A geometric approach for the control of switched and LPV systems

by

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Ph.D.

A DISSERTATION SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF
D.Sc.

AT

HUNGARIAN ACADEMY OF SCIENCES

Budapest

2010

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Typesetting in LaTeX2$_\epsilon$: Camera ready by the author

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Acknowledgements

The thesis contains the results of an intensive research activity performed during the last decade. Throughout his work the author has been supported by many people, especially by the members of the Systems and Control Laboratory (SCL) of Computer and Automation Research Institute of the Hungarian Academy of Sciences (MTA-SZTAKI). Special thanks to MTA-SZTAKI which provided excellent conditions of research which helped to accomplish this work in the defined quality.

The author is deeply indebted to Prof. József Bokor whose stimulating suggestions and encouragement was indispensable during this period. The author would like to express his sincere gratitude to Prof. Tibor Vámos and Prof. László Keviczky academicians their encouraging remarks and for the many useful comments. Discussions with Prof. Rudolf E. Kalman and Prof. Michael Athans have a great impact on the scientific view of the author. The stimulating personal communications with Prof. Gary Balas, head of the Department of Aerospace and Mechanics, University of Minnesota were always a source of encouragement for the research.

The author expresses his gratitude to all members of SCL. Special thanks to Dr. Péter Gáspár for the fruitful discussions and valuable suggestions that considerably improved the content and presentation quality of this work. The author is also grateful to Prof. Ferenc Szigeti for the fruitful discussions along the interesting mathematical aspects of the research.

It is also to be noted that the research has been supported by various grants and organizations. Special thanks to the Hungarian Scientific Research Fund (OTKA), the National Research and Development Foundation (NKFP) which supported projects between 2000 and 2008, the NASA Langley grant with Dr. Christine M. Belcastro as technical monitor, the grant of the Air Force Office of Scientific Research, Air Force Material Command, USAF (EOARD) with Dr. Paul Losiewicz as technical monitor, and the support of the National Science Foundation.

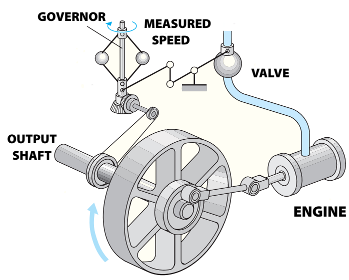
1 Introduction

In general terms, control theory can be described as the study of how to design the process of influencing the behavior of a physical system to achieve a desired goal. An *open-loop control* is one in which the control input is not affected in any way by the actual (measured) outputs. If the system changes during the operational time then the *control performance* can be severely reduced. In a *closed-loop system* the control input is affected by the measured outputs, i. e., a *feedback* is being applied to that system. Very often a reference input is given, which is directly related to the desired value of system outputs, and the purpose of the controller will be to minimize the error between the actual system output and the desired (reference input) value.

While construction of automatic machines dates back at least to the time of ancient greeks, as far as *automatic control* is concerned, a historical example cited in many texts is James Watt's fly-ball governor from the 18th century, where the control objective is to ensure that the speed of rotation is approximately constant. As the fly-balls rotate so they determine, via the valve, how much steam is supplied; the faster the rotation – the less steam is supplied. The rate of steam supplied then governs, via the piston and flywheel, the speed of rotation of the fly-balls. Although tight limits of operation, in terms of speed variation, can be obtained with such a device a possible disadvantage of the feedback scheme is shown: oscillations can occur in the system output, i. e., the speed of rotation, which would not occur if the system were connected in open-loop mode.

Heron of Alexandria

In the second half of the 19th century J. C. Maxwell developed a theoretical framework for such regulators by means of a differential equation analysis relating to performance of the overall system, thereby explaining in mathematical terms reasons for oscillations within the system. It was gradually found that Maxwell's governor equations were more widely applicable and could be used to describe phenomena in other systems, an example being piloting (steering) of ships.



Watt's regulator

A common feature with these systems was the employment of information feedback in order to achieve a controlling action, an idea that was widely exploited in the technological achievements of the 20th century. As usual, armed conflicts has a great impact on the technological innovation and the last century was not in need of such opportunities. In particular, the Second World War provided an ideal breeding ground for further developments in automatic control. In this period was developed the *classical control* theory, represented, among others, by the work of H. W. Bode, A. Kolmogorov, H. Nyquist, L. Pontryagin and N. Wiener that rely heavily on the spectral properties of the signals and reflects frequency-domain concepts.

In the 1960s the influence of space flight was felt, with optimization techniques gaining in prominence, while digital control also became widespread due to computers. *Modern control theory* has been developed to cope with the increasing complexity of multiple-input-multiple-output (MIMO) control systems. Unlike classical control theory that is based on frequency-domain analysis, modern control theory is based on time-domain analysis and synthesis using state variables.

R. E. Kalman was the leader in the development of a rigorous theory of control systems. His research in fundamental systems concepts, such as the formulation and study of most fundamental state-space notions, controllability and observability, helped put on a solid theoretical basis some of the most important engineering systems structural aspects. While some of these concepts were also encountered in other contexts, such as optimal control theory, it was Kalman who recognized the central role what they play in systems analysis. The paradigms formulated by Kalman and the basic results he established have become an intrinsic part of the foundations of control and systems theory. The author of this thesis has had the opportunity to meet professor Kalman during his regular visits at MTA SZTAKI. The discussions with him on the topics related to controllability have made a great influence on the research.

There are two main features in the analysis of a control system: system modeling, which means expressing the physical system under examination in terms of a model (or models) which can be readily dealt with and understood, and the design stage, in which a suitable control strategy is both selected and implemented in order to achieve a desired system performance. Forming a mathematical model which represents the characteristics of a physical system is crucially important as far as the further analysis of that system is concerned.

Controllability and *observability* are the main issues in the analysis of a system before deciding the best control strategy to be applied, or whether it is possible to control or stabilize the system. Controllability is related to the possibility of forcing the system into a particular state by applying an appropriate control signal while observability is related to the possibility of reconstructing, through output measurements, the state of a system.

The model should not be over simple so that important properties of the system are not included, something that would lead to an incorrect analysis or an inadequate controller design. In some cases the nonlinear characteristics are so important that they must be dealt with directly, and this can be quite a complex procedure.

Supersonic flight

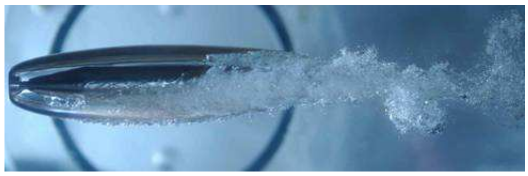
As an illustrative example for the importance of adequate modeling and that of controllability recall the story of the supersonic aircraft: at the mid forties designing a supersonic controllable airframe was the problem for aeronautical engineers. The main issue is called shock stall, and it's what happens when a control surface approaches the speed of sound. A shockwave forms around the control surface, rendering it useless, and the pilot has no way to control the aircraft. In distinction from the subsonic aircraft in which the system of longitudinal control was quite simple and the favorable conditions of controllability fully assured a rigid kinematic coupling of the control stick with the elevator, the control system on supersonic machines is more complex. In modern fighters, an adequate effectiveness of the horizontal tail empennage at supersonic flight modes is achieved only in the presence, in the control system, of a fully rotatable control surface, i. e., of a controlled stabilizer. Longitudinal controllability is provided by the elevator and by a stabilizer which is adjustable during flight.

Motivated by the need of dealing with physical systems that exhibit a more complicated behavior that those normally described by classical continuous and discrete time domains, hybrid systems have become very popular nowadays. In particular, there has been a relevant interest in the analysis and synthesis of so-called *switching systems* intended as the simplest class of hybrid systems.

A switching system is composed of a family of different (smooth) dynamic modes such that the switching pattern gives continuous, piecewise smooth trajectories. Moreover, we assume that one and only one mode is active at each time instant.

Controllability of switching systems has been investigated mostly for the case when arbitrary switching is possible (open–loop switching) and the objective is to design a proper switching sequence to ensure controllability or stability of (usually) piecewise linear systems, see Altafini (2002), Sun et al. (2003), Xie and Wang (2002), Yang (2002), or Sontag and Qiao (1999), for recurrent neural networks. In these investigations the control input set for the individual modes is assumed to be unconstrained.

Bimodal systems are special classes of switching systems, where the switch from one mode to the other one depends on the state (closed–loop switching). In the simplest case the switching condition is described by a hypersurface $\mathcal{C}$ in the state space.



Supersonic torpedo

My interest in this topic was triggered by the controllability analysis of a high–speed supercavitating vehicle. Supercavitation is a means of drag reduction in water, wherein a body is enveloped in a gas layer in order to reduce skin friction. After a suitable feedback linearization of the highly nonlinear dynamics the longitudinal motion of this device can be cast as a bimodal piecewise linear system, Balas et al. (2006); Vanek et al. (2007). A fundamental achievement of this study was that for a certain class of bimodal systems controllability question can be reduced to the problem of controllability of sign constrained open–loop switching system, Bokor, Balas and Szabó (2006).

One of the most elementary constrained controllability problems is that of the single-input-single-output (SISO) linear time invariant (LTI) system, with nonnegative inputs, see Saperstone (1973) for details. The multi-input LTI case, i. e., a special sign constrained switching problem, was solved in Brammer (1972) and Korobov (1980), for further insights see Stern and Heymann (1975), Pachter and Jacobson (1978), Hajek et al. (1992). Constrained controllability results for the linear time varying case with continuous right hand side can be found e. g., in Schmitendorf and Barmish (1980).

From practical point of view it is important to know if controllability can be performed using a finite number of switchings. It is known that for the unconstrained case and for the constrained case when the small time controllability property holds or the dynamics is continuous the answer is affirmative, Lee and Markus (1967), Sun and Ge (2005), Krastanov and Veliov (2005), moreover in all these cases there exist a bound for the number of switchings.

The first part of the thesis focuses on the controllability problem of LTI switched systems driven by sign constrained control. After recalling some fundamental results from geometrical control theory it will be proved that if the system is globally controllable then one can always achieve controllability by applying only a finite number of switchings, moreover, as in the unconstrained situation, the number of necessary switchings is bounded.

The first part of the thesis also provides a global controllability condition that can be applied for input sign constrained systems. In contrast to the unconstrained case where pure Lie algebraic methods, see, e.g., Jurdjevic (1997); Agrachev and Sachkov (2004), can be used efficiently to obtain global controllability conditions, in the input sign constrained problem methods borrowed from the theory of differential inclusions, Aubin and Cellina (1984); Wolenski (1990), and convex processes, Frankowska et al. (1986), have been proven to be efficient in obtaining global controllability condition formulated in algebraic terms.

For LTI systems $\dot{x} = Ax + Bu$ controllability is intimately related to stabilizability in that the former implies the later, moreover stabilizability can be achieved by applying a linear state feedback $u = Kx$, that can be computed relative easily. Similar result, with a suitable set of linear state feedbacks, is valid for the case when the inputs are sign constrained, see Smirnov (1996) and Krastanov and Veliov (2003).

Stability issues of switched systems, especially switched linear systems, have been of increasing interest in the recent decade, see for example Dayawansa and Martin (1999), Liberzon and Morse (1999), Liberzon et al. (1999), Liberzon (2003), Lin and Antsaklis (2005b), Sun and Ge (2005).

In the study of the stability of switched systems one may consider switched systems governed by given switching signals or one may try to synthesize stabilizing switching signals for a given collection of dynamical systems. Concerning the first class a lot of papers focus on the asymptotic stability analysis for switched homogeneous linear systems under arbitrary switching (strong stability, robust stabilization), and provide necessary and sufficient conditions, see Blanchini (1995), Agrachev and Liberzon (1999), Liu and Molchanov (2002).

The requirement of (robust) stability imposes very strict conditions on the dynamics, e.g., all the subsystems must be stable or stabilizable. Even under this condition, one has, in general, further restrictions on the allowable switching frequency (dwell time), determined by the spectrum of the matrices, Wirth (2005).

For strongly stabilizable linear controlled switching systems the feedback control always can be chosen as a "patchy", linear variable structure controller, see Blanchini (1995). The control is defined by a conic partition $\mathbb{R}^n = \bigcup_{k=1}^{N} \mathcal{C}_k$ of the state space while on each cone $\mathcal{C}_k$ the feedback is linear, i.e., it is given by $u = F_k x$.

In the more general situation, when one has unstable modes, more severe conditions on the switching sequence have to be imposed. In this respect one of the most elusive problems is the switched stabilizability problem, i.e., under what condition is it possible to stabilize a switched system by properly designing autonomous (event driven) switching control laws. For autonomous switchings the vector field changes discontinuously when the state hits certain "boundaries". This problem corresponds to the weak asymptotic stability notion of the associated differential inclusions.

Based on the ideas presented in Molchanov and Pyatnitskiy (1989) it was proved that the (weak) asymptotic stabilizability of switched autonomous linear systems by means of an event driven switching strategy can be formulated in terms of a conic partition of the state space, see Lin and Antsaklis (2004), Lin and Antsaklis (2005a). This result can be seen as a generalization of the corresponding theorem for strong stability. However, in contrast to the strong stability results, the corresponding Lyapunov function is not always convex, see Blanchini and Savorgnan (2006).

The second part of the thesis gives an extension of the fundamental LTI stabilizability results for the weak stabilizability of the class of completely controllable linear switching systems, where the control inputs might also be sign constrained, i.e., it is shown that a completely controllable linear switching system is closed–loop stabilizable, moreover, the stabilization can be performed by using a generalized piecewise linear feedback.

Despite the fact that linear switched systems are time varying nonlinear systems, their controllability and stabilizability properties can be described entirely in terms of the system matrices by using matrix algebraic manipulations. This property does not hold for general LTV systems. There is a notion, however, that survives the extension from LTI to LTV: the concept of the *invariant subspace*. The germ of this notion was related first to the study of controllability, where the reachability set behaves as the minimal set invariant to the action of the (controlled) dynamics. For LTV systems the reachability set is a subspace which induces a controllability decomposition of the system. Later on variants of the concept and the corresponding decompositions has been proved very useful in solving design problems.

This geometric view, i.e., the idea of invariance and invariant subspaces, relates the controllability study of switched systems to the topics from the second half of the thesis. The developed techniques and algorithms that leads to the specific invariant subspaces, hence, to the specific state space decompositions, make the glue that unifies the different problems like controllability, detection filter design or tracking control at an applicational level. Thus the geometric approach provides a common framework in which all these problems can be handled.

Designing a controller for systems with widely varying nonlinear and/or parameter-dependent dynamics is a major area of research in control theory. A general theory for the robust control of nonlinear systems is not computationally tractable and useful progress requires an intermediate level of complexity that, for example, incorporates scheduling requirements whilst remaining computationally tractable.

For LTI systems the concept of certain invariant subspaces and the corresponding global decompositions of the state equations induced by these invariant subspaces was one of the main thrusts for the development of geometric methods for solutions to problems of disturbance decoupling or non-interacting control, see Wonham (1985). In the so called *geometrical approach* to some fundamental problems of LTI control theory, such as disturbance decoupling, unknown input observer design, fault detection, a central role is played by the (A, B)-invariant and (C, A)-invariant subspaces and certain controllability and unobservability subspaces, Wonham (1985); Massoumnia (1986); Edelmayer et al. (1997).

Gain-scheduling is a technique widely used to control such systems in a variety of engineering applications. The gains of the gain-scheduled controllers are typically chosen using linear control design techniques and is a two step process. First, several operating points are selected to cover the range of system dynamics. At each of these points, the designer makes an LTI approximation to the plant and designs a linear compensator for each linearized plant. This process gives a set of linear feedback control laws that perform satisfactorily when the closed-loop system is operated near the respective operating points. A global nonlinear controller for the nonlinear system is then obtained by interpolating, or scheduling, the gains from the local operating point designs. The designer typically cannot assess a priori the stability, robustness, and performance properties of gain-scheduled controller designs. The above method represents the classical gain-scheduling method and has immediate application in flight control.

Linear parameter varying (LPV) modeling techniques have gained a lot of interest, especially those related to vehicle and aerospace control, Becker and Packard (1994); Fiahlo and Balas (1997); Barker and Balas (1999); Marcos and Balas (2001); Szászi et al. (2005). LPV systems have recently become popular as they provide a systematic means of computing gain-scheduled controllers. In this framework the system dynamics are written as a linear state-space model with the coefficient matrices functions of external scheduling variables. Assuming that these scheduling variables remain in some given range then analytical results can guarantee the level of closed loop performance and robustness. The parameters are not uncertain and can often be measured in real-time during system operation. However, it is generally assumed that the parameters vary slowly in comparison to the dynamics of the system. LPV based gain-scheduling approaches are replacing ad-hoc techniques and are becoming widely used in control design.

Many of the control system design techniques using LPV models can be cast or recast as convex problems that involve linear matrix inequalities (LMI). Significant progress has been made recently in the use of LMI and $\mathbf{H}_\infty$ optimization in gain-scheduled control. One such control design technique, described by Apkarian et al. (1995), is the Lyapunov function/quadratic $\mathbf{H}_\infty$ approach wherein a single Lyapunov function is sought to bound the performance of the LPV system. Such a framework generally has a strong form of robust stability with respect to time-varying parameters. However, due to the continuous variation of scheduling parameters,such a synthesis approach is generally associated with a convex feasibility problem with infinite constraints imposed on the LMI formulation. This problem can be addressed by using affine LPV modeling that reduces the infinite constraints imposed on the LMI formation to a finite number. Such a modeling approach has been used to solve design problems by Packard and Becker (1992); Sun and Postlethwaite (1998).

The above pure LPV model is not quite matched to the flight control problem where the scheduling variables are in fact system states (e.g., airspeed and angle of attack), rather than bounded external variables. An approach to this problem is to generate so-called quasi-LPV models, which are applicable when the scheduling variables are measured states, the dynamics are linear in the inputs and other states, and there exist inputs to regulate the scheduling variables to arbitrary equilibrium values.

In a more general context such robust control problems – both analysis and synthesis – can be formulated using a generalized plant technique based on a linear fractional transformation (LFT) description of the uncertain LPV system, see, e.g., Iwasaki and Hara (1998); Iwasaki and Shibata (2001); Wu (2001). The controller synthesis leads to bilinear matrix inequalities (BMI) but often it is possible to reduce the problem to the solution of a finite set of LMIs, for details see e.g., Scherer et al. (1997); Scherer (2001); Wu (2001); Gyurkovics and Takács (2009).

These methods concentrate on robust performance, hence, robust stability of the controlled system. However, a series of control tasks can be solved efficiently by exploiting the inner structure present in the dynamics, i.e., to make use of specific invariant manifolds of the controlled system. Nonlinear systems can be studied using tools from differential geometry, when the central role is played by the concept of *invariant distributions*. From the geometric viewpoint results of the classical linear control can be seen as special cases of more general nonlinear results, for details see Isidori (1989) and Nijmeijer and van der Schaft (1990). Due to the computational complexity involved, these nonlinear methods have limited applicability in practice.

The third part of the thesis extends the notions of different LTI invariant subspaces to (quasi) parameter-varying systems by introducing the notion of *parameter-varying* $(\mathcal{A}, \mathcal{B})$*-invariant* and *parameter-varying* $(\mathcal{C}, \mathcal{A})$*-invariant* subspaces. In introducing the various parameter-varying invariant subspaces an important goal was to set notions that lead to computationally tractable algorithms for the case when the parameter dependency of the system matrices is affine.

In general it is a hard task to give an exhaustive characterization for the solution of the fundamental problems such as the disturbance decoupling problem (DDP) or the fundamental problem of residual generation (FPRG) even in the LPV case. However, since the main ingredient in the solution of these problems are certain local decomposition theorems – in observable and unobservable subsystems, for example – using suitable invariant subspaces instead of the distributions or codistributions one can get sufficient conditions for solvability that can be useful in practical engineering applications.

Concerning the structure of the presentation: in order to highlight the motivation background of these research activities, the thesis starts with a motivation chapter that revolves around the classical topic of controllability of a linear time varying system. It is concluded that despite the exhaustive characterization of the controllability property in mathematical terms, the problem remains undecidable in any practical sense in terms of the initial data of the system. Besides giving this negative result this chapter also provides the germs that leads to the notions that has a real impact for a series of engineering control design problems.

The first two parts of the thesis are dedicated to the controllability and stabilizability problems related to switched linear systems, possibly with sign constrained control inputs. Despite to its inherent nonlinear nature, the class of linear switched systems provides an example for time varying systems whose controllability can be decided by using the initial data of the problem (the system matrices) in algebraic terms. Moreover, it turns out that the transparent relation between controllability and stabilizability met in the LTI context remains valid for this class, too.

The third part of the thesis provides the notion of parameter varying invariant subspaces as an extension of the corresponding ideas that has already been proven to be useful in the LTI context. These invariant subspaces provides a viable alternative of the more complex objects such as the corresponding invariant distributions and codistributions of the full nonlinear framework. Efficient algorithms are provided to compute these subspaces. This is the engineering answer to the challenge of the decomposition problem sketched in the introductory chapter.

The last part of the thesis presents some of the application examples, in which the geometric techniques, the introduced invariant subspaces and the corresponding algorithmic tools can be efficiently used.

Concerning hybrid systems, the thesis is concluded by stating the controllability result related to bimodal piecewise linear systems. This application covers almost all topics contained in the thesis: in order to put the problem in a quasi canonical form and to reduce the original task to an open-loop input sign constrained linear switched controllability problem, notions related to parameter varying invariant subspaces and invertibility are applied while the obtained controllability question is answered based on the results established in the first part of the thesis.

Design for an active suspension system for heavy vehicles and the controllability study for a high speed supercavitating underwater vehicle made the engineering applicational background for this research, see Bokor, Balas and Szabó (2006); Bokor, Szabó and Balas (2006a,b, 2007); Gáspár et al. (2009a).

This chapter is followed by applications, such as invertibility tasks and different decoupling problems that heavily rely on the state decompositions induced by certain robust invariant subspaces. The procedure to obtain the dynamical inverse of affine LPV systems is emphasized, since reconstruction of unknown inputs is an important task for either control applications or for fault detection filter design.

Concerning real engineering applications related to these methods reconfigurable fault detection controls of vehicle dynamical systems has to be mentioned, e. g., FDI filter design for a Boeing 747 aircraft, fleet control of road vehicles, fault tolerant active suspension design, see Balas et al. (2002, 2004); Szabó et al. (2003); Gáspár, Szabó and Bokor (2008); Gáspár, Szederkényi, Szabó and Bokor (2008b); Gáspár, Szabó and Bokor (2008f); Gáspár et al. (2009). The developed algorithms were also successfully applied in the dynamic inversion based controller design for stabilizing the primary circuit pressurizer at the Paks Nuclear Power Plant in Hungary during 2004-2005, see, e. g., Szabó et al. (2005); Gáspár et al. (2006).

To improve readability the new scientific results are listed in a separate chapter, while the corresponding publications of the author are contained at the end of the thesis in a separate list. Trying not to bloat the main text with unnecessary details the additional notations and facts related to the main material are placed in the Appendix.

2 Motivation

Let us consider the state dynamics of a controlled linear time varying (LTV) system:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{2.1}$$

where $x(t) \in \mathcal{X} \subset \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input while the initial condition is $x_0 = x(t_0)$. The measured signals are obtained by a linear readout map $y(t) = C(t)x(t)$, with $y \in \mathbb{R}^p$.

Our interest in such models is motivated by the fact that nonlinear dynamics can be often cast as an LTV system

$$\dot{x}(t) = A(\rho(y))x(t) + B(\rho(y))u(t) \tag{2.2}$$

by choosing a suitable set of *scheduling functions* ρ that depend only on measured variables y, i.e., its values are available in operational time. These models are called quasi linear parameter varying (qLPV) systems. A special case is when the dependence from the scheduling variables is affine, i.e.,

$$\begin{aligned} A(\rho(t)) &= A_0 + \rho_1(t)A_1 + \ldots + \rho_N(t)A_N, \\ B(\rho(t)) &= B_0 + \rho_1(t)B_1 + \ldots + \rho_N(t)B_N. \end{aligned} \tag{2.3}$$

For the sake of notational simplicity, in what follows, the time dependency of the matrices will be dropped ($A(\rho) := A(\rho(t))$) where it is possible.

Properties of some hybrid dynamics can also be analyzed in this framework. Hybrid systems involve two kinds of variables: continuous-valued and discrete-valued Branicky (1998). We focus on controlled switching linear hybrid systems where all mode switches are controllable, the dynamical subsystem within each mode has a linear time invariant form, the admissible region of operation within each mode is the whole state and input space, and there are no discontinuous state jumps. This model fits into the logic based switched system framework (Liberzon; 2003). This class of linear switched systems can be viewed as:

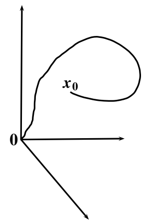
$$\dot{x}(t) = A(\sigma(t))x(t) + B(\sigma(t))u(t) \tag{2.4}$$

where $\sigma : \mathbb{R}^+ \to \mathbb{N}$ is a piecewise constant switching function, i.e., the matrices $A(\sigma)$ and $B(\sigma)$ are piecewise constant.

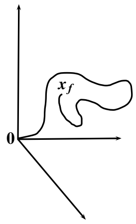
2.1 Controllability

One of the main questions of system theory is to determine wether the system is controllable and/or is observable.

A state x_0 is said to be controllable at time t_0 if there exist a control function $u(t)$ that steers the system into the origin in finite time; a state x_f is said to be reachable if the system can be steered from the origin into x_f in finite time. If the property holds for every state x and every t_0 then the system will be called controllable (reachable). The system (2.1) is called observable on a finite interval $[t_0, T]$ if any initial state x_0 at t_0 can be determined from knowledge of the system output $y(t)$ and input $u(t)$ over the given interval.



A controllable state



A reachable state

The controllability subspace is denoted by $\mathcal{C}$, while the reachability subspace by $\mathcal{R}$, respectively. For linear systems (complete) controllability and reachability are equivalent, i.e., the system is completely controllable if and only if $\mathcal{C} = \mathcal{R} = \mathcal{X}$.

Analogously $\mathcal{U}$ ($\mathcal{O}$) denotes the unobservability (observability) subspace; $\mathcal{U}$ is the set of all initial states that cannot be recognized from the output function. The system is observable if and only if $\mathcal{U} = 0$, i.e., $\mathcal{O} = \mathcal{X}$.

A convenient way to study all solutions of a linear equation on the interval $[\sigma, \tau]$, for all possible initial values simultaneously, is to introduce the corresponding transition matrix $\Phi(\tau, \sigma)$[1]:

$$x(\tau) = \Phi(\tau, \sigma)x(\sigma) + \int_\sigma^\tau \Phi(\tau, t)B(t)u(t)\mathrm{d}t = \Phi(\tau, \sigma)(x_0 + \int_\sigma^\tau \Phi(\sigma, t)B(t)u(t)\mathrm{d}t).$$

Applying the time varying coordinate change $z = \Phi(\sigma, t)x$ in the state space, the dynamic equation transforms into:

$$\dot{z} = \Phi(\sigma, t)B(t)u(t).$$

Thus in this new coordinate system controllability reduces to the solvability study of the equation:

$$z_0 = -\int_\sigma^\tau \Phi(\sigma, t)B(t)u(t)\mathrm{d}t$$

for a suitable finite τ. If we denote by $\mathcal{C}_\tau$ the set of states controllable at τ then $\mathcal{C}_\tau$ is a (closed) subspace, moreover $\mathcal{C}_{\tau_1} \subset \mathcal{C}_{\tau_2}$ for $\tau_1 < \tau_2$. Since the image space of the corresponding integral operator is finite dimensional, if the system is controllable there must be a finite[2] $\bar{\tau} > 0$ such that $\mathcal{C}_{\bar{\tau}} = \mathbb{R}^n$.

Hence, the controllability problem of an LTV system has been reduced to the question wether the finite rank operator $\mathfrak{L} : \mathbf{L}_2([\sigma, \bar{\tau}], \mathbb{R}^m) \to \mathbb{R}^n$ defined as $\mathfrak{L}u = \int_\sigma^{\bar{\tau}} \Phi(\sigma, t)B(t)u(t)\mathrm{d}t$ is onto. These type of linear operators have a nice theory: it is immediate that the adjoint operator

[1] $\Phi(t, t_0)$ is nonsingular and $\Phi(t, t_0) = X(t)X^{-1}(t_0)$ with $\dot{X}(t) = A(t)X(t),\ X(t_0) = \mathbb{I},\ X(t) \in \mathbb{R}^{n \times n}$.

[2] Although it is essential for the reasoning, this statement is often missing from many of the available control textbooks.

$\mathfrak{L}^* : \mathbb{R}^n \to \mathbf{L}_2^*([\sigma, \bar{\tau}], \mathbb{R}^m)$ can be identified with $\mathfrak{L}^* x = B^*(t)\Phi^*(\sigma, t)x$ and that $\mathfrak{L}$ is onto if and only if $\mathfrak{L}\mathfrak{L}^* > 0$. So, the fundamental result (Kalman; 1960) concerning controllability of the LTV system (2.1) can be stated as the equivalence of the following statements:

Kalman's Controllability Theorem: *There exist a* $\tau > 0$ *such that*

K1. the controllability Grammian $W(\sigma, \tau) = \int_\sigma^\tau \Phi(\sigma, s)B(s)B^*(s)\Phi^*(\sigma, s)ds$ *is positive definite;*

K2. there is no nonzero vector $p \in \mathbb{R}^n$ *such that* $\langle p, \Phi(\sigma, t)b_i(t)\rangle = 0$, *for* $t \in [\sigma, \tau]$, *and* $i = 1, \cdots, m$.

It is a standard result (Silverman and Meadows; 1967) that one can derive a rank condition[3] that guarantees controllability while it does not involve integration and it can be obtained directly from the initial data matrices $(A(t), B(t))$:

Silverman & Meadows Controllability Test: *if* (2.1) *is analytic on an interval* $\mathcal{J}$ *and* t *is an arbitrary fixed element of* $\mathcal{J}$, *then the system is completely controllable on every nontrivial subinterval of* $\mathcal{J}$ *if and only if*

$$\textbf{\textit{rank}} \begin{bmatrix} B_0(t) & B_1(t) & \cdots & B_k(t) \end{bmatrix} = n, \tag{2.5}$$

for some integer k, *where*

$$B_0(t) := B(t), \quad B_{i+1}(t) := A(t)B_i(t) - \frac{d}{dt}B_i(t). \tag{2.6}$$

If the analyticity condition is dropped, then the rank condition is only sufficient.

The problem is that it is hard to compute the rank of a time varying matrix, and we have no information about how to compute the controllability decomposition of the system[4].

Kalman's controllability result also reveals a structural property of linear systems: namely, by applying a suitable – in general time-varying – state transformation these systems decompose into a controllable an a purely uncontrollable part. To see this, suppose that there are at most r vectors $p_i \in \mathcal{X}$, $\langle p_i, \Phi(\sigma, s)B(s)\rangle = 0$, $s \in [\sigma, \tau]$. Chose them such that $\Pi^*\Pi = \mathbb{I}_r$, where $\Pi = [X^*(\sigma)p_i]$. Consider $n - r$ vectors $\lambda_i \in \mathcal{X}$ orthogonal on p_i, such that $\Lambda^*\Lambda = \mathbb{I}_{n-r}$, where $\Lambda = [X^*(\sigma)\lambda_i]$. Then, the time varying matrix $z = Tx$ with $T(t) = \begin{bmatrix} \Pi^* \\ \Lambda^* \end{bmatrix} X^{-1}(t)$ transforms system (2.1) into the *controllability decomposition* form:

$$\dot{z}_1(t) = 0 \tag{2.7}$$

$$\dot{z}_2(t) = \Lambda^* X^{-1}(t)B(t)u. \tag{2.8}$$

with the uncontrollable mode $z_1(t) = \Pi^* X^{-1}(t)x(t)$ and with the completely controllable mode $z_2(t) = \Lambda^* X^{-1}(t)x(t)$. In other words, the reachable set is *invariant* to the action of the controlled dynamics. The notion of invariance met in this context plays a central role in the investigations of geometric systems theory and it has been proven to be very useful in solving a series of control problems.

[3] For more details see the comments of Section A.2 of the Appendix.

[4] However, one can give a condition, when the system is surely uncontrollable: namely when the determinant of the matrix composed by the vectors $B_i(t)$ vanishes.

Linear time invariant (LTI) systems

In the framework of LTI theory the question of controllability can be decided by consulting the dimension of the reachability subspace, that can be computed easily from the initial data (A, B) of the problem, i.e.,

$$\mathcal{R} = \sum_{k=0}^{n-1} \mathbf{Im} A^k B. \tag{2.9}$$

Practically, the dimension of $\mathcal{R}$ is equal to the rank of the matrix whose columns are selected properly from those of the matrices $A^k B$, where $k = 1, \cdots, n-1$. This condition is called *Kalman rank condition*.

The reachability set is a subspace and knowing this subspace one can decompose the system in a controllable and an uncontrollable part by using a state transformation that does not depend on t. Moreover, for LTI systems the different stabilizability properties are strictly related to controllability.

For practical reasons it would be very convenient to give – if it is possible – a characterization of the controllability property for a larger class of systems by using only matrix manipulations and to construct the controllability decomposition by using a time independent state transformation matrix.

2.2 Controllability of linear affine systems

For affine time dependency $A(t) = \sum_{i=1}^{N} \rho_i(t) A_i$ the fundamental matrix can be given, at least locally, in terms of the *coordinates of second kind* (Wei and Norman; 1964), i.e., the solutions of the Wei–Norman equation:

$$\dot{g}(t) = (\sum_{i=1}^{K} e^{\Gamma_1 g_1} \cdots e^{\Gamma_{i-1} g_{i-1}} E_{ii})^{-1} \rho(t), \quad g(0) = 0. \tag{2.10}$$

Here $\rho(t) = [\rho_1(t), \ldots, \rho_N(t)]^T$ and $\{\hat{A}_1, \ldots, \hat{A}_K\}$ is a basis of the Lie-algebra $\mathcal{L}(A_1, \ldots, A_N)$, the structure matrices $\Gamma_i = [\gamma_{i,j}^l]_{l,j=1,\cdots,K}$ of the algebra are given by $[\hat{A}_i, \hat{A}_j] = \sum_{l=1}^{K} \gamma_{i,j}^l \hat{A}_l$ and E_{ii} is the matrix with a single nonzero unitary entry at the i-th diagonal element.

Locally, the fundamental matrix is given by the expression:

$$\Phi(t) = e^{g_1(t)\hat{A}_1} e^{g_2(t)\hat{A}_2} \cdots e^{g_n(t)\hat{A}_n}, \tag{2.11}$$

and generally it is not available in closed form.

Exploiting the affine structure and using the Peano–Baker formula for the transition matrix one can prove the following result:

Lemma 1: *For affine linear systems the points attainable from the origin are those from the subspace* $\mathcal{R}_{(\mathcal{A},\mathcal{B})}$ *given by:*

$$\mathcal{R}_{(\mathcal{A},\mathcal{B})} = \boldsymbol{span}\{\prod_{j=1}^{J} A_{l_j}^{i_j} B_k \mid J \geq 0,\ l_j, k \in \{0, \cdots, N\},\ i_j \in \{0, \cdots, n-1\}\}, \tag{2.12}$$

i.e., $\mathcal{R} \subset \mathcal{R}_{(\mathcal{A},\mathcal{B})}$.

Moreover, if one consider the finitely generated Lie-algebra $\mathcal{L}(A_0, \ldots, A_N)$ *which contains the matrices* $A_0, \ldots, A_N$, *and a basis* $\hat{A}_1, \ldots, \hat{A}_K$ *of this algebra, then*

$$\mathcal{R}_{(\mathcal{A},\mathcal{B})} = \sum_{l=0}^{N} \sum_{n_1=0}^{n-1} \ldots \sum_{n_K=0}^{n-1} \boldsymbol{Im}\,(\hat{A}_1^{n_1} \ldots \hat{A}_K^{n_K} B_l).$$

A direct consequence of this fact is that if the inclusion $\mathcal{R}_{\mathcal{A},\mathcal{B}} \subset \mathbb{R}^n$ is strict, i.e, if $\mathcal{R}_{\mathcal{A},\mathcal{B}}$ is a proper subspace, then the system (2.1) cannot be completely controllable.

The main question is that under which condition is the reachability set equal to the Lie algebra, i.e., when we have $\mathcal{R} = \mathcal{R}_{\mathcal{A},\mathcal{B}}$. In what follows, if this property holds, then the system will be called *c-excited*. Characterization of this property by using only the initial data seems to be difficult. However, from condition $K2.$ of the Kalman's controllability result, one has the following statement:

Proposition 1: *A system is c-exciting if and only if the following implication holds: there exist a nonzero* $\xi \in \mathbb{R}^n$ *such that*

$$B(t)^* \Phi^*(t_0, t)\xi = 0$$

for all $t \in [t_0, T]$ *implies that*

$$\mathcal{R}^*_{\mathcal{A},\mathcal{B}} \Phi^*(t_0, t)\xi = 0$$

for all $t \in [t_0, T]$.

It is clear, that for c-excited systems controllability is guaranteed if the relation $\mathcal{R}_{\mathcal{A},\mathcal{B}} = \mathbb{R}^n$, i.e., the *multivaraiable Kalman rank condition*, holds. Moreover, if the rank condition does not hold, for this class of sytems one can construct the controllability decomposition by using a time independent state transformation matrix that depends only on the matrix Lie algebra.

Therefore it would be useful to give a condition that uses the original data only to decide wether a system is c-exciting or not. Unfortunately, such a condition has not been available yet.

In Szigeti (1992) a sufficient condition for for a system to be c-excited is given by the following property:

> **Szigeti, Controllability Test:** *The system* $\dot{x} = A(t)x + Bu$ *with affine time dependency is c–persistently excited on* $[t_0, T]$ *if from the equalities*
>
> $$B^* A^*_{i_1} \cdots A^*_{i_l} A(t)^* \Phi^*(t_0, t)p = 0 \tag{2.13}$$
>
> *follows*
>
> $$B^* A^*_{i_1} \cdots A^*_{i_l} A^*_j \Phi^*(t_0, t)p = 0, \quad j = 0, \cdots, N, \tag{2.14}$$
>
> *where* p *is a no nonzero vector in* $\mathbb{R}^n$.

This property was characterized indirectly, in terms of the coordinates of second kind, i.e., the solutions of the Wei–Norman equation in Szigeti et al. (1995):

Szigeti, c-excitedness Test: *Let ρ_i be smooth functions. If the components of the fundamental solutions of the linear affine differential equation are differential–algebraically independent, i.e., there is no non–trivial polynomial differential equation*

$$P(g, \dot{g}, \cdots, g^{(q)}) = 0,$$

then the multivariable Kalman rank condition is equivalent to the controllability of system.

c-excited systems versus linear independency

Let us consider systems with constant B and such that $A(t)$ has an affine structure; then the fundamental matrix $Q(t)$ can be written as

$$Q(t) = \sum_{n_1=0}^{n-1} \dots \sum_{n_K=0}^{n-1} \hat{A}_1^{n_1} \dots \hat{A}_K^{n_K} \psi_{n_1,\cdots,n_K}(t). \tag{2.15}$$

Introducing the multi-index notation $\hat{A}^{\mathrm{i}} := \hat{A}_1^{i_1} \dots \hat{A}_K^{i_K}$, with $\mathbf{K} := \{0, 1, \cdots n-1\}^K$ and $\mathbf{i} := (i_1, \cdots, i_K)$, let us choose a linearly independent set of matrices from the set $\{\hat{A}^{\mathrm{i}} \,|\, \mathbf{i} \in \mathbf{K}\}$, say $\{\hat{A}^{\mathrm{j}} \,|\, \mathbf{j} \in \mathrm{j},\ \mathrm{j} \subset \mathbf{K}\}$. For the sake of simplicity, let us assume that $\mathbb{I}$ is a member of this basis, i.e., one can impose the condition that $[\varphi_{\mathrm{j}}(0)]_{\mathrm{j}\in\mathrm{j}}$ is the first canonical unit vector. With these notations, one has

$$Q(t) = \sum_{\mathrm{j}\in\mathrm{j}} \hat{A}^{\mathrm{j}} \varphi_{\mathrm{j}}(t). \tag{2.16}$$

The system $\{\varphi_{\mathrm{j}}(\sigma) \,|\, \mathbf{j} \in \mathbf{j}\}$ is not necessarily linearly independent and it can be obtained as the first column of the fundamental matrix associated to the equation

$$\dot{\tilde{Q}} = -\tilde{Q} \Lambda(t), \quad \tilde{Q}(0) = \mathbb{I}, \tag{2.17}$$

where $\Lambda(t)$ is a structure matrix[5]depending on the matrix Lie algebra and on the parameter functions $\rho(t)$. Note, that from this derivation the system $\{\varphi_{\mathrm{j}}(\sigma) \,|\, \mathbf{j} \in \mathbf{j}\}$ is not necessarily unique, but our choice satisfy (A.6).

Since the subspace $\mathcal{R}_{\mathcal{A},\mathcal{B}}$ is exactly the image space of the matrix

$$R_{\mathcal{A},\mathcal{B}} := [\, \hat{A}_{\mathrm{j}} B \,]_{\mathrm{j}\in\mathrm{j}}, \tag{2.18}$$

one can obtain the expression

$$W(\sigma, \tau) = R_{\mathcal{A},\mathcal{B}} (\int_\sigma^\tau [\, \varphi_{\mathrm{j}}(s) \,]_{\mathrm{j}\in\mathrm{j}} [\, \varphi_{\mathrm{j}}(s) \,]_{\mathrm{j}\in\mathrm{j}}^* \mathrm{d}s) R_{\mathcal{A},\mathcal{B}}^*.$$

[5]For details see the Appendix, Subsection A.1.

It is clear that if the system $\{\varphi_{\mathrm{j}}(\tau) \mid \mathrm{j} \in \mathrm{j}\}$ is linearly independent then $\mathbf{rank}\, W(\sigma, \tau) = \mathbf{rank}\, R_{\mathcal{A},\mathcal{B}}$, i.e., the system is c-exciting.

Suppose now that $\mathbf{rank}\, R_{\mathcal{A},\mathcal{B}} = m$, where $m \leq n$, and let us consider the singular value decomposition $R_{\mathcal{A},\mathcal{B}} = USV^*$ of this matrix. Then

$$\mathbf{rank}\, W(\sigma, \tau) = \mathbf{rank}\, [\,\mathbb{I}_m\, 0\,](\int_\sigma^\tau [\,\tilde{\varphi}_{\mathrm{j}}(s)\,]_{\mathrm{j}\in\mathrm{j}}[\,\tilde{\varphi}_{\mathrm{j}}(s)\,]^*_{\mathrm{j}\in\mathrm{j}}\mathbf{d}s)[\,\mathbb{I}_m\, 0\,]^*,$$

where $[\,\tilde{\varphi}_{\mathrm{j}}(s)\,]_{\mathrm{j}\in J} = V^*[\,\varphi_{\mathrm{j}}(s)\,]_{\mathrm{j}\in\mathrm{j}}$. This set of functions can be chosen as the first column of the fundamental matrix associated to the equation:

$$\dot{\Pi} = -\bar{\Lambda}(t)\Pi \quad \Pi(0) = V^*, \tag{2.19}$$

with $\bar{\Lambda}(t) = V^*\Lambda(t)V$. It follows that if the functions $\{\tilde{\varphi}_0, \cdots, \tilde{\varphi}_m\}$ are linearly independent, then $\mathbf{rank}\, W(\sigma, \tau) = \mathbf{rank}\, R_{\mathcal{A},\mathcal{B}}$. Putting these facts together, one has the following result:

Proposition 2: *The time varying system is c-excited if and only if the functions $\{\tilde{\varphi}_0, \cdots, \tilde{\varphi}_m\}$ are linearly independent, where $m = \boldsymbol{rank}\, R_{\mathcal{A},\mathcal{B}}$.*

Remark 1: *As an example, for LTI systems one has $Q(t) = \sum_{j=0}^{n-1} A^j \varphi_j(t)$. Suppose, that $A^n = \sum_{k=0}^{n-1} -\alpha_k A^k$. Then, the matrix Λ is given by*

$$\Lambda = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -\alpha_0 \\ 1 & 0 & 0 & \cdots & 0 & -\alpha_1 \\ 0 & 1 & 0 & \cdots & 0 & -\alpha_2 \\ \vdots & & & & & \\ 0 & 0 & 0 & \cdots & 1 & -\alpha_{n-1} \end{bmatrix},$$

and the system $\{\varphi_j(t) \mid j = 0, \cdots, n-1\}$ is the first column of the fundamental matrix of the equation $\dot{\tilde{Q}} = \Lambda\tilde{Q}$, $\tilde{Q}(0) = \mathbb{I}$. By an elementary argument one can show that these function are are always linearly independent, i.e., the LTI system is always c-excited, regardless to the matrix B.

For an affine LTV system if the functions $[\,\psi_{\mathrm{j}}(s)\,]_{\mathrm{j}\in\mathrm{j}}$ are not linearly independent, the c-excitedness property depends on $B(t)$, too.

One can derive[6] an explicit expression, i.e.,

$$\varphi_{\mathrm{j}}(t) = \sum_{\mathrm{n}\in\mathbf{N}} \alpha_{\mathrm{j}}^{\mathrm{n}}\gamma_{\mathrm{n}}(t). \tag{2.20}$$

between the functions $[\,\varphi_{\mathrm{j}}(s)\,]_{\mathrm{j}\in\mathrm{j}}$ and the coordinate functions g_i of the Wei–Norman formula. This expression makes possible, in principle, the verification whether these functions are linear independent. However, the computational burden and the encountered numerical problems are so high that a practical application of the method for a real-sized application is out of the question.

To conclude this chapter a (negative) example is presented in order to demonstrate through a nonlinear dynamics, put into an affine qLPV form of (9.10), the importance of the c-excitedness property of the scheduling variables for controllability.

[6]For details see the Appendix, Subsection A.1.

An illustrative example:

Let us consider the system

$$\begin{aligned} \dot{x}_1 &= x_1 x_2 + x_2 \\ \dot{x}_2 &= u \end{aligned} \tag{2.21}$$

that can be rewritten as $\dot{x} = A_0 + \rho A_1 + Bu$, where

$$A_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and with $\rho = x_2$.

Since $A_0 B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ one has $\mathbf{dim} R_{\mathcal{A},\mathcal{B}} = 2$, i.e., the Kalman rank condition holds.

Applying the Silverman Meadows approach, one has $B_0 = B$ and $B_1 = A_0 B$, i.e., $\mathbf{rank}\,[B_0\ B_1] = 2$, that shows that the system is controllable for any $\rho(t)$.

Using the Wei–Normann theory, one has $[A_0, A_1] = -A_0$, i.e., $\gamma_{01}^0 = -1$, $\gamma_{10}^0 = 1$ and the rest of the $\gamma_{ij}^l = 0$. It follows that

$$\Gamma_1 = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \ \Gamma_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \text{ i.e.,} \quad e^{\Gamma_1 t} = \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix}, \ e^{\Gamma_2 t} = \begin{bmatrix} e^t & 0 \\ 0 & 1 \end{bmatrix}.$$

From

$$E_{11} + e^{\Gamma_1 g_1} E_{22} = \begin{bmatrix} 1 & -g_1 \\ 0 & 1 \end{bmatrix},$$

it follows that the Wei–Normann equations are

$$\begin{aligned} \dot{g}_1 &= \rho g_1 + 1 \\ \dot{g}_2 &= \rho. \end{aligned}$$

The fundamental solution is given by $\Phi(t) = e^{\Gamma_1 A_0} e^{\Gamma_2 A_1}$, i.e., $\Phi(t) = \begin{bmatrix} e^{g_2} & g_1 \\ 0 & 1 \end{bmatrix}$.

If the system is uncontrollable, according to the Kalman condition there should be a nonzero vector ξ such that $B^* \Phi^{-*}(t) \xi = 0$ for all t, i.e., a number ν must exists such that $e^{-g_2} g_1 + \nu = 0$. But such a number does not exists[7], hence the system should be controllable.

However, it is immediate that $x_1 = -1$ is an uncontrollable manifold of system (2.21).

The reason why these tests fail relays in the fact that the uncontrollable manifold, i.e., $(-1, x_2)$ is not a subspace, while in the linear case the set of uncontrollable points is always a subspace.

If one shift the system from the equilibria point $(-1, 0)$, to $(0, 0)$, i.e., apply a (time-varying) change of coordinates $z_1 = x_1 + 1, \ z_2 = x_2$, then one has the system $\dot{z}_1 = z_1 z_2, \quad \dot{z}_2 = u$, with $\bar{A}(\rho) = \begin{bmatrix} \rho & 0 \\ 0 & 0 \end{bmatrix}$, and $\rho = z_2 = x_2$, that is clearly uncontrollable.

[7]Otherwise $\frac{\mathrm{d}}{\mathrm{dt}}(e^{-g_2} g_1) = 0$, i.e., $-\dot{g}_2 g_1 + \dot{g}_1 = 0$; but the left hand side is 1.

2.3 Conclusions

One of the main motivation doing this "tour de force" in this introductory chapter along a classical topic of linear control theory was to illustrate that the controllability problem cannot be tackled in a mathematical completeness and rigor even for linear systems, if the system is time varying. The situation is even worse if the dynamics is actually nonlinear, but cast as a qLPV system. This stays in contrast to the familiar framework of LTI systems where the answer to the fundamental problem concerning controllability is very accessible and transparent. Filling the gap between these two extremities was one of the motivation backgrounds of my research.

The short review presented on the previous sections, however, reveals that the simplicity of the time invariant results might be regained in that of a splitting of the state space in a surely uncontrollable mode and a mode, that might be controllable. Controllability of this mode cannot be inferred, in general, only if some additional conditions on the parameters are fulfilled (c-excitedness). Moreover, the simple example at the end of the chapter warn us on the inherent limitations of the approach when trying to extend it for nonlinear systems. It turns out, however, that for the class of linear switched systems (2.4), that despite of the linearity of the individual modes are true nonlinear systems, there always exists a c-exited switching sequence, which explains the effectiveness of the geometric (algebraic) treatment presented in the next chapter.

Concerning the (q)LPV systems (2.3) with affine parameter dependence the main issue is the problem of finding a time independent – and global – state transformation that splits the state space into modes that has specific properties – in these examples potentially controllable/uncontrollable modes. Concentrating on a rigorous proof of the controllability of the potentially uncontrollable mode is futile: not only due to the encountered mathematical difficulty of the computations but also due to the inherent uncertainty present in every practical model used in a nontrivial engineering application.

Affine (polytopic) LPV models considered in the thesis reflect the fact that the structure of the model (the system matrices) is known while the scheduling variables are often given by some approximations or lookup–tables. Therefore trying to test wether they fulfill some differential polynomial relations, in order to check a c-exitedness property for controllability, is not reasonable. The most that can be supposed, in general, that they are linearly independent, a condition that will be exploited in the forthcoming chapters.

This fact motivates our desire in finding certain "robust" invariant subspaces that often provides acceptable (sufficient) conditions to obtain an engineering solution for a series of basic control problems. What we apparently miss in these constructions, i.e., the knowledge of controllability/observability, might cause problems at a different (higher) level of the design: namely, in obtaining stable controllers or filters. Lacking of a stable design might be a clear indication that our assumptions on the c-exitedness of the scheduling variables might not hold, or, more likely, our techniques to ensure stability are too conservative. Hence, a different approach should be used.

The proposed geometric framework based on parameter varying invariant subspaces provides an example for a strategy, in which giving up to get the complete mathematical solution of the problem but not sacrificing the mathematical correctness in following a more "rough" route to an acceptable result leads to a useful, engineering design.

Part I

Controllability

3 Linear switched systems

A switching system is composed of a family of different (smooth) dynamic modes such that the switching pattern gives continuous, piecewise smooth trajectories. We assume that one and only one mode is active at each time instant. During the last decade there has been a considerably interest in the analysis and synthesis of *linear switched systems*, intended as the simplest class of hybrid systems.

A lot of work has been done to address the fundamental questions of control theory – controllability, observability, stabilizability – that were reported in a series of papers, Liberzon and Agrachev (2001); Ge et al. (2001); Xie et al. (2002a); Cheng and Chen (2003); Lin and Antsaklis (2005b); Sun et al. (2003) and monographs like Liberzon (2003); Sun and Ge (2005), just to list a few of them.

Controllability of switched systems has been investigated mostly for the case when arbitrary switching is possible (open–loop switching) and the objective is to design a proper switching sequence to ensure controllability or stability of (usually) piecewise linear systems, see Altafini (2002), Sun et al. (2003), Xie and Wang (2002), Yang (2002), or Sontag and Qiao (1999) for recurrent neural networks. Usually the input set $\mathcal{U}$ is assumed to be unconstrained, i.e., $\mathcal{U} = \mathbb{R}^m$, however for certain systems, e.g., in process engineering applications where the inputs cannot be negative due to physical reasons, the sign constrained case $\mathcal{U} = \mathbb{R}^m_+$ is more relevant.

For LTI systems the controllability question was entirely solved. Moreover there is a controllability condition that describes both the unconstrained and constrained problems, Korobov (1980); Frankowska et al. (1986). It turns out that a condition of the same type can be also formulated for switching systems. The elaboration of the solution to the controllability problem gives an opportunity to revise the main tools applied to the investigation of linear switched systems and to reveal facts and relations that remain hidden in previous works. In the elaboration of the topic advanced techniques like the geometric control theory of Jurdjevic (1997); Grasse and Sussmann (1990); Agrachev and Sachkov (2004), nonsmooth analysis and differential inclusions of Aubin and Cellina (1984); Wolenski (1990); Dontchev and Lempio (1992); Smirnov (2002) met the more elementary techniques of Wonham (1985).

3.1 General considerations

Consider the class of (open-loop) linear switched[1] systems:

$$\dot{x}(t) = A(\sigma(t))x(t) + B(\sigma(t))u(t) \tag{3.1}$$

where $x \in \mathbb{R}^n$ is the state variable and $u \in \mathfrak{U}$ is the input variable. $\sigma : \mathbb{R}^+ \to \mathfrak{S}$ is a measurable switching function mapping the positive real line into $\mathfrak{S} = \{1, \cdots, s\}$, i.e., the matrices $A(\sigma)$ and $B(\sigma)$ are measurable. The input set might be unconstrained $\mathfrak{U} = \mathbb{R}^m$ or constrained $\mathfrak{U} = \mathbb{R}^m_+$.

A solution (Carathéodory) of (3.1) on an interval I is an almost everywhere differentiable function $\varphi(t) : I \to \mathbb{R}^n$ that satisfies (3.1) *a.e.* on I. A state $x \in \mathbb{R}^n$ is *controllable* at time t_0, if there exist a time instant $t_f > t_0$, a (measurable) switching function $\sigma : [t_0, t_f] \to \mathfrak{S}$, and a bounded measurable input function $u : [t_0, t_f] \to \mathfrak{U}$ such that $x(t_f; t_0, x, u, \sigma) = 0$. A state $x \in \mathbb{R}^n$ is *reachable* at time t_0, if there exist a time instant $t_f > t_0$, a switching function $\sigma : [t_0, t_f] \to \mathfrak{S}$, and a bounded measurable input function $u : [t_0, t_f] \to \mathfrak{U}$ such that $x(t_f; t_0, 0, u, \sigma) = x$. We will term as reachability set the set ($\mathcal{R}$) of points reachable from the origin, and as controllability set ($\mathcal{C}$) the set of points from which the origin is reachable.

Following classical lines, (3.1) is said to be *completely controllable*[2] if every point in the state space is reachable from any other point in the state space by using bounded measurable controls and a suitable switching function.

A *trajectory* of the switching system (3.1) will be defined as follows: let $x(t)$ be an absolutely continuous function. We say that $x(t)$ is a (admissible) trajectory of the system (3.1) on $[t_0, t_f]$ if there exists a finite subdivision $t_0 < t_1 < \cdots < t_{N-1} < t_N = t_f$ of the interval $[t_0, t_f]$, such that on each subinterval (t_{k-1}, t_k) there exists an admissible function u_k such that one has $\dot{x} = A_k x + B_k u_k$.

The set of admissible inputs depends on the specific application: usually it is fixed to be the set of piecewise constant functions, but could be the set of sufficiently smooth functions, too. The notion of the trajectory excludes problematic situations from open– loop switching, like Zeno behavior, that might appear, however in closed–loop switching systems. In practical problems besides the left continuity of the switching signal it is often required that any time interval within which σ is constant is no less than a proper positive scalar $T_\delta > 0$, which is called the *dwell time*. Therefore it is an important issue how complete controllability by trajectories, i.e., using piecewise constant switching, is related to complete controllability by measurable switchings.

[1]The fact that the switching signal can be chosen and in particular, can be set to be a specific one, motivates that the term *switched* is preferred against *switching*.

[2]In Sun and Ge (2005) complete observability and reconstructibility are defined along classical lines as dual notions for complete reachability (controllability). Since these notions guarantees the possibility to recover the initial state only for some switching trajectories they does not cover the situation needed in practice. The requirement to reconstruct the state regardless the switching signal implies the complete observability of the individual modes. Therefore in this work we does not investigate problems related to this topic.

3.2 Switching systems and vector fields

The concept of *control system* plays a central role in the geometric theory of nonlinear control. A control system is a collection $\mathfrak{F}$ of smooth vector fields depending on independent parameters $w = [w_1, \cdots, w_m] \in \mathfrak{W} \subset \mathbb{R}^m$, called control inputs, such that $w(t)$ belongs to a suitable class of real valued functions, called admissible controls, Agrachev and Sachkov (2004). Usually it is supposed that the state space M is an n-dimensional real analytic manifold.

Associated with the control system $\mathfrak{F}$ denote by $\mathcal{A}_{\mathfrak{F}}(x,t)$ the set of all elements attainable from x at time t. For each $x \in M$, $\mathcal{A}_{\mathfrak{F}}(x) = \cup_{t\geq 0}\mathcal{A}_{\mathfrak{F}}(x,t)$. To a controlled nonlinear system $\dot{x} = f(x,u)$ can be associated in a natural way the collection of vector fields

$$V_f = \{f_u \,|\, u \in \mathfrak{U}\},$$

that can be used, e.g., in a Lie algebraic treatment, quite suitable for unconstrained problems and small time local controllability problems[3].

An important object of the controllability study of nonlinear systems is the set of (positive) orbits[4] $\Phi^q_{\tau,x_0}(\omega)(T) = e^{f_{u_q}t_q}e^{f_{u_{q-1}}t_{q-1}}\cdots e^{f_{u_2}t_2}e^{f_{u_1}t_1}x_0$, where $e^{f_u t}x_0$ is the solution of the equation $\dot{\xi} = f_u(\xi)$, $\xi(0) = x_0$, and $\tau = (t_1, t_2, \cdots t_q)$, $t_i \geq 0$ with $T = \sum_{j=1}^q t_j$ while $\omega = (u_1, u_2 \cdots u_q) \in \mathfrak{U}^q$, $f_{u_i} \in \mathcal{F}$. Observe that an orbit can be interpreted as a possible trajectory corresponding to a switched system formed by the modes $\dot{x} = f_{u_i}(x)$. Starting from this idea, a switched system can be considered as a nonlinear polysystem of the form

$$\dot{x} = f(x(t), w(t)), \quad x(0) = 0 \tag{3.2}$$

where in general, it is assumed that $x \in M$ and $f(., w)$, $w \in \mathfrak{W}$ is an analytic (smooth) vector field on M. The benefit of this interpretation is that the controllability study of switched systems with unconstrained inputs can be placed in the framework of the nonlinear geometric control theory. The aim of this section is to show that the powerful techniques of the general theory provides an elegant and transparent tool which can be applied efficiently in the controllability study of switched systems.

We would like to decide (global) controllability by just examining the vector fields that define a control system without the necessity of obtaining solutions of any kind of the given system. It turns out that it is possible possible to "expand" the available vector fields, e.g., by convexification, without changing the system itself, obtaining equivalent descriptions of the same system.

To introduce more and more redundancy in this description – by enlarging the set of vector fields that describes the system – is very useful in deciding the controllability question. This goal can be achieved by using the procedure of Lie extension, sketched in the next section.

Lie saturate

The Lie bracket of two vector fields f and g is denoted by $[f, g]$. Under the Lie bracket, and the pointwise addition, the space of all analytic vector fields on M becomes a Lie algebra; $Lie(\mathfrak{F})$

[3]A system is small-time locally controllable from the initial state x_0 if the reachable set from x_0 in time at most $T > 0$ contains x_0 in its interior for each $T > 0$, i.e., $x_0 \in \mathbf{int}\mathcal{A}_{\mathfrak{F}}(x_0, t)$ for all $t > 0$.

[4]For the notation and for additional details see, e.g., Jurdjevic (1997) and the Appendix.

denotes the subalgebra generated by $\mathfrak{F}$. For each $q \in M$, $Lie_q(\mathfrak{F})$ is a subspace of T_qM, the tangent space of M at q. A set of vector fields $\mathfrak{F}$ on a connected smooth manifold M is called *bracket-generating* (full-rank) if $Lie_q\mathfrak{F} = T_qM$ for all $q \in M$.

Families of vector fields $\mathfrak{F}$ and $\mathfrak{G}$ are said to be (strongly) *equivalent* if $Lie(\mathfrak{F}) = Lie(\mathfrak{G})$ and $\overline{\mathcal{A}_{\mathfrak{F}}(q,T)} = \overline{\mathcal{A}_{\mathfrak{G}}(q,T)}$ for all $q \in M$ and for all $T > 0$, where the overbar denotes the closure of the sets. The Lie Saturate $LS(\mathfrak{F})$ of a family of vector fields $\mathfrak{F}$ is the union of families strongly equivalent to $\mathfrak{F}$.

In general it is difficult to construct the Lie saturate explicitly, however one can construct a completely ascending family of compatible vector fields – *Lie extension* – starting from a given set $\mathfrak{F}$ of vector fields. A vector field f is called compatible with the system $\mathfrak{F}$ if $\mathcal{A}_{\mathfrak{F}\cup f}(q) \subset \overline{\mathcal{A}_{\mathfrak{F}}(q)}$ for all $q \in M$. Since $LS(\mathfrak{F})$ is a closed convex positive cone in $Lie(\mathfrak{F})$, a possibility to obtain compatible vector fields is extension by convexification, see Jurdjevic (1997): for $f_1, f_2 \in \mathfrak{F}$ and any nonnegative functions $\alpha_1, \alpha_2 \in C^\infty(M)$ the vector fields $\alpha_1 f_1 + \alpha_2 f_2$ is compatible with $\mathfrak{F}$. If $LS(\mathfrak{F})$ contains a vector space $\mathcal{V}$, then $Lie(\mathcal{V}) \subset LS(\mathfrak{F})$.

The importance of Lie extension is given by the following result, Agrachev and Sachkov (2004):

Lie Saturates – a Controllability Test: *If $\mathcal{F}$ is a bracket-generating system such that the positive convex cone generated by $\mathfrak{F}$, i.e.,*

$$cone(\mathfrak{F}) = \{\sum_{i=1}^{k} \alpha_i f_i \mid f_i \in \mathfrak{F}, \alpha_i \in C^\infty(M), \alpha_i \geq 0, k \in \mathbb{N}\}$$

is symmetric, i.e., $cone(\mathfrak{F}) = cone(-\mathfrak{F})$, then $\mathfrak{F}$ is completely controllable.

Let us apply this result to the unconstrained situation: by constructing the Lie extension of the vector field $\mathfrak{F} = \{A_i x + B_i u \mid \underline{u \in \mathfrak{U}}\}$, one can observe that $B_i u$ is compatible with $\mathfrak{F}$, i.e., $B_i u \in LS(\mathfrak{F})$. Indeed, $B_i u \in \overline{co(\mathfrak{F})}$, since $B_i u = \lim_{\lambda\to\infty} \frac{1}{\lambda}(A_i x + \lambda B_i u)$. If there is a vector $v \in LS(\mathfrak{F})$ such that $-v \in LS(\mathfrak{F})$, then $\pm A_i v \in LS(\mathfrak{F})$, too, see Jurdjevic (1997).

Then for the unconstrained case a necessary and sufficient condition for controllability can be formulated as:

Proposition 3: *The unconstrained switching system is controllable if and only if*

$$\boldsymbol{rank}\ \mathcal{R}_{\mathcal{A},\mathcal{B}} = n, \tag{3.3}$$

i.e., the multivariable Kalman rank condition*, holds, where the subspace $\mathcal{R}_{\mathcal{A},\mathcal{B}}$ is defined as*

$$\mathcal{R}_{(\mathcal{A},\mathcal{B})} = \boldsymbol{span} \left\{ \prod_{j=1}^{J} A_{l_j}^{i_j} B_k \mid k = 1, \cdots, s \right\} \tag{3.4}$$

where $J \geq 0$, $l_j \in \{0, \cdots, s\}$, $i_j \in \{0, \cdots, n-1\}$. Moreover, if one considers the finitely generated Lie-algebra $\mathcal{L}(A_0, \ldots, A_s)$ which contains $A_0, \ldots, A_s$, and a basis $\hat{A}_1, \ldots, \hat{A}_K$ of this algebra, then

$$\mathcal{R}_{\mathcal{A},\mathcal{B}} = \sum_{k=0}^{s} \sum_{n_1=0}^{n-1} \cdots \sum_{n_K=0}^{n-1} Im\ (\hat{A}_1^{n_1} \ldots \hat{A}_K^{n_K} B_k). \tag{3.5}$$

We obtained here a stronger result than that of Lemma 1. Controllability of unconstrained switched systems can be determined based on the system matrices only. As an early contribution of the author to the field this controllability result was derived in Stikkel et al. (2004) by using different, matrix Lie algebraic, techniques. In that context it was also stressed that the subspace $\mathcal{R}_{\mathcal{A},\mathcal{B}}$ is the minimal subspace invariant for all of the A_is containing the subspace $\mathcal{B} = \cup_{i=1}^{s} Im B_i$, see e.g., Balas et al. (2003). Using this fact one can obtain a controllability decomposition analogous to the corresponding LTV result.

Further results on how linear switched systems can be related to linear time varying systems are presented in the next section. It is also stressed that for controllable systems the condition on the finiteness of the switching numbers can be relaxed by admitting merely measurable switching rules. One of the main technical benefit of this fact is that it permits the use of nonsmooth analysis (differential inclusions) without the additional condition of piecewise continuity, required by the concept of trajectory, of the switching rules. This property will be exploited later, in the study of controllability with sign constrained control inputs.

3.3 Finite number of switchings, sampling

Let us denote by $F_w^t x_0$ the solution of the equation $\dot{\xi} = f_w(\xi)$, $\xi(0) = x_0$ on the interval $[0, t]$. Then for a given vector field $\mathfrak{F}$ one can consider the associated trajectories (positive orbits), i.e.,

$$\Phi_{\omega,\tau}^{q,T}(x_0) := F_{w_q}^{t_q} F_{w_{q-1}}^{t_{q-1}} \cdots F_{w_2}^{t_2} F_{w_1}^{t_1} x_0$$

where $\tau = (t_1, t_2, \cdots t_q)$, $t_i \geq 0$ with $T = \sum_{j=1}^{q} t_j$ and $f_{w_i} \in \mathfrak{F}$ corresponding to the sequence of piecewise constant controls $\omega = (w_1, w_2 \cdots w_q) \in \mathfrak{W}^q$.

For a switched linear system $f_{w_i}(x) = A_{s_i}x + B_{s_i}u_i$, with $w_i = (s_i, u_i)$. We will suppress the switching sequence $\sigma = (s_1, s_2, \cdots, s_q)$ from the notation and denote the flow by $\Phi_\tau^q x_0$ for fixed $\mu = (u_1, u_2, \cdots, u_q)$ and by $\Phi_\mu^q x_0$ for fixed τ.

A point $y \in M$ is called *normally reachable* from an $x \in M$ if there exist a flow such that $\Phi_{\bar{\tau}}^q x = y$ and the mapping $\tau \in \mathbb{R}_+^q \to \Phi_\tau^q(x)$, which is defined in an open neighborhood of $\bar{\tau}$, has rank $n = \mathbf{dim} M$ at $\bar{\tau}$. The system is *normally controllable* if y is normally reachable from x for every $x, y \in M$.

Proposition 4: *If the switching system* (3.1) *is globally controllable than it is also globally controllable by using piecewise constant switching functions, i.e., using only a finite number of switchings.*

Moreover, there exist a bound for the necessary number of switchings, that depends only on the system matrices and $\mathfrak{U}$. *There exist a universal (finite) switching sequence* σ *such that the time varying system*

$$\dot{x} = A(\sigma)x + B(\sigma)u$$

is globally controllable.

Proof: The first part of the assertion follows from a fundamental result on controllability, see Theorem 4.3 in Sussmann (1976), i.e., for a system of $\mathcal{C}^r$ vector fields $\mathfrak{F}$ controllability is equivalent to normal controllability.

For the second part recall that the reachability set $\mathcal{R}_\sigma$ associated to a switching sequence σ is a pointed cone. In particular from normal controllability follows that the origin is normally accessible from itself, hence there is a neighborhood of the origin (a ball) that is also normally accessible by the same switching sequence – by varying the time sequence τ. It follows that the pointed cone $\mathcal{R}_\sigma$ contains a ball around the origin, i.e., $\mathcal{R}_\sigma = \mathbb{R}^n$. Since σ contains a finite number of switchings our assertion is proved.

Remark 2: *The content of Proposition 4 is that one can concentrate on the global controllability problem in general, i.e., admitting measurable controls, which is a common setting for studying controllability and the existence of nice controls (e.g. piecewise constant, non–Zeno) is automatically guaranteed.*

Note that for a fixed $r > 0$ by taking sufficiently large but fixed inputs it is possible to reach all the points of the ball having radius r by controlling the system only with the individual time length t_i of the switching sequence. Actually all $\mathbb{R}^n$ is reachable by having only a finite set of controls and a periodic application of the sequence σ with suitable time instances τ_k. (It is a sort of a bang–bang property.)

In the definition of normal reachability the control input sequence μ is fixed while the switching times may vary in a certain neighborhood of τ. It turns out that the rank of the analogous map $\mu \in \mathfrak{U}^N \to \Phi_\mu^N(x)$ is also significant and it is closely related to the controllability of the sampled system, in general, for details see Sontag (1983); Sontag and Sussmann (1982); Sontag (1986). A point $y \in M$ will be called called *full rank reachable* from an $x \in M$ if there exist a flow such that $\Phi_{\bar{\mu}}^N x = y$ and the mapping $\mu \in \mathfrak{U}^N \to \Phi_\mu^N(x)$, which is defined in an open neighborhood of $\bar{\mu}$, has rank $n = \mathbf{dim} M$ at $\bar{\mu}$.

Proposition 5: *For the globally controllable linear switching system* (3.1) *for arbitrary point pairs* (x, y) *one has that* y *is full rank reachable from* x*. Moreover, every point pair can be joined in a full rank reachable way by using the same sequence (*σ *and* τ *fixed).*

Proof: For unconstrained linear switching systems the assertion is well known, see e.g. Sun and Ge (2002) or Sun and Ge (2005). The constrained case can be reduced to the unconstrained result and Proposition 1: let us consider a point that is full rank reachable from the origin with positive controls. Since the constrained controllable system is also unconstrained controllable, such a point clearly exists. However, by controllability, the origin can be reached from the point z by using a finite switching sequence. By joining these two finite sequences one has that an open neighborhood of the origin is full rank reachable from the origin. Since the reachability set $\mathcal{R}_\sigma$ is a pointed cone that contains a ball it follows that $\mathcal{R}_\sigma = \mathbb{R}^n$.

The next small example illustrates the difference between the concept of "time topology" related to $\Phi_\tau^q(x)$ and the "input topology" related to $\Phi_\mu^q(x)$ – see Sontag (1984) for the terminology.

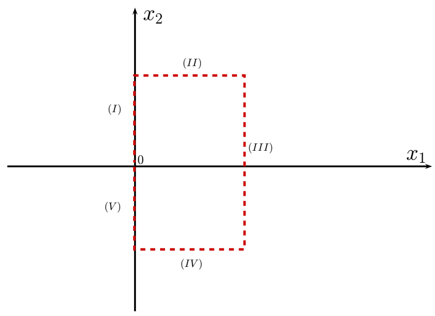
Example: *Let us consider the switched system defined by the modes* $A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $b_1 = 0$ *and* $A_2 = 0$, $b_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, *respectively.*

The corresponding flows are

$$F_1^t(x) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} x$$

and

$$F_{2,u}^t(x) = x + tu \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$



The flow $\Phi_\tau^5(0)$

It follows that for any $u > 0$ *and* $t > 0$ *with the switching sequence* $\sigma = (2, 1, 2, 1, 2)$, *input sequence* $\mu = (u, 0, -2u, 0, u)$ *and time sequence* $\bar{\tau} = (t, t, t, t, t)$ *the flow*

$$\begin{aligned} \Phi_\tau^5(0) &= F_{2,u}^{t_5} \circ F_1^{t_4} \circ F_{2,-2u}^{t_3} \circ F_2^{t_2} \circ F_{2,u}^{t_1}(0) = \\ &= t_1 u \begin{bmatrix} t_2 + t_4 \\ 1 \end{bmatrix} - 2t_3 u \begin{bmatrix} t_4 \\ 1 \end{bmatrix} + t_5 u \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{aligned}$$

has full rank at $\bar{\tau}$ *with* $\Phi_{\bar{\tau}}^5(0) = 0$.

For any $t > 0$ *with the switching sequence* $\sigma = (2, 1, 2)$, *input sequence* $\mu = (u_1, 0, u_2)$ *and time sequence* $\tau = (t, t, t)$ *the flow*

$$\Phi_\mu^3(x) = F_{2,u_2}^t \circ F_2^t \circ F_{2,u_1}^t(0) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} t^2 & 0 \\ t & t \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

has full rank at any $\bar{\mu}$ *with* $\Phi_{\bar{\mu}}^3(x) = y$ ($\bar{\mu} = (0, 0)$ *for x=y=0).*

Observe that in the input topology, i.e., for the discretized system, the design problem is linear in the unknown variables. This fact motivates that in the investigations of linear switched systems the usage of this topology is preponderant.

From Proposition 5 it is immediate that for sufficiently small sampling times the sampled system is also completely controllable – which is already known from the general theory – which is quite involved in this respect, see e.g. Sontag (1983); Petreczky (2006).

In what follows a more constructive proof of Proposition 5 will be presented: actually the result is a consequence of the similar fact that holds for the LTI systems, i.e., for the minimal A invariant set containing $\mathcal{V}$ ($< A, \mathcal{V} >$) for almost all $t \in \mathbb{R}$ one has

$$< A, \mathcal{V} > = < e^{At}, \mathcal{V} > . \tag{3.6}$$

The content of the assertion is that there is a switching sequence σ and times τ such that one has

$$\prod_{i=1}^{N} \bar{A}_{s_i} x_o + \mathcal{C}_\tau^\sigma u = x_f \tag{3.7}$$

where $u = [u_1^T, \ldots, u_N^T]^T$, the l^{th} column of $\mathcal{C}_\tau^\sigma$ is $\bar{A}_{s_N} \cdots \bar{A}_{s_{l+1}} B_{s_l}$ with $(\bar{A}_i, \bar{B}_i)$ corresponding to the t_i sampled linear system (A_i, B_i) and with $\bar{A}_{s_{N+1}} = \mathbb{I}$ such that the matrix $\mathcal{C}_\tau^\sigma$ is of full rank. Let us denote by $\bar{A}_\sigma = \prod_{i=1}^N \bar{A}_{s_i}$.

To obtain the constrained result let us consider a point that is full rank reachable from the origin. Such a point clearly exists, e.g., $z = \mathcal{C}_{\tau_1}^{\sigma_1} e$ from (3.7), the vector e having ones for its components. However, by controllability, the origin can be reached from the point z by using a finite switching sequence, say $(\sigma_2, \tau_2, \tilde{u}^{(2)})$. By joining these two finite sequences one has that an open neighborhood of the origin is full rank reachable from the origin. Since the reachability set $\mathcal{R}_\sigma$ is a pointed cone that contains a ball it follows that $\mathcal{R}_\sigma = \mathbb{R}^n$.

It is instructive to detail these ideas: in the first step one can build a sequence such that $\mathcal{C}_{\tau_1}^{\sigma_1}$ is of full rank and $\bar{A}_{\sigma_2}\mathcal{C}_{\tau_1}^{\sigma_1} e + \mathcal{C}_{\tau_2}^{\sigma_2}\tilde{u}^{(2)} = 0$. Moreover, the equation $\bar{A}_{\sigma_2}\bar{A}_{\sigma_1} x_0 + \bar{A}_{\sigma_2}\mathcal{C}_{\tau_1}^{\sigma_1} u^1 = x_f$ has an unconstrained solution $u^{(1)}$ for arbitrary (x_0, x_f). Then for sufficiently large λ the components of $u_c^1 = u^1 + \lambda e$ are all nonnegative, e.g., for $\lambda = \max\{|u_i^1| \mid u_i < 0\}$, hence $\bar{A}_{\sigma_2}\bar{A}_{\sigma_1} x_0 + \bar{A}_{\sigma_2}\mathcal{C}_{\tau_1}^{\sigma_1} u_c^1 + \lambda \mathcal{C}_{\tau_2}^{\sigma_2}\tilde{u}^{(2)} = x_f$. As a consequence, for both cases there is a switching sequence σ and time sequence τ such that one has (3.7) with $u \in \mathfrak{U}^N$.

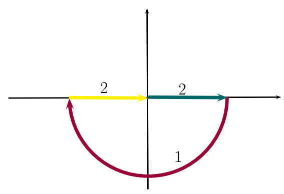
The construction is illustrated through the following small example:

Example: *Let us consider the switched systems described by the two modes:*

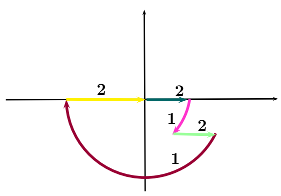
$$A_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \; b_1 = 0, \quad and \quad A_2 = 0, \; b_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The flow corresponding to the time topology is generated by the switching sequence $\sigma = (2, 1, 2)$ and fix input sequence $w = (u_1, 0, u_2)$, respectively:

$$x_f = \begin{bmatrix} \cos t_2 & \sin t_2 \\ -\sin t_2 & \cos t_2 \end{bmatrix} x_0 + \begin{bmatrix} \cos t_2 & \sin t_2 \\ -\sin t_2 & \cos t_2 \end{bmatrix} \begin{bmatrix} t_1 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} t_2 \\ 0 \end{bmatrix} u_2$$



Time topology



Input topology

The flow corresponding to the input topology is generated by the switching sequence $\sigma = (2, 1, 2, 1, 2)$ and fixed switching time sequence $\tau = (\sqrt{2}, \frac{\pi}{4}, \sqrt{2}, \pi - \arctan(1/3), \sqrt{10})$:

$$x_f = \bar{A} x_0 + \bar{A} \begin{bmatrix} 1 & \sqrt{2} \\ -1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} \sqrt{10} \\ 0 \end{bmatrix} u_3,$$

where $\bar{A} = \begin{bmatrix} -0.9487 & 0.3162 \\ -0.3162 & 0.9487 \end{bmatrix}$. *Accordingly, the full rank matrix $\mathcal{C}_\tau^\sigma$ is given by*

$$\mathcal{C}_\tau^\sigma = \begin{bmatrix} -1.2649 & -0.9487\sqrt{2} & \sqrt{10} \\ -1.2649 & -0.3162\sqrt{2} & 0 \end{bmatrix}.$$

Thus we obtain a generalization of Theorem 1 from Yoshida et al. (2003) derived for positively controlled discrete LTI systems:

Corollary 1: *The sign constrained linear switching system* (3.1) *is completely controllable if and only if there exist* $\sigma = (\sigma_1, \sigma_2)$ *and* $\tau = (\tau_1, \tau_2)$

- $\mathcal{C}_{\tau_1}^{\sigma_1}$ *has full rank (i.e. the unconstrained linear switching system is completely controllable)*
- *equation* $\bar{A}_{\sigma_2}\mathcal{C}_{\tau_1}^{\sigma_1}u^1 + \mathcal{C}_{\tau_2}^{\sigma_2}u^{(2)} = 0$ *has a solution such that* u^1 *is positive and* u^2 *is nonnegative.*

Corollary 2: *For every completely controllable linear switching system* (3.1) *the sampled discrete–time system is also completely controllable for suitable sampling rates.*

As a consequence one has the following embedding/restriction, see Stikkel et al. (2004) for further details:

Corollary 3: *For every completely controllable linear switching system* (3.1) *one can associate – not necessary a unique – completely controllable periodic linear time varying system* $\dot{x} = A(t)x + B(t)$.

One can relate this result with c-exitedness. Linear switched systems are c-excited, i.e., there is a switching sequence and corresponding switching times such that the resulting time varying system will be c-excited. This fact explains from another point of view why controllability of linear switched systems can be decided by a multivariate rank condition.

The switching sequence of Proposition 5 can be determined relative easily. The non uniqueness comes from the fact that one has more switching sequences σ such that $\mathcal{R}_\sigma = \mathbb{R}^n$. For discrete time systems – with nonsingularA_i matrices – the core of the solution is to determine a sequence $\sigma = (s_1, \ldots, s_N)$ such that the matrix $\mathcal{C}_\sigma$ has rank equal to n where

$$\mathcal{C}_\sigma = \begin{bmatrix} A_{s_N} \cdots A_{s_2} B_{s_1} & \ldots & A_{s_N} B_{s_{N-1}} & B_{s_N} \end{bmatrix}. \tag{3.8}$$

For the continuous time case one can use the matrices of the zero-older hold discretized systems instead. Actually this step can be skipped because the algorithms instead of doing a blind search based on (3.8) uses the corresponding invariant subspaces. Different techniques exists to determine such sequences, for details see e.g. Xie and Wang (2003b); Jia et al. (2007); Ji et al. (2007).

However it is an open question that for a given controllable linear switched system what is the sequence σ containing the minimal number of switches (of minimal length) such that $\mathbf{rank}\,\mathcal{C}_\sigma = n$. It is obvious that performing a search on a finite, but possible very big, set such a sequence can be obtained. The point is if there exist a characterization of the "optimal" sequence that would facilitate to find it efficiently. To illustrate the idea: for the multi input LTI system (A, B) the controllability indices shows where the switch in the "input" direction (actuator) should be performed; these indices can be determined by a suitable ranking of the vectors $A^k b_j$ and a basis selection procedure, see Wonham (1985). Such a transparent algorithm to determine the extended "controllability indices" is missing yet. These problems are significant for the control synthesis problems, e.g., stabilizability, which will be detailed in the next part of the thesis.

Controllability of linear switched systems was an intensively researched area, thus, besides our approach, the multivariable Kalman rank condition was obtained in a series of other papers using algebraic techniques, see e.g. Sun and Ge (2002); Sun et al. (2003); Xie et al. (2002b); Xie and

Wang (2003a). These papers basically uses the identity formulated in (3.6). The equivalence of the controllability of the continuous time system and the discrete time system obtained by sampling, however, was not realized in these works.

A contribution of the author of the thesis to this topic was to observe and exploit this equivalence which, together with the invariance property of $\mathcal{R}_{\mathcal{A},\mathcal{B}}$, provides a common framework for the study of discrete-time and continuous-time switched systems. This property was intensively used in the stabilizability study of these systems.

Relation (3.5) can be obtained by using the general differential geometric approach, see e.g. Szigeti (1992); Cheng (2005); Petreczky (2006) or equivalently the geometric control theory of Jurdjevic (1997). We do not insist further in this direction. The main reason to abandon the technique based on the vector field description is that it is hard to obtain useful conditions for complete controllability for switched systems with sign constrained inputs, see e.g. Bokor, Szabó and Balas (2007) for further details. A result that gives a necessary and sufficient condition for the small time controllability, i.e., controllability in an arbitrary small time, of the constrained switching system and uses Lie algebraic ideas is Veliov and Krastanov (1986) and Krastanov and Veliov (2005). These results are quite restrictive, since small time controllability requires that the convex cone generated by B_i contain a subspace, i.e., $\overline{co}(\cup_{i=1}^{s} B_i) - \overline{co}(\cup_{i=1}^{s} B_i) \neq \emptyset$.

These observations motivates the necessity to search for other methods in order to obtain a useful algorithm that might test controllability in the sign constrained case. This will be done in the next chapter.

4 Linear switched systems with sign constrained inputs

In practical applications there are often constrains that are imposed to the control input of the systems. The most widely studied case is when the inputs are constrained to a ball of given radius (bounded inputs). The obstruction caused by this type of constraint to (global) controllability is revealed by the equation (3.7): it is immediate that we always have both (small time) local reachability and (small time) local controllability in a neighborhood of the origin, however, in general it is not possible to extend this property to the entire state space, i.e., the system is not globally controllable, in general.

The case when the inputs are sign constrained is more difficult. It differs from the bounded input constraint in that even (small time) local controllability does not hold, in general, the system might be globally controllable. As an example, consider the switched system with two modes $\dot{x} = u$ and $\dot{x} = -u$, with $u \geq 0$. It is not hard to figure out that the system is globally controllable, see Figure 4.1 – for illustration purposes the points x_0, x_f from the line are slightly misplaced.

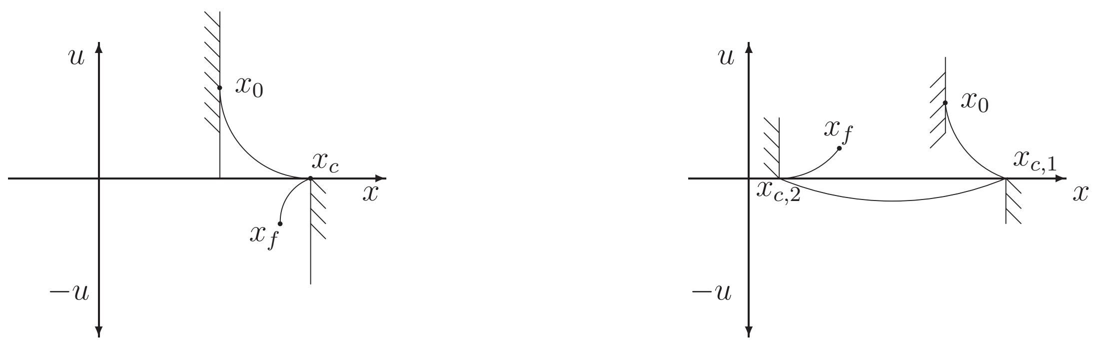


Figure 4.1: From the given point the shaded area cannot be reached directly

This fact explains why the usual differential-geometric (Lie algebraic) techniques fail in obtaining useful controllability conditions. As in the previous chapter our goal is to decide controllability by just examining the vector fields that define a control system without the necessity of obtaining solutions of any kind of the given system.

4.1 Differential inclusions

By the Filippov–Ważewski relaxation theorem the solution set defined by (3.1) is dense in the set of relaxed solutions, i.e., the solutions of the differential inclusion whose right hand side is the convex hull of the original set valued map, see e.g., Aubin and Cellina (1984). This implies that the corresponding attainable sets coincides. Hence, instead of (3.1) one can consider the controllability problem associated to the convexified differential inclusion $\dot{x} \in \mathcal{A}_c(x)$, where $\mathcal{A}_c(x) = \sum_{i=1}^{s} \alpha_i(A_i x + B_i u)$ and $\alpha_i \geq 0$ and $\sum_{i=1}^{s} \alpha_i = 1$.

Generalization of the LTI systems, which maintains some fundamental properties of the class, is the concept of convex processes. A closed convex process A is a set-valued map whose graph is a closed convex cone and that it is strict if its domain is the whole space. With a strict closed convex process A one can associate the Cauchy problem for the differential inclusion:

$$\dot{x}(t) \in A(x(t)), \quad x(0) = 0,$$

for details see Aubin and Cellina (1984). In this framework the class of LTI systems with sign (cone) constrained inputs: $\dot{x} \in \mathcal{A}(x) = \{Ax + \mathfrak{C}\}$, with $\mathfrak{C} = \mathbb{R}_+^m$ can be naturally cast and a fundamental controllability result was obtained, Frankowska et al. (1986), that contains result of Kalman, Kalman (1960), for the unconstrained case and also the results reported in Brammer (1972) and Korobov (1980) for the constrained input case.

Let us consider the differential inclusion $\dot{x} \in F(x)$, $x(0) = \xi$ and the corresponding reachable set $\mathcal{R}^T(\xi) = \{x(T) \,|\, x(0) = \xi,\ x \text{ is a solution}\}$. If F has nonempty, compact, convex values and is locally Lipschitz then by using the Euler discretization of the inclusion one has

$$\mathcal{R}^T(\xi) = \lim_{N \to \infty} (I + \frac{T}{N} F)^N(\xi) := [Exp\ F](T\xi),$$

where the limit is in the sense of Kuratowski, for definitions and details see Wolenski (1990).

Extending this result, Proposition 2 of Cabot and Seeger (2006) shows that for a positively homogeneous inclusion, ($F(\alpha) = \alpha F(x),\ \alpha > 0$), one has

$$[Exp\ F](t\xi) = \xi + \sum_{k=1}^{\infty} \frac{t^k}{k!} F^k(\xi), \tag{4.1}$$

where $F^k = F \circ F \circ \cdots \circ F$. This exponentiation formula was the main tool in obtaining the controllability result of sign constrained linear switched systems that will be detailed in the next section.

4.2 Controllability analysis

Even the differential inclusion related to a linear switched system (3.1) does not define a convex process, a controllability result of the same type still remains valid:

Proposition 6: *The following conditions are equivalent:*

a) the switching system $\dot{x} = A_i x + B_i u,\ i \in \{1, \cdots, s\}, \quad u \in \mathfrak{U}$ *is controllable,*

b) for the associated differential inclusion $\dot{x} \in A_c(x)$ *one has* $A_c^k(0) = (-A)_c^k(0) = \mathbb{R}^n$ *for some* $k \geq 1$.

Proof: The assertion follows by applying (4.1) for the differential inclusion defined by $\mathcal{A}_c$ which is a positively homogeneous inclusion with closed, convex values, hence $\alpha A_c^k(0) = A_c^k(0)$ for any $\alpha > 0$. Since $A_c^k(x) = co\{A_i\}A_c^{k-1}(x) + A_c(0)$ it follows that $A_c^{k-1}(0) \subset A_c^k(x)$ and that $A_c^k(x)$ is a closed convex cone.

It follows that for the reachability set $\mathcal{R}$ one has

$$\mathcal{R} = \cup_{T \geq 0} \mathcal{R}^T(0) = \lim_{N \to \infty} \sum_{k=1}^{N} A_c^k(0) = \lim_{N \to \infty} A_c^N(0).$$

Since the series $A_c^k(0)$ is an increasing sequence of closed convex cones it follows, that if $\mathcal{R} = \mathbb{R}^n$, then there is a finite index M such that $\mathcal{R} = A_c^M(0)$. Since controllability of $\mathcal{A}_c(x)$ is equivalent to reachability of $-\mathcal{A}_c(x)$, it follows the condition $b.)$ of the proposition.

Introducing the notation $\mathbf{co}\{V_j\}$ for the convex hull of the subsets $V_j \subset \mathbb{R}^n$, then the sets $A_p^k := A_c^k(0)$ and $A_m^k := (-A)_c^k(0)$ can be computed using the following algorithm:

General Controllability Algorithm (GCA):

$$\mathbf{U} = \mathbf{co}\{B_i \mathfrak{U} \mid i = 1, \cdots, s\} \tag{4.2}$$

$$A_p^1 = \mathbf{U}, \quad A_m^1 = -\mathbf{U}, \tag{4.3}$$

$$A_p^{k+1} = \mathbf{co}\{A_i A_p^k + B_i \mathfrak{U} \mid i = 1, \cdots, s\}, \tag{4.4}$$

$$A_m^{k+1} = \mathbf{co}\{-A_i A_m^k - B_i \mathfrak{U} \mid i = 1, \cdots, s\}. \tag{4.5}$$

Example 1: *To illustrate the results let us consider the system*

$$A_1 = 0,\ B_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \ B_2 = 0,$$

$$A_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \ B_3 = 0.$$

Applying the algorithm one can find that $A_p^k = A_m^k$ *with* $k = 4$*, i.e., the system is globally controllable.*

Summary

Concerning controllability properties of switched linear time invariant systems the following results were established:

Full rank reachability: Chapter 3, Proposition 5

For a completely controllable linear switching system for arbitrary point pairs (x, y) one has that y is full rank/normally reachable from x. Moreover, every point pair can be joined in a full rank/normal reachable way by using a fixed sequence, i.e., $(\sigma, \tau)/(\sigma, \mu)$ fixed.

Finite switching number: Chapter 3, Proposition 4, Corollary 1, 2, 3

A completely controllable linear switching system is also globally controllable by using piecewise constant switching functions, i.e., using only a finite number of switchings. Moreover, there exist a bound for the necessary number of switchings, that depends only on the system matrices and $\mathfrak{U}$. There exist a universal (finite) switching sequence σ such that the time varying system $\dot{x} = A(\sigma)x + B(\sigma)u$ is globally controllable.

Sign constrained inputs: Chapter 3 and 4, Proposition 3, 6, Algorithm GCA

A complete–controllability condition has been formulated for linear switching systems controlled by sign constrained inputs. The condition – a generalization of the multivariable Kalman rank condition – is expressed in algebraic terms and an algorithm to test it is also provided.

The material covered by these chapters was published in the papers Bokor and Szabó (2003a); Bokor, Szabó and Szigeti (2007c); Stikkel et al. (2004); Szabó (2009).

The theoretical results concerning controllability were used in engineering applications related to fault tolerant and reconfigurable control of vehicles, Gáspár, Szabó and Bokor (2008c,a); Gáspár, Szabó, Szederkényi and Bokor (2008d). In cooperation with the Department of Aerospace and Mechanics, University of Minnesota the developed algorithms were successfully applied in the controllability problem related to the longitudinal dynamics of a supercavitating torpedo, Bokor, Balas and Szabó (2006).

Part II

Stabilizability

5 Stabilizability of completely controllable linear switched systems

The concept of stabilizability is related to the property that there exists a state dependent control law (closed-loop) which, starting from any initial state, asymptotically drives the system into the equilibrium (the origin). This concept expresses the requirements imposed by practical applications to an automatic control solution and it is a corner-stone of every control design algorithm.

For controlled LTI and LTV systems controllability is intimately related to stabilizability in that the former implies the later, moreover stabilizability can be achieved by applying a linear state feedback. Similar result, with a suitable set of linear state feedbacks, is valid for LTI systems when the inputs are sign constrained, see Smirnov (1996) and Krastanov and Veliov (2003).

For general nonlinear systems, however, there is no such result. Controllability ensures that from every initial state the system can be driven to the origin in finite time by using a suitable control. It is not known, in general, whether among these controls there exists at least one which is uniformly bounded by the norm of the initial condition. If this property holds, the system is called *asymptotically controllable*, and despite its name the concept is related to stabilizability rather than controllability, see Clarke et al. (1997). Moreover, it turns out that asymptotic controllability is not only equivalent to stabilizability but also guarantees – under fairly mild conditions – the existence of a not too pathological feedback and control Lyapunov function, see Ancona and Bressan (1999), Kellett and Teel (2004), Rifford (2002).

Unfortunately, these results are hard to be applied in practice to construct directly the required feedback, i.e., to obtain the closed–loop switching strategy and necessary control inputs or even to infer that the control inputs are given by linear feedbacks. Concerning linear switched systems, they are essentially nonlinear, even the individual dynamics are linear. This fact makes the stabilizability problem of linear switched systems nontrivial.

5.1 Asymptotic controllability and weak stabilizability

The zero solution of the differential inclusion $\dot{x} \in A_c(x)$ is called asymptotically weakly stable if there exists a solution $x(t)$ such that for any $\epsilon > 0$ there is a $\delta > 0$ and $\Delta > 0$ such that if $||x(0)|| < \delta$ then $||x(t)|| < \epsilon$ holds for all $t \geq 0$ and if $||x(0)|| < \Delta$ then $\lim_{t\to\infty} x(t) = 0$ holds.

In order to prove stabilizability of completely controllable linear switching systems it is sufficient to show that they are globally asymptotically controllable.

Lemma 2: *A completely controllable linear switching system is globally asymptotically controllable.*

Proof: Let us consider the unit sphere $\mathcal{S}$ and a point $x \in \mathcal{B}$. By complete controllability it follows that there is a finite switching sequence $\tau_x = (\tau_{L_x}, \ldots, \tau_2, \tau_1)$ and a bounded measurable control sequence (actually a piecewise constant control) $u_x = (u_{L_x}, \ldots, u_2, u_1) \in \Omega^{L_x}$ such that the corresponding trajectory steers the point x to the origin, i.e.,

$$\Phi(\tau_x, u_x)x = \prod_{j=1}^{L_x} e^{(A_{l_j}\xi + B_{l_j}u_j)\tau_j} x = 0,$$

where, for notational convenience $e^{(A_{l_j}\xi + B_{l_j}u_j)\tau}\zeta$ denotes the flow associated to the vector field $A_{l_j}\xi + B_{l_j}u_j$ that passes through the initial state ζ at $t = 0$.

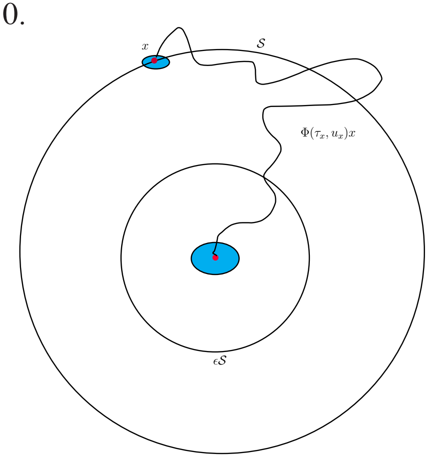
By the continuity of the map $\Phi(\tau_x, u_x)$ for the fixed pair (τ_x, u_x) for every $\epsilon > 0$ there is a neighborhood $\mathcal{V}_x$ of x such that

$$||\Phi(\tau_x, u_x)\xi|| < \epsilon, \quad \forall\, \xi \in \mathcal{V}_x,$$

hence for all $\xi \in \mathcal{W}_x = \mathcal{V}_x \cap \mathcal{S}$.

Since the unit sphere is compact, there is a finite covering $\mathcal{S} = \cup_{j \in J} \mathcal{W}_{x_j}$. It follows that there is a control strategy that maps the unit sphere into the sphere with radius $\epsilon < 1$ defined by this finite partition.

Since the linear maps $\Phi(\tau_{x_j}, u_{x_j})$ are bounded one has a uniform bound for the "overshoot",



Asymptotic controllability

$$\Theta = \max_{j \in J} ||\Phi(\tau_{x_j}, u_{x_j})||.$$

Since the vector fields are linear the reachable spaces are cones, therefore the control strategy can be extended from the unit sphere to the whole state space, i.e., one can construct a trajectory with the bound $||x(t)|| < \Theta||x_0||$ that converges to the origin. It follows that a completely controllable linear switching system is globally asymptotically controllable.

Proposition 7: *The completely controllable linear switching system* (3.1) *is closed–loop stabilizable.*

Remark 3: *For discrete–time linear switched systems with unconstrained inputs the assertion of Proposition 7 was proved recently, see Xie and Wang (2005). The switching strategy in the proposed solution is a periodic one, based on the universal switching sequence. In contrast to the continuous time case the proof is constructive, moreover the necessary linear feedbacks can be obtained by a linear matrix inequality.*

The continuous–time result for the unconstrained input case can be obtained directly from the discrete–time one by using the fact that generically the discretized linear switched system preserves the complete controllability property, see Sun and Ge (2005). The resulting control will be a stabilizing control with a periodic (open–loop) switching strategy and a "feedback–like" control for u – a feedback implemented in a sample and hold way.

The assertion of Proposition 7, however, is also valid for the sign constrained control input case, when the proof based on the discrete–time result is not applicable.

5.2 Stabilizability by Generalized Piecewise Linear Feedback

Given an autonomous linear switching system

$$\dot{x} = A_i x, \quad i \in S$$

it is a nontrivial task to decide if the system is (weakly) stabilizable or not, in general. There are only a few sufficient conditions that guarantee stabilizability and provide a relatively simple closed-loop switching strategy. One such situation is when the convex hull of the system matrices contains a stable (Hurwitz) matrix, i.e., when there are $\alpha_i > 0,\ \sum_{i=1}^{s} \alpha_i = 1$ such that $\sum_{i=1}^{s} \alpha_i A_i$ is stable.

For the non-autonomous case with unconstrained inputs it is known that if the sum of the individual controllability subspaces gives the whole state space, then there are linear state feedbacks $u = K_i x$ such that the resulting linear switching system

$$\dot{x} = (A_i + B_i K_i)x, \quad i \in S$$

is stable with a suitable closed–loop switching strategy, see Sun and Ge (2005). It is not hard to figure out that the required condition is sufficient to guarantee that for any convex combination $\alpha_i > 0,\ \sum_{i=1}^{s} \alpha_i = 1$ there exist feedbacks K_i such that $\sum_{i=1}^{s} \alpha_i (A_i + B_i K_i)$ is stable.

As it can be concluded through simple examples, see Sun and Ge (2005), there are completely controllable switching systems that are not stabilizable by merely applying a single linear state feedback for the individual subsystems. However, as it will be shown in this Section, if the number of linear feedbacks is increased, one can obtain a set of autonomous linear systems that are (weakly) stabilizable.

For a given set of non–autonomous (controlled) linear switched systems (3.1) we call *Generalized Piecewise Linear Feedback Stabilizability* (GPLFS) the problem of finding a closed-loop switching strategy with

- suitable linear feedbacks $u_i = K_{l_i} x, \quad i \in S$
- a switching law $\kappa(x) \in S,\ x \in \mathbb{R}^n$

that (weakly)stabilizes the system.

The reasoning behind introducing the concept of generalized piecewise linear feedback stabilizability is to separate the task of finding a suitable switching strategy and that of finding suitable control inputs with low complexity that stabilizes the system in closed–loop.

The main idea is to substitute the original stabilizable non-autonomous system by a stabilizable autonomous linear switched system that might contain more modes then the original one, by applying as control inputs a number of suitable static linear control feedbacks.

Proposition 8: *The completely controllable linear switching system* (3.1) *is generalized piecewise linear feedback stabilizable.*

Proof: In proving the assertion we will apply ideas of the Nagano–Sussmann–Jurdjevic theory of attainability.

The first observation is that the vector field

$$f(x) = \{f_u(x) = Ax + Bu\}$$

can be replaced by the vector field

$$\mathcal{F}(x) = \{f_K(x) = Ax + BKx\},$$

if $x \neq 0$. Indeed, for any $u \in \Omega$ one can chose a nonzero component x_i of x and a $K = [k_{l,j}]$ such that $k_{l,j} = 0$ if $j \neq i$ and $k_{l,i} = \frac{u_l}{x_i}$, then $u = Kx$. Actually one has

$$F(x) = \mathcal{F}(x), \quad \text{if} \quad x \neq 0.$$

Moreover, for any $y, z \in \mathbb{R}^n \setminus 0$ there is a trajectory of the original system that does not pass through the origin. This follows from the fact that the origin is normally reachable from any point, see Grasse and Sussmann (1990), Sussmann (1987). Then by the surjective mapping theorem, Bartle (1976), follows that a neighborhood of the origin is reachable by the same switching sequence. Hence, by the linearity of the vector fields, the whole space is reachable with the given switching sequence.

Since the trajectory $x(t)$ does not pass through the origin, the original vector fields ($F(x)$) can be replaced by the new one ($\mathcal{F}(x)$). Moreover, since a given component of $x(t)$ might vanish only a finite times on a finite interval, it follows that the controls K_i of the vector field $F_K(x)$ are piecewise continuous. It follows that every point pair of the manifold $\mathbb{R}^n \setminus 0$ can be joined by a trajectory corresponding to the vector field $\mathcal{F}$ by admissible controls.

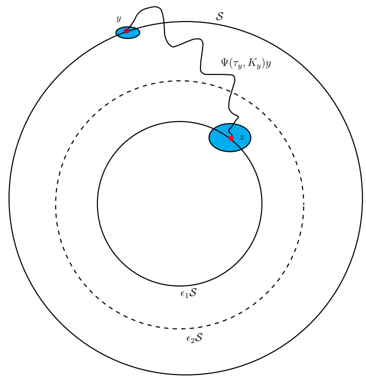
It follows that the vector field $\mathcal{F}$ is completely controllable on the manifold $\mathbb{R}^n \setminus 0$. Since complete controllability implies controllability by piecewise constant controls, see Grasse (1985), Grasse and Sussmann (1990), it follows that every point pair of the space $\mathbb{R}^n \setminus 0$ can be joined by a trajectory corresponding to a suitable autonomous switched systems $A_l + B_l K_l$.

Remark 4: *Complete controllability of the vector field $\mathcal{F}$ has a very intuitive geometrical background. Since the solutions of a linear autonomous differential equations realizes some rotations and dilations/compressions in $\mathbb{R}^n$, it means that for a given point pair (y, z) it is possible to select a finite set of feedbacks such that the resulting set of autonomous systems transform the point y into z for a suitable (finite) switching sequence.*

In order to show that it is possible to select a finite set of autonomous systems that has the (weak) stabilizability property, the compactness argument applied in the proof of Lemma 2 can be repeated.

Indeed, selecting a point y on the unit sphere $\mathcal{S}$ and fixing a point z on the sphere ϵ_1, there is a trajectory formed by suitable autonomous systems $A_l + B_l K_l$ that steers y to z, i.e.,

$$\Psi(\tau_y, K_y)y = \prod_{j=1}^{L_y} e^{(A_{l_j} + B_{l_j} K_j)\tau_j} y = z.$$



Closed–loop stabilizability

By continuity of $\Psi(\tau_y, K_y)$ for fixed τ_y and K_y there is a neighborhood of y that is mapped in a sufficiently small neighborhood of z, such that $||\Psi(\tau_y, K_y)\xi|| < \epsilon_2$, with $0 < \epsilon_1 < \epsilon_2 < 1$.

These neighborhoods form a covering of the unit sphere, from which it is possible to select a finite one. It follows that it is possible to select a finite set of linear static state feedbacks such, that the resulting set of autonomous system is stabilizable.

Concerning the switching strategy the existence of the suitable closed–loop switching rule is guaranteed by the general results for nonlinear globally asymptotically controllable systems, Rifford (2002). However, for nonautonomous switching systems with unconstrained controls slightly more can be asserted.

In Lin and Antsaklis (2007) it was shown that the existence of an asymptotically stabilizing switching strategy (without sliding motion) of an autonomous linear switched system implies the existence of a conic partition based switching law which globally asymptotically stabilizes the closed–loop switching system. The control is defined by a conic partition $\mathbb{R}^n = \bigcup_{l=1}^{L} \mathcal{C}_l$ of the state space while on each cone $\mathcal{C}_l$ the system defined by $A_{i_l} + B_{i_l} K_l$ with $i_l \in S$ is active.

Remark 5: *Since for linear autonomous switching systems asymptotical stability and exponential stability are equivalent, see Sun and Ge (2005), Proposition 7 shows that completely controllable linear switching systems with (unconstrained input) are exponentially stabilizable.*

The sign constrained case is more delicate. The resulting autonomous systems correspond to certain regions of the state space, i.e., the resulting switching system is an autonomous state constrained linear switching system. Therefore the result from Sun and Ge (2005) is not applicable directly and the case needs further investigation.

Remark 6: *Proposition 8 guarantees the generalized piecewise linear feedback stabilizability but does not give a method to compute such feedbacks. However – for the unconstrained input case – the property of complete controllability is feedback invariant. It is known that any controllable unconstrained multi–input linear switching system can be changed into a controllable single–input system via suitable non–regular state feedbacks, see Sun and Ge (2005). Moreover, the controllable single–input system can be put into the form* $(A_1, b_1), A_2, \cdots, A_s$. *Proposition 7 guarantees that by these transformations not only controllability but also stabilizability is preserved. Hence one can obtain a switching system with a reduced complexity for which one might find suitable stabilizing feedbacks more easily, e.g. the resulting BMI or LMI equations in finding suitable piecewise quadratic Lyapunov functions will be simpler.*

Besides the fact that stabilization schemes with state depending switching rules are hard to construct these schemes might not be robust against the quantization errors introduced by a sampled implementation.

From a more general perspective the difficulties encountered at the feedback stabilization of switching systems are not surprising. For continuous–time control systems the existence of smooth Lyapunov functions implies that the differential inclusion satisfy a certain covering condition – an extension of Brockett's "covering condition" from continuous feedback stabilization theory, Clarke et al. (1998). However, robustness of the feedback scheme and the existence of a smooth control Lyapunov function are closely related, see Ledyaev and Sontag (1999). Moreover, in general, stabilizable switched linear systems does not have a convex Lyapunov function, see Blanchini and Savorgnan (2006).

In contrast to the pure continuous–time approach, discrete–time asymptotic controllability implies smooth control Lyapunov function. Moreover, robustness can be induced via a sample–and–hold control. For details see Kellett and Teel (2004).

The results of the previous section gives an opportunity to verify these claims for the class of unconstrained linear switching systems (3.1).

By choosing a nonsingular Schur-stable matrix A_d, one can explicitly construct the inputs that stabilize the time–varying systems obtained by a periodic repetition of the sequence σ defined in Proposition 5 by choosing the sequence of inputs as follows:

$$u^{x_0} = (\mathcal{C}^{\sigma}_{\tau})^{\dagger}(A_d - \bar{A}_{\sigma})x_0, \tag{5.1}$$

where $M^{\dagger}$ denotes a generalized inverse of M. Considering linear feedbacks, i.e., the closed-loop matrix $A_c = \prod_{i=1}^{N}(\bar{A}_{s_i} + \bar{B}_{s_i}K_i)$, one has $A_d = A_c$ provided that the system

$$\tilde{K}_i = K_i \prod_{j=1}^{i-1}(\bar{A}_{s_i} + \bar{B}_{s_i}K_i) \tag{5.2}$$

is solvable for $\tilde{K}_i = P_i(\mathcal{C}^{\sigma}_{\tau})^{\dagger}(A_d - \bar{A}_{\sigma})$ with P_i the projection that gives the i^{th} input from (5.1). This is equivalent with the assertion that the resulting feedback sequence is such that $\bar{A}_{s_i} + \bar{B}_{s_i}K_i$ is nonsingular.

It is not true, in general, that for an arbitrary nonsingular A_d (5.2) always has a solution. This can be seen through the following counterexample :

consider the discrete–time linear switching system

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{5.3}$$

which is completely controllable. One has $\sigma = (2, 1)$ *and* $\mathcal{C}^{\sigma} = [A_2B_1\ B_2] = \mathbb{I}_2$.

By choosing the nonsingular Schur matrix $A_d = \begin{bmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}$ *one can obtain the gains* $K_1 = \tilde{K}_1 = [1\ \frac{1}{2}]$ *and* $\tilde{K}_2 = [-\frac{1}{2}\ 0]$. *Since* $A_1 + B_1K_1 = \begin{bmatrix} 0 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix}$ *there does not exist* K_2 *such that the relation* $A_d = (A_2 + B_2K_2)(A_1 + B_1K_1)$ *holds.*

Despite this fact there always exist feedback gains such that A_c is a (nonsingular) Schur matrix. A sketch of the proof is as follows: fix a nonsingular Schur matrix A_d, i.e., $||A_d|| < \frac{\lambda}{2}$with $\lambda < 1$, and compute the sequence $\tilde{K}_i$. For any invertible matrix A and matrices U, V one has that $\det(\mathbf{A} + \mathbf{U}\mathbf{V}^{\mathrm{T}}) = \det(\mathbf{I} + \mathbf{V}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{U})\det(\mathbf{A})$. Moreover, the determinant is a continuous function of the matrix components, hence if at a given step $\bar{A}_{s_i} + \bar{B}_{s_i}K_i$ would be nonsingular then K_i can be perturbed to $\hat{K}_i = K_i + \epsilon K_{i,\epsilon_i}$ such that $\bar{A}_{s_i} + \bar{B}_{s_i}\hat{K}_i$ is nonsingular and $\epsilon_i > 0$ is arbitrarily small. Finally one get the matrix

$$A_c = \prod_{i=1}^{N}(\bar{A}_{s_i} + \bar{B}_{s_i}\hat{K}_i) = A_d + \sum_{i=1}^{N}\epsilon_i \bar{B}_{s_i} K_{i,\epsilon}\tilde{A}_i \tag{5.4}$$

with $\tilde{A}_i = \prod_{j=1}^{i-1}(\bar{A}_{s_i} + \bar{B}_{s_i}\hat{K}_i)$, $\tilde{A}_1 = \mathbb{I}$.

By choosing $\epsilon_i \le \frac{\lambda}{2N\|B_{s_i}K_{i,\epsilon}\tilde{A}_i\|}$ one has $\|A_c\| \le \lambda$, i.e., A_c is a nonsingular Schur matrix.

Observe that the number of modes needed for the stabilization is bounded by the length of the switching sequence σ. This fact motivates the interest in finding efficiently the shortest sequence.

Remark 7: *For the constrained case the formula* (3.7) *does not leads directly to a stabilizing feedback solution of the time–varying system. Moreover, since for any nonzero vector* k *the term* $k^T x$ *cannot be nonnegative for all* $x \in \mathbb{R}^n$ *it is immediate that a suitable partition of the state space is also needed, i.e., the solution will be an event driven switching strategy.*

An LMI condition can be given for the synthesis of the stabilizing feedback gains of unconstrained controllable discrete–time linear switching systems. Moreover, this result can be directly applied for the stabilization of sampled unconstrained controllable linear switching systems.

This section will be concluded by a slightly extended version of the result, by setting LMIs that provide robust stabilization for uncertain systems.

Proposition 9: *Suppose that the uncertain discrete–time switching system* $x_{k+1} = A_i(\Delta)x_k + B_i(\Delta)u_k$, $u_k \in \mathbb{R}^m$ *is controllable and suppose that there exist a switching sequence* $\sigma = (s_1, \cdots, s_M)$ *such that* $\mathcal{R}_\sigma = \mathbb{R}^n$ *independently of* Δ.

Then there exist a positive definite matrix S*, nonsingular matrices* V_i *and matrices* F_i *such that the following LMI is feasible.*

$$\begin{bmatrix} S & A_{s_M}V_M + B_{s_M}F_M & \dots & 0 & 0 \\ (\bullet)^T & V_M + V_M^T & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & V_2 + V_2^T & A_{s_1}V_M + B_{s_1}F_1 \\ 0 & 0 & \dots & (\bullet)^T & V_1 + V_1^T - S \end{bmatrix} > 0$$

The system can be stabilized with the periodic switching signal defined by σ *and the state feedback gains given by* $K_i = F_i V_i^{-1}$, $i = 1, \cdots, M$.

Proof: The proof of the assertion is based by a recursive application of the elimination lemma, see Boyd et al. (1994). Denote by $\sigma_i = (s_i, \cdots, s_M)$, then $\bar{A}_{\sigma_i} = \bar{A}_i \bar{A}_{\sigma_{i-1}}$ with $\bar{A}_i = A_{s_i} + B_{s_i}K_i$. By assumption there is an $S > 0$ such that $\bar{A}_{\sigma_1} S \bar{A}_{\sigma_1}^T - S < 0$ which can be written as

$$\begin{bmatrix} \mathbb{I} & -\bar{A}_M(\Delta) \end{bmatrix} \begin{bmatrix} -S & 0 \\ 0 & \bar{A}_{\sigma_{M-1}}(\Delta) S \bar{A}_{\sigma_{M-1}}^T(\Delta) \end{bmatrix} \begin{bmatrix} \mathbb{I} \\ -\bar{A}_M^T(\Delta) \end{bmatrix} < 0 \tag{5.5}$$

By the elimination lemma this inequality is equivalent with:

$$\begin{bmatrix} -S & 0 \\ 0 & \bar{A}_{\sigma_{M-1}}(\Delta) S \bar{A}^T_{\sigma_{M-1}}(\Delta) \end{bmatrix} + Sym\{\begin{bmatrix} -\bar{A}_M(\Delta) \\ -\mathbb{I} \end{bmatrix} V_M \begin{bmatrix} 0 & \mathbb{I} \end{bmatrix}\} \tag{5.6}$$

Repeating the procedure one can obtain the assertion of the proposition.

Remark 8: *Having a polytopic uncertainty, i.e., $A(\Delta) = A_0 + \delta_1 A_1 + \cdots + \delta_k A_K$, the LMIs of Proposition 9 form a finite set of conditions that can be easily solved.*

Proposition 10: *Completely controllable linear switched systems can be piecewise linear feedback stabilized using a periodic switching sequence.*

Proof: For the proof recall that the reachable set of the Euler discretized differential inclusion approaches uniformly well the reachable set of the original inclusion $\mathcal{A}_c$, see Proposition 5.3 in Wolenski (1990), i.e., for a given $\epsilon > 0$ and for all ξ in a compact set there is an N_0 independent of ξ such that for each $N > N_0$, $0 \leq j \leq N$ one has $\mathbf{dist}(\mathcal{R}^{jh}(\xi), (I + \frac{T}{N}\mathcal{A}_c)^j(\xi)) < \epsilon$, where **dist** is the Hausdorff distance. Since for almost all τ one has $< A, \mathcal{V} >=< \mathbb{I} + \tau A, \mathcal{V} >$, the Euler discretized system will be also completely controllable, moreover their is a common stabilizing sequence σ for the two systems. For the discretized system one can design feedbacks that ensure arbitrary high decay rates of the closed–loop system. By choosing sufficiently small τ the point $\tilde{A}_\sigma \xi$ will be in a sufficiently small neighborhood of $\bar{A}_\sigma \xi$, where $\tilde{A}_i = A_i + B_i K_i$ and $\bar{A}_i = \mathbb{I} + \tau \tilde{A}_i$ and $\bar{A}_\sigma$ is a Schur matrix. It follows that the matrix $e^{\tau \tilde{A}_{s_N}} \cdots e^{\tau \tilde{A}_{s_1}}$ will be also a Schur matrix. This proves the assertion.

Example

This chapter will be concluded by an illustrative example, which is based on a problem setting borrowed from the book of Sun and Ge (2005).

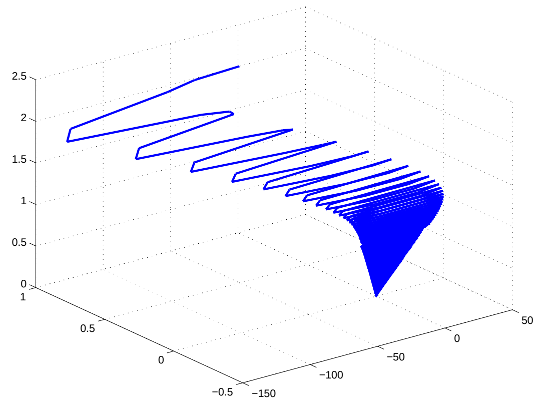
Let us consider the controlled linear switching system $\dot{x} = A_i x + B_i u$, $i \in \{1, 2\}$ defined by

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix},$$

which was exposed in Sun and Ge (2005) as a system which is globally controllable but that is not trivial to stabilize since $< A_1, B_1 > + < A_2, B_2 > \neq \mathbb{R}^3$ and the individual dynamics have a common unstable mode. By applying the methods presented in this chapter, however, it is possible to construct a homogeneous linear switched system by applying suitable linear state feedbacks. Moreover, this switched system can be stabilized by applying a periodic switching law.

One can figure out that $\mathcal{R}_\sigma = \mathbb{R}^3$ for the switching sequence $\sigma = (1, 1, 1, 2, 1)$.
Solving the LMIs the stabilizing feedback gains are:

$$k_1 = 10^4 \begin{bmatrix} -0.1086 & -8.6083 & 1.4443 \end{bmatrix}$$
$$k_2 = 10^5 \begin{bmatrix} -0.0112 & -1.2846 & 0.2099 \end{bmatrix}$$
$$k_3 = 10^4 \begin{bmatrix} -0.1081 & -6.6929 & -7.1136 \end{bmatrix}$$
$$k_4 = 10^3 \begin{bmatrix} -1.0000 & 0.0000 & -0.0000 \end{bmatrix}$$



GPLF stabilizability

The feedbacks were designed for the Euler discretization corresponding to the sampled time of $\tau = 0.001$ sec, while the simulation was started from the initial point $x_0 = [1\ 1\ 2]$.

The overshoots are due to the unstability present in the individual modes that acts as a performance barrier in these type of problems. Even some preliminary results concerning the LQ control of discrete–time switched systems are reported recently in Zhang, Abate, Vitus and Hu (2009) and Zhang, Hu and Abate (2009), there are no reliable design algorithms for feedback stabilization, in general. The stabilizability result presented here makes possible to extend the discrete–time results to continuous–time design problems.

It is a subject of further research to investigate the optimal performance level achievable by certain configurations and to determine how it can be imposed additional performance requirements in the design process.

Summary

Concerning stabilizability properties of switched linear time invariant systems the following results were established:

Stabilizability Chapter 5, Proposition 7

A completely controllable linear switching system is globally asymptotically controllable, hence it is closed–loop stabilizable.

Linear feedback Chapter 5, Proposition 8

The completely controllable linear switching system (3.1) is generalized piecewise linear feedback stabilizable.

Time driven stabilization Chapter 5, Proposition 9, 10

Completely controllable linear switched systems can be piecewise linear feedback stabilized using a periodic switching sequence. The feedback gains can be computed by obtaining a switching sequence that realizes the complete controllability and by solving an LMI.

The topic of the chapter is covered by the following papers Szabó, Bokor and Balas (2007, 2008); Szabó (2009); Szabó, Bokor and Balas (2009c).

The motivating engineering problems that provide, among others, the applicational background of these stabilizability results were related to fault-tolerant reconfigurable control with multiple, possibly conflicting performance specifications, see, e.g., Bokor, Szabó, Nádai and Rudas (2007b); Gáspár et al. (2009,a), and the control of the supercavitating torpedo..

Part III

Geometric theory of LPV systems: Robust Invariance

6 Parameter-varying invariant subspaces

As it was shown in the first part of the thesis controllability and stabilizability properties of switched linear systems can be described entirely in terms of the system matrices by using matrix algebraic manipulations. While this property does not hold for general LTV systems, there is a notion that survives the transition from LTI to LTV: the concept of the *invariant subspace*. The notion has already occurred in the controllability study, where the reachability set behaves as the minimal set invariant to the action of the (controlled) dynamics. For unconstrained linear systems the reachability set is a subspace which induces a controllability decomposition of the system. This geometric view, i.e., the idea of invariance and invariant subspaces, relates the controllability study of switched systems to the topics that will be presented in this part of the thesis. The developed techniques and algorithms that leads to the specific invariant subspaces, hence, to the specific state space decompositions, make the glue that unifies the different problems like controllability, detection filter design or tracking control at an applicational level. Thus the geometric approach provides a common framework in which all these problems can be handled.

For LTI systems the concept of certain invariant subspaces and the corresponding global decompositions of the state equations induced by these invariant subspaces was one of the main thrusts for the development of geometric methods for solutions to problems of disturbance decoupling or non-interacting control, see Wonham (1985). The mathematically dual concepts of (A, B)–invariance (controlled–invariance) and of (C, A)–invariance (conditioned–invariance) play an important role in the geometric theory of LTI systems. These concepts were used to study some fundamental problems of LTI control theory, e.g., Wonham (1985); Massoumnia (1986); Basile and Marro (2002).

Linear time varying (LTV) case and nonlinear systems can be studied using tools from differential geometry, when the central role is played by the concept of invariant distributions and much more complex mathematical objects given by the locally controlled or conditioned invariant distribution(or codistribution) algorithms. From a geometric viewpoint results of the classical linear control can be seen as special cases of these more general nonlinear results, for details see, e.g., Isidori (1989); Nijmeijer and van der Schaft (1990); De Persis and Isidori (2000). Due to the computational complexity involved, these nonlinear methods have limited applicability in practice. The main problem that arises in practical situations is that either one cannot perform the computations or one cannot verify the conditions under the given algorithms provide the desired results.

If certain conditions are fulfilled, e.g., if the parameter functions are differential algebraically independent, then the parameter invariant subspaces, that will be introduced in this chapter, coincide with the corresponding invariant distribution or codistribution, respectively. However, to give sufficient conditions for the solution of certain state feedback and observer filter design problems it is enough that some decompositions of the state equations could be performed. The parameter–varying versions of these invariant spaces are suitable objects to define the required decompositions, therefore they can play the same role in the solution of the fundamental problems, such as disturbance decoupling(DDP), unknown input observer design, fault detection (FPRG), as their counterparts in the time invariant context.

6.1 Invariant subspaces for time varying systems

Before the introduction of the invariance notion that best suits the parameter varying framework let us recall some the corresponding term used in the general nonlinear context: a distribution Δ is said to be *invariant*[1] under a vector field f if for $\tau \in \Delta$ one has $[f, \tau] \in \Delta$, or shortly, $[f, \Delta] \subset \Delta$. Dealing with codistributions, Ω is said to be invariant[2] under the vector field f if for $\omega \in \Omega$ one has $L_f \omega \in \Omega$ or shortly $L_f \Omega \subset \Omega$.

By doing a usual augmentation, see e.g., Hermann and Krener (1977), of the original state space to $\xi := [t, x]^T$, an LPV system can be viewed as an affine nonlinear system:

$$\dot{\xi} = g_0(\xi) + \sum_{i=1}^{m} g_i(\xi) u_i$$
$$y = h(\xi)\xi,$$

where $g_0(\xi)$ denotes $\begin{bmatrix} 1 \\ A(\rho)x \end{bmatrix}$, $g_i(\xi)$ is the vector $\begin{bmatrix} 0 \\ B_i(\rho) \end{bmatrix}$ with $B_i(\rho)$ the i^{th} column of $B(\rho)$ and $h(\xi) = [0 \;\; C(\rho)]$.

Restricting the investigations to linear subspaces, as special instances of distributions, i.e., with some subspace $\mathcal{V}$ of $\mathbb{R}^n$ $\Delta(\xi) = \begin{bmatrix} 0 \\ \mathcal{V} \end{bmatrix}$, then Δ will be invariant under the vector fields g_i if and only if $\partial_\xi g_i \Delta(\xi) \subset \Delta(\xi)$, for all i and ξ. Performing the computations one has that Δ is an invariant distribution for the action of the vector fields g_i if and only if $A(\rho)\mathcal{V} \subset \mathcal{V}$ for all $\rho \in \mathcal{P}$. Using a similar argument, one can get the analogous relations for the corresponding codistributions.

These facts motivate the introduction of the following notion:

Definition 1: *A subspace $\mathcal{V}$ is called parameter-varying invariant subspace (or shortly $\mathcal{A}$–invariant subspace) for the family of the linear maps $A(\rho)$ if*

$$A(\rho)\mathcal{V} \subset \mathcal{V} \quad \textit{for all } \rho \in \mathcal{P}. \tag{6.1}$$

As for the LTI case an $\mathcal{A}$-invariant subspace $\mathcal{V}$ induces a splitting $x = \bar{x} + \tilde{x}$ of the state space

[1] Let $\Delta_{\mathcal{V}}(x) = \mathcal{V}$ be a constant distribution, where $\mathcal{V}$ is a subspace of $\mathbb{R}^n$ and $f_A(x) = Ax$ be a linear vector field. Since $[f_A, v](x) = -Av$ for all $v \in \mathcal{V}$ and $x \in \mathbb{R}^n$, we get back the usual invariance notion for subspaces, i.e., $A\mathcal{V} \subset \mathcal{V}$.

[2] Now let $\Omega_{\mathcal{W}}(x) = \mathcal{W}_c$ be a constant codistribution, where $\mathcal{W}_c$ is a subspace of $(\mathbb{R}^n)^*$ and the vector field $f_A(x) = Ax$ is linear then we get back the invariance notion of subspaces in $(\mathbb{R}^n)^*$, i.e., $A^T \mathcal{W} \subset \mathcal{W}$. Recall that $\mathcal{W}_c A \subset \mathcal{W}_c$ and we identify $\mathcal{W}_c \subset (\mathbb{R}^n)^*$ with $\mathcal{W} \subset \mathbb{R}^n$ in a usual way, i.e., if $\mathcal{W} = \mathbf{Im}\, W$ than $\mathcal{W}_c = \mathbf{Im}\, W^T$.

with $\bar{x} = \mathbf{P}_{\mathcal{V}}$ and $\tilde{x} = \mathbf{P}_{\mathcal{V}^\perp}$ such that the system $\dot{x} = A(\rho)x$ will have the form

$$\dot{\bar{x}} = \bar{A}(\rho)\bar{x} + \tilde{A}_1(\rho)\tilde{x} \tag{6.2}$$

$$\dot{\tilde{x}} = \tilde{A}_2(\rho)\tilde{x}. \tag{6.3}$$

where

$$\bar{A}(\rho) = A(\rho)|_{\mathcal{V}}, \tag{6.4}$$

is the restriction of $A(\rho)$ to the subspace $\mathcal{V}$.

The main point here is the fact that the state transform $x = T\begin{bmatrix}\bar{x}\\ \tilde{x}\end{bmatrix}$ defined by $T = \begin{bmatrix}\mathcal{V} & \mathcal{V}^\perp\end{bmatrix}$ leads to the splitting

$$A \xrightarrow{TAT^{-1}} \underbrace{\begin{bmatrix} \bar{A} \\ 0 \end{bmatrix}}_{\mathcal{V}} \underbrace{\begin{bmatrix} \tilde{A}_1 \\ \tilde{A}_2 \end{bmatrix}}_{\mathcal{V}^\perp} \begin{matrix} \}\mathcal{V} \\ \}\mathcal{V}^\perp \end{matrix}$$

and this splitting is independent of the actual values of the parameters ρ, i.e., it can be performed offline. This fact has a great impact on the applicability of the newly introduced concept for design problems.

6.2 Controlled invariance

Let us observe, that if $\mathcal{V}$ is an $\mathcal{A}$-invariant subspace and $\mathbf{Im}\, B(\rho) \subset \mathcal{V}$ for all $\rho \in \mathcal{P}$ then the system $\dot{x} = A(\rho)x + B(\rho)u$ can be decomposed as

$$\dot{\bar{x}} = \bar{A}(\rho)\bar{x} + \tilde{A}_1(\rho)\tilde{x} + \overline{B}(\rho)u \tag{6.5}$$

$$\dot{\tilde{x}} = \tilde{A}_2(\rho)\tilde{x}, \tag{6.6}$$

An involutive distribution Δ is said to be *controlled invariant* on an open set U if

$$[g_i, \Delta](x) \subset \Delta(x) + G(x), \quad i = 0, 1, \ldots, m,\ x \in U.$$

or shortly $[g_i, \Delta] \subset \Delta + G$, assuming that Δ, G and $\Delta + G$ are nonsingular, where G denotes the distribution $\mathbf{span}\{g_1, \ldots, g_m\}$. For the covectorial version: a codistribution Ω is said to be controlled invariant if

$$L_{g_i}(\Omega \cap G^\perp) \subset \Omega, \qquad i = 0, 1, \ldots, m.$$

If the intersection may fail to be smooth, then L_{g_i} is only defined on the smooth codistributions of the intersection.

Using again the augmented state space and the distribution $\begin{bmatrix}0\\ \mathcal{V}\end{bmatrix}$ one can show that when $B(\rho) = \mathcal{B}$ then $\mathcal{V} \subset \mathbb{R}^n$ is controlled invariant subspace (distribution) if and only if $A(\rho)\mathcal{V} \subset \mathcal{V} + \mathcal{B}$ for all $\rho \in \mathcal{P}$.

These facts motivate the introduction of the following notion:

Definition 2: *The subspace $\mathcal{V}$ is called a parameter-varying (A, B)–invariant subspace (or shortly $(\mathcal{A}, \mathcal{B})$–invariant subspace) if for all $\rho \in \mathcal{P}$ any of the following equivalent conditions holds :*

$$A(\rho)\mathcal{V} \subset \mathcal{V} + \mathcal{B}(\rho), \tag{6.7}$$

and there exists a mapping (a state feedback) $F \circ \rho : [0, T] \to \mathbb{R}^{m \times n}$ such that:

$$(A(\rho) + B(\rho)F(\rho))\mathcal{V} \subset \mathcal{V}, \tag{6.8}$$

where $\mathcal{B}(\rho)$ denotes $\boldsymbol{Im}\, B(\rho)$.

Dealing with parametric uncertainties a similar concept was introduced in Basile and Marro (1987), called *robust controlled invariant* subspace. If one sets the gain matrix to be constant then the resulting subspace will be more restrictive, this approach was used in Bhattacharyya (1983) and Otsuka (2000), and was termed as *generalized controllability (A, B)–invariant subspace*.

It is obvious that the subspace $\mathcal{R}_{(\mathcal{A},\mathcal{B})}$ in (3.4) is $A(\rho)$ invariant, i.e.,

$$A(\rho)\mathcal{R}_{(\mathcal{A},\mathcal{B})} \subseteq \mathcal{R}_{(\mathcal{A},\mathcal{B})}, \quad \text{for all} \quad \rho \in \mathcal{P}, \tag{6.9}$$

moreover, one has that for the induced decomposition $\mathcal{R}_{(\mathcal{A},\mathcal{B})} = \mathcal{R}_{(\bar{\mathcal{A}},\bar{\mathcal{B}})}$. Actually $\mathcal{R}_{(\mathcal{A},\mathcal{B})}$ is the minimal $A(\rho)$ invariant subspace containing $\mathcal{B}$.

The set of all $\mathcal{A}$–invariants containing $\mathcal{B}$ is a nondistributive lattice with respect to the set operations $\subseteq$, $\cup$, $\cap$. The supremum of the lattice is the entire state space $\mathcal{X}$, while the infimum is the intersection of all the $\mathcal{A}$–invariants containing $\mathcal{B}$. It will be called, the *minimal $\mathcal{A}$–invariant subspace containing* $\mathcal{B}$, which is also an $(\mathcal{A}, \mathcal{B})$–invariant subspace, and it will be denoted by $\langle \mathcal{A} | \mathcal{B} \rangle$.

As for the LTI systems (6.8) guarantees that with a suitable state feedback $u = F(\rho)x + v$ equation (6.5) can be rendered diagonal, i.e.,

$$\dot{x} = A(\rho)x + B(\rho)u \quad \xrightarrow[u=F(\rho)x+v]{TAT^{-1},\, TB} \quad \begin{array}{lcll} \dot{\bar{x}} & = & \bar{A}(\rho)\bar{x} + & \qquad +\overline{B}(\rho)v \\ \dot{\tilde{x}} & = & \qquad \tilde{A}_2(\rho)\tilde{x}, & \end{array} \tag{6.10}$$

$\tilde{x}$ being an uncontrollable mode. However, as it was shown in Section 2.1 in general controllability of $\bar{x}$ can be asserted only if $\mathcal{V} = \langle \mathcal{A} | \mathcal{B} \rangle$ and the c-persistency property holds.

The set of all $(\mathcal{A}, \mathcal{B})$–invariant subspaces contained in a given subspace $\mathcal{K}$, is an upper semilattice with respect to subspace addition. This semilattice admits a maximum which will be denoted by $\mathcal{V}^*$.

As far as the LPV case is concerned it was found that the following definition would be usable for the generalization of the concept of the controllability subspace:

Definition 3: *A subspace $\mathcal{R}$ is called parameter-varying controllability subspace if there exists a constant matrix K and a parameter varying matrix $F : [0, T] \to \mathbb{R}^{m \times n}$ such that*

$$\mathcal{R} = \langle \mathcal{A} + B\mathcal{F} | \boldsymbol{Im}\, BK \rangle, \tag{6.11}$$

where the notation $\mathcal{A} + B\mathcal{F}$ stems for the system $A(\rho) + BF(\rho)$ with $\boldsymbol{Im}\, B(\rho) = \boldsymbol{Im}\, B$.

Analogously with the corresponding LTI results the following properties hold:

Proposition 11: *If* $\hat{\mathcal{B}} \subset \mathcal{B}$ *and* $\mathcal{R} = \langle \mathcal{A} | \mathcal{B} \rangle$ *then* $\mathcal{R} = \langle \mathcal{A} | \mathcal{B} \cap \mathcal{R} \rangle$. *Conversely, if one has* $\mathcal{R} = \langle \mathcal{A} | \mathcal{B} \cap \mathcal{R} \rangle$ *then there exists an input mixing map* $K : \mathcal{U} \to \mathcal{U}$ *for which* $\mathcal{R} = \langle \mathcal{A} | \boldsymbol{Im}\, BK \rangle$.

Proposition 12: *A subspace* $\mathcal{R}$ *is parameter-varying controllability subspace if and only if there exists* $F : [0, T] \to \mathbb{R}^{m \times n}$ *such that*

$$\mathcal{R} = \langle \mathcal{A} + B\mathcal{F} | \mathcal{B} \cap \mathcal{R} \rangle.$$

If the parameter dependence is affine, then considering a fixed subspace $\mathcal{R} \subset \mathcal{X}$ and defining the class

$$\Gamma = \{\mathcal{Z} : \mathcal{Z} = \mathcal{R} \cap (\sum_{i=0}^{N} A_i \mathcal{Z} + \mathcal{B})\}.$$

one has that:

Lemma 3: *There exists a unique minimal element* $\mathcal{Z}^*$ *of* Γ.

Moreover it can be shown that:

Proposition 13: *A subspace* $\mathcal{R} \subset \mathcal{X}$ *is a parameter-varying controllability subspace if and only if it is* $(\mathcal{A}, \mathcal{B})$*-invariant and* $\mathcal{R} = \mathcal{Z}^*$.

6.3 Conditioned invariance

The dual notion of controlled invariance is conditioned invariance which can be defined as follows: a distribution Δ is said to be *conditioned invariant* on an open set U if it satisfies $[g_i, \Delta \cap \mathbf{Ker}\, \mathrm{d}h](x) \subset \Delta(x)$, or shortly $[g_i, \Delta \cap \mathbf{Ker}\, \mathrm{d}h] \subset \Delta$ for $i = 0, 1, \ldots, m$, $x \in U$. For the covectorial version: a codistribution Ω is said to be conditioned invariant if $L_{g_i} \Omega \subset \Omega + \mathbf{span}\, \mathrm{d}h\}$ for $i = 0, 1, \ldots, m$.

Considering a subspace $\mathcal{W}$ for affine parameter dependence one hast that for any $w \in \mathcal{W} \cap \mathbf{Ker}\, C$

$$\frac{\partial (A(\rho)x)v}{\partial x} = A(\rho)v + \sum_{i=1}^{N} A_i x \frac{\partial \rho_i}{\partial y} C v = A(\rho)v,$$

and

$$\frac{\partial B_i(\rho)v}{\partial x} = \sum_{i=1}^{N} B_i \frac{\partial \rho_i}{\partial y} C v = 0.$$

Using the augmented state space, the distribution $\begin{bmatrix} 0 \\ \mathcal{W} \end{bmatrix}$ and considering the case $C(\rho) = C$ it follows that for LPV systems with affine parameter dependence $\mathcal{W} \subset \mathbb{R}^n$ is a conditioned invariant subspace if and only if $A(\rho)(\mathcal{W} \cap \mathbf{Ker}\, C) \subset \mathcal{W}$ for all $\rho \in \mathcal{P}$.

This fact leads us to the introduction of the following notion:

Definition 4: *The subspace $\mathcal{W}$ is called a parameter-varying (C, A)–invariant subspace (or shortly $(\mathcal{C}, \mathcal{A})$–invariant subspace) if for all $\rho \in \mathcal{P}$ any of the following equivalent conditions holds:*

$$A(\rho)(\mathcal{W} \cap \mathcal{C}(\rho)) \subset \mathcal{W} \tag{6.12}$$

and there exists a mapping $G \circ \rho : [0, T] \to \mathbb{R}^{n \times p}$ such that:

$$(A(\rho) + G(\rho)C(\rho))\mathcal{W} \subset \mathcal{W}. \tag{6.13}$$

where $\mathcal{C}(\rho)$ denotes $\mathbf{Ker}\, C(\rho)$.

The set of all $\mathcal{A}$–invariants contained in $\mathcal{C}$ is a nondistributive lattice with respect to to the set operations $\subseteq$, $\cup$, $\cap$. The infimum of the lattice is clearly $\{0\}$, while the supremum is the sum of all the $\mathcal{A}$–invariants contained in $\mathcal{C}$. It will be called the *maximal $\mathcal{A}$–invariant contained in $\mathcal{C}$*, which is also a $(\mathcal{C}, \mathcal{A})$–invariant subspace, and it will be denoted by $\langle \mathcal{C} | \mathcal{A} \rangle$.

As for the LTI systems (6.12) guarantees that the following splitting holds:

$$\begin{array}{rcl} \dot{x} & = & A(\rho)x \\ y & = & Cx \end{array} \xrightarrow{TAT^{-1},\, CT^{-1}} \begin{array}{rcl} \dot{\bar{x}} & = & \bar{A}(\rho)\bar{x} \\ \dot{\tilde{x}} & = & \tilde{A}_{21}(\rho)\bar{x} + \tilde{A}_{22}(\rho)\tilde{x} \\ y & = & \bar{C}\bar{x} \end{array} \tag{6.14}$$

$\tilde{x}$ being an unobservable mode. In general, however, observability of $\bar{x}$ can be asserted only if $\mathcal{W} = \langle \mathcal{C} | \mathcal{A} \rangle$ and the c-persistency property holds.

Moreover, with a suitable output injection $G(\rho)y$ one has $\tilde{A}_{21}(\rho)\bar{x} = G(\rho)y = G(\rho)C\bar{x}$, i.e.,

$$\dot{\tilde{x}} = \tilde{A}_{22}(\rho)\tilde{x} + G(\rho)y. \tag{6.15}$$

The dual notion of parameter-varying controllability subspace is the following:

Definition 5: *A subspace $\mathcal{S}$ is called parameter-varying unobservability subspace if there exists a constant output mixing matrix H and a parameter varying output injection gain $G : [0, T] \to \mathbb{R}^{n \times p}$ such that*

$$\mathcal{S} = \langle \mathbf{Ker}\, HC | \mathcal{A} + \mathcal{G}C \rangle, \tag{6.16}$$

where $\mathcal{A} + \mathcal{G}C$ denotes the system $A(\rho) + G(\rho)C$.

The family of parameter-varying unobservability subspaces containing a given subspace $\mathcal{L}$ is closed under subspace intersection. The minimal element of this family will be denoted by $\mathcal{S}_*$.

7 Parameter-varying invariant subspace algorithms

In Basile and Marro (1987) an algorithm was given to determine the robust controlled invariant subspace for arbitrary parameter dependence, however, since the number of conditions is not finite, the algorithm, in general is not applicable in practice. Therefore, from a practical point of view it is an important question to characterize these parameter-varying subspaces by a finite number of conditions.

It turns out that this is possible for the class of LPV systems, where the parameter dependency is affine. To impose this requirement is not too restrictive: even the true parameter dependency is more general, e.g., is given by a linear fractional transform, commonly used relaxation techniques that are used to obtain stability will embed it in a finitely generated (polytopic) convex set. But this convexified set can be always associated with an affine parameter dependence.

7.1 Affine parameter dependency

Assuming an affine parameter dependency of the state matrix, i.e., $A(\rho) = \sum_{i=1}^{N} \rho_i A_i$, it is immediate that if the inclusions hold for all A_i, then they hold also for all $\rho \in \mathcal{P}$. It is not so straightforward under which conditions the reverse implication is true, too.

A sufficient condition that characterizes property can be given as:

Lemma 4: *If the functions $\rho_1, \ldots, \rho_N$ are linearly independent over $\mathbb{R}$ then $A(\rho)\mathcal{V} \subset \mathcal{W} \;\; \forall \rho \in \mathcal{P}$ if and only if*

$$A_i \mathcal{V} \subset \mathcal{W}, \quad i = 0, \ldots, N. \tag{7.1}$$

In what follows, as otherwise is not stated, an affine parameter dependence is assumed. We are interested in finding supremal $\mathcal{A}$-invariant subspaces in a given subspace $\mathcal{K}$ or containing a given subspace $\mathcal{L}$. As far as the first purpose is concerned, by applying Lemma 4, one can formulate the $\mathcal{A}$-$\mathcal{I}$nvariant $\mathcal{S}$ubspace $\mathcal{A}$lgorithm over $\mathcal{L}$ as:

$$\mathcal{AISAL}: \quad \mathcal{V}_0 = \mathcal{L}, \quad \mathcal{V}_{k+1} = \mathcal{L} + \sum_{i=0}^{N} A_i \mathcal{V}_k, \quad k \geq 0, \tag{7.2}$$

$$\mathcal{V}^* = \lim_{k \to \infty} \mathcal{V}_k. \tag{7.3}$$

Obviously the algorithm will stop after a finite number of steps, i.e., $\mathcal{V}^* = \mathcal{V}_{n-1}$.

Proposition 14: *The subspace $\mathcal{V}^*$ given by (7.2) is such that*

$$\mathcal{L} \subset \mathcal{V}^*, \quad \mathcal{V}^* \text{ is } \mathcal{A}\text{-invariant}$$

and assuming that the parameters are c-excited, it is minimal with these properties.

Proof The invariance property is satisfied by construction. For minimality let us consider a subspace $\mathcal{S}$ for which the properties claimed by the Proposition holds. From Lemma 4 it follows that $A_i \mathcal{S} \subset \mathcal{S}$ is true for all i. It follows by induction that $\mathcal{V}_k \subset \mathcal{S}$ for all k: the $k = 0$ case is obvious and suppose that $\mathcal{V}_k \subset \mathcal{S}$ holds for an arbitrarily fixed k, then:

$$\mathcal{V}_{k+1} = \mathcal{L} + \sum_{i=0}^{N} A_i \mathcal{V}_k \subset \mathcal{L} + \sum_{i=0}^{N} A_i \mathcal{S} \subset \mathcal{L} + \sum_{i=0}^{N} \mathcal{S} \subset \mathcal{S}.$$

It follows that $\mathcal{V}^* \subset \mathcal{S}$, hence $\mathcal{V}^* = \mathcal{S}$.

Similar to the linear case the subspace $\mathcal{V}^*$ is denoted by $\langle \mathcal{A} | \mathcal{L} \rangle$.

By duality, one has the $\mathcal{A}$-$\mathcal{I}$nvariant $\mathcal{S}$ubspace $\mathcal{A}$lgorithm in $\mathcal{K}$, i.e.,

$$\mathcal{AISAK}: \quad \mathcal{W}_0 = \mathcal{K}, \quad \mathcal{W}_{k+1} = \mathcal{K} \cap \bigcap_{i=0}^{N} A_i^{-1} \mathcal{W}_k, \quad k \geq 0, \tag{7.4}$$

$$\mathcal{W}^* = \lim_{k \to \infty} \mathcal{W}_k. \tag{7.5}$$

The subspace $\mathcal{W}^*$ will be denoted by $\langle \mathcal{K} | \mathcal{A} \rangle$.

The corresponding version of Proposition 1. follows by duality, and can be stated as:

Proposition 15: *The subspace $\mathcal{W}^*$ given by (7.4) is such that*

$$\mathcal{W}^* \subset \mathcal{K} \quad \mathcal{W}^* \text{ is } \mathcal{A}\text{-invariant}$$

and assuming that the parameters are c-excited, it is maximal with these properties.

The set of all $(\mathcal{A}, \mathcal{B})$-invariant subspaces contained in a given subspace $\mathcal{K}$, is an upper semilattice with respect to subspace addition. This semilattice admits a maximum which can be computed from the $(\mathcal{A}, \mathcal{B})$-$\mathcal{I}$nvariant $\mathcal{S}$ubspace $\mathcal{A}$lgorithm ($\mathcal{ABISA}$):

$$\mathcal{ABISA} \quad \mathcal{V}_0 = \mathcal{K} \tag{7.6}$$

$$\mathcal{V}_{k+1} = \mathcal{K} \cap \bigcap_{i=0}^{N} A_i^{-1} (\mathcal{V}_k + \mathcal{B}). \tag{7.7}$$

The limit of this algorithm will be denoted by $\mathcal{V}^*$ and its calculation needs at most n steps.

The set of all $(\mathcal{C}, \mathcal{A})$-invariant subspaces containing a given subspace $\mathcal{L}$, is a lower semilattice with respect to subspace intersection. This semilattice admits a minimum which can be computed using the $(\mathcal{C}, \mathcal{A})$-$\mathcal{I}$nvariant $\mathcal{S}$ubspace $\mathcal{A}$lgorithm ($\mathcal{CAISA}$) (note that $\mathcal{C} = \mathbf{Ker}\, C$):

$$\mathcal{CAISA} \quad \mathcal{W}_0 = \mathcal{L}, \quad \mathcal{W}_{k+1} = \mathcal{L} + \sum_{i=0}^{N} A_i(\mathcal{W}_k \cap \mathcal{C}). \tag{7.8}$$

The limit of this algorithm will be denoted by $\mathcal{W}^*$. It takes at most n steps to compute.

As in the classical case, it can be seen that the family of controllability subspaces contained in a given subspace $\mathcal{K}$ is closed under subspace addition. Hence this family has a maximal element which can be computed from the parameter-varying $\mathcal{C}$ontrollability $\mathcal{S}$ubspace $\mathcal{A}$lgorithm:

$$\mathcal{CSA}: \quad \mathcal{R}_0 = 0, \quad \mathcal{R}_{k+1} = \mathcal{V}^* \cap \left(\sum_{i=0}^{N} A_i \mathcal{R}_k + \mathcal{B} \right) \tag{7.9}$$

$$\mathcal{R}^* = \lim_{k \to \infty} \mathcal{R}_k \tag{7.10}$$

where $\mathcal{V}^*$ is computed by $\mathcal{ABISA}$.

Proposition 16: *The subspace $\mathcal{R}^*$ is the largest parameter–varying controllability subspace in $\mathcal{C}$.*

Proposition 13 reveals that for a fixed $(\mathcal{A}, \mathcal{B})$-invariant $\mathcal{R} \subset \mathcal{X}$ the minimum $\mathcal{Z}_*$ of the set

$$\Gamma = \{\mathcal{Z} : \mathcal{Z} = \mathcal{R} \cap (\sum_{i=0}^{N} A_i \mathcal{Z} + \mathcal{B})\}$$

is a parameter-varying controllability subspace. The minimal element $\mathcal{Z}_*$ can be computed from the following algorithm:

$$\mathcal{Z}_0 = 0, \quad \mathcal{Z}_{k+1} = \mathcal{R} \cap (\sum_{i=0}^{N} A_i \mathcal{S}_k + \mathcal{B}). \tag{7.11}$$

The family of unobservability subspaces associated to an LPV system containing a given subspace $\mathcal{L}$ is closed under subspace intersection. The minimal element $\mathcal{S}_*$ of this family is the result of the $\mathcal{U}$nobservability $\mathcal{S}$ubspace $\mathcal{A}$lgorithm ($\mathcal{USA}$) :

$$\mathcal{USA}: \quad \mathcal{S}_0 = \mathcal{X}, \quad \mathcal{S}_{k+1} = \mathcal{W}^* + \left(\bigcap_{i=0}^{N} A_i^{-1} \mathcal{S}_k \cap \mathcal{C} \right) \tag{7.12}$$

$$\mathcal{S}_* = \lim_{k \to \infty} \mathcal{S}_k \tag{7.13}$$

where $\mathcal{W}^*$ is computed by $\mathcal{CAISA}$.

Remark 9: *Under the conditions of the Lemma 4 the subspaces $\mathcal{W}^*$ and $\mathcal{S}_*$ are exactly the distributions that can be obtained by the maximal conditioned invariant distribution algorithm and minimal unobservability distribution algorithm, see Isidori (1989); De Persis and Isidori (2000).*

The benefit of this approach is that these algorithms use only linear algebraic tools avoiding the complexity of dealing with vector space distributions and associated Lie - product calculations.

Summary

Concerning parameter invariant invariant subspaces the following results were established:

Invariant subspaces Chapter 6, Lemma 4, Proposition 11, 12, 13

An extension was given of the classical invariant subspaces – such as controlled invariant, conditioned invariant, controllability and unobservability subspaces – defined for LTI systems to a parameter-varying context, i.e., for LPV systems.

Invariance Algorithms Chapter 7, Proposition 14, 15, 16, Algorithm AISAL, AISAK, ABISA, CAISA, CSA, USA

If the parameter dependence is affine, a series of algorithms is provided for the effective computation of the parameter-varying invariant subspaces. These algorithms are formulated in terms of the original data, i.e., the state space matrices, uses only matrix manipulations and terminates in a number of finite steps.

The material covered by these chapters was published in the papers Balas et al. (2002, 2003); Bokor, Szabó and Stikkel (2002a); Szabó et al. (2002).

Results of the research and the developed LPV algorithms were directly applied in solving vehicle control problems, such as the FDI filter design for a Boeing 747 aircraft, see Bokor, Szabó and Balas (2002c); Bokor, Szabó and Stikkel (2002a); Stikkel et al. (2003).

Part IV

Application of geometric analysis and design for hybrid and LPV systems

8 Bimodal systems

Bimodal systems are special classes of switched systems governed by event-driven switchings, where the switch from one mode to the other is performed in closed-loop, i.e., in the simplest case the switching condition is described by a hypersurface in the state space. The controllability study of event-driven switched systems is very involved, since, in general, not even the well-posedness of the system, i.e., the existence and uniqueness of the solutions starting from any initial condition, is guaranteed.

The study of bimodal systems was motivated by an application representing a true emerging technology, related to the linearized longitudinal motion of a high speed supercavitating vehicle. There are more common examples, however, for a bimodal behavior, e.g., the dynamics of a hydraulic actuator in an active suspension system. The research revealed that for a wide class of bimodal systems the controllability can be cast in terms of the behavior of an associated open-loop switch system that has sign constrained control inputs, i.e., the controllability conditions can be tested in practice by using matrix algebraic tools. In this study the geometric view and the tools concerning robust invariant subspaces have been proven to be very useful. In what follows a detailed presentation of the results is provided.

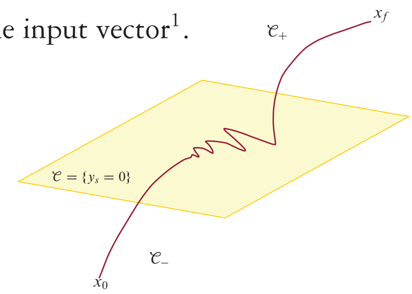
8.1 Problem formulation

Consider a *bimodal piecewise linear system*, i.e., a division of the state space by a hyperplane $\mathcal{C}$. The dynamics valid within each region is

$$\dot{x}(t) = \begin{cases} A_1 x(t) + B_1 u(t) & \text{if } x \in \mathcal{C}_-, \\ A_2 x(t) + B_2 u(t) & \text{if } x \in \mathcal{C}_+, \end{cases} \tag{8.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector and $u(t) \in \mathcal{U} \subset \mathbb{R}^m$ is the input vector[1].

The initial state of the system at time t_0 is determined by the initial state $x_0 = x(t_0)$ and the initial mode $s_0 \in \{1, 2\}$ in which the system is found at t_0. $\mathcal{C}$ denotes the hyperplane $\mathbf{Ker}\, C = \{x \mid Cx = 0\}$ and let $\mathcal{C}_\pm$ denote the half spaces $\mathcal{C}_+ = \{ x \mid Cx \geq 0 \}$ and $\mathcal{C}_- = \{ x \mid Cx \leq 0 \}$. The state matrices are constant and of compatible dimensions, B_1, B_2 having full column rank. $y_s = Cx$ defines the decision vector.



Bimodal system

[1] One can consider a number of different inputs for each mode. For sake of simplicity we chose $m_1 = m_2 = m$ but this does not affect the generality of the results.

Let us suppose that the *relative degree* corresponding to the output y_s and the ith mode is r_i, i.e., $y_s^{(k)} = CA_i^k x$, $k < r_i$ and $y_s^{(r_i)} = CA_i^{r_i} x + CA_i^{r_i-1} B_i u$ with $CA_i^{r_i-1} B_i \neq 0$, see Isidori (1989). It is reasonable to assume that $r_i < n$, otherwise it would follows that y_s fulfills a homogeneous differential equation, defined by the characteristic polynomial of A_i. In this case the i^{th} mode would not be able to leave the points of the hypersurface $\mathcal{C}$, characterized by $y_s = 0$, i.e., such a system would not be well–posed nor completely controllable.

If $r_i < n$ then the system is *right invertible*. Right invertibility denotes the possibility of imposing any sufficiently smooth output function by a suitable input function, starting at the zero state. It turns out that this property is related to $\mathcal{S}_{i,*}$, i.e., the minimal (C_i, A_i)—invariant subspace containing $Im B_i$. On the other hand *left invertibility*, i.e., the property that for every admissible y_s corresponds uniquely an input u, is closely related to the subspace $\mathcal{V}_i^*$, the maximal (A_i, B_i)—invariant subspace contained in $\mathcal{C}$.

For linear systems the points of $\mathcal{V}_i^*$ are not visible by the output. Only the orthogonal projection of the state on the subspace $\mathcal{V}_i^{*,\perp}$ can be deduced from the output and its derivatives, moreover this is the largest subspace where the orthogonal projection of the state can be recognized solely from the output. If the state is known, the orthogonal projection of the input can be determined modulo $B_i^{-1,T} \mathcal{V}_i^*$, see Basile and Marro (1973).

Having a single output, in order to remove the ambiguity in the right inverse, one can always redefine the inputs of the system. Indeed, define an input transformation $M_i u = \begin{bmatrix} \tilde{u}_i \\ w_i \end{bmatrix}$ such that $B_i M_i^{-1} = \begin{bmatrix} \tilde{B}_i & b_i \end{bmatrix}$ with $CA_i^{r_i-1} \tilde{B}_i = 0$ and $CA_i^{r_i-1} b_i = 1$, e.g., by considering the basis $\{b_i, \tilde{b}_{i,j} = b_{i,j} - CA_i^{r_i-1} b_{i,j} b_i,\ j = 2, \cdots, m\}$ in $\mathbf{Im}\, B_i$. Then the single input single output (SISO) subsystem (A_i, b_i, C) is left and right invertible, i.e., $\tilde{\mathcal{V}}_i^* \cap \tilde{\mathcal{S}}_{i,*} = 0$ and $\tilde{\mathcal{V}}_i^* + \tilde{\mathcal{S}}_{i,*} = \mathbb{R}^n$, where the invariant subspaces correspond to the SISO system, while the remaining subsystem $(A_i, \tilde{B}_i, C)$ is not invertible.

The invariant subspace $\mathcal{V}_i^*$ produces a decomposition of the state corresponding to the i^{th}, i.e., the system can be transformed[2] into :

$$\begin{bmatrix} \dot{\eta}_i \\ \dot{\xi}_i \end{bmatrix} = \begin{bmatrix} P_i \eta_i + R_i y_s + Q_i \tilde{u}_i \\ A_{r_i} \xi_i + B_{r_i} v_i \end{bmatrix} \tag{8.2}$$

$$y_s = C_{r_i} \xi_i, \tag{8.3}$$

where $\eta_i \in \mathcal{V}_i^*$ and the subsystem for ξ_i is a chain of integrators with $B_{r_i} = [1\, 0 \cdots 0]^T$ and $C_i = [0 \cdots 0\, 1]$. The inputs v_i and w_i are related as $v_i = CA_i^r x + w_i$.

Since y_s is common for both systems, if $r_1 = r_2 = r$ then $\xi_1 = \xi_2 = \xi$. Recall that the components of ξ are formed by y_s and its derivatives up to order $r - 1$. It follows that the complementer subspaces (zero dynamics) have the same dimension, i.e., there exist a basis

[2]The transformation is a special case of the one used for the dynamical inversion of the systems, which is presented in details in the next chapter, Section 9.2, applied for a SISO setting.

transformation T such that $\eta_2 = T\eta_1 = T\eta$. In this case the bimodal system can be written as

$$\dot{\eta} = \begin{cases} P_1\eta + R_1 y_s + Q_1\tilde{u}_1 & \text{if} \quad y_s \geq 0 \\ P_2\eta + R_2 y_s + Q_2\tilde{u}_2 & \text{if} \quad y_s \leq 0 \end{cases} \tag{8.4}$$

$$\dot{\xi} = \begin{cases} A_r\xi + B_r v_1 & \text{if} \quad y_s \geq 0 \\ A_r\xi + B_r v_2 & \text{if} \quad y_s \leq 0 \end{cases} \tag{8.5}$$

Remark 10: *Observe that the required transformation can be performed by the same change of base in the state space. e.g.,* $\begin{bmatrix} \eta \\ \xi \end{bmatrix} = Tx$*, where for the last rows of* T *are chosen the vectors* $CA_2^j,\ j = 0, \cdots, r-1$*. However the feedback to obtain the desired structure might differ. The input transformations are also different, in general; this difference is reflected in the notation* u_1, u_2 *and* v_1, v_2*, respectively.*

Since the decomposition – i.e., the transformation T *– depends only on* C*,* A *and* r*, the choice of the input transformation does not play any role in the validity of the controllability results.*

In the case when $r_1 \neq r_2$ such a splitting is not possible but the system can be transformed into (suppose that $r_1 < r_2$):

$$\dot{\eta} = \begin{cases} P_1\eta + R_1 y_s + Q_1\tilde{u}_1 & \text{if} \quad y_s \geq 0 \\ P_2\eta + R_2 y_s + Q_2\tilde{u}_2 + Q_3 v_2 & \text{if} \quad y_s \leq 0 \end{cases} \tag{8.6}$$

$$\dot{\xi} = \begin{cases} A_r\xi + B_r v_1 & \text{if} \quad y_s \geq 0 \\ A_r\xi + B_r\bar{\eta} & \text{if} \quad y_s \leq 0, \end{cases}. \tag{8.7}$$

where $\bar{\eta}$ denotes the last component of η.

In contrast to the previous situation, in this case the subsystem ξ, hence the decision variable y_s, cannot be controlled independently from the subsystem η in both modes. Moreover, in the first mode the only way to control the higher order derivatives of y_s is through the inputs $\tilde{u}_1$. This fact makes the study of the controllability problem for these systems, in general, more difficult.

Here it is addressed the case when $r_i = r$, for which the system is always well posed, see Imura (2003). For sake of simplicity the results will be presented for the case when $r = 1$, i.e.,

$$\dot{\eta} = \begin{cases} P_1\eta + R_1 y_s + Q_1 u & \text{if} \quad y_s \geq 0 \\ P_2\eta + R_2 y_s + Q_2 u & \text{if} \quad y_s \leq 0 \end{cases}. \tag{8.8}$$

$$\dot{y}_s = v, \tag{8.9}$$

but the assertions remain valid for the general case.

8.2 The controllability result

The controllability question of the bimodal system can be reduced to the question of controllability/reachability of the origin through the closed-loop switchings allowed by the switching surface

$\mathcal{C}$. Due to the fact that the bimodal system is not a linear system, the affirmative answer given on this question is not completely trivial.

The reference Veliov and Krastanov (1986) deals directly with problems described by (8.8) and (8.9), while Çamlibel et al. (2004) assumes only single input left and right–invertible systems whose dynamics are smooth, i.e., continuous along the trajectories. In this case one has $A_1x + B_1u = A_2x + B_2u$, for all $x \in \mathcal{C}, u \in \mathcal{U}$. It follows that $A_2 = A_1 - KC$ and $B_1 = B_2 = B$ for a suitable matrix K, i.e., one has $P_1 = P_2 = P$ and $Q_1 = Q_2 = 0$ in (8.8).

Note, that in Proposition 3 the subspace $\mathcal{R}_{\mathcal{A},\mathcal{B}}$ is the minimal subspace invariant for all of the A_is containing $\mathcal{B} = \sum_{i=0}^{s} \mathbf{Im}\, B_i$. Thus the bimodal system can be transformed, via a state transform and suitable feedbacks, to

$$\dot{\eta}_1 = \begin{cases} P_{1,1}\eta_1 + \tilde{R}_1 y_s + \tilde{Q}_1 u_1 & \text{if} \quad y_s \geq 0 \\ P_{2,1}\eta_1 + \tilde{R}_2 y_s + \tilde{Q}_2 u_2 & \text{if} \quad y_s \leq 0 \end{cases}, \tag{8.10}$$

$$\dot{\eta}_2 = \begin{cases} P_{1,2}\eta_2 + R_1 y_s & \text{if} \quad y_s \geq 0 \\ P_{2,2}\eta_2 + R_2 y_s & \text{if} \quad y_s \leq 0 \end{cases}, \tag{8.11}$$

$$\dot{y}_s = v, \tag{8.12}$$

where, by Proposition 3, subsystem (8.10) is controllable on $\mathcal{C}$ using open–loop switchings. It follows that this decomposition can be viewed as a controllability decomposition of the bimodal LTI system where the study of the controllability of the original bimodal system reduces to controllability of the bimodal system formed by (8.11) and (8.12).

Remark 11: *When $y_s = 0$, i.e., on $\mathcal{C}$, subsystems* (8.10) *does not contain y_s and the switching law must be defined externally. However for linear switching systems there exist a universal switching sequence that provides complete controllability, hence the switching sequence is fixed and a fundamental solution of* (8.10) *(as a linear time varying system) is well defined. Therefore, by linearity, the controllability of* (8.10) *is not affected by the values of y_s.*

Lemma 5: *The bimodal system* (8.8), (8.9) *is completely controllable if and only if the subsystem defined by* (8.11), (8.12) *is completely controllable.*

Proof The necessity is obvious. For the sufficiency it is enough to consider the reachability case, i.e., the situation when $x_0 = 0$ and a given x_f is to be reached. Decompose x_f into $\eta_{1,f}$ and $(\eta_{2,f}, y_{s,f})$ according to (8.10) and (8.11),(8.12). Since the open–loop linear switched system (8.10) is completely controllable, there is a finite switching sequence, see Sun and Ge (2005), and suitable inputs u_1,u_2 that steers the origin according to (8.10) to $\eta_{1,f}$. Let us denote these inputs by u_η. The switching sequence can be realized by a suitable y_s^η that has sign changes at the required time instances, e.g., a modulated sine signal. Let us denote by v_η one of the controls that realizes y_s^η. By linearity and complete controllability of (8.11), (8.12) there are points $(\eta_{2,o}, y_{s,o})$ from which the system (8.11), (8.12) is steered into $(\eta_{2,f}, y_{s,f})$ applying v_η. Let us denote by v_o the input that steers (8.11),(8.12) from the origin to $(\eta_{2,o}, y_{s,o})$. During this the inputs u_i are maintained at zero, i.e., $\eta_1 = 0$. It follows that applying the inputs $(0, u_\eta)$, (v_o, v_η) one can steer the origin into $\eta_{1,f}$ and $(\eta_{2,f}, y_{s,f})$, i.e., into x_f.

Having the decomposition (8.11),(8.12) for a bimodal system it is immediate that if the system is controllable then the input constrained open–loop switching system of the type

$$\dot{\eta} = P_i \eta + \bar{R}_i w, \quad i \in \{ 1, 2 \}, \quad w \geq 0 \tag{8.13}$$

with $\bar{R}_i = (-1)^{i+1} R_i$ is also controllable. Consulting the result of Çamlibel et al. (2004), i.e., the case $P_1 = P_2$, it is apparent that the controllability condition of the bimodal system is equivalent to the input constrained controllability condition of the corresponding open–loop system given by (8.13). It is less apparent, but this consequence also holds for the case presented in Veliov and Krastanov (1986).

8.3 A separation theorem

The bimodal system (8.11), (8.12) can be seen as a dynamic extension[3] of

$$\dot{\eta}_2 = P_{i,2} \eta_2 + \bar{R}_{i,2} w, \quad i \in \{ 1, 2 \}, \quad w \geq 0. \tag{8.14}$$

Controllability of the dynamically extended system, provided that the original system was controllable, is by far non–trivial issue though for smooth vector fields it was proved in Sussmann (1991); Sontag and Qiao (1999). For linear systems it is straightforward for unconstrained input case. This can be verified by checking the Kalman rank condition of the extended system, however this result cannot be directly applied here, since the input is signed constrained.

Lemma 6: *If the points η_0 and η_f can be connected by a trajectory of the linear system $\dot{\eta} = P\eta + Rw$ using nonnegative control $w \geq 0$ then, for a given r, they can be also connected using a smooth nonnegative control $\omega \geq 0$ with prescribed end points, i.e., $\omega^{(k)}(0) = \omega_{0,k}$ and $\omega^{(k)}(T_f) = \omega_{T_f,k}$ for $k = 0, 1, \cdots, r$.*

Proof The proof of the assertion is an adaptation of the proof for controllability by smooth controls given in Chapter 5., Theorem 4 of Jurdjevic (1997). The main points of the proof are the following: for a linear system every accessible point is normally accessible (i.e., by using piecewise constant controls). Consider the control formed by the sequence $(\mathbf{w}, \hat{\mathbf{t}}) = \{(w_1, \hat{t}_1), \cdots, (w_F, \hat{t}_F))\}$ where the control $w_i \geq 0$ is applied for a duration of $\hat{t}_i$, that steers η_0 to η_f. By the inverse mapping theorem, there exist functions t_i defined on a neighborhood V of η_f such that the sequence $\{(w_1, t_1(z)), \cdots, (w_F, t_F(z)))\}$ steers η_0 to z for all $z \in V$. Denote by $\tau_i = \sum_{l=1}^{i} t_l$ and for any sufficiently small $\epsilon > 0$ consider the smooth nonnegative control $\omega(t, z, \epsilon)$ defined by

$$\omega(t, z, \epsilon) = \begin{cases} \beta_1(t, z) & \text{if } t \in [0, \epsilon] \\ w_1 & \text{if } t \in [\epsilon, \tau_1(z) - \epsilon] \\ w^* & \text{if } t \in [\tau_1(z) - \epsilon, \tau_1(z) + \epsilon] \\ & \vdots \\ w_F & \text{if } t \in [\tau_{F-1}(z) + \epsilon, \tau_F(z) - \epsilon] \\ \beta_F(t, z) & \text{if } t \in [\tau_F(z) - \epsilon, \tau_F(z)], \end{cases} \tag{8.15}$$

[3] See Isidori (1989) for details.

where $w^* = (1 - \alpha_1(t,z))w_1 + \alpha_1(t,z)w_2$ and α_i, β_j are smooth, nonnegative, increasing functions in t for each z in the interval $[\tau_i^-, \tau_i^+] := [\tau_i(z) - \epsilon, \tau_i(z) + \epsilon]$, with end conditions $\alpha_i(\tau, z) = 0$, $\partial^k_{t^k}\alpha_i(\tau, z) = 0$ at $\tau \in \{\tau_i^-, \tau_i^+\}$ and $k \geq 1$. The same end conditions are imposed for β_1 at t_1^+ and for β_F at t_F^-. We also impose $\partial^k_{t^k}\beta_1(0,z) = \omega_{0,k}$ and $\partial^k_{t^k}\beta_F(\tau_F(z), z) = \omega_{F,k}$ for $k = 0, 1, \cdots, r$. If we denote the associated integral curve by $\Phi_\epsilon(t)$ then one has that $\lim_{\epsilon \to 0} \Phi_\epsilon(\tau_F(z)) = z$ in some neighborhood of η_f.

The assertion of the Lemma follows by a fix point argument, for details see Jurdjevic (1997) or Sussmann (1991).

Using this lemma the main controllability result for the given bimodal system can be formulated as:

Proposition 17: *The bimodal system given by* (8.11) *and* (8.12) *is controllable if and only if the input constrained open–loop switching system* (8.14) *is controllable.*

Using this result controllability can be decided by using the result of Proposition 6 and the Controllability Algorithm of Section 4.2.

Remark 12: *The assertion of Proposition 17 remains valid for q'LPV systems, too. If the dynamics depends affinely on the scheduling variables, the reduction of the bimodal systems to the form given by* (8.10), (8.11), (8.12) *can be performed by using the algorithms of Chapter 7.*

This section is concluded by an example to illustrate the content of the controllability result an the role of the separation lemma in the construction.

Example 2: *Let us consider the system:*

$$\dot{x} = \begin{cases} P_1 x + R_1 y_s & \text{if} \quad y_s \geq 0 \\ P_2 x + R_2 y_s & \text{if} \quad y_s \leq 0 \end{cases}, \tag{8.16}$$

$$\dot{y}_s = u, \tag{8.17}$$

where

$$P_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad R_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and

$$P_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad R_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

According to Proposition 17 controllability of the original system is equivalent to controllability of the sign constraint switched system:

$$\dot{\eta} = P_i \eta + \bar{R}_i w. \quad w \geq 0 \tag{8.18}$$

It is not hard to figure out that the coordinates corresponding to trajectories of the individual subsystems can be obtained as:

$$\eta_{1,1}(t) = \eta_{1,1}(t_0) + \eta_{1,2}(t_0)t + \int_0^t \int_0^\tau w(\sigma)d\sigma d\tau$$
$$\eta_{1,2}(t) = \eta_{1,2}(t_0) + \int_0^t w(\tau)d\tau,$$

and

$$\eta_{2,1}(t) = e^t \eta_{2,1}(t_0)$$
$$\eta_{2,2}(t) = e^t \eta_{2,2}(t_0) - e^t \int_0^t v(\tau)d\tau,$$

with $w(\tau) = e^\tau v(\tau)$.

Let us apply the following control strategy: fix $t_1 > 0$ *and steer the second subsystem with constant control* $v \geq 0$ *then apply the first subsystem for a time* t_2 *with constant control* $w \geq 0$.

One has the following system of equations:

$$\begin{aligned} \eta_{2,1}(t_1) &= e^{t_1}\eta_{2,1}(t_0), \\ \eta_{2,2}(t_1) &= e^{t_1}\eta_{2,2}(t_0) - t_1 e^{t_1} v, \\ \eta_{1,1}(t_2) - \eta_{2,1}(t_1) &= \eta_{2,2}(t_1) + \frac{1}{2} w t_2^2, \\ \eta_{1,2}(t_2) - \eta_{2,2}(t_1) &= w t_2. \end{aligned}$$

One has

$$t_2 = 2\frac{\eta_{1,1}(t_2) - \eta_{2,1}(t_1)}{\eta_{1,2}(t_2) + \eta_{2,2}(t_1)}, \quad w = \frac{\eta_{1,2}^2(t_2) - \eta_{2,2}^2(t_1)}{2(\eta_{1,1}(t_2) - \eta_{2,1}(t_1))},$$

i.e.,

$$\begin{aligned} \tilde{\eta}_1^f - \eta_1^0 - \eta_2^0 &= \frac{1}{2}\tilde{w} t_2, \\ \eta_{2,2}(t_1) &= e^{t_1}\eta_{2,2}(t_0) - t_1 e^{t_1} v, \\ \tilde{\eta}_2^f - \eta_2^0 &= \tilde{w} - t_1 e^{t_1} v, \end{aligned}$$

with $\tilde{\eta}_i^f = e^{-t_1}\eta f_i$ *and* $\tilde{w} = e^{-t_1} w t_2$. *This equation can be solved satisfying the nonnegativity constraint for a suitable choice of* t_1 *and* v *for any* η_0 *and* η_f. *Therefore the input constrained open–loop switching system* (8.18) *is controllable.*

In order to prove complete controllability for the original bimodal system, we have to ensure that (8.18) *can be controlled with inputs that has arbitrarily prescribed end conditions. By linearity it is sufficient to ensure null end conditions for the input* w.

By replacing the piecewise constant inputs by $w \to w\varphi_t(\tau)$ *and* $v \to v\varphi_t(\tau)$ *where the function defined as*

$$\varphi_t(\tau) = \frac{6}{t^2}\tau(t-\tau)$$

is nonnegative on $[0, t]$ *and fulfills the end–point conditions* $\varphi_t(0) = \varphi_t(t) = 0$ *and has*

$$\int_0^t \varphi_t(\tau)d\tau = t, \quad \int_0^t \int_0^\tau \varphi_t(\sigma)d\sigma d\tau = \frac{t^2}{2},$$

one obtain the same equations for t_1, t_2, v, w*, i.e., it follows that the bimodal system is also controllable.*

8.4 Stabilizability of bimodal systems

The bimodal system (8.1) is said to be stabilizable if any initial state can be asymptotically steered to the origin by a suitable admissible input u, i.e., for all $x_0 \in \mathbb{R}^n$ there exist a solution $x(t)$ of the bimodal system such that $\lim_{t\to\infty} x(t) = 0$.

Let us first examine bimodal systems with continuous dynamics. In view of Proposition 17 these systems are equivalent with an LTI system with two sign constrained inputs. Starting from this observation one has the following result:

Proposition 18: *If the bimodal system has continuous dynamics, i.e.,* $P_1 = P_2 = P$*, then the bimodal system* (8.11)*,* (8.12) *is stabilizable if and only if the corresponding sign constrained open–loop switching system is stabilizable.*

Proof: The necessity is obvious. For sufficiency let us recall the following basic fact: for a stabilizable LTI system, in particular for the sign constrained system $\dot{x} = Px + [\bar{R}_1\ \bar{R}_2]w,\ w \geq 0$, there exist numbers $\alpha > 0$ and $\gamma > 0$ such that for any point x_0 a trajectory of system satisfying the condition $x(0) = x_0$ and

$$||x(t)|| \leq \alpha ||x_0|| e^{-\gamma t} \quad t \geq 0 \tag{8.19}$$

can be found, see Smirnov (1996). It follows, that for this trajectory one has $\int_0^\infty w < \infty$, i.e., $\lim_{t\to\infty} w(t) = 0$. Moreover this w can be chosen to be continuous, see Smirnov (1996), i.e.,, the implied switching sequence induced by the sign changes of w is piecewise constant. Then the construction leading to Proposition 17 can be applied and it follows, that the corresponding bimodal system, which has the state $\begin{bmatrix} x \\ w \end{bmatrix}$, is also stable.

In Heemels et al. (1998) one can find the following characterization of the stabilizability of a sign constrained LTI system:

HT: The system

$$\dot{x} = Px + Rw \quad w \in \mathbb{R}^2_+$$

is stabilizable if and only if

- the unconstrained system is stabilizable and
- all real eigenvectors v of P^T corresponding to a nonnegative eigenvalue of P^T have the property that $R^T v$ has both positive and negative components.

Remark 13: *An equivalent result was given in Smirnov (1996), where a method for the construction of the stabilizing feedback was also presented.*

If the more severe conditions of small time local controllability are satisfied, then Lipschitz continuous piecewise linear stabilizing feedback can be constructed, see Krastanov and Veliov (2003).

The conditions HT are satisfied for controllable systems.

The general case is more difficult. We conclude this section with a result that provides a sufficient condition for stabilizability:

Proposition 19: *If the bimodal system* (8.11)*,* (8.12) *is globally controllable, then it is asymptotically stabilizable.*

Proof: By controllability one has that from any initial state x_0 there is a control that steers the point to the origin in a finite time, say T. By the finite switching property, see Proposition 4, at time T a well defined system is active. Setting the input $u = 0$ for $T > 0$ the system is maintained in the origin, i.e., the system is stable.

Summary

The engineering applications that provide the motivation background for the research of bimodal systems were related to control of the hydraulic actuator of an active suspension system and the controllability study for a high speed supercavitating underwater vehicle, see Bokor, Szabó and Balas (2006b, 2007); Gáspár, Szabó and Bokor (2008a); Gáspár et al. (2009a).

Controllability decomposition Chapter 8, Lemma 5, Proposition 17 , 18 and 19

> A controllability decomposition was established for bimodal systems that have a well defined relative degree. It was shown that such a bimodal system is completely controllable if and only if a given subsystem of the controllability decomposition is completely controllable. It turns out that the latter is equivalent to the controllability of an input constrained open–loop switching system. If the bimodal system is globally controllable, then it is asymptotically stabilizable.

Additional details can be found in the papers Bokor, Szabó and Balas (2007, 2006a,b); Bokor and Szabó (2009).

9 Inversion of LPV systems

The solution of the problem of dynamic inversion of systems gave rise to considerable attention in the control literature: in his classical paper Silverman (1969) considered the properties and calculation of the inverse of LTI systems guaranteeing neither minimality nor stability properties of the resulting inverse system. The problem was also considered by Fliess (1986) for nonlinear input-output systems. For certain classes of nonlinear state space representations Isidori (1989) provided algorithms and also sufficient or necessary conditions of invertibility.

There are two aspects concerning dynamical system inversion: *left invertibility*, which is related to unknown input observability – the target application field being fault detection filter design – and *right invertibility*, related to the solution of output tracking control problems. Dynamic inversion based controllers are popular in aerospace control, see, e.g., Morton et al. (1996); Looye and Joos (2001).

This chapter provides a geometric view of dynamic inversion of LPV systems. In contrast to the pseudo-inversion techniques, in the proposed method the availability of the full state measurements is not assumed, instead, it is supposed that measured outputs, and possibly some of their derivatives are available, for which the resulting system is minimum phase and left (right) invertible. For output tracking a two degree of freedom controller structure is proposed, where the first part is an inversion based controller making the linearization of the plant while the second controller, using an error feedback, achieves the required stability properties.

The algorithm was successfully applied in the dynamic inversion based controller design for stabilizing the primary circuit pressure at the Paks Nuclear Power Plant in Hungary in 2004-2005, see, e.g., Szabó et al. (2005). This controller implementation (together with other important reconstruction steps) largely contributed to the possibility that the average thermal power of the plant units could be increased by 1-2 %.

The general nonlinear setting

Let us consider the nonlinear input affine system Σ,

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i \qquad (9.1)$$
$$y = h(x),$$

with $y = [\, y_j \,]_{j=1,p}$ and $h(x) = [\, h_j(x) \,]_{j=1,p}$, respectively. It is reasonable to assume that the rank of $g = [\, g_i \,]_{i=1,m}$ is m and that the rank of h is p.

The problem when the outputs – and possible its derivatives – are measured and the unknown input is to be determined involves the notion of the *left invertibility* of the system. We are going to construct another dynamic system

$$\dot{\zeta}(t) = \varphi(\zeta, y, \dot{y}, \ldots, u, \dot{u}, \ldots)$$
$$u(t) = \omega(\zeta, y, \dot{y}, \ldots, u, \dot{u}, \ldots)$$

with outputs u and inputs $\vartheta = (\tilde{y}, \tilde{u})$ that contains the measurements of the signals u, y and possible their time derivatives.

Let us recall, that the system (9.1) is *(left)invertible* at x_0, if the output functions corresponding to the initial state x_0 and distinct admissible controls u are different. A system is called *strongly invertible* if there exist an open and dense submanifold of the state manifold on which the system is invertible.

Left invertibility can be characterized more completely by using algebraic techniques, for more details see, e.g., Zheng and Cao (1993); Conte et al. (2006). However, for practical purposes design algorithms based on a geometrical framework are often more suitable.

A dual problem is to find a suitable input signal that produces a desired behavior of the outputs, i.e., output tracking, is related to the concept of *right invertibility*. A dynamical system is right invertible at x_0 if the rank of its input-output map at this point is p, i.e., the number of outputs (to be tracked), see Nijmeijer (1986).

9.1 A geometrical framework

Let us recall, first, some elementary definitions and facts from Isidori (1989) and Nijmeijer (1991). A smooth connected submanifold M which contains the point x_0 is said to be *locally controlled invariant* at x_0 if their is a smooth feedback $u(x)$ and a neighborhood U_0 of x_0 such that the vector field $\tilde{f}(x) = f(x) + g(x)u(x)$ is tangent to M for all $x \in M \cap U_0$, i.e. M is locally invariant under $\tilde{f}$.

An *output zeroing submanifold* of Σ is a smooth connected submanifold M with contains x_0 and satisfy:

1. for all $x \in M$ one has $h(x) = 0$,

2. M is locally controlled invariant at x_0.

This means that for some choice of the feedback control $u(x)$ the trajectories of Σ which start in M stay in M for all t in a neighborhood of $t_0 = 0$ and the corresponding output is identically zero. Such a submanifold Z^* can be determined by a "zero dynamics algorithm", Nijmeijer and van der Schaft (1990).

If in addition

$$\dim \mathbf{span}\{ g_i(x_0) \,|\, i = 1, m\} = m, \tag{9.2}$$

and $\dim \mathbf{span}\{ g_i(x) \,|\, i = 1, m\} \cap T_x Z^*$ is constant for all $x \in Z^*$ then Z^* is a *locally maximal output zeroing submanifold*. Moreover, if

$$\dim \mathbf{span}\{ g_i(x) \,|\, i = 1, m\} \cap T_x Z^* = 0, \tag{9.3}$$

then there is a unique smooth feedback u^* such that $f^*(x) := f(x) + g(x)u^*(x)$ is tangent to Z^*. An algorithm for computing Z^* for a general case can be found in Isidori (1989) and Nijmeijer (1991). In some cases, however, Z^* can be determined relative easily relating it to the maximal controlled invariant distribution Δ^* contained in $\mathbf{Ker}\, \mathbf{d}h$, given by the controlled invariant codistribution algorithm ($\Delta^* = \Omega_*^{\perp}$), namely $\Delta^*(x) = T_x Z^*$, for details see D.3 and Isidori (1989).

An important case when this relation holds is the set of LTI systems and the class of systems that have a vector relative degree. The concept of relative degree plays a key role in several control problems both for linear and nonlinear systems. In particular, the computation of the relative degree and the derivation of consequent normal forms for nonlinear systems, represents key design step in order to solve successfully several control problems, like disturbance decoupling, feedback linearization and system inversion problems.

A multivariable nonlinear system has a vector relative degree $r = \{r_1, \cdots, r_p\}$ at a point x_0 if

i. $L_{g_j} L_f^k h_i(x) = 0$ for $j = 1, \cdots, m, \quad i = 1, \cdots, p$, and $k < r_i - 1$.

ii. the matrix

$$A(x) := \begin{bmatrix} L_{g_1} L_f^{r_1 - 1} h_1(x) & \cdots & L_{g_m} L_f^{r_1 - 1} h_1(x) \\ \cdots & \cdots & \cdots \\ L_{g_1} L_f^{r_p - 1} h_p(x) & \cdots & L_{g_m} L_f^{r_p - 1} h_p(x) \end{bmatrix} \tag{9.4}$$

has rank m for left invertibility (p for right invertibility) at x_0.

For further usage let us denote by

$$B(x) := \begin{bmatrix} L_f^{r_1} h_1(x) \\ \vdots \\ L_f^{r_p} h_p(x) \end{bmatrix}. \tag{9.5}$$

If condition (**ii.**) does not hold but there exist numbers r_i with property (**i.**) then they are called *relative orders* of the system (9.1).

Lemma 7: *Let us suppose that the system* (9.1) *has relative degree. Then the row vectors*

$$dh_1(x_0), \cdots, dL_f^{r_1-1}h_1(x_0), \cdots, dh_p(x_0), \cdots, dL_f^{r_p-1}h_p(x_0) \tag{9.6}$$

are linearly independent.

Conditions (9.2) and (9.3) can be interpreted as a special property of (left) invertibility of the system Σ. Our interest in the determination of the output zeroing manifold is motivated by the role played by these notions in the question of invertibility and the construction of the reduced inverse of linear and nonlinear controlled systems.

The characterization of right invertibility, related to the number of zeros at infinity, is analogous, for details see Nijmeijer (1986).

Nonlinear systems with vector relative degree

In this section the construction of a left inverse of a nonlinear system is presented – the construction of a right inverse is similar, hence, it is left out.

As it was already stated, if $\mathbf{rank}A(x) = m$ then

$$Z^* = \{x \mid L_f^k h_i = 0,\ i = 1, \cdots, p \quad k = 0, \cdots, r_i - 1\}$$

and the maximal controlled invariant distribution in **Ker** $\mathbf{d}h$ is

$$V^* = \mathbf{Ker\,span}\{\, dL_f^k h_i,\ i = 1, \cdots, p \quad k = 0, \cdots, r_i - 1\},$$

see also Nijmeijer (1991). Moreover the feedback $u^*(x) = \alpha(x)$ is the solution of an equation $A(x)\alpha(x) = B(x)$.

Let us denote by $\xi = (\xi^i)_{i=1,p} = \Xi(x)$ the diffeomorphism defined by $\xi^i = (L_f^k h_i(x))_{k=0,r_i-1}$. It is a standard computation, that

$$\dot{\xi}^i = A^i \xi^i + B^i y_i^{(r_i)},$$

where A^i, B^i are in the Brunowsky form ($\xi_1^i = y_i$).

Let us complete $\Xi(x)$ to a diffeomorphism on X:

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \Phi(x) := \begin{bmatrix} \Xi(x) \\ \Lambda(x) \end{bmatrix}.$$

Since $\partial_x \Xi = [\mathbf{d}L_f^k h_i]$, one has

$$\dot{\xi} = [\mathbf{d}L_f^k h_i] f|_{\Phi^{-1}} + [\mathbf{d}L_f^k h_i] g|_{\Phi^{-1}} u,$$

i.e., maintaining the nonzero rows:

$$[\dot{\xi}_{r_i}^i] = B|_{\Phi^{-1}} + A|_{\Phi^{-1}} u, \tag{9.7}$$

and

$$\dot{\eta} = \partial_x \Lambda f|_{\Phi^{-1}} + \partial_x \Lambda g|_{\Phi^{-1}} u. \tag{9.8}$$

The zero dynamics[1] can be obtained by

$$\dot{\eta} = \partial_x \Lambda f|_{\Phi^{-1}} + \partial_x \Lambda g \alpha|_{\Phi^{-1}}, \tag{9.9}$$

putting $\xi = 0$.

Finally, the *output equations* of the dynamic inverse are

$$u(t) = A^{-1}\begin{bmatrix} \xi \\ \eta \end{bmatrix}\left(y^{(r)} - L_f^r h \begin{bmatrix} \xi \\ \eta \end{bmatrix}\right)$$

and one can get the (*minimal*) inverse dynamics as

$$\dot{\eta} = f(\xi, \eta),$$

where ξ contains the corresponding output derivatives. Observe that the inverse does not inherit the structure of the original system, i.e., it is not necessarily input affine.

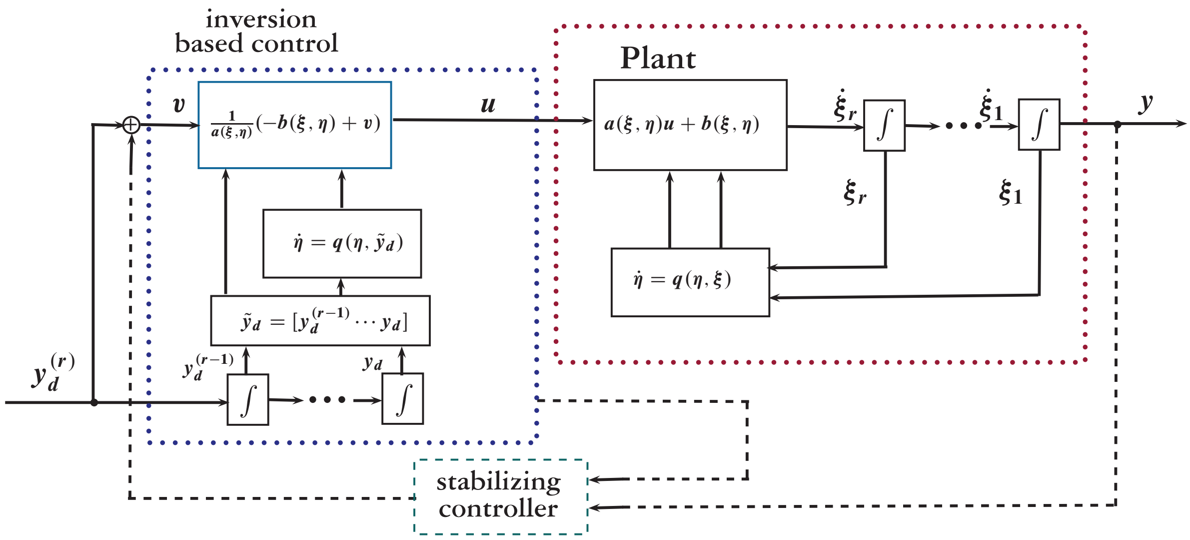


Figure 9.1: Inversion based control: general scheme

The main difficulty in the construction of the dynamical inverse in this general nonlinear context consists in obtaining and handling the time varying coordinate transform $\Phi(x)$ with its splitting

[1] If g is involutive, then one can choose $d\Lambda \subset g^{\perp}$, and then $\dot{\eta} = \partial_x \Lambda f|_{\Phi^{-1}}$.

in $\Xi(x)$ and $\Lambda(x)$. This is a state dependent nonlinear transformation, and the construction of the suitable extension requires, in general, solution of partial differential equations, hence, it is necessary to know the full state vector of the system. The linearized system will be a chain of integrators and the actual input of the linearizing controller will be the derivative, with order equal to the relative degree of the system, of the desired output. For a schematic view of this approach for a SISO system see Figure 9.1.

Even if all the data required for the implementation of dynamical the inverse is available the method might be useless in practice. Invertibility does not involve the knowledge of the initial condition but for the implementation it plays an implicit role. The zero dynamics should be stable because it cannot be influenced by output injection since it is not observable for the outputs used in the inversion process.

The next section will provide a method for a class of LPV systems when the entire construction can be performed based on a suitable parameter varying conditioned invariant subspace.

9.2 Dynamic inverse of LPV systems

Let us consider the class of LPV systems with m inputs and p outputs, that can be described as:

$$\dot{x}(t) = A(\rho(t))x(t) + B(\rho(t))u(t) \tag{9.10}$$

$$y(t) = Cx(t) \tag{9.11}$$

where

$$A(\rho(t)) = A_0 + \rho_1(t)A_1 + \ldots + \rho_N(t)A_N, \tag{9.12}$$

$$B(\rho(t)) = B_0 + \rho_1(t)B_1 + \ldots + \rho_N(t)B_N, \tag{9.13}$$

$$\tag{9.14}$$

and the dimension of the state space is supposed to be n. It is assumed that each parameter ρ_i ranges between known external values $\rho_i(t) \in [\underline{\rho}_i, \overline{\rho}_i]$ and the parameter set that contains all $(\rho_1(t), \cdots, \rho_N(t))$, where $t \in [0, T]$ will be denoted by $\mathcal{P}$. For the sake of notational simplicity the time dependency of the matrices will be omitted ($A(\rho) := A(\rho(t))$) where it is possible.

It is not hard to figure out that in the LTI case $T_x Z^* = V^*$, where V^* is the maximal (A, B)-invariant subspace contained in $\mathbf{Ker}\ C$ while for the LPV case if some technical conditions for the parameter functions (persistency) are fulfilled, then $T_x Z^* = \mathcal{V}^*$, where $\mathcal{V}^*$ is the maximal $(\mathcal{A}, \mathcal{B})$-invariant subspace contained in $\mathcal{C} = \mathbf{Ker}\ C$. The minimal $(\mathcal{C}, \mathcal{A})$-invariant subspace containing $\mathcal{B} = \mathbf{Im}\, B$ is denoted by $\mathcal{S}_*$.

Left and right invertibility of LPV system can be characterized in geometric terms as follows:

Proposition 20: *The LPV system* (9.10),(9.11) *is left-invertible if*

$$\mathcal{V}^* \cap \mathcal{B} = 0. \tag{9.15}$$

The system is right invertible if

$$\mathcal{S}_* + \mathcal{C} = \mathcal{X}. \tag{9.16}$$

Let us observe, that if conditions (9.15) are fulfilled, one can always choose a coordinate transform of the form

$$z = Tx, \text{ where } \quad T = \begin{bmatrix} \mathcal{V}^{*\perp} \\ \Lambda \end{bmatrix}, \; \Lambda \subset \mathcal{B}^{\perp}.$$

Accordingly, the system will be decomposed into:

$$\begin{aligned} \dot{\xi} &= A_{11}(t)\xi + A_{12}(t)\eta + \bar{B}(t)u \\ \dot{\eta} &= A_{21}(t)\xi + A_{22}(t)\eta \\ y &= \bar{C}\xi. \end{aligned} \tag{9.17}$$

It follows, that applying a suitable feedback

$$u = F_2(t)\eta + v, \tag{9.18}$$

that makes the subspace $\mathcal{V}^*$ be $(\mathcal{A} + \mathcal{B}F, \mathcal{B})$ invariant, one can obtain the system:

$$\dot{\xi} = A_{11}(t)\xi + \bar{B}v \tag{9.19}$$

$$y = \bar{C}\xi. \tag{9.20}$$

Maximality of $\mathcal{V}^*$ ensures that both ξ and v can be expressed as functions of y and its derivatives. By introducing the notation $\tilde{y} = \mathcal{S}\xi$, where

$$\tilde{y} = \left[y_1, \cdots, y_1^{(r_1-1)}, \cdots, y_p, \cdots, y_p^{(r_p-1)} \right]^T$$

one has $v = \bar{B}^{\{-1\}}\mathcal{S}^{-1}(\dot{\tilde{y}} - \dot{\mathcal{S}}\mathcal{S}^{-1}\tilde{y} - \mathcal{S}A_{11}\mathcal{S}^{-1}\tilde{y})$, i.e.,

$$\dot{\eta} = A_{22}\eta + A_{21}\mathcal{S}^{-1}\tilde{y} \tag{9.21}$$

$$u = F_2\eta + \bar{B}^{\{-1\}}\mathcal{S}^{-1}(\dot{\tilde{y}} - \dot{\mathcal{S}}\mathcal{S}^{-1}\tilde{y} - \mathcal{S}A_{11}\mathcal{S}^{-1}\tilde{y}). \tag{9.22}$$

The coordinate transform $\mathcal{S}(t)$ can be obtained by applying the recursive algorithm defined by:

$$\begin{aligned} S_i^0(t) &= c_i, \\ S_i^{k+1}(t) &= \dot{S}_i^k(t) + S_i^k(t)A_{11}(t), \end{aligned}$$

see, e.g., Silverman and Meadows (1967).

Remark 14: *It is clear that the method presented above can be also applied for nonlinear dynamics cast as quasi LPV systems with affine parameter dependence. One can observe that to compute the matrix $\mathcal{S}(t)$ one needs certain derivatives of the parameter functions $\rho_i(y)$, i.e., certain derivatives of the output y, but the order of these derivatives are bounded by* $\max_i r_i$.

The main result of this section can be formalized as:

Proposition 21: *If the the LPV system* (9.10)-(9.11) *has a relative degree, i.e., conditions* (9.15) *are fulfilled, the system has a well defined left dynamical inverse of the form* (9.21)-(9.22).

Moreover, if the parameter dependence is affine $\mathcal{V}^*$, the maximal $(\mathcal{A}, \mathcal{B})$-invariant subspace contained in **Ker** C*, and the transformation matrix $\mathcal{S}$ needed to obtain the dynamical inverse can be computed in finite steps.*

9.3 Inversion based output tracking controller

Since the condition of right-invertibility is equivalent to $\mathcal{V}^* + \mathcal{S}_* = \mathcal{X}$, one has the dual result as:

Proposition 22: *If the the LPV system* (9.10)-(9.11) *has a relative degree and condition* (9.16) *is fulfilled, the system has a well defined right dynamical inverse of the form* (9.21)-(9.22).

Moreover, if the parameter dependence is affine the dynamical inverse, i.e., the output tracking controller can be computed in finite steps.

The right inverse is realizable in exactly the same way as the (left)inverse system. The input u corresponding to the desired output is not unique, in general. The difference between any two admissible input corresponds to a zero-state motion on $R_{\mathcal{V}^*} = \mathcal{V}^* \cap \mathcal{S}_*$ which does not affect the output. A common solution is to set to zero the input components which, expressed in a suitable basis, correspond to forcing actions belonging to $\mathcal{V}^* \cap \mathcal{B}$.

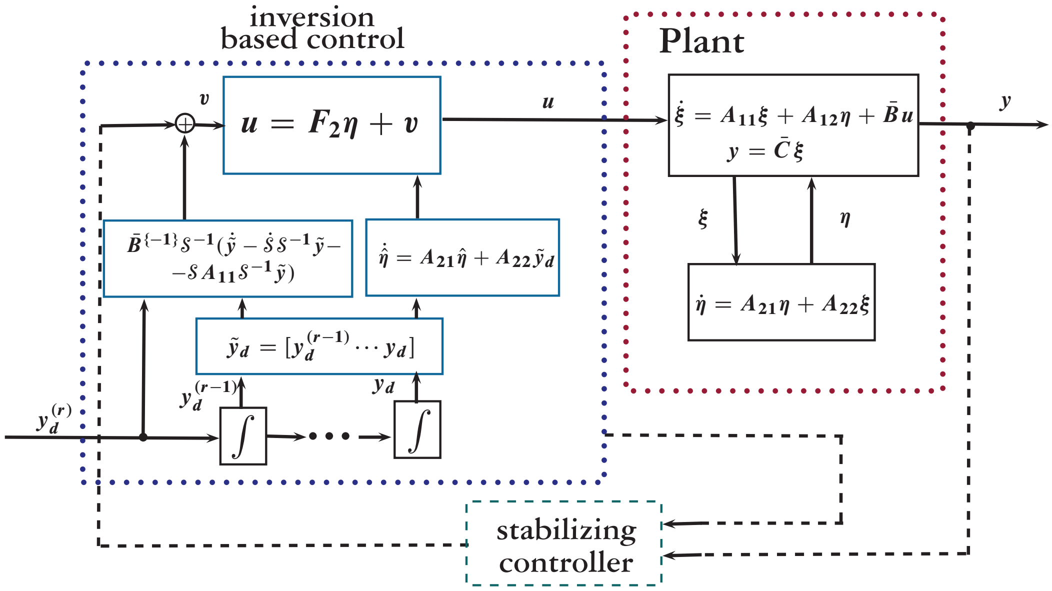


Figure 9.2: Inversion based tracking controller

Applying the dynamic inversion algorithm, one can obtain a system that realizes the tracking if the initial conditions are known. Let us denote the outputs to be tracked by y_d. Due to the effect caused by the unknown initial condition, there will be an error of the estimated state η. Introducing an outer-loop based on error feedback, one can obtain the following structure for the tracking controller, see Figure 9.2:

$$\begin{aligned}\dot{\bar{\eta}} &= A_{22}\bar{\eta} + A_{21}\mathcal{S}^{-1}\tilde{y}_d + \Gamma_1\tilde{e}\\ \bar{u} &= F_2\bar{\eta} + \lambda(\tilde{y}_d) + \Gamma_2\tilde{e},\end{aligned} \tag{9.23}$$

with $\lambda(\tilde{y}_d) = \bar{B}^{\{-1\}}\mathcal{S}^{-1}(\dot{\tilde{y}} - \dot{\mathcal{S}}\mathcal{S}^{-1}\tilde{y} - \mathcal{S}A_{11}\mathcal{S}^{-1}\tilde{y})$, the tracking error $e = \hat{y} - y_d$ and the possibly parameter dependent gain matrices Γ_1 and Γ_2.

Let us denote by $e_\xi = \hat{\xi} - \xi_d$ and $e_\eta = \hat{\eta} - \bar{\eta}$ and recall that $\tilde{e} = \mathcal{S}e_{x_1}$. Then the error dynamics can be expressed as:

$$\dot{e}_\xi = (A_{11} + \bar{B}\Gamma_2\mathcal{S})e_\xi + A_{12}e_\eta \tag{9.24}$$

$$\dot{e}_\eta = (A_{21} + \Gamma_1\mathcal{S})e_\xi + A_{22}e_\eta \tag{9.25}$$

$$\tilde{e} = \mathcal{S}e_\xi. \tag{9.26}$$

Actually the decay rate of e_η cannot be increased – the dynamics determined by A_{22} should be stable – therefore a convenient choice is $\Gamma_1 = -A_{21}\mathcal{S}^{-1}$. The gain Γ_2 is tuned to obtain a desired decay rate for e_ξ, this can be done by solving a suitable set of LMIs, see also Section 10.3.

In implementing the tracking control a problem might be that $\tilde{e}$ is not available for the measurement. If a state observer is available, then the inversion scheme can be replaced by the combination of this observer and the linearization feedback. Such a state observer can be design if additional measured outputs are available, say:

$$z = C_2x = C_{21}\xi + C_{22}\eta, \tag{9.27}$$

that makes the plant fully observable. Then, the inversion is achieved by the following dynamical system:

$$\begin{aligned}\dot{\bar{w}} &= (A - K\bar{C} + BF)\bar{w} + K\bar{y} + B\lambda(\tilde{y}_d) + \Gamma_1\tilde{e}\\ \bar{u} &= F\bar{w} + \lambda(\tilde{y}_d) + \Gamma_2\tilde{e}.\end{aligned} \tag{9.28}$$

where $\bar{C}^T = [\, C^T \; C_2^T \,]$ and $\bar{y} = [y\; z]^T$.

The additional degree of freedom can be used to improve the performance properties – estimation time, disturbance rejection – of the unknown input observer or of the output tracking controller, respectively.

Example

As an illustrative example for the LPV inversion scheme let us consider the following linearized parameter varying model:

$$\begin{aligned}\dot{x}(t) &= A(\rho)x(t) + Bv(t)\\ y(t) &= Cx(t),\end{aligned}$$

where $A(\rho) = A_0 + \rho_1 A_1 + \rho_2 A_2$. The state matrices are:

$$A_0 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0\\ 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & -1 & 1 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 0\\ 0 & 1\\ 0 & 1\\ 0 & 0\\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Applying the $\mathcal{ABISA}$ algorithm one has $\mathcal{V}^* = \mathbf{Im}\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T$ and the corresponding state transform can be chosen as:

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\\ 0 & -1 & 1 & 0 & 0 \end{bmatrix}, \quad \text{i.e.,} \quad T^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 1\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Accordingly the the system splits as

$$\begin{bmatrix} A_{11}^0 & A_{12}^0\\ A_{21}^0 & A_{22}^0 \end{bmatrix} = \left[\begin{array}{cccc|c} -1 & 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0\\ 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & -1 & 0\\ \hline 0 & 0 & 0 & 0 & -1 \end{array}\right], \quad \begin{bmatrix} A_{11}^1 & A_{12}^1\\ A_{21}^1 & A_{22}^1 \end{bmatrix} = \left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 1\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0\\ \hline 0 & 0 & 0 & -1 & 0 \end{array}\right],$$

$$\begin{bmatrix} A_{11}^2 & A_{12}^2\\ A_{21}^2 & A_{22}^2 \end{bmatrix} = \left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ \hline 1 & 0 & 0 & 0 & 0 \end{array}\right], \quad \begin{bmatrix} \bar{B}\\ 0 \end{bmatrix} = \left[\begin{array}{cc} 1 & 0\\ 0 & 1\\ 0 & 0\\ 0 & 0\\ \hline 0 & 0 \end{array}\right],$$

$$\begin{bmatrix} \bar{C} & 0 \end{bmatrix} = \left[\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 1 & 0 \end{array}\right].$$

The matrix $F(\rho) = F_0 + \rho_1 F_1 + \rho_2 F_2$, is given by

$$F_0 = 0, \quad F_1 = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad F_2 = 0.$$

The transformation $\mathcal{S}(\rho) = S_0 + \rho_1 S_1 + \rho_2 S_2$, where

$$S_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad S_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

maps ξ to $\tilde{y} = \begin{bmatrix} y_1 & \dot{y}_1 & y_2 & y_3 \end{bmatrix}^T$.

One can figure out that

$$\mathcal{S}^{-1}(t) = \begin{bmatrix} \frac{1}{\rho_1} & \frac{1}{\rho_1} & -\frac{\rho_2}{\rho_1} & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \dot{\mathcal{S}}(t)\mathcal{S}^{-1}(t) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{\dot{\rho}_1}{\rho_1} & \frac{\dot{\rho}_1}{\rho_1} & -\frac{\dot{\rho}_1\rho_2}{\rho_1} + \dot{\rho}_2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

It follows, that

$$\mathcal{S}A_{11}\mathcal{S}^{-1} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} + \rho_1 \begin{bmatrix} \frac{1}{\rho_1} & \frac{1}{\rho_1} & -\frac{\rho_2}{\rho_1} & 0 \\ \rho_2 - \rho_1 - \frac{1}{\rho_1} & -\frac{1}{\rho_1} & \frac{\rho_2}{\rho_1} & 0 \\ 1 & 0 & 0 & 0 \\ \frac{1}{\rho_1} & \frac{1}{\rho_1} & -\frac{\rho_2}{\rho_1} & 0 \end{bmatrix} + \rho_2 \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$\bar{B}^{-r}\mathcal{S}^{-1} = \begin{bmatrix} \frac{1}{\rho_1} & \frac{1}{\rho_1} & -\frac{\rho_2}{\rho_1} & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Finally, for the unknown input observer, i.e., the left inverse system one has

$$\dot{\eta} = -\eta + (\frac{\rho_2}{\rho_1} - \rho_1)y_1 + \frac{\rho_2}{\rho_1}\dot{y}_1 - \frac{\rho_2^2}{\rho_1}y_2,$$

and

$$\hat{\nu} = \begin{bmatrix} -\rho_1 \\ 0 \end{bmatrix} \eta + \begin{bmatrix} \frac{1}{\rho_1} & \frac{1}{\rho_1} & -\frac{\rho_2}{\rho_1} & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \left(\begin{bmatrix} \dot{y}_1 \\ \ddot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & 0 \\ \rho_1\rho_2 - \rho_1^2 + \frac{\dot{\rho}_1 - \rho_1}{\rho_1} & \frac{\dot{\rho}_1 - \rho_1}{\rho_1} & \frac{\dot{\rho}_2\rho_1 - \dot{\rho}_1\rho_2}{\rho_1} & \rho_1^2 \\ \rho_1 & 0 & -1 & 0 \\ 1 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ \dot{y}_1 \\ y_2 \\ y_3 \end{bmatrix} \right).$$

During the simulation the parameters vary as on Figure 9.3 and some measurement noise was also considered. The applied and reconstructed inputs are depicted on Figure 9.4.

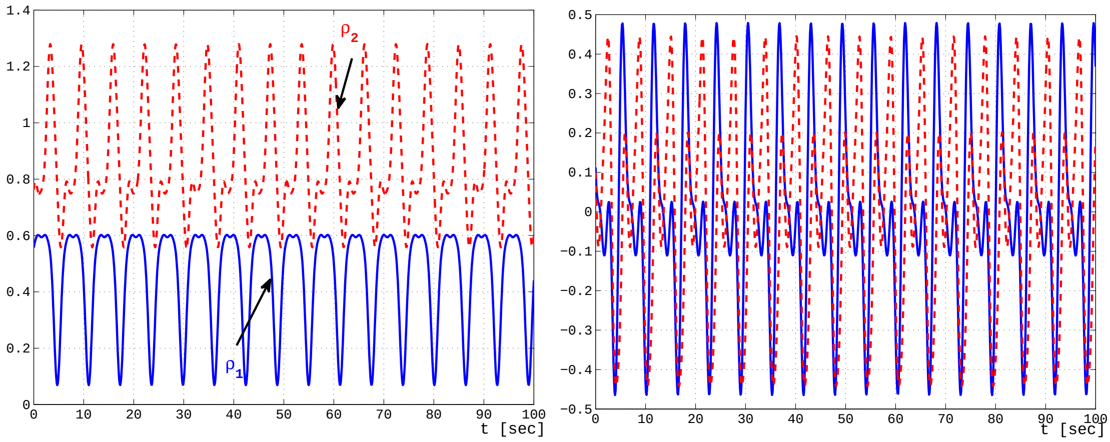


Figure 9.3: Parameters ρ_1 and ρ_2 (dashed) and its derivatives

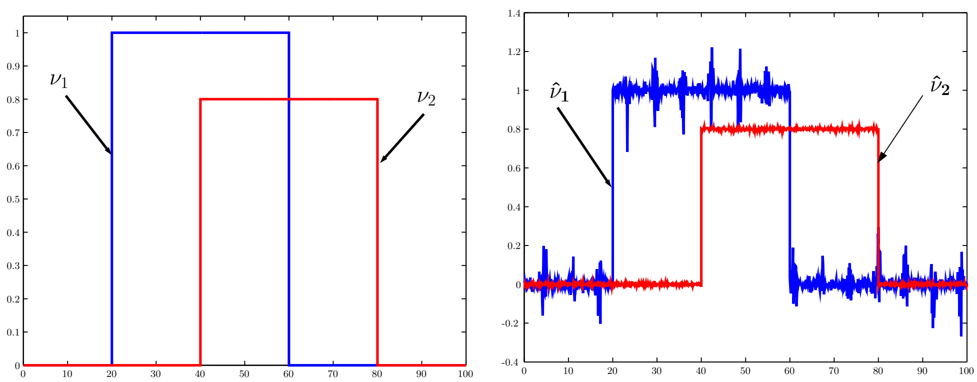


Figure 9.4: Applied and reconstructed inputs

Since $\mathcal{S}_* = \mathbf{Im}\begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ one has $\mathcal{S}_* + \mathcal{V}^* \neq \mathcal{X}$, i.e., the right invertibility condition is not fulfilled, as it was expected.

To make the system right invertible consider the first two outputs only, i.e., $y_t = C_t x$ with

$C_t = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$. With this setting one has $\mathcal{V}^* = \mathbf{Im} \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T$ and $\mathcal{S}_* = \mathbf{Im} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, i.e., $\mathcal{S}_* + \mathcal{V}^* = \mathcal{X}$. The corresponding state transform can be chosen as:

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad \text{i.e.,} \quad T^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Accordingly the the system splits as

$$\begin{bmatrix} A_{11}^0 & A_{12}^0 \\ A_{21}^0 & A_{22}^0 \end{bmatrix} = \left[\begin{array}{ccc|cc} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{array}\right], \quad \begin{bmatrix} A_{11}^1 & A_{12}^1 \\ A_{21}^1 & A_{22}^1 \end{bmatrix} = \left[\begin{array}{ccc|cc} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

$$\begin{bmatrix} A_{11}^2 & A_{12}^2 \\ A_{21}^2 & A_{22}^2 \end{bmatrix} = \left[\begin{array}{ccc|cc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right], \quad \begin{bmatrix} \bar{B} \\ 0 \end{bmatrix} = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ \hline 0 & 0 \\ 0 & 0 \end{array}\right],$$

$$\begin{bmatrix} \bar{C} & 0 \end{bmatrix} = \left[\begin{array}{ccc|cc} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right].$$

One has $\mathcal{S}(\rho) = \begin{bmatrix} 0 & 0 & 1 \\ \rho_1 & \rho_2 & -1 \\ 0 & 1 & 0 \end{bmatrix}$ that maps ξ to $\tilde{y} = \begin{bmatrix} y_1 & \dot{y}_1 & y_2 \end{bmatrix}^T$.

One can figure out that

$$S^{-1}(t) = \begin{bmatrix} \frac{1}{\rho_1} & \frac{1}{\rho_1} & -\frac{\rho_2}{\rho_1} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \dot{S}(t)S^{-1}(t) = \begin{bmatrix} 0 & 0 & 0 \\ \frac{\dot{\rho}_1}{\rho_1} & \frac{\dot{\rho}_1}{\rho_1} & -\frac{\dot{\rho}_1\rho_2}{\rho_1} + \dot{\rho}_2 \\ 0 & 0 & 0 \end{bmatrix},$$

while

$$\mathcal{S}A_{11}\mathcal{S}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ \rho_1\rho_2 - 1 & -2 & 0 \\ \rho_1 & 0 & -1 \end{bmatrix}, \quad \bar{B}^{-r}\mathcal{S}^{-1} = \begin{bmatrix} \frac{1}{\rho_1} & \frac{1}{\rho_1} & -\frac{\rho_2}{\rho_1} \\ 0 & 0 & 1 \end{bmatrix}.$$

The output tracking controller has the form:

$$\dot{\zeta} = -\zeta + \left[\begin{array}{ccc} \frac{\rho_2}{\rho_1} - \rho_1 & \frac{\rho_2}{\rho_1} & -\frac{\rho_2\rho_2}{\rho_1} \end{array} \right] \tilde{y}$$

$$u = -\rho_1\zeta + \left[\begin{array}{ccc} \frac{1}{\rho_1} & \frac{1}{\rho_1} & -\frac{\rho_2}{\rho_1} \\ 0 & 0 & 1 \end{array} \right] \left(\dot{\tilde{y}} - \left[\begin{array}{ccc} 0 & 1 & 0 \\ \rho_1\rho_2 + \frac{\dot{\rho}_1 - \rho_1}{\rho_1} & \frac{\dot{\rho}_1 - 2\rho_1}{\rho_1} & \frac{\dot{\rho}_2\rho_1 - \dot{\rho}_1\rho_2}{\rho_1} \\ \rho_1 & 0 & -1 \end{array} \right] \tilde{y} \right) + \Gamma \tilde{y},$$

with the gain $\Gamma = \left[\begin{array}{ccc} \frac{-100}{\rho_1} & \frac{-100}{\rho_1} & \frac{100\rho_2}{\rho_1} \\ -\rho_1 & 0 & -50 \end{array} \right]$.

The results of the simulation are depicted on Figure 9.5.

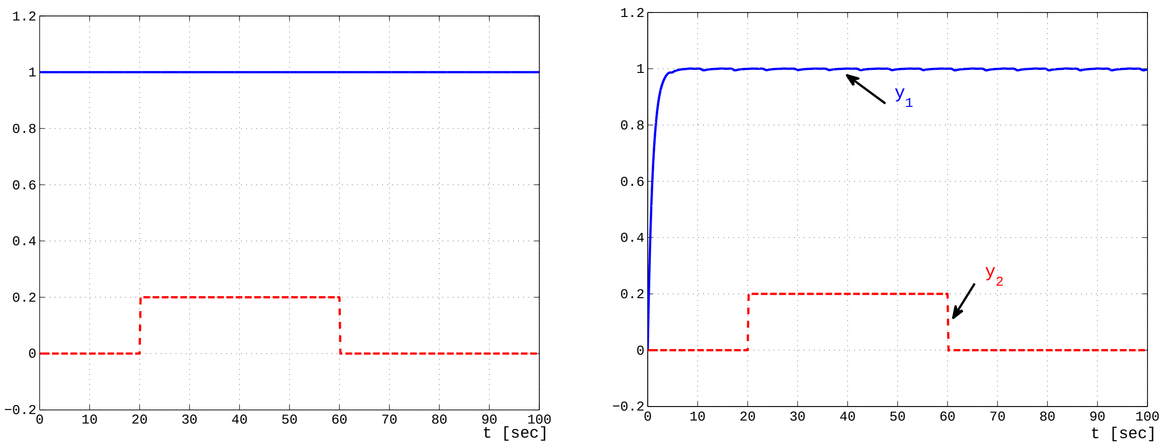


Figure 9.5: Desired and actual outputs

Summary

Dynamical inversion of qLPV systems Chapter 9, Proposition 20, 21, 22

> An algorithm was established for the computation of the dynamical inverse of linear parameter varying systems. The method can be applied for the class of nonlinear systems that can be cast in the qLPV form.
>
> Based on the dynamic inversion method a design algorithm for an unknown input observer, and for an output tracking controller was given.
>
> If the parameter dependency is affine the algorithm provides the state matrices of the dynamic inverse by using only a finite number of matrix manipulations.

Further details can be found in the papers Balas et al. (2004); Edelmayer et al. (2003, 2004, 2009); Szabó et al. (2003a).

The results were used in engineering applications, such as reconfigurable fault detection controls of vehicle, fault tolerant active suspension design, see Szabó et al. (2003); Gáspár, Szabó and Bokor (2007, 2008f); Gáspár et al. (2009). The developed algorithms were also successfully applied in the dynamic inversion based controller design for stabilizing the primary circuit pressurizer at the Paks Nuclear Power Plant Hungary, see Gáspár et al. (2006); Szabó et al. (2005).

10 Decoupling in FDI and control

Up to this point applicability of the geometrical concepts has been manifesting through a more theoretical context, where the geometric tools were hidden in the derivations that lead to the solutions. To conclude the last part of the thesis two additional applications are presented in this chapter in order to provide a more direct example for the usability of the geometric view. These apllications represent the two facades of the very same problem of decoupling: in the first application the fault to be detected is decoupled from the other, nuisance, faults while in the second application the classical problem of disturbance decoupling is tackled. The proposed solutions are extensions of the classical LTI methods to the LPV framework based on the suitable introduced parameter varying invariant subspaces and the induced state decompositions.

Fundamental problem of residual generation (FPRG)

Let us consider the following LTI system, that has two failure events:

$$\dot{x}(t) = Ax(t) + Bu(t) + L_1 m_1(t) + L_2 m_2(t) \tag{10.1}$$
$$y(t) = Cx(t), \tag{10.2}$$

then the task to design a residual generator that is sensitive to L_1 and insensitive to L_2 is called the fundamental problem of residual generation (FPRG). More precisely, one has to design a residual generator with outputs r such that if $m_1 \neq 0$ then $r \neq 0$ and if $m_1 = 0$ then $\lim_{t\to\infty} \|r(t)\| = 0$, i.e., a stability condition is required.

In the solution of this problem a central role is played by the (C, A)–invariant subspaces and certain unobservability subspaces, Massoumnia (1986); Massoumnia et al. (1989) or observability codistributions, De Persis and Isidori (2000, 2001), in the nonlinear version of this problem.

As it is well known, for LTI models, a subspace $\mathcal{W}$ is (C, A)–invariant if $A(\mathcal{W} \cap \mathbf{Ker}\, C) \subset \mathcal{W}$ that is equivalent with the existence of a matrix G such that $(A + GC)\mathcal{W} \subset \mathcal{W}$. A (C, A)–unobservability subspace $\mathcal{U}$ is a subspace such that there exist matrices G and H with the property that $(A + GC)\mathcal{U} \subset \mathcal{U}$, i.e., $\mathcal{U}$ is (C, A)–invariant, and $\mathcal{U} \subset \mathbf{Ker}\, HC$. The family of (C, A)–unobservability subspaces containing a given set $\mathcal{L}$ has a minimal element $\mathcal{U}^*$.

Let us denote by $\mathcal{S}^*$ the smallest unobservability subspace containing $\mathcal{L}_2$, where $\mathcal{L}_i = ImL_i$. Then one has the following result, Massoumnia (1986):

MA: A FPRG has a solution if and only if $\mathcal{S}^* \cap \mathcal{L}_1 = 0$, moreover, if the problem has a solution, the dynamics of the residual generator can be assigned arbitrary.

Given the residual generator in the form

$$\dot{w}(t) = Nw(t) - Gy(t) + Fu(t) \tag{10.3}$$
$$r(t) = Mw(t) - Hy(t), \tag{10.4}$$

then H is a solution of $\mathbf{Ker}\, HC = \mathbf{Ker}\, C + \mathcal{S}^*$, and M is the unique solution of $MP = HC$, where P is the projection $P : \mathcal{X} \to \mathcal{X}/\mathcal{S}^*$. Let us consider a G_0 such that $(A + G_0C)\mathcal{S}^* \subset \mathcal{S}^*$ and denote by $A_0 = A + G_0C|_{\mathcal{X}/\mathcal{S}^*}$. Then there is a G_1 such that $N = A_0 + G_1M$ has prescribed eigenvalues. Then set $G = PG_0 + G_1H$ and $F = PB$.

Extending this result to the case with multiple events one has the extension of the fundamental problem of residual generation (EFPRG), that has a solution if and only if $\mathcal{S}_i^* \cap \mathcal{L}_i = 0$, where $\mathcal{S}_i^*$ is the smallest unobservability subspace containing $\overline{\mathcal{L}}_i := \sum_{j \neq i} \mathcal{L}_j$.

These ideas were also applied to nonlinear systems, and a similar condition was obtained for the solvability of the FPRG problem in terms of the observability codistributions, see Hammouri et al. (1999); De Persis and Isidori (2002).

In what follows, this result will be extended to the LPV systems where the state matrix depends affinely on the parameter vector and quasi LPV systems, where the parameters depends on measurable outputs.

10.1 FPRG for LPV systems

Let us consider the class of linear parameter–varying systems of which state matrix depends affinely on the parameter vector will be considered. This class of systems can be described as:

$$\begin{aligned} \dot{x}(t) &= A(\rho)x(t) + B(\rho)u(t) + \sum_{j=1}^{m} L_j(\rho)v_j(t) \\ y(t) &= Cx(t), \end{aligned} \tag{10.5}$$

where v_j are the failures to be detected, C is right invertible,

$$A(\rho) = A_0 + \rho_1 A_1 + \cdots + \rho_N A_N, \tag{10.6}$$
$$B(\rho) = B_0 + \rho_1 B_1 + \cdots + \rho_N B_N, \tag{10.7}$$
$$L_j(\rho) = L_{j,0} + \rho_1 L_{j,1} + \cdots + \rho_N L_{j,N}, \tag{10.8}$$

and ρ_i are time varying parameters. It is assumed that each parameter ρ_i and its derivatives $\dot{\rho}_i$ ranges between known extremal values $\rho_i(t) \in [-\overline{\rho}_i, \overline{\rho}_i]$ and $\dot{\rho}_i(t) \in [-\overline{\dot{\rho}}_i, \overline{\dot{\rho}}_i]$, respectively. Let us denote this parameter set by $\mathcal{P}$.

The assertion of MA remains valid also for the LPV systems (10.5), i.e.,

Proposition 23: *For the LPV systems (10.5) one can design a – not necessarily stable – residual generator of type*

$$\dot{w}(t) = N(\rho)w(t) - G(\rho)y(t) + F(\rho)u(t) \tag{10.9}$$
$$r(t) = Mw(t) - Hy(t), \tag{10.10}$$

if and only if for the smallest (parameter varying) unobservability subspace $\mathcal{U}^*$ *containing* $\mathcal{L}_2$ *one has* $\mathcal{U}^* \cap \mathcal{L}_1 = 0$, *where* $\mathcal{L}_i = \cup_{j=0}^{N} \textbf{\textit{Im}}\, L_{i,j}$.

Proof Let H be the solution of $\mathbf{Ker}\, HC = \mathbf{Ker}\, C + \mathcal{U}^*$, and M is the unique solution of $MP = HC$, where P is the projection $P : \mathcal{X} \to \mathcal{X}/\mathcal{U}^*$. By the definition of the unobservabilitry subspaces there is a matrix $G_0(\rho)$ such that $(A(\rho) + G_0(\rho)C)\mathcal{U}^* \subset \mathcal{U}^*$ holds. Then set $A_0(\rho) = A(\rho) + G_0(\rho)C|_{\mathcal{X}/\mathcal{U}^*}$, $N(\rho) = A_0(\rho)$ and $F = PB(\rho)$.

One can compute an acceptable $G_0(\rho)$ as follows: let H_1 be the matrix that completes H to a nonsingular matrix and let us consider a matrix K_1 that has as raws the coordinates of the basis vectors for $\mathcal{X} \ominus \mathcal{U}^*$. Let us denote by

$$K = \begin{bmatrix} K_1 \\ H_1 C \\ K_3 \end{bmatrix},$$

where K_3 is an arbitrary matrix that makes K nonsingular.

Then

$$KA(\rho)K^{-1} = \begin{bmatrix} A_{11}(\rho) & A_{12}(\rho) & 0 \\ A_{21}(\rho) & A_{22}(\rho) & A_{23}(\rho) \\ A_{31}(\rho) & A_{32}(\rho) & A_{33}(\rho) \end{bmatrix},$$

and the matrix $G_0(\rho)$ can be chosen as

$$G_0(\rho) = K^{-1} \begin{bmatrix} 0 & -A_{12}(\rho) \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} H \\ H_1 \end{bmatrix}.$$

Example: As an illustrative example let us consider the following linearized parameter varying model of the longitudinal dynamics of an aircraft:

$$\begin{aligned} \dot{x}(t) &= A(\rho)x(t) + Bu(t) + L_1 v_1(t) + L_2 v_2(t) \\ y(t) &= Cx(t), \end{aligned}$$

where $A(\rho) = A_0 + \rho_1 A_1 + \rho_2 A_2$. It is assumed that the parameter ρ_1 and ρ_2 vary in the intervals $[-0.3, 0.3]$ and $[-0.6, 0.6]$, respectively, see Figure 10.1.

The state matrices are:

$$A_0 = \begin{bmatrix} -1.05 & -2.55 & 0 & 0 & -169.66 & -0.0091 \\ 2.55 & -1.05 & 0 & 0 & 57.09 & 0.0017 \\ 0 & 0 & -77.53 & 39.57 & 0 & 0 \\ 0 & 0 & 0 & -20.20 & 0 & 0 \\ 0 & 0 & -8.80 & 0 & -20.20 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.1000 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -4.49 \\ 0 \\ 0 \end{bmatrix}$$

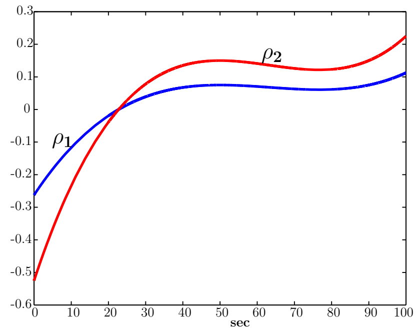


Figure 10.1: Scheduling variables for the simulation

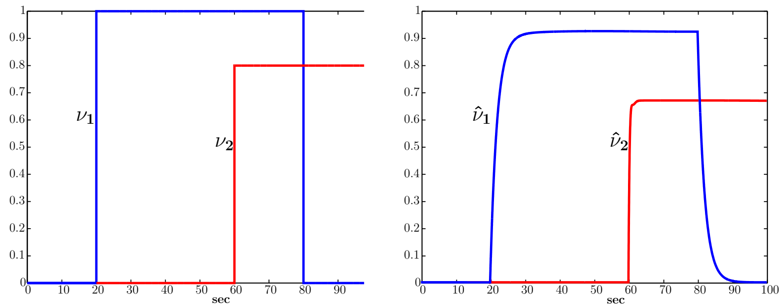


Figure 10.2: Fault signals and the estimated residuals

$$L_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1.00 \\ 0 \end{bmatrix}, L_2 = \begin{bmatrix} 3.55 & 2.41 \\ -0.55 & 8.04 \\ 0 & 0 \\ 0 & 0 \\ -0.02 & 0.56 \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} -0.01 & 0.1 & 0.07 & 0 & 0.0 & -0.000 \\ -0.48 & -0.6 & 0.00 & 0 & -49.5 & -0.002 \\ 0.03 & 0.1 & -0.06 & 0 & -0.0 & 0.000 \\ 0.26 & -0.1 & 0.01 & 0 & 0.0 & -0.000 \end{bmatrix}$$

The simulation results are depicted on Figure 10.2.

10.2 Disturbance decoupling

The scope of this section is to solve the disturbance decoupling problem(DDP), see Wonham (1985), for LPV systems.

Consider the system

$$\dot{x} = A(\rho)x + B(\rho)u + S(\rho)q \tag{10.11}$$
$$y = Cx \tag{10.12}$$

where q represents a disturbance and the matrix $S(\rho)$ has the same affine parameter dependent structure like $A(\rho)$.

By using the concept of generalized controllability (A, B)–invariant subspaces, Bhattacharyya (1983), one can provide a possible solution to the DDP problem, see Otsuka (2000). A generalized (A, B)–invariant subspace can be viewed as a special $(\mathcal{A}, \mathcal{B})$–invariant subspace, where the gain matrix F is independent of the parameters.

As it is shown in the following example, there are $(\mathcal{A}, \mathcal{B})$–invariant subspaces that are not generalized (A, B)–invariant: let us consider

$$A(\rho) = \begin{bmatrix} 1 & \rho \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathcal{V} = \operatorname{Im} \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{10.13}$$

Then there exists $F(\rho)$, namely $[0 \quad -\rho]$, such that

$$(A(\rho) + BF(\rho))\mathcal{V} \subset \mathcal{V}, \quad \rho \in \mathcal{P}, \tag{10.14}$$

but there does not exist a parameter independent F, such that $A(\rho) + BF$ satisfy the invariance property. Therefore by using parameter varying invariant subspaces one might obtain less conservative solutions for the disturbance decoupling problem.

The aim of the proposed algorithm is to find, if it is possible, $F : [0, T] \to \mathbb{R}^{m \times n}$, such that

$$\langle \mathcal{A} + B\mathcal{F} | \mathcal{S} \rangle \subset \mathcal{C}. \tag{10.15}$$

Here $\mathcal{S}$ represents all possible images of matrices $S(q)$ which can occur on the time interval $[0, T]$. Practically we have to use

$$\mathcal{S} = \sum_{\rho \in \mathcal{P}} S(\rho) \tag{10.16}$$

to be sure that all possible disturbances will be decoupled. Another practical assumption is that $B(\rho) = B$. Then the following theorem holds:

Proposition 24: *Let us denote by $\mathcal{V}^*$ the maximal $(\mathcal{A}, \mathcal{B})$–invariant subspace contained in $\mathcal{C}$. Then the DDP problem is solvable if and only if*

$$\mathcal{S} \subset \mathcal{V}^*. \tag{10.17}$$

In contrast to the LTI case, when stabilizability is guaranteed by certain pole allocation properties, in the LPV case the problem of stability is more involved. A common stabilization strategy in these schemes is to suppose a Lyapunov function of certain type – usually a quadratic Lyapunov function defined by a constant positive definite matrix – and to find the stabilizing feedback gains starting from the corresponding analysis equations. This method is sketched in the next section.

10.3 The question of stability

An (q)LPV system is said to be quadratically stable if there exist a matrix $P = P^T > 0$ such that

$$A(\rho)^T P + PA(\rho) < 0 \tag{10.18}$$

for all the parameters $\rho \in \mathcal{P}$. A necessary and sufficient condition for a system to be quadratically stable is that the condition (10.18) holds for all the corner points of the parameter space, i.e., one can obtain a finite system of LMI's that has to be fulfilled for $A(\rho)$ with a suitable positive definite matrix P, see Apkarian et al. (1995).

In order to obtain a quadratically stable residual generator one can set $N(\rho) = A_0(\rho)+G(\rho)M$ in (10.9), where $G(\rho) = G_0 + \rho_1 G_1 + \cdots \rho_N G_N$ is determined such that the LMI defined in (10.18), i.e.,

$$(A_0(\rho) + G(\rho)M)^T P + P(A_0(\rho) + G(\rho)M) < 0$$

holds for suitable $G(\rho)$ and $P = P^T > 0$. By introducing the auxiliary variable $K(\rho) = G(\rho)P$, one has to solve the following set of LMIs on the corner points of the parameter space:

$$A_0(\rho)^T P + PA_0(\rho) + M^T K(\rho)^T + K(\rho)M < 0.$$

Remark 15: *If **Ker** $C \subset \mathcal{U}^*$ then one can choose $G(\rho)$ such that the matrix $N(\rho)$ be parameter independent with arbitrary eigenvalues, since the equation $G(\rho)CU = UT - A(\rho)U$ has a solution for arbitrary T, where U is the insertion map of $\mathcal{X}/\mathcal{U}^*$.*

This method for quadratic stabilization can be also used for computing the gains Γ_i for the inversion based tracking controller (9.23).

11 New Scientific Results

11.1 Controllability

Consider the (open-loop) linear switched system $\Sigma_{(\mathfrak{S},\mathfrak{U})}$:

$$\dot{x}(t) = A(\sigma(t))x(t) + B(\sigma(t))u(t) \tag{11.1}$$

where $x \in \mathbb{R}^n$ is the state variable, $u \in \mathfrak{U} \subset \mathbb{R}^m$ is the input variable $\sigma : \mathbb{R}^+ \to \mathfrak{S}$ is a measurable switching function mapping the positive real line into $\mathfrak{S} = \{1, \cdots, s\}$, i.e., the matrices $A(\sigma)$, $B(\sigma)$ and $C(\sigma)$ are measurable. The input set might be unconstrained $\mathfrak{U} = \mathbb{R}^m$ or constrained $\mathfrak{U} = \mathbb{R}^m_+$. A finite switching sequence is $\sigma = (s_1, s_2, \cdots, s_q)$ while the corresponding switching time sequence is denoted by $\tau = (t_1, t_2, \cdots t_q)$, $t_i < 0$ and the control input sequence is $\mu = (u_1, u_2, \cdots, u_q)$.

In what follows the main results of the research concerning controllability of this system and reported in the Thesis are presented:

Thesis 1 (**Chapter 3 and 4, Propositions 3, 4, 5, 6 and Corollary 1, 2, 3).**
Concerning controllability properties of switched linear time invariant systems the following results were established:

Full rank reachability *For a completely controllable linear switching system for arbitrary point pairs (x, y) one has that y is full rank/normally reachable from x. Moreover, every point pair can be joined in a full rank/normal reachable way by using a fixed sequence, i.e., $(\sigma, \tau)/(\sigma, \mu)$ fixed.*

Finite switching number *A completely controllable linear switching system is also globally controllable by using piecewise constant switching functions, i.e., using only a finite number of switchings. Moreover, there exist a bound for the necessary number of switchings, that depends only on the system matrices and $\mathfrak{U}$. There exist a universal (finite) switching sequence σ such that the time varying system $\dot{x} = A(\sigma)x + B(\sigma)u$ is globally controllable.*

Sign constrained inputs *A complete–controllability condition has been formulated for linear switching systems controlled by sign constrained inputs. The condition – a generalization of the multivariable Kalman rank condition – is expressed in algebraic terms and al algorithm is also provided.*

For every completely controllable linear switching system (11.1) the sampled discrete–time system is also completely controllable for suitable sampling rates. Moreover for every completely controllable linear switching system one can associate – not necessarily a unique – completely controllable periodic linear time varying system $\dot{x} = A(t)x + B(t)$. The non–uniqueness comes from the fact that one has more switching sequences that σ such that for the reachability subspace $\mathcal{R}_\sigma = \mathbb{R}^n$ holds.

The solution set of of (11.1) is dense in the set of the relaxed solutions, i.e., the solutions of the the convexified differential inclusion $\dot{x} \in A_c(x)$, where $A_c(x) = \sum_{i=1}^{s} \alpha_i (A_i x + B_i u)$ and $\alpha_i \geq 0$ and $\sum_{i=1}^{s} \alpha_i = 1$. Hence, the corresponding attainable sets coincide.

It was shown that the following conditions are equivalent:

a) the switching system $\dot{x} = A_i x + B_i u,\ i \in \{1, \cdots, s\}, \quad u \in \mathfrak{U}$ is controllable,

b) the associated differential inclusion $\dot{x} \in A_c(x),\ x(0) = 0$ is controllable,

c) for some $k \geq 1$, one has $A_c^k(0) = (-A)_c^k(0) = \mathbb{R}^n$.

Introducing the notation $\mathbf{co}\{V_j\}$ for the convex hull of the subsets $V_j \subset \mathbb{R}^n$, then the sets $A_p^k := A_c^k(0)$ and $A_m^k := (-A)_c^k(0)$ can be computed using the following algorithm:

General Controllability Algorithm:

$$
\begin{aligned}
\mathbf{U} &= \mathbf{co}\{B_i \mathfrak{U} \mid i = 1, \cdots, s\} \\
A_p^1 &= \mathbf{U}, \quad A_m^1 = -\mathbf{U}, \\
A_p^{k+1} &= \mathbf{co}\{A_i A_p^k + B_i \mathfrak{U} \mid i = 1, \cdots, s\}, \\
A_m^{k+1} &= \mathbf{co}\{-A_i A_m^k - B_i \mathfrak{U} \mid i = 1, \cdots, s\}.
\end{aligned}
$$

If the system is completely controllable then there is a k such that $A_p^k = A_m^k$.

Additional details can be found in the papers Bokor and Szabó (2003a); Bokor, Szabó and Szigeti (2007c); Stikkel et al. (2004); Szabó (2009).

The theoretical results concerning controllability in engineering application related to fault tolerant and reconfigurable control of vehicles, Gáspár, Szabó and Bokor (2008c,a); Gáspár, Szabó, Szederkényi and Bokor (2008d). In cooperation with the Department of Aerospace and Mechanics, University of Minnesota the developed algorithms were successfully applied in the controllability problem related to the longitudinal dynamics of a supercavitating torpedo, Bokor, Balas and Szabó (2006).

11.2 Stabilizability

The zero solution of the differential inclusion $\dot{x} \in A_c(x)$ is called asymptotically (weakly) stable if there exists a solution $x(t)$ such that for any $\epsilon > 0$ there is a $\delta > 0$ and $\Delta > 0$ such that if $||x(0)|| < \delta$ then $||x(t)|| < \epsilon$ holds for all $t \geq 0$ and if $||x(0)|| < \Delta$ then $\lim_{t \to \infty} x(t) = 0$ holds.

For a given set of non–autonomous (controlled) linear switched systems (11.1) we call *Generalized Piecewise Linear Feedback Stabilizability* (GPLFS) the problem of finding a closed-loop switching strategy with

- suitable linear feedbacks $u_i = K_{l_i} x, \quad i \in S$
- a switching law $\kappa(x) \in S, \ x \in \mathbb{R}^n$

that (weakly)stabilizes the system.

Thesis 2 (Chapter 5, Propositions 7,8, 9, 10).
Concerning stabilizability properties of switched linear time invariant systems the following results were established:

Stabilizability *A completely controllable linear switching system is globally asymptotically controllable, hence it is closed–loop stabilizable.*

Linear feedback *The completely controllable linear switching system* (11.1) *is generalized piecewise linear feedback stabilizable.*

Time driven stabilization *Completely controllable linear switched systems can be piecewise linear feedback stabilized using a periodic switching sequence. The feedback gains can be computed by obtaining a switching sequence that realizes the complete controllability and by solving an LMI.*

The reasoning behind introducing the concept of generalized piecewise linear feedback stabilizability is to separate the task of finding a suitable switching strategy and that of finding suitable control inputs with low complexity that stabilizes the system in closed–loop.

The main idea is to substitute the original stabilizable non–autonomous system by a stabilizable autonomous linear switched system that might contain more modes then the original one, by applying as control inputs a number of suitable static linear control feedbacks.

It was shown that for a controllable uncertain discrete–time switching system $x_{k+1} = A_i(\Delta)x_k + B_i(\Delta)u_k, \ u_k \in \mathbb{R}^m$ and a corresponding switching sequence $\sigma = (s_1, \cdots, s_M)$ such that $\mathcal{R}_\sigma = \mathbb{R}^n$ independently of Δ, there exist a positive definite matrix S, nonsingular matrices V_i and matrices F_i such that the following LMI is feasible:

$$\begin{bmatrix} S & A_{s_M} V_M + B_{s_M} F_M & \dots & 0 & 0 \\ (\bullet)^T & V_M + V_M^T & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & V_2 + V_2^T & A_{s_1} V_M + B_{s_1} F_1 \\ 0 & 0 & \dots & (\bullet)^T & V_1 + V_1^T - S \end{bmatrix} > 0$$

The system can be stabilized with the periodic switching signal defined by σ and the state feedback gains given by $K_i = F_i V_i^{-1}, \ i = 1, \cdots, M$.

Additional details can be found in the papers Szabó, Bokor and Balas (2007, 2008); Szabó (2009); Szabó, Bokor and Balas (2009c).

The motivating engineering problems that provide, among others, the applicational background of these stabilizability results were related to fault-tolerant reconfigurable control with multiple, possibly conflicting performance specifications, see Bokor, Szabó, Nádai and Rudas (2007b); Gáspár et al. (2009,a).

11.3 Parameter varying invariant subspaces

Thesis 3 (Chapter 6 and 7, Lemma 4, Proposition 11, 12, 13, 14, 15, 16, Algorithm AISAL, AISAK, ABISA, CAISA, CSA, USA).
An extension was given of the classical invariant subspaces defined for LTI systems to a parameter–varying context, i.e., for LPV systems. Moreover, if the parameter dependence is affine, a series of algorithms are provided for the effective computation of these subspaces.

Invariant subspaces: For LPV systems a subspace $\mathcal{V}$ is called parameter-varying invariant subspace for the family of the linear maps $A(\rho)$ (or shortly $\mathcal{A}$-invariant subspace) if

$$A(\rho)\mathcal{V} \subset \mathcal{V} \quad \text{for all } \rho \in \mathcal{P}.$$

Furthermore, if $\mathcal{B}$ denotes $\mathbf{Im}\, B(\rho)$, a subspace $\mathcal{V}$ is called parameter-varying (A, B)–invariant subspace (or shortly $(\mathcal{A}, \mathcal{B})$-invariant subspace) if for all $\rho \in \mathcal{P}$ any of the following equivalent conditions holds :

$$A(\rho)\mathcal{V} \subset \mathcal{V} + \mathcal{B}(\rho);$$

there exists a mapping $F \circ \rho : [0, T] \to \mathbb{R}^{m\times n}$ such that:

$$(A(\rho) + B(\rho)F(\rho))\mathcal{V} \subset \mathcal{V}.$$

The dual notion is the following: if $\mathcal{C}(\rho)$ denotes $\mathbf{Ker}\, C(\rho)$, a subspace $\mathcal{W}$ is called parameter-varying (C, A)-invariant subspace (or shortly $(\mathcal{C}, \mathcal{A})$-invariant subspace) if for all $\rho \in \mathcal{P}$ any of the following equivalent conditions holds:

$$A(\rho)(\mathcal{W} \cap \mathcal{C}(\rho)) \subset \mathcal{W};$$

there exists a mapping $G \circ \rho : [0, T] \to \mathbb{R}^{n\times p}$ such that:

$$(A(\rho) + G(\rho)C(\rho))\mathcal{W} \subset \mathcal{W}.$$

These definitions are suitable for quasi-LPV (qLPV) systems, too.

For the LPV case and with a constant B matrix one can get the following definition for the controllability subspace: $\mathcal{R}$ is called parameter-varying controllability subspace if there exists a constant matrix K and a parameter–varying matrix $F : [0, T] \to \mathbb{R}^{m\times n}$ such that

$$\mathcal{R} = \langle \mathcal{A} + B\mathcal{F} | \mathbf{Im}\, BK \rangle,$$

where the maximal $\mathcal{A}$-invariant subspace contained in a constant subspace $\mathcal{K}$ is denoted by $\langle \mathcal{K} | A(\rho) \rangle$. The family of controllability subspaces contained in a given subspace $\mathcal{K}$ has a maximal element $\mathcal{R}^*$.

The dual notion of parameter-varying controllability subspace is the following: $\mathcal{S}$ is called an unobservability subspace associated to an LPV system if there exists a constant matrix H and a parameter–varying matrix $G : \mathcal{P} \to \mathbb{R}^{n\times p}$ such that

$$\mathcal{S} = \langle \mathbf{Ker}\, HC | A(\rho) + G(\rho)C \rangle.$$

The family of unobservability subspaces associated to an LPV system containing a given subspace $\mathcal{L}$ has a minimal element denoted by $\mathcal{S}_*$.

Invariance Algorithms: from a practical point of view it is an important question to characterize these subspaces associated to an LPV system by a finite number of conditions. Assuming an affine structure of the matrix $A(\rho)$ one has the following algorithms for finding supremal $\mathcal{A}$-invariant subspaces in a given subspace $\mathcal{K}$ or containing a given subspace $\mathcal{L}$:

$\mathcal{A}$-$\mathcal{I}$nvariant $\mathcal{S}$ubspace $\mathcal{A}$lgorithm over $\mathcal{L}$

$$\begin{aligned} \mathcal{AISAL}: \quad & \mathcal{V}_0 = \mathcal{L}, \quad \mathcal{V}_{k+1} = \mathcal{L} + \sum_{i=0}^{N} A_i \mathcal{V}_k, \quad k \geq 0, \\ & \mathcal{V}^* = \lim_{k\to\infty} \mathcal{V}_k. \end{aligned}$$

Obviously the algorithm will stop after a finite number of steps, i.e., $\mathcal{V}^* = \mathcal{V}_{n-1}$. Similar to the LTI case the subspace $\mathcal{V}^*$ is denoted by $\langle \mathcal{A} | \mathcal{L} \rangle$'

By duality, one has the $\mathcal{A}$-$\mathcal{I}$nvariant $\mathcal{S}$ubspace $\mathcal{A}$lgorithm in $\mathcal{K}$:

$$\begin{aligned} \mathcal{AISAK}: \quad & \mathcal{W}_0 = \mathcal{K}, \quad \mathcal{W}_{k+1} = \mathcal{K} \cap \bigcap_{i=0}^{N} A_i^{-1} \mathcal{W}_k, \quad k \geq 0, \\ & \mathcal{W}^* = \lim_{k\to\infty} \mathcal{W}_k, \end{aligned}$$

where $A_i^{-1}\mathcal{W}_k$ denotes the inverse image of $\mathcal{W}_k$ under A_i^{-1}. The subspace $\mathcal{W}^*$ is denoted by $\langle \mathcal{K} | \mathcal{A} \rangle$.

The set of all $(\mathcal{A}, \mathcal{B})$-invariant subspaces contained in a given subspace $\mathcal{K}$, is an upper semilattice with respect to subspace addition which admits a maximum that can be computed using the $(\mathcal{A}, \mathcal{B})$-$\mathcal{I}$nvariant $\mathcal{S}$ubspace $\mathcal{A}$lgorithm:

$$\mathcal{ABISA}: \quad \mathcal{V}_0 = \mathcal{K}, \quad \mathcal{V}_{k+1} = \mathcal{K} \cap \bigcap_{i=0}^{N} A_i^{-1}(\mathcal{V}_k + \mathcal{B}).$$

The limit of this algorithm is denoted by $\mathcal{V}^*$ and its calculation needs at most n steps. The set of all $(\mathcal{C}, \mathcal{A})$-invariant subspaces – note that $\mathcal{C} = \mathbf{Ker}\, C$ – containing a given subspace $\mathcal{L}$, is a lower semilattice with respect to subspace intersection. This semilattice admits a minimum which can be computed from the $(\mathcal{C}, \mathcal{A})$-$\mathcal{I}$nvariant $\mathcal{S}$ubspace $\mathcal{A}$lgorithm that can be obtained by duality from the $\mathcal{ABISA}$ algorithm:

$$\mathcal{CAISA} \quad \mathcal{W}_0 = \mathcal{L}, \quad \mathcal{W}_{k+1} = \mathcal{L} + \sum_{i=0}^{N} A_i(\mathcal{W}_k \cap \mathcal{C}).$$

The limit of this algorithm will be denoted by $\mathcal{W}_*$. The minimal element of the family of parameter-varying unobservability subspaces containing a given subspace $\mathcal{L}$ can be computed as the result of the parameter-varying $\mathcal{U}$nobservability $\mathcal{S}$ubspace $\mathcal{A}$lgorithm ($\mathcal{USA}$) :

$$\mathcal{USA}: \quad \mathcal{S}_0 = \mathcal{X}, \quad \mathcal{S}_{k+1} = \mathcal{W}_* + \left(\bigcap_{i=0}^{N} A_i^{-1} \mathcal{S}_k \cap \mathcal{C} \right)$$

$$\mathcal{S}_* = \lim_{k \to \infty} \mathcal{S}_k$$

where $\mathcal{W}_*$ is computed by $\mathcal{CAISA}$.

By duality one can obtain the parameter-varying $\mathcal{C}$ontrollability $\mathcal{S}$ubspace $\mathcal{A}$lgorithm.

Additional details can be found in the papers Balas et al. (2002, 2003); Bokor, Szabó and Stikkel (2002a); Szabó et al. (2002).

Results of the research and the developed LPV algorithms were directly applied in solving vehicle control problems, such as the FDI filter design for a Boeing 747 aircraft, see Bokor, Szabó and Balas (2002c); Bokor, Szabó and Stikkel (2002a); Stikkel et al. (2003).

11.4 Dynamical inversion of LPV systems

Let us consider the class of LPV systems with m inputs and p outputs that can be described as:

$$\begin{aligned} \dot{x}(t) &= A(\rho(t))x(t) + B(\rho(t))u(t) \\ y(t) &= Cx(t) \end{aligned}$$

where

$$\begin{aligned} A(\rho(t)) &= A_0 + \rho_1(t)A_1 + \ldots + \rho_N(t)A_N, \\ B(\rho(t)) &= B_0 + \rho_1(t)B_1 + \ldots + \rho_N(t)B_N, \end{aligned}$$

and the dimension of the state space is supposed to be n.

It is assumed that each parameter ρ_i ranges between known external values $\rho_i(t) \in [\underline{\rho}_i, \overline{\rho}_i]$ and the parameter set that contains all $(\rho_1(t), \cdots, \rho_N(t))$, where $t \in [0, T]$ will be denoted by $\mathcal{P}$.

Thesis 4 (**Chapter 9, Proposition 20, 21, 22).**
A method was established for the computation of the dynamical inverse system corresponding to the class of (q)LPV systems with affine parameter dependence. Based on the dynamical inversion algorithm an unknown input observer and a controller that solves the output tracking problem was designed.

The LPV system is left-invertible if

$$\mathcal{V}^* \cap \mathcal{B} = 0,$$

and it is right invertible if

$$\mathcal{S}_* + \mathcal{V}^* = \mathcal{X},$$

where $\mathcal{V}^*$ is the maximal $(\mathcal{A}, \mathcal{B})$-invariant subspace contained in $\mathcal{C} = \mathbf{Ker}\, C$. The minimal $(\mathcal{C}, \mathcal{A})$-invariant subspace containing $\mathcal{B} = \mathbf{Im}\, B$ is denoted by $\mathcal{S}_*$.

To construct the dynamical inverse, one can always choose a coordinate transform of the form

$$z = Tx, \text{ where } \quad T = \begin{bmatrix} \mathcal{V}^{*\perp} \\ \Lambda \end{bmatrix}, \Lambda \subset \mathcal{B}^\perp.$$

Accordingly, the system will be decomposed to:

$$\begin{aligned} \dot{\xi} &= A_{11}(t)\xi + A_{12}(t)\eta + \bar{B}(t)u \\ \dot{\eta} &= A_{21}(t)\xi + A_{22}(t)\eta \\ y &= \bar{C}\xi. \end{aligned}$$

Follows, that applying a suitable feedback

$$u = F_2(t)\eta + v,$$

such that $\mathcal{V}^*$ is $(\mathcal{A} + \mathcal{B}F, \mathcal{B})$ invariant, one can obtain the system:

$$\begin{aligned} \dot{\xi} &= A_{11}(t)\xi + \bar{B}v \\ y &= \bar{C}\xi. \end{aligned}$$

By the maximality of $\mathcal{V}^*$ follows that both ξ and v can be expressed as functions of y and its derivatives. With $\tilde{y} = \mathcal{S}\xi$, where

$$\tilde{y} = \left[y_1, \cdots, y_1^{(\gamma_1)}, \cdots, y_p, \cdots, y_p^{(\gamma_p)} \right]^T$$

one has $v = \bar{B}^{-1}\mathcal{S}^{-1}(\dot{\tilde{y}} - \dot{\mathcal{S}}\mathcal{S}^{-1}\tilde{y} - \mathcal{S}\bar{A}_{11}\mathcal{S}^{-1}\tilde{y})$, i.e.,

$$\begin{aligned} \dot{\eta} &= A_{22}\eta + A_{21}\xi \\ u &= F_2\eta + \bar{B}^{-1}\mathcal{S}^{-1}(\dot{\tilde{y}} - (\dot{\mathcal{S}}\mathcal{S}^{-1} + \mathcal{S}A_{11}\mathcal{S}^{-1})\tilde{y}). \end{aligned}$$

Given an output y_d to be tracked the tracking controller has the following structure:

$$\begin{aligned} \dot{\bar{\eta}} &= A_{22}\bar{\eta} + A_{21}\mathcal{S}^{-1}\tilde{y}_d + \Gamma_1\tilde{e} \\ \bar{u} &= F_2\bar{\eta} + \lambda(\tilde{y}_d) + \Gamma_2\tilde{e}, \end{aligned}$$

with $\lambda(\tilde{y}_d) = \bar{B}^{\{-1\}}\mathcal{S}^{-1}(\dot{\tilde{y}} - \dot{\mathcal{S}}\mathcal{S}^{-1}\tilde{y} - \mathcal{S}A_{11}\mathcal{S}^{-1}\tilde{y})$ and suitable, possibly parameter varying, gains Γ_1, Γ_2 that are selected to improve the performance of the controller.

Further details can be found in the papers Balas et al. (2004); Edelmayer et al. (2003, 2004, 2009); Szabó et al. (2003a).

The results were used in engineering applications, such as reconfigurable fault detection controls of vehicle, fault tolerant active suspension design, see Szabó et al. (2003); Gáspár, Szabó and Bokor (2007, 2008f); Gáspár et al. (2009). The developed algorithms were also successfully applied in the dynamic inversion based controller design for stabilizing the primary circuit pressurizer at the Paks Nuclear Power Plant Hungary, see Gáspár et al. (2006); Szabó et al. (2005).

11.5 Bimodal systems

Thesis 5 (Chapter 8, Lemma 5, Proposition 17 , 18 and 19).
A controllability decomposition was established for bimodal systems that have a well defined relative degree. It was shown that such a bimodal system is completely controllable if and only if a given subsystem of the controllability decomposition is completely controllable. It turns out that the latter is equivalent to the controllability of an input constrained open–loop switching system. If the bimodal system is globally controllable, then it is asymptotically stabilizable.

A *bimodal piecewise linear system* is a switching system where the switching law defined by a division of the state space by a hyperplane $\mathcal{C}$, i.e.,

$$\dot{x}(t) = \begin{cases} A_1 x(t) + B_1 u(t) & \text{if } x \in \mathcal{C}_-, \\ A_2 x(t) + B_2 u(t) & \text{if } x \in \mathcal{C}_+. \end{cases}$$

The initial state of the system at t_0 is determined by the initial state $x_0 = x(t_0)$ and the initial mode $s_0 \in \{1, 2\}$ in which the system is found at t_0. If $y_s = Cx$ defines the decision vector then $\mathcal{C} = \mathbf{Ker}\, C = \{x \mid Cx = 0\}$, $\mathcal{C}_+ = \{\, x \mid Cx \geq 0 \,\}$ and $\mathcal{C}_- = \{\, x \mid Cx \leq 0 \,\}$, respectively. The state matrices are constant and of compatible dimensions, B_1, B_2 having full column rank.

Let us suppose that the relative degree corresponding to the output y_s and the ith mode is r_i, i.e., $y_s^{(k)} = CA_i^k x$, $k < r_i$ and $y_s^{(r_i)} = CA_i^{r_i} x + CA_i^{r_i - 1} B_i u$ with $CA_i^{r_i - 1} B_i \neq 0$. If $r_1 = r_2 = r$ – when the system is always well posed – the the bimodal system can be written as

$$\dot{\eta} = \begin{cases} P_1 \eta + R_1 y_s + Q_1 \tilde{u}_1 & \text{if} \quad y_s \geq 0 \\ P_2 \eta + R_2 y_s + Q_2 \tilde{u}_2 & \text{if} \quad y_s \leq 0 \end{cases}$$
$$\dot{\xi} = \begin{cases} A_r \xi + B_r v_1 & \text{if} \quad y_s \geq 0 \\ A_r \xi + B_r v_2 & \text{if} \quad y_s \leq 0 \end{cases} .$$

The bimodal system can be transformed, via a state transform and suitable feedbacks, to

$$\dot{\eta}_1 = \begin{cases} P_{1,1} \eta_1 + \tilde{R}_1 y_s + \tilde{Q}_1 u_1 & \text{if} \quad y_s \geq 0 \\ P_{2,1} \eta_1 + \tilde{R}_2 y_s + \tilde{Q}_2 u_2 & \text{if} \quad y_s \leq 0 \end{cases} , \tag{11.2}$$
$$\dot{\eta}_2 = \begin{cases} P_{1,2} \eta_2 + R_1 y_s & \text{if} \quad y_s \geq 0 \\ P_{2,2} \eta_2 + R_2 y_s & \text{if} \quad y_s \leq 0 \end{cases} , \tag{11.3}$$
$$\dot{y}_s = v, \tag{11.4}$$

where the subsystem (11.2) is controllable on $\mathcal{C}$ using open–loop switchings. Thus, this decomposition can be viewed as a controllability decomposition of the bimodal LTI system where the study of the controllability of the original bimodal system reduces to controllability of the bimodal system formed by (11.3) and (11.4).

The bimodal system (11.3), (11.4) can be seen as a dynamic extension of

$$\dot{\eta}_2 = P_{i,2} \eta_2 + \bar{R}_{i,2} w, \quad i \in \{\, 1, 2 \,\}, \quad w \geq 0. \tag{11.5}$$

If the points η_0 and η_f can be connected by a trajectory of the linear system $\dot{\eta} = P\eta + Rw$ using nonnegative control $w \geq 0$ then, for a given r, they can be also connected using a smooth nonnegative control $\omega \geq 0$ with prescribed end points, i.e., $\omega^{(k)}(0) = \omega_{0,k}$ and $\omega^{(k)}(T_f) = \omega_{T_f,k}$ for $k = 0, 1, \cdots, r$. It follows that controllability of (11.3),(11.4) is equivalent to controllability of (11.5). Moreover the bimodal system (11.3), (11.4) is stabilizable if and only if the corresponding sign constrained open–loop switching system (11.5) is stabilizable.

If the bimodal system has continuous dynamics, i.e., $P_1 = P_2 = P$, then the system

$$\dot{x} = Px + Rw \quad w \in \mathbb{R}^2_+$$

is stabilizable if and only if the unconstrained system is stabilizable and all real eigenvectors v of P^T corresponding to a nonnegative eigenvalue of P^T have the property that $R^T v$ has both positive and negative components.

Additional details can be found in the papers Bokor, Szabó and Balas (2007, 2006a,b); Bokor and Szabó (2009).

The engineering applications that provide the motivation background for the research of bimodal systems were related to control of the hydraulic actuator of an active suspension system and the controllability study for a high speed supercavitating underwater vehicle, see Bokor, Szabó and Balas (2006b, 2007); Gáspár, Szabó and Bokor (2008a); Gáspár et al. (2009a).

12 Conclusions

Demands imposed by a series of engineering applications have motivated the controllability and stabilizability study concerning hybrid systems of the thesis. Although the presented results are formulated in theoretical terms they were successfully used, however, in practice as the numerous applicational examples contained in the corresponding publications illustrates. The design of an active suspension system for heavy vehicles and problems related to fault-tolerant reconfigurable control with multiple, possibly conflicting performance specifications provided the main field where the result were applied, Bokor, Szabó, Nádai and Rudas (2007b); Gáspár, Szabó and Bokor (2008); Gáspár, Szederkényi, Szabó and Bokor (2008b); Szabó, Bokor and Balas (2008); Bokor, Szabó and Nádai (2009a); Gáspár et al. (2009).

The target of the research presented by the thesis is placed at the forefront of modern control theory. The work extends the formulation of basic properties of LTI control systems originated from R.E. Kalman, such as controllabiliy and stabilizability, to a special class of switched systems, the bimodal systems. A main result of the research states that controllability of bimodal systems is equivalent to controllability of a corresponding open-loop switched system having sign constraint control inputs. Moreover, using geometric tools an algebraic condition that describes controllability and extends the Kalman rank test was given.

The study of bimodal systems was motivated by an application representing a true emerging technology, Bokor, Balas and Szabó (2006); Bokor, Szabó and Balas (2006a,b, 2007). The research concerning controllability study of a high speed supercavitating underwater vehicle was done in cooperation with the Department of Aerospace and Mechanics, University of Minnesota, headed by Prof. Gary Balas. The research was supported by the Office of Naval Research through the project "Stability and Control of Very High Speed Cavity Running Bodies". The designed control algorithms were applied on the special test-field of the project in Minneapolis.

The basic geometric view present in the controllability study is more accentuated in the second half of the thesis. This part presents results concerning parameter-varying invariant subspaces, a concept obtained by extending the notion of invariant subspaces of LTI systems and applying it to the class of (q)LPV systems. In the solution of engineering applications it is a central issue to use efficient design tools. Dealing with problems raised by the practice the work aims to provide applicable solutions, both theoretical procedures and practical algorithms. The research work revealed new methods which has been proven to be useful in the design of control solutions that satisfy more efficiently the practical need for robust and fault-tolerant systems. Results of the LTI control theory were extended to LPV systems which makes possible the application of current efficient optimization techniques based on LMIs.

The thesis provides a set of algorithms to obtain the different parameter-varying invariant subspaces for the case of affine parameter dependence. Based on these tools in the (q)LPV context solutions to a series of basic control problems, such as unknown input detectability, disturbance decoupling, output tracking, are presented. In the development of these design methods the geometric approach has played a central role.

Results of the research and the developed LPV algorithms were directly applied in solving vehicle control problems, such as preventing lane departure by asymmetric braking or rollover prevention. As an example, a (q)LPV based detection algorithm was provided that finally in the fault detection for the longitudinal dynamics of the airplanes (Boeing 747) was applied, Bokor, Szabó and Stikkel (2002a); Balas et al. (2004); Szabó et al. (2003); Bokor and Szabó (2009).

An other application example is from process engineering. Based on the developed dynamical inversion techniques a set-point tracking control was designed for stabilizing the primary circuit pressurizer at the Paks Nuclear Power Plant, Szabó et al. (2005); Gáspár et al. (2006). The implemented controller is still in operation on all four blocks of the power plant. Using the new controller the variation of the pressure in the primary circuit was reduced from the maximal interval of 1 bar to 0.25 bar in a wide range of operational conditions. The implemented control scheme demonstrates that significant improvement of the performance can be achieved by a combined application of accurately identified mathematical models and controllers based on modern principles without the need for a costly change of the technological environment.

Since the basic topics of control theory, such as controllability, geometrical system theory, are revisited by the research work, the provided theoretical methods and practical algorithms can be used through the educational activity. The results demonstrate directly the applicability and impact of theoretical concepts to the solution of practical, engineering problems.

The thesis contains the results of a research work that lasts a decade. However, there are still a lot of problems related to this relatively narrow field, motivated by real world applications, to solve. Considering a quadratic performance criteria for controlled linear switched systems is a relatively new topic, where only a few preliminary results for discrete time switched systems are available. An extension of the bimodal class, the cone-wise systems, i.e., systems with a state space having a conic partition and on each of the individual partitions the dynamics being linear, is also a recent topic with some early results for the planar setting. Related to robust invariant subspaces, a combination of the geometric based methods with other techniques that aims robustness and less conservative solutions, is a current research topic. There are also a series of problems concerning reachability set computations and controllability problems combined with a required performance level for the reconfiguration of controls, cast as (q)LPV systems. In the solution of these problems the starting points are the techniques and methods used through the thesis.

Part V

Appendix

A Linear time varying systems

Let us consider the state dynamics of a controlled linear time varying (LTV) system:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{A.1}$$

where $x(t) \in \mathcal{X} \subset \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input while the initial condition is $x_0 = x(t_0)$. The measured signals are obtained by a linear readout map $y(t) = C(t)x(t)$, with $y \in \mathbb{R}^p$.

A convenient way to study all solutions of a linear equation on the interval $[\sigma, \tau]$, for all possible initial values simultaneously, is to introduce the corresponding transition matrix $\Phi(\tau, \sigma)$:

$$x(\tau) = \Phi(\tau, \sigma)x(\sigma) + \int_\sigma^\tau \Phi(\tau, t)B(t)u(t)\mathrm{d}t = \Phi(\tau, \sigma)(x_0 + \int_\sigma^\tau \Phi(\sigma, t)B(t)u(t)\mathrm{d}t),$$

where $\Phi(t, t_0)$ is nonsingular and $\Phi(t, t_0) = X(t)X^{-1}(t_0)$ with $\dot{X}(t) = A(t)X(t),\ X(t_0) = \mathbb{I},\ X(t) \in \mathbb{R}^{n\times n}$. The inverse map $Q(t) = X^{-1}(t)$ obeys to the equation $\dot{Q}(t) = -Q(t)A(t)$ with $Q(t_0) = \mathbb{I}$.

A diffeomorphism $T(t)$ defines a time varying coordinate change[1] $z = Tx$ in the state space. The dynamic equation transforms as:

$$\dot{z} = (\dot{T}T^{-1} + TAT^{-1})z + TBu.$$

By using the Lyapunov transformation defined by $Q(t) = X^{-1}(t)$, one has the equivalent system $\dot{z} = Q(t)B(t)u(t)$. Recall that $\Phi(\sigma, t) = X(\sigma)X^{-1}(t)$, i.e.,

$$\dot{\bar{z}} = \Phi(\sigma, t)B(t)u(t),$$

with $\bar{z} = X(\sigma)z$.

Thus in this new coordinate system controllability reduces to the solvability study of the equation:

$$\bar{z}_0 = -\int_\sigma^\tau \Phi(\sigma, t)B(t)u(t)\mathrm{d}t$$

for a suitable finite τ.

[1] Lyapunov transformation; the corresponding dynamics are called Lyapunov equivalent.

A.1 Linear affine dynamics

For affine time dependency $A(t) = \sum_{i=1}^{N} \rho_i(t) A_i$ the fundamental matrix can be given, at least locally, in terms of the *coordinates of second kind* (Wei and Norman; 1964), i.e., the solutions of the Wei–Norman equation:

$$\dot{g}(t) = (\sum_{i=1}^{K} e^{\Gamma_1 g_1} \cdots e^{\Gamma_{i-1} g_{i-1}} E_{ii})^{-1} \rho(t), \quad g(0) = 0. \tag{A.2}$$

Here $\rho(t) = [\rho_1(t), \ldots, \rho_N(t)]^T$ and $\{\hat{A}_1, \ldots, \hat{A}_K\}$ is a basis of the Lie-algebra $\mathcal{L}(A_1, \ldots, A_N)$, the structure matrices $\Gamma_i = [\gamma_{i,j}^l]_{l,j=1,\cdots,K}$ of the algebra are given by $[\hat{A}_i, \hat{A}_j] = \sum_{l=1}^{K} \gamma_{i,j}^l \hat{A}_l$ and E_{ii} is the matrix with a single nonzero unitary entry at the i-th diagonal element.

Locally, the fundamental matrix is given by the expression:

$$\Phi(t) = e^{g_1(t)\hat{A}_1} e^{g_2(t)\hat{A}_2} \cdots e^{g_n(t)\hat{A}_n}, \tag{A.3}$$

and generally it is not available in closed form.

c-excited systems

Exploiting the affine structure and using the Peano–Baker formula for the transition matrix, i.e.,

$$\Phi(t,\sigma) = \mathbb{I} + \int_{\sigma}^{t} A(s_1) ds_1 + I_1(t,\sigma) + \cdots + I_l(t,\sigma) + \cdots ,$$

where

$$I_l(t,\sigma) = \int_{\sigma}^{t} \cdots \int_{\sigma}^{s_l} A(s_1) \cdots A(s_{l+1}) ds_{l+1} \cdots ds_1,$$

one can give an upper bound of the reachability (sub)space.

Let us consider systems with constant B and such that $A(t)$ has an affine structure; then the fundamental matrix $Q(t)$ can be written as

$$Q(t) = \sum_{n_1=0}^{n-1} \cdots \sum_{n_K=0}^{n-1} \hat{A}_1^{n_1} \ldots \hat{A}_K^{n_K} \psi_{n_1,\cdots,n_K}(t). \tag{A.4}$$

Introducing the multi-index notation $\hat{A}^{\mathrm{i}} := \hat{A}_1^{i_1} \ldots \hat{A}_K^{i_K}$, with $\mathbf{K} := \{0, 1, \cdots n-1\}^K$ and $\mathbf{i} := (i_1, \cdots, i_K)$, let us choose a linearly independent set of matrices from the set $\{\hat{A}^{\mathrm{i}} \,|\, \mathbf{i} \in \mathbf{K}\}$, say $\{\hat{A}^{\mathrm{j}} \,|\, \mathbf{j} \in \mathbf{j},\ \mathbf{j} \subset \mathbf{K}\}$. For the sake of simplicity, let us assume that $\mathbb{I}$ is a member of this basis, i.e., one can impose the condition that $[\varphi_{\mathrm{j}}(0)]_{\mathrm{j}\in\mathrm{j}}$ is the first canonical unit vector. With these notations, one has

$$Q(t) = \sum_{\mathrm{j}\in\mathrm{j}} \hat{A}^{\mathrm{j}} \varphi_{\mathrm{j}}(t). \tag{A.5}$$

Note, that the system $\{\varphi_{\mathrm{j}}(\sigma)\,|\,\mathrm{j}\in\mathbf{j}\}$ is not necessarily linearly independent.

Recall that $\sum_{i=0}^{N} A_i\rho_i(t) = \sum_{i=0}^{K}\hat{A}_i\hat{\rho}_i(t)$ and denote by Λ_i the matrices for which $\hat{A}_i[\hat{A}_{\mathrm{j}}]_{\mathrm{j}\in\mathrm{j}} = [\hat{A}_{\mathrm{j}}]_{\mathrm{j}\in\mathrm{j}}(\mathbb{I}_n\otimes\Lambda_i)$. Then

$$A(t)[\hat{A}_{\mathrm{j}}]_{\mathrm{j}\in\mathrm{j}} = [\hat{A}_{\mathrm{j}}]_{\mathrm{j}\in\mathrm{j}}(\mathbb{I}_n\otimes\Lambda(t)), \tag{A.6}$$

where $\Lambda(t) = \sum_{i=0}^{K}\hat{\rho}_k(t)\Lambda_i$.

Putting all these things together, it follows that the system $\{\varphi_{\mathrm{j}}(\sigma)\,|\,\mathrm{j}\in\mathbf{j}\}$ is the first column of the fundamental matrix associated to the equation

$$\dot{\tilde{Q}} = -\tilde{Q}\Lambda(t), \quad \tilde{Q}(0) = \mathbb{I}. \tag{A.7}$$

Note, that from this derivation the system $\{\varphi_{\mathrm{j}}(\sigma)\,|\,\mathrm{j}\in\mathbf{j}\}$ is not necessarily unique, but our choice satisfy (A.6).

Since

$$X(\sigma)^{-1}W(\sigma,\tau)X(\sigma)^{-*} = \int_{\sigma}^{\tau}[\,\hat{A}_{\mathrm{j}}B\,]_{\mathrm{j}\in\mathrm{j}}[\,\varphi_{\mathrm{j}}(s)\,]_{\mathrm{j}\in\mathrm{j}}[\,\varphi_{\mathrm{j}}(s)\,]^{*}_{\mathrm{j}\in\mathrm{j}}[\,\hat{A}_{\mathrm{j}}B\,]^{*}_{\mathrm{j}\in\mathrm{j}}\mathrm{d}s,$$

and subspace $\mathcal{R}_{\mathcal{A},\mathcal{B}}$ is exactly the image space of the matrix

$$R_{\mathcal{A},\mathcal{B}} := [\,\hat{A}_{\mathrm{j}}B\,]_{\mathrm{j}\in\mathrm{j}}. \tag{A.8}$$

one has

$$W(\sigma,\tau) = R_{\mathcal{A},\mathcal{B}}(\int_{\sigma}^{\tau}[\,\varphi_{\mathrm{j}}(s)\,]_{\mathrm{j}\in\mathrm{j}}[\,\varphi_{\mathrm{j}}(s)\,]^{*}_{\mathrm{j}\in\mathrm{j}}\mathrm{d}s)R^{*}_{\mathcal{A},\mathcal{B}}.$$

It is clear that if the system $\{\varphi_{\mathrm{j}}(\tau)\,|\,\mathrm{j}\in\mathbf{j}\}$ is linearly independent then $rank W(\sigma,\tau) = rank R_{\mathcal{A},\mathcal{B}}$, i.e., the system is c-exciting.

Suppose now that $rank R_{\mathcal{A},\mathcal{B}} = m$, where $m\le n$, and let us consider the singular value decomposition $R_{\mathcal{A},\mathcal{B}} = USV^*$ of this matrix. Then

$$rank W(\sigma,\tau) = rank[\,\mathbb{I}_m\,0\,](\int_{\sigma}^{\tau}[\,\tilde{\varphi}_{\mathrm{j}}(s)\,]_{\mathrm{j}\in\mathrm{j}}[\,\tilde{\varphi}_{\mathrm{j}}(s)\,]^{*}_{\mathrm{j}\in\mathrm{j}}\mathrm{d}s)[\,\mathbb{I}_m\,0\,]^{*},$$

where $[\,\tilde{\varphi}_{\mathrm{j}}(s)\,]_{\mathrm{j}\in\mathrm{J}} = V^*[\,\varphi_{\mathrm{j}}(s)\,]_{\mathrm{j}\in\mathrm{j}}$. This set of functions can be chosen as the first column of the fundamental matrix associated to the equation:

$$\dot{\Pi} = -\bar{\Lambda}(t)\Pi \quad \Pi(0) = V^*, \tag{A.9}$$

with $\bar{\Lambda}(t) = V^*\Lambda(t)V$. It follows that if the functions $\{\tilde{\varphi}_0,\cdots,\tilde{\varphi}_m\}$ are linearly independent, then $rank W(\sigma,\tau) = rank R_{\mathcal{A},\mathcal{B}}$.

To conclude this section, a relation will be investigated between the functions $[\,\varphi_{\mathrm{j}}(s)\,]_{\mathrm{j}\in\mathrm{j}}$ and the coordinate functions g_i of the Wei–Norman formula. In the special case when $g(t) = t$ and A has

n distinct eigenvalues the computation of the coefficients α_i in $e^{tA} = \sum_{i=0}^{n-1} \alpha_i(t) A^i$, is relatively easy, see e.g., Vidyasagar (1970):

$$\alpha_{i-1}(t) = \sum_{j=1}^{n} \left(\frac{\sum_{k=i}^{n} a_k \lambda_j^{k-i}}{\sum_{l=1}^{n} l a_l \lambda_j^{l-1}} \right) e^{\lambda_j t}.$$

with $P(s) = \sum_{i=0}^{n} a_i s^i = \prod_{j=1}^{n} (s - \lambda_j)$, the characteristic polynomial of A, and λ_j the distinct eigenvalues.

The general case is more involved; let

$$P(s) = (s - \lambda_1)^{q_1+1} \cdots (z - \lambda_k)^{q_k+1}$$

be the characteristic polynomial of the complex square matrix A, $P_p(s) := \frac{Q(s)}{(s-\lambda_p)^{q_p+1}}$. Let $b_{p,n}$ be the n-th Taylor coefficient of $\frac{1}{P_p(s)}$ at $s = \lambda_p$. Consider an entire function, i.e., a complex-valued function that is holomorphic over the whole complex plane, f and let $Q(f(s), s) \in \mathbb{C}[s]$ be $P(s)$ times the singular part of $\frac{f(s)}{Q(s)}$.

Hermite Lemma: *A result due to Hermite reveals that:*

h1. $b_{p,n} = (-1)^n \sum_{\substack{\beta_p = 0 \\ |\beta| = n}} \prod_{\substack{j=1,\dots,k \\ j \neq p}} \binom{q_j + \beta_j}{q_j} \frac{1}{(\lambda_p - \lambda_j)^{q_j+1+\beta_j}}$ *where* $|\beta| := \beta_1 + \cdots + \beta_k$ *and* β *runs over* $\mathbb{N}^k$,

h2. $Q(f(s), s) = \sum_{p=1}^{k} \sum_{q=0}^{q_p} \sum_{j=0}^{q} \frac{f^{(j)}(\lambda_p)}{j!} b_{p,q-j} (s - \lambda_p)^q P_p(s)$,

h3. $f(A) = Q(f(s), A)$.

Accordingly, then one can define functions $\varphi_j : \mathbb{R} \to \mathbb{C}$ for $0 \le j < d$ such that

$$Q(e^{ts}, s) = \varphi_{n-1}(t) s^{n-1} + \cdots + \varphi_1(t) s + \varphi_0(t).$$

By the Wei-Norman theorem, at least locally, the computation of the fundamental matrix $X(t)$ can be done by the product of matrices of the form $e^{g(t)A}$, namely

$$X(t) = e^{g_1(t)\hat{A}_1} e^{g_2(t)\hat{A}_2} \cdot \ldots \cdot e^{g_l(t)\hat{A}_l}.$$

Substituting the formulae for each of the the matrix exponentials, i.e., $e^{g_j(t)\hat{A}_j} = \sum_{k=0}^{n-1} \gamma_k^j(t) \hat{A}_j^k$, where $\gamma_k^j(t) = \varphi_{k,j}(g_j(t))$, one has that

$$X(t) = \sum_{n_1=1}^{n-1} \cdots \sum_{n_l=1}^{n-1} \gamma_{n_1}^1(t) \cdot \ldots \cdot \gamma_{n_l}^l(t) \hat{A}_1^{n_1} \cdot \ldots \cdot \hat{A}_l^{n_l}.$$

Expressing the products in the basis determined by the multi-index sets $\mathbf{J}$, and $\mathbf{N}$, respectively, i.e., $A^{\mathrm{n}} = \sum_{\mathrm{j} \in \mathrm{j}} \hat{A}^{\mathrm{j}} \alpha_{\mathrm{j}}^{\mathrm{n}}$, one has $X(t) = \sum_{\mathrm{j} \in \mathrm{j}} \sum_{\mathrm{n} \in \mathbf{N}} \alpha_{\mathrm{j}}^{\mathrm{n}} \gamma_{\mathrm{n}}(t) A^{\mathrm{j}}$, with $\gamma_{\mathrm{n}}(t) = \gamma_{n_1}^1(t) \cdot \ldots \cdot \gamma_{n_l}^l(t)$, i.e.,

$$\varphi_{\mathrm{j}}(t) = \sum_{\mathrm{n} \in \mathbf{N}} \alpha_{\mathrm{j}}^{\mathrm{n}} \gamma_{\mathrm{n}}(t). \tag{A.10}$$

This expression makes possible, in principle, the verification whether these functions are linear independent. However, the computational burden and the encountered numerical problems are so high that a practical application of the method for a real-sized application is out of the question.

A.2 Connection to the general nonlinear theory

Time varying systems can be viewed as input affine nonlinear systems, by augmenting the state with the time variable as $\xi := [t,\ x^t]^T$ and rewriting the system equations as:

$$\dot{\xi} = g_0(\xi) + \sum_{i=1}^{m} g_i(\xi)u_i,$$

with $g_0(\xi) = \begin{bmatrix} 1 \\ A(t)x \end{bmatrix}$, $g_i(\xi) = \begin{bmatrix} 0 \\ B_i(t) \end{bmatrix}$, and B_i is the ith column of B.

A distribution Δ will be *invariant* on an open set U under the vector fields g_i if and only if $[g_i, \eta_j] = \frac{\partial \eta_j}{\partial \xi} g_i - \frac{\partial g_i}{\partial \xi} \eta_j \in \Delta(\xi)$, for all $\eta_j \in \Delta$ and $\xi \in U$, where $\eta_j,\ j = 1, \cdots dim(\Delta)$ are vector fields locally spanning Δ, see Remark 6.1 on pp. 44 of Isidori (1989).

Controllability depends on the rank of the smallest distribution that contains g and is invariant under the vector field f, given by the following algorithm : $\Delta_0 = g, \quad \Delta_{i+1} = \Delta_i + [f, \Delta_i]$ as the limiting distribution of $\Delta^* = \lim_{i \to \infty} \Delta_i$.

For the linear affine system the distribution Δ_i is spanned exactly by the vectors $B_i(t)$ given by the Silverman–Meadows algorithm.

If Δ^* is involutive, by the Frobenius theorem, one can determine the transformation that decomposes system equations in the controllability form. To do this, it is necessary to solve partial differential equations of the form $(\partial_x \lambda)\delta_j = 0$, where $\{\ \delta_j\}$ span the distribution Δ^*, for details see Isidori (1989).

B Vector Fields

$f : \mathbb{R}^n \to \mathbb{R}^n$ is a *smooth vector field* if all of its coordinate functions are real valued functions of $x^T = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^T$ with continuous partial derivatives of any order. These mappings may be represented in the form of n-dimensional column vectors of real valued functions. The dual object is called a *covector field*, which is a smooth mapping assigning to each point x an element of the dual space $(\mathbb{R}^n)^*$. A special covector field is the so-called *differential* of a real-valued function λ defined on an open subset U of $\mathbb{R}^n$:

$$d\lambda(x) := \frac{\partial \lambda}{\partial x} := \begin{bmatrix} \frac{\partial \lambda}{\partial x_1} & \frac{\partial \lambda}{\partial x_2} & \dots & \frac{\partial \lambda}{\partial x_n} \end{bmatrix}.$$

The derivative of λ along f is defined as

$$L_f \lambda(x) := d\lambda \cdot f = \sum_{i=1}^{n} \frac{\partial \lambda}{\partial x_i} f_i(x),$$

which is a real-valued function. The *Lie product* $[f, g]$ of two vector fields f and g is a vector field of the form

$$[f, g](x) = \frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x).$$

The last operation of frequent use involves a covector field ω and a vector field f:

$$L_f \omega(x) = f^T(x) \left[\frac{\partial \omega^T}{\partial x} \right]^T + \omega(x) \frac{\partial f}{\partial x}$$

and the result is a covector field, the derivative of ω along f.

The differential operations introduced above can be related to each other in the following way:

- if α, β are real-valued functions and f, g are vector fields then

$$[\alpha f, \beta g](x) = \alpha(x)\beta(x)[f, g](x) + (L_f \beta(x))\alpha(x) g(x) - (L_g \alpha(x))\beta(x) f(x),$$

- if α, β are real-valued functions f a vector field and ω a covector field then

$$\begin{aligned} L_{\alpha f} \beta\omega(x) =& \alpha(x)\beta(x)(L_f \omega(x)) + \beta(x)\langle \omega(x), f(x) \rangle d\alpha(x) \\ &+ (L_f \beta(x))\alpha(x)\omega(x). \end{aligned}$$

Suppose we have d smooth vector fields $f_1, \dots f_d$, all defined on the same open set U. The vectors $f_1(x), \dots, f_d(x)$ span a subspace of $\mathbb{R}^n$:

$$\Delta(x) := \text{span}\,\{f_1(x), \dots, f_d(x)\}$$

which is called a *smooth distribution*. Starting from the dual objects, if we have $\omega_1, \dots, \omega_d$ smooth covector fields we can define a subspace of $(\mathbb{R}^n)^*$:

$$\Omega(x) := \text{span}\,\{\omega_1(x), \dots, \omega_d(x)\}$$

and this mapping is called a *smooth codistribution*.

A distribution Δ is said to be **involutive** if $\tau_1, \tau_2 \in \Delta$ implies that $[\tau_1, \tau_2] \in \Delta$.

Let Δ be a distribution, the *annihilator* of Δ at point x is

$$\Delta^{\perp}(x) := \{w^* \in (\mathbb{R}^n)^* : w^*(v) = 0 \text{ for all } v \in \Delta(x)\}.$$

Analogously, for a codistribution Ω one has

$$\Omega^{\perp}(x) := \{v \in \mathbb{R}^n : w^*(v) = 0 \text{ for all } w^* \in \Omega(x)\}.$$

It is well-known that the dual space of $\mathbb{R}^n$ is isomorph with itself, i.e., for every $w^* \in (\mathbb{R}^n)^*$ there exists one and only one $w \in \mathbb{R}^n$ such that

$$w^*(v) = \langle w, v\rangle := w^T v.$$

The effect of a linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ on covectors, i.e., on the elements of $(\mathbb{R}^n)^*$ can be expressed as:

$$\tilde{A} : (\mathbb{R}^n)^* \to (\mathbb{R}^n)^* \qquad \tilde{A}(w^*) = wA.$$

Hence the notion $\mathcal{W}A$ makes sense for any "cosubspace" of covectors $\mathcal{W}$. This cosubspace, generated by the covectors $w_1^T, \dots, w_k^T$, is:

$$\mathcal{W} = \text{span}\{w_1^T, \dots, w_k^T\}$$

and it is often identified with the row image of the matrix W, with rows $w_1^T, \dots, w_k^T$.

A control system on a smooth n-dimensional manifold M is a collection $\mathcal{F}$ of smooth vector fields depending on independent parameters $w = [w_1, \cdots, w_m] \in \Xi \subset \mathbb{R}^m$ called control inputs such that $w(t)$ belongs to a suitable class of real valued functions $\mathcal{W}$, called admissible controls.

A dynamical system can be considered as a nonlinear polysystem of the form

$$\dot{x} = f(x(t), w(t)), \quad x(0) = 0, \tag{B.1}$$

where in general, it is assumed that $x \in M$ and $f(., w)$, $w \in \Xi$ is an analytic (smooth) vector field on M. It is supposed that M is an n-dimensional real analytic manifold (para-compact and connected).

Associated with the system (B.1), denote by $\mathcal{A}_{\mathcal{F}}(x,t)$ the set of all elements attainable from x at time t. For each $x \in M$, $\mathcal{A}_{\mathcal{F}}(x) = \cup_{t\geq 0}\mathcal{A}_{\mathcal{F}}(x,t)$.

Under the Lie bracket, and the pointwise addition, the space of all analytic vector fields on M becomes a Lie algebra; $Lie(\mathcal{F})$ denotes the subalgebra generated by $\mathcal{F}$. For each $q \in M$, $Lie_q(\mathcal{F})$ is a subspace of T_qM, the tangent space of M at q. A set of vector fields $\mathcal{F}$ on a connected smooth manifold M is called *bracket-generating* (full-rank) if $Lie_q\mathcal{F} = T_qM$ for all $q \in M$.

Families of vector fields $\mathcal{F}$ and $\mathcal{G}$ are said to be (strongly) *equivalent* if $Lie(\mathcal{F}) = Lie(\mathcal{G})$ and $\overline{\mathcal{A}_{\mathcal{F}}(q,T)} = \overline{\mathcal{A}_{\mathcal{G}}(q,T)}$ for all $q \in M$ and for all $T > 0$, where the overbar denotes the closure of the sets. The Lie Saturate $LS(\mathcal{F})$ of a family of vector fields $\mathcal{F}$ is the union of families strongly equivalent to $\mathcal{F}$.

In general it is difficult to construct the Lie saturate explicitly, however one can construct a completely ascending family of compatible vector fields – *Lie extension* – starting from a given set $\mathcal{F}$ of vector fields. A vector field f is called compatible with the system $\mathcal{F}$ if $\mathcal{A}_{\mathcal{F}\cup f}(q) \subset \overline{\mathcal{A}_{\mathcal{F}}(q)}$ for all $q \in M$. Since $LS(\mathcal{F})$ is a closed convex positive cone in $Lie(\mathcal{F})$, a possibility to obtain compatible vector fields is extension by convexification, see Jurdjevic (1997): for $f_1, f_2 \in \mathcal{F}$ and any nonnegative functions $\alpha_1, \alpha_2 \in C^{\infty}(M)$ the vector fields $\alpha_1 f_1 + \alpha_2 f_2$ is compatible with $\mathcal{F}$. If $LS(\mathcal{F})$ contains a vector space $\mathcal{V}$, then $Lie(\mathcal{V}) \subset LS(\mathcal{F})$.

B.1 Normal controllability

Let us denote by $e^{f_w t}x_0$ the solution of the equation $\dot{\xi} = f_w(\xi)$, $\xi(0) = x_0$. Then for a given vector field $\mathcal{F}$ one can consider the (positive) orbits of the vector field, i.e.,

$$\Phi^q_{\tau,x_0}(\omega)(T) = e^{f_{w_q}t_q}e^{f_{w_{q-1}}t_{q-1}}\cdots e^{f_{w_2}t_2}e^{f_{w_1}t_1}x_0$$

where $\tau = (t_1, t_2, \cdots t_q)$, $t_i \geq 0$ with $T = \sum_{j=1}^{q} t_j$ and $\omega = (w_1, w_2 \cdots w_q) \in \Xi^q$, $f_{w_i} \in \mathcal{F}$. We will use Φ^q_τ for $\Phi^q_{\tau,0}(\omega)$ with fixed ω.

A point $y \in M$ is called *normally reachable* from an $x \in M$ if there exist a finite sequence of vector fields $\{f_i, i = 1, \cdots q\}$ and $\bar{\tau} \in \mathbb{R}^q_+$ such that $\Phi^q_{\bar{\tau},x} = y$ and the mapping $\tau \in \mathbb{R}^q_+ \to \Phi^q_{\tau,x}$, which is defined in an open neighborhood of $\bar{\tau}$, has rank $n = \mathbf{dim}M$ at $\bar{\tau}$.

As a consequence of the surjective mapping theorem, Bartle (1976) Theorem 41.6, one has that there is a neighborhood V of y such that the points $z \in V$ are normally reachable points from x. Let us denote by $\mathcal{N}(x)$ the set of normally reachable points. It follows that if $\mathcal{N}(x)$ is not empty, then it has a nonempty interior. A fundamental result is Theorem 4.3 in Sussmann (1976):

Theorem 1: *Let $\mathcal{F}$ be a system of $\mathcal{C}^r$ vector fields on the $\mathcal{C}^{r+1}$ manifold M, $1 \leq r \leq \infty$. Then the following conditions are equivalent:*

i. $\mathcal{F}$ is controllable

ii. $\mathcal{F}$ is normally controllable

iii. M is connected and, for every $x \in M$, x is normally accessible from x.

Remark 16: *Further details concerning the relation between controllability and normal controllability can be found in Grasse (1985), too. In Grasse and Sussmann (1990) it is proved that globally controllable smooth systems are controllable by using piecewise constant controls. The key point here is that for a globally controllable system every point has the normal accessibility property. Actually the interior points of the reachability set are reachable by piecewise constant controls, for details see Sussmann (1987).*

B.2 Convex processes

A convex process A from $\mathbb{R}^n$ to itself is a set-valued map satisfying $\lambda A(x)+\mu A(y) \subset A(\lambda x+\mu y)$ for all $\lambda, \mu \geq 0$, or, equivalently, a set-valued map whose graph is a convex cone. A convex process is closed if its graph is closed and that it is strict if its domain is the whole space. With a strict closed convex process A one can associate the Cauchy problem for the differential inclusion: $\dot{x}(t) \in A(x(t)),\ x(0) = 0$, for details see Filipov (1960) and Aubin and Cellina (1984).

If $G \subset \mathbb{R}^n$, let us denote by G^+ its (positive) polar cone defined by

$$G^+ = \{p \in \mathbb{R}^n \mid \langle p, x\rangle \geq 0,\ \forall x \in G\}.$$

The transpose A^* of A is defined as the set-valued map defined by $p \in A^*(q) \Leftrightarrow \forall (x, y) \in Graph(A),\ \langle p, x\rangle \leq \langle q, y\rangle$. For $\lambda \in \mathbb{R}$ the eigenvectors v of A^* are the nonzero solutions of the inclusion $\lambda v \in A^*(v)$.

Motivated by the terminology used for linear systems we say that A satisfies the *rank condition* if the subspace spanned by the cone $A^k(0)$ is the whole space for some integer $k \geq 1$.

Theorem 2 (Frankowska et al. (1986)): *The following conditions are equivalent:*

a) the differential inclusion $\dot{x}(t) \in A(x(t)),\ x(0) = 0$ is controllable,

b) the differential inclusion is controllable at some time $T > 0$,

c) the rank condition is satisfied and A^* has no eigenvectors,

d) for some $k \geq 1$, one has $A^k(0) = (-A)^k(0) = \mathbb{R}^n$.

Controllability of a linear control system is equivalent to the controllability of the differential inclusion defined by $\dot{x}(t) \in Ax(t) + U,\ x(0) = 0$, with $U = \overline{co}(B\Omega)$ is a closed convex cone of controls, where $\overline{co}(S)$ denotes the closure of the convex hull of the set S, see Aubin and Cellina (1984). The adjoint inclusion is $-\dot{q}(t) \in A^T q(t),\ q(t) \in U^+$, see Frankowska et al. (1986).

C Geometry of LTI systems

Let us consider the LTI control system

$$\dot{x} = Ax + Bu$$

with the output

$$y = Cx$$

It is assumed that columns of the matrix $B \in \mathbb{R}^{n\times m}$ and the rows of the matrix C are linearly independent.

C.1 Brunovsky canonical form

The set of points that lies on the same trajectory with the origin is called the reachability (controllability) subspace. Let us denote the controllability subspace of the pair (A, B) by $\mathcal{R}(A, B)$.

The state space of the system is partitioned by the manifolds of type $x + \mathcal{R}(A, B)$, where $x \in X$. By definition, the points of two different manifolds cannot be joined by a trajectory.

Any controllable linear system can be effectively transformed to Brunovsky canonical form, Brunovsky (1970) by feedback and a change of state and input coordinates as asserted in the next result:

Theorem 3 (Generalized Brunovsky canonical form): *For every pair* (A, B)*, there exists a unique sequence of integers* $k = (k_1, k_2, \cdots k_m)$ *satisfying*

$$k_1 \geq k_2 \geq \cdots \geq k_m, \quad k_1 + k_2 + \cdots + k_m = n_c,$$

a linear transformation G *and invertible linear transformations* F *and* H *such that the pair*

$$(\tilde{A}, \tilde{B}) = (F(A - BH^{-1}G)F^{-1}, FBH^{-1})$$

is in Brunovsky normal form, i.e.,

$$\tilde{A} = \begin{bmatrix} A_{k_1} & 0 & \cdots & 0 & 0 \\ 0 & A_{k_2} & \cdots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \cdots & A_{k_m} & 0 \\ 0 & 0 & \cdots & 0 & A_J \end{bmatrix} \quad \tilde{B} = \begin{bmatrix} B_{k_1} & 0 & \cdots & 0 & 0 \\ 0 & B_{k_2} & \cdots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \cdots & B_{k_m} & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

with

$$A_{k_i} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \quad B_{k_i} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

and A_J a block diagonal matrix with Jordan blocks.

The Generalized Brunovsky form reveals two kinds of complete invariants: the controllability indices k_i of the controllable part of (A, B) and the invariant factors of the uncontrollable part of (A, B) shown in A_j.

An LTI system with unconstrained inputs is not only controllable on its controllability subspace, but it can also be driven on any sufficiently smooth trajectory that lies in the controllability subspace. However, if there is a constrain on the input u, this property might not hold, as the following small example shows:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u, \end{aligned}$$

if $u \geq 0$, then x_1 cannot be decreased.

Considering the measured output $y = Cx$ then by using feedback, and output injection one can obtain the Morse canonical form, i.e.,

$$(\tilde{A}, \tilde{B}, \tilde{C}) = (F(A + BH + KC)F^{-1}, FBH, GCF^{-1})$$

where

$$\tilde{A} = \begin{bmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & A_3 & 0 \\ 0 & 0 & 0 & A_4 \end{bmatrix} \quad \tilde{B} = \begin{bmatrix} 0 & 0 \\ B_2 & 0 \\ 0 & 0 \\ 0 & B_4 \end{bmatrix} \quad \tilde{C} = \begin{bmatrix} 0 & 0 & C_3 & 0 \\ 0 & 0 & 0 & C_4 \end{bmatrix}$$

with $(A_i, B_i), i = 2, 4$ Note that the transformed system is related to original system by an invertible state/input transformation U:

$$\begin{bmatrix} \tilde{x} \\ \tilde{u} \end{bmatrix} = \begin{bmatrix} F & 0 \\ G & H \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = U \begin{bmatrix} x \\ u \end{bmatrix}.$$

If there is a constraint on u, the feedback might not be implemented, so the system is equivalent through similarity with a canonical form that also contains coupling terms.

C.2 Controlled and conditioned invariance

In the absence of control action a subspace of the state space $\mathcal{X}$ is a locus of trajectories if and only if it is an A-invariant[1]set. The extension of this property to the case in which the control is

[1]For the details concerning the notions and propositions used in this section the interested reader is sent to Basile and Marro (2002) and Wonham (1985).

present and suitably used to steer the state along a convenient trajectory leads to the concept of (A, B)-*controlled invariant* subspace $\mathcal{V}$ defined as:

$$A\mathcal{V} \subset \mathcal{V} + \mathcal{B}, \quad \mathcal{B} = \mathbf{Im}\, B.$$

The dual of a controlled invariant subspace is an (A, C)—*conditioned invariant* subspace $\mathcal{S}$, which is defined as:

$$A(\mathcal{S} \cap \mathcal{C}) \subset \mathcal{S}, \quad \mathcal{C} = \mathbf{Ker}\, C.$$

The set of all (A, B)—controlled invariants $\mathcal{V}_{\mathcal{E}}$ contained in a given subspace $\mathcal{E}$ is an upper semilattice that admits a supremum, the *maximal* (A, B)—*controlled invariant* contained in $\mathcal{E}$, which will be denoted by $\mathcal{V}^*_{\mathcal{E}} = maxV(A, B, \mathcal{E})$. Similarly the set of all (A, C)—conditioned invariants $\mathcal{S}_{\mathcal{D}}$ containing a given subspace $\mathcal{D}$ is a lower semilattice that admits an infimum, the *minimal* (A, C)—*conditioned invariant* containing $\mathcal{D}$, which will be denoted by $\mathcal{S}^*_{\mathcal{D}} = \min S(A, C, \mathcal{D})$. These subspaces can be determined by efficient algorithms in finite steps.

A trajectory of the pair (A, B) can be controlled on $\mathcal{E}$ if and only if its initial state belongs to a controlled invariant contained in $\mathcal{E}$, hence in $\mathcal{V}_{\mathcal{E}}$. In general, for any initial state belonging to a controlled invariant $\mathcal{V}_{\mathcal{E}}$, it is possible not only to continuously maintain the state on $\mathcal{V}_{\mathcal{E}}$ by means of a suitable control action, but also to leave $\mathcal{V}_{\mathcal{E}}$ with a trajectory on $\mathcal{E}$ and to pass to some other controlled invariant contained in $\mathcal{E}$. On the other hand there exist controlled invariants that are closed with respect to the control, i.e., that cannot be exited by means of any trajectory on $\mathcal{E}$: these will be called self-bounded with respect to $\mathcal{E}$. An (A, B)—controlled invariant $\mathcal{V}$ contained in a subspace $\mathcal{E}$ is said to be self-bounded with respect to $\mathcal{E}$ if $\mathcal{V}^*_{\mathcal{E}} \cap \mathcal{B} \subset \mathcal{V}$.

The duals of the self-bounded controlled invariants are the self-hidden conditioned invariants: an (A, C)—conditioned invariant $\mathcal{S}$ containing a subspace $\mathcal{D}$ is said to be self-hidden with respect to $\mathcal{D}$ if $\mathcal{S} \subset \mathcal{S}^*_{\mathcal{D}} + \mathcal{C}$.

In general, however, it is not possible to reach any point of a controlled invariant from any other point (in particular, from the origin) by a trajectory completely belonging to it. In other words, given a subspace $\mathcal{E}$, by leaving the origin with trajectories belonging to $\mathcal{E}$, hence to $\mathcal{V}_{\mathcal{E}}$, (the maximal (A, B)—controlled invariant contained in $\mathcal{E}$), it is not possible to reach any point of $\mathcal{V}_{\mathcal{E}}$, but only a subspace of it, which is called the reachable set on $\mathcal{E}$ and denoted by $\mathcal{R}_{\mathcal{E}}$. It can be proved that $\mathcal{R}_{\mathcal{E}} = \mathcal{V}^*_{\mathcal{E}} \cap \mathcal{S}$ with $\mathcal{S} = \min \mathcal{S}(A, \mathcal{E}, \mathcal{B})$.

Let us denote by $\mathcal{V}^* = \max \mathcal{V}(A, B, \mathcal{C})$ the maximal (A, B)-controlled invariant subspace contained in $\mathcal{C}$ and by $\mathcal{S}_* = \min \mathcal{S}(A, C, \mathcal{B})$ the minimal (A, C)-conditioned invariant subspace containing $\mathcal{B}$.

Theorem 4 (Four Map Theorem): *Let us consider the state transformation* $\xi = T^{-1}x$ *defined by*

$$T = \begin{bmatrix} T_1 & T_2 & T_3 & T_4 \end{bmatrix},$$

with $\mathbf{Im}\ T_1 = \mathcal{V}^* \cap \mathcal{S}_*$ *and* $\mathbf{Im}\ [T_1\, T_2] = \mathcal{V}^*$, $\mathbf{Im}\ [T_1\, T_3] = \mathcal{S}_*$. *Then*

$$T^{-1}AT = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ 0 & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ 0 & 0 & A_{43} & A_{44} \end{bmatrix} \quad T^{-1}B = \begin{bmatrix} B_1 \\ 0 \\ B_3 \\ 0 \end{bmatrix} \quad CT = \begin{bmatrix} 0 & 0 & C_3 & C_4 \end{bmatrix},$$

where $A_{23} = \bar{A}_{23}C_3$ and $A_{43} = \bar{A}_{43}C_3$. Moreover by a suitable feedback A_{31} and A_{32} can be zeroed out.

C.3 Left and right invertibility

It is well known that the response of the triple (A, B, C) is related to initial state $x(0)$ and control function $u(t)$ by

$$y(t) = \Psi^{x_0}_{(A,B,C)}u = Ce^{At}x(0) + C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau = \Psi_{(A,B,C)}x(0) + \Phi^0_{(A,B,C)}u.$$

The term system invertibility denotes the possibility of reconstructing the input from the output function; more precisely the term *invertibility* refers to unknown-input invertibility, i.e., to the invertibility of map $\Phi^0_{(A,B,C)}$ such that $u(t) = (\Phi^0_{(A,B,C)})^{-1}\Phi^0_{(A,B,C)}(u(t))$.

When (A, C) is not *observable*(*reconstructable*), the initial or final state can be determined modulo the subspace

$$\mathbf{Ker}\, \Psi_{(A,B,C)} = \mathcal{Q}$$

where $\mathcal{Q}$ denotes the maximal $A-$invariant subspace contained in $\mathcal{C}$, which is called *unobservability subspace* (*unreconstructability subspace*). This means that the state canonical projection on $\mathcal{X}/\mathcal{Q}$ can be determined from the output function. $\mathcal{Q}$ is the locus of the free motions corresponding to the output function identically zero. A dynamical system is completely *unknown-input state observable* by means of differentiators if it is possible to determine its state x when an arbitrary short output segment y is given.

The subspace of unknown input state observability by means of differentiators is

$$\min \mathcal{S}(A^T, \mathbf{Ker}\, B^T, \mathbf{Im}\, C^T) = \max \mathcal{V}^{\perp}(A, B, C)$$

and the subspace of functional input observability is $B^T \min \mathcal{S}(A^T, \mathbf{Ker}\, B^T, \mathbf{Im}\, C^T)$. The orthogonal projection of the state on the subspace $\mathcal{V}^{*,\perp}$ can be deduced from the output and from its derivatives, moreover this is the greatest subspace where the orthogonal projection of the state can be recognized solely from the output. If the state is known the orthogonal projection of the input can be determined on $B^T\mathcal{V}^{*,\perp}$ and it cannot be recognized a greater subspace (it can be determined modulo $B^{-1,T}\mathcal{V}^*$).

Definition 6: *Assume that B has maximal rank. The system (A, B, C) with $x(0) = 0$ is said to be invertible (left-invertible) if, given any output function $y(t)$ defined on $[0, t_1]$, $t_1 > 0$ belonging to $\mathbf{Im}\, \Phi^0_{(A,B,C)}$, there exists a unique input function $u(t)$ such that $\Phi^0_{(A,B,C)}u(t) = y(t)$ holds, i.e., $\mathbf{Ker}\, T^0_{(A,B,C)} = 0$.*

The triple (A, B, C), with B having maximal rank, is unknown-state (zero-state) invertible if and only if it is unknown-state, unknown-input (zero-state, unknown-input) completely reconstructable.

A dynamic system exists which, connected to the system output and with initial state suitably set as a linear function of the system state (which is assumed to be known), provides tracking of the system state modulo $\mathcal{S}_*$. This system is is not necessarily stable. The observer equations, expressed in the basis that corresponds to the transformation $T = [T_1\ T_2]$, with $\mathbf{Im}\, T_1 = \mathcal{S}_*$, can be written as:

$$\begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u + \begin{bmatrix} G1 \\ G_2 \end{bmatrix} y,$$

where G is such that $(A + GC)\mathcal{S}_* \subset \mathcal{S}_*$. If the observer initial state is set according to $\eta(0) = T^{-1}x(0)$ a state estimate modulo $\mathcal{S}_*$ is derived.

An algebraic reconstructor with differentiators provides as output a state estimate z_1 modulo $\mathcal{V}^*$ and works if neither the initial state nor the input function is known, while the dynamic tracking device provides as z_2 a state estimate modulo $\mathcal{S}_*$, but requires the initial state to be known. A state estimate modulo $\mathcal{V}^* \cap \mathcal{S}_*$ is obtained as a linear function of the outputs of both devices, i.e., $z = Mz_1 + Nz_2$. This state reconstructor provides the maximal information on the system state when the input function is unknown and the initial state known, by observing the output in any nonzero time interval.

The term *functional controllability* denotes the possibility of imposing any sufficiently smooth (piecewise differentiable at least n times) output function by a suitable input function, starting at the zero state. Starting from the identity $y(t) = \Phi^0_{(A,B,C)}(\Phi^0_{(A,B,C)})^{-1}(y(t))$ it is also called *right invertibility*.

For multi input single output (MISO) systems a formal definition can be given as:

Definition 7: *Assume that C has maximal rank. The system $(A, B, C)$is said to be functionally controllable (right-invertible) if there exists an integer $\rho \geq 1$ such that, given any output function y with ρth derivative piecewise continuous and such that $y(0) = 0, \cdots y^{(\rho)}(0) = 0$, there exists at least one input function u such that $\Phi^0_{(A,B,C)}u = y$ holds. The minimum value of ρ satisfying the above statement is called the relative degree of the system.*

In order to define the *relative degree* for MIMO systems in geometric terms the following extension of functional output controllability is introduced:

Definition 8 (Constrained Functional Output Controllability): *A subspace $\mathcal{Y}^{(h)}$ is said to be a functional output controllability subspace with respect to the hth derivative if the output of the triple (A, B, C) can be driven along any trajectory y such that $y \in \mathcal{Y}^{(h)}$ with the hth derivative piecewise continuous.*

This is possible exactly when there exist an $(A, B)-$controlled invariant subspace $\mathcal{V}$ such that $\mathcal{Y}^{(h)} = C\mathcal{V}$. Let us consider $\mathcal{E} = C^{-1}\mathcal{Y}^{(h)}$ and $\mathcal{V}^{(h)}_{\mathcal{E}}$, the maximal $(A, B)-$controlled invariant subspace contained in $\mathcal{E}$ such that the output can be driven on $C\mathcal{V}^{(h)}_{\mathcal{E}}$ along any trajectory y with piecewise continuous hth derivative for all the initial states $x(0) \in \mathcal{V}^{(h)}_{\mathcal{E}}$.

Definition 9 (Multivariable Relative Degree): *The relative degree ρ_i of output y_i is defined as $\rho_i = h$ (if exists), where $\mathcal{Y}^{(h)} = C\mathcal{V}^{(h)}_{\mathcal{E}}$ assuming that $\mathcal{Y}^{(h)} = \{y \mid y_k = 0,\ k \neq i\}$.*

The functional controller is realizable in exactly the same way as the (left)inverse system, i.e., by a state reconstructor completed with a further differentiator stage and an algebraic part. Its dynamic part is asymptotically stable if and only if all the invariant zeros of (A, B, C) are stable. In this case, however, the input u corresponding to the desired output is not unique, in general. The difference between any two admissible input corresponds to a zero-state motion on $R_{\mathcal{V}^*} = \mathcal{V}^* \cap \mathcal{S}_*$ which does not affect the output, so that the functional controller can be realized to provide any one of the admissible inputs, for instance by setting to zero input components which, expressed in a suitable basis, correspond to forcing actions belonging to $\mathcal{V}^* \cap Im B$.

Left and right invertibility can be characterized in geometric terms as follows:

Theorem 5: *Triple* (A, B, C) *is left-invertible if and only if*

$$\mathcal{V}^* \cap \mathcal{B} = 0.$$

Condition of left-invertibility is equivalent to $\mathcal{V}^* \cap \mathcal{S}_* = 0$.

Theorem 6: *Let* $\mathcal{C} := \textit{Ker}\, C$. *Triple* (A, B, C) *is right-invertible if and only if*

$$\mathcal{S}_* + \mathcal{C} = X.$$

Condition of right-invertibility is equivalent to $\mathcal{V}^* + \mathcal{S}_* = X$.

D Invariant distributions and codistributions

Let Δ be a distribution defined on an open set U. We are interested in finding the smallest distribution, which is invariant under given vector fields $(\tau_1, \ldots, \tau_q)$ and which is denoted by the symbol $\langle \tau_1, \ldots, \tau_q | \Delta \rangle$. Given a distribution Δ and a set $\tau_1, \ldots, \tau_q$ of vector fields we define the nondecreasing sequence of distributions:

$$\begin{aligned} \Delta_0 &= \Delta \\ \Delta_k &= \Delta_{k-1} + \sum_{i=1}^{q} [\tau_i, \Delta_{k-1}], \end{aligned} \tag{D.1}$$

i.e., for all k one has that $\Delta_k \subset \langle \tau_1, \ldots, \tau_k | \Delta \rangle$. If there exists an integer k^* such that $\Delta_{k^*} = \Delta_{k^*+1}$ then $\Delta_{k^*} = \langle \tau_1, \ldots, \tau_k | \Delta \rangle$.

Let Ω be a codistribution defined on an open set U and we are interested in finding the smallest codistribution, which is invariant under the given vector fields $(\tau_1, \ldots, \tau_q)$ and which is denoted by the symbol $\langle \tau_1, \ldots, \tau_q | \Omega \rangle$. Given a codistribution Ω and a set $\tau_1, \ldots, \tau_q$ of vector fields we define the dual version of (D.1), i.e.,

$$\begin{aligned} \Omega_0 &= \Omega \\ \Omega_k &= \Omega_{k-1} + \sum_{i=1}^{q} L_{\tau_i} \Omega_{k-1}. \end{aligned} \tag{D.2}$$

Then for all k one has $\Omega_k \subset \langle \tau_1, \ldots, \tau_k | \Omega \rangle$ while $\Omega_{k^*} = \langle \tau_1, \ldots, \tau_k | \Omega \rangle$ provided that there exists an integer k^* such that $\Omega_{k^*} = \Omega_{k^*+1}$.

Example 3: *In the special case of LTI systems the algorithm (D.1) ends up with the well-known controllable subspace of the system:*

$$\Delta_{n-1}(x) = \boldsymbol{Im}\,[B \;\; AB \;\; \ldots \;\; A^{n-1}B], \qquad x \in \mathbb{R}^n$$

Considering the dual case let Ω_0 be the codistribution spanned by the row vectors $c_1, \ldots, c_p$ of C, the *algorithm (D.2) ends up with the subspace:*

$$\Omega_{n-1}(x) = \textit{span}\,\{c_1,\dots,c_p,c_1A,\dots,c_pA,\dots,c_1A^{n-1},\dots,c_pA^{n-1}\} =$$

$$= \tilde{\boldsymbol{Im}} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = \boldsymbol{Im}\,(C^T\ A^TC^T\ \dots\ (A^T)^{n-1}C^T).$$

By duality $\Omega_{n-1}^{\perp}(x)$ is the largest distribution invariant under the vector field f_A and contained in the distribution $\Omega_0^{\perp}(x)$. Moreover, by construction, at each $x \in \mathbb{R}^n$,

$$\Omega_0^{\perp}(x) = \boldsymbol{Ker}\,C$$

$$\Omega_{n-1}^{\perp}(x) = \boldsymbol{Ker} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}.$$

Example 4: *As far as bilinear systems are concerned, denoting by $\tau_i(x) = A_i x,\ i = 0,1,\dots,m$ one can get*

$$\Delta_k = \Delta_{k-1} + \sum_{i=0}^{m} A_i \Delta_{k-1}$$

yielding

$$\Delta_{n-1} = \sum_{l=0}^{n-1} \sum_{j_i\in\{0,\dots,m\},\ i=1,\dots,l} A_{j_1}\dots A_{j_l}\Delta$$

where the algorithm was initialized at a constant distribution Δ.

Starting from the constant codistribution $\Omega_0 = \tilde{\boldsymbol{Im}}\,C = \boldsymbol{Im}\,C^T$, one has

$$\Omega_k = \Omega_{k-1} + \sum_{i=0}^{m} \Omega_{k-1} A_i.$$

Let $p_1(x),\dots,p_d(x)$ be a set of smooth vector fields defined on an open set U, set $P =$ span$\{p_1,\dots,p_d\}$ and consider the nondecreasing sequence of distributions defined as follows:

$$S_0 = \overline{P}$$

$$S_k = \overline{S}_{k-1} + \sum_{i=0}^{m} [g_i, \overline{S}_{k-1} \cap \mathbf{Ker}\,\mathrm{d}h\}]$$

where $\overline{S}$ denotes the involutive closure of S.

Suppose there exists an integer k^* such that $S_{k^*+1} = \overline{S}_{k^*}$ and set $\Sigma_*^P = \overline{S}_{k^*}$. Then Σ_*^P is the minimal conditioned invariant and involutive distribution containing P. This algorithm is called termed as *conditioned invariant distribution algorithm*.

Example 5: *By setting*

$$g_0(x) = Ax, \quad g_1(x) = B, \quad h(x) = Cx$$

one has

$$[g_0, \overline{S}_{k-1} \cap \mathbf{Ker}\, C](x) = A(S_{k-1} \cap \mathbf{Ker}\, C),$$

thus one can obtain the well-known (C, A)-invariant subspace algorithm for LTI systems:

$$\begin{aligned} S_0 &= \mathcal{P} \\ S_k &= S_{k-1} + A(S_{k-1} \cap \mathbf{Ker}\, C). \end{aligned}$$

Example 6: *For bilinear systems, i.e., $g_i(x) = A_i x$ it follows that:*

$$\begin{aligned} S_0 &= \mathcal{P} \\ S_k &= S_{k-1} + \sum_{i=0}^{m} A_i(S_{k-1} \cap \mathbf{Ker}\, C). \end{aligned}$$

Example 7: *Using the augmented state space $\xi = [t, x]^T$ one can obtain the algorithm*

$$\begin{aligned} S_0(\xi) &= \mathcal{P} \\ S_k(\xi) &= S_{k-1}(\xi) + \left(\frac{\partial}{\partial t} - A(\pi(t))\right)(S_{k-1}(\xi) \cap \mathbf{Ker}\, C), \end{aligned}$$

for a linear time varying dynamics, a readout map with constant C matrix and a constant distribution $\mathcal{P}$.

The dual is the *controlled invariant distribution algorithm* which is defined via codistributions:

$$\begin{aligned} \Omega_0 &= \mathbf{span}\, \mathrm{d}h \\ \Omega_k &= \Omega_{k-1} + \sum_{i=0}^{m} L_{g_i}(\Omega_{k-1} \cap G^{\perp}). \end{aligned} \tag{D.3}$$

Suppose there exists an integer k^* such that $\Omega_{k^*+1} = \Omega_{k^*}$. Then $\Omega_k = \Omega_{k^*}$, for all $k > k^*$ and if $\Omega_{k^*} \cap G^{\perp}$ and $\Omega_{k^*}^{\perp}$ are smooth, then $\Omega_{k^*}^{\perp}$ is the maximal controlled invariant smooth distribution contained in $\mathbf{Ker}\, \mathbf{d}h$.

Example 8: *Considering LTI systems, the algorithm*

$$\begin{aligned} \Omega_0 &= \tilde{\mathbf{Im}}\ C = \mathbf{Im}\, C^T \\ \Omega_k &= \Omega_{k-1} + (\Omega_{k-1} \cap \mathbf{Ker}\, B^T)A, \end{aligned}$$

ends up in the minimal (B^T, A^T)-invariant subspace over $\mathbf{Im}\, C^T$ so its dual is the maximal (A, B)-invariant subspace in $\mathbf{Ker}\, C$.

Remark 17: *The derivation of the time-dependent form (i.e., in the augmented state space) of the controlled invariant distribution algorithm (D.3) will end up in*

$$\tilde{\Omega}_{k+1}(\xi) = \boldsymbol{span}\{\boldsymbol{dh}\} + (\tilde{\Omega}_k \cap \mathcal{B}^{\perp})A(\rho),$$

provided that there exists k^* *such that* $\Omega_{k^*+1} = \Omega_{k*}$. *Then* $\tilde{\Omega}_{k*}^{\perp}$ *will be the maximal controlled invariant distribution in* $\boldsymbol{Ker}\{\boldsymbol{dh}\}$ *which contains* $G = \boldsymbol{span}\{g_1, \ldots, g_m\}$. *Considering constant codistributions in each step we get the dual form of (7.7):*

$$\tilde{\Omega}_{k+1} = \boldsymbol{span}\{\boldsymbol{dh}\} + \sum_{i=0}^{N}(\tilde{\Omega}_k \cap \mathcal{B}^{\perp})A_i.$$

Let Θ be a fixed codistribution and define the nondecreasing sequence of codistributions as:

$$\begin{aligned} Q_0 &= \Theta \cap \mathbf{span}\,\mathrm{d}h \\ Q_{k+1} &= \Theta \cap \left(\sum_{i=0}^{m} L_{g_i} Q_k + \mathbf{span}\,\mathrm{d}h\right). \end{aligned} \tag{D.4}$$

Suppose that all the codistributions of this sequence are nonsingular, i.e., there exists an integer $k^* \leq n-1$ such that $Q_k = Q_{k^*}$ for all $k > k^*$, set $\Omega^* = Q_{k^*}$ and use the notation:

$$\Omega^* = \text{o.c.a.}(\Theta)$$

where o.c.a. stands for *observability codistribution algorithm*. Then

$$\begin{aligned} Q_0 &= \Omega^* \cap \mathbf{span}\,\mathrm{d}h\} \\ Q_{k+1} &= \Omega^* \cap \left(\sum_{i=0}^{m} L_{g_i} Q_k + \mathbf{span}\,\mathrm{d}h\right). \end{aligned}$$

provided that all the codistributions generated by the observability codistribution algorithm are nonsingular. As a consequence o.c.a.$(\Omega^*) = \Omega^*$ and if Θ is conditioned invariant, so is the codistribution Ω^*.

Ω is said to be a *observability codistribution* if fulfills the relations:

$$\begin{aligned} L_{g_i}\Omega \subset \Omega + \mathbf{span}\,\mathrm{d}h, \quad i = 0, 1, \ldots, m \\ \text{o.c.a.}(\Omega) = \Omega. \end{aligned}$$

The distribution Δ is called *unobservability distribution* if its annihilator $\Omega = \Delta^{\perp}$ is an observability codistribution. If the algorithm (D.4) is initialized at $(\Sigma_*^P)^{\perp}$, then o.c.a.$((\Sigma_*^P)^{\perp})$ is an observability codistribution contained in $\mathcal{P}^{\perp}$. Moreover, it is the largest codistribution having this property.

Example 9: *Let us consider the nonlinear system*

$$\dot{x} = A_0 x + \sum_{i=1}^{m} u_i A_i x + l(x)m + \sum_{i=1}^{d} p_i(x) w_i$$
$$y = Cx$$

with the assumption that

$$\mathcal{P} = \textit{span}\,\{p_1, \ldots, p_d\}$$

is independent of x. Then the observability codistribution algorithm will be read as:

$$Q_0 = \Theta \cap \tilde{\boldsymbol{Im}}\,C$$
$$Q_{k+1} = \Theta \cap \left(\sum_{i=0}^{m} Q_k A_i + \tilde{\boldsymbol{Im}}\,C \right). \tag{D.5}$$

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