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Dr. Michael Makai
Secretary, Committee for Doctoral Awards
Hungarian Academy of Sciences

Dr. Makai:

I am writing in response your request for an evaluation of the thesis submitted by Lajos Soukup to your committee. Let me begin with my estimation that this thesis represents the fruits of a coherent research project leading to important results in a central area of set theory. It is a body of work of which any mathematician with an established career, such as Lajos Soukup, may well be proud.

The majority of the results in the thesis have to do with understanding the possible structure of locally compact, scattered spaces and all the result are, ultimately, motivated by such questions. The notion of a scattered space has its roots in the early investigations of Cantor on the question posed by Riemann and Heine: If the trigonometric series $\sum_n e^{-inx}$ converges to $f(x)$ at each point x, is the series uniquely determined. Cantor showed that the answer is yes, and that it is even possible to relax the hypothesis of convergence everywhere to convergence off a finite set and, moreover, “finite" could be relaxed further to what we would now call finite Cantor–Bendixson height. The ensuing notion of a scattered space continues to play an important role in many areas of analysis and it is this notion that is at the heart of Soukup’ s thesis.

However, in order to understand where the work described in this thesis fits into modern set theory one has to look at some early work of Baumgartner and Shelah that produced the first examples of the types of spaces in which Soukup is interested. In particular, they established the consistency of the existence of a locally compact, scattered space of height ω_2 all of whose levels in the Cantor-Bendixson hierarchy are countable. Perhaps even more significant than their example was the method by which Baumgartner and Shelah obtained it. Extending a method used by Baumgartner considerably earlier to construct an uncountable family of uncountable subsets of ω_1 whose pairwise intersections are finite, they were able to construct a partial order that satisfies the countable chain condition that forces their space to exist and has the countable chain condition. The last part is the hard part and this is where the notion of a Δ function plays an important role. It is this notion of a Δ function that Soukup has analyzed very carefully and about which he has gained a number of new insights such as in Section 1.2 and especially in Section 1.7. Section 1.11 of the thesis examines another method for constructing locally compact, scattered spaces. Soukup and Juhász have devised a novel method for constructing initially ω_1-compact spaces that are not compact. Applying this forcing together with techniques of Koszmider, Soukup is able to modify a construction of Rabus to answer a question of Dow. However, once again Δ functions play an important role, but this time it is necessary for Soukup to add extra properties to them in order for his argument to succeed.

It is important to note that Soukup’s work on Δ functions has not been done in isolation; rather this has been a very active area of set theory for many years and Soukup’s contributions should be seen as an important component of a large body of work with applications to many parts of set theory as well as other areas of mathematics. For example, the ρ functions introduced by Todorcevic that have had such a tremendous influence on combinatorial set theory and geometric Banach space theory can be seen as improved versions of Δ functions.

The first half of the thesis contains many other results related to locally compact, scattered spaces that exhibit Soukup’s scholarly overview of the subject. He describes very well the previous state of knowledge and explains where his results fit into the growth of this knowledge. The second half of the thesis deals more with combinatorial and set theoretic questions that have arisen in the study of the locally compact, scattered spaces that are the topic of the first half.

One major direction of research described in the second half of the thesis has to do with isolating useful combinatorial principles that hold in the models of set theory obtained by adding sufficiently many Cohen reals for the Continuum Hypothesis to fail. Soukup's goal is to find principles powerful enough to derive even the most subtle consequences of these Cohen models. While it might seem that this is more of an aesthetic endeavour than one with much in the way of mathematical consequences, this would be a wrong conclusion to draw. Much of the influence of set theory on other areas of mathematics has come from the formulation of axioms which non-set theorists are able to apply to the topics of their research — Martin's Axiom being a prime example. In spite of its broad usefulness, the Cohen real model has no such axiomatization and so Soukup's work is very likely to have significant impact in areas outside of set theory.

However, his combinatorial formulations are interesting in their own right. The same is true for the applications he finds of his principles. Theorem 2.1.24 can be considered an extension of a classical result of Kunen. Theorem 2.1.31 yields a result of Just and was the original motivation for this project. Section 2.1.4 refers to some recent results in the theory of Banach spaces. While this brief section does not constitute a significant part of the thesis, I think it is worth mentioning because it already hints at how the combinatorial principle $\hat{C}(\kappa)$ isolated by Soukup in the Cohen real model could be of interest beyond set theory.

Section 2.3 looks at the (weak) Freeze–Nation property in the light of some weak combinatorial principles. The $\square$ principles were introduced by Jensen half a century ago as part of a project to find combinatorial principles that capture the essence of Gödel' s constructible universe — a project very similar in spirit to Soukup's project on capturing the combinatorial essence of the Cohen real model. These principles have fascinated set theorists ever since their introduction and we are currently in a period of great interest in their weaker forms. Just one of the reasons for interest in $\square$ sequences is their relation to the already mentioned ρ functions of Todorcevic that can be viewed as generalizations of the Δ functions that play such an important role in the first half of Soukup's thesis. However, Soukup has not chosen to pursue this connection in the second half. Rather, his contributions are directly in the mainstream of current research in this area, research done by leading set theorists such as Magidor, Foreman, Shelah as well as a host of researchers just starting their careers.

The third significant body of work in the second half of the thesis concerns weak versions of the guessing principle $\diamondsuit$. The axiom Soukup calls MA(countable) is really just the following strengthening of the usual Baire Category Theorem: If $\mathcal{S}$ is a family of meagre subsets of $\mathbb{R}$ of cardinality less than $2^{\aleph_0}$ then $\cup\mathcal{S} \neq \mathbb{R}$. This version of the Baire Category Theorem usually implies the negation of even weak guessing principles. However, Soukup is able to find a model where the strengthened Baire Category Theorem holds along with a weak guessing principle. His model has to add Cohen real, but provably can not be the usual Cohen real model. The model he obtains is essentially a mixed support product. These have been studied in the past, but Soukup's approach is novel and unique, as are the results he obtains in his model.

Before concluding I would like to touch upon the broader implications of the work described in this thesis. I have already mentioned the connections with classical analysis through the early work of Cantor on trigonometric series. However, the notion of a derived set has myriad applications to analysis as well as other branches of mathematics. In functional analysis especially one finds that derived sets often play a role in describing and classifying spaces and so it is is very reasonable to expect that the very detailed sort of analysis carried out by Soukup will eventually find uses in this sort of work. One should also be aware that scattered spaces have already found many uses as the X in the study of $C(X)$ — the work of Koszmider being an important example. Soukup's preliminary work in this direction indicates his awareness of the possible impact his research may have in the study of Banach spaces and functional analysis in general.

In conclusion, I believe this thesis contains original work that has substantially enriched our understanding of locally compact, scattered spaces and the combinatorics associated with these. Furthermore, I believe that this work will have impact beyond the boundaries of set theory. This being the case, I urge the Committee for Doctoral Awards to set a date for public debate of this thesis.

Sincerely,

Juris Steprāns
Professor of Mathematics