dc_101_10

1–MOTIVES AND ALBANESE MAPS IN ARITHMETIC GEOMETRY

Tamás Szamuely

A doctoral dissertation

submitted to the Hungarian Academy of Sciences

Budapest, 2011

Contents

dc_101_10

Introduction

This dissertation is centered around abelian varieties and their generalizations. Abelian varieties are central objects in algebraic and arithmetic geometry: they are projective varieties with a geometrically defined (commutative) group law. The simplest examples of abelian varieties are elliptic curves. It has been known for a long time that if one fixes a base point O on a smooth cubic in the projective plane, one can use secant and tangent lines to define on its points an addition law satisfying the axioms for abelian groups. This additional group structure has a great influence on the geometry of the curve but also on its arithmetic. For instance, if the equation of the curve has rational coefficients and the point O has rational coordinates, then the rational points form a subgroup in the group of real or complex points, and this subgroup is finitely generated by a famous theorem of Mordell.

Abelian varieties are also naturally associated with curves of higher genus. Over an algebraically closed field linear equivalence classes of degree zero divisors on a smooth projective curve X correspond to points of an abelian variety J_X canonically attached to X, called the Jacobian of the curve. If we fix a point O of X, sending a point P to the class of the divisor $P-O$ defines a morphism $\alpha_O : X \to J_X$ which is an isomorphism in genus 1 and an embedding in higher genus. It can also be characterized by a remarkable universal property: every morphism from X to an abelian variety sending O to zero factors uniquely through α_O.

It turns out that an abelian variety Alb_X satisfying the above universal property exists for a variety X of arbitrary dimension. It is called the Albanese variety of X; the map $\alpha_O : X \to \mathrm{Alb}_X$ sending the distinguished point O to zero is the Albanese map. Though not as intimately related to X as the Jacobian is to a curve, the Albanese variety still captures a lot of geometric information.

On a higher-dimensional smooth variety divisors do not come from points any more, but from codimension 1 subvarieties. However, there is still a connection with abelian varieties. If X is a smooth projective variety over an algebraically closed field, linear equivalence classes of divisors correspond to points of a (non-connected) group scheme Pic_X whose identity component Pic^0_X is an abelian variety, the Picard variety of X. For X a curve this is of course the Jacobian; the other components of Pic_X correspond bijectively to nonzero integers indexed by the degree. If A is an abelian variety, then Pic^0_A is called the dual abelian variety of A and is usually denoted by A^*. One has a canonical isomorphism $(A^*)^* \cong A$; the Jacobian of a curve is self-dual. For X arbitrary a theorem going back to Severi states that the dual of the Picard variety Pic^0_X is none but the Albanese variety Alb_X.

Roughly speaking, the main theme of this dissertation is the extension of known geometric and arithmetic results about abelian varieties to semi-abelian varieties. A semi-abelian variety is a commutative group variety that is an extension of an abelian variety by an algebraic torus; the latter term means an affine group variety that over the algebraic closure of the base field becomes a finite product of copies of the multiplicative group. Semi-abelian varieties abound in nature: for instance, the Jacobian of an open (affine) curve is a semi-abelian variety. Also, Serre [59] has shown that to every variety over an algebraically closed field one can attach a generalized Albanese variety $\widetilde{\mathrm{Alb}}_X$ which is universal for morphisms to semi-abelian varieties. These are interesting for open varieties: if U is an open subvariety of a smooth projective variety X, then they have the same Albanese variety in the classical sense, but in the generalized sense $\widetilde{\mathrm{Alb}}_X = \mathrm{Alb}_X$ is the abelian variety quotient of $\widetilde{\mathrm{Alb}}_U$ which has a toric part in general.

When proving theorems about semi-abelian varieties the difficulty is that in most cases one cannot reduce them to the extreme cases of abelian varieties and tori; the additional difficulty is created by the fact that the extension of the abelian variety by the torus is nontrivial in general. To put it in a more highbrow way, an abelian variety (e.g. the Jacobian of a curve) is a basic example of a pure motive, whereas a semi-abelian variety gives rise to a mixed motive. A further step of generalization comes when one considers 1-motives in the sense of Deligne [16]: these are not group schemes any more but certain 2-term complexes of such. They arise naturally when one wants to generalize the construction of the dual abelian variety to the semi-abelian case. In a certain sense they yield the simplest category of mixed motives with good properties.

In this dissertation we focus on three different, though interrelated, questions concerning the geometry and arithmetic of semi-abelian varieties and 1-motives.

1. *Serre's generalized Albanese map.* We study the generalized Albanese map of Serre [59] on smooth quasi-projective varieties. The main result is a generalization of a famous theorem of Roitman [55] to open subvarieties of smooth projective varieties. Our method also yields a new conceptual proof of the result of Roitman and has inspired subsequent research on the Albanese functor.

2. *Arithmetic duality theorems for 1-motives.* We consider Galois hypercohomology groups of 1-motives defined over number fields and their completions and prove several duality theorems about them. These theorems constitute a sym-

metric common generalization of classical results by Cassels [11], Tate [70] and Tate–Nakayama [68] on the cohomology of abelian varieties and tori and thus unify the basic cohomological results on connected commutative group varieties over number fields.

3. *Rational points on principal homogeneous spaces of semi-abelian varieties.* A central topic in Diophantine geometry is the study of local-global principles for rational points on varieties defined over number fields. We investigate this question for principal homogeneous spaces (a.k.a torsors) under semi-abelian varieties and prove a common generalization of results by Manin [42] and Sansuc [57], thereby settling a long-standing open question in the field. Our proof relies on the duality theorems of part 2 and also on a construction from part 1. This theorem was something of a 'missing link' in the arithmetic of torsors under group varieties and had a considerable impact on further research.

In the following three sections, which correspond to the three chapters of the main text, we give a more detailed discussion of the three topics above, including precise statements of the main results and some indications about methods and applications.

0.1 On the Albanese map and Suslin homology

We begin by explaining the classical theorem of A. A. Roitman [55]. Consider a variety X over an algebraically closed field k. Fixing a base point O gives rise to an Albanese map $\alpha_O : X \to \mathrm{Alb}_X$ that is by its very definition universal for morphisms of X to abelian varieties that send O to the zero point. One can make this map independent of the base point O as follows. Consider the group $\mathscr{Z}(X)^0$ of formal $\mathbf{Z}$-linear combinations $\Sigma n_i P_i$ of points of X satisfying $\sum n_i = 0$. Extending the map α_O to $\mathscr{Z}(X)^0$ yields a map $\alpha_X : \mathscr{Z}(X)^0 \to \mathrm{Alb}_X$ that does not depend on 0 any more. This map is known to factor through *rational equivalence*: two elements of $\mathscr{Z}(X)^0$ are called rationally equivalent if their difference comes from divisors on normalizations of curves on X. The quotient of $\mathscr{Z}(X)^0$ modulo rational equivalence is usually denoted by $CH_0(X)^0$; it is the degree zero part of the *Chow group of zero-cycles.* Roitman's theorem can now be stated as follows.

Theorem 0.1.1 *For X smooth and projective the Albanese map*

$$\alpha_X : CH_0(X)^0 \to \mathrm{Alb}_X$$

induces an isomorphism on torsion elements of order prime to the characteristic of k.

dc_101_10

Roitman's result was later completed by Milne [45] in the case of characteristic $p > 0$: he showed that the isomorphism also holds on the subgroup of elements of p-power order. As a consequence of these results one obtains that the n-torsion subgroup of $CH_0(X)^0$ is finite for each $n > 0$; indeed, this is known to hold for an abelian variety.

Notice that for X a smooth projective curve the Albanese variety is none but the Jacobian of X and $CH_0(X)^0$ is the degree zero part of the Picard group, so the map α_X itself is an isomorphism. However, already for surfaces examples of Mumford show that the Albanese map can have *uncountable* kernel. Therefore it is quite remarkable that it at least detects torsion classes in the Chow group.

Jointly with M. Spieß we have proven in [65] the following generalization to semi-abelian varieties. Assume that X is smooth and projective, and $U \subset X$ is an open subvariety. Then one can consider the generalized Albanese map $U \to \widetilde{\mathrm{Alb}}_U$ of Serre [59]; by definition it is the universal map for morphisms of U to semi-abelian varieties that send some fixed base point to zero. As above, it induces a canonical map $\mathscr{Z}(U)^0 \to \widetilde{\mathrm{Alb}}_U$.

Next, as a generalization of $CH_0(X)^0$ to the open case we consider a quotient $h_0(U)^0$ of $\mathscr{Z}(U)^0$ called the degree zero part of the 0-th algebraic singular homology (or *Suslin homology*) group. In the paper [66] it was introduced in a more general framework, but here is an elementary description. The group $h_0(U)$ can be defined as the quotient of $\mathscr{Z}(U)$ (the free abelian group generated by the closed points of U) by the subgroup generated by elements of the form $i_0^*(Z) - i_1^*(Z)$, where $i_\nu : U \to U \times \mathbf{A}^1$ $(\nu = 0,1)$ stand for the inclusions $x \mapsto (x,\nu)$ and Z runs through all closed irreducible subvarieties of $U \times \mathbf{A}^1$ such that the projection $Z \to \mathbf{A}^1$ is finite and surjective. There is a natural degree map $\mathscr{Z}(U) \to \mathbf{Z}$ given by the formula

$$\sum_i n_i P_i \mapsto \sum_i n_i.$$

Using the fact that the projections $Z \to \mathbf{A}^1$ are finite and flat it is not hard to check that the degree map factors through $h_0(U)$, and we define $h_0(U)^0$ as the kernel of the induced map. This definition gives back $CH_0(X)^0$ in the case $U = X$.

It can be shown that the map $\mathscr{Z}(U)^0 \to \widetilde{\mathrm{Alb}}_U$ factors through $h_0(U)^0$, so we can finally state:

Theorem 0.1.2 (= Theorem 1.1.1) *For U an open subvariety of a smooth projective variety defined over an algebraically closed field k the generalized Albanese map*

$$h_0(U)^0 \to \widetilde{\mathrm{Alb}}_U$$

induces an isomorphism on torsion elements of order prime to the characteristic of k.

An attractive feature of the above generalization of Roitman's theorem is that the proof is new even in the case $U = X$ and is very conceptual. In fact, its basic idea can be simply summarized in the following commutative diagram:

$$\begin{array}{ccc} h_1(U,\mathbf{Z}/n) & \longrightarrow & {}_nh_0(U)^0 \\ \cong\Big\downarrow & & \Big\downarrow \\ \mathrm{Hom}(H^1_{\text{ét}}(U,\mathbf{Z}/n),\mathbf{Z}/n) & \longrightarrow & {}_n\widetilde{\mathrm{Alb}}_U(k). \end{array}$$

In this diagram the right vertical map is our generalized Albanese map restricted to the n-torsion subgroup of $h_0(U)^0$, where n is an integer prime to the characteristic of k. The left vertical map is a basic comparison isomorphism, proven in [66], relating the *first* Suslin homology group with finite coefficients to the first étale cohomology of U (over $k = \mathbf{C}$ the latter is just the usual singular cohomology group). The upper map is a boundary map coming from a long exact homology sequence and the bottom map expresses a more-or-less well-known relation of the generalized Albanese variety to the first étale cohomology; it is a generalization of the classical fact that on a curve H^1-classes with finite coefficients come from torsion points of the Jacobian.

Now the proof is just this: the diagram commutes, the upper map is surjective and the bottom map is an isomorphism after passing to the direct limit over powers of n. Hence so is the right vertical map. Of course, checking the commutativity of the diagram is the hard part. It involves, among other things, an interpretation of the Albanese map in Voevodsky's derived category [71] of motivic complexes, which has proven to be fruitful in later research.

By contrast, previous proofs of Roitman's theorem (the original one, but also that of S. Bloch [5]) involved several ad hoc arguments, whereas our proof reduces the statement to a basic cohomological comparison isomorphism. It later inspired Barberi-Viale and Kahn [2] to put the theory into an even more general framework which allows them to remove the assumption that U admits a smooth compactification (namely X). This of course improves the result only in positive characteristic where resolution of singularities is not known at present.

We have also proven the following complement (which was in fact the starting point of the research project).

Theorem 0.1.3 (= Theorem 1.1.2) *Let k be the algebraic closure of a finite field, and U an open subvariety of a smooth projective variety defined over k. Then the generalized Albanese map*

$$h_0(U)^0 \to \widetilde{\mathrm{Alb}}_U$$

induces an isomorphism of torsion groups.

Of course, the prime-to-the-characteristic part follows from the generalized Roitman theorem above once we know the elementary fact that the group $h_0(U)^0$ is torsion over the algebraic closure of a finite field. However, the p-part is not covered by the previous theorem.

The method of proof is completely different, and relies on a result of arithmetic nature: class field theory for tame coverings of varieties over finite fields [58]. Recently, this result was reproven in an elementary way by the late G. Wiesend; see our report [67].

0.2 Arithmetic duality theorems for 1-motives

Duality theorems for the Galois cohomology of commutative group schemes over local and global fields are among the most fundamental results in arithmetic. Let us briefly and informally recall some of the most famous ones.

Perhaps the earliest such result is the following. Consider a p-adic field K (i.e. a finite extension of $\mathbf{Q}_p$ for some prime p) and an algebraic torus T defined over K; this is a commutative group scheme that over the algebraic closure becomes isomorphic to a finite direct power of the multiplicative group $\mathbf{G}_m$. Denote by Y^* the character group of T. Consider the Galois cohomology groups $H^i(K,T)$ and $H^{2-i}(K,Y^*)$ of these group schemes for $i = 0,1,2$ (see e.g. [23], [62]), related by the cup-product

$$H^i(K,T) \times H^{2-i}(K,Y^*) \to H^2(K,\mathbf{G}_m)$$

coming from the pairing $T \times Y^* \to \mathbf{G}_m$. Here $H^2(K,\mathbf{G}_m)$ is none but the Brauer group of the p-adic field K which is isomorphic to $\mathbf{Q}/\mathbf{Z}$ by a famous theorem of Hasse. Therefore we get canonical pairings

$$H^i(K,T) \times H^{2-i}(K,Y^*) \to \mathbf{Q}/\mathbf{Z}$$

for $i = 0,1,2$. The Tate-Nakayama duality theorem (whose original form can be found in [68]) asserts that these pairings become perfect if in the cases $i \neq 1$ we replace the groups H^0 by their profinite completions. This theorem subsumes the reciprocity isomorphism of local class field theory which is equivalent to the case $i = 0$, $T = \mathbf{G}_m$.

Next, in his influential Bourbaki exposé [69], Tate observed that given an abelian variety A over K, the Poincaré pairing between A and its dual A^* together with the isomorphism $H^2(K,\mathbf{G}_m) \cong \mathbf{Q}/\mathbf{Z}$ enables one to construct similar pairings

$$H^i(K,A) \times H^{1-i}(K,A^*) \to \mathbf{Q}/\mathbf{Z}$$

for $i = 0,1$, and he proved that these pairings are also perfect.

The last result we recall is also due to Tate. Consider now an abelian variety A over a number field k, and denote by $\text{Ш}^1(A)$ its Tate-Shafarevich group. By definition, this group consists of those classes in the Galois cohomology group $H^1(k,A)$ that become trivial when restricted to each completion of k. According to a widely believed conjecture (which has been verified in some cases) this is a finite abelian group.

Now Tate constructed a duality pairing

$$\text{Ш}^1(A) \times \text{Ш}^1(A^*) \to \mathbf{Q}/\mathbf{Z}$$

(generalizing earlier work of Cassels [11] on elliptic curves) and announced in [70] that this pairing is nondegenerate modulo divisible subgroups. If one assumes the finiteness of $\text{Ш}^1(A)$, divisible subgroups are trivial and one obtains a perfect pairing of finite abelian groups. A detailed proof of this duality theorem first appeared in Milne's book [47]. Similar results for tori are attributed to Kottwitz in the literature; indeed, the references [38] and [39] contain such statements, but without (complete) proofs. The monographs [47] and [49] contain proofs in some cases.

In the joint work [30] with D. Harari we established common generalizations of the results mentioned above for *1-motives* in the sense of Deligne [16]. We recall the definition: a 1-motive over a field F is a two-term complex M of F-group schemes $[Y \to G]$ (placed in degrees -1 and 0), where Y is the F-group scheme associated to a finitely generated free abelian group equipped with a continuous $\text{Gal}(F)$-action and G is a semi-abelian variety over F, i.e. an extension of an abelian variety A by a torus T. Every 1-motive M as above has a *Cartier dual* $M^* = [Y^* \to G^*]$ generalizing the duals seen above in the cases $M = [0 \to T]$ and $M = [0 \to A]$. A key example is the Cartier dual of a 1-motive of the form $[0 \to G]$, where G is a semi-abelian variety with toric part T and abelian quotient A. In this case the Cartier dual is a 1-motive of the form $[Y^* \to A^*]$, where Y^* is the character group of T and A^* the dual abelian variety of A. There is no intelligent way of defining the dual of G as a group scheme.

The above duality construction together with arithmetic results enable one to construct duality pairings relating the cohomology of M and M^* over local and global fields. However, one has to use Galois *hyper*cohomology as we are dealing with complexes of group schemes and not group schemes any more.

Let us now state the main results concerning these. Over local fields, we prove:

Theorem 0.2.1 (= Theorem 2.1.1) *Let K be a local field, and let $M = [Y \to G]$ be a 1-motive over K. For $i = -1, 0, 1, 2$ there are canonical pairings*

$$\mathbf{H}^i(K,M) \times \mathbf{H}^{1-i}(K,M^*) \to \mathbf{Q}/\mathbf{Z}$$

inducing perfect pairings between

1. *the profinite group* $\mathbf{H}^{-1}_{\wedge}(K,M)$ *and the discrete group* $\mathbf{H}^2(K,M^*)$*;*
2. *the profinite group* $\mathbf{H}^0(K,M)^{\wedge}$ *and the discrete group* $\mathbf{H}^1(K,M^*)$.

Here the groups $\mathbf{H}^0(K,M)^{\wedge}$ and $\mathbf{H}^{-1}_{\wedge}(K,M)$ are obtained from the corresponding hypercohomology groups by certain completion procedures. We also have a generalization of the above theorem to 1-motives over so-called henselian local fields of mixed characteristic, and a result showing that in the duality pairing the unramified parts of the cohomology are exact annihilators of each other.

Now let M be a 1-motive over a number field k. For all $i \geq 0$ define the Tate-Shafarevich groups

$$Ш^i(M) = \mathrm{Ker}\,[\mathbf{H}^i(k,M) \to \prod_v \mathbf{H}^i(k_v,M)]$$

where the product is taken over completions of k at all (finite and infinite) places of k. Our main result can then be summarized as follows.

Theorem 0.2.2 (= Theorem 2.1.2) *Let k be a number field and M a 1-motive over k. There exist canonical pairings*

$$Ш^i(M) \times Ш^{2-i}(M^*) \to \mathbf{Q}/\mathbf{Z}$$

for $i = 0,1$.

For $i = 1$ *the pairing is non-degenerate modulo maximal divisible subgroups.*

For $i = 0$ *it is a perfect pairing between a compact and a discrete topological group, provided that we replace* $Ш^0(M)$ *by a certain modification* $Ш^0_{\wedge}(M)$*, and assume the finiteness of* $Ш^1(A)$ *for the abelian quotient A.*

See the beginning of Section 2.6 for the definition of $Ш^0_{\wedge}(M)$. If one accepts the conjecture that the (usual) Tate-Shafarevich group of an abelian variety is finite, it can be shown that for $i = 1$ the pairing above is a perfect pairing of finite groups.

The proof of the theorem is rather technical. In the most important case $i = 1$ it proceeds by constructing first some pairings in étale cohomology and proving duality theorems for these. They are then shown to induce duality results on Galois cohomology. As the definition of the pairings is rather abstract, the following proposition is by no means obvious.

Proposition 0.2.3 (= Proposition 2.7.1) *In the case* $i = 1$ *and* $M = [0 \to A]$ *with A an abelian variety the pairing above coincides with the classical Cassels–Tate pairing used in [11] and [70].*

In case the reader is willing to digest more cohomological theorems, here are two more of them. They are generalizations of classical results known as the Poitou–Tate and the Cassels–Tate exact sequences, respectively.

Theorem 0.2.4 (= Theorem 2.6.6) *Let M be a 1-motive over a number field k. Assume that $Ш^1(A)$ and $Ш^1(A^*)$ are finite, where A is the abelian variety corresponding to M. Then there is a twelve-term exact sequence of topological abelian groups*

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{H}^{-1}(k,M)^{\wedge} \arrow[r, "\gamma_2{}^D"] & \prod_{v\in\Omega_k}\mathbf{H}^2(k_v,M_k^*)^D \arrow[r, "\beta_2{}^D"] & \mathbf{H}^2(k,M^*)^D \arrow[d] \\
 & \mathbf{H}^1(k,M^*)^D \arrow[d] & \mathbf{P}^0(M)^{\wedge} \arrow[l, "\gamma_0"'] & \mathbf{H}^0(k,M)^{\wedge} \arrow[l, "\beta_0"'] \\
 & \mathbf{H}^1(k,M) \arrow[r, "\beta_1"] & \mathbf{P}^1(M)_{\mathrm{tors}} \arrow[r, "\gamma_1"] & (\mathbf{H}^0(k,M^*)^D)_{\mathrm{tors}} \arrow[d] \\
0 & \mathbf{H}^{-1}(k,M^*)^D \arrow[l] & \bigoplus_{v\in\Omega_k}\mathbf{H}^2(k_v,M) \arrow[l, "\gamma_2"'] & \mathbf{H}^2(k,M) \arrow[l, "\beta_2"']
\end{tikzcd}$$

where the groups $\mathbf{P}^i$ are certain restricted topological products of hypercohomology groups, the maps β_i are restriction maps, the maps γ_i are induced by local duality and the unnamed maps by the global duality results above.

Theorem 0.2.5 (= Theorem 3.1.2) *Under the assumptions of the previous theorem there is an exact sequence of topological abelian groups*

$$0 \to \overline{\mathbf{H}^0(k,M)} \to \prod_{v\in\Omega}\mathbf{H}^0(k_v,M) \to Ш^1_{\omega}(M^*)^D \to Ш^1(M) \to 0.$$

Here $\overline{\mathbf{H}^0(k,M)}$ denotes the closure of the diagonal image of $\mathbf{H}^0(k,M)$ in the topological product of the $\mathbf{H}^0(k_v,M)$, and $A^D := \mathrm{Hom}(A,\mathbf{Q}/\mathbf{Z})$ for a discrete abelian group A. (By convention, for v archimedean we take here the modified (Tate) hypercohomology groups instead of the usual ones.) Finally, the group $Ш^1_{\omega}(M^*)$ consists of those classes in the Galois hypercohomology group $\mathbf{H}^1(k,M^*)$ that become trivial when restricted to *all but finitely many* completions of k.

There has been a fair amount of later research developing the results in this section. C. González-Avilés [24] has extended the main results to the function field case. For the function field of a curve defined over a finite field of characteristic p our proofs carry over to treat the prime-to-p torsion part of the cohomology groups involved. González-Avilés was able to prove an analogue of the $i = 1$ case of Theorem 0.2.2 for the p-primary torsion part. In their paper [25] González-Avilés and Tan have also extended the Poitou–Tate exact sequence (Theorem 0.2.4) and the Cassels–Tate dual exact sequence (Theorem 0.2.5) to positive characteristic. They have moreover constructed a variant of the latter sequence that does not rest upon a finiteness assumption on Tate–Shafarevich groups (but is maybe less suitable for applications).

Another kind of generalization was proven by Peter Jossen in his thesis [34] written under my supervision. He defined a 1-motive *with torsion* to be a morphism $Y \to G$, where Y is an extension of a lattice by a finite flat group scheme and G is an extension of an abelian scheme by a group scheme X that is itself an extension of a finite flat group scheme by a torus. Jossen extended the theory of Deligne 1-motives to 1-motives with torsion, including Cartier duality and ℓ-adic realizations. He was then able to prove the analogue of Theorem 0.2.2 for 1-motives with torsion. This theorem yields a common generalization of all previously known duality results over number fields, including Poitou–Tate duality for finite group schemes which was not covered by Theorem 0.2.2.

0.3 Local-global principles for 1-motives

The duality theorems of the previous section have stimulated a fair amount of subsequent research besides those already mentioned. We now present a major application, again based on a joint paper [31] with D. Harari. It concerns local-global principles for points on torsors under semi-abelian varieties. A torsor, or a principal homogeneous space, under a k-group variety G is a k-variety X equipped with an action of G that becomes simply transitive over the algebraic closure; in particular, over the algebraic closure X becomes isomorphic to G as a variety. Basic examples are curves of genus 1 over non-algebraically closed fields: if they have a point, they are elliptic curves and hence abelian varieties; if not, they are torsors under their Jacobians which are elliptic curves. There exist classical examples of curves of genus 1 over a number field k that have points over all completions but not over k (e.g. the plane curve with homogeneous equation $3x^3 + 4y^3 + 5z^3 = 0$). Such curves are usually referred to as counterexamples to the Hasse principle (which holds if the existence of local points implies the existence of a global point).

Here we study the failure of the Hasse principle for rational points on torsors under general semi-abelian varieties over a number field k. There is a general method going back to the 1970 ICM lecture of Manin [42] that justifies the existence of counterexamples in many (though not all) cases. To explain it, we need to introduce two more notions. One is the set $X(\mathbf{A}_k)$ of adelic points of a variety X; its elements are sequences of points (P_v) of X over each completion k_v such that for all but finitely many finite places v the point P_v is actually a v-integral point. The other is the notion of the Brauer group $\mathrm{Br}X$ of a scheme S. We shall not give the definition here, but for our purposes it suffices to know that $S \mapsto \mathrm{Br}S$ is a contravariant functor on the category of schemes which sends the spectrum of a field F to the Brauer group $\mathrm{Br}F$ of F. Recall that for a completion k_v of k at a finite place we have an isomorphism $\mathrm{Br}k_v \cong \mathbf{Q}/\mathbf{Z}$ by local class field theory; for $k_v = \mathbf{R}$ we have $\mathrm{Br}\mathbf{R} \cong \mathbf{Z}/2\mathbf{Z}$ which we may view as a subgroup of $\mathbf{Q}/\mathbf{Z}$. Now for a smooth k-variety X Manin defines a pairing

$$X(\mathbf{A}_k) \times \mathrm{Br}X \to \mathbf{Q}/\mathbf{Z}, \quad [(P_v), \alpha] \mapsto \sum \alpha(P_v)$$

where the evaluation map $\alpha \mapsto \alpha(P_v)$ is induced by contravariant functoriality of $\mathrm{Br}X$ and the sum is taken inside $\mathbf{Q}/\mathbf{Z}$ (it is known to be finite). If the sequence (P_v) is the diagonal image of a k-rational point, then the pairing with any $\alpha \in \mathrm{Br}X$ gives zero by the global reciprocity law of class field theory. So denoting by $X(\mathbf{A}_k)^{\mathrm{Br}}$ the left kernel of the above pairing we have the implication $X(\mathbf{A}_k)^{\mathrm{Br}} = \emptyset \Rightarrow X(k) = \emptyset$. This is the Manin obstruction to the Hasse principle. It is said to be the only obstruction if the converse implication holds.

It is often interesting to restrict the Manin pairing to subquotients of $\mathrm{Br}X$. We shall be interested in the subquotient $Б(X)$ defined as follows. Consider the natural maps $\mathrm{Br}k \xrightarrow{\pi} \mathrm{Br}X \xrightarrow{\rho} \mathrm{Br}(X \times_k \bar{k})$ and set $\mathrm{Br}_a X := \ker(\rho)/\mathrm{im}(\pi)$. Then take $Б(X) \subset \mathrm{Br}_a(X)$ to be the subgroup of locally trivial elements. As the image of $\mathrm{Br}k$ in $\mathrm{Br}X$ pairs trivially with adelic points (again by the global reciprocity law), the Manin pairing induces a pairing with $\mathrm{Br}_a X$ and finally with $Б(X)$. We still have of course $X(\mathbf{A}_k)^{Б} = \emptyset \Rightarrow X(k) = \emptyset$, with $X(\mathbf{A}_k)^{Б}$ defined similarly as $X(\mathbf{A}_k)^{\mathrm{Br}}$. The group $Б(X)$ is often more interesting than $\mathrm{Br}X$ because if one assumes that the Tate-Shafarevich group of the Albanese variety of X is finite, it is also finite, and in some cases even explicitly computable. This gives a practical way for verifying the failure of the Hasse principle in the cases where the Manin obstruction coming from $Б(X)$ is the only one.

The main theorem of [31] now states:

Theorem 0.3.1 (= Theorem 3.1.1) *Given a torsor X under a semi-abelian variety G over a number field whose abelian quotient has finite Tate–Shafarevich group, we have $X(\mathbf{A}_k)^{Б} \neq \emptyset \Rightarrow X(k) \neq \emptyset$, i.e. the Manin obstruction associated with $Б(X)$ is the only obstruction to the Hasse principle.*

This result was known for $G = A$ an abelian variety (Manin himself) or G a torus (Sansuc [57]) but the general case is considerably harder. It was in fact, a long-standing open question; see e.g. Skorobogatov's book ([63], p. 133).

The main idea of the proof is (as already in Manin's case) to relate the Manin pairing

$$X(\mathbf{A}_k) \times Б(X) \to \mathbf{Q}/\mathbf{Z} \tag{0.1}$$

to the Cassels–Tate type pairing

$$Ш(M) \times Ш(M^*) \to \mathbf{Q}/\mathbf{Z} \tag{0.2}$$

for the 1–motive $M = [0 \to G]$ and to use the non-degeneracy of the latter pairing proven in Theorem 0.2.2. More precisely, denote by $\langle\ ,\ \rangle_M$ the first pairing and by $\langle\ ,\ \rangle_{CT}$ the second. The method is to construct a map $\iota : Ш(M^*) \to Б(X)$ such that for all adelic points (P_v) of X and all $\alpha \in Б(X)$ the formula

$$\langle (P_v), \iota(\alpha)\rangle_M = \langle [X], \alpha\rangle_{CT} \tag{0.3}$$

holds. To understand the formula, note first that the torsor X is known to have a cohomology class $[X]$ in the group $H^1(k,G) = \mathbf{H}^1(k,M)$; it is a trivial class if and only if X has a k-point (see e.g. [63], pp. 18–19). Hence the assumption $X(\mathbf{A}_k) \neq \emptyset$ implies that $[X] \in Ш^1(M)$. The left hand side does not depend on the choice of (P_v) because elements of $Б(X)$ are ‘locally constant’ by definition. Now assume that the map ι exists and formula (0.3) holds. Then the assumption $X(\mathbf{A}_k)^{Б} \neq \emptyset$ together with (0.3) implies that $[X]$ is orthogonal to the whole of $Ш^1(M^*)$ under the pairing $\langle\ ,\ \rangle_{CT}$. Thus $[X] = 0$ by Theorem 0.2.2, i.e. $X(k) \neq \emptyset$.

It took us considerable time to figure out the right way to define the map ι. We finally discovered that the key was given by the duality between generalized Albanese and Picard varieties which was already used in the proof of Theorem 0.1.2. We also needed a new cohomological interpretation of the Manin pairing $\langle\ ,\ \rangle_M$ which has found other applications since.

We have also proven a similar result for weak approximation of adelic points by rational points. The question is whether the set $X(k)$ of rational points are dense in $X(\mathbf{A}_k)$ for the restricted product topology. To study it, one works with a modified version of the Manin pairing, namely with the induced pairing

$$X(k_\Omega) \times \mathrm{Br}_{\mathrm{nr}} X \to \mathbf{Q}/\mathbf{Z},$$

where k_Ω is the topological direct product of all completions of k, and $\mathrm{Br}_{\mathrm{nr}} X$ is the unramified Brauer group of X; it can be defined as the Brauer group of any smooth compactification of X. One may also work with subgroups of $\mathrm{Br}_{\mathrm{nr}} X$, such as $\mathrm{Br}_{\mathrm{nr}\,1} X := \ker(\mathrm{Br}_{\mathrm{nr}} X \to \mathrm{Br}_{\mathrm{nr}}(X \times_k \bar{k}))$. Finally, for a smooth k-group scheme G there is yet another variant, which is the one we shall use:

$$\prod_{v \in \Omega} G(k_v) \times \mathrm{Br}_{\mathrm{nr}\,1} G \to \mathbf{Q}/\mathbf{Z}. \tag{0.4}$$

Here we have taken the same convention at the archimedean places as in Theorem 0.2.5 above. Concerning this pairing one has the following result:

Theorem 0.3.2 (= Theorem 3.6.1) *Let G be a semi-abelian variety defined over k. Assuming that the abelian quotient has finite Tate–Shafarevich group, the left kernel of the pairing (0.4) is contained in the closure of the diagonal image of $G(k)$.*

Actually, this theorem was first proven in [29]. However, the techniques used in the proof of Theorem 0.3.1 together with exact sequence 0.2.5 enabled us to give another, shorter proof.

Theorem 0.3.1 gave rise to several applications by other mathematicians. Borovoi, Colliot-Thélène and Skorobogatov [8] have generalized it to homogeneous spaces under an arbitrary connected algebraic group. The precise statement is the same as in Theorem 0.3.1, except that G is a connected algebraic group, and X is a homogeneous space of G whose geometric points have connected stabilizers. There is, however, an additional restriction on the number field k: it must be totally imaginary. In fact, the same paper contains a quite surprising example ([8], Proposition 3.16) of a connected non-commutative and non-linear algebraic group over $\mathbf{Q}$ for which the statement fails. This shows that over arbitrary number fields general connected algebraic groups behave differently from commutative or linear ones.

The proof uses techniques going back to Borovoi's papers [6] and [7] to reduce to the case of a torsor under a semi-abelian variety, where our Theorem 0.3.1 can be applied.

In fact, Borovoi, Colliot-Thélène and Skorobogatov formulated their result in a different but equivalent way, in terms of the *elementary obstruction* of Colliot-Thélène and Sansuc [15]. By definition, this obstruction is the extension class $ob(X)$ of $\mathrm{Gal}\,(\bar{k}|k)$-modules

$$0 \to \bar{k}^{\times} \to \bar{k}(X)^{\times} \to \bar{k}(X)^{\times}/\bar{k}^{\times} \to 0 \tag{0.5}$$

where k is a perfect field, X is an arbitrary smooth geometrically integral k-variety and $\bar{k}(X)^{\times}$ is the group of invertible rational functions on $X \times_k \bar{k}$. An easy argument in Galois cohomology (see e.g. [63], p. 27) shows that a k-rational point induces a Galois-equivariant splitting of the above extension. Thus nontriviality of $ob(X)$ is an obstruction to the existence of a k-point.

For varieties over a number field possessing an adelic point the triviality of $ob(X)$ is equivalent to the triviality of the pairing (0.1) under the assumption that the Tate–Shafarevich group of the Albanese variety of X is finite (which conjecturally is the case). This was shown in Wittenberg's paper [72]. We go into some details as the argument uses Albanese maps and Theorem 0.3.1. We have seen in section 1 that in the case when k is algebraically closed there exists a semi-abelian variety $\widetilde{\mathrm{Alb}}_X$ attached to X which is universal for morphisms of X to semi-abelian varieties. Over a general k the generalized Albanese variety $\widetilde{\mathrm{Alb}}_X$ still exists: it is a semi-abelian variety over k that comes equipped with a canonical k-torsor Alb^1_X which is universal for morphisms of X to torsors under semi-abelian varieties. The main geometric result of [72] is the proof that the triviality of the torsor Alb^1_U for all Zariski dense open subsets $U \subset X$ implies the triviality of $ob(X)$. By a short argument due to Colliot-Thélène, this in turn implies the triviality of the pairing (0.1) if k is a number field and the said Tate–Shafarevich group is finite. Assuming conversely that (0.1) is trivial, one easily shows using Theorem 0.3.1 as well as the fact ([18], Lemma 3.4) that for each U as above the map $\text{Б}(X) \to \text{Б}(U)$ is an isomorphism that Alb^1_U is trivial for all dense open subsets U.

We thus see that the results of Sections 2 and 3 imply interesting arithmetic properties of general varieties via the generalized Albanese map.

Chapter 1

On the Albanese map and Suslin homology

1.1 Introduction

This chapter is an almost identical reproduction of my joint paper [65] with Michael Spieß. I have made a correction communicated to me by the late J. van Hamel shortly before his untimely death and added Remark 1.5.6 addressing a subsequent improvement of the main theorem.

Consider an algebraically closed field k of characteristic $p \geq 0$ and a smooth connected quasi-projective k-variety X. When X is in fact projective, a famous theorem due to A. A. Roitman ([55], see also [5]) asserts that the Albanese map

$$alb_X : CH_0(X)^0 \longrightarrow \mathrm{Alb}_X(k) \tag{1.1}$$

from the Chow group of zero-cycles of degree 0 on X to the group of k-points of the Albanese variety induces an isomorphism on prime-to-p torsion subgroups (later J. S. Milne proved that the isomorphism holds for p-primary torsion subgroups as well, cf. [45]). As a well-known counter-example of Mumford shows, in dimensions greater than one the map alb_X itself is not an isomorphism in general. Still, Kato and Saito ([37], Section 10) have established the bijectivity of alb_X in the case when k is the algebraic closure of a finite field (in fact, in this case both groups are torsion). Moreover, bijectivity over $k = \overline{\mathbf{Q}}$ has been conjectured by Bloch and Beilinson, as a consequence of some expected standard features of the conjectural category of mixed motives over $\overline{\mathbf{Q}}$.

In this chapter we present a new conceptual approach to the theorem of Roitman which at the same time yields a generalization to the case when X is not necessarily projective but admits a smooth compactification. Here the natural target for the Albanese map is the generalized Albanese variety introduced by Serre [59]. If X is a curve, this variety is a generalized Jacobian in the sense of Rosenlicht [56] and for X proper it is the usual Albanese. In general, it is a semi-abelian variety universal for morphisms of X into semi-abelian varieties; it is related to the Picard variety by a duality theorem (see sections 1.3 and 1.4 for more details).

The generalization of alb_X to this context is a map

$$alb_X : h_0(X)^0 \longrightarrow \mathrm{Alb}_X(k) \tag{1.2}$$

where the group on the left is the degree zero part of Suslin's 0-th algebraic singular homology group defined in [66]; it coincides with $CH_0(X)^0$ when X is proper (see section 1.2 for the precise definition). The map (1.2) first appeared in the 1998 preprint version of N. Ramachandran's paper [53]; we give a simple proof for the "reciprocity law" implying its existence in Section 1.3.

Now we can state our main result.

Theorem 1.1.1 *Let k be an algebraically closed field of characteristic $p \geq 0$ and let X be a smooth connected quasi-projective variety over k. Assume that there exists a smooth projective connected k-variety $\mathfrak{X}$ containing X as an open subscheme. Then the Albanese map (1.2) induces an isomorphism on prime-to-p torsion subgroups.*

Note that the required smooth compactification $\mathfrak{X}$ exists if k is of characteristic 0 or if X is of dimension ≤ 3 and $p \geq 5$, by virtue of the desingularization theorems of Hironaka and Abhyankar.

Our method for proving Theorem 1.1.1 is new even in the proper case and is (at least to our feeling) more conceptual than the previous ones. The proof exploits the comparison maps $h^i(X, \mathbf{Z}/n\mathbf{Z}) \to H^i_{\text{ét}}(X, \mathbf{Z}/n\mathbf{Z})$ relating algebraic singular cohomology to étale cohomology according to Suslin and Voevodsky [66] for any n prime to p; by the main result of *loc. cit.* these maps are isomorphisms. We reduce the proof of our theorem to the case $i = 1$ of this fundamental result by showing that, in that case, taking the dual of the inverse map for all n and passing to the direct limit one obtains the restriction of the map (1.2) to the prime-to-p torsion subgroup of $h_0(X)$. One of the basic observations for proving this identification, which may be of independent interest, is that thanks to its functoriality and homotopy invariance properties, the (generalized) Albanese variety can be regarded as an object of Voevodsky's triangulated category of effective motivic complexes $DM_-^{eff}(k)$ and in fact for smooth varieties the Albanese map can be interpreted as a morphism in this category.

Over the algebraic closure of a finite field we can prove somewhat more:

Theorem 1.1.2 *We keep the hypotheses of Theorem 1.1.1 and assume moreover that k is the algebraic closure of a finite field. Then (1.2) is an isomorphism of torsion groups.*

The proof of this theorem is more traditional: in fact, it is a direct generalization of the argument given in ([37], section 10), using the "tamely ramified class field theory" developed in [58].

Finally it should be mentioned that during recent years fruitful efforts have been made for generalising Roitman's theorem to singular complex projective varieties (see [4] and the references quoted there). Our generalization seems to be unrelated to this theory except perhaps in the case when X is the complement of the singular locus of a complex projective variety.

A word on notation: For an abelian group A and a nonzero integer n we denote by ${}_nA$ the n-torsion subgroup of A and we write A/n as a shorthand for A/nA. For a prime number ℓ we let $A\{\ell\}$ be the ℓ-primary component of the torsion subgroup of A.

1.2 Review of algebraic singular homology

This section and the next are devoted to the definition of the map (1.2) and the groups involved. We begin by recalling the definition of the algebraic singular homology groups introduced in [66]. In this section k may stand for an arbitrary perfect field.

For an integer $n \geq 1$ consider the algebraic n-simplex

$$\Delta^n = \operatorname{Spec} k[T_0, \ldots, T_n]/(T_0 + \ldots + T_n - 1).$$

If X is a k-variety (i.e. an integral separated scheme of finite type over k), denote by $C_n(X)$ the free abelian group generated by those integral closed subschemes Z of $X \times \Delta^n$ for which the projection $Z \to \Delta^n$ is finite and surjective. Any nondecreasing map $\alpha : \{0, 1, \ldots, m\} \to \{0, 1, \ldots, n\}$ induces a morphism $\Delta^m \to \Delta^n$ and thus a homomorphism $\alpha^* : C_n(X) \to C_m(X)$ via pull-back of cycles. These maps endow the set of the $C_n(X)$ with the structure of a simplicial abelian group; we denote by $C_\bullet(X)$ the associated chain complex. For an abelian group A the n-th algebraic singular homology group $h_n(X, A)$ of X with coefficients in A is defined as the n-th homology of the complex $C_\bullet(X) \otimes A$ and the n-th algebraic singular cohomology $h^n(X, A)$ as the n-th cohomology of $\operatorname{Hom}(C_\bullet(X, \mathbf{Z}), A)$. For $A = \mathbf{Z}$ we shall simply write $h_n(X)$ for $h_n(X, \mathbf{Z})$ etc.

The group $h_0(X)$ has the following concrete description. Let $\mathscr{Z}(X)$ be the free abelian group with basis the set X_0 of closed points of X. Then $h_0(X)$ is the quotient of $\mathscr{Z}(X)$ by the submodule $\mathscr{R}$ generated by $i_0^*(Z) - i_1^*(Z)$, where $i_\nu : X \to X \times \mathbf{A}^1$ $(\nu = 0, 1)$ stand for the inclusions $x \mapsto (x, \nu)$ and Z runs through all closed integral subschemes of $X \times \mathbf{A}^1$ such that the projection $Z \to \mathbf{A}^1$ is finite and surjective. There is a natural degree map $\mathscr{Z}(X) \longrightarrow \mathbf{Z}$ given by the formula

$$\sum_i n_i P_i \mapsto \sum_i n_i [k(P_i) : k],$$

of which we denote the kernel by $\mathscr{Z}(X)^0$. Using the fact that the projections $Z \to \mathbf{A}^1$ are finite and flat, one checks that $\mathscr{Z}(X)^0$ contains $\mathscr{R}$; the quotient $\mathscr{Z}(X)^0/\mathscr{R}$ will be denoted by $h_0(X)^0$.

For the proofs we shall also need a sheafified version of the above construction. For this, denote by Sm/k the category of smooth schemes of finite type over k. Let $\mathscr{F}$ be an abelian presheaf on Sm/k, i.e. a contravariant functor from Sm/k to the category of abelian groups. For any $m \geq 0$ we may define a presheaf $\mathscr{F}_m$ by the rule $\mathscr{F}_m(X) = \mathscr{F}(X \times \Delta^m)$. Together with the operations induced from the cosimplicial scheme $\Delta^\bullet$ these presheaves assemble to form a simplicial presheaf whose associated chain complex we denote by $C_\bullet(\mathscr{F})$. If $\mathscr{F}$ is *homotopy invariant*, i.e. if the natural map $\mathscr{F}(X) \to \mathscr{F}(X \times \mathbf{A}^1)$ is an isomorphism for all $X \in Sm/k$, then the augmentation map $C_\bullet(\mathscr{F}) \to \mathscr{F}$ given by the identity in degree 0 is a map of complexes and in fact a quasi-isomorphism (here we view $\mathscr{F}$ as a complex concentrated in degree 0). Indeed, in view of the canonical isomorphism $\Delta^n \cong \mathbf{A}^n$ in this case $C_\bullet(\mathscr{F})$ is none but the complex associated to the constant simplicial presheaf defined by $\mathscr{F}$.

We also recall the notion of *presheaves with transfers* from Section 2 of [71]. These are contravariant additive functors with values in abelian groups from the category $SmCor(k)$ whose objects are smooth schemes of finite type over k and where a morphism from an object X to an object Y is a *finite correspondence,* i.e. an element of the free abelian group $c(X,Y)$ generated by those integral closed subschemes Z of $X \times Y$ for which the projection $Z \to X$ is finite and surjective over a component of X. (*Note:* This definition of presheaves with transfers differs from the one in the earlier paper [66] whose results we shall use in the sequel, but the two definitions are equivalent.) Now the link with the algebraic singular complex is the following. For a separated k-scheme X the rule $U \mapsto c(U,X)$ defines a presheaf with transfers on which we denote by $\mathbf{Z}_{tr}(X)$; actually it is a sheaf for the étale topology on $SmCor(k)$. Then by definition $C_\bullet(\mathbf{Z}_{tr}(X))(k) = C_\bullet(X)$.

1.3 The generalized Albanese map

In this section we explain the construction of the generalized Albanese maps on two levels of generality: first, in order to keep technicalities to a minimum, we construct the map (1.2) over an algebraically closed k as stated in the introduction, and then we explain a sheafified version over an arbitrary perfect field.

So we begin by working over an algebraically closed field k and recalling the notion of the generalized Albanese variety Alb_X of a variety X, as introduced in [59]. It is a semiabelian variety satisfying the following universal property: for every k-point P of X there is a morphism $\iota_P : X \to \mathrm{Alb}_X$ such that $\iota_P(P) = 0$ and if (B, f) is a pair consisting of a semiabelian variety B and a morphism $f : X \to B$

mapping P to 0_B there is a unique morphism $g : \mathrm{Alb}_X \to B$ of group schemes with $g \circ \iota_P = f$. Note that the maps ι_P satisfy the formula

$$\iota_P(Q) = \iota_P(R) + \iota_R(Q) \tag{1.3}$$

for any k-points P, Q, R of X.

If X is proper, then Alb_X is the Albanese variety in the classical sense. If X is a curve, it coincides with Rosenlicht's generalized Jacobian for the modulus defined by the sum of points at infinity.

The assignment $X \mapsto \mathrm{Alb}_X$ is a covariant functor for arbitrary morphisms of varieties. Moreover, there is also a contravariant functoriality of Alb_X with respect to finite flat morphisms $f : X \to Y$ which we now briefly explain. Mapping a closed point Q of Y to the pull-back zero-cycle $f^*(Q)$ defines a morphism of Y into the d-fold symmetric product $Sym^d(X)$, where d is the degree of f. On the other hand, for a fixed k-point P of Y the zero-cycle $f^*(P) = P_1 + \cdots + P_d$ defines a morphism $Sym^d(X) \to \mathrm{Alb}_X$ via the sum of the maps ι_{P_i} $(1 \le i \le d)$. The composite of these two maps sends P to 0 in Alb_X, hence by definition of Alb_Y factors as the composite of ι_P with a morphism $f^* : \mathrm{Alb}_Y \to \mathrm{Alb}_X$. Using formula (1.3) one checks that f^* is independent of the choice of P; it is the map we were looking for.

We denote by

$$a_X : \mathscr{Z}(X)^0 \longrightarrow \mathrm{Alb}_X(k)$$

the homomorphism

$$\sum_i n_i P_i \mapsto \sum_i n_i(\iota_P(P_i))$$

for some $P \in X(k)$; again the map is independent of the choice of P by formula (1.3). For a morphism $f : X \to Y$ of varieties the diagram

$$\begin{array}{ccc} \mathscr{Z}(X)^0 & \xrightarrow{a_X} & \mathrm{Alb}_X(k) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ \mathscr{Z}(Y)^0 & \xrightarrow{a_Y} & \mathrm{Alb}_Y(k) \end{array} \tag{1.4}$$

commutes where the vertical maps are induced by f through covariant functoriality. Similarly, for a finite flat $f : X \to Y$ the diagram

$$\begin{array}{ccc} \mathscr{Z}(Y)^0 & \xrightarrow{a_Y} & \mathrm{Alb}_Y(k) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ \mathscr{Z}(X)^0 & \xrightarrow{a_X} & \mathrm{Alb}_X(k) \end{array} \tag{1.5}$$

commutes.

dc_101_10

Using these functoriality properties we can give an easy proof of the following "reciprocity law" which immediately yields the existence of the map alb_X as in (1.2).

Lemma 1.3.1 *With notations as above, the subgroup $\mathscr{R} \subset \mathscr{Z}(X)^0$ is contained in the kernel of the map a_X.*

Proof. Let $Z \subseteq X \times \mathbf{A}^1$ be a closed integral subscheme such that the projection $q : Z \to \mathbf{A}^1$ is finite and surjective (hence also flat, its target being a regular integral scheme of dimension 1) and denote by $p : Z \to X$ the other projection. By the commutativity of (1.4) and (1.5) we have

$$a_X(i_0^*(Z) - i_1^*(Z)) = a_X(p_*(q^*((0)-(1)))) = p_*(q^*(a_{\mathbf{A}^1}((0)-(1)))).$$

Since the generalized Albanese of $\mathbf{A}^1$ is trivial (any map of $\mathbf{A}^1$ into a semi-abelian variety being constant), it follows that the left hand side is 0. □

Now we treat the sheafification of the above construction, over an arbitrary perfect base field k. For this purpose it is convenient to replace Alb_X by the "Albanese scheme" $\widetilde{\mathrm{Alb}}_X$ considered in [53] (where it is denoted by A_X; for the following facts, see his Definition 1.5 and the subsequent discussion). For geometrically connected X it is a smooth commutative group scheme which is an extension of the constant group scheme $\mathbf{Z}$ by a semi-abelian variety $\widetilde{\mathrm{Alb}}_X^0$. Any k-point of X (if exists) defines a splitting, i.e. an isomorphism $\mathbf{Z} \times \widetilde{\mathrm{Alb}}_X^0 \cong \widetilde{\mathrm{Alb}}_X$. For k algebraically closed, the variety $\widetilde{\mathrm{Alb}}_X^0$ is none but our Alb_X considered above. The scheme $\widetilde{\mathrm{Alb}}_X$ comes equipped with a canonical morphism $\iota : X \to \widetilde{\mathrm{Alb}}_X$ satisfying an appropriate universal property.

Now for simplicity we restrict to the case when X is *smooth*, which is sufficient for the applications we have in mind. Consider the abelian presheaf on Sm/k represented by the group scheme $\widetilde{\mathrm{Alb}}_X$ which we also denote by $\widetilde{\mathrm{Alb}}_X$. It is a sheaf for the étale (even the $fppf$) topology.

Lemma 1.3.2 *The étale sheaf $\widetilde{\mathrm{Alb}}_X$ is a homotopy invariant presheaf with transfers.*

Proof. Homotopy invariance is again a consequence of the fact that there is no non-constant map $\mathbf{A}^1 \to \widetilde{\mathrm{Alb}}_X$. To construct transfer maps, we can work more generally with an arbitrary commutative group scheme G. Take $X, Y \in Sm/k$ and let $Z \subset X \times Y$ be a closed integral subscheme finite and surjective over a component of X. As explained before Theorem 6.8 of [66], to X one can associate a canonical map $\alpha_Z : X \to Sym^d(Y)$, where d is the degree of the projection $Z \to X$.

Now given a map $Y \to G$, it induces a map $Sym^d(Y) \to Sym^d(G)$, whence we obtain the required map $X \to G$ by composing by α_Z on the left and by the summation map on the right. □

The lemma implies that there is a unique map of presheaves

$$\mathbf{Z}_{tr}(X) \to \widetilde{\mathrm{Alb}}_X \tag{1.6}$$

which maps the correspondence associated to the identity map $id : X \to X$ to the Albanese map $\iota \in \widetilde{\mathrm{Alb}}_X(X)$. By applying the functor $C_\bullet(\ )$ we get a map $C_\bullet(\mathbf{Z}_{tr}(X)) \to C_\bullet(\widetilde{\mathrm{Alb}}_X)$. Composing it with the augmentation map on the right (existing by homotopy invariance of $\widetilde{\mathrm{Alb}}_X$; see the previous section) yields the map of complexes of étale sheaves with transfers

$$C_\bullet(\mathbf{Z}_{tr}(X)) \longrightarrow \widetilde{\mathrm{Alb}}_X. \tag{1.7}$$

Here we again consider $\widetilde{\mathrm{Alb}}_X$ as a complex concentrated in degree 0. Since this is a morphism of complexes, it factors through the 0-th homology presheaf $H_0(C_\bullet(\mathbf{Z}_{tr}(X)))$; as $\widetilde{\mathrm{Alb}}_X$ is an étale sheaf, it even factors through the associated étale sheaf $H_0(C_\bullet(\mathbf{Z}_{tr}(X)))_{\acute{e}t}$. We remark for later use that the map (1.6) can be obtained as a composite of (1.7) with the natural morphism of complexes $\mathbf{Z}_{tr}(X) \to C_\bullet(\mathbf{Z}_{tr}(X))$ (again with $\mathbf{Z}_{tr}(X)$ placed in degree 0 on the left).

Passing to sections over k and taking homology, we get a map $h_0(X) \to \widetilde{\mathrm{Alb}}_X(k)$, and, in the presence of a k-point, a map as in (1.2) which agrees with the previous one for k algebraically closed. In other words, the existence of the map (1.7) subsumes a sheafified version of the reciprocity law (perceptive readers have already noted the similarity of the argument with the proof of Lemma 1.3.1). The existence of the map (1.2) over a perfect base field, as demonstrated here, will be used in the proof of Theorem 1.1.2.

Remark 1.3.3 In the terminology of [71], Lemma 1.3.2 states that the sheaf $\widetilde{\mathrm{Alb}}_X$ defines an object in the category $DM_-^{eff}(k)$ of effective motivic complexes; on the other hand, $C_\bullet(\mathbf{Z}_{tr}(X))$ is precisely the motivic complex that Voevodsky associates to the smooth variety X. Therefore the map (1.7), which was shown above to be a morphism in $DM_-^{eff}(k)$, can be regarded as the "motivic interpretation" of the Albanese map.

1.4 Relation to tame abelian covers

Assume now we are in the situation of Theorem 1.1.1. In this case Alb_X has been described by Serre in his exposé [60] as an extension of the abelian variety $\mathrm{Alb}_{\mathfrak{X}}$

by a torus T whose rank is equal to the rank of the subgroup B_X of divisors on $\mathfrak{X}$ which are algebraically equivalent to zero and whose support is contained in $\mathfrak{X}-X$. As a consequence of this result one gets for any n prime to p, just as in the proper case (see [35], Lemma 5 or [45], p. 273), an injection of the dual group of ${}_n\mathrm{Alb}_X(k)$ into $H^1_{\text{ét}}(X,\mathbf{Z}/n)$ with a finite cokernel of order bounded independently of n.

The construction of this injection is completely analogous to the proper case, only technically a bit more involved. Consider the group C_X of irreducible divisors of $\mathfrak{X}$ supported in $\mathfrak{X}\setminus X$. Composing the projection $\mathrm{Pic}\,(\mathfrak{X})\to NS(\mathfrak{X})$ to the Néron-Severi group by the natural map

$$C_X \to \mathrm{Pic}\,(\mathfrak{X}) \tag{1.8}$$

associating to a divisor its class one gets an exact sequence

$$0\to B_X\to C_X\to S'\to 0$$

with the appropriate subgroup S' of $NS(\mathfrak{X})$. Denote by $M^*(X)$ the complex of smooth commutative group schemes (concentrated in degrees 0 and 1) associated to (1.8). By restriction to B_X we get another complex $[B_X\to \mathrm{Pic}^0(\mathfrak{X})]$ which we denote by $M^1(X)$. The above considerations give a distinguished triangle in the derived category of bounded complexes of smooth commutative group schemes

$$M^1(X)\longrightarrow M^*(X)\longrightarrow S[-1]\longrightarrow M^1(X)[1].$$

where S is a finitely generated constant commutative group scheme. By taking k-valued points (this is an exact functor since k is algebraically closed), tensoring with $\mathbf{Z}/n$ in the derived sense and passing to cohomology we obtain the exact sequence

$$\begin{aligned}0\longrightarrow H^0(M^1(X)(k)\otimes^{\mathbf{L}}\mathbf{Z}/n)\longrightarrow H^0(M^*(X)(k)\otimes^{\mathbf{L}}\mathbf{Z}/n)\longrightarrow \mathrm{Tor}(S,\mathbf{Z}/n)\\ \longrightarrow H^1(M^1(X)(k)\otimes^{\mathbf{L}}\mathbf{Z}/n).\end{aligned} \tag{1.9}$$

Here the last term vanishes by the following easy lemma:

Lemma 1.4.1 *Let k be an algebraically closed field and M a complex of commutative k-group schemes concentrated in degrees 0 and 1 whose degree 1 term is smooth and connected. Then $H^1(M(k)\otimes^{\mathbf{L}}\mathbf{Z}/n)=0$ for all integers $n\neq 0$.*

Proof. This boils down to the divisibility of the group of rational points of a smooth connected commutative group scheme over an algebraically closed field. We leave the details to the reader. □

Now assume for a moment that X is proper or the complement of a divisor in $\mathfrak{X}$. As remarked by Ramachandran (in (2-30) of the preprint version of [53]), the same argument as that for curves given on p. 70 of [16] gives a canonical isomorphism

$$H^0(M^*(X)(k) \otimes^{\mathbf{L}} \mathbf{Z}/n) \xrightarrow{\cong} H^1_{\text{ét}}(X, \mu_n). \tag{1.10}$$

For the convenience of the reader we recall the definition of (1.10). The target can be identified with group of isomorphism classes of pairs $(\mathscr{L}, \psi)$ consisting of a line bundle $\mathscr{L}$ on X and an isomorphism $\psi : \mathscr{L}^{\otimes n} \to \mathscr{O}_X$. On the other $H^0(M^*(X)(k) \otimes^{\mathbf{L}} \mathbf{Z}/n)$ can be described as the group of equivalence classes of pairs $(\bar{\mathscr{L}}, D)$ where $\bar{\mathscr{L}}$ is a line bundle on $\mathfrak{X}$ and $D \in C_X$ with $\bar{\mathscr{L}}^{\otimes n} \cong \mathscr{O}(D)$, two such pairs $(\bar{\mathscr{L}}_1, D_1)$, $(\bar{\mathscr{L}}_2, D_2)$ being equivalent if there exist $D_3 \in C_X$ such that $\bar{\mathscr{L}}_1 \otimes \bar{\mathscr{L}}_2^{-1} \cong \mathscr{O}(D_3)$ and $D_1 - D_2 = nD_3$. Given a pair $(\bar{\mathscr{L}}, D)$ we choose an isomorphism $\bar{\psi} : \bar{\mathscr{L}}^{\otimes n} \to \mathscr{O}(D)$. The map (1.10) is given by sending the class of $(\bar{\mathscr{L}}, D)$ to the isomorphism class of $(\bar{\mathscr{L}}\,|_X, \bar{\psi}\,|_X)$.

As explained by ([53], Theorem 2.3), the main result of [60] can be reinterpreted by saying that the Cartier dual of the complex $M^1(X)$ regarded as a 1-motive (cf. [16], Chapter 10 for this terminology) is the 1-motive $[0 \to \mathrm{Alb}_X]$; in particular, the toric part T of Alb_X has character group B_X. Since the construction of ([16], 10.2.5 and 10.2.11) puts into duality the "n-adic realizations" $T_{\mathbf{Z}/n\mathbf{Z}}M$ and $T_{\mathbf{Z}/n\mathbf{Z}}M^\vee$ of a 1-motive M and of its Cartier dual $M^\vee$ (this is a generalization of the classical fact that the duality between an abelian variety and its dual induces a duality on n-torsion points), as a consequence we get a canonical isomorphism

$$H^0(M^1(X)(k) \otimes^{\mathbf{L}} \mathbf{Z}/n) \xrightarrow{\cong} \mathrm{Hom}({}_n\mathrm{Alb}_X(k), \mu_n). \tag{1.11}$$

Hence we have almost proven

Proposition 1.4.2 *Let X and $\mathfrak{X}$ be as in Theorem 1.1.1. For every integer n prime to p the above construction gives an exact sequence*

$$0 \longrightarrow \mathrm{Hom}({}_n\mathrm{Alb}_X(k), \mathbf{Z}/n) \longrightarrow H^1_{\acute{e}t}(X, \mathbf{Z}/n) \longrightarrow \mathrm{Hom}(\mu_n, S) \longrightarrow 0. \tag{1.12}$$

Proof. When $X = \mathfrak{X}$ or the complement of a divisor, this follows from the above considerations after twisting by μ_n. In the general case, we may find an open subscheme X' of $\mathfrak{X}$ containing X which is the complement of a divisor in $\mathfrak{X}$ and such that the codimension of $X' - X$ in X' is at least 2. Then we have canonical isomorphisms $\mathrm{Alb}_X \cong \mathrm{Alb}_{X'}$ (see [53], Corollary 2.4) and $H^1_{\text{ét}}(X, \mathbf{Z}/n) \cong H^1_{\text{ét}}(X', \mathbf{Z}/n)$ (a consequence of Zariski-Nagata purity; see [28], exposé X for an exposition in the language of étale covers) and therefore the construction of the exact sequence for X reduces to that for X' using contravariant functoriality. □

Corollary 1.4.3 *The dual of the first map of the proposition induces a canonical isomorphism*

$$\mathrm{Hom}(H^1_{\acute{e}t}(X,\mathbf{Z}_\ell),\mathbf{Q}_\ell/\mathbf{Z}_\ell) \cong \mathrm{Alb}_X(k)\{\ell\}$$

for any prime number $\ell \neq p$.

Proof. The group S being finitely generated, its Tate module is trivial. Therefore passing to the inverse limit by making n run over powers of ℓ in (1.12) yields an isomorphism between the limit of the first two terms. The corollary follows by dualising. □

In the remainder of this section, which will only be needed for the proof of Theorem 1.1.2, we strengthen the result of the proposition to obtain a description of the abelianized tame fundamental group $\pi_1^{t,ab}(X)$ of X. By definition, this group classifies finite abelian Galois covers of $\mathfrak{X}$ which are étale over X and tamely ramified at codimension 1 points of $\mathfrak{X}\setminus X$ (i.e. the ramification index at such a point is prime to p and the extension of its residue field is separable). One has a direct sum decomposition

$$\pi_1^{t,ab}(X) \cong \pi_1^{ab}(X)(p') \oplus \pi_1^{ab}(\mathfrak{X})(p)$$

where the symbols (p') and (p) stand for the maximal profinite prime-to-p (resp. p) quotients of the groups in question. Indeed, any finite abelian Galois cover of X of order prime to p extends to a tamely ramified cover of $\mathfrak{X}$ by normalization; for the p-part, notice that any abelian cover of $\mathfrak{X}$ which is of p-power degree, étale over X and tamely ramified in codimension 1 must be étale in codimension 1, hence étale by Zariski-Nagata purity. Since the fundamental group is a birational invariant of projective varieties, the above decomposition shows that $\pi_1^{t,ab}(X)$ depends only on X but not on the compactification $\mathfrak{X}$. Needless to say, all these notions and facts are valid more generally over any perfect base field in place of k.

Proposition 1.4.4 *Under the assumptions of the previous proposition, there is an exact sequence*

$$0 \longrightarrow T \longrightarrow \pi_1^{t,ab}(X) \longrightarrow T(\mathrm{Alb}_X) \longrightarrow 0 \tag{1.13}$$

where $T(\mathrm{Alb}_X)$ denotes the full Tate module of Alb_X and T is a finite abelian group whose twisted dual can be described as follows: its prime-to-p part is isomorphic to that of the finite torsion subgroup of the group S considered above and its p-part is isomorphic to the p-primary torsion subgroup of $NS(\mathfrak{X})$.

Proof. We use the decomposition of $\pi_1^{t,ab}(X)$ recalled above. The assertion for the prime-to-p part follows from the previous proposition by dualising and passing to the limit. For the p-part we note that $T_p(\mathrm{Alb}_X) \cong T_p(\mathrm{Alb}_{\mathfrak{X}})$, the toric part of Alb_X having no p-primary torsion, so the result follows from the analogous statement for $\mathfrak{X}$ proven in ([35], Lemma 5). □

1.5 The generalization of Roitman's theorem

Keeping the assumptions of the previous section, we now prove Theorem 1.1.1. The proof involves the verification of some delicate compatibilities (Proposition 1.5.1 and Lemma 1.5.3) which occupy much of this section. We therefore offer alternative arguments in Remarks 1.5.4 and 1.5.5. In the first of these we explain how Lemma 1.5.3 can be avoided by using a counting argument. In Remark 1.5.5 we give a second shorter proof of the theorem – based on a hypersurface section argument – which circumvents the use of both 1.5.1 and 1.5.3. We note, however, our firm belief that from the conceptual point of view the optimal proof passes through the checking of compatibilities and not through the shortcuts.

We begin with some preliminary observations. For any positive integer n prime to p the long exact sequence

$$\ldots \longrightarrow h_i(X) \xrightarrow{n} h_i(X) \longrightarrow h_i(X, \mathbf{Z}/n) \longrightarrow h_{i-1}(X) \xrightarrow{n} \ldots$$

yields a surjection

$$h_1(X, \mathbf{Z}/n) \longrightarrow {}_n h_0(X). \tag{1.14}$$

On the other hand, we have a chain of isomorphisms

$$h_1(X, \mathbf{Z}/n) \cong \mathrm{Hom}(h^1(X, \mathbf{Z}/n), \mathbf{Z}/n) \cong \mathrm{Hom}(H^1_{\text{ét}}(X, \mathbf{Z}/n), \mathbf{Z}/n), \tag{1.15}$$

the first by the very definition of the groups in question (note that $\mathbf{Z}/n$ is injective as a $\mathbf{Z}/n$-module) and the second by the comparison theorem of Suslin and Voevodsky (see [66], Corollary 7.8 for the argument in characteristic 0; for the modifications in positive characteristic using de Jong's work on alterations, cp. [22], Theorem 3.2).

Let

$$\mathrm{Hom}({}_n\mathrm{Alb}_X(k), \mathbf{Z}/n) \to H^1_{\text{ét}}(X, \mathbf{Z}/n) \tag{1.16}$$

be the map given by "pulling back covers from Alb_X to X". Indeed, each $\phi : {}_n\mathrm{Alb}_X(k) \to \mathbf{Z}/n$ gives an étale $\mathbf{Z}/n$-cover of Alb_X by pushing out the extension

$$0 \to {}_n\mathrm{Alb}_X \to \mathrm{Alb}_X \xrightarrow{n} \mathrm{Alb}_X \to 0$$

via the map of group schemes associated to ϕ, hence defines a class in the group $H^1_{\text{ét}}(\mathrm{Alb}_X, \mathbf{Z}/n)$. Whence a map $\mathrm{Hom}({}_n\mathrm{Alb}_X, \mathbf{Z}/n) \to H^1_{\text{ét}}(\mathrm{Alb}_X, \mathbf{Z}/n)$, which by composition with the map induced on cohomology by a canonical map $X \to \mathrm{Alb}_X$ yields the map (1.16).

Now we can state the key result:

dc_101_10

Proposition 1.5.1 *For any positive integer n prime to p we have a commutative diagram*

$$\begin{array}{ccc} h_1(X,\mathbf{Z}/n) & \longrightarrow & {}_nh_0(X) \\ \cong\Big\downarrow & & \Big\downarrow alb_X \\ \mathrm{Hom}(H^1_{\acute{e}t}(X,\mathbf{Z}/n),\mathbf{Z}/n) & \longrightarrow & {}_n\mathrm{Alb}_X(k) \end{array} \tag{1.17}$$

where the upper horizontal, left vertical and bottom horizontal maps are respectively (1.14), (1.15) and the dual of (1.16).

For the proof of the proposition we need the following technical statements about abelian groups whose formal proof will be left to the reader.

Lemma 1.5.2

1. *For any abelian group A and integer $n > 0$ there is a canonical isomorphism*

   $$\mathrm{Hom}({}_nA,\mathbf{Z}/n) \cong \mathrm{Ext}^1(A,\mathbf{Z}/n).$$

2. *Let $(C_\bullet, d)$ be a homological complex of free abelian groups concentrated in nonnegative degrees. Then the natural map*

   $$H_1(C_\bullet \otimes \mathbf{Z}/n) \to {}_nH_0(C_\bullet)$$

   coming from tensoring by the exact sequence $0 \to \mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}/n \to 0$ can be identified with the natural map

   $$H_1(C_\bullet \otimes \mathbf{Z}/n) \to \mathrm{Tor}(H_0(C_\bullet),\mathbf{Z}/n) \tag{1.18}$$

   coming from computing the Tor-group using the free resolution $d(C_1) \to C_0$ of $H_0(C_\bullet)$.

3. *With the previous notations, the natural map*

   $$\mathrm{Ext}^1(H_0(C_\bullet),\mathbf{Z}/n) \to \mathrm{Ext}^1(C_\bullet,\mathbf{Z}/n)$$

   induced by the truncation map $C_\bullet \to H_0(C_\bullet)$ can be identified (using statement 1. and the self-injectivity of the ring $\mathbf{Z}/n$) with the image of the map (1.18) under the functor $\mathrm{Hom}(\ ,\mathbf{Z}/n)$.

Proof of Proposition 1.5.1. We prove the commutativity of the dual diagram

$$\begin{array}{ccc} \mathrm{Hom}({}_n\mathrm{Alb}_X(k),\mathbf{Z}/n) & \longrightarrow & H^1_{\acute{e}t}(X,\mathbf{Z}/n) \\ \Big\downarrow & & \Big\downarrow\cong \\ \mathrm{Hom}({}_nh_0(X),\mathbf{Z}/n) & \longrightarrow & h^1(X,\mathbf{Z}/n) \end{array}$$

which, using Lemma 1.5.2 (1), can be rewritten as

$$\begin{array}{ccc} \mathrm{Ext}^1(\widetilde{\mathrm{Alb}}_X(k),\mathbf{Z}/n) & \longrightarrow & H^1_{\text{ét}}(X,\mathbf{Z}/n) \\ \big\downarrow & & \big\downarrow \cong \\ \mathrm{Ext}^1(h_0(X),\mathbf{Z}/n) & \longrightarrow & \mathrm{Ext}^1(C_\bullet(X),\mathbf{Z}/n) \end{array}$$

where the Ext-groups are taken with respect to the category of abelian groups (there was no harm in replacing Alb_X by $\widetilde{\mathrm{Alb}}_X$ since $\mathbf{Z}$ is torsion-free). Using Lemma 1.5.2 the bottom horizontal map can then be identified as coming from the natural truncation map. Now we apply the rigidity theorem of Suslin-Voevodsky ([66], Theorem 4.5) to the three Ext-groups and a standard comparison theorem to the fourth group to obtain a diagram

$$\begin{array}{ccc} \mathrm{Ext}^1_{\text{ét}}(\widetilde{\mathrm{Alb}}_X,\mathbf{Z}/n) & \longrightarrow & \mathrm{Ext}^1_{\text{ét}}(\mathbf{Z}(X),\mathbf{Z}/n) \\ \big\downarrow & & \big\downarrow \cong \\ \mathrm{Ext}^1_{\text{ét}}(H_0(C_\bullet(\mathbf{Z}_{tr}(X)))_{\text{ét}},\mathbf{Z}/n) & \longrightarrow & \mathrm{Ext}^1_{\text{ét}}(C_\bullet(\mathbf{Z}_{tr}(X)),\mathbf{Z}/n) \end{array} \tag{1.19}$$

where the Ext-groups are now taken on the étale site of Sm/k, the subscript *ét* means sheafification for the étale topology and $\mathbf{Z}(X)$ is the étale sheaf whose sections over a smooth k-scheme Y are given by the free abelian group with basis $\mathrm{Hom}(Y,X)$. Note that the rigidity theorem was applicable to the upper left group by virtue of Lemma 1.3.2 and to the two lower ones by ([66], Corollary 7.5). Now to finish the proof, we claim that the above diagram is induced by applying the functor $\mathrm{Ext}^1_{\text{ét}}(\ ,\mathbf{Z}/n)$ to the commutative diagram of complexes of sheaves

$$\begin{array}{ccc} C_\bullet(\mathbf{Z}_{tr}(X)) & \longrightarrow & H_0(C_\bullet(\mathbf{Z}_{tr}(X)))_{\text{ét}} \\ \big\uparrow & & \big\downarrow \\ \mathbf{Z}_{tr}(X) & \longrightarrow & \widetilde{\mathrm{Alb}}_X \end{array}$$

whose existence was established in Section 3 (the map on the left inducing the inverse of the isomorphism marked in (1.19)) .

The identification of the bottom horizontal and left vertical arrows in (1.19) follows from the functoriality of the rigidity isomorphism. As for the upper horizontal map, note first that it is well known to be induced by the map of étale sheaves $\mathbf{Z}(X) \to \widetilde{\mathrm{Alb}}_X$ which factors through the natural inclusion $\mathbf{Z}(X) \to \mathbf{Z}_{tr}(X)$ by Lemma 1.3.2. Now by ([66], Corollary 10.10) the natural map

$$\mathrm{Ext}^1_{\text{ét}}(\mathbf{Z}_{tr}(X),\mathbf{Z}/n) \to \mathrm{Ext}^1_{\text{ét}}(\mathbf{Z}(X),\mathbf{Z}/n)$$

can be identified with the map

$$\mathrm{Ext}^1_{qfh}(\mathbf{Z}_{tr}(X)_{qfh},\mathbf{Z}/n) \to \mathrm{Ext}^1_{qfh}(\mathbf{Z}(X)_{qfh},\mathbf{Z}/n),$$

where the subscript qfh denotes sheafification for the so-called qfh-topology introduced in *loc. cit.* , which is finer than the étale topology. But the latter map is an isomorphism, for by ([66], Theorem 6.7) $\mathbf{Z}_{tr}(X)$ can be identified, after localization by the characteristic p, with $\mathbf{Z}(X)_{qfh}$. This finishes the identification of the upper horizontal map, and for the right vertical map one first uses the same argument to pass from $\mathbf{Z}(X)$ to $\mathbf{Z}_{tr}(X)$, whereupon the result follows from the construction of the isomorphism $h^1(X,\mathbf{Z}/n) \to H^1_{\text{ét}}(X,\mathbf{Z}/n)$ in the proof of Theorem 7.5 in [66] (from which one sees that it can be identified with the map induced by $\mathbf{Z}_{tr}(X) \to C_*(\mathbf{Z}_{tr}(X))$ on Ext^1-groups). □

Proof of Theorem 1.1.1. It is enough to consider ℓ-primary torsion for a prime $\ell \neq p$. It then suffices to see that by making n vary among powers of ℓ and passing to the direct limit we get a diagram whose bottom horizontal map is an isomorphism. Indeed, since the left vertical map is also an isomorphism and the upper horizontal map is surjective, by commutativity all maps in the diagram (1.17) must become isomorphisms in the limit.

The most natural way to prove that the bottom horizontal map induces an isomorphism in the limit is to identify it with the map between the first two terms of exact sequence (1.12) and apply Corollary 1.4.3. This identification is well known in the proper case and we give now a detailed sketch of checking it in general. Alternatively, one can avoid checking this compatibility by arguing as in Remark 1.5.4 below.

Lemma 1.5.3 *The map (1.16) coincides with the map between the first two terms in (1.12) given in the last section.*

Proof. We may again assume that X is proper or the complement of a divisor and argue about the map (1.16) twisted by μ_n. The map (1.16) associates to $\phi \in \mathrm{Hom}({}_n\mathrm{Alb}_X(k),\mu_n)$ an extension of the group scheme Alb_X by μ_n and hence also an extension E of Alb_X by $\mathbf{G}_m$, corresponding to a line bundle $\mathscr{L}$ with an isomorphism $\psi : \mathscr{L}^{\otimes n} \to \mathscr{O}_{\mathrm{Alb}_X}$ (the latter is a consequence of the fact that the n-fold sum of the extension $0 \to \mathbf{G}_m \to E \to \mathrm{Alb}_X \to 0$ is canonically isomorphic to the trivial extension). Pulling back $\mathscr{L}$ and ψ to X we get a pair $(\mathscr{L}_X,\psi_X)$ which defines an element ξ of $H^1(X,\mu_n)$, the image of ϕ by (1.16). Now since the natural map $Ext^1(\mathrm{Alb}_{\mathfrak{X}},\mathbf{G}_m) \to Ext^1(\mathrm{Alb}_X,\mathbf{G}_m)$ is surjective (the toric part T of Alb_X having no non-trivial extensions by $\mathbf{G}_m$), there is some extension $\bar{E}$ of $\mathrm{Alb}_{\mathfrak{X}}$ by $\mathbf{G}_m$ (defining a line bundle $\bar{\mathscr{L}}$) of which E is the pullback to Alb_X. Since $Ext^1(\mathrm{Alb}_{\mathfrak{X}},\mathbf{G}_m) \cong Pic^0(\mathrm{Alb}_{\mathfrak{X}})$, the isomorphism class of the pullback $\bar{\mathscr{L}}_{\mathfrak{X}}$ of $\bar{\mathscr{L}}$ to $\mathfrak{X}$ lies in $Pic^0(\mathfrak{X})$ and hence $\bar{\mathscr{L}}_{\mathfrak{X}}^{\otimes n} \cong \mathscr{O}_{\mathfrak{X}}(D)$ with some $D \in B_X$. As in the previous section, the pair $(\bar{\mathscr{L}}_{\mathfrak{X}},D)$ defines an element of $H^0(M^1(X)(k) \otimes^{\mathbf{L}} \mathbf{Z}/n) \subset H^0(M^*(X)(k) \otimes^{\mathbf{L}} \mathbf{Z}/n)$ which, by construction, is mapped to ξ under (1.10). On the other hand, one sees by going through Serre's

dc_101_10

duality construction that the element of $H^0(M^1(X)(k) \otimes^{\mathbf{L}} \mathbf{Z}/n)$ represented by $(\bar{\mathscr{L}}_{\mathfrak{X}}, D)$ is exactly the image of ϕ under (1.11). This completes the proof of the lemma and thereby that of Theorem 1.1.1. □

Remark 1.5.4 Alternatively, one may prove that the bottom horizontal map in (1.17) induces an isomorphism in the limit as follows. Consider the dual map $\pi_1^{ab}(X)/n \to {}_n\mathrm{Alb}_X$ which is known to be surjective by ([59], Théorème 10). For $n = \ell^m$ this is none but the surjection $\pi_1^{ab}(X)(\ell) \to T_\ell \mathrm{Alb}_X$ tensored by $\mathbf{Z}/\ell^m$ (where (ℓ) denotes the maximal pro-ℓ quotient). This latter surjection must have finite kernel for by Corollary 1.4.3 its domain and target are finitely generated $\mathbf{Z}_\ell$-modules of the same rank. Hence the modulo n map has a finite kernel of order bounded independently of n, from which we conclude by the same argument as in Corollary 1.4.3.

Remark 1.5.5 One can give a quicker, albeit less conceptual proof of Theorem 1.1.1 which avoids the verification of commutativity in (1.17), in the spirit of the simplified version of Bloch's approach to Roitman's theorem given in [12]. Indeed, using the horizontal and left vertical maps in (1.17) and passing to the limit one gets a surjection $\mathrm{Alb}_X(k)\{\ell\} \to h_0(X)\{\ell\}$. Since Alb_X is a semiabelian variety, both groups here must be isomorphic to some finite direct power of $\mathbf{Q}_\ell/\mathbf{Z}_\ell$, so that for any $m > 0$ the groups ${}_{\ell^m}\mathrm{Alb}_X(k)$ and ${}_{\ell^m}h_0(X)$ are finite and the order of the second one doesn't exceed that of the first. So by comparing orders, we are done once we show the surjectivity of alb_X on ℓ^m-torsion. This is achieved by induction on dimension starting from the case of curves treated in [41] and ([66], Theorem 3.1). For the inductive step, taking into account the covariant functoriality of alb_X, it suffices to prove the surjectivity of ${}_{\ell^m}\mathrm{Alb}_Y(k) \to {}_{\ell^m}\mathrm{Alb}_X(k)$ for an appropriate smooth closed subvariety $Y \subsetneq X$, or else, using the injectivity part of Proposition 1.4.2, the injectivity of $H^1_{\text{ét}}(X, \mathbf{Z}/\ell^m) \to H^1_{\text{ét}}(Y, \mathbf{Z}/\ell^m)$. To choose Y, we may assume as before that the complement Z of X in $\mathfrak{X}$ is empty or has pure codimension one. Then by the Bertini theorems we may find a smooth connected hyperplane section $\mathfrak{Y}$ of $\mathfrak{X}$ that cuts each component of Z smoothly and away from the intersections. Putting $Y = \mathfrak{Y} \cap X$ and $W = \mathfrak{Y} \cap Z$, the claim then follows from the injectivity of the first and third vertical maps in the commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^1_{\text{ét}}(\mathfrak{X}, \mathbf{Z}/\ell^m) & \longrightarrow & H^1_{\text{ét}}(X, \mathbf{Z}/\ell^m) & \longrightarrow & H^0_{\text{ét}}(Z, \mathbf{Z}/\ell^m(-1)) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & H^1_{\text{ét}}(\mathfrak{Y}, \mathbf{Z}/\ell^m) & \longrightarrow & H^1_{\text{ét}}(Y, \mathbf{Z}/\ell^m) & \longrightarrow & H^0_{\text{ét}}(W, \mathbf{Z}/\ell^m(-1))
\end{array}$$

whose exact rows are Gysin sequences. Indeed, the injectivity of the first arrow is classical (it follows e.g. by Poincaré duality from the weak Lefschetz Theorem), and that of the third follows from the choice of $\mathfrak{Y}$, each component of Z containing at least one component of W.

Remark 1.5.6 Some years following the publication of the paper [65] L. Barbieri-Viale and B. Kahn [2] made an improvement of Theorem 1.1.1: they could prove it for an arbitrary smooth variety without assuming that it is quasi-projective and has a smooth compactification. What made this possible is their theory of Albanese and Picard 1–motives to which they could also associate objects in Voevodsky's derived category of motivic complexes. Thereby they had at their disposal a vast generalization of Serre's duality statement between the generalized Albanese variety and the 1-motive $M^1(X)$. Also, by working directly in the motivic category they could avoid checking the necessary identifications between maps coming from different theories. The argument of the proof itself is basically the same but their version is much more streamlined.

1.6 Proof of Theorem 1.1.2

Assume now that k is the algebraic closure of a finite field $\mathbf{F}$ and denote by G the Galois group $Gal(k|\mathbf{F})$. Before embarking on the proof of Theorem 1.1.2, we remark that, as the perceptive reader has surely noticed, in this case one can immediately show by reduction to the case of curves that the groups whose isomorphism we are to establish are both torsion. Hence in this case the prime-to-p part of Theorem 1.1.2 is equivalent to Theorem 1.1.1. But in the proof below we shall use a different method (and thus give another proof of Theorem 1.1.1 in this special case which works also for the p-part) originating in an argument of [37].

By extending $\mathbf{F}$ if necessary we may assume that there are varieties $X_{\mathbf{F}} \subset \mathfrak{X}_{\mathbf{F}}$ defined over $\mathbf{F}$ such that $X_{\mathbf{F}}$ has an $\mathbf{F}$-rational point and $X_{\mathbf{F}} \times_{\mathbf{F}} k \cong X$, $\mathfrak{X}_{\mathbf{F}} \times k \cong \mathfrak{X}$. Similarly to section 10 of [37], the key to the proof of Theorem 1.1.2 is the exact sequence (1.13) which in this case is in fact an exact sequence of G-modules.

Recall from Section 4 that the abelianized tame fundamental group can be defined for $X_{\mathbf{F}}$ as well. Moreover, there is a natural projection $\pi_1^{t,ab}(X_{\mathbf{F}}) \to G \cong \widehat{\mathbf{Z}}$ whose kernel $\pi_1^{t,ab}(X_{\mathbf{F}})^0$ can be identified with the coinvariants of $\pi_1^{t,ab}(X)$ under the action of G. Therefore taking coinvariants under Frobenius in the exact sequence (1.13) yields the sequence

$$0 \to T_G \longrightarrow \pi_1^{t,ab}(X_{\mathbf{F}})^0 \longrightarrow \mathrm{Alb}_{X_{\mathbf{F}}}(\mathbf{F}) \longrightarrow 0 \tag{1.20}$$

(for exactness note that a semi-abelian variety over a finite field has only finitely many rational points and therefore the Frobenius acting on its Tate module has no eigenvalue 1). There are similar exact sequences over each finite extension $\mathbf{F}'$ of $\mathbf{F}$ which naturally form a direct system. (One way to see this is that coinvariants under Frobenius form the first Galois cohomology group over a finite field, so the maps in the direct system are just the restriction maps.) The direct limit of the finite groups $T_{Gal(k|\mathbf{F}')}$ is trivial (this is a general fact for the first cohomology of a finite Galois module over any field; over a sufficiently large extension such a module becomes isomorphic to a sum of $\mathbf{Z}/m$'s and one may conclude e.g. by using Kummer and Artin-Schreier theory).

Now by the main result of [58] the middle group in (1.20) is isomorphic to $h_0(X_{\mathbf{F}})^0$ by means of a reciprocity map $h_0(X_{\mathbf{F}})^0 \to \pi_1^{t,ab}(X_{\mathbf{F}})$ which sends the class of a closed point of X to the class of its Frobenius. Using this isomorphism and taking the direct limit over finite extensions of $\mathbf{F}$ as above we thus get an isomorphism $h_0(X)^0 \cong \mathrm{Alb}_X(k)$.

It remains to see that it is induced by the Albanese map. For this it suffices to consider the image of the class of a zero-cycle of the form $P_1 - P_2$ and we may replace k by the finite extension of $\mathbf{F}$ over which both P_1 and P_2 are defined. Moreover, by using the covariant functoriality of the Albanese map and of the reciprocity map we may assume that P_1 and P_2 both lie on some smooth curve C and check the required compatibility for C, but this is a well-known property of Lang's class field theory (see [61]).

Finally we note that the above proof has the following interesting by-product, generalising the similar statement proved in [37] for the proper case:

Corollary 1.6.1 *With notations as above, the natural map $h_0(X_{\mathbf{F}}) \to h_0(X)$ has a finite kernel isomorphic to the group T introduced in Proposition 1.4.4.*

Chapter 2

Arithmetic duality theorems for 1-motives

2.1 Introduction

Duality theorems for the Galois cohomology of commutative group schemes over local and global fields are among the most fundamental results in arithmetic. Let us briefly and informally recall some of the most famous ones.

Perhaps the earliest such result is the following. Given an algebraic torus T with character group Y^* defined over a p-adic field K, cup-products together with the isomorphism $\mathrm{Br}(K) = H^2(K, \mathbf{G}_m) \cong \mathbf{Q}/\mathbf{Z}$ given by the invariant map of the Brauer group of K define canonical pairings

$$H^i(K,T) \times H^{2-i}(K,Y^*) \to \mathbf{Q}/\mathbf{Z}$$

for $i = 0,1,2$. The Tate-Nakayama duality theorem (whose original form can be found in [68]) then asserts that these pairings become perfect if in the cases $i \neq 1$ we replace the groups H^0 by their profinite completions. Note that this theorem subsumes the reciprocity isomorphism of local class field theory which is the case $i = 0$, $T = \mathbf{G}_m$.

Next, in his influential exposé [69], Tate observed that given an abelian variety A, the Poincaré pairing between A and its dual A^* enables one to construct similar pairings

$$H^i(K,A) \times H^{1-i}(K,A^*) \to \mathbf{Q}/\mathbf{Z}$$

for $i = 0,1$ and he proved that these pairings are also perfect.

The last result we recall is also due to Tate. Consider now an abelian variety A over a number field k, and denote by $Ш^1(A)$ the Tate-Shafarevich group formed by isomorphism classes of torsors under A that split over each completion of k. Then Tate constructed a duality pairing

$$Ш^1(A) \times Ш^1(A^*) \to \mathbf{Q}/\mathbf{Z}$$

(generalising earlier work of Cassels on elliptic curves) and announced in [70] that this pairing is nondegenerate modulo divisible subgroups or else, if one believes the widely known conjecture on the finiteness of $Ш^1(A)$, it is a perfect pairing of finite abelian groups. Similar results for tori are attributed to Kottwitz in the literature; indeed, the references [38] and [39] contain such statements, but without (complete) proofs.

In this chapter we establish common generalizations of the results mentioned above for *1-motives.* Recall that according to Deligne, a 1-motive over a field F is a two-term complex M of F-group schemes $[Y \to G]$ (placed in degrees -1 and 0), where Y is the F-group scheme associated to a finitely generated free abelian group equipped with a continuous $\mathrm{Gal}(F)$-action and G is a semi-abelian variety over F, i.e. an extension of an abelian variety A by a torus T. As we shall recall in the next section, every 1-motive M as above has a *Cartier dual* $M^* = [Y^* \to G^*]$ equipped with a canonical (derived) pairing $M \otimes^{\mathbf{L}} M^* \to \mathbf{G}_m[1]$ generalising the ones used above in the cases $M = [0 \to T]$ and $M = [0 \to A]$. This enables one to construct duality pairings for the Galois hypercohomology groups of M and M^* over local and global fields.

Let us now state the main results. In Section 2.3, we shall prove:

Theorem 2.1.1 *Let K be a local field and let $M = [Y \to G]$ be a 1-motive over K. For $i = -1, 0, 1, 2$ there are canonical pairings*

$$\mathbf{H}^i(K, M) \times \mathbf{H}^{1-i}(K, M^*) \to \mathbf{Q}/\mathbf{Z}$$

inducing perfect pairings between

1. *the profinite group $\mathbf{H}^{-1}_{\wedge}(K, M)$ and the discrete group $\mathbf{H}^2(K, M^*)$;*
2. *the profinite group $\mathbf{H}^0(K, M)^{\wedge}$ and the discrete group $\mathbf{H}^1(K, M^*)$.*

Here the groups $\mathbf{H}^0(K, M)^{\wedge}$ and $\mathbf{H}^{-1}_{\wedge}(K, M)$ are obtained from the corresponding hypercohomology groups by certain completion procedures explained in Section 2.3. We shall also prove there a generalization of the above theorem to 1-motives over henselian local fields of mixed characteristic and show that in the duality pairing the unramified parts of the cohomology are exact annihilators of each other.

Now let M be a 1-motive over a number field k. For all $i \geq 0$ define the Tate-Shafarevich groups

$$Ш^i(M) = \mathrm{Ker}\,[\mathbf{H}^i(k, M) \to \prod_v \mathbf{H}^i(\hat{k}_v, M)]$$

where the product is taken over completions of k at all (finite and infinite) places of k. Our most important result can then be summarized as follows.

Theorem 2.1.2 *Let k be a number field and M a 1-motive over k. There exist canonical pairings*

$$Ш^i(M) \times Ш^{2-i}(M^*) \to \mathbf{Q}/\mathbf{Z}$$

for $i = 0, 1$.

For $i = 1$ *the pairing is non-degenerate modulo maximal divisible subgroups.*

For $i = 0$ *it is a perfect pairing between a compact and a discrete topological group, provided that we replace* $Ш^0(M)$ *by a certain modification* $Ш^0_\wedge(M)$*, and assume the finiteness of* $Ш^1(A)$ *for the abelian quotient A.*

See the beginning of Section 2.6 for the definition of $Ш^0_\wedge(M)$. Assuming the finiteness of the (usual) Tate-Shafarevich group of an abelian variety one derives that for $i = 1$ the pairing is a perfect pairing of finite groups.

The pairings used here can be defined purely in terms of Galois cohomology (see Section 2.7); however, to prove the duality isomorphisms we first construct pairings using étale cohomology in Sections 2.4 and 2.5, and then in the section 2.7 we compare them to the Galois-cohomological one which in the case of abelian varieties gives back the classical construction of Tate.

Finally, in Section 2.6 we establish a twelve-term Poitou-Tate type exact sequence similar to the one for finite modules, assuming the finiteness of the Tate-Shafarevich group. The reader is invited to look up the precise statement there.

Since the chapter title contains the word "motive", it is appropriate to explain our motivations for establishing the generalizations offered here. The first of these should be clear from the above: working in the context of 1-motives gives a unified and symmetric point of view on the classical duality theorems cited above and gives more complete results than those known before. As an example, we may cite the duality between $Ш^i(T)$ and $Ш^{3-i}(Y^*)$ ($i = 1, 2$) for an algebraic torus T with character group Y^* which is a special case of Theorem 2.1.2 above (see Section 2.5); it is puzzling to note that the reference [49] only contains the case $i = 1$, whereas [47] only the case $i = 2$. Another obvious reason is that due to recent spectacular progress in the theory of mixed motives there has been a regain of interest in 1-motives as well; indeed, the category of 1-motives over a field (with obvious morphisms) is equivalent, up to torsion, to the subcategory of the triangulated category of mixed motives (as defined, e.g., by Voevodsky) generated by motives of varieties of dimension at most 1.

But there is a motivation coming solely from the arithmetic duality theory. In fact, if one tries to generalize the classical duality theorems of Tate to a semi-abelian variety G, one is already confronted to the fact that the only reasonable definition for the dual of G is the dual $[Y^* \to A^*]$ of the 1-motive $[0 \to G]$, where actually A^* is the dual of the abelian quotient of G and Y^* is the character group of its toric part. Duality results of this type are needed for the study of the arithmetic of G over local and global fields.

This chapter is based on my joint paper [30] with David Harari. The original version unfortunately contained some inaccuracies and gaps – this seems to be a plague affecting almost all publications on the subject. The present text therefore incorporates the corrections published in the 2009 corrigendum to [30]; it also contains a last section reviewing subsequent developments.

Some notation and conventions

Let B be an abelian group. For each integer $n>0$, $B[n]$ stands for the n-torsion subgroup of B and B_{tors} for the whole torsion subgroup of B. We shall often abbreviate the quotient B/nB by B/n. For any prime number ℓ, we denote by $B\{\ell\}$ the ℓ-primary torsion subgroup of B and by $\bar{B}\{\ell\}$ the quotient of $B\{\ell\}$ by its maximal divisible subgroup. Also, we denote by $B^{(\ell)}$ the ℓ-adic completion of B, i.e. the projective limit $\varprojlim B/\ell^n B$.

For a topological group B, we denote by $B^{\wedge}$ the completion of B with respect to *open* subgroups of finite index (in the discrete case this is the usual profinite completion of B). We set B^D for the group of *continuous* homomorphisms $B\to\mathbf{Q}/\mathbf{Z}$ (in the discrete case these are just all homomorphisms). We equip B^D with the compact-open topology. The topological group B is *compactly generated* if B contains a compact subset K such that K generates B as a group. A continuous morphism $f: B\to C$ of topological groups is *strict* if the image of any open subset of B is an open subset of $\mathrm{Im}\, f$ for the topology induced by C.

2.2 Preliminaries on 1-motives

Let S be a scheme. Denote by $\mathscr{F}_S$ the category of *fppf* sheaves of abelian groups over S (when $S=\mathrm{Spec}\, K$ is the spectrum of a field, we shall write $\mathscr{F}_K$ for $\mathscr{F}_S$). Write $\mathscr{C}^b(\mathscr{F}_S)$ for the category of bounded complexes of *fppf* sheaves over S and $\mathscr{D}^b(\mathscr{F}_S)$ for the associated derived category. Recall (e.g. from [54]) that a *1-motive M* over S consists of the following data :

- An S-group scheme Y which is étale locally isomorphic to $\mathbf{Z}^r$ for some $r\geq 0$.
- A commutative S-group scheme G fitting into an exact sequence of S-groups

  $$0\to T\to G\xrightarrow{p} A\to 0$$

  where T is an S-torus and A an abelian scheme over S.
- An S-homomorphism $u: Y\to G$.

The 1-motive M can be viewed as a complex of *fppf* S-sheaves $[Y \xrightarrow{u} G]$, with Y put in degree -1 and G in degree 0, and also as an object of the derived category $\mathscr{D}^b(\mathscr{F}_S)$. It is equipped with a 3-term weight filtration: $W_i(M) = 0$ for $i \leq -3$, $W_{-2}(M) = [0 \to T]$, $W_{-1}(M) = [0 \to G]$ and $W_i(M) = M$ for $i \geq 0$. From this we shall only need the 1-motive $M/W_{-2}(M)$, i.e. the complex $[Y \xrightarrow{h} A]$, where $h = p \circ u$. By [54], Proposition 2.3.1, we can identify morphisms of 1-motives in $\mathscr{C}^b(\mathscr{F}_S)$ and $\mathscr{D}^b(\mathscr{F}_S)$.

To each 1-motive M one can associate a *Cartier dual* M^* by the following construction which we briefly recall. Denote by Y^* the group of characters of T, by A^* the abelian scheme dual to A, and by T^* the S-torus with character group Y. According to the generalized Barsotti-Weil formula ([51], III.18), A^* represents the functor $S' \mapsto \mathrm{Ext}_{S'}(A, \mathbf{G}_m)$ on $\mathscr{F}_S$. Writing $M' = M/W_{-2}M$, one deduces from this that the functor $S' \mapsto \mathrm{Ext}_{S'}(M', \mathbf{G}_m)$ on $\mathscr{C}^b(\mathscr{F}_S)$ is representable by an S-group scheme G^* which is an extension of A^* by T^*. One calls the 1-motive $[0 \to G^*]$ the *(Cartier) dual of* M'. Pulling back the *Poincaré biextension* ([47], p. 395) on $A \times A^*$ to $A \times G^*$ one gets a *biextension* $\mathscr{P}'$ of the 1-motives M' and $[0 \to G^*]$ by $\mathbf{G}_m$, which is a $\mathbf{G}_m$-torsor over $A \times G^*$ whose pullback to $Y \times G^*$ by the natural map $Y \times G^* \to A \times G^*$ is trivial (cf. [16], 10.2.1 for this definition).

To treat the general case, consider M as an extension of M' by T. Any element of $Y^* = \mathrm{Hom}_S(T, \mathbf{G}_m)$ then induces by pushout an extension of M' by $\mathbf{G}_m$, i.e. an element of G^*. Whence a map $Y^* \to G^*$ which is in fact a map of S-group schemes; we call the associated 1-motive M^* the *(Cartier) dual of* M. The pull-back $\mathscr{P}$ of the biextension $\mathscr{P}'$ from $A \times G^*$ to $G \times G^*$ becomes trivial over $G \times Y^*$ when pulled back by the map $G \times Y^* \to G \times G^*$, hence defines a biextension of M and M^* by $\mathbf{G}_m$ (again in the sense of [16]).

According to the formula

$$\mathrm{Biext}_S(M, M^*, \mathbf{G}_m) \cong \mathrm{Hom}_{\mathscr{D}^b(\mathscr{F}_S)}(M \otimes^{\mathbf{L}} M^*, \mathbf{G}_m[1])$$

of ([16], 10.2.1), the biextension $\mathscr{P}$ defines a map

$$\Phi_M : M \otimes^{\mathbf{L}} M^* \to \mathbf{G}_m[1] \tag{2.1}$$

in $\mathscr{D}^b(\mathscr{F}_S)$, whence pairings

$$\mathbf{H}^i(S, M) \times \mathbf{H}^j(S, M^*) \to H^{i+j+1}(S, \mathbf{G}_m) \tag{2.2}$$

for each $i, j \geq 0$.

(Except when explicitly specified, the cohomology groups in this paper are relative to the étale topology; here we can work with either the étale or the *fppf* topology because G is smooth over S and Y is étale locally constant).

Remark 2.2.1 Two special cases of this construction are classical :

- $Y = A = 0$, $M = [0 \to T]$, $M^* = Y^*[1]$. Then the pairing (2.2) is just the cup-product $H^r(S,T) \times H^{s+1}(S,Y^*) \to H^{r+s+1}(S,\mathbf{G}_m)$. (Similarly for $T = A = 0$, $M = Y[1]$, $M^* = T^*$.)
- $Y = T = 0$, $M = [0 \to A]$, $M^* = [0 \to A^*]$. Then (2.2) is the well-known pairing in the cohomology of abelian varieties coming from the generalized Barsotti-Weil formula (compare [47], p. 243 and Chapter III, Appendix C).

Recall finally ([16], 10.1.5 and 10.1.10) that for any integer n invertible on S and 1-motive M one has an "n-adic realization", namely the finite sheaf (or group scheme) defined by

$$T_{\mathbf{Z}/n\mathbf{Z}}(M) = H^0(M[-1] \otimes^{\mathbf{L}} \mathbf{Z}/n\mathbf{Z}),$$

which can be explicitly calculated using the flat resolution $[\mathbf{Z} \xrightarrow{n} \mathbf{Z}]$ of $\mathbf{Z}/n\mathbf{Z}$. The pairing (2.1) then induces a perfect pairing

$$T_{\mathbf{Z}/n\mathbf{Z}}(M) \otimes T_{\mathbf{Z}/n\mathbf{Z}}(M^*) \to \mu_n \tag{2.3}$$

where μ_n is the sheaf of n-th roots of unity. The classical case is when M is of the form $[0 \to A]$ with A an abelian variety, where we find the well-known Weil pairing.

We finish this section by introducing some notation: for each prime number ℓ invertible on S, we denote by $T(M)\{\ell\}$ the direct limit of the $T_{\mathbf{Z}/\ell^n\mathbf{Z}}(M)$ over all $n > 0$ and by $T_\ell(M)$ their inverse limit. The piece of notation $T(M)_{\mathrm{tors}}$ stands for the direct sum (taken over all primes ℓ invertible on S) of the groups $T_{\mathbf{Z}/\ell^n\mathbf{Z}}(M)$.

2.3 Local results

In this section S is the spectrum of a field K, complete with respect to a discrete valuation and with finite residue field. In particular K is a p-adic field if $\operatorname{char} K = 0$, and is isomorphic to the field $\mathbf{F}_q((t))$ for some finite field $\mathbf{F}_q$ if $\operatorname{char} K > 0$. We let $\mathscr{O}_K$ denote the ring of integers of K and $\mathbf{F}$ its residue field.

Lemma 2.3.1 *For a 1-motive $M = [Y \to G]$ over K, we have*

- $\mathbf{H}^{-1}(K,M) \cong \operatorname{Ker}[H^0(K,Y) \to H^0(K,G)]$*, a finitely generated free abelian group;*
- $\mathbf{H}^2(K,M) \cong \operatorname{Coker}[H^2(K,Y) \to H^2(K,G)]$*;*
- $\mathbf{H}^i(K,M) = 0 \qquad i \neq -1,0,1,2.$

Proof: The field K has strict Galois cohomological dimension 2 ([47], I.1.12). Since G is smooth, $H^i(K,G) = 0$ for any $i > 2$; by [47], I.2.1, we also have $H^i(K,Y) = 0$ for $i > 2$, whence the last equality. For the first two, use moreover the distinguished triangle

$$Y \to G \to M \to Y[1] \tag{2.4}$$

in $\mathscr{C}^b(\mathscr{F}_K)$. □

Using the trace isomorphism $H^2(K,\mathbf{G}_m) \cong \mathbf{Q}/\mathbf{Z}$ of local class field theory, the pairing (2.2) of the previous section induces bilinear pairings

$$\mathbf{H}^i(K,M) \times \mathbf{H}^{1-i}(K,M^*) \to \mathbf{Q}/\mathbf{Z} \tag{2.5}$$

for all integers i (by the previous lemma, they are trivial for $i \neq -1,0,1,2$).

For $i = -1,1,2$, we endow the group $\mathbf{H}^i(K,M)$ with the discrete topology. To topologize $\mathbf{H}^0(K,M)$ we proceed as follows. The exact triangle (2.4) yields an exact sequence of abelian groups

$$0 \to L \to G(K) \to \mathbf{H}^0(K,M) \to H^1(K,Y) \to H^1(K,G) \tag{2.6}$$

where $L := H^0(K,Y)/\mathbf{H}^{-1}(K,M)$ is a discrete abelian group of finite type. We equip $I = G(K)/\mathrm{Im}\,(L)$ with the quotient topology (note that in general it is not Hausdorff). The cokernel of the map $G(K) \to \mathbf{H}^0(K,M)$ being finite (as $H^1(K,Y)$ itself is finite by [62], II.5.8 *iii*)), we can define a natural topology on $\mathbf{H}^0(K,M)$ by taking as a basis of open neighbourhoods of zero the open neighbourhoods of zero in I (this makes I an open subgroup of finite index of $\mathbf{H}^0(K,M)$).

Already in the classical duality theorem for tori over local fields one has to take the profinite completion on H^0 in order to obtain a perfect pairing. However, for the generalizations we have in mind a nuisance arises from the fact that the completion functor is not always left exact, even if one works only with discrete lattices and p-adic Lie groups. As a simple example, consider $K = \mathbf{Q}_p$ ($p \geq 3$) and the injection $\mathbf{Z} \hookrightarrow \mathbf{Q}_p^\times$ given by sending 1 to $1+p$. Here the induced map on completions $\widehat{\mathbf{Z}} \to (\mathbf{Q}_p^\times)^\wedge$ is not injective (because $\mathbf{Q}_p^\times \simeq \mathbf{Z} \times \mathbf{F}_p^\times \times \mathbf{Z}_p$ and the image of $\mathbf{Z}$ lands in the $\mathbf{Z}_p$-component).

Bearing this in mind, for a 1-motive $M = [Y \to G]$ we denote by $\mathbf{H}^{-1}_\wedge(K,M)$ the kernel of the map $H^0(K,Y)^\wedge \to H^0(K,G)^\wedge$ coming from $Y \to G$. There is always a surjection $\mathbf{H}^{-1}(K,M)^\wedge \to \mathbf{H}^{-1}_\wedge(K,M)$ but it is not an isomorphism in general; the previous example comes from the 1-motive $[\mathbf{Z} \to \mathbf{G}_m]$.

However, we shall also encounter a case where the completion functor behaves well.

Lemma 2.3.2 *Let G be a semi-abelian variety over the local field K, with abelian quotient A and toric part T. Then the natural sequence*

$$0 \to T(K)^\wedge \to G(K)^\wedge \to A(K)^\wedge \to H^1(K,T)^\wedge$$

is exact. Moreover, $G(K) \hookrightarrow G(K)^\wedge$ and $(G(K)^\wedge)^D = G(K)^D$.

Here in fact we have $A(K)^\wedge = A(K)$ (the group $A(K)$ being compact and completely disconnected, hence profinite) and $H^1(K,T)^\wedge = H^1(K,T)$ by finiteness of $H^1(K,T)$ ([47], I.2.3).

Proof: To begin with, the maps between completions are well defined because the maps $T(K) \to G(K)$, $G(K) \to A(K)$, and $A(K) \to H^1(K,T)$ are continuous (by [43], I.2.1.3, $T(K)$ is closed in $G(K)$ and the image of $G(K)$ is open in $A(K)$ by the implicit function theorem). The theory of Lie groups over a local field shows that $G(K)$ is locally compact, completely disconnected, and compactly generated; we conclude with the third part of the proposition proven in the appendix. $\square$

Now we can state the main result of this section.

Theorem 2.3.3 *Let $M = [Y \to G]$ be a 1-motive over the local field K. The pairing (2.5) induces a perfect pairing between*

1. *the profinite group $\mathbf{H}^{-1}_\wedge(K,M)$ and the discrete group $\mathbf{H}^2(K,M^*)$;*
2. *the profinite group $\mathbf{H}^0(K,M)^\wedge$ and the discrete group $\mathbf{H}^1(K,M^*)$.*

In the special cases $M = [0 \to T]$ or $M = [Y \to 0]$ we recover Tate-Nakayama duality for tori over K ([62], II.5.8 and [47], I.2.3 for the positive characteristic case) and in the case $M = [0 \to A]$ we recover Tate's p-adic duality theorem for abelian varieties and its generalization to the positive characteristic case due to Milne ([69], [47], Cor. I.3.4, and Theorem III.7.8).

Proof: For the first statement, set $M' := M/W_{-2}M$. The dual of M' is of the form $[0 \to G^*]$, where G^* is an extension of A^* by T^*. Via the pairing (2.5) for $i = -1, 0$, we obtain a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathbf{H}^{-1}_\wedge(K,M') & \longrightarrow & H^0(K,Y)^\wedge & \longrightarrow & H^0(K,A)^\wedge \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & H^2(K,G^*)^D & \longrightarrow & H^2(K,T^*)^D & \longrightarrow & H^1(K,A^*)^D
\end{array}$$

The first line of this diagram is exact by definition, and the second one is exact because it is the dual of an exact sequence of discrete groups (recall that

$H^2(K,A^*) = 0$ by [62], II.5.3, Prop. 16 and [47], III.7.8). By Tate duality for abelian varieties and Tate-Nakayama duality for tori, the last two vertical maps are isomorphisms, hence the same holds for the first one.

Now using Lemma 2.3.2 we get that the map $H^0(K,Y) \to H^0(K,G)$ induces a map $\mathbf{H}^{-1}_\wedge(K,M') \to T(K)^\wedge$ with kernel $\mathbf{H}^{-1}_\wedge(K,M)$. ¿From the definition of M' we get a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{H}^{-1}_\wedge(K,M) \arrow[r] \arrow[d] & \mathbf{H}^{-1}_\wedge(K,M') \arrow[r] \arrow[d] & H^0(K,T)^\wedge \arrow[d] \\
0 \arrow[r] & \mathbf{H}^2(K,M^*)^D \arrow[r] & H^2(K,G^*)^D \arrow[r] & H^2(K,Y^*)^D
\end{tikzcd}$$

whence we conclude as above that the left vertical map is an isomorphism, by the first part and Tate-Nakayama duality. Then $H^2(K,M^*) \cong H^{-1}_\wedge(K,M)^D$ follows by dualising, using the isomorphism $H^2(K,M^*)^{DD} \cong H^2(K,M^*)$ for the discrete torsion group $H^2(K,M^*)$.

For the second statement, we also begin by working with M'. Using the pairings (2.5) and Lemma 2.3.2 (applied to G^*), we get a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & T^*(K)^\wedge \arrow[r] \arrow[d] & G^*(K)^\wedge \arrow[r] \arrow[d] & A^*(K)^\wedge \arrow[r] \arrow[d] & H^1(K,T^*)^\wedge \arrow[d] \\
0 \arrow[r] & H^2(K,Y)^D \arrow[r] & \mathbf{H}^1(K,M')^D \arrow[r] & H^1(K,A)^D \arrow[r] & H^1(K,Y)^D
\end{tikzcd}$$

Using the local dualities for (A,A^*) and (Y,T^*), this implies that the map $G^*(K)^\wedge \to \mathbf{H}^1(K,M')^D$ is an isomorphism.

Now the distinguished triangle $T \to M \to M' \to T[1]$ in $\mathscr{C}^b(\mathscr{F}_K)$ induces the following commutative diagram with exact rows:

$$\begin{tikzcd}
H^0(K,Y^*)^\wedge \arrow[r] \arrow[d] & G^*(K)^\wedge \arrow[r] \arrow[d] & \mathbf{H}^0(K,M^*)^\wedge \arrow[r] \arrow[d] & H^1(K,Y^*) \arrow[r] \arrow[d] & H^1(K,G^*) \arrow[d] \\
H^2(K,T)^D \arrow[r] & \mathbf{H}^1(K,M')^D \arrow[r] & \mathbf{H}^1(K,M)^D \arrow[r] & H^1(K,T)^D \arrow[r] & \mathbf{H}^0(K,M')^D
\end{tikzcd}$$

Here the exactness of the rows needs some justification. The upper row is exact without completing the first three terms. Completion in the third term is possible by finiteness of the fourth, and completion in the first two terms is possible because the map $G^*(K) \to \mathbf{H}^0(K,M^*)$ is open with finite cokernel by definition of the topology on the target. In the lower row dualization behaves well because the first four terms are duals of discrete torsion groups.

By Tate-Nakayama duality for tori and what we have already proven, the first, second and fourth vertical maps are isomorphisms. To derive an isomorphism in the middle it remains to prove the injectivity of the fifth map.

This in turn follows from the commutative diagram with exact rows (where again we have used the finiteness of $H^1(K,T^*)$ and of $H^1(K,Y)$):

$$\begin{array}{ccccccc} A^*(K) & \longrightarrow & H^1(K,T^*) & \longrightarrow & H^1(K,G^*) & \longrightarrow & H^1(K,A^*) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^1(K,A)^D & \longrightarrow & H^1(K,Y)^D & \longrightarrow & \mathbf{H}^0(K,M')^D & \longrightarrow & A(K)^D \end{array}$$

Here the first, second and fourth vertical maps are isomorphisms by local duality for tori and abelian varieties. Again, exactness at the third term of the lower row follows from the definition of the topology on $\mathbf{H}^0(K,M')$. Finally the map $\mathbf{H}^0(K,M^*)^\wedge \to \mathbf{H}^1(K,M)^D$ is an isomorphism and applying this statement to M^* instead of M, we obtain the theorem. □

Remark 2.3.4 If K is of characteristic zero, any subgroup of finite index of $T(K)$ is open (cf. [47], p.32). It is easy to see that in this case $\mathbf{H}^0(K,M)^\wedge$ is just the profinite completion of $\mathbf{H}^0(K,M)$.

Next we state a version of Theorem 2.3.3 for henselian fields that will be needed for the global theory.

Theorem 2.3.5 *Let F be the field of fractions of a henselian discrete valuation ring R with finite residue field and let M be a 1-motive over F. Assume that F is of characteristic zero. Then the pairing (2.5) induces perfect pairings*

$$\mathbf{H}^{-1}_\wedge(F,M) \times \mathbf{H}^2(F,M^*) \to \mathbf{Q}/\mathbf{Z}$$

$$\mathbf{H}^0(F,M)^\wedge \times \mathbf{H}^1(F,M^*) \to \mathbf{Q}/\mathbf{Z}$$

where $\mathbf{H}^{-1}_\wedge(F,M) := \mathrm{Ker}\,[H^0(F,Y)^\wedge \to G(F)^\wedge]$ *and* $^\wedge$ *means profinite completion.*

Remarks 2.3.6

1. Denoting by K the completion of F, the group $\mathbf{H}^0(F,M)$ injects into $\mathbf{H}^0(K,M)$ by the lemma below, hence it is natural to equip $\mathbf{H}^0(F,M)$ with the topology induced by $\mathbf{H}^0(K,M)$. But we shall also show that $\mathbf{H}^0(F,M)$ and $\mathbf{H}^0(K,M)$ have the same profinite completion, hence by Remark 2.3.4 the profinite completion of $\mathbf{H}^0(F,M)$ coincides with its completion with respect to open subgroups of finite index. Therefore there is no incoherence in the notation.

2. In characteristic $p > 0$, the analogue of Theorem 2.3.5 is not clear because of the p-part of the groups. Compare [47], III.6.13.

Taking the first remark into account, Theorem 2.3.5 immediately results from Theorem 2.3.3 via the following lemma.

Lemma 2.3.7 *Keeping the assumptions of the theorem, denote by K the completion of F. Then the natural map $\mathbf{H}^i(F,M) \to \mathbf{H}^i(K,M)$ is an injection for $i = 0$ inducing an isomorphism $\mathbf{H}^0(F,M)^\wedge \to \mathbf{H}^0(K,M)^\wedge$ on completions, and an isomorphism for $i \geq 1$.*

Proof: For any $n > 0$, the canonical map $G(F)/n \to G(K)/n$ is surjective, for $G(F)$ is dense in $G(K)$ by Greenberg's approximation theorem [26], and $nG(K)$ is an open subgroup in $G(K)$. But this map is also injective, for any point P in $G(K)$ with $nP \in G(F)$ is locally given by coordinates algebraic over F, but F is algebraically closed in K (apply e.g. [48], Theorem 4.11.11 and note that F is of characteristic 0), hence $P \in G(F)$. Since Y is locally constant in the étale topology over $\operatorname{Spec} F$, we have $H^i(F,Y) = H^i(K,Y)$ for each $i \geq 0$. The case $i = 0$ of the lemma follows from these facts by dévissage.

To treat the cases $i > 0$, recall first that multiplication by n on G is surjective in the étale topology. Therefore

$$H^i(F,G)[n] = \operatorname{coker}[H^{i-1}(F,G)/n \to H^i(F,G[n])]$$

for $i \geq 1$, and similarly for $H^i(K,G)$. Moreover, $H^i(F,G[n]) = H^i(K,G[n])$ because $G[n]$ is locally constant in the étale topology (note that F and K have the same absolute Galois group). Starting from the isomorphism $G(F)/n \cong G(K)/n$ already proven, we thus obtain isomorphisms of torsion abelian groups $H^i(F,G) \simeq H^i(K,G)$ for any $i \geq 1$ by induction on i, which together with the similar isomorphisms for Y mentioned above yield the statement by dévissage. □

We shall also need the following slightly finer statement.

Proposition 2.3.8 *Keeping the notations above, equip $\mathbf{H}^0(F,M^*)$ with the topology induced by $\mathbf{H}^0(K,M^*)$. Then $\mathbf{H}^0(F,M^*)$ and $\mathbf{H}^0(F,M^*)^\wedge$ have the same continuous dual. Moreover, the pairing (2.5) yields an isomorphism*

$$\mathbf{H}^1(F,M) \cong \mathbf{H}^0(F,M^*)^D.$$

Proof: We use the exact sequence

$$H^0(F,Y^*) \to G^*(F) \to \mathbf{H}^0(F,M^*) \to H^1(F,Y^*) \to H^1(F,G^*)$$

and similarly for K instead of F. Since $G^*(F)$ is a dense subgroup of $G^*(K)^\wedge$, they have the same dual. Similarly, we see that the map $H^0(F,Y^*) \to H^0(K,Y^*)^\wedge$ induces an isomorphism on duals using the fact that $H^0(K,Y^*)$ is of finite type. Thus the first statement follows from the exact sequence using the finiteness of the groups in $H^1(F,Y^*) \hookrightarrow H^1(K,Y^*)$ and Lemma 2.3.7. The second statement now follows from Theorem 2.3.3, again using Lemma 2.3.7. □

For the global case, we shall also need a statement for the real case. Consider a 1-motive $M_{\mathbf{R}}$ over the spectrum of the field $\mathbf{R}$ of real numbers. As in the classical cases, the duality results in the previous section extend in a straightforward fashion to this situation, provided that we replace usual Galois cohomology groups by Tate modified groups. Denote by $\Gamma_{\mathbf{R}} = \mathrm{Gal}\,(\mathbf{C}/\mathbf{R}) \simeq \mathbf{Z}/2$ the Galois group of $\mathbf{R}$. Let $\mathscr{F}^\bullet$ be a bounded complex of $\mathbf{R}$-groups. For each $i \in \mathbf{Z}$, the modified hypercohomology groups $\widehat{\mathbf{H}}^i(\mathbf{R},\mathscr{F}^\bullet)$ are defined in the usual way: for each term $\mathscr{F}^i$ of $\mathscr{F}$, we take the standard Tate complex associated to the $\Gamma_{\mathbf{R}}$-module $\mathscr{F}^i(\mathbf{C})$ (cf. [47], pp. 2–3); then we obtain Tate hypercohomology groups via the complex associated to the arising double complex. From the corresponding well-known results in Galois cohomology, it is easy to see that $\widehat{\mathbf{H}}^i(\mathbf{R},\mathscr{F}^\bullet) = \mathbf{H}^i(\mathbf{R},\mathscr{F}^\bullet)$ for $i \geq 1$ if $\mathscr{F}^\bullet$ is concentrated in nonpositive degrees, and that $\widehat{\mathbf{H}}^i(\mathbf{R},\mathscr{F}^\bullet)$ is isomorphic to $\widehat{\mathbf{H}}^{i+2}(\mathbf{R},\mathscr{F}^\bullet)$ for any $i \in \mathbf{Z}$. Recall also that the Brauer group $\mathrm{Br}\,\mathbf{R}$ is isomorphic to $\mathbf{Z}/2\mathbf{Z} \subset \mathbf{Q}/\mathbf{Z}$ via the local invariant.

Now we have the following analogue of Theorem 2.3.3:

Proposition 2.3.9 *Let $M_{\mathbf{R}} = [Y_{\mathbf{R}} \to G_{\mathbf{R}}]$ be a 1-motive over $\mathbf{R}$. Then the cup-product pairing induces a perfect pairing of finite 2-torsion groups*

$$\widehat{\mathbf{H}}^0(\mathbf{R},M_{\mathbf{R}}) \times \widehat{\mathbf{H}}^1(\mathbf{R},M^*_{\mathbf{R}}) \to \mathbf{Z}/2\mathbf{Z}$$

Proof: Let $T_{\mathbf{R}}$ (resp. $A_{\mathbf{R}}$) be the torus (resp. the abelian variety) corresponding to $G_{\mathbf{R}}$. In the special cases $M_{\mathbf{R}} = T_{\mathbf{R}}$, $M_{\mathbf{R}} = A_{\mathbf{R}}$, $M_{\mathbf{R}} = Y_{\mathbf{R}}[1]$, the result is known ([47], I.2.13 and I.3.7). Now the proof by devissage consists exactly of the same steps as in Theorem 2.3.3, except that we don't have to take any profinite completions, all occurring groups being finite. □

In Section 5 we shall need the fact that when M is a 1-motive over a local field K which extends to a 1-motive over $\mathrm{Spec}\,\mathscr{O}_K$, the unramified parts of the cohomology are exact annihilators of each other in the local duality pairing for $i=1$ (see [62], II.5.5, [49], Theorem 7.2.15 and [47], III.1.4 for analogues for finite modules). More precisely, let $\mathscr{M} = [\mathscr{Y} \to \mathscr{G}]$ be a 1-motive over

dc_101_10

Spec $\mathscr{O}_K$ and $M = [Y \to G]$ the restriction of $\mathscr{M}$ to Spec K. Denote by $\mathbf{H}^0_{\mathrm{nr}}(K,M)$ and $\mathbf{H}^1_{\mathrm{nr}}(K,M^*)$ the respective images of the maps $\mathbf{H}^0(\mathscr{O}_K,\mathscr{M}) \to \mathbf{H}^0(K,M)$ and $\mathbf{H}^1(\mathscr{O}_K,\mathscr{M}^*) \to \mathbf{H}^1(K,M^*)$. To make the notation simpler, we still let $\mathbf{H}^0_{\mathrm{nr}}(K,M)^\wedge$ denote the image of $\mathbf{H}^0_{\mathrm{nr}}(K,M)^\wedge$ in $\mathbf{H}^0(K,M)^\wedge$. (We work with complete fields since this is what will be needed later; the henselian case is similar in mixed characteristic.)

Theorem 2.3.10 *In the above situation,* $\mathbf{H}^0_{\mathrm{nr}}(K,M)^\wedge$ *and* $\mathbf{H}^1_{\mathrm{nr}}(K,M^*)$ *are the exact annihilators of each other in the pairing*

$$\mathbf{H}^0(K,M)^\wedge \times \mathbf{H}^1(K,M^*) \to \mathbf{Q}/\mathbf{Z}$$

induced by (2.5).

Proof: The restriction of the local pairing to $\mathbf{H}^0_{\mathrm{nr}}(K,M) \times \mathbf{H}^1_{\mathrm{nr}}(K,M^*)$ is zero because $H^2(\mathscr{O}_K,\mathbf{G}_m) \cong H^2(\mathbf{F},\mathbf{G}_m) = 0$. Thus it is sufficient to show that the maps

$$\mathbf{H}^0(K,M)^\wedge/\mathbf{H}^0_{\mathrm{nr}}(K,M)^\wedge \to \mathbf{H}^1_{\mathrm{nr}}(K,M^*)^D,$$
$$\mathbf{H}^1(K,M^*)/\mathbf{H}^1_{\mathrm{nr}}(K,M^*) \to \mathbf{H}^0_{\mathrm{nr}}(K,M)^D$$

are injective, where we have equipped $\mathbf{H}^1_{\mathrm{nr}}(K,M^*)$ with the discrete topology and $\mathbf{H}^0_{\mathrm{nr}}(K,M)$ with the topology induced by that on $\mathbf{H}^0(K,M)$.
Denote by $\mathscr{T}$ (resp. T) the torus and by $\mathscr{A}$ (resp. A) the abelian scheme (resp. abelian variety) corresponding to $\mathscr{G}$ (resp. G). We need the following lemma presumably well known to the experts.

Lemma 2.3.11 *In the Tate-Nakayama pairing*

$$H^2(K,Y) \times H^0(K,T^*) \to \mathbf{Q}/\mathbf{Z},$$

the exact annihilator of $H^0_{\mathrm{nr}}(K,T^*)$ *is* $H^2_{\mathrm{nr}}(K,Y)$.

Proof: Let $n > 0$. We work in flat cohomology. The exact sequence of *fppf* sheaves

$$0 \to \mathscr{T}[n] \to \mathscr{T} \to \mathscr{T} \to 0$$

and the cup-product pairings induce a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
 & & H^1_{fppf}(\mathscr{O}_K,\mathscr{Y}/n\mathscr{Y}) & \longrightarrow & H^2_{fppf}(\mathscr{O}_K,\mathscr{Y})[n] & & \\
 & & \downarrow & & \downarrow & & \\
 & & H^1_{fppf}(K,Y/nY) & \longrightarrow & H^2_{fppf}(K,Y)[n] & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^1_{fppf}(\mathscr{O}_K,\mathscr{T}^*[n])^D & \longrightarrow & H^0(\mathscr{O}_K,\mathscr{T}^*)^D & &
\end{array}$$

where all groups are given the discrete topology. The zero at lower left comes from the vanishing $H^1_{fppf}(\mathscr{O}_K, \mathscr{T}^*) \cong H^1_{fppf}(\mathbf{F}, \widetilde{T}^*) = 0$ (where $\widetilde{T}^*$ stands for the special fibre of $\mathscr{T}^*$), which is a consequence of Lang's theorem ([40], Theorem 2 and [46], III.3.11).

Now by [47], III.1.4. and III.7.2., the left column is exact. Therefore the right column is exact as well. To see that this implies the statement it remains to note that since $\mathscr{Y}$ is a smooth group scheme over $\operatorname{Spec} \mathscr{O}_K$, its étale and flat cohomology groups are the same and moreover they are all torsion in positive degrees. □

We resume the proof of Theorem 2.3.10. The weight filtration on M, the cup-product pairings and the inclusion $\mathscr{O}_K \subset K$ induce a commutative diagram with exact rows (here the groups in the lower row are given the discrete topology)

$$\begin{tikzcd}
 & & \mathbf{H}^1(\mathscr{O}_K, \mathscr{M}) \arrow[r] \arrow[d] & \mathbf{H}^2(\mathscr{O}_K, \mathscr{Y}) \arrow[r] \arrow[d] & 0 \\
 & H^1(K,G) \arrow[r] \arrow[d] & \mathbf{H}^1(K,M) \arrow[r] \arrow[d] & H^2(K,Y) \arrow[d] & \\
0 \arrow[r] & \mathbf{H}^0(\mathscr{O}_K, [\mathscr{Y}^* \to \mathscr{A}^*])^D \arrow[r] & \mathbf{H}^0(\mathscr{O}_K, \mathscr{M}^*)^D \arrow[r] & H^0(\mathscr{O}_K, \mathscr{T}^*)^D &
\end{tikzcd} \tag{2.7}$$

where the two zeros come from the vanishing of the groups $H^1(\mathscr{O}_K, \mathscr{T}^*)$ and $H^2(\mathscr{O}_K, \mathscr{G}) = H^2(\mathbf{F}, \widetilde{G})$; the second vanishing follows from the fact that $\widetilde{G}(\bar{\mathbf{F}})$ is torsion and $\mathbf{F}$ is of cohomological dimension 1.

Next, observe that the map $\mathbf{H}^0(\mathscr{O}_K, [\mathscr{Y}^* \to \mathscr{A}^*]) \to \mathbf{H}^0(K, [Y^* \to A^*])$ is an isomorphism. Indeed, by dévissage this reduces to showing that the natural maps $H^0(\mathscr{O}_K, \mathscr{A}) \to H^0(K, A)$ and $H^1(\mathscr{O}_K, \mathscr{Y}) \to H^1(K, Y)$ are isomorphisms. The first isomorphism follows from the properness of the abelian scheme $\mathscr{A}$. For the second, denote by $\mathscr{O}_K^{nr}$ the strict henselization of $\mathscr{O}_K$ and by K^{nr} its fraction field. Then the Hochschild-Serre spectral sequence induces a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & H^1(\mathbf{F}, H^0(\mathscr{O}_K^{nr}, \mathscr{Y})) \arrow[r] \arrow[d] & H^1(\mathscr{O}_K, \mathscr{Y}) \arrow[r] \arrow[d] & H^0(\mathbf{F}, H^1(\mathscr{O}_K^{nr}, \mathscr{Y})) \arrow[d] \\
0 \arrow[r] & H^1(\mathbf{F}, H^0(K^{nr}, Y)) \arrow[r] & H^1(K, Y) \arrow[r] & H^0(\mathbf{F}, H^1(K^{nr}, Y))
\end{tikzcd}$$

Here the group at top right vanishes because $\mathscr{O}_K^{nr}$ is acyclic for étale cohomology. Also, since $\mathscr{O}_K^{nr}$ is simply connected for the étale topology, the sheaf $\mathscr{Y}$ is isomorphic to a (torsion free) constant sheaf $\mathbf{Z}^r$, whence the vanishing of the group at bottom right. For the same reason, both groups on the left are isomorphic to

$H^1(\mathrm{Gal}(\mathbf{F}'/\mathbf{F}), H^0(\mathscr{O}_K^{nr}, \mathscr{Y}))$, where $\mathbf{F}'$ is a finite extension trivialising the action of $\mathrm{Gal}(\bar{F}|F)$ on $H^0(\mathscr{O}_K^{nr}, \mathscr{Y})$, whence the claim.

This being said, we conclude from Proposition 2.3.8 (which also holds in positive characteristic over complete fields) that the left vertical map in diagram (2.7) is injective. On the other hand the right column is exact by Lemma 2.3.11. Hence the middle column, i.e. the sequence

$$\mathbf{H}^1(\mathscr{O}_K, M) \to \mathbf{H}^1(K, M) \to \mathbf{H}^0(\mathscr{O}_K, \mathscr{M}^*)^D$$

is exact. Since the map $\mathbf{H}^1(K,M) \to \mathbf{H}^0(\mathscr{O}_K, \mathscr{M}^*)^D$ factors through the map $\mathbf{H}^1(K,M) \to (\mathbf{H}^0_{\mathrm{nr}}(K, M^*)^\wedge)^D$, the sequence

$$\mathbf{H}^1(\mathscr{O}_K, M) \to \mathbf{H}^1(K, M) \to (\mathbf{H}^0_{\mathrm{nr}}(K, M^*)^\wedge)^D$$

(of which we knew before that it is a complex) is exact as well. Dualising this exact sequence of discrete groups, we obtain from Theorem 2.3.3 that the sequence

$$\mathbf{H}^0_{\mathrm{nr}}(K, M^*)^\wedge \to \mathbf{H}^0(K, M)^\wedge \to \mathbf{H}^1(\mathscr{O}_K, M)^D$$

is exact and the theorem is proven. □

2.4 Global results : étale cohomology

Let k be a number field with ring of integers $\mathscr{O}_k$. Denote by Ω_k the set of places of k and $\Omega_k^\infty \subset \Omega_k$ the subset of real places. Let k_v be the completion of k at v if v is archimedean, and the field of fractions of the henselization of the local ring of $\mathrm{Spec}\,\mathscr{O}_k$ at v if v is finite. In the latter case, the piece of notation $\hat{k}_v$ stands for the completion of k at v. We denote by U an open subscheme of $\mathrm{Spec}\,\mathscr{O}_k$ and by Σ_f the set of finite places coming from closed points outside U.

In this section and the next, every abelian group is equipped with the discrete topology; in particular $B^\wedge$ denotes the profinite completion of B and $B^D := \mathrm{Hom}(B, \mathbf{Q}/\mathbf{Z})$ (even if B has a natural nondiscrete topology).

We need the notion of 'cohomology groups with compact support' $\mathbf{H}^i_c(U, \mathscr{F}^\bullet)$ for cohomologically bounded complexes of abelian sheaves $\mathscr{F}^\bullet$ on U. For k totally imaginary, these satisfy $\mathbf{H}^i_c(U, \mathscr{F}^\bullet) = \mathbf{H}^i(\mathrm{Spec}\,\mathscr{O}_k, j_!\mathscr{F}^\bullet)$, where $j : U \to \mathrm{Spec}\,\mathscr{O}_k$ is the inclusion map. In the general case this equality holds up to a finite 2-group. More precisely, there exists a long exact sequence (infinite in both directions)

$$\cdots \to \mathbf{H}^i_c(U, \mathscr{F}^\bullet) \to \mathbf{H}^i(U, \mathscr{F}^\bullet) \to \bigoplus_{v \in \Sigma_f} \mathbf{H}^i(k_v, \mathscr{F}^\bullet) \oplus \bigoplus_{v \in \Omega_k^\infty} \widehat{\mathbf{H}}^i(k_v, \mathscr{F}^\bullet) \to \mathbf{H}^{i+1}_c(U, \mathscr{F}^\bullet) \to \ldots \tag{2.8}$$

where for $v \in \Omega_k^\infty$ the notation $\widehat{\mathbf{H}}^i(k_v, \mathscr{F}^\bullet)$ stands for Tate (modified) cohomology groups of the group $\mathrm{Gal}(\bar{k}_v/k_v) \cong \mathbf{Z}/2\mathbf{Z}$ and where we have abused notation in denoting the pullbacks of $\mathscr{F}^\bullet$ under the maps $\mathrm{Spec}\, k_v \to \mathrm{Spec}\, \mathscr{O}_k$ by the same symbol.

In the literature, two constructions for the groups $\mathbf{H}^i_c(U, \mathscr{F}^\bullet)$ have been proposed, by Kato [36] and by Milne [47], respectively. The two definitions are equivalent (though we could not find an appropriate reference for this fact). We shall use Kato's construction which we find more natural and which we now copy from [36] for the convenience of the reader.

First, for an abelian sheaf $\mathscr{G}$ on the big étale site of $\mathrm{Spec}\,\mathbf{Z}$, one defines a complex $\widehat{\mathscr{G}}^\bullet$ as follows. Denote by $a : \mathrm{Spec}\,\mathbf{C} \to \mathrm{Spec}\,\mathbf{Z}$ the canonical morphism and by $\sigma : a_*a^*\mathscr{G} \to a_*a^*\mathscr{G}$ the canonical action of the complex conjugation viewed as an element of $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$. Now put $\widehat{\mathscr{G}}^0 = \mathscr{G} \oplus a_*a^*\mathscr{G}$ and $\widehat{\mathscr{G}}^i = a_*a^*\mathscr{G}$ for $i \in \mathbf{Z} \setminus \{0\}$. One defines the differentials d^i of the complex $\widehat{\mathscr{G}}^\bullet$ as follows:

$$d^{-1}(x) = (0, (\sigma - \mathrm{id})(x)); \qquad d^0(x,y) = b(x) + (\sigma + \mathrm{id})(y),$$

where $b : \mathscr{G} \to a_*a^*\mathscr{G}$ is the adjunction map; otherwise, set $d^i = \sigma + \mathrm{id}$ for i even and $d^i = \sigma - \mathrm{id}$ for i odd. This definition extends to bounded complexes $\mathscr{G}^\bullet$ on the big étale site of $\mathrm{Spec}\,\mathbf{Z}$ in the usual way: construct the complex $\widehat{\mathscr{G}^i}$ for each term $\mathscr{G}^i$ of the complex and then take the complex associated to the arising double complex. Finally for U and $\mathscr{F}^\bullet$ as above, one sets

$$\mathbf{H}^i_c(U, \mathscr{F}^\bullet) := \mathbf{H}^i(\mathrm{Spec}\,\mathbf{Z}, \widehat{\mathbf{R}f_!\mathscr{F}^\bullet}),$$

where $f : U \to \mathrm{Spec}\,\mathbf{Z}$ is the canonical morphism. From this definition one infers that for an open immersion $j_V : V \to U$ and a complex $\mathscr{F}^\bullet_V$ of sheaves on V one has $\mathbf{H}^i_c(V, \mathscr{F}^\bullet_V) \cong \mathbf{H}^i_c(U, j_{V!}\mathscr{F}^\bullet_V)$; therefore, setting $\mathscr{F}^\bullet_V = j_V^*\mathscr{F}^\bullet$ one obtains a canonical map

$$\mathbf{H}^i_c(V, \mathscr{F}^\bullet_V) \to \mathbf{H}^i_c(U, \mathscr{F}^\bullet)$$

coming from the morphism of complexes $j_{V!}j_V^*\mathscr{F}^\bullet \to \mathscr{F}^\bullet$. This covariant functoriality for open immersions will be crucial for the arguments in the next section.

Finally, we remark that for cohomologically bounded complexes $\mathscr{F}^\bullet$ and $\mathscr{G}^\bullet$ of étale sheaves on U, one has a cup-product pairing

$$\mathbf{H}^i(U, \mathscr{F}^\bullet) \times \mathbf{H}^j_c(U, \mathscr{G}^\bullet) \to \mathbf{H}^{i+j+1}_c(U, \mathscr{F}^\bullet \otimes^{\mathbf{L}} \mathscr{G}^\bullet). \tag{2.9}$$

Indeed, for f as above (which is quasi-finite by definition), one knows from general theorems of étale cohomology that the complexes $\mathbf{R}f_*\mathscr{F}^\bullet$ and $\mathbf{R}f_!\mathscr{F}^\bullet$ are cohomologically bounded (and similarly for $\mathscr{G}^\bullet$, $\mathscr{F}^\bullet \otimes^{\mathbf{L}} \mathscr{G}^\bullet$), and that there exists a canonical pairing $\mathbf{R}f_*\mathscr{F}^\bullet \otimes^{\mathbf{L}} \mathbf{R}f_!\mathscr{G}^\bullet \to \mathbf{R}f_!(\mathscr{F}^\bullet \otimes^{\mathbf{L}} \mathscr{G}^\bullet)$. One then uses the simple remark that any derived pairing $\mathscr{A}^\bullet \otimes^{\mathbf{L}} \mathscr{B}^\bullet \to \mathscr{C}^\bullet$ of cohomologically bounded complexes of étale sheaves on Spec $\mathbf{Z}$ induces a pairing $\mathscr{A}^\bullet \otimes^{\mathbf{L}} \widehat{\mathscr{B}^\bullet} \to \widehat{\mathscr{C}^\bullet}$.

Remark 2.4.1 In [73], Zink defines modified cohomology groups $\widehat{H}^i(U,\mathscr{F})$ which take the real places into account and satisfy a localization sequence for cohomology. He applies this to prove the Artin-Verdier duality theorem for finite sheaves in the case $U = \operatorname{Spec}\mathscr{O}_k$ (where cohomology and compact support cohomology coincide). For general U, however, one needs the groups $H^i_c(U,\mathscr{F})$ which are the same as the groups $\widehat{H}^i(\operatorname{Spec}\mathscr{O}_k, j_!\mathscr{F})$ in his notation.

This being said, we return to 1-motives.

Lemma 2.4.2 *Let M be a 1-motive over U.*

1. *The groups $\mathbf{H}^i(U,M)$ are torsion for $i \geq 1$ and so are the groups $\mathbf{H}^i_c(U,M)$ for $i \geq 2$.*
2. *For any ℓ invertible on U, the groups $\mathbf{H}^i(U,M)\{\ell\}$ ($i \geq 1$) are of finite cotype. Same assertion for the groups $\mathbf{H}^i_c(U,M)\{\ell\}$ ($i \geq 2$).*
3. *The group $\mathbf{H}^0(U,M)$ is of finite type.*

Proof: For the first part of (1) note that, with the notation of Section 2.2, the group $H^i(U,A)$ is torsion ([47], II.5.1), and so are $H^i(U,T)$ and $H^{i+1}(U,Y)$ by [47], II.2.9. The second part then follows from exact sequence (2.8) and the local facts.

By this last argument, for (2) it is again enough to prove the first statement. To do so, observe that $H^i(U,A)\{l\}$ is of finite cotype ([47], II.5.2) and for each positive integer n, there are surjective maps

$$H^i(U,Y/l^nY) \to H^{i+1}(U,Y)[l^n], \quad H^i(U,T[l^n]) \to H^i(U,T)[l^n]$$

whose sources are finite by [47], II.3.1.

To prove (3), one first uses for each $n > 0$ the surjective map from the finite group $H^0(U,Y/nY)$ onto $H^1(U,Y)[n]$. It shows that the finiteness of $H^1(U,Y)$ follows if we know that $H^1(U,Y) = H^1(U,Y)[n]$ for some n. Using a standard restriction-corestriction argument, this follows from the fact that $H^1(V,\mathbf{Z}) = 0$ for any normal integral scheme V ([1], IX.3.6 (ii)). It remains to note that the groups $H^0(U,A) = H^0(k,A)$ and $H^0(U,T)$ are of finite type, by the Mordell-Weil Theorem and Dirichlet's Unit Theorem, respectively (for the latter observe that $H^0(U,T)$ injects into $H^0(V,T) \cong (H^0(V,\mathbf{G}_m))^r$, where V/U is an étale covering trivialising T). □

Remark 2.4.3 The structure of the group $\mathbf{H}^1_c(U,M)$ is a bit more complicated: it is an extension of a torsion group (whose ℓ-part is of finite cotype for all ℓ invertible on U) by a quotient of a profinite group.

Now, as explained on p. 159 of [36], combining a piece of the long exact sequence (2.8) for $\mathscr{F}^\bullet = \mathbf{G}_m$ with the main results of global class field theory yields a canonical (trace) isomorphism

$$H^3_c(U,\mathbf{G}_m) \cong \mathbf{Q}/\mathbf{Z}.$$

Also, we have a natural compact support version

$$\mathbf{H}^i(U,M) \times \mathbf{H}^j_c(U,M^*) \to \mathbf{H}^{i+j+1}_c(U,\mathbf{G}_m)$$

of the pairing (2.2), constructed using the pairing (2.9) above. Combining the two, we get canonical pairings

$$\mathbf{H}^i(U,M) \times \mathbf{H}^{2-i}_c(U,M^*) \to \mathbf{Q}/\mathbf{Z}$$

defined for $-1 \le i \le 3$. For any prime number ℓ invertible on U, restricting to ℓ-primary torsion and modding out by divisible elements (recall the notations from the beginning of the paper) induces pairings

$$\overline{\mathbf{H}^i(U,M)}\{\ell\} \times \overline{\mathbf{H}^{2-i}_c(U,M^*)}\{\ell\} \to \mathbf{Q}/\mathbf{Z}. \tag{2.10}$$

Theorem 2.4.4 *For any 1-motive M and any ℓ invertible on U, the pairing (2.10) is non-degenerate for $0 \le i \le 2$.*

Note that the two groups occurring in the pairing (2.10) are finite by Lemma 2.4.2 (2).

Proof: This is basically the argument of ([47], II.5.2 (b)). Let n be a power of ℓ. Tensoring the exact sequence

$$0 \to \mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}/n\mathbf{Z} \to 0$$

by M in the derived sense and passing to étale cohomology over U induces exact sequences

$$0 \to \mathbf{H}^{i-1}(U,M) \otimes \mathbf{Z}/n\mathbf{Z} \to \mathbf{H}^{i-1}(U,M \otimes^{\mathbf{L}} \mathbf{Z}/n\mathbf{Z}) \to \mathbf{H}^i(U,M)[n] \to 0$$

Now $M \otimes^{\mathbf{L}} \mathbf{Z}/n\mathbf{Z}$ viewed as a complex of étale sheaves has trivial cohomology in degrees other than -1; indeed, with the notation of Section 1, the group Y is torsion free and multiplication by n on the group scheme G is surjective in the étale topology. Therefore, using the notation of Section 1, we may rewrite the previous sequence as

$$0 \to \mathbf{H}^{i-1}(U,M) \otimes \mathbf{Z}/n\mathbf{Z} \to H^i(U, T_{\mathbf{Z}/n\mathbf{Z}}(M)) \to \mathbf{H}^i(U,M)[n] \to 0. \qquad (2.11)$$

Write $T(M)\{\ell\}$ for the direct limit of the groups $T_{\mathbf{Z}/n\mathbf{Z}}(M)$ as n runs through powers of ℓ and $T_\ell(M)$ for their inverse limit. For each $r \geq 0$, $H^r(U, T_\ell(M))$ stands for the inverse limit of $H^r(U, T_{\mathbf{Z}/n\mathbf{Z}}(M))$ (n running through powers of l), and similarly for compact support cohomology. Passing to the direct limit in the above sequence then induces an isomorphism

$$\overline{H^i(U, T(M)\{\ell\})} \cong \overline{\mathbf{H}^i(U,M)\{\ell\}}.$$

Now by Artin-Verdier duality for finite sheaves ([73], [47], II.3; see also [44] in the totally imaginary case) the first group here is isomorphic (via a pairing induced by (2.10)) to the dual of the group $H_c^{3-i}(U, T_\ell(M^*))\{\ell\}$.

Working with the analogue of exact sequence (2.11) for compact support cohomology and passing to the inverse limit over n using the finiteness of the groups $H_c^{3-i}(U, T_{\mathbf{Z}/n\mathbf{Z}}(M^*))$, we get isomorphisms

$$\mathbf{H}_c^{2-i}(U,M^*)^{(\ell)}\{\ell\} \cong H_c^{3-i}(U, T_\ell(M^*))\{\ell\}$$

using the torsion freeness of the ℓ-adic Tate module of the group $\mathbf{H}_c^{3-i}(U,M^*)$. Finally, we have $\overline{\mathbf{H}_c^{2-i}(U,M^*)\{\ell\}} \cong \mathbf{H}_c^{2-i}(U,M^*)^{(\ell)}\{\ell\}$ by the results in Lemma 2.4.2 and Remark 2.4.3. □

From now on, we shall make the convention (to ease notation) that for any archimedean place v and each $i \in \mathbf{Z}$, $\mathbf{H}^i(k_v, M)$ means the *modified* group $\widehat{\mathbf{H}}^i(k_v, M)$ (In particular it is zero if v is complex, and it is a finite 2-torsion group if v is real).

Following [47], II.5, we define for $i \geq 0$

$$D^i(U,M) = \mathrm{Ker}\,[\mathbf{H}^i(U,M) \to \prod_{v\in\Sigma} \mathbf{H}^i(k_v,M)]$$

where the finite subset $\Sigma = \Sigma_f \cup \Omega_\infty \subset \Omega_k$ consists of the real places and the primes of $\mathscr{O}_k$ which do not correspond to a closed point of U. For $i \geq 0$ we also have

$$D^i(U,M) = \mathrm{Im}\,[\mathbf{H}_c^i(U,M) \to \mathbf{H}^i(U,M)]$$

by the definition of compact support cohomology. By Lemma 2.4.2, $D^1(U,M)$ is a torsion group and $D^1(U,M)\{\ell\}$ is of finite cotype. Now Theorem 2.4.4 has the following consequence:

Corollary 2.4.5 *Under the notation and assumptions of Theorem 2.4.4, there is a pairing*

$$D^1(U,M)\{\ell\} \times D^1(U,M^*)\{\ell\} \to \mathbf{Q}/\mathbf{Z} \tag{2.12}$$

whose left and right kernels are respectively the divisible subgroups of the two groups.

Proof: As in the proof of [47], Corollary II.5.3, we use the commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & D^1(U,M)\{\ell\} & \longrightarrow & \mathbf{H}^1(U,M)\{\ell\} & \longrightarrow & \bigoplus_{v\in\Sigma}\mathbf{H}^1(k_v,M) \\
 & & & & \downarrow & & \downarrow \\
0 & \longrightarrow & D^1(U,M^*)^D & \longrightarrow & \mathbf{H}^1_c(U,M^*)^D & \longrightarrow & \bigoplus_{v\in\Sigma}\mathbf{H}^0(k_v,M^*)^D
\end{array}$$

whose exact rows come from the definition of the group D^1 and whose vertical maps are induced by the pairings (2.10) and (2.5).

The diagram defines a map $D^1(U,M)\{\ell\} \to D^1(U,M^*)^D$. The right vertical map is injective by Proposition 2.3.9 and the second statement in Proposition 2.3.8 (indeed for v finite, $\mathbf{H}^0(k_v,M^*)$ is now equipped with the discrete topology, which is finer than the topology defined in Proposition 2.3.8). Given an element in the kernel of the map $D^1(U,M)\{\ell\} \to D^1(U,M^*)^D$, its image in $\mathbf{H}^1(U,M)\{\ell\}$ lies in the kernel of $\mathbf{H}^1(U,M)\{\ell\} \to (\mathbf{H}^1_c(U,M^*)\{\ell\})^D$ which is divisible by Theorem 2.4.4. To finish the proof of the corollary it thus suffices to prove the proposition below. □

Proposition 2.4.6 *If n is a power of ℓ, and a is an element of $D^1(U,M)$ that is n-divisible in $\mathbf{H}^1(U,M)$ and orthogonal to $D^1(U,M^*)[n]$, then a is n-divisible in $D^1(U,M)$.*

To prove the proposition we need an analogue of [47], I.6.15.

Lemma 2.4.7 *Let n be an integer invertible on U, and $S_n(U,M)$ the kernel of the map $H^1(U,T_{\mathbf{Z}/n\mathbf{Z}}(M)) \to \oplus_{v\in\Sigma}\mathbf{H}^1(k_v,M)$. If a is an element of the direct sum $\oplus_{v\in\Sigma}H^1(k_v,T_{\mathbf{Z}/n\mathbf{Z}}(M))$ orthogonal to the image of $S_n(U,M^*)$ in the group $\oplus_{v\in\Sigma}H^1(k_v,T_{\mathbf{Z}/n\mathbf{Z}}(M^*))$, then a is the sum of the coboundary of an element in $\oplus_{v\in S}H^0(k_v,M)$ and of the restriction of an element in $H^1(U,T_{\mathbf{Z}/n\mathbf{Z}}(M))$.*

The proof of the lemma is an application of Poitou–Tate duality for finite modules and runs as in *loc. cit.*, except that the dual of $\mathbf{H}^1(k_v,M)$ is the profinite completion of $\mathbf{H}^0(k_v,M)$, but the image of both the completed and uncompleted groups in the finite group $\oplus_{v\in\Sigma}\mathbf{H}^1(k_v,T_{\mathbf{Z}/n\mathbf{Z}}(M))$ is the same. Also, in place

dc_101_10

of the map γ_1 there it is more convenient to use the composite of the coboundary map $\oplus_{v\in\Sigma}H^1(k_v, T_{\mathbf{Z}/n\mathbf{Z}}(M)) \to H^2_c(U, T_{\mathbf{Z}/n\mathbf{Z}}(M))$ in the localization exact sequence for compact support cohomology with the Artin–Verdier isomorphism $H^2_c(U, T_{\mathbf{Z}/n\mathbf{Z}}(M)) \cong H^1(U, T_{\mathbf{Z}/n\mathbf{Z}}(M^*))^D$.

Proof of Proposition 2.4.7: Consider the commutative exact diagram

$$\begin{array}{ccccc} & & H^1_c(U,M) & \longrightarrow & H^1(U,M) \\ & & \downarrow n & & \downarrow n \\ \bigoplus_{v\in\Sigma} H^0(k_v,M) & \longrightarrow & H^1_c(U,M) & \longrightarrow & H^1(U,M) \\ \downarrow & & \downarrow & & \downarrow \\ \bigoplus_{v\in\Sigma} H^1(k_v, T_{\mathbf{Z}/n\mathbf{Z}}(M)) & \longrightarrow & H^2_c(U, T_{\mathbf{Z}/n\mathbf{Z}}(M)) & \longrightarrow & H^2(U, T_{\mathbf{Z}/n\mathbf{Z}}(M)) \end{array}$$

Let a be an element of $D^1(U,M) = \mathrm{im}\,(\mathbf{H}^1_c(U,M) \to \mathbf{H}^1(U,M))$ arising as $a = na_1$ with $a_1 \in H^1(U,M)$. By definition a comes from some $\tilde{a}$ in $H^1_c(U,M)$ whose image in $H^2_c(U, T_{\mathbf{Z}/n\mathbf{Z}}(M))$ will be denoted by a_2. By functoriality, given $a' \in D^1(U,M^*)[n]$, the value $\langle a, a'\rangle$ of the Cassels–Tate pairing equals that of the Artin–Verdier pairing $[a_2, b']$, where $b' \in H^1(U, T_{\mathbf{Z}/n\mathbf{Z}}(M^*))$ is a preimage of a'. A diagram chasing now shows that a_2 comes from $(c_v) \in \oplus_{v\in\Sigma}H^1(k_v, T_{\mathbf{Z}/n\mathbf{Z}}(M))$. It follows that $[a_2, b']$ equals the sum of the local pairings $\langle c_v, b'_v\rangle_v$ for $v \in S$, where b'_v is the image of b' in $H^1(k_v, T_{\mathbf{Z}/n\mathbf{Z}}(M^*))$.

Our assumption that $\langle a, a'\rangle = 0$ for all $a' \in D^1(U,M^*)[n]$ thus implies that (c_v) satisfies the assumptions of the lemma, and hence up to modifying it by an element of $\oplus_{v\in\Sigma}H^0(k_v,M)$ (which does not change a), we may assume that (c_v) comes from $H^1(U, T_{\mathbf{Z}/n\mathbf{Z}}(M))$, and hence $\tilde{a}$ maps to 0 in $H^2_c(U, T_{\mathbf{Z}/n\mathbf{Z}}(M))$. By the diagram this means that $\tilde{a}$ is divisible by n in $H^1_c(U,M)$, and hence so is a in $D^1(U,M)$. □

We can make Corollary 2.4.7 more precise under an additional assumption. Let A_k denote the generic fibre of A and

$$Ш^1(A_k) := \mathrm{Ker}\,[H^1(k, A_k) \to \prod_{v\in\Omega_k} H^1(\hat{k}_v, A_k)]$$

its Tate-Shafarevich group. According to a well-known conjecture this group should be finite.

Proposition 2.4.8 *Let M be a 1-motive over U and ℓ a prime number invertible on U. Assume that $Ш^1(A_k)\{\ell\}$ and $Ш^1(A^*_k)\{\ell\}$ are finite. Then the pairing*

$$D^1(U,M)\{\ell\} \times D^1(U,M^*)\{\ell\} \to \mathbf{Q}/\mathbf{Z}$$

of Corollary 2.4.5 is a perfect pairing of finite groups.

Proof: Using Corollary 2.4.5, it is sufficient to prove that $D^1(U,M)\{\ell\}$ is finite. We have a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
H^1(U,Y) & \longrightarrow & H^1(U,G) & \longrightarrow & \mathbf{H}^1(U,M) & \longrightarrow & H^2(U,Y) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\bigoplus_{v\in\Sigma} H^1(k_v,Y) & \longrightarrow & \bigoplus_{v\in\Sigma} H^1(k_v,G) & \longrightarrow & \bigoplus_{v\in\Sigma} \mathbf{H}^1(k_v,M) & \longrightarrow & \bigoplus_{v\in\Sigma} H^2(k_v,Y)
\end{array}$$

An inspection of the diagram reveals that for the finiteness of $D^1(U,M)\{\ell\}$ it suffices to show (using finiteness of Σ) the finiteness of the torsion groups $D^1(U,G)\{\ell\}$, $H^1(k_v,Y)$ and $D^2(U,Y)$, respectively.
For the first, note that the assumption on $Ш^1(A_k)$ implies that $D^1(U,A)\{\ell\}$ is finite by [47], II.5.5, whence the required finiteness follows from the finiteness of $H^1(U,T)$ ([47], II.4.6). We have seen the second finiteness several times when discussing local duality. The finiteness of $D^2(U,Y)$ follows from that of $H^2_c(U,Y)$ which is is dual to the finite group $H^1(U,T)$ by a result of Deninger ([47], II.4.6). (One can also prove the finiteness of $D^2(U,Y)$ by the following more direct reasoning: using a restriction-corestriction argument, one reduces to the case $Y = \mathbf{Z}$. Then $D^2(U,\mathbf{Z}) = D^1(U,\mathbf{Q}/\mathbf{Z})$ is the dual of the Galois group of the maximal abelian extension of k unramified over U and totally split at the places outside U, and hence is finite by global class field theory.)

□

Remark 2.4.9 The same argument shows that the finiteness of $Ш^1(A_k)$ implies the finiteness of $D^1(U,M)$.

For $i = 0$ we have the following consequence of Theorem 2.4.4. Set

$$D^0_\wedge(U,M) := \ker(\mathbf{H}^0(U,M) \to \bigoplus_{v\in\Sigma} \mathbf{H}^0(k_v,M)^\wedge).$$

Corollary 2.4.10 *Under the notation and assumptions of Theorem 2.4.4, there is a pairing*

$$D^0_\wedge(U,M)\{\ell\} \times D^2(U,M^*)\{\ell\} \to \mathbf{Q}/\mathbf{Z} \tag{2.13}$$

whose left and right kernels are respectively the divisible subgroups of the two groups.

Proof: Consider the commutative exact diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & D^0_\wedge(U,M)\{\ell\} & \longrightarrow & \mathbf{H}^0(U,M)\{\ell\} & \longrightarrow & \bigoplus_{v\in\Sigma} \mathbf{H}^0(k_v,M)^{(l)} \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \overline{D^2(U,M^*)}\{\ell\}^D & \longrightarrow & \overline{\mathbf{H}^2_c(U,M^*)}\{\ell\}^D & \longrightarrow & \bigoplus_{v\in\Sigma} \mathbf{H}^1(k_v,M^*)\{\ell\}^D
\end{array}$$

In the lower row the notation as in Theorem 2.4.4; its exactness comes from the fact that the groups being of finite cotype, we have

$$\overline{D^2(U,M^*)}\{\ell\} = D^2(U,M^*)\{\ell\}/\ell^N, \overline{\mathbf{H}^2_c(U,M^*)}\{\ell\} = \mathbf{H}^2_c(U,M^*)\{\ell\}/\ell^N$$

for N large enough. Now by Theorem 2.4.4 the middle vertical map is an isomorphism (recall that $\mathbf{H}^0(U,M)\{\ell\}$ is finite by Lemma 2.4.2 (3)), and by Theorem 2.3 the right vertical map is injective. □

Remark 2.4.11 In the case $M = [0 \to T]$, the two corollaries above give back (part of) [47], Corollary II.4.7, itself based on the main result of Deninger [17] (which we do not use here).

2.5 Global results: relation to Galois cohomology

The notation and assumptions are the same as in the previous section. In particular U is an affine open subset of $\operatorname{Spec} \mathscr{O}_k$ and $\Sigma \subset \Omega_k$ consists of the real places and the finite places of $\operatorname{Spec} \mathscr{O}_k \setminus U$. Fix an algebraic closure $\bar{k}$ of k (corresponding to a geometric point $\bar{\eta}$ of U). Let $\Gamma_\Sigma = \pi_1(U,\bar{\eta})$ be the Galois group of the maximal subfield k_Σ of $\bar{k}$ such that the extension k_Σ/k is unramified outside Σ. When G_k is the restriction of a U-group scheme G to $\operatorname{Spec} k$, we shall write $H^i(\Gamma_\Sigma, G_k)$ for $H^i(\Gamma_\Sigma, G_k(k_\Sigma))$.

We begin with the following analogue of [47], II.2.9:

Proposition 2.5.1 *Let $M = [Y \to G]$ be a 1-motive over U and M_k its restriction to* $\operatorname{Spec} k$. *Let ℓ be a prime number invertible on U. Then:*

1. *The natural map $\mathbf{H}^i(U,M)\{\ell\} \to \mathbf{H}^i(\Gamma_\Sigma, M_k)\{\ell\}$ is an isomorphism for $i > 1$. For $i = 1$ it is an isomorphism for U sufficiently small.*
2. *The natural map $\mathbf{H}^1(U,M)\{\ell\} \to \mathbf{H}^1(k,M_k)\{\ell\}$ is injective for U sufficiently small.*
3. *The natural map $\mathbf{H}^0(U,M) \to \mathbf{H}^0(k,M_k)$ is injective.*

Proof: To prove (1) in the case $i > 1$, pass to the direct limit over powers of ℓ in exact sequence (2.11). Then Lemma 2.4.2 (1) implies that it is sufficient to prove the statement for M replaced by $T(M)\{\ell\}$, which follows from [47], II.2.9 by passing to the limit. In the case $i = 1$ one unscrews M using the weight filtration reduces the statement to the case where $M = [0 \to T]$ is a torus using

dc_101_10

[47], II.2.9, II.5.5. as well as the fact that $A(k) = H^0(U,A) = H^0(\Gamma_\Sigma, A_k)$ for the abelian scheme A with generic fibre A_k.

To handle the case $i = 1$ for a torus, one first observes that the statement holds for a norm torus $R_{K|k}\mathbf{G}_m$ for some finite extension $K|k$, because $H^1(U, R_{K|k}\mathbf{G}_m) = \mathrm{Pic}(U \times_k K)$ is zero for U sufficiently small. The statement then follows for quasi-trivial tori, i.e. finite products of norm tori. Now let T be arbitrary. By Ono's lemma ([50], Theorem 1.5.1), there exist $m > 0$ and a quasi-trivial k-torus R_k such that $T_k^m \times R_k$ is isogenous to a quasi-trivial torus. As the statements to be proven are compatible with products and we have just shown them for R_k, we may replace T_k by $T_k^m \times R_k$ and therefore assume that there is an exact sequence

$$0 \to F \to R \to T \to 0$$

with F finite étale over U and R_k quasi-trivial. Now the result follows from the associated long exact sequence using the case $i = 2$, the case of a quasi-trivial torus, and [47], II.2.9.

For (2), it is sufficient by (1) to show that the canonical map $\mathbf{H}^1(\Gamma_\Sigma, M_k) \to \mathbf{H}^1(k, M_k)$ is injective. Again using the weight filtration on M, one reduces this to the injectivity of $\mathbf{H}^1(\Gamma_\Sigma, G_k) \to \mathbf{H}^1(k, G_k)$, the injectivity of $H^2(\Gamma_\Sigma, Y_k) \to H^2(k, Y_k)$ and the surjectivity of $H^1(\Gamma_\Sigma, Y_k) \to H^1(k, Y_k)$ (Note that $Y_k(\bar{k}) = Y_k(k_\Sigma)$ because there exists a finite étale covering $\widetilde{U}/U$ such that $Y \times_U \widetilde{U}$ is constant). The two injectivities are consequences of the restriction-inflation sequence for H^1 in Galois cohomology, noting the isomorphisms

$$H^2(\Gamma_\Sigma, Y_k) \cong H^1(\Gamma_\Sigma, Y_k(\bar{k}) \otimes \mathbf{Q}/\mathbf{Z}) \quad \text{and} \quad H^2(k, Y_k) \cong H^1(k, Y_k(\bar{k}) \otimes \mathbf{Q}/\mathbf{Z})$$

for the lattice $Y_k(\bar{k}) = Y_k(k_\Sigma)$. The surjectivity follows from the triviality of the abelian group $H^1(\mathrm{Gal}\,(\bar{k}/k_\Sigma), Y_k(\bar{k})) = \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}\,(\bar{k}/k_\Sigma), Y_k(\bar{k}))$, the Galois group $\mathrm{Gal}\,(\bar{k}/k_\Sigma)$ being profinite and $Y_k(\bar{k})$ torsion-free.

Statement (3) is obvious for $M = [0 \to G]$ (a morphism from U to G is trivial if and only if it is trivial at the generic point). Using the weight filtration, it is sufficient to prove that the map $H^0(U,Y) \to H^0(k, Y_k)$ is surjective and the map $H^1(U,Y) \to H^1(k, Y_k)$ is injective. Actually we have isomorphisms $H^0(U,Y) \cong H^0(k, Y_k)$ and $H^0(U, Y/nY) \cong H^0(k, Y_k/nY_k)$ for each $n > 0$ because Y and Y/nY are locally constant in the étale topology. The injectivity of the map $H^1(U,Y) \to H^1(k, Y_k)$ now follows from the commutative exact diagram

$$\begin{array}{ccccccc}
H^0(U,Y) & \longrightarrow & H^0(U,Y/nY) & \longrightarrow & H^1(U,Y)[n] & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
H^0(k,Y_k) & \longrightarrow & H^0(k,Y_k/nY_k) & \longrightarrow & H^1(k,Y_k)[n] & \longrightarrow & 0.
\end{array}$$

$\square$

For a 1-motive M_k over k and $i \geq 0$, define the *Tate-Shafarevich groups*

$$Ш^i(M_k) = \operatorname{Ker}[\mathbf{H}^i(k, M_k) \to \prod_{v \in \Omega_k} \mathbf{H}^i(\hat{k}_v, M_k)]$$

If M_k is the restriction of a 1-motive M defined over U, we also define

$$Ш^i_\Sigma(M_k) := \operatorname{Ker}[\mathbf{H}^i(\Gamma_\Sigma, M_k) \to \prod_{v \in \Sigma} \mathbf{H}^i(\hat{k}_v, M_k)]$$

Remark 2.5.2 By Lemma 2.3.7, we can replace $\hat{k}_v$ by k_v in the definition of $Ш^i$ for $i \geq 0$.

In the remaining of this section, we prove Theorem 2.1.2. As in ([47], II.6), the idea is to identify the Tate-Shafarevich groups with the groups $D^i(U, M)$ considered in the previous section for U sufficiently small. But a difficulty is that when Y is not trivial, the restriction of $D^i(U, M)$ to $\mathbf{H}^i(V, M)$ for $V \subset U$ need not be a subset of $D^i(V, M)$, as the following example shows.

Example 2.5.3 Let k be a totally imaginary number field with $\operatorname{Pic} \mathscr{O}_k \neq 0$ and $M := \mathbf{Z}[1]$. Then for $U = \operatorname{Spec} \mathscr{O}_k$ we have $D^1(U, M) = H^1(U, \mathbf{Q}/\mathbf{Z}) \neq 0$ by global class field theory. Let $\alpha \neq 0$ in $H^1(U, \mathbf{Q}/\mathbf{Z})$. Then there is a finite place v coming from a closed point of U such that the restriction of α to $H^1(k_v, \mathbf{Q}/\mathbf{Z})$ is nonzero, for otherwise we would get $\alpha = 0$ by Chebotarev's density theorem. Therefore the restriction of α to $H^1(V, M)$ does not belong to $D^1(V, M)$ for $V = U - \{v\}$.

Thus the identification of $Ш^1(M_k)$ with some $D^1(U, M)$ for suitable U is not straightforward, in contrast to the case of abelian varieties. The following proposition takes care of this problem.

Note that by Proposition 2.5.1, any $D^1(U, M)\{\ell\}$ is naturally a subgroup of $H^1(k, M_k)$ for ℓ invertible on U.

Proposition 2.5.4 *Let $M = [Y \to G]$ be a 1-motive over U and ℓ a prime invertible on U. There exists an open subset $U_0 \subset U$ such that for any open subset $U_1 \subset U_0$, the group $D^1(U_1, M)\{\ell\}$ as a subgroup of $H^1(k, M_k)$ is contained in $Ш^1(M_k)\{\ell\}$.*

For the proof of this proposition we need two preliminary lemmas.

Lemma 2.5.5 *Let v be a finite place of U and $\mathscr{O}_v$ the henselization of the local ring of U at v. Then the natural map $H^2(\mathscr{O}_v, Y) \to H^2(k_v, Y)$ is injective when restricted to ℓ-primary torsion.*

Note that the groups occurring in the lemma are torsion groups (for the first one, this follows from Lemma 2.4.2 (1) applied to $M = Y[1]$ and passing to the limit).

dc_101_10

Proof: By the localization sequence for the pair $\operatorname{Spec} k_v \subset \operatorname{Spec} \mathscr{O}_v$ it is enough to prove triviality of the group $H^2_v(\mathscr{O}_v, Y)[\ell^n]$ for any n. This group is a quotient of $H^1_v(\mathscr{O}_v, Y/\ell^n Y)$ which, according to [47], II.1.10 (a), is a finite group dual to $H^2(\mathscr{O}_v, \underline{Hom}(Y/\ell^n Y, \mathbf{G}_m))$. But for any finite sheaf F over $\mathscr{O}_v$ of order prime to the characteristic of the residue field $\kappa(v)$ of v, we have $H^2(\mathscr{O}_v, F) = 0$ because $H^2(\mathscr{O}_v, F) = H^2(\kappa(v), \widetilde{F})$ by [46], III.3.11 a) (where $\widetilde{F}$ is the restriction of F to $\operatorname{Spec} \kappa(v)$), and $\kappa(v)$ is of cohomological dimension 1. □

Lemma 2.5.6 *There exists an open subset $U_0 \subset U$ such that for any open subset $U_1 \subset U_0$, the group $\text{Ш}^2(Y_k)\{\ell\}$ contains $D^2(U_1, Y)\{\ell\}$.*

Note that according to Proposition 2.5.1, for any open subset $V \subset U$ the group $D^2(V, Y)\{\ell\} = D^1(V, Y[1])\{\ell\}$ identifies with a subgroup of $H^2(k, Y)$.

Proof: By definition we have $\text{Ш}^2(Y_k)\{\ell\} \supset \bigcap_{V \subset U} D^2(V, Y)\{\ell\}$. The group $D^2(V, Y)$ is finite for each open subset $V \subset U$ (cf. proof of Proposition 2.4.8). Therefore there exist finitely many open subsets $V_1, \dots, V_r$ of U such that

$$\text{Ш}^2(Y_k)\{\ell\} \supset \bigcap_{i=1}^{r} D^2(V_i, Y)\{\ell\}.$$

Let U_0 be the intersection $\bigcap_{i=1}^{r} V_i$. As the $H^2_c(\_\_, Y)$ are covariantly functorial for open immersions $U_1 \subset U_2$ (see the previous section), for $i = 1, \dots, r$ there are natural maps $H^2_c(U_0, Y) \to H^2_c(V_i, Y) \to H^2(k, Y)$ which factor through $D^2(U_0, Y)$, so that we get inclusions $D^2(U_0, Y) \hookrightarrow D^2(V_i, Y)$ and finally $D^2(U_0, Y)\{\ell\} \hookrightarrow \text{Ш}^2(k, Y)\{\ell\}$. The above of course holds for any $U_1 \subset U_0$ instead of U_0. □

Proof of Proposition 2.5.4: Choose U_0 as in the previous lemma and take $U_1 \subset U_0$. It suffices to show that for each closed point $v \in U_1$ the group $D^1(U_1, M)\{\ell\}$ maps to 0 by the restriction map $\mathbf{H}^1(U_1, M) \to \mathbf{H}^1(k_v, M)$. Now there is a commutative diagram with exact rows

$$\begin{array}{ccccc}
H^1(U_1, G) & \longrightarrow & \mathbf{H}^1(U_1, M) & \longrightarrow & H^2(U_1, Y) \\
\downarrow & & \downarrow & & \downarrow \\
H^1(\mathscr{O}_v, G) & \longrightarrow & \mathbf{H}^1(\mathscr{O}_v, M) & \xrightarrow{\alpha} & H^2(\mathscr{O}_v, Y) \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \beta} \\
H^1(k_v, G) & \longrightarrow & \mathbf{H}^1(k_v, M) & \xrightarrow{\gamma} & H^2(k_v, Y)
\end{array}$$

Here the group $H^1(\mathscr{O}_v, G)$ is zero because G is smooth and connected over U_1, hence $H^1(\mathscr{O}_v, G) = H^1(\mathbf{F}_v, \widetilde{G})$ (cf. [46], III.3.11 a)) is trivial by Lang's Theorem.

So the map α in the diagram is injective, but when restricted to ℓ-primary torsion, the map β is injective as well, by Lemma 2.5.5. Therefore the image of the map $D^1(U_1,M)\{\ell\} \to \mathbf{H}^1(k_v,M)$ injects into $H^2(k_v,Y)$ by γ, so we reduce to the triviality of the composite map $D^1(U_1,M)\{\ell\} \to H^2(k_v,Y)$. This in turn follows from Lemma 2.5.6 as the map factors through $D^2(U_1,Y)\{\ell\}$ by the right half of the diagram. □

We are now able to prove one half of Theorem 2.1.2:

Theorem 2.5.7 *Let M_k be a 1-motive over k. Then there exists a canonical pairing*

$$Ш^1(M_k) \times Ш^1(M_k^*) \to \mathbf{Q}/\mathbf{Z}$$

whose kernels are the maximal divisible subgroups of each group.

Proof: We construct the pairing separately for each prime ℓ. Let U be an open subset of $\operatorname{Spec}\mathscr{O}_k$ such that M_k is the restriction of a 1-motive M defined over U and ℓ is invertible on U. Take a subset U_0 as in Proposition 2.5.4. We contend that the inclusion $D^1(U_0,M)\{\ell\} \subset Ш^1(M_k)\{\ell\}$ furnished by the proposition is in fact an equality, from which the theorem will follow by Corollary 2.4.5. Indeed, any element of $Ш^1(M_k)\{\ell\}$ is contained in some $D^1(V,M)\{\ell\}$, where we may assume $V \subset U_0$. But $D^1(V,M)\{\ell\} \subset D^1(U_0,M)\{\ell\} \subset Ш^1(M_k)$, by the same argument as in the end of the proof of Lemma 2.5.6.

□

Corollary 2.5.8 *Let M_k be a 1-motive over k. If $Ш^1(A_k)$ and $Ш^1(A_k^*)$ are finite, then there is a perfect pairing of finite groups*

$$Ш^1(M_k) \times Ш^1(M_k^*) \to \mathbf{Q}/\mathbf{Z}$$

Proof: Apply Theorem 2.5.7, Proposition 2.4.8, and Remark 2.4.9. □

Remark 2.5.9 In the case $M_k = [0 \to T_k]$, we recover the duality between $Ш^1(T_k)$ and $Ш^2(Y_k^*)$, where Y_k^* is the module of characters of the torus T_k. See [68] and [49], VIII.6.8.

2.6 The Poitou-Tate exact sequence

We keep notation from the previous sections. In particular, k is a number field, M_k is a 1-motive over k, and U is an open subset of $\operatorname{Spec}\mathscr{O}_k$ such that M_k extends to a 1-motive M over U. For each finite place v of k, we denote by $\widehat{\mathscr{O}}_v$ the ring of integers of the completion $\hat{k}_v$. The groups $\mathbf{H}^i(k,M_k)$ $(-1 \le i \le 2)$ and $\mathbf{H}^i(\hat{k}_v,M_k)$ for $i \ge 1$ $(v \in \Omega_k)$ are equipped with the discrete topology. For $i \ge 0$, we define $\mathbf{P}^i(M_k)$ as the restricted product over $v \in \Omega_k$ of the $\mathbf{H}^i(\hat{k}_v,M_k)$, with respect to the images of $\mathbf{H}^i(\widehat{\mathscr{O}}_v,M)$ for finite places v of U. The groups $\mathbf{P}^i(F_k)$ are defined similarly for any étale sheaf F_k over $\operatorname{Spec} k$. Clearly all these groups are independent of the choice of U. We equip them with their restricted product topology. Note that $\mathbf{P}^2(M_k)$ is the direct sum of the groups $\mathbf{H}^2(\hat{k}_v,M_k)$ for all places v: indeed, for each finite place v of U, we have $\mathbf{H}^2(\hat{\mathscr{O}}_v,M) = \mathbf{H}^2(\mathbf{F}_v,\widetilde{M})$, and we have already seen that this last group vanishes. We have natural restriction maps $\beta_i : \mathbf{H}^i(k,M_k) \to \mathbf{P}^i(M_k)$ for $i = 1,2$; their kernels are precisely the groups $\text{Ш}^i(M_k)$.

In this section we establish a Poitou-Tate type exact sequence for M_k. A technical complication for our considerations to come arises from the fact that one is forced to work with two kinds of completions: the first one is what we have used up till now, i.e. the inverse limit $A^\wedge$ of all open subgroups of finite index of a topological abelian group A, and the second is the inverse limit $A_\wedge$ of the quotients A/n for all $n > 0$. These are not the same in general (indeed, the latter is not necessarily profinite). But it is easy to see that $(A_\wedge)^\wedge$ is naturally isomorphic to $A^\wedge$; in particular, we have a natural map $A_\wedge \to A^\wedge$.

The natural map $\mathbf{H}^0(k,M_k) \to \mathbf{P}^0(k,M_k)$ therefore gives rise to a commutative diagram between the two different kinds of completions:

$$\begin{array}{ccc} \mathbf{H}^0(k,M_k)_\wedge & \xrightarrow{\theta_0} & \mathbf{P}^0(k,M_k)_\wedge \\ \downarrow & & \downarrow \\ \mathbf{H}^0(k,M_k)^\wedge & \xrightarrow{\beta_0} & \mathbf{P}^0(k,M_k)^\wedge. \end{array} \tag{2.14}$$

Denote by $\text{Ш}^0_\wedge(M_k)$ the kernel of the above map θ_0. We first prove the following duality result analogous to [47], I.6.13 (b).

Proposition 2.6.1 *Let $M_k = [Y_k \to G_k]$ be a 1-motive over k. Assume that $\text{Ш}^1(A_k)$ is finite. Then there is a perfect pairing*

$$\text{Ш}^0_\wedge(M_k) \times \text{Ш}^2(M_k^*) \to \mathbf{Q}/\mathbf{Z}$$

where the first group is compact and the second is discrete.

Remarks 2.6.2

1. The special case $M_k = Y_k[1]$ corresponds to the duality between the groups $\text{Ш}^1(Y_k)$ and $\text{Ш}^2(T_k^*)$, where T_k^* is the torus with character module Y_k; compare [47], I.4.20 (a). Indeed, in this case the groups $\mathbf{H}^0(k,M_k)$ and $\mathbf{P}^0(k,M_k)$ are finite, so $\text{Ш}^0_\wedge(M_k) \cong \text{Ш}^1(Y_k)$.

dc_101_10

2. The proof below will show that if one only assumes the finiteness of the ℓ-primary torsion part of $\text{Ш}^1(A_k)$, then one gets a similar duality between the ℓ-primary torsion part of $\text{Ш}^2(M_k^*)$ and a group $\text{Ш}^0_{(\ell)}(M_k)$ defined similarly to $\text{Ш}^0_\wedge(M_k)$ but taking only ℓ-adic completions.

3. In the paper [30] we guessed that $\text{Ш}^2(M_k)$ is finite, as it is finite for $M_k = T_k$ and even trivial for $M_k = A_k$ and $M_k = Y_k[1]$. This finiteness was proven in important mixed cases in the thesis of P. Jossen [34], among them 1-motives of the form $[Y_k \to A_k]$, with A_k a geometrically simple abelian variety. He also proved that in these cases there is a perfect pairing

$$\text{Ш}^0(M_k) \times \text{Ш}^2(M_k^*) \to \mathbf{Q}/\mathbf{Z}$$

of finite groups (in fact, the finiteness of $\text{Ш}^0(M_k)$ can be proven to hold in general). However, he recently produced an example where $\text{Ш}^2(M_k)$ is infinite!

For the proof of the proposition we first show a lemma.

Lemma 2.6.3 *For any $n > 0$, the natural map*

$$\mathbf{P}^0(M_k)/n \to \prod_v \mathbf{H}^0(\hat{k}_v, M_k)/n$$

is injective and its image is the restricted product of the groups $\mathbf{H}^0(\hat{k}_v, M_k)/n$ with respect to the subgroups $\mathbf{H}^0(\hat{\mathscr{O}}_v, M_k)/n$.

Proof: Take an element $x = (x_v) \in \mathbf{P}^0(M_k)$ and assume x_v is in $n\mathbf{H}^0(\hat{k}_v, M)$ for all v. For almost all v, it also comes from $\mathbf{H}^0(\hat{\mathscr{O}}_v, M)$, hence in fact it comes from $n\mathbf{H}^0(\hat{\mathscr{O}}_v, M)$, for in the exact commutative diagram

$$\begin{array}{ccccc}
\mathbf{H}^0(\widehat{\mathscr{O}}_v, M) & \xrightarrow{n} & \mathbf{H}^0(\widehat{\mathscr{O}}_v, M) & \longrightarrow & H^1(\widehat{\mathscr{O}}_v, T_{\mathbf{Z}/n\mathbf{Z}}(M)) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{H}^0(\hat{k}_v, M_k) & \xrightarrow{n} & \mathbf{H}^0(\hat{k}_v, M_k) & \longrightarrow & H^1(\hat{k}_v, T_{\mathbf{Z}/n\mathbf{Z}}(M_k))
\end{array}$$

the third vertical map is injective, $H^1_v(\widehat{\mathscr{O}}_v, T_{\mathbf{Z}/n\mathbf{Z}}(M))$ being trivial by the same argument as in the proof of Lemma 2.5.5. But this means $x \in n\mathbf{P}^0(M_k)$. The second statement is obvious.

□

Proof of Proposition 2.6.1: Using exact sequence (2.11) we get a commutative diagram with exact rows for each $n > 0$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{H}^0(k,M_k)/n & \longrightarrow & H^1(k,T_{\mathbf{Z}/n\mathbf{Z}}(M_k)) & \longrightarrow & \mathbf{H}^1(k,M_k)[n] & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbf{H}^0(\hat{k}_v,M)/n & \longrightarrow & H^1(\hat{k}_v,T_{\mathbf{Z}/n\mathbf{Z}}(M_k)) & \longrightarrow & \mathbf{H}^1(\hat{k}_v,M_k)[n] & \longrightarrow & 0
\end{array}$$

and a similar exact diagram holds with $\widehat{\mathscr{O}}_v$ instead of $\hat{k}_v$. Taking restricted products and using the above lemma, we get a commutative exact diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{H}^0(k,M_k)/n & \longrightarrow & H^1(k,T_{\mathbf{Z}/n\mathbf{Z}}(M_k)) & \longrightarrow & \mathbf{H}^1(k,M_k)[n] & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbf{P}^0(M_k)/n & \longrightarrow & \mathbf{P}^1(T_{\mathbf{Z}/n\mathbf{Z}}(M_k)) & \longrightarrow & \mathbf{P}^1(M_k)[n] & \longrightarrow & 0
\end{array}$$

If we pass to the inverse limit over all n, the two lines remain left exact, whence a commutative exact diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{H}^0(k,M_k)_\wedge & \longrightarrow & H^1(k,T(M_k)) & \longrightarrow & I_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\theta_0} & & \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\beta_1} & & \\
0 & \longrightarrow & \mathbf{P}^0(M_k)_\wedge & \longrightarrow & \mathbf{P}^1(T(M_k)) & \longrightarrow & I_2 & \longrightarrow & 0
\end{array} \tag{2.15}$$

where we define $\mathbf{P}^1(T(M_k))$ (resp. $H^1(k,T(M_k))$) as the inverse limit of the groups $\mathbf{P}^1(T_{\mathbf{Z}/n\mathbf{Z}}(M_k))$ (resp. $H^1(k,T_{\mathbf{Z}/n\mathbf{Z}}(M_k))$). Here I_1, I_2 are respectively subgroups of the full Tate modules $T(H^1(k,M_k))$ and $T(\mathbf{P}^1(M_k))$. In particular, $\operatorname{Ker}\beta_1$ is a subgroup of $T(Ш^1(M_k))$ because the inverse limit functor is left exact.

But $T(Ш^1(M_k))$ is zero thanks to the finiteness assumption on $Ш^1(A_k)$ (which implies the finiteness of $Ш^1(M_k)$ as in Corollary 2.5.8).

Therefore we obtain that $Ш^0_\wedge(M_k) = \operatorname{Ker}\theta_0$ is isomorphic to $\operatorname{Ker}\theta$. By Poitou-Tate duality for finite modules ([49], VIII.6.8), there is a perfect pairing between the latter group and $Ш^2(T(M_k^*)_{\mathrm{tors}})$.

We conclude by observing that $Ш^2(T(M_k^*)_{\mathrm{tors}})$ is also $Ш^2(M_k^*)_{\mathrm{tors}}$ by the same argument as in [47], I.6.8 (use the analogue of the exact sequence (2.11) for M_k over $\operatorname{Spec} k$ with $i = 2$ and pass to the limit).

□

We now return to diagram (2.14) and prove:

Proposition 2.6.4 *Keep the finiteness assumption on* $Ш^1(A_k)$*. Then, with notations as in diagram (2.14), the natural map* $\ker\theta_0 \to \ker\beta_0$ *is an isomorphism.*

By virtue of the proposition, we may employ the notation $Ш^0_\wedge(M_k)$ for $\ker\beta_0$ as well and use the resulting duality. The finiteness assumption on $Ш^1(A_k)$ is presumably superfluous here but we did not succeed in removing it (and use it elsewhere anyway).

For the proof we need a lemma about abelian groups.

Lemma 2.6.5 *Let A be a discrete abelian group of finite exponent n. Then the intersection of the finite index subroups of A is trivial.*

Proof: Consider the profinite group $A^D = \mathrm{Hom}(A, \mathbf{Z}/n)$. As $A^{DD} = A$, the statement is equivalent to saying that any character of A^D vanishing on all finite subgroups of A^D is trivial. This holds because all finitely generated subgroups of A^D are finite. □

Proof of Proposition 2.6.4: We begin by showing that the vertical maps in diagram (2.14) above are injective, whence the injectivity of the map $\ker\theta_0 \to \ker\beta_0$. For the left one, note that the topology on $\mathbf{H}^0(k,M)$ being discrete, injectivity means that any element of $\mathbf{H}^0(k,M)$ which is nontrivial modulo n for some n gives a nonzero element in some finite quotient – this holds by the lemma above.

For injectivity of the map $\mathbf{P}^0(k,M)_\wedge \to \mathbf{P}^0(k,M)^\wedge$, take an element $x = (x_v)$ in $\mathbf{P}^0(M_k)$ not lying in $n\mathbf{P}^0(M_k)$. We have to find an open subgroup of finite index avoiding x. By Lemma 2.6.3, there is a local component x_v not lying in $n\mathbf{H}^0(\hat{k}_v, M)$. We thus get that for some v, our x is not contained in the inverse image in $\mathbf{P}^0(M_k)$ of the subgroup $n\mathbf{H}^0(\hat{k}_v, M) \subset \mathbf{H}^0(\hat{k}_v, M)$, which is open of finite index, by definition of the topology on $\mathbf{H}^0(\hat{k}_v, M)$.

For the surjectivity of the map $\ker\theta_0 \to \ker\beta_0$, remark first that $\ker\theta_0$ is a profinite group, being dual to the torsion group $Ш^2(M_k)$ by the previous proposition.

Therefore $(\ker\theta_0)^\wedge = \ker\theta_0$, so by completing the exact sequence

$$0 \to \ker\theta_0 \to \mathbf{H}^0(k, M_k)_\wedge \to \mathrm{im}\,\theta_0 \to 0 \tag{2.16}$$

we get an exact sequence $\ker\theta_0 \to \mathbf{H}^0(k, M_k)^\wedge \to (\mathrm{im}\,\theta_0)^\wedge \to 0$.

To conclude, it is thus enough to show the injectivity of the natural map $(\mathrm{im}\,\theta_0)^\wedge \to \mathbf{P}^0(M_k)^\wedge$. By the above considerations, $\mathbf{P}^0(M_k)_\wedge$ injects into its completion $\mathbf{P}^0(M_k)^\wedge$, so the completion of the subgroup $\mathrm{im}\,\theta_0$ is simply its closure in $\mathbf{P}^0(M_k)^\wedge$, whence the claim. But there is a subtle point here: in this argument, $\mathrm{im}\,\theta_0$ is equipped with the subspace topology inherited from $\mathbf{P}^0(M_k)_\wedge$, whereas in exact sequence (2.16) it carried the quotient topology from $\mathbf{H}^0(M_k)_\wedge$; we have to check that the two topologies are the same, or in other words that the morphism θ_0 is strict. For every $n > 0$, the groups $\mathbf{H}^0(k, M_k)/n$ and $\mathbf{P}^0(M_k)/n$

are locally compact (the latter thanks to Lemma 2.6.3). Moreover, the morphism $f_n : \mathbf{H}^0(k,M_k)/n \to \mathbf{P}^0(M_k)/n$ is strict for every $n > 0$. Indeed, the morphisms $\mathbf{H}^0(k,M_k)/n \to H^1(k,T_{\mathbf{Z}/n\mathbf{Z}}(M_k))$ and $\mathbf{P}^0(M_k)/n \to \mathbf{P}^1(T_{\mathbf{Z}/n\mathbf{Z}}(M_k))$ are strict (thanks to [33], Theorem 5.29 and to diagram (2.15)), hence the image of $\mathbf{H}^0(k,M_k)/n$ in $\mathbf{P}^0(M_k)/n$ identifies with a subspace of the image of the group $H^1(k,T_{\mathbf{Z}/n\mathbf{Z}}(M_k))$ in $\mathbf{P}^1(T_{\mathbf{Z}/n\mathbf{Z}}(M_k))$, which is discrete. Using the Poitou-Tate exact sequence for finite modules ([49], VIII.6.13) we get that this image is a closed subset of $\mathbf{P}^1(T_{\mathbf{Z}/n\mathbf{Z}}(M_k))$, hence the morphism $H^1(k,T_{\mathbf{Z}/n\mathbf{Z}}(M_k)) \to \mathbf{P}^1(T_{\mathbf{Z}/n\mathbf{Z}}(M_k))$ is strict (again by [33], Theorem 5.29). Now θ_0 is obtained as the projective limit of the strict morphisms f_n with $\ker f_n$ finite and $H^0(k,M_k)/n$ discrete; it is not difficult to check that this implies that θ_0 is strict. □

We can now state the main result of this section.

Theorem 2.6.6 (Poitou-Tate exact sequence) *Let M_k be a 1-motive over k. Assume that $Ш^1(A_k)$ and $Ш^1(A_k^*)$ are finite, where A_k is the abelian variety corresponding to M_k. Then there is a twelve term exact sequence of topological groups*

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{H}^{-1}(k,M_k)^\wedge \arrow[r, "\gamma_2{}^D"] & \prod_{v\in\Omega_k}\mathbf{H}^2(\hat{k}_v,M_k^*)^D \arrow[r, "\beta_2{}^D"] & \mathbf{H}^2(k,M_k^*)^D \arrow[d] \\
& \mathbf{H}^1(k,M_k^*)^D \arrow[d] & \mathbf{P}^0(M_k)^\wedge \arrow[l, "\gamma_0"'] & \mathbf{H}^0(k,M_k)^\wedge \arrow[l, "\beta_0"'] \\
& \mathbf{H}^1(k,M_k) \arrow[r, "\beta_1"] & \mathbf{P}^1(M_k)_{\mathrm{tors}} \arrow[r, "\gamma_1"] & (\mathbf{H}^0(k,M_k^*)^D)_{\mathrm{tors}} \arrow[d] \\
0 & \mathbf{H}^{-1}(k,M_k^*)^D \arrow[l] & \bigoplus_{v\in\Omega_k}\mathbf{H}^2(\hat{k}_v,M_k) \arrow[l, "\gamma_2"'] & \mathbf{H}^2(k,M_k) \arrow[l, "\beta_2"']
\end{tikzcd} \tag{2.17}$$

where the maps β_i are the restriction maps defined at the beginning of this section, the maps γ_i are induced by the local duality theorem of Section 2, and the unnamed maps come from the global duality results of Proposition 2.6.1 (completed by Proposition 2.6.4) and Corollary 2.5.8.

Remarks 2.6.7

1. In the above sequence the group $\mathbf{P}^1(M_k)_{\mathrm{tors}}$ is equipped with the discrete topology, and *not* the subspace topology from $\mathbf{P}^1(M_k)$.

2. The sequence (2.17) is completely symmetric in the sense that if we replace M_k by M_k^* and dualize, we obtain exactly the same sequence.

dc_101_10

3. If we only assume the finiteness of $Ш^1(A_k)\{\ell\}$ and $Ш^1(A_k^*)\{\ell\}$ for some prime number ℓ, then the analogue of the exact sequence (2.17) still holds, with profinite completions replaced by ℓ-adic completions and the torsion groups involved by their ℓ-primary part.

4. Some special cases of the theorem are worth noting. For $M_k = [0 \to A_k]$ we get the ten-term exact sequence of [47], I.6.14. For $M_k = [0 \to T_k]$ one can show (see proof of Proposition 2.6.9 below) that β_2^D is an isomorphism, hence we obtain a classical (?) nine-term exact sequence; the case $M_k = Y_k[1]$ is symmetric (compare [47], I.4.20).

The proof will use the following lemma.

Lemma 2.6.8 *Let M_k be a 1-motive over k. Then $\mathbf{H}^0(k, M_k)_{\mathrm{tors}}$ is finite.*

Proof: Let $M'_k = M_k/W_{-2}(M_k) = [Y_k \to A_k]$. From the exact sequence

$$H^0(k,Y_k) \to H^0(k,A_k) \to \mathbf{H}^0(k,M'_k) \to H^1(k,Y_k)$$

we deduce that $\mathbf{H}^0(k,M'_k)$ is of finite type because $H^0(k,A_k)$ is of finite type (Mordell-Weil theorem) and $H^1(k,Y_k)$ is finite (because $Y_k(\bar{k})$ is a lattice). There is also an exact sequence

$$\mathbf{H}^{-1}(k,M'_k) \to H^0(k,T_k) \to \mathbf{H}^0(k,M_k) \to \mathbf{H}^0(k,M'_k)$$

where T_k is the torus corresponding to M_k. It is therefore sufficient to show that the torsion subgroup of the group $B := T_k(k)/\mathrm{im}\,\mathbf{H}^{-1}(k,M'_k)$ is finite. Choose generators $a_1,\dots,a_r$ of the lattice $\mathbf{H}^{-1}(k,M'_k)$, and let $b_1,\dots,b_r$ be their images in $T_k(k)$. We can find an open subset U of $\mathrm{Spec}\,\mathscr{O}_k$ such that T_k extends to a torus T over U, and $b_i \in H^0(U,T)$ for any $i \in \{1,\dots,r\}$. In this way, each $x \in B_{\mathrm{tors}}$ is the image in B of an element $y \in T_k(k)$ for which $y^n \in H^0(U,T)$ for some $n > 0$. Let V/U be an étale covering such that T splits over V, i.e. it becomes isomorphic to some power $\mathbf{G}_m^N$. If L denotes the fraction field of V, we have $T_k(L)/H^0(V,T) \cong (L^\times/H^0(V,\mathbf{G}_m))^N$ which is naturally a subgroup of the free abelian group $\mathrm{Div}(V)^N$; in particular, it has no torsion. Therefore, since $y^n \in H^0(V,T)$ via the inclusion $H^0(U,T) \to H^0(V,T)$ we get that $y \in H^0(V,T) = H^0(V,\mathbf{G}_m)^N$. Let H be the subgroup of $H^0(V,T)$ generated by the y's; since $H^0(V,\mathbf{G}_m)$ is of finite type by Dirichlet's Unit Theorem, so is H. We thus get a surjection from the finitely generated group H to the torsion group B_{tors}, whence the claim. □

Proof of Theorem 2.6.6: The first line is dual to the last one, so for its exactness it is enough to show the exactness of the latter. We proceed as in [47], I.6.13 (b).

For each $n > 0$, we have an exact commutative diagram (using the exact sequence (2.11) and the Poitou-Tate sequence for the finite module $T_{\mathbf{Z}/n\mathbf{Z}}(M_k)$):

$$\begin{tikzcd}
 & & 0 \arrow[d] & \\
 & & \mathbf{H}^0(k,M_k^*)[n]^D \arrow[d] & \\
H^2(k,T_{\mathbf{Z}/n\mathbf{Z}}(M_k)) \arrow[r] \arrow[d] & \bigoplus_{v\in\Omega_k} H^2(\hat{k}_v,T_{\mathbf{Z}/n\mathbf{Z}}(M_k)) \arrow[r] \arrow[d] & H^0(k,T_{\mathbf{Z}/n\mathbf{Z}}(M_k^*))^D \arrow[r] \arrow[d] & 0 \\
\mathbf{H}^2(k,M_k)[n] \arrow[r] & \bigoplus_{v\in\Omega_k} \mathbf{H}^2(\hat{k}_v,M_k)[n] \arrow[r] & (\mathbf{H}^{-1}(k,M_k^*)/n)^D \arrow[d] & \\
 & & 0 &
\end{tikzcd}$$

By Lemma 2.6.8, the full Tate module $T(\mathbf{H}^0(k,M_k^*))$ is trivial. Therefore taking the inductive limit over all n, we obtain the exact commutative diagram, where the right vertical map is an isomorphism:

$$\begin{tikzcd}
H^2(k,T(M_k)) \arrow[r] \arrow[d] & \bigoplus_{v\in\Omega_k} H^2(\hat{k}_v,T(M_k)) \arrow[r] \arrow[d] & H^0(k,T(M_k^*))^D \arrow[r] \arrow[d] & 0 \\
\mathbf{H}^2(k,M_k) \arrow[r] & \bigoplus_{v\in\Omega_k} \mathbf{H}^2(\hat{k}_v,M_k) \arrow[r] & (\mathbf{H}^{-1}(k,M_k^*)^{\wedge})^D \arrow[r] & 0
\end{tikzcd}$$

We remark that the first two vertical maps are also isomorphisms by the same argument as at the end of the proof of Proposition 2.6.1. Since the first row is exact by the Poitou-Tate sequence for finite modules ([49], VIII.6.13), so is the second one. Therefore the exactness of the last line of (2.17) (and hence that of the first) follows noting that the dual of the profinite completion of the lattice $\mathbf{H}^{-1}(k,M_k^*)$ is the same as the dual of $\mathbf{H}^{-1}(k,M_k^*)$ itself.

To prove exactness of the second line, note first that it is none but the profinite completion of the sequence

$$H^0(k,M_k)_\wedge \stackrel{\theta_0}{\to} \mathbf{P}^0(M_k)_\wedge \stackrel{\gamma_0'}{\to} H^1(k,M_k^*)^D, \tag{2.18}$$

where the map γ_0' is induced by local duality, taking Lemma 2.6.3 into account. First we show that this latter sequence is a complex. The map γ_0' has the following concrete description at a finite level: for $\alpha = (\alpha_v) \in \mathbf{P}^0(M_k)/n$ and $\beta \in \mathbf{H}^1(k,M_k^*)[n]$, denote by β_v the image of β in $\mathbf{H}^1(\hat{k}_v,M_k^*)[n]$. Then $[\gamma_0'(\alpha)](\beta)$ is the sum over all v of the elements $j_v(\alpha_v \cup \beta_v)$, where $(\alpha_v \cup \beta_v) \in H^2(\hat{k}_v,\mu_n)$ via the pairing $M \times M^* \to \mathbf{G}_m[1]$, and j_v is the local invariant. (The sum is finite by virtue of the property $H^2(\widehat{\mathscr{O}}_v, \mu_n) = 0$ because the elements α_v and β_v are unramified for almost all v.) Then $\gamma_0' \circ \theta_0 = 0$ follows (after passing to the limit) from the reciprocity law of global class field theory, according to which the sequence

$$0 \to H^2(k, \mu_n) \to \bigoplus_{v \in \Omega_k} H^2(\hat{k}_v, \mu_n) \stackrel{\sum j_v}{\to} \mathbf{Z}/n \to 0 \tag{2.19}$$

is a complex.

For the exactness of the sequence (2.18) recall that, as remarked at the end of the proof of Proposition 2.6.4, diagram (2.15) and the Poitou-Tate sequence for finite modules imply that $\operatorname{im} \theta_0$ is the kernel of the composed map

$$\mathbf{P}^0(M_k)_\wedge \to \mathbf{P}^1((T(M_k)) \to H^1(k, T(M_k^*)_{\mathrm{tors}})^D.$$

The claimed exactness then follows from the commutative diagram

$$\begin{array}{ccc}
\mathbf{P}^0(M_k)_\wedge & \longrightarrow & \mathbf{P}^1(T(M_k)) \\
\Big\downarrow{\scriptstyle \gamma_0'} & & \Big\downarrow \\
\mathbf{H}^1(k, M_k^*)^D & \longrightarrow & \mathbf{H}^1(k, T(M_k^*)_{\mathrm{tors}})^D
\end{array}$$

whose commutativity arises from the compatibility of the duality pairings for 1-motives and their "n-adic" realizations via the Kummer map.

Now we show that the profinite completion of sequence (2.18), i.e. the second row of diagram (2.17) remains exact. This follows by an argument similar to the one at the end of the proof of Proposition 2.6.4, once having checked that $\operatorname{im} \gamma_0'$ is closed in $\mathbf{H}^1(k, M_k^*)^D$ and $(\operatorname{im} \gamma_0')^\wedge = \operatorname{im} \gamma_0'$. To see this, note that by applying the snake lemma to diagram (2.15) we get that $\operatorname{Coker} \theta_0$ (with the quotient topology) injects as a closed subgroup into $\operatorname{Coker} \theta$. But one sees using the Poitou-Tate sequence for finite modules that the latter group is profinite, hence so is $\operatorname{im} \gamma_0'$; in particular, it is compact and hence closed in $\mathbf{H}^1(k, M_k^*)^D$.

Next, remark that by definition of the restricted product topology and Theorem 2.3.10, the dual of the group $\mathbf{P}^0(M_k)$ (equipped with the restricted product topology) is $\mathbf{P}^1(M_k^*)$. Thus the dual of $\mathbf{P}^0(M_k)^\wedge$ is $\mathbf{P}^1(M_k^*)_{\mathrm{tors}}$. Therefore we obtain the third line by dualizing the second one (and exchanging the roles of M and M^*), which consists of profinite groups.

Finally, the exactness of the sequence (2.17) at the "corners" follows immediately, in the first two rows, from the dualities $\text{Ш}^0_\wedge(M_k) \cong \text{Ш}^2(M_k^*)^D$ (Proposition 2.6.1 combined with Proposition 2.6.4) and $\text{Ш}^1(M_k) \cong \text{Ш}^1(M_k^*)^D$ (Corollary 2.5.8), respectively; the remaining corner is dual to the first one.

□

We conclude with the following complement.

dc_101_10

Proposition 2.6.9 *Let M_k be a 1-motive over k. Then the natural map $\mathbf{H}^i(k,M_k) \to \bigoplus_{v\in\Omega_\infty} \mathbf{H}^i(k_v,M_k)$ is an isomorphism for $i \geq 3$.*

Proof: When $M = [0 \to G]$, this follows immediately by devissage from [47], I.4.21, and I.6.13 (c). To deal with the general case, it is sufficient to show that the map $f_i : H^i(k,Y_k) \to \bigoplus_{v\in\Omega_{\mathbf{R}}} H^i(k_v,Y_k)$ is an isomorphism for $i \geq 3$. Using the exact sequence

$$0 \to H^i(k,Y_k)/n \to H^i(k,Y_k/nY_k) \to H^{i+1}(k,Y_k)[n] \to 0$$

for each $n > 0$, we reduce to the case $i = 3$. The last line of the Poitou-Tate exact sequence for $M_k = Y_k[1]$ yields the surjectivity of f_3 because $\hat{k}_v$ is of strict cohomological dimension 2 for v finite. On the other hand, we remark that $H^3(k,\mathbf{Z}) = 0$ ([47], I.4.17), hence $H^3(k,Y_k) = H^3(k,Y_k)[n]$ for some $n > 0$ by a restriction-corestriction argument. In particular the divisible subgroup of $H^3(k,Y_k)$ is zero. The injectivity of f_3 now follows from Proposition 2.6.1 applied to $M_k = T_k^*$, where T_k^* is the torus with module of characters Y_k. □

2.7 Comparison with the Cassels-Tate pairing

In this section, we give a definition of the pairing of Theorem 2.1.2 purely in terms of Galois cohomology and show that in the case $M = [0 \to A]$ it reduces to the classical Cassels-Tate pairing for abelian varieties.

The idea is to use the *diminished cup-product* construction discovered by Poonen and Stoll (see [52], pp. 1117–1118). One could present it in a general categorical setting but for the ease of exposition we stick to the special situation we have. So assume given three exact sequences

$$0 \to M_1 \to M_2 \to M_3 \to 0,$$

$$0 \to N_1 \to N_2 \to N_3 \to 0,$$

$$0 \to P_1 \to P_2 \to P_3 \to 0$$

where the M_i, N_i, P_i are complexes of abelian étale sheaves over some scheme S. Assume further given a pairing $M_2 \otimes^{\mathbf{L}} N_2 \to P_2$ in the derived category that maps $M_1 \otimes^{\mathbf{L}} N_1$ to P_1. Then one has a pairing

$$\mathrm{Ker}\,[\mathbf{H}^i(S,M_1) \to \mathbf{H}^i(S,M_2)] \times \mathrm{Ker}\,[\mathbf{H}^j(S,N_1) \to \mathbf{H}^j(S,N_2)] \to \mathbf{H}^{i+j-1}(S,P_3)$$

defined as follows. Any element a of $\ker[\mathbf{H}^i(S,M_1) \to \mathbf{H}^i(S,M_2)]$ comes from an element b of $\mathbf{H}^{i-1}(S,M_3)$. Since the above pairing induces a pairing $M_3 \otimes^{\mathbf{L}} N_1 \to P_3$, our b can be cupped with $a' \in \ker[\mathbf{H}^j(S,N_1) \to \mathbf{H}^j(S,N_2)]$ to get an element in $\mathbf{H}^{i+j-1}(S,P_3)$. It follows from the fact that a' maps to 0 in $\mathbf{H}^j(S,N_2)$ that this definition does not depend on the choice of b.

Now apply this in the following situation. Let $\mathbf{A}_k$ be the adèle ring of the number field k, $S=\operatorname{Spec} k$ and M_k a 1-motive over k. Consider étale sheaves over $\operatorname{Spec} k$ as $\operatorname{Gal}(k)$-modules and put $M_1=M_k(\bar k)$, $M_2=M_k(\bar k\otimes_k \mathbf{A}_k)$, $N_1=M_k^*(\bar k)$, $N_2=M_k^*(\bar k\otimes_k \mathbf{A}_k)$, $P_1=\mathbf{G}_m(\bar k)[1]$, $P_2=\mathbf{G}_m(\bar k\otimes_k \mathbf{A}_k)[1]$ and define M_3, N_3, P_3 to make the above sequences exact. The pairing $M_k\otimes^{\mathbf{L}} M_k^*\to \mathbf{G}_m[1]$ induces the pairing $M_2\otimes^{\mathbf{L}} N_2\to P_2$ required in the above situation. Note that we have

$$Ш^i(M_k)=\operatorname{Ker}[\mathbf{H}^i(k,M_1)\to \mathbf{H}^i(k,M_2)],$$

$$Ш^j(M_k^*)=\operatorname{Ker}[\mathbf{H}^j(k,N_1)\to \mathbf{H}^j(k,N_2)].$$

Finally, class field theory tells us that the cokernel of the map $H^2(k,\mathbf{G}_m)\to H^2(k,\mathbf{G}_m(\bar k\otimes_k \mathbf{A}_k))$ is isomorphic to $\mathbf{Q}/\mathbf{Z}$; indeed, one has

$$H^2(k,\mathbf{G}_m(\bar k\otimes_k \mathbf{A}_k))=\bigoplus_v H^2(k_v,\mathbf{G}_m)$$

by Shapiro's lemma (and the isomorphisms $H^2(k_v,\mathbf{G}_m)\cong H^2(\hat k_v,\mathbf{G}_m)$, cf. Lemma 2.3.7), so the claim follows from the exactness of the sequence (2.19) ([49], Theorem 8.1.17) in view of the vanishing of $H^3(k,\mathbf{G}_m)$ ([49], 8.3.10 (iv)).

Putting all this together, we get that the diminished cup-product construction yields for $i=j=1$ pairings

$$Ш^1(M_k)\times Ш^1(M_k)\to \mathbf{Q}/\mathbf{Z} \tag{2.20}$$

for $i=j=1$.

Proposition 2.7.1 *The above pairing coincides with those constructed in Section 2.5.*

Proof: Assume M_k extends to a 1-motive M over an open subset $U\subset \operatorname{Spec} k$; let Σ denote the finite set of places of k which are real or coming from closed points of $\operatorname{Spec}\mathscr{O}_k$ outside U. Apply the diminished cup-product construction for $S=U$, $M_1=M$, $M_2=p_*p^*M$, $N_1=M^*$, $N_2=p_*p^*M^*$, $P_1=\mathbf{G}_m[1]$, $P_2=p_*p^*\mathbf{G}_m[1]$, where $p:\oplus_{v\in\Sigma}\operatorname{Spec} k_v\to U$ is the canonical morphism (here we use the conventions of Section 2.4 for k_v). Note that, in the notation of Section 2.4,

$$D^i(U,M)=\ker[\mathbf{H}^i(U,M_1)\to \mathbf{H}^i(U,M_2)]$$

and

$$D^j(U,M^*)=\ker[\mathbf{H}^j(U,N_1)\to \mathbf{H}^j(U,N_2)].$$

Moreover, one has

$$\operatorname{coker}[H^2(U,\mathbf{G}_m) \to H^2(U,p_*p^*\mathbf{G}_m)] \cong H^3_c(U,\mathbf{G}_m) \cong \mathbf{Q}/\mathbf{Z}$$

where the first isomorphism comes from Shapiro's lemma in the étale setting and the second from class field theory, both combined with the fundamental long exact sequence of compact support cohomology (the second isomorphism was already used to construct the pairings of Section 2.4). For $i = j = 1$ one thus gets a pairing

$$D^1(U,M) \times D^1(U,M) \to \mathbf{Q}/\mathbf{Z}$$

which is manifestly the same as that of Corollary 2.4.5. Moreover, over a sufficiently small U these are by construction compatible with the above definition of the pairing (2.20). □

Proposition 2.7.2 *For $M_k = [0 \to A_k]$ the above pairing reduces to the classical Cassels-Tate pairing.*

Proof: This was basically proven in [52]. There one finds four equivalent definitions of the Cassels-Tate pairing, two of them using the diminished cup-product construction for explicitly defined pairings $A_k(\bar{k}) \times A^*_k(\bar{k}) \to \mathbf{G}_m[1]$. For instance, one of them (called the Albanese-Picard pairing by [52]) is given as follows: replace $A_k(\bar{k})$ with the quasi-isomorphic complex $C_1 = [Y(A) \to Z(A)]$, where $Z(A)$ is the group of zero-cycles on $A_{\bar{k}} = A_k \times_k \bar{k}$ and $Y(A)$ is the kernel of the natural summation map $Z(A) \to A(\bar{k})$, and replace $A^*_k(\bar{k})$ by the complex

$$C_2 = [\bar{k}(A_{\bar{k}})^\times/\bar{k}^\times \to \mathrm{Div}^0(A_{\bar{k}})],$$

where Div^0 stands for divisors algebraically equivalent to 0. Now define a map $C_1 \otimes C_2 \to \mathbf{G}_m(\bar{k})[1]$ using the partially defined pairings

$$Z(A) \times [\bar{k}(A_{\bar{k}})^\times/\bar{k}^\times] \to \bar{k}^\times$$

and

$$\mathrm{Div}^0(A_{\bar{k}}) \times Y(A) \to \bar{k}^\times$$

where the first is defined on ([52], p. 1116) using the Poincaré divisor on $A \times A^*$ and the second by evaluation. The resulting pairing $A_k(\bar{k}) \times A^*_k(\bar{k}) \to \mathbf{G}_m[1]$ is one of the classical definitions of the duality pairing for abelian varieties over an algebraically closed field. □

Remark 2.7.3 Similarly, one verifies that in the case $M_k = [0 \to T_k]$ one gets the usual pairing between the Tate-Shafarevich groups of the torus T_k and its character module.

Remark 2.7.4 In [20], Flach defines a Cassels-Tate pairing for finite dimensional continuous ℓ-adic representations of the Galois group of a number field k that stabilize some lattice. Comparing the above construction with his shows that in the case when the representation comes from the ℓ-adic realization of a 1-motive M_k the two pairings are compatible. In the special cases $M_k = [\mathbf{Z} \to 0]$ and $M_k = [0 \to A_k]$ this is already pointed out in Flach's paper (see also [21] for the latter case). So our result can be interpreted as a "motivic version" of Flach's pairing for motives of dimension one.

2.8 Further developments

Cristian González-Avilés has extended the main results of this chapter to the function field case. For a function field of characteristic p our proofs carry over to treat the prime-to-p torsion part of the cohomology groups involved. Concerning the p-part, González-Avilés proved the following analogue of the $i = 1$ case of Theorem 2.1.2:

Theorem 2.8.1 (González-Avilés [24], Theorem 2.3) *Let k be a global function field of characteristic $p > 0$, and M a 1-motive over k. There exists a canonical pairing*

$$Ш^1(M)\{p\} \times Ш^1(M^*)\{p\} \to \mathbf{Q}/\mathbf{Z}$$

which is non-degenerate modulo maximal divisible subgroups.

Here Tate–Shafarevich groups are defined in the same way as above, but using flat cohomology instead of étale cohomology. The proof follows a similar overall pattern to ours, but several technical modifications are needed to adapt the argument to flat cohomology.

In their paper [25] González-Avilés and Tan have also extended the Poitou–Tate exact sequence (Theorem 2.6.6) and the Cassels–Tate dual exact sequence (Theorem 3.1.2 in the next chapter) to positive characteristic. They have moreover constructed a variant of the latter sequence that does not rest upon a finiteness assumption on Tate–Shafarevich groups (but is maybe less suitable for applications).

Another kind of generalization was proven by Peter Jossen in his thesis written under my supervision. He generalized our duality theorems to *1-motives with torsion*. He defined a 1-motive with torsion to be a morphism $Y \to G$, where Y is an extension of a lattice by a finite flat group scheme and G is an extension of an abelian scheme by a group scheme X that is itself an extension of a finite flat group scheme by a torus. There is a natural weight filtration on such objects and morphisms are required to preserve it. Jossen extended the theory of Deligne 1-motives to 1-motives with torsion, including Cartier duality and ℓ-adic realizations. His main result is:

Theorem 2.8.2 (Jossen [34], Chapter 4) *Let k be a number field, and $M = [Y \to G]$ a 1-motive with torsion over k. There exist canonical pairings*

$$Ш^1(M) \times Ш^1(M^*) \to \mathbf{Q}/\mathbf{Z}$$

which are non-degenerate modulo maximal divisible subgroups.

This theorem yields a common generalization of all previously known duality results over number fields, including Poitou–Tate duality for finite group schemes which was not covered by Theorem 2.1.2.

As mentioned briefly in Remark 2.6.2 (3), Jossen has also made important progress concerning the finiteness of $Ш^2(M)$ for a 1-motive M (with or without torsion). He has recently found a surprising example where this group is infinite. However, he had previously also proven that finiteness holds in the cases where G is a geometrically simple abelian variety or a 1-dimensional torus. As the finiteness of $Ш^0(M^*)$ is easy to prove in general, in these cases one obtains a perfect pairing $Ш^0(M^*) \times Ш^2(M) \to \mathbf{Q}/\mathbf{Z}$ of finite groups. This finiteness result also has interesting 'concrete' consequences for the arithmetic of abelian varieties.

Appendix : Completion of topological abelian groups

In this appendix we collect some (probably well-known) results that we needed in the paper.

Proposition *Let $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ be an exact sequence in the category of topological abelian groups.*

1. *If i is strict with open image, then the map $B^D \to A^D$ is surjective.*
2. *Assume that the map $p : B \to C$ is open. Then the sequences*

$$A^\wedge \to B^\wedge \to C^\wedge \to 0$$

$$0 \to C^D \to B^D \to A^D$$

are exact.

3. *Assume that p is open and i is strict with closed image. Suppose further that B is Hausdorff, locally compact, completely disconnected, and compactly generated. Then the sequences*

$$0 \to A^\wedge \to B^\wedge \to C^\wedge \to 0$$

$$0 \to C^D \to B^D \to A^D \to 0$$

are exact, the map $B \to B^\wedge$ is injective, and $(B^\wedge)^D = B^D$.

Proof: 1. If i is strict and $i(A)$ is open in B, then any continuous homomorphism $A \to \mathbf{Q}/\mathbf{Z}$ induces a continuous homomorphism $s : i(A) \to \mathbf{Q}/\mathbf{Z}$ which in particular has open kernel. By divisibility of $\mathbf{Q}/\mathbf{Z}$ this extends to a homomorphism $\bar{s} : B \to \mathbf{Q}/\mathbf{Z}$. Since $i(A)$ is open in B, $\operatorname{Ker}\bar{s}$ is also open in B being a union of cosets of $\operatorname{Ker} s$, whence the continuity of $\bar{s}$.

2. To get the first exact sequence, the only nontrivial point consists of proving that an element $b \in B^\wedge$ whose image in $C^\wedge$ is trivial comes from $A^\wedge$. Let $B' \subset B$ be an open subgroup of finite index. Then $p(B') \subset C$ is of finite index and is open because p is open. Since i is continuous, $(A \cap B') \subset A$ is open and of finite index as well. Now we use the exact sequence

$$0 \to A/(A \cap B') \to B/B' \to C/p(B') \to 0$$

where B' runs over the finite index open subgroups of B.

The second exact sequence follows immediately from the fact that the group $C \simeq B/i(A)$ is equipped with the quotient topology (p being an open map).

3. Let us show the left exactness of the first sequence. Since i is strict with closed image, we can assume that A is a closed subgroup of B with the induced topology. We have to show that if $A' \subset A$ is open and of finite index, then there exists an open subgroup of finite index $B' \subset B$ such that $B' \cap A \subset A'$. Replacing A and B by A/A' and B/A' (equipped with the quotient topology), we reduce to the case when A is a finite subgroup and we must show that B contains a finite index open subgroup B' with $B' \cap A = \{0\}$. To do so, it is sufficient (using the finiteness of A) to prove that the intersection of all finite index open subgroups of B is zero, that is $B \hookrightarrow B^\wedge$. By [33], II.9.8. (in the special case when B is completely disconnected), B is topologically isomorphic to a product $\mathbf{Z}^b \times K$ with K compact (hence profinite), thus $B \hookrightarrow B^\wedge = \widehat{Z}^b \times K$.

This also shows that $(B^\wedge)^D = B^D$. The projection L of A on $\mathbf{Z}^b \subset B$ is a discrete lattice and A is topologically isomorphic to $L \times (A \cap K)$, hence $(A^\wedge)^D = A^D$. This proves the exactness of the second sequence. $\square$

dc_101_10

Chapter 3

Local-global principles for 1-motives

3.1 Introduction

In this chapter we use the duality theorems of the previous chapter to prove some results related to 1-motives over number fields. The main object of study is the Manin obstruction to the Hasse principle and weak approximation on torsors under a semi-abelian variety over a number field k (i.e. a commutative k-group scheme which is an extension of an abelian variety by a torus).

We briefly recall the basic idea of the Manin obstruction; see the Introduction for more details. Given a smooth, geometrically integral variety X over our number field k, one considers the ring of adeles $\mathbf{A}_k$ of k, and defines a pairing

$$X(\mathbf{A}_k) \times \operatorname{Br} X \to \mathbf{Q}/\mathbf{Z} \tag{3.1}$$

by evaluating elements of the cohomological Brauer group $\operatorname{Br} X := H^2_{\text{ét}}(X, \mathbf{G}_m)$ at each component and then taking the sum of local invariants (which is known to be finite; see e.g. [63], p. 101). By global class field theory, the diagonal image of $X(k)$ in $X(\mathbf{A}_k)$ is contained in the subset $X(\mathbf{A}_k)^{\operatorname{Br}}$ of adeles annihilated by the above pairing. Consequently, the emptiness of $X(\mathbf{A}_k)^{\operatorname{Br}}$ is an obstruction to the Hasse principle if $X(\mathbf{A}_k)$ itself is nonempty.

In our case, a special role is played by a subquotient $\text{Б}(X)$ of $\operatorname{Br} X$ defined as follows. Fix an algebraic closure $\bar{k}$ of k. First one defines the subgroup $\operatorname{Br}_1 X \subset \operatorname{Br} X$ as the kernel of the natural map $\operatorname{Br} X \to \operatorname{Br}(X \times_k \bar{k})$, and then defines $\operatorname{Br}_a X$ as the quotient of $\operatorname{Br}_1 X$ modulo the image of the map $\operatorname{Br} k \to \operatorname{Br} X$. Finally one puts

$$\text{Б}(X) := \operatorname{Ker}(\operatorname{Br}_a X \to \prod_{v \in \Omega} \operatorname{Br}_a(X \times_k k_v)),$$

where Ω denotes the set of all places of k. (Note that some authors use the notation $\text{Б}(X)$ for the preimage of our $\text{Б}(X)$ in $\operatorname{Br} X$.) The pairing (3.1) manifestly factors through the image of $\operatorname{Br} k$ in $\operatorname{Br} X$, and hence induces a pairing

$$X(\mathbf{A}_k) \times \text{Б}(X) \to \mathbf{Q}/\mathbf{Z}. \tag{3.2}$$

Our first main result in this chapter concerns a semi-abelian variety G over k, i.e. an extension of an abelian variety A by a torus T. Recall that the *Tate-Shafarevich* group of A is defined as the kernel

$$Ш(A) := \ker(H^1(k,A) \to \prod_{v\in\Omega} H^1(k_v,A))$$

of the natural restriction map in Galois cohomology.

Theorem 3.1.1 *Let G be a semi-abelian variety defined over k, and let X be a k-torsor under G. Assume that the Tate-Shafarevich group of the abelian quotient A of G is finite. If there is an adelic point of X annihilated by all elements of $Б(X)$ under the pairing (3.2), then X has a k-rational point.*

The theorem answers positively a question raised by Skorobogatov in [63] (p. 133, question 1) for semi-abelian varieties. The analogous result for connected linear algebraic groups has been known for a long time (see [57]).

A reason why it is more interesting to work with the subquotient $Б(X)$ instead of the whole group $\mathrm{Br}\,X$ is that under the assumptions of the theorem it is finite (see Remark 3.4.7 below). In this respect Theorem 3.1.1 improves the main result of [29] that shows that a statement as in Theorem 3.1.1 holds even for arbitrary connected algebraic groups, provided that one replaces $Б(X)$ with the unramified part of $\mathrm{Br}_1 X$, which is a much bigger group.

Though Theorem 3.1.1 has been known for quite some time in the extreme cases $G = A$ (Manin [42]) and $G = T$ (Sansuc [57]), the general case does not follow from these, and is substantially more difficult. Our approach is based on a strategy similar to the proofs in the extreme cases, but it is more conceptual and avoids some rather involved cocycle calculations that made earlier texts hard to follow (at least for us). Among new ingredients we use the theory of generalised Albanese varieties and 1-motives, and a new interpretation of the pairing (3.2) as a cup-product in étale hypercohomology. The latter fact is valid for an arbitrary smooth variety and is of independent interest (see Section 3.3).

To state the second main result of the chapter, let $M = [Y \to G]$ be a 1-motive over k (i.e. a complex of k-group schemes placed in degrees $(-1,0)$ with Y étale locally isomorphic to $\mathbf{Z}^r$ for some $r \geq 0$ and G a semiabelian variety). Denote the dual 1-motive ([16], 10.2.1) by $M^* = [Y^* \to G^*]$. For each positive integer i denote by $Ш^i(M)$ (resp. $Ш^i_\omega(M)$) the subgroup of $\mathbf{H}^i(k,M)$ consisting of those elements whose restriction to $\mathbf{H}^i(k_v,M)$ is zero for all places (resp. for all but finitely many places) of k. In Section 3.5 we shall prove the following generalisation of the classical Cassels-Tate dual exact sequence to 1-motives.

Theorem 3.1.2 *Assume that the Tate-Shafarevich group* $Ш(A)$ *of the abelian quotient of* G *is finite. Then there is an exact sequence of abelian groups*

$$0 \to \overline{\mathbf{H}^0(k,M)} \to \prod_{v\in\Omega} \mathbf{H}^0(k_v,M) \to Ш^1_\omega(M^*)^D \to Ш^1(M) \to 0,$$

where $\overline{\mathbf{H}^0(k,M)}$ *denotes the closure of the diagonal image of* $\mathbf{H}^0(k,M)$ *in the topological product of the* $\mathbf{H}^0(k_v,M)$*, and* $A^D := \mathrm{Hom}(A,\mathbf{Q}/\mathbf{Z})$ *for a discrete abelian group* A*. By convention, for* v *archimedean we take here the modified (Tate) hypercohomology groups instead of the usual ones.*

The third (resp. fourth) maps in the above sequence are induced by the local (resp. global) duality pairings of Chapter 2 (see Section 3.5 for more details), and the topology on $\mathbf{H}^0(k_v,M)$ was defined in §2 of the same reference. A similar statement for connected linear algebraic groups was proven by Sansuc ([57], Theorem 8.12). Our method yields a new proof of his result in the case of tori.

The case $M = [0 \to G]$ of this exact sequence allows us to give a new short proof of the weak approximation part of the main result of [29] in the crucial case of a semi-abelian variety. For the precise statement, see Section 3.6.

The bulk of this chapter is a slightly modified version of [31]. In the last section we review further progress based on the main results of that paper.

3.2 Preliminaries on the Brauer group

As a preparation for the proof of Theorem 3.1.1, we collect here some auxiliary statements concerning the Brauer group. Statements 3.2.2–3.2.4 will not serve until Section 3.4.

We investigate the exact sequence of complexes of $\mathrm{Gal}(\bar{k}|k)$-modules

$$0 \to [\bar{k}^\times \to 0] \to [\bar{k}(X)^\times \to \mathrm{Div}\overline{X}] \to [\bar{k}(X)^\times/\bar{k}^\times \to \mathrm{Div}\overline{X}] \to 0 \tag{3.3}$$

for an arbitrary smooth geometrically integral variety X over a field k. Here, as usual, $\overline{X}$ stands for $X \times_k \bar{k}$, $\mathrm{Div}\overline{X}$ for the group of divisors on $\overline{X}$ and $\bar{k}(X)$ for its function field. In accordance with our conventions for 1-motives, the above complexes are placed in degrees -1 and 0.

Lemma 3.2.1 *There are canonical isomorphisms*

$$\mathbf{H}^1(k,[\bar{k}(X)^\times \to \mathrm{Div}\overline{X}]) \cong \mathrm{Br}_1 X$$

and, assuming $H^3(k,\bar{k}^\times) = 0$*,*

$$\mathbf{H}^1(k,[\bar{k}(X)^\times/\bar{k}^\times \to \mathrm{Div}\overline{X}]) \cong \mathrm{Br}_a X.$$

Proof: For the first isomorphism we use the long exact hypercohomology sequence associated with the distinguished triangle

$$\bar{k}(X)^\times \to \mathrm{Div}\,\overline{X} \to [\bar{k}(X)^\times \to \mathrm{Div}\,\overline{X}] \to \bar{k}(X)^\times[1]$$

and the fact that the permutation module $\mathrm{Div}\,\overline{X}$ has trivial H^1 to obtain

$$\mathbf{H}^1(k, [\bar{k}(X)^\times \to \mathrm{Div}\,\overline{X}]) \cong \ker\left(H^2(k, \bar{k}(X)^\times) \to H^2(k, \mathrm{Div}\,\overline{X})\right).$$

The latter group is identified in [63] with $\mathrm{Br}_1 X$: see the second column of the exact diagram (4.17) on p. 72. The injectivity of the map

$$\mathrm{Br}_1 X \to \ker\left(H^2(k, \bar{k}(X)^\times) \to H^2(k, \mathrm{Div}\,\overline{X})\right)$$

constructed there follows again from the vanishing of $H^1(k, \mathrm{Div}\,\overline{X})$, and the assumption $\bar{k}[X]^\times = \bar{k}^\times$ made in the reference is not used at this point.

Another (?) argument is to use the quasi-isomorphism of Galois modules

$$[\bar{k}(X)^\times \to \mathrm{Div}\,\overline{X}] \xrightarrow{\sim} \tau_{\leq 1}\mathbf{R}\pi_*\mathbf{G}_m[1]$$

constructed in [9], Lemma 2.3, where $\pi : \overline{X} \to \mathrm{Spec}\,\bar{k}$ is the natural projection. It yields an isomorphism of $\mathbf{H}^1(k, [\bar{k}(X)^\times \to \mathrm{Div}\,\overline{X}])$ with $\mathbf{H}^2(k, \tau_{\leq 1}\mathbf{R}\pi_*\mathbf{G}_m)$, in turn isomorphic to $\ker(H^2(X, \mathbf{G}_m) \to H^0(k, H^2(\overline{X}, \mathbf{G}_m))$ by a Hochschild-Serre argument.

The second isomorphism of the lemma follows from the first one in view of the long exact cohomology sequence associated with (3.3) and the assumption $H^3(k, \bar{k}^\times) = 0$. □

The complex of Galois modules $[\bar{k}(X)^\times/\bar{k}^\times \to \mathrm{Div}\,\overline{X}]$ was considered in several recent papers, in particular in Borovoi and van Hamel [9] and Colliot-Thélène [14]. The following property seems to have been observed by all of us:

Lemma 3.2.2 *Assume that X has a smooth compactification X^c over k. Then there is a natural Galois-equivariant quasi-isomorphism of complexes*

$$[\bar{k}(X)^\times/\bar{k}^\times \to \mathrm{Div}\,\overline{X}] \simeq [\mathrm{Div}^\infty\,\overline{X}^c \to \mathrm{Pic}\,\overline{X}^c],$$

where $\mathrm{Div}^\infty\,\overline{X}^c$ denotes the group of divisors on $\overline{X}^c := X^c \times_k \bar{k}$ supported in $\overline{X}^c \setminus \overline{X}$, and the maps in both complexes are divisor maps.

Proof: We consider the map of two-term complexes

$$\begin{array}{ccc} [\bar{k}(X)^\times/\bar{k}^\times & \longrightarrow & \mathrm{Div}\,\overline{X}] \\ \big\downarrow & & \big\downarrow \\ [\mathrm{Div}^\infty\overline{X}^c & \longrightarrow & \mathrm{Pic}\,\overline{X}^c] \end{array} \tag{3.4}$$

where the right vertical map is induced by sending a codimension 1 point on $\overline{X}$ to the class of the corresponding point of $\overline{X}^c$ *with a minus sign.* This sign convention implies that we indeed have a map of complexes, and it is a quasi-isomorphism by construction. □

A combination of the two previous lemmas yields:

Corollary 3.2.3 *If moreover k satisfies $H^3(k,\bar{k}^\times)=0$, we have a natural isomorphism*

$$\mathrm{Br}_a X \cong \mathbf{H}^1(k,[\mathrm{Div}^\infty\overline{X}^c \to \mathrm{Pic}\,\overline{X}^c]).$$

If k is of characteristic 0, the smooth compactification X^c always exists thanks to Hironaka's theorem on resolution of singularities. In particular, the statement of the corollary holds over number fields and their completions, since they are of characteristic 0 and satisfy the cohomological condition.

Remarks 3.2.4

1. Note that when X is proper, the isomorphism of the corollary is just the well-known identification $\mathrm{Br}_a X \cong H^1(k,\mathrm{Pic}\,\overline{X})$ induced by the Hochschild-Serre spectral sequence.

2. We shall also need a sheafified version of Lemma 3.2.2 over sufficiently small nonvoid open subsets of $U \subset \mathrm{Spec}\,\mathcal{O}_k$, where $\mathcal{O}_k$ denotes the ring of integers of k, as usual. For U sufficiently small the k-varieties X and X^c extend to smooth U-schemes $\mathcal{X}$ and $\mathcal{X}^c$, with $\mathcal{X}^c$ projective over U, and $\mathcal{X}$ open in $\mathcal{X}^c$. We shall work on the *big étale site* of U restricted to the subcategory Sm/U of smooth U-schemes of finite type. Consider the étale sheaf $\mathrm{Div}_{\mathcal{X}^c/U}$ associated with the presheaf $S \mapsto \mathrm{Div}(\mathcal{X}^c \times_U S/S)$ of relative Cartier divisors on this site (see e.g. [10] pp. 212–213, for a discussion of relative effective Cartier divisors, and then take group completion). It contains as subsheaves the sheaf $\mathrm{Div}_{\mathcal{X}/U}$ of relative divisors on $\mathcal{X} \times_U S/S$ and the sheaf $\mathrm{Div}^\infty_{\mathcal{X}^c/U}$ of relative divisors on $\mathcal{X}^c \times_U S/S$ with support in $(\mathcal{X}^c \setminus \mathcal{X}) \times_U S$. There is also the sheaf $\mathrm{Pic}_{\mathcal{X}^c/U}$ given by the relative Picard functor. Finally, denote by $\mathcal{K}^\times_{\mathcal{X}}$ the étale sheaf on Sm/U associated with the presheaf sending S to the group of rational functions on

dc_101_10

$\mathscr{X} \times_U S$ that are regular and invertible on a dense open subset of each fibre of the projection onto S. It contains as a subsheaf the pullbacks of invertible functions on S; we identify it with the sheaf $\mathbf{G}_m$ on the big étale site of Sm/U. We contend that we may lift the quasi-isomorphism (3.4) to a quasi-isomorphism

$$\begin{array}{ccc} [\mathscr{K}_{\mathscr{X}}^{\times}/\mathbf{G}_m & \longrightarrow & \mathrm{Div}_{\mathscr{X}/U}] \\ \downarrow & & \downarrow \\ [\mathrm{Div}^{\infty}_{\mathscr{X}^c/U} & \longrightarrow & \mathrm{Pic}_{\mathscr{X}^c/U}] \end{array}$$

of complexes of étale sheaves on Sm/U. The definition of the morphism is as above (one works with the inclusion $\mathscr{X} \times_U S \subset \mathscr{X}^c \times_U S$ for each object S of Sm/U). To check that it is a quasi-isomorphism, we may restrict to the small étale site of each S, and then look at stalks at geometric points. The statement to be checked then becomes a variant of Lemma 3.2.2 over a strictly henselian base, which is proven in the same way.

3.3 Reinterpretations of the Brauer-Manin pairing

In this section we use exact sequence (3.3) and Lemma 3.2.1 to give other formulations of the Brauer-Manin pairing

$$X(\mathbf{A}_k) \times \text{Б}(X) \to \mathbf{Q}/\mathbf{Z}.$$

Our X is still an arbitrary smooth geometrically integral variety over a number field k, and we assume $X(\mathbf{A}_k) \neq \emptyset$.

We start with a couple of well-known observations. Since the elements of $\text{Б}(X)$ are locally constant by definition, the maps

$$\text{б} : \text{Б}(X) \to \mathbf{Q}/\mathbf{Z} \tag{3.5}$$

given by evaluation on an adelic point (P_v) do not depend on the choice of (P_v), so defining the pairing is equivalent to defining the map б. There is a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Br}\, k & \longrightarrow & \mathrm{Br}_1 X & \longrightarrow & \mathrm{Br}_a X & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \bigoplus_{v\in\Omega} \mathrm{Br}\, k_v & \longrightarrow & \bigoplus_{v\in\Omega} \mathrm{Br}_1 X_v & \longrightarrow & \bigoplus_{v\in\Omega} \mathrm{Br}_a X_v & \longrightarrow & 0 \end{array} \tag{3.6}$$

where $X_v := X \times_k k_v$, and the first map in the bottom row is injective because our assumption that $X(\mathbf{A}_k) \neq \emptyset$ implies that each map $\mathrm{Br}\, k_v \to \mathrm{Br}_1(X \times_k k_v)$ is injective. The first map in the top row is then injective because so is the left vertical map, by the Hasse principle for Brauer groups. Applying the snake lemma to the diagram we thus have a map

$$\text{Б}(X) = \ker(\mathrm{Br}_a X \to \bigoplus_{v\in\Omega} \mathrm{Br}_a X_v) \to \mathrm{coker}\,(\mathrm{Br}\, k \to \bigoplus_{v\in\Omega} \mathrm{Br}\, k_v) \cong \mathbf{Q}/\mathbf{Z}.$$

Lemma 3.3.1 *The above map equals the map* $\text{б} : \text{Б}(X) \to \mathbf{Q}/\mathbf{Z}$.

Proof: For $\alpha \in \text{Б}(X)$ the value $\text{б}(\alpha)$ is defined by lifting first α to $\alpha' \in \mathrm{Br}_1 X$, then sending α' to an element of $\oplus_v \mathrm{Br}\, k_v$ via a family of local sections $(s_v : \mathrm{Br}_1 X \to \mathrm{Br}\, k_v)$ determined by an adelic point of X, and finally taking the sum of local invariants. Since each s_v factors through $\mathrm{Br}_1(X \times_k k_v)$, this yields the same element as the snake lemma construction. □

Now observe that in view of Lemma 3.2.1 one may also obtain the diagram (3.6) by taking the long exact hypercohomology sequence coming from the diagram

$$\begin{array}{ccccccc}
0 \to [\bar{k}^\times \to 0] \to & & [\bar{k}(X)^\times \to \mathrm{Div}\overline{X}] & \to & [\bar{k}(X)^\times/\bar{k}^\times \to \mathrm{Div}\overline{X}] \to 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 \to \bigoplus_{v\in\Omega} [\bar{k}_v^\times \to 0] \to & & \bigoplus_{v\in\Omega} [\bar{k}_v(X)^\times \to \mathrm{Div}\overline{X}_v] & \to & \bigoplus_{v\in\Omega} [\bar{k}_v(X_v)^\times/\bar{k}_v^\times \to \mathrm{Div}\overline{X}_v] \to 0,
\end{array}$$

where $\overline{X}_v := X \times_k \bar{k}_v$. The zeros on the right in (3.6) come from the fact that the groups $H^3(k, \mathbf{G}_m)$ and $H^3(k_v, \mathbf{G}_m)$ all vanish.

Remark 3.3.2 Note in passing that the sections s_v used in the above proof come from Galois-equivariant splittings

$$[\bar{k}_v(X)^\times \to \mathrm{Div}(X \times_k \bar{k}_v)] \to [\bar{k}_v^\times \to 0] \tag{3.7}$$

of the base change of the extension (3.3) to $\bar{k}_v$. As maps of complexes, the latter are given in degree -1 by a natural splitting of the inclusion map $\bar{k}_v^\times \to \bar{k}_v(X)^\times$ coming from P_v as constructed e.g. in ([63], Theorem 2.3.4 (b)), and in degree 0 by the zero map. In particular, the extension (3.3) is locally split.

As in Remark 3.2.4 (2), we now pass to sheaves over the étale site of Sm/U, where $U \subset \mathrm{Spec}\, \mathscr{O}_k$ is a suitable open subset. We can then extend the upper row of the last diagram to an exact sequence

$$0 \to \mathbf{G}_m[1] \to \mathscr{K}\mathscr{D}(\mathscr{X}) \to \mathscr{K}\mathscr{D}'(\mathscr{X}) \to 0 \tag{3.8}$$

of complexes of étale sheaves on Sm/U, where

$$\mathscr{KD}(\mathscr{X}) := [\mathscr{K}_{\mathscr{X}}^{\times} \to \mathrm{Div}_{\mathscr{X}/U}]$$

and

$$\mathscr{KD}'(\mathscr{X}) := [\mathscr{K}_{\mathscr{X}}^{\times}/\mathbf{G}_m \to \mathrm{Div}_{\mathscr{X}/U}].$$

By Lemma 3.2.2 we have $\mathrm{Br}_a X \cong H^1(k, [\bar{k}(X)^\times/\bar{k}^\times \to \mathrm{Div}\overline{X}])$, so each element of $\mathrm{Br}_a X$ comes from an element in $\mathbf{H}^1(U, \mathscr{KD}'(\mathscr{X}))$. Now assume moreover $\alpha \in \mathrm{Br}_a X$ is locally trivial, i.e. lies in $\text{Б}(X)$. For a finite place v of k we have $\mathbf{H}^1(k_v^h, j_v^{h*}\mathscr{KD}'(\mathscr{X})) \cong \mathbf{H}^1(k_v, j_v^*\mathscr{KD}'(\mathscr{X}))$, where k_v^h is the henselisation of k at v, and $j_v^h : \mathrm{Spec}\, k_v^h \to U$ as well as $j_v : \mathrm{Spec}\, k_v \to U$ are the natural maps. This is shown using the quasi-isomorphism of Lemma 3.2.2, and then reasoning as in the proof of ([30], Lemma 2.7). Next recall that in the case when k is totally imaginary the arithmetic compact support hypercohomology $\mathbf{H}_c^1(U, \mathscr{F})$ of a complex of sheaves $\mathscr{F}$ is defined by $\mathbf{H}_c^i(\mathrm{Spec}\, \mathscr{O}_k, j_!\mathscr{F})$, where $j : U \to \mathrm{Spec}\, \mathscr{O}_k$ is the natural inclusion. It fits into a long exact sequence

$$\cdots \to \mathbf{H}_c^1(U, \mathscr{F}) \to \mathbf{H}^1(U, \mathscr{F}) \to \bigoplus_{v \in \mathrm{Spec}\, \mathscr{O}_k \setminus U} \mathbf{H}^1(k_v^h, j_v^{h*}\mathscr{F}) \to \dots$$

In the general case there are corrective terms coming from the real places; see the discussion at the beginning of §3 in [30] (but note the misprint in formula (8) there: the $\hat{k}_v$ should be k_v in that paper's notation). It then follows from the above discussion that we may lift $\alpha \in \text{Б}(X)$ to an element $\alpha_U \in \mathbf{H}_c^1(U, \mathscr{KD}'(\mathscr{X}))$ for sufficiently small U.

There is a cup-product pairing

$$\cup : \mathrm{Ext}^1_{Sm/U}(\mathscr{KD}'(\mathscr{X}), \mathbf{G}_m[1]) \times \mathbf{H}_c^1(U, \mathscr{KD}'(\mathscr{X})) \to H_c^3(U, \mathbf{G}_m) \cong \mathbf{Q}/\mathbf{Z},$$

where the Ext-group is taken in the category of étale sheaves on Sm/U, and the last isomorphism comes from global class field theory (see [47], p. 159). We shall be interested in the class $\mathscr{E}_X \cup \alpha_U$, where $\mathscr{E}_X$ is the class of the sequence (3.8)in $\mathrm{Ext}^1_{Sm/U}(\mathscr{KD}'(\mathscr{X}), \mathbf{G}_m[1])$. Note that there is a commutative diagram

$$\begin{array}{ccccc}
\mathrm{Ext}^1_{Sm/U}(\mathscr{KD}'(\mathscr{X}), \mathbf{G}_m[1]) & \times & \mathbf{H}_c^1(U, \mathscr{KD}'(\mathscr{X})) & \to & H_c^3(U, \mathbf{G}_m) \cong \mathbf{Q}/\mathbf{Z}, \\
\downarrow & & \downarrow \cong & & \downarrow \mathrm{id} \\
\mathrm{Ext}^1_U(g_*\mathscr{KD}'(\mathscr{X}), \mathbf{G}_m[1]) & \times & \mathbf{H}_c^1(U, g_*\mathscr{KD}'(\mathscr{X})) & \to & H_c^3(U, \mathbf{G}_m) \cong \mathbf{Q}/\mathbf{Z},
\end{array}$$

where g_* is the natural pushforward (or restriction) map from the étale site of Sm/U to the small étale site of U, and the Ext-group in the bottom row is an Ext-group for étale sheaves on U. The left vertical map exists because the functor g_* is exact (and $\mathbf{G}_m$ as an étale sheaf on U is the pushforward by g of the $\mathbf{G}_m$ on Sm/U). The middle isomorphism comes from the fact that the hypercohomology of complexes of sheaves on the big étale site of U equals the hypercohomology on the small étale site. So instead of $\mathscr{E}_X$ we may work with its image $g_*\mathscr{E}_X$ in $\mathrm{Ext}^1_U(g_*\mathscr{K}\mathscr{D}'(\mathscr{X}),\mathbf{G}_m[1])$, and omit the g_* from the notation when no confusion is possible. Note that the generic stalk of $g_*\mathscr{E}_X$ is the class of the extension (3.3) in the group $\mathrm{Ext}^1_k([\bar{k}(X)^\times/\bar{k}^\times \to \mathrm{Div}\,\overline{X}],\bar{k}^\times[1])$.

Proposition 3.3.3 *With notations as above, we have*

$$\mathscr{E}_X \cup \alpha_U = \text{б}(\alpha).$$

Before starting the proof, note that though one has several choices for α_U, the cup-product depends only on α. Indeed two choices of α_U differ by an element of the direct sum of the groups $\mathbf{H}^0(k_v^h,\mathscr{K}\mathscr{D}'(\mathscr{X}))$, and the cup-product of each such group with $\mathrm{Ext}^1_U(\mathscr{K}\mathscr{D}'(\mathscr{X}),\mathbf{G}_m[1])$ factors through the cup-product with $\mathrm{Ext}^1_{k_v^h}(j_v^*\mathscr{K}\mathscr{D}'(\mathscr{X}),\mathbf{G}_m[1])$. But the image of the class $g_*\mathscr{E}_X$ in these groups is 0, because the extension is locally split (Remark 3.3.2).

Proof: In order to avoid complicated notation, we do the verification in the case when U is totally imaginary and the simpler definition of compact support cohomology is available, and leave the general case to anxious readers. We may work over the small étale site of U by the previous observations; in particular, we identify the complexes $\mathscr{K}\mathscr{D}(\mathscr{X})$ and $\mathscr{K}\mathscr{D}'(\mathscr{X})$ with their images under g_*.

The cup-product $\mathscr{E}_X \cup \alpha_U$ is none but the image of α by the boundary map $\mathbf{H}^1_c(U,\mathscr{K}\mathscr{D}'(\mathscr{X})) \to H^3_c(U,\mathbf{G}_m)$ coming from the long exact hypercohomology sequence associated with (3.8). Now consider the commutative exact diagram

$$\begin{array}{ccccccccc}
0 & \to & \mathbf{G}_m[1] & \to & \mathscr{K}\mathscr{D}(\mathscr{X}) & \to & \mathscr{K}\mathscr{D}'(\mathscr{X}) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \bigoplus_{v\notin U} j_{v*}j_v^*\mathbf{G}_m[1] & \to & \bigoplus_{v\notin U} j_{v*}j_v^*\mathscr{K}\mathscr{D}(\mathscr{X}) & \to & \bigoplus_{v\notin U} j_{v*}j_v^*\mathscr{K}\mathscr{D}'(\mathscr{X}) & \to & 0
\end{array}$$

dc_101_10

of complexes of étale sheaves on U, and denote the cones of the vertical maps by $\mathscr{C}$, $\mathscr{C}_{\mathscr{K}}$ and $\mathscr{C}'_{\mathscr{K}}$, respectively. The group $\mathbf{H}^1(U, \mathscr{C}'_{\mathscr{K}}[-1])$ may be identified with $\mathbf{H}^1_c(U, \mathscr{K}\mathscr{D}'(\mathscr{X}))$ (apply, for instance, [47], Lemma II.2.4 and its proof with our U as V and our $\operatorname{Spec}\mathscr{O}_k$ as U), so we may view α_U as an element of the former group. The cup-product $\mathscr{E}_X \cup \alpha_U$ maps to a class in $\mathbf{H}^2(U, \mathscr{C}[-1]) = \mathbf{H}^1(U, \mathscr{C})$. But when one makes U smaller and smaller and passes to the limit, this class yields an element in the cokernel of the map $\operatorname{Br} k \to \oplus_v \operatorname{Br} k_v^h$ which (noting the isomorphism $\operatorname{Br} k_v^h \cong \operatorname{Br} k_v$) is precisely the one obtained by the snake-lemma construction at the beginning of this section. It remains to apply Lemma 3.3.1. □

3.4 Proof of Theorem 1

We now prove Theorem 3.1.1, of which we take up the notation and assumptions. As in the case of abelian varieties, the idea is to relate the Brauer-Manin pairing for a torsor X under a semi-abelian variety G to a Cassels-Tate type pairing. In our case it is the generalised pairing for 1-motives

$$\langle\, , \rangle : Ш(M) \times Ш(M^*) \to \mathbf{Q}/\mathbf{Z}$$

defined in [30], where $Ш(M^*)$ is the Tate-Shafarevich group attached to the dual 1-motive $M^* = [\widehat{T} \to A^*]$ of $M = [0 \to G]$. For the various generalities about 1-motives used here and in the sequel, we refer to the first section of [30].

To relate the two pairings, we shall construct a map

$$\iota : Ш(M^*) \to Б(X)$$

and prove that the equality

$$\langle [X], \beta \rangle = б(\iota(\beta)) \tag{3.9}$$

holds for all $\beta \in Ш(M^*)$ up to a sign. Theorem 3.1.1 will then follow from the non-degeneracy of the Cassels-Tate pairing proven in [30].

To construct the map ι we proceed as follows. Recall that for a smooth quasi-projective variety $\overline{V}$ over a field of characteristic 0 there exists a generalised Albanese variety Alb_V introduced in [59] over an algebraically closed field, and in [53] in general. It is a semi-abelian variety, and according to a result of Severi generalised by Serre [60] and amplified in [53] the Cartier dual of the 1-motive $[0 \to \mathrm{Alb}_V]$ is $[\mathrm{Div}^{\infty,\mathrm{alg}}_{V^c} \to \mathrm{Pic}^0_{V^c}]$. Here V^c is a smooth compactification of V, and the term $\mathrm{Div}^{\infty,\mathrm{alg}}_{V^c}$ is the group of divisors on V^c algebraically equivalent to 0 and supported in $V^c \setminus V$ viewed as an étale locally constant group scheme. In the case $V = X$ we have $\mathrm{Alb}_V = G$ by definition, and therefore

$$M^* = [\mathrm{Div}^{\infty,\mathrm{alg}}_{X^c} \to \mathrm{Pic}^0_{X^c}]. \tag{3.10}$$

Since there is a natural map of complexes of k-group schemes

$$[\mathrm{Div}^{\infty,\mathrm{alg}}_{X^c} \to \mathrm{Pic}^0_{X^c}] \to [\mathrm{Div}^{\infty}_{X^c} \to \mathrm{Pic}_{X^c}], \tag{3.11}$$

passing to hypercohomology yields a map

$$\mathbf{H}^1(k, M^*) \to \mathbf{H}^1(k, DP(X^c))$$

with the notation

$$DP(X^c) := [\mathrm{Div}^{\infty}_{X^c} \to \mathrm{Pic}_{X^c}].$$

Over a number field k the group $\mathbf{H}^1(k, DP(X^c))$ is isomorphic to $\mathrm{Br}_a X$ by Corollary 3.2.3, and the same holds over the completions of k. Since the above map is manifestly functorial for field extensions, we obtain the required map ι by restricting to locally constant elements.

We can thus rewrite the map ι as

$$\iota : Ш(M^*) \to Ш(DP(X^c)).$$

The previous construction also yields a dual map

$$\mathbf{H}^1(k, Hom(DP(X^c), \mathbf{G}_m[1])) \to \mathbf{H}^1(k, M) \tag{3.12}$$

by applying the functor $Hom(\quad, \mathbf{G}_m[1])$ to the map (3.11) and taking hypercohomology (recall that $M \cong Hom(M^*, \mathbf{G}_m[1])$; see the remark below). Restricting to locally trivial elements yields a map

$$\iota^D : Ш(Hom(DP(X^c), \mathbf{G}_m[1])) \to Ш(M).$$

Remark 3.4.1 The *Hom*-functor used in the above formulas is the internal Hom-functor in the bounded derived category of sheaves on the big étale site of $\mathrm{Spec}\, k$ restricted to the full subcategory Sm/k of smooth k-schemes. It may also be viewed as H^0 of $\mathbf{R}Hom$, the total derived functor of the internal Hom in the category of sheaves on the said site. The other H^i's are the higher Ext^i's coming from this internal Hom.

The Barsotti-Weil formula $A^* \cong Ext^1(A, \mathbf{G}_m)$ for abelian schemes holds in this context, because (as O. Wittenberg kindly explained to us) the proof of [51], Corollary 17.5 carries over from the *fpqc* site to the big étale site in the case of smooth group schemes. Hence so does the isomorphism $M \cong Hom(M^*, \mathbf{G}_m[1])$ used above. Note here that since the duality between M and M^* comes from the derived pairing $M \otimes^{\mathbf{L}} M^* \to \mathbf{G}_m[1]$, one a priori only has $M \cong \mathbf{R}Hom(M^*, \mathbf{G}_m[1])$, but the higher Ext^i's vanish (see the end of the proof of Lemma 3.4.4 below).

dc_101_10

We shall also need versions of the maps constructed above over a suitable $U \subset \operatorname{Spec} \mathscr{O}_k$. Over U sufficiently small the complex $DP(X^c)$, viewed as a complex of étale sheaves on Sm/k, extends to a complex

$$\mathscr{D}\mathscr{P}(\mathscr{X}^c) := [\mathrm{Div}^\infty_{\mathscr{X}^c/U} \to \mathrm{Pic}_{\mathscr{X}^c/U}],$$

where we have used the notations of Remark 3.2.4 (2). Shrinking U if necessary, we can also extend the 1-motive M to a 1-motive $\mathscr{M}$ over U. The main point is then:

Lemma 3.4.2 *The dual 1-motive $\mathscr{M}^* = [\mathscr{Y} \to \mathscr{A}^*]$ is isomorphic to the 1-motive $[\mathrm{Div}^{\infty,\mathrm{alg}}_{\mathscr{X}^c/U} \to \mathrm{Pic}^0_{\mathscr{X}^c/U}]$ over sufficiently small U, where $\mathrm{Div}^{\infty,\mathrm{alg}}_{\mathscr{X}^c/U}$ is the inverse image of $\mathrm{Pic}^0_{\mathscr{X}^c/U}$ in $\mathrm{Div}^\infty_{\mathscr{X}^c/U}$.*

Here $\mathrm{Pic}^0_{\mathscr{X}^c/U} \subset \mathrm{Pic}_{\mathscr{X}^c/U}$ is the subsheaf of elements whose restriction to each fibre of the map $\mathscr{X}^c \to U$ lies in Pic^0 of the fibre.

Proof: This should be part of a duality theory of Albanese and Picard 1-motives for smooth quasi-projective schemes over U. Since we do not know an adequate reference for this, we have chosen to circumvent the problem as follows. The group schemes $\mathrm{Pic}^0_{\mathscr{X}^c/U}$ and $\mathscr{A}^*$ are both smooth group schemes of finite type over U, whereas $\mathrm{Div}^{\infty,\mathrm{alg}}_{\mathscr{X}^c/U}$ and $\mathscr{Y}$ are both character groups of U-tori, so maps defined between their generic fibres extend to maps over suitable U. □

Corollary 3.4.3 *Over suitable $U \subset \operatorname{Spec} \mathscr{O}_k$ the map ι lifts to a map*

$$\iota_U : \mathbf{H}^1_c(U, \mathscr{M}^*) \to \mathbf{H}^1_c(U, \mathscr{D}P(\mathscr{X}^c)),$$

and the map (3.12) extends to

$$\iota_U^D : H^1(U, Hom(\mathscr{D}P(\mathscr{X}^c), \mathbf{G}_m[1])) \to \mathbf{H}^1(U, \mathscr{M}).$$

Proof: The map (3.11) extends to a map of complexes

$$[\mathrm{Div}^{\infty,\mathrm{alg}}_{\mathscr{X}^c/U} \to \mathrm{Pic}^0_{\mathscr{X}^c/U}] \to \mathscr{D}P(\mathscr{X}^c),$$

so by the lemma we dispose of a map $\mathscr{M}^* \to \mathscr{D}P(\mathscr{X}^c)$. The required maps are obtained by passing to cohomology. □

In the previous section we worked with a certain extension class $\mathscr{E}_X$ in the group $\mathrm{Ext}^1_{Sm/U}(\mathscr{K}\mathscr{D}'(\mathscr{X}), \mathbf{G}_m[1])$. According to Remark 3.2.4 (2) the Ext-group here is isomorphic to $\mathrm{Ext}^1_{Sm/U}(\mathscr{D}\mathscr{P}(\mathscr{X}^c), \mathbf{G}_m[1])$. The next lemma will imply that over sufficiently small U we may identify $\mathscr{E}_X$ with a class $\mathscr{E}'_{\mathscr{X}}$ in the group $\mathbf{H}^1(U, Hom(\mathscr{D}\mathscr{P}(\mathscr{X}^c), \mathbf{G}_m[1])$.

Lemma 3.4.4 *There are canonical isomorphisms*

$$\mathrm{Ext}^j_{Sm/k}(DP(X^c),\mathbf{G}_m[1]) \cong H^j(k, Hom(DP(X^c),\mathbf{G}_m[1]))$$

for all $j > 0$.

Proof: We start with the isomorphism

$$\mathrm{Ext}^j_{Sm/k}(DP(X^c),\mathbf{G}_m[1]) \cong \mathbf{H}^j(k, \mathbf{R}Hom_{Sm/k}(DP(X^c),\mathbf{G}_m[1])) \tag{3.13}$$

coming from the derived category analogue of the spectral sequence for composite functors; the functor $\mathbf{R}Hom$ was explained in Remark 3.4.1. It shows that the lemma follows if we prove that the restrictions of the sheaves

$$Ext^{i-1}_{Sm/k}(DP(X^c),\mathbf{G}_m[1]) = Ext^i_{Sm/k}(DP(X^c),\mathbf{G}_m)$$

to the small étale site of Spec (k) are trivial for $i > 1$ and $i = 0$. We are thus reduced to checking the triviality of the Galois modules $\mathrm{Ext}^i_{Sm/\bar{k}}(DP(X^c)_{\bar{k}},\mathbf{G}_m) = 0$ for $i > 1$ and $i = 0$. We drop the subscripts in the rest of the proof.

Observe first that the cokernel of the map of complexes (3.11) is quasi-isomorphic to the complex $[0 \to B(X^c)]$, where $B(X^c)$ is the quotient of the Néron-Severi-group of X^c modulo the subgroup of classes coming from divisors at infinity; in particular, its $\bar{k}$-points form a finitely generated abelian group. Hence the group $\mathrm{Ext}^i(B(X^c),\mathbf{G}_m)$ is trivial for $i > 0$ (see [63], Sublemma 2.3.8). Therefore the distinguished triangle coming from (3.11) shows that it is enough to prove $\mathrm{Ext}^i([\mathrm{Div}^{\infty,\mathrm{alg}}_{X^c} \to \mathrm{Pic}^0_{X^c}],\mathbf{G}_m) = 0$ for $i > 1$ and $i = 0$, which is the same as proving $\mathrm{Ext}^i(M^*,\mathbf{G}_m) = 0$ by the isomorphism (3.10). The case $i = 0$ then follows from the fact that every morphism from A^* to $\mathbf{G}_m$ is trivial. For the case $i > 1$ we remark that the stupid filtration on $M^* = [\widehat{T} \to A^*]$ induces an exact sequence

$$\mathrm{Ext}^i(A^*,\mathbf{G}_m) \to \mathrm{Ext}^i(M^*,\mathbf{G}_m) \to \mathrm{Ext}^{i-1}(\widehat{T},\mathbf{G}_m).$$

Here the terms at the two extremities are trivial for $i > 1$ (the left one by [51], Prop. 12.3), hence so is the middle one. □

Remarks 3.4.5

1. Here it was crucial to work with extensions over the big étale site; over the small étale site of $\bar{k}$ the group $\mathrm{Ext}^i(A^*,\mathbf{G}_m)$ is trivial even for $i = 1$.

2. In the course of the proof we also established canonical isomorphisms

$$\mathrm{Ext}^j_{Sm/k}(M^*,\mathbf{G}_m[1]) \cong H^j(k, Hom(M^*,\mathbf{G}_m[1]))$$

for all $j > 0$.

Now denote by E_X the image of the class $\mathscr{E}_X$ of the previous section in the group $\mathrm{Ext}^j_{Sm/k}(DP(X^c), \mathbf{G}_m[1])$. Via the isomorphism of the lemma it corresponds to a class E'_X in $H^1(k, Hom_{Sm/k}(DP(X^c), \mathbf{G}_m[1]))$. We may extend the latter to a class in $H^1(U, Hom_{Sm/U}(\mathscr{D}P(\mathscr{X}^c), \mathbf{G}_m[1])$ over a sufficiently small $U \subset \mathrm{Spec}\,(k)$. There is a natural map

$$H^1(U, Hom_{Sm/U}(\mathscr{D}P(\mathscr{X}^c), \mathbf{G}_m[1]) \to \mathrm{Ext}^1_{Sm/U}(\mathscr{D}\mathscr{P}(\mathscr{X}^c), \mathbf{G}_m[1])$$

coming from the analogue of (3.13) over U. By shrinking U if necessary we may assume that the image of $\mathscr{E}'_X$ by this map is $\mathscr{E}_X$, since the two classes coincide at the generic point.

Applying the map ι^D_U to $\mathscr{E}'_X$ we obtain a class in $\mathbf{H}^1(U, \mathscr{M})$. Over the generic point $\iota^D_U(\mathscr{E}'_X)$ restricts to the image of E'_X by the map (3.12). We can identify the latter as follows.

Lemma 3.4.6 *The image of E'_X by the map (3.12) equals (up to a sign) the class of X in $H^1(k, G) = \mathbf{H}^1(k, M)$.*

The lemma should be true over an arbitrary field of characteristic 0. It is known in the two extreme cases $G = A$ and $G = T$ (see references in the proof below); we leave the general case to the reader as a challenge. The following proof, which is sufficient for our purposes, works under the assumptions of Theorem 3.1.1 (i.e. over a number field, assuming $X(\mathbf{A}_k) \neq \emptyset$ and the finiteness of Ш(A)). Also, as O. Wittenberg pointed out to us, Corollary 4.2.4 of [72] implies that E'_X maps to 0 in $H^1(k, G)$ if and only if $[X] = 0$, which is also sufficient for the proof of Theorem 3.1.1 given below.

Proof of Lemma 3.4.6. Thanks to Proposition 2.1 of [64] the case $G = A$ is known, and we may complete the proof of Theorem 3.1.1 given below in this special case. Thus we are allowed to apply Theorem 3.1.1 to the pushforward torsor p_*X under A (here of course $p : G \to A$ is the natural projection map), and conclude that it is trivial. The exact sequence

$$H^1(k, T) \xrightarrow{i_*} H^1(k, G) \xrightarrow{p_*} H^1(k, A)$$

then implies that $X = i_*Y$ for some k-torsor Y under T, where $i : T \to G$ is the natural inclusion.

The map (3.12) factors through $H^1(k, Hom(M^*, \mathbf{G}_m[1]))$ by construction, and by Remark 3.4.5 (2) we may identify the image of E'_X in the latter group with a class $E^0_X \in \mathrm{Ext}^1(M^*, \mathbf{G}_m[1])$. By performing the same construction for the torsor Y we obtain a class $E^0_Y \in \mathrm{Ext}^1(\widehat{T}[1], \mathbf{G}_m[1])$. According to [72], Proposition 4.1.4 applied with $V = Y$ and $W = X$ we have $E^0_X = i^*E^0_Y$, where $i^* : \mathrm{Ext}^1(\widehat{T}[1], \mathbf{G}_m[1]) \to$

$\mathrm{Ext}^1(M^*,\mathbf{G}_m[1])$ is the natural map induced by the projection $M^* \to \widehat{T}[1]$. (Note that this equality is not completely obvious, because the map $Y \to X$ is not dominating.) But for $G = T$ the lemma is known over an arbitrary field ([63], Lemma 2.4.3), so the image of E_Y^0 in $H^1(k,T)$ is $[Y]$ up to a sign. The image of $[Y]$ in $H^1(k,G)$ is $[X]$, so the lemma in the general case follows from the commutativity of the diagram

$$\begin{array}{ccc} H^1(k,T) & \longrightarrow & H^1(k,G) \\ \uparrow & & \uparrow \\ \mathrm{Ext}^1(\widehat{T}[1],\mathbf{G}_m[1]) & \longrightarrow & \mathrm{Ext}^1(M^*,\mathbf{G}_m[1]). \end{array}$$

$\square$

Proof of Theorem 3.1.1. As already remarked at the beginning of this section, for the proof of the theorem it will suffice to verify formula (3.9) for the class of X in Ш(M), i.e. the equality $\langle [X],\beta\rangle = б(\iota(\beta))$ up to a sign for all $\beta \in$ Ш(M^*). Indeed, our assumption that X has an adelic point orthogonal to Б(X) implies the triviality of the map б, so the right hand side of the formula is 0 for all β in Ш(M^*). But under the finiteness assumption on Ш(A) the Cassels-Tate pairing $\langle\, ,\rangle$ is non-degenerate ([30], Corollary 4.9), so $[X] = 0$, i.e. X has a k-rational point.

We now verify formula (3.9). Consider the cup-product pairing

$$H^1(U,Hom(\mathscr{DP}(X^c),\mathbf{G}_m[1])) \times \mathbf{H}^1_c(U,\mathscr{DP}(X^c)) \to H^3_c(U,\mathbf{G}_m)$$

and recall that we have defined above a class $\mathscr{E}'_X$ in $H^1(U,Hom(\mathscr{DP}(X^c),\mathbf{G}_m[1]))$. By construction, taking the product of $\mathscr{E}'_X$ with some $\alpha_U \in \mathbf{H}^1_c(U,\mathscr{DP}(X^c))$ under this pairing is the same as the element $\mathscr{E}_X \cup \alpha_U$ considered in Proposition 3.3.3. So applying the proposition we obtain the equality $\mathscr{E}'_X \cup \alpha_U = б(\alpha)$ in the case when α_U maps to a locally trivial element in $\mathbf{H}^1(k,DP(X^c))$.

Moreover, using the maps constructed in Corollary 3.4.3 we have a diagram

$$\begin{array}{ccccc} \mathbf{H}^1(U,\mathscr{M}) & \times & \mathbf{H}^1_c(U,\mathscr{M}^*) & \to & H^3_c(U,\mathbf{G}_m) \\ \iota_U^D \uparrow & & \downarrow\ \iota_U & & \downarrow \mathrm{id} \\ H^1(U,Hom(\mathscr{DP}(X^c),\mathbf{G}_m[1])) & \times & \mathbf{H}^1_c(U,\mathscr{DP}(X^c)) & \to & H^3_c(U,\mathbf{G}_m) \end{array}$$

where the horizontal maps are cup-product pairings. The diagram commutes by construction, and the image $\iota^D(E'_X)$ of the element $\iota_U^D(\mathscr{E}'_X)$ in $H^1(k,M) = H^1(k,G)$

is the class $[X]$ up to a sign by the previous lemma. By Corollary 4.3 of [30] each element $\beta \in Ш(M^*)$ comes from some $\beta' \in \mathbf{H}^1_c(U, \mathscr{M}^*)$ for U sufficiently small, and moreover the value of the upper pairing on $(\iota^D_U(\mathscr{E}'_X), \beta')$ equals the value of the Cassels-Tate pairing on $(\iota^D(E'_X), \beta)$, i.e. on $([X], \beta)$ up to a sign. The commutativity of the diagram together with the arguments of the previous paragraph implies that this value equals $б(\iota(\beta))$. This proves formula (3.9), and thereby the theorem. □

Remark 3.4.7 As a complement to the theorem, we justify here a claim made in the introduction, namely that the group $Б(X)$ is *finite*. In [8], Proposition 2.14 the finiteness of $Б(V)$ is verified for a smooth *proper* V such that $Ш(\mathrm{Pic}^0 V)$ is finite. To deduce the statement for our X, apply this result with $V = X^c$, a smooth compactification of X. The condition on the Tate-Shafarevich group holds because $\mathrm{Pic}^0(V)$ is the Picard variety of the Albanese variety of V (theorem of Severi), and the latter is none but A (see [59], [60] for these facts). It remains to add that $Б(V) \cong Б(X)$ in view of [57], (6.1.4).

To conclude this section we mention a variant of Theorem 3.1.1 that deals with points of X over the direct product k_Ω of all completions of k instead of $\mathbf{A}_k$. In this situation we look at a modified version of the Brauer-Manin pairing, namely the induced pairing

$$X(k_\Omega) \times (\mathrm{Br}_{\mathrm{nr}} X / \mathrm{Br}\, k) \to \mathbf{Q}/\mathbf{Z}, \tag{3.14}$$

where $\mathrm{Br}_{\mathrm{nr}} X$ is the unramified Brauer group of X, which may be defined as the Brauer group of a smooth compactification V of X. Since $Б(X) \cong Б(V)$ as in the remark above, the group $Б(X)$ is contained in $\mathrm{Br}_{\mathrm{nr}} X / \mathrm{Br}\, k$.

Corollary 3.4.8 *Let G be a semi-abelian variety defined over k, and let X be a k-torsor under G. Assume that the Tate-Shafarevich group of the abelian quotient of G is finite. If there is a point of $X(k_\Omega)$ annihilated by all elements of $Б(X)$ under the pairing (3.14), then X has a k-rational point.*

The corollary immediately follows from Theorem 3.1.1 and the following lemma:

Lemma 3.4.9 *Let X be a smooth geometrically integral variety defined over k. If there is a point of $X(k_\Omega)$ orthogonal to $Б(X)$ under the pairing (3.14), then there is also an adelic point on X orthogonal to $Б(X)$ under the pairing (3.2).*

Proof: From Chow's lemma we know that X contains a quasi-projective open subset U. Choose a finite set S of places of k such that the pair $U \subset X$ extends to a pair of smooth schemes $\mathscr{U} \subset \mathscr{X}$ over $\operatorname{Spec}(\mathscr{O}_{k,S})$ with U quasi-projective, where $\mathscr{O}_{k,S}$ is the ring of S-integers of k. From the Lang–Weil estimates and Hensel's lemma we know that by enlarging S if necessary we have $\mathscr{U}(\mathscr{O}_v) \neq \emptyset$ for $v \notin S$, and hence the same holds for $\mathscr{X}$. Now if $(P_v) \in X(k_\Omega)$ is orthogonal to $\text{Б}(X)$, we replace P_v by an $\mathscr{O}_v$-point P'_v of $\mathscr{X}$ for $v \notin S$. Then (P'_v) is an adelic point of X, and this adelic point remains orthogonal to $\text{Б}(X)$ because elements of $\text{Б}(X)$ induce constant elements of $\operatorname{Br}(X \times_k k_v)$ for every place v. □

3.5 The Cassels-Tate dual exact sequence for 1-motives

In this section we prove Theorem 3.1.2, of which we take up the notation. Recall that by convention for an archimedean place v of k the notation $\mathbf{H}^0(k_v, M)$ stands for the Tate group $\widehat{\mathbf{H}}^0(k_v, M)$, which is a 2-torsion finite group. Also, recall from ([30], §2) that the group $\mathbf{H}^0(k_v, M)$ is equipped with a natural topology. In the case $M = G$ and v finite, this is just the usual v-adic topology on $H^0(k_v, G) = G(k_v)$, but in general the topology on $\mathbf{H}^0(k_v, M)$ is not Hausdorff.

We denote by $\overline{\mathbf{H}^0(k, M)}$ the closure of the diagonal image of $\mathbf{H}^0(k, M)$ in the topological direct product of the $\mathbf{H}^0(k_v, M)$. The local pairings $(\ ,\ )_v$ of ([30], §2) induce a map

$$\theta : \prod_{v \in \Omega} \mathbf{H}^0(k_v, M) \to Ш^1_\omega(M^*)^D$$

defined by

$$\theta((m_v))(\alpha) = \sum_{v \in \Omega} (m_v, \alpha_v)_v,$$

where α_v is the image of $\alpha \in Ш^1_\omega(M^*)$ in $\mathbf{H}^1(k_v, M)$ (the sum is finite by definition of $Ш^1_\omega(M^*)$). On the other hand, the analogue of Cassels-Tate pairing for 1-motives ([30], Theorem 4.8) and the inclusion $Ш^1(M^*) \subset Ш^1_\omega(M^*)$ induce a map

$$p : Ш^1_\omega(M^*)^D \to Ш^1(M)$$

We have thus defined all maps in the sequence

$$0 \to \overline{\mathbf{H}^0(k, M)} \to \prod_{v \in \Omega} \mathbf{H}^0(k_v, M) \xrightarrow{\theta} Ш^1_\omega(M^*)^D \xrightarrow{p} Ш^1(M) \to 0$$

and our task is to prove its exactness.

We shall need several intermediate results. The first one is the following well-known lemma, for which we give a proof by lack of a reference.

Lemma 3.5.1 *Let Y be a k-group scheme étale locally isomorphic to $\mathbf{Z}^r$ for some $r > 0$. Then the group $\text{Ш}^2_\omega(Y)$ is finite.*

Here by definition $\text{Ш}^2_\omega(Y) := \text{Ш}^1_\omega([Y \to 0])$, with the notation of the introduction.

Proof: Let L be a finite Galois extension of k that splits Y. Since $\text{Ш}^2_\omega(\mathbf{Z}) = \text{Ш}^1_\omega(\mathbf{Q}/\mathbf{Z})$ is zero by Chebotarev's density theorem, we obtain that $\text{Ш}^2_\omega(Y)$ is a subgroup of $H^2(\mathrm{Gal}\,(L|k), Y)$, which is a torsion group annihilated by $n = [L:k]$. The boundary map

$$H^1(\mathrm{Gal}\,(L|k), Y/nY) \to H^2(\mathrm{Gal}\,(L|k), Y)$$

obtained from the exact sequence of $\mathrm{Gal}\,(L|k)$-modules

$$0 \to Y \to Y \to Y/nY \to 0$$

is therefore surjective. Since $\mathrm{Gal}\,(L|k)$ and Y/nY are finite, the lemma follows. □

Now return to the situation above, and recall from [30], Theorem 2.3 and Remark 2.4 that the local pairings $(\ ,\ )_v$ used in the definition of θ actually factor through the profinite completion $\mathbf{H}^0(k_v, M)^\wedge$ of $\mathbf{H}^0(k_v, M)$, hence θ extends to $\mathbf{H}^0(k_v, M)^\wedge$. Technical complications will arise from the fact that the topology on $\mathbf{H}^0(k_v, M)$ is in general finer than the topology induced by the profinite topology of $\mathbf{H}^0(k_v, M)^\wedge$. For instance, this is the case for $M = [0 \to T]$ with T a torus.

Lemma 3.5.2 *The groups $\prod_{v\in\Omega} \mathbf{H}^0(k_v, M)^\wedge$ and $\prod_{v\in\Omega} \mathbf{H}^0(k_v, M)$ have the same image by θ.*

Proof: For v archimedean, the group $\mathbf{H}^0(k_v, M)$ is finite, hence it is the same as its profinite completion, so we can concentrate on the finite places. We proceed by dévissage, starting with the case $M = [0 \to G]$. Let v be a finite place of k. Since A is proper, we have $H^0(k_v, A) = A(k_v) = H^0(k_v, A)^\wedge$. Using the exact sequences

$$0 \to T(k_v) \to G(k_v) \to A(k_v) \to H^1(k_v, T)$$

$$0 \to T(k_v)^\wedge \to G(k_v)^\wedge \to A(k_v) \to H^1(k_v, T)$$

(cf. [30], Lemma 2.2), we obtain that

$$\prod_{v\in\Omega} H^0(k_v, G)^\wedge = \left\{ g + t : g \in \mathrm{im}\left(\prod_{v\in\Omega} H^0(k_v, G)\right), t \in \prod_{v\in\Omega} H^0(k_v, T)^\wedge \right\}.$$

Therefore it is sufficient to prove the statement for $G = T$. But this follows from the facts that $Ш^1_\omega(M^*) = Ш^2_\omega(Y^*)$ is finite (by the previous lemma), and each $H^0(k_v, T)$ is dense in $H^0(k_v, T)^\wedge$.

The same method reduces the general case to the case $M = [0 \to G]$, using the exact sequences ([30], p. 101):

$$H^0(k_v, G) \to \mathbf{H}^0(k_v, M) \to H^1(k_v, Y) \to H^1(k_v, G)$$

$$H^0(k_v, G)^\wedge \to \mathbf{H}^0(k_v, M)^\wedge \to H^1(k_v, Y) \to H^1(k_v, G)$$

□

Denote by $Ш^1_S(M^*)$ the kernel of the diagonal map

$$\mathbf{H}^1(k, M^*) \to \prod_{v \notin S} \mathbf{H}^1(k_v, M^*).$$

As above, the local pairings induce maps

$$\theta_S : \prod_{v \in S} \mathbf{H}^0(k_v, M) \to Ш^1_S(M^*)^D$$

and

$$\widehat{\theta}_S : \prod_{v \in S} \mathbf{H}^0(k_v, M)^\wedge \to Ш^1_S(M^*)^D.$$

Proposition 3.5.3 *Let S be a finite set of places of k. Assume that the Tate-Shafarevich group $Ш(A)$ of the abelian quotient of G is finite.*

1. *The sequence*

$$\mathbf{H}^0(k, M)^\wedge \to \prod_{v \in S} \mathbf{H}^0(k_v, M)^\wedge \xrightarrow{\widehat{\theta}_S} Ш^1_S(M^*)^D \tag{3.15}$$

   is exact.

2. *Denote by $\overline{\mathbf{H}^0(k, M)}_S$ the closure of the diagonal image of $\mathbf{H}^0(k, M)$ in $\prod_{v \in S} \mathbf{H}^0(k_v, M)$. Then the sequence*

$$0 \to \overline{\mathbf{H}^0(k, M)}_S \to \prod_{v \in S} \mathbf{H}^0(k_v, M) \xrightarrow{\theta_S} Ш^1_S(M^*)^D \tag{3.16}$$

   is exact.

Proof: 1. Let $\mathbf{P}^1(M^*)$ be the restricted product of the $\mathbf{H}^1(k_v, M^*)$ (cf. [30], §5). By the Poitou-Tate exact sequence for 1-motives ([30], Th. 5.6), there is an exact sequence

$$\mathbf{H}^1(k, M^*) \to \mathbf{P}^1(M^*)_{\text{tors}} \to (\mathbf{H}^0(k, M)^D)_{\text{tors}}.$$

(Recall that this uses the finiteness of the Tate-Shafarevich group of A^*, which is equivalent to that of A by [47], Remark I.6.14(c)). Sending an element of $\prod_{v \in S} \mathbf{H}^1(k_v, M^*)$ to $\mathbf{P}^1(M^*)_{\text{tors}}$ via the map

$$(m_v)_{v \in S} \mapsto ((m_v), 0, 0, \ldots)$$

yields an exact sequence of discrete torsion groups

$$Ш^1_S(M^*) \to \prod_{v \in S} \mathbf{H}^1(k_v, M^*) \to (\mathbf{H}^0(k, M)^D)_{\text{tors}}.$$

We claim that the required exact sequence is the dual of the above. Indeed, the dual of the discrete torsion group $\mathbf{H}^1(k_v, M^*)$ is the profinite group $\mathbf{H}^0(k_v, M)^\wedge$ by the local duality theorem ([30], Th. 2.3), and the dual of the discrete torsion group $(\mathbf{H}^0(k, M)^D)_{\text{tors}}$ is the profinite completion $\mathbf{H}^0(k, M)^\wedge$ of $\mathbf{H}^0(k, M)$, because $(\mathbf{H}^0(k, M)^D)_{\text{tors}}$ is nothing but the direct limit (over the subgroups $I \subset \mathbf{H}^0(k, M)$ of finite index) of the groups $\text{Hom}(\mathbf{H}^0(k, M)/I, \mathbf{Q}/\mathbf{Z})$.

2. Consider the commutative diagram

$$\begin{array}{ccccc} \mathbf{H}^0(k, M) & \xrightarrow{j} & \prod_{v \in S} \mathbf{H}^0(k_v, M) & \xrightarrow{\theta_S} & Ш^1_S(M^*)^D \\ \downarrow & & \downarrow & & \downarrow \text{id} \\ \mathbf{H}^0(k, M)^\wedge & \xrightarrow{j^\wedge} & \prod_{v \in S} \mathbf{H}^0(k_v, M)^\wedge & \xrightarrow{\widehat{\theta}_S} & Ш^1_S(M^*)^D \end{array} \tag{3.17}$$

The second line is exact by what we have just proven, so the first line is a complex. Hence so is the sequence (3.16) by continuity of $\widehat{\theta}_S$. Denote by J the closure of the image of j in the above diagram. Set

$$C := \prod_{v \in S} \mathbf{H}^0(k_v, M)/J,$$

and equip C with the quotient topology. In particular, C is a Hausdorff topological group (because J is closed).

Consider now the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & J & \longrightarrow & \prod_{v \in S} \mathbf{H}^0(k_v, M) & \longrightarrow & C \\ & & \downarrow & & \downarrow & & \downarrow \\ & & J^\wedge & \longrightarrow & \prod_{v \in S} \mathbf{H}^0(k_v, M)^\wedge & \longrightarrow & C^\wedge. \end{array} \tag{3.18}$$

dc_101_10

Assume for the moment that the right vertical map here is injective. We can then derive the exactness of sequence (3.15) as follows. The first line of diagram (3.18) is exact by definition, and the second line is a complex because it is the completion of an exact sequence. Since the second line of diagram (3.17) is exact, the image of an element $x \in \ker(\theta_S)$ in $\ker(\widehat{\theta_S})$ comes from $H^0(k,M)^\wedge$, hence from $J^\wedge$. A diagram chase in (3.18) then shows $x \in J$, which is what we wanted to prove.

Now the injectivity of the right vertical map in (3.18) follows from statement (3) of the Appendix to [30], of which we have to check the assumptions. The last horizontal map above is an open mapping because it is a quotient map. The group C is Hausdorff, locally compact and totally disconnected by construction; it remains to check that it is also compactly generated (i.e. it is generated as a group by the elements of a compact subset). This is because by §2 of [30] the group $\mathbf{H}^0(k_v,M)$ has a finite index open subgroup that is a topological quotient of $H^0(k_v,G)$, so C has a finite index open subgroup C' that is a quotient of the product of the $H^0(k_v,G)$ for $v \in S$. Since each $H^0(k_v,G)$ is compactly generated (this follows from the theory of p-adic Lie groups) and C' is Hausdorff, we obtain that C', and hence C, are compactly generated.

□

Proof of Theorem 3.1.2. Let us start by proving the exactness of the sequence

$$\prod_{v\in\Omega} \mathbf{H}^0(k_v,M) \to Ш^1_\omega(M^*)^D \to Ш^1(M) \to 0. \tag{3.19}$$

The sequence

$$0 \to Ш^1(M^*) \to Ш^1_\omega(M^*) \to \bigoplus_{v\in\Omega} \mathbf{H}^1(k_v,M^*) \tag{3.20}$$

is exact by definition. By the local duality theorem for 1-motives ([30], Theorem 2.3 and Proposition 2.9), the dual of each group $\mathbf{H}^1(k_v,M^*)$ is $\mathbf{H}^0(k_v,M)^\wedge$, and by the global duality theorem ([30], Corollary 4.9), the dual of $Ш^1(M^*)$ is $Ш^1(M)$ under our finiteness assumption on Tate-Shafarevich groups. Therefore the dual of (3.20) is the exact sequence

$$\prod_{v\in\Omega} \mathbf{H}^0(k_v,M)^\wedge \to Ш^1_\omega(M^*)^D \to Ш^1(M) \to 0,$$

and Lemma 3.5.2 gives the exactness of (3.19).

It remains to prove the exactness of the sequence

$$0 \to \overline{\mathbf{H}^0(k,M)} \to \prod_{v\in\Omega} \mathbf{H}^0(k_v,M) \to Ш^1_\omega(M^*)^D.$$

But this sequence is obtained by applying the (left exact) inverse limit functor (over all finite subsets $S \subset \Omega$) to the exact sequences (3.16). Indeed, by definition of the direct product topology the inverse limit of the groups $\overline{\mathbf{H}^0(k,M)_S}$ is $\overline{\mathbf{H}^0(k,M)}$, and the inverse limit of the groups $Ш^1_S(M^*)^D$ is the dual of the direct limit of the discrete torsion groups $Ш^1_S(M^*)$, i.e. the dual of $Ш^1_\omega(M^*)$. □

3.6 Obstruction to weak approximation

In the study of weak approximation on a variety X one works with a modified version of the Brauer-Manin pairing, namely with the induced pairing

$$X(k_\Omega) \times \mathrm{Br}_{\mathrm{nr}} X \to \mathbf{Q}/\mathbf{Z},$$

already encountered at the end of Section 3.4, where k_Ω is the topological direct product of all completions of k, and $\mathrm{Br}_{\mathrm{nr}} X$ is the unramified Brauer group of X. One may also work with subgroups of $\mathrm{Br}_{\mathrm{nr}} X$, such as $\mathrm{Br}_{\mathrm{nr}\,1} X := \ker(\mathrm{Br}_{\mathrm{nr}} X \to \mathrm{Br}_{\mathrm{nr}}(X \times_k \bar{k}))$. Finally, for a smooth k-group scheme G there is yet another variant, which is the one we shall use:

$$\prod_{v \in \Omega} H^0(k_v, G) \times \mathrm{Br}_{\mathrm{nr}\,1} G \to \mathbf{Q}/\mathbf{Z}. \tag{3.21}$$

Here we have taken the same convention at the archimedean places as in Theorem 3.1.2 proven above. Concerning this pairing one has the following result, first proven in [29]:

Theorem 3.6.1 *Let G be a semi-abelian variety defined over k. Assuming that the abelian quotient has finite Tate-Shafarevich group, the left kernel of the pairing (3.21) is contained in the closure of the diagonal image of $G(k)$.*

This result was proven in *loc. cit.* for arbitrary connected algebraic groups, but the key case is that of a semi-abelian variety. We now show that the statement can be easily derived from Theorem 3.1.2 as follows. The Brauer-Manin pairing induces a map

$$\prod_{v \in \Omega} H^0(k_v, G) \to (\mathrm{Br}_{\mathrm{nr}\,1} G/\mathrm{Br}\, k)^D.$$

Going through the construction of the map ι at the beginning of Section 3.4 with $Ш_\omega$ in place of $Ш$ we obtain a map $\iota_\omega : Ш^1_\omega(M^*) \to Б_\omega(G)$, where $Б_\omega(G) \subset \mathrm{Br}_a G$ is the subgroup of elements that are locally trivial for almost all places for k. Using the inclusion $Б_\omega(X) \subset \mathrm{Br}_{\mathrm{nr}} X/\mathrm{Br}\, k$ resulting from ([57], 6.1.4) we thus obtain a map $r : Ш^1_\omega(M^*) \to \mathrm{Br}_{\mathrm{nr}\,1}\, G/\mathrm{Br}\, k$, whence a diagram

$$\begin{tikzcd}
 & & (\mathrm{Br}_{\mathrm{nr}\,1} G/\mathrm{Br}\, k)^D \arrow[d, "\iota_\omega^D"] \\
\overline{G(k)} \arrow[r] & \prod_{v\in\Omega} H^0(k_v, G_v) \arrow[ur] \arrow[r] & Ш^1_\omega(M^*)^D
\end{tikzcd}$$

where $G_v := G \times_k k_v$. If we prove that the triangle commutes, the theorem follows, since the bottom row is exact by Theorem 3.1.2.

We shall prove the commutativity of the dualized diagram

$$\begin{tikzcd}
Ш^1_\omega(M^*) \arrow[r] \arrow[d, "\iota_\omega"'] & H^0(k_v, G_v)^D \\
\mathrm{Br}_{\mathrm{nr}\,1} G/\mathrm{Br}\, k \arrow[ur]
\end{tikzcd} \tag{3.22}$$

for all places v. Here the horizontal map is induced by local duality, so it is in fact enough to consider the finitely many nonzero images of an element in $Ш^1_\omega(M^*)$ by the restriction maps $Ш^1_\omega(M^*) \to H^1(k_v, M^*)$ and show that the diagram

$$\begin{tikzcd}
\mathbf{H}^1(k_v, M^*) \arrow[r] \arrow[d] & H^0(k_v, G_v)^D \\
\mathrm{Br}_a\, G_v \arrow[ur]
\end{tikzcd}$$

commutes, where the diagonal map is induced by the evaluation pairing

$$G(k_v) \times \mathrm{Br}\, G_v \to \mathrm{Br}\, k_v \cong \mathbf{Q}/\mathbf{Z}, \tag{3.23}$$

and we view $\mathrm{Br}_a\, G_v$ as a subgroup of $\mathrm{Br}\, G_v$ thanks to the splitting of the map $\mathrm{Br}\, k_v \to \mathrm{Br}\, G_v$ coming from the zero section of G_v.

To do so, return to the beginning of Section 3.3 and observe that in the case $X = G$ the maps (3.7) actually assemble to a pairing of complexes of Galois modules

$$[0 \to G(\bar{k}_v)] \otimes_{\mathbf{Z}} [\bar{k}_v(G)^\times \to \mathrm{Div}(G \times_k \bar{k}_v)] \to [\bar{k}_v^\times \to 0].$$

The sections $\bar{k}_v(G)^\times \to \bar{k}_v^\times$ used in this construction are not canonical, but the pairing becomes canonical at the level of the derived category, again by the argument in ([63], Theorem 2.3.4 (b)). We thus obtain a cup-product pairing

$$H^0(k_v, G) \times \mathbf{H}^1(k_v, [\bar{k}_v(G)^\times \to \mathrm{Div}(G \times_k \bar{k}_v)]) \to \mathrm{Br}\, k_v$$

that identifies via Lemma 3.2.1 with the restriction of the local pairing (3.23) to $\mathrm{Br}_1 G_v$ by the argument at the beginning of Section 3.3. On the other hand, we may lift the map $\mathbf{H}^1(k_v, M^*) \to \mathrm{Br}_a G_v$ to a map $\mathbf{H}^1(k_v, M^*) \to \mathrm{Br}_1 G_v$ via the zero section as above, and then the claim follows from the commutativity of the diagram of cup-product pairings

$$\begin{array}{ccccc} \mathbf{H}^0(k_v, M) & \times & \mathbf{H}^1(k_v, M^*) & \to & \mathrm{Br}\, k_v \\ \mathrm{id} \downarrow & & \downarrow & & \downarrow \mathrm{id} \\ H^0(k_v, G) & \times & \mathrm{Br}_1 G_v & \to & \mathrm{Br}\, k_v. \end{array}$$

3.7 Further developments

Borovoi, Colliot-Thélène and Skorobogatov have generalized Theorem 3.1.1 to homogeneous spaces under an arbitrary connected algebraic group. The precise statement is the following.

Theorem 3.7.1 ([8], Theorem 3.14) *Let G be a connected linear algebraic group defined over a totally imaginary number field k, and let X be a homogeneous space of G whose geometric points have connected stabilisers. Assume that the Tate-Shafarevich group of the abelian quotient A of G is finite. If there is an adelic point of X annihilated by all elements of Б (X) under the pairing (3.2), then X has a k-rational point.*

The proof uses techniques going back to Borovoi's papers [6] and [7] to reduce to the case of a torsor under a semi-abelian variety, where our Theorem 3.1.1 can be applied. Note, however, the additional assumption that k is totally imaginary. In fact, the same paper contains a quite surprising example ([8], Proposition 3.16) of a connected non-commutative and non-linear algebraic group over $\mathbf{Q}$ for which the statement fails. This shows that over arbitrary number fields general connected algebraic groups behave differently from commutative or linear ones.

In fact, Borovoi, Colliot-Thélène and Skorobogatov formulated their result in a different but equivalent way, in terms of the *elementary obstruction* of Colliot-Thélène and Sansuc [15]. By definition, this obstruction is the extension class

$ob(X)$ of $\mathrm{Gal}(\bar{k}|k)$-modules

$$0 \to \bar{k}^\times \to \bar{k}(X)^\times \to \bar{k}(X)^\times/\bar{k}^\times \to 0 \tag{3.24}$$

where k is a perfect field, X is an arbitrary smooth geometrically integral k-variety and $\bar{k}(X)^\times$ is the group of invertible rational functions on $X \times_k \bar{k}$. An easy argument in Galois cohomology (see e.g. [63], p. 27) shows that a k-rational point induces a Galois-equivariant splitting of the above extension. Thus nontriviality of $ob(X)$ is an obstruction to the existence of a k-point.

In fact, the triviality of $ob(X)$ is equivalent to the triviality of the pairing (3.2), under a finiteness assumption on the appropriate Tate–Shafarevich group. This is shown in Wittenberg's paper [72] by relating both properties to a third one concerning the *periods* of open subsets of X. We now explain the concepts involved in some detail. We have seen in Chapter 1 that in the case when k is algebraically closed there exists a semi-abelian variety Alb_X attached to X which is universal for morphisms of X to semi-abelian varieties. Over a general k the generalized Albanese variety Alb_X still exists: it is a semi-abelian variety over k that comes equipped with a canonical k-torsor Alb^1_X which is universal for morphisms of X to torsors under semi-abelian varieties. The existence of Alb_X follows from the statement over algebraically closed fields by Galois descent and some additional arguments in positive characteristic (see [18] and the appendix to [72]). The order of the class of the torsor Alb^1_X in the group $H^1(k, \mathrm{Alb}_X)$ is called the *period* of X. If X has a k-point, then so does Alb^1_X via the map $X \to \mathrm{Alb}^1_X$, so the period is 1. This gives a necessary condition for the existence of a k-point. One can show using an elementary moving lemma argument as in ([13], p. 599) that the existence of a k-point on X in fact implies that the period of every dense open subset $U \subset X$ is 1.

Now we can state:

Theorem 3.7.2 ([72], Theorem 3.3.2) *Let X be a smooth geometrically integral variety over a number field k with $X(\mathbf{A}_k) \neq \emptyset$. Assume that the Tate-Shafarevich group of the Albanese variety of X is finite. Then the following are equivalent.*

(1) There is a point in $X(\mathbf{A}_k)$ annihilated by the pairing (3.2).

(2) The sequence (3.24) splits, i.e. $ob(X)$ is trivial.

(3) The period of every dense open subset $U \subset X$ is 1.

In fact, the implication (3) ⇒ (2) holds over arbitrary fields, without the arithmetic assumptions of the theorem; this is one of the main results of Wittenberg's paper [72]. The other two implications are of arithmetic nature. That (2) implies

dc_101_10

(1) was proven in ([8], Theorem 2.12); the proof is short and is based on the global reciprocity law of class field theory.

The implication $(1) \Rightarrow (3)$ is essentially Theorem 1.1 of [18]. As its proof rests on Theorem 3.1.1, we give a sketch. One first uses the fact that $X(\mathbf{A}_k) \neq \emptyset$ implies $U(\mathbf{A}_k) \neq \emptyset$ for every Zariski open subset $U \subset X$. Indeed, there is an étale map $\lambda : X \to V$ onto an open subset V of affine space; over each completion k_v this map is an isomorphism in a v-adic analytic neighbourhood of a k_v-point of X by the inverse function theorem, but such a neighbourhood meets every Zariski open $U \subset X$. Next one shows ([18], Lemma 3.4) that for each U as above the map $\text{Б}(X) \to \text{Б}(U)$ is an isomorphism. Since the elements of $\text{Б}(X)$ are locally constant, it follows that U inherits condition (1) from X. But then using the map $U \to \mathrm{Alb}^1_U$ we see that Alb^1_U also satisfies (1). By Theorem 3.1.1 there is a k-point on Alb^1_U, i.e. U has period 1.

In their recent paper [19] Esnault and Wittenberg establish the equivalence of yet another condition with the above three. It concerns the exact sequence of absolute Galois groups

$$1 \to \mathrm{Gal}\,(\overline{k(X)}|\bar{k}(X)) \to \mathrm{Gal}\,(\overline{k(X)}|k(X)) \to \mathrm{Gal}\,(\bar{k}|k) \to 1.$$

Taking the pushout by projection $\mathrm{Gal}\,(\overline{k(X)}|\bar{k}(X)) \to \mathrm{Gal}\,(\overline{k(X)}|\bar{k}(X))^{\mathrm{ab}}$ onto the maximal abelian profinite quotient one obtains an exact sequence of profinite groups

$$1 \to \mathrm{Gal}\,(\overline{k(X)}|\bar{k}(X))^{\mathrm{ab}} \to \Gamma \to \mathrm{Gal}\,(\bar{k}|k) \to 1. \tag{3.25}$$

Theorem 3.7.3 *Under the assumptions of Theorem 3.7.2, conditions (1)–(3) are equivalent to*

(4) The extension (3.25) of profinite groups splits.

Here implication $(4) \Rightarrow (3)$ is proven by an argument inspired by the proofs of the implications $(2) \Rightarrow (1) \Rightarrow (3)$ above; in particular, the final step is again given by Theorem 3.1.1. The authors prove $(2) \Rightarrow (4)$ by a general argument in Galois cohomology valid over an arbitrary field.

Remark 3.7.4 In Section 4 of [32] David Harari and I have proven that for an arbitrary field k and an arbitrary smooth geometrically integral k-variety the vanishing of $ob(X)$ implies the splitting of the exact sequence

$$1 \to \pi_1^{\mathrm{ab}}(X \times_k \bar{k}) \to \Pi \to \mathrm{Gal}\,(\bar{k}|k) \to 1 \tag{3.26}$$

obtained as above by pushout via an abelianization map from the exact sequence of profinite groups

$$1 \to \pi_1(X \times_k \bar{k}) \to \pi_1(X) \to \mathrm{Gal}\,(\bar{k}|k) \to 1. \tag{3.27}$$

Here $\pi_1(X)$ denotes Grothendieck's algebraic fundamental group introduced in [27] (with respect to a fixed geometric base point which we omitted from the notation). As $\pi_1(X)$ is a quotient of $\mathrm{Gal}\,(\overline{k(X)}|k(X))$, this statement also follows from the general form of implication $(2) \Rightarrow (4)$ above. However, we were more interested in fundamental groups as $\pi_1(X \times_k \bar{k})$ is known to be topologically finitely generated over a field of characteristic 0, whereas $\mathrm{Gal}\,(\overline{k(X)}|k(X))$ is huge.

Finally, we note that the splitting of (3.26) does not imply the vanishing of $ob(X)$; there are examples of simply connected varieties with nontrivial elementary obstruction.

dc_101_10

Bibliography

[1] M. Artin, A. Grothendieck, J.-L. Verdier (eds.), *Théorie des topos et cohomologie étale des schémas* (SGA 4), Lecture Notes in Math. **269**, Springer-Verlag 1972.

[2] L. Barbieri-Viale, B. Kahn, On the derived category of 1-motives, eprint `arXiv:1009.1900`.

[3] V. G. Berkovich, Duality theorems in Galois cohomology of commutative algebraic groups, *Selecta Math. Soviet.* **6** (1987), 201–296.

[4] J. Biswas, V. Srinivas, Roitman's theorem for singular complex varieties, *Compositio Math.* **119** (1999), 213–237.

[5] S. Bloch, Torsion algebraic cycles and a theorem of Roitman, *Compositio Math.* **39** (1979), 107–127.

[6] M. Borovoi, Abelianization of the second nonabelian Galois cohomology, *Duke Math. J.* 72 (1993), 217-239.

[7] M. Borovoi, The BrauerManin obstruction for homogeneous spaces with connected or abelian stabilizer, *J. reine angew. Math.* 473 (1996) 181-194.

[8] M. Borovoi, J.-L. Colliot-Thélène, A. N. Skorobogatov, The elementary obstruction for homogeneous spaces, *Duke Math. J.* 141 (2008), 321–364.

[9] M. Borovoi, J. van Hamel, Extended Picard complexes and linear algebraic groups, *J. reine angew. Math.* 627 (2009), 53–82.

[10] S. Bosch, W. Lütkebohmert, M. Raynaud, *Néron models*, Springer-Verlag, Berlin, 1990.

[11] J. W. S. Cassels, Arithmetic on curves of genus 1, IV: Proof of the Hauptvermutung, *J. reine angew. Math.* 211 (1962), 95112; VII: The dual exact sequence, *ibid.* 216 (1964), 150-158.

[12] J-L. Colliot-Thélène, *Cycles algébriques de torsion et K-théorie algébrique*. In: Arithmetic Algebraic Geometry. Lect. Notes in Math. **1553**, Springer Verlag 1993, 1–49.

[13] J-L. Colliot-Thélène, Un théorème de finitude pour le groupe de Chow des zéro-cycles d'un groupe algébrique linéaire sur un corps p-adique, *Invent. Math.* 159 (2005) 589–606.

[14] J-L. Colliot-Thélène, Résolutions flasques des groupes linéaires connexes, *J. reine angew. Math.* 618 (2008), 77–133.

[15] J-L. Colliot-Thélène, J-J. Sansuc, La descente sur les variétés rationnelles II, *Duke Math. J.* **54** (1987), 375–492.

[16] P. Deligne, Théorie de Hodge III, *Inst. Hautes Études Sci. Publ. Math.* **44** (1974), 5–77.

[17] C. Deninger, An extension of Artin-Verdier duality to non-torsion sheaves, *J. reine angew. Math.* **366** (1986), 18–31.

[18] D. Eriksson, V. Scharaschkin, On the Brauer–Manin obstruction for zero-cycles on curves, *Acta Arithmetica* 135 (2008), 99–110.

[19] H. Esnault, O. Wittenberg, Abelian birational sections, *J. Amer. Math. Soc.* 23 (2010), 713-724.

[20] M. Flach, A generalisation of the Cassels-Tate pairing, *J. reine angew. Math.* **412** (1990), 113–127.

[21] J.-M. Fontaine, B. Perrin-Riou, Autour des conjectures de Bloch-Kato: cohomologie galoisienne et valeurs de fonctions L, in U. Jannsen et al. (eds.) *Motives*, Proc. Symp. Pure Math. **55**, vol. 1., pp. 599–706.

[22] T. Geisser, *Applications of de Jong's theorem on alterations*, in: Resolution of Singularities. Progress in Math. **181**, Birkhäuser 2000, 299–314.

[23] P. Gille, T. Szamuely, *Central Simple Algebras and Galois Cohomology*, Cambridge Studies in Advanced Mathematics, vol. 101, Cambridge University Press, 2006.

[24] C. González-Avilés, Arithmetic duality theorems for 1-motives over function fields, *J. reine angew. Math.* 632 (2009), 203-231.

[25] C. González-Avilés, K. S. Tan, The generalized Cassels-Tate dual exact sequence for 1-motives, *Math. Res. Letters* 16 (2009), 827–839.

[26] M. J. Greenberg, Rational points in Henselian discrete valuation rings, *Inst. Hautes Études Sci. Publ. Math.* **31** (1966), 59–64.

[27] A. Grothendieck, *Revêtements étales et groupe fondamental* (SGA 1), Lecture Notes in Mathematics, vol. 224, Springer-Verlag, Berlin-New York, 1971. New annotated edition: Société Mathématique de France, Paris, 2003.

[28] A. Grothendieck, *Cohomologie des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA2)*. North-Holland, 1968.

[29] D. Harari, The Manin obstruction for torsors under connected algebraic groups, *Intern. Math. Res. Not*., 2006, article ID 68632, 13 pages.

[30] D. Harari, T. Szamuely, Arithmetic duality theorems for 1-motives, *J. reine angew. Math.* **578** (2005), 93–128. Corrections: *J. reine angew. Math.* 632 (2009), 233–236.

[31] D. Harari, T. Szamuely, Local-global principles for 1-motives, *Duke Math. J.* 143 (2008), 531-557.

[32] D. Harari, T. Szamuely, Galois sections for abelianized fundamental groups, *Math. Ann.* 344 (2009), 779–800.

[33] E. Hewitt, K. Ross, *Abstract harmonic analysis I*, Springer-Verlag 1963.

[34] P. Jossen, On the arithmetic of 1-motives, PhD thesis, Central European University, Budapest, 2009.

[35] N. Katz and S. Lang, Finiteness theorems in geometric class field theory, *L'Enseign. Math.* **27** (1981), 285–314.

[36] K. Kato, A Hasse principle for two dimensional global fields, *J. reine angew. Math.* **366** (1986), 142–183.

[37] K. Kato, S. Saito, Unramified class field theory of arithmetical surfaces, *Ann. of Math.* **118** (1983), 241–275.

[38] R. Kottwitz, Stable trace formula: cuspidal tempered terms. *Duke Math. J.* **51** (1984), 611–650.

[39] R. Kottwitz, D. Shelstad, *Foundations of twisted endoscopy,* Astérisque **255**, 1999.

[40] S. Lang, Algebraic groups over finite fields, *Amer. J. Math.* **78** (1956), 555–563.

dc_101_10

[41] S. Lichtenbaum, Suslin homology and Deligne 1-motives, in: *Algebraic K-Theory and Algebraic Topology* (P. G. Goerss and J. F. Jardine, eds.). NATO ASI Series (1993), 189–197.

[42] Yu. Manin, Le groupe de Brauer–Grothendieck en géométrie diophantienne, in *Actes du Congrès International des Mathématiciens (Nice, 1970)*, Tome 1, Gauthier-Villars, Paris, 1971, pp. 401–411.

[43] G. A. Margulis, *Discrete subgroups of semisimple Lie groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) **17**, Springer-Verlag, Berlin, 1991.

[44] B. Mazur, Notes on étale cohomology of number fields, *Ann. Sci. Éc. Norm. Sup.* **6** (1973), 521–556.

[45] J. S. Milne, Zero cycles on algebraic varieties in nonzero characteristic: Rojtman's theorem, *Compositio Math.* **47** (1982), 271–287.

[46] J.S. Milne, *Étale Cohomology*, Princeton University Press, 1980.

[47] J.S. Milne, *Arithmetic Duality Theorems*, Academic Press, 1986.

[48] M. Nagata, *Theory of commutative fields*, Translations of Mathematical Monographs, vol. 125, American Mathematical Society, 1993.

[49] J. Neukirch, A. Schmidt, K. Wingberg : *Cohomology of number fields*, Grundlehren der Math. Wiss. **323**, Springer-Verlag, 2000.

[50] T. Ono, Arithmetic of algebraic tori, *Ann. of Math. (2)* **74** (1961), 101–139.

[51] F. Oort, *Commutative group schemes*, Lecture Notes in Math. **15**, Springer-Verlag 1966.

[52] B. Poonen, M. Stoll, The Cassels-Tate pairing on polarized abelian varieties, *Ann. of Math.* **150** (1999), 1109–1149.

[53] N. Ramachandran, Duality of Albanese and Picard 1-motives, *K-theory* **21** (2001), 271–301.

[54] M. Raynaud, 1-motifs et monodromie géométrique, in *Périodes p-adiques*, Astérisque **223** (1994), 295–319.

[55] A. A. Roitman, The torsion of the group of 0-cycles modulo rational equivalence, *Ann. of Math.* **111** (1980), 553–569.

[56] M. Rosenlicht, Generalized Jacobian varieties, *Ann. of Math.* **59** (1954), 505–530.

[57] J.-J. Sansuc, Groupe de Brauer et arithmétique des groupes algébriques linéaires sur un corps de nombres, *J. reine angew. Math.* **327** (1981), 12–80.

[58] A. Schmidt, M. Spieß, Singular homology and class field theory for varieties over finite fields, *J. reine angew. Math.* **527** (2000), 13–36.

[59] J-P. Serre, Morphismes universels et variété d'Albanese, *Séminaire Chevalley*, année 1958/58, exposé 10.

[60] J-P. Serre, Morphismes universels et différentielles de troisième espèce, *Séminaire Chevalley*, année 1958/59, exposé 11.

[61] J-P. Serre, *Groupes algébriques et corps de classes*. Hermann, 1959.

[62] J-P. Serre, *Cohomologie Galoisienne* (cinquième édition, révisée et complétée), Lecture Notes in Math. **5**, Springer Verlag, 1994.

[63] A. N. Skorobogatov, *Torsors and rational points*, Cambridge University Press, 2001.

[64] A. N. Skorobogatov, On the elementary obstruction to the existence of rational points (Russian), *Mat. Zametki* **81** (2007), 112–124; English translation: *Math. Notes* **81** (2007) 97–107.

[65] M. Spieß, T. Szamuely, On Albanese maps for smooth quasi-projective varieties, *Math. Ann.* **325** (2003), 1–17.

[66] A. Suslin, V. Voevodsky, Singular homology of abstract algebraic varieties, *Invent. Math.* **123** (1996), 61–94.

[67] T. Szamuely, Corps de classes des schémas arithmétiques, Séminaire Bourbaki, exposé 1006, mars 2009, Astérisque 332 (2010), 257–286.

[68] J. Tate, The cohomology groups of tori in finite Galois extensions of number fields, *Nagoya Math. J.* **27** (1966), 709–719.

[69] J. Tate, WC-groups over p-adic fields, *Séminaire Bourbaki*, année 1957/58, exposé 156.

[70] J. Tate, Duality theorems in Galois cohomology over number fields, *Proc. Internat. Congr. Mathematicians (Stockholm, 1962)*, 288–295.

[71] V. Voevodsky *Triangulated categories of motives over a field*. In V. Voevodsky, A. Suslin, E. M. Friedlander: Cycles, Transfers, and Motivic Homology Theories. Annals of Math. Studies **143**, Princeton University Press, 2000, 188–238.

[72] O. Wittenberg, On Albanese torsors and the elementary obstruction to the existence of 0-cycles of degree 1, *Math. Ann.* 340 (2008), 805–838.

[73] Th. Zink, Appendix 2 in K. Haberland, *Galois cohomology of number fields*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1978.