+1}$.

For each $2 \leq p < q \leq r$ we can deduce that $B_p \subseteq B_q$. If not there exists $r_1 \in B_p \backslash B_q$ so that all columns of sum p have 1 in row r_1 and all but one column of sum q has 0 in row r_1. We may choose $r_2 \in A_q \backslash r_1$ since $|A_q| > 3$ and so at least one column of sum p has 1 in row r_2 and all but one column of sum q has 0 in row r_2. This yields two columns α, β as described and a contradiction so that we may assume $B_p \subseteq B_q$. By reordering rows (A_{r+1}) if necessary we may assume $A_2 = \{1, 2, \dots, r+1\}$, $A_3 = \{1, 2, \dots, r\}$, $A_r = \{1, 2, 3\}$, $B_2 = \{r+2\}$, $B_3 = \{r+1, r+2\}$, ...,$B_r = \{4, 5, \dots, r+2\}$ and $C_2 = C_3 = \cdots = C_r = \{r+3, r+4, \dots, m\}$.

This completes the argument that the structure of the matrix A is determined by each of the $m-1$ choices for r. □

We are unable to determine an exact bound for F_{2110} and instead indicate some good constructions in the following theorem . The problem has a decidedly design theoretic flavor and appears quite complicated despite the simple appearance of F_{2110}.

Theorem 2.4.8 $\mathsf{forb}(m, F_{2110}) \le 2\binom{m}{2} + \binom{m}{1} + \binom{m}{0}$ *and for infinitely many* m, $\mathsf{forb}(m, F_{2110}) \ge \frac{29}{21}\binom{m}{2} + \binom{m}{1} + \binom{m}{0}$

Proof of Theorem 2.4.8 Let A be an m-rowed simple matrix with no F_{2110}. Following the decomposition

$$A = \begin{bmatrix} 11\cdots1 & 00\cdots0 \\ A_1 & A_0 \end{bmatrix} \tag{2.64}$$

we note that A_1 is simple and has no configuration F_{1110} and so has at most $\mathsf{forb}(m-1, F_{1110}) = 2(m-1)$ columns. Similarly A_0 has no F_{2110} and so has at most $\mathsf{forb}(m-1, F_{2110})$. By induction we deduce that $\mathsf{forb}(m, F_{2110}) \le 2(m-1) + 2\binom{m-1}{2} + \binom{m}{1} + \binom{m}{0} = 2\binom{m}{2} + \binom{m}{1} + \binom{m}{0}$, establishing the upper bound.

Delete the column of column sums $0, 1, 2, m$ from A. Considering each remaining column as a clique on its rows containing 1's, then the maximal cliques/columns form a packing of K_m. This suggests constructions as follows: There is a 2-$(7, 3, 1)$-design (the Fano plane) and for infinitely many m, there is a 2-$(m, 7, 1)$-design. For this we need the result of Wilson [Wil75]. We then form a matrix A by taking the columns (of sum 7) corresponding to the blocks of the 2-$(m, 7, 1)$ design. Then for each of the $\frac{1}{7\times3}\binom{m}{2}$ columns of sum 7, add the 7 columns of sum 3 corresponding to the blocks of the 2-$(7, 3, 1)$-design on the rows where the column of sum 7 has 1's. We have constructed $(\frac{1}{21} + \frac{1}{3})\binom{m}{2}$ columns of sum 3 and 7 to which we may add the $\binom{m}{2} + \binom{m}{1} + \binom{m}{0}$ columns of sum 2,1,0 obtaining a matrix with no F_{2110}. □

We could improve slightly on $\frac{29}{21}$ and the construction as follows. There is a 2-$(91, 7, 1)$-design and for infinitely many m, there is an 2-$(m, 91, 1)$-design. For this we need the result of Wilson [Wil75]. We then form a matrix A by taking the columns (of sum 91) corresponding to the blocks of the 2-$(m, 91, 1)$ design. Then for each of the $\frac{1}{91\times45}\binom{m}{2}$ columns of sum 91, add the 195 columns of sum 7 corresponding to the blocks of the 2-$(91, 7, 1)$-design on the rows where the column of sum 91 has 1's. Then for each of the $\frac{1}{7\times 3}\binom{m}{2}$ columns of sum 7, add the 7 columns of column sum 3 corresponding to the blocks of the 2-$(7, 3, 1)$-design on the rows where the column of sum 7 has 1's.

It is clear that for larger m we can get a larger multiple of $\binom{m}{2}$ by seeking a 2-$(k, 91, 1)$-design and an 2-$(m, k, 1)$-design. And we could continue this recursion to get larger multiples (of $\binom{m}{2}$). Also, it is not necessary to use block designs to get good constructions although the idea of using a packing remains important. At this stage it appears that $\frac{\mathsf{forb}(m,F_{2110})}{\binom{m}{2}}$ is increasing as $m \to \infty$. We do not conjecture a limiting value.

2.5 Partition critical hypergraphs

The proof of Theorem 2.3.11, in particular that of Lemma 2.3.12 lead to the concept of *partition critical hypergraphs* that are generalizations of *color critical hypergraphs*. These latter ones have been investigated extensively. Our interest here is in the maximum number of edges in a k-uniform ℓ-critical hypergraph.

Definition 2.5.1 *A k-uniform hypergraph $\mathcal{H}$ is* ℓ-critical *if it is not $\ell - 1$-colorable, but deleting any edge or vertex results in a $\ell - 1$-colorable hypergraph.*

Toft proved [Tof73] that for $k, \ell > 3$ fixed, $n \to \infty$, there $\exists k$-uniform ℓ-critical hypergraph on n vertices of size $\Omega(n^k)$. But all 3-critical k-uniform hypergraphs have size $o(n^k)$. He asked: What is the maximum size of a 3-critical k-uniform hypergraph? Lovász [Lov76] gave the following upper bound.

Theorem 2.5.2 *Let $\mathcal{H}$ be a 3-critical k-uniform hypergraph on an n-element underlying set. Then*

$$|E(\mathcal{H})| \leq \binom{n}{k-1}. \tag{2.65}$$

Partition critical hypergraphs are generalizations of color critical ones.

Definition 2.5.3 *A k-uniform hypergraph $\mathcal{E} \subseteq \binom{[n]}{k}$ on an underlying set X of n elements is called* partition critical *if the followings hold. There exists an ordering $E_1, E_2, \ldots E_t$ of $\mathcal{E}$ and a prescribed partition $A_i \cup B_i = E_i$ ($A_i \cap B_i = \emptyset$) for each member of $\mathcal{E}$ such that for all $i = 1, 2, \ldots, t$ there exists a partition $C_i \cup D_i = X$ ($C_i \cap D_i = \emptyset$), such that $E_i \cap C_i = A_i$ and*

$E_i \cap D_i = B_i$, but $E_j \cap C_i \neq A_j$ and $E_j \cap C_i \neq B_j$ for all $j < i$. (That is, the ith partition cuts the ith set as it is prescribed, but does not cut any earlier set properly.

A 3-critical hypergraph is certainly partition critical, as well. Indeed, for an arbitrary ordering of the edges $E_1, E_2, \dots E_t$ of $\mathcal{E}$, the partition $A_i = E_i$, $B_i = \emptyset$ works for all edges. A middle stage between 3-critical hypergraphs and partition critical hypergraphs is the concept of *ordered 3-critical* hypergraphs.

Definition 2.5.4 *A k-uniform partition critical hypergraph is called* ordered 3-critical *if the prescribed partition is $A_i = E_i$ and $B_i = \emptyset$ for all edges $E_i \in \mathcal{E}$. In other words, there is an ordering of the edges and a partition for every edge, such that the partition belonging to E_i cuts every edge E_j to nonempty parts for $j < i$, and does not cut E_i.*

Thus the following theorem is a strengthening of Lovász' theorem.

Theorem 2.5.5 *Let $(X, \mathcal{E})$ be an ordered 3-critical k-uniform hypergraph on the n-element underlying set $X = \{1, 2, \dots, n\}$. Then*

$$|\mathcal{E}| \leq \binom{n}{k-1}. \tag{2.66}$$

If $(X, \mathcal{E})$ is partition critical, then

$$|\mathcal{E}| \leq \binom{n-1}{k-1} + \binom{n-1}{k-2} + \ldots + \binom{n-1}{0}. \tag{2.67}$$

holds. This bound is sharp in the sense that for all $n \geq 2k-1$ and $k \geq 2$ there exist partition critical k-uniform hypergraphs of size $\binom{n}{k-1}$.

Note that a partition critical or an ordered 3-critical hypergraph can be two-colorable.

The proof of the bound (2.66) is based on the polynomial method outlined in [FHW06], while (2.67) is obtained by a refinement of the argument in [AFFS05]. The construction is very recent [FS09].

Proof of Theorem2.5.5 Let us first consider the inequality (2.66). We define n-variable polynomials $P_i(x_1, x_2, \dots x_n) \in \mathbb{R}[x_1, x_2, \dots x_n]$ for all $E_i \in \mathcal{E}$, and $Q_H(x_1, x_2, \dots x_n) \in \mathbb{R}[x_1, x_2, \dots x_n]$ for all $H \subset X = \{1, 2, \dots, n\}$ with $|H| \leq k-2$. Let P_i be defined by

$$P_i(x_1, x_2, \dots x_n) = \prod_{1 \leq m \leq k-1} \left(\left(\sum_{v \in C_i} x_v \right) - m \right), \tag{2.68}$$

where C_i is one side of the partition $C_i \cup D_i = X$ that belongs to edge E_i according to Definition 2.5.4. On the other hand, Q_H is defined by

$$Q_H(x_1, x_2, \ldots x_n) = \prod_{h \in H} x_h \left(\sum_{j=1}^{n} x_i - k \right). \tag{2.69}$$

Let $\hat{Y}$ denote the characteristic vector of subset $Y \subseteq X$. According to Definition 2.5.4 $P_j(\widehat{E_i}) = 0$, if $i < j$ but $P_j(\widehat{E_j}) \neq 0$. Indeed, $P_j(\widehat{E_i}) = \prod_{1 \le m \le k-1} (|C_j \cap E_i| - m)$. Since the partition $C_j \cup D_j = X$ cuts E_i in proper nonempty subsets, $1 \le |C_j \cap E_i| \le k - 1$ for $i < j$. Similarly, $Q_H(\hat{Y}) \neq 0$ iff $H \subseteq Y$ and $|Y| \neq k$. Now let $\tilde{P}_i(x_1, x_2, \ldots x_n)$ be the polynomial obtained from P_i by expanding the products and the repeatedly replacing higher order factor x_v^2 by x_v for all $1 \le v \le n$. $\tilde{P}_i$ is multilinear of degree at most $k - 1$, furthermore for any subset $Y \subseteq X$ we have $\tilde{P}_i(\hat{Y}) = P_i(\hat{Y})$. Let $\tilde{Q}_H$ be obtained from Q_H by the same reduction as above. $\tilde{Q}_H$ is also multilinear of degree at most $k - 1$ and $\tilde{Q}_H(\hat{Y}) = Q_H(\hat{Y})$ for any subset $Y \subseteq X$.

We claim that the system of polynomials $\mathcal{P} = \{\tilde{Q}_H \colon H \subset X, \ |H| \le k - 2\} \cup \{\tilde{P}_i \colon 1 \le i \le t\}$ is linearly independent in the space of multilinear polynomials of degree at most $k-1$ of n variables. Indeed, order the polynomials as follows. Put first $\tilde{Q}_H$ in decreasing order of the size of H. Then put $\tilde{P}_i$ for $1 \le i \le t$. Suppose in contrary, that there exists a non-trivial linear combination

$$\sum_{H \subset X, \ |H| \le k-2} \lambda_H \tilde{Q}_H + \sum_{i=1}^{t} \beta_i \tilde{P}_i = 0 \tag{2.70}$$

that results in the zero polynomial. Consider the last non-zero coefficient according to the order defined above. If that is λ_H for some H, then evaluate (2.70) at $\hat{H}$. Since for any $\tilde{Q}_K$ earlier in the order than $\tilde{Q}_H$ we have $\tilde{Q}_K(\hat{H}) = 0$. the value of (2.70) at $\hat{H}$ is $\lambda_H \tilde{Q}_H(\hat{H}) \neq 0$, a contradiction. Similarly, if the last non-zero coefficient is β_j for some j, then evaluate (2.70) at $\widehat{E_j}$. $\tilde{Q}_H(\widehat{E_j}) = 0$, since $|E_j| = k$. On the other hand, $\tilde{P}_i(\widehat{E_j}) = 0$ for $i < j$, as it was observed above. Thus, the value of (2.70) at $\widehat{E_j}$ is $\tilde{P}_j(\widehat{E_j}) \neq 0$, a contradiction again.

Hence the number of polynomials in $\mathcal{P}$ is at most the dimension of the linear space of multilinear polynomials of degree at most $k-1$ of n variables. Thus,

$$|\{\tilde{Q}_H \colon H \subset X, \ |H| \le k - 2\}| + t = |\mathcal{P}| \le \binom{n}{k-1} + \binom{n}{k-2} + \ldots + \binom{n}{0}, \tag{2.71}$$

which implies (2.66).

For arbitrary prescribed partitions (2.67) is proved using also the polynomial method, but with completely different polynomials. Namely, define a polynomial $p_i(\underline{x}) \in \mathbb{R}[x_1, x_2, \ldots, x_m]$ for each E_i as follows.

$$p_i(x_1, x_2, \ldots, x_m) = \prod_{a \in A_i}(1 - x_a) \prod_{b \in B_i} x_b + (-1)^{k+1} \prod_{a \in A_i} x_a \prod_{b \in B_i}(1 - x_b) \quad (2.72)$$

Polynomials defined by (2.72) are multilinear of degree at most $k-1$, since the product $\prod_{e \in E_i} x_e$ cancels by the coefficient $(-1)^{k+1}$. It can be easily checked that $p_j(\widehat{C}_i) = 0$ if $j < i$ and $p_i(\widehat{C}_i) \neq 0$. Let us assume without loss of generality that the partitions $C_i \cup D_i = X$ are so that $n \in D_i$ holds for every $i = 1, 2, \ldots, t$. Let polynomials q_i be defined by

$$q_i(x_1, x_2, \ldots, x_{n-1}) = p_i(x_1, x_2, \ldots, x_n)|_{x_n=0} \in \mathbb{R}[x_1, x_2, \ldots, x_{n-1}]. \quad (2.73)$$

Let $C_i' = C_i|_{\{1,2,\ldots,n-1\}}$. Then $q_j(\widehat{C_i'}) = p_j(\widehat{C}_i)$ for all $j \leq i$. Thus the polynomials defined in (2.73) are linearly independent similarly to those described in (2.68) and (2.69), (2.67) follows.

The following is a construction of partition critical k-uniform hypergraph of size $\binom{n}{k-1}$. The difference of this and the upper bound (2.67) is of one smaller order of magnitude than the bound itself.

The following proposition is easy exercise.

Proposition 2.5.6 *Let $a \leq b \leq \frac{m}{2}$. There exists a matching from $\binom{[m]}{a}$ to $\binom{[m]}{b}$ so that if $A \in \binom{[m]}{a}$ is matched to $B \in \binom{[m]}{b}$ then $A \subseteq B$.* □

The edge set $\mathcal{E}$ is a disjoint union $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2 \cup \ldots \cup \mathcal{E}_k$ where $\mathcal{E}_i$ is on the underlying set $X_i = \{i, i+1, \ldots, n\}\,(\subset X)$. Let $\mathcal{E}_i$ consist of the k-sets of X_i matched by Proposition2.5.6 to the collection of $k-i+1$-sets of X_i that contain the element i. Thus, $|\mathcal{E}_i| = \binom{n-i}{k-i}$. If $F \in \mathcal{E}_i$, then there exists $i \in G_F \subset X_i$, such that $|G_F| = k - i + 1$ and $G_F \subseteq F$. Let the partition prescribed to F be $F = (G_F \setminus \{i\}) \cup (F \setminus G_F \cup \{i\})$. The partition of the underlying set X that belongs to $F \in \mathcal{E}_i$ is $X = (G_F \setminus \{i\}) \cup (X \setminus G_F \cup \{i\})$. The ordering of edges in $\mathcal{E}$ is that $E \in \mathcal{E}_i$ is before of $F \in \mathcal{E}_j$ if $i < j$, within the same $\mathcal{E}_i$ arbitrary. We claim that $X, \mathcal{E}$ is partition critical with the given partitions.

Let us first consider edges E and F such that $E \in \mathcal{E}_i$ and $F \in \mathcal{E}_j$ with $i < j$. The prescribed partition of E is $E = (G_E \setminus \{i\}) \cup (E \setminus G_E \cup \{i\})$, while the partition of X belonging to F is $X = (G_F \setminus \{j\}) \cup (X \setminus G_F \cup \{j\})$. $k - j = |G_F \setminus \{j\}| < |G_E \setminus \{i\}| = k - i$, hence $(G_F \setminus \{j\}) \cap E \neq G_E \setminus \{i\}$. On the other hand, $i \in (E \setminus G_E \cup \{i\}$ but $i \notin G_F \setminus \{j\})$, thus $E \setminus G_E \cup \{i\} \neq E \cap (G_F \setminus \{j\})$. On the other hand, if E and F belong to the same $\mathcal{E}_i$, then clearly $(G_F \setminus \{i\}) \cap E \neq G_E \setminus \{i\}$ since $G_F \neq G_E$. Furthermore, $E \setminus G_E \cup \{i\} \neq E \cap (G_F \setminus \{i\})$, since i is contained in the left hand side, but not in the right hand side.

Thus, if E is before F in the ordering of the edges, then the partition of X belonging to F does not cut E properly.

The size of $\mathcal{E}$ is

$$|\mathcal{E}| = \sum_{i=1}^{k} |\mathcal{E}_i| = \sum_{i=1}^{k} \binom{n-i}{k-i} = \binom{n}{k-1}. \tag{2.74}$$

It is clear that one cannot add any new edges after the existing ones in the order, since any partition of X cuts some edges in $\mathcal{E}$ in the prescribed way. We conjecture that the construction is extremal, that is the upper bound for a k-uniform partition critical hypergraph $(X, \mathcal{E})$ on the underlying set $X = \{1, 2, \ldots, n\}$ is actually $\binom{n}{k-1}$.

Chapter 3

VC-Dimension of Antichains

3.1 Motivation, introduction

In this chapter we explore a conjecture of Frankl [Fra89], that brings together two classic extremal set theoretical results. One is Theorem 2.1.3, the other one is Sperner's famous theorem [Spe28].

Theorem 3.1.1 (Sperner) *Let $\mathcal{A}$ be an antichain in $2^{[m]}$. Then*

$$|\mathcal{A}| \leq \binom{m}{\lceil \frac{m}{2} \rceil}. \tag{3.1}$$

Equality holds if and only if

$$\mathcal{A} = \binom{[m]}{\lceil \frac{m}{2} \rceil} \text{ or } \binom{[m]}{\lfloor \frac{m}{2} \rfloor}. \tag{3.2}$$

□

We say a family of sets $\mathcal{F} \subseteq 2^{[m]}$ has a *trace* K_k if there is a set $S \subseteq [m]$ with $|S| = k$ so that $|\{F \cap S \mid F \in \mathcal{F}\}| = 2^k$. In this case S is said to be *shattered* by $\mathcal{F}$. The *Vapnik-Chervonenkis dimension (VC-dimension)* of a set system $\mathcal{F}$ is the largest k such that $\mathcal{F}$ has trace K_k, that is the size of the largest set shattered by $\mathcal{F}$. The VC-dimension plays a significant role in machine learning theory, for example "The theory of learning based on the VC-dimension predicts that the behavior of the difference between training error and test error as a function of the training set size is characterized by a single quantity – the VC-dimension – which characterizes the machine's capacity" Vapnik, 1982 see [Vap82].

Frankl posed the following problem [Fra89]

Conjecture 3.1.2 *Let $\mathcal{F} \subseteq 2^{[m]}$ be an antichain with no trace K_k and $m \geq 2k$. Then*

$$|\mathcal{F}| \leq \binom{m}{k-1}. \tag{3.3}$$

As a partial answer, Frankl [Fra89] proved the following which handles the cases $k = 2, 3$.

Theorem 3.1.3 *Let $\mathcal{F} \subseteq 2^{[m]}$ be an antichain with no trace K_k and $m \geq 2k - 2$. Then*

$$|\mathcal{F}| \leq \sum_{0 \leq i < k/3} \binom{m}{k-1-3i}. \tag{3.4}$$

There is a natural correspondence between an $\mathcal{F} \in 2^{[m]}$ and a simple $m \times |\mathcal{F}|$ matrix $A = (a_{ij})$ given by $a_{ij} = 1$ if and only if $i \in j$th set of $\mathcal{F}$. We will shift between these two views of the same object during this chapter. We use the notation $|A|$ to denote the number of columns of A and hence $|\mathcal{F}|$. It is useful to allow a matrix to have 0 columns. The trace idea translates easily and refers to a *configuration* K_k. The matrix notation is often convenient. For example, for any $k \times 1$ $(0,1)$-column α, the matrix of all columns with no submatrix α also achieves the bound (2.1) (Theorem 2.4 [AF86]). There are many other examples achieving equality e.g Frankl [Fra89].

We establish Conjecture 3.1.2 for $k = 4$ in Section 3.2 by establishing that equality is achieved in (3.3) for $k = 2, 3$ and $m \geq 2k - 2$ if and only if

$$\mathcal{F} = \binom{[m]}{k-1} \text{ or } \binom{[m]}{m-k+1}. \tag{3.5}$$

We define

$$\mathsf{sh}(\mathcal{F}) = \{S \subseteq [m] : \mathcal{F} \text{ shatters } S\}. \tag{3.6}$$

We obtain such properties as $\mathsf{sh}(\mathsf{sh}(\mathcal{F})) = \mathsf{sh}(\mathcal{F})$ and that $\mathsf{sh}(\mathcal{F})$ is a *down-set* i.e. if $A \in \mathsf{sh}(\mathcal{F})$ and $B \subseteq A$ then $B \in \mathsf{sh}(\mathcal{F})$. The following result is due to Pajor [Paj85]. The proof could be recovered from Bollobás el at [BLR89]

Theorem 3.1.4 *Let $\mathcal{F}$ be a family of subsets of $[m]$. Then*

$$|\mathsf{sh}(\mathcal{F})| \geq |\mathcal{F}|. \tag{3.7}$$

This immediately yields Theorem 2.1.3.

At one point we thought that perhaps if $|\mathcal{F}|$ is strictly less than $|\mathsf{sh}(\mathcal{F})|$ then you could add to $\mathcal{F}$ new sets without creating new shattered sets. Tom Drummond [Dru], a graduate student at Curtin U. of Tech. offered the following counterexample of 12 subsets of $\{1, 2, 3, 4\}$ in incidence matrix form.

$$\begin{bmatrix} 0&0&0&0&0&0&0&1&1&1&1&1 \\ 0&0&0&1&1&1&1&0&0&0&1&1 \\ 0&1&1&0&0&1&1&0&1&1&0&0 \\ 1&0&1&0&1&0&1&1&0&1&0&1 \end{bmatrix} \tag{3.8}$$

Nonetheless there is a version of shattering that does result in equality in a version of (3).

Definition 3.1.5 *We define the concept of order-shattered in an inductive way on the size of S. This is just the set version of $C(s)$ given in [AA95]. For $S = \emptyset$ all we need is a single set from $\mathcal{F}$. We say that $S = \{s_1, s_2, \ldots, s_k\}$ is order-shattered by $\mathcal{F}$ if, when $s_1 < s_2 < \cdots < s_k$, there are $2^{|S|}$ sets of $\mathcal{F}$ divided into two families $\widetilde{\mathcal{F}_0}, \widetilde{\mathcal{F}_1}$ so that if we define $T = \{s_k+1, s_k+2, \ldots, m\}$ (possibly $T = \emptyset$) we have that*

$$T \cap C = T \cap D \text{ for all } C \in \widetilde{\mathcal{F}_0},\ D \in \widetilde{\mathcal{F}_1}, \tag{3.9}$$

$$\{s_k\} \cap C = \emptyset, \{s_k\} \cap D = \{s_k\} \text{ for all } C \in \widetilde{\mathcal{F}_0},\ D \in \widetilde{\mathcal{F}_1} \tag{3.10}$$

and both $\widetilde{\mathcal{F}_0}, \widetilde{\mathcal{F}_1}$ individually order-shatter $(S - \{s_k\})$.

We define

$$\mathsf{osh}(\mathcal{F}) = \{S \subseteq [m] : \mathcal{F} \text{ order-shatters } S\}. \tag{3.11}$$

We note $\mathsf{osh}(\mathcal{F}) \subseteq \mathsf{sh}(\mathcal{F})$, $\mathsf{osh}(\mathcal{F})$ is a downset, and $\mathsf{osh}(\mathsf{osh}(\mathcal{F})) = \mathsf{osh}(\mathcal{F})$.

Theorem 3.1.6 *Let $\mathcal{F}$ be a family of subsets of $[m]$. Then*

$$|\mathsf{osh}(\mathcal{F})| = |\mathcal{F}|. \tag{3.12}$$

We will give an induction proof of Theorem 3.1.6 in Section 3.3. Lajos Rónyai gave another proof of this theorem based on an algebraic interpretation of $\mathsf{osh}(\mathcal{F})$ [ARS02]. In fact, $\mathsf{osh}(\mathcal{F})$ comes up naturally in the context of Gröbner bases and resulted in a fruitful research area see [HR03b, BRR06, HR03a, FR03, HR06, BHR08].

Note that this equality provides a strengthening of Theorem 2.1.3, since we can weaken the hypothesis on $\mathcal{F}$ to having no order-shattered set of size k. To place the concept of order-shattered in context, consider the paper of Bollobás et al [BLR89]. They define that $S \subseteq [m]$ is *strongly traced* by $\mathcal{F}$ if there exists a $B \subseteq [m] - S$ with

$$\{E \cap S : E \in \mathcal{F},\ E \cap ([m] - S) = B\} = 2^S. \tag{3.13}$$

Define $\mathsf{st}(\mathcal{F})$ to be the family of subsets S which are strongly traced by $\mathcal{F}$. The concept is somehow complementary (dual?) to shattering since

$$2^{[m]} - \mathsf{sh}(\mathcal{F}) = \{[m] - S \,:\, S \in \mathsf{st}(2^{[m]} - \mathcal{F})\}. \tag{3.14}$$

From the definition we see that

$$\mathsf{st}(\mathcal{F}) \subseteq \mathsf{osh}(\mathcal{F}) \subseteq \mathsf{sh}(\mathcal{F}), \tag{3.15}$$

and hence, using (3.12), we have the 'reverse Sauer' inequality stated by Bollobás and Radcliffe ([BR95]) that $|\mathsf{st}(\mathcal{F})| \le |\mathcal{F}|$. Unfortunately if we weaken the hypothesis on $\mathcal{F}$ in Theorem 2.1.3 to having no strongly traced set of size k we cannot obtain the same bound (2.1).

3.2 Sperner families of small VC-dimension

We let $f(m,k,l)$ denote the maximum number of subsets of $[m]$ with no trace K_k and no chain of size $l+1$. This was introduced by Frankl in the following conjecture.

Conjecture 3.2.1 (Frankl [Fra89]) *Assume $m + l \ge 2k$. Then*

$$f(m,k,l) \le \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{k-l}. \tag{3.16}$$

For $l \ge k$ we easily deduce $f(m,k,l)$ from the bound (2.1). The following lemma was found by R. Israel [Isr] which extends the argument in [Fra89].

Lemma 3.2.2

$$f(m,k,l) \le f(m-1,k-1,l-1) + f(m-1,k,2l). \tag{3.17}$$

Proof of Lemma 3.2.2 Let A be a simple matrix on m rows with no configuration K_k and no chain of size $l+1$. Decompose A in the standard way like in (2.6)

$$A = \begin{bmatrix} 00\cdots0 & 11\cdots1 \\ B_1B_2 & B_2B_3 \end{bmatrix} \tag{3.18}$$

where B_2 consists of those columns that are repeated when the first row of A is removed. We see that $[B_1B_2B_3]$ is simple and has no K_k. Also it cannot have a chain of length $2l+1$ since then either $[B_1B_2]$ or $[B_2B_3]$ will have a chain of length $l+1$ contradicting our choice of A. Thus $[B_1B_2B_3]$ has at most $f(m-1,k,2l)$ columns. We see that B_2 has no chain of length l since that yields a chain of length $l+1$ in A. Also B_2 has no K_{k-1} since

$$K_k = \begin{bmatrix} 00\cdots 0 & 11\cdots 1 \\ K_{k-1} & K_{k-1} \end{bmatrix} \tag{3.19}$$

Thus B_2 has at most $f(m-1,k-1,l-1)$ columns and we obtain (3.5). □

Repeating (3.17) we get $f(m,k,1) = f(m-2,k-1,1) + f(m-2,k,4)$ but to prove Conjecture 3.1.2(in view of Conjecture 3.2.1) we would like a 3 in place of the 4. A quick consequence of this lemma proves Conjecture 3.2.1 in one case.

Theorem 3.2.3 *We have*

$$f(m,k,k-1) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{1} \tag{3.20}$$

Moreover if $\mathcal{F} \subseteq 2^{[m]}$ has $|\mathcal{F}| = f(m,k,k-1)$, no trace K_k and no chain of length k then

$$\mathcal{F} = \binom{[m]}{k-1} + \binom{[m]}{k-2} + \cdots + \binom{[m]}{1} \tag{3.21}$$

or the (0,1)-complement.

Proof of Theorem 3.2.3 Let A be the matrix associated with $\mathcal{F}$. Use the standard decomposition of (2.6) and note that the argument of Lemma 3.2.2 yields

$$|B_2| \le \binom{m-1}{k-2} + \binom{m-1}{k-3} + \cdots + \binom{m-1}{1}, \tag{3.22}$$

$$|[B_1B_2B_3]| \le \binom{m-1}{k-1} + \binom{m-1}{k-2} + \cdots + \binom{m-1}{0}. \tag{3.23}$$

Thus the bound (3.20) follows and so an extremal family has equality in (3.22,3.23). We may assume by induction on k and without loss of generality that

$$B_2 = \binom{[m-1]}{k-2} + \binom{[m-1]}{k-3} + \cdots + \binom{[m-1]}{1}. \tag{3.24}$$

But then B_3 has at most one column, the column of all 0's since a column with between 1 and $k-2$ 1's violates simplicity of A and any column of at least $k-1$ 1's yields a chain of size $k+1$ in A ($k-1$ from B_2 under 0's and 1 from B_2 under 1's and 1 from B_3). By induction on m

$$|[B_1B_2]| \le \binom{m-1}{k-1} + \binom{m-1}{k-2} + \cdots + \binom{m-1}{1}, \tag{3.25}$$

and by equality in (3.23) we obtain $|B_3| = 1$ and equality in (3.25). Thus by induction, either

$$[B_1B_2] = \binom{[m-1]}{k-1} + \binom{[m-1]}{k-2} + \cdots + \binom{[m-1]}{1} \tag{3.26}$$

or the (0,1)-complement. The latter is impossible so $B_1 = \binom{[m-1]}{k-1}$ and $B_3 = \binom{[m-1]}{0}$ and so A has the desired form. □

Later in this section we see that the case of equality for $f(m,k,k)$ behaves entirely differently.

Theorem 3.2.4 *Let $\mathcal{F} \subseteq 2^{[m]}$ be an antichain with no trace K_2. Then*

$$|\mathcal{F}| \leq m \tag{3.27}$$

with equality if and only if

$$\mathcal{F} = \binom{[m]}{1} \text{ or } \binom{[m]}{m-1} \tag{3.28}$$

Proof of Theorem 3.2.4 The result is true for $m = 2, 3$ using Theorem 3.1.1 alone. Now repeat the standard decomposition (2.6) for the second row but using the fact you have an antichain so that for example, $B_2 = \emptyset$. Then

$$A = \begin{bmatrix} 00\cdots0 & 00\cdots0 & 11\cdots1 & 11\cdots1 \\ 00\cdots0 & 11\cdots1 & 00\cdots0 & 11\cdots1 \\ C_1 & C_2C_3 & C_3C_4 & C_5 \end{bmatrix} \tag{3.29}$$

Now $[C_1C_2C_3C_4C_5]$ is simple and has no K_2 and C_3 is simple, an antichain and has no K_1. Thus by induction

$$|C_1C_2C_3C_4C_5| \leq \binom{m-2}{1} + \binom{m-2}{0} = m-1, \tag{3.30}$$

$$|C_3| \leq 1. \tag{3.31}$$

This yields the bound of (3.27) with equality if and only if we have equalities in both (3.30) and (3.31), so in particular $|C_3| = 1$. Thus to avoid K_2 in the first two rows we must have $C_1 = \emptyset$ or $C_5 = \emptyset$. Assume, without loss of generality, that $C_5 = \emptyset$. We may decompose A as follows

$$A = \begin{bmatrix} 00\cdots0 & 00\cdots0 & 11\cdots1 \\ 00\cdots0 & 11\cdots1 & 00\cdots0 \\ D & \alpha\alpha\cdots\alpha & \beta\beta\cdots\beta \\ \gamma\gamma\cdots\gamma & E & \gamma\gamma\cdots\gamma \\ \delta\delta\cdots\delta & \epsilon\epsilon\cdots\epsilon & F \end{bmatrix}. \tag{3.32}$$

We have chosen D to consist of the rows of C_1 containing a [01] (which forces α and β in order to avoid a K_2) and then we choose E similarly and F is in the remaining rows. It is obtained from $|C_3| = 1$, that $\alpha = \beta$.

We allow matrices to have 0 rows in this decomposition where D is $l_1 \times k_1$, E is $l_2 \times k_2$, F is $l_3 \times k_3$ and each are simple, form an antichain and have no K_2. We note that $l_1 + l_2 + l_3 = m - 2$ and $k_1 + k_2 + k_3 = m$. Using induction on (3.27), we deduce two of D, E, F are 0×1(!) and the remaining one is $(m-2) \times (m-2)$.

Consider the case where F is $(m-2) \times (m-2)$, hence by (3.28) each row of F has a [01] and so $\delta = \epsilon$. Then the two columns not from F form a chain in A, a contradiction. The same happens with E, so assume D is $(m-2) \times (m-2)$. Using induction and (3.28) we have that $D = \binom{[m-2]}{1}$ or $\binom{[m-2]}{m-3}$. In the former case we quickly deduce that α (and β) is all 0's to avoid a chain of two columns and so (3.28) holds for A. In the latter case we can assume $m - 2 \geq 3$ and so to avoid having a K_2 the vector α cannot have two 0's but then a chain of two columns will be formed, a contradiction. $\square$

Theorem 3.2.5 *Let $\mathcal{F} \subseteq 2^{[m]}$ be an antichain with no trace K_3. Assume $m \geq 3$. Then*

$$|\mathcal{F}| \leq \binom{m}{2} \tag{3.33}$$

with equality if and only if

$$\mathcal{F} = \binom{[m]}{2} \text{ or } \binom{[m]}{m-2} \tag{3.34}$$

Proof of Theorem 3.2.5 The result is true for $m = 3, 4, 5$ simply using Theorem 3.1.1. For $m \geq 6$ use the decomposition of (3.29). Now $[C_1C_2C_3C_4C_5]$ is simple and has no K_3. Also C_3 is simple, an antichain, and has no K_2. Thus by induction

$$|C_1C_2C_3C_4C_5| \leq \binom{m-2}{2} + \binom{m-2}{1} + \binom{m-2}{0}, \tag{3.35}$$

$$|C_3| \leq \binom{m-2}{1}. \tag{3.36}$$

This immediately yields the bound of (3.33) with equality if and only if we have equality in (3.35) and (3.36). By Theorem 3.2.4, C_3 is $\binom{[m-2]}{1}$ or the (0,1)-complement. Without loss of generality assume the former. Then because A is an antichain, $C_2, C_4 = \emptyset$. Also C_5 is either a column of 0's or is empty. Now C_1 is simple, an antichain with no K_3, and so by induction

$|C_1| \leq \binom{m-2}{2}$. In order to have equality in (3.35) we deduce $|C_1| = \binom{m-2}{2}$ and $|C_5| = 1$ so either $C_1 = \binom{[m-2]}{2}$ or the complement and so C_5 is a column of 0's. Thus $C_1 = \binom{[m-2]}{2}$, the case $C_1 = \binom{[m-2]}{m-4}$ yielding a K_3 in rows 3,4,5 for $m - 4 > 2$. □

Theorem 3.2.6 *Let $\mathcal{F} \subseteq 2^{[m]}$ be an antichain with no trace K_4. Assume $m \geq 5$. Then*

$$|\mathcal{F}| \leq \binom{m}{3} \tag{3.37}$$

Proof of Theorem 3.2.6 The result is true for $m = 5, 6, 7$ using Theorem 3.1.1 alone. Assume $m \geq 8$ and let A be the matrix associated with $\mathcal{F}$. Use the decomposition of (3.29). Now $[C_1C_2C_3C_4C_5]$ is simple and has no K_3. Also C_3 is simple, an antichain, and has no K_2. Thus by induction

$$|C_1C_2C_3C_4C_5| \leq \binom{m-2}{3} + \binom{m-2}{2} + \binom{m-2}{1} + \binom{m-2}{0}, \tag{3.38}$$

$$|C_3| \leq \binom{m-2}{2}. \tag{3.39}$$

If the inequality in (3.39) is strict then the inequality (3.37) holds as desired. So assume (3.39) holds with equality. Then by Theorem 3.2.5, without loss of generality we may assume $C_3 = \binom{[m]}{2}$. Since A is an antichain, $C_2 = \emptyset$ and $C_4 = \emptyset$. Also C_5 is an antichain contained in $\binom{[m-2]}{1} \cup \binom{[m-2]}{0}$ and hence $|C_5| \leq \binom{m}{1}$. Similarly C_1 is an antichain with no K_4 and so by induction $|C_1| \leq \binom{m-2}{3}$. But now (3.37) follows. □

3.2.1 A Surprising Construction.

In contrast to the previous results which tend to support the conjecture, the following result perhaps casts doubt on the conjecture.

Theorem 3.2.7 *Let m, k, t be given satisfying $t + k - 1 \leq m$ and $k \geq 1$. There exists an $m \times f(m, k, k)$ simple matrix $A(m, k, t)$ with no configuration K_k and column sums $t, t+1, t+2, \ldots, t+k-1$ (hence no chain of size $k+1$).*

Proof of Theorem3.2.7 We use induction on m with the additional hypothesis for $k \geq 2, t \geq 1$ that

$$A(m, k-1, t) \subseteq A(m, k, t-1). \tag{3.40}$$

We use the base cases:

$$A(m,k,0) = \binom{[m]}{k-1} \cup \binom{[m]}{k-2} \cup \cdots \cup \binom{[m]}{0} \text{ for } k \geq 2 \tag{3.41}$$

$$A(m,1,t) = \begin{bmatrix} \vec{1}_t \\ \vec{0}_{m-t} \end{bmatrix}, \tag{3.42}$$

where $\vec{a}_p$ is the vector of p a's. We can easily handle the case $m = 1$ since either $k = 1$ or $t = 0$. The inductive construction for $k \geq 2$ and $t \geq 1$ is

$$A(m,k,t) = \begin{bmatrix} 11\cdots1 & 00\cdots0 \\ A(m-1,k,t-1) & A(m-1,k-1,t) \end{bmatrix} \tag{3.43}$$

Note that $k+(t-1) = (k-1)+t \leq m$. Now consider $m > 1$. By construction (3.43) $A(m,k,t)$ has $f(m-1,k,k) + f(m-1,k-1,k-1) = f(m,k,k)$ columns and is simple with column sums $t, t+1, \ldots, t+k-1$. If $A(m,k,t)$ has a K_k using the first row then it must have a K_{k-1} under the 0's in $A(m-1,k-1,t)$, a contradiction. If $A(m,k,t)$ has a K_k not using the first row then, since $A(m-1,k-1,t) \subseteq A(m-1,k,t-1)$, we must have a K_k in $A(m-1,k,t-1)$, a contradiction.

To verify (3.40), first consider $t = 1$. Then (3.41) ensures that $A(m,k-1,1) \subseteq A(m,k,0)$. For $k = 2$ note that

$$A(m,2,t) = \begin{bmatrix} 11\cdots1 & 00\cdots0 \\ A(m-1,2,t-1) & A(m-1,1,t) \end{bmatrix} \tag{3.44}$$

$$\text{and } A(m,1,t+1) = \begin{bmatrix} 11\cdots1 \\ A(m-1,1,t) \end{bmatrix} \tag{3.45}$$

We use induction to obtain $A(m-1,1,t) \subseteq A(m-1,2,t-1)$, then (3.40) follows for $k = 2$. For $k > 2$ and $t > 1$ we use the inductive constructions for $A(m,k-1,t+1), A(m,k,t)$. By induction, $A(m-1,k-1,t) \subseteq A(m-1,k,t-1)$ and $A(m-1,k-2,t+1) \subseteq A(m-1,k-1,t)$ so that $A(m,k-1,t+1) \subseteq A(m,k,t)$. □

This parallels the inductive construction of Anstee and Murty [AM85] for matrices of the same size with no trace $\binom{[k]}{t}$. Its interest is in showing that matrices achieving the bound for $f(m,k,k)$ are hardly unique which is in considerable contrast to Theorems 3.1.1,3.2.4,3.2.5.

An alternative construction follows from the tower

$$A(p,1,k) \subseteq A(p,2,k-1) \subseteq \cdots \subseteq A(p,k,1) \tag{3.46}$$

with $p = m-k-t$. Taking the (0,1)-complement of the tower with p replaced by $k+t$ we get the tower

$$B(k+t,1,t) \subseteq B(k+t,2,t) \subseteq \cdots \subseteq B(k+t,k,t) \tag{3.47}$$

Let K_k^l denote the matrix of $\binom{[k]}{l}$ and let $K_k^{\geq l}$ denote the matrix of $\cup_{i=l}^k \binom{[k]}{i}$. We obtain a new construction achieving equality for $f(m, 2k+1, 2k+1)$ where $p = m-k-t$ as follows:

$$\left[\begin{array}{cccccccccc} K_{k+t}^{\geq t} & K_{k+t}^{t-1} & K_{k+t}^{t-2} & & K_{k+t}^{t-k} & B(k+t,k,t) & B(k+t,k+1,t) & & B(k+t,1,t) \\ \times & \times & \times & \cdots & \times & \times & \times & \cdots & \times \\ K_p^{\leq k} & A(p,k,1) & A(p,k-1,2) & & A(p,1,k) & K_p^{k+1} & K_p^{k+2} & & K_p^{2k} \end{array}\right] \tag{3.48}$$

Here we use the cross product construction defined in Definition 2.2.1. It is easy to verify that the matrix is simple, has no K_{2k+1} and column sums $t, t+1, \ldots, t+2k$. The number of columns is

$$\begin{array}{c} \left(\sum_{i=0}^k \binom{k+t}{i}\right)\left(\sum_{i=0}^k \binom{m-k-t}{i}\right) + \sum_{i=1}^k \binom{k+t}{k+i}\left(\sum_{j=0}^{k-i}\binom{m-k-t}{j}\right) \\ + \sum_{i=1}^k \binom{m-k-t}{k+i}\left(\sum_{j=0}^{k-i}\binom{k+t}{j}\right) = \sum_{i=0}^{2k}\binom{m}{i} = f(m,2k+1,2k+1). \end{array} \tag{3.49}$$

3.3 Order shattering

Let us first give a proof of the most remarkable property of osh.

Proof of Theorem 3.1.6 Use induction on m and secondarily on $|\mathcal{F}|$, the result being easy for $m = 1$ as long as you remember the empty set. Now consider a family $\mathcal{F}$ of subsets of $[m]$. If no member of $\mathcal{F}$ contains m, then $\mathcal{F} \subset 2^{[m-1]}$ and use induction. If all elements of $\mathcal{F}$ contain m then if we define $\mathcal{F}' = \{E - m : E \in \mathcal{F}\}$, we find $\mathsf{osh}(\mathcal{F}) = \mathsf{osh}(\mathcal{F}')$ and can use induction on m. Otherwise split $\mathcal{F}$ into two new families of subsets of $[m]$:

$$\mathcal{F}_0' = \{E : E \in \mathcal{F},\ m \notin \mathcal{F}\}, \tag{3.50}$$

$$\mathcal{F}_1' = \{E : E \in \mathcal{F},\ m \in \mathcal{F}\}. \tag{3.51}$$

We note that $\mathsf{osh}(\mathcal{F}_0') \cup \mathsf{osh}(\mathcal{F}_1') \subseteq \mathsf{osh}(\mathcal{F}) \cap 2^{[m-1]}$ and deduce from (3.9),(3.10) that $\mathsf{osh}(\mathcal{F})$ is the disjoint union of $\mathsf{osh}(\mathcal{F}_0') \cup \mathsf{osh}(\mathcal{F}_1')$ and $\{S \cup \{m\} : S \in \mathsf{osh}(\mathcal{F}_0') \cap \mathsf{osh}(\mathcal{F}_1')\}$. By induction ($|\mathcal{F}_0'| < |\mathcal{F}|, |\mathcal{F}_1'| < |\mathcal{F}|$) $|\mathsf{osh}(\mathcal{F}_0')| = |\mathcal{F}_0'|$ and $|\mathsf{osh}(\mathcal{F}_1')| = |\mathcal{F}_1'|$. Hence

$$|\mathsf{osh}(\mathcal{F})| = |\mathsf{osh}(\mathcal{F}_0') \cup \mathsf{osh}(\mathcal{F}_1')| + |\mathsf{osh}(\mathcal{F}_0') \cap \mathsf{osh}(\mathcal{F}_1')| \tag{3.52}$$

$$= |\mathsf{osh}(\mathcal{F}_0')| + |\mathsf{osh}(\mathcal{F}_1')| = |\mathcal{F}_0'| + |\mathcal{F}_1'| = |\mathcal{F}| \tag{3.53}$$

□

Reordering the elements of $[m]$ does not affect $\mathsf{sh}(\mathcal{F})$ or $\mathsf{st}(\mathcal{F})$(except in the trivial way) but in general could affect $\mathsf{osh}(\mathcal{F})$ (and almost certainly the columns which order-shatter a given set). Cases where $|\mathsf{sh}(\mathcal{F})| = |\mathcal{F}|$ (or $|\mathsf{st}(\mathcal{F})| = |\mathcal{F}|$) and hence $\mathsf{osh}(\mathcal{F}) = \mathsf{sh}(\mathcal{F})$ (from (3.15)) have interest in that $\mathsf{osh}(\mathcal{F})$ is preserved (up to the reordering).

A consequence of the equality in (3.12) is that for $E \in \mathcal{F}$ there is a unique set $S(E)$ so that $S(E) \in \mathsf{osh}(\mathcal{F}) - \mathsf{osh}(\mathcal{F} - E)$. Any reasonable converse is not true, namely one can construct examples where there are maximal order-shattered sets S of $\mathcal{F}$ for whom the deletion of any single set from $\mathcal{F}$ leaves S order-shattered. But if we choose a special maximal element of $\mathsf{osh}(\mathcal{F})$ we obtain a result. Define a subset S of $\mathsf{osh}(\mathcal{F})$ in a family to be *lexically largest* if the function $f(S) = \sum_{i\in S} 2^{m-i}$ is largest among all sets in $\mathsf{osh}(\mathcal{F})$.

Proposition 3.3.1 *Let S be the lexically largest set in $\mathsf{osh}(\mathcal{F})$. Then there is a unique subset of $\mathcal{F}$ (of cardinality $2^{|S|}$) which order-shatters S.*

Proof of Proposition 3.3.1 Let $S = \{s_1, s_2, \ldots, s_t\}$. Assume there are two subsets $\mathcal{F}', \mathcal{F}'' \subseteq \mathcal{F}$ each of size $2^{|S|}$ and each order-shattering S. Let $D' = E' \cap \{s_t+1, s_t+2, \ldots, m\}$ for any $E' \in \mathcal{F}'$ and let $D'' = E'' \cap \{s_t+1, s_t+2, \ldots, m\}$ for any $E'' \in \mathcal{F}''$. If $D' \neq D''$ then let $i \in (D' - D'') \cup (D'' - D')$ yet $j \cap D' = j \cap D''$ for $j > i$. But then $\mathcal{F}' \cap \mathcal{F}'' = \emptyset$ and so $S \cup i \in \mathsf{osh}(\mathcal{F})$, a contradiction.

If $s_t = m$ or $D' = D''$, then let $\mathcal{F}_0' = \{E \in \mathcal{F}' : s_t \notin E\}$ and $\mathcal{F}_0'' = \{E \in \mathcal{F}'' : s_t \notin E\}$. Let $T = S - s_t$. Then $T \in \mathsf{osh}(\mathcal{F}_0')$ and $T \in \mathsf{osh}(\mathcal{F}_0'')$. By the choice of S, for any S' satisfying $T \subseteq S' \subseteq \{1, 2, \ldots, s_t - 1\}$ we must have $S' = T$ or $S' \notin \mathsf{osh}(\mathcal{F})$. Thus T is lexically largest in $\mathsf{osh}(\mathcal{F}_0' \cup \mathcal{F}_0'')$. But then $\mathcal{F}_0' = \mathcal{F}_0''$. Similarly $\mathcal{F}_1' = \mathcal{F}_1''$ and so $\mathcal{F}' = \mathcal{F}''$. □

3.3.1 Uniform Families

In general, an ℓ-uniform set system can shatter any subset of $[m]$ of size at most the ℓ. Remarkably, many subsets of $[m]$ cannot be order shattered, and we use this to reprove and strengthen the following result.

Theorem 3.3.2 *[Frankl,Pach[FP84]]Let $\mathcal{F}$ be a uniform system with no shattered set of size k. Then*

$$|\mathcal{F}| \leq \binom{m}{k-1}. \tag{3.54}$$

We provide a proof that weakens the hypothesis to no order shattered set of size k.

Lemma 3.3.3 *Let i, l, m be given. Let S be a set with*

$$|\{j \in S : j \leq 2i-1\}| \geq i. \tag{3.55}$$

Then $S \notin \mathsf{osh}(\binom{[m]}{l})$ for any l.

Proof of Lemma 3.3.3 We may assume $S \subseteq [2i-1]$ since otherwise we may replace S by $S \cap [2i-1]$ and if $S \cap [2i-1] \notin \mathsf{osh}(\binom{[m]}{l})$ then $S \notin \mathsf{osh}(\binom{[m]}{l})$. Assume there is an $\mathcal{F} \subseteq \binom{[m]}{l}$, $|\mathcal{F}| = 2^{|S|}$ with $S \in \mathsf{osh}(\mathcal{F})$. Let A, B be the two elements of $\mathcal{F}$ satisfying

$$A \cap S = S, \quad B \cap S = \emptyset \tag{3.56}$$

Now $|S| \geq i$ and so $|A \cap [2i-1]| \geq i$ and $|B \cap [2i-1]| \leq i-1$. By order shattering, $|A \cap ([m]-[2i-1])| = |B \cap ([m]-[2i-1])|$. But then $|A| > |B|$, a contradiction. □

By Lemma 3.3.3, any set $S \in \mathsf{osh}(\binom{[m]}{l})$ has the property

$$|\{j \in S : j \leq 2i-1\}| \leq i-1 \tag{3.57}$$

for $i = 1, 2, \ldots, m/2$. We call (3.57) the ballot property in that a sequence $a_1 a_2 \ldots a_m$ of 1's and -1's has the ballot sequence property if every partial sum $a_1 + a_2 + \cdots + a_t \geq 0$ for $t \in [m]$. If we let $f(p, q)$ be the number of ballot sequences of p 1's and q -1's then from, for example Riordan [Rio68],

$$f(p,q) = \frac{p+1-q}{p+1}\binom{p+q}{q} = \binom{p+q}{q} - \binom{p+q}{q-1}. \tag{3.58}$$

Lemma 3.3.4 *The number of sets $S \in \mathsf{osh}(\binom{[m]}{l})$ of size at most $k-1$ ($k \leq l$) is at most $\binom{m}{k-1}$*

Proof of Lemma 3.3.4 Using (3.58) we compute

$$\begin{aligned} f(m-k+1, k-1) + f(m-k+2, k-2) + \cdots + f(m, 0) = \\ \left[\binom{m}{k-1} - \binom{m}{k-2}\right] + \left[\binom{m}{k-2} - \binom{m}{k-3}\right] + \\ \cdots + \left[\binom{m}{1} - \binom{m}{0}\right] + \binom{m}{0} \quad = \quad \binom{m}{k-1} \end{aligned} \tag{3.59}$$

□

In fact we get equality by considering the family $\mathcal{F} = \binom{[m]}{k-1}$ which has $\mathsf{osh}(\mathcal{F}) \subseteq \binom{[m]}{k-1} \cup \binom{[m]}{k-2} \cup \cdots \cup \binom{[m]}{0}$, since $\mathcal{F}$ does not shatter a k-set and yet $\binom{m}{k-1} = |\mathcal{F}| = |\mathsf{osh}(\mathcal{F})|$. This family also shows Theorem 3.3.2 is best possible.

Proof of Theorem 3.3.2 Since $\mathcal{F}$ shatters no k-set, then $\mathsf{osh}(\mathcal{F}) \subseteq \binom{[m]}{k-1} \cup \binom{[m]}{k-2} \cup \cdots \cup \binom{[m]}{0}$. By Lemma 3.3.4, $|\mathsf{osh}(\mathcal{F})| \leq \binom{m}{k-1}$. We now apply our equality (3.12) to obtain (3.54). □

3.3.2 Antichains

Conjecture 3.1.2 of Frankl [Fra89] attacks the case where $\mathcal{F}$ is not uniform but merely required to be an antichain. Frankl conjectured the bound on the size of an antichain (in $2^{[m]}$) which does not shatter a k-set to be the same as in Theorem 3.3.2. We are able to find the following simple characterization of those sets which can be order shattered by an antichain.

Theorem 3.3.5 *Let $S = \{s_1, s_2, \ldots, s_k\}$ with $s_1 < s_2 < \cdots < s_k$. Then there in an antichain $\mathcal{A}$ with $S \in \mathsf{osh}(\mathcal{A})$ if and only if*

$$f(S) = \sum_{i=1}^{k} \frac{1}{2^{s_i - i}} < 1. \tag{3.60}$$

In particular, the set $\{3, 4, 5\}$ is an example of a set which can be order shattered by an antichain but not by a uniform family. As a result, the bounds we could obtain by this analysis will fall short of Conjecture 3.1.2 but the characterization is remarkably simple and perhaps the result will find application in an attack on Frankl's conjecture. Some notation will be needed for our constructions. Let U, V be two sets with $U \cap V = \emptyset$. Now for two families of subsets $\mathcal{A} \subseteq 2^U, \mathcal{B} \subseteq 2^V$ we define

$$\mathcal{A} \times \mathcal{B} = \{A \cup B : A \in \mathcal{A}, B \in \mathcal{B}\} \tag{3.61}$$

We will also use the notation for a family consisting of a single set $A \subseteq U$ as

$$A \times \mathcal{B} = \{A \cup B : B \in \mathcal{B}\} \tag{3.62}$$

and we will use this sometimes in cases where $A = \emptyset$. A stricter rule would be to write $\{\emptyset\}$ as a family of sets whose only element is $\emptyset$ but this seems unnecessary. We will provide the proof from the following series of Lemmas.

Lemma 3.3.6 *Let $1 \leq s_1 < s_2 < \cdots < s_k$ be integers and $l \geq 2$. If there is an antichain $\mathcal{A}$ with $S = \{s_1, s_2, \ldots, s_{k-1}, s_k + 1, s_k + 2, \ldots, s_k + l\} \in \mathsf{osh}(\mathcal{A})$ then there is an antichain $\mathcal{A}'$ with $S' = \{s_1, s_2, \ldots, s_{k-1}, s_k, s_k + 2, s_k + 3, \ldots, s_k + 2\lceil (l+1)/2 \rceil - 2\} \in \mathsf{osh}(\mathcal{A}')$.*

Proof of Lemma 3.3.6 Assume $|\mathcal{A}| = 2^{k-1+l}$ and $\mathcal{A} \subseteq 2^{[s_k+l]}$. Since $S \in \mathsf{osh}(\mathcal{A})$, we can partition $\mathcal{A}$ into 2^l families $\mathcal{A}(C_i)$, one for each $C_i \subseteq \{s_k+1, s_k+2, \ldots, s_k+l\}$, each of the form

$$\mathcal{A}(C_i) = \mathcal{A}_i \times B_i \times C_i \tag{3.63}$$

where $\mathcal{A}_i \subseteq 2^{[s_k-1]}$ and $\{s_1, s_2, \ldots, s_{k-1}\} \in \mathsf{osh}(\mathcal{A}_i)$, $B_i \subseteq \{s_k\}$.

We take any tower of length $l+1$ of the C_i's e.g. $\emptyset \subset \{s_k+1\} \subset \{s_k+1, s_k+2\} \subset \cdots \subset \{s_k+1, s_k+2, \ldots, s_k+l\}$. We deduce that at least

$$p = \left\lceil \frac{l+1}{2} \right\rceil \tag{3.64}$$

have the same set B_i (i.e. p indices in which $B_i = \emptyset$ or p indices in which $B_i = \{s_k\}$). Let the indexing of the sets be reordered so we obtain

$$C_1 \subset C_2 \subset C_3 \cdots \subset C_p \quad \text{and} \quad B_1 = B_2 = \cdots = B_p \tag{3.65}$$

We now use the fact $\mathcal{A}$ is an antichain to deduce that for any pair i, j with $1 \le i < j \le p$, and for any set $D \in \mathcal{A}_i$ and any set $E \in \mathcal{A}_j$ we must have $D \setminus E \neq \emptyset$.

Let $T = \{s_k+2, s_k+3, \ldots, s_k+2p-2\}$. We now form the desired antichain $\mathcal{A}'$ consisting of all the following subsets of $[s_k+2p-2]$.

$$\begin{array}{l}
\left(\mathcal{A}_1 \times \emptyset \times \{s_k+1\} \times \binom{T}{0}\right) \\
\bigcup \left(\mathcal{A}_2 \times \{s_k\} \times \{s_k+1\} \times \binom{T}{0}\right) \\
\bigcup \left(\mathcal{A}_2 \times \emptyset \times \{s_k+1\} \times \binom{T}{1}\right) \\
\bigcup \left(\mathcal{A}_3 \times \{s_k\} \times \{s_k+1\} \times \binom{T}{1}\right) \\
\vdots \\
\bigcup \left(\mathcal{A}_{p-1} \times \emptyset \times \{s_k+1\} \times \binom{T}{p-2}\right) \\
\bigcup \left(\mathcal{A}_p \times \{s_k\} \times \{s_k+1\} \times \binom{T}{p-2}\right) \\
\text{and also} \\
\left(\mathcal{A}_1 \times \emptyset \times \emptyset \times \binom{T}{p-1}\right) \\
\bigcup \left(\mathcal{A}_2 \times \{s_k\} \times \emptyset \times \binom{T}{p-1}\right) \\
\bigcup \left(\mathcal{A}_2 \times \emptyset \times \emptyset \times \binom{T}{p}\right) \\
\bigcup \left(\mathcal{A}_3 \times \{s_k\} \times \emptyset \times \binom{T}{p}\right) \\
\vdots \\
\bigcup \left(\mathcal{A}_{p-1} \times \emptyset \times \emptyset \times \binom{T}{2p-3}\right) \\
\bigcup \left(\mathcal{A}_p \times \{s_k\} \times \emptyset \times \binom{T}{2p-3}\right)
\end{array} \tag{3.66}$$

Thus for $0 \le i \le p-2$, we have the sets $\mathcal{A}_{i+1} \times \emptyset \times \{s_k+1\} \times \binom{T}{i}$ and $\mathcal{A}_{i+2} \times \{s_k\} \times \{s_k+1\} \times \binom{T}{i}$ and for $p-1 \le i \le 2p-3$, we have the sets $\mathcal{A}_{i-p+2} \times \emptyset \times \emptyset \times \binom{T}{i}$ and $\mathcal{A}_{i-p+3} \times \{s_k\} \times \emptyset \times \binom{T}{i}$. Thus $S' \in \mathsf{osh}(\mathcal{A}')$. Some careful analysis verifies that $\mathcal{A}'$ is an antichain. $\square$

Corollary 3.3.7 *Let $1 \le s_1 < s_2 < \cdots < s_k$ be integers and $l \ge 2$. If there is an antichain $\mathcal{A}$ with $S = \{s_1, s_2, \ldots, s_{k-1}, s_k+1, s_k+2, \ldots, s_k+l\} \in \mathsf{osh}(\mathcal{A})$ then there is an antichain $\mathcal{A}'$ with $S' = \{s_1, s_2, \ldots, s_{k-1}, s_k, s_k+1, s_k+2, \ldots, s_k+\lceil (l+1)/2 \rceil - 2\} \in \mathsf{osh}(\mathcal{A}')$.*

Proof of Corollary 3.3.7 Apply Lemma 3.3.6 repeatedly noting that the final segment of l consecutive numbers is reduced to $l-2$ if l is odd (where $l-2$ is odd) and to $l-1$ if l is even (where $l-1$ is odd). The last step is to note that for $l=1$, i.e., the final segment of consecutive elements is of length 1, we can conclude that if $S = \{s_1, s_2, \ldots, s_{j-1}, s_j+1\} \in \mathsf{osh}(\mathcal{A})$ then $\{s_1, s_2, \ldots, s_{j-1}\} \in \mathsf{osh}(\mathcal{A})$, where $s_{j-1} = s_k + \lceil (l+1)/2 \rceil - 2$. $\square$

Corollary 3.3.8 *Let $1 \le s_1 < s_2 < \cdots < s_k$ be integers and $l \ge 2^g$. If there is an antichain $\mathcal{A}$ with $S = \{s_1, s_2, \ldots, s_{k-1}, s_k+g, s_k+g+1, \ldots, s_k+g+l-1\} \in \mathsf{osh}(\mathcal{A})$ then there is an antichain $\mathcal{A}'$ with $S' = \{s_1, s_2, \ldots, s_{k-1}, s_k, s_k+1, s_k+2, \ldots, s_k+\lceil (l+1)/2^g \rceil - 2\} \in \mathsf{osh}(\mathcal{A}')$.*

Proof of Corollary 3.3.8 Apply Corollary 3.3.7 g times. Use the fact that $\lfloor \lfloor p/q \rfloor /2 \rfloor = \lfloor p/(2q) \rfloor$ and that $\lceil (\lceil (l+1)/2^k \rceil - 1 + 1)/2 \rceil - 1 = \lceil (l+1)/2^{k+1} \rceil - 1$. $\square$

Lemma 3.3.9 *Let $1 \le s_1 < s_2 < \cdots < s_k$ be integers and $l \ge 1$. If there is an antichain $\mathcal{A}$ with $S = \{s_1, s_2, \ldots, s_{k-1}, s_k, s_k+g+1, \ldots, s_k+g+l\} \in \mathsf{osh}(\mathcal{A})$ then there is an antichain $\mathcal{A}'$ with $S' = \{s_1, s_2, \ldots, s_{k-1}, s_k+g, s_k+g+1, \ldots, s_k+g+2^g+\lceil (l+1)/2^g \rceil 2^g - 2\} \in \mathsf{osh}(\mathcal{A}')$.*

Proof of Lemma 3.3.9 Assume $|\mathcal{A}| = 2^{k+l}$ and $\mathcal{A} \subseteq 2^{[s_k+g+l]}$. Since $S \in \mathsf{osh}(\mathcal{A})$, we can partition $\mathcal{A}$ into 2^l families $\mathcal{A}(C_i)$, one for each $C_i \subseteq \{s_k+g+1, s_k+2, \ldots, s_k+g+l\}$, each of the form

$$\mathcal{A}(C_i) = (\mathcal{A}_{i,\emptyset} \times \emptyset \times B_i \times C_i) \cup \left(\mathcal{A}_{i,\{s_k\}} \times \{s_k\} \times B_i \times C_i\right). \tag{3.67}$$

where $\mathcal{A}_{i,D} \subseteq 2^{[s_k-1]}$ and $\{s_1, s_2, \ldots, s_{k-1}\} \in \mathsf{osh}(\mathcal{A}_{i,D})$ for $D = \emptyset$ or $\{s_k\}$ and $B_i \subseteq \{s_k+1, s_k+2, \ldots, s_k+g\}$. As before, there is a tower of the C_i's of length $l+1$, say $C_1 \subset C_2 \subset \cdots C_{l+1}$, of which $\lceil (l+1)/2^g \rceil$ will have identical B_j's since there are only 2^g choices for B_i. Reorder the indices so that

$$C_1 \subset C_2 \subset \cdots \subset C_{\lceil (l+1)/2^g \rceil} \qquad B_1 = B_2 = \cdots = B_{\lceil (l+1)/2^g \rceil}. \tag{3.68}$$

Now using $p = \lceil (l+1)/2^g \rceil + 1$ we define $\mathcal{F}_i = \mathcal{A}_{i,\emptyset}$ for $i = 1, 2, \ldots, p-1$ and then define $\mathcal{F}_p = \mathcal{A}_{p-1,\{s_k\}}$. Thus $\{s_1, s_2, \ldots, s_{k-1}\} \in \mathsf{osh}(\mathcal{F}_i)$ for $i = 1, 2, \ldots, p$ and moreover for $1 \leq i < j \leq p$, we have, because $\mathcal{A}$ is an antichain, that for $E \in \mathcal{F}_i$ and $F \in \mathcal{F}_j$ we must have $E \setminus F \neq \emptyset$.

Choose an ordering of the 2^g subsets of $\{s_k, s_k+1, \ldots s_k+g-1\}$ as $G_0, G_1, \ldots, G_{2^g-1}$ so that for $0 \leq k < j \leq 2^g - 1$ we have $G_k \setminus G_j \neq \emptyset$ (i.e. order so $|G_0| \geq |G_1| \geq \ldots \geq |G_{2^g-1}|$). Let $T = \{s_k+g, s_k+g+1, \ldots, s_k+g+p2^g-2\}$ Our new antichain is

$$\bigcup_{j=0}^{2^g-1} \bigcup_{i=1}^{p} \left\{ \mathcal{F}_i \times G_j \times \binom{T}{pj+i-1} \right\} \tag{3.69}$$

Some case checking is required to see that it is an antichain and order shatters S'. □

Lemma 3.3.10 *Let $1 \leq s_1 < s_2 < \cdots < s_k$ be integers. If there is an antichain $\mathcal{A}$ with $S = \{s_1, s_2, \ldots, s_{k-1}, s_k\} \in \mathsf{osh}(\mathcal{A})$ then there is an antichain $\mathcal{A}'$ with $S' = \{s_1, s_2, \ldots, s_{k-1}, s_k, s_k+g+1, s_k+g+2, \ldots, s_k+g+2^g-1\} \in \mathsf{osh}(\mathcal{A}')$.*

Proof of Lemma 3.3.10 Order the 2^g subsets of $\{s_k+1, s_k+2, \ldots s_k+g\}$ as $G_0, G_1, \ldots, G_{2^g-1}$ so that for $0 \leq k < j \leq 2^g - 1$ we have $G_k \setminus G_j \neq \emptyset$. Let $T = \{s_k+g+1, s_k+g+2, \ldots, s_k+g+2^g-1\}$. Then we form the antichain $\mathcal{A}'$ as

$$\bigcup_{i=0}^{2^g-1} \left(\mathcal{A} \times G_i \times \binom{T}{i} \right) \tag{3.70}$$

□

Corollary 3.3.11 *Let $1 \leq s_1 < s_2 < \cdots < s_k$ be integers and $t \geq 1$. If there is an antichain $\mathcal{A}$ with $S = \{s_1, s_2, \ldots, s_{k-1}, s_k, s_k+1, s_k+2, \ldots, s_k+t\} \in \mathsf{osh}(\mathcal{A})$ then there is an antichain $\mathcal{A}'$ with $S' = \{s_1, s_2, \ldots, s_{k-1}, s_k, s_k+g+1, \ldots, s_k+g+(t+1)2^g-1\} \in \mathsf{osh}(\mathcal{A}')$.*

Proof of Corollary 3.3.11 Apply Lemma 3.3.10, then apply Lemma 3.3.9 t times. □

Lemma 3.3.12 *There exists an antichain $\mathcal{A}$ with $S = \{g+1, g+2, \ldots, g+2^g-1\} \in \mathsf{osh}(\mathcal{A})$.*

Proof of Lemma 3.3.12 Use the sets $G_0, G_1, \ldots$ defined in Lemma 3.3.10. Then

$$\mathcal{A} = \cup_{i=0}^{2^g-1}(G_i \times \binom{S}{i}). \tag{3.71}$$

□

Lemma 3.3.13 *There does not exist an antichain $\mathcal{A}$ with $S = \{g+1, g+2, \ldots, g+2^g\} \in \mathsf{osh}(\mathcal{A})$.*

Proof of Lemma 3.3.13 Assume such an $\mathcal{A}$ exists with $|\mathcal{A}| = 2^{2^g}$. Then $\mathcal{A}$ may be decomposed into 2^{2^g} sets $B_i \cup C_i$ where $B_i \subseteq [g]$ and C_i is one of the 2^{2^g} subsets of S. We see that there is a tower of 2^g+1 sets $C_0 \subset C_1 \subset C_2 \subset \cdots C_{2^g}$ (with $C_0 = \emptyset$ and $C_{2^g} = S$) and yet there are only 2^g choices for any B_i. Thus there are indices $k < l$ for which $C_k \subset C_l$ and yet $B_k = B_l$. Thus $\mathcal{A}$ is not an antichain. □

Proof of Theorem 3.3.5 We may encode any set $S = \{s_1, s_2, \ldots, s_k\}$ as a sequence $g_1, b_1, g_2, b_2, \ldots, g_j, b_j$ where we think of g_i as the length of the ith gap and b_i as the length of the ith block of consecutive entries. Thus $S = \{g_1+1, g_1+2, \ldots g_1+b_1, g_1+b_1+g_2+1, g_1+b_1+g_2+2, \ldots, g_1+b_1+g_2+b_2, \ldots\}$. We prove the result by induction on j. The cases with $j = 1$ are handled by Lemmas 3.3.12,3.3.13. We assume $j > 1$. Two cases are distinguished.
Case 1. $b_j \leq 2^{g_j} - 1$. If there exists antichain $\mathcal{A}$ so that the set S encoded by $g_1, b_1, g_2, b_2, \ldots, g_j, b_j$ is in $\mathsf{osh}(\mathcal{A})$, then the set S' encoded by $g_1, b_1, g_2, b_2, \ldots, g_{j-1}, b_{j-1}$ is in $\mathsf{osh}(\mathcal{A})$, as well. Using Lemma 3.3.10 we obtain that if S' is in $\mathsf{osh}(\mathcal{A})$, then there exists an antichain $\mathcal{A}'$, such that the set encoded by $g_1, b_1, g_2, b_2, \ldots, g_j, 2^{g_j} - 1$ is in $\mathsf{osh}(\mathcal{A}')$. Using $b_j \leq 2^{g_j} - 1$, we obtain that the set $S \in \mathsf{osh}(\mathcal{A})$ if and only if there is an antichain $\mathcal{A}'$ with the set $S' \in \mathsf{osh}(\mathcal{A}')$. It is not hard to see that $f(S') < 1$ if and only if $f(S) < 1$.
Case 2. $b_j > 2^{g_j} - 1$. Using Corollary 3.3.8, we see that if there exists an antichain $\mathcal{A}$ so that the set S encoded by $g_1, b_1, g_2, b_2, \ldots, g_j, b_j$ is in $\mathsf{osh}(\mathcal{A})$ then there is an antichain $\mathcal{A}'$ with the set S' encoded by $g_1, b_1, \ldots, g_{j-1}, b_{j-1} + \lceil (b_j+1)/2^{g_j} \rceil - 1$ in $\mathsf{osh}(\mathcal{A}')$. But then, by Corollary 3.3.11, there is an antichain $\mathcal{A}''$ with set encoded by $g_1, b_1, \ldots, g_{j-1}, b_{j-1}, g_j, \lceil (b_j+1)/2^{g_j} \rceil 2^{g_j} - 1$ in $\mathsf{osh}(\mathcal{A}'')$. Noting that $\lceil (b_j+1)/2^{g_j} \rceil 2^{g_j} - 1 \geq b_j$, we deduce that there exists an antichain $\mathcal{A}$ so that the set $S \in \mathsf{osh}(\mathcal{A})$ if and only if there is an antichain $\mathcal{A}'$ with the set $S' \in \mathsf{osh}(\mathcal{A}')$. Now we apply induction on the set S' encoded by $g_1, b_1, \ldots, g_{j-1}, b_{j-1} + \lceil (b_j+1)/2^{g_j} \rceil - 1$ to obtain that there is an antichain $\mathcal{A}'$ with the set $S' \in \mathsf{osh}(\mathcal{A}')$ if and only if $f(S') < 1$. One can also verify

that $f(S') < 1$ if and only if $f(S) < 1$. Note that $f(S') < 1$ if and only if $f(S') \leq 1 - 2^{-(g_1+g_2+\cdots+g_{j-1})}$ and so we obtain the result. $\square$

Chapter 4

Database Matrices

4.1 The mathematical model used

The *relational model* of a database was introduced by Codd [Cod70] in 1970. The main idea is that data is stored in *relations*, where coordinates or columns correspond to *attributes*, and tuples or rows correspond to data of an individual. To make it precise, let us assume that a countably infinite set $\mathcal{A}$ of attribute names is given, furthermore, for every $A \in \mathcal{A}$ its domain – also a countably infinite set that is a set of elementary values that the attribute can take values from – $\mathsf{dom}(A)$ is assigned. A *relational schema* $\mathbf{R}$ is a finite subset $\mathbf{R} = \{A_1, A_2, \ldots, A_n\}$ of $\mathcal{A}$. A *relation of the schema* $\mathbf{R}$ is a finite collection R of mappings $r \colon \mathbf{R} \to \cup_{i=1}^{n} \mathsf{dom}(A_i)$ with the property that $r(A_i) \in \mathsf{dom}(A_i)$. Such an r is called a *tuple* or *row* of the the relation R. Let us note that the present definition of relation differs from the usual in the sense that the order of attribute values (entries of a tuple) is not important. As an example, consider the following schema.

Employee(Name,Mother's name,Social Security Number,Post,Salary)

The domain of attributes *Name* and *Mother's name* is the set of finite character strings (more precisely its subset containing all possible names). The domain of *Social Security Number* is the set of integers satisfying certain formal and parity check requirements. The attribute *Post* can take values from the set {Director, Section chief, System integrator, Programmer, Receptionist, Janitor, Handyman}. An *instance* of a schema $\mathbf{R}$ is a relation R. A typical row of a relation of the Employee schema could be

(John Brown,Camille Parker,184-83-2010,Programmer,$172,000)

There can be dependencies between different data of a relation. For example, in an instance of the Employee schema the value of Social Security Number determines all other values of a row. Similarly, the pair (Name,Mother's name) is a unique identifier. Naturally, it may occur that some set of attributes do not determine all attributes of a record uniquely, just some of its subsets.

A relational schema has several *integrity constraints* attached. Two main types of these are *tuple generating* and *equality generating* dependencies. A tuple generating dependency deduces the existence of a certain tuple from the existence of some others. On the other hand, an equality generating dependency's conclusion is the equality of certain values in tuples on the condition of the existence of other specific records. The most important kind of the latter is *functional dependency.* Let U and V be two sets of attributes. V *functionally depends* on U, $U \to V$ in notation, means that whenever two records are identical in the attributes belonging to U, then they must agree in the attribute belonging to V, as well. The equality generating dependency form of the above definition is as follows

$$\forall r_1, r_2 \in R\colon r_1[U] = r_2[U] \Rightarrow r_1[V] = r_2[V]. \tag{4.1}$$

Here $r[X]$ denotes the restriction of mapping R to the set X.

It is interesting from the point of view of schema design that given a collection Σ of functional dependencies, what other dependencies hold in a database instance that satisfies Σ. The functional dependency $U \to V$ is *logically implied* by Σ, in notation $\Sigma \models U \to V$, if each instance of $\mathbf{R}$ that satisfies all dependencies of Σ also satisfies $U \to V$. The *closure* of a set Σ of functional dependencies is the set Σ^+ given by

$$\Sigma^+ = \{U \to V \colon \Sigma \models U \to V\}. \tag{4.2}$$

It is easy to see that the operation defined in (4.2) is really a closure operation, that is

1. $\Sigma \subseteq \Sigma^+$,
2. If $\Sigma \subseteq \Gamma$, then $\Sigma^+ \subseteq \Gamma^+$,
3. $(\Sigma^+)^+ = \Sigma^+$.

A way of solving the problem of implication is the construction of an *Armstrong instance* for Σ, that is a database that satisfies a functional dependency $X \to Y$ if and only if $\Sigma \models X \to Y$. Silva and Melkanoff [SM81] developed a design aid that for a collection of functional and multivalued dependencies as input presents an Armstrong instance for that set. The existence of Armstrong instance for a set of functional dependencies was proved by Armstrong

[Arm74] and Demetrovics [Dem79]. Later Fagin [Fag82] gave a necessary and sufficient condition for general dependencies.

Since the size of Σ^+ can be an exponential function of the size of Σ, another way of representing functional dependencies is needed. Let X^+ denote the *closure* of the set of attributes $X \subseteq \mathbf{R}$ with respect to the family of functional dependencies Σ, that is

$$X^+ = \{A \in \mathbf{R} \colon \Sigma \models X \to A\}. \tag{4.3}$$

Again, as it is proven in e.g. [DK81], the operation defined in (4.3) is a closure operation. The following is a fundamental observation.

Lemma 4.1.1 *Let Σ be a collection of functional dependencies over the schema $\mathbf{R}$, furthermore let $X, Y \subseteq \mathbf{R}$. Then*

$$\Sigma \models X \to Y \iff Y \subseteq X^+. \tag{4.4}$$

Lemma 4.1.1 means that there is a one-to-one correspondence between systems of functional dependencies of schema $\mathbf{R}$ and closure operations on the set of attributes of $\mathbf{R}$. This can be used to characterize Armstrong instances of Σ [DK81].

Proposition 4.1.2 *Let Σ be a collection of functional dependencies over the schema $\mathbf{R}$. The relation R is an Armstrong instance for Σ iff the following two conditions hold:*

1. $\forall X \subset \mathbf{R}\ \forall r_1, r_2 \in R \colon r_1[X] = r_2[X] \Longrightarrow r_1[X^+] = r_2[X^+]$, *and*
2. $\forall X \subset \mathbf{R}\ \forall A \notin X^+\ \exists r_1, r_2 \in R \colon r_1[X] = r_2[X]$ *and* $r_1[A] \neq r_2[A]$.

(4.5)

A database relation R can be represented by a matrix M. The columns correspond to the attributes (in an arbitrarily fixed order) and the rows are the tuples of R. The closure defined by a matrix M is given by 1. and 2. of (4.5). That is, an attribute A is in the closure X^+ iff whenever two rows of M agree on X then they agree on A, as well. For a given matrix M with columns labeled by $\mathbf{R}$ let the closure operation on the subsets of $\mathbf{R}$ defined by M denoted by $\mathcal{C}_M$. That is $\mathcal{C}_M \colon 2^{\mathbf{R}} \to 2^{\mathbf{R}}$ is given by $\mathcal{C}_M(X) = \{A \in \mathbf{R} \colon X \to A\}$.

It is an interesting measure of complexity of systems of functional dependencies the minimum size of an Armstrong instance. The following function was introduced in [DK81].

Definition 4.1.3 *Let $\mathcal{C}$ be a closure on $\mathbf{R}$. Then let*

$$s(\mathcal{C}) = \min_{M \colon \mathcal{C}_M = \mathcal{C}} \{\textit{number of rows in } M\}. \tag{4.6}$$

Since all along this chapter the only thing interesting about attribute values is their equality or non-equality, we may assume without loss of generality that the domain of each attribute is $\mathbb{N}$, the set of natural numbers. Thus database instances for us are integer matrices with non-negative entries. In the next section a survey of known combinatorial results about $s(C)$ is given and its influence on design theory is shown. In the third section a generalization of functional dependencies, *branching dependencies* are discussed. It is shown that many interesting and hard combinatorial problems arise in connection of the existence of Armstrong instances of branching dependencies. In the solutions we apply a wide variety of methods including Lovász' theorem on k-trees, finite projective planes, Hamiltonian type theorems. These latter resulted in a new type of coding problem. In the last section another coding type question is discussed that arose from the study of of Armstrong instances of bounded domains.

4.2 Minimum representations of closures

It is very hard to determine $s(\mathcal{C})$ for an arbitrary closure $\mathcal{C}$. However, there are nice combinatorial results for certain closures.

Definition 4.2.1 *Let $\mathcal{C}_n^k$ denote the following closure on* $\mathbf{R}$*:*

$$\mathcal{C}_n^k(X) = \begin{cases} X & \text{if } |X| < k \\ \mathbf{R} & \text{otherwise.} \end{cases} \tag{4.7}$$

The following lemma gives a general lower bound for $s(\mathcal{C}_n^k)$.

Lemma 4.2.2 ([DK81])

$$\binom{s(\mathcal{C}_n^k)}{2} \geq \binom{n}{k-1}. \tag{4.8}$$

Proof of Lemma 4.2.2 Suppose, that M represents $\mathcal{C}_n^k$ and let $|A| = k-1$ be a subset of $\mathbf{R}$, furthermore let $b \notin A$. Then by the definition of $\mathcal{C}_n^k$, $A \not\to b$, i.e., there is a pair of rows i and j, such that they are identical in A, but different in b. If there is another $k-1$-subset B of $\mathbf{R}$ such that i and j are identical on B, as well, then $A \cup B \not\to b$ would hold, but $|A \cup B| \geq k$, so by the definition of $\mathcal{C}_n^k$ this cannot happen. Thus, we can assign distinct pairs of rows to distinct $k-1$-subsets of columns. □

The exact value of $s(\mathcal{C}_n^k)$ is determined for certain values of k.

Theorem 4.2.3 ([DK81]) *The following equalities hold:*

$$\begin{array}{ll} a)\ s(\mathcal{C}_n^1) = 2, & c)\ s(\mathcal{C}_n^{n-1}) = n, \\ b)\ s(\mathcal{C}_n^2) = \left\lceil \frac{1+\sqrt{1+8n}}{2} \right\rceil, & d)\ s(\mathcal{C}_n^n) = n+1. \end{array} \tag{4.9}$$

We give the proof of Case b) as an example.

Proof of Case b) of Theorem 4.2.3 Let $s = s(\mathcal{C}_n^2)$. Lemma 4.2.2 gives $\binom{s}{2} \geq n$. Note that the number of the right hand side of equality in Case b) is the smallest s satisfying the previous inequality. If s is such, then we construct a matrix M with s rows such that $\mathcal{C}_M = \mathcal{C}_n^2$ as follows:

$$M = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 1 & 1 & 1 & \dots & 1 \\ 0 & 2 & 2 & \dots & 2 & 0 & 0 & 0 & \dots & 2 \\ 3 & 0 & 3 & \dots & 3 & 0 & 3 & 3 & \dots & 3 \\ 4 & 4 & 0 & \dots & 4 & 4 & 0 & 4 & \dots & 4 \\ 5 & 5 & 5 & \dots & 5 & 5 & 5 & 0 & \dots & 5 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ s & s & s & \dots & 0 & s & s & s & \dots & s \end{pmatrix}. \tag{4.10}$$

There is a pair of zeros in every column of M such that for different columns the zeros are in different pairs of rows, which implies that every one-element subset of $\mathbf{R}$ is closed. This can be done by the choice of s. On the other hand, no two rows agree in more than one column, so if $A \subseteq \mathbf{R}$ with $|A| > 1$, then $\mathcal{C}_M(A) = \mathbf{R}$. $\square$

Let us note that in Case d) of Theorem 4.2.3 Lemma 4.2.2 yields only $s(\mathcal{C}_n^n) > \sqrt{2n}$, hence some other tricks are needed to prove $n+1$ as a lower bound.

Let us now consider the case $k = 3$. From Lemma 4.2.2 we obtain that

$$\binom{s(\mathcal{C}_n^3)}{2} \geq \binom{n}{2}, \tag{4.11}$$

hence $s = s(\mathcal{C}_n^3) \geq n$. Equality holds if we can construct an $n \times n$ matrix M such that:

1) for any distinct $a, b, c \in \mathbf{R}$ there are two rows equal in columns a and b, but different in c.
2) for any distinct $a, b, c \in \mathbf{R}$ there are no two rows equal in all of them.

Consider the dual problem. A column naturally determines a partition of the set Y of rows, by the equalities of its entries. We say that a partition *covers* the pair (α, β) $(\alpha, \beta \in Y,\ \alpha \neq \beta)$ iff α and β are in the same class of the partition. We can state the previous two properties as follows.

Find n partitions of Y ($|Y| = n$) such that:
1') for any two partitions there exists a pair (α, β) covered by both,
2') no pair (α, β) is covered by three different partitions.

However, the number of pairs of partitions is also $\binom{n}{2}$ and different pairs of partitions cannot cover the same pair of elements by 2'). Thus, we may conclude that 1') and 2') (consequently 1) and 2)) are equivalent to:

(i) for any two partitions there is exactly one pair of elements, which is covered by both,
(ii) each pair of elements is covered by exactly two different partitions.

Definition 4.2.4 *A collection of partitions satisfying (i) and (ii) is called an* orthogonal double cover*.*

The following conjecture was formulated in [DFK85]. (It was posed in other terms, since the notion of orthogonal double cover was introduced later, in [GGM94].)

Conjecture 4.2.5 *There exists an orthogonal double cover of the n-element set by n partitions provided $n \geq 7$.*

In the same paper they proved that Conjecture 4.2.5 is true for certain n's.

Theorem 4.2.6 ([DFK85]) *Conjecture 4.2.5 is true if $n = 12r + 1$ or $n = 12r + 4$.*

In the proof they used a theorem of Hanani to construct special type of partitions, namely each partition consisted of one 1-element class and $4r$ ($4r + 1$, resp.) 3-element classes. This motivated the following conjecture.

Conjecture 4.2.7 ([DFK85]) *If $n = 3r+1$, then there exists an orthogonal double cover of the n-element set by n partitions that have one 1-element class and r of the 3-element classes.*

Note that the two conjectures are independent in the sense that the solution of one of them does not imply the solution of the other. The first result about these conjectures was negative. In 1987 Rausche [Rau87] observed that Conjecture 4.2.7 is not true for $n = 10$. However, that turned out to be the only "bad case". Ganter and Gronau and later Yeow Meng Chee [Che92] independently proved the following.

Theorem 4.2.8 ([GG87]) *Conjecture 4.2.7 is true for $n \geq 13$.*

The first conjecture was decided affirmatively, as well. Bennett and Wu proved the following theorem.

Theorem 4.2.9 ([BW90]) *Conjecture 4.2.5 is true.*

If we have an orthogonal double cover by partitions, then we can define a graph for each partition. The vertex set is the underlying set **R**, the edges are the pairs covered by that partition. These graphs are unions of disjoint cliques. Furthermore, in the case of Conjecture 4.2.7 these graphs are pairwise isomorphic, namely they are unions of r K_3's and an isolated point. This observation motivated the following definition.

Definition 4.2.10 *A collection of n pairwise isomorphic graphs with the same vertex set V, where $|V| = n$ and $G_i = (V, E_i)$ for $i = 1, 2, \ldots, n$, is called an* orthogonal double cover by graphs *iff*

1) *each edge of K_n is contained in exactly two of the E_i's,*
2) $|E_i \cap E_j| = 1$ *for $i \neq j$.*

With this concept Theorem 4.2.9 states that there exists a double cover by graphs where the $G_i = K_1 + r * K_3$. Gronau, Mullin and Schellenberg [GMS95] proved a conjecture of Chung and West [CD94] stating that there is an orthogonal double cover by graphs where each G_i has maximum degree at most two. A sharpening of this result was given by Ganter, Gronau and Mullin.

Theorem 4.2.11 ([GGM94]) *For all $n \geq 4$, $n \neq 8$ there is an orthogonal double cover by graphs where each G_i consists of the isolated vertex i and a union of disjoint cycles of length 3,4 or 5 only.*

An extensive survey of orthogonal double covers is found in [Gro02]

The exact value of $s(\mathcal{C}_n^k)$ is not known for $k > 3$. However, if k is fixed, then its asymptotic behavior is known.

Theorem 4.2.12 ([DFK85]) *If k is fixed and $n > n_0(k)$, then*

$$c_1(k)n^{\frac{k-1}{2}} \leq s(\mathcal{C}_n^k) \leq c_2(k)n^{\frac{k-1}{2}}. \tag{4.12}$$

The lower bound in Theorem 4.2.12 follows from Lemma 4.2.2. The upper bound is proven by a construction involving polynomials over a finite field. Füredi proved some bounds for the "other end" of the range of k.

Theorem 4.2.13 ([Für90]) *If k is fixed and $n > n_0(k)$, then*

$$c_3(k)n^{\frac{2k+1}{3}} \leq s(\mathcal{C}_n^{n-k}) \leq c_4(k)n^k. \tag{4.13}$$

The following concept allows us to find $s(\mathcal{C})$ for infinitely many closures.

Definition 4.2.14 *Let $\mathcal{L}$ and $\mathcal{N}$ be closures on the ground sets U and V, respectively, with $U \cap V = \emptyset$. The* direct product *of $\mathcal{L}$ and $\mathcal{N}$ is the closure on the ground set $U \cup V$ defined by*

$$(\mathcal{L} \times \mathcal{N})(A) = \mathcal{L}(A \cap U) \cup \mathcal{N}(A \cap V) \quad \text{for} \quad A \subseteq U \cup V. \tag{4.14}$$

The size of a minimum representation of a direct product of closures can be calculated provided that the minimum representation known for the members of the product.

Theorem 4.2.15 ([DFK85])

$$s(\mathcal{C}_1 \times \mathcal{C}_2) = s(\mathcal{C}_1) + s(\mathcal{C}_2) - 1. \tag{4.15}$$

Theorem 4.2.15 provides an alternative proof for Case d) of Theorem 4.2.3, one has only to observe that $\mathcal{C}_n^n = \mathcal{C}_{n-1}^{n-1} \times \mathcal{C}_1^1$.

4.3 Branching Dependencies

The general concept we shall study is the (p,q)-dependency ($1 \le p \le q$ integers).

Definition 4.3.1 *Let $\mathbf{R}$ be a database schema, R be an instance of $\mathbf{R}$, $X \subseteq \mathbf{R}$, $a \in \mathbf{R}$, furthermore let $1 \le p \le q$ be integers. R is said to satisfy the (p,q)-dependency $X \xrightarrow{(p,q)} a$ if for arbitrarily chosen $q+1$ tuples $r_1, r_2, \ldots r_{q+1} \in R$ we have that $\forall x \in X\colon |\{r_1[x], r_2[x], \ldots, r_{q+1}[x]\}| \le p$ implies that $|\{r_1[a], r_2[a], \ldots, r_{q+1}[a]\}| \le q$.*

The meaning of Definition 4.3.1 is as follows. An attribute $a \in \mathbf{R}$ (p,q)-depends on the set X if there exist no $q+1$ tuples in the relation R such that they take at most p distinct values on each attribute belonging to X, but all distinct values in a.

Note, that (p,q)-dependency is a generalization of functional dependency, namely functional dependency $X \to a$ holds iff $X \xrightarrow{(1,1)} a$ $(1,1)$-dependency is satisfied.

As a motivation or example for (p,q)-dependency consider the following. Let us suppose that the database consists of the trips of an international transport truck, more precisely, the names of the countries the truck enters. For the sake of simplicity, let us suppose, that the truck goes through exactly four countries in each trip, (counting the start and endpoints, too) and does not enter a country twice during one trip. Suppose furthermore, that there are 30 possible countries and one country has at most five neighbors. Let

A_1, A_2, A_3, A_4 denote the first, second, third and fourth country as attributes. It is easy to see that $A_1 \overset{(1,5)}{\longrightarrow} A_2$, $\{A_1, A_2\} \overset{(1,4)}{\longrightarrow} A_3$ and $\{A_2, A_3\} \overset{(1,4)}{\longrightarrow} A_4$. Now, we can decrease the range of the elements of the stored database matrix. The range of each element of the original matrix consists of 30 values, names of countries or some codes of them (5 bits each, at least). Let us store a little table ($30 \times 5 \times 5 = 750$ bits) that contains a numbering of the neighbors of each country, which assigns to them the numbers 0,1,2,3,4 in some order. Now we can replace attribute A_2 by these numbers (A_2^*), because the value of A_1 gives the starting country and the value of A_2^* determines the second country with the help of the little table. The same holds for the attribute A_3, but we can decrease the number of possible values even further, if we give a table of numbering the possible third countries for each A_1, A_2 pair. In this case, the attribute A_3^* can take only 4 different values. The same holds for A_4, too. That is, while each element of the original matrix could be encoded by 5 bits, now for the cost of two little auxiliary tables we could decrease the length of the elements in the second column to 3 bits, and that of the elements in the third and fourth columns to 2 bits.

It is easy to see, that the same idea can be applied in each case when we store the paths of a graph, whose maximal degree is much less than the number of its vertices or when we want to store the sequence of states of a process, where the number of all possible states is much larger, than the number of possible successor states of a state or in any case when there hold many $(1, q)$-dependencies, where q is small.

Similarly to closure in case of functional dependencies, the *extension* of a set of attributes can be defined with respect to a collection Σ of (p, q)-dependencies.

Definition 4.3.2 *Let Σ be a collection of (p, q)-dependencies over the schema $\mathbf{R}$. Let us suppose that $1 \leq p \leq q$. The mapping $\mathcal{J}_{\Sigma pq} \colon 2^{\mathbf{R}} \to 2^{\mathbf{R}}$ is defined by*

$$\mathcal{J}_{\Sigma pq}(A) = \left\{ b \colon \Sigma \models A \overset{(p,q)}{\longrightarrow} b \right\}. \tag{4.16}$$

We collect two important properties of the mapping $\mathcal{J}_{\Sigma pq}$ in the following proposition, see [DKS92].

Proposition 4.3.3 *Let $\mathbf{R}$, Σ, p and q as above. Furthermore, let $A, B \subseteq \mathbf{R}$. Then*

$$\begin{array}{rl} (i) & A \subseteq \mathcal{J}_{\Sigma pq}(A) \\ (ii) & A \subseteq B \Longrightarrow \mathcal{J}_{\Sigma pq}(A) \subseteq \mathcal{J}_{\Sigma pq}(B). \end{array} \tag{4.17}$$

Set functions satisfying (i) and (ii) of (4.17) are called *extensions.* Note that the major difference between an extension and a closure is that the former is not idempotent. Naturally arises the question whether extensions and (p,q)-dependencies are in such close connection as closures and functional dependencies are. The definition of an Armstrong instance can be applied for a collection Σ of (p,q)-dependencies, as well. An extension $\mathcal{N}\colon 2^{\mathbf{R}} \to 2^{\mathbf{R}}$ is said to be *(p,q)-representable* if there exists a collection Σ of (p,q)-dependencies over $\mathbf{R}$ that has an Armstrong instance and $\mathcal{N} = \mathcal{J}_{\Sigma pq}$.

The question of (p,q)-representability and the size of the minimal Armstrong instance of extensions and closures leads to beautiful and hard combinatorial problems.

4.3.1 Existence question of an Armstrong instance

The major obstacle on the way of establishing the existence of an Armstrong instance of a given extension $\mathcal{N}$ is that (p,q)-dependencies have no such nice and short axiomatization as functional dependencies have with the Armstrong Axioms. Another problem is that extensions have much less structure theory than closures do have. In general we can prove the following.

Theorem 4.3.4 ([DKS92]) *Let $\mathcal{N}$ be an extension on subsets of $\mathbf{R}$ satisfying $\mathcal{N}(\emptyset) = \emptyset$. Then $\mathcal{N}$ (p,q)-representable if one of the following holds.*

$$\begin{array}{rl} (i) & p = 1 \text{ and } 1 < q \text{ or} \\ (ii) & p = 2 \text{ and } 3 < q \text{ or} \\ (iii) & 2 < p \text{ and } p^2 - p - 1 < q. \end{array} \tag{4.18}$$

Proof of Theorem 4.3.4 We will construct construct an Armstrong instance in the form of a database matrix. Let us call a sequence of subsets $\emptyset \neq A_1 \subset A_2 \subset \cdots \subset A_k$ of $\mathbf{R}$ a *chain* if the following two conditions hold:

$$\begin{array}{ll} (i) & \mathcal{N}(A_i) = A_{i+1} \quad (1 \le i < k) \\ (ii) & \mathcal{N}(A_k) = A_k. \end{array} \tag{4.19}$$

For such a chain L we construct the matrix $M(z,r,L)$ shown in Table 4.3.1 Each column of the matrix begins with some z's, then from a certain position the natural numbers come in increasing order: $z,\ldots,z,z+1,z+2,\ldots.$ The columns of $A_i \setminus A_{i-1}$ $(1 < i \le k)$ are all identical and the same holds for the columns of A_1 and $\mathbf{R} \setminus A_k$, respectively. The columns of the latter consist of $z, z+1, z+2, \ldots.$ On the other hand, columns of A_1 consist of all z's. Columns of $A_2 \setminus A_1$ are shifted in comparison to columns of A_1 by r, i.e. the number of z's at the beginning is r less than that in columns of A_1, but their last element is $r+z$. In general, columns of $A_i \setminus A_{i-1}$ are shifted in comparison to those of $A_{i-1} \setminus A_{i-2}$ by r $(1 < i \le k)$. However, columns

$$\begin{array}{cccccc}
A_1 & A_2 \setminus A_1 & A_3 \setminus A_2 & \dots & A_k \setminus A_{k-1} & \mathbf{R} \setminus A_k \\
\\
z & z & z & \dots & z & z \\
z & z & z & \dots & z & z+1 \\
z & z & z & \dots & z & z+2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
z & z & z+r & \dots & z+(k-2)r & z+kr \\
z & z+1 & z+r+1 & \dots & z+(k-2)r+1 & z+kr+1 \\
z & z+2 & z+r+2 & \dots & z+(k-2)r+2 & z+kr+2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
z & z+r & z+2r & \dots & z+(k-1)r & z+(k+1)r
\end{array} \tag{4.20}$$

Table 4.1: The matrix $M(z, r, L)$.

of $\mathbf{R} \setminus A_k$ are shifted by $2r$ in comparison to $A_k \setminus A_{k-1}$. According to the definition of a chain, $A_i \setminus A_{i-1}$ $(1 \le i \le k)$ cannot be empty, but $\mathbf{R} \setminus A_k$ can be. In the latter case the matrix does not contain such columns. We shall only use the following easily checked properties of this matrix.

Fact 4.3.5 *1. If two positions in a column of $A_i \setminus A_{i-1}$ $(1 < i \le k)$ contain the same element, then any column of $A_j \setminus A_{j-1}$ contains identical element in those two positions for all $j < i$. ($A_0 = \emptyset$ by assumption.)*

2. *Choosing a z in a column of $A_i \setminus A_{i-1}$ there can stand only z or $z+1$ or $z+2$ or ... or $z+r$ in the same position of a column of $A_{i+1} \setminus A_i$. However, if we choose a number s different from z in a column of $A_i \setminus A_{i-1}$, then only $s+r$ can stand in the same position of a column of $A_{i+1} \setminus A_i$.*

3. *For $k \ge j > i+1 \ge 2$ we can find $2r+1$ different numbers (namely $z, z+1, \dots, z+2r$) in a column of $A_j \setminus A_{j-1}$ so that only z's stand in the same positions of a column of $A_i \setminus A_{i-1}$.*

4. *We can find $2r+1$ different numbers (namely $z, z+1, \dots, z+2r$) in a column of $\mathbf{R} \setminus A_k$ so that only z's stand in the same positions of a column of $A_i \setminus A_{i-1}$ for $1 \le i \le k$.*

Let $\mathcal{L} = \{L_1, L_2, \dots, L_m\}$ be a set of chains which satisfies that for every pair A, b $(A \subseteq \mathbf{R},\ b \in \mathbf{R})$ satisfying $A \ne \emptyset$, $b \notin \mathcal{N}(A)$ there is a chain L_j and a set A_i in that chain satisfying

$$A \subseteq A_i \qquad \text{and} \qquad b \notin \mathcal{N}(A_i). \tag{4.21}$$

We obtain such a set of chains for example, if we take all possible nonempty subsets of $\mathbf{R}$ as A_1. For every chain L_i we construct p matrices $M(z_1^i, r, L_i)$, $M(z_2^i, r, L_i), \ldots, M(z_p^i, r, L_i)$. We choose the numbers z_j^i so that a natural number can occur in at most one of these matrices. We write the matrices one under the other to obtain the matrix $\mathcal{M}(r)$. If some column contains less than $q+1$ different symbols, then we repeat $M(z_1^1, r, L_1)$ enough times with all different z's to obtain at least $q+1$ different symbols in every column. Let $\mathcal{J}_{\mathcal{M}(r)pq}$ denote the extension obtained from the system of (p,q)-dependencies whose Armstrong instance is $\mathcal{M}(r)$. We claim that for a suitable choice of r, $\mathcal{J}_{\mathcal{M}(r)pq} = \mathcal{N}$ holds. This is true if

1. $b \notin \mathcal{N}(A)$ implies that $b \notin \mathcal{J}_{\mathcal{M}(r)pq}(A)$ and
2. $b \in \mathcal{N}(A)$ implies that $b \in \mathcal{J}_{\mathcal{M}(r)pq}(A)$.

1) Let us suppose first that $b \notin \mathcal{N}(A)$ for some $A \subset \mathbf{R}$. If $A = \emptyset$, then $b \notin \mathcal{J}_{\mathcal{M}(r)pq}(\emptyset)$ follows from the fact that there are at least $q+1$ different symbols in any column of $\mathcal{M}(r)$. However, if $A \neq \emptyset$, then there exists a chain L_j and a set A_i of that chain satisfying (4.21). We have that $b \notin \mathcal{N}(A_i) = A_{i+1}$, so $b \in A_f \setminus A_{f-1}$, $k \geq f > i+1$ or $b \in \mathbf{R} \setminus A_k$ holds. In the first case we use 3. of Fact 4.3.5 and in the second case we use 4. of Fact 4.3.5 to choose altogether $p(2r+1)$ rows from $M(z_1^j, r, L_j), M(z_2^j, r, L_j), \ldots, M(z_p^j, r, L_j)$ so that they contain at most p different symbols in columns of $A \subseteq A_i$, but they contain all different symbols in the column b. Thus, if

$$p(2r+1) \geq q+1, \tag{4.22}$$

then $b \notin \mathcal{J}_{\mathcal{M}(r)pq}(A)$ holds.

2) Let us suppose now that $b \in \mathcal{N}(A)$. $\mathcal{N}(\emptyset) = \emptyset$ implies that $A \neq \emptyset$. Let us consider an arbitrary chain L_v from $\mathcal{L}$: $A_1, A_2, \ldots A_k$. Let $i = i(L_v) = k+1$ if $A \cap (\mathbf{R} \setminus A_k) \neq \emptyset$. On the other hand, if $A \cap (\mathbf{R} \setminus A_k) = \emptyset$, then let i be the largest index that $A \cap (A_i \setminus A_{i-1})$ is nonempty. $A \subseteq A_i$ implies that $b \in \mathcal{N}(A) \subseteq \mathcal{N}(A_i) = A_{i+1}$ for $i < k$. For $i = k$ we have that $b \in \mathcal{N}(A_i) = A_i$. Applying 2. of Fact 4.3.5, this implies that if there are at most t different symbols in a column of A in the matrix $M(z_f^v, r, L_v)$, then in the column b there can stand only $t+r$ different values.

Let us choose $q+1$ rows that contain at most p different values in columns of A. These rows could be chosen from at most p different matrices $M(z_f^v, r, L_v)$. Suppose that they are chosen in fact from u ($u \leq p$)

different matrices. Because there are different symbols in different matrices, we have that in the columns of A there can only stand at most $p-u+1$ different symbols in one matrix, which implies that in one matrix at most $p-u+1+r$ different values are in the column b. Altogether there are at most $u(p-u+1+r)$ different symbols in column b in the u different matrices of type $M(z_f^v, r, L_v)$. If $r \geq p-2 > 0$, then this number is maximal for $u = p$. Thus, if $r \geq p-2 > 0$ and

$$p(r+1) \leq q, \tag{4.23}$$

then $b \in \mathcal{J}_{\mathcal{M}(r)pq}(A)$ follows.

It easy to check that for the pairs p, q satisfying (4.18) one can find r which simultaneously satisfies (4.22) and (4.23). □

Closures are special extensions. We can use some structure theory to prove the following. Details are included in [DKS92].

Theorem 4.3.6 *Let $\mathcal{L}$ be a closure on $\mathbf{R}$. If $p = 1$ or $p = 2$ and $p \leq q$, or $3 \leq p$ and $\left(\frac{p+1}{2}\right)^2 \leq q$, then $\mathcal{L}$ is (p,q)-representable.*

Theorem 4.3.6 allows $p = q$ in some cases while Theorem 4.3.4 does not. This is not surprising in the light of the next proposition.

Proposition 4.3.7 *Let $\mathbf{R}$ be a relational database schema, Σ a collection of (p,p)-dependencies for some $p \geq 1$. Then in addition to (i) and (ii) of (4.17),*

$$(iii) \quad \mathcal{J}_{\Sigma pp}\left(\mathcal{J}_{\Sigma pp}(A)\right) = \mathcal{J}_{\Sigma pp}(A) \tag{4.24}$$

holds, as well for $A \subseteq \mathbf{R}$.

Thus, (p,p)-representable extensions are closures. Every closure is (p,p)-representable for $p = 1, 2$ by Theorem 4.3.6. It was shown in [DKS92], that there exist a closure which is (p,p)-representable exactly for $p = 1, 2$ only. This motivates the following definition.

Definition 4.3.8 ([sS98]) *Let $\mathcal{L}$ be a closure on the set $\mathbf{R}$. The* spectrum $\mathsf{sp}(\mathcal{L})$ *of $\mathcal{L}$ is defined as follows.*

$$q \in \mathsf{sp}(\mathcal{L}) \iff \mathcal{L} \text{ is } (q,q) - \text{representable} \tag{4.25}$$

Note that $\mathsf{sp}(\mathcal{L}) \subseteq \mathbb{N}$.

It is very hard to determine the spectrum of an arbitrary closure. However, for *uniform* closures given in Definition 4.2.1 we can give a complete characterization. The result is quiet surprising in the following sense. If an instance R of a database schema $\mathbf{R}$ satisfies $X \xrightarrow{(p,q)} a$ for some $X \subset \mathbf{R}$ and

$a \in \mathbf{R}$, furthermore the tuples of R take at least $q+1$ distinct values in each attribute, then R satisfies $X \overset{(p-1,q-1)}{\longrightarrow} a$, as well. Thus one would expect the spectrum of a closure being an interval of natural numbers. However, in [sS98] we proved the following.

Theorem 4.3.9 *Let $n \geq k^2(k-1)$. Then the spectrum $\mathsf{sp}\left(\mathcal{C}_n^k\right)$ of $\mathcal{C}_n^k$ is given by:*

$$\mathsf{sp}\left(\mathcal{C}_n^k\right) = \{1, 2, \ldots, k-1\} \cup \{p \colon \exists s \in \mathbb{N}\; p+1-\left\lceil \frac{p+1}{s} \right\rceil = k-1\}. \quad (4.26)$$

(4.26) gives a spectrum that consists of an interval and some "sporadic points". In particular, $\mathcal{C}_n^{10}$ is $(17,17)$-representable but neither $(16,16)$- nor $(15,15)$-representable. It is also $(18,18)$-representable, but it is not (p,p)-representable for $p > 18$.

The proof of Theorem 4.3.9 consists of two parts. First, constructions show that the claimed numbers are in the spectrum, then some combinatorial arguments rule out the others. We give the proof through several lemmas.

Proof of Theorem 4.3.9 Let the $m \times n$ matrix M (p,p)-represent the closure $\mathcal{L}$ on $\mathbf{R}$. A mapping w from the edges of the complete graph K_m to the subsets of $\mathbf{R}$ can be defined, as follows. The vertices of K_m are identified with the set of rows of M. For an edge $e = \{i, j\}$ of K_m, let $w(e)$ be the set of positions where rows i and j agree. If $A \subset \mathbf{R}$ and $b \in \mathbf{R}$ such that $b \notin \mathcal{L}(A)$, then there exist $p+1$ rows $r_1, r_2, \ldots, r_{p+1}$ that contain at most p distinct values in columns of A but they are all different in column b. Equivalently, $b \notin \bigcup_{1 \leq i < j \leq p+1} w(\{r_i, r_j\}) \supset A$. The next lemma, which is an equivalent formulation of Theorem 2.12 of [DKS92] is explained by the above observation.

Lemma 4.3.10 *Let $\mathcal{L}$ be a closure on $\mathbf{R}$. $\mathcal{L}$ is p-representable if and only if there exists a mapping $w \colon E(K_m) \longrightarrow 2^{\mathbf{R}}$ of the edges of K_m for some m (where $w(e)$ is called the* weight *of edge e) that satisfies the following two properties:*

1. *For any three edges e_1, e_2, e_3 forming a triangle, $w(e_i) \cap w(e_j) \subseteq w(e_k)$ holds for any permutation (i,j,k) of $(1,2,3)$.*

2. *For any $p+1$ vertices of K_m, the union weights of edges spanned by these vertices is closed by $\mathcal{L}$, and every closed set of $\mathcal{L}$ can be obtained as intersections of sets of this type.*

Condition 1. is the necessary and sufficient condition for the existence of a matrix with prescribed edge weights, while condition 2. is that of the (p,p)-representation. In what follows, edges of K_m of empty weight will be omitted for the sake of simplicity, i.e. weightings of not necessarily complete graphs will be given with the understanding that edges not mentioned have empty weight.

The following result of Rucinski and Vince [RV86] is needed for constructions. A graph G of $e(G)$ edges and $v(G)$ vertices is called *balanced* if $e(G)/v(G) \geq e(H)/v(H)$ holds for every subgraph H of G. G is called *strongly balanced* if $e(G)/(v(G)-1) \geq e(H)/(v(H)-1)$ holds for every subgraph H of G. A strongly balanced graph is clearly balanced.

Theorem 4.3.11 ([RV86]) *There exists a strongly balanced graph with v vertices and e edges if and only if* $1 \leq v-1 \leq e \leq \binom{v}{2}$.

Lemma 4.3.12 $\mathcal{C}_n^k$ *is p representable if* $p \leq k-2$.

Proof of Lemma 4.3.12 We may assume without loss of generality that $p > 2$ by Theorem 4.3.6. Let $k-1 = a\binom{p+1}{2}+b$ where $0 \leq a$ and $0 \leq b < \binom{p+1}{2}$ are integers. Suppose first, that $b \geq p$. Let G be a balanced graph of $p+1$ vertices and b edges provided by Theorem 4.3.11. For every $k-1$-element subset of $\mathbf{R}$ we take K_{p+1} so that edges corresponding to edges of G are weighted by $a+1$-element subsets, the remaining ones by a-element subsets, such that the weights of edges are pairwise disjoint sets, and their union is the given $k-1$-element subset of $\mathbf{R}$. We claim that the disjoint union of these weighted complete graphs satisfy the conditions of Lemma 4.3.10.

It is clear that Condition 1. is satisfied, because weights of adjacent edges are pairwise disjoint sets. Also clear is that every $k-1$-element subset of $\mathbf{R}$ occurs as union of weights of edges spanned by some $p+1$-element subset of vertices. The only thing to check is that larger subsets of $\mathbf{R}$ do not occur this way. Let us suppose that the $p+1$-element subset of vertices U is the union of sets U_i, $i=1,2,\ldots,t$, where U_i's are the intersections of U with the weighted complete graphs. Let $u_i = |U_i|$, furthermore let e_i be the number of edges of the subgraph of balanced graph G spanned by vertices corresponding to U_i. Then $e_i/u_i \leq b/(p+1)$ is satisfied. The cardinality e of the union of the weights of edges spanned by U can bounded from above, as follows:

$$\begin{aligned} e &\leq a\sum_{i=1}^{t}\binom{u_i}{2}+\sum_{i=1}^{t}e_i \\ &\leq a\binom{p+1}{2}+\sum_{i=1}^{t}\frac{e_i}{u_i}u_i \\ &\leq a\binom{p+1}{2}+\sum_{i=1}^{t}\frac{b}{p+1}u_i \\ &= a\binom{p+1}{2}+b = k-1 \end{aligned} \tag{4.27}$$

On the other hand, if $b < p$, then $a > 0$ is satisfied. Let $k-1-p = (a-1)\binom{p+1}{2} + c$. Then $c \geq p$ holds. Let us consider two graphs, G and H, on the same $p+1$ vertices, where G is a balanced graph with c edges, and H is a path (which is clearly balanced). For every $k-1$-element subset of $\mathbf{R}$ we take K_{p+1} so that edges corresponding to edges of $G \cap H$ are weighted by $a+1$-element subsets, those corresponding to edges of $G \setminus H$ and $H \setminus G$ are weighted by a-element subsets, the remaining ones by $a-1$-element subsets, such that the weights of edges are pairwise disjoint sets, and their union is the given $k-1$-element subset of $\mathbf{R}$. That the disjoint union of these weighted complete graphs satisfies the conditions of Lemma 4.3.10 can be proved by a similar argument to the one above. □

Lemma 4.3.13 *If*

$$p+1-\left\lceil\frac{p+1}{s}\right\rceil = k-1 \qquad \textit{for } s>1 \tag{4.28}$$

then $p \in \mathsf{sp}\left(\mathcal{C}_n^k\right)$

Proof of Lemma 4.3.13 Take $\binom{n}{s-1}$ paths of s vertices whose edges have one element weights so that each $s-1$-element subset occurs as union of elements of a path. Any $p+1$ vertices span a forest that has at least $\lceil\frac{p+1}{s}\rceil$ components, so at most $k-1$ edges. □

Note, that in Lemma 4.3.13 $s \leq p+1$ may be assumed. Any $s \geq p+1$ gives the same $p = k-1$ case.

The general pattern of the non-representability proofs is that a minimal (non-decreasable) representing matrix is assumed, then it is shown that it must contain identical rows that clearly contradicts to its minimality. Detailed proofs of the following statements are in [sS98]. We include here only the proof of Lemma 4.3.18, the "converse" of Lemma 4.3.13

Lemma 4.3.14 *Let* $p \geq 2k-1$. *If* $n \geq k^2(k-1)$, *then* $\mathcal{C}_n^k$ *is not* (p,p)-*representable.*

Proposition 4.3.15 *If the matrix* M (p,p)-*represents* $\mathcal{C}_n^k$ *and minimal subject this condition, then the weight of an edge* $w(e)$ *is at most* $k-1$-*element set.*

Proposition 4.3.16 *Let* $p \leq 2k-4$ *and* $n \geq (k-1)(2k-3)$. *Let* M (p,p)-*represent* $\mathcal{C}_n^k$ *and let* M *be minimal subject to this condition. Then for any* $p+1$ *rows* $r_1, r_2, \ldots, r_{p+1}$,

$$\left|\bigcup_{1\leq i<j\leq p+1} w(\{r_i,r_j\})\right| \leq k-1. \tag{4.29}$$

Proposition 4.3.17 *Let* $2\lfloor\frac{p+1}{2}\rfloor \geq k$ *and suppose that* $\mathcal{C}_n^k$ *is* (p,p)*-representable. Suppose furthermore that* $p \leq 2\,k-4$ *and* $n \geq (k-1)\,(2k-3)$*. Then there exists* $n' \geq n-k+1$ *such that* $\mathcal{C}_{n'}^k$ *is* (p,p)*-represented so that each edge weight is at most one element set.*

Lemma 4.3.18 *Let* $n \geq (k-1)\,(2k-3)$ *and suppose that there exists integer* $s>1$ *such that*

$$p+1-\left\lceil\frac{p+1}{s}\right\rceil < k-1 < p+1-\left\lceil\frac{p+1}{s+1}\right\rceil. \tag{4.30}$$

Then $\mathcal{C}_n^k$ *is* not (p,p)*-representable.*

Proof of Lemma 4.3.18 Let us suppose indirectly that $\mathcal{C}_n^k$ is p-represented by $m\times n$ matrix M. We may assume without loss of generality that each edge weight of K_m is at most one element set according to Proposition 4.3.17. In the following "number of edges" means "number of edges of pairwise different weights" for the sake of simplicity. If there are more than one edges of the same non-empty weight in a sub-K_{p+1}, then an arbitrary one of them can be picked.

Each $k-1$-element subset of $\mathbf{R}$ must occur as union of weights of edges of a sub-K_{p+1}. By the condition on k and p, the edges of non-empty but pairwise different weight of such a sub-K_{p+1} span a graph that has a non-tree component or a tree component of size at least $s+1$. Such a component is called *big*. Let $B_1, B_2, \ldots, B_z$ be big components of different sub-K_{p+1}'s corresponding to pairwise disjoint $k-1$-element subsets. A $p+1$- vertices subgraph is constructed as follows. First, take as many non-tree components as possible, then big tree components, until the number of vertices reaches $p+1$. Let this graph be H, and suppose the number of vertices of H covered by non-tree components is d, and let $u=p+1-d$. Then the number of edges $e(H)$ of H satisfies

$$e(H) \geq d+u+\left\lceil\frac{u}{s+1}\right\rceil \geq p+1-\left\lceil\frac{p+1}{s+1}\right\rceil > k-1, \tag{4.31}$$

that contradicts to Proposition 4.3.16. □

4.3.2 Minimum representations

The minimum size of an Armstrong instance of a system of (p,q)-dependencies or an extension $\mathcal{N}$, or a closure $\mathcal{L}$ is a good measure of the complexity of the object in question. Of course in case of extensions or closures we may speak about (p, q)-complexity, since a given extension or closure could be (p, q)-represented for various values of p and q. Since the question of minimum representation is is hard already for $p = q = 1$, that is the functional dependency case, and for arbitrary extensions and closures even the (p, q)-representability question is complex enough, one cannot expect general results here. Thus, we only treat uniform closures. Nevertheless, the problems arising are combinatorially very interesting, they have a design-theoretic flavor, and we apply a broad range of methods. On the other hand, one of our constructions gave rise to a new type of coding problem.

First, we give two simple general results, an upper bound and a lower bound.

Definition 4.3.19 *Let $s_{pq}(\mathcal{N})$ denote the minimum number of rows of a matrix that (p, q)-represents $\mathcal{N}$, for an extension $\mathcal{N}$. If $\mathcal{N}$ is not (p, q)-representable, then we put $s_{pq}(\mathcal{N}) = \infty$.*

The following general upper bound is an easy corollary of Theorem 4.3.4.

Proposition 4.3.20 *Let $\mathcal{N}$ be an extension on $\mathbf{R}$ with $\mathcal{N}(\emptyset) = \emptyset$ and let (p, q) satisfy one of $(i) - (iii)$ of Theorem 4.3.4. assume that $|\mathbf{R}| = n$. Then*

$$s_{pq}(\mathcal{N}) \leq q(n + 1)2^n. \tag{4.32}$$

The proof of Lemma 4.2.2 can be easily adapted to show the following generalization.

Lemma 4.3.21 *Let us assume that $\mathcal{C}_n^k$ is (p, q)-representable. Then*

$$\binom{s_{pq}(\mathcal{C}_n^k)}{q+1} \geq \binom{n}{k-1}. \tag{4.33}$$

In some cases we could show that Lemma 4.33 gives the right order of magnitude. The constructions involve finite projective planes in one case, Hamiltonian theorem in another case [DKS98].

Theorem 4.3.22

$$3^{\frac{1}{3}}\, n^{\frac{2}{3}} + O(n^{\frac{1}{3}}) < s_{22}(\mathcal{C}_n^3) < \frac{3}{4^{\frac{1}{3}}}\, n^{\frac{2}{3}} + o(n^{\frac{2}{3}}). \tag{4.34}$$

The following proposition is an easy exercise, but we need it for the proof of Theorem 4.3.22.

Proposition 4.3.23 *The point-line pairs (P, l) $(P \in l)$ of the projective plane $PG(2, q)$ can be colored with $q + 1$ colors so that pairs with the same first or second coordinates receive distinct colors.*

Proof of Theorem 4.3.22 The lower bound follows from Lemma 4.3.21. The upper bound will be proved by a construction. We will construct a bipartite graph $G(A, B, E)$ with color classes A and B ($|A|+|B| = r$), where the set of edges E is a union of matchings $T_1, T_2, \ldots, T_t$. Let $V(T_j)$ denote the set of vertices covered by T_j. G will satisfy the following three properties:

$$\begin{array}{lll} (i) & V(T_i) \cap V(T_j) \neq \emptyset \quad \text{for any } i, j, & \\ (ii) & T_i \cap T_j = \emptyset \quad \text{for } i \neq j & \text{(no edge is covered twice)}, \\ (iii) & \forall C \in \binom{A \cup B}{3} \quad \binom{C}{2} \not\subseteq \bigcup T_j & \text{(no triangle).} \end{array} \tag{4.35}$$

Suppose for a moment that $G(A, B, E)$ is constructed. The $r \times t$ matrix M showing the upper bound is constructed as follows. The columns of M will be indexed by the matchings, while the rows will be indexed by the points of the bipartite graph. In a column indexed by some T_i we will have identical elements for the row pairs determined by the edges of T_i, different identical pairs for different edges, the other elements will be pairwise distinct and distinct from these pairs. In other words, columns of M correspond to partitions into two and one element classes. We claim, that this matrix $(2, 2)$-represents $\mathcal{C}_t^3$. Let T_x denote the matching corresponding to column x of M. Indeed, by property (i) of (4.35) there exist three rows u, v, w for any pair (a, b) of columns that contain at most two different entries in these columns. If there were a third column c also containing at most two distinct values in u, v and w, then by (ii) the equal entries in a, b and c must be in pairwise distinct pairs of rows. So, with $C = \{u, v, w\}$, $\binom{C}{2} \subseteq T_a \cup T_b \cup T_c$ would hold, that contradicts (iii). This proves that for every two–element subset $A \subset \mathbf{R}$ $\mathcal{J}_{M22}(A) = A$. The same argument shows that if $D \subset \mathbf{R}$ with $|D| > 2$, then there exist no three rows containing at most two different entries in each column from D, hence $\mathcal{J}_{M22}(D) = \mathbf{R}$. $\mathcal{J}_{M22}(\emptyset) = \emptyset$ and $\mathcal{J}_{M22}(\{a\}) = \{a\}$ for all $a \in \mathbf{R}$ follows from (ii) of Proposition 4.3.3.

Now the only thing left is to construct $G(A, B, E)$ with $r \sim ct^{\frac{2}{3}}$. Let A be the point set of $PG(2, q)$ and let $C = \{1, 2, \ldots, q+1\}$ be a $q+1$-element set. $q^2 + q + 1$ matchings can be constructed using Proposition 4.3.23, as follows. The matching T_l will correspond to line l of $PG(2, q)$, namely if P is a point incident to l and the color of (P, l) is i, then T_l contains the edge (P, i), so $|T_l| = q + 1$. The graph $G(A, C, \bigcup T_l)$ satisfies (ii) of (4.35), which follows from Proposition 4.3.23. Finally, any bipartite graph satisfies (iii) of (4.35) trivially.

If B is a union of k pairwise disjoint copies of C ($C_1, C_2, \ldots, C_k$) and the above matchings from A are constructed for each copy C_i, then it is not hard to see that the graph $G(A, B, \bigcup T_j)$ satisfies (i)-(iii) of (4.35). For example, $V(T_i)$ and $V(T_j)$ intersect in A, because $V(T_i) \cap A$ is a line of $PG(2, q)$, for all i.

This G results in an $r \times t$ matrix M, where $r = q^2 + q + 1 + k\,(q+1)$ and $t = k\,(q^2 + q + 1)$. This gives $r \sim 3\,q^2$ and $t \sim 2\,q^3$ if $k = 2\,q$. □

The exact value of $s_{pq}(\mathcal{C}_n^k)$ is known in a few cases only [DKS95] and [DKS98].

Theorem 4.3.24

$$\begin{array}{ll} \textit{(pq1)} & s_{pq}(\mathcal{C}_n^1) = q+1, \\ \textit{(222)} & s_{22}(\mathcal{C}_n^2) = 2n \textit{ for } n > 5, \\ \textit{(ppn)} & s_{pp}(\mathcal{C}_n^n) = \min\left\{\nu \textit{ integer: } \binom{\nu-1}{p} \geq n\right\}, \\ \textit{(122)} & s_{12}(\mathcal{C}_n^2) = \min\left\{s \textit{ integer: } \binom{s}{3} \geq 2\,n\right\} \textit{for } n > 452. \end{array} \tag{4.36}$$

The lower bound in (pq1) of Theorem 4.3.24 is an easy consequence of Lemma 4.3.21. The upper bound is given by a a $q+1 \times n$ matrix with all entries equal to i in row i.

A matrix M of $2n$-rows $(2,2)$-representing $\mathcal{C}_n^2$ can be constructed as follows. Rows $2i-1$ and $2i$ will contain 0 in column i and $2i-1$ and $2i$ in other columns, respectively. If $A \subset \mathbf{R}$ has more than one element, then there exist no three rows of M containing at most two different values in columns of A implying $\mathcal{J}_{M22}(A) = \mathbf{R}$. On the other hand, for any pair of one element subsets $\{i\}$ and $\{j\}$ of $\mathbf{R}$ rows $2i-1, 2i$ and $2j$ show that $\{i\} \overset{(2,2)}{\not\longrightarrow} j$ in M.

In order to prove that we need at least $2n$ rows to $(2,2)$-represent $\mathcal{C}_n^2$, let us assume that M is a representing matrix of minimum number of rows. As in the proof of Lemma 4.3.21, for every column there exist three rows that contain at most two different values in that column. That is, for every column, there is a pair of rows that agree on that column. We claim that these pairs are disjoint for different columns, which proves (222) of Theorem 4.3.24. The details are in [DKS95].

Proof of (ppn) of Theorem 4.3.24 Let us first prove the upper bound by a construction. Assume that $\binom{v-1}{p} \geq n$. Construct a matrix M of v rows and n columns as follows. The first row consists of all 0's. Then assign a distinct p-element subset of the remaining $v-1$ rows to every column, and put the numbers $1, 2, \ldots, p$ in them, respectively. The remaining entries are 0s. We show the case $p=2$, $n=6$ and $v=5$.

$$\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 0 & 2 & 2 \end{array} \tag{4.37}$$

Let us now assume that $b \notin A \subset \mathbf{R}$. Then there are $p+1$ distinct entries in column b in row 0 and the p rows assigned to b, while 0 occurs at least twice in these rows in columns of A. This means that $b \notin \mathcal{J}_{Mpp}(A)$, i.e. every subset A of $\mathbf{R}$ is closed under $\mathcal{J}_{Mpp}$, so $\mathcal{J}_{Mpp} = \mathcal{C}_n^n$.

On the other hand, let M (p,p)-represent $\mathcal{C}_n^n$ and let V be its set of rows. Every $n-1$-element set is closed in $\mathcal{L}_n^n$, thus there exist $p+1$ rows for any column $b \in \mathbf{R}$ such that they contain $p+1$ different entries in b but at most p distinct ones in each of the remaining columns. Thus these $p+1$-element row sets are all different, let S_b denote the one belonging to column b. We may assume without loss of generality that for every b the numbers $0,1,2,\ldots,p$ are standing in b and in the rows of S_b. Now let us change all entries of M which are not between 0 and p (inclusive) to 0. It is easy to see that the obtained matrix still (p,p)-represents $\mathcal{C}_n^n$, but now exactly $p+1$ different entries occur in each column. Let us consider the hypergraph $\mathcal{V} = (V, \{S_b \colon b \in \mathbf{R}\})$. $\mathcal{V}$ is $p+1$-uniform and there exists a partition of the vertex set V into $p+1$ classes for every edge S_b that completely cuts S_b but does not cut completely any other edge. This latter partition can be constructed according to the numbers occurring in column b. Such a hypergraph is called $p+1$-*forest*. Lovász [Lov79] proved that the maximum number of edges of a k-forest on m vertices is $\binom{m-1}{k-1}$. Now $\mathcal{V}$ is a $p+1$-forest on v points with n edges, so Lovász's result gives

$$n \le \binom{v-1}{p}. \tag{4.38}$$

□

We prove the upper bound in (122) of Theorem 4.3.24 via construction. In fact, we consider the number of rows m to be given, and construct $n = \left\lfloor \binom{m}{3}/2 \right\rfloor$ columns so that the $(1,2)$-dependency in that matrix will be exactly $\mathcal{C}_n^2$. The construction is based on the following theorem, which leads to coding theory type generalizations.

Theorem 4.3.25 ([DKS98]) *Let $|X| = n$ and $2k > q$. The family of all q-subsets of X can be partitioned into unordered pairs (except possibly one if $\binom{n}{q}$ is odd), so that paired q-subsets are disjoint and if A_1, B_1 and A_2, B_2 are two such pairs with $|A_1 \cap A_2| \ge k$, then $|B_1 \cap B_2| < k$, provided $n > n_0(k,q)$.*

Let us suppose, that m is an integer that satisfies $\binom{m}{3} \ge 2n$. A matrix with m rows and n columns will be constructed that $(1,2)$-represents $\mathcal{C}_n^2$. Let us denote the set of rows by X. Apply Theorem 4.3.25 with $q=3$ and $k=2$ to obtain disjoint pairs of 3-subsets of X. There are $\left\lfloor \binom{m}{3}/2 \right\rfloor$, that is, at least n such pairs. Choose n of them. We construct a column from such a pair, as follows. Put 1's in the rows indexed by the first 3-set, 2's in the rows indexed by the second one, and all different entries, that are at least 3, in the other positions.

If a and b are two distinct columns, then there are no 3 rows that agree in both a and b, because we used all distinct 3-subsets of rows, hence $\{a,b\} \xrightarrow{(1,2)} \mathbf{R}$. On the other hand, if a is constructed from the pair of 3-subsets A_1, A_2 and b is constructed from B_1, B_2, then either $|A_1 \cap B_1| < 2$ or $|A_2 \cap B_2| < 2$, so there are 3 rows which contain all identical entries in column a, but all distinct ones in column b, hence $a \overset{(1,2)}{\not\longrightarrow} b$. □

Theorem 4.3.25 is proved using the following Hamiltonian type theorem.

Theorem 4.3.26 ([DKS98]) *Let $G_0 = (V, E_0)$ and $G_1 = (V, E_1)$ be simple graphs on the same vertex set $|V| = N$, such that $E_0 \cap E_1 = \emptyset$. The 4-tuple (x, y, z, v) is called an* alternating cycle *if (x, y) and (z, v) are in E_0 and (y, z) and (x, v) are in E_1. Let r be the minimum degree of G_0 and let s be the maximum degree of G_1. Suppose, that*

$$2\,r - 8\,s^2 - s - 1 > N, \tag{4.39}$$

then there is a Hamiltonian cycle in G_0 such that if (a, b) and (c, d) are both edges of the cycle, then (a, b, c, d) is not an alternating cycle.

The pairs of disjoint q-subsets are obtained from neighboring vertices of a Hamiltonian cycle of type above. G_0 and G_1 are as follows. The vertex set V consists of the q-subsets of X, $|V| = \binom{n}{q} = N$. Two q-subsets are adjacent in G_0 if their intersection is empty, while two q-subsets are adjacent in G_1 if they intersect in at least k elements.

Note that Lemma 4.3.21 gives only $\binom{s_{12}(\mathcal{C}_n^2)}{3} \geq n$ as a lower bound. To obtain the one in (4.36), we introduced the concept of *indicator triplets* in [DKS98]. Suppose, that M is a matrix of m rows and n columns that $(1,2)$-represents $\mathcal{C}_n^2$. Each column of M determines a partition of the row set $\{1, 2, \ldots, m\}$ according to which entries are the same. The partition corresponding to column i is denoted by Π_i. A triplet $\{i, j, k\}$ is an *indicator* for the partition (column) Π_t if there is another column u such that i, j, k are in the same class of Π_t but are in three different classes in Π_u. (That is, the triplet of rows shows that $t \overset{(1,2)}{\not\longrightarrow} u$.)

Fact 4.3.27 *A triplet can be an indicator for at most one column.*

Fact 4.3.28 *For any pair of columns t and u, there is an indicator triplet $\{i, j, k\}$ for Π_t such that i, j and k are in three different classes of Π_u.*

Partition Π_t is called of *first kind*, iff there exist at least two different indicator triplets for Π_t. Otherwise, the partition is called of *second kind*.

Proposition 4.3.29 *Let Π_u be a partition of second kind. Then the elements i, j, k of the indicator triple of Π_u are all in different classes in any other partition Π_t.*

If not all three elements were in different classes of Π_t, then an other triplet should show that $u \overset{(1,2)}{\not\longrightarrow} t$, so Π_u would not be of second kind. □

As a corollary, we obtain that the indicator triplets of partitions of second kind form an at most 1-intersecting system. We need the following easy lemma.

Lemma 4.3.30 *Let $\mathcal{T} = \{T_1, T_2, \dots, T_k\}$ be an at most 1-intersecting system of triplets of M. Then there exists a collection $\mathcal{S}$ of k triplets of M such that each member of $\mathcal{S}$ 2-intersects at least one member of $\mathcal{T}$.*

Let $\mathcal{S}$, $|\mathcal{S}| = s$ be the system of such triplets that 2-intersect at least one member of $\mathcal{T}$. We use double counting, namely we count the number of pairs (T, S), $T \in \mathcal{T}$, $S \in \mathcal{S}$ and $|S \cap T| = 2$. On one hand, counting by the T's, it is $3\,k\,(m-3)$. On the other hand, for each S there are at most $3\,(m-3)$ T's that 2-intersects, so $s\,3\,(m-3)$ is at least as large as the number to count, which imply $s \geq k$. □

According to Fact 4.3.27 all indicator triplets are different. Partitions of the first kind each use at least two of them. The indicator triplets of partitions of the second kind can be matched with triplets of M so that matched pairs 2-intersect, by Lemma 4.3.30 and Hall's condition. These matched triplets cannot coincide with some indicator triplet by Fact 4.3.28, so we have found two "own" triplets for each partition of second kind, as well. This proves $\binom{m}{3} \geq 2\,n$. □

Coding type questions

Enomoto and Katona [EK01] realized that Theorem 4.3.25 really speaks about a certain kind of distance-like concept. Define the *closeness* of the pairs $\{A_1, B_1\}$ and $\{A_2, B_2\}$ by

$$\gamma(\{A_1, B_1\}, \{A_2, B_2\}) = \max\{|A_1 \cap A_2| + |B_1 \cap B_2|, |A_1 \cap B_2| + |B_1 \cap A_2|\} \tag{4.40}$$

It is clear that $|A_1 \cap A_2| \geq k$ and $|B_1 \cap B_2| \geq k$ imply $\gamma((A_1, B_1), (A_2, B_2)) \geq 2k$ for sets satisfying $A_1 \cap B_1 = A_2 \cap B_2 = \emptyset$, therefore the following theorem is really a sharpening of Theorem 4.3.25 .

Theorem 4.3.31 ([EK01]) *Let $|X| = n$. The family of all k-element subsets of X can be partitioned into disjoint pairs (except possibly one if $\binom{n}{k}$ is odd), so that $\gamma(\{A_1, B_1\}, \{A_2, B_2\}) \leq k$ holds for any two such pairs $\{A_1, B_1\}$ and $\{A_2, B_2\}$, provided $n > n_0(k)$.*

The proof of Theorem 4.3.31 follows the line of that of Theorem 4.3.25, the difference is that a strengthening of Theorem 4.3.26 is needed, which involves weighted Hamiltonian cycles. Define $\delta(\{A_1, B_1\}, \{A_2, B_2\}) = 2k - \gamma(\{A_1, B_1\}, \{A_2, B_2\})$. This is a "distance" in the "space" of all disjoint pairs of k-element subsets of X. Theorem 4.3.31 answers a coding type question, how many elements can be chosen from this space with large pairwise distances. The distance above can be equivalently formulated as follows. Let $n, k \in \mathbb{N}$ with $2k \leq n$ and X be an n-set. Consider

$$\mathcal{R} := \left\{ \{A, B\} \subseteq \binom{X}{k} \mid A \cap B = \emptyset \right\}, \tag{4.41}$$

consisting of all unordered pairs of disjoint k-element subsets of X. The function

$$\begin{aligned} d^{\mathcal{R}}\colon\quad & \mathcal{R} \times \mathcal{R} \to \{0, 1, ..., 2k\}, \\ & (\{A, B\}, \{S, T\}) \mapsto \min\{|A \setminus S| + |B \setminus T|, |A \setminus T| + |B \setminus S|\} \end{aligned} \tag{4.42}$$

is a metric on $\mathcal{R}$. The finite metric space $(\mathcal{R}, d^{\mathcal{R}})$, called Enomoto-Katona space, was motivated by our construction in Theorem 4.3.25, and discussed in Katona et al. [BK01, BKL, KS04] as well as in Quistorff [Qui05, Qui09]. Recently, it was mentioned in a monograph by Deza/Deza [DD06] which might become a standard reference.

A main issue concerning this space is the *coding type problem*, i.e. the determination of the maximum cardinality of a code consisting of unordered pairs of subsets far away from each other.

4.4 Armstrong codes

All papers cited above assumed that the *domain* of each attribute is unbounded, countably infinite. However, in the study of *Higher Order Datamodel* [HLS04, Sal04, SS06, SS08b] the question of bounded domains arises naturally. In fact, if a minimal key system contains only *counter attributes*, then the possible number of tuples in an Armstrong instance is bounded from above. Another reason to consider bounded domains comes from real life databases. In many cases the domain of an attribute is a well defined finite set, for example in car rental, the class of cars can take values from the set {subcompact, compact, mid-size, full-size, SUV, sports car, van}. Same kind of finiteness may occur in case of job assignments, schedules, etc.

It is natural to ask what can be said about Armstrong instances if attribute A_i has a domain of size ℓ_i. The main question investigated in this paper was introduced in [SS08b].

Definition 4.4.1 *Let $q > 1$ and $k > 1$ be given natural numbers. Let $f(q,k)$ be the maximum such n that there exists an Armstrong instance using at most q symbols for the closure $\mathcal{C}_n^k$.*

It is clear that for a meaningful Armstrong instance we need at least two distinct symbols, so $q > 1$ is necessary. On the other hand the minimal Armstrong instance for $\mathcal{C}_n^1$ uses only two symbols for arbitrary n [DK81], hence $f(q,k)$ is well defined only for $k > 1$. The following basic fact is known [DK81].

Proposition 4.4.2 *R is an Armstrong instance for $\mathcal{C}_n^k$ if and only if the following two properties hold:*

*(**K**) there exist no two rows of R that agree in at least k positions,*

*(**A**) for every $A \subset R$, $|A| = k-1$, there exist two rows of R that agree in all positions of A.*

It is helpful to view an Armstrong instance for $\mathcal{C}_n^k$ using q symbols as a q-ary code $\mathcal{C}$ of length n, where codewords are the tuples, or rows of the instance. Using Proposition 4.4.2

(**md**) $\mathcal{C}$ has minimum distance at least $n-k+1$ by (**K**).

(**di**) For any set of $k-1$ coordinates there exist two codewords that agree exactly there by (**A**).

A $k-1$-set of coordinates can be considered as a direction, so in $\mathcal{C}$ the minimum distance is *attained in all directions.* A q-ary code of length n satisfying the two properties above is called an *Armstrong(q,k,n)-code.* Thus, $f(q,k)$ is the largest n such that an Armstrong(q,k,n)-code exists. Note that the $(k+1)\times(k+1)$ identity matrix is an Armstrong($2,k,k+1$)-code, so Armstrong codes *do* exist. We proved the following lower and upper bounds in [GOHKSS08].

Theorem 4.4.3 *1. Given $q > 4$, there is k_0 such that for every $k > k_0$ and for every $n < \frac{1}{2}k\log q$ we have $n \leq f(q,k)$.*

2. There exists k_0 and $c > 1$ constants, that for $k > k_0$, and $\lfloor ck \rfloor \leq f(2,k)$.

3. Let $q > 1$ and $k > 2$ be integers. Then

$$f(q,k) \leq q(k-1)\left(1 + \frac{q-1}{\sqrt{\frac{2(qk-q-k+2)^{k-1}}{(k-1)!}} - q}\right) \tag{4.43}$$

holds.

4. *If* $5 \leq k$ *and* $2 \leq q$ *then the upper bound in (4.43)can be improved to*

$$f(q,k) \leq q(k-1) \tag{4.44}$$

with the following exceptions: $(k,q) = (5,2),(5,3),(5,4),(5,5),(6,2)$.

The lower bounds were given by greedy construction. The main advantage of the second lower bound is that it gives a constant larger than 1, while the identity matrix construction does not. In order to prove the upper bounds we give two estimates on n that are functions of the number of codewords: $a_{q,k}(m)$ being a decreasing, while $b_{q,k}(m)$ being an increasing function of m. Therefore, if α is the solution of the equation

$$a_{q,k}(m) = b_{q,k}(m) \tag{4.45}$$

in m then $a_{q,k}(\alpha) = b_{q,k}(\alpha)$ is a universal (independent of m) upper bound for n. The paper [GOHKSS08] also contains an exact and an almost exact bound.

Proposition 4.4.4 $f(q,2) = \binom{q+1}{2}$ *and* $f(q,3) \leq 3q-1$.

Interestingly enough, Theorem 4.2.8 gives a lower bound for $f(q,3)$. The Armstrong instance provided there has $r+1$ symbols in every column, and has $3r+1$ columns. That is, $q = r+1$ and $n = 3q-2$. We believe that this is the right answer, since that is a solution of the minimum representation of $\mathcal{C}_n^3$. Nevertheless, the proof of Proposition 4.4.4 has no room for for improvement.

It was clear that the lower bound given in Theorem 4.4.3 can be improved, but constructions are hard to come by. On the other hand, the upper bound (4.44) seems nice enough to be sharp. However, we could improve on both in [SS08a].

Theorem 4.4.5 *For* $k > k_0(q)$ *we have*

$$\frac{\sqrt{q}}{e}k < f(q,k) < (q - \log q)k. \tag{4.46}$$

Proof of Theorem 4.4.5 The idea of the upper bound is to embed an Armstrong(q,k,n)-code into an $n' = (q-1)n$-dimensional space as a spherical code and use existing bounds for the size spherical codes of given minimum distance. On the other hand, an old result of Demetrovics and Katona [DK81] gives a lower bound for the size of an Armstrong(q,k,n)-code. Comparing the two estimates results in the lower bound for c, where $k-1 = cn$.

It is not hard to see that if k is fixed and an Armstrong(q,k,n)-code exists for some $k < n$, then Armstrong(q,k,n')-codes also exist for all $k < n' < n$.

Let $\mathcal{C}$ be an Armstrong(q,k,n)-code of size $m = |\mathcal{C}|$. Let $\ell = k-1$. Using (**di**) and the argument of [DK81],

$$\binom{n}{\ell} \leq \binom{m}{2} \tag{4.47}$$

is obtained. Let $s\colon \{0,1,\ldots,q-1\} \to \mathbb{R}^{q-1}$ be a bijective mapping of the q symbols to the vertices of a regular simplex centered at the origin. Extend this mapping to codewords by juxtaposition of coordinates of vectors that are images of symbols of codewords under s. Thus each codeword of $\mathcal{C}$ is mapped to a vector from $\mathbb{R}^{(q-1)n}$ and we normalize them so they are unit vectors. Let $\mathcal{D}$ be the spherical code obtained. Using the minimum distance of $\mathcal{C}$ we obtain that $\mathcal{D}$ has minimum angle ϕ with $\cos\phi = \frac{\ell q - n}{(q-1)n}$ and $\sin(\frac{\phi}{2}) = \sqrt{\frac{q(n-k+1)}{2(q-1)n}}$. By (4.47) and the upper bound of Rankin [Ran55]

$$A(n,\phi) \leq \sqrt{\frac{\pi}{2} n^3 \cos\phi}\left(\sqrt{2}\sin(\frac{\phi}{2})\right)^{-n}(1+o(1)) \tag{4.48}$$

on the maximum size of a spherical code in n dimension with minimum angle ϕ. Applying that $m \leq A((q-1)n,\phi)$ we obtain

$$\sqrt{2\binom{n}{\ell}} < m \leq \sqrt{\frac{\pi}{2}(q-1)^3 n^3 \frac{\ell q - n}{(q-1)n}}\left(\sqrt{\frac{q(n-k+1)}{(q-1)n}}\right)^{-(q-1)n}(1+o(1)). \tag{4.49}$$

Writing $\ell = cn$ and using the approximation of $\binom{n}{cn}$ (4.49) yields

$$\sqrt{2}\left(\frac{1}{c^c(1-c)^{1-c}}\right)^{\frac{n}{2}} < \sqrt{\frac{\pi}{2}}(q-1)n\sqrt{(cq-1)n}\left(\sqrt{\frac{q-1}{q(1-c)}}\right)^{(q-1)n}. \tag{4.50}$$

Now, (4.50) can only hold for large enough n if

$$\frac{1}{c^c(1-c)^{1-c}} < \left(\frac{q-1}{q(1-c)}\right)^{q-1}. \tag{4.51}$$

It is easy to see that for $c = \frac{1}{q}$ LHS>1 and RHS=1 in (4.51). However, that only gives the upper bound established in [GOHKSS08]. With considerably more effort it can be shown that LHS>RHS in (4.51) for $c = \frac{1}{q-\log q}$, as well. Indeed, let $x = q - \log q$, $c = \frac{1}{x}$ and consider x as a function of variable q. LHS>RHS in (4.51) means

$$\frac{1}{\left(\frac{1}{x}\right)^{\frac{1}{x}}\left(\frac{x-1}{x}\right)^{\frac{x-1}{x}}} > \left(1-\frac{1}{q}\right)^{q-1}\left(\frac{x-1}{x}\right)^{1-q}. \tag{4.52}$$

Now, (4.52) can be written as

$$\frac{x}{(x-1)^{\frac{x-1}{x}}} > \left[\left(1-\frac{1}{q}\right)\left(1+\frac{1}{x-1}\right)\right]^{q-1}. \tag{4.53}$$

taking the logarithm of both sides of (4.53)

$$\log x - \frac{x-1}{x}\log(x-1) > (q-1)\left[\log\left(1-\frac{1}{q}\right)+\log\left(1+\frac{1}{x-1}\right)\right] \tag{4.54}$$

is obtained. Both sides of (4.54) is differentiated with respect to q to show that the difference LHS−RHS is an increasing function of q. For small values of q the validity of (4.51) is checked by computer. The derivative of LHS is

$$\frac{1}{x}\cdot x' - \frac{x-1}{x}\cdot\frac{1}{x-1}\cdot x' - \frac{-1}{(x-1)^2}\cdot x'\cdot\log(x-1), \tag{4.55}$$

where $x' = 1-\frac{1}{q}$. Similarly, the derivative of the RHS is

$$\begin{aligned}\log\left(1-\tfrac{1}{q}\right) &+ \log\left(1+\tfrac{1}{x-1}\right)\\ &+ (q-1)\tfrac{1}{(1-\frac{1}{q})}\cdot\tfrac{-1}{q^2}+(q-1)\tfrac{1}{1+\frac{1}{x-1}}\cdot\tfrac{-1}{(x-1)^2}\cdot x'.\end{aligned} \tag{4.56}$$

Simplifying the inequality (4.55) − (4.56) ≥ 0

$$\frac{1}{(x-1)^2}\cdot\frac{q-1}{q}\cdot\log(x-1)+\frac{1}{q}+\frac{(q-1)^2}{q(x-1)x} \geq \log\left(1-\frac{1}{q}\right)+\log\left(1+\frac{1}{x-1}\right) \tag{4.57}$$

is obtained. The right hand side of (4.57) can be estimated as

$$\log(q-1)-\log q+\log x-\log(x-1) = -\int_{q-1}^{q}\frac{dy}{y}+\int_{x-1}^{x}\frac{dy}{y} < \frac{-1}{q}+\frac{1}{x-1}. \tag{4.58}$$

Considering that the left hand side of (4.57) is $\frac{1}{q}$ plus two positive numbers, it is enough to see that

$$\frac{1}{q} \geq \frac{-1}{q}+\frac{1}{x-1}. \tag{4.59}$$

However, (4.59) holds if $q \geq 2\log q + 2$, which is true as long as $q \geq 8$. Solving for c the equality of LHS and RHS in (4.51) numerically in the range of $q = 2, 3, \ldots, 200$ suggests that the limitation of the method above is really $\frac{1}{c} = q - \log q$.

We will use the Lovász Local Lemma, that is a probabilistic construction will be given to prove the lower bound. For each $|K| = k-1$ subset of coordinate positions a pair of codewords (A_1^K, A_2^K) is chosen randomly that agree exactly at those positions. That is, in each coordinate position each symbol is chosen with probability $\frac{1}{q}$, and the choices are pairwise independent for distinct positions. Consider events $v(A_i^K, A_j^L)$ where $i, j \in \{1,2\}$ and $K \neq L$ are $k-1$-sets of coordinate positions, that the two codewords agree in at least k coordinates. Two such events $v(A_i^K, A_j^L)$ and $v(A_i^{K'}, A_j^{L'})$ are independent if $\{K, L\} \cap \{K', L'\} = \emptyset$. Define the dependency graph $G = (V, E)$ by V being the set of events $v(A_i^K, A_j^L)$, and $v(A_i^K, A_j^L)$ and $v(A_i^{K'}, A_j^{L'})$ are connected by an edge if $\{K, L\} \cap \{K', L'\} \neq \emptyset$. Thus the degree of $v(A_i^K, A_j^L)$ in the dependency graph is $4\binom{n}{k-1} - 4$. On the other hand,

$$\text{Prob}(v(A_i^K, A_j^L)) = \sum_{\ell=k}^{n} \binom{n}{\ell} \left(\frac{1}{q}\right)^{\ell} \left(\frac{q-1}{q}\right)^{n-\ell} = B(k, n, \frac{1}{q}). \tag{4.60}$$

By the well-known Chernoff bound $B(k, n, p) \leq \left(\frac{np}{k}\right)^k e^{k-np}$ if $k > np$ that in our case means $k > \frac{n}{q}$, which can be assumed without loss of generality. If $4\binom{n}{k-1} B(k, n, \frac{1}{q}) < \frac{1}{e}$, then

$$\text{Prob}\left(\bigcap \overline{v(A_i^K, A_j^L)}\right) > 0 \tag{4.61}$$

is obtained using Lovász' Local Lemma, consequently an Armstrong(q, k, n)-code code exists with these parameters. Writing $n = ck$, $4\binom{n}{k-1} B(k, n, \frac{1}{q}) < \frac{1}{e}$ becomes

$$\left[\frac{1}{\left(\frac{1}{c}\right)^{\frac{1}{c}} \left(1 - \frac{1}{c}\right)^{1-\frac{1}{c}}}\right]^{ck} \left(\frac{c}{q}\right)^k e^{k\left(1-\frac{c}{q}\right)} < \frac{1}{4e}. \tag{4.62}$$

The LHS of (4.62) is a complete kth power, thus if

$$\frac{1}{\frac{1}{c}\left(1 - \frac{1}{c}\right)^{c-1}} \left(\frac{c}{q}\right) e^{\left(1-\frac{c}{q}\right)} < 1, \tag{4.63}$$

then (4.62) holds for $k > k_0$. Since $\left(\frac{c}{c-1}\right)^{c-1} < e$, (4.63) holds if $\frac{c^2}{q} e^{2-\frac{c}{q}} < 1$. This latter one is certainly true for $c < \frac{\sqrt{q}}{e}$. □

4.4.1 Constructions of binary Armstrong codes

Using simulated annealing Andries Brouwer found extremal examples of binary Armstrong codes. As it was shown by A. Keszler in her diploma thesis using computer that Armstrong$(2, n-2, n)$-codes do not exist for $n \leq 8$, the case $n = 9$ is extremal. Later Brouwer found a description, as well, using Steiner triple systems.

Proposition 4.4.6 ([Bro08]) *Armstrong*$(2,7,9)$*- and Armstrong*$(2,7,10)$*-codes exist.*

Proof of Proposition 4.4.6 Take the unique S(3,4,10) and delete a point and add the all-0 vector. This is an Armstrong$(2,7,9)$-code. Take the unique S(3,4,10) and the all-0 vector. This is an Armstrong$(2,7,10)$-code. □

We can construct Armstrong$(2,n-2,n)$-codes for any $n > 19$.

Proposition 4.4.7 ([BS]) *There exists Armstrong*$(2,n-2,n)$*-codes for any* $n > 19$*.*

Proof of Proposition 4.4.7 First we partition the triplets from an n-set into n collections with the property that two triplets in the same collection intersect in at most 1 point. The triplet $\{p,q,r\}$ goes into collection $\mathcal{T}_i$ iff $p+q+r \equiv i \pmod{n}$. Now let $\vec{c}_0, \vec{c}_1, \dots, \vec{c}_{n-1}$ be vectors formed from n rows of a (0,1)-Hadamard matrix of smallest possible order that is at least n by taking the first n coordinates of each, respectively. Since $n \geq 20$, the vectors $\vec{c}_i$ have pairwise Hamming distance at least 9. Our code $\mathcal{C}$ consists of the codewords $\vec{c}_0, \vec{c}_1, \dots, \vec{c}_{n-1}$, and for every $T \in \mathcal{T}_i$ the codeword $\vec{c}_i + \vec{t}$ where $\vec{t}$ is the characteristic vector of triple T. It is straightforward to check that $\mathcal{C}$ is an Armstrong$(2,n-2,n)$-code. □

This "skeleton-code" approach can be extended for Armstrong$(2,k,n)$-codes where $n-k=m$ is small. If $m=3$, we can partition the quadruplets of an n-set into n classes that two quadruplets in the same collection intersect in at most 2 elements by putting quadruple $\{p,q,r,a\}$ in collection $p+q+r+s \pmod{n}$. Using Hadamard matrices, we can find n codewords of mutual distance 12 if $n=24$ or $n \geq 26$. For general m we can prove the following.

Theorem 4.4.8 *Let* $n-k=2m$ *or* $n-k=2m-1$ *and* $m>1$*. Then an Armstrong*$(2,k,n)$*-code exists if* $n \geq 8m\log m$*.*

Proof of Theorem 4.4.8 We apply the skeleton-code method. That is the $n-k+1$-tuples (subsets) of the n-set are partitioned into A classes $\mathcal{T}_i$ $(i=0,1,\dots,A-1)$ so that the symmetric difference of two tuples in the same class is at least $n-k+1$. Then A codewords $\vec{c}_0, \vec{c}_1, \dots, \vec{c}_{A-1}$ of length n are selected of pairwise distance at least $3(n-k+1)$. The Armstrong$(2,k,n)$-code $\mathcal{C}$ will consists of the codewords $\vec{c}_0, \vec{c}_1, \dots, \vec{c}_{A-1}$, and for every $T \in \mathcal{T}_i$ the codeword $\vec{c}_i + \vec{t}$ where $\vec{t}$ is the characteristic vector of $n-k+1$-tuple T. If T and T' are $n-k+1$-tuples of the same class $\mathcal{T}_i$, then the distance of $\vec{c}_i + \vec{t}$ and $\vec{c}_i + \vec{t'}$ is exactly the size of the symmetric difference of T and T', that is at least $n-k+1$. If T and T' are of different classes, say $T \in \mathcal{T}_i$ and $T' \in \mathcal{T}_j$, then the distance of $\vec{c}_i + \vec{t}$ and $\vec{c}_j + \vec{t'}$ is at least $3(n-k+1) - |T| - |T'| \geq n-k+1$. Thus the minimum distance of code $\mathcal{C}$ is $n-k+1$ that is attained in every $n-k+1$-tuple of coordinates T between codewords $\vec{c}_i$ and $\vec{c}_i + \vec{t}$, that is $\mathcal{C}$ is an Armstrong$(2,k,n)$-code.

In order to find the partition of the $n-k+1$-tuples, let p be the smallest prime number not less than n. (It is known that $p \leq n + n^{3/5}$.) Let $n - k + 1 = 2m + 1$, (the $n-k = 2m-1$ case is similar). The $2m+1$-tuple $\underline{a} = (a_1, a_2, \ldots, a_{2m+1})$ is mapped to $(\sigma_1(\underline{a}), \sigma_2(\underline{a}), \ldots, \sigma_m(\underline{a})) \in \mathbb{F}_p^m$, where $\sigma_i(\underline{a})$ is the i^{th} symmetric function. We claim that if two $2m+1$-tuples mapped to the same m-tuple $\mod p$, then they differ in at least $m+1$ positions, so the symmetric difference of them is at least $n-k+1$. Indeed, let $\underline{a} = (a_1, a_2, \ldots, a_{2m+1})$ and $\underline{b} = (b_1, b_2, \ldots, b_{2m+1})$ be mapped to the same m-tuple, that is assume that $\sigma_i(\underline{a}) = \sigma_i(\underline{b}$ for $i = 1, 2, \ldots, m$. Consider the polynomials $a(x) = \prod_{i=1}^{2m+1}(x - a_i)$ and $b(x) = \prod_{i=1}^{2m+1}(x - b_i)$. Then by the agreement of the symmetric functions $a(x) - b(x)$ is of degree m. On the other hand, $c(x) = \prod_{c \in \underline{a} \cap \underline{b}}(x - c) \mid a(x) - b(x)$. This implies that the degree of $c(x)$ is at most m, so the tuples $\underline{a}$ and $\underline{b}$ differ in at least $m+1$ positions. A partition class of the $n-k+1$-tuples consists of those that map to the same m-tuple $\mod p$. The number of the latter vectors is p^m, so we need that many codewords of length n of pairwise distance at least $3(n-k+1) = 3(2m+1)$. Now, if $M < p^m$ codewords, then the spheres of radius $3(2m+1)$ around them cover at most $Mn^{3(2m+1)}$ points, so if $p^m n^{3(2m+1)} < 2^n$, then we can find a skeleton code using a simple greedy argument. This latter inequality is true if $n \geq 8m \log m$. □

4.5 A discrepancy type result

This section contains some extremal results motivated also by databases, but this time the motivation is data retrieval optimization.

Information systems often use the two dimensional screen as a tool for retrieval of detailed data that is associated with a specific part of the 2D-screen. A standard example is a geographic database, where first a low resolution map is displayed on the 2D-screen and then the user specifies a part of the map that is to be displayed in higher resolution. Another application is when pictures of famous sightseeing spots of an area are to be displayed. Efficient support of such queries is quite important for image databases in particular, and for browsing geographically referenced information in general. In the Alexandria Digital Library project [Smi96] a large satellite image is divided into tiles and each tile is decomposed using wavelet decomposition [VH92]. A wavelet decomposition of an image results in a lower resolution image of the original one together with higher order coefficients that can be used to retrieve higher resolution versions of the same image. Similar approaches are common to other systems for browsing large image databases [FBF$^+$94, OS95]. A user would usually browse the lower resolution images fast and then specify areas to be displayed in higher resolution. This requires the retrieval of the higher resolution components for the various tiles that overlap with the specific region.

In the present section the model introduced in [AGA97] is analyzed further. It is assumed that data is associated with the tiles of a two dimensional grid. The data corresponding to individual tiles is usually large, so it is preferable to store them on parallel I/O devices in such a way, that for a given query, retrieval from these parallel devices can occur concurrently. The ideal situation, when information related to each individual tile could be stored on a distinct I/O device, and hence data for any query could be retrieved concurrently, is not realizable in general because the number of tiles is much larger, than the number of I/O devices available. Thus, an idea is to "spread out" data over the available I/O devices as evenly as possible (to achieve good parallel speedup). In the following, a measure of optimality of data allocation is defined as smallest possible discrepancy in the number of access requests for different I/O devices for any rectangular set of tiles. Upper bounds for this discrepancy of $O(\log(m))$ are derived using latin squares in Theorem 4.5.4. Asymptotically matching lower bounds are given for these latin square constructions. The results show that excellent parallel speedup is possible.

This section is based on a conference paper [ADKS00] and the journal article [AS07]. Subsequent to the publication of [ADKS00], Doerr et al [DHW04] obtained alternative proofs yielding similar bounds and proving the lower bound in general. We believe the latin square techniques in this section are interesting as latin square results and may shed further light on how to obtain small discrepancy allocations.

The mathematical problem

Consider an $n_1 \times n_2$ array, whose elements (i,j) where $1 \leq i \leq n_1$ and $1 \leq j \leq n_2$ are called *tiles*. Given two tiles (i_1,j_1) and (i_2,j_2), where $i_1 \leq i_2$ and $j_1 \leq j_2$, two dimensional query rectangle is defined by

$$\mathcal{R} = \mathcal{R}[(i_1,j_1),(i_2,j_2)] = \{(i,j)\colon i_1 \leq i \leq i_2 \text{ and } j_1 \leq j \leq j_2\}. \tag{4.64}$$

This represents a rectangle, whose opposite corners are (i_1,j_1) and (i_2,j_2) and the *area* is $(i_2-i_1+1)\,(j_2-j_1+1)$ (the number of tiles contained in the rectangle). To each tile (i,j) in $\mathcal{R}$ is assigned a number $f(i,j)$ from the set $\{1, 2, \ldots, m\}$. The number $f(i, j)$ refer to one of m available I/O devices on which the information related to the given tile is stored and f is called an *m-assignment* for the array. We define the multiplicity of a symbol k in a rectangle $\mathcal{R}$ as

$$d_{\mathcal{R},f}(k) = |\{(i, j) \,:\, (i, j) \in \mathcal{R},\; f(i, j) = k\}| \tag{4.65}$$

An m-assignment is called *d-discrepancy assignment* if for any given rectangle $\mathcal{R} = \mathcal{R}[(i_1, j_1), (i_2, j_2)]$

$$\max_{\mathcal{R}}\{ \max_{i:\,1\le i\le m} d_{\mathcal{R},f}(i) - \min_{j:\,1\le j\le m} d_{\mathcal{R},f}(j)\} \le d \tag{4.66}$$

holds. Clearly, d-discrepancy m-assignments with small d are sought for efficient retrieval of data using as many I/O devices concurrently, as possible. The *optimality* $d(m)$ of m is the minimum d, such that a d-discrepancy m-assignment exists for arbitrary n_1 and n_2. There can only be 0-discrepancy assignments for the trivial case $m = 1$. In [AGA97], 1-discrepancy m-assignments were called *strictly optimal* and the following was proved.

Theorem 4.5.1 ([AGA97]) *A 1-discrepancy m-assignment exists for an $n_1 \times n_2$ array $\mathcal{R}$ iff one of the following conditions holds:*

$$\begin{array}{ll} - & \min\{n_1, n_2\} \le 2, \\ - & m \in \{1, 2, 3, 5\}, \\ - & m \ge n_1 n_2 - 2, \\ - & m = n_1 n_2 - 4 \text{ and } \min\{n_1, n_2\} = 3, \\ - & m = 8 \text{ and } n_1 = n_2 = 4. \end{array} \tag{4.67}$$

Corollary 4.5.2 *$d(m) \ge 2$ if $m \notin \{1, 2, 3, 5\}$.*

Let us consider m-assignments for $\infty \times \infty$ arrays for $m \ge 2$. We define a variation on the discrepancy introduced that measures the difference of the multiplicity of a symbol as compared to its expected number. The resulting discrepancy measures are at most the previous measures and at least half the previous measures. Let

$$\mathsf{disc}_{\mathcal{R}}(k) = |d_{\mathcal{R},f}(k) - \frac{1}{m}\text{area } \mathcal{R}|, \tag{4.68}$$

$$\mathsf{disc}(\mathcal{R}) = \max_{k} \mathsf{disc}_{\mathcal{R}}(k) \tag{4.69}$$

and

$$\mathsf{disc}(f) = \max_{\mathcal{R},k} \mathsf{disc}_{\mathcal{R}}(k). \tag{4.70}$$

For the f we consider, this max is well defined. It is natural to then define

$$\mathsf{disc}(m) = \min_f \mathsf{disc}(f). \tag{4.71}$$

Note that $\frac{1}{2}d(m) \leq \mathsf{disc}(m) \leq d(m)$ and we focus on $\mathsf{disc}(m)$ in this section. We could extend the definition of $\mathsf{disc}(\mathcal{R})$ to $\mathsf{disc}(S)$ for some arbitrary set of positions $S \subseteq \infty \times \infty$. For example, if $\mathcal{R}_1$ and $\mathcal{R}_2$ are two disjoint rectangles then we have $\mathsf{disc}(\mathcal{R}_1 \cup \mathcal{R}_2) \leq \mathsf{disc}(\mathcal{R}_1) + \mathsf{disc}(\mathcal{R}_2)$. Of course the use of rectangles in (4.70) is an essential part of the problem.

Good m-assignments are constructed using design techniques for latin squares with discrepancy $O(\log(m))$. Theorem 4.5.7 shows that if we have good n- and m-assignments from latin squares then we can find a good nm-assignment. Then using transversals in latin squares the construction is extended for all orders.

Let us recall that a *latin square* L of order m is an $m \times m$ array consisting of m different symbols, $\{1, 2, \ldots, m\}$, such that each symbol occurs in each row and column exactly once. We can associate an entry function $f : \{1, 2, \ldots, m\} \times \{1, 2, \ldots, m\} \longrightarrow \{1, 2, \ldots, m\}$ where $f(i, j)$ is the entry of L in row i and column j. This notation is a useful way to encode certain manipulations we will perform. A *transversal* T of L is a set of m positions $T \subseteq \{1, 2, \ldots, m\} \times \{1, 2, \ldots, m\}$ containing pairwise different entries, with $T = \{(1, t(1)), (2, t(2)), \ldots (m, t(m))\}$ having exactly one position from each row and column (i.e. $(t(1), t(2), \ldots, t(m))$ is a permutation of $\{1, 2, \ldots, m\}$ and $\{f(i, j) : (i, j) \in T\} = \{1, 2, \ldots, m\}$). Note that not all latin squares have transversals.

An m-assignment for the $\infty \times \infty$ array can be generated from L by 'stamping' L on the infinite array as follows:

$$f(i', j') = f(i, j) \text{ if and only if } i' \equiv i (\text{mod } m), j' \equiv j (\text{mod } m). \tag{4.72}$$

We refer to $\mathsf{disc}(L)$ as the discrepancy of the infinite m-assignment generated by f as above and will use notations where f is replaced by L such as $d_{\mathcal{R},L}(k)$, $\mathsf{disc}(L)$.

Proposition 4.5.3 *Consider an m-assignment generated by L_m. For any $a \times b$ rectangle $\mathcal{R}$, there is a $c \times d$ rectangle $\mathcal{R}'$ with $c, d < m$ so that $\mathsf{disc}_{\mathcal{R}}(k) = \mathsf{disc}_{\mathcal{R}'}(k)$ for each $k \in \{1, 2, \ldots, m\}$.*

Proof of Proposition 4.5.3 Let $\mathcal{R}$ be an $a \times b$ rectangle. If $a \geq m$, then we split $\mathcal{R}$ into the $m \times b$ rectangle $\mathcal{R}_1$ consisting of the first m rows of $\mathcal{R}$ and the $(a - m) \times b$ rectangle $\mathcal{R}_2$. For each $k \in \{1, 2, \ldots, m\}$, we have

$\mathsf{disc}_{\mathcal{R}_1}(k) = 0$, since each column of $\mathcal{R}_1$ is a permutation of $\{1, 2, \ldots, m\}$. Thus $\mathsf{disc}_{\mathcal{R}}(k) = \mathsf{disc}_{\mathcal{R}_2}(k)$. We can continue in this way reducing the number of rows to less than m. The same can be done by columns. □

Hence we may compute discrepancy entirely using rectangles of largest dimension less than m.

Consider a transversal T_m of a latin square L_m. It is natural to think of the positions of the transversal as being the positions of a single symbol in some alternate latin square and so we can imagine T_m stamped over the $\infty \times \infty$ array as $T_m(\infty)$ as done in (4.72). We define

$$\mathsf{disc}(T_m) = \max_{\mathcal{R}} \left| |\mathcal{R} \cap T_m(\infty)| - \frac{1}{m}(\text{area } \mathcal{R}) \right| \tag{4.73}$$

The main theorem of this section is the following.

Theorem 4.5.4 ([AS07]) *For each positive integer m, there exists a latin square L_m of order m with $\mathsf{disc}(L_m) \leq 15 \log_3(m)$*

We derive from a result of Schmidt [Sch72]:

Theorem 4.5.5 *There is a c' so that for $m \geq 2$ and any latin square L_m of order m, $\mathsf{disc}(L_m) \geq c' \log(m)$.*

Multiplication and Addition.

We give a way to build a latin square of order nm of small discrepancy from latin squares of order n, m each of small discrepancy. In addition we introduce a way to take a latin square of order m of small discrepancy and obtain latin squares of orders $m+1, m-1$ of small discrepancy. The former requires a transversal of small discrepancy and the latter requires a special '2×2' property.

Definition 4.5.6 *Let L_n, L_m be latin squares of orders n,m with entry functions f, g respectively. Define $L_n \times L_m = L_{nm}$ to be the latin square of order nm with entry function h defined as*

$$h(i + (j-1)m, k + (l-1)n) = n(g(i,l) - 1) + f(j,k) \tag{4.74}$$

for $1 \leq i, l \leq m$ and $1 \leq j, k \leq n$.

Roughly speaking we form an $nm \times m$ array by taking n copies of L_m one on top of the other. For the first copy of L_m we take the 1×1 blocks containing i and stretch it to a $n \times 1$ array containing $(n(i-1) + f(1,1), n(i-$

$1) + f(1,2), \ldots, n(i-1) + f(1,n))$ where $(f(1,2), f(1,2), \ldots, f(1,n))$ is the first row of L_n and is a permutation of $\{1, 2, \ldots, n\}$. For the jth copy of L_m we take the 1×1 blocks containing i and stretching each to a $n \times 1$ array $(n(i-1) + f(j,1), n(i-1) + f(j,2), \ldots, n(i-1) + f(j,n))$ where $(f(j,1), f(j,2), \ldots, f(j,n))$ is the jth row of L_n and consequently, it is a permutation of $\{1, 2, \ldots, n\}$. The following examples give L_2, $L_4 = L_2 \times L_2$, $L_8 = L_2 \times L_4$, $L_{16} = L_2 \times L_8$.

$$L_2 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \; L_4 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \\ 2 & 1 & 4 & 3 \\ 4 & 3 & 2 & 1 \end{bmatrix}, \; L_8 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 & 7 & 8 & 5 & 6 \\ 7 & 8 & 5 & 6 & 3 & 4 & 1 & 2 \\ 2 & 1 & 4 & 3 & 6 & 5 & 8 & 7 \\ 6 & 5 & 8 & 7 & 2 & 1 & 4 & 3 \\ 4 & 3 & 2 & 1 & 8 & 7 & 6 & 5 \\ 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{bmatrix}, \tag{4.75}$$

$$L_{16} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 1 & 2 & 3 & 4 & 13 & 14 & 15 & 16 & 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 & 9 & 10 & 11 & 12 & 5 & 6 & 7 & 8 & 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 & 7 & 8 & 5 & 6 & 11 & 12 & 9 & 10 & 15 & 16 & 13 & 14 \\ 11 & 12 & 9 & 10 & 15 & 16 & 13 & 14 & 3 & 4 & 1 & 2 & 7 & 8 & 5 & 6 \\ 7 & 8 & 5 & 6 & 3 & 4 & 1 & 2 & 15 & 16 & 13 & 14 & 11 & 12 & 9 & 10 \\ 15 & 16 & 13 & 14 & 11 & 12 & 9 & 10 & 7 & 8 & 5 & 6 & 3 & 4 & 1 & 2 \\ 2 & 1 & 4 & 3 & 6 & 5 & 8 & 7 & 10 & 9 & 12 & 11 & 14 & 13 & 16 & 15 \\ 10 & 9 & 12 & 11 & 14 & 13 & 16 & 15 & 2 & 1 & 4 & 3 & 6 & 5 & 8 & 7 \\ 6 & 5 & 8 & 7 & 2 & 1 & 4 & 3 & 14 & 13 & 16 & 15 & 10 & 9 & 12 & 11 \\ 14 & 13 & 16 & 15 & 10 & 9 & 12 & 11 & 6 & 5 & 8 & 7 & 2 & 1 & 4 & 3 \\ 4 & 3 & 2 & 1 & 8 & 7 & 6 & 5 & 12 & 11 & 10 & 9 & 16 & 15 & 14 & 13 \\ 12 & 11 & 10 & 9 & 16 & 15 & 14 & 13 & 4 & 3 & 2 & 1 & 8 & 7 & 6 & 5 \\ 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 16 & 15 & 14 & 13 & 12 & 11 & 10 & 9 \\ 16 & 15 & 14 & 13 & 12 & 11 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{bmatrix}. \tag{4.76}$$

The discrepancies are 1, 2, 2, 3 respectively when measured as in (4.66), much less than bounds provided by the following theorem.

Theorem 4.5.7 $\mathsf{disc}(L_n \times L_m) \leq \mathsf{disc}(L_n) + \mathsf{disc}(L_m) + 4.$

Proof of Theorem 4.5.7 We first consider an $m \times n$ rectangle $\mathcal{R} = \mathcal{R}[(1 + km, 1 + ln), (m + km, n + ln)]$, which we can consider *aligned* with the construction of (4.74). Each symbol occurs exactly once in $\mathcal{R}$ and so $\mathsf{disc}(\mathcal{R}) = 0$

Now consider a large rectangle $\mathcal{R} = \mathcal{R}[(i_1, j_1), (i_4, j_4)]$ which we split into 9 rectangles by defining $i_2 \equiv 1(\text{mod } m)$, $0 \le i_2 - i_1 < m$, $i_3 \equiv m(\text{mod } m)$, $0 \le i_4 - i_3 < m$, $j_2 \equiv 1(\text{mod } n)$, $0 \le j_2 - j_1 < m$, $j_3 \equiv n(\text{mod } n)$, $0 \le j_4 - j_3 < m$. In what follows, we assume that $i_1 < i_2 < i_3 < i_4$ and $j_1 < j_2 < j_3 < j_4$ since cases of equality can be easily handled. The following is a map of the 9 rectangles

$$\mathcal{R} = \begin{matrix} \mathcal{R}_2 & \mathcal{R}_6 & \mathcal{R}_3 \\ \mathcal{R}_8 & \mathcal{R}_1 & \mathcal{R}_9 \\ \mathcal{R}_4 & \mathcal{R}_7 & \mathcal{R}_5 \end{matrix} \tag{4.77}$$

The central rectangle $\mathcal{R}_1 = \mathcal{R}[(i_2, j_2), (i_3, j_3)]$ has $\mathsf{disc}(\mathcal{R}_1) = 0$ by our observation about aligned rectangles. The corner blocks $\mathcal{R}_2 = \mathcal{R}[(i_1, j_1), (i_2 - 1, j_2 - 1)]$, $\mathcal{R}_3 = \mathcal{R}[(i_1, j_3 + 1), (i_2 - 1, j_4)]$, $\mathcal{R}_4 = \mathcal{R}[(i_3 + 1, j_1), (i_4, j_2 - 1)]$, $\mathcal{R}_5 = \mathcal{R}[(i_3 + 1, j_3 + 1), (i_4, j_4)]$ are each contained in an aligned rectangle and so $\mathsf{disc}(\mathcal{R}_2), \mathsf{disc}(\mathcal{R}_3), \mathsf{disc}(\mathcal{R}_4), \mathsf{disc}(\mathcal{R}_5) \le 1$. The middle top and middle bottom blocks are $\mathcal{R}_6 = \mathcal{R}[(i_1, j_2), (i_2 - 1, j_3)]$, $\mathcal{R}_7 = \mathcal{R}[(i_3 + 1, j_2), (i_4, j_3)]$. We claim $\mathsf{disc}(\mathcal{R}_6 \cup \mathcal{R}_7) \le \mathsf{disc}(L_m)$. We define related rectangles from the m-assignment associated with L_m as $\mathcal{R}'_6 = \mathcal{R}[(m + i_1 - i_2 + 1, (j_2 - 1)/n), (m, (j_3/n) - 1)]$, $\mathcal{R}'_7 = \mathcal{R}[(m + 1, (j_2 - 1)/n), (i_4 - i_3 + m, (j_3/n) - 1)]$ where we need that $m + i_1 - i_2 + 1 \equiv i_1(\text{mod } m)$, $m \equiv i_2 - 1(\text{mod } m)$, $m + 1 \equiv i_3 + 1(\text{mod } m)$, $m + i_4 - i_3 \equiv i_4(\text{mod } m)$. We have $\mathsf{disc}(\mathcal{R}_6 \cup \mathcal{R}_7) = \mathsf{disc}(\mathcal{R}'_6 \cup \mathcal{R}'_7)$. From (4.74), we check

$$d_{\mathcal{R}_6, L_n \times L_m}(n(i-1) + j) = d_{\mathcal{R}'_6, L_m}(i) \tag{4.78}$$

for each j, $1 \le j \le n$. Also

$$\frac{1}{nm}(\text{area } \mathcal{R}_6) = \frac{1}{m}(\text{area } \mathcal{R}'_6). \tag{4.79}$$

We have the similar equations for $\mathcal{R}'_7$. Now $\mathcal{R}'_6$ and $\mathcal{R}'_7$ can be viewed as a single rectangle $\mathcal{R}'_{67} = \mathcal{R}[(m + i_1 - i_2 + 1, (j_2 - 1)/n), (i_4 - i_3 + m, (j_3/n) - 1)]$ and so using the m-assignment from L_m, $\mathsf{disc}(\mathcal{R}'_{67}) \le \mathsf{disc}(L_m)$.

We proceed in a similar way for the middle left and middle right rectangles $\mathcal{R}_8 = \mathcal{R}[(i_2, j_1), (i_3, j_2 - 1)]$ and $\mathcal{R}_9 = \mathcal{R}[(i_2, j_3 + 1), (i_3, j_4)]$. We claim $\mathsf{disc}(\mathcal{R}_8 \cup \mathcal{R}_9) \le \mathsf{disc}(L_n)$. We define related rectangles from the n-assignment associated with L_n as $\mathcal{R}'_8 = \mathcal{R}[((i_2 - 1)/m, n + j_1 - j_2 + 1), ((i_3/m) - 1, n)]$, $\mathcal{R}'_9 = \mathcal{R}[((i_2 - 1)/m, n + 1), ((i_3/m) - 1, n + j_4 - j_3)]$. From (4.74), we check

$$d_{\mathcal{R}_8, L_n \times L_m}(n(i-1) + j) = d_{\mathcal{R}'_8, L_n}(j) \tag{4.80}$$

and also

$$\frac{1}{nm}(\text{area } \mathcal{R}_8) = \frac{1}{n}(\text{area } \mathcal{R}'_8). \tag{4.81}$$

We have the similar equations for $\mathcal{R}'_9$. Now $\mathcal{R}'_8$ and $\mathcal{R}'_9$ can be viewed as a single rectangle $\mathcal{R}'_{89} = \mathcal{R}[((i_2-1)/m, n+j_1-j_2+1), ((i_3/m)-1, n+j_4-j_3)]$ and so using the m-assignment from L_m, $\mathsf{disc}(\mathcal{R}'_{89}) \leq \mathsf{disc}(L_m)$.

We now can estimate $\mathsf{disc}(\mathcal{R})$ as at most $\sum_{i=1}^{9} \mathsf{disc}(\mathcal{R}_i)$ and obtain the desired inequality. □

This result would provide families of latin squares of order p^t whose discrepancy is logarithmic in the order of the latin square. Given that we can find latin squares of discrepancy 1 for orders 2,3,5 one might be led to consider products using L_2, L_3, L_5. We can extend the multiplication idea (and its proof) to transversals where we can view a transversal as the set of positions of a single symbol in an alternate latin square.

Theorem 4.5.8 *Let T_n be a transversal of a latin square L_n and let T_m be a transversal of a latin square L_m. Then $T_n \times T_m$ is a transversal of $L_n \times L_m$ with* $\mathsf{disc}(T_n \times T_m) \leq \mathsf{disc}(T_n) + \mathsf{disc}(T_m) + 4$.

Proposition 4.5.9 *Let L_n be a latin square of order n with entry function f and a transversal $T_n = \{(1, t(1)), (2, t(2)), \ldots, (n, t(n))\}$. Then we can form a new latin square L_{n+1} from L_n, T with entry function g given as*

$$g(i,j) = \begin{cases} f(i,j) & 1 \leq i,j \leq n, (i,j) \notin T \\ n+1 & (i,j) \in T \text{ or } i = j = n+1 \\ f(i,t(i)) & 1 \leq i \leq n, j = n+1 \\ f(t^{-1}(j), j) & i = n+1, 1 \leq j \leq n \end{cases} \tag{4.82}$$

Moreover $\mathsf{disc}(L_{n+1}) \leq \max\{\mathsf{disc}(L_n) + 3, \mathsf{disc}(T_n) + 3\}$

Proof of Proposition 4.5.9 It is easy to check we have a latin square.

We consider the $(n+1)$-assignment for L_{n+1} and any rectangle $\mathcal{R}$ of largest dimension n and so area at most n^2. We let $\mathcal{R}'$ denote the rectangle formed from $\mathcal{R}$ by deleting from the array (and so possibly $\mathcal{R}$) any row r where $r \equiv n+1 (\text{mod } n+1)$ or any column c where $c \equiv n+1 (\text{mod } n+1)$ (there is at most one such row and one such column in $\mathcal{R}$)).

We will be considering the $(n+1)$-assignment from L_{n+1} for $\mathcal{R}$ and the n-assignment from L_n for $\mathcal{R}'$.

Consider a symbol $i \in \{1, 2, \ldots, n\}$. We can show

$$|d_{\mathcal{R}, L_{n+1}}(i) - d_{\mathcal{R}', L_n}(i)| \leq 1 \tag{4.83}$$

since at most a single occurrence of i in $\mathcal{R}'$ can be from $T_n(\infty)$ and would be replaced by $n+1$ in $\mathcal{R}$. A symbol i can appear at most once in a row r of $\mathcal{R}$ where $r \equiv n+1 (\text{mod } n+1)$ and at most once in a column c where $c \equiv n+1(\text{mod } n+1)$. In the case that both occur, then that will imply that there will be an occurrence of i in $\mathcal{R}'$ replaced by $n+1$ in $\mathcal{R}$ by (4.82). We have that

$$\left|\frac{1}{n+1}(\text{area } \mathcal{R}) - \frac{1}{n}(\text{area } \mathcal{R}')\right| \leq 2, \tag{4.84}$$

since area $\mathcal{R}' \leq$ area $\mathcal{R} \leq$ area $\mathcal{R}' + 2n + 1$. Also we have

$$|d_{\mathcal{R}',L_n}(i) - \frac{1}{n}(\text{area } \mathcal{R}')| \leq \mathsf{disc}(L_n). \tag{4.85}$$

We conclude that

$$\left|d_{\mathcal{R},L_{n+1}}(i) - \frac{1}{n+1}(\text{area } \mathcal{R})\right| \leq \mathsf{disc}(L_n) + 3 \tag{4.86}$$

Now for symbol $n+1$, we have $|T_n(\infty) \cap \mathcal{R}'| \leq d_{\mathcal{R},L_{n+1}}(n+1) \leq |T_n(\infty) \cap \mathcal{R}'| + 1$, with equality if we have some pair i, j with $i \equiv j \equiv n+1(\text{mod } n+1)$ and $(i, j) \in \mathcal{R}$. Also $\left||T_n(\infty) \cap \mathcal{R}'| - \frac{1}{n}\text{area } \mathcal{R}'\right| \leq \mathsf{disc}(T_n)$ and so

$$\left|d_{\mathcal{R},L_{n+1}}(n+1) - \frac{1}{n+1}(\text{area } \mathcal{R})\right| \leq \mathsf{disc}(T_n) + 3. \tag{4.87}$$

This yields our bound. □

We can extend this argument to obtain a transversal of the new latin square.

Proposition 4.5.10 *Let T_n be a transversal of a latin square L_n of order n and let L_{n+1} be the latin square obtained as in Proposition 4.5.9. Then T_{n+1} defined as $T_n \cup (n+1, n+1)$ is a transversal of L_{n+1} with* $\mathsf{disc}(T_{n+1}) \leq \mathsf{disc}(T_n) + 3$. □

Subtracting one can be somewhat like undoing the operation of adding one. We say that a latin square L_n of order n with $f(n, n) = n$ has the '2×2' *property* if when $f(i, j) = n = f(n, n)$ then $f(i, n) = f(n, j)$.

Proposition 4.5.11 *Let L_n be a latin square of order n with entry function f. Assume $f(n, n) = n$ and L_n has the 2×2 property. Then we can form a new latin square L_{n-1} from L_n with entry function h given as*

$$h(i,j) = \begin{cases} f(i,j) & 1 \leq i, j \leq n-1, f(i,j) \neq n, \\ f(i,n) & 1 \leq i \leq n-1, f(i,j) = n. \end{cases}$$

Moreover

$$\mathsf{disc}(L_{n-1}) \leq \mathsf{disc}(L_n) + 3. \tag{4.88}$$

Proof of Proposition 4.5.11 It is easy to check we have a latin square. Consider any rectangle $\mathcal{R} \subseteq \infty \times \infty$ of largest dimension $n-2$. Let $\mathcal{R}'$ be the smallest rectangle containing $\mathcal{R}$ after we insert into the array new rows between a pair of rows $k, k+1$ where $k \equiv n-1 (\text{mod } n-1)$ and new columns between a pair of columns $k, k+1$ where $k \equiv n-1 (\text{mod } n-1)$. One can think in the reverse direction: $\mathcal{R}'$ is the smallest rectangle so that if you delete those rows and columns of index $\equiv n (\text{mod } n)$ from $\mathcal{R}'$ (as one would do in Proposition 4.5.9), one obtains $\mathcal{R}$. We add one copy of i to a position of L_n formerly occupied by n and delete a copy of i in row n and a copy of i in column n in forming L_{n-1} from L_n. Thus for $i \in \{1, 2, \ldots, n-1\}$ we have $|d_{\mathcal{R}', L_n}(i) - d_{\mathcal{R}, L_{n-1}}(i)| \leq 1$ because the dimensions are at most $n-2$ and if we have lost two copies of i as we go from $\mathcal{R}'$ to $\mathcal{R}$, then it must be that we also added one copy. Also $|d_{\mathcal{R}', L_n}(i) - \frac{1}{n}\text{area}(\mathcal{R}')| \leq \mathsf{disc}(L_n)$ and $|\frac{1}{n}\text{area}(\mathcal{R}') - \frac{1}{n-1}\text{area}(\mathcal{R})| \leq 2$ using $|\text{area}(\mathcal{R})| \leq (n-1)^2$. Thus we get the desired inequality $\mathsf{disc}(L_{n-1}) \leq \mathsf{disc}(L_n) + 3$. □

Note that the matrices we determined using the recursive product $L_2 \times (L_2 \times (L_2 \cdots))$ remarkably have the desired 2×2 property allowing us to easily construct latin squares of order $2^n - 1$ of small discrepancy.

Proof of Main Theorem

In the previous part constructions allowing 'multiplication','add one' and 'subtract one' of latin square orders were given. We can now provide a proof of the main result.

Proof of Theorem 4.5.4 We use the induction hypothesis that for each n, we can find a latin square L_n and a transversal T_n of L_n with

$$\mathsf{disc}(L_n) \leq 15 \log_3(n), \quad \mathsf{disc}(T_n) \leq 15 \log_3(n). \tag{4.89}$$

This is trivially true for small n. We then construct three latin square, transversal pairs L_{3n}, T_{3n}, L_{3n+1}, T_{3n+1}, L_{3n+2}, T_{3n+2} satisfying the induction hypothesis each with discrepancy at most $15 \log_3(n) + 15 = 15 \log_3(3n)$. The desired inequality now follows.

Let L_3 be the order 3 latin square given by (4.90) with 3 disjoint transversals $T_3(1) = \{(1,3), (2,2), (3,1)\}$, $T_3(2) = \{(1,2), (2,1), (3,3)\}$, $T_3(3) = \{(1,1), (2,3), (3,2)\}$ with $\mathsf{disc}(L_3) = 1 = \mathsf{disc}(T_3(i))$, for $i = 1, 2, 3$.

$$L_3 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}. \tag{4.90}$$

We form $L_{3n} = L_3 \times L_n$ and obtain 3 disjoint transversals as $T_{3n}(k) = T_3(k) \times T_n$ for $k = 1, 2, 3$, where $\mathsf{disc}(L_{3n}) \leq \mathsf{disc}(L_n) + 5$, $\mathsf{disc}(T_{3n}(k)) \leq \mathsf{disc}(T_n) + 5$ for $i = 1, 2, 3$ using Theorems 4.5.7,4.5.8. We will choose $T_{3n} = T_{3n}(1)$. We can now form L_{3n+1} from L_{3n}, T_{3n} using Proposition 4.5.9 and obtain that

$$\mathsf{disc}(L_{3n+1}) \leq \max\{\mathsf{disc}(L_{3n}), \mathsf{disc}(T_{3n})\} + 3 \leq \max\{\mathsf{disc}(L_n), \mathsf{disc}(T_n)\} + 8. \tag{4.91}$$

By Proposition 4.5.10 we can form a transversal $T_{3n+1} = T_{3n}(2) \cup (3n+1, 3n+1)$ of L_{3n+1} which has

$$\mathsf{disc}(T_{3n+1}) \leq \mathsf{disc}(T_{3n}) + 3 \leq \mathsf{disc}(T_n) + 8. \tag{4.92}$$

We could try to repeat using the remaining transversal $T_{3n}(3)$ but some modest alterations are required since, if we derive L_{3n+2} from L_{3n+1} using T_{3n+1} in Proposition 4.5.9, then $T_{3n}(3) \cup \{(3n+1, 3n+1), (3n+2, 3n+2)\}$ is *not* a transversal of that choice for L_{3n+2} and it seems hard to select any transversal, much less one of small discrepancy.

Consider a symbol i in a position $(j, (k+2)/3)$ of T_n, where $k \equiv 1 (\mathrm{mod}\ 3)$. After multiplication by L_3, this becomes inside L_{3n}:

$$\begin{array}{cccc}
 & \mathsf{col}\ k & \mathsf{col}\ k+1 & \mathsf{col}\ k+2 \\
\mathsf{row}\ j & 3i-2 & 3i-1 & 3i \\
 & \vdots & \vdots & \vdots \\
\mathsf{row}\ j+n & 3i & 3i-2 & 3i-1 \\
 & \vdots & \vdots & \vdots \\
\mathsf{row}\ j+2n & 3i-1 & 3i & 3i-2
\end{array} \tag{4.93}$$

(k was chosen as it was to make the column labels easy). The entries are transformed further in L_{3n+1}:

$$\begin{array}{cccccc}
 & \mathsf{col}\ k & \mathsf{col}\ k+1 & \mathsf{col}\ k+2 & & \mathsf{col}\ 3n+1 \\
\mathsf{row}\ j & 3i-2 & \underline{3i-1} & \underline{\underline{3n+1}} & \dots & 3i \\
 & \vdots & \vdots & \vdots & & \vdots \\
\mathsf{row}\ j+n & \underline{3i} & 3n+1 & 3i-1 & \dots & \underline{\underline{3i-1}} \\
 & \vdots & \vdots & \vdots & & \vdots \\
\mathsf{row}\ j+2n & 3n+1 & \underline{\underline{3i}} & \underline{3i-2} & \dots & 3i-1 \\
 & \vdots & \vdots & \vdots & & \vdots \\
\mathsf{row}\ 3n+1 & \underline{\underline{3i-1}} & 3i-2 & 3i & \dots & \underline{3n+1}
\end{array} \tag{4.94}$$

Here the single underlined entries are from the transversal $T_{3n}(2)$. We switch these with the doubly underlined positions and alter T_{3n+1} as follows

$$\begin{aligned} T'_{3n+1} &= T_{3n}(2)\backslash\{(j,k+1),(j+n,k),(j+2n,k+2)\} \\ &\cup \{(j,k+2),(j+n,3n+1),(j+2n,k+1),(3n+1,k)\} \end{aligned} \tag{4.95}$$

which is seen to be another transversal of L_{3n+1}. We compute

$$\mathsf{disc}(T'_{3n+1}) \leq \mathsf{disc}(T_{3n+1}) + 4 \tag{4.96}$$

since a rectangle $\mathcal{R}$ of largest dimension at most $3n$ will have

$$|\mathcal{R} \cap T_{3n+1}(\infty)| - 4 \leq |\mathcal{R} \cap T'_{3n+1}(\infty)| \leq |\mathcal{R} \cap T_{3n+1}(\infty)| + 4. \tag{4.97}$$

Thus $\mathsf{disc}(T'_{3n+1}) \leq \mathsf{disc}(T_n) + 12$. Using Proposition 4.5.9 we compute L_{3n+2} from L_{3n+1} and T'_{3n+1} with

$$\begin{aligned} \mathsf{disc}(L_{3n+2}) &\leq \max\{\mathsf{disc}(L_{3n+1}, \mathsf{disc}(T'_{3n+1})\} + 3 \\ &\leq \max\{\mathsf{disc}(L_n + 8, \mathsf{disc}(T'_n) + 12\} + 3 \\ &\leq \max\{\mathsf{disc}(L_n) + 11, \mathsf{disc}(T_n + 15)\} \end{aligned} \tag{4.98}$$

The only thing left to finish the proof is to find a good transversal of L_{3n+2}. Now, in L_{3n+2} we have:

$$\begin{array}{lcccccc} & \mathsf{col}\ k & \mathsf{col}\ k+1 & \mathsf{col}\ k+2 & & \mathsf{col}\ 3n+1 & \mathsf{col}\ 3n+2 \\ \mathsf{row}\ j & 3i-2 & \underline{3i-1} & 3n+2 & \ldots & 3i & 3n+1 \\ & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \mathsf{row}\ j+n & \underline{3i} & 3n+1 & 3i-1 & \ldots & 3n+2 & 3i-2 \\ & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \mathsf{row}\ j+2n & 3n+1 & 3n+2 & \underline{3i-2} & \ldots & 3i-1 & 3i \\ & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \mathsf{row}\ 3n+1 & 3n+2 & 3i-2 & 3i & \ldots & \underline{3n+1} & 3i-1 \\ \mathsf{row}\ 3n+2 & 3i-1 & 3i & 3n+1 & \ldots & 3i-2 & \underline{3n+2} \end{array} \tag{4.99}$$

The underlined entries and the rest of the transversal $T_{3n}(3)$ form a low discrepancy transversal T_{3n+2} of L_{3n+2}.

$$\begin{aligned} &T_{3n+2} = T_{3n}(3)\backslash\{(j,k),(j+n,k+2),(j+2n,k+1)\} \\ &\cup\{(j,k+1),(j+n,k),(j+2n,k+2),(3n+1,3n+1),(3n+2,3n+2)\} \end{aligned} \tag{4.100}$$

We have $\mathsf{disc}(T_{3n}(3)) \leq \mathsf{disc}(T_n) + 5$ and $\mathsf{disc}(T_{3n+2}) \leq \mathsf{disc}(T_n) + 14$, using Proposition 4.5.10 twice (which adds 6 to the bound) and then noting that 3 entries have been added and three entries removed (which adds a further 3 to the bound). This completes our verification of the induction hypothesis for $3n, 3n+3, 3n+2$. □

A Lower Bound

In this section we use the following deep result of Schmidt [Sch72] to prove that Theorem 4.5.4 is best possible for latin square type assignments.

Theorem 4.5.12 *There exists an absolute constant c so that if P is an arbitrary set of N points in the unit square $[0,1)^2$, then there exists a rectangle $B \subset [0,1)^2$ with sides parallel to the coordinate axes such that*

$$\Big| |P \cap B| - N\, area(B) \Big| > c \log N. \tag{4.101}$$

To prove a lower bound for the discrepancy of an assignment it is enough to consider a finite part of it, in our case, the generating latin square.

Proof of Theorem 4.5.5 Let us partition the unit square into m^2 little squares of side $1/m$. Consider entry t of L and put a point in the center of the little square in the ith row and jth column if the (i,j) entry of L is equal to t. Apply Theorem 4.5.12 to find subrectangle B. We may assume that B's sides coincide with the sides of the little squares by either increasing or decreasing the area by at most $4/m$. Then inequality (4.101) states that the deviation of entry t from the expected value in the subrectangle of L corresponding to B is at least $c \log m$. Choose c' to accommodate both terms. □

Doerr et al [DHW04] have verified that $\mathsf{disc}(m)$ is $\Omega(\log m)$ over all m-assignments.

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