Report on the dissertation

Subconvex bounds for automorphic L-functions and applications

by Gergely Harcos

In his dissertation Harcos treats a very important topic in analytic number theory, that of bounds for L-functions. The L-functions in question are Dirichlet series that encode important arithmetic information, just as Riemann's zeta function encodes information about prime numbers, or Dirichlet's original L-functions encode everything about primes in arithmetic progressions. It is extremely important to understand the analytic behavior of these L-functions, such as their analytic continuation, functional equation, their special values or their growth in the various parameters that define them. The problem of the analytic continuation and functional equation is, when properly formulated, is equivalent to the identification of the L-function with automorphic L-functions. These automorphic L-functions are Dirichlet series that encode information about special functions on Lie groups, and therefore can be accessed by a variety of tools of algebraic, analytic and geometric nature. The importance of these question is well represented by recent Fields medal honors, the proof of Fermat's last theorem, or their dominating presence in the Milleneum prize problems.

The problem on growth estimates for L-functions, which from now on will always be assumed to be automorphic, is what is essential for the arithmetic applications. One gets bounds for L-functions from the convexity version of the Phragmen-Lindelof principle. It is very usual that these estimates using classical ideas are just short of achieving the desired goal and any improvements whatsoever will lead to the solution of a long standing open problem. Important applications of breaking convexity include proving equidistribution results for points on the sphere cut out by rays given by rational coefficients, estimating the smallest norm of ideals satisfying certain generalized congruence conditions, various problems on diophantine equations and geometric problems on the Bolyai-Lobachevsky plane.

It must be clear then for all that bounds for L-functions especially breaking convexity bounds is being pursued very actively at the most prestigious institutions around the world. Harcos is one of the leaders of the theory of sub-convexity bounds. In the last decade he has made very important contributions to this field. His achievements in this very competitive field not only advanced our understanding but also promoted the eminence of Hungarian mathematics.

The dissertation only contains some selected results of Harcos in this field. These are sub-convex bounds for L-functions of modular forms (Theorem 2), for L-functions of modular forms twisted by a Dirichlet character (Theorem 1) or twisted by another modular form (Theorem 3).

These are extremely important results. I start with consequences of the first two results. They lead to improved bounds for Fourier coefficients of half integral weight modular forms, improved estimates for equidistribution of points on ellipsoids and representations of a natural number by a ternary quadratic form, as well as improved bounds for Dedekind zeta functions and Hecke L-functions. There are also other applications such as to the Bloch-Kato conjecture on the central values of L-functions. These are the most basic objects of analytic number theory showing the power and richness of sub-convex bounds once again.

The third main result is about Rankin-Selberg L-functions that are constructed via the Dirichlet's series

$$\sum_n \frac{a(n)b(n)}{n^s}$$

where $a(n), b(n)$ are the Hecke parameters of two modular forms f and g. These can be holomorphic as in the classical approach of Rankin and Selberg or Maass eigenforms of the hyperbolic Laplace-Beltrami operator. Harcos and Philippe in an extremely important and highly cited paper proved that the Rankin-Selberg L-functions can be bounded better than the convexity bound in the level. The level is roughly the smallest integer q so that the modularity holds with extra conditions modulo q.

Theorem 3 has applications in the equidistribution of Heegner points and closed geodesics on the modular surface $SL_2(\mathbf{Z})\backslash\mathcal{H}$. These points and cycles are associated to narrow ideal classes in quadratic extensions of $\mathbf{Q}$. Their distribution was first attacked by Linnik via ergodic theoretic techniques but the problem was fully solved by Duke using modular forms. The sub-convexity bound by Harcos and Michel allows one to refine these equidistributions by considering proper subgroups of the class group.

Finally I remark that the ergodic theory approach has been extended by Ratner and Margulis and their school and there is now a fruitful interaction between the two groups where the theorems of Harcos, especially Theorem 2, has proved extremely useful.

The proofs are too technical to even outline here. However the underlying ideas one classical, one very new, are simple. The classical idea is that in families of L-functions averages over the family are usually easier to estimate and give exactly the right order of magnitude what could be achieved if the Lindelof hypothesis were assumed to hold for individual members of the family. The new idea is that by using a suitable weight function one can amplify the contribution coming from individual members of the family hence breaking the convexity bound. In this generality this of course is a castle in the air and the art is to make this approach realizable in concrete situations. This art of amplification is exactly where Harcos and his collaborators made great progress. I should mention that there are other other approaches using representation theoretic tools, also due to Harcos, but he very modestly only mentions them in a footnote.

The whole dissertation is filled with very important results. The theorems and the proofs are tour de force and completely changed the landscape of the analytic theory of automorphic L-functions. These results from the basis of further work of a great number of other mathematicians in the field and elsewhere. For these reasons I very strongly recommend the acceptance of the dissertation.

Bp. ~~Arpad Toth~~
2011. nov. 18.

Tóth Árpád