Cardinal Sequences and Combinatorial Principles

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CHAPTER 1

Cardinal sequences

1.1. Introduction and Summary

Scattered spaces. Definitions and basic facts. A topological space is called *scattered* if its every non-empty subspace has an isolated point. Since

- the union of two scattered spaces is scattered, and
- the increasing union of scattered, open subspaces is scattered,

using Zorn lemma one can prove that every topological space X has a unique partition into two subspaces,

$$X = P \cup S, \tag{1.1}$$

such that S is an open, scattered subspace, and P is *perfect*, i.e., closed and dense-in-itself. (Of course, P or S can be empty.)

This observation explains why it is natural to investigate the structures of scattered spaces. In the first chapter of the dissertation we study the structure of scattered spaces. To do so, we assign invariants to the scattered spaces, and we investigate the possible values of these invariants.

The Cantor-Bendixson Hierarchy. All topological spaces will be assumed to be infinite Hausdorff in this thesis.

Since a space is scattered iff it is right-separated, we have $|X| \leq \mathrm{w}(X)$ for scattered spaces.

Given a topological space X, for each ordinal number α, the *α-th derived set* of X, $X^{(\alpha)}$, is defined as follows: $X^{(0)} = X$, $X^{(\alpha+1)}$ is the derived set of $X^{(\alpha)}$, i.e. the collection of all limit points of $X^{(\alpha)}$, and if α is limit then $X^{(\alpha)} = \cap_{\nu<\alpha} X^{(\nu)}$. Since $X^{(\alpha)} \supseteq X^{(\beta)}$ for $\alpha < \beta$ we have a minimal ordinal α such that $X^{(\alpha)} = X^{(\alpha+1)}$. This ordinal α, denoted by $\mathrm{ht}(X)$, is called the *Cantor-Bendixson height*, or just the *height* of X. Clearly the subspace $X^{(\alpha)}$ does not have any isolated points, it is dense-in-itself. The derived sets are all closed, so $X^{(\alpha)}$ is perfect. Moreover, $Y = X \setminus X^{(\alpha)}$ is scattered and so it has cardinality $\leq \mathrm{w}(X)$. (So $P = X^{(\alpha)}$ and $S = X \setminus X^{(\alpha)}$ in 1.1) This observation yields the Cantor-Bendixson theorem: every space of countable weight can be represented as the union of two disjoint subspaces, of which one is perfect and the other is countable. G. Cantor and I. Bendixson proved this fact independently in 1883 for subsets of the real line.

Historically, the investigation of scattered spaces was started by Cantor. He proved, in [**32**], that if the partial sums of a trigonometric series

$$a_0/2 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

converge to zero except possibly on a set of points of finite scattered height, then all coefficients of the series must be zero.

Cardinal sequences of scattered spaces. We will assign invariants to scattered spaces.

Denote by $\mathrm{I}(Y)$ the isolated points of a topological space Y. For each ordinal α define the α-th *Cantor-Bendixson level* of a topological space X, $\mathrm{I}_\alpha(X)$, as follows:

$$\mathrm{I}_\alpha(X) = \mathrm{I}\big(X \setminus \cup\{\mathrm{I}_\beta(X) : \beta < \alpha\}\big).$$

Clearly $\mathrm{I}_\alpha(X) = X^{(\alpha)} \setminus X^{(\alpha+1)}$, and so if X is scattered, then

$$\mathrm{ht}(X) = \min\{\alpha : \mathrm{I}_\alpha(X) = \emptyset\}. \tag{1.2}$$

Write $\mathrm{I}_{<\lambda}(X) = \bigcup_{\alpha<\lambda} \mathrm{I}_\alpha(X)$.

Observe that X is scattered iff $X = \mathrm{I}_{<\mathrm{ht}(X)}(X)$ iff $X^{(\mathrm{ht}(X))} = \emptyset$.

Given a scattered space X, define the *width* of X, $\mathrm{wd}(X)$, as follows:

$$\mathrm{wd}(X) = \sup\{|I_\alpha(X)| \colon \alpha < \mathrm{ht}(X)\}. \tag{1.3}$$

The *cardinal sequence* of a scattered space X, $\mathrm{CS}(X)$, is the sequence of the cardinalities of its Candor-Bendixson levels, i.e.

$$\mathrm{CS}(X) = \langle\ |\mathrm{I}_\alpha(X)| : \alpha < \mathrm{ht}(X)\ \rangle. \tag{1.4}$$

We shall use the notation $\langle\kappa\rangle_\alpha$ to denote the constant κ-valued sequence of length α. As usual, the concatenation of a sequence f of length α and of a sequence g of length β is denoted by $f^\frown g$. So the domain of $h = f^\frown g$ is $\alpha+\beta$, $h(\xi) = f(\xi)$ for $\xi < \alpha$, and $h(\alpha+\eta) = g(\eta)$ for $\eta < \beta$.

By definition, if $\mathrm{CS}(X) = \langle\kappa\rangle_\alpha$ then X has height α and width κ.

Cardinal sequences of Boolean algebras. A Boolean algebra $\mathbf{B}$ is *superatomic* iff every homomorphic image of $\mathbf{B}$ is atomic. We abbreviate "superatomic Boolean algebra" to "sBA".

The essential questions concerning sBA-s were asked by Tarski and Mostowski, [**87**], in 1939. The main categories of results are: cardinal invariants, isomorphism types and automorphism groups. The major cardinal invariants of sBA-s were also introduced in [**87**] in the following way.

Definition 1.1.1. Let $\mathbf{A}$ be a Boolean algebra.

(a) Let $\mathrm{At}(\mathbf{A})$ be the atoms of $\mathbf{A}$.
(b) If $\mathcal{J}$ is an ideal in $\mathbf{A}$, then let $\mathcal{J}^*$ be the ideal generated by

$$\mathcal{J} \cup \{x \in \mathbf{A} : x/\mathcal{J} \in \mathrm{At}(\mathbf{A}/\mathcal{J})\}.$$

(c) Let $\mathcal{J}_0(\mathbf{A}) = \{0_\mathbf{A}\}$, $\mathcal{J}_{\alpha+1}(\mathbf{A}) = \mathcal{J}_\alpha(\mathbf{A})^*$, if α is limit then $\mathcal{J}_\alpha(\mathbf{A}) = \bigcup_{\alpha'<\alpha} \mathcal{J}_{\alpha'}(\mathbf{A})$.
(d) The *height* of $\mathbf{A}$, $\mathrm{ht}(\mathbf{A})$, is the least δ with $\mathcal{J}_\delta(\mathbf{A}) = \mathcal{J}_{\delta+1}(\mathbf{A})$.
(e) $\mathrm{wd}_\alpha(\mathbf{A}) = |\mathrm{At}(\mathbf{A}/\mathcal{J}_\alpha(\mathbf{A})|$,
(f) The *cardinal sequence* of $\mathbf{A}$, $\mathrm{CS}(\mathbf{A})$, is defined as follows:

$$\mathrm{CS}(\mathbf{A}) = \langle \mathrm{wd}_\alpha(\mathbf{A}) : \alpha < \mathrm{ht}(\mathbf{A})\rangle\,. \tag{1.5}$$

By [**68**, Proposition 17.8], $\mathbf{A}$ is superatomic iff there is an ordinal α such that $\mathbf{A} = \mathcal{J}_\alpha(\mathbf{A})$.

A *Boolean space* is a compact, 0-dimensional, Hausdorff space. The Stone duality establishes a 1–1 correspondence between Boolean spaces and Boolean algebras.

Under Stone duality, homomorphic images of a Boolean algebra $\mathbf{A}$ correspond to closed subspaces of its Stone space $S(\mathbf{A})$, and atoms of $\mathbf{A}$ correspond to isolated points of $S(\mathbf{A})$. Moreover, if $\mathbf{B}$ is a superatomic Boolean algebra, then the dual space of $\mathbf{B}^{(\alpha)}$ is $(S(\mathbf{B}))^{(\alpha)}$ (see [**68**, Construction 17.7]). So we have the following fact.

Fact* 1.1.2. *A Boolean algebra* $\mathbf{B}$ *is superatomic iff its dual space* $S(\mathbf{B})$ *is scattered. Moreover,* $\mathrm{ht}(\mathbf{B}) = \mathrm{ht}(S(\mathbf{B}))$*, and* $\mathrm{CS}(\mathbf{B}) = \mathrm{CS}(S(\mathbf{B}))$.

Historically, the cardinal sequences of Boolean algebras were defined earlier: this notion was introduced by Day, [**33**], in 1967. The notion of cardinal sequences of topological spaces was introduced by LaGrange.

The beginning. By a classical result of S. Mazurkiewicz and J. Sierpiński, (see [**86**] or [**68**, Theorem 17.11]) a countable, superatomic Boolean algebra $\mathbf{B}$ is determined completely by its cardinal sequence!

Theorem* 1.1.3. *If* $\mathbf{B}$ *is a countable, superatomic Boolean algebra, then* $\mathrm{CS}(\mathbf{B}) = \langle\beta\rangle_\omega{}^\frown\langle n\rangle$*, where* $\beta+1 = \mathrm{ht}(\mathbf{B})$*, and* $n = |\mathrm{wd}_\beta(X)|$ *is a natural number. Moreover, in this case* $\mathbf{B}$ *is homeomorphic to Boolean algebra of the clopen subsets of the compact ordinal space* $\omega^\beta \cdot n + 1$*, i.e.* $S(\mathbf{B}) = \omega^\beta \cdot n + 1$.

What about uncountable sBA-s? An sBA is called *thin-tall* iff it has width ω and height at least ω_1. The question of whether thin-tall sBA can exist was asked by Telgarsky in 1968. This question was the actual starting point of the modern investigation of sBA-s.

Cardinal sequences of LCS* spaces. As we remarked, the study of cardinal sequences was actually originated in the theory of superatomic Boolean algebras. Since superatomic Boolean algebras correspond to compact scattered spaces via Stone duality, the efforts were concentrated on the study of cardinal sequences of compact scattered spaces.

Reduction. We say that a space X is an *LCS*-space* iff it is locally compact, scattered, but not compact. [1]

A locally compact scattered space X is non-compact iff either $\mathrm{ht}(X)$ is a limit ordinal, or $\mathrm{ht}(X) = \delta + 1$ and the top level of X, $\mathrm{I}_\delta(X)$, is infinite.

The height of a compact scattered space X is a successor ordinal, $\mathrm{ht}(X) = \delta + 1$, and $I_\delta(X)$ is finite. So $Y = X \setminus I_\delta(X)$ is an LCS* space, moreover $\mathrm{CS}(X) = \mathrm{CS}(Y)^\frown\langle n\rangle$, where $n = |I_\delta(X)|$.

On the other hand, if Y is an LCS* space, and X is the disjoint union of n copies of the one point compactification αY of Y, then X is compact and $\mathrm{CS}(X) = \mathrm{CS}(Y)^\frown\langle n\rangle$. So we have

$$\{\mathrm{CS}(X) : X \text{ is a compact space}\} = \{\mathrm{CS}(Y)^\frown\langle n\rangle : Y \text{ is an LCS}^* \text{ space}, n \in \omega\}. \tag{1.6}$$

Hence, instead of compact, scattered spaces we can study the cardinal sequences of LCS* spaces.

So Telgarsky's question can be reformulated as follows:

Telgarsky's Question: *Is there an LCS* space with cardinal sequence $\langle\omega\rangle_{\omega_1}$?*

His question was answered by Rajagopalan.

Theorem* 1.1.4 (Rajagopalan, 1976, **[91]**)**.** *There is an LCS* space X with* $\mathrm{CS}(X) = \langle\omega\rangle_{\omega_1}$.

There are strong limitations on cardinal sequences of LCS* spaces.

Fact* 1.1.5. *The cardinality of a scattered T_3, in particular of an LCS* space X is at most $2^{|\mathrm{I}(X)|}$, hence $\mathrm{ht}(X) < (2^{|\mathrm{I}(X)|})^+$. Hence if $\mathfrak{s} = \langle\kappa_\alpha : \alpha < \delta\rangle$ is the cardinal sequence of a scattered regular space, then for each $\alpha < \beta < \delta$ we have $\kappa_\beta \le 2^{\kappa_\alpha}$ and $|\delta \setminus \alpha| \le 2^{\mathfrak{s}(\alpha)}$.*

The next definition simplifies the formulation of certain results.

Definition 1.1.6. We let $\mathcal{C}(\alpha)$ denote the class of all cardinal sequences of length α of LCS*-spaces. We also put, for any fixed infinite cardinal λ,

$$\mathcal{C}_\lambda(\alpha) = \{s \in \mathcal{C}(\alpha) : s(0) = \lambda \wedge \forall\beta < \alpha\ [s(\beta) \ge \lambda]\}. \tag{1.7}$$

In 1978, Juhász and Weiss improved the above mentioned result of Rajagopalan.

Theorem* 1.1.7 (**Juhász, Weiss, [59]**)**.** $\langle\omega\rangle_\alpha \in \mathcal{C}(\alpha)$ *for each* $\alpha < \omega_2$.

By Fact 1.1.5, if CH holds then $\langle\omega\rangle_{\omega_2} \notin \mathcal{C}(\omega_2)$. Juhász and Weiss, **[59]**, asked if it is consistent that $\langle\omega\rangle_\alpha \in \mathcal{C}(\alpha)$. Just proved that the failure of CH is not enough to get this consistency.

Theorem* 1.1.8 (**Just, [64]**)**.** *In the Cohen model there is no LCS*-space X with* $\mathrm{CS}(X) = \langle\omega\rangle_{\omega_2}$.

In 1985, Baumgartner and Shelah gave an affirmative answer to the question of Juhász and Weiss.

Theorem* 1.1.9 (**Baumgartner, Shelah, [25]**)**.** *It is consistent that there is an LCS*-space X with* $\mathrm{CS}(X) = \langle\omega\rangle_{\omega_2}$.

Their proof was based on the construction of a Δ-*function*.

Definition 1.1.10 (**[25]**)**.** Let $f : [\omega_2]^2 \to [\omega_2]^{\le\omega}$ be a function with $f\{\alpha,\beta\} \subset \alpha \cap \beta$ for $\{\alpha,\beta\} \in [\omega_2]^2$. (1) We say that two finite subsets x and y of ω_2 are *good for f* provided that for $\alpha \in x \cap y$, $\beta \in x \setminus y$ and $\gamma \in y \setminus x$ we always have

(a) $\alpha < \beta, \gamma \Longrightarrow \alpha \in f\{\beta,\gamma\}$,
(b) $\alpha < \beta \Longrightarrow f\{\alpha,\gamma\} \subset f\{\beta,\gamma\}$,
(c) $\beta < \gamma \Longrightarrow f\{\alpha,\beta\} \subset f\{\alpha,\gamma\}$.

[1] In the literature an *LCS space* is defined as a locally compact, scattered space.

(2) The function f is a Δ-*function* if every uncountable family of finite subsets of ω_2 contains two elements x and y which are are good for f.
(3) The function f is a *strong* Δ-*function* if every uncountable family $\mathcal{A}$ of finite subsets of ω_2 contains an uncountable subfamily $\mathcal{B}$ such that any two sets x and y from $\mathcal{B}$ are good for f.

To get Theorem 1.1.9 actually the following statements were proved:

- It is consistent that there is a Δ-function.
- If there is a Δ-function, the in some c.c.c generic extension of the ground model there is an LCS*-space X with $\mathrm{CS}(X) = \langle\omega\rangle_{\omega_2}$.

It is worth to mention that there is a strong Δ-function in L.

Theorem* 1.1.11 (Velickovic)**.** *If $\square_{\omega_1}$ holds then there is a strong Δ-function.*

Two types of questions were considered in connection with cardinal sequences:

(1) Given a sequence s of cardinals of length α decide whether $s \in \mathcal{C}(\alpha)$ in a certain/in some model of ZFC.
(2) Characterize $\mathcal{C}(\alpha)$ or $\mathcal{C}_\lambda(\alpha)$ in certain models of ZFC.

There are many results concerning Type 1 problems in the literature. In Section 1.2 we also consider such a problem. By Fact 1.1.5, $\mathrm{ht}(X) < (2^{|I(X)|})^+$ for any LCS* space X. Especially, if $\mathrm{I}_0(X)$ is countable, then $\mathrm{ht}(X) < (2^\omega)^+$. It is not hard to prove (see e.g. Theorem 1.2.3 below) that the estimate above is sharp for LCS* spaces with countably many isolated points:

- *for each $\alpha < (2^\omega)^+$ there is an LCS* space X with $\mathrm{ht}(X) = \alpha$ and $|\,\mathrm{I}_0(X)| = \omega$.*

Much less is known about LCS* spaces with ω_1 isolated points, for example it is a long standing open problem whether there is, in ZFC, an LCS* space of height ω_2 and width ω_1. In fact, as was noticed by Juhász in the mid eighties, even the much simpler question if there is a ZFC example of an LCS* space of height ω_2 with only ω_1 isolated points, turned out to be surprisingly difficult. In section 1.2 we prove the main result of [**9**], and in Theorem 1.2.1 we give an affirmative answer to the above question of Juhász: *There is, in ZFC, an LCS* space of height ω_2 and having ω_1 isolated points.* Although it is a ZFC result, but the main ingredient of the proof, Theorem 1.2.2, is a construction under CH!

To prove this result we had to develop a new amalgamation method of construction of LCS* spaces.

In Sections 1.3, 1.4, 1.5 and 1.7 we consider "Type 2" problems, i.e. we try to give full characterization of the elements of certain classes $\mathcal{C}_\lambda(\alpha)$.

For countable sequences LaGrange [**79**], for sequences of length ω_1 Juhász and Weiss, [**61**], give full characterization:

Theorem* 1.1.12 ([**61**])**.** *$\langle\kappa_\xi : \xi < \omega_1\rangle \in \mathcal{C}(\omega_1)$ iff $\kappa_\eta \le \kappa_\xi^\omega$ holds whenever $\xi < \eta < \omega_1$.*

It follows that cardinal arithmetic alone decides whether a sequence of cardinals of length ω_1 belongs to $\mathcal{C}(\omega_1)$ or not. The situation changes dramatically for longer sequences, in fact already for sequences of length $\omega_1 + 1$. For example, the question if $\langle\omega\rangle_{\omega_1} {}^\frown\langle\omega_2\rangle_1 \in \mathcal{C}(\omega_1 + 1)$ is not decided by the following cardinal arithmetic: $2^\omega = \omega_2$ and $2^\kappa = \kappa^+$ for all $\kappa > \omega$ (see Just[**64**] and Roitman[**93**]).

However, as we showed in [**14**], the elements of $\mathcal{C}(\alpha)$ can be characterized for all $\alpha < \omega_2$ if we assume GCH. Section 1.3 contains this result. In order to characterize those sequences of length $< \omega_2$ which are cardinal sequences of LCS spaces, it suffices to characterize the classes $\mathcal{C}_\lambda(\alpha)$ for any ordinal $\alpha < \omega_2$ and any infinite cardinal λ. In fact, this follows from the general reduction Theorem 1.3.2 that is valid in ZFC.

The promised GCH characterization of the classes $\mathcal{C}_\lambda(\alpha)$ is given in Theorem 1.3.4. The main ingredient of the proof is Theorem 1.3.5 which is generalization of the main construction of Theorem 1.2.2.

In Section 1.4 we consider the classes $\mathcal{C}_\lambda(\alpha)$ for $\alpha \ge \omega_2$ under GCH. In 1.4.1 we define a family $\mathcal{D}_\lambda(\alpha)$ of sequences of cardinals. Using elementary topological considerations, this family is a natural "upper bound" of $\mathcal{C}_\lambda(\alpha)$, i.e. GCH implies that $\mathcal{C}_\lambda(\alpha) \subseteq \mathcal{D}_\lambda(\alpha)$.

We conjecture that GCH implies that $\mathcal{C}_\lambda(\alpha) = \mathcal{D}_\lambda(\alpha)$, but in [**15**] we could prove just a weaker statement. Namely, as we prove it in Section 1.4, for each regular cardinal λ it is consistent that GCH holds and $\mathcal{C}_\lambda(\alpha) = \mathcal{D}_\lambda(\alpha)$, see Theorem 1.4.3.

Using this result, in Theorem 1.4.4 we characterize those sequences of regular cardinals which can be obtained as a cardinal sequence of an LCS* space in some model of ZFC + GCH.

To prove our theorems we should introduce the notion of universal spaces in 1.4.2 which is the main tool of some proofs in the forthcoming sections.

In Sections 1.5 and 1.7 we investigate the class $\mathcal{C}_\omega(\omega_2)$.

Baumgartner and Shelah proves that it is consistent that $\langle\omega\rangle_{\omega_2} \in \mathcal{C}_\omega(\omega_2)$. Building on their method, Bagaria, [**21**], proved that $\mathcal{C}_\omega(\omega_2) \supseteq \{s \in {}^{\omega_2}\{\omega, \omega_1\} : s(0) = \omega\}$ is also consistent. However, he used $MA_{\aleph_2}$ in his argument, and $MA_{\aleph_2}$ implies $2^{\omega_0} \geq \omega_3$, and if $2^{\omega_0} = \omega_\alpha$, then the natural "upper bound" of $\mathcal{C}_\omega(\omega_2)$ is a much larger family of sequences:

$$\mathcal{C}_\omega(\omega_2) \subseteq \{s \in {}^{\omega_2}\{\omega_\nu : \nu \leq \alpha\} : s(0) = \omega\}. \tag{1.8}$$

These results naturally raised the questions whether we may have equality in (1.8). In Theorem 1.5.10 we answer in the positive: it is consistent that $2^\omega = \omega_2$ and

$$\mathcal{C}_\omega(\omega_2) = \{s \in {}^{\omega_2}\{\omega, \omega_1, \omega_2\} : s(0) = \omega\}.$$

In Section 1.7 we improve the above mention result: we prove Theorem 1.6.1 which claims that it is consistent that 2^ω is as large as you wish and every sequence $\mathbf{s} = \langle s_\alpha : \alpha < \omega_2\rangle$ of infinite cardinals with $s_\alpha \leq 2^\omega$ is the cardinal sequence of some LCS space.

For a long time ω_2 was a mystique barrier in both height and width. In that section we can construct wider spaces. However we should pay a price: by forcing we should introduce a morass type structure which helps us to obtain a very strong Δ-like functions in a generic extension which makes possible to force our desired space.

In Section 1.6 we prove certain stepping up theorems.

In a classical theorem, Roitman proved that it is consistent that $\langle\omega\rangle_{\omega_1} {}^\frown\langle\omega_2\rangle \in \mathcal{C}_\omega(\omega_1 + 1)$. Combining Koszmider's strongly functions and Martinez's orbit methods in Section 1.8 we improve Roitman's result. We show e.g. that for each $\beta < \omega_3$ with $\mathrm{cf}(\beta) = \omega_2$ it is consistent that 2^{ω_1} is as large as you wish and $\langle\omega_1\rangle_\beta {}^\frown\langle 2^{\omega_1}\rangle \in \mathcal{C}_{\omega_1}(\beta + 1)$.

Since the classes of the regular, of the zero-dimensional, and of the locally compact scattered spaces are different, it was a natural question what is the relationship between their cardinal sequences. Actually, Juhász raised the question whether there is a regular space with cardinal sequence $\langle\omega\rangle_{2^\omega}$ in ZFC. In [**25**, Theorem 10.1] the authors answered his question positively. for each $\alpha < (2^\omega)^+$ there is a regular (even zero-dimensional) scattered space with cardinal sequence $\langle\omega\rangle_\alpha$.

In Section 1.9 we succeeded in giving a complete characterization of the cardinal sequences of both T_3 and zero-dimensional T_2 scattered spaces. Although the classes of the regular and of the zero-dimensional scattered spaces are different, it will turn out that they yield the same class of cardinal sequences.

E. van Douwen and, independently, A. Dow [**34**] have observed that under CH an initially ω_1-compact T_3 space of countable tightness is compact. (A space X is initially κ-compact if any open cover of X of size $\leq \kappa$ has a finite subcover, or equivalently any subset of X of size $\leq \kappa$ has a complete accumulation point). Naturally, the question arose whether CH is needed here, i.e. whether the same is provable just in ZFC. The question became even more intriguing when in [**23**] , D. Fremlin and P. Nyikos proved the same result from PFA. Quite recently, A. V. Arhangel'skiĭhas devoted the paper [**20**] to this problem, in which he has raised many related problems as well.

In [**90**] M. Rabus has answered the question of van Douwen and Dow in the negative. He constructed by forcing a Boolean algebra B such that the Stone space $St(B)$ includes a counterexample X of size ω_2 to the van Douwen–Dow question, in fact $St(B)$ is the one point compactification of X, hence X is also locally compact.

In Sections 1.10, we give an alternative forcing construction of counterexamples to the van Douwen–Dow question, We directly force a topology τ on ω_2 such that in the generic extension V^P out space $X = \langle \omega_2, \tau \rangle$ is normal, locally compact, 0-dimensional, Frechet-Urysohn, initially ω_1-compact and non-compact space X of size ω_2 having the following property: for every open (or closed) set A in X we have $|A| \leq \omega_1$ or $|X \setminus A| \leq \omega_1$.

In [**20**, problem 12] Arhangel'skii asks if it is provable in ZFC that a normal, first countable initially ω_1-compact space is necessarily compact. We could not completely answer this question, but we show that the Frechet-Urysohn property (which is sort of half-way between countable tightness and first countability) in not enough to get compactness.

To achieve that we constructed a further generic extension of the model V^P in which X becomes Frechet-Urysohn but its other properties are preserved, for example, X remains initially ω_1-compact and normal.

Arhangel'skii raised the question, [**20**, problem 3], whether CH can be weakened to $2^\omega < 2^{\omega_1}$ in the theorem of van Douwen and Dow? We shall answer this question in the negative: theorem 1.10.34 implies that the existence of a counterexample to the van Douwen–Dow question is consistent with practically any cardinal arithmetic that violates CH.

Improving the results of section 1.10, in Section 1.11 we answer a question of Arhangel'skii,[**20**, problem 12]: we force a first countable, normal, locally compact, initially ω_1-compact and non-compact space $X = \langle \omega_2 \times 2^\omega, \tau \rangle$.

Actually, Alan Dow conjectured that applying the method of [**74**] (that "turns" a compact space into a first countable one) to the space of Rabus in [**90**] yields an ω_1-compact but non-compact first countable space. How one can carry out such a construction was outlined by the second author in the preprint [**71**]. However, [**71**] only sketches some arguments as the language adopted there, which follows that of [**90**], does not seem to allow direct combinatorial control over the space which is forced. This explains why the second author hesitated to publish [**71**].

One missing element of [**71**] was a language similar to that of [**11**] which allows working with the points of the forced space in a direct combinatorial way. In section 1.11 we combine the approach of [**11**] with the ideas of [**71**] to obtain directly an ω_1-compact but non-compact first countable space. Consequently, our proofs follow much more closely the arguments of [**11**] than those of [**90**] or their analogues in [**71**].

As before, we again use a Δ-function to make our forcing CCC but we need both CH and a Δ-function with some extra properties to obtain first countability.

It is immediate from the countable compactness of X that its one-point compactification X^* is not first countable. In fact, one can show that the character of the point at infinity $*$ in X^* is ω_2. As X is initially ω_1-compact, this means that every (transfinite) sequence converging from X to $*$ must be of type cofinal with ω_2. Since X is first countable, this trivially implies that there is no non-trivial converging sequence of type ω_1 in X^*. In other words: the convergence spectrum of the compactum X^* omits ω_1. As far as we know, this is the first and only (consistent) example of this sort.

Combinatorial principles. In the second chapter of the dissertation we consider combinatorial principles.

The last 40 years have seen a furious activity in proving results that are independent of the usual axioms of set theory, that is ZFC. As the methods of these independence proofs (e.g. forcing or the fine structure theory of the constructible universe) are often rather sophisticated, while the results themselves are usually of interest to "ordinary" mathematicians (e.g. topologists or analysts), it has been natural to try to isolate a relatively small number of principles, i.e. independent statements that a) are *simple* to formulate and b) are *useful* in the sense that they have many interesting consequences. Most of these statements, we think by necessity, are of combinatorial nature, hence they have been called combinatorial principles. The best known example is the Martin's Axiom: to prove the consistency of this axiom you need to know the iterated forcing, but to apply this axiom in algebra, topology or combinatorics it is enough to know elementary set-theory.

The above formulated criteria a) and b) as to what constitutes a combinatorial principle are often contrary to each other: for more usefulness one often has to sacrifice some simplicity. It is

not clear whether an ideal balance exists between them. It is up to the reader to judge if we have come close to this balance.

As we mentioned, Just proved that in the Cohen model neither $\langle\omega\rangle_{\omega_2}$ nor $\langle\omega\rangle_{\omega_1}{}^\frown\langle\omega_2\rangle$ is a cardinal sequence of an LCS* space. The two proofs are based on similar arguments. Using the same ideas we could prove other statements in the Cohen model, so it was a natural idea to investigate if we can combine these arguments into a "combinatorial principle". That was the starting point of the investigation of combinatorial principles in the Cohen model. As we will see, we could find principles which are (a) true in the Cohen model, and (b) which implies the above mentioned results of Just.

In section 2.1 we present several new combinatorial principles that are all statements about $\mathcal{P}(\omega)$, the power set of the natural numbers. In fact, they all concern matrices of the form $\langle A(\alpha,n) : \langle\alpha,n\rangle \in \kappa\times\omega\rangle$, where $A(\alpha,n) \subset \omega$ for each $\langle\alpha,n\rangle \in \kappa\times\omega$, and, in the interesting cases, κ is a regular cardinal with $c = 2^\omega \geq \kappa > \omega_1$.

We show that these statements are valid in the generic extensions obtained by adding any number of Cohen reals to any ground model V, assuming that the parameter κ is a regular and ω-inaccessible cardinal in V (i.e. $\lambda < \kappa$ implies $\lambda^\omega < \kappa$).

Then we present a large number of consequences of these principles, some of them combinatorial but most of them topological, mainly concerning separable and/or countably tight topological spaces. (This, of course, is not surprising because these are objects whose structure depends basically on $\mathcal{P}(\omega)$.)

Recently we could find an application of our principles in the theory of Banach spaces, [**1**].

Using a much finer analysis of the Cohen model, in Section 2.2 we recall the main result of [**10**]: in the Cohen model an LCS* may have at most ω_1 many countable Cantor-Bendixson level.

We conjecture that in ZFC an LCS* may have at most ω_2 many countable Cantor-Bendicty level. In his celebrated theorem, Shelah proves that $\max\operatorname{pcf}\{\aleph_n : n < \omega\} < \aleph_{\omega_4}$. One of the central question of set theory is to improve this result. If this conjecture is true, then $\max\operatorname{pcf}\{\aleph_n : n < \omega\} < \aleph_{\omega_3}$.

I jointed to the investigation of the weak Freeze-Nation (wFN) property independently from the research of the combinatorial principles. However, at some point we realized that the wFN property of the poset $\langle\mathcal{P},\subset\rangle$ itself has numerous interesting combinatorial and topological consequence, so it can be considered as a combinatorial principle. This observation explains why we discuss the wFN property of certain posets in 3 sections.

Definition 1.1.13. A poset $P = \langle P,\leq\rangle$ has the *weak Freese-Nation (wFN)* property iff there is a function $f : P \to \left[P\right]^\omega$ such that

- if p, $q \in P$ and $p \leq q$ then there is $r \in f(p)\cap f(q)$ with $p \leq r \leq q$.

Let $Q \subseteq P$. Write $Q \leq_\sigma P$ if for each $p \in P$ the set $\{q \in Q : q \leq p\}$ has a countable cofinal subset, and the set $\{q \in Q : q \geq p\}$ has a countable coinitial subset.

These notions were introduced by Heindorf and Shapiro [**51**].

By [**46**], the poset P has the wFN property iff the family $\{Q \in \left[P\right]^{\omega_1} : Q \leq_\sigma P\}$ contains a club subfamily. In [**7**], we prove that a weaker assumption on the class $\{Q \in \left[P\right]^{\omega_1} : Q \leq_\sigma P\}$ is enough to derive the wFN property of P. This result is included in section 2.3.

We introduce a very weak box principle $\square^{***}_\mu$, and prove Theorem 2.3.14: assume that λ is a cardinal with

(i) $\operatorname{cf}([\mu]^\omega,\subseteq) = \mu$ if $\omega_1 < \mu < \lambda$ and $\operatorname{cf}(\mu) \geq \omega_1$,
(ii) $\square^{***}_\mu$ holds for each $\mu < \lambda$ with cofinality ω.

Then for each poset P of cardinality $\leq \lambda$ the following are equivalent:

(w1) P has the wFN property,
(w2) for each large enough regular cardinal χ, if $M \prec \mathcal{H}(\chi)$, P, $\omega_1 \in M$, and M is the union of an ω_1 chain of countable elementary submodels of $\mathcal{H}(\chi)$ then $P\cap M \leq_\sigma P$.

We proved that the assumption of $\square^{***}$ can not be omitted: if GCH holds and the Chang conjecture $(\aleph_{\omega+1},\aleph_\omega) \twoheadrightarrow (\aleph_1,\aleph_0)$. holds then the poset $([\aleph_\omega]^{\aleph_0},\subseteq)$ does not have the wFN property.

Let us remark that if $2^\omega = \omega_1$ then (w2) holds for $([\aleph_\omega]^{\aleph_0}, \subseteq)$. So GCH is not enough to prove the equivalence of (w1) and (w2)

In section 2.4 we solve some questions concerning wFN properties of Boolean algebras. While in section 2.3 we proved that $\langle \mathcal{P}(\omega), \subset \rangle$ *has the wFN property* if we add arbitrary number of Cohen reals to a "nice" model of ZFC, (e.g. to, L), in Section 2.5 we proved that L can not be replaced by an arbitrary model of ZFC which satisfies GCH.

In Section 2.5 we show that the assumption $\langle \mathcal{P}(\omega), \subset \rangle$ *has the wFN property* can be considered as a useful combinatorial principle. Indeed, we proved, that if $\langle \mathcal{P}(\omega), \subset \rangle$ has the wFN property, then

(a) there is no $\aleph_2$-Luzin gap, and ω_2 is not embeddable into $\langle \mathcal{P}(\omega), \subseteq^* \rangle$;
(b) $\langle \omega \rangle_{\omega_1} {}^\frown \langle \omega_2 \rangle \notin \mathcal{C}(\omega_1 + 1)$ and $\langle \omega \rangle_{\omega_2} \notin \mathcal{C}(\omega_2)$;
(c) $non(\mathcal{M}) = \omega_1$ and $cov(\mathcal{M}) > \omega_1$;
(d) $\mathfrak{a} = \omega_1$,
(e) any ω_1-fold cover of the real real by closed sets can be partitioned into ω_1 disjoint subcover.

The last application is from [**2**].

So far we investigated principles which hold in the Cohen model. However, there are other principles which can not hold in the Cohen model, e.g. $\clubsuit$ or $\spadesuit$.

Our starting point in the last section was to find a generic extension which allows to blow up the continuum without collapsing cardinals in such a way that $\spadesuit$ holds in the generic extension.

To handle this problem we developed a new type of product of posets.

As it turned out, this new product is suitable to obtain models in which seemingly inconsistent statements hold. The basic idea is to add Cohen reals in such a way that in the generic extension there is no generic filter for the poset $Fn(\omega_1, 2; \omega)$.

We formulate just our most interesting result. As usual, $MA(countable)$ denotes the statement that Martin's Axiom holds for countable posets. So we proved that it is consistent that you can obtain a model without collapsing cardinals in such a way that in the generic extension 2^ω is as large as you wish, and both $MA(countable)$ and $\spadesuit$ hold.

1.2. A tall space with small bottom

(This section is based on [9])

By theorems 1.1.7, 1.1.8 and 1.1.9,

- for each $\alpha < \omega_2$ there is an LCS* space X with $\mathrm{CS}(X) = \langle \omega \rangle_\alpha$,
- ZFC+ $\neg$CH does not decide if there is an LCS* space X with $\mathrm{CS}(X) = \langle \omega \rangle_{\omega_2}$.

The estimate $\mathrm{ht}(X) < (2^{|I(X)|})^+$ is sharp for LCS* spaces with countably many isolated points by theorem 1.2.3 below:

- *for each $\alpha < (2^\omega)^+$ there is an LCS* space X with $\mathrm{ht}(X) = \alpha$ and $|\mathrm{I}_0(X)| = \omega$.*

Much less is known about LCS* spaces with ω_1 isolated points, for example it is a long standing open problem whether there is, in ZFC, an LCS* space of height ω_2 and width ω_1. In fact, as was noticed by Juhász in the mid eighties, even the much simpler question if there is a ZFC example of an LCS* space of height ω_2 with only ω_1 isolated points, turned out to be surprisingly difficult. On the other hand, Martínez in [**84**, theorem 1] proved that

- *it is consistent that for each $\alpha < \omega_3$ there is a LCS* space X with $\mathrm{CS}(X) = \langle \omega_1 \rangle_\alpha$.*

As the main result of [**9**], we gave an affirmative answer to the above question of Juhász:

Theorem 1.2.1. *There is, in ZFC, an LCS* space of height ω_2 and having ω_1 isolated points.*

The main ingredient of the proof of Theorem 1.2.1 is the following result.

Theorem 1.2.2. *If $\kappa^{<\kappa} = \kappa$ then there is an LCS* space X of height κ^+ with $|\mathrm{I}_0(X)| = \kappa$.*

In particular, if $2^\omega = \omega_1$ then the above result yields an LCS* space X with $\mathrm{ht}(X) = \omega_2$ and $|\mathrm{I}_0(X)| = \omega_1$. That such a space also exist under $\neg$CH, hence in ZFC, follows from the following result.

Theorem 1.2.3. *For each $\alpha < (2^\omega)^+$ there is a locally compact, scattered space X_α with $|X_\alpha| = |\alpha| + \omega$, $\mathrm{ht}(X_\alpha) = \alpha$ and $|\mathrm{I}_0(X_\alpha)| = \omega$.*

Proof. We do induction on α. If $\alpha = \beta + 1$ then we let X_α be the 1-point compactification of the disjoint topological sum of countably many copies of X_β.

If α is limit then we first fix an almost disjoint family $\{A_\beta : \beta < \alpha\} \subset \left[\omega\right]^\omega$, which is possible by $|\alpha| \leq 2^\omega$. Applying the inductive hypothesis, for each $\beta < \alpha$ we can also fix a locally compact scattered space X_β of cardinality $\omega + |\beta|$ and height β such that $\mathrm{I}_0(X_\beta) = A_\beta$ and $X_\beta \cap X_\gamma = A_\beta \cap A_\gamma$ for $\{\beta, \gamma\} \in \left[\alpha\right]^2$. Now amalgamate the spaces X_β as follows: consider the topological space $X = \langle \cup_{\beta<\alpha} X_\beta, \tau\rangle$ where τ is the topology generated by $\cup_{\beta<\alpha}\tau_{X_\beta}$. Since $A_\beta \cap A_\gamma$ is a finite and open subspace of both X_β and X_γ it follows that each X_β is an open subspace of X. Consequently, X is LCS* with countably many isolated points, and $\mathrm{ht}(X) = \sup_{\beta<\alpha} \mathrm{ht} X_\beta = \alpha$. □

Proof of Theorem 1.2.1 using Theorem 1.2.2 . If $2^\omega = \omega_1$, then theorem 1.2.2 gives such a space.

If $2^\omega > \omega_1$ then $(2^\omega)^+ \geq \omega_3$ and so according to theorem 1.2.3 for each $\alpha < \omega_3$ there is locally compact, scattered space of height α and with countably many isolated points. □

The rest of this section is devoted to the proof of Theorem 1.2.2.

Definition 1.2.4. Given a family of sets $\mathcal{A}$ we define the topological space $X(\mathcal{A}) = \langle \mathcal{A}, \tau_\mathcal{A}\rangle$ as follows: $\tau_\mathcal{A}$ is the coarsest topology in which the sets $\mathrm{U}_\mathcal{A}(A) = \mathcal{A} \cap \mathcal{P}(A)$ are clopen for each $A \in \mathcal{A}$, in other words: $\{\mathrm{U}_\mathcal{A}(A), \mathcal{A} \setminus \mathrm{U}_\mathcal{A}(A) : A \in \mathcal{A}\}$ is a subbase for $\tau_\mathcal{A}$.

We shall write $\mathrm{U}(A)$ instead of $\mathrm{U}_\mathcal{A}(A)$ if $\mathcal{A}$ is clear from the context.

Clearly $X(\mathcal{A})$ is a 0-dimensional T_2-space. A family $\mathcal{A}$ is called *well-founded* iff $\langle \mathcal{A}, \subset\rangle$ is well-founded. In this case we can define the rank-function $\mathrm{rk} : \mathcal{A} \longrightarrow On$ as usual:

$$\mathrm{rk}(A) = \sup\{\mathrm{rk}(B) + 1 : B \in \mathcal{A} \ \wedge \ B \subsetneq A\},$$

and write $\mathrm{R}_\alpha(\mathcal{A}) = \{A \in \mathcal{A} : \mathrm{rk}(A) = \alpha\}$.

The family $\mathcal{A}$ is said to be $\cap$-*closed* iff $A \cap B \in \mathcal{A} \cup \{\emptyset\}$ whenever $A, B \in \mathcal{A}$.

It is easy to see that if $\mathcal{A}$ is $\cap$-closed, then a neighbourhood base in $X(\mathcal{A})$ of $A \in \mathcal{A}$ is formed by the sets

$$\mathrm{W}(A; B_1, \ldots, B_n) = \mathrm{U}(A) \setminus \bigcup_1^n \mathrm{U}(B_i),$$

where $n \in \omega$ and $B_i \subsetneq A$ for $i = 1, \ldots, n$. (For $n = 0$ we have $\mathrm{W}(A) = \mathrm{U}(A)$.)

The following simple result enables us to obtain LCS* spaces from certain families of sets. Let us point out, however, that not every LCS* space is obtainable in this manner, but we do not dwell upon this because we will not need it.

Lemma 1.2.5 ([**9**]). *Assume that $\mathcal{A}$ is both $\cap$-closed and well-founded. Then $X(\mathcal{A})$ is an LCS* space.* □

To simplify notation, if $X(\mathcal{A})$ is scattered then we write $\mathrm{I}_\alpha(\mathcal{A}) = \mathrm{I}_\alpha(X(\mathcal{A}))$.

Clearly each minimal element of $\mathcal{A}$ is isolated in $X(\mathcal{A})$; more generally we have $\alpha \leq \mathrm{rk}(A)$ if $A \in \mathrm{I}_\alpha(\mathcal{A})$, as is shown by an easy induction on $\mathrm{rk}(A)$.

Example 1.2.6. *Assume that $\langle T, \prec\rangle$ is a well-ordering, $\mathrm{tp}\,\langle T, \prec\rangle = \alpha$, and let $\mathcal{A}$ be the family of all initial segments of $\langle T, \prec\rangle$, i. e. $\mathcal{A} = \{T\} \cup \{T_x : x \in T\}$, where $T_x = \{t \in T : t \prec x\}$. Then $\mathcal{A}$ is well-founded, $\cap$-closed and it is easy to see that $X(\mathcal{A}) \cong \alpha + 1$, i.e. the space $X(\mathcal{A})$ is homeomorphic to the space of ordinals up to and including α.*

Example 1.2.6 above shows that, in general, $\mathrm{R}_\alpha(\mathcal{A})$ and $\mathrm{I}_\alpha(\mathcal{A})$ may differ even for $\alpha = 0$. Indeed, if x is the successor of y in $\langle T, \prec\rangle$ then T_x is isolated in $X(\mathcal{A})$ because $\{T_x\} = W(T_x; T_y) = \mathrm{U}_\mathcal{A}(T_x) \setminus \mathrm{U}_\mathcal{A}(T_y)$ is open, but $\mathrm{rk}(T_x) = \mathrm{tp}(T_x) > 0$. However, for a wide class of families, the two kinds of levels do agree. Let us call a well-founded family $\mathcal{A}$ rk-*good* iff the following condition is satisfied:

$$\forall A \in \mathcal{A}\ \forall \alpha < \mathrm{rk}(A)\ |\{A' \in \mathcal{A} : A' \subset A \wedge \mathrm{rk}(A') = \alpha\}| \geq \omega.$$

Then we have the following result.

Lemma 1.2.7 ([**9**]). *If $\mathcal{A}$ is a well-founded, $\cap$-closed and* rk*-good family, then* $\mathrm{I}_\alpha(\mathcal{A}) = \mathrm{R}_\alpha(\mathcal{A})$ *for each α.* □

Example 1.2.8. *For a fixed cardinal κ and any ordinal $\gamma < \kappa^+$ we define the family $\mathcal{E}^\gamma \subset \mathcal{P}(\kappa^\gamma)$ as follows:*

$$\mathcal{E}^\gamma = \Big\{\big[\kappa^{1+\alpha}\cdot\xi, \kappa^{1+\alpha}\cdot(\xi+1)\big) : \ \alpha \le \gamma,\ \kappa^{1+\alpha}\cdot(\xi+1) \le \kappa^{1+\gamma}\Big\}.$$

Of course, throughout this definition exponentiation means ordinal exponentiation.

$\mathcal{E}^\gamma$ is clearly well-founded, $\cap$-closed, moreover $\mathrm{rk}\Big(\big[\kappa^{1+\alpha}\cdot\xi, \kappa^{1+\alpha}\cdot(\xi+1)\big)\Big) = \alpha$*, hence $\mathcal{E}^\gamma$ is also* rk*-good. Consequently $X(\mathcal{E}^\gamma)$ is an LCS space of height $\gamma+1$ in which the αth level is* $\Big\{\big[\kappa^{1+\alpha}\cdot\xi, \kappa^{1+\alpha}\cdot(\xi+1)\big) \ : \kappa^{1+\alpha}\cdot(\xi+1) \le \kappa^{1+\gamma}\Big\}$*, i. e. all levels except the top one are of size κ.*

To get an LCS* space of height κ^+ with "*few*" isolated points, our plan is to *amalgamate* the spaces $\{X(\mathcal{E}^\gamma) : \gamma < \kappa^+\}$ into one LCS* space X in such a way that $|\,\mathrm{I}_0(X)| \le \kappa^{<\kappa}$. The following definition describes a situation in which such an amalgamation can be done.

Definition 1.2.9. A system of families $\{\mathcal{A}_i : i \in I\}$ is called *coherent* iff $A \cap B \in \mathcal{A}_i \cup \{\emptyset\}$ whenever $\{i,j\} \in \big[I\big]^2$, $A \in \mathcal{A}_i$ and $B \in \mathcal{A}_j$.

To simplify notation, we introduce the following convention. Whenever the system of families $\{\mathcal{A}_i : i \in I\}$ is given, we will write $\mathrm{U}_i(A)$ for $\mathrm{U}_{\mathcal{A}_i}(A)$, and τ_i for $\tau_{\mathcal{A}_i}$. If the family $\mathcal{A}$ is defined then we will write $\mathrm{U}(A)$ for $\mathrm{U}_{\mathcal{A}}(A)$, and τ for $\tau_{\mathcal{A}}$.

Lemma 1.2.10 ([**9**].). *Assume that $\{\mathcal{A}_i : i \in I\}$ is a coherent system of well-founded, $\cap$-closed families and $\mathcal{A} = \cup\{\mathcal{A}_i : i \in I\}$. Then for each $i \in I$ and $A \in \mathcal{A}_i$ we have $\mathrm{U}(A) = \mathrm{U}_i(A)$, $\mathcal{A}$ is also well-founded and $\cap$-closed, moreover $\tau_i \restriction \mathrm{U}(A) = \tau \restriction \mathrm{U}(A)$. Consequently each $X(\mathcal{A}_i)$ is an open subspace of $X(\mathcal{A})$ and thus $\{X(\mathcal{A}_i) : i \in I\}$ forms an open cover of $X(\mathcal{A})$.* □

Given a system of families $\{\mathcal{A}_i : i \in I\}$ we would like to construct a coherent system of families $\{\widehat{\mathcal{A}_i} : i \in I\}$ such that $\mathcal{A}_i$ and $\widehat{\mathcal{A}_i}$ are isomorphic for all $i \in I$. A sufficient condition for when this can be done will be given in lemma 1.2.12 below.

First, however, we need a definition. While reading it, one should remember that an ordinal is identified with the family of its proper initial segments.

Definition 1.2.11. Given a limit ordinal ρ and a family $\mathcal{A}$ with $\rho \subset \mathcal{A} \subset \mathcal{P}(\rho)$, let us define the family $\widehat{\mathcal{A}}$ as follows. Consider first the function $\mathrm{k}_{\mathcal{A}}$ on ρ determined by the formula $\mathrm{k}_{\mathcal{A}}(\eta) = \mathrm{U}_{\mathcal{A}}(\eta+1)$ for $\eta \in \rho$ and put

$$\widehat{\mathcal{A}} = \{\mathrm{k}''_{\mathcal{A}}\, A : A \in \mathcal{A}\}.$$

Since $\rho \subset \mathcal{A}$, for each $\eta \in \rho$ we clearly have $\cup\,\mathrm{U}_{\mathcal{A}}(\eta) = \eta$ and so $\mathrm{k}_{\mathcal{A}}(\eta) = \mathrm{U}_{\mathcal{A}}(\eta+1) \ne \mathrm{U}_{\mathcal{A}}(\xi+1) = \mathrm{k}_{\mathcal{A}}(\xi)$ whenever $\{\eta,\xi\} \in \big[\rho\big]^2$. Consequently, $\mathrm{k}_{\mathcal{A}}$ is a bijection that yields an isomorphism between $\mathcal{A}$ and $\widehat{\mathcal{A}}$ (and so the spaces $X(\mathcal{A})$ and $X(\widehat{\mathcal{A}})$ are homeomorphic).

If the system of families $\{\mathcal{A}_i : i \in I\}$ is given, then we write k_i for $\mathrm{k}_{\mathcal{A}_i}$ for each $i \in I$.

If $\mathcal{A} \subset \mathcal{P}(\rho)$ and $\xi \le \rho$ then we let

$$\mathcal{A} \restriction \xi = \{A \cap \xi : A \in \mathcal{A}\}.$$

For $\mathcal{A}_0 \ne \mathcal{A}_1 \subset \mathcal{P}(\rho)$ we let

$$\Delta(\mathcal{A}_0, \mathcal{A}_1) = \min\{\delta : \mathcal{A}_0 \restriction \delta \ne \mathcal{A}_1 \restriction \delta\}.$$

Clearly we always have $\Delta(\mathcal{A}_0, \mathcal{A}_1) \le \rho$. If, in addition, $\rho + 1 \subset \mathcal{A}_0 \cap \mathcal{A}_1$, moreover both $\mathcal{A}_0$ and $\mathcal{A}_1$ are $\cap$-closed then we also have

$$\Delta(\mathcal{A}_0, \mathcal{A}_1) = \min\{\delta : \mathrm{U}_0(\delta) \ne \mathrm{U}_1(\delta)\},$$

because then $\mathcal{A}_i \restriction \delta = \mathrm{U}_i(\delta)$ whenever $i \in 2$ and $\delta \le \rho$.

Lemma 1.2.12 ([9]). *Assume that κ is a cardinal, $\{\mathcal{A}_i : i \in I\} \subset \mathcal{PP}(\kappa)$ are $\cap$-closed families, $\kappa \dot{+} 1 \subset \mathcal{A}_i$ for each $i \in I$, and $\Delta(\mathcal{A}_i, \mathcal{A}_j)$ is a successor ordinal whenever $\{i, j\} \in \left[I\right]^2$. Then the system $\{\widehat{\mathcal{A}_i} : i \in I\}$ is coherent.* □

More is needed still if we want the "amalgamated" family to provide us a space with a small bottom, i.e. having not too many isolated points. This will be made clear by the following lemma.

Lemma 1.2.13. *Let κ be a cardinal and $\{\mathcal{A}_i : i \in I\} \subset \mathcal{PP}(\kappa)$ be a system of families such that*

(i) *$\kappa \dot{+} 1 \subset \mathcal{A}_i$ and $\mathcal{A}_i$ is well-founded and $\cap$-closed for each $i \in I$,*
(ii) *$\Delta(\mathcal{A}_i, \mathcal{A}_j)$ is a successor ordinal for each $\{i, j\} \in \left[I\right]^2$.*

Then

(a) *the system $\{\widehat{\mathcal{A}_i} : i \in I\}$ is coherent and thus $\mathcal{A} = \bigcup\{\widehat{\mathcal{A}_i} : i \in I\}$ is well-founded, $\cap$-closed and $X(\mathcal{A})$ is covered by its open subspaces $\{X(\widehat{\mathcal{A}_i}) : i \in I\}$.*

If, in addition, we also have

(iii) *$\mathrm{I}_0(\mathcal{A}_i) \subset \left[\kappa\right]^{<\kappa}$ for each $i \in I$,*

and

(iv) *$|\,\mathrm{U}_i(\xi)| < \kappa$ for each $i \in I$ and $\xi \in \kappa$,*

then

(b) $\mathrm{I}_0(\mathcal{A}) \subset \left[\left[[\kappa]^{<\kappa}\right]^{<\kappa}\right]^{<\kappa}$.

Proof of lemma 1.2.13. The system $\{\widehat{\mathcal{A}_i} : i \in I\}$ is coherent by lemma 1.2.12, thus (a) holds by lemma 1.2.10.

Consequently we have

$$\mathrm{I}_0(\mathcal{A}) = \bigcup\{\mathrm{I}_0(\widehat{\mathcal{A}_i}) : i \in I\}.$$

Now if $A \in \mathrm{I}_0(\mathcal{A}_i)$ and $\eta \in \kappa$ then $|A| < \kappa$ by (iii) and $\mathrm{U}_i(\eta) \in \left[\left[\kappa\right]^{<\kappa}\right]^{<\kappa}$ by (iv), hence

$$\mathrm{k}_i'' A = \{\mathrm{U}_i(\eta + 1) : \eta \in A\} \in \left[\left[\left[\kappa\right]^{<\kappa}\right]^{<\kappa}\right]^{<\kappa}.$$

This, by $\mathrm{I}_0(\widehat{\mathcal{A}_i}) = \{\mathrm{k}_i'' A : A \in \mathrm{I}_0(\mathcal{A}_i)\}$, proves (b). □

Definition 1.2.14. If ρ is an ordinal and $\mathcal{A} \subset \mathcal{P}(\rho)$ let us put

$$\mathcal{A}^* = \{A \cap \xi : A \in \mathcal{A} \wedge \xi \leq \rho\} = \mathcal{A} \cup \{A \cap \xi : A \in \mathcal{A} \wedge \xi < \rho\}.$$

Definition 1.2.15. A family $\mathcal{A}$ is called *chain-closed* if for each non-empty $\mathcal{B} \subset \mathcal{A}$ if $\mathcal{B}$ is ordered by $\subset$ (i.e. if $\mathcal{B}$ is a chain) then $\cup\mathcal{B} \in \mathcal{A}$.

Lemma 1.2.16 ([9]). *If ρ is an ordinal and $\mathcal{A}_0, \mathcal{A}_1 \subset \mathcal{P}(\rho)$ are chain-closed, $\cap$-closed and well-founded families such that ${\mathcal{A}_0}^* \neq {\mathcal{A}_1}^*$ then $\Delta({\mathcal{A}_0}^*, {\mathcal{A}_1}^*)$ is a successor ordinal.* □

The last result shows us that the operation * is useful because its application yields us families that satisfy condition (ii) of lemma 2.10. On the other hand, the following result tells us that the LCS spaces associated with certain families modified by * do not differ significantly from the spaces given by the original families, moreover they also satisfy condition (iii) of lemma 2.10.

Lemma 1.2.17. *Let κ be a cardinal and $\mathcal{A} \subset \left[\kappa\right]^{\kappa}$ be well-founded and $\cap$-closed. Then so is $\mathcal{A}^*$, moreover*

(*a*) *$X(\mathcal{A})$ is a closed subspace of $X(\mathcal{A}^*)$,*
(*b*) $\mathrm{I}_0(\mathcal{A}) \subseteq \mathrm{I}_\kappa(\mathcal{A}^*)$.
(*c*) $\mathrm{ht}(\mathcal{A}^*) \geq \kappa \dot{+} \mathrm{ht}(\mathcal{A})$,
(*d*) $\mathrm{I}_0(\mathcal{A}^*) \subset \left[\kappa\right]^{<\kappa}$.

Proof of lemma 1.2.17. We shall write $\mathrm{U}(A)$ for $\mathrm{U}_{\mathcal{A}}(A)$, and $\mathrm{U}_*(A)$ for $\mathrm{U}_{\mathcal{A}^*}(A)$.

First observe that because

$$\mathrm{U}_*(A) \cap \mathcal{A} = \begin{cases} \mathrm{U}(A) & \text{if } A \in \mathcal{A}, \\ \emptyset & \text{if } A \in \mathcal{A}^* \setminus \mathcal{A}, \end{cases}$$

$X(\mathcal{A})$ is a closed subspace of $X(\mathcal{A}^*)$, hence (a) holds.

Now let $A \in \mathrm{I}_0(\mathcal{A})$. Then there are $B_1, \dots, B_n \in \mathrm{U}(A) \setminus \{A\}$ such that

$$\{A\} = \mathrm{W}(A; B_1, \dots, B_n) = \mathrm{U}(A) \setminus \bigcup_{i=1}^{n} \mathrm{U}(B_i).$$

Since here $B_i \subsetneq A$ and $|A| = \kappa$, we can fix $\eta \in A$ such that $(A \cap \eta) \not\subset B_i$ for every $i = 1, \dots, n$.

Now consider the basic neighbourhood

$$\mathcal{Z} = \mathrm{W}_*(A; A \cap \eta, B_1, \dots, B_n) = \mathrm{U}_*(A) \setminus \mathrm{U}_*(A \cap \eta) \setminus \bigcup_{i=1}^{n} \mathrm{U}_*(B_i)$$

of A in $X(\mathcal{A}^*)$. We claim that $\mathcal{Z} = \{A \cap \xi : \eta < \xi \leq \kappa\}$. The inclusion $\supset$ is clear from the choice of η. On the other hand, if $C \cap \xi \in \mathcal{Z}$ with $C \in \mathcal{A}$ and $\xi \leq \kappa$, then $C \cap \xi \subset A$ hence $C \cap \xi = A \cap C \cap \xi$, so as $\mathcal{A}$ is $\cap$-closed we can assume that $C \subseteq A$. If we had $C \neq A$ then $\{A\} = \mathrm{W}(A; B_1, \dots, B_n)$ would imply $C \subset B_i$ for some i, hence $C \cap \xi \in \mathrm{U}_*(B_i)$ and so $C \cap \xi \notin \mathcal{Z}$, a contradiction, thus we must have $C = A$. Moreover, since $\mathrm{U}_*(A \cap \eta) \supset \{A \cap \nu : \nu \leq \eta\}$, we must also have $\xi > \eta$.

By example 1.2.6 we have $X(\mathcal{Z}) \cong \kappa + 1$. Moreover, the topologies $\tau_{\mathcal{Z}}$ and $\tau_{\mathcal{A}^*} \restriction \mathcal{Z}$ coincide because the above argument also shows that for each $C \in \mathcal{A}$ and $\zeta \leq \kappa$ we have

$$\mathrm{U}_*(C \cap \zeta) \cap \mathcal{Z} = \mathrm{U}_*(A \cap C \cap \zeta) \cap \mathcal{Z} = \begin{cases} \mathrm{U}_{\mathcal{Z}}(A \cap \zeta) & \text{if } A \subset C \text{ and } \zeta > \eta; \\ \emptyset & \text{otherwise.} \end{cases}$$

Hence $X(\mathcal{Z}) \cong \kappa \dot{+} 1$ is a clopen subspace of $X(\mathcal{A}^*)$ and so $\{A\} = \mathrm{I}_\kappa(\mathcal{Z}) = \mathrm{I}_\kappa(\mathcal{A}^*) \cap \mathcal{Z}$, what proves (b).

(c) follows immediately from (a) and (b).

Finally, $\mathrm{I}_0(\mathcal{A}^*) \subset \mathrm{I}_{<\kappa}(\mathcal{A}^*) \subset (\mathcal{A}^* \setminus \mathcal{A}) \subset \big[\kappa\big]^{<\kappa}$, as follows immediately from (b), proving (d). □

Before we could apply the amalgamation result to the families $\mathcal{E}^\gamma$, however, we need some further preparation that will be useful in ensuring the fulfillment of condition (iv) in 1.2.13.

Definition 1.2.18. A family $\mathcal{A}$ is called *tree-like* iff $A \cap A' \neq \emptyset$ implies that $A \subset A'$ or $A' \subset A$, whenever $A, A' \in \mathcal{A}$.

It is easy to see that the families $\mathcal{E}^\gamma$ given in example 1.2.8 are both tree-like and chain-closed. Also, tree-like families are clearly $\cap$-closed.

Lemma 1.2.19 ([**9**]). *If δ is an ordinal and $\mathcal{A} \subset \mathcal{P}(\delta)$ is tree-like, well-founded and chain-closed then so is $\mathcal{A} \restriction \xi$ for each $\xi \leq \delta$.* □

Definition 1.2.20. Given a family $\mathcal{A} \subset \mathcal{P}(\delta)$ and $\alpha, \beta \in \delta$ let us put

$$\mathrm{S}^{\mathcal{A}}(\alpha, \beta) = \cup\{A \in \mathcal{A} : \alpha \in A \text{ and } \beta \notin A\}.$$

Lemma 1.2.21. *Assume that δ is an ordinal and $\mathcal{A} \subset \mathcal{P}(\delta)$ is a tree-like, well-founded and chain-closed family with $\delta \in \mathcal{A}$. Then*

$$\mathcal{A} \setminus \{\emptyset\} = \{\delta\} \cup \{\mathrm{S}^{\mathcal{A}}(\alpha, \beta) : \alpha, \beta \in \delta\} \setminus \{\emptyset\}.$$

Consequently, $|\mathcal{A}| \leq |\delta|^2$.

Proof of lemma 1.2.21. Given $\alpha, \beta \in \delta$, the family $\mathcal{S} = \{A \in \mathcal{A} : \alpha \in A, \beta \notin A\}$ is ordered by $\subset$ because $\mathcal{A}$ is tree-like. Thus either $\mathcal{S} = \emptyset$ and so $\mathrm{S}^{\mathcal{A}}(\alpha, \beta) = \cup\mathcal{S} = \emptyset$, or if $\mathcal{S} \neq \emptyset$ then $\mathrm{S}^{\mathcal{A}}(\alpha, \beta) = \cup\mathcal{S} \in \mathcal{A}$, for $\mathcal{A}$ is chain-closed.

Assume now that $A \in \mathcal{A} \setminus \{\emptyset, \delta\}$ and let $\mathcal{D} = \{D \in \mathcal{A} : A \subsetneq D\}$. Clearly $\delta \in \mathcal{D}$. Since $\mathcal{A}$ is tree-like, $\mathcal{D}$ is ordered by $\subset$, so it has a $\subset$-least element, say D, because $\langle \mathcal{A}, \subset \rangle$ is also

well-founded. Pick $\beta \in D \setminus A$ and let $\alpha \in A$. We claim that $A = \mathrm{S}^{\mathcal{A}}(\alpha, \beta)$. Clearly $A \subset \mathrm{S}^{\mathcal{A}}(\alpha, \beta)$ because $\alpha \in A$ and $\beta \notin A$. On the other hand, if $A' \in \mathcal{A}$, $\alpha \in A'$ and $\beta \notin A'$ then either $A' \subset A$ or $A \subset A'$ because $\mathcal{A}$ is tree-like. But $\beta \notin A'$ implies that $A' \notin \mathcal{D}$, i.e. $A \subsetneq A'$ can not hold. Thus $A' \subset A$ and so $\mathrm{S}^{\mathcal{A}}(\alpha, \beta) = A$ is proved. $\square$

Now we are ready to collect the fruits of all the preparatory work.

Proof of theorem 1.2.2. For each $\gamma < \kappa^+$ consider the well-founded, $\cap$-closed, rk-good family $\mathcal{E}^\gamma$ constructed in example 1.2.8:

$$\mathcal{E}^\gamma = \Big\{ \big[\kappa^{1+\alpha} \cdot \xi, \kappa^{1+\alpha} \cdot (\xi + 1)\big) : \ \alpha \leq \gamma, \ \kappa^{1+\alpha} \cdot \xi < \kappa^\gamma \Big\}.$$

Fix a bijection $f_\gamma : \kappa^\gamma \longrightarrow \kappa$, and let $\mathcal{F}_\gamma = \{f_\gamma{}''E : E \in \mathcal{E}^\gamma\}$, i.e. $\mathcal{F}_\gamma$ is simply an isomorphic copy of $\mathcal{E}^\gamma$ on the underlying set κ. As $\mathcal{E}^\gamma$ is also chain-closed and tree-like, hence so is $\mathcal{F}_\gamma$.

We shall now show that the $*$-modified families $\{\mathcal{F}_\gamma{}^* : \gamma < \kappa^+\}$ satisfy conditions (i)-(iv) of lemma 1.2.13. Since $\kappa \in \mathcal{F}_\gamma$ it follows that $\kappa \dot{+} 1 \subset \mathcal{F}_\gamma{}^*$ and so (i) is true. For $\{\gamma, \delta\} \in \big[\kappa^+\big]^2$, the height of $X(\mathcal{E}^\gamma)$ is $\gamma + 1$ and the height of $X(\mathcal{E}^\delta)$ is $\delta + 1$, hence $\mathcal{E}^\gamma$ and $\mathcal{E}^\delta$ are not isomorphic. Thus $\mathcal{F}_\gamma \neq \mathcal{F}_\delta$ and so $\mathcal{F}_\gamma{}^* \neq \mathcal{F}_\delta{}^*$ as well because $\mathcal{F}_\gamma = \mathcal{F}_\gamma{}^* \cap \big[\kappa\big]^\kappa$ and $\mathcal{F}_\delta = \mathcal{F}_\delta{}^* \cap \big[\kappa\big]^\kappa$. Hence $\Delta(\mathcal{F}_\gamma{}^*, \mathcal{F}_\delta{}^*)$ is a successor ordinal by lemma 1.2.16, i.e. (ii) is satisfied.

(iii) holds by 1.2.17.(d.)

To show (iv), let us fix $\xi < \kappa$. Then $\mathrm{U}_{\mathcal{F}_\gamma{}^*}(\xi) = \mathcal{F}_\gamma{}^* \restriction \xi = \cup\{\mathcal{F}_\gamma \restriction \zeta : \zeta \leq \xi\}$ where $|\mathcal{F}_\gamma \restriction \zeta| \leq |\zeta|^2$ for all $\zeta \leq \xi$ by lemmas 1.2.19 and 1.2.21, consequently $|\mathcal{F}_\gamma{}^* \restriction \xi| \leq |\xi|^3 < \kappa$.

Thus we may apply lemma 1.2.13 to the family $\mathcal{F} = \cup\{\widehat{\mathcal{F}_\gamma{}^*} : \gamma < \kappa^+\}$ and conclude that the space $X = X(\mathcal{F})$ is LCS, moreover $|\,\mathrm{I}_0(X)| \leq \big((\kappa^{<\kappa})^{<\kappa}\big)^{<\kappa} = \kappa$. Since for every $\gamma \in \kappa^+$ the space $X(\mathcal{F}_\gamma{}^*)$ is an open subspace of X, we have $\mathrm{ht}(X) \geq \mathrm{ht}(X(\mathcal{F}_\gamma{}^*)) > \gamma$, consequently $\mathrm{ht}(X) \geq \kappa^+$. $\square$

1.3. Cardinal sequences of length $< \omega_2$ under GCH

(This section is based on [14])

The cardinal sequences of LCS* spaces with height ω_1 has the following characterization in ZFC:

Theorem 1.3.1 (Juhasz, Weiss, **[61]**). *$\langle \kappa_\xi : \xi < \omega_1 \rangle \in \mathcal{C}(\omega_1)$ iff $\kappa_\eta \leq \kappa_\xi^\omega$ holds whenever $\xi < \eta < \omega_1$.*

It follows that cardinal arithmetic alone decides whether a sequence of cardinals of length ω_1 belongs to $\mathcal{C}(\omega_1)$ or not. The situation changes dramatically for longer sequences, in fact already for sequences of length $\omega_1 + 1$. For example, the question if $\langle \omega \rangle_{\omega_1} \frown \langle \omega_2 \rangle_1 \in \mathcal{C}(\omega_1 + 1)$ is not decided by the following cardinal arithmetic: $2^\omega = \omega_2$ and $2^\kappa = \kappa^+$ for all $\kappa > \omega$ (see Just**[64]** and Roitman**[93]**).

However, as we showed in **[14]**, the elements of $\mathcal{C}(\alpha)$ can be characterized for all $\alpha < \omega_2$ if we assume GCH. This section contains that result.

In order to characterize those sequences of length $< \omega_2$ which are cardinal sequences of LCS* spaces, it suffices to characterize the classes $\mathcal{C}_\lambda(\alpha)$ for any ordinal $\alpha < \omega_2$ and any infinite cardinal λ. In fact, this follows from the following general reduction theorem that is valid in ZFC.

Theorem 1.3.2 (**[14]**). *For any ordinal α and any sequence f of cardinals of length α the following are equivalent:*

(1) $f \in \mathcal{C}(\alpha)$

(2) for some natural number n there is a decreasing sequence $\lambda_0 > \lambda_1 > \cdots > \lambda_{n-1}$ of infinite cardinals and there are ordinals $\alpha_0, \ldots \alpha_{n-1}$ such that $\alpha = \alpha_0 + \cdots + \alpha_{n-1}$ and $f = f_0 \frown f_1 \frown \cdots \frown f_{n-1}$ with $f_i \in \mathcal{C}_{\lambda_i}(\alpha_i)$ for each $i < n$.

We prove Theorem 1.3.2 at the end of this section.

From here on let us assume GCH. Our aim is to characterize the classes $\mathcal{C}_\lambda(\alpha)$ with $\alpha < \omega_2$. (For an ordinal α and a set B, as usual, we let ${}^\alpha B$ denote the set of all sequences of length α

taking values in B.) Now, for any $s \in {}^{\alpha}\{\lambda, \lambda^+\}$ we write

$$\mathrm{A}_\lambda(s) = \{\beta \in \alpha : s(\beta) = \lambda\} = s^{-1}\{\lambda\}.$$

Definition 1.3.3. If α is any ordinal, a subset $L \subset \alpha$ is called *κ-closed in α*, where κ is an infinite cardinal, iff $\sup \langle \alpha_i : i < \kappa \rangle \in L \cup \{\alpha\}$ for each increasing sequence $\langle \alpha_i : i < \kappa \rangle \in {}^{\kappa}L$. The set L is *$< \lambda$-closed in α* provided it is κ-closed in α for each cardinal $\kappa < \lambda$. We say that L is *successor closed in α* if $\beta + 1 \in L \cup \{\alpha\}$ for all $\beta \in L$.

We are now ready to present the promised GCH characterization of the classes $\mathcal{C}_\lambda(\alpha)$ and consequently, in view of 1.3.2, the characterization of $\mathcal{C}(\alpha)$ for all $\alpha < \omega_2$.

Theorem 1.3.4. *Assume GCH and fix $\alpha < \omega_2$.*

(i) $\mathcal{C}_\omega(\alpha) = \{s \in {}^{\alpha}\{\omega, \omega_1\} : s(0) = \omega\}$.

(ii) If $\lambda > \mathrm{cf}(\lambda) = \omega$,

$$\mathcal{C}_\lambda(\alpha) = \{s \in {}^{\alpha}\{\lambda, \lambda^+\} : s(0) = \lambda \text{ and } \mathrm{A}_\lambda(s) \text{ is } \omega_1\text{-closed in } \alpha\}.$$

(iii) If $\mathrm{cf}(\lambda) = \omega_1$,

$$\mathcal{C}_\lambda(\alpha) = \{s \in {}^{\alpha}\{\lambda, \lambda^+\} : s(0) = \lambda \text{ and } \mathrm{A}_\lambda(s) \text{ is both } \omega\text{-closed and successor-closed in } \alpha\}.$$

(iv) If $\mathrm{cf}(\lambda) > \omega_1$,

$$\mathcal{C}_\lambda(\alpha) = \{\langle \lambda \rangle_\alpha\}.$$

The main ingredient of the proof is the following statement:

Theorem 1.3.5. *Let λ be a cardinal with $\mu = \mathrm{cf}(\lambda) > \omega$ and satisfying $\lambda = \lambda^{<\mu}$. Then for any cardinal κ with $\lambda < \kappa \leq \lambda^\mu$ and for every ordinal $\alpha < \mu^+$ with $\mathrm{cf}(\alpha) = \mu$ we have $\langle \lambda \rangle_\alpha {}^\frown \langle \kappa \rangle_{\mu^+} \in \mathcal{C}(\mu^+)$.*

To prove theorems above we need some preparation.

If X is a scattered space and $x \in X$ then we write $\mathrm{ht}(x, X) = \alpha$ iff $x \in I_\alpha(X)$. Trivially, then

$$\mathrm{ht}(X) = \min\{\beta : \forall x \in X[\mathrm{ht}(x, X) < \beta]\}.$$

It is obvious that if $Y \subset X$ then $\mathrm{ht}(x, X) \geq \mathrm{ht}(x, Y)$ whenever $x \in Y$, and if Y is also *open* in X then actually $\mathrm{ht}(x, X) = \mathrm{ht}(x, Y)$. On the other hand, for the points of X outside of Y one can get the following upper bound.

Fact 1.3.6. *If Y is an open subspace of the scattered space X then for every point $x \in X \setminus Y$ we have $\mathrm{ht}(x, X) \leq \mathrm{ht}(Y) + \mathrm{ht}(x, X \setminus Y)$. Consequently, $\mathrm{ht}(X) \leq \mathrm{ht}(Y) + \mathrm{ht}(X \setminus Y)$.*

Indeed, this can be proved by a straight-forward transfinite induction on $\mathrm{ht}(x, X)$, using $Y \subset \mathrm{I}_{<\mathrm{ht}(Y)}(X)$.

It is well-known that any ordinal, as an ordered topological space, is locally compact and scattered. It is easy to see that if $\alpha < \beta$ are ordinals then $\mathrm{ht}(\alpha, \beta) = \gamma$ iff α can be written in the form $\omega^\gamma \cdot (2\delta + 1)$, or equivalently, γ is minimal such that α can be written as $\alpha = \varepsilon + \omega^\gamma$. Note that in the notation $\mathrm{ht}(\alpha, \beta)$ the ordinals play a double role: α is considered as a "point" in the set β. Using the above characterization of the Cantor-Bendixson levels of ordinal spaces, it is easy to show that for any infinite cardinal λ and for any ordinal $\alpha < \lambda^+$ we have $\langle \lambda \rangle_\alpha \in \mathcal{C}(\alpha)$.

This section is based on [**14**] which is is a natural sequel to [**61**], so now we recall a few general statements concerning cardinal sequences from [**61**] that will be needed later.

Fact* 1.3.7 ([**61**, lemma 1])**.** *If $s \in \mathcal{C}(\beta)$ then $|\beta| \leq 2^{s(0)}$ and $s(\alpha) \leq 2^{s(0)}$ for each $\alpha < \beta$.*

Fact* 1.3.8 ([**61**, lemma 2])**.** *If $s \in \mathcal{C}(\beta)$ and $\alpha + 1 < \beta$ then $s(\alpha + 1) \leq s(\alpha)^\omega$.*

Fact* 1.3.9 ([**61**, lemma 3])**.** *If $s \in \mathcal{C}(\beta)$, $\delta < \beta$ is a limit ordinal and C is any cofinal subset of δ, then*

$$s(\delta) \leq \prod\{s(\alpha) : \alpha \in C\} \ .$$

We shall also need the following general construction from [**61**] that is used to obtain an LCS* space by gluing together certain others.

Lemma 1.3.10 ([**61**, lemma 7]). *Let X be an LCS* space with a closed discrete subset S such that for each $s \in S$ there is given a sequence $\langle U_{s,n} : n \in \omega\rangle$ of pairwise disjoint compact open subsets of $X \setminus S$, converging to the point s. Also, for each $s \in S$ let Y_s be a separable LCS* space such that the collection of spaces $\{X\} \cup \{Y_s : s \in S\}$ is disjoint. Then there is an LCS* space Z with the following three properties (i) - (iii):*

(i) $Z = (X \setminus S) \cup \bigcup\{Y_s : s \in S\}$ *with $X \setminus S$ as an open subspace and each Y_s as a closed subspace. Moreover, $\{Y_s : s \in S\}$ forms a discrete collection in Z.*

(ii) $\mathrm{ht}(x, Z) = \mathrm{ht}(x, X)$ *for $x \in X \setminus S$.*

(iii) $\mathrm{ht}(y, Z) = \delta_s + \mathrm{ht}(y, Y_s)$ *for $y \in Y_s$, where δ_s is the least ordinal δ such that the set $\{n < \omega : U_{s,n} \cap I_\delta(X) \neq \emptyset\}$ is finite. (Clearly, $\delta_s \leq \mathrm{ht}(s, X)$.)*

Definition 1.3.11. For any family of sets $\mathcal{A}$ we define the topological space $X(\mathcal{A}) = \langle \mathcal{A}, \tau_{\mathcal{A}}\rangle$ as follows: $\tau_{\mathcal{A}}$ is the coarsest topology on $\mathcal{A}$ such that the sets $\mathrm{U}_{\mathcal{A}}(A) = \mathcal{A} \cap \mathcal{P}(A)$ are clopen for each $A \in \mathcal{A}$. In other words: $\{\mathrm{U}_{\mathcal{A}}(A), \mathcal{A} \setminus \mathrm{U}_{\mathcal{A}}(A) : A \in \mathcal{A}\}$ is a subbase for $\tau_{\mathcal{A}}$.

Clearly $X(\mathcal{A})$ is a 0-dimensional T_2-space.

A family $\mathcal{A}$ is called *well-founded* iff the partial order $\langle \mathcal{A}, \subset\rangle$ is well-founded. $\mathcal{A}$ is said to be $\cap$-*closed* iff $A \cap B \in \mathcal{A} \cup \{\emptyset\}$ whenever $A, B \in \mathcal{A}$.

It is easy to see that if $\mathcal{A}$ is $\cap$-closed, then a neighbourhood base of $A \in \mathcal{A}$ in the space $X(\mathcal{A})$ is formed by the clopen sets

$$\mathrm{W}_{\mathcal{A}}(A; B_1, \ldots, B_n) = \mathrm{U}_{\mathcal{A}}(A) \setminus \bigcup_1^n \mathrm{U}_{\mathcal{A}}(B_i),$$

where $n \in \omega$ and $B_i \subsetneq A$ for $i = 1, \ldots, n$. (For $n = 0$ we have $\mathrm{W}_{\mathcal{A}}(A) = \mathrm{U}_{\mathcal{A}}(A)$.) We shall write $\mathrm{U}(A)$ instead of $\mathrm{U}_{\mathcal{A}}(A)$ if $\mathcal{A}$ is clear from the context, and similarly for the W's.

The following statement, that was proved in [**9**, Lemma 2.2], shows the relevance of the above concepts to the subject matter of this section.

Fact 1.3.12. *Assume that $\mathcal{A}$ is both $\cap$-closed and well-founded. Then $X(\mathcal{A})$ is an LCS* space.*

To simplify notation, if $X(\mathcal{A})$ is scattered then we write $\mathrm{I}_\alpha(\mathcal{A})$ instead of $\mathrm{I}_\alpha(X(\mathcal{A}))$, and $\mathrm{I}_{<\alpha}(\mathcal{A})$ instead of $\cup\{\mathrm{I}_\zeta(\mathcal{A}) : \zeta < \alpha\}$. In the same spirit, for $A \in \mathcal{A}$ we sometimes write $\mathrm{ht}(A, \mathcal{A})$ instead of $\mathrm{ht}(A, X(\mathcal{A}))$.

We shall say that $\mathcal{A}$ is an *ordinal family* if all members of $\mathcal{A}$ are sets of ordinal numbers, moreover $\mathcal{A}$ is both $\cap$-closed and well-founded. (As usual, we shall denote by On the class of all ordinals.) The following definition makes sense for any ordinal family $\mathcal{A}$ and will play an important role in our construction.

If $\mathcal{A}$ is an ordinal family and ξ is any ordinal then we let

$$\mathcal{A} \restriction \xi = \{A \cap \xi : A \in \mathcal{A}\}$$

and

$$\mathcal{A}^* = \bigcup\{\mathcal{A} \restriction \xi : \xi \in On\}.$$

So $\mathcal{A}^*$ is simply the family consisting of all initial segments of all members of $\mathcal{A}$. Clearly,

$$\mathcal{A}^* = \mathcal{A} \cup \{A \cap \xi : A \in \mathcal{A} \wedge \xi \in A\}.$$

It is easy to see that if $\mathcal{A}$ is an ordinal family then so is $\mathcal{A}^*$, hence both $X(\mathcal{A})$ and $X(\mathcal{A}^*)$ are LCS* spaces. A key ingredient in our construction is, just like in [**9**], the clarification of the relationship between these two spaces. The following technical lemma will play a significant role in this. As indicated above, we shall write $\mathrm{U}(A)$ instead of $\mathrm{U}_{\mathcal{A}}(A)$, $\mathrm{U}_*(A)$ instead of $\mathrm{U}_{\mathcal{A}^*}(A)$, and similarly for the W's.

Lemma 1.3.13 ([**14**]). *Let $\mathcal{A}$ be an ordinal family. Then for any $A \in \mathcal{A}$ we have*

$$\mathrm{ht}(\mathrm{U}_*(A)) \leq \sup\{\mathrm{ht}(\mathrm{U}_*(A')) : A' \in \mathrm{U}(A) \setminus \{A\})\} + \mathrm{ht}(\mathrm{tp}\, A + 1) \quad \square$$

What we shall really need in our construction is the following corollary of lemma 1.3.13.

Lemma 1.3.14 ([**14**]). *Let $\mathcal{A}$ be an ordinal family such that, for a fixed indecomposable $\alpha \in On$, we have $|\mathcal{A}| < \mathrm{cf}(\alpha)$ and $\mathrm{ht}(\mathrm{tp}\, A) < \alpha$ for all $A \in \mathcal{A}$. Then $\mathrm{ht}(X(\mathcal{A}^*)) < \alpha$.* $\square$

Now we shall prove a result showing that, for certain ordinal families $\mathcal{H}$, the space $X(\mathcal{H})$ is a very special subspace of $X(\mathcal{H}^*)$. This result naturally corresponds to [**9**, lemma 2.14]. If ρ is an ordinal and $L \subset On$ then we write

$$(L)^\rho = \{K \subset L : \operatorname{tp} K = \rho\},$$

moreover

$$(L)^{<\rho} = \bigcup\{(L)^\alpha : \alpha < \rho\}.$$

Lemma 1.3.15 ([**14**]). *Let ρ be an indecomposable ordinal such that $\operatorname{ht}(\rho) = \alpha$ is also indecomposable, $\operatorname{cf}(\alpha) = \operatorname{cf}(\rho)$, moreover $\operatorname{ht}(\xi) < \alpha$ for all $\xi < \rho$. Let $\mathcal{H} \subset (\rho)^\rho$ be an ordinal family such that $\left|\mathcal{H} \restriction \xi\right| < \operatorname{cf}(\alpha)$ for all $\xi < \rho$. Then $X(\mathcal{H})$ forms a "tail" of $X(\mathcal{H}^*)$ in the following sense:*

(*a*) *$X(\mathcal{H})$ is a closed subspace of $X(\mathcal{H}^*)$,*
(*b*) *$\mathrm{I}_\beta(\mathcal{H}) = \mathrm{I}_{\alpha+\beta}(\mathcal{H}^*)$ for all $\beta < \operatorname{ht}(X(\mathcal{H}))$,*
(*c*) *$\mathrm{I}_{<\alpha}(\mathcal{H}^*) = \mathcal{H}^* \setminus \mathcal{H}$.* □

Definition 1.3.16. Two families of sets $\mathcal{A}_0$ and $\mathcal{A}_1$ are said to be *coherent* iff $A_0 \cap A_1 \in (\mathcal{A}_0 \cap \mathcal{A}_1) \cup \{\emptyset\}$ whenever $A_i \in \mathcal{A}_i$ for $i < 2$. A system of families $\{\mathcal{A}_i : i \in I\}$ is *coherent* iff $\mathcal{A}_i$ and $\mathcal{A}_j$ are coherent for each pair $\{i, j\} \in \left[I\right]^2$.

If $\{\mathcal{A}_i : i \in I\}$ is a coherent system of well-founded and $\cap$-closed families then we can "amalgamate" the spaces $\{X(\mathcal{A}_i) : i \in I\}$ as follows: According to [**9**, lemma 7], then $\mathcal{A} = \cup\{\mathcal{A}_i : i \in I\}$ is also well-founded and $\cap$-closed, hence $X(\mathcal{A})$ is also an LCS* space; moreover $\{X(\mathcal{A}_i) : i \in I\}$ forms an open cover of $X(\mathcal{A})$.

Next we introduce a method that transforms an ordinal family $\mathcal{A}$ into another, isomorphic family $\widehat{\mathcal{A}}$. We do this because, for certain systems of ordinal families $\{\mathcal{A}_i : i \in I\}$, the new system $\{\widehat{\mathcal{A}_i} : i \in I\}$ turns out to be coherent. Since $X(\mathcal{A}_i)$ and $X(\widehat{\mathcal{A}_i})$ are clearly homeomorphic, in this way the spaces $\{X(\mathcal{A}_i) : i \in I\}$ may be amalgamated in two steps. The definition of the operation $\mathcal{A} \mapsto \widehat{\mathcal{A}}$ given below is a slight generalization of the one given in definition 2.8 from [**9**].

The next definition is a generalization of Definition 1.2.11.

Definition 1.3.17. Let us be given a set of ordinals $L \subset On$ of limit order type and a family $\mathcal{A}$ with $L^\circ \subset \mathcal{A} \subset \mathcal{P}(L)$. We first define the map $\mathrm{k}_\mathcal{A}$ on L by the formula $\mathrm{k}_\mathcal{A}(\eta) = \mathcal{A} \restriction \eta + 1$ for $\eta \in L$; then we define the map $\chi_\mathcal{A}$ on $\mathcal{A}$ by putting $\chi_\mathcal{A}(A) = \mathrm{k}''_\mathcal{A} A$, i. e. $\chi_\mathcal{A}(A)$ is the $\mathrm{k}_\mathcal{A}$-image of $A \in \mathcal{A}$. Finally, we define the family $\widehat{\mathcal{A}}$ by

$$\widehat{\mathcal{A}} = \{\chi_\mathcal{A}(A) : A \in \mathcal{A}\}.$$

Let us remark that $L \cap (\eta + 1) \in L^\circ \subset \mathcal{A}$ for each $\eta \in L$, hence we have

$$\mathrm{k}_\mathcal{A}(\eta) = \mathcal{A} \restriction \eta + 1 = \mathrm{U}_\mathcal{A}(L \cap (\eta + 1)).$$

By the same token, for any $\eta \in L$ we also have

$$\max[\cup \mathrm{k}_\mathcal{A}(\eta)] = \max[L \cap (\eta + 1)] = \eta,$$

consequently $\mathrm{k}_\mathcal{A}$ is a bijection between the sets L and $\chi_\mathcal{A}(L)$. Therefore, $\chi_\mathcal{A}$ is indeed an isomorphism between the partial orders $\langle \mathcal{A}, \subset\rangle$ and $\langle \widehat{\mathcal{A}}, \subset\rangle$, and so the spaces $X(\mathcal{A})$ and $X(\widehat{\mathcal{A}})$ are homeomorphic. Note, however, that the sets $\chi_\mathcal{A}(A)$ from $\widehat{\mathcal{A}}$ are not sets of ordinals anymore. This is the price we have to pay for transforming the ordinal family $\mathcal{A}$ into the coherent family $\widehat{\mathcal{A}}$.

Next, if $\mathcal{A}_0 \neq \mathcal{A}_1$ are two families of sets of ordinals then we let

$$\Delta(\mathcal{A}_0, \mathcal{A}_1) = \min\{\delta : \mathcal{A}_0 \restriction \delta \neq \mathcal{A}_1 \restriction \delta\}.$$

Since ${\mathcal{A}_i}^* \restriction \xi = \cup\{\mathcal{A}_i \restriction \eta : \eta \leq \xi\}$, we clearly have

$$\Delta(\mathcal{A}_0, \mathcal{A}_1) = \Delta({\mathcal{A}_0}^*, {\mathcal{A}_1}^*),$$

provided that ${\mathcal{A}_0}^* \neq {\mathcal{A}_1}^*$ holds as well.

The following lemma is a strengthening of [**9**, lemma 2.9]. It gives us a condition under which the transforms $\widehat{\mathcal{A}}$ and $\widehat{\mathcal{B}}$ of two families $\mathcal{A}$ and $\mathcal{B}$, respectively turn out to be coherent.

Lemma 1.3.18 ([**14**]). *Assume that L and M are two sets of ordinals of limit order type such that $L \cap M$ is a proper initial segment of both L and M, moreover $L^\circ \subset \mathcal{A} \subset \mathcal{P}(L)$ and $M^\circ \subset \mathcal{B} \subset \mathcal{P}(M)$ are $\cap$-closed families. If $\Delta(\mathcal{A}, \mathcal{B}) = \delta + 1$ is a successor ordinal then the families $\widehat{\mathcal{A}}$ and $\widehat{\mathcal{B}}$ are coherent.* $\square$

We can prove the result that was formulated at the beginning of this section.

Proof of 1.3.5. First we show that it suffices to prove the theorem in the case when α is also indecomposable. Indeed, let $\alpha < \mu^+$ be arbitrary with $\mathrm{cf}(\alpha) = \mu$. We may then write $\alpha = \alpha' + \alpha''$ where α'' is indecomposable and, of course, $\mathrm{cf}(\alpha) = \mathrm{cf}(\alpha'') = \mu$. We may apply lemma 1.3.20 to the sequences $\langle\lambda\rangle_{\alpha'} \in \mathcal{C}(\alpha')$ and $\langle\lambda\rangle_{\alpha''} {}^\frown \langle\kappa\rangle_{\mu^+} \in \mathcal{C}(\mu^+)$ to conclude that

$$\langle\lambda\rangle_\alpha {}^\frown \langle\kappa\rangle_{\mu^+} = \langle\lambda\rangle_{\alpha'} {}^\frown \langle\lambda\rangle_{\alpha''} {}^\frown \langle\kappa\rangle_{\mu^+} \in \mathcal{C}(\mu^+).$$

So assume now that α is indecomposable and let $\rho = \omega^\alpha$. Clearly then ρ is also indecomposable, $\mathrm{cf}\,\rho = \mathrm{cf}(\alpha) = \mu$, moreover $\mathrm{ht}(\rho) = \alpha$ and $\mathrm{ht}(\xi) < \alpha$ for all $\xi < \rho$. Let $\langle\nu_\zeta : \zeta < \mu\rangle$ be a strictly increasing sequence of limit ordinals cofinal in ρ. As ρ is indecomposable and $\mathrm{cf}(\rho) = \mu$, for every set $a \in [\mu]^\mu$ we have $\sum\{\nu_\zeta : \zeta \in a\} = \rho$.

Next we fix a *disjoint* family $\{K_t : t \in {}^{<\mu}\lambda\}$ of intervals of ordinals such that for any $t, s \in {}^{<\mu}\lambda$ we have

(a) $\mathrm{tp}(K_t) = \nu_{\mathrm{dom}\, t}$,
(b) if s is a proper initial segment of t then $\sup K_s < \min K_t$.

We also choose a family of functions $\mathcal{G} \subset {}^\mu\lambda$ with $|\mathcal{G}| = \kappa$ and for every $g \in \mathcal{G}$ put

$$L_g = \bigcup\{K_{g\restriction\xi} : \xi \in \mu\}.$$

Then we have

(I) $\mathrm{tp}\, L_g = \rho$ for each $g \in \mathcal{G}$,
(II) $L_g \cap L_h$ is a proper initial segment of both L_g and L_h whenever $\{g, h\} \in \big[\mathcal{G}\big]^2$.

In the proof of the main result of [**9**], namely Theorem 2.19, we constructed, for any fixed cardinal μ and for all ordinals $\gamma < \mu^+$, ordinal families $\mathcal{F}_\gamma$ such that the following five conditions were satisfied:

(i) $\mu \in \mathcal{F}_\gamma \subset \big[\mu\big]^\mu$,
(ii) $\mathcal{F}_\gamma$ is well-founded and $\cap$-closed,
(iii) $\mathrm{ht}\, X(\mathcal{F}_\gamma) = \gamma + 1$,
(iv) $\Delta(\mathcal{F}_\gamma, \mathcal{F}_\delta) = \Delta(\mathcal{F}^*_\gamma, \mathcal{F}^*_\delta)$ is a successor ordinal if $\gamma \neq \delta$,
(v) $\big|\mathcal{F}_\gamma \restriction \xi\big| \leq |\xi| + \omega < \mu$ for each $\xi < \mu$.

We shall also make use of these families $\{\mathcal{F}_\gamma : \gamma < \mu^+\}$, more precisely some transformed versions of them, in the present proof. To this end, we first fix a function $\Gamma : \mathcal{G} \longrightarrow \mu^+$ such that $|\Gamma^{-1}\{\gamma\}| = \kappa$ for each $\gamma \in \mu^+$. This is possible because $|\mathcal{G}| = \kappa \geq \mu^+$.

Fix $g \in \mathcal{G}$ and for all $F \subset \mu$ put

$$\varphi_g(F) = \cup\{K_{g\restriction\xi} : \xi \in F\};$$

then we define

$$\mathcal{H}_g = \{\varphi_g(F) : F \in \mathcal{F}_{\Gamma(g)}\}.$$

Note that, by (i), for each $g \in \mathcal{G}$ we have $\mathcal{H}_g \subset (L_g)^\rho$.

It is obvious that the map φ_g induces an inclusion-preserving isomorphism between the families $\mathcal{F}_{\Gamma(g)}$ and $\mathcal{H}_g$ (i. e. between the partial orders $\big\langle\mathcal{F}_{\Gamma(g)}, \subset\big\rangle$ and $\langle\mathcal{H}_g, \subset\rangle$), consequently the spaces $X(\mathcal{F}_{\Gamma(g)})$ and $X(\mathcal{H}_g)$ are homeomorphic. It is also easy to check that, for each $g \in \mathcal{G}$, the ordinal family $\mathcal{H}_g$ satisfies all the requirements of lemma 1.3.15, or actually of its more general version formulated in the remark made afterwards. In view of this and property (iii), we may sum up the relevant properties of the spaces $X({\mathcal{H}_g}^*)$ as follows.

Fact 1.3.18.1. *$X(\mathcal{H}_g)$ is a closed subspace of $X({\mathcal{H}_g}^*)$ and we have*

(*a*) $\mathrm{ht}(X({\mathcal{H}_g}^*)) = \alpha + \Gamma(g) + 1$,
(*b*) $\mathrm{I}_{\alpha+\beta}({\mathcal{H}_g}^*) = \mathrm{I}_\beta(\mathcal{H}_g)$ *for all* $\beta < \mathrm{ht}(X(\mathcal{H}_g)) = \Gamma(g) + 1$,

(c) $\mathrm{I}_{<\alpha}({\mathcal{H}_g}^*) = {\mathcal{H}_g}^* \setminus \mathcal{H}_g$.

Our aim is to amalgamate the spaces $\{X({\mathcal{H}_g}^*) : g \in \mathcal{G}\}$ but to do that we shall have to transform the families ${\mathcal{H}_g}^*$ by means of the operation "hat" described in lemma 1.3.18.

Since $\mu \in \mathcal{F}_{\Gamma(g)}$, we have ${L_g}^\circ \subset {\mathcal{H}_g}^* \subset \mathcal{P}(L_g)$ for each $g \in \mathcal{G}$. Also, if $\{g,h\} \in \big[\mathcal{G}\big]^2$ then $L_g \cap L_h$ is a proper initial segment of both L_g and L_h by (II). Consequently, by lemma 1.3.18, the system $\{\widehat{\mathcal{H}_g^*} : g \in \mathcal{G}\}$ will be proven to be coherent once the following claim is established.

Claim 1.3.18.1. *$\Delta({\mathcal{H}_g}^*, {\mathcal{H}_h}^*)$ is a successor ordinal for each $\{g,h\} \in \big[\mathcal{G}\big]^2$.*

Proof of the claim. Let $\xi < \mu$ be minimal such that $g \restriction \xi \neq h \restriction \xi$, then we clearly have

$$\eta = \min(L_g \triangle L_h) = \min(K_{g\restriction\xi} \cup K_{h\restriction\xi}).$$

(As usual, we are using $\triangle$ to denote symmetric difference.) Now, if $\mathcal{F}_{\Gamma(g)} \restriction \xi = \mathcal{F}_{\Gamma(h)} \restriction \xi$ (this happens for instance if $\Gamma(g) = \Gamma(h)$) then $\mathcal{H}_g \restriction \eta = \mathcal{H}_h \restriction \eta$, and so we also have $\mathcal{H}_g^* \restriction \eta = \mathcal{H}_h^* \restriction \eta$. On the other hand, $K_{g\restriction\xi} \cap K_{h\restriction\xi} = \emptyset$ implies that

$$(L_g \cap L_h) \cup \{\eta\} \in (\mathcal{H}_g^* \restriction \eta+1) \triangle (\mathcal{H}_h^* \restriction \eta+1),$$

hence $\Delta({\mathcal{H}_g}^*, {\mathcal{H}_h}^*) = \eta + 1$, and we are done.

Thus we can assume that $\mathcal{F}_{\Gamma(g)} \restriction \xi \neq \mathcal{F}_{\Gamma(h)} \restriction \xi$. In this case, by property (iv), we know that $\Delta(\mathcal{F}_{\Gamma(g)}, \mathcal{F}_{\Gamma(h)}) \leq \xi$ is a successor ordinal, say $\delta + 1$. But then there is a set $A \in \mathcal{F}_{\Gamma(g)} \restriction \delta = \mathcal{F}_{\Gamma(h)} \restriction \delta$ such that

$$A \cup \{\delta\} \in (\mathcal{F}_{\Gamma(g)} \restriction \delta+1) \triangle (\mathcal{F}_{\Gamma(h)} \restriction \delta+1).$$

Write $\sigma = \min K_{g\restriction\delta} (= \min K_{h\restriction\delta})$ and put

$$D = \cup\{K_{g\restriction\zeta} : \zeta \in A\} \cup \{\sigma\},$$

then clearly

$$D \in ({\mathcal{H}_g}^* \restriction \sigma+1) \triangle ({\mathcal{H}_h}^* \restriction \sigma+1).$$

On the other hand, $\mathcal{F}_{\Gamma(g)} \restriction \delta = \mathcal{F}_{\Gamma(h)} \restriction \delta$ and $g \restriction \delta = h \restriction \delta$ together imply $\mathcal{H}_g \restriction \sigma = \mathcal{H}_h \restriction \sigma$ and so ${\mathcal{H}_g}^* \restriction \sigma = {\mathcal{H}_h}^* \restriction \sigma$. Thus we have $\Delta({\mathcal{H}_g}^*, {\mathcal{H}_h}^*) = \sigma + 1$, completing the proof. $\square$

Consequently, we may now apply lemma 1.3.18 to conclude that the family $\{\widehat{\mathcal{H}_g^*} : g \in \mathcal{G}\}$ is coherent. Therefore, by [**9**, Lemma 2.7], the family $\mathcal{H} = \bigcup\{\widehat{\mathcal{H}_g^*} : g \in \mathcal{G}\}$ is well-founded and $\cap$-closed, and so the amalgamation $X(\mathcal{H})$ is an LCS* space that is covered by its open subspaces $\{X(\widehat{\mathcal{H}_g^*}) : g \in \mathcal{G}\}$. As a consequence of this we have

$$\mathrm{I}_\beta(\mathcal{H}) = \bigcup\{\mathrm{I}_\beta(\widehat{\mathcal{H}_g^*}) : g \in \mathcal{G}\} \tag{‡}$$

for any ordinal β. Since $X(\widehat{\mathcal{H}_g^*})$ is homeomorphic to $X(\mathcal{H}_g^*)$ we obviously have from fact 1.3.18.1 that $\mathrm{ht}(X(\mathcal{H})) = \mu^+$.

Our aim now is to determine the sizes of the levels $\mathrm{I}_\beta(\mathcal{H})$ of the LCS* space $X(\mathcal{H})$. To simplify notation, for each $g \in \mathcal{G}$ we shall denote the maps $\mathrm{k}_{\mathcal{H}_g^*}$ and $\chi_{\mathcal{H}_g^*}$, both defined in 1.3.17, by k_g and χ_g, respectively.

Claim 1.3.18.2. *$|\mathrm{I}_\beta(\mathcal{H})| = \kappa$ whenever $\alpha \leq \beta < \mu^+$.*

Proof. Let γ be the ordinal such that $\alpha + \gamma = \beta$. Then for every $g \in \mathcal{G}$ with $\Gamma(g) \geq \gamma$ we can apply lemma 1.3.15 to the family $\mathcal{H}_g$ to conclude that $\mathrm{I}_\beta({\mathcal{H}_g}^*) = \mathrm{I}_\gamma(\mathcal{H}_g) \neq \emptyset$.

On the other hand, for any $G \in \mathcal{H}_g$ we have $G \in (L_g)^\rho$ and so G is cofinal in L_g, while $L_g \cap L_h$ is bounded in both L_g and in L_h whenever $\{g,h\} \in \big[\mathcal{G}\big]^2$. So if $G \in \mathcal{H}_g$ and $H \in \mathcal{H}_h$ are arbitrary then

$$\chi_g(G) \cap \chi_h(H) \subset \chi_g(L_g \cap L_h) = \mathrm{k}_g''(L_g \cap L_h)$$

implies $\chi_g(G) \neq \chi_h(H)$. In other words, the families

$$\{\chi_g(G) : G \in \mathcal{H}_g\} = \mathrm{I}_{\geq\alpha}(\widehat{\mathcal{H}_g^*})$$

are pairwise disjoint as g ranges over $\mathcal{G}$. Since we have $\mathrm{I}_\beta(\mathcal{H}) \supset \mathrm{I}_\beta(\widehat{\mathcal{H}_g^*})$ for $g \in \mathcal{G}$ by (‡), we conclude that

$$|\,\mathrm{I}_\beta(\mathcal{H})| \geq |\{g \in \mathcal{G} : \Gamma(g) \geq \gamma\}| = \kappa.$$

But by $|\mathcal{H}| = \kappa$ we must have equality here. □

Next we show, in a single step, that $|\,\mathrm{I}_\beta(\mathcal{H})| \leq \lambda$ for each $\beta < \alpha$.

Claim 1.3.18.3. $|\,\mathrm{I}_{<\alpha}(\mathcal{H})| \leq \lambda$.

Proof of 1.3.18.3. To each $S \in \mathrm{I}_{<\alpha}(\mathcal{H})$ we may assign a quadruple $F(S)$ as follows. First pick $g \in \mathcal{G}$ with $S \in \mathrm{I}_{<\alpha}(\widehat{\mathcal{H}_g^*})$. Then we have $S = \chi_g(T)$ for some $T \in \mathrm{I}_{<\alpha}(\mathcal{H}_g^*)$. By (1.3.15)(c) this set T must be bounded in L_g, so we can fix $\xi < \mu$ such that $T \subset \bigcup\{K_{g\restriction\zeta} : \zeta < \xi\}$. Let us put then

$$F(S) = \left\langle \xi,\ g \restriction \xi,\ \mathcal{F}_{\Gamma(g)} \restriction \xi,\ T \right\rangle.$$

We shall now show that F is injective, i.e. S can be recovered from the assigned quadruple $F(S)$.

Indeed, both the sequence $\langle K_{g\restriction\zeta} : \zeta < \xi\rangle$ and the ordinal $\eta = \min K_{g\restriction\xi}$ are obviously determined by the map $g \restriction \xi$. Next, the family $\mathcal{H}_g^* \restriction \eta$ is determined by the sequence $\langle K_{g\restriction\zeta} : \zeta < \xi\rangle$ and the family $\mathcal{F}_{\Gamma(g)} \restriction \xi$ because we clearly have

$$\mathcal{H}_g^* \restriction \eta = \left\{\cup\{K_{g\restriction\zeta} : \zeta \in A\} : A \in \mathcal{F}_{\Gamma(g)} \restriction \xi\right\}^*.$$

It is easy to see that the family $\mathcal{H}_g^* \restriction \eta$ determines the map $\mathrm{k}_g \restriction \eta$ and consequently $\chi_g \restriction (\mathcal{H}_g^* \restriction \eta)$ as well. But $S = \chi_g(T)$ where we have $T \in \mathcal{H}_g^* \restriction \eta$, and so we are done.

Therefore, to conclude, it suffices to prove that there are at most λ many quadruples of the form $F(S)$. To see this, first note that we have μ many choices for ξ. Next, since $\lambda^{<\mu} = \lambda$ we have λ many choices for $g \restriction \xi$. By (v) we have

$$\mathcal{F}_{\Gamma(g)} \restriction \xi \in \left[\mathcal{P}(\xi)\right]^{\leq|\xi+\omega|},$$

consequently there are at most $2^{|\xi+\omega|} \leq \lambda^{|\xi+\omega|} = \lambda$ many choices for $\mathcal{F}_{\Gamma(g)} \restriction \xi$. Finally, it is easy to see that

$$|\mathcal{H}_g^* \restriction \eta| \leq |\cup\{K_{g\restriction\zeta} : \zeta < \xi\}| \cdot |\mathcal{F}_{\Gamma(g)} \restriction \xi| \leq \mu,$$

hence, for fixed ξ, $g \restriction \xi$, and $\mathcal{F}_\Gamma(g) \restriction \xi$, there are at most μ many choices for T. All this together clearly gives us that

$$|\{F(S) : S \in \mathrm{I}_{<\alpha}(\mathcal{H})\}| = |\,\mathrm{I}_{<\alpha}(\mathcal{H})\}| \leq \lambda.$$

□

We are now almost finished with the proof of theorem 1.3.5: the LCS* space $X = X(\mathcal{H})$ satisfies $|\,\mathrm{I}_\beta(X)| = \kappa$ for all $\alpha \leq \beta < \mu^+$ and $|\,\mathrm{I}_\beta(X)| \leq \lambda$ for all $\beta < \alpha$. Thus if Y is the disjoint topological sum of λ many copies of X then Y is an LCS* space with

$$\mathrm{CS}(Y) = \langle\lambda\rangle_\alpha \frown \langle\kappa\rangle_{\mu^+}.$$

□

Proof of Theorem 1.3.4. It follows immediately from 1.3.7 and GCH that

$$\mathcal{C}_\lambda(\alpha) \subset {}^\alpha\{\lambda, \lambda^+\}.$$

The first case, $\lambda = \omega$, follows immediately from [**61**, Theorem 9] that actually implies

$${}^\alpha\{\omega, \omega_1\} \subset \mathcal{C}(\alpha)$$

in ZFC.

Now consider the second case: $\lambda > \mathrm{cf}(\lambda) = \omega$. Let $\langle\lambda_n : n < \omega\rangle$ be an increasing sequence of cardinals cofinal in λ.

Necessity: Assume $s \in \mathcal{C}_\lambda(\alpha)$ and fix an LCS* space X with cardinal sequence s. Suppose $\beta < \alpha$, $\mathrm{cf}(\beta) = \omega_1$, $\mathrm{A}_\lambda(s) \cap \beta$ is cofinal in β. We have to show that $\beta \in \mathrm{A}_\lambda(s)$. Let $\{\beta_\eta : \eta < \omega_1\} \subset \mathrm{A}_\lambda(s)$ be an increasing sequence cofinal in β. For each $x \in \mathrm{I}_\beta(X)$ let U_x be a compact open neighbourhood of x such that

$$U_x \setminus \{x\} \subset \cup\{\mathrm{I}_\xi(X) : \xi < \beta\}.$$

For each $\eta < \omega_1$ pick a point $p(x,\eta) \in U_x \cap \mathrm{I}_{\beta_\eta}(X)$. Then the sequence $\langle p(x,\eta) : \eta < \omega_1 \rangle$ converges to x. Now since $|\,\mathrm{I}_{\beta_\eta}(X)| = s(\beta_\eta) = \lambda$ for all $\eta < \omega_1$ we have that the set

$$S = \cup\{\mathrm{I}_{\beta_\eta}(X) : \eta < \omega_1\}$$

has size λ. Let $S = \cup\{S_n : n < \omega\}$ where $|S_n| = \lambda_n$ for each $n < \omega$. For each $x \in \mathrm{I}_\beta(X)$ there must be some $n < \omega$ such that

$$S_n \cap \{p(x,\eta) : \eta < \omega_1\}$$

is uncountable and so $x \in \overline{S_n}$. However, according to the GCH, we have $|\overline{S_n}| \le \lambda_n^+ < \lambda$ for each $n < \omega$, consequently

$$s(\beta) = |\,\mathrm{I}_\beta(X)| \le |\cup\{\overline{S_n} : n \in \omega\}| \le \sup_{n<\omega} \lambda_n^+ = \lambda,$$

i.e. $\beta \in \mathrm{A}_\lambda(s)$. This completes the necessary part of the second case.

Sufficiency: We first handle some specific sequences $s \in {}^\alpha\{\lambda, \lambda^+\}$. In particular, as was noted in the introduction, the constant sequence $\langle\lambda\rangle_\alpha$ is a member of $\mathcal{C}_\lambda(\alpha)$.

Claim 1.3.18.4. *If* $0 < \beta, \gamma < \omega_2$ *and* $\mathrm{cf}(\beta) < \omega_1$ *then*

$$\langle\lambda\rangle_\beta {}^\frown \langle\lambda^+\rangle_\gamma \in \mathcal{C}_\lambda(\beta+\gamma).$$

Proof of the claim. First we do the case $\gamma = 1$, that is we construct an LCS* space Z with $\mathrm{CS}(Z) = \langle\lambda\rangle_\beta {}^\frown \langle\lambda^+\rangle_1$.

If $\mathrm{cf}(\beta) = \omega$ let $\langle\beta_n : n < \omega\rangle$ be an increasing sequence converging to β. If $\beta = \rho + 1$ let $\beta_n = \rho$ for each $n < \omega$. For each $n < \omega$ and each ordinal μ with $\lambda_n \le \mu < \lambda_{n+1}$ let Z_μ be a copy of the one point compactification of an LCS* space of height β_n such that $\mathrm{CS}(Z_\mu) = \langle\lambda\rangle_{\beta_n}$ and $\{Z_\mu : \mu < \lambda\}$ is disjoint. Let $\{a_\eta : \eta < \lambda^+\}$ be a collection of almost disjoint subsets of λ such that $|a_\eta \cap (\lambda_{n+1} \setminus \lambda_n)| = 1$ for all $\eta < \lambda^+$ and $n < \omega$. (The existence of such an almost disjoint family of size λ^+ is well-known.) $\{p_\eta : \eta < \lambda^+\}$ be a set of new points and set

$$Z = \{p_\eta : \eta < \lambda^+\} \cup \bigcup\{Z_\mu : \mu < \lambda\}.$$

Let τ be the topology on Z generated by all sets which are open in any Z_μ along with all sets of the form

$$\{p_\eta\} \cup \bigcup\{Z_\mu : \mu \in a_\eta \text{ and } \mu > \lambda_m\}$$

for $\eta < \lambda^+$ and $m < \omega$. It is straightforward to show that $\langle Z, \tau\rangle$ is an LCS* space of height $\beta + 1$ with $\mathrm{CS}(Z,\tau) = \langle\lambda\rangle_\beta {}^\frown \langle\lambda^+\rangle_1$.

Now, if $\gamma > 1$, then we extend this space Z using lemma 1.3.10 with the choices $S = \{p_\eta : \eta < \lambda^+\}$, $U_{p_\eta,n} = Z_\zeta$ where $\{\zeta\} = a_\eta \cap [\lambda_n, \lambda_{n+1})$, and each Y_{p_η} as an LCS* space of height γ satisfying $\mathrm{CS}(Y_{p_\eta}) = \langle\omega\rangle_\gamma$, i. e. $|\,\mathrm{I}_\xi(Y_{p_\eta})| = \omega$ for all $\xi < \gamma$. Note that for each $\eta < \lambda^+$ we have $\delta_{p_\eta} = \beta$ here, consequently we thus obtain an LCS* space with cardinal sequence $\langle\lambda\rangle_\beta {}^\frown \langle\lambda^+\rangle_\gamma$. □

Now let $s \in {}^\alpha\{\lambda, \lambda^+\}$ be an arbitrary sequence such that $s(0) = \lambda$ and $\mathrm{A}_\lambda(s)$ is ω_1-closed in α. Our aim is to construct an LCS* space X with s as its cardinal sequence.

Let $C = \alpha \setminus \mathrm{A}_\lambda(s) = \{\gamma < \alpha : s(\gamma) = \lambda^+\}$. For each $\gamma \in C$ let $\beta_\gamma = \min\{\beta \in C : [\beta, \gamma] \subset C\}$. By the choice of s we have $\beta_\gamma > 0$ and $\mathrm{cf}(\beta_\gamma) < \omega_1$.

By claim 1.3.18.4 above, for each $\gamma \in C$ there is an LCS* space X_γ with $\mathrm{CS}(X_\gamma) = \langle\lambda\rangle_{\beta_\gamma} {}^\frown \langle\lambda^+\rangle_{(\gamma+1)\dot{-}\beta_\gamma}$.

The final X that we require is simply the disjoint topological sum of $\{X_\gamma : \gamma \in C\} \cup \{Y\}$, where $\mathrm{CS}(Y) = \langle\lambda\rangle_\alpha$.

Now consider the third case: $\mathrm{cf}(\lambda) = \omega_1$.

Necessity: Let $s \in \mathcal{C}_\lambda(\alpha)$. By 1.3.8 and the GCH we see that if $\beta + 1 < \alpha$ and $s(\beta) = \lambda$ then $s(\beta+1) \le s(\beta)^{\aleph_0} = \lambda$ as well, i. e. $\mathrm{A}_\lambda(s)$ is successor closed in α. Similarly, 1.3.9 and the GCH together guarantee that if $\beta < \alpha$, $\mathrm{cf}(\beta) = \omega$, and $\mathrm{A}_\lambda(s) \cap \beta$ is cofinal in β then $s(\beta) \le \lambda^\omega = \lambda$, that is $\mathrm{A}_\lambda(s)$ is indeed ω-closed in α. So we have completed the necessary part of this case.

The proof of sufficiency makes essential use of the following proposition that is an immediate corollary to theorem 1.3.5 .

Proposition 1.3.19. *Let λ be a cardinal with $\operatorname{cf}\lambda = \omega_1$ and satisfying $\lambda^\omega = \lambda$. Then for every ordinal $\gamma < \omega_2$ with $\operatorname{cf}\gamma = \omega_1$ we have*

$$\langle\lambda\rangle_\gamma \frown \langle\lambda^+\rangle_{\omega_2} \in \mathcal{C}_\lambda(\omega_2).$$

Now let $s \in {}^\alpha\{\lambda, \lambda^+\}$ be such that $s(0) = \lambda$ and $\mathrm{A}_\lambda(s)$ is both ω-closed and successor closed in α. Let $B = \alpha \setminus \mathrm{A}_\lambda(s) = \{\beta < \alpha : s(\beta) = \lambda^+\}$. For $\beta \in B$ let $\gamma_\beta = \min\{\gamma \in B : [\gamma, \beta] \subset B\}$. Since $s(0) = \lambda$ and $\mathrm{A}_\lambda(s)$ is both ω-closed and successor closed, we have

$$\operatorname{cf}(\gamma_\beta) = \omega_1 = \operatorname{cf}(\lambda)$$

for each $\beta \in B$.

Thus, using proposition 1.3.19, we may fix for each $\beta \in B$ an LCS* space X_β with $\mathrm{CS}(X_\beta) = \langle\lambda\rangle_{\gamma_\beta} \frown \langle\lambda^+\rangle_{\nu_\beta}$, where ν_β is chosen so as to satisfy $\beta + 1 = \gamma_\beta + \nu_\beta$. Now, let X be the disjoint topological sum of the family of spaces

$$\{Y\} \cup \{X_\beta : \beta \in B\} ,$$

where Y is an LCS* space with $\mathrm{CS}(Y) = \langle\lambda\rangle_\alpha$. Since $|B| \le \omega_1 \le \lambda$, it is clear that $\mathrm{CS}(X) = s$.

Finally, consider the case when $\operatorname{cf}(\lambda) > \omega_1$. For any $s \in \mathcal{C}_\lambda(\alpha)$ we may then use facts 1.3.8 and 1.3.9 along with the GCH to inductively show that $s(\xi) = \lambda$ for all $\xi < \alpha$, hence $s = \langle\lambda\rangle_\alpha$. Since $\langle\lambda\rangle_\alpha \in \mathcal{C}_\lambda(\alpha)$ is clear, we are done. □

1.3.1. The reduction theorem.

Proof of Theorem 1.3.2. We first prove that the condition is necessary, so fix $f \in \mathcal{C}(\alpha)$. Let us say that $\beta < \alpha$ is a *drop point* in f if for all $\gamma < \beta$ we have $f(\gamma) > f(\beta)$. Clearly f has only finitely many, say n, drop points; let $\{\beta_i : i < n\}$ enumerate all of them in increasing order. (In particular, we have then $\beta_0 = 0$.) For each $i < n$ let us set $\lambda_i = f(\beta_i)$, then we clearly have $\lambda_0 > \lambda_1 > \cdots > \lambda_{n-1}$.

For each $i < n-1$ let $\alpha_i = \beta_{i+1} - \beta_i$ be the unique ordinal such that $\beta_i + \alpha_i = \beta_{i+1}$, moreover define the sequence f_i on α_i by the stipulation

$$f_i(\xi) = f(\beta_i + \xi)$$

for all $\xi < \alpha_i$. Similarly, let $\alpha_{n-1} = \alpha - \beta_{n-1}$ be the unique ordinal such that $\beta_{n-1} + \alpha_{n-1} = \alpha$, and define f_{n-1} on α_{n-1} by the stipulation

$$f_{n-1}(\xi) = f(\beta_{n-1} + \xi)$$

for all $\xi < \alpha_{n-1}$. Now, it is obvious that we have $f = f_0 \frown f_1 \frown \cdots \frown f_{n-1}$ where $f_i \in \mathcal{C}_{\lambda_i}(\alpha_i)$ for each $i < n$.

We shall now prove that the condition is also sufficient. In fact, this will follow from the next lemma.

Lemma 1.3.20. *If $f \in \mathcal{C}(\alpha)$, $g \in \mathcal{C}(\beta)$, and $f(\nu) \ge g(0)$ for each $\nu < \alpha$ then $f \frown g \in \mathcal{C}(\alpha + \beta)$.*

Indeed, given the sequences f_i for $i < n$, this lemma enables us to inductively define for every $i = 0, \dots, n-1$ an LCS* spaces Z_i with cardinal sequence $f_0 \frown \cdots \frown f_i$ because

$$f_i(0) = \lambda_i < \lambda_{i-1} = \min\{f_j(\nu) : j < i, \nu < \alpha_j\}.$$

In particular, then we have $\mathrm{CS}(Z_{n-1}) = f_0 \frown f_1 \frown \cdots \frown f_{n-1}$. □1.3.2

Proof of lemma 1.3.20. Let Y be an LCS* space with cardinal sequence g and satisfying $\mathrm{I}_\beta(Y) = \emptyset$. Next fix LCS* spaces X_y for all $y \in \mathrm{I}_0(Y)$, each having the cardinal sequence f and satisfying $\mathrm{I}_\alpha(X_y) = \emptyset$. Assume also that the family $\{Y\} \bigcup \{X_y : y \in Y\}$ is disjoint.

We then define the space $Z = \langle Z, \tau\rangle$ as follows. Let us first set

$$Z = Y \cup \bigcup\{X_y : y \in I_0(Y)\}.$$

For any subset $V \subset Y$ we let

$$Z(V) = V \cup \bigcup\{X_y : y \in \mathrm{I}_0(Y) \cap V\},$$

moreover we put

$$\mathcal{T} = \{W : W \text{ is compact open in some } X_y\}.$$

Then the family

$$\mathcal{B} = \mathcal{T} \cup \{Z(V) \setminus \cup\mathcal{T}' : V \text{ is compact open in } Y \text{ and } \mathcal{T}' \in \left[\mathcal{T}\right]^{<\omega}\}$$

clearly covers Z and is closed under finite intersections, hence it forms a base for the topology τ on Z that it generates .

Since $\mathcal{T} \subset \tau$ and $B \cap X_y$ is open in X_y for all $B \in \mathcal{B}$, each X_y is an open subspace of Z. We also have

$$\{B \cap Y : B \in \mathcal{B}\} = \{V \subset Y : V \text{ is compact open in } Y\},$$

and the latter is a base of Y, hence Y is a closed subspace of Z. It easily follows then that any non-empty subspace $A \subset Z$ has an isolated point, hence Z is scattered. It is also easy to check that Z is Hausdorff because so are Y and all the X_y.

So to see that Z is LCS*, it remains to check that it is locally compact. Now, let V be compact open in Y, we claim that then $Z(V)$ is compact in Z. Indeed, this can be proved by a straight-forward transfinite induction on

$$\sigma(V) = \max\{\mathrm{ht}(z, Y) : z \in V\}.$$

It clearly follows from this that all members of $\mathcal{B}$ are compact in Z, hence Z is locally compact.

Note that for any isolated point y of Y we have that

$$Z(\{y\}) = \{y\} \cup X_y,$$

as a subspace of Z, is the one-point compactification of X_y. This clearly implies that $\mathrm{ht}(y, Z) = \alpha$. From this, with an easy transfinite induction, one can prove that for all points $z \in Y$ we have

$$\mathrm{ht}(z, Z) = \alpha + \mathrm{ht}(z, Y).$$

On the other hand, since each X_y is an open subspace of Z, it follows that for every point $x \in X_y$ we have

$$\mathrm{ht}(x, Z) = \mathrm{ht}(x, X_y) < \alpha.$$

Consequently, for each $\nu < \alpha$ we have

$$\mathrm{I}_\nu(Z) = \bigcup\{\mathrm{I}_\nu(X_y) : y \in \mathrm{I}_0(Y)\}.$$

This implies that $\mathrm{ht}(Z) = \alpha + \beta$, and if $\nu < \alpha$ then

$$|\,\mathrm{I}_\nu(Z)| = |\,\mathrm{I}_0(Y)| \cdot f(\nu) = g(0) \cdot f(\nu) = f(\nu).$$

Moreover, if $\eta < \beta$ then $|\,\mathrm{I}_{\alpha+\eta}(Z)| = |\,\mathrm{I}_\eta(Y)| = g(\eta)$, consequently $\mathrm{CS}(Z) = f \frown g$. $\square_{1.3.20}$

1.4. Cardinal sequences of length $\geq \omega_2$ under GCH

(This section is based on [15])

Having given a full characterization, under GCH, of the classes $\mathcal{C}(\alpha)$ for all $\alpha < \omega_2$, it is now natural to raise the following question: Can this GCH characterization be extended to longer sequences, i. e. to $\mathcal{C}(\omega_2)$ and beyond? This question, however, remains open. In fact, we do not even know if GCH implies that the constant sequence $\langle\omega_1\rangle_{\omega_2}$ belongs to $\mathcal{C}(\omega_2)$, although this is known to be consistent with GCH.

A partial answer was given in [**15**].

Definition 1.4.1. For a cardinal λ and ordinal $\delta < \lambda^{++}$ we define $\mathcal{D}_\lambda(\delta)$ as follows: if $\lambda = \omega$,

$$\mathcal{D}_\omega(\delta) = \{f \in {}^\delta\{\omega, \omega_1\} : f(0) = \omega\},$$

and if λ is uncountable,

$$\mathcal{D}_\lambda(\delta) = \{s \in {}^\delta\{\lambda, \lambda^+\} : s(0) = \lambda, s^{-1}\{\lambda\} \text{ is } < \lambda\text{-closed and successor-closed in } \delta\}.$$

In [**14**, Theorem 4.1] it was proved that if GCH holds then

$$\mathcal{C}_{\omega_1}(\delta) \subseteq \mathcal{D}_{\omega_1}(\delta),$$

and we have equality for $\delta < \omega_2$. In Theorem 1.4.3 we show that it is consistent with GCH that we have equality for each $\delta < \omega_3$.

To formulate our results we need to introduce some more notation.

Definition 1.4.2. An LCS* space X is called $\mathcal{C}_\lambda(\alpha)$-*universal* iff $\mathrm{CS}(X) \in \mathcal{C}_\lambda(\alpha)$ and for each sequence $s \in \mathcal{C}_\lambda(\alpha)$ there is an open subspace Y of X with $\mathrm{CS}(Y) = s$.

In this section we prove the following result:

Theorem 1.4.3. *If κ is an uncountable regular cardinal with $\kappa^{<\kappa} = \kappa$ and $2^\kappa = \kappa^+$ then for each $\delta < \kappa^{++}$ there is a κ-complete κ^+-c.c poset P of cardinality κ^+ such that in V^P*

$$\mathcal{C}_\kappa(\delta) = \mathcal{D}_\kappa(\delta)$$

and there is a $\mathcal{C}_\kappa(\delta)$-universal LCS* space.

How do the universal spaces come into the picture? The first idea to prove the consistency of $\mathcal{C}_\lambda(\alpha) = \mathcal{D}_\lambda(\alpha)$ is to try to carry out an iterated forcing. For each $f \in \mathcal{D}_\lambda(\alpha)$ we can try to find a poset P_f such that

$$1_{P_f} \Vdash \text{There is an LCS}^* \text{ space } X_f \text{ with cardinal sequence } f.$$

Since typically $|X_f| = \lambda^+$, if we want to preserve the cardinals and CGH we should try to find a λ-complete, λ^+-c.c. poset P_f of cardinality λ^+. In this case forcing with P_f introduces λ^+ new subsets of λ because P_f has cardinality λ^+. However $|\mathcal{D}_\lambda(\alpha)| = \lambda^{++}$! So the length of the iteration is at least λ^{++}, hence in the final model the cardinal λ will have $\lambda^+ \cdot \lambda^{++} = \lambda^{++}$ many new subsets, i.e. $2^\lambda > \lambda^+$.

A $\mathcal{C}_\lambda(\delta)$-universal space has cardinality λ^+ so we may hope that there is a λ-complete, λ^+-c.c. poset P of cardinality λ^+ such that V^P contains a $\mathcal{C}_\lambda(\delta)$-universal space. In this case $(2^\lambda)^{V^P} \leq ((|P|^\lambda)^\lambda)^V = \lambda^+$. So in the generic extension we might have GCH.

Theorem 1.4.3 yields the following characterization:

Theorem 1.4.4. *Under GCH for any sequence f of regular cardinals of length α the following statements are equivalent:*

(A) $f \in \mathcal{C}(\alpha)$ in some cardinal preserving and GCH-preserving generic-extension of the ground model.

(B) for some natural number n there are infinite regular cardinals $\lambda_0 > \lambda_1 > \cdots > \lambda_{n-1}$ and ordinals $\alpha_0, \ldots, \alpha_{n-1}$ such that $\alpha = \alpha_0 + \cdots + \alpha_{n-1}$ and $f = f_0 \frown f_1 \frown \cdots \frown f_{n-1}$ where each $f_i \in \mathcal{D}_{\lambda_i}(\alpha_i)$.

Proof. (A) clearly implies (B) by Theorem 1.3.2.

Assume now that (B) holds. Without loss of generality, we may suppose that $\lambda_{n-1} = \omega$. Since the notion of forcing defined in Theorem 1.4.3 preserves GCH, we can carry out a cardinal-preserving and GCH-preserving iterated forcing of length $n-1$, $\langle P_m : m < n-1 \rangle$, such that for $m < n-1$

$$V^{P_m} \models \mathcal{C}_{\lambda_m}(\alpha_m) = \mathcal{D}_{\lambda_m}(\alpha_m).$$

Put $k = n-2$, $\beta = \alpha_0 + \cdots + \alpha_k$ and $g = f_0 \frown f_1 \frown \cdots \frown f_k$. Since $f_m \in \mathcal{D}_{\lambda_m}(\alpha_m) \cap V$, in V^{P_k} we have $f_m \in \mathcal{C}_{\lambda_m}(\alpha_m)$ for each $m < n-1$. Hence in V^{P_k} we have $g \in \mathcal{C}(\beta)$ by [**14**, Lemma 2.2]. Also, by using [**61**, Theorem 9], we infer that $f_{n-1} \in \mathcal{C}(\alpha_{n-1})$ in ZFC. Then as $f = g \frown f_{n-1}$, in V^{P_k} we have $f \in \mathcal{C}(\alpha)$ again by [**14**, Lemma 2.2]. $\square_{1.4.4}$

For an ordinal $\delta < \kappa^{++}$ let $\mathcal{L}^\delta_\kappa = \{\alpha < \delta : \mathrm{cf}(\alpha) \in \{\kappa, \kappa^+\}\}$.

Definition 1.4.5. An LCS* space X is called $\mathcal{L}^\delta_\kappa$-good iff X has a partition $X = Y \cup^* \bigcup^* \{Y_\zeta : \zeta \in \mathcal{L}^\delta_\kappa\}$ such that

(1) Y is an open subspace of X, $\mathrm{CS}(Y) = \langle \kappa \rangle_\delta$,

(2) $Y \cup Y_\zeta$ is an open subspace of X with $\mathrm{CS}(Y \cup Y_\zeta) = \langle \kappa \rangle_\zeta \frown \langle \kappa^+ \rangle_{\delta - \zeta}$.

Clearly Theorem 1.4.3 follows immediately from Theorem 1.4.6 and Proposition 1.4.7 below.

Theorem 1.4.6. *If κ is an uncountable regular cardinal with $\kappa^{<\kappa} = \kappa$ then for each $\delta < \kappa^{++}$ there is a κ-complete κ^+-c.c poset $\mathcal{P}$ of cardinality κ^+ such that in $V^{\mathcal{P}}$ there is an $\mathcal{L}^\delta_\kappa$-good space.*

Proposition 1.4.7. *Let κ be an uncountable regular cardinal, $\delta < \kappa^{++}$ and X be an $\mathcal{L}^\delta_\kappa$-good space. Then for each $s \in \mathcal{D}_\kappa(\delta)$ there is an open subspace Z of X with $\mathrm{CS}(Z) = s$. Especially, under GCH an $\mathcal{L}^\delta_\kappa$-good space is $\mathcal{C}_\kappa(\delta)$-universal.*

Proof of Proposition 1.4.7 . Let $J = s^{-1}\{\kappa^+\} \cap \mathcal{L}^\delta_\kappa$. For each $\zeta \in J$ let

$$f(\zeta) = \min((\delta + 1) \setminus (s^{-1}\{\kappa^+\} \cup \zeta)).$$

Let

$$Z = Y \cup \bigcup\{\mathrm{I}_{<f(\zeta)}(;)Y \cup Y_\zeta : \zeta \in J\}.$$

Since $Y \cup Y_\zeta$ is an open subspace of X it follows that $\mathrm{I}_{<f(\zeta)}(Y \cup Y_\zeta)$ is an open subspace of Z. Hence for every $\alpha < \delta$

$$\begin{aligned}\mathrm{I}_\alpha(Z) &= \mathrm{I}_\alpha(Y) \cup \bigcup\{\mathrm{I}_\alpha(\mathrm{I}_{<f(\zeta)}(Y \cup Y_\zeta)) : \zeta \in J\} \\ &= \mathrm{I}_\alpha(Y) \cup \bigcup\{\mathrm{I}_\alpha(Y \cup Y_\zeta) : \zeta \in J, \zeta \le \alpha < f(\zeta)\}. \quad (1.9)\end{aligned}$$

Since $[\zeta, f(\zeta)) \subset s^{-1}\{\kappa^+\}$ for $\zeta \in J$ it follows that if $s(\alpha) = \kappa$ then $\mathrm{I}_\alpha(Z) = \mathrm{I}_\alpha(Y)$, and so

$$|\,\mathrm{I}_\alpha(Z)| = |\,\mathrm{I}_\alpha(Y)| = \kappa. \quad (1.10)$$

If $s(\alpha) = \kappa^+$, let $\zeta_\alpha = \min\{\zeta \le \alpha : [\zeta, \alpha] \subset s^{-1}\{\kappa^+\}\}$. Then $\zeta_\alpha \in J$ because $s(0) = \kappa$ and $s^{-1}\{\kappa\}$ is $< \kappa$-closed and successor-closed in δ. Thus $\zeta_\alpha \le \alpha < f(\zeta_\alpha)$ and so

$$|\,\mathrm{I}_\alpha(Z)| \ge |\,\mathrm{I}_\alpha(Y \cup Y_{\zeta_\alpha})| = \kappa^+. \quad (1.11)$$

Since $|Z| \le |X| = \kappa^+$ we have $|\,\mathrm{I}_\alpha(Z)| = \kappa^+$. Thus $\mathrm{CS}(Z) = s$. $\square_{1.4.7}$

The rest of this section is devoted to the proof of Theorem 1.4.6, so κ is an uncountable regular cardinal with $\kappa^{<\kappa} = \kappa$, and $\delta < \kappa^{++}$ is an ordinal.

If $\alpha \le \beta$ are ordinals let

$$[\alpha, \beta) = \{\gamma : \alpha \le \gamma < \beta\}. \quad (1.12)$$

We say that I is an *ordinal interval* iff there are ordinals α and β with $I = [\alpha, \beta)$. Write $I^- = \alpha$ and $I^+ = \beta$.

If $I = [\alpha, \beta)$ is an ordinal interval let $\mathrm{E}(I) = \{\varepsilon^I_\nu : \nu < \mathrm{cf}(\beta)\}$ be a cofinal closed subset of I having order type $\mathrm{cf}\,\beta$ with $\alpha = \varepsilon^I_0$ and put

$$\mathcal{E}(I) = \{[\varepsilon^I_\nu, \varepsilon^I_{\nu+1}) : \nu < \mathrm{cf}\,\beta\} \quad (1.13)$$

provided β is a limit ordinal, and let $\mathrm{E}(I) = \{\alpha, \beta'\}$ and put

$$\mathcal{E}(I) = \{[\alpha, \beta'), \{\beta'\}\} \quad (1.14)$$

provided $\beta = \beta' + 1$.

Define $\{\mathcal{I}_n : n < \omega\}$ as follows:

$$\mathcal{I}_0 = \{[0, \delta)\} \text{ and } \mathcal{I}_{n+1} = \bigcup\{\mathcal{E}(I) : I \in \mathcal{I}_n\}. \quad (1.15)$$

Put $\mathbb{I} = \bigcup\{\mathcal{I}_n : n < \omega\}$. Note that $\mathbb{I}$ is a *cofinal tree of intervals* in the sense defined in **[84]**. Then, for each $\alpha < \delta$ we define

$$\mathrm{n}(\alpha) = \min\{n : \exists I \in \mathcal{I}_n \text{ with } I^- = \alpha\}, \quad (1.16)$$

and for each $\alpha < \delta$ and $n < \omega$ we define

$$\mathrm{I}(\alpha, n) \in \mathcal{I}_n \text{ such that } \alpha \in \mathrm{I}(\alpha, n). \quad (1.17)$$

Proposition 1.4.8. *Assume that $\zeta < \delta$ is a limit ordinal. Then, there is a $j(\zeta) \in \omega$ and an interval $J(\zeta) \in \mathcal{I}_{j(\zeta)}$ such that ζ is a limit point of $E(J(\zeta))$. Also, we have $\mathrm{n}(\zeta) - 1 \le j(\zeta) \le \mathrm{n}(\zeta)$, and $j(\zeta) = n(\zeta)$ if $cf(\zeta) = \kappa^+$.*

Proof. Clearly $j(\zeta)$ and $\mathrm{J}(\zeta)$ are unique if defined.

If there is an $I \in \mathcal{I}_{\mathrm{n}(\zeta)}$ with $I^+ = \zeta$ then $J(\zeta) = I$, and so $j(\zeta) = \mathrm{n}(\zeta)$. If there is no such I, then ζ is a limit point of $\mathrm{E}(\mathrm{I}(\zeta, \mathrm{n}(\zeta) - 1))$, so $J(\zeta) = \mathrm{I}(\zeta, \mathrm{n}(\zeta) - 1)$ and $j(\zeta) = \mathrm{n}(\zeta) - 1$.

Assume now that $\mathrm{cf}(\zeta) = \kappa^+$. Then $\zeta \in \mathrm{E}(\mathrm{I}(\zeta, \mathrm{n}(\zeta) - 1))$, but $|\,\mathrm{E}(\mathrm{I}(\zeta, \mathrm{n}(\zeta) - 1)) \cap \zeta| \leq \kappa$, so ζ can not be a limit point of $\mathrm{E}(\mathrm{I}(\zeta, \mathrm{n}(\zeta) - 1))$. Therefore, it has a predecessor ξ in $\mathrm{E}(\mathrm{I}(\zeta, \mathrm{n}(\zeta) - 1))$, i.e $[\xi, \zeta) \in \mathcal{I}_{\mathrm{n}(\zeta)}$, and so $J(\zeta) = [\xi, \zeta)$ and $j(\zeta) = \mathrm{n}(\zeta)$. □

Example 1.4.9. *Put* $\delta = \omega_2 \cdot \omega_2 + 1$. *We define*

$E([0, \delta)) = \{0, \omega_2 \cdot \omega_2\}$,

$E([0, \omega_2 \cdot \omega_2)) = \{\omega_2 \cdot \xi : 0 \leq \xi < \omega_2\}$,

$E([\omega_2 \cdot \xi, \omega_2 \cdot (\xi + 1))) = \{\zeta : \omega_2 \cdot \xi \leq \zeta < \omega_2 \cdot (\xi + 1)\}$,

$E(\{\zeta\}) = \{\zeta\}$ *for each* $\zeta \leq \omega_2 \cdot \omega_2$.

Then, we have $\mathrm{n}(\omega_2 \cdot \omega_2) = 1$, $\mathrm{n}(\omega_2 \cdot \omega_1) = 2$, $\mathrm{n}(\omega_2 \cdot \omega_1 + \omega) = 3$. *Also, we have* $j(\omega_2 \cdot \omega_2) = \mathrm{j}(\omega_2 \cdot \omega_1) = 1$ *and* $J(\omega_2 \cdot \omega_2) = J(\omega_2 \cdot \omega_1) = [0, \omega_2 \cdot \omega_2)$.

If $\mathrm{cf}(J(\zeta)^+) \in \{\kappa, \kappa^+\}$, we denote by $\{\epsilon_\nu^\zeta : \nu < \mathrm{cf}(J(\zeta)^+)\}$ the increasing enumeration of $\mathrm{E}(J(\zeta))$, i.e. $\epsilon_\nu^\zeta = \varepsilon_\nu^{J(\zeta)}$ for $\nu < \mathrm{cf}(J(\zeta)^+)$.

Now if $\zeta < \delta$, we define the *basic orbit* of ζ (with respect to $\mathbb{I}$) as

$$\mathrm{o}(\zeta) = \bigcup \{(\mathrm{E}(\mathrm{I}(\zeta, m)) \cap \zeta) : m < \mathrm{n}(\zeta)\}. \tag{1.18}$$

Note that this is the notion of orbit used in [**84**] in order to construct by forcing an LCS* space X such that $\mathrm{CS}(X) = \langle \kappa \rangle_\eta$ for any specific regular cardinal κ and any ordinal $\eta < \kappa^{++}$. However, this notion of orbit can not be used to construct an LCS* space X such that $\mathrm{CS}(X) = \langle \kappa \rangle_{\kappa^+} {}^\frown \langle \kappa^+ \rangle$. To check this point, assume on the contrary that such a space X can be constructed by forcing from the notion of a basic orbit. Then, since the basic orbit of κ^+ is $\{0\}$, we have that if x, y are any two different elements of $I_{\kappa^+}(X)$ and U, V are basic neighbourhoods of x, y respectively, then $U \cap V \subset I_0(X)$. But then, we deduce that $|I_1(X)| = \kappa^+$.

However, we will show that a refinement of the notion of basic orbit can be used to proof Theorem 1.6.

If $\zeta < \delta$ with $\mathrm{cf}\,\zeta \geq \kappa$, we define the *extended orbit* of ζ by

$$\overline{\mathrm{o}}(\zeta) = \mathrm{o}(\zeta) \cup (\mathrm{E}(J(\zeta)) \cap \zeta). \tag{1.19}$$

Consider the tree of intervals defined in Example-2.2. Then, we have $\mathrm{o}(\omega_2 \cdot \omega_1) = \overline{o}(\omega_2 \cdot \omega_1) = \{\omega_2 \cdot \xi : 0 \leq \xi < \omega_1\}$, $o(\omega_2 \cdot \omega_2) = \{0\}$, $\overline{o}(\omega_2 \cdot \omega_2) = \{\omega_2 \cdot \xi : 0 \leq \xi < \omega_2\}$.

Note that if $\zeta < \delta$, the basic orbit of ζ is a set of cardinality at most κ (see [**84**, Proposition 1.3]). Then, it is easy to see that for any $\zeta < \delta$ with $\mathrm{cf}\,\zeta \geq \kappa$, the extended orbit of ζ is a cofinal subset of ζ of cardinality $\mathrm{cf}\,\zeta$.

In order to define the desired notion of forcing, we need some preparations. The underlying set of the desired space will be the union of a collection of blocks.

Let

$$\mathbb{B} = \{S\} \cup \{\langle \zeta, \eta \rangle : \zeta < \delta, \mathrm{cf}\,\zeta \in \{\kappa, \kappa^+\}, \eta < \kappa^+\}. \tag{1.20}$$

Let

$$B_S = \delta \times \kappa \tag{1.21}$$

and

$$B_{\zeta,\eta} = \{\langle \zeta, \eta \rangle\} \times [\zeta, \delta) \times \kappa \tag{1.22}$$

for $\langle \zeta, \eta \rangle \in \mathbb{B} \setminus \{S\}$.

Let

$$X = \bigcup \{B_T : T \in \mathbb{B}\}. \tag{1.23}$$

The underlying set of our space will be X. We should produce a partition $X = Y \cup^* \bigcup^* \{Y_\zeta : \zeta \in \mathcal{L}_\kappa^\delta\}$ such that

(1) Y is an open subspace of X with $\mathrm{CS}(Y) = \langle \kappa \rangle_\delta$,

(2) $Y \cup Y_\zeta$ is an open subspace of X with $\mathrm{CS}(Y \cup Y_\zeta) = \langle \kappa \rangle_\zeta {}^\frown \langle \kappa^+ \rangle_{\delta-\zeta}$.

We will have $Y = B_S$, $Y_\zeta = \bigcup\{B_{\zeta,\eta} : \eta < \kappa^+\}$ for $\zeta \in \mathcal{L}^\delta_\kappa$.

Let

$$\pi : X \longrightarrow \delta \text{ such that } \begin{array}{l} \pi(\langle \alpha, \nu \rangle) = \alpha, \\ \pi(\langle \zeta, \eta, \alpha, \nu \rangle) = \alpha. \end{array} \tag{1.24}$$

Let

$$\pi_- : X \longrightarrow \delta \text{ such that } \begin{array}{l} \pi_-(\langle \alpha, \nu \rangle) = \alpha, \\ \pi_-(\langle \zeta, \eta, \alpha, \nu \rangle) = \zeta. \end{array} \tag{1.25}$$

Define

$$\pi_B : X \longrightarrow \mathbb{B} \text{ by the formula } x \in B_{\pi_B(x)}. \tag{1.26}$$

Define the *block orbit* function $\mathrm{o_B} : \mathbb{B} \setminus \{S\} \longrightarrow [\delta]^{\leq\kappa}$ as follows:

$$\mathrm{o_B}(\langle \zeta, \eta \rangle) = \begin{cases} \overline{\mathrm{o}}(\zeta) & \text{if } \mathrm{cf}\,\zeta = \kappa, \\ \mathrm{o}(\zeta) \cup \{\epsilon^\zeta_\nu : \nu < \eta\} & \text{if } \mathrm{cf}\,\zeta = \kappa^+. \end{cases} \tag{1.27}$$

That is, if $\mathrm{cf}\,\zeta = \kappa^+$ then

$$\mathrm{o_B}(\langle \zeta, \eta \rangle) = \overline{\mathrm{o}}(\zeta) \cap \epsilon^\zeta_\eta.$$

Finally we define the *orbits* of the elements of X as follows:

$$\mathrm{o}^* : X \longrightarrow [\delta]^{\leq\kappa} \text{ such that } \begin{array}{l} \mathrm{o}^*(\langle \alpha, \nu \rangle) = \mathrm{o}(\alpha), \\ \mathrm{o}^*(\langle \zeta, \eta, \alpha, \nu \rangle) = \mathrm{o_B}(\langle \zeta, \eta \rangle) \cup (\mathrm{o}(\alpha) \setminus \zeta). \end{array} \tag{1.28}$$

Let $\Lambda \in \mathbb{I}$ and $\{x, y\} \in [X]^2$. We say that Λ *isolates x from y* if

(i) $\Lambda^- < \pi(x) < \Lambda^+$,
(ii) $\Lambda^+ \leq \pi(y)$ provided $\pi_B(x) = \pi_B(y)$,
(iii) $\Lambda^+ \leq \pi_-(y)$ provided $\pi_B(x) \neq \pi_B(y)$.

Now, we define the poset $\mathcal{P} = \langle P, \leq \rangle$ as follows: $\langle A, \preceq, i \rangle \in P$ iff

(P1) $A \in [X]^{<\kappa}$.
(P2) $\preceq$ is a partial order on A such that $x \preceq y$ implies $x = y$ or $\pi(x) < \pi(y)$.
(P3) Let $x \preceq y$.
  (a) If $\pi_B(y) = \langle \zeta, \eta \rangle$ and $\zeta \leq \pi(x)$ then $\pi_B(x) = \pi_B(y)$.
  (b) If $\pi_B(y) = \langle \zeta, \eta \rangle$ and $\zeta > \pi(x)$ then $\pi_B(x) = S$.
  (c) If $\pi_B(y) = S$ then $\pi_B(x) = S$.
(P4) $\mathrm{i} : [A]^2 \longrightarrow A \cup \{\mathrm{undef}\}$ such that for each $\{x, y\} \in [A]^2$ we have

$$\forall a \in A([a \preceq x \wedge a \preceq y] \text{ iff } a \preceq \mathrm{i}\{x, y\}).$$

(P5) $\forall \{x, y\} \in [A]^2$ if x and y are $\preceq$-incomparable but $\preceq$-compatible, then $\pi(\mathrm{i}\{x, y\}) \in \mathrm{o}^*(x) \cap \mathrm{o}^*(y)$.
(P6) Let $\{x, y\} \in [A]^2$ with $x \preceq y$. Then:
  (a) If $\pi_B(x) = S$ and $\Lambda \in \mathbb{I}$ isolates x from y, then there is $z \in A$ such that $x \preceq z \preceq y$ and $\pi(z) = \Lambda^+$.
  (b) If $\pi_B(x) \neq S$, $\pi(x) \neq \pi_-(x)$ and $\Lambda \in \mathbb{I}$ isolates x from y, then there is $z \in A$ such that $x \preceq z \preceq y$ and $\pi(z) = \Lambda^+$.

The ordering on P is the extension: $\langle A, \preceq, \mathrm{i} \rangle \leq \langle A', \preceq', \mathrm{i}' \rangle$ iff $A' \subset A$, $\preceq' = \preceq \cap (A' \times A')$, and $\mathrm{i}' \subset \mathrm{i}$.

By using (P3), we obtain:

Claim 1.4.10. Assume that x, y, z and Λ are as in (P6). Then we have:
(a) If $\pi_B(x) = \pi_B(y)$, then $\pi_B(z) = \pi_B(x) = \pi_B(y)$.
(b) If $\pi_B(x) \neq \pi_B(y)$ and $\Lambda^+ < \pi_-(y)$, then $\pi_B(z) = \pi_B(x)$.
(c) If $\pi_B(x) \neq \pi_B(y)$ and $\Lambda^+ = \pi_-(y)$, then $\pi_B(z) = \pi_B(y)$.

Since $\kappa^{<\kappa} = \kappa$ implies $(\kappa^+)^{<\kappa} = \kappa^+$, we have that the cardinality of P is κ^+. Then, using the arguments of **[84]** it is enough to prove that Lemmas 1.4.11, 1.4.12 and 1.4.13 below hold.

Lemma 1.4.11 ([**15**]). *$\mathcal{P}$ is κ-complete.* □

Lemma 1.4.12 ([**15**]). *$\mathcal{P}$ satisfies the κ^+-c.c.*

Lemma 1.4.13 ([**15**]). *Assume that $p = \langle A, \preceq, \mathrm{i}\rangle \in P$, $x \in A$, and $\alpha < \pi(x)$. Then there is $p' = \langle A', \preceq', \mathrm{i}'\rangle \in P$ with $p' \leq p$ and there is $b \in A' \setminus A$ with $\pi(b) = \alpha$ such that $b \preceq' y$ iff $x \preceq y$ for $y \in A$.* □

The statement of Lemma 1.4.11 clearly holds; and the proof of lemma 1.4.13 is standard.

However, the proof of of Lemma 1.4.12 is the real burden of [**15**]. Basically it is a very complicated amalgamation argument which is 14 page long. The detailed proof can be found in [**15**].

1.5. Universal spaces

(This section is based on [16])

The notion of universal spaces was introduced in Definition 1.4.2. We conjecture that there are always universal spaces, but we can prove just the following special cases:

Theorem 1.5.1 ([**16**]).

(1) There is a $\mathcal{C}_\omega(\omega_1)$-universal LCS space.*
(2) (GCH) For every infinite cardinal λ and every ordinal $\delta < \omega_2$, there is a $\mathcal{C}_\lambda(\delta)$-universal LCS space.* □

The proofs of the above statements are based on a carefull analysis of the construction of LCS* spaces in [**61**] and in [**14**]. The details can be found in [**16**].

What about $\mathcal{C}_\omega(\omega_2)$? Baumgartner and Shelah introduced the notion of Δ-functions in [**25**, Section 8]. In that paper they also proved that (a) *the existence of a Δ-function is consistent with ZFC + GCH*, (b) *if there is a Δ-function then $\langle\omega\rangle_{\omega_2} \in \mathcal{C}_\omega(\omega_2)$ holds in a* "natural" *c.c.c forcing extension of the ground model.* "Natural" means that the elements of the posets are just finite approximations of the locally compact right-separating neighbourhoods of the points of the desired space. Building on their method, Bagaria, [**21**], proved that

$$\mathcal{C}_\omega(\omega_2) \supseteq \{s \in {}^{\omega_2}\{\omega, \omega_1\} : s(0) = \omega\} \tag{†}$$

is also consistent. More precisely, he showed that if there is a Δ-function and $MA_{\aleph_2}$ holds (which is a consistent assumption), then (†) above holds.

However, $MA_{\aleph_2}$ implies $2^{\omega_0} \geq \omega_3$, and if $2^{\omega_0} = \omega_\alpha$, then the natural "upper bound" of $\mathcal{C}_\omega(\omega_2)$ is a much larger family of sequences:

$$\mathcal{C}_\omega(\omega_2) \subseteq \{s \in {}^{\omega_2}\{\omega_\nu : \nu \leq \alpha\} : s(0) = \omega\}. \tag{‡}$$

These results naturally raised the following questions.

Problem 1. *Does $\langle\omega\rangle_{\omega_2} \in \mathcal{C}_\omega(\omega_2)$ imply (†), or even*

$$\mathcal{C}_\omega(\omega_2) \supseteq \{s \in {}^{\omega_2}\{\omega, \omega_1, \omega_2\} : s(0) = \omega\}. \tag{$*$}$$

Although these questions are still open we proved Theorem 1.5.10 claiming that if there is a "natural" poset P such that $\langle\omega\rangle_{\omega_2} \in \mathcal{C}_\omega(\omega_2)$ holds in V^P, then there is a natural poset Q such that $(*)$ holds in V^Q. Moreover,

Theorem 1.5.2 ([**16**]). Con*(ZFC)* $\longrightarrow$ Con*(ZFC + $2^\omega = \omega_2$ + there is a $\mathcal{C}_\omega(\omega_2)$-universal LCS* space witnessing that $\mathcal{C}_\omega(\omega_2)$ is as large as possible, i.e.*

$$\mathcal{C}_\omega(\omega_2) = \{s \in {}^{\omega_2}\{\omega, \omega_1, \omega_2\} : s(0) = \omega\}.$$

Before proving Theorem 1.5.2 we need some preparation.

Let $T_0 = \{0\} \times \omega$, $T_\alpha = \{\alpha\} \times \omega_2$ for $1 \leq \alpha < \omega_2$, and

$$T = \bigcup\{T_\alpha : \alpha < \omega_2\}.$$

Let $\pi : T \to \omega_2$ be the natural projection: $\pi(\langle\alpha, \xi\rangle) = \alpha$.

Definition 1.5.3. Define the poset $\mathcal{P}^* = \langle P^*, \preceq \rangle$ as follows. The underlying set P^* consists of triples $p = \langle a_p, \leq_p, i_p \rangle$ satisfying the following requirements:

(1) $a_p \in \big[T\big]^{<\omega}$,
(2) $\leq_p$ is a partial ordering on a_p with the property that if $x <_p y$ then $x \in \omega_2 \times \omega$ and $\pi(x) < \pi(y)$,
(3) $i_p : \big[a_p\big]^2 \to \big[a_p\big]^{<\omega}$ is such that
  (3.1) if $\{x, y\} \in \big[a_p\big]^2$ then
    (3.1.1) if $x, y \in \omega_2 \times \omega$ and $\pi(x) = \pi(y)$ then $i_p\{x, y\} = \emptyset$,
    (3.1.2) if $x <_p y$ then $i_p\{x, y\} = \{x\}$.
  (3.2) if $\{x, y\} \in \big[a_p\big]^2$ and $z \in a_p$ then
    $\big((z \leq_p x \wedge z \leq_p y) \text{ iff } \exists t \in i\{x, y\}\ z \leq_p t\big)$.

Set $p \preceq q$ iff $a_p \supseteq a_q$, $\leq_p \restriction a_q = \leq_q$ and $i_p \restriction \big[a_q\big]^2 = i_q$.

Let $P^*_\omega = \{p \in P^* : a_p \subseteq \omega_2 \times \omega\}$ and $\mathcal{P}^*_\omega = \langle P^*_\omega, \preceq \rangle$.

Consider a function $d : \big[\omega_2\big]^2 \to \big[\omega_2\big]^{\leq\omega}$. An element $p \in P^*_\omega$ is *d-good* iff

$(\star_d)$ if $\{x, y\} \in \big[a_p\big]^2$, $\pi(x) < \pi(y)$ and $x \not<_p y$ then

$$\pi'' i_p\{x, y\} \subseteq d(\pi(x), \pi(y)).$$

Let P^*_d be the family of d-good elements of P^*_ω and put $\mathcal{P}^*_d = \langle P^*_d, \preceq \rangle$.

Observation 1.5.4. *Our poset P^*_d is just "the poset P defined from d" in* [**25**, Section 7]. *(Stipulation $(\star_d)$ corresponds to (3.1.3), the other requirements have the same numbering here as in* [**25**]*.)*

A condition $r \in P$ is an *amalgamation of conditions p and q* iff $r \prec p$ and $r \prec q$, $a_r = a_p \cup a_q$, and $\leq_r$ is the partial ordering on a_r generated by $\leq_p \cup \leq_q$. Let $Q \subseteq P^*$. The poset $\mathcal{Q} = \langle Q, \prec \rangle$ has the *amalgamation property* iff every uncountable subset of Q contains two elements which have an amalgamation in Q.

Clearly the amalgamation property implies the countable chain condition.

Baumgartner and Shelah proved, [**25**, Theorem 8.1], that *if $d : \big[\omega_2\big]^2 \to \big[\omega_2\big]^{\leq\omega}$ is a Δ-function then $\mathcal{P}^*_d$ has the countable chain condition.* Actually, they proved the following:

Proposition* 1.5.5. *If d is a Δ-function then $\mathcal{P}^*_d$ has the amalgamation property.*

For a condition $p \in P^*$ and $x \in T \setminus a_p$ define the condition $q = p \uplus \{x\}$, as follows. Let $a_q = a_p \cup \{x\}$. Put $u \leq_q t$ iff $u \leq_p t$ or $u = t = x$. Let $i_q\{u, t\} = i_p\{u, t\}$ unless u or t is x. Let $i_q\{u, x\} = \emptyset$. Clearly $q = p \uplus \{x\} \in P^*$.

Let $P' \subseteq P^*$. The poset $\mathcal{P}' = \langle P', \preceq \rangle$ has the *density property D* (or the *density property D_ω*) iff $p \uplus \{x\} \in P$ for each $p \in P$ and for each $x \in (T) \setminus a_p$ (or for each $x \in (\omega_2 \times \omega) \setminus a_p$, respectively).

For $p \in P^*$, $y \in a_p$ and $x \in (\omega_2 \times \omega) \setminus a_p$ with $\pi(x) < \pi(y)$ define the condition $q = p \uplus_y \{x\}$ as follows. Let $a_q = a_p \cup \{x\}$. Put $u \leq_q t$ iff $u \leq_p t$, or $u = x$ and $y \leq_p t$. Let $i_q\{u, t\} = i_p\{u, t\}$ unless u or t, say t, is x. Let $i_q\{x, u\} = x$ if $x \leq_q u$ and $i_q\{x, u\} = \emptyset$ otherwise.

Since x is a minimal element in $\leq_q$ we have $q = p \uplus_y \{x\} \in P^*$.

Let $P \subseteq P^*$. The poset $\mathcal{P} = \langle P, \prec \rangle$ has the *density property E* iff $p \uplus_x \{y\} \in P$ for each $p \in P$, $x \in a_p$ and $y \in (\omega_2 \times \omega) \setminus a_p$ with $\pi(y) < \pi(x)$.

The following claim is straightforward from the definition of $\mathcal{P}^*_d$.

Proposition 1.5.6. *For each function $d : \big[\omega_2\big]^2 \to \big[\omega_2\big]^{\leq\omega}$ the poset $\mathcal{P}^*_d$ has the density properties D_ω and E.*

Definition 1.5.7. (a) Let $Q \subseteq P^*_\omega$. We say that the poset $\mathcal{Q} = \langle Q, \preceq \rangle$ is a *BS-poset* iff $\mathcal{Q}$ has the amalgamation property, and the density properties D_ω and E.

(b) Let $P \subseteq P^*$. We say that the poset $\mathcal{P} = \langle P, \preceq \rangle$ is a *U-poset* iff $\mathcal{P}$ has the amalgamation property, and the density properties D and E.

In [**25**, Section 9] Baumgartner and Shelah also proved that

Proposition 1.5.8. *It is consistent that $2^\omega \leq \omega_2$ and there is a Δ-function d.*

Putting together Propositions 1.5.5, 1.5.6 and 1.5.8 we obtain

Proposition 1.5.9. *It is consistent that $2^\omega \leq \omega_2$ and there is a BS-poset $\mathcal{Q} = \langle Q, \preceq \rangle$.*

As we will see, Theorem 1.5.2 follows almost immediately from the Proposition above and from the next two theorems.

Theorem 1.5.10. *If there is a BS-poset then there is a U-poset as well.*

Theorem 1.5.11. *If $\mathcal{P}$ is a U-poset then in $V^{\mathcal{P}}$ there is an LCS space X such that $\mathrm{CS}(X) = \langle \omega \rangle \frown \langle \omega_2 \rangle_{\omega_2} \in \mathcal{C}_\omega(\omega_2)$, and for every $s \in {}^{\omega_2}\{\omega, \omega_1, \omega_2\}$ with $s(0) = \omega$ there is an open subspace $Y \subseteq X$ with $\mathrm{CS}(Y) = s$.*

Proof of Theorem 1.5.2 from 1.5.9, 1.5.10, and 1.5.11. By Proposition 1.5.9 and Theorem 1.5.10 we can assume that in the ground model we have $2^\omega \leq \omega_2$ and there is a U-poset $\mathcal{P} = \langle P, \preceq \rangle$. We show that the model $V^{\mathcal{P}}$ satisfies the requirements.

Since $|P| = \omega_2$, $\mathcal{P}$ satisfies c.c.c and $2^\omega \leq \omega_2$, we have $(2^\omega)^{V^{\mathcal{P}}} \leq ((|\mathcal{P}|^\omega)^\omega)^V = \omega_2$.

By Theorem 1.5.11, in $V^{\mathcal{P}}$ there is an LCS space X such that $\mathrm{CS}(X) \in \mathcal{C}_\omega(\omega_2)$ and

$$\{\mathrm{CS}(Y) : Y \subseteq X \text{ is open } \} \supseteq \{s \in {}^{\omega_2}\{\omega, \omega_1, \omega_2\} : s(0) = \omega\}. \tag{$\bullet$}$$

Since $|X| \geq \omega_2$ and $|\,\mathrm{I}_0(X)| = \omega$, we have $2^\omega \geq \omega_2$ in $V^{\mathcal{P}}$. So $2^\omega = \omega_2$ in $V^{\mathcal{P}}$.

Thus

$$\mathcal{C}_\omega(\omega_2) \subseteq \{s \in {}^{\omega_2}\{\omega, \omega_1, \omega_2\} : s(0) = \omega\}. \tag{$\circ$}$$

But ($\bullet$) and ($\circ$) together yield that

$$\mathcal{C}_\omega(\omega_2) = \{s \in {}^{\omega_2}\{\omega, \omega_1, \omega_2\} : s(0) = \omega\},$$

and that X is a $\mathcal{C}_\omega(\omega_2)$-universal space. $\square_{1.5.2}$

Proof of Theorem 1.5.11. Let $\mathcal{G}$ be a $\mathcal{P}$-generic filter. Recall that $T_0 = \{0\} \times \omega$, $T_\alpha = \{\alpha\} \times \omega_2$ for $1 \leq \alpha < \omega_2$, and $T = \bigcup\{T_\alpha : \alpha < \omega_2\}$ Let

$$\leq_{\mathcal{G}} = \bigcup\{\leq_p : p \in \mathcal{G}\}$$

and for each $x \in T$ put

$$U(x) = \{z \in T : z \leq_{\mathcal{G}} x\}.$$

Let $X = \langle T, \tau \rangle$ be the LCS* space generated by the family $\{U(x) : x \in T\}$. Density properties D and E imply that $\mathrm{I}_\alpha(X) = T_\alpha$ for $\alpha < \omega_2$.

Let $s \in {}^{\omega_2}\{\omega, \omega_1, \omega_2\}$ with $s(0) = \omega$. Put

$$Y = \{\langle \alpha, \xi \rangle : \alpha < \omega_2, \xi < s(\alpha)\}.$$

If $x \in \mathrm{I}_\alpha(X)$ then $U(x) \setminus \{x\} \subseteq \alpha \times \omega$. Hence

$$Y = \bigcup\{U(y) : y \in Y\},$$

therefore Y is open. Thus $\mathrm{I}_\alpha(Y) = \mathrm{I}_\alpha(X) \cap Y$, that is, $\mathrm{I}_\alpha(Y) = \{\alpha\} \times s(\alpha)$. Thus the cardinal sequence of Y is exactly s. $\square_{1.5.11}$

Proof of Theorem 1.5.10. Fix an injective function $\varphi : \omega_2 \times \omega_2 \xrightarrow{1-1} \omega_2 \times \omega$ such that

(A) if $\pi(x) < \pi(y)$ and $x \in \omega_2 \times \omega$ then $\pi(\varphi(x)) < \pi(\varphi(y))$,
(B) if $x \neq y$ then $\pi(\varphi(x)) = \pi(\varphi(y))$ iff $\pi(x) = \pi(y)$ and $x, y \in \omega_2 \times \omega$.

"Lift" this φ to a function $\varphi : P^* \to P^*_\omega$ in the natural way: for $p = \langle a_p, \leq_p, i_p\rangle \in P^*$ let $\varphi(p) = \left\langle a_{\varphi(p)}, \leq_{\varphi(p)}, i_{\varphi(p)}\right\rangle$, where $a_{\varphi(p)} = \varphi'' a_p$, $\varphi(x) \leq_{\varphi(p)} \varphi(y)$ iff $x \leq_p y$ and $i_{\varphi(p)}\{\varphi(x), \varphi(y)\} = \varphi'' i_p\{x, y\}$.

Let $\mathcal{Q} = \langle Q, \preceq\rangle$ be a BS-poset. Take

$$P = \{p \in P^* : \varphi(p) \in Q\}$$

and $\mathcal{P} = \langle P, \preceq\rangle$.

We claim that $\mathcal{P}$ is a U-poset. For the details see [**16**].

$\square_{1.5.10}$

1.6. A lifting theorem

(This section is based on [18])

To simplify the formulation of the forthcoming results denote by $\mathcal{THIN}(\alpha)$ the statement that there is an LCS* space with cardinal sequence $\langle\omega\rangle_\alpha$.

Baumgartner and Shelah proved that if there is a Δ-function then $\mathcal{THIN}(\omega_2)$ holds in a *natural* c.c.c forcing extension, where the meaning of "natural" was explained in the previous section. Building on their method, but using much more involved combinatorics, Martinez [**84**] proved that if there is a strong Δ-function, then for each $\delta < \omega_3$ there is a c.c.c poset P_δ such that $\mathcal{THIN}(\delta)$ holds in V^{P_δ}. These results naturally raised the following problem.

Problem 2. *Does $\mathcal{THIN}(\omega_2)$ imply $\mathcal{THIN}(\delta)$ for each $\delta < \omega_3$?*

Although this question remains still open we prove a "lifting theorem" (Theorem 1.6.21) claiming that if there is a natural poset P_{ω_2} such that $\mathcal{THIN}(\omega_2)$ holds in $V^{P_{\omega_2}}$, then for each $\delta < \omega_3$ there is a natural poset P_δ such that $\mathcal{THIN}(\delta)$ holds in V^{P_δ}: the posets used by Martinez can be constructed directly from the poset applied by Baumgartner and Shelah without even mentioning the Δ-function. Moreover, our lifting theorem works for each cardinal κ, not only for ω_2! Since there is no Δ-function on ω_3 you can not expect to apply the method of Baumgartner and Shelah to prove $\mathcal{THIN}(\omega_3)$. However, if anybody can construct a "natural" c.c.c poset P such that $\mathcal{THIN}(\omega_3)$ holds in V^P then our theorem gives immediately the consistency of $\mathcal{THIN}(\alpha)$ for each $\alpha < \omega_4$.

To formulate our statement more precisely we introduce some notation, so we postpone the formulation of our main result till theorem 1.6.15.

First we recall some definition and results.

To formulate our result we neeed some preparation. Fix a cardinal $\kappa \geq \omega$ and let $\pi : \kappa^+ \times \omega \longrightarrow \kappa^+$ be the natural projection: $\pi(\langle\alpha, n\rangle) = \alpha$.

Define the poset $P^0 = \left\langle P^0, \prec\right\rangle$ as follows. The underlying set P^0 consists of triples $\langle a, \leq, i\rangle$ satisfying the following requirements:

(i) $a \in \left[\kappa^+ \times \omega\right]^{<\omega}$,
(ii) $\leq$ is a partial ordering on a,
(iii) $\forall\{x, y\} \in \left[a\right]^2$ if $x \leq y$ the $\pi(x) < \pi(y)$,
(iv) $i : \left[a\right]^2 \longrightarrow \mathcal{P}(a)$ is a function,
(v) $\forall\{x, y\} \in \left[a\right]^2$ if $\pi(x) = \pi(y)$ then $i\{x, y\} = \emptyset$,
(vi) $\forall\{x, y\} \in \left[a\right]^2$ if $x \leq y$ then $i\{x, y\} = \{x\}$.

Write $p = \langle a^p, \leq^p, i^p\rangle$ for $p \in P^0$. Define the function $h^p : a^p \longrightarrow \mathcal{P}(a^p)$ by the formula $h^p(x) = \{y \in a^p : y \leq^p x\}$. For $b \subset a^p$ write $h^p[b] = \bigcup\{h^p(x) : x \in b\}$.

Let $p \prec q$ iff $a^q \subset a^p$,
$\leq^q = \leq^p \cap (a^q \times a^q)$,
$i^q \subset i^p$.

Clearly $\prec$ is a partial ordering on P^0.

Let

$$P^* = \big\{\langle a, \leq, i\rangle \in P^0 : \forall \{x,y\} \in \big[a\big]^2\ \forall z \in a \\ (z \leq x \wedge z \leq y) \text{ iff } \exists t \in i\{x,y\}\ z \leq t.\big\}$$

Fact 1.6.1. *For $p \in P^0$,*

$$p \in P^* \text{ iff } \forall \{x,y\} \in \big[a^p\big]^2\ h^p(x) \cap h^p(y) = h^p[i^p\{x,y\}].$$

The elements of P^* can be considered as the natural finite approximations of an LCS*-order on $\kappa^+ \times \omega$ and the witnessing function i.

Definition 1.6.2. Two condition $p, q \in P^0$ are *twins* iff (i)–(ii) below hold, where $a = a^p \cap a^q$:

(i) $\leq^p \restriction a = \leq^q \restriction a$,
(ii) $i^p \restriction \big[a\big]^2 = i^q \restriction \big[a\big]^2$.

Definition 1.6.3. Let $p, q \in P^0$ be twins. A condition $r \in P^0$ is an *amalgamation of p and q* iff

(a) $a^r = a^p \cup a^q$
(b) $\leq^r$ is the partial ordering on a^r generated by $\leq^p \cup \leq^q$,
(c) $i^r \supset i^p \cup i^q$.

Let

$$\mathfrak{amalg}(p,q) = \{r : r \text{ is an amalgamation of } p \text{ and } q\}.$$

When we speak about amalgamations of two conditions we will always assume that these conditions are twins.

Fact 1.6.4. *If $r \in P^0$ is an amalgamation of p and q, then*

(1) $\leq^r \restriction a^p = \leq^p$,
(2) $r \prec p$ and $r \prec q$,
(3) If $x \in a^p$ and $y \in a^q$ then $x \leq^r y$ iff there is $z \in a^p \cap a^q$ such that $x \leq^p z \leq^q y$.

Fact 1.6.5. *If $r \in P^0$ is an amalgamation of p and q, moreover $p, q \in P^*$ then*

$$\forall \{x,y\} \in \big[a^p\big]^2 \cup \big[a^q\big]^2\ h^r(x) \cap h^r(y) = h^r[i\{x,y\}].$$

Proof. See in [**18**]. □

For $A \subset \kappa^+$ let

$$P^*_A = \{p \in P^* : a^p \subset A \times \omega\}.$$

Next we introduce three properties, (K^+), D^A_1 and D^A_2, of posets $\langle P, \prec\rangle$, where $P \subset P^*_A$ for some $A \subset \kappa^+$. The first one is a strong version of property (K), the two others are density requirements.

Definition 1.6.6. Let $P \subset P^*$. The poset $\mathcal{P} = \langle P, \prec\rangle$ has *property (K^+)* iff

$$\forall S \in \big[P\big]^{\omega_1}\ \exists T \in \big[S\big]^{\omega_1}\ \forall \{p,q\} \in \big[T\big]^2\ p \text{ and } q \text{ have an amalgamation in } P.$$

Definition 1.6.7. For a condition $p \in P^0$ and $x \in (\kappa^+ \times \omega) \setminus a^p$ define $q = p \uplus \{x\} \in P^0$ as follows:

- $a^q = a^p \cup \{x\}$,
- $\leq^q = \leq^p \cup \{\langle x, x\rangle\}$,
- $i^p \subset i^q$,
- $i^q\{x,y\} = \emptyset$ for $y \in a^p$.

Fact 1.6.8. *$p \uplus \{x\} \in P^*_A$ for each $p \in P^*_A$ and $x \in (A \times \omega) \setminus a^p$.*

Definition 1.6.9. Let $P \subset P^*_A$. The poset $\mathcal{P} = \langle P, \prec\rangle$ has property D^A_1 iff

$$p \uplus \{x\} \in P \text{ for each } p \in P \text{ and } x \in (A \times \omega) \setminus a^p.$$

Definition 1.6.10. For $p \in P_A$, $x \in a^p$, $y_0, y_1, \ldots, y_{n-1} \in (A \times \omega) \setminus a^p$ with $\pi(y_0) < \pi(y_1) < \ldots \pi(y_{n-1}) < \pi(x)$ define the condition $q = p \uplus_x \langle y_0, \ldots, y_{n-1}\rangle \in P^0$ as follows:

- $a^q = a^p \cup \{y_0, \ldots, y_{n-1}\}$,

- $\leq^q = \leq^p \cup \{\langle y_i, y_j : i \leq j < n\rangle\} \cup \{\langle y_i, z\rangle : z \in a^p, x \leq^p z\}$,
- $i^p \subset i^q$,
- $i^q\{y_i, y_j\} = y_{\min(i,j)}$,
- $i^q\{y_i, z\} = \begin{cases} y_i & \text{if } x \leq^p z \\ \emptyset & \text{otherwise} \end{cases}$ for $z \in a^p$.

Fact 1.6.11. *If $p \in P^*_A$, $x \in a^p$, $y_0, y_1, \ldots, y_{n-1} \in (A \times \omega) \setminus a^p$ with $\pi(y_0) < \pi(y_1) < \ldots \pi(y_{n-1}) < \pi(x)$, then $q = p \uplus_x \langle x_0, \ldots, x_{n-1}\rangle \in P^*_A$.*

Definition 1.6.12. Let $P \subset P^*_A$. The poset $\mathcal{P} = \langle P, \prec\rangle$ has property D^A_2 iff

> $\forall\{\alpha, \beta\} \in A$, $\alpha < \beta$, there is a finite set of ordinals $L^P(\alpha, \beta) = \{\alpha_0, \ldots, \alpha_{n-1}\} \in \left[A\right]^{<\omega}$ such that $\alpha = \alpha_0 < \alpha_1 < \ldots \alpha_{n-1} < \beta$ and if $p \in P$, $x \in a^p$ with $\pi(x) = \beta$ and $x_i \in (A \times \omega) \setminus a^p$ with $\pi(x_i) = \alpha_i$ for $i < n$, then $p \uplus_x \langle y_0, \ldots y_{n-1}\rangle \in P$.

Definition 1.6.13. Let $A \subset \kappa^+$ and $P \subset P^*$. The poset $P = \langle P, \prec\rangle$ is *A-nice* iff $P \subset P^*_A$ and P has properties (K^+), (D^A_1) and (D^A_2). For $\delta < \kappa^+$ let $\mathcal{NAT}(\delta)$ be the statement that there is δ-nice poset P_δ.

Proposition 1.6.14. *If a poset P is δ-nice then P has property (K) and $\mathcal{THIN}(\delta)$ holds in V^P.*

Proof. By fact 1.6.4(2) property (K^+) implies property (K). Let $G \subset P$ be a generic filter. Put $A = \bigcup\{a^p : p \in G\}$, $i = \bigcup\{i^p : p \in G\}$ and $\prec = \bigcup\{\prec^p : p \in G\}$. Then $A = \delta \times \omega$ by (D^δ_1). By Lemma 1.8.6 it is enough to prove that $\langle \delta \times \omega, \prec\rangle$ is a $\langle\delta\rangle_\omega$-poset (see definition 1.8.5).

Since $A = \delta \times \omega$, 1.8.5(1) holds. The partial ordering $\prec$ satisfies 1.8.5(2) because every $p \in P$ satisfies (iii). The function $i : \left[\delta \times \omega\right]^2 \longrightarrow \left[\delta \times \omega\right]^{<\omega}$ satisfies 1.8.5(3) because every element of P is in P^*. Finally 1.8.5(4) holds because (D^2_δ) can be applied in a suitable density argument. Thus $\langle \delta \times \omega, \prec\rangle$ is an $\langle\delta\rangle_\omega$-poset, hence there is an LCS* space with character sequence $\langle\delta\rangle_\omega$ by Lemma 1.8.6. $\square_{1.6.14}$

After this preparation we are able to formulate the main lifting theorem.

Theorem 1.6.15. *$\mathcal{NAT}(\kappa)$ implies $\mathcal{NAT}(\delta)$ for each cardinal κ and ordinal $\delta < \kappa^+$.*

First, in lemma 1.6.16 below, we show that our lifting theorem works downwards. Although $\mathcal{THIN}(\kappa)$ clearly implies $\mathcal{THIN}(\delta)$ for $\delta < \kappa$ we should prove $\mathcal{NAT}(\delta)$ for $\delta < \kappa$ as well, because we will use the posets witnessing this to prove $\mathcal{NAT}(\gamma)$ for $\gamma \geq \kappa$.

If $p \in P^0$ and $I \subset \kappa^+$ let

$$p \restriction I = \left\langle a^p \cap (I \times \omega), \leq^p \restriction (I \times \omega), i^p \restriction \left[I \times \omega\right]^2 \right\rangle.$$

Observe that

- $p \restriction I \in P^0$ iff $i^p\{x, y\} \subset I \times \omega$ for each $\{x, y\} \in \left[a^p \cap (I \times \omega)\right]^2$,
- if $p \in P^*$ and $p \restriction I \in P^0$ then $p \restriction I \in P^*$.

Lemma 1.6.16 ([18]). *$\mathcal{NAT}(\kappa)$ implies $\mathcal{NAT}(\delta)$ for $\delta < \kappa$.* $\square$

Proof of theorem 1.6.15. Since we know the statement for $\delta < \kappa$ we prove the theorem by induction on $\delta \geq \kappa$. When we constructed P_δ we will also have $P_A \subset P^*_A$ for each $A \subset \kappa^+$ with order type δ such that $\mathcal{P}_A = \langle P_A, \prec\rangle$ has properties (K^+), (D^A_1) and (D^A_2).

We will write $L^A(\alpha, \beta)$ for $L^{\mathcal{P}_A}(\alpha, \beta)$. Let $L^A(\alpha, \alpha) = \emptyset$.

Successor step:

Assume that $\mathcal{P}_\delta$ is constructed. Then we can get $\mathcal{P}_{\delta+1}$ as follows.

A p is in $P_{\delta+1}$ iff

(i) $p \in P^*_{\delta+1}$,

(ii) $p \restriction \delta \in P_\delta$,

(iii) $\forall\{x, y\} \in \left[a^p\right]^2$ if $\pi(x) < \delta$ and $\pi(y) = \delta$ then either $i\{x, y\} = x$ (i.e. $x \leq^p y$) or $i\{x, y\} = \emptyset$ (i.e. $h^p(x) \cap h^p(y) = \emptyset$).

The next lemma claims that $\mathcal{P}_{\delta+1} = \langle P_\delta, \prec\rangle$ works:

Lemma 1.6.17 ([18]). *$\mathcal{P}_{\delta+1}$ satisfies (K^+), $(D^{\delta+1}_1)$. and $(D^{\delta+1}_2)$.* $\square_{1.6.15}$

The successor step is done.

Limit step:

Assume that δ is limit ordinal, and $\mathcal{P}_A$ is constructed for each $A \subset \kappa^+$ with order type $< \delta$.
Fix a club $C \subset \delta$, $C = \{\gamma_\zeta : \zeta < \mathrm{cf}(\delta)\}$. Let $I_\zeta = [\gamma_\zeta, \gamma_{\zeta+1})$ for $\zeta < \mathrm{cf}(\delta)$.
Let $\rho : \delta \longrightarrow \mathrm{cf}(\delta)$ s.t. $\rho(\alpha) = \zeta$ iff $\alpha \in I_\zeta$.
Let $p \in P_\delta$ iff

(δ1) $p \in P^*_\delta$,
(δ2) $p \restriction C \in P_C$,
(δ3) $p \restriction I_\zeta \in P_{I_\zeta}$ for each $\zeta < \mathrm{cf}(\delta)$,
(δ4) $\forall x, y \in a^p$ if $x \leq^p y$, $\gamma_\zeta < \pi(x) < \gamma_{\zeta+1} \leq \pi(y)$ then $\exists u \in a^p$ $x \leq^p u \leq^p y$ and $\pi(u) = \gamma_{\zeta+1}$
(δ5) $\forall x, y \in a^p$ if $x \leq^p y$, $\pi(x) < \gamma_\xi \leq \pi(y) < \gamma_{\xi+1}$ then $\exists v \in a^p$ $x \leq^p v \leq^p y$ and $\pi(v) = \gamma_\xi$
(δ6) $\forall x, y \in a^p$, $\gamma_\zeta \leq \pi(x) < \gamma_{\zeta+1} \leq \gamma_\xi \leq \pi(y) < \gamma_{\xi+1}$, $x \not\leq^p y$ then

$$i^p\{x, y\} \subset \bigcup\{i^p\{u, v\} : u \leq^p x, v \leq^p y, \pi(u) = \gamma_\zeta, \pi(v) = \gamma_\xi\}.$$

We show that $\mathcal{P}_\delta = \langle P_\delta, \prec\rangle$ works, i.e. it satisfies properties (K^+), (D^δ_1) and (D^δ_2).

Lemma 1.6.18. *$\mathcal{P}_\delta$ satisfies (K^+).*

Proof. Let $\{p_\nu : \nu < \omega_1\} \in \left[P_\delta\right]^{\omega_1}$, $p_\nu = \langle a_\nu, \leq_\nu, i_\nu\rangle$, $h_\nu = h^{p_\nu}$. Let $c_\nu = \{\eta < \mathrm{cf}(\delta) : a_\nu \cap I_\eta \neq \emptyset\}$. By thinning out the sequence $\{p_\nu : \nu < \omega_1\}$ we can assume that

(a) $\{a_\nu : \nu \in \omega_1\}$ forms a Δ-system with kernel d,
(b) there is a partial ordering $\leq^d$ on d such that $\leq_\nu \restriction d = \leq^d$ for each $\nu \in \omega_1$,
(c) $\{c_\nu : \nu < \omega_1\}$ forms a Δ-system with kernel c.
(d) $\forall \eta \in c\ \exists e_\eta\ \forall \nu \in \omega_1\ a_\nu \cap (\{\gamma_\eta, \gamma_\eta + 1\} \times \omega) = e_\eta$
(e) $\forall \eta \in c\ \forall \{\nu, \mu\} \in \left[\omega_1\right]^2$ the conditions $p_\nu \restriction I_\eta$ and $p_\mu \restriction I_\eta$ have an amalgamation $r^\eta_{\nu,\mu} = \left\langle a^\eta_{\nu,\mu}, \leq^\eta_{\nu,\mu}, i^\eta_{\nu,\mu}\right\rangle$ in P_{I_η}.
(f) $\forall \{\nu, \mu\} \in \left[\omega_1\right]^2$ the conditions $p_\nu \restriction C$ and $p_\mu \restriction C$ have an amalgamation $r^C_{\nu,\mu} = \left\langle a^C_{\nu,\mu}, \leq^C_{\nu,\mu}, i^C_{\nu,\mu}\right\rangle$ in P_C.
(g) $i_\nu\{x, y\} = i_\mu\{x, y\}$ for each $\{x, y\} \in \left[d\right]^2$ and $\{\nu, \mu\} \in \left[\omega_1\right]^2$.

To ensure (g) fix $\{x, y\} \in \left[d\right]^2$. If $\rho(x) = \rho(y) = \eta$ then (g) holds by (e): $i_\nu\{x, y\} = i_\mu\{x, y\}$. If $\{\pi(x), \pi(y)\} \in \left[C\right]^2$ then $i_\nu\{x, y\} = i_\mu\{x, y\} \stackrel{\mathrm{def}}{=} i^C\{x, y\}$ by (f). If $\eta = \rho(x) \neq \rho(y) = \sigma$ then by (δ6) we have

$$i_\nu\{x, y\} \subset \bigcup\{i_\nu\{u, v\} : u \leq_\nu x, v \leq_\nu y, \pi(u) = \gamma_{\rho(x)}, \pi(v) = \gamma_{\rho(y)}\} \subset \bigcup\{i^C\{u, v\} : u \in e_{\rho(x)}, v \in e_{\rho(y)}\},$$

i.e. $i_\nu\{x, y\}$ is a subset of a fixed finite set for each $\nu \in \omega_1$. So, by thinning out our sequence we can guarantee that (g) holds.

Claim 1.6.18.1. *p_ν and p_μ are twins for each $\{\nu, \mu\} \in \left[\omega_1\right]^2$.*

Fix $\{\nu, \mu\} \in \left[\omega_1\right]^2$. Define $r = \langle a, \leq, i\rangle \in P^0$ as follows:

(r1) $a = a_\nu \cup a_\mu$,
(r2) $\leq$ is the partial ordering on a generated by $\leq_\nu \cup \leq_\mu$,

(r3)

$$i\{x,y\} = \begin{cases} i_\nu\{x,y\} & \text{if } \{x,y\} \in \left[a_\nu\right]^2, \\ i_\mu\{x,y\} & \text{if } \{x,y\} \in \left[a_\mu\right]^2, \\ i^C_{\nu,\mu}\{x,y\} & \text{if } \{x,y\} \in \left[C\right]^2, \\ i^\eta_{\nu,\mu}\{x,y\} & \text{if } \{x,y\} \in \left[I_\eta\right]^2, \\ M(x,y) & \text{otherwise,} \end{cases}$$

where

$$M(x,y) = \bigcup\{i\{u,v\} : \{u,v\} \in \left[a\right]^2, u \le x, v \le y, \pi(u) = \gamma_{\rho(x)}, \pi(v) = \gamma_{\rho(y)}\}.$$

Claim 1.6.18.2. *$r \in P^*_\delta$ is an amalgamation of p_ν and p_μ.*

Proof. The proof is quite boring and technical. See in [**18**]. □

Hence $\mathcal{P}_\delta$ satisfies (K^+). □$_{1.6.18}$

Lemma 1.6.19. *$\mathcal{P}_\delta$ satisfies (D^δ_1).*

Proof. Assume that $p \in P_\delta$ and $z \in (\delta \times \omega) \setminus a^p$. Let $q = p \uplus \{x\}$. We need to show that $q \in P_\delta$, i.e., q satisfies $(\delta 1)$–$(\delta 6)$. For further details see [**18**]. □

Lemma 1.6.20. *P_δ satisfies (D^δ_2).*

Proof. If $\{\alpha,\beta\} \in \left[I_\eta\right]^2$ for some η then let $L^\delta(\alpha,\beta) = L^{I_\eta}(\alpha,\beta)$. Otherwise, if $\alpha \in I_\eta$, $\beta \in I_\sigma$, $\eta < \sigma$, then let $\alpha^+ = \min(C \setminus \alpha + 1)$ and put

$$L^\delta(\alpha,\beta) = \{\alpha,\alpha^+\} \cup L^C(\alpha^+,\gamma_\sigma) \cup L^{I_\sigma}(\gamma_\sigma,\beta).$$

Enumerate $L^P(\alpha,\beta)$ as $\alpha = \alpha_0 < \alpha_1 < \cdots < \alpha_{n-1} < \beta$. Let $p \in P_\delta$, $z \in a^p$ with $\pi(z) = \beta$ and $z_i \in (\delta \times \omega) \setminus a^p$ with $\pi(z_i) = \alpha_i$ for $i < n$. Let $q = p \uplus_z \langle z_0, \dots z_{n-1}\rangle$.

We should show that $q \in P_\delta$, i.e. q satisfies $(\delta 1)$–$(\delta 6)$. The details of this quite long argument can be found in [**18**]. □

Thus the limit step is done as well, which completes the inductive construction, so theorem 1.6.15 is proved. □$_{1.6.15}$

Now we can conclude the section.

Theorem 1.6.21. *If there is a κ-nice poset P for some regular cardinal κ then there is a c.c.c poset Q such that $\mathcal{THIN}(\delta)$ holds in V^Q for each $\delta < \kappa^+$.*

Proof. Using theorem 1.6.15 we fix, for each $\delta < \kappa^+$, a δ-nice poset P_δ. Let Q be the finite-support product of $\{P_\delta : \delta < \kappa^+\}$. Since every P_δ has property (K), so has Q.

Let $\mathcal{G}$ be a Q-generic filter and let $\delta < \kappa^+$ be arbitrary. Then $\mathcal{G}_\delta = \{p(\delta) : p \in \mathcal{G} \wedge \delta \in \operatorname{dom} p\}$ is a P_δ-generic filter, hence $\mathcal{THIN}(\delta)$ holds in $V[\mathcal{G}_\delta]$ winessed by some space X_δ by proposition 1.6.14. Since $V[\mathcal{G}_\delta] \subset V[\mathcal{G}]$ the space X_δ witnesses $\mathcal{THIN}(\delta)$ in $V[\mathcal{G}]$. □

1.7. Wide scattered spaces and morasses

(This section is based on [19])

By Baumgartner and Shelah, [**25**], it is relatively consistent with ZFC that $\langle\omega\rangle_{\omega_2} \in \mathcal{C}(\omega_2)$. Refining their argument, first Bagaria, [**21**], proved that ${}^{\omega_2}\{\omega,\omega_1\} \subset \mathcal{C}(\omega_2)$ in some ZFC model, then we showed in [**16**]that $2^\omega = \omega_2$ and ${}^{\omega_2}\{\omega,\omega_1,\omega_2\} \subset \mathcal{C}(\omega_2)$ is also consistent.

For a long time ω_2 was a mystique barrier in both height and width. In this section we can construct wider spaces.

Theorem 1.7.1. *If GCH holds and $\lambda \geq \omega_2$ is a regular cardinal, then in some cardinal preserving generic extension $2^\omega = \lambda$ and every sequence $\mathbf{s} = \langle s_\alpha : \alpha < \omega_2 \rangle$ of infinite cardinals with $s_\alpha \leq \lambda$ is the cardinal sequence of some locally compact scattered space.*

We will find the suitable generic extension in three steps:

(I) The first extension adds a "*strongly stationary strong* (ω_1, λ)*-semimorass*" to the ground model (see Definition 1.7.2 and Theorem 1.7.4).
(II) Using that strong semimorass the second extension adds a $\Delta(\omega_2 \times \lambda)$*-function* to the first extension (see Definition 1.7.12 and Theorem 1.7.13).
(III) Using the $\Delta(\omega_2 \times \lambda)$-function we add an "*LCS* space with stem*" to the second model and we show that those2 space alone guarantees that every sequence $\mathbf{s} = \langle s_\alpha : \alpha < \omega_2 \rangle$ of infinite cardinals with $s_\alpha \leq \lambda$ is the cardinal sequence of some locally compact scattered space (see Theorem 1.7.26).

Steps (I) and (II) are based on works of P. Koszmider, see **[75]** and **[72]**.

1.7.1. Strong semimorasses. If ρ is a function and X is set, write $\rho[X] = \{\rho(\xi) : \xi \in X\}$.

If X and Y are sets of ordinals with $\mathrm{tp}(X) = \mathrm{tp}(Y)$, denote the unique order preserving bijection between X and Y by $\rho_{X,Y}$.

For $X \in \left[\lambda\right]^\omega$ and $\mathcal{F} \subset \left[\lambda\right]^\omega$ let $\mathcal{F} \restriction X = \{Y \in \mathcal{F} : Y \subsetneq X\}$.

If X, X_1 and X_2 are sets of ordinals, we write

$X = X_1 \oplus X_2$ iff $\mathrm{tp}(X_1) = \mathrm{tp}(X_2)$, $X = X_1 \cup X_2$ and $\rho_{X_1,X_2} \restriction X_1 \cap X_2 = \mathrm{id}$;
$X = X_1 \odot X_2$ iff $\mathrm{tp}(X_1) = \mathrm{tp}(X_2)$, $X = X_1 \cup X_2$ and $X_1 \cap X_2 < X_1 \setminus X_2 < X_2 \setminus X_1$;

and

$X = X_1 \otimes X_2$ iff $X = X_1 \oplus X_2$ and $X \cap \omega_2 = (X_1 \cap \omega_2) \odot (X_2 \cap \omega_2)$.

In **[75]** Koszmider introduced the notion of semimorasses and proved several properties concerning that structures. Unfortunately, in our proof we need structures with a bit stronger properties.

Definition 1.7.2. Let $\omega_1 \leq \lambda$ be a cardinal. A family $\mathcal{F} \subset \left[\lambda\right]^\omega$ is a *strong* (ω_1, λ)*-semimorass* iff

(M1) $\langle \mathcal{F}, \subseteq \rangle$ is *well-founded*, (and so we have the *rank* function on $\mathcal{F}$),
(M2) $\mathcal{F}$ is *locally small*, i.e. $|\mathcal{F} \restriction X| \leq \omega$ for each $X \in \mathcal{F}$.
(M3) $\mathcal{F}$ is *homogeneous*, i.e. $\forall X, Y \in \mathcal{F}$ if $\mathrm{rank}(X) = \mathrm{rank}(Y)$ then $\mathrm{tp}(X) = \mathrm{tp}(Y)$ and $\mathcal{F} \restriction Y = \{\rho_{X,Y}[Z] : Z \in \mathcal{F} \restriction X\}$.
(M4) $\mathcal{F}$ is *directed*, i.e. $\forall X, Y \in \mathcal{F}\ (\exists Z \in \mathcal{F})\ X \cup Y \subset Z$.
(M5) $\mathcal{F}$ is *strongly locally semidirected*, i.e. $\forall X \in \mathcal{F}$ either (a) or (b) holds:
  (a) $\mathcal{F} \restriction X$ is directed,
  (b) $\exists X_1, X_2 \in \mathcal{F}$ $\mathrm{rank}(X_1) = \mathrm{rank}(X_2)$, $X = X_1 \otimes X_2$, and $\mathcal{F} \restriction X = (\mathcal{F} \restriction X_1) \cup (\mathcal{F} \restriction X_2) \cup \{X_1, X_2\}$.
(M6) $\mathcal{F}$ *covers* λ, i.e. $\cup\mathcal{F} = \lambda$.

If in (M5)(b) we weaken the assumption $X = X_1 \otimes X_2$ to $X = X_1 \oplus X_2$ then we obtain the definition of an (ω_1, λ)*-semimorass* (see **[75**, Definition 1]). Moreover, a strong (ω_1, ω_2)-semimorass is just Velleman's simplified (ω_1, ω_2)-morass.

Definition 1.7.3. A family $\mathcal{F} \subset \left[\lambda\right]^\omega$ is *strongly stationary* iff for each function $c : \left[\mathcal{F}\right]^{<\omega} \to \left[\lambda\right]^\omega$ there are stationary many $X \in \mathcal{F}$ such that X is *c-closed*, i.e. $c(X^*) \subset X$ for each $X^* \in \left[\mathcal{F} \restriction X\right]^{<\omega}$.

Theorem 1.7.4. *If $2^\omega = \omega_1 < \lambda = \lambda^{\omega_1}$ then there is a σ-complete ω_2-c.c. forcing notion P such that*

$$V^P \models \text{ "}\lambda^{\omega_1} = \lambda \textit{ and there is a strongly stationary strong } (\omega_1, \lambda)\textit{-semimorass } \mathcal{F}.\text{ "}$$

We say that a family $p \subset \left[\lambda\right]^\omega$ is *neat* iff $\cup p = \cup(p \setminus \{\cup p\})$.

Proof of Theorem 1.7.4. Define $P = \langle P, \leq \rangle$ as follows. Let

$$P = \{p \subset \left[\lambda\right]^\omega : |p| \leq \omega, \cup p \in p, p \text{ is neat and satisfies (M1)–(M5)}\}.$$

Write $\mathrm{supp}(p) = \cup p$ for $p \in P$. Clearly $\mathrm{supp}(p)$ is the $\subset$-largest element of p. Put

$$p \leq q \text{ iff } \mathrm{supp}(q) \in p \wedge q = (p \restriction \mathrm{supp}(q)) \cup \{\mathrm{supp}(q)\}. \tag{1.29}$$

P is σ-complete. Indeed, if $p_0 \geq p_1 \geq p_2 \dots$ then let

$$p = \cup_{n<\omega} p_n \cup \{\cup_{n<\omega} \mathrm{supp}(p_n)\}.$$

Then $p \in P$ and $p \leq p_n$ for each n.

Definition 1.7.5. We say that two conditions p and p' are twins iff

(i) $\mathrm{tp}(\mathrm{supp}(p)) = \mathrm{tp}(\mathrm{supp}(p'))$,
(ii) $\mathrm{supp}(p) \cup \mathrm{supp}(p') = \mathrm{supp}(p) \otimes \mathrm{supp}(p')$,
(iii) $p' = \rho_{\mathrm{supp}(p),\mathrm{supp}(p')}[p]$.

Lemma 1.7.6. *If p and p' are twins then they have a common extension in P*

Proof. Write $D = \mathrm{supp}(p)$ and $D' = \mathrm{supp}(p')$. Put $r = p \cup p' \cup \{D \cup D'\}$. We show that r is a common extension of p and p'.
Claim: *$p = (r \restriction D) \cup \{D\}$ and $p' = (r \restriction D') \cup \{D'\}$.*

Indeed, assume that $X \in r \restriction D$. Then $X \in p$ or $X \in p'$. If $X \in p'$ then $X \subset D'$, and so $X \subset D \cap D'$. Since $\rho_{D,D'} \restriction D \cap D' = \mathrm{id}$ it follows that $X = \rho^{-1}_{D,D'}[X] \in p$. So $r \restriction D \subset p$, which proves the Claim.

First we check that that $r \in P$. (M1) and (M2) are clear. Since $\mathrm{supp}(r) = \mathrm{supp}(p) \cup \mathrm{supp}(p')$, r is neat. r has the largest element $\mathrm{supp}(r) = D \cup D' \in r$, and so (M4) also holds. In (M5) we have just one new instance $X = \mathrm{supp}(r)$. But in this case the choice $X_1 = D$ and $X_2 = D'$ works. To check (M3) assume that $X, Y \in r$, $\mathrm{rank}(X) = \mathrm{rank}(Y)$. If $X, Y \in p$ or $X, Y \in p'$ then we can apply that p and p' satisfy (M3). So we can assume that $X \in p \setminus p'$ and $Y \in p' \setminus p$. Let $X' = \rho_{D,D'}[X]$. Then $\mathrm{rank}(X') = \mathrm{rank}(X) = \mathrm{rank}(Y)$ and $X', Y \in p'$. Since p' satisfies (M3), we have $\mathrm{tp}(X') = \mathrm{tp}(Y)$, and so $\mathrm{tp}(X) = \mathrm{tp}(Y)$. Since $\rho_{X',Y} : p' \restriction X' \to p' \restriction Y$ is an isomorphism, and $\rho_{X,Y} = \rho_{D,D'} \circ \rho_{X',Y}$ it follows that $\rho_{X,Y} : p \restriction X \to p' \restriction Y$ is also an isomorphism. However $: p \restriction X = r \restriction X$ and $p' \restriction Y = r \restriction Y$ by the Claim, and so $\rho_{X,Y} : r \restriction X \to r \restriction Y$ is also an isomorphism, which proves (M3).

Finally $r \leq p, p'$ follows immediately from the Claim. □

Lemma 1.7.7. *P satisfies ω_2-c.c.*

Proof. Assume that $\{r_\alpha : \alpha < \omega_2\} \subset P$. Write $D_\alpha = \mathrm{supp}(r_\alpha)$ for $\alpha < \omega_2$. By standard argument we can find $I \in [\omega_2]^{\omega_2}$ such that

(a) $\{D_\alpha : \alpha \in I\}$ forms a Δ-system with kernel D, and $\mathrm{tp}(D_\alpha) = \mathrm{tp}(D_\beta)$ for $\alpha, \beta \in I$,
(b) For $\alpha < \beta \in I$ we have $D \cap \omega_2 < D_\alpha \setminus D < D_\beta \setminus D$,
(c) $\rho_{D_\alpha,D_\beta}[D_\alpha \cap \omega_2] = D_\beta \cap \omega_2$ and $\rho_{D_\alpha,D_\beta} \restriction D = \mathrm{id}$,
(d) $r_\beta = \{\rho_{D_\alpha,D_\beta}[X] : X \in r_\alpha\}$.

Then for each $\alpha \neq \beta \in I$ the conditions r_α and r_β are twins, so they are compatible by Lemma 1.7.6. □

Lemma 1.7.8. *(a) $\forall \alpha \in \omega_2$*

$$D_\alpha = \{p \in P : \mathrm{supp}(p) \cap (\omega_2 \setminus \alpha) \neq \emptyset\}$$

is dense in P.
(b) $\forall \beta \in \lambda \setminus \omega_2$

$$E_\beta = \{p \in P : \beta \in \mathrm{supp}(p)\}$$

is dense in P.

Proof. (a) For each $q \in P$ and $\alpha < \omega_2$ there is q' such that q and q' are twins and $\mathrm{supp}(q') \cap (\omega_2 \setminus \alpha) \neq \emptyset$. Then q and q' has a common extension $p \in D_\alpha$ by Lemma 1.7.6.
(b) For all q and $\beta \in \lambda \setminus \omega_2$ there is q' such that q and q' are twins and $\beta \in \mathrm{supp}(q')$. Then the common extension p of q and q' is in E_β. □

Let $\mathcal{G}$ be a P-generic filter over V, and put $\mathcal{F}' = \cup\mathcal{G}$ and $F = \cup\mathcal{F}'$. Then $\mathcal{F}' \subset \big[\lambda\big]^\omega$ and so $F \subset \lambda$. By the previous lemma, $F \supset \lambda \setminus \omega_2$ and $|F \cap \omega_2| = \omega_2$. So $\mathcal{F} = \{\rho_{F,\lambda}[X] : X \in \mathcal{F}'\}$ is a strong (ω_1, λ)-semimorass. To complete the proof of 1.7.4 it is enough to prove the following lemma.

Lemma 1.7.9. $V^P \models$ *"$\mathcal{F}$ is strongly stationary".*

Proof. It is enough to prove that if

$$q \Vdash \dot{\mathcal{C}} \subset \big[F\big]^\omega \text{ is club and } \dot{c} : \big[\mathcal{F}'\big]^{<\omega} \to \big[F\big]^\omega$$

then there are $p \le q$ and $C \in \big[\lambda\big]^\omega$ such that $p \Vdash$ "$\check{C} \in \mathcal{F}' \cap \dot{\mathcal{C}}$ is $\dot{c}$-closed".

First we need a claim.

Claim 1.7.9.1. *If $p \Vdash \check{A} \in \big[F\big]^\omega$ then $\exists p' \le p$ such that $A \subset \mathrm{supp}(p')$.*

Proof. If $\alpha \in A$ and $p \Vdash \check{\alpha} \in F$ then there are $p' \le p$ and $p'' \in P$ such that $\alpha \in \mathrm{supp}(p'')$ and $p' \Vdash p'' \in F$. Then p' and p'' have a common extension q, and then $\alpha \in \mathrm{supp}(q)$ and $q \Vdash \check{\alpha} \in F$.

Since P is σ-complete and A is countable, we are done by a straightforward induction. □

We will choose a decreasing sequence $\langle p_n : n < \omega\rangle \subset P$ and an increasing sequence $\langle C_n : n < \omega\rangle \subset \big[\lambda\big]^\omega$ as follows. Let $C_0 = \emptyset$ and $p_0 = q$. If p_n and C_n are given, let $Z_n \supset C_n \cup \mathrm{supp}(p_n)$ and $p'_n \le p_n$ s.t.

$$p'_n \Vdash \cup\{\dot{c}(X) : X \in \big[\check{p}_n\big]^{<\omega}\} \subset Z_n \in \big[F\big]^\omega.$$

Let $p_{n+1} \le p'_n$ and $C_{n+1} \supset Z_n \cup C_n$ such that $C_{n+1} \subset \mathrm{supp}(p_{n+1})$ and $p_{n+1} \Vdash \check{C}_{n+1} \in \dot{\mathcal{C}}$.

Having constructed the sequence finally put $C = \cup\{C_n : n < \omega\}$ and $p = \cup_{n<\omega} p_n \cup \{C\}$. Then $p \in P$, $p \le q$, $C = \mathrm{supp}(p)$. Since $p \Vdash$ "$\check{C}_n \in \dot{\mathcal{C}}$ *and* $\dot{\mathcal{C}}$ *is club*", we have $p \Vdash$ "$\check{C} \in \dot{\mathcal{C}}$. Since $p \Vdash \dot{c}''\big[[p_n]^{<\omega}\big] \subset \mathrm{supp}(p_{n+1})$, we have $p \Vdash \dot{C}$ *is $\dot{c}$-closed.* Moreover $p \Vdash p \subset \mathcal{F}'$, so $p \Vdash \check{C} \in \mathcal{F}'$.

Putting these together we obtain that p and C have the desired properties, which proves the lemma. □

Since $(\lambda^{\omega_1})^{V[\mathcal{F}]} \le ((|P| + \lambda)^{\omega_1})^V = (\lambda^{\omega_1})^V = \lambda$, the proof of Theorem 1.7.4 is complete. □

Next we investigate some properties of strong semimorasses.

Lemma 1.7.10. *Let $\mathcal{F} \subset \big[\lambda\big]^\omega$ be a strongly stationary strong (ω_1, λ)-semimorass.*

(1) If $X, Y \in \mathcal{F}$, $\mathrm{rank}(X) = \mathrm{rank}(Y)$, $\alpha \in X \cap Y \cap \omega_2$, then $X \cap \alpha = Y \cap \alpha$.

(2) If $X, Y \in \mathcal{F}$, $X \subset Y$ and $\mathrm{rank}(X) < \alpha < \mathrm{rank}(Y)$ then there is $Z \in \mathcal{F}$ such that $\mathrm{rank}(Z) = \alpha$ and $X \subset Z \subset Y$.

(3) If $X \in \mathcal{F}$ and $\mathrm{rank}(X) < \alpha < \omega_1$ then there is $Z \in \mathcal{F}$ such that $\mathrm{rank}(Z) = \alpha$ and $X \subset Z$.

(4) If $X, Y \in \mathcal{F}$, $\mathrm{rank}(X) \le \mathrm{rank}(Y)$, and $\alpha \in X \cap Y \cap \omega_2$, then $X \cap \alpha \subset Y \cap \alpha$.

Proof. (1) We prove the statement by induction on the minimal rank of $Z \in \mathcal{F}$ with $Z \supset X \cup Y$.

If rank of Z is minimal, then clearly $Z = Z_1 \otimes Z_2$ where $X \subset Z_1$ and $Y \subset Z_2$. Let $X' = \rho_{Z_1,Z_2}[X] \in \mathcal{F} \restriction Z_2$. Since $\alpha \in Z_1 \cap Z_2 \cap \omega_2$, we have $Z_1 \cap \alpha = Z_2 \cap \alpha$ and so $\rho_{Z_1,Z_2} \restriction (\alpha+1) = \mathrm{id}$. Thus $X' \cap \alpha = X \cap \alpha$ and $\alpha \in X'$. Since $X', Y \in \mathcal{F} \restriction Z_2$, $\alpha \in X' \cap Y$ and $\mathrm{rank}(Z_2) < \mathrm{rank}(Z)$, by the inductive assumption we have $X' \cap \alpha = Y \cap \alpha$. Thus $X \cap \alpha = Y \cap \alpha$.

(2) Easy by straightforward induction on $\mathrm{rank}(Y)$.

(3) By straightforward induction on α there is $Y \in \mathcal{F}$ such that $X \subset Y$ and $rank(Y) \ge \alpha$. Then apply (2).

(4) By (3) there is $Y' \supset X$ such that $\mathrm{rank}(Y) = \mathrm{rank}(Y')$. Then apply (1) for Y and Y'. □

In [**72**] Koszmider proved several statements for Velleman's simplified morasses. Here we need similar results for strong semimorasses. The following lemma corresponds to [**72**, Fact 2.6-Fact 2.7].

Lemma 1.7.11. *Let $\mathcal{F} \subset \big[\lambda\big]^\omega$ be a strongly stationary strong (ω_1, λ)-semimorass. Assume $\lambda^\omega = \lambda$, fix an injective function $c : \mathcal{F} \to \lambda$, and consider the stationary set*

$$\mathcal{F}' = \{X \in \mathcal{F} : c(X^*) \in X \text{ for each } X^* \in \mathcal{F} \restriction X\}. \tag{1.30}$$

Assume that $\mathcal{F}, \mathcal{F}', c \in M \prec H(\theta)$, $|M| = \omega$, *and* $M \cap \lambda \in \mathcal{F}'$. *Then*

(1) $\mathcal{F} \restriction M \cap \lambda \subset M$.

(2) $\mathrm{rank}(M \cap \lambda) = M \cap \omega_1$.

(3) If $Y \in \mathcal{F}$ *with* $\mathrm{rank}(Y) < \delta = M \cap \omega_1$ *then there is* $Z \in M \cap \mathcal{F}$ *such that* $(M \cap \lambda) \cap Y \subset Z$, *and* $\mathrm{rank}(Z) = \mathrm{rank}(Y)$.

(4) If $\mathcal{A} \in [\mathcal{F}]^{<\omega}$ *then there is* $Z \in \mathcal{F} \cap M$ *such that*

$$\cup\{X \cap M : X \in \mathcal{A}, \mathrm{rank}(X) < M \cap \omega_1\} \subset Z. \tag{1.31}$$

Proof. (1) Let $X \in \mathcal{F} \restriction M \cap \lambda$, i.e. $X \in \mathcal{F}$ and $X \subsetneq M \cap \lambda$. Then $X \subsetneq M \cap \lambda \in \mathcal{F}'$ implies $\alpha = c(X) \in M \cap \lambda$. But $c, \alpha \in M$ and c is injective, so $X = c^{-1}\{\alpha\} \in M$.
(2) If $X \subsetneq M \cap \lambda$, $X \in \mathcal{F}$, then $X \in M$ by (1) and so $\mathrm{rank}(X) \in M \cap \omega_1$. Thus $\mathrm{rank}(M \cap \lambda) \leq M \cap \omega_1$.

Assume that $\alpha < M \cap \omega_1$. Then

$$M \models \text{“}\exists X \in \mathcal{F}\ \mathrm{rank}(X) = \alpha.\text{”}$$

Thus there is $X \in M \cap \mathcal{F}$ such that $\mathrm{rank}(X) = \alpha$. Hence $\mathrm{rank}(M \cap \lambda) \geq M \cap \omega_1$.
(3) There is $Y' \supset Y$, $Y' \in \mathcal{F}$ and $\mathrm{rank}(Y') = \mathrm{rank}(M \cap \lambda)$. Let $Z = \rho_{Y', M \cap \lambda}[Y]$. Since $Y \cap (M \cap \lambda) \subset Y' \cap (M \cap \lambda)$ and $\rho_{Y', M \cap \lambda} \restriction Y' \cap (M \cap \lambda) = \mathrm{id}$, we have $Z \supset Y \cap (M \cap \lambda)$.
(4) Just apply (3) and the fact that $\mathcal{F}$ is directed. □

1.7.2. A $\Delta(\omega_2 \times \lambda)$-function. Let $\lambda \geq \omega_2$ be an infinite cardinal and let $\pi : \omega_2 \times \lambda \to \omega_2$ be the natural projection: $\pi(\langle \xi, \alpha \rangle) = \xi$.

Definition 1.7.12. (1) Assume that f is a function such that $\mathrm{dom}(f) \subset [\omega_2 \times \lambda]^2$ and $f\{x, y\} \in [\pi(x) \cap \pi(y)]^{<\omega}$ for each $\{x, y\} \in \mathrm{dom}(f)$. We say that two finite subsets d and d' of $\omega_2 \times \lambda$ are *good for* f provided $[d \cup d']^2 \subset \mathrm{dom}(f)$ and $\forall x \in d' \setminus d\ \forall y \in d \setminus d'\ \forall z \in d \cap d' \cap (\omega_2 \times \omega)$
(S1) if $\pi(z) < \pi(x), \pi(y)$ then $\pi(z) \in f\{x, y\}$,
(S2) if $\pi(z) < \pi(y)$ then $f\{x, z\} \subset f\{x, y\}$,
(S3) if $\pi(z) < \pi(x)$ then $f\{y, z\} \subset f\{x, y\}$.
(2) A function $f : [\omega_2 \times \lambda]^2 \to [\omega_2]^{<\omega}$ is a $\Delta(\omega_2 \times \lambda)$-*function* iff $f\{x, y\} \subset \min(\pi(x), \pi(y))$ and for each sequence $\{d_\alpha : \alpha < \omega_1\} \subset [\omega_2 \times \lambda]^{<\omega}$ there are $\alpha \neq \beta$ such that d_α and d_β are good for f.

Remark . The assumption $|f\{x, y\}| < \omega$, instead of the usual $|f\{x, y\}| \leq \omega$, is not a misprint.

Theorem 1.7.13. *If* $2^\omega = \omega_1 < \lambda = \lambda^{\omega_1}$ *and there is a strongly stationary strong* (ω_1, λ)-*semimorass, then in some cardinal preserving generic extension* $\lambda^{\omega_1} = \lambda$ *and there is a* $\Delta(\omega_2 \times \lambda)$-*function.*

Proof. To start with fix a strongly stationary strong (ω_1, λ)-semimorass $\mathcal{F} \subset [\lambda]^\omega$. We can assume that

$$\omega \subset X \text{ for each } X \in \mathcal{F}. \tag{1.32}$$

Fix an injective function $c : \mathcal{F} \to \lambda$, and consider the stationary set

$$\mathcal{F}' = \{X \in \mathcal{F} : c(X^*) \in X \text{ for each } X^* \in \mathcal{F} \restriction X\}. \tag{1.33}$$

Definition 1.7.14. We define a poset $P = \langle P, \leq \rangle$ as follows: P consists of triples $p = \langle a, f, \mathcal{A} \rangle$, where $a \in [\omega_2 \times \lambda]^{<\omega}$, $f : [a]^2 \to \mathcal{P}(\pi[a])$ with $f\{s, t\} \subset \min(\pi(s), \pi(t))$, $\mathcal{A} \in [\mathcal{F}]^{<\omega}$ such that

$$\forall s, t \in a\ \forall X \in \mathcal{A} \text{ if } s, t \in X \times X \text{ then } f\{s, t\} \subset X. \tag{1.34}$$

Write $p = \langle a_p, f_p, \mathcal{A}_p \rangle$ for $p \in P$. Put $p \leq q$ iff $a_p \supset a_q$, $f_p \supset f_q$ and $\mathcal{A}_p \supset \mathcal{A}_q$.

For $p \in P$ let

$$\mathrm{supp}(p_\nu) = \cup a_\nu \cup \cup \mathcal{A}_\nu.$$

Clearly $\mathrm{supp}(p) \in [\lambda]^{\leq\omega}$.

If ρ is a function and $x = \langle a, b \rangle \in \mathrm{dom}(\rho)^2$, let $\bar{\rho}(x) = \langle \rho(a), \rho(b) \rangle$. We say $p, q \in P$ are *twins* iff

(A) $\mathrm{supp}(p)$ and $\mathrm{supp}(q)$ have the same order type,
(B) the unique $<_{On}$-preserving bijection ρ between $\mathrm{supp}(p)$ and $\mathrm{supp}(q)$ gives an *isomorphism* between p and q, i.e.
  (a) $a_q = \bar{\rho}[a_p]$,
  (b) $\{\rho[X] : X \in \mathcal{A}_p\} = \mathcal{A}_q$,
  (c) for each $\{s,t\} \in \left[a_p\right]^2$, $\rho[f_p\{s,t\}] = f_q\{\bar{\rho}(s), \bar{\rho}(t)\}$.

Definition 1.7.15 ([**75**, Definition 22])**.** Let $\mathcal{K} \subset \left[\lambda\right]^\omega$. A poset P is *$\mathcal{K}$-proper* iff for some large enough regular cardinal θ if M is a countable elementary submodel of $\mathcal{H}(\theta)$ with $P \in M$ and $M \cap \lambda \in \mathcal{K}$ then for each $p \in M \cap P$ there is an (M,P)-generic $q \leq p$.

Lemma 1.7.16 ([**75**, Fact 23])**.** *If $\mathcal{K} \subset \left[\lambda\right]^\omega$ is stationary and a poset P is $\mathcal{K}$-proper, then forcing with P preserves ω_1.*

Definition 1.7.17 ([**75**, Definition 24])**.** Assume that P is a poset, $M \prec \mathcal{H}(\theta)$, $|M| = \omega$, $q \in P$, and $P, \pi_1, \dots, \pi_n \in M$. We say that *the formula $\Phi(x, \pi_1, \dots, \pi_n)$ well-reflects q in M* iff
(1) $\mathcal{H}(\theta) \models \Phi(q, \pi_1, \dots, \pi_n)$,
(2) if $s \in M \cap P$ and $M \models \Phi(s, \pi_1, \dots \pi_n)$ then q and s are compatible in P.

Definition 1.7.18 ([**75**, Definition 25])**.** Assume that P is a poset, $\mathcal{K} \subset \left[\lambda\right]^\omega$. We say that P is *simply $\mathcal{K}$-proper* if the following holds: for some/each large enough regular cardinal θ
IF
  (i) $M \prec \mathcal{H}(\theta)$, $|M| = \omega$,
  (ii) $p \in P$, $P, p, \mathcal{K} \in M$,
  (iii) $M \cap \lambda \in \mathcal{K}$,
THEN there is $p_0 \leq p$ such that for each $q \leq p_0$ some formula $\Phi(x, \pi_1, \dots, \pi_n)$ well-reflects q in M.

By lemmas [**75**, Fact 23 and Lemma 26] we have

Lemma 1.7.19. *If $\mathcal{K} \subset \left[\lambda\right]^\omega$ is stationary and a poset P is simply $\mathcal{K}$-proper, then forcing with P preserves ω_1.*

To show that ω_1 is preserved we prove the following lemma.

Lemma 1.7.20. *P is simply $\mathcal{F}'$-proper.*

Actually we will prove some stronger statement. To formulate it we need some preparation. If $M \prec \mathcal{H}(\theta)$, $p \in P \cap M$, $M \cap \lambda \in \mathcal{F}$ and $q \in P$ let

$$p^M = \langle a_p, f_p, \mathcal{A}_p \cup \{M \cap \lambda\}\rangle$$

and

$$q \restriction M = \langle a_q \cap M, f_q \restriction M, \mathcal{A}_q \cap M\rangle\,.$$

Lemma 1.7.21. *(1) If $M \prec \mathcal{H}(\theta)$ and $p \in P \cap M$ then $p^M \in P$. (2) If $q \leq p^M$ then $q \restriction M \in P \cap M$ as well.*

Proof. (1) We should only check (1.34) for p^M. Assume that $s, t \in a_p$ and $X \in \mathcal{A}_p \cup \{M \cap \lambda\}$. Since $p \in P$, we can assume $X = M \cap \lambda$. However $s, t \in M$, and so $f_p\{s,t\} \in M$ as well by $p \in M$. Since $|f_p\{s,t\}| \leq \omega$, it follows $f_p\{s,t\} \subset M \cap \lambda = X$ which was to be proved.
(2) It is straightforward that $q \restriction M \in P$. To show $q \restriction M \in M$ we should check that $f_q \restriction M \in M$. So assume that $s, t \in a_q \cap M$. Then $s, t \in (M \cap \lambda) \times (M \cap \lambda)$ and $M \cap \lambda \in \mathcal{A}_{p^M} \subset \mathcal{A}_q$. So, by (1.34), $f_q\{s,t\} \subset M \cap \lambda$. Since $f_q\{s,t\}$ is finite, we have $f_q\{s,t\} \in M$. □

Lemma 1.7.22. *Assume that $M \prec \mathcal{H}(\theta)$, $|M| = \omega$, $P, \mathcal{F} \in M$, $p \in P \cap M$, $M \cap \lambda \in \mathcal{F}'$. Let $\delta = \mathrm{rank}(M \cap \lambda) = M \cap \omega_1$. Assume that $Z \in M \cap \mathcal{F}$ such that*

$$Z \supset \cup\{X \cap M : X \in \mathcal{A}_q, \mathrm{rank}(X) < \delta\}. \tag{1.35}$$

Let $\Phi(x, Z, q \restriction M)$ be the following formula:

$$\text{“ } x \in P,\ x \leq q \restriction M,\ (a_x \setminus a_{q\restriction M}) \cap (Z \times Z) = \emptyset.\text{”} \tag{1.36}$$

Then
(1) $\Phi(q, Z, q \restriction M)$ *holds.*
(2) If $s \in M$, $M \models \Phi(s, Z, q \restriction M)$ *and*

$$h : [a_s \setminus a_{q\restriction M}, a_q \setminus a_{q\restriction M}] \to \mathcal{P}(\pi[a_s \cup a_q])$$

such that

$$h\{x, y\} \subset \min(\pi(x), \pi(y)) \cap \bigcap\{X \in A_q : x, y \in X \times X, \mathrm{rank}(X) \geq \delta\}, \tag{1.37}$$

then $r = \langle a_s \cup a_q, f_s \cup f_q \cup h, \mathcal{A}_s \cup \mathcal{A}_q\rangle \in P$ *is a common extension of* q *and* s.
(3) $\Phi(x, Z, q \restriction M)$ *well reflects* q *in* M.

Proof. (1) Since $q \restriction M \in P$ by Lemma 1.7.21(2), we have $q \leq q \restriction M$ by the definition of the relation $\leq$. Since $Z \in M$, we have $Z \times Z \subset M \times M \subset M$ and $a_q \setminus a_{q\restriction M} = a_q \setminus M$.
(2) To show that $r \in P$ we need to check that $r = \langle a_r, f_r, \mathcal{A}_r\rangle$ satisfies (1.34). So let $x, y \in a_r$, $X \in \mathcal{A}_r$.

Case 1. $x, y \in a_s$ *and* $X \in \mathcal{A}_s$ *or* $x, y \in a_q$ *and* $X \in \mathcal{A}_q$.

Then everything is fine, because $s, q \in P$.

Case 2. $\{x, y\} \in \left[a_s\right]^2$, $x \in a_s \setminus a_q$ *and* $X \in \mathcal{A}_q$.

If $\mathrm{rank}(X) < \delta$ then $(a_s \setminus a_q) \cap (X \times X) = \emptyset$ by (1.35), so $x \notin X \times X$. Thus (1.34) is void.

If $\mathrm{rank}(X) \geq \delta$ then $\nu = \min(\pi(x), \pi(y)) \in M \cap X$, so $M \cap \nu \subset X \cap \nu$ by Lemma 1.7.10(4). Thus $f_r\{x, y\} = f_s\{x, y\} \cap \nu \subset M \cap \nu \subset X$.

Case 3. $\{x, y\} \in \left[a_q\right]^2$, $x \in a_q \setminus a_s$ *and* $X \in \mathcal{A}_s$.

Since $(a_q \setminus a_s) \cap M = \emptyset$, it is not possible that $X \in \mathcal{A}_s$. Then $x \in a_q \setminus a_s = a_q \setminus a_{q\restriction M}$, so $x \notin M$. However $X \subset M$ and so $x \notin X \times X$, so (1.34) is void.

Case 4. $x \in a_q \setminus a_s$ *and* $y \in a_s \setminus a_q$.

Then the assumption concerning h in (1.37) is stronger than (1.34). Indeed, if $y \in a_s \setminus a_q$ then $y \notin Z \times Z$. So if $y \in X \times X$ for some $X \in \mathcal{A}_q$ then $\mathrm{rank}(X) \geq \delta$.

(3) Define the function

$$h : [a_s \setminus a_{q\restriction M}, a_q \setminus a_{q\restriction M}] \to \left[\omega_2\right]^{<\omega}$$

by $h\{x, y\} = \emptyset$. Then (1.37) holds, so s and q are comparable by (2), which was to be proved. □

Proof of Lemma 1.7.20. We can apply Lemma 1.7.22 because by Lemma 1.7.11 we can pick $Z \in M \cap \mathcal{F}$ such that $Z \supset \cup\{X \cap M : X \in \mathcal{A}_q, \mathrm{rank}(X) < \delta\}$. □

Lemma 1.7.23. P *satisfies* ω_2*-c.c.*

Proof. Let $\{p_\nu : \nu < \omega_2\} \subset P$. Put $p_\nu = \langle a_\nu, f_\nu, \mathcal{A}_\nu\rangle$. Recall that $\mathrm{supp}(p_\nu) = \cup a_\nu \cup \cup\mathcal{A}_\nu$.We can assume that

(i) $\{\mathrm{supp}(p_\nu) : \nu < \omega_2)\}$ forms a Δ-system with kernel D
(ii) the conditions are pairwise twins witnessed by functions $\rho_{\nu,\mu} : \mathrm{supp}(p_\nu) \to \mathrm{supp}(p_\mu)$.

Fix $\nu < \mu < \omega_2$. Define the function e as follows:

$$\mathrm{dom}(e) = [a_\nu \setminus a_\mu, a_\mu \setminus a_\nu] \text{ and } e\{s, t\} = \emptyset.$$

We claim that

$$r = \langle a_\nu \cup a_\mu, f_\nu \cup f_\mu \cup e, \mathcal{A}_\nu \cup \mathcal{A}_\mu\rangle \tag{1.38}$$

is a common extension of p_ν and p_μ. We need to show that $r \in P$. Since p_ν and p_μ are twins, we should check only (1.34). So let $t, s \in a_\nu \cup a_\mu$ and $X \in \mathcal{A}_\nu \cup \mathcal{A}_\mu$ with $s, t \in X \times X$. We can assume e.g. $X \in \mathcal{A}_\nu$. Since $X \subset \mathrm{supp}(p_\nu)$, we have $s, t \in \mathrm{supp}(p_\nu) \times \mathrm{supp}(p_\nu)$. Then $\mathrm{supp}(p_\nu) \times \mathrm{supp}(p_\nu) \cap a_\mu \subset a_\nu$ because $\mathrm{supp}(p_\nu) \times \mathrm{supp}(p_\nu) \cap a_\mu \subset D$ and a_ν, a_μ are twins. Thus we have $s, t \in a_\nu$, and so we are done because p_ν satisfies (1.34). □

To complete the proof of Theorem 1.7.13 we claim that if $\mathcal{G}$ is a P-generic filter then the function

$$f = \cup\{f_p : p \in \mathcal{G}\} \tag{1.39}$$

is a $\Delta(\omega_2 \times \lambda)$-function.

Assume that

$$p \Vdash \{\dot{d}_\xi : \xi < \omega_1\} \subset \left[\omega_2 \times \lambda\right]^{<\omega}.$$

We can assume

$$p \Vdash \{\dot{d}_\xi : \xi < \omega_1\} \text{ is a } \Delta\text{-system with kernel } \check{d}.$$

Assume $M \prec H(\theta)$, $|M| = \omega$, $p, \mathcal{F}' \left\langle \dot{d}_\xi : \xi < \omega_1 \right\rangle \in M$ and $X_0 = M \cap \lambda \in \mathcal{F}'$. Let

$$p^M = \langle a_p, f_p, \mathcal{A}_p \cup \{X_0\}\rangle .$$

Let $q \leq p^M$, $\xi_1 < \omega_1$ and $e_1 \in \left[\omega_2 \times \lambda\right]^{<\omega}$ that

$$q \Vdash \dot{d}_{\xi_1} = \check{e}_1 \wedge (e_1 \setminus d) \cap M = \emptyset \wedge e_1 \subset a_q.$$

Put $\delta = \text{rank}(M \cap \lambda)$. By Lemma 1.7.11 we can pick $Z \in M \cap \mathcal{F}$ such that

$$Z \supset \cup\{X \cap M : X \in \mathcal{A}_q, \text{rank}(X) < \delta\}.$$

Consider the following formula $\Psi(x, \xi, e)$ with free variables x, ξ and e and parameters $Z, q \restriction M, \left\langle \dot{d}_\xi : \xi < \omega_1 \right\rangle, d \in M$:

$$\Phi(x, Z, q \restriction M) \wedge \xi \in \omega_1 \wedge (x \Vdash \dot{d}_\xi = e) \wedge e \subset a_x \wedge (e \setminus d) \subset a_x \setminus a_{q\restriction M}$$

where the formula $\Phi(x, Z, q \restriction M)$ was defined in (1.36) in Lemma 1.7.22. Then $\Psi(q, \xi_1, e_1)$ holds. Thus $\exists x \exists \xi \exists e \Psi(x, \xi, e)$ also holds. Since the parameters are all in M, we have

$$M \models \exists x\ \exists \xi\ \exists e\ \Psi(x, \xi, e). \tag{1.40}$$

Thus there are $s, \xi_2, e_2 \in M$ such that

$$\Phi(s, Z, q \restriction M) \wedge (s \Vdash \dot{d}_{\xi_2} = e_2) \wedge e_2 \subset a_s \wedge (e_2 \setminus d) \subset a_s \setminus a_{q\restriction M}.$$

Since $e_2 \setminus d \subset a_s \setminus a_{q\restriction M}$ and $(a_s \setminus a_{q\restriction M}) \cap Z \times Z = \emptyset$ by $\Phi(s, Z, q \restriction M)$, we have

$$(e_2 \setminus d) \cap Z \times Z = \emptyset.$$

Define the function

$$h : [a_s \setminus a_{q\restriction M}, a_q \setminus a_{q\restriction M}] \to \mathcal{P}(\pi[a_s \cup a_q])$$

by the formula

$$h\{x, y\} = \pi[a_s \cup a_q] \cap \min(\pi(x), \pi(y)) \cap \bigcap\{X \in \mathcal{A}_q : x, y \in X \times X, \text{rank}(X) \geq \delta\}. \tag{1.41}$$

So $h\{x, y\}$ is as large as it is allowed by (1.37).

Then, by Lemma 1.7.22, the condition $r = \langle a_r, f_r, \mathcal{A}_r\rangle$, where $a_r = a_s \cup a_q$, $f_r = f_s \cup f_q \cup h$ and $\mathcal{A}_r = \mathcal{A}_s \cup \mathcal{A}_q$, is a common extension of q and s.

Lemma 1.7.24. *e_1 and e_2 are good for f_r.*

Proof. We should check conditions (S1)-(S3).

Assume that $z \in e_1 \cap e_2 \cap (\omega_2 \times \omega)$, $x \in e_1 \setminus e_2 \subset a_q \setminus a_{q\restriction M}$, and $y \in e_2 \setminus e_1 \subset a_s \setminus a_{q\restriction M}$. Observe that $z, y \in M$, and so $\pi(z), \pi(y) \in M$ as well.

(S1): Assume that $\pi(z) < \pi(x), \pi(y)$.

We should show that $\pi(z) \in f_r\{x, y\}$. However, $f_r\{x, y\}$ was defined by (1.41). So we should show that

$$\text{if } X \in \mathcal{A}_q,\ \text{rank}(X) \geq \delta,\ x, y \in X \times X \text{ then } \pi(z) \in X.$$

Since $\pi(y) \in M \cap X \cap \omega_2$ and $\text{rank}(M \cap \lambda) \leq \text{rank}(X)$ we have $M \cap \pi(y) \subset X \cap \pi(y)$ by Lemma 1.7.10(4). Since $\pi(z) < \pi(y)$ and $\pi(z) \in M$ it follows that $\pi(z) \in X$.

(S2): Assume that $\pi(z) < \pi(y)$.

We need to show that $f_r\{x,z\} \subset f_r\{x,y\}$. Since $f_r\{x,z\} = f_q\{x,z\}$ and $f_r\{x,y\}$ was defined by (1.41) we should show that

$$\text{if } X \in \mathcal{A}_q,\ \mathrm{rank}(X) \geq \delta,\ x,y \in X \times X \text{ then } f_q\{x,z\} \subset X.$$

Since $\pi(y) \in M \cap X$ we have $\pi(z) \in M \cap \pi(y) \subset X \cap \pi(y)$ by Lemma 1.7.10(4).

Since $\pi(z) \in X$, $z \in \omega_2 \times \omega$ and $\omega \subset X$ by (1.32), it follows that $z \in X \times X$. Since $x, z \in X \times X$ and $X \in \mathcal{A}_q$, we have $f_q\{x,z\} \subset X$ by (1.34), which was to be proved.

(S3): Assume that $\pi(z) < \pi(x)$.

We need to show that $f_r\{y,z\} \subset f_r\{x,y\}$. Since $f_r\{y,z\} = f_s\{y,z\}$ and $f_r\{x,y\}$ was defined by (1.41) we should show that

$$\text{if } X \in \mathcal{A}_q,\ \mathrm{rank}(X) \geq \delta,\ x,y \in X \times X \text{ then } f_s\{y,z\} \subset X.$$

Since $z, y \in M$ we have $f_s\{y,z\} \subset M$.

Moreover $y \in X \times X$, and so $\pi(y) \in M \cap X \cap \omega_2$, which implies $M \cap \pi(y) \subset X \cap \pi(y)$ by Lemma 1.7.10(4) .

Thus $f_s\{y,z\} = f_s\{y,z\} \cap \pi(y) \subset M \cap \pi(y) \subset X \cap \pi(y) \subset X$, which was to be proved. □

Since $r \Vdash \dot{d}_{\xi_1} = \check{e}_1 \wedge \dot{d}_{\xi_2} = \check{e}_2 \wedge f \supset \check{f}_r$, by Lemma 1.7.24 $r \Vdash$ "$\dot{d}_{\xi_1}$ and $\dot{d}_{\xi_2}$ are good for f". So f is a $\Delta(\omega_2 \times \lambda)$-function in $V[\mathcal{G}]$.

Since $|P| \leq \lambda$ and so $(\lambda^{\omega_1})^{V[\mathcal{G}]} \leq ((|P| + \lambda)^{\omega_1})^V = (\lambda^{\omega_1})^V = \lambda$, the proof of Theorem 1.7.13 is complete. □

1.7.3. Space construction. Assume that X is a scattered space. We say that a subspace $Y \subset X$ is a *stem* of X provided

(i) $ht(Y) = ht(X)$,
(ii) $X \setminus Y$ is closed discrete in X.

Clearly (ii) holds iff every $x \in X$ has a neighborhood U_x such that $U_x \setminus \{x\} \subset Y$.

Proposition 1.7.25. *Assume that X is an LCS* space, $Y \subset X$ is a stem,* $\mathrm{CS}(X) = \langle \kappa_\nu : \nu < \mu \rangle$ *and* $\mathrm{CS}(Y) = \langle \lambda_\nu : \nu < \mu \rangle$. *Then*

$$\{\mathrm{CS}(Z) : Y \subset Z \subset X\} = \{s \in {}^{\mu}Card : \lambda_\nu \leq s(\nu) \leq \kappa_\nu \text{ for each } \nu < \mu\}. \quad (1.42)$$

Proof. Assume that $s \in {}^{\mu}\mathrm{Card}$ such that $\lambda_\nu \leq s(\nu) \leq \kappa_\nu$ for each $\nu < \mu$. For $\nu < \mu$ pick $Z_\nu \in [I_\nu(X)]^{s(\nu)}$ with $Z_\nu \supset I_\nu(Y)$. Put $Z = \bigcup\{Z_\nu : \nu < \mu\}$. Since $Y \subset Z$ and Y is a stem, we have $I_\nu(Z) = Z_\nu$ for $\nu < \mu$, and so $\mathrm{CS}(Z) = s$. □

Theorem 1.7.26. *If there is a $\Delta(\omega_2 \times \lambda)$-function, then there is a c.c.c poset P such that in V^P there is an LCS space X with stem Y such that* $\mathrm{CS}(X) = \langle \lambda \rangle_{\omega_2}$ *and* $\mathrm{CS}(Y) = \langle \omega \rangle_{\omega_2}$.

Corollary 1.7.27. *If there is a $\Delta(\omega_2 \times \lambda)$-function, then there is a c.c.c poset P such that in V^P every sequence $\mathbf{s} = \langle s_\alpha : \alpha < \omega_2 \rangle$ of infinite cardinals with $s_\alpha \leq \lambda$ is the cardinal sequence of some locally compact scattered space.*

Proof of Theorem 1.7.26. Instead of constructing the topological space directly, we actually produce a certain "graded poset" which guarantees the existence of the desired locally compact scattered space. We use the ideas from **[21]** to formulate the properties of our required poset.

Definition 1.7.28. Given two sequences $\mathfrak{t} = \langle \kappa_\alpha : \alpha < \delta \rangle$ and $\mathfrak{s} = \langle \lambda_\alpha : \alpha < \delta \rangle$ of infinite cardinals with $\lambda_\alpha \leq \kappa_\alpha$, we say that a poset $\langle T, \prec \rangle$ is a $\mathfrak{t}$-*poset with an* $\mathfrak{s}$-*stem* iff the following conditions are satisfied:

(T1) $T = \bigcup\{T_\alpha : \alpha < \delta\}$ where $T_\alpha = \{\alpha\} \times \kappa_\alpha$ for each $\alpha < \delta$. Let $S_\alpha = \{\alpha\} \times \lambda_\alpha$, and $S = \bigcup\{S_\alpha : \alpha < \delta\}$.
(T2) For each $s \in T_\alpha$ and $t \in T_\beta$, if $s \prec t$ then $\alpha < \beta$ and $s \in S_\alpha$.
(T3) For every $\{s,t\} \in \left[T\right]^2$ there is a finite subset $i\{s,t\}$ of S such that for each $u \in T$:

$$(u \preceq s \wedge u \preceq t) \text{ iff } u \preceq v \text{ for some } v \in i\{s,t\}.$$

(T4) For $\alpha < \beta < \delta$, if $t \in T_\beta$ then the set $\{s \in S_\alpha : s \prec t\}$ is infinite.

Lemma 1.7.29. *If there is a $\mathfrak{t}$-poset with an $\mathfrak{s}$-stem then there is an LCS* space X with stem Y such that* $\mathrm{CS}(X) = \mathfrak{t}$ *and* $\mathrm{CS}(Y) = \mathfrak{s}$.

Indeed, if $\mathcal{T} = \langle T, \prec \rangle$ is an $\mathfrak{s}$-poset, we write $U_{\mathcal{T}}(x) = \{y \in T : y \preceq x\}$ for $x \in T$, and we denote by $X_{\mathcal{T}}$ the topological space on T whose subbase is the family

$$\{U_{\mathcal{T}}(x), T \setminus U_{\mathcal{T}}(x) : x \in T\}, \tag{1.43}$$

then $X_{\mathcal{T}}$ is our desired LCS* space with stem.

So, to prove Theorem 1.7.26 it will be enough to show that a $\langle \lambda \rangle_{\omega_2}$-poset with an $\langle \omega \rangle_{\omega_2}$-stem may exist.

We follow the ideas of **[25]** to construct P. Fix a $\Delta(\omega_2 \times \lambda)$-function $f : \big[\omega_2 \times \lambda\big]^2 \to \big[\omega_2\big]^{<\omega}$.

Definition 1.7.30. Define the poset $\mathcal{P} = \langle P, \preceq \rangle$ as follows. The underlying set P consists of triples $p = \langle a_p, \leq_p, i_p \rangle$ satisfying the following requirements:

(1) $a_p \in \big[\omega_2 \times \lambda\big]^{<\omega}$,
(2) $\leq_p$ is a partial ordering on a_p with the property that if $x <_p y$ then $x \in \omega_2 \times \omega$ and $\pi(x) < \pi(y)$,
(3) $i_p : \big[a_p\big]^2 \to \big[a_p\big]^{<\omega}$ is such that
  (3.1) if $\{x, y\} \in \big[a_p\big]^2$ then
    (3.1.1) if $x, y \in \omega_2 \times \omega$ and $\pi(x) = \pi(y)$ then $i_p\{x, y\} = \emptyset$,
    (3.1.2) if $x <_p y$ then $i_p\{x, y\} = \{x\}$,
    (3.1.3) if x and y are $<_p$-incomparable, then
      $i_p\{x, y\} \subset f\{x, y\} \times \omega$.
  (3.2) if $\{x, y\} \in \big[a_p\big]^2$ and $z \in a_p$ then
    $\big((z \leq_p x \wedge z \leq_p y) \text{ iff } \exists t \in i_p\{x, y\}\ z \leq_p t\big)$.

Set $p \preceq q$ iff $a_p \supseteq a_q$, $\leq_p \restriction a_q = \leq_q$ and $i_p \restriction \big[a_q\big]^2 = i_q$.

Lemma 1.7.31. *P satisfies ω_1-c.c..*

Proof. Let $\{p_\nu : \nu < \omega_1\} \subset P$, $p_v = \langle a_\nu, \leq_\nu, i_\nu \rangle$. By thinning out our sequence we can assume that

(i) $\{a_\nu : \nu < \omega_1\}$ forms a Δ-system with kernel a'.
(ii) $i_\nu \restriction \big[a'\big]^2 = i$.
(iii) $\leq_\nu \restriction a' \times a' = \leq$.
(iv) for each $\nu < \mu < \omega_1$ there is a bijection $\rho_{\nu,\mu} : a_\nu \to a_\mu$ such that
  (a) $\rho_{\nu,\mu} \restriction a' = \mathrm{id}$
  (b) $\pi(x) \leq \pi(y)$ iff $\pi(\rho_{\nu,\mu}(x)) \leq \pi(\rho_{\nu,\mu}(y))$,
  (c) $x \leq_\nu y$ iff $\rho_{\nu,\mu}(x) \leq_\mu \rho_{\nu,\mu}(y)$,
  (d) $x \in \omega_2 \times \omega$ iff $\rho_{\nu,\mu}(x) \in \omega_2 \times \omega$,
  (e) $\rho_{\nu,\mu}[i_\nu\{x, y\}] = i_\mu\{\rho_{\nu,\mu}(x), \rho_{\nu,\mu}(y)\}$.

Now it follows from condition (3.1) and condition (iv) that if $\nu < \mu < \omega_2$ and $\{x, y\} \in \big[a'\big]^2$ then $i_\nu\{x, y\} = i_\mu\{x, y\}$.

Since f is a $\Delta(\omega_2 \times \lambda)$-function there is $\nu < \mu < \omega_1$ such that a_ν and a_μ are good for f, i.e. (S1)–(S3) hold. Define $r = \langle a, \leq, i \rangle$ as follows:

(a) $a = a_\nu \cup a_\mu$,
(b) $x \leq y$ iff $x \leq_\nu y$ or $x \leq_\mu y$ or there is $s \in a_\nu \cap a_\mu$ such that $x \leq_\nu s \leq_\mu y$ or $x \leq_\mu s \leq_\nu y$,
(c) $i \supset i_\nu \cup i_\mu$,
(d) for $x \in a_\nu \setminus a_\mu$ and $y \in a_\mu \setminus a_\nu$, if x and y are $\leq$-incomparable then

$$i\{x, y\} = \big(f\{x, y\} \times \omega\big) \cap \{t \in a : t \leq x \wedge t \leq y\}. \tag{1.44}$$

(e) for $\{x, y\} \in \big[a\big]^2$ with $x < y$, $i\{x, y\} = \{x\}$.

We claim that $r \in P$.

By the construction, we have $\leq \restriction a_\nu \times a_\nu = \leq_\nu$ and $\leq \restriction a_\mu \times a_\mu = \leq_\mu$.

Claim: $\leq$ is a partial order.

We should check only the transitivity. Assume $x \leq y \leq z$. If $x \leq_\nu y \leq_\nu z$ or $x \leq_\mu y \leq_\mu z$ then we are done. Assume that $x \leq_\nu u \leq_\mu y \leq_\mu z$ for some $u \in a_\nu \cap a_\mu$. Then $x \leq_\nu u \leq_\mu z$ so $x \leq z$.

If $x \leq_\nu u \leq_\mu y \leq_\mu t \leq_\nu z$ for some $u, t \in a_\nu \cap a_\mu$, then $u \leq_\mu t$, which implies $u \leq_\nu t$. Thus $x \leq_\nu u \leq_\nu t \leq_\nu z$ and so $x \leq_\nu z$, and hence $x \leq z$.

The other cases are similar to these ones.

(3.1.3) holds by the construction of i.

To show that p is a condition we should finally check (3.2). Let $x, y \in a$ be $\leq$-incomparable elements. It is clear that if $u \leq t$ for some $t \in i\{x, y\}$ then $u \leq x$ and $\leq y$. So we should check that

$(*)$ if $z \leq x$ and $z \leq y$ then there is $t \in i\{x, y\}$ such that $z \leq t$.

If $x, y, z \in a_\nu$ or $x, y, z \in a_\mu$ then it is clear because $p_\nu, p_\mu \in P$.

Case 1. *$x, y \in a_\nu$ and $z \in a_\mu \setminus a_\nu$.*

Subcase 1.1 *$x, y \in a_\nu \setminus a_\mu$.*

There are $x', y' \in a_\nu \cap a_\mu$ such that $z \leq_\mu x' \leq_\nu x$ and $z \leq_\mu y' \leq_\nu y$. Then there is $t' \in i_\mu\{x', y'\}$ such that $z \leq_\mu t'$. Then $t' \in a_\nu \cap a_\mu$, so $t' \leq_\nu x, y$. Thus there is $t \in i_\nu\{x, y\}$ such that $t' \leq_\nu t$, and so $z \leq t$. Since $i\{x, y\} = i_\nu\{x, y\}$, we are done.

Subcase 1.2 *$x \in a_\nu \setminus a_\mu$ and $y \in a_\nu \cap a_\mu$*

Put $y' = y$, then proceed as in Subcase 1.1.

Case 2. *$x, z \in a_\nu \setminus a_\mu$ and $y \in a_\mu \setminus a_\nu$.*

Then $z \leq_\nu y' \leq_\mu y$ for some $y' \in a_\nu \cap a_\mu$. Then there is $t \in i_\nu\{x, y'\}$ such that $z \leq_\nu t$. Clearly $t \leq x, y$. We show that $t \in i\{x, y\}$.

If $t = y'$ then $t \leq x, y$ and $\pi(t) \in f\{x, y\}$ by (S1). Thus $t \in i\{x, y\}$.

Assume that $t <_\nu y'$. Then $\pi(t) \in f\{x, y'\} \subset f\{x, y\}$ by (S2), because $y' \in a_\nu \cap a_\mu$ and $\pi(y') < \pi(y)$. Thus $t \in i\{x, y\}$ by (1.44). □

Assume that $\mathcal{G}$ is a $\mathcal{P}$-generic filter. We claim that if we take

$$\ll = \cup\{\leq_p : p \in \mathcal{G}\}.$$

then $\langle \omega_2 \times \lambda, \ll \rangle$ is a $\langle \lambda \rangle_{\omega_2}$-poset with an $\langle \omega \rangle_{\omega_2}$-stem. By standard density arguments, $\ll$ is a partial order on $\omega_2 \times \lambda$ which satisfies (T4). Moreover, every $p \in P$ satisfies (2), so (T2) also holds. Finally the function

$$i = \bigcup\{i_p : p \in \mathcal{G}\}$$

witnesses (T3) because every $p \in P$ satisfies (3.2). □

1.8. Spaces constructed from strongly unbounded function

(This section is based on [17])

By using the combinatorial notion of the *new Δ property (NDP)* of a function, it was proved by Roitman that the existence of an LCS* spaces with cardinal sequence $\langle \omega \rangle_{\omega_1} {}^\frown \langle \omega_2 \rangle$ is consistent with ZFC (see **[94]** and **[92]**). (Actually, she considered superatomic Boolean algebras, but in this thesis we mainly use the language of topology) It is worth to mention that **[94]** was the first paper in which such a special function was used to guarantee the chain condition of a certain poset. Roitman's result was generalized in **[66]**, where for every infinite regular cardinal κ, it was proved that the existence of an LCS* space with cardinal sequence $\langle \kappa \rangle_{\kappa^+} {}^\frown \langle \kappa^{++} \rangle$ is consistent with ZFC. Then, our aim here is to prove the following stronger result.

Theorem 1.8.1. *Assume that κ, λ are infinite cardinals such that $\kappa^{+++} \leq \lambda$, $\kappa^{<\kappa} = \kappa$ and $2^\kappa = \kappa^+$. Then for each ordinal η with $\kappa^+ \leq \eta < \kappa^{++}$ and $\mathrm{cf}(\eta) = \kappa^+$, in some cardinal-preserving generic extension there is an LCS space with cardinal sequence $\langle \kappa \rangle_\eta {}^\frown \langle \lambda \rangle$.*

Corollary 1.8.2. *$\langle \omega \rangle_{\omega_1} {}^\frown \langle \omega_3 \rangle \in \mathcal{C}(\omega + 1)$ is consistent with ZFC. $\langle \omega_1 \rangle_{\omega_2} {}^\frown \langle \omega_4 \rangle \in \mathcal{C}(\omega_1 + 1)$ is also consistent.*

In order to prove Theorem 1.8.1, we shall use the main result of [**75**]. Assume that κ, λ are infinite cardinals such that κ is regular and $\kappa < \lambda$. We say that a function $F : [\lambda]^2 \to \kappa^+$ is a κ^+*-strongly unbounded function on* λ iff for every ordinal $\delta < \kappa^+$, every cardinal $\nu < \kappa$ and every family $A \subseteq [\lambda]^\nu$ of pairwise disjoint sets with $|A| = \kappa^+$, there are different $a, b \in A$ such that $F\{\alpha, \beta\} > \delta$ for every $\alpha \in a$ and $\beta \in b$.

Theorem* 1.8.3 (Koszmider, [**75**].). *If κ, λ are infinite cardinals such that $\kappa^{+++} \leq \lambda$, $\kappa^{<\kappa} = \kappa$ and $2^\kappa = \kappa^+$, then there is a κ-closed and cardinal-preserving partial order that forces the existence of a κ^+- strongly unbounded function on λ.*

So, in order to prove Theorem 1.8.1 it is enough to show the following result.

Theorem 1.8.4. *Assume that κ, λ are infinite cardinals with $\kappa^{+++} \leq \lambda$ and $\kappa^{<\kappa} = \kappa$, and η is an ordinal with $\kappa^+ \leq \eta < \kappa^{++}$ and $\mathrm{cf}(\eta) = \kappa^+$. Assume that there is a κ^+- strongly unbounded function on λ. Then, there is a cardinal-preserving partial order that forces the existence of an LCS* space with cardinal sequence $\langle \kappa \rangle_\eta {}^\frown \langle \lambda \rangle$.*

In [**66**], [**84**], [**94**] and in many other papers, the authors proved the existence of certain LCS* space in such a way that instead of constructing the spaces directly, they actually produced certain "graded posets" which guaranteed the existence of the wanted LCS* space. From these constructions, Bagaria, [**21**], extracted the following notion and proved the Lemma 1.8.6 below which was implicitly used in many earlier papers.

Definition 1.8.5 ([**21**]). Given a sequence $\mathfrak{s} = \langle \kappa_\alpha : \alpha < \delta \rangle$ of infinite cardinals, we say that a poset $\langle T, \prec \rangle$ is an $\mathfrak{s}$*-poset* iff the following conditions are satisfied:

(1) $T = \bigcup \{T_\alpha : \alpha < \delta\}$ where $T_\alpha = \{\alpha\} \times \kappa_\alpha$ for each $\alpha < \delta$.
(2) For each $s \in T_\alpha$ and $t \in T_\beta$, if $s \prec t$ then $\alpha < \beta$.
(3) For every $\{s, t\} \in \left[T\right]^2$ there is a finite subset $\mathrm{i}\{s, t\}$ of T such that for each $u \in T$:

$$(u \preceq s \wedge u \preceq t) \text{ iff } u \preceq v \text{ for some } v \in \mathrm{i}\{s, t\}.$$

(4) For $\alpha < \beta < \delta$, if $t \in T_\beta$ then the set $\{s \in T_\alpha : s \prec t\}$ is infinite.

Lemma 1.8.6 ([**21**, Lemma 1]). *If there is an $\mathfrak{s}$-poset then there is an LCS* space with cardinal sequence $\mathfrak{s}$.*

Actually, if $\mathcal{T} = \langle T, \prec \rangle$ is an $\mathfrak{s}$-poset, we write $U_{\mathcal{T}}(x) = \{y \in T : y \preceq x\}$ for $x \in T$, and we denote by $X_{\mathcal{T}}$ the topological space on T whose subbase is the family

$$\{U_{\mathcal{T}}(x), T \setminus U_{\mathcal{T}}(x) : x \in T\}, \tag{1.45}$$

then $X_{\mathcal{T}}$ is a locally compact, Hausdorff, scattered space whose cardinal sequence is $\mathfrak{s}$.

So, to prove Theorem 1.8.4 it will be enough to show that $\langle \kappa \rangle_\eta {}^\frown \langle \lambda \rangle$-posets may exist for κ, η and λ as above.

The organization of this section is as follows. In Subsection 2, we shall prove Theorem 1.8.4 for the special case in which $\kappa = \omega$ and $\lambda \geq \omega_3$, generalizing in this way the result proved by Roitman in [**94**]. In Subsection 1.8.2, we shall define the combinatorial notions that make the proof of Theorem 1.8.4 work. And in Section 1.8.3, we shall present the proof of Theorem 1.8.4.

1.8.1. Generalization of Roitman's Theorem. In this subsection, our aim is to prove the following result.

Theorem 1.8.7. *Let λ be a cardinal with $\lambda \geq \omega_3$. Assume that there is an ω_1-strongly unbounded function on λ. Then, in some cardinal-preserving generic extension for each ordinal η with $\omega_1 \leq \eta < \omega_2$ and $\mathrm{cf}(\eta) = \omega_1$ there is an LCS* space with cardinal sequence $\langle \omega \rangle_\eta {}^\frown \langle \lambda \rangle$.*

The theorem above is a bit stronger than Theorem 1.8.4 for $\kappa = \omega$, because the generic extension does not depend on η. However, as we will see, its proof is much simpler than the proof of the general case.

By Lemma 1.8.6, it is enough to construct a c.c.c. poset $\mathcal{P}$ such that in $V^{\mathcal{P}}$ for each $\eta < \omega_2$ with $\mathrm{cf}(\eta) = \omega_1$ there is an $\langle \omega \rangle_\eta {}^\frown \langle \lambda \rangle$-poset.

For $\eta = \omega_1$ it is straightforward to obtain a suitable $\mathcal{P}$: all we need is to plug Kosmider's strongly unbounded function into the original argument of Roitman. For $\omega_1 < \eta < \omega_2$ this simple approach does not work, but we can use the "stepping-up" method of Er-rhaimini and Veličkovic from [**41**]. Using this method, it will be enough to construct a single $\langle\omega\rangle_{\omega_1}{}^\frown\langle\lambda\rangle$-poset (with some extra properties) to obtain $\langle\omega\rangle_\eta{}^\frown\langle\lambda\rangle$-posets for each $\eta < \omega_2$ with $\mathrm{cf}(\eta) = \omega_1$.

To start with, we adapt the notion of a skeleton introduced in [**41**] to the cardinal sequences we are considering.

Definition 1.8.8. Assume that $\mathcal{T} = \langle T, \prec\rangle$ is an $\mathfrak{s}$-poset such that $\mathfrak{s}$ is a cardinal sequence of the form $\langle\kappa\rangle_\mu{}^\frown\langle\lambda\rangle$ where κ, λ are infinite cardinals with $\kappa < \lambda$ and μ is a non-zero ordinal. Let i be the infimum function associated with $\mathcal{T}$. Then for $\gamma < \mu$ we say that T_γ, the γth-level of $\mathcal{T}$, is a *bone level* iff the following holds:

(1) $i\{s, t\} = \emptyset$ for every $s, t \in T_\gamma$ with $s \neq t$.
(2) If $x \in T_{\gamma+1}$ and $y \prec x$ then there is a $z \in T_\gamma$ with $y \preceq z \prec x$.

We say that $\mathcal{T}$ is a *μ-skeleton* iff T_γ is a bone level of $\mathcal{T}$ for each $\gamma < \mu$.

The next statement can be proved by a straightforward modification of the proof of [**41**, Theorem 2.8].

Theorem 1.8.9. *Let κ, λ be infinite cardinals. If there is a $\langle\kappa\rangle_{\kappa^+}{}^\frown\langle\lambda\rangle$-poset which is a κ^+-skeleton, then for each $\eta < \kappa^{++}$ with $cf(\eta) = \kappa^+$ there is a $\langle\kappa\rangle_\eta{}^\frown\langle\lambda\rangle$-poset.*

So, to get Theorem 1.8.7 it is enough to prove the following result.

Theorem 1.8.10. *Let λ be a cardinal with $\lambda \geq \omega_3$. Assume that there is an ω_1- strongly unbounded function on λ. Then, in some c.c.c. generic extension there is an $\langle\omega\rangle_{\omega_1}{}^\frown\langle\lambda\rangle$-poset which is an ω_1-skeleton.*

Let $F : [\lambda]^2 \to \omega_1$ be an ω_1- strongly unbounded function on λ. In order to prove Theorem 1.8.10, we shall define a c.c.c. forcing notion $\mathcal{P} = \langle P, \leq\rangle$ that adjoins an $\mathfrak{s}$-poset $\mathcal{T} = \langle T, \preceq\rangle$ which is an ω_1-skeleton, where $\mathfrak{s}$ is the cardinal sequence $\langle\omega\rangle_{\omega_1}{}^\frown\langle\lambda\rangle$.

So, the underlying set of the required $\mathfrak{s}$-poset is the set $T = \bigcup\{T_\alpha : \alpha \leq \omega_1\}$ where $T_\alpha = \{\alpha\} \times \omega$ for $\alpha < \omega_1$ and $T_{\omega_1} = \{\omega_1\} \times \lambda$. If $s = (\alpha, \nu) \in T$, we write $\pi(s) = \alpha$ and $\xi(s) = \nu$.

Then, we define the poset $\mathcal{P} = \langle P, \leq\rangle$ as follows. We say that $p = \langle X, \preceq, i\rangle \in P$ iff the following conditions hold:

(P1) X is a finite subset of T.
(P2) $\preceq$ is a partial order on X such that $s \prec t$ implies $\pi(s) < \pi(t)$.
(P3) $i : [X]^2 \to [X]^{<\omega}$ is an infimum function, that is, a function such that for every $\{s, t\} \in [X]^2$ we have:

$$\forall x \in X([x \preceq s \wedge x \preceq t] \text{ iff } x \preceq v \text{ for some } v \in i\{s, t\}).$$

(P4) If $s, t \in X \cap T_{\omega_1}$ and $v \in i\{s, t\}$, then $\pi(v) \in F\{\xi(s), \xi(t)\}$.
(P5) If $s, t \in X$ with $\pi(s) = \pi(t) < \omega_1$, then $i\{s, t\} = \emptyset$.
(P6) If $s, t \in X$, $s \prec t$ and $\pi(t) = \alpha + 1$, then there is a $u \in X$ such that $s \preceq u \prec t$ and $\pi(u) = \alpha$.

Now, we define $\leq$ as follows: $\langle X', \preceq', i'\rangle \leq \langle X, \preceq, i\rangle$ iff $X \subseteq X'$, $\preceq = \preceq' \cap (X \times X)$ and $i \subseteq i'$.

We will need condition (P4) in order to show that $\mathcal{P}$ is c.c.c.

Lemma 1.8.11. *Assume that $p = \langle X, \preceq, i\rangle \in P$, $t \in X$, $\alpha < \pi(t)$ and $n < \omega$. Then, there is a $p' = \langle X', \preceq', i'\rangle \in P$ with $p' \leq p$ and there is an $s \in X' \setminus X$ with $\pi(s) = \alpha$ and $\xi(s) > n$ such that, for every $x \in X$, $s \preceq' x$ iff $t \preceq' x$.*

Proof. Let $L = \{\alpha\} \cup \{\xi : \alpha < \xi < \pi(t) \wedge \exists j < \omega\ \xi + j = \pi(t)\}$. Let $\alpha = \alpha_0, \ldots, \alpha_\ell$ be the increasing enumeration of L. Since X is finite, we can pick an $s_j \in T_{\alpha_j} \setminus X$ with $\xi(s_j) > n$ for $j \leq \ell$. Let $X' = X \cup \{s_j : j \leq \ell\}$ and let

$$\prec' = \prec \cup \{(s_j, y) : j \leq l, t \preceq y\} \cup \{(s_j, s_k) : j < k \leq \ell\}.$$

Now, we put $i'\{x,y\} = i\{x,y\}$ if $x,y \in X$, $i'\{s_j,y\} = \{s_j\}$ if $t \preceq y$, $i'\{s_j,s_k\} = s_{\min(j,k)}$, and $i'\{s_j,y\} = \emptyset$ otherwise. Clearly, $\langle X', \preceq', i'\rangle$ is as required. □

Lemma 1.8.12. *If $\mathcal{P}$ preserves cardinals, then $\mathcal{P}$ adjoins an $\langle\omega\rangle_{\omega_1}{}^\frown\langle\lambda\rangle$-poset which is an ω_1-skeleton.*

Proof. Let $\mathcal{G}$ be a $\mathcal{P}$-generic filter. We put $p = \langle X_p, \preceq_p, i_p\rangle$ for $p \in \mathcal{G}$. By Lemma 10 and standard density arguments, we have

$$T = \bigcup\{X_p : p \in \mathcal{G}\}, \tag{1.46}$$

and taking

$$\preceq = \bigcup\{\preceq_p : p \in \mathcal{G}\}, \tag{1.47}$$

the poset $\langle T, \preceq\rangle$ is an $\langle\omega\rangle_{\omega_1}{}^\frown\langle\lambda\rangle$-poset. Especially, Lemma 1.8.11 ensures that $\langle T, \preceq\rangle$ satisfies (4) in Definition 1.8.5. Properties (P5) and (P6) guarantee that $\langle T, \preceq\rangle$ is an ω_1-skeleton. □

Now, we prove the key lemma for showing that $\mathcal{P}$ adjoins the required poset.

Lemma 1.8.13. *$\mathcal{P}$ is c.c.c.*

Proof. Assume that $R = \langle r_\nu : \nu < \omega_1\rangle \subseteq P$ with $r_\nu \neq r_\mu$ for $\nu < \mu < \omega_1$. For $\nu < \omega_1$, write $r_\nu = \langle X_\nu, \preceq_\nu, \mathrm{i}_\nu\rangle$ and put $L_\nu = \pi[X_\nu]$. By the Δ-System Lemma, we may suppose that the set $\{X_\nu : \nu < \omega_1\}$ forms a Δ-system with root X^*. By thinning out R again if necessary, we may assume that $\{L_\nu : \nu < \omega_1\}$ forms a Δ-system with root L^* in such a way that $X_\nu \cap T_\alpha = X_\mu \cap T_\alpha$ for every $\alpha \in L^* \setminus \{\omega_1\}$ and $\nu < \mu < \omega_1$. Without loss of generality, we may assume that $\omega_1 \in L^*$. Since $\beta \setminus \alpha$ is a countable set for $\alpha, \beta \in L^*$ with $\alpha < \beta < \omega_1$, we may suppose that $L^* \setminus \{\omega_1\}$ is an initial segment of L_ν for every $\nu < \omega_1$. Of course, this may require a further thinning out of R. Now, we put $Z_\nu = X_\nu \cap T_{\omega_1}$ for $\nu < \omega_1$. Without loss of generality, we may assume that the domains of the forcing conditions of R have the same size and that there is a natural number $n > 0$ with $|Z_\nu \setminus X^*| = |Z_\mu \setminus X^*| = n$ for $\nu < \mu < \omega_1$. We consider in T_{ω_1} the well-order induced by λ. Then, by thinning out R again if necessary, we may assume that for every $\{\nu,\mu\} \in [\omega_1]^2$ there is an order-preserving bijection $h = h_{\nu,\mu} : L_\nu \to L_\mu$ with $h \restriction L^* = L^*$ that lifts to an isomorphism of X_ν with X_μ satisfying the following:

(A) For every $\alpha \in L_\nu \setminus \{\omega_1\}$, $h(\alpha,\xi) = (h(\alpha),\xi)$.

(B) h is the identity on X^*.

(C) For every $i < n$, if x is the ith-element in $Z_\nu \setminus X^*$ and y is the ith-element in $Z_\mu \setminus X^*$, then $h(x) = y$.

(D) For every $x, y \in X_\nu$, $x \preceq_\nu y$ iff $h(x) \preceq_\mu h(y)$.

(E) For every $\{x,y\} \in [X_\nu]^2$, $h[i_\nu\{x,y\}] = i_\mu\{h(x),h(y)\}$.

Now, we deduce from condition (P4) and the fact that R is uncountable that if $\{x,y\} \in [X^*]^2$ then $i_\nu\{x,y\} \subseteq X^*$ for every $\nu < \omega_1$. So if $\{x,y\} \in [X^*]^2$, then $i_\nu\{x,y\} = i_\mu\{x,y\}$ for $\nu < \mu < \omega_1$.

Let $\delta = \max(L^* \setminus \{\omega_1\})$. Since F is an ω_1-strongly unbounded function on λ, there are ordinals ν, μ with $\nu < \mu < \omega_1$ such that if we put $a = \{\xi \in \lambda : (\omega_1,\xi) \in Z_\nu \setminus X^*\}$ and $a' = \{\xi \in \lambda : (\omega_1,\xi) \in Z_\mu \setminus X^*\}$, then $F\{\xi,\xi'\} > \delta$ for every $\xi \in a$ and every $\xi' \in a'$. Our purpose is to prove that r_ν and r_μ are compatible in $\mathcal{P}$. We put $p = r_\nu$ and $q = r_\mu$. And we write $p = \langle X_p, \preceq_p, \mathrm{i}_p\rangle$ and $q = \langle X_q, \preceq_q, \mathrm{i}_q\rangle$. Then, we define the extension $r = \langle X_r, \preceq_r, i_r\rangle$ of p and q as follows. We put $X_r = X_p \cup X_q$. We define $\preceq_r = \preceq_p \cup \preceq_q$. Note that $\preceq_r$ is a partial order on X_r, because $L^* \setminus \{\omega_1\}$ is an initial segment of $\pi[X_p]$ and $\pi[X_q]$. Now, we define the infimum function i_r. Assume that $\{x,y\} \in [X_r]^2$. We put $i_r\{x,y\} = i_p\{x,y\}$ if $x,y \in X_p$, and $i_r\{x,y\} = i_q\{x,y\}$ if $x,y \in X_q$. Suppose that $x \in X_p \setminus X_q$ and $y \in X_q \setminus X_p$. Note that x,y are not comparable in $\langle X_r, \preceq_r\rangle$ and there is no $u \in (X_p \cup X_q) \setminus X^*$ such that $u \preceq_r x, y$. Then, we define $i_r\{x,y\} = \{u \in X^* : u \prec_r x, y\}$. It is easy to check that $r \in P$, and so $r \leq p, q$. □

After finishing the proof of Theorem 1.8.4 for $\kappa = \omega$, try to prove it for $\kappa = \omega_1$. So, assume that $2^\omega = \omega_1$, $\omega_4 \leq \lambda$, and there is an ω_2-strongly unbounded function on λ. We want to find $\langle\omega_1\rangle_\eta{}^\frown\langle\lambda\rangle$-posets for each ordinal $\eta < \omega_3$ with $\mathrm{cf}(\eta) = \omega_2$ in some cardinal-preserving generic

extension. Since the "stepping-up" method of Er-rhaimini and Veličkovic worked for $\kappa = \omega$, it is natural to try to apply Theorem 1.8.9 for the case $\kappa = \omega_1$. That is, we can try to find a cardinal-preserving generic extension that contains an $\langle\omega_1\rangle_{\omega_2}{}^\frown\langle\lambda\rangle$-poset which is an ω_2-skeleton. For this, first we should consider the forcing construction given in [**66**, Section 4] to add an $\langle\omega_1\rangle_{\omega_2}{}^\frown\langle\omega_3\rangle$-poset, and then try to extend this construction to add the required ω_2-skeleton. However, the construction from [**66**] is σ-complete and requires that CH holds in the ground model. Then, the following results show that the forcing construction of an $\langle\omega_1\rangle_{\omega_2}{}^\frown\langle\lambda\rangle$-poset which is an ω_2-skeleton is quite hopeless, at least by using the standard forcing from [**66**].

If X is the topological space associated with a skeleton and $x \in X$, we denote by $t(x, X)$ the tightness of x in X.

Proposition 1.8.14. *Assume that $\mathcal{T} = \langle T, \prec\rangle$ is a μ-skeleton, $\alpha < \mu$ and $x \in I_{\alpha+1}(X_{\mathcal{T}})$. Then, $t(x, X_{\mathcal{T}}) = \omega$.*

Proof. Assume that $A \subseteq T$ and $x \in A'$. We can assume that $a \prec x$ for each $a \in A$.

Let

$$U = \{u \in I_\alpha(X_{\mathcal{T}}) : u \prec x \wedge \exists a_u \in A\ a_u \preceq u\}. \tag{1.48}$$

Since $y \prec x$ iff $y \preceq u$ for some $u \prec x$ with $u \in I_\alpha(X_{\mathcal{T}})$, the set U is infinite.

Pick $V \in \big[U\big]^\omega$, and put $B = \{a_v : v \in V\}$. We claim that $x \in B'$. Indeed, if $y \prec x$ then there is a $u \in I_\alpha(X_{\mathcal{T}})$ such that $y \preceq u \prec x$. So $|\{b \in B : b \preceq y\}| \leq 1$. Hence $y \notin B'$. However, B has an accumulation point because $B \subseteq U_{\mathcal{T}}(x)$ and $U_{\mathcal{T}}(x)$ is compact in $X_{\mathcal{T}}$. So, B should converge to x. □

Corollary 1.8.15. *If $\mathcal{T}$ is a μ-skeleton, then $\mu \leq |I_0(X_{\mathcal{T}})|^\omega$. Especially, under CH an $\langle\omega_1\rangle_{\omega_2}{}^\frown\langle\lambda\rangle$-poset can not be an ω_2-skeleton.*

Thus, we are unable to use Theorem 1.8.9 to prove Theorem 1.8.4 even for $\kappa = \omega_1$. Instead of this stepping-up method, in the next two sections we will construct $\langle\omega_1\rangle_\eta{}^\frown\langle\lambda\rangle$-posets directly using the method of orbits from [**84**]. This method was used to construct by forcing $\langle\omega_1\rangle_\eta$-posets for $\omega_2 \leq \eta < \omega_3$. It is not difficult to get an $\langle\omega_1\rangle_{\omega_2}$-poset by means of countable "approximations" of the required poset. However, for $\omega_2 \leq \eta < \omega_3$ we need the notion of orbit and a much more involved forcing to obtain $\langle\omega_1\rangle_\eta$-posets (see [**84**]).

1.8.2. Combinatorial notions. In this section, we define the combinatorial notions that will be used in the proof of Theorem 1.8.4.

If α, β are ordinals with $\alpha \leq \beta$ let

$$[\alpha, \beta) = \{\gamma : \alpha \leq \gamma < \beta\}. \tag{1.49}$$

We say that I is an *ordinal interval* iff there are ordinals α and β with $\alpha \leq \beta$ and $I = [\alpha, \beta)$. Then, we write $I^- = \alpha$ and $I^+ = \beta$.

Assume that $I = [\alpha, \beta)$ is an ordinal interval. If β is a limit ordinal, let $\mathrm{E}(I) = \{\varepsilon^I_\nu : \nu < \mathrm{cf}(\beta)\}$ be a cofinal closed subset of I having order type $\mathrm{cf}\,(\beta)$ with $\alpha = \varepsilon^I_0$, and then put

$$\mathcal{E}(I) = \{[\varepsilon^I_\nu, \varepsilon^I_{\nu+1}) : \nu < \mathrm{cf}\,(\beta)\}. \tag{1.50}$$

If $\beta = \beta' + 1$ is a successor ordinal, put $\mathrm{E}(I) = \{\alpha, \beta'\}$ and

$$\mathcal{E}(I) = \{[\alpha, \beta'), \{\beta'\}\}. \tag{1.51}$$

Now, for an infinite cardinal κ and an ordinal η with $\kappa^+ \leq \eta < \kappa^{++}$ and $\mathrm{cf}(\eta) = \kappa^+$, we define $\mathbb{I}_\eta = \bigcup\{\mathcal{I}_n : n < \omega\}$ where:

$$\mathcal{I}_0 = \{[0, \eta)\} \text{ and } \mathcal{I}_{n+1} = \bigcup\{\mathcal{E}(I) : I \in \mathcal{I}_n\}. \tag{1.52}$$

Note that $\mathbb{I}_\eta$ is a cofinal tree of intervals in the sense defined in [**84**]. So, the following conditions are satisfied:

(i) For every $I, J \in \mathbb{I}_\eta$, $I \subseteq J$ or $J \subseteq I$ or $I \cap J = \emptyset$.
(ii) If I, J are different elements of $\mathbb{I}_\eta$ with $I \subseteq J$ and J^+ is a limit, then $I^+ < J^+$.
(iii) $\mathcal{I}_n$ partitions $[0, \eta)$ for each $n < \omega$.

(iv) $\mathcal{I}_{n+1}$ refines $\mathcal{I}_n$ for each $n < \omega$.
(v) For every $\alpha < \eta$ there is an $I \in \mathbb{I}_\eta$ such that $I^- = \alpha$.

Then, for each $\alpha < \eta$ and $n < \omega$ we define $\mathrm{I}(\alpha, n)$ as the unique interval $I \in \mathcal{I}_n$ such that $\alpha \in I$. And for each $\alpha < \eta$ we define $n(\alpha)$ as the least natural number n such that there is an interval $I \in \mathcal{I}_n$ with $I^- = \alpha$. So if $n(\alpha) = k$, then for every $m \geq k$ we have $I(\alpha, m)^- = \alpha$.

Assume that $\alpha < \eta$. If $m < \mathrm{n}(\alpha)$, we define $o_m(\alpha) = \mathrm{E}(\mathrm{I}(\alpha, m)) \cap \alpha$. Then, we define the *orbit* of α (with respect to $\mathbb{I}_\eta$) as

$$o(\alpha) = \bigcup \{o_m(\alpha) : m < \mathrm{n}(\alpha)\}. \tag{1.53}$$

For basic facts on orbits and trees of intervals, we refer the reader to [**84**, Section 1]. In particular, we have $|o(\alpha)| \leq \kappa$ for every $\alpha < \eta$.

We write $E([0, \eta)) = \{\varepsilon_\nu : \nu < \kappa^+\}$.

Claim 1.8.15.1. *$o(\varepsilon_\nu) = \{\varepsilon_\zeta : \zeta < \nu\}$ for $\nu < \kappa^+$.*

Proof. Clearly $I(\varepsilon_\nu, 0) = [0, \eta)$ and $I(\varepsilon_\nu, 1) = [\varepsilon_\nu, \varepsilon_{\nu+1})$. So $n(\varepsilon_\nu) = 1$. Thus $o(\varepsilon_\nu) = o_0(\varepsilon_\nu) = E(I(\varepsilon_\nu, 0)) \cap \varepsilon_\nu = E([0, \eta)) \cap \varepsilon_\nu = \{\varepsilon_\zeta : \zeta < \nu\}$. □

For $\alpha < \beta < \eta$ let

$$j(\alpha, \beta) = \max\{j : I(\alpha, j) = I(\beta, j)\}, \tag{1.54}$$

and put

$$J(\alpha, \beta) = I(\alpha, j(\alpha, \beta) + 1). \tag{1.55}$$

For $\alpha < \eta$ let

$$J(\alpha, \eta) = I(\alpha, 1). \tag{1.56}$$

Claim 1.8.15.2. *If $\varepsilon_\zeta \leq \alpha < \varepsilon_{\zeta+1} \leq \beta \leq \eta$, then $J(\alpha, \beta) = [\varepsilon_\zeta, \varepsilon_{\zeta+1})$.*

Proof. For $\beta = \eta$, $J(\alpha, \beta) = I(\alpha, 1) = [\varepsilon_\zeta, \varepsilon_{\zeta+1})$.

Now assume that $\beta < \eta$. Since $I(\alpha, 0) = I(\beta, 0) = [0, \eta)$, but $I(\alpha, 1) = [\varepsilon_\zeta, \varepsilon_{\zeta+1})$ and $I(\beta, 1) = [\varepsilon_\xi, \varepsilon_{\xi+1})$ for some ε_ξ with $\varepsilon_{\zeta+1} \leq \varepsilon_\xi$, we have $j(\alpha, \beta) = 0$ and so $J(\alpha, \beta) = [\varepsilon_\zeta, \varepsilon_{\zeta+1})$. □

1.8.3. Proof of the Main Theorem 1.8.4. Suppose that κ, λ are infinite cardinals with $\kappa^{+++} \leq \lambda$ and $\kappa^{<\kappa} = \kappa$, η is an ordinal with $\kappa^+ \leq \eta < \kappa^{++}$ and $\mathrm{cf}(\eta) = \kappa^+$, and there is a κ^+-strongly unbounded function on λ. We will use a refinement of the arguments given in [**84**] and [**66**, Section 4].

First, we define the underlying set of our construction. For every ordinal $\alpha < \eta$, we put $T_\alpha = \{\alpha\} \times \kappa$. And we put $T_\eta = \{\eta\} \times \lambda$. We define $T = \bigcup\{T_\alpha : \alpha \leq \eta\}$. Let $T_{<\eta} = T \setminus T_\eta$. If $s = (\alpha, \nu) \in T$, we write $\pi(s) = \alpha$ and $\xi(s) = \nu$.

We put $\mathbb{I} = \mathbb{I}_\eta$. Also, we define $E = E([0, \eta)) = \{\varepsilon_\nu : \nu < \kappa^+\}$. Since there is a κ^+-strongly unbounded function on λ and $\mathrm{cf}(\eta) = \kappa^+$ there is a function $F : \left[\lambda\right]^2 \to E$ such that

($\star$) For every ordinal $\gamma < \eta$ and every family $A \subseteq [\lambda]^{<\kappa}$ of pairwise disjoint sets with $|A| = \kappa^+$, there are different $a, b \in A$ such that $F\{\alpha, \beta\} > \gamma$ for every $\alpha \in a$ and $\beta \in b$.

Let $\Lambda \in \mathbb{I}$ and $\{s, t\} \in [T]^2$ with $\pi(s) < \pi(t)$. We say that Λ *isolates s from t* iff $\Lambda^- < \pi(s) < \Lambda^+$ and $\Lambda^+ \leq \pi(t)$.

Now we define the poset $\mathcal{P} = \langle P, \leq \rangle$ as follows. We say that $p = \langle X, \preceq, i \rangle \in P$ iff the following conditions hold:

(P1) $X \in [T]^{<\kappa}$.
(P2) $\preceq$ is a partial order on X such that $s \prec t$ implies $\pi(s) < \pi(t)$.
(P3) $\mathrm{i} : \left[X\right]^2 \to X \cup \{\mathrm{undef}\}$ is an infimum function, that is, a function such that for every $\{s, t\} \in \left[X\right]^2$ we have:

$$\forall x \in X([x \preceq s \wedge x \preceq t] \text{ iff } x \preceq \mathrm{i}\{s, t\}).$$

(P4) If $s, t \in X$ are compatible but not comparable in $\langle X, \preceq \rangle$, $v = i\{s,t\}$ and $\pi(s) = \alpha_1$, $\pi(t) = \alpha_2$ and $\pi(v) = \beta$, we have:
  (a) If $\alpha_1, \alpha_2 < \eta$, then $\beta \in o(\alpha_1) \cap o(\alpha_2)$.
  (b) If $\alpha_1 < \eta$ and $\alpha_2 = \eta$, then $\beta \in o(\alpha_1) \cap E$.
  (c) If $\alpha_1 = \eta$ and $\alpha_2 < \eta$, then $\beta \in o(\alpha_2) \cap E$.
  (d) If $\alpha_1 = \alpha_2 = \eta$, then $\beta \in F\{\xi(s), \xi(t)\} \cap E$.

(P5) If $s, t \in X$ with $s \preceq t$ and $\Lambda = J(\pi(s), \pi(t))$ isolates s from t, then there is a $u \in X$ such that $s \preceq u \preceq t$ and $\pi(u) = \Lambda^+$.

Now, we define $\leq$ as follows: $\langle X', \preceq', \mathrm{i}' \rangle \leq \langle X, \preceq, \mathrm{i} \rangle$ iff $X \subseteq X'$, $\preceq = \preceq' \cap (X \times X)$ and $\mathrm{i} \subseteq \mathrm{i}'$.

Lemma 1.8.16. *Assume that $p = \langle X, \preceq, \mathrm{i} \rangle \in P$, $t \in X$, $\alpha < \pi(t)$ and $\nu < \kappa$. Then, there is a $p' = \langle X', \preceq', \mathrm{i}' \rangle \in P$ with $p' \leq p$ and there is an $s \in X' \setminus X$ with $\pi(s) = \alpha$ and $\xi(s) > \nu$ such that, for every $x \in X$, $s \preceq' x$ iff $t \preceq' x$.*

Proof. Since $|X| < \kappa$, we can take an $s \in T_\alpha \setminus X$ with $\xi(s) > \nu$. Let $\{I_0, \ldots, I_n\}$ be the list of all the intervals in $\mathbb{I}$ that isolate s from t in such a way that $I_0^+ > I_1^+ > \cdots > I_n^+$. Put $\gamma_i = I_i^+$ for $i \leq n$. We take points $c_i \in T \setminus X$ with $\pi(c_i) = \gamma_i$ for $i \leq n$. Let $X' = X \cup \{s\} \cup \{c_i : i \leq n\}$ and let

$$\prec' = \prec \cup \{\langle s, c_i \rangle : i \leq n\} \cup \{\langle s, y \rangle : t \preceq y\} \cup \{\langle c_j, c_i \rangle : i < j\} \cup \{\langle c_i, y \rangle : i \leq n, t \preceq y\}.$$

Note that, for $z \in X'$ and $y \in \{s\} \cup \{c_i : i \leq n\}$, either z and y are comparable or they are incompatible with respect to $\preceq'$. So, the definition of i' is clear.

Finally, observe that p' satisfies (P5) because if $x \prec' y$ with $x \in \{s\} \cup \{c_i : i \leq n\}$, $y \in X'$ and $J(\pi(x), \pi(y))$ isolates x from y then either $J(\pi(x), \pi(y)) = I_k$ for some $0 \leq k \leq n$ or $J(\pi(x), \pi(y)) = J(\pi(t), \pi(y))$. But if $J(\pi(x), \pi(y)) = I_k$, then c_k witnesses (P5) for x and y; and if $J(\pi(x), \pi(y)) = J(\pi(t), \pi(y))$, we are done by condition (P5) for p. □

For $p \in P$ we write $p = \langle X_p, \preceq_p, i_p \rangle$, $Y_p = X_p \cap T_{<\eta}$ and $Z_p = X_p \cap T_\eta$.

Lemma 1.8.17. *If $\mathcal{P}$ preserves cardinals, then forcing with $\mathcal{P}$ adjoins an LCS* space with cardinal sequence $\langle \kappa \rangle_\eta {}^\frown \langle \lambda \rangle$.*

Proof. Let $\mathcal{G}$ be a $\mathcal{P}$-generic filter. Then

$$T = \bigcup \{X_p : p \in \mathcal{G}\}, \tag{1.57}$$

and taking

$$\preceq = \bigcup \{\preceq_p : p \in \mathcal{G}\} \tag{1.58}$$

the poset $\langle T, \preceq \rangle$ is a $\langle \kappa \rangle_\eta {}^\frown \langle \lambda \rangle$-poset. Especially, Lemma 1.8.16 guarantees that $\langle T, \prec \rangle$ satisfies (4) from Definition 1.8.5. So, by Lemma 1.8.6, in $V[\mathcal{G}]$ there is an LCS* space with cardinal sequence $\langle \kappa \rangle_\eta {}^\frown \langle \lambda \rangle$. □

To complete our proof we should check that forcing with P preserves cardinals. It is straightforward that $\mathcal{P}$ is κ-closed. The burden of our proof is to verify the following statement, which completes the proof of Theorem 1.8.4.

Lemma 1.8.18. *$\mathcal{P}$ has the κ^+-chain condition.*

Define the subposet $\mathcal{P}_\eta = \langle P_\eta, \leq_\eta \rangle$ of $\mathcal{P}$ as follows:

$$P_\eta = \{p \in P : X_p \subseteq \eta \times \kappa\}, \tag{1.59}$$

and let $\leq_\eta = \leq \restriction P_\eta$. The poset $\mathcal{P}_\eta$ was defined in [**84**, Definition 2.1], and it was proved that $\mathcal{P}_\eta$ satisfies the κ^+-chain condition. In [**84**, Lemmas 2.5 and 2.6] it was shown that every set $R \in \left[P_\eta\right]^{\kappa^+}$ has a linked subset of size κ^+. Actually, a stronger statement was proved, and we will use that statement to prove Lemma 1.8.18. However, before doing so, we need some preparation.

Definition 1.8.19. Suppose that $g : A \to B$ is a bijection, where $A, B \in \left[T\right]^{<\kappa}$. We say that g is *adequate* iff the following conditions hold:

(1) $g[A \cap T_{<\eta}] = B \cap T_{<\eta}$ and $g[A \cap T_\eta] = B \cap T_\eta$.
(2) For every $s, t \in A$, $\pi(s) < \pi(t)$ iff $\pi(g(s)) < \pi(g(t))$.
(3) For every $s = \langle \alpha, \nu \rangle \in A \cap T_{<\eta}$, $g(\alpha, \nu) = (\beta, \zeta)$ implies $\nu = \zeta$.
(4) For every $s, t \in A \cap T_\eta$, $\xi(s) < \xi(t)$ iff $\xi(g(s)) < \xi(g(t))$.

For $A, B \subseteq T_{<\eta}$, this definition is just [**84**, Definition 2.2].

Definition 1.8.20. A set $Z \subseteq P$ is *separated* iff the following conditions are satisfied:

(1) $\{X_p : p \in Z\}$ forms a Δ-system with root X.
(2) For each $\alpha < \eta$, either $X_p \cap T_\alpha = X \cap T_\alpha$ for every $p \in Z$, or there is at most one $p \in Z$ such that $X_p \cap T_\alpha \neq \emptyset$.
(3) For every $p, q \in Z$ there is an adequate bijection $h_{p,q} : X_p \to X_q$ which satisfies the following:
   (a) For any $s \in X$, $h_{p,q}(s) = s$.
   (b) If $s, t \in X_p$, then $s \prec_p t$ iff $h_{p,q}(s) \prec_q h_{p,q}(t)$.
   (c) If $s, t \in X_p$, then $h_{p,q}(\mathrm{i}_p\{s, t\}) = \mathrm{i}_q\{h_{p,q}(s), h_{p,q}(t)\}$.

For $Z \subseteq P_\eta$, this definition is just [**84**, Definition 2.3].

Lemma 1.8.21. *Assume that $Z \in \left[P\right]^{\kappa^+}$ is separated and X is the root of the Δ-system $\{X_p : p \in Z\}$. If s, t are compatible but not comparable in $p \in Z$ and $s \in X \cap T_{<\eta}$, then $\mathrm{i}_p\{s, t\} \in X$.*

Proof. Assume that s, t are compatible but not comparable in $p \in Z$ and $s \in X \cap T_{<\eta}$. Assume that $\mathrm{i}_p\{s, t\} \notin X$. Then since

$$\{\mathrm{i}_q\{s, h_{p,q}(t)\} : q \in Z\} = \{h_{p,q}(\mathrm{i}_p\{s, t\}) : q \in Z\}, \tag{1.60}$$

the elements of $\{\mathrm{i}_q\{s, h_{p,q}(t)\} : q \in Z\}$ are all different. But this is impossible, because $\pi(\mathrm{i}_q\{s, h_{p,q}(t)\}) \in o(s)$ for all $q \in Z$ and $|o(s)| \leq \kappa$. □

In [**84**, Lemmas 2.5 and 2.6], as we explain in the Appendix of this section, actually the following statement was proved.

Proposition 1.8.22. *For each subset $R \in \left[P_\eta\right]^{\kappa^+}$ there is a separated subset $Z \in \left[R\right]^{\kappa^+}$ and an ordinal $\gamma < \eta$ such that every $p, q \in Z$ have a common extension $r \in P_\eta$ such that the following holds:*

(R1) $\sup\ \pi[X_r \setminus (X_p \cup X_q)] < \gamma$.
(R2) *(a) $y \prec_r s$ iff $y \prec_r h_{p,q}(s)$ for each $s \in X_p$ and $y \in X_r \setminus (X_p \cup X_q)$,*
(b) $s \prec_r y$ iff $h_{p,q}(s) \prec_r y$ for each $s \in X_p$ and $y \in X_r \setminus (X_p \cup X_q)$,
(c) if $s \prec_r y$ for $s \in X_p \cup X_q$ and $y \in X_r \setminus (X_p \cup X_q)$, then there is a $w \in X_p \cap X_q$ with $s \preceq_r w \prec_r y$,
(d) for $s \in X_p \setminus X_q$ and $t \in X_q \setminus X_p$,

$$\begin{aligned} &s \prec_r t \text{ iff } \exists u \in X_p \cap X_q \text{ such that } s \prec_p u \prec_q t, \\ &t \prec_r s \text{ iff } \exists u \in X_p \cap X_q \text{ such that } t \prec_q u \prec_p s. \end{aligned} \tag{1.61}$$

After this preparation, we are ready to prove Lemma 1.8.18.

Proof of Lemma 1.8.18. We will argue in the following way. Assume that $R = \langle r_\nu : \nu < \kappa^+ \rangle \subseteq P$, where $r_\nu = \langle X_\nu, \preceq_\nu, \mathrm{i}_\nu \rangle$. For each $\nu < \kappa^+$ we will "push down" r_ν into P_η, more precisely, we will construct an isomorphic copy $r'_\nu \in P_\eta$ of r_ν. Using Proposition 1.8.22 we can find a separated subfamily $\{r'_\nu : \nu \in K\}$ of size κ^+ and an ordinal $\gamma < \eta$ such that for each $\nu, \mu \in K$ with $\nu \neq \mu$ there is a condition $r'_{\nu,\mu} \in P_\eta$ such that $r'_{\nu,\mu} \leq_\eta r'_\nu, r'_\mu$ and (R1)–(R2) hold, especially

$$\sup \pi[X'_{\nu,\mu} \setminus (X'_\nu \cup X'_\mu)] < \gamma. \tag{1.62}$$

Let X be the root of $\{X_\nu : \nu < \kappa^+\}$, $Y = X \setminus T_\eta$ and $\gamma_0 = \max(\gamma, \sup \pi[Y])$. Since F is κ^+-strongly unbounded, there are $\nu, \mu \in K$ with $\nu < \mu$ such that

$$\forall s \in (X_\nu \setminus X_\mu) \cap T_\eta \quad \forall t \in (X_\mu \setminus X_\nu) \cap T_\eta \quad F\{\xi(s), \xi(t)\} > \gamma_0. \tag{1.63}$$

Then we will be able to "pull back" $r' = r'_{\nu,\mu}$ into P to get a condition $r = r_{\nu,\mu}$ which is a common extension of r_ν and r_μ. Let us remark that r will not be an isomorphic copy of r', rather r will be a "homomorphic image" of r'.

Now we carry out our plan.

Since $\kappa^{<\kappa} = \kappa$, by thinning out our sequence we can assume that R itself is a separated set. So $\{X_r : r \in R\}$ forms a Δ-system with kernel $\bar{X}$. We write $\bar{Y} = \bar{X} \cap T_{<\eta}$ and $\bar{Z} = \bar{X} \cap T_\eta$.

Recall that $E = E([0,\eta)) = \{\varepsilon_\zeta : \zeta < \kappa^+\}$ is a closed unbounded subset of η.

Fix $\nu < \kappa^+$. Write $Y_\nu = X_\nu \cap T_{<\eta}$ and $Z_\nu = X_\nu \cap T_\eta$. Pick a limit ordinal $\zeta(\nu) < \kappa^+$ such that:

(i) $\sup(\pi[Y_\nu]) < \varepsilon_{\zeta(\nu)}$,
(ii) $\zeta(\mu) < \zeta(\nu)$ for $\mu < \nu$.

Let $\theta = \mathrm{tp}(\xi[Z_\nu])$ and $\alpha = \varepsilon_{\zeta(\nu)}$. We put $Z'_\nu = \{\langle \alpha, \xi\rangle : \xi < \theta\}$. Clearly, $Z'_\nu \subseteq T_{\varepsilon_{\zeta(\nu)}}$ and $\mathrm{tp}(\xi[Z'_\nu]) = \mathrm{tp}(\xi[Z_\nu])$. We consider in Z'_ν and Z_ν the well-orderings induced by κ and λ respectively. Put $X'_\nu = Y_\nu \cup Z'_\nu$, and let $g_\nu : X'_\nu \to X_\nu$ be the natural bijection, i.e. $g_\nu \restriction Y_\nu = id$ and $g_\nu(s) = t$ if for some $\xi < \mathrm{tp}(\xi[Z_\nu])$ s is the ξ-element in Z'_ν and t is the ξ-element in Z_ν.

Let $\bar{Z}'_\nu = g_\nu^{-1}\bar{Z}$. We define the condition $r'_\nu = \left\langle X'_\nu, \preceq'_\nu, \mathrm{i}'_\nu \right\rangle \in P_\eta$ as follows: for $s, t \in X'_\nu$ with $s \neq t$ we put

$$s \prec'_\nu t \text{ iff } g_\nu(s) \prec_\nu g_\nu(t), \tag{1.64}$$

and

$$\mathrm{i}'_\nu\{s,t\} = \mathrm{i}_\nu\{g_\nu(s), g_\nu(t)\}. \tag{1.65}$$

Claim 1.8.22.1. $r'_\nu \in P_\eta$.

Proof. (P1), (P2) and (P3) are clear because g_ν is an isomorphism between $r'_\nu = \left\langle X'_\nu, \preceq'_\nu, \mathrm{i}'_\nu \right\rangle$ and $r_\nu = \langle X_\nu, \preceq_\nu, \mathrm{i}_\nu\rangle$, moreover $\pi(s) < \pi(t)$ iff $\pi(g_\nu(s)) < \pi(g_\nu(t))$.

(P4) Since $X'_\nu \subseteq T_{<\eta}$ we should check just (a). So assume that $s', t' \in X'_\nu$ are compatible but not comparable in $\langle X'_\nu, \leq'_\nu\rangle$ and $v' = \mathrm{i}'_\nu\{s', t'\}$. Put $s = g_\nu(s')$, $t = g_\nu(t')$. Since $g_\nu \restriction Y_\nu = id$, we can assume that $\{s', t'\} \notin \left[Y_\nu\right]^2$, e.g. $s' \in Z'_\nu$ and so $s \in Z_\nu$.

First observe that $v' \in Y_\nu$, so $v' = g_\nu(v')$.

If $t' \in Y_\nu$, then $t' = g_\nu(t')$, and $v' = \mathrm{i}_\nu\{s, t'\}$. By applying (P4)(c) in r_ν for s and t' we obtain

$$\pi(v') \in E \cap o(\pi(t')) \subseteq E \cap \varepsilon_{\zeta(\nu)} \cap o(\pi(t')) = o(\pi(s')) \cap o(\pi(t')) \tag{1.66}$$

because $o(\pi(s')) = E \cap \varepsilon_{\zeta(\nu)}$ by Claim 1.8.15.1.

If $t' \in Z'_\nu$, then $t = g_\nu(t') \in Z_\nu \subseteq T_\eta$. Since $v' = i'_\nu\{s', t'\} = i_\nu\{s, t\}$, applying (P4)(d) in r_ν for s and t we obtain

$$\pi(v') \in F\{\xi(s), \xi(t)\} \cap E \cap \varepsilon_{\zeta(\nu)} \subseteq E \cap \varepsilon_{\zeta(\nu)} = o(\pi(s')) \cap o(\pi(t'))$$

because $o(\pi(s')) = o(\pi(t')) = E \cap \varepsilon_{\zeta(\nu)}$ by Claim 1.8.15.1.

(P5) Assume that $s', t' \in X'_\nu$, $s' \prec'_\nu t'$ and $\Lambda = J(\pi(s'), \pi(t'))$ isolates s' from t'. Then $s' \in Y_\nu$, so $g_\nu(s') = s'$. Since $g_\nu \restriction Y_\nu = id$, we can assume that $\{s', t'\} \notin \left[Y_\nu\right]^2$, i.e. $t' \in Z'_\nu$.

Write $t = g_\nu(t')$. Since $\pi(t') = \varepsilon_{\zeta(\nu)} \in E$, by Claim 1.8.15.2, $J(\pi(s'), \pi(t')) = J(\pi(s'), \pi(t)) = [\varepsilon_\zeta, \varepsilon_{\zeta+1}) = I(\pi(s'), 1)$, where $\varepsilon_\zeta \leq \pi(s') < \varepsilon_{\zeta+1}$. Applying (P5) in r_ν for s' and t, we obtain a $v \in Y_\nu$ such that $\pi(v) = \Lambda^+$ and $s' \prec_\nu v \prec_\nu t$. Then $g_\nu(v) = v$, so $s' \prec'_\nu v \prec'_\nu t'$, which was to be proved. $\square$

Now applying Proposition 23 to the family $\{r'_\nu : \nu < \kappa^+\}$, there are $K \in \left[\kappa^+\right]^{\kappa^+}$ and $\gamma < \eta$ such that $\{r'_\nu : \nu \in K\}$ is separated and for every $\nu, \mu \in K$ with $\nu \neq \mu$ there is a common extension $r' \in P_\eta$ of r'_ν and r'_μ such that (R1)-(R2) hold. Let $\gamma_0 = \max(\gamma, \sup \pi[\bar{Y}])$. Recall that $\bar{Y}$ is the root of the Δ-system $\{Y_\nu : \nu \in \kappa^+\}$. For $\nu < \mu < \kappa^+$ we denote by $h'_{\nu,\mu}$ the adequate bijection $h_{r'_\nu, r'_\mu}$.

Since F satisfies $(\star)$, there are $\nu, \mu \in K$ with $\nu \neq \mu$ such that for each $s \in (Z_\nu \setminus Z_\mu)$ and $t \in (Z_\mu \setminus Z_\nu)$ we have

$$F\{\xi(s), \xi(t)\} > \gamma_0. \tag{1.67}$$

We show that the conditions r_ν and r_μ have a common extension $r = \langle X, \preceq, \mathrm{i}\rangle \in P$.

Consider a condition $r' = \langle X', \preceq', \mathrm{i}'\rangle$ which is a common extension of r'_ν and r'_μ and satisfies (R1)–(R2). We define the condition $r = \langle X, \preceq, \mathrm{i}\rangle$ as follows. Let

$$X = (X' \setminus (Z'_\nu \cup Z'_\mu)) \cup (Z_\nu \cup Z_\mu). \tag{1.68}$$

Write $U = X' \setminus (Z'_\nu \cup Z'_\mu) = X \setminus (Z_\nu \cup Z_\mu)$ and $V = X' \setminus (X'_\nu \cup X'_\mu)$. Clearly, $V \subseteq U$. We define the function $h : X' \to X$ as follows:

$$h = g_\nu \cup g_\mu \cup (id \restriction U). \tag{1.69}$$

Then h is well-defined, h is onto, $h \restriction X' \setminus (\bar{Z}'_\nu \cup \bar{Z}'_\mu)$ is injective, and $h[\bar{Z}'_\nu] = h[\bar{Z}'_\mu] = \bar{Z}$.

Now, if $s, t \in X$ we put

$$s \prec t \text{ iff there is a } t' \in X' \text{ with } h(t') = t \text{ and } s \prec' t'. \tag{1.70}$$

Finally, we define the meet function i on $\left[X\right]^2$ as follows:

$$\mathrm{i}\{s,t\} = \max_{\prec'}\{\mathrm{i}'\{s',t'\} : \ h(s') = s \text{ and } h(t') = t\}. \tag{1.71}$$

We will prove in the following claim that the definition of the function i is meaningful. Then, the proof of Lemma 1.8.18 will be complete as soon as we verify that $r \in P$ and $r \leq r_\nu, r_\mu$.

Claim 1.8.22.2. i *is well-defined by (1.71), moreover* $\mathrm{i} \supseteq \mathrm{i}_\nu \cup \mathrm{i}_\mu$.

Proof. We need to verify that the maximum in (1.71) does exist when we define $\mathrm{i}\{s,t\}$. So, suppose that $\{s,t\} \in [X]^2$.

If $\{s,t\} \in \left[X \setminus \bar{Z}\right]^2$ then there is exactly one pair (s',t') such that $h(s') = s$ and $h(t') = t$, and hence there is no problem in (1.71). So if $\{s,t\} \in \left[X_\nu\right]^2$ then $\mathrm{i}\{s,t\} = \mathrm{i}'\{s',t'\} = \mathrm{i}_\nu\{s,t\}$ by the construction of r'_ν. If $\{s,t\} \in \left[X_\mu\right]^2$ proceeding similarly we obtain $\mathrm{i}\{s,t\} = \mathrm{i}'\{s',t'\} = \mathrm{i}_\mu\{s,t\}$.

So we can assume that e.g. $s \in \bar{Z}$. Then $h^{-1}(s) = \{s', s''\}$ for some $s' \in \bar{Z}'_\nu$ and $s'' \in \bar{Z}'_\mu$.

First assume that $t \notin \bar{Z}$, so there is exactly one $t' \in X'$ with $h(t') = t$. We distinguish the following cases.

Case 1. $t \in V$.

Note that since $t \in V$, $t = t'$. We show that $\mathrm{i}'\{s',t\} = \mathrm{i}'\{s'',t\}$.

Let $v = \mathrm{i}'\{s',t\}$. Assume that $v \in X'_\nu \cup X'_\mu$. Then, by (R2)(c), $v \prec' t$ and $t \in V$ imply that there is a $w \in \bar{Y} = X'_\nu \cap X'_\mu$ such that $v \preceq' w \prec' t$. Thus $v = \mathrm{i}'\{s',w\}$ and $\mathrm{i}'\{s',w\} = \mathrm{i}'_\nu\{s',w\} = \mathrm{i}_\nu\{s,w\} \in \bar{Y}$ by Lemma 1.8.21 for $w \in \bar{Y}$. Clearly, $v \prec' t, s''$. Hence $v \preceq' \mathrm{i}'\{s'',t\}$.

Now assume that $v \in V$. Then $v \prec' s'$ implies $v \prec' h'_{\nu,\mu}(s') = s''$ by (R2)(a). So $v \prec' t, s''$, thus $\mathrm{i}'\{s',t\} \preceq' \mathrm{i}'\{s'',t\}$.

So, in both cases $\mathrm{i}'\{s',t\} \preceq' \mathrm{i}'\{s'',t\}$. But s' and s'' are symmetrical, hence $\mathrm{i}'\{s'',t\} \preceq' \mathrm{i}'\{s',t\}$, and so we are done.

Case 2. $t \in X_\nu \setminus \bar{Z}$.

We show that in this case $\mathrm{i}'\{s'',t'\} \preceq' \mathrm{i}'\{s',t'\}$.

Let $v = \mathrm{i}'\{s'',t'\}$. If $v \in V$, then $v \prec' s''$ and $h'_{\nu,\mu}(s') = s''$ imply $v \prec' s'$ by (R2)(a). Thus $v \preceq' s', t'$, and so $v \preceq' \mathrm{i}'\{s',t'\}$.

Now assume that $v \in X'_\nu \cup X'_\mu$. If $v \in \bar{Y} = X'_\nu \cap X'_\mu$, then $v \prec' s'$, so $v \prec' \mathrm{i}'\{s',t'\}$. We show that it is not possible that $v \notin \bar{Y}$. For this, assume that $v \in (X'_\nu \cup X'_\mu) \setminus \bar{Y}$. Without loss of generality, we may suppose that $v \in X'_\nu \setminus X'_\mu$. Then, by (R2)(d), there is a $w \in \bar{Y}$ such that $v \prec' w \prec' s''$. Thus $v = \mathrm{i}'\{w,t'\} = \mathrm{i}'_\nu\{w,t'\} \in \bar{Y}$ by Lemma 1.8.21.

Moreover, $\{s,t\} \in \left[X_\nu\right]^2$ and $\mathrm{i}\{s,t\} = \mathrm{i}'\{s',t'\} = \mathrm{i}_\nu\{s,t\}$ because $g_\nu(s') = h(s') = s$ and $g_\nu(t') = h(t') = t$.

Case 3. $t \in X_\mu \setminus \bar{Z}$.

Proceeding as in Case 2, we can show that $\mathrm{i}'\{s',t'\} \preceq' \mathrm{i}'\{s'',t'\} = \mathrm{i}_\mu\{s,t\}$.

Finally, assume that $t \in \bar{Z}$. Then $h^{-1}(t) = \{t', t''\}$ for some $t' \in \bar{Z}'_\nu$ and $t'' \in \bar{Z}'_\mu$.

Note that by Cases (2) and (3),

$$\mathrm{i}'\{s'',t'\} \preceq' \mathrm{i}'\{s',t'\} \text{ and } \mathrm{i}'\{s',t''\} \preceq' \mathrm{i}'\{s'',t''\}.$$

Since $\mathrm{i}'\{s',t'\} = \mathrm{i}_\nu\{s,t\} = \mathrm{i}_\mu\{s,t\} = \mathrm{i}'\{s'',t''\}$ by the construction of r'_ν and r'_μ, we have

$$\mathrm{i}'\{s',t'\} = \mathrm{i}'\{s'',t''\} = \max_{\prec'}(\mathrm{i}'\{s',t'\}, \mathrm{i}'\{s'',t'\}, \mathrm{i}'\{s',t''\}, \mathrm{i}'\{s'',t''\}). \tag{1.72}$$

Moreover, in this case $\{s,t\} \in \left[X_\nu\right]^2 \cap \left[X_\mu\right]^2$ and we have just proved that $\mathrm{i}\{s,t\} = \mathrm{i}_\nu\{s,t\} = \mathrm{i}_\mu\{s,t\}$. □

By Claim 1.8.22.2 above, r is well-defined. Since $\mathrm{i} \supseteq \mathrm{i}_\nu \cup \mathrm{i}_\mu$, it is easy to check that if $r \in P$ then $r \leq r_\nu, r_\mu$. So, the following claim completes the verification of the chain condition.

Claim 1.8.22.3. $r \in P$.

Proof. (P1) and (P2) are clear.

(P3) Assume that $\{s,t\} \in [X]^2$. Without loss of generality, we may assume that s,t are compatible but not comparable in $\langle X, \preceq\rangle$. Note that by (1.70), (1.71) and condition (P3) for r', we have $\mathrm{i}\{s,t\} \prec s,t$. So, we have to show that if $v \prec s,t$ then $v \preceq \mathrm{i}\{s,t\}$.

Assume that $v \prec s,t$. Then, $v \in U$ and there are $s',t' \in X'$ such that $h(s') = s$, $h(t') = t$ and $v \prec' s',t'$. By (P3) for r', $v \preceq' \mathrm{i}'\{s',t'\}$. Now as $v, \mathrm{i}'\{s',t'\}, \mathrm{i}\{s,t\} \in U$ and $h \restriction U = id$, we infer from (1.71) that $v \preceq' \mathrm{i}'\{s',t'\} \preceq' \mathrm{i}\{s,t\}$ and hence $v \preceq \mathrm{i}\{s,t\}$.

(P4) Assume that $s,t \in X$ are compatible but not comparable in $\langle X, \preceq\rangle$. Let $v = \mathrm{i}\{s,t\}$.

(a) In this case $\pi(s), \pi(t) < \eta$. Then $s,t \in X \setminus (Z_\nu \cup Z_\mu) = U$, so $h(s) = s$ and $h(t) = t$. Thus $\mathrm{i}\{s,t\} = \mathrm{i}'\{s,t\}$. Hence, it follows from condition (P4)(a) for r' that $\pi(\mathrm{i}\{s,t\}) \in o(s) \cap o(t)$.

(b) In this case $\pi(s) < \eta$ and $\pi(t) = \eta$. Then $s \in X \setminus (Z_\nu \cup Z_\mu) = U$ and $t \in Z_\nu \cup Z_\mu$.

By (1.71) and Claim 1.8.22.2, there is a $t^* \in Z'_\nu \cup Z'_\mu$ such that $h(t^*) = t$ and $\mathrm{i}\{s,t\} = \mathrm{i}'\{s,t^*\}$.

Now, applying (P4)(a) for r', we infer that $\pi(v) \in o(s) \cap o(t^*)$. Since $\pi(t^*) \in E$, we have $o(t^*) \subseteq E$ by Claim 1.8.15.1. Then we deduce that $\pi(v) \in o(s) \cap E$, which was to be proved.

(c) The same as (b).

(d) In this case $\pi(s) = \pi(t) = \eta$. If $\{s,t\} \in \left[Z_\nu\right]^2$ then $\mathrm{i}\{s,t\} = \mathrm{i}_\nu\{s,t\}$, and by (P4)(d) for r_ν, we deduce that $\pi(\mathrm{i}\{s,t\}) \in F\{\xi(s), \xi(t)\} \cap E$. A parallel argument works if $s,t \in Z_\mu$.

So we can assume that $s \in Z_\nu \setminus Z_\mu$ and $t \in Z_\mu \setminus Z_\nu$. Note that there are a unique $s' \in Z'_\nu$ with $h(s') = s$ and a unique $t' \in Z'_\mu$ with $h(t') = t$. Then, $v = \mathrm{i}\{s,t\} = \mathrm{i}'\{s',t'\} \in U$. Hence either $v \in V$, or $v \in X_\nu \cup X_\mu$ and in this case there is a $w \in X_\nu \cap X_\mu$ with $v \preceq' w$ by (R2)(d).

In both cases $\pi(v) \leq \gamma_0$. Note that, applying (P4)(a) in r' for s', t' and $v = \mathrm{i}'\{s',t'\}$, we obtain $\pi(v) \in o(s') \cap o(t')$. Since $\pi(s'), \pi(t') \in E$ we have $o(s') \cup o(t') \subseteq E$ by Claim 1.8.15.1. Thus $\pi(v) \in E$. And since $\pi(v) \leq \gamma_0$, we have $\pi(v) \in F\{\xi(s), \xi(t)\} \cap E$, which was to be proved.

(P5) Assume that $s,t \in X$, $s \prec t$ and $\Lambda = J(\pi(s), \pi(t))$ isolates s from t. Then $s \notin T_\eta$, so $h(s) = s$.

If $t \notin T_\eta$ then $h(t) = t$, so we are done because r' satisfies (P5).

Assume that $t \in T_\eta$. As $s \prec t$, there is a $t' \in T_{\varepsilon_{\zeta(\nu)}} \cup T_{\varepsilon_{\zeta(\mu)}}$ such that $h(t') = t$ and $s \prec' t'$. Since $\pi(t') \in E$, by Claim 1.8.15.2 we have $J(\pi(s), \pi(t')) = I(\pi(s), 1) = J(\pi(s), \pi(t))$. Applying (P5) in r' for s and t', we obtain a $v \in X'$ such that $s \prec' v \preceq' t'$ and $\pi(v') = \Lambda^+$. But as $\zeta(\nu), \zeta(\mu)$ are limit ordinals, we have $v \prec' t'$, and hence $v \in X' \setminus (Z'_\nu \cup Z'_\mu) = U$. Then $h(v) = v$, so $s \prec v \prec t$, which was to be proved. □

Hence we have proved that $\mathcal{P}$ satisfies the κ^+-chain condition, which completes the proof of Theorem 1.8.4. □

1.8.4. Appendix. We explain in detail how Proposition 23 was proved in [**84**].

Assume that $Z \subseteq P_\eta$ is a separated set. Let $\bar{X}$ be the root of $\{X_p : p \in Z\}$. For every $n \in \omega$ and every $I \in \mathcal{I}_n$ with $\mathrm{cf}(I^+) = \kappa^+$, we define $\xi(I) =$ the least ordinal γ such that $\varepsilon^I_\gamma \supseteq \pi[\bar{X}] \cap I$ and we put $\gamma(I) = \varepsilon^I_{\xi(I)+\kappa}$. Now for every $\alpha < \eta$, if there is an $n < \omega$ and an interval $I \in \mathcal{I}_n$ with $\mathrm{cf}(I^+) = \kappa^+$ such that $\alpha \in I$ and $\gamma(I) \leq \alpha$, we consider the least natural number k with this property and write $I(\alpha) = I(\alpha, k)$. Otherwise, we write $I(\alpha) = \{\alpha\}$. Then we say that Z is

pairwise equivalent iff for every $p, q \in Z$ and every $s \in X_p$, $I(\pi(s)) = I(\pi(h_{p,q}(s)))$. In [**84**], the following two lemmas were proved:

Lemma 1.8.23 ([**84**, Lemma 2.5])**.** *Every set in* $\left[P_\eta\right]^{\kappa^+}$ *has a pairwise equivalent subset of size* κ^+.

Lemma 1.8.24 ([**84**, Lemma 2.6])**.** *A pairwise equivalent set* $Z \subseteq P_\eta$ *of size* κ^+ *is linked.*

To get Proposition 23 we explain that the proof of [**84**, Lemma 2.6] actually gives the following statement:

If $Z \subseteq P_\eta$ *is a pairwise equivalent set of size* κ^+*, then there is an ordinal* $\gamma < \eta$ *such that every* $p, q \in Z$ *have a common extension* $r \in P_\eta$ *satisfying* $(R1)$-$(R2)$.

As above, we denote by $\bar{X}$ the root of $\{X_p : p \in Z\}$. Assume that $p, q \in Z$ with $p \neq q$. First observe that the ordering $\prec_r$ is defined in [**84**, Definition 2.4]. For this, adequate bijections $g_1 : X_r \setminus (X_p \cup X_q) \to X_p \setminus \bar{X}$ and $g_2 : X_r \setminus (X_p \cup X_q) \to X_q \setminus \bar{X}$ are considered in such a way that $g_2 = h_{p,q} \circ g_1$. Then since $g_2 = h_{p,q} \circ g_1$, [**84**, Definition 2.4](b) and (c) imply (R2)(a) and [**84**, Definition 2.4](d) and (f) imply (R2)(b). Also, (R2)(c) follows directly from [**84**, Definition 2.4](d) and (f), and (R2)(d) is just [**84**, Definition 2.4](e) and (g). So, we have verified (R2).

To check (R1), i.e. to get the right γ we need a bit more work. Let

$$\mathcal{J} = \{I(\pi(s)) : s \in X_p\} \tag{1.73}$$

where $p \in Z$. Since Z is pairwise equivalent, $\mathcal{J}$ does not depend on the choice of $p \in Z$. For every $I \in \mathbb{I}_\eta$ with $\mathrm{cf}(I^+) = \kappa^+$ we can choose a set $D(I) \in [E(I) \cap \gamma(I)]^\kappa$ unbounded in $\gamma(I)$. We claim that

$$\gamma = \sup(\bigcup\{D(I) : I \in \mathcal{J}\}) + 1 \tag{1.74}$$

works.

First observe that $\gamma < \eta$, because $\mathrm{cf}(\eta) = \kappa^+$, $|\mathcal{J}| < \kappa$ and $|D(I)| = \kappa$ for any $I \in \mathcal{J}$.

Now assume that $p, q \in Z$ with $p \neq q$. Write $L_p = \pi[X_p]$, $L_q = \pi[X_q]$ and $\bar{L} = \pi[\bar{X}]$. Let $\{\alpha_\xi : \xi < \delta\}$ and $\{\alpha'_\xi : \xi < \delta\}$ be the strictly increasing enumerations of $L_p \setminus \bar{L}$ and $L_q \setminus \bar{L}$ respectively. In the proof of [**84**, Lemma 2.6], for each $\xi < \delta$ an element $\beta_\xi \in D(I(\alpha_\xi)) = D(I(\alpha'_\xi))$ was chosen, and then a condition $r \leq_\eta p, q$ was constructed in such a way that $X_r = X_p \cup X_q \cup Y$ where $Y \cap (X_p \cup X_q) = \emptyset$ and $\pi[Y] = \{\beta_\xi : \xi < \delta\}$. Then since $\{\beta_\xi : \xi < \delta\} \subseteq \bigcup\{D(I) : I \in \mathcal{J}\}$, we infer that

$$\sup \pi[X_r \setminus (X_p \cup X_q)] = \sup \pi[Y] < \gamma, \tag{1.75}$$

which was to be proved.

1.9. Regular and zero-dimensional spaces

(This section is based on [10])

Since the classes of the regular, of the zero-dimensional, and of the locally compact scattered spaces are different, it was a natural question what is the relationship between their cardinal sequences. Actually, Juhász raised the question whether there is a regular space with cardinal sequence $\langle \omega \rangle_{2^\omega}$ in ZFC. In [**25**, Theorem 10.1] the authors answered his question positively.

Theorem* 1.9.1. *For each* $\alpha < (2^\omega)^+$ *there is a regular (even zero-dimensional) scattered space with cardinal sequence* $\langle \omega \rangle_\alpha$.

In [**10**] we succeeded in giving a complete characterization of the cardinal sequences of both T_3 and zero-dimensional T_2 scattered spaces. Although the classes of the regular and of the zero-dimensional scattered spaces are different, it will turn out that they yield the same class of cardinal sequences.

For any regular, scattered space X we have $|X| \leq 2^{|I_0(X)|}$, hence for such a space X its cardinal sequence s satisfies $length(s) < (2^{|I_0(X)|})^+$ and $s(\alpha) \leq 2^{s(\beta)}$ whenever $\beta < \alpha$. We shall show below that these properties of a sequence s actually characterize the cardinal sequences of regular scattered spaces.

In [**25**], for each $\gamma < (2^\omega)^+$, a 0-dimensional, scattered space of height γ and width ω was constructed. The next lemma generalizes that construction.

For an infinite cardinal κ, let S_κ be the following family of sequences of cardinals:

$$S_\kappa = \big\{\langle \kappa_\alpha : \alpha < \delta\rangle : \delta < (2^\kappa)^+,\ \kappa_0 = \kappa \text{ and } \kappa \le \kappa_\alpha \le 2^\kappa \text{ for each } \alpha < \delta\ \big\}.$$

Lemma 1.9.2. *For any infinite cardinal κ and $s \in S_\kappa$ there is a 0-dimensional scattered space X with $\mathrm{CS}(X) = s$.*

Proof. Let $s = \langle \kappa_\alpha : \alpha < \delta\rangle \in S_\kappa$. Write $X = \bigcup\big\{\{\alpha\} \times \kappa_\alpha : \alpha < \delta\big\}$. Since $|X| \le 2^\kappa$ we can fix an independent family $\{F_x : x \in X\} \subset \big[\kappa\big]^\kappa$.

The underlying set of our space is X and and the topology τ on X is given by declaring for each $x = \langle \alpha, \xi\rangle \in X$ the set

$$U_x = \{x\} \cup (\alpha \times F_x)$$

to be clopen, i.e. $\{U_x, X \setminus U_x : x \in X\}$ is a subbase for τ.

The space X is clearly 0-dimensional and T_2.

Claim 1.9.2.1. *If $x = \langle \beta, \xi\rangle \in U \in \tau$ and $\alpha < \beta$ then $U \cap (\{\alpha\} \times \kappa_\alpha)$ is infinite.*

Proof of the claim. We can find disjoint sets $A, B \in \big[X \setminus \{x\}\big]^{<\omega}$ such that

$$x \in U_x \cap \bigcap_{y \in A} U_y \setminus \bigcup_{z \in B} U_z \subset U.$$

Observe that if $\langle \gamma, \xi\rangle \in A$ then $\beta < \gamma$. Thus

$$U \cap (\{\alpha\} \times \kappa_\alpha) \supset \{\alpha\} \times \Big(\bigcap_{y \in A \cup \{x\}} F_y \setminus \bigcup_{z \in B} F_z\Big),$$

and the set on the right side is infinite because $\{F_x : x \in X\}$ was chosen to be independent. □

To complete our proof, by induction on $\alpha < \kappa$, we verify that $\mathrm{I}_\alpha(X) = \{\alpha\} \times \kappa_\alpha$, hence $\mathrm{CS}(X) = s$. Assume that this is true for $\nu < \alpha$. If $x \in \{\alpha\} \times \kappa_\alpha$ then

$$U_x \cap \big(X \setminus \bigcup_{\nu < \alpha} \mathrm{I}_\nu(X)\big) = \{x\},$$

hence $\{\alpha\} \times \kappa_\alpha \subset \mathrm{I}_\alpha(X)$. On the other hand, if $x = \langle \beta, \xi\rangle \in X$ with $\beta > \alpha$ and $U \in \tau$ is a neighbourhood of x, then, by the claim above, $U \cap (\{\alpha\} \times \kappa_\alpha)$ is infinite, hence x is not isolated in $X \setminus \bigcup_{\nu<\alpha} \mathrm{I}_\nu(X)$, i.e., $x \notin \mathrm{I}_\alpha(X)$. Thus $\mathrm{I}_\alpha(X) = \{\alpha\} \times \kappa_\alpha$. □

Theorem 1.9.3. *For any sequence s of cardinals the following statements are equivalent:*

(1) $s = CS(X)$ for some regular scattered space X,
(2) $s = CS(X)$ for some 0-dimensional scattered space X,
(3) for some natural number m there are infinite cardinals $\kappa_0 > \kappa_1 > \cdots > \kappa_{m-1}$ and for all $i < m$ sequences $s_i \in S_{\kappa_i}$ such that $s = s_0{}^\frown s_1{}^\frown \ldots {}^\frown s_{m-1}$ or $s = s_0{}^\frown s_1{}^\frown \ldots {}^\frown s_{m-1}{}^\frown \langle n\rangle$ for some natural number $n > 0$.

Proof.
(1)$\Longrightarrow$ (3)
By induction on j we choose ordinals $\nu_j < \mathrm{ht}(X)$ and cardinals κ_j such that $\nu_0 = 0$ and $\kappa_0 = |\mathrm{I}_0(X)|$, moreover, for $j > 0$ with κ_{j-1} infinite

$$\nu_j = \min\big\{\nu \le \mathrm{ht}(X) : |\mathrm{I}_\nu(X)| < \kappa_{j-1}\big\},$$

and $\kappa_j = |\mathrm{I}_{\nu_j}(X)|$. We stop when κ_m is finite. For each $j < m$ let $\delta_j = \nu_{j+1} \dot{-} \nu_j$. Then the sequence $s_j = \big\langle |\mathrm{I}_{\nu_j+\delta}(X)| : \delta < \delta_j\big\rangle$ is in S_{κ_j}. Thus $\mathrm{CS}(X) = s_0{}^\frown s_1{}^\frown \ldots {}^\frown s_{m-1}$ provided $\kappa_m = 0$ (i.e. $\mathrm{I}_{\nu_m}(X) = \emptyset$) and $\mathrm{CS}(X) = s_0{}^\frown s_1{}^\frown \ldots {}^\frown s_{m-1}{}^\frown \langle \kappa_m\rangle$ when $0 < \kappa_m < \omega$.

(3)$\Longrightarrow$ (2)
First we prove this implication for sequences s of the form $s_0{}^\frown s_1{}^\frown \ldots {}^\frown s_{m-1}$ by induction on m. If $s \in S_{\kappa_o}$ then the statement is just lemma 1.9.2

Assume now that $s = s_0{}^\frown s_1{}^\frown \dots {}^\frown s_{m-1}$, where $\kappa_0 > \kappa_1 > \dots > \kappa_{m-1}$ and $s_i \in S_{\kappa_i}$ for $i < m$.

According to lemma 1.9.2 there is a 0-dimensional space Y with cardinal sequence s_{m-1}. Using the inductive assumption we can also fix pairwise disjoint 0-dimensional topological spaces $X_{y,n}$ for $\langle y, n\rangle \in I_0(Y) \times \omega$, each having the cardinal sequence $s' = s_0{}^\frown s_1{}^\frown \dots {}^\frown s_{m-2}$. We then define the space $Z = \langle Z, \tau\rangle$ as follows. Let

$$Z = Y \cup \bigcup\{X_{y,n} : y \in I_0(Y), n < \omega\}.$$

A set $U \subset Z$ is in τ iff

(i) $U \cap Y$ is open in Y,
(ii) $U \cap X_{y,n}$ is open in $X_{y,n}$ for each $\langle y, n\rangle \in I_0(Y) \times \omega$,
(iii) if $y \in I_0(Y) \cap U$ then there is $m < \omega$ such that $\bigcup\{X_{y,n} : m < n < \omega\} \subset U$.

If U is a clopen subset of Y and $n < \omega$ then it is easy to check that

$$Z(U, n) = U \cup \bigcup\{X_{y,m} : y \in \mathrm{I}_0(Y) \cap U, n < m < \omega\}$$

is clopen in Z. Hence

$$\mathcal{B} = \{Z(U, n) : U \subset Y \text{ is clopen}, n < \omega\} \cup \{T : T \text{ is a clopen subset of some } X_{y,n}\}$$

is a clopen base of Z and so Z is 0-dimensional.

Let $\delta' = \mathrm{length}(s')$ and $\delta = \mathrm{length}(s)$.

Claim 1.9.3.1. $\mathrm{I}_\alpha(Z) = \bigcup\{\mathrm{I}_\alpha(X_{y,n}) : \langle y, n\rangle \in \mathrm{I}_0(Y) \times \omega\}$ *for* $\alpha < \delta'$.

Proof of the claim 1.9.3.1. Since $X_{y,n}$ is an open subspace of Z it follows that $\mathrm{I}_\alpha(X_{y,n}) \subset \mathrm{I}_\alpha(Z)$. On the other hand,

$$Y \subset \overline{\bigcup\{\mathrm{I}_\alpha(X_{y,n}) : \langle y, n\rangle \in \mathrm{I}_0(Y) \times \omega\}}^Z,$$

hence $Y \cap \mathrm{I}_\alpha(Z) = \emptyset$. □

Since, by claim 1.9.3.1,

$$Z \setminus \bigcup_{\alpha<\delta'} \mathrm{I}_\alpha(Z) = Y,$$

it follows that for $\delta' \le \alpha < \delta$ we have

$$\mathrm{I}_\alpha(Z) = \mathrm{I}_{\alpha \dot{-} \delta'}(Y). \qquad (*)$$

Thus $Z = \bigcup_{\alpha<\delta} \mathrm{I}_\alpha(Z)$, hence Z is a scattered space of height δ.

If $\alpha < \delta'$ then, by claim 1.9.3.1,

$$|\,\mathrm{I}_\alpha(Z)| = |\,\mathrm{I}_0(y)| \cdot \omega \cdot s'(\alpha) = \kappa_{m-1} \cdot \omega \cdot s'(\alpha) = s'(\alpha) = s(\alpha).$$

If $\delta' \le \alpha < \delta$ then, by $(*)$, $|\,\mathrm{I}_\alpha(Z)| = |\,\mathrm{I}_{\alpha \dot{-} \delta'}(Y)| = s_{m-1}(\alpha \dot{-} \delta') = s(\alpha)$, consequently $\mathrm{CS}(Z) = s$.

Thus we proved the statement for sequences of the form $s_0{}^\frown \dots {}^\frown s_{m-1}$.

If $s = s_0{}^\frown \dots {}^\frown s_{m-1}{}^\frown \langle n\rangle$ then writing $s' = s_0{}^\frown \dots {}^\frown s_{m-1}$ we can first find pairwise disjoint 0-dimensional scattered spaces $X_{i,m}$, $\langle i, m\rangle \in n \times \omega$ each having cardinal sequence s'. Let

$$Z = \{x_i : i < n\} \cup \bigcup\{X_{\langle i,m\rangle} : i < n, m < \omega\}.$$

Declare a set $U \subset Z$ open iff

(i) $U \cap X_{i,m}$ is open in $X_{i,m}$ for each $\langle i, m\rangle \in n \times \omega$,
(ii) if $x_i \in U$ then there is $n_i < \omega$ such that $\bigcup\{X_{i,m} : n_i < m < \omega\} \subset U$.

Then Z is 0-dimensional, and

$$\mathrm{I}_\alpha(Z) = \begin{cases} \bigcup\{\mathrm{I}_\alpha(X_{i,m}) : i < n, m < \omega\} & \text{if } \alpha < \mathrm{length}(s'), \\ \{x_i : i < n\} & \text{if } \alpha = \mathrm{length}(s'). \end{cases}$$

Hence again Z is a scattered space with $\mathrm{CS}(Z) = s$.

(2)$\Longrightarrow$ (1) is trivial. □

1.10. Initially ω_1-compact spaces

(This section is based on [11])

Improving a result of M. Rabus we force a normal, locally compact, 0-dimensional, Frechet-Urysohn, initially ω_1-compact and non-compact space X of size ω_2 having the following property: for every open (or closed) set A in X we have $|A| \leq \omega_1$ or $|X \setminus A| \leq \omega_1$.

1.10.1. Introduction. E. van Douwen and, independently, A. Dow **[34]** have observed that under CH an initially ω_1-compact T_3 space of countable tightness is compact. (A space X is initially κ-compact if any open cover of X of size $\leq \kappa$ has a finite subcover, or equivalently any subset of X of size $\leq \kappa$ has a complete accumulation point). Naturally, the question arose whether CH is needed here, i.e. whether the same is provable just in ZFC. The question became even more intriguing when in **[23]** , D. Fremlin and P. Nyikos proved the same result from PFA. Quite recently, A. V. Arhangel'skiĭhas devoted the paper **[20]** to this problem, in which he has raised many related problems as well.

In **[90]** M. Rabus has answered the question of van Douwen and Dow in the negative. He constructed by forcing a Boolean algebra B such that the Stone space $St(B)$ includes a counterexample X of size ω_2 to the van Douwen–Dow question, in fact $St(B)$ is the one point compactification of X, hence X is also locally compact. The forcing used by Rabus is closely related to the one due to J. Baumgartner and S. Shelah in **[25]**, which had been used to construct a thin very tall superatomic Boolean algebra. In particular, Rabus makes use of a Δ-function f with some extra properties that are satisfied if f is obtained by the original, rather sophisticated forcing argument of Shelah from **[25]**.

In this section we give an alternative forcing construction of counterexamples to the van Douwen–Dow question, which we think is simpler, more direct and more intuitive than the one in **[90]**. First of all, we directly force a topology τ_f on ω_2 that yields an example from a Δ-function (with no extra properties) in the ground model which also satisfies CH. There is a wide variety of such ground models since they are easily obtained when one forces a Δ-function or because $\square_{\omega_1}$ implies the existence of a Δ-function (cf. **[25]**).

Both in **[25]** and **[90]** the main use of the Δ-function f is to suitably restrict the partial order of finite approximations to a structure on ω_2 so as to become c.c.c. This we do as well, but in the proof of the countable compactness of τ_f we also need the following simple result that yields an additional property of Δ-functions provided CH also holds.

Lemma 1.10.1 ([**11**]). *Assume that CH holds, f is a Δ-function, $\{c_\alpha : \alpha < \omega_2\}$ are pairwise disjoint finite subsets of ω_2 and $B \in \left[\omega_2\right]^\omega$. Then for each $n \in \omega$ there are distinct ordinals $\alpha_0, \alpha_1, \ldots, \alpha_{n-1} \in \omega_2$ such that*

$$B \subset \bigcap \{f(\xi, \eta) : \xi \in c_{\alpha_i}, \eta \in c_{\alpha_j}, i < j < n\}.$$

□

The following, even simpler, result about arbitrary functions $f : \left[\omega_2\right]^2 \longrightarrow \left[\omega_2\right]^{\leq\omega}$ with $f\{\alpha, \beta\} \subset \alpha \cap \beta$ for $\{\alpha, \beta\} \in \left[\omega_2\right]^2$ will also be needed.

Lemma 1.10.2. *If f is a function as above then for each $K, K' \in \left[\omega_2\right]^{\leq\omega}$ there is a countable set $\mathrm{cl}_f(K, K') \subset \omega_2$ such that*

(a) $K \subset \mathrm{cl}_f(K, K')$, $\sup K = \sup \mathrm{cl}_f(K, K')$,
(b) $\forall \xi \in \mathrm{cl}_f(K, K')\ \forall \eta \in \mathrm{cl}_f(K, K') \cup K'\ (f\{\xi, \eta\} \subset \mathrm{cl}_f(K, K'))$.

□

The topology τ_f that we will construct on ω_2 is right separated (in the natural order of ω_2) and is also locally compact and 0-dimensional. Thus for each $\alpha \in \omega_2$ one can fix a compact (hence closed) and open neighbourhood $\mathrm{H}(\alpha)$ of α such that $\max \mathrm{H}(\alpha) = \alpha$. Conversely, if we can fix for each $\alpha \in \omega_2$ such a right-separating compact open neighbourhood $\mathrm{H}(\alpha)$ then the family $\{\mathrm{H}(\alpha) : \alpha < \omega_2\}$ determines the whole topology τ on ω_2. In fact, using the notation $\mathrm{U}(\alpha, b) = \mathrm{H}(\alpha) \setminus \bigcup\{\mathrm{H}(\beta) : \beta \in b\}$, it is easy to check that for each $\alpha \in \omega_2$ the family $\mathcal{B}_\alpha =$

$\{\mathrm{U}(\alpha, b) : b \in [\alpha]^{<\omega}\}$ is a τ-neighbourhood base of α. Therefore, our notion of forcing consists of finite approximations to a family $\mathcal{H} = \{\mathrm{H}(\alpha) : \alpha < \omega_2\}$ like above.

Now, if $\mathcal{H} = \{\mathrm{H}(\alpha) : \alpha < \omega_2\}$ is as required and $\beta < \alpha < \omega_2$ then either (i) $\beta \in \mathrm{H}(\alpha)$ or (ii) $\beta \notin \mathrm{H}(\alpha)$. If (i) holds then $\mathrm{H}(\beta) \setminus \mathrm{H}(\alpha)$, if (ii) holds then $\mathrm{H}(\beta) \cap \mathrm{H}(\alpha)$ is a compact open subset of β, hence there is a finite subset of β, call it $i\{\alpha, \beta\}$, such that this set is covered by $\mathrm{H}[i\{\alpha, \beta\}] = \bigcup\{\mathrm{H}(\gamma) : \gamma \in i\{\alpha, \beta\}\}$ It may come as a surprise, but the existence of such a function i is also sufficient to insure that the collection $\mathcal{H}$ be as required. More precisely, we have the following result.

Definition 1.10.3. If $\mathcal{H} = \{\mathrm{H}(\alpha) : \alpha \in \omega_2\}$ is a family of subsets of ω_2 such that $\max \mathrm{H}(\alpha) = \alpha$ for each $\alpha \in \omega_2$ then we denote by $\tau_{\mathcal{H}}$ the topology on ω_2 generated by $\mathcal{H} \cup \{\omega_2 \setminus H : H \in \mathcal{H}\}$ as a subbase.

Clearly, $\tau_{\mathcal{H}}$ is a 0-dimensional, Hausdorff and right separated topology in which the elements of $\mathcal{H}$ are clopen.

Theorem 1.10.4 ([**11**]). *Assume that $\mathcal{H}$ is as in definition 1.10.3 above and there is a function $i : [\omega_2]^2 \longrightarrow [\omega_2]^{<\omega}$ satisfying $i\{\alpha, \beta\} \subset \alpha \cap \beta$ for each $\{\alpha, \beta\} \in [\omega_2]^2$ such that if $\beta < \alpha$ then $\beta \in \mathrm{H}(\alpha)$ implies $\mathrm{H}(\beta) \setminus \mathrm{H}(\alpha) \subset \mathrm{H}[i\{\alpha, \beta\}]$ and $\beta \notin \mathrm{H}(\alpha)$ implies $\mathrm{H}(\beta) \cap \mathrm{H}(\alpha) \subset \mathrm{H}[i\{\alpha, \beta\}]$. Then each $\mathrm{H}(\alpha)$ is compact in the topology $\tau_{\mathcal{H}}$, hence $\tau_{\mathcal{H}}$ is locally compact.* □

It is now very natural to try to force a generic 0-dimensional, locally compact and right separated topology on ω_2 by finite approximations (or pieces of information) of $\mathcal{H}$ and i. As was already mentioned, the Δ-function f comes into the picture when one wants to make this forcing c.c.c. The technical details of this are done in section 1.10.2.

We call the family $\mathcal{H}$ *coherent* if $\beta \in \mathrm{H}(\alpha)$ implies $\mathrm{H}(\beta) \subset \mathrm{H}(\alpha)$. Clearly, this makes things easier because then $\mathrm{H}(\beta) \setminus \mathrm{H}(\alpha) = \emptyset$, hence there is no problem covering it, the requirement on i is only that if $\beta \notin \mathrm{H}(\alpha)$ and $\beta < \alpha$ then $\mathrm{H}(\beta) \cap \mathrm{H}(\alpha) \subset \mathrm{H}[i\{\alpha, \beta\}]$. The original forcing of Baumgartner and Shelah from [**25**] (when translated to scattered, i.e. right separated, locally compact spaces rather than superatomic Boolean algebras) actually produced such a coherent family $\mathcal{H}$. This is interesting because if $\mathcal{H}$ is coherent and $\tau_{\mathcal{H}}$ is separable, which we have almost automatically if $\mathcal{H}$ is obtained generically, then $\tau_{\mathcal{H}}$ is also countably tight!

Theorem 1.10.5. *If there is a coherent family $\mathcal{H}$ of right separating compact open sets for a separable topology τ on ω_2 then $\mathrm{t}(\langle \omega_2, \tau \rangle) = \omega$.*

Proof. Let $X = \langle \omega_2, \tau \rangle$. Then for each $\alpha \in \omega_2$ we have $\mathrm{t}(\alpha, X) = \mathrm{t}(\alpha, \mathrm{H}(\alpha))$, hence it suffices to prove $\mathrm{t}(\alpha, \mathrm{H}(\alpha)) = \omega$. If we had $\mathrm{t}(\alpha, \mathrm{H}(\alpha)) = \omega_1$ then we could find $A \subset \mathrm{H}(\alpha)$ such that $\alpha \in \overline{A}$ but $\alpha \notin \overline{B}$ for every countable $B \subset A$. Next we can find a strictly increasing sequence $S = \{x_\nu : \nu < \omega_1\}$ in A such that $x_\nu > \sup F_\nu$, where $F_\nu = \overline{\{x_\mu : \mu < \nu\}}$. This construction can be carried out as $\sup F_\nu < \alpha$ because $\alpha \notin F_\nu$, F_ν is compact and every initial segment of $\mathrm{H}(\alpha)$ is open and so one of them should cover F_ν. Now for each $\nu < \omega_1$ there is a finite subset b_ν of F_ν such that $F_\nu \subset \mathrm{H}[b_\nu]$. If $\mu < \nu$ then $b_\mu \subset F_\mu \subset F_\nu \subset \mathrm{H}[b_\nu]$ implies that for each $\beta \in b_\mu$ we have $\mathrm{H}(\beta) \subset \mathrm{H}[b_\nu]$ by the coherence of $\mathcal{H}$, hence $\mathrm{H}[b_\mu] \subset \mathrm{H}[b_\nu]$. But $x_\mu \in \mathrm{H}[b_\nu] \setminus \mathrm{H}[b_\mu]$ for each $\mu < \nu \in \omega_1$, hence the $\mathrm{H}[b_\nu]$'s yield a strictly increasing ω_1-sequence of clopen sets in a separable space, which is a contradiction completing the proof. □

Ironically, this general result that gives countable tightness so easily cannot be used in our construction because we had to abandon the coherency of $\mathcal{H}$ in our effort to insure countable compactness (implied by the initial ω_1-compactness) of $\tau_{\mathcal{H}}$.

We mentioned above that our examples, by genericity, are separable. But this is not a coincidence. It is well-known and very easy to prove that if X is an initially ω_1-compact space then $\mathrm{t}(X) \leq \omega$ implies that X has no uncountable free sequence. (Moreover, if X is T_3 the converse of this is also true.) Hence the following easy, but perhaps not widely known, result immediately implies that any non-compact, initially ω_1-compact space of countable tightness contains a countable subset whose closure is not compact. Thus if there is a counterexample to the van Douwen–Dow question then there is also a separable one.

Lemma 1.10.6 ([**11**]). *If Y is a non-compact topological space, then for some ordinal μ the space Y contains a free sequence $\{y_\xi : \xi < \mu\} \subset Y$ with non-compact closure.* □

Note that under CH the weight of a separable T_3 space is $\leq \omega_1$, and an initially ω_1-compact space of weight $\leq \omega_1$ is compact, hence the CH result of van Douwen and Dow is a trivial consequence of 1.10.6. Arhangel'skii raised the question, [**20**, problem 3], whether in this CH can be weakened to $2^\omega < 2^{\omega_1}$? We shall answer this question in the negative: theorem 1.10.34 implies that the existence of a counterexample to the van Douwen–Dow question is consistent with practically any cardinal arithmetic that violates CH.

In [**20**, problem 17] Arhangel'skii asked if it is provable in ZFC that an initially ω_1-compact subspace of a T_3 space of countable tightness is always closed. (Clearly this is so under CH or PFA, or in general if the answer to the van Douwen–Dow question is "yes".) In view of our next result both Rabus' and our spaces give a negative answer to this question. More generally we have the following result.

Theorem 1.10.7. *If X is a locally compact counterexample to van Douwen–Dow then the one-point compactification $\alpha X = X \cup \{p\}$ of X also has countable tightness. On the other hand, X is an initially ω_1-compact non-closed subset of αX.*

Proof. Let $A \subset X$ be such that $p \in \overline{A}$ (i.e. $\overline{A}^X$ is not compact). By lemma 1.10.6 and our preceding remark then there is a countable set $S \subset \overline{A}^X$ such that $\overline{S}^X$ is not compact. But by $\mathrm{t}(X) = \omega$ then there is a countable $T \subset A$ for which $S \subset \overline{T}^X$, hence $\overline{T}^X$ is non-compact as well, so $p \in \overline{T}$. Consequently we have $\mathrm{t}(p, \alpha X) = \omega$ and so $\mathrm{t}(\alpha X) = \omega$. □

1.10.2. The forcing construction. The following notation will be used in the definition of the poset P_f. Given a function h and $a \subset \mathrm{dom}(h)$ we write $h[a] = \cup\{h(\xi) : \xi \in a\}$. Given non-empty sets x and y of ordinals with $\sup x \neq \sup y$ let

$$x * y = \begin{cases} x \cap y & \text{if } \sup x \notin y \text{ and } \sup y \notin x, \\ x \setminus y & \text{if } \sup x \in y, \\ y \setminus x & \text{if } \sup y \in x. \end{cases}$$

Definition 1.10.8. For each function $f : \left[\omega_2\right]^2 \longrightarrow \left[\omega_2\right]^{\leq\omega}$ satisfying $f(\alpha, \beta) \subset \alpha \cap \beta$ for any $\{\alpha, \beta\} \in \left[\omega_2\right]^2$ we define a poset $P_f = \langle P_f, \leq\rangle$ as follows. The underlying set of P_f is the family of triples $p = \langle a, h, i\rangle$ for which

(i) $a \in \left[\omega_2\right]^{<\omega}$, $h : a \longrightarrow \mathcal{P}(a)$ and $i : \left[a\right]^2 \longrightarrow \mathcal{P}(a)$ are functions,
(ii) $\max h(\xi) = \xi$ for each $\xi \in a$,
(iii) $i(\xi, \eta) \subset f(\xi, \eta)$ for each $\{\xi, \eta\} \in \left[a\right]^2$,
(iv) $h(\xi) * h(\eta) \subset h[i(\xi, \eta)]$ for each $\{\xi, \eta\} \in \left[a\right]^2$.

We will often write $p = \langle a^p, h^p, i^p\rangle$ for $p \in P_f$.
For $p, q \in P_f$ let $p \leq q$ if $a^p \supset a^q$, $h^p(\xi) \cap a^q = h^q(\xi)$ for $\xi \in a^q$, and $i^p \supset i^q$. If $p \in P_f$, $\alpha \in a^p$, $b \subset a^p \cap \alpha$, let us write $u^p(\alpha, b) = h^p(\alpha) \setminus h^p[b]$.

Lemma 1.10.9. *For each $\alpha < \omega_2$ the set $D_\alpha = \{p \in P_f : \alpha \in a^p\}$ is dense in P_f.* □

Definition 1.10.10. If $\mathcal{G}$ is a P_f-generic filter over V, in $V[\mathcal{G}]$ we can define the topological space $X_f[\mathcal{G}] = X_f = \langle\omega_2, \tau_f\rangle$ as follows. For $\alpha \in \omega_2$ put $\mathrm{H}(\alpha) = \bigcup\{h^p(\alpha) : p \in \mathcal{G} \wedge \alpha \in a^p\}$, let $\mathcal{H} = \{\mathrm{H}(\alpha) : \alpha < \omega_2\}$ and let $\tau_f = \tau_{\mathcal{H}}$ as defined in 1.10.3, that is, τ_f is the topology on ω_2 generated by $\mathcal{H} \cup \{\omega_2 \setminus H : H \in \mathcal{H}\}$ as a subbase.

If $\mathcal{G}$ is a P_f-generic filter over V then by lemma 1.10.9 we have $\bigcup\{a^p : p \in \mathcal{G}\} = \omega_2$, and for each $\alpha < \omega_2$ $\max \mathrm{H}(\alpha) = \alpha$ and $\mathrm{H}(\alpha)$ is clopen in X_f. Thus X_f is 0-dimensional and right separated. Of course, neither f nor i is needed for this. As was explained in section 1.10.1, we need f to be a Δ-function in order to make P_f c.c.c (which insures that no cardinal is collapsed), and the function i is used to make X_f also locally compact.

Theorem 1.10.11. *If CH holds and f is a Δ-function, then P_f satisfies the c.c.c and $V^{P_f} \models$ "$X_f = \langle \omega_2, \tau_f \rangle$ is a 0-dimensional, right separated, locally compact space having the following properties:*

(i) $t(X_f) = \omega$,
(ii) $\forall A \in [\omega_2]^{\omega_1}\ \exists \alpha \in \omega_2\ |A \cap H(\alpha)| = \omega_1$
(iii) $\forall A \in [\omega_2]^{\omega}$ *($\overline{A}$ is compact or $|\omega_2 \setminus \overline{A}| \leq \omega_1$)."*

Consequently, in V^{P_f}, X_f is a locally compact, normal, countably tight, initially ω_1-compact but non-compact space.

Proof of theorem 1.10.11. To show that P_f satisfies c.c.c we will proceed in the following way. We first formulate when two conditions p and p' from P_f are called *good twins* (definition 1.10.12), then we construct the *amalgamation* $r = p + p'$ of p and p' (definition 1.10.13) and show that r is a common extension of p and p' in P_f. Finally we prove in lemma 1.10.15 that every uncountable family of conditions contains a couple of elements which are good twins.

Definition 1.10.12. Let $p = \langle a, h, i \rangle$ and $p' = \langle a', h', i' \rangle$ be from P_f. We say that p and p' are *good twins* provided

(1) p and p' are *twins*, i.e., $|a| = |a'|$ and the natural order-preserving bijection $e = e_{p,p'}$ between a and a' is an isomorphism between p and p':
  (i) $h'(e(\xi)) = e''h(\xi)$ for each $\xi \in a$,
  (ii) $i'(e(\xi), e(\eta)) = e''i(\xi, \eta)$ for each $\{\xi, \eta\} \in \big[a\big]^2$,
  (iii) $e(\xi) = \xi$ for each $\xi \in a \cap a'$,
(2) $i(\xi, \eta) = i'(\xi, \eta)$ for each $\{\xi, \eta\} \in \big[a \cap a'\big]^2$,
(3) a and a' are good for f.

Let us remark that, in view of (ii) and (iii), condition 2 can be replaced by "$i(\xi, \eta) \subset a \cap a'$ for each $\{\xi, \eta\} \in \big[a \cap a'\big]^2$".

Definition 1.10.13. If $p = \langle a, h, i \rangle$ and $p' = \langle a', h', i' \rangle$ are good twins we define the *amalgamation* $r = \langle b, g, j \rangle$ *of p and p'* as follows:

Let $b = a \cup a'$. For $\xi \in h[a \cap a'] \cup h'[a \cap a']$ define

$$\delta_\xi = \min\{\delta \in a \cap a' : \xi \in h(\delta) \cup h'(\delta)\}.$$

Now, for any $\xi \in b$ let

$$g(\xi) = \begin{cases} h(\xi) \cup h'(\xi) & \text{if } \xi \in a \cap a', \\ h(\xi) \cup \{\eta \in a' \setminus a : \delta_\eta \in h(\xi)\} & \text{if } \xi \in a \setminus a', \\ h'(\xi) \cup \{\eta \in a \setminus a' : \delta_\eta \in h'(\xi)\} & \text{if } \xi \in a' \setminus a. \end{cases} \qquad (\bullet)$$

Finally for $\{\xi, \eta\} \in \big[b\big]^2$ let

$$j(\xi, \eta) = \begin{cases} i(\xi, \eta) & \text{if } \xi, \eta \in a, \\ i'(\xi, \eta) & \text{if } \xi, \eta \in a', \\ f(\xi, \eta) \cap b & \text{otherwise.} \end{cases} \qquad (\bullet\bullet)$$

(Observe that j is well-defined because 1.10.12.(2) holds.)
We will write $r = p + p'$ for the amalgamation of p and p'.

Lemma 1.10.14. *If p and p' are good twins then their amalgamation, $r = p + p'$, is a common extension of p and p' in P_f.*

Proof. First we prove two claims.

Claim 1.10.14.1. *Let $\eta \in a$ and $\delta \in a \cap a'$. Then $\eta \in h(\delta)$ if and only if δ_η is defined and $\delta_\eta \in h(\delta)$. (Clearly, we also have a symmetric version of this statement for $\eta \in a'$.)*

Proof of claim 1.10.14.1. Assume first $\eta \in h(\delta)$. Then δ_η is defined and clearly $\delta_\eta \in h(\delta)$ if $\delta_\eta = \delta$. So assume $\delta_\eta \neq \delta$. Since $i(\delta_\eta, \delta) \subset a \cap a'$ and $\max i(\delta_\eta, \delta) < \delta_\eta$ we have $\eta \notin h[i(\delta_\eta, \delta)]$ by the choice of δ_η. Thus from $p \in P_f$ we have

$$\eta \notin h(\delta_\eta) * h(\delta). \qquad (\dagger)$$

Then $h(\delta_\eta) * h(\delta) \neq h(\delta_\eta) \cap h(\delta)$ by (†). Since $\eta \in h(\delta)$ implies $\delta_\eta < \delta$, we actually have $h(\delta_\eta) * h(\delta) = h(\delta_\eta) \setminus h(\delta)$. Thus $\delta_\eta \in h(\delta)$ by the definition of the operation $*$.

On the other hand, if $\delta_\eta \in h(\delta)$, then either $\delta_\eta = \delta$ or $h(\delta_\eta) * h(\delta) = h(\delta_\eta) \setminus h(\delta)$. Thus $\eta \in h(\delta)$ because in the latter case again $\eta \notin h[i(\delta_\eta, \delta)]$, hence (†) holds. □

Claim 1.10.14.2. *If $\xi \in a \cap a'$ then*

$$g(\xi) = h(\xi) \cup \{\eta \in a' \setminus a : \delta_\eta \in h(\xi)\} = h'(\xi) \cup \{\eta \in a \setminus a' : \delta_\eta \in h'(\xi)\}.$$

Proof of claim 1.10.14.2. Conditions 1.10.12.1(i) and (iii) imply $h(\xi) \cap a \cap a' = h'(\xi) \cap a \cap a'$ and so

$$g(\xi) = h(\xi) \cup h'(\xi) = h(\xi) \cup ((a' \setminus a) \cap h'(\xi)).$$

By claim 1.10.14.1 we have

$$(a' \setminus a) \cap h'(\xi) = \{\eta \in a' \setminus a : \delta_\eta \in h'(\xi)\}.$$

But by 1.10.12.(1) we have $\delta_\eta \in h(\xi)$ iff $\delta_\eta \in h'(\xi)$, hence it follows that

$$g(\xi) = h(\xi) \cup \{\eta \in a' \setminus a : \delta_\eta \in h(\xi)\}.$$

The second equality follows analogously. □

Next we check $r \in P_f$. Conditions 1.10.8.(i)–(iii) for r are clear by the construction. So we should verify 1.10.8.(iv).

Let $\xi \neq \eta \in b$ and $\alpha \in g(\xi) * g(\eta)$. We need to show that $\alpha \in g[j(\xi, \eta)]$. We will distinguish several cases.

Case 5. *$\xi, \eta \in a$ (or $\xi, \eta \in a'$).*

Since $g(\xi) \cap a = h(\xi)$ and $g(\eta) \cap a = h(\eta)$ we have $(g(\xi) * g(\eta)) \cap a = h(\xi) * h(\eta)$ by the definition of operation $*$. Thus $(g(\xi) * g(\eta)) \cap a \subset h[i(\xi, \eta)] = g[j(\xi, \eta))] \cap a \subset g[j(\xi, \eta)]$. So we can assume that $\alpha \in a' \setminus a$. We know that δ_α is defined because $\alpha \in g(\xi) \cup g(\eta)$ is also satisfied. Since $\alpha \in g(\xi)$ iff $\delta_\alpha \in h(\xi)$ and $\alpha \in g(\eta)$ iff $\delta_\alpha \in h(\eta)$ by (•) it follows that $\delta_\alpha \in h(\xi) * h(\eta)$. Thus there is $\nu \in i(\xi, \eta)$ such that $\delta_\alpha \in h(\nu)$. But $i(\xi, \eta) = j(\xi, \eta)$ and $\alpha \in g(\nu)$ by (•). Thus $\alpha \in g[j(\xi, \eta)]$.

Case 6. *$\xi \in a \setminus a'$ and $\eta \in a' \setminus a$.*

We can assume that $\alpha \in a$, since the $\alpha \in a'$ case is done symmetrically.

Subcase 6.1. *$g(\xi) * g(\eta) = g(\eta) \setminus g(\xi)$.*

Then $\alpha \in \ g(\eta)$ and $\eta \in \ g(\xi)$ so δ_α and δ_η are both defined and $\delta_\alpha \in h'(\eta)$, $\delta_\eta \in h(\xi)$ hold, hence $\alpha \leq \delta_\alpha < \eta < \delta_\eta < \xi$. But a and a' are good for f, so by 1.1.10(a) we have $\delta_\alpha \in f(\eta, \xi) \cap b = j(\xi, \eta)$. Thus $\alpha \in h(\delta_\alpha) \subset g(\delta_\alpha) \subset g[j(\xi, \eta)]$ which was to be proved.

Subcase 6.2. *$g(\xi) * g(\eta) = g(\xi) \cap g(\eta)$ or $g(\xi) * g(\eta) = g(\xi) \setminus g(\eta)$.*

Since now $\alpha \in g(\xi) * g(\eta) \subset g(\xi)$, by the definition of the operation $*$ we have

$$|\{\alpha, \xi\} \cap g(\eta)| = 1. \tag{1.76}$$

Thus, by the definition of $g(\eta)$, $\delta^* = \min\{\delta \in a \cap a' : \alpha \in h(\delta) \vee \xi \in h(\delta)\}$ is well-defined and $\delta^* < \eta$. If $\delta^* < \xi$ then $\alpha \in h(\delta^*)$ and by 1.1.10(a) we have $\delta^* \in f(\xi, \eta) \cap b = j(\xi, \eta)$ for a and a' are good for f, and so $\alpha \in g[j(\xi, \eta)]$.

Thus we can assume $\xi < \delta^*$. We know that $\delta^* = \delta_\alpha$ or $\delta^* = \delta_\xi$ by the choice of δ^*, but $\delta_\alpha = \delta_\xi$ is impossible by (1.76). Thus

$$|\{\alpha, \xi\} \cap h(\delta^*)| = 1. \tag{1.77}$$

Since $\alpha \in g(\xi)$ implies $\alpha \in h(\xi)$ and we have $\xi < \delta^*$, (1.77) implies $\alpha \in h(\xi) * h(\delta^*)$ and so $\alpha \in h[i(\xi, \delta^*)]$. But $i(\xi, \delta^*) \subset f(\xi, \delta^*) \subset f(\xi, \eta)$ because a and a' are good for f, so 1.1.10(b) or (c) may be applied. Consequently, we have $i(\xi, \delta^*) \subset j(\xi, \eta)$ by (••). Hence $\alpha \in g[j(\xi, \eta)]$ which was to be proved.

Since we investigated all the cases it follows that r satisfies 1.10.8.(iv), that is, $r \in P_f$. Since $r \leq p, q$ are clear from the construction, the lemma is proved. □

Lemma 1.10.15. *Every uncountable family $\mathcal{F}$ of conditions in P_f contains a couple of elements which are good twins. Consequently, P_f satisfies c.c.c.*

Proof. By standard counting arguments $\mathcal{F}$ contains an uncountable subfamily $\mathcal{F}'$ such that every pair $p \neq p' \in \mathcal{F}'$ satisfies 1.10.12.(1)-(2). But f is a Δ-function, so there are $p \neq p' \in \mathcal{F}'$ such that a^p and $a^{p'}$ are good for f, i.e. p and p' satisfies 1.10.12.(3), too. In other words, p and p' are good twins and so $r = p + p'$ is a common extension of p and p' in P_f. □

Let $\mathcal{G}$ be a P_f-generic filter over V. As in definition 1.10.10, let $\mathrm{H}(\alpha) = \bigcup\{h^p(\alpha) : p \in \mathcal{G} \wedge \alpha \in a^p\}$ for $\alpha \in \omega_2$, and let τ_f be the topology on ω_2 generated by $\{\mathrm{H}(\alpha) : \alpha \in \omega_2\} \cup \{\omega_2 \setminus \mathrm{H}(\alpha) : \alpha \in \omega_2\}$ as a subbase. Put $i = \bigcup\{i^p : p \in \mathcal{G}\}$.

Since X_f is generated by a clopen subbase and $\max(H(\alpha)) = \alpha$ for each $\alpha \in \omega_2$ by 1.10.8(ii), it follows that X_f is 0-dimensional and right separated in its natural well-order.

The following proposition is clear by 1.10.8.(iv) and by the definition of H and i.

Proposition . $\mathrm{H}(\alpha) * \mathrm{H}(\beta) \subset \mathrm{H}[i(\alpha, \beta)]$ *for $\alpha < \beta < \omega_2$. So by 1.10.4 every* $\mathrm{H}(\alpha)$ *is a compact open set in X_f.*

Definition 1.10.16. For $\alpha \in \omega_2$ and $b \in \left[\alpha\right]^{<\omega}$ let

$$\mathrm{U}(\alpha, b) = \mathrm{H}(\alpha) \setminus \mathrm{H}[b]$$

and let

$$\mathcal{B}_\alpha = \{\mathrm{U}(\alpha, b) : b \in \left[\alpha\right]^{<\omega}\}.$$

By theorem 1.10.4 every $\mathrm{H}(\alpha)$ is compact and $\mathcal{B}_\alpha$ is a neighborhood base of α in X_f. Thus X_f is locally compact and the neighbourhood base $\mathcal{B}_\alpha$ of α consists of compact open sets.

Unfortunately, the family $\mathcal{H} = \{\mathrm{H}(\alpha) : \alpha < \omega_2\}$ is not coherent, so we can't apply theorem 1.10.5 to prove that X_f is countably tight. It will however follow from the following result.

Lemma 1.10.17. *In V^{P_f}, if a sequence $\{z_\zeta : \zeta < \omega_1\} \subset \mathrm{H}(\beta)$ converges to β, then there is some $\xi < \omega_1$ such that $\beta \in \overline{\{z_\zeta : \zeta < \xi\}}$.*

Proof. Assume on the contrary that for each $\xi < \omega_1$ we can find a finite subset $b_\xi \subset \beta$ such that $\{z_\zeta : \zeta < \xi\} \cap \mathrm{U}(\beta, b_\xi) = \emptyset$, that is, $\{z_\zeta : \zeta < \xi\} \subset \mathrm{H}[b_\xi]$.

Fix now a condition $p \in P_f$ which forces the above described situation and decides the value of β. Then, for each $\xi < \omega_1$ we can choose a condition $p_\xi \leq p$ which decides the value of z_ξ and b_ξ. We can assume that $\{a^{p_\xi} : \xi < \omega_1\}$ forms a Δ-system with kernel D, $z_\xi \in a^{p_\xi} \setminus D$ and that $z_\xi < z_\eta$ for $\xi < \eta < \omega_1$.

Claim . *Assume that $\xi < \eta < \omega_1$, p_ξ and p_η are good twins and $r = p_\xi + p_\eta$. Then $r \Vdash$"$z_\xi \in \mathrm{H}[D \cap \beta]$".*

Proof of the claim. Indeed, $z_\xi \in h^r[b_\eta]$ because $r \leq p_\xi, p_\eta$. Since $z_\xi \in a^{p_\xi} \setminus a^{p_\eta}$, (•) and claim 1.10.14.2 imply that $z_\xi \in h^r[b_\eta]$ holds if and only if $\delta_{z_\xi} \in D = a^{p_\xi} \cap a^{p_\eta}$ is defined and $\delta_{z_\xi} \in h^{p_\eta}[b_\eta]$. Since $b_\eta \subset \beta$, we also have $\delta_{z_\xi} < \beta$ and so $z_\xi \in h^r[D \cap \beta]$. □

Applying lemma 1.10.15 to appropriate final segments of $\{p_\xi : \xi < \omega_1\}$ we can choose, by induction on $\mu < \omega_1$, pairwise different ordinals $\mu < \xi_\mu < \eta_\mu < \omega_1$ with $\eta_\mu < \xi_\nu$ if $\mu < \nu$ such that p_{ξ_μ} and p_{η_μ} are good twins. Let $r_\mu = p_{\xi_\mu} + p_{\eta_\mu}$. Since P_f satisfies c.c.c there is a condition $q \leq p$ such that $q \Vdash$" $|\{\mu \in \omega_1 : r_\mu \in \mathcal{G}\}| = \omega_1$". Thus, by the claim $q \Vdash$"$|\{z_\xi : \xi < \omega_1\} \cap \mathrm{H}[D \cap \beta]| = \omega_1$", i.e. the neighbourhood $\mathrm{U}(\beta, D \cap \beta)$ of β misses uncountably many of the points z_ξ which contradicts that β is the limit of this sequence. □

Corollary 1.10.18. $\mathrm{t}(X_f) = \omega$.

Proof. Assume on the contrary that $\alpha \in \omega_2$ and $\mathrm{t}(\alpha, X_f) = \mathrm{t}(\alpha, \mathrm{H}(\alpha)) = \omega_1$. Then there is an ω-closed set $Y \subset \mathrm{H}(\alpha)$ that is not closed. Since the subspace $H(\alpha)$ is compact and right separated and so it is pseudo-radial , for some regular cardinal κ there is a sequence $\{z_\xi : \xi < \kappa\} \subset Y$ which converges to some point $\beta \in H(\alpha) \setminus Y$. Since Y is ω-closed and $|Y| \leq |\mathrm{H}(\alpha)| = \omega_1$ we have $\kappa = \omega_1$. By lemma 1.10.17 there is some $\xi < \omega_1$ with $\beta \in \overline{\{z_\zeta : \zeta < \xi\}} \subset Y$ contradicting $\beta \notin Y$. □

Lemma 1.10.19. *In V^{P_f}, for each uncountable $A \subset X_f$ there is $\beta \in \omega_2$ such that $|A \cap \mathrm{H}(\beta)| = \omega_1$.*

Proof. Assume that $p \Vdash$ "$\dot{A} = \{\dot{\alpha}_\xi : \xi < \omega_1\} \in \left[\omega_2\right]^{\omega_1}$". For each $\xi < \omega_1$ pick $p_\xi \leq p$ and $\alpha_\xi \in \omega_2$ such that $p_\xi \Vdash \dot{\alpha}_\xi = \hat{\alpha}_\xi$. Since P_f satisfies c.c.c we can assume that the α_ξ are pairwise different. Let $\sup\{\alpha_\xi : \xi < \omega_1\} < \beta < \omega_2$. Now for each $\xi < \omega_1$ define the condition $q_\xi \leq p_\xi$ by the stipulations $a^{q_\xi} = a^{p_\xi} \cup \{\beta\}$, $h^{q_\xi}(\beta) = a^{q_\xi}$ and $i^{q_\xi}(\nu, \beta) = \emptyset$ for $\nu \in a^{p_\xi}$. Then $q_\xi \in P_f$ and $q_\xi \Vdash \dot{\alpha}_\xi \in \mathrm{H}(\beta)$. But P_f satisfies c.c.c, so there is $q \leq p$ such that $q \Vdash$ "$|\{\xi \in \omega_1 : q_\xi \in \mathcal{G}\}| = \omega_1$". Thus $q \Vdash$ "$|A \cap \mathrm{H}(\beta)| = \omega_1$." □

Since every $\mathrm{H}(\beta)$ is compact, lemma 1.10.19 above clearly implies that X_f is ω_1-compact, i.e. every subset $S \subset X_f$ of size ω_1 has a complete accumulation point.

Now we start to work on (iii): in V^{P_f} the closure of any countable subset Y of X_f is either compact or it contains a final segment of ω_2. If Y is also in the ground model, then actually the second alternative occurs and this follows easily from the next lemma.

Lemma 1.10.20. *If $p \in P_f$, $\beta \in a^p$, $b \subset a^p \cap \beta$, $\alpha \in \beta \setminus a^p$, then there is a condition $q \leq p$ such that $\alpha \in u^q(\beta, b)$.*

Proof. Define the condition $q \leq p$ by the following stipulations: $a^q = a^p \cup \{\alpha\}$, $h^q(\alpha) = \{\alpha\}$,

$$h^q(\nu) = \begin{cases} h^p(\nu) \cup \{\alpha\} & \text{if } \beta \in h^p(\nu) \\ h^p(\nu) & \text{if } \beta \notin h^p(\nu) \end{cases}$$

for $\nu \in a^p$, and let $i^q \supset i^p$ and $i^q(\alpha, \nu) = \emptyset$ for $\nu \in a^p$.

To show $q \in P_f$ we need to check only 1.10.8(iv). Assume that $\alpha \in h^q(\nu) * h^q(\mu)$. Then by the construction of q we have $\beta \in h^p(\nu) * h^p(\mu)$. Thus there is $\xi \in i^p(\nu, \mu)$ with $\beta \in h^p(\xi)$. But then $\alpha \in h^q(\xi)$, so by $i^q(\nu, \mu) = i^p(\nu, \mu)$ we have $\alpha \in h^q[i^q(\nu, \mu)]$. In view of $h^p(\nu) * h^p(\mu) \subset h^p[i^p(\nu, \mu)]$ we are done. Thus $q \in P_f$, $q \leq p$ and clearly $\alpha \in u^q(\beta, b)$, so we are done. □

This lemma yields the following corollary.

Corollary 1.10.21. *If $Z \in \left[\omega_2\right]^\omega \cap V$ and $\beta \in \omega_2 \setminus \sup Z$ then $\beta \in \overline{Z}$.*

Proof. Let $\mathrm{U}(\beta, b)$ be a neighbourhood of β, $b \in \left[\beta\right]^{<\omega}$. Since $p \Vdash$"$\mathrm{U}(\beta, b) \supset u^p(\beta, b)$" for each $p \in P_f$ and the set

$$D_{\beta,b,Z} = \{q \in P_f : u^q(\beta, b) \cap Z \neq \emptyset\}$$

is dense in P_f by the previous lemma, it follows that $\mathrm{U}(\beta, b)$ intersects Z. Consequently $\beta \in \overline{Z}$. □

The space X_f is right separated, i.e. scattered, so we can consider its Cantor-Bendixson hierarchy. According to corollary 1.10.21 for each $\alpha < \omega_2$ the set $A_\alpha = [\omega\alpha, \omega\alpha + \omega)$ is a dense set of isolated points in $X_f \lceil (\omega_2 \setminus \omega\alpha)$. Thus the αth Cantor-Bendixson level of X_f is just A_α. Therefore X_f is a thin very tall, locally compact scattered space in the sense of [**93**]. Let us emphasize that CH was not needed to get this result, hence we have also given an alternative proof of the main result of [**25**].

Now we continue to work on proving property 1.10.11(iii) of X_f.

Given $p \in P_f$ and $b \subset a^p$ let $p \lceil b = \left\langle b, h, i^p \lceil \left[b\right]^2 \right\rangle$ where h is the function with $\mathrm{dom}(h) = b$ and $h(\xi) = h^p(\xi) \cap b$ for $\xi \in b$. Let us remark that $p \lceil b$ in not necessarily in P_f. In fact, $p \lceil b \in P_f$ if and only if $i^p(\xi, \eta) \subset b$ for each $\xi \neq \eta \in b$. Especially, if b is an initial segment of a^p, then $p \lceil b \in P_f$. The order $\leq$ of P_f can be extended in a natural way to the restrictions of conditions: if p and q are in P_f, $b \subset a^p$, $c \subset a^q$, define $p \lceil b \leq q \lceil c$ iff $b \supset c$, $h^p(\xi) \cap c = h^q(\xi) \cap c$ for each $\xi \in c$, and $i^p \lceil \left[c\right]^2 = i^q \lceil \left[c\right]^2$. Clearly if $p \lceil b \in P_f$ and $q \lceil c \in P_f$ then the two definitions of $\leq$ coincide.

Definition 1.10.22. Let $p, p' \in P_f$ with $a^p = a^{p'}$. We write $p \prec p'$ if for each $\alpha \in a^p$ and $b \subset a^p \cap \alpha$ we have $u^p(\alpha, b) \subset u^{p'}(\alpha, b)$.

The following technical result will play a crucial role in the proof of 1.10.11(iii). Part (c) in it will enable us to "insert" certain things in $\mathrm{H}(\gamma_0)$ in a non-trivial way. But there is a price we have to pay for this: this is the point where the coherency of the $\mathrm{H}(\alpha)$ has to be abandoned. Part (d) will be needed in section 1.10.3.

Lemma 1.10.23. *Assume that* $s = \langle a^s, h^s, i^s\rangle \in P_f$, $a^s = S \cup E \cup F$, $Q \subset S$, $S <_{\mathrm{On}} E <_{\mathrm{On}} F$, $E = \{\gamma_i : i < k\}$, $\gamma_0 < \gamma_1 < \cdots < \gamma_{k-1}$, $F = \{\gamma_{i,0}, \gamma_{i,1} : i < k\}$, *moreover*

(i) $\forall i < k\ h^s(\gamma_{i,0}) \cap h^s(\gamma_{i,1}) = h^s[Q \cup E]$,

(ii) $\forall i < k\ \forall \xi \in S\ f(\xi, \gamma_i) = f(\xi, \gamma_{i,0}) = f(\xi, \gamma_{i,1})$.

Then there is a condition $r = \langle a^r, h^r, i^r\rangle$ *with* $a^r = S \cup E$ *such that*

(a) $r \le s\lceil S$,

(b) $r \le s\lceil(Q \cup E)$,

(c) $S \setminus h^s[Q \cup E] \subset h^r(\gamma_0)$,

(d) $s\lceil(S \cup E) \prec r$.

Proof. Let $a^r = S \cup E$ and write $C = S \setminus h^s[Q \cup E]$. For $\xi \in a^r$ we set

$$h^r(\xi) = \begin{cases} h^s(\xi) \cup C & \text{if } \xi = \gamma_i \text{ and } \gamma_0 \in h^s(\gamma_i), \\ h^s(\xi) & \text{otherwise.} \end{cases}$$

For $\xi \ne \eta \in a^r$ we let

$$i^r(\xi, \eta) = \begin{cases} i^s(\xi, \eta) & \text{if } \xi, \eta \in Q \cup E \text{ or } \xi, \eta \in S, \\ f(\xi, \eta) \cap a^r & \text{otherwise.} \end{cases}$$

Finally let $r = \langle a^r, h^r, i^r\rangle$.

We claim that r satisfies the requirements of the lemma. a, b and c are clear from the definition of r, once we establish that $r \in P_f$. To see that it suffices to check only 1.10.8.(iv) because the other requirements are clear from the construction of r. So let $\xi < \eta \in a^r$. We have to show $h^r(\xi) * h^r(\eta) \subset h^r[i^r(\xi, \gamma_i)]$.

If $\xi, \eta \in S$, then $h^r(\xi) * h^r(\eta) \subset h^r[i^r(\xi, \eta)]$ holds because $r\lceil S = s\lceil S \in P_f$.

So we can assume that $\eta = \gamma_i$ for some $i < k$.

Case 1. $\xi \in S$.

Subcase 1.1. $\xi \notin h^r(\gamma_i)$, *hence* $h^r(\xi) * h^r(\gamma_i) = h^r(\xi) \cap h^r(\gamma_i)$.

In this case we also have $h^s(\xi) * h^s(\gamma_i) = h^s(\xi) \cap h^s(\gamma_i$ and so

$$h^s(\xi) \cap h^s(\gamma_i) \subset h^s[i^s(\xi, \gamma_i)] \subset h^r[i^r(\xi, \gamma_i)] \tag{1.78}$$

for $i^r(\xi, \gamma_i) \supset i^s(\xi, \gamma_i,)$. If $\gamma_0 \notin h^s(\gamma_i)$ then $h^r(\gamma_i) = h^s(\gamma_i)$ and since $h^r(\xi) = h^s(\xi)$ we have $h^r(\xi) * h^r(\gamma_i) = h^s(\xi) \cap h^s(\gamma_i) \subset h^r[i^r(\xi, \gamma_i)]$ by (1.78). Assume now that $\gamma_0 \in h^s(\gamma_i)$. Thus $h^r(\gamma_i) = h^s(\gamma_i) \cup C$, and so $\xi \notin C$, that is $\xi \in h^s[Q \cup E]$. Then $h^r(\xi) * h^r(\gamma_i) = h^r(\xi) \cap h^r(\gamma_i) = (h^s(\xi) \cap h^s(\gamma_i)) \cup (h^s(\xi) \cap C)$. By (1.78) above it is enough to show that $h^s(\xi) \cap C \subset h^r[i^r(\xi, \gamma_i)]$. Since $h^s(\xi) \cap C = \emptyset$ for $\xi \in Q$ we can assume that $\xi \notin Q$. By (i) $h^s[Q \cup E] = h^s(\gamma_{i,0}) \cap h^s(\gamma_{i,1})$ and so

$$h^s(\xi) \cap C = h^s(\xi) \setminus h^s[Q \cup E] = (h^s(\xi) \setminus h^s(\gamma_{i,0})) \cup (h^s(\xi) \setminus h^s(\gamma_{i,1})). \tag{1.79}$$

Since $\xi \in h^s[Q \cup E] \subset h^s(\gamma_{i,j})$ for $j = 0, 1$ we have

$$h^s(\xi) \setminus h^s(\gamma_{i,j}) = h^s(\xi) * h^s(\gamma_{i,j}) \subset h^s[i^s(\xi, \gamma_{i,j})]. \tag{1.80}$$

By (ii), $i^s(\xi, \gamma_{i,j}) \subset f(\xi, \gamma_{i,j}) \cap a^s = f(\xi, \gamma_i) \cap a^s$. Since $\xi \notin Q$ it follows that $f(\xi, \gamma_i) \cap a^s = i^r(\xi, \gamma_i)$. Thus from (1.80) we obtain

$$h^s(\xi) \setminus h^s(\gamma_{i,j}) \subset h^s[i^r(\xi, \gamma_i)] = h^r[i^r(\xi, \gamma_i)]. \tag{1.81}$$

Putting (1.79) and (1.81) together we get $h^s(\xi) \cap C \subset h^r[i^r(\xi, \gamma_i)]$ which was to be proved.

Subcase 1.2. $\xi \in h^r(\gamma_i)$, *hence* $h^r(\xi) * h^r(\gamma_i) = h^r(\xi) \setminus h^r(\gamma_i)$.

If $\xi \in h^s(\gamma_i)$ then since $h^s(\xi) = h^r(\xi)$ we have $h^r(\xi) * h^r(\gamma_i) = h^r(\xi) \setminus h^r(\gamma_i) \subset h^s(\xi) \setminus h^s(\gamma_i) \subset h^s[i^s(\xi, \gamma_i)] \subset h^r[i^r(\xi, \gamma_i)]$ and we are done.

So we can assume that $\xi \notin h^s(\gamma_i)$ and so $h^r(\gamma_i) \ne h^s(\gamma_i)$. By the construction of r, we have $\gamma_0 \in h^s(\gamma_i)$, $h^r(\gamma_i) = h^s(\gamma_i) \cup C$ and so $\xi \in C$, i.e $\xi \notin h^s[Q \cup E]$. By (i) we can assume that $\xi \notin h^s(\gamma_{i,0})$. So $s \in P_f$ implies

$$h^s(\xi) \cap h^s(\gamma_{i,0}) = h^s(\xi) * h^s(\gamma_{i,0}) \subset h^s[i^s(\xi, \gamma_{i,0})] \tag{1.82}$$

We have

$$h^r(\xi) * h^r(\gamma_i) = h^s(\xi) \setminus (h^s(\gamma_i) \cup C) \subset h^s(\xi) \setminus C \tag{1.83}$$

and applying (i) again

$$h^s(\xi) \setminus C = h^s(\xi) \cap h^s[Q \cup E] \subset h^s(\xi) \cap h^s(\gamma_{i,0}). \tag{1.84}$$

By (ii), $i^s(\xi, \gamma_{i,0}) \subset f(\xi, \gamma_{i,0}) \cap a^s = f(\xi, \gamma_i) \cap a^s$. Since $\xi \notin Q \subset h^s[Q \cup E]$ it follows that $f(\xi, \gamma_i) \cap a^s = i^r(\xi, \gamma_i)$ and so (1.82)–(1.84) together yield

$$h^r(\xi) * h^r(\gamma_i) \subset h^r[i^r(\xi, \gamma_i)] \tag{1.85}$$

which was to be proved.

Case 2. *$\xi = \gamma_j$ for some $j < i$.*

Since $i^s(\gamma_j, \gamma_i) = i^r(\gamma_j, \gamma_i)$ we have

$$h^s(\gamma_j) * h^s(\gamma_i) \subset h^r[i^r(\gamma_i, \gamma_j)]. \tag{1.86}$$

It is easy to check that

$$h^r(\gamma_j) * h^r(\gamma_i) = \begin{cases} h^s(\gamma_j) * h^s(\gamma_i) & \text{if } \gamma_0 \notin h^s(\gamma_j) * h^s(\gamma_i), \\ h^s(\gamma_j) * h^s(\gamma_i) \cup C & \text{if } \gamma_0 \in h^s(\gamma_j) * h^s(\gamma_i). \end{cases} \tag{1.87}$$

So we are done if $\gamma_0 \notin h^s(\gamma_j) * h^s(\gamma_i)$. Assume $\gamma_0 \in h^s(\gamma_j) * h^s(\gamma_i)$. Then there is $\gamma_l \in i^s(\gamma_j, \gamma_i)$ with $\gamma_0 \in h^s(\gamma_l)$. Thus, by the construction of r we have

$$C \subset h^r(\gamma_l) \subset h^r[i^r(\gamma_j, \gamma_i)] \tag{1.88}$$

But (1.87) and (1.88) together imply what we wanted.

Thus we proved $r \in P_f$.

Clearly r satisfies 1.10.23.(a)–(c). To check 1.10.23.(d) write $s' = s\lceil(S \cup E)$ and let $\alpha \in S \cup E$ and $b \subset (S \cup E) \cap \alpha$. We need to show that $u^{s'}(\alpha, b) \subset u^r(\alpha, b)$. Since $S \cup E$ is an initial segment of a^s, we have $u^{s'}(\alpha, b) = u^s(\alpha, b)$. If $\alpha \in S$, then also $u^{s'}(\alpha, b) = u^r(\alpha, b)$, so we can assume $\alpha \in E$.

Let $\xi \in u^{s'}(\alpha, b) = u^s(\alpha, b)$. Then $\xi \in h^s(\alpha) \subset h^s[E]$, and hence $\xi \notin C$. But $h^r[b] \setminus h^s[b] \subset C$, more precisely, it is empty or just C. Since $\xi \in u^s(\alpha, b)$, it follows that $\xi \notin h^s[b]$ and so $\xi \notin h^r[b]$ because $\xi \notin C$. Thus $\xi \in u^r(\alpha, b)$. Hence r satisfies (d).

The lemma is proved. □

Lemma 1.10.24. *In V^{P_f}, if $Y \subset \omega_2$ is countable, then either $\overline{Y}$ is compact or $|\omega_2 \setminus \overline{Y}| \leq \omega_1$.*

Proof. Assume that $1_{P_f} \Vdash$"$\dot{Y} = \{\dot{y}_n : n \in \omega\} \subset \omega_2$". For each $n \in \omega$ fix a maximal antichain $C_n \subset P_f$ such that for each $p \in C_n$ there is $\alpha \in a^p$ with $p \Vdash$"$\dot{y}_n = \hat{\alpha}$". Let $A = \bigcup\{a^p : p \in \bigcup_{n<\omega} C_n\}$. Since every C_n is countable by c.c.c we have $|A| = \omega$.

Assume also that $1_{P_f} \Vdash$ "$\overline{\dot{Y}}$ is not compact", that is, Y can not be covered by finitely many $H(\delta)$ in V^{P_f}.

Let

$$I = \{\delta < \omega_2 : \exists p \in P_f \ p \Vdash \text{“}\delta \notin \overline{\dot{Y}}\text{”}\}.$$

Clearly $1_{P_f} \Vdash \ \omega_2 \setminus I \subset \overline{\dot{Y}}$. Since $\omega_2 \setminus I$ is in the ground model, by corollary 1.10.21 it is enough to show that $\omega_2 \setminus I$ is infinite. Actually we will prove much more:

Claim . *I is not stationary in ω_2.*

Assume on the contrary that I is stationary. Let us fix, for each $\delta \in I$, a condition $p_\delta \in P_f$ and a finite set $D_\delta \in \left[\delta\right]^{<\omega}$ such that $p_\delta \Vdash$ "$\dot{Y} \cap \mathrm{U}(\delta, D_\delta) = \emptyset$". For each $\delta \in I$ let $Q_\delta = a^{p_\delta} \cap \delta$ and $E_\delta = a^{p_\delta} \setminus \delta$. We can assume that $D_\delta \subset Q_\delta$ and $\sup A < \delta$ for each $\delta \in I$.

Let $B_\delta = \mathrm{cl}_f(A \cup Q_\delta, E_\delta)$ for $\delta \in I$ (see 1.10.2). For each $\delta \in I$ the set B_δ is countable with $\sup(B_\delta) = \sup(A \cup Q_\delta)$, so we can apply Fodor's pressing down lemma and CH to get a stationary set $J \subset I$ and a countable set $B \subset \omega_2$ such that $B_\delta = B$ for each $\delta \in J$.

By thinning out J and with a further use of CH we can assume that for a fixed $k \in \omega$ we have

(1) $E^\delta = \{\gamma_i^\delta : i < k\}$ for $\delta \in J$, $\gamma_0^\delta < \gamma_1^\delta < \cdots < \gamma_{k-1}^\delta$,
(2) $f(\xi, \gamma_i^\delta) = f(\xi, \gamma_i^{\delta'})$ for each $\xi \in B$, $\delta, \delta' \in J$ and $i < k$.

Let $\delta = \min J$, $D = D_\delta$, $E = E_\delta$, $p = p_\delta$, $Q = Q_\delta$. By lemma 1.10.1 there are ordinals $\delta_j \in J$ with $\delta < \delta_0 < \delta_1 < \cdots < \delta_{2k-1}$ such that

$$B \cup E \subset \bigcap \{f(\xi, \eta) : \xi \in E_{\delta_i}, \eta \in E_{\delta_j}, i < j < 2k\}. \tag{$\star$}$$

For $i < k$ and $j < 2$ let $\gamma_i = \gamma_i^\delta$ and $\gamma_{i,j} = \gamma_i^{\delta_{2i+j}}$. Let $F = \{\gamma_{i,j} : i < k, j < 2\}$.

We know that $a^p = Q \cup E$. Define the condition $q \in P_f$ by the following stipulations:

(i) $a^q = a^p \cup F$ and $q \le p$,
(ii) $h^q(\gamma_{i,j}) = \{\gamma_{i,j}\} \cup a^p$ for $\langle i, j\rangle \in k \times 2$
(iii) $i^q(\gamma_{i_0,j_0}, \gamma_{i_1,j_1}) = a^p$ for $\langle i_0, j_0\rangle \ne \langle i_1, j_1\rangle \in k \times 2$,
(iv) $i^q(\xi, \gamma_{i,j}) = \emptyset$ for $\xi \in a^p$ and $\langle i, j\rangle \in k \times 2$.

Since $a^p \subset B \cup E$, $(\star)$ implies that $q \in P_f$.

Since $1_{P_f} \Vdash$ "$\mathrm{H}[Q \cup E] \not\supset \dot{Y}$", there is a condition $t \le q$, a natural number n and an ordinal α such that $t \Vdash$ "$\alpha = \dot{y}_n$" but $\alpha \in a^t \setminus h^t[Q \cup E]$. Since C_n is a maximal antichain we can assume that $t \le v$ for some $v \in C_n$.

Let $s = t\lceil(B \cup E \cup F)$. Then $s \in P_f$ because for each pair $\xi < \eta \in a^s$ if $\xi \in B$ then $i^t(\xi, \eta) \subset f(\xi, \eta) \subset B$ and so $i^t(\xi, \eta) \subset a^s$, and if $\xi, \eta \in E \cup F$ then $i^t(\xi, \eta) = i^q(\xi, \eta) \subset Q \cup E \subset a^s$. Moreover $s \le v$ because $a^v \subset B$. Thus $s \Vdash$ "$\alpha = \dot{y}_n$" and $\alpha \notin h^s[Q \cup E]$. Let $S = a^s \cap B$.

Since $i^s(\gamma_{i,0}, \gamma_{i,1}) = i^q(\gamma_{i,0}, \gamma_{i,1}) = Q \cup E$, and $\gamma_{i,j} \notin h^s(\gamma_{i,1-j})$ we have

$$h^s(\gamma_{i,0}) \cap h^s(\gamma_{i,1}) = h^s(\gamma_{i,0}) * h^s(\gamma_{i,1}) \subset h^s[Q \cup E]. \tag{1.89}$$

Moreover, if $\xi \in Q \cup E$ and $j < 2$ then $\xi \in h^q(\gamma_{i,j}) \subset h^s(\gamma_{i,j})$ and $i^s(\xi, \gamma_{i,j}) = \emptyset$, consequently

$$h^s(\xi) \setminus h^s(\gamma_{i,j}) = h^s(\xi) * h^s(\gamma_{i,j}) \subset h^s[i^s(\xi, \gamma_{i,j})] = \emptyset. \tag{1.90}$$

Putting (1.89) and (1.90) together it follows that $h^s(\gamma_{i,0}) \cap h^s(\gamma_{i,1}) = h^s[Q \cup E]$.

Thus we can apply 1.10.23 to get a condition r such that $r \le s\lceil(Q \cup E) = p$, $r \le s\lceil S \le v$ and $\alpha \in S \setminus h^s[Q \cup E] \subset h^s(\gamma_0) = h^s(\delta)$. Since $D \subset Q$, we have $\alpha \in h^s(\delta) \setminus h^s[D]$.

Thus

$$r \Vdash \alpha \in \dot{Y} \cap (H(\delta) \setminus \mathrm{H}[D]).$$

On the other hand $r \Vdash \dot{Y} \cap (H(\delta) \setminus \mathrm{H}[D]) = \emptyset$ because $r \le p$. With this contradiction the claim is proved and this completes the proof of the lemma. □

Clearly, lemma 1.10.24 implies that X_f is countably compact.

Corollary 1.10.25. *If $F \subset X$ is closed (or open), then either $|F| \le \omega_1$ or $|X \setminus F| \le \omega_1$.*

Proof. If $|F| = \omega_2$ then F is not compact, so by lemma 1.10.6 F contains a free sequence Y with non-compact closure. But F is initially ω_1-compact and countably tight, so Y is countable. Consequently, we have $|\omega_2 \setminus \overline{Y}| \le \omega_1$ by lemma 1.10.24 and so $|X \setminus F| \le \omega_1$. □

Corollary 1.10.26. *X_f is normal and $\mathrm{z}(X_f) = \mathrm{hd}(X_f) \le \omega_1$.*

Proof. To show that X_f is normal let F_0 and F_1 be disjoint closed subsets of X_f. Since at least one of them is compact by lemma 1.10.24 they can be separated by open subsets of X_f because X_f is T_3.

Concerning the hereditarily density of X_f it follows easily from corollary 1.10.25 that X_f does not contain a discrete subspace of size ω_2. But X_f is right separated, so all the left separated subspaces of X_f are of size $\le \omega_1$, that is , $\mathrm{z}(X) \le \omega_1$. □

Thus theorem 1.10.11 is proved. □

We know that the space X_f is not automatically hereditarily separable, so the following question of Arhangel'skiĭ, [**20**, problem 5], remains unanswered: Is it true in ZFC that every hereditarily separable, initially ω_1-compact space is compact?

As we have seen our space X_f is normal. However, we don't know whether X_f is or can be made hereditarily normal, i.e. T_5. This raises the following problem.

Problem 3. *Is it provable in ZFC that every T_5, countably tight, initially ω_1-compact space is compact?*

1.10.3. Making X_f Frechet-Urysohn. In [**20**, problem 12] Arhangel'skiĭasks if it is provable in ZFC that a normal, first countable initially ω_1-compact space is necessarily compact. We could not completely answer this question, but in this subsection we show that the Frechet-Urysohn property (which is sort of half-way between countable tightness and first countability) in not enough to get compactness.

To achieve that we want to find a further extension of the model V^{P_f} in which X_f becomes Frechet-Urysohn but its other properties are preserved, for example, X_f remains initially ω_1-compact and normal. Since X_f is countably tight and $\chi(X_f) \leq \omega_1$ it is a natural idea to make X_f Frechet-Urysohn by constructing a generic extension of V^{P_f} in which X_f remains countably tight and $\mathrm{p} > \omega_1$, i.e. $\mathrm{MA}_{\omega_1}(\sigma\text{-centered})$ holds.

The standard c.c.c poset P which forces $p > \omega_1$ is obtained by a suitable finite support iteration of length 2^{ω_1}. During this iteration in the αth step we choose a non-principal filter $\mathcal{F} \subset \mathcal{P}(\omega)$ generated by at most ω_1 elements and we add a new subset A of ω to the αth intermediate model so that A is almost contained in every element of $\mathcal{F}$, i.e. $A \setminus F$ is finite for each $F \in \mathcal{F}$. It is well-known and easy to see that P has property K. Thus, by theorem 1.10.27 below, X_f remains countably tight in $V^{P_f * R}$ and so indeed X_f becomes Frechet-Urysohn in that model. Moreover, theorem 1.10.28 implies that the ω_1-compactness of X_f is also preserved. Unfortunately, we could not prove that forcing with P preserves the countable compactness of X_f.

So, instead of aiming at $p > \omega_1$ we will consider only those filters during the iteration which are needed in proving the Frechet-Urysohn property of X_f. As we will see, we can handle these filters in such a way that our iterated forcing R preserves not only the countable compactness of X_f but also property (iii) from theorem 1.10.11: in $V^{P_f * R}$ the closure of any countable subset of X_f is either compact or contains a final segment of ω_2. Of course, this will insure the preservation of the normality of X_f as well.

We start with the two easy theorems, promised above, about the preservation of countable tightness and ω_1-compactness of X_f under certain c.c.c forcings.

Theorem 1.10.27. *If the topological space X is right separated, compact, countably tight and the poset R has property K then forcing with R preserves the countable tightness of X.*

Proof. First we recall that X remains compact (and clearly right separated) in any extension of the ground model by [**60**, lemma 7]. Since $\mathrm{F}(X) = \mathrm{t}(X)$ for compact spaces, assume indirectly that $1_R \Vdash$ "$\{\dot{z}_\xi : \xi < \omega_1\} \subset X$ *is a free sequence*". For every $\xi < \omega_1$ we have that $1_R \Vdash$"$\overline{\{\dot{z}_\zeta : \zeta < \xi\}}$ *and* $\overline{\{\dot{z}_\zeta : \xi \leq \zeta < \omega_1\}}$ *are disjoint compact sets*" and X is T_3, so we can fix a condition $p_\xi \in P$, open sets U_ξ and V_ξ from the ground model and a point $z_\xi \in X$ such that $\overline{U_\xi} \cap \overline{V_\xi} = \emptyset$ and

$$p_\xi \Vdash \text{“ } \{\dot{z}_\zeta : \zeta < \xi\} \subset U_\xi,\ \{\dot{z}_\zeta : \xi \leq \zeta < \omega_1\} \subset V_\xi \text{ and } \dot{z}_\xi = z_\xi\text{.”}$$

Since R has property K, there is an uncountable set $I \subset \omega_1$ such that the conditions $\{p_\xi : \xi \in I\}$ are pairwise compatible.

We claim that the sequence $\{z_\xi : \xi \in I\}$ is an uncountable free sequence in the ground model which contradicts $\mathrm{F}(X) = \mathrm{t}(X) = \omega$. Indeed let $\xi \in I$. If $\zeta \in I \cap \xi$, then p_ξ and p_ζ has a common extension q in P and we have

$$q \Vdash \text{“}\dot{z}_\zeta = z_\zeta \text{ and } \{\dot{z}_\eta : \eta < \xi\} \subset \dot{U}_\xi\text{.”}$$

Hence $z_\zeta \in U_\xi$. Similarly for $\zeta \in I \setminus \xi$ we have $z_\zeta \in V_\xi$. Therefore $\overline{U_\xi}$ and $\overline{V_\xi}$ separate $\overline{\{z_\zeta : \zeta \in I \cap \xi\}}$ and $\overline{\{z_\zeta : \zeta \in I \setminus \xi\}}$ which implies that $\{z_\xi : \xi \in I\}$ is really free. □

Theorem 1.10.28. *Forcing with a c.c.c poset R over V^{P_f} preserves property 1.10.11.(ii) of the space X_f, i.e. for each uncountable $A \subset X_f$ there is $\beta \in \omega_2$ such that $A \cap H(\beta)$ is uncountable.*

Proof. We work in V^{P_f}. Assume that $r \Vdash_R$ "$\dot{A} = \{\dot{\alpha}_\xi : \xi < \omega_1\} \in \left[X_f\right]^{\omega_1}$". For each $\xi < \omega_1$ pick a condition $r_\xi \leq r$ from R which decides the value of $\dot{\alpha}_\xi$, $r_\xi \Vdash_R$ "$\dot{\alpha}_\xi = \alpha_\xi$". Since R satisfies c.c.c, $\{\alpha_\xi : \xi \in \omega_1\}$ is uncountable, hence as X_f has property (ii) in V^{P_f}, for some $\beta < \omega_2$ the set

$I = \mathrm{H}(\beta) \cap \{\alpha_\xi : \xi < \omega_1\}$ is also uncountable. Since R satisfies c.c.c there is a condition $q \leq r$ in R such that $q \Vdash_R$ "$|\{\xi \in I : r_\xi \in \mathcal{G}\}| = \omega_1$", where $\mathcal{G}$ is the R-generic filter over V^{P_f}. Thus $q \Vdash_R$ "$|\dot{A} \cap \mathrm{H}(\beta)| = \omega_1$" which was to be proved. □

Of course, theorem 1.10.28 implies that forcing with any c.c.c poset R preserves the ω_1-compactness of X_f. It is much harder to find a property of a poset R which guarantees that forcing with R over V^{P_f} preserves the countable compactness of X_f. We will proceed in the following way. In definition 1.10.29 we formulate when a poset $\dot{R}$ is called *nice (over P_f)*, and then in theorem 1.10.30 we show that forcing with a nice poset preserves not only the countable compactness of X_f, but also property 1.10.11(iii): the closure of any countable subset of X_f is either compact or contains a final segment of ω_2. Finally, in definitions 1.10.31 and 1.10.32 we describe a class of finite support iterated forcings, which by theorem 1.10.33 are nice and have property K , and then in theorem 1.10.34 we show that forcing with a suitable member of this class makes X_f Frechet-Urysohn.

Definition 1.10.29. Let $\dot{R}$ be a name for a poset in V^{P_f}. We say that $\dot{R}$ is *nice (over P_f)* if there is a dense subset $\mathcal{D}$ of the iteration $P_f * \dot{R}$ with the following property: If $\langle p_0, r_0\rangle, \langle p_1, r_1\rangle \in \mathcal{D}$, $p, p' \in P_f$ are such that $p \leq p_0, p_1$, $p' \leq p_0, p_1$, and

$$p \Vdash \text{“}r_0 \text{ and } r_1 \text{ are compatible in } \dot{R}\text{”},$$

moreover we have either $p \leq p'$ or $p \prec p'$ (see definition 1.10.22) then we also have

$$p' \Vdash \text{“}r_0 \text{ and } r_1 \text{ are compatible in } \dot{R}\text{”}.$$

Theorem 1.10.30. *If CH holds in V, f is a Δ-function and $\dot{R}$ is a P_f-name for a c.c.c poset which is nice over P_f, then 1.10.11.(iii) is preserved by forcing with $\dot{R}$, i.e.*

$$V^{P_f * \dot{R}} \models \forall Y \in \left[X_f\right]^\omega (\ \overline{Y} \textit{ is compact or } |X_f \setminus \overline{Y}| \leq \omega_1\ ).$$

Proof. The proof is quite long and technical. See in [**11**]. □

Definition 1.10.31. Assume that $A \in \left[X_f\right]^\omega$ and $\alpha \in \overline{A}$. We define the poset $Q(A, \alpha)$ as follows. Its underlying set is $\left[A\right]^{<\omega} \times \left[\alpha\right]^{<\omega}$. If $\langle s, C\rangle$ and $\langle s'C'\rangle$ are conditions, let $\langle s, C\rangle \leq \langle s'C'\rangle$ if and only if $s \supset s'$, $C \supset C'$ and $s \setminus s' \subset \mathrm{U}(\alpha, C')$. For $q \in Q(A, \alpha)$ write $q = \langle s^q, C^q\rangle$ and $\mathrm{supp}(q) = s^q \cup C^q$.

It is well-known and easy to see that if $\mathcal{G}$ is a $Q(A, \alpha)$-generic filter, then $S = \bigcup\{s^q : q \in \mathcal{G}\}$ is a sequence from A which converges to α, i.e. every open neighbourhood of α contains all but finitely many points of S.

Clearly $Q(A, \alpha)$ is σ-centered and well-met. In fact, if $p_0 = \langle s_0, C_0\rangle$ and $p_1 = \langle s_1, C_1\rangle$ are compatible, then $p_0 \wedge p_1 = \langle s_0 \cup s_1, C_0 \cup C_1\rangle$.

Definition 1.10.32. A finite support iterated forcing $\langle R_\xi : \xi \leq \kappa\rangle$ over V^{P_f} is called an *FU-iteration* if for each $\xi < \kappa$ we have

$$1_{P_f * R_\xi} \Vdash R_{\xi+1} = R_\xi * Q(\dot{A}, \dot{\alpha}) \text{ for some } \dot{A} \in \left[X_f\right]^\omega \text{ and } \dot{\alpha} \in \overline{\dot{A}}.$$

Since every $Q(A, \alpha)$ is σ-centered, it is clear that any FU-iteration is c.c.c, in fact it even has property K. The really important, and much less trivial, property of them is given in our next result.

Theorem 1.10.33. *Any FU-iteration is nice over P_f.*

Proof. This proof is rather long and technical. See in [**11**]. □

Now we are ready to obtain the main result of this section.

Theorem 1.10.34. *Assume that CH holds in the ground model V, there is a Δ-function f and λ is a cardinal such that $\omega_1 < \lambda = \lambda^\omega$. Then in V^{P_f} there is an FU-iteration $\dot{R}_\lambda$ of length λ such that in $V^{P_f * \dot{R}_\lambda}$ the space X_f is Frechet-Urysohn and satisfies 1.10.11(i)–(iii), moreover $V^{P_f * \dot{R}_\lambda} \models$"$2^\kappa = (\lambda^\kappa)^V$ for each cardinal $\kappa \geq \omega$".*

Proof. It is standard to proof together the pieces we obtained so far. For details, see [**11**]. □

Theorem 1.10.34 answers a question raised by Arhangel'skiĭ, [**20**, problem 3], in the negative: CH can not be weakened to $2^\omega < 2^{\omega_1}$ in the theorem of van Douwen and Dow. In fact, if we have a ZFC model V in which CH holds and $\lambda = \lambda^\omega \leq 2^{\omega_1}$, then we can find a cardinal preserving generic extension W of V which contains a Frechet-Urysohn and normal counterexample to the van Douwen–Dow question, $(2^\omega)^W = \lambda$, moreover $(2^\kappa)^W = (2^\kappa)^V$ for each $\kappa \geq \omega_1$.

Let us finish by formulating the following higher cardinal version of the van Douwen–Dow problem:

Problem 4. *Is it provable in ZFC that an initially ω_2-compact T_3 space of countable tightness is compact ?*

The main problem is trying to use out approach that worked for ω_2 (instead of ω_3) here is that no Δ-function may exist for ω_3! This problem has come up already in the efforts trying to lift the result of [**25**] from ω_2 to ω_3.

1.11. First countable, initially ω_1-compact spaces

(This section is based on [8])

Improving the results from the previous section, we force a first countable, normal, locally compact, initially ω_1-compact and non-compact space $X = \langle \omega_2 \times 2^\omega, \tau \rangle$.

Actually, Alan Dow conjectured that applying the method of [**74**] (that "turns" a compact space into a first countable one) to the space of Rabus in [**90**] yields an ω_1-compact but non-compact first countable space. How one can carry out such a construction was outlined by the second author in the preprint [**71**]. However, [**71**] only sketches some arguments as the language adopted there, which follows that of [**90**], does not seem to allow direct combinatorial control over the space which is forced. This explains why the second author hesitated to publish [**71**].

One missing element of [**71**] was a language similar to that of [**11**] which allows working with the points of the forced space in a direct combinatorial way. In this section we combine the approach of [**11**] with the ideas of [**71**] to obtain directly an ω_1-compact but non-compact first countable space. Consequently, our proofs follow much more closely the arguments of [**11**] than those of [**90**] or their analogues in [**71**].

As before, we again use a Δ-function to make our forcing CCC but we need both CH and a Δ-function with some extra properties to obtain first countability.

It is immediate from the countable compactness of X that its one-point compactification X^* is not first countable. In fact, one can show that the character of the point at infinity $*$ in X^* is ω_2. As X is initially ω_1-compact, this means that every (transfinite) sequence converging from X to $*$ must be of type cofinal with ω_2. Since X is first countable, this trivially implies that there is no non-trivial converging sequence of type ω_1 in X^*. In other words: the convergence spectrum of the compactum X^* omits ω_1. As far as we know, this is the first and only (consistent) example of this sort.

1.11.1. A general construction.

First we introduce a general method to construct locally compact, zero-dimensional spaces. This generalizes the method for the construction of locally compact right-separated (i.e. scattered) spaces that was described in [**11**].

Definition 1.11.1. Let ϑ be an ordinal, X be a 0-dimensional space, and fix a clopen subbase (i.e. a subbase consisting of clopen sets) $\mathcal{S}$ of X such that $X \in \mathcal{S}$ and

$$S \in \mathcal{S} \setminus \{X\} \text{ implies } (X \setminus S) \in \mathcal{S}. \tag{1.91}$$

Let $\mathrm{K} : \vartheta \times \mathcal{S} \longrightarrow \mathcal{P}(\vartheta)$ be a function satisfying

$$\mathrm{K}(\delta, S) \subset \mathrm{K}(\delta, X) \subset \delta, \tag{1.92}$$

and for any $\delta \in \vartheta$ and $S \in \mathcal{S}$ set

$$U(\delta, S) = (\{\delta\} \times S) \cup (\mathrm{K}(\delta, S) \times X). \tag{1.93}$$

We shall denote by τ_{K} the topology on $\vartheta \times X$ generated by the family

$$\mathcal{U}_{\mathrm{K}} = \{U(\delta, S)\,,\ (\vartheta \times X) \setminus U(\delta, S) : \delta < \vartheta, S \in \mathcal{S}\} \tag{1.94}$$

as a subbase. Write $\mathcal{X}_{\mathrm{K}} = \langle \vartheta \times X, \tau_{\mathrm{K}} \rangle$.

If a is a set of ordinals and s is an arbitrary set we write

$$a \otimes a \otimes s = \{\langle \zeta, \xi, \sigma \rangle : \zeta, \xi \in a, \zeta < \xi, \sigma \in s\}. \tag{1.95}$$

Theorem 1.11.2. *(1) Assume that ϑ, X, $\mathcal{S}$ and K are as in definition 1.11.1 above. Then the space $\mathcal{X}_K = \langle \vartheta \times X, \tau_K \rangle$ is 0-dimensional and Hausdorff and the subspace $\{\alpha\} \times X$ is homeomorphic to X for each $\alpha < \vartheta$.*
(2) Assume, in addition, that X is compact and
(K1) if $S \cap S' = \emptyset$ then $\mathrm{K}(\delta, S) \cap \mathrm{K}(\delta, S') = \emptyset$,
(K2) if $X = \cup \mathcal{S}'$ for some $\mathcal{S}' \in \left[\mathcal{S}\right]^{<\omega}$ then

$$\mathrm{K}(\delta, X) = \cup\{\mathrm{K}(\delta, S) : S \in \mathcal{S}'\},$$

(K3) there is a function i with $\mathrm{dom}(i) = \vartheta \otimes \vartheta \otimes \mathcal{S}$ such that for each $\langle \delta, \delta', S \rangle \in \vartheta \otimes \vartheta \otimes \mathcal{S}$ we have
(i1) $i(\delta, \delta', S) \in \left[\delta\right]^{<\omega}$ and
(i2) $\mathrm{K}(\delta, X) * \mathrm{K}(\delta', S) \subset \cup\{\mathrm{K}(\nu, X) : \nu \in i(\delta, \delta', S)\}$,

where

$$\mathrm{K}(\delta, X) * \mathrm{K}(\delta', S) = \begin{cases} \mathrm{K}(\delta, X) \cap \mathrm{K}(\delta', S) & \text{if } \delta \notin \mathrm{K}(\delta', S), \\ \mathrm{K}(\delta, X) \setminus \mathrm{K}(\delta', S) & \text{if } \delta \in \mathrm{K}(\delta', S). \end{cases} \tag{1.96}$$

Then all members of $\mathcal{U}_{\mathrm{K}}$ are compact, hence $\mathcal{X}_{\mathrm{K}}$ is locally compact.

Proof. The argument is quite long, but more or less standard. See the details in [**8**]. □

To describe a natural base of the space $\mathcal{X}_{\mathrm{K}}$, we fix some more notation. For $\delta < \vartheta$, $\mathcal{S}' \in \left[\mathcal{S}\right]^{<\omega}$ and $F \in \left[\delta\right]^{<\omega}$ we shall write

$$B(\delta, \mathcal{S}', F) = \cap\{U(\delta, S) : S \in \mathcal{S}'\} \setminus U[F].$$

For a point $x \in X$ we set $\mathcal{S}(x) = \{S \in \mathcal{S} : x \in S\}$, moreover we put

$$\mathcal{B}(\delta, x) = \{B(\delta, \mathcal{S}',\ F) : \mathcal{S}' \in \left[\mathcal{S}(x)\right]^{<\omega}, F \in \left[\delta\right]^{<\omega}\}. \tag{1.97}$$

Lemma 1.11.3. *Assume that ϑ, X, $\mathcal{S}$ and K are as in part (2) of the previous theorem 1.11.2. Then for each $\delta < \vartheta$ and $x \in X$ the family $\mathcal{B}(\delta, x)$ forms a neighbourhood base of the point $\langle \delta, x \rangle$ in $\mathcal{X}_{\mathrm{K}}$.*

Proof. See in [**8**]. □

As we already mentioned above, our construction of the locally compact spaces $\mathcal{X}_{\mathrm{K}}$ generalizes the construction of locally compact right-separated spaces given in [**11**]. In fact, the latter is the special case when X is a singleton space (and $\mathcal{S}$ is the only possible subbase $\{X\}$). We may actually say that in the space $\mathcal{X}_{\mathrm{K}}$ the compact open sets $U(\delta)$ right separate the copies $\{\delta\} \times X$ of X rather than the points.

Actually, a locally compact, right separated, and initially ω_1-compact but non-compact space cannot be first countable. (Indeed, this is because the scattered height of such a space must exceed ω_1.) So the transition to a more complicated procedure is necessary if we want to keep our example locally compact.

We now present a much more interesting example of our general construction, where X will be the Cantor set $\mathbb{C}$ and $\mathcal{S}$ will be a natural subbase. For technical reasons, we put $\mathbb{C} = 2^{\mathbb{N}}$ instead of 2^{ω}, where $\mathbb{N} = \omega \setminus \{0\}$.

The clopen subbase $\mathcal{S}$ of $\mathbb{C}$ is the one that determines the product topology and is defined as follows. If $n > 0$ and $\varepsilon < 2$ then let $[n, \varepsilon] = \{f \in \mathbb{C} : f(n) = \varepsilon\}$. We then put

$$\mathcal{S} = \{[n, \varepsilon] : n > 0, i < 2\} \cup \{\mathbb{C}\}.$$

Then $\mathcal{S}$ satisfies 1.11.1.(1.91), moreover if $\mathcal{S}' \subset \mathcal{S} \setminus \{\mathbb{C}\}$ covers $\mathbb{C} = 2^{\mathbb{N}}$ then there is $n \in \mathbb{N}$ such that both $[n, 0], [n, 1] \in \mathcal{S}'$.

In order to apply our general scheme, we still need to fix an ordinal ϑ, a function $\mathrm{K} : \vartheta \times \mathcal{S} \longrightarrow \mathcal{P}(\vartheta)$ satisfying 1.11.1.(2), and another function i with $\mathrm{dom}(i) = \vartheta \otimes \vartheta \otimes \mathcal{S}$ such that all the requirements of theorem 1.11.1 are satisfied. In our present particular case this may be achieved in a slightly different form that turns out to be simpler and more convenient for the purposes of our forthcoming forcing argument.

If h is a function and $a \subset \mathrm{dom}(h)$ we write $h[a] = \cup\{h(\xi) : \xi \in a\}$ (this piece of notation has been used before). If x and y are two non-empty sets of ordinals with $\sup x < \sup y$ then we let

$$x * y = \begin{cases} x \cap y & \text{if } \sup x \notin y, \\ x \setminus y & \text{if } \sup x \in y. \end{cases}$$

Note that this operation $*$ is not symmetric, on the contrary, if $x * y$ is defined then $y * x$ is not.

Definition 1.11.4. A pair of functions $H : \vartheta \times \omega \longrightarrow \mathcal{P}(\vartheta)$ and $i : \vartheta \otimes \vartheta \otimes \omega \longrightarrow \big[\vartheta\big]^{<\omega}$ are said to be *ϑ-suitable* if the following three conditions hold for all $\alpha, \beta \in \vartheta$ and $n \in \omega$:

(H1) $\alpha \in H(\alpha, n) \subset H(\alpha, 0) \subset \alpha + 1$,
(H2) $i(\alpha, \beta, n) \in \big[\alpha\big]^{<\omega}$,
(H3) if $\alpha < \beta$ then $H(\alpha, 0) * H(\beta, n) \subset H[i(\alpha, \beta, n)]$.

Concerning (H3) note that we have

$$\max H(\alpha, 0) = \alpha < \max H(\beta, n) = \beta,$$

hence $H(\alpha, 0) * H(\beta, n)$ is defined.

Given a ϑ-suitable pair (H, i) as above, let us define the functions

$$\mathrm{K} : \vartheta \times \mathcal{S} \longrightarrow \mathcal{P}(\vartheta) \ \text{ and } \ i' : \vartheta \otimes \vartheta \otimes \mathcal{S} \longrightarrow \big[\vartheta\big]^{<\omega}$$

as follows:

$$\mathrm{K}(\alpha, \mathbb{C}) = H(\alpha, 0) \cap \alpha, \tag{1.98}$$

$$\mathrm{K}(\alpha, [n, 1]) = H(\alpha, n) \cap \alpha, \tag{1.99}$$

$$\mathrm{K}(\alpha, [n, 0]) = H(\alpha, 0) \setminus H(\alpha, n), \tag{1.100}$$

$$i'(\alpha, \beta, \mathbb{C}) = i(\alpha, \beta, 0), \tag{1.101}$$

$$i'(\alpha, \beta, [n, \varepsilon]) = i(\alpha, \beta, 0) \cup i(\alpha, \beta, n). \tag{1.102}$$

It is straightforward to check then that K and i' satisfy all the requirements of theorem 1.11.2. Because of this, with some abuse of notation, we shall denote the topology τ_{K} also by τ_H and the space $\langle \vartheta \times \mathbb{C}, \tau_{\mathrm{K}} \rangle$ by $\mathcal{X}_H$.

For our subbasic compact open sets we have

$$U(\alpha) = U(\alpha, \mathbb{C}) = H(\alpha, 0) \times \mathbb{C}, \tag{1.103}$$

and to simplify notation we write

$$U(\alpha, [n, \varepsilon]) = U(\alpha, n, \varepsilon). \tag{1.104}$$

Using this terminology, we may now formulate lemma 1.11.3 for this example in the following manner.

Lemma 1.11.5. *If (H, i) is an ϑ-suitable pair then for every $\langle \alpha, x \rangle \in \vartheta \times \mathbb{C}$ the compact open sets*

$$B(\alpha, x, n, F) = \bigcap \{U(\alpha, j, x(j)) \ : 1 \leq j \leq n, \ \} \setminus U[F]$$

with $n \in \mathbb{N}$ and $F \in [\alpha]^{<\omega}$ form a neighbourhood base of the point $\langle \alpha, x \rangle$ in the space X_H.

What we are set out to do now is to force an ω_2-suitable pair (H, i) such that the space $\mathcal{X}_H$ is as required. As mentioned, for this we need a special kind of Δ-function and this will be discussed in the next subsection.

1.11.2. Δ-functions.

Definition 1.11.6. Let $f : [\omega_2]^2 \longrightarrow [\omega_2]^{\leq\omega}$ be a function with $f(\{\alpha,\beta\}) \subset \alpha \cap \beta$ for $\{\alpha,\beta\} \in [\omega_2]^2$. Actually, in what follows, we shall simply write $f(\alpha,\beta)$ instead of $f(\{\alpha,\beta\})$.

We say that two finite subsets x and y of ω_2 are *very good for* f provided that for $\tau, \tau_1, \tau_2 \in x \cap y$, $\alpha \in x \setminus y$, $\beta \in y \setminus x$ and $\gamma \in (x \setminus y) \cup (y \setminus x)$ we always have

$\Delta 1)$ $\tau < \alpha, \beta \Longrightarrow \tau \in f(\alpha,\beta)$,
$\Delta 2)$ $\tau < \alpha \Longrightarrow f(\tau,\beta) \subset f(\alpha,\beta)$,
$\Delta 3)$ $\tau < \beta \Longrightarrow f(\tau,\alpha) \subset f(\beta,\alpha)$,
$\Delta 4)$ $\gamma, \tau_1 < \tau_2 \Longrightarrow f(\gamma,\tau_1) \subset f(\gamma,\tau_2)$.
$\Delta 5)$ $\tau_1 < \gamma < \tau_2 \Longrightarrow \tau_1 \in f(\gamma,\tau_2)$.

The sets x and y are said to be *good for* f iff $\Delta 1)$–$\Delta 3)$ hold.

We say that $f : [\omega_2]^2 \longrightarrow [\omega_2]^{\leq\omega}$ with $f(\alpha,\beta) \subset \alpha \cap \beta$ is a *very good* Δ*-function*, if every uncountable family of finite subsets of ω_2 contains two sets x and y which are very good for f.

We will prove in Lemma 1.11.8 that it is consistent with CH that there is a very good Δ-function.

In the proof of the countable compactness of our space we shall need the following simple consequence of [**11**, Lemma 1.2] that yields an additional property of Δ-functions provided that CH holds.

Lemma 1.11.7. *Assume that CH holds, f is a Δ-function, and $B \in [\omega_2]^\omega$. Then for any finite collection $\{T_i : i < m\} \subset [\omega_2]^{\omega_2}$ we may select a strictly increasing sequence $\langle \gamma_i : i < m\rangle$ with $\gamma_i \in T_i$ such that $B \subset f(\gamma_i,\gamma_j)$ whenever $i < j < m$.* $\square$

Now, we have come to the main result of this section.

Lemma 1.11.8. *It is consistent with CH that there is a very good Δ-function.*

Proof of Lemma 1.11.8. There are several natural ways of constructing such a very good Δ-function f. One can do it by forcing, following and modifying a bit the construction given in [**25**]. One can use Velleman's simplified morasses (see [**104**]) and put

$$f(\alpha,\beta) = X \cap \beta$$

where X is an element of minimal rank of the morass that contains α and β.

In this section we chose to follow Todorčevič's approach that uses his canonical coloring $\rho\colon [\omega_2]^2 \to \omega_1$ obtained from a $\square_{\omega_1}$-sequence (see [**103**, 7.3.2 and 7.4.8]). From this coloring ρ he defines f by

$$f(\alpha,\beta) = \{\xi < \alpha : \rho(\xi,\beta) \leq \rho(\alpha,\beta)\}$$

and proves that this f is a Δ-function in our terminology of 1.11.6 (see [**103**, 7.4.9 and 7.4.10]). (We should warn the reader, however, that he calls this a D-function instead of a Δ-function in [**103**].)

He also establishes the following canonical inequalities for ρ (see [**103**, 7.3.7 and 7.3.8]):

$$(i) \qquad |\{\xi < \alpha : \rho(\xi,\alpha) \leq \nu\}| < \omega_1$$

$$(ii) \qquad \rho(\alpha,\gamma) \leq \max\{\rho(\alpha,\beta), \rho(\beta,\gamma)\}$$

$$(iii) \qquad \rho(\alpha,\beta) \leq \max\{\rho(\alpha,\gamma), \rho(\beta,\gamma)\}$$

for $\alpha < \beta < \gamma < \omega_2$ and $\nu < \omega_1$. We will now use these inequalities to prove that this f is even a very good Δ-function.

Let $\mathcal{A}$ be an uncountable family of finite subsets of ω_2. Note that it is enough to find an uncountable $\mathcal{A}' \subseteq \mathcal{A}$ such that $\Delta 4)$ and $\Delta 5)$ of 1.11.6 hold for every two elements of $\mathcal{A}'$, since then we may apply to $\mathcal{A}'$ the fact that f is a Δ-function to obtain two elements of $\mathcal{A}$ that are very good for f.

We may assume w.l.o.g. that $\mathcal{A}$ forms a Δ-system with root $\Delta \subseteq \omega_2$. Note that then the set

$$D = \{\xi \in \omega_2 : \exists \tau_1, \tau_2, \tau_3 \in \Delta, \quad \xi < \tau_1, \quad \rho(\xi,\tau_1) \leq \rho(\tau_2,\tau_3)\}$$

is countable by (i). Define $\mathcal{A}' \subseteq \mathcal{A}$ to be the set of all elements $a \in \mathcal{A}$ which satisfy $(a-\Delta)\cap D = \emptyset$. The countability of D implies that $\mathcal{A}'$ is uncountable, moreover we have

$$\rho(\gamma, \tau_1) > \rho(\tau_2, \tau_3) \tag{1}$$

for all $\tau_1, \tau_2, \tau_3 \in \Delta$ and $\gamma \in a - \Delta$ with $a \in \mathcal{A}'$ and $\gamma < \tau_1$.

Now we prove that both $\Delta 4$) and $\Delta 5$) of 1.11.6 hold for every two sets $x, y \in \mathcal{A}'$ which will complete the proof of the lemma. Let $\tau_1, \tau_2 \in \Delta = x \cap y$ and $\gamma \in (x \setminus y) \cup (y \setminus x)$.

Note that if $\tau_1, \gamma < \tau_2$, then

$$\rho(\gamma, \tau_1) \leq \rho(\gamma, \tau_2). \tag{2}$$

This follows from (iii) and (1).

Now we prove $\Delta 4$). Consider two cases. First when $\tau_1 < \gamma < \tau_2$. Assume $\xi \in f(\tau_1, \gamma)$, that is $\xi < \tau_1$ and

$$\rho(\xi, \gamma) \leq \rho(\tau_1, \gamma), \tag{3}$$

By (ii) we have $\rho(\xi, \tau_2) \leq \max(\rho(\xi, \gamma), \rho(\gamma, \tau_2))$ which by (3) is less or equal to $\max(\rho(\tau_1, \gamma), \rho(\gamma, \tau_2)) = \rho(\gamma, \tau_2)$ by (2). But this means that $\xi \in f(\gamma, \tau_2)$ and so gives the inclusion of $\Delta 4$).

The second case is when $\gamma < \tau_1 < \tau_2$. Assume $\xi \in f(\gamma, \tau_1)$, that is $\xi < \gamma$ and

$$\rho(\xi, \tau_1) \leq \rho(\gamma, \tau_1)\,. \tag{4}$$

By (ii) we have that $\rho(\xi, \tau_2) \leq \max(\rho(\xi, \tau_1), \rho(\tau_1, \tau_2))$ which by (4) is less or equal to $\max(\rho(\gamma, \tau_1), \rho(\tau_1, \tau_2))$. But we have

$$\max(\rho(\gamma, \tau_1), \rho(\tau_1, \tau_2)) \leq \rho(\gamma, \tau_2)$$

by (1) and (2), hence $\rho(\xi, \tau_2) \leq \rho(\gamma, \tau_2)$ and so $\xi \in f(\gamma, \tau_2)$ that again gives the inclusion of $\Delta 4$).

Finally, we prove $\Delta 5$). Assume $\tau_1 < \gamma < \tau_2$, then by (1) we have $\rho(\tau_1, \tau_2) \leq \rho(\gamma, \tau_2)$ and so the definition of f gives that $\tau_1 \in f(\gamma, \tau_2)$, as required in $\Delta 5$).

□

1.11.3. The forcing notion. Now we describe a natural notion of forcing with finite approximations that produces an ω_2-suitable pair (H, i). The forcing depends on a parameter f that will be chosen to be a very good Δ-function, like the one constructed in 1.11.8.

Definition 1.11.9. For each function $f : \left[\omega_2\right]^2 \longrightarrow \left[\omega_2\right]^{\leq\omega}$ satisfying $f(\alpha, \beta) \subset \alpha \cap \beta$ for any $\{\alpha, \beta\} \in \left[\omega_2\right]^2$ we define the poset $(P_f, \leq)$ as follows. The elements of P_f are all quadruples $p = \langle a, h, n, i\rangle$ satisfying the following five conditions (P1) – (P5):

(P1) $a \in [\omega_2]^{<\omega}$, $n \in \omega$, $h : a \times n \to \mathcal{P}(a)$, $i : a \otimes a \otimes n \to \mathcal{P}(a)$,
(P2) $\max h(\xi, j) = \xi$ for all $\langle \xi, j\rangle \in a \times n$,
(P3) $h(\xi, j) \subset h(\xi, 0)$ for all $\langle \xi, j\rangle \in a \times n$,
(P4) $i(\xi, \eta, j) \subseteq f(\xi, \eta)$ whenever $\langle \xi, \eta, j\rangle \in a \otimes a \otimes n$,
(P5) if $\langle \xi, \eta, j\rangle \in a \otimes a \otimes n$ then $h(\xi, 0) * h(\eta, j) \subset h[i(\xi, \eta, j)]$,

where, with some abuse of our earlier notation, we write

$$h[b] = \cup\{h(\alpha, 0) : \alpha \in b\} \tag{1.105}$$

for $b \subset a$. We say that $p \leq q$ if and only if $a_p \supseteq a_q$, $n_p \geq n_q$, $h_p(\xi, j) \cap a_q = h_q(\xi, j)$ for all $\langle \xi, j\rangle \in a_q \times n_q$, moreover $i_p \supset i_q$.

Assume that the sets

$$D_{\alpha,n} = \{p \in P_f : \alpha \in a_p \text{ and } n < n_p\}$$

are dense in P_f for all pairs $\langle \alpha, n\rangle \in \omega_2 \times \omega$. Then if $\mathcal{G}$ is a P_f-generic filter over V we may define , in $V[\mathcal{G}]$, the function H with $\operatorname{dom} H = \omega_2 \times \omega$ and the function i with $\operatorname{dom}(i) = \omega_2 \otimes \omega_2 \otimes \omega$ as follows:

$$H(\alpha, n) = \cup\{h_p(\alpha, n) : p \in \mathcal{G}, \langle \alpha, n\rangle \in \operatorname{dom}(h_p)\}, \tag{1.106}$$

$$i = \cup\{i_p : p \in \mathcal{G}\}. \tag{1.107}$$

Theorem 1.11.10. *Assume that CH holds and f is a very good Δ-function. Then P_f is CCC and (H, i) is an ω_2-suitable pair in $V[\mathcal{G}]$. Moreover, the locally compact, 0-dimensional, and Hausdorff space $\mathcal{X}_H = \langle \omega_2 \times \mathbb{C}, \tau_H \rangle$ defined as in 1.11.4 satisfies, in $V[\mathcal{G}]$, the following properties:*

(i) $U(\delta) = H(\delta, 0) \times \mathbb{C}$ *is compact open for each* $\delta \in \omega_2$,
(ii) $\mathcal{X}_H$ *is first countable,*
(iii) $\forall A \in [\omega_2 \times \mathbb{C}]^{\omega_1}\ \exists \alpha \in \omega_2\ |A \cap U(\alpha)| = \omega_1$,
(iv) $\forall Y \in [\omega_2 \times \mathbb{C}]^{\omega}$ *either the closure* $\overline{Y}$ *is compact or there is* $\alpha < \omega_2$ *such that* $(\omega_2 \setminus \alpha) \times \mathbb{C} \subset \overline{Y}$.

Consequently, $\mathcal{X}_H$ is a locally compact, 0-dimensional, normal, first countable, initially ω_1-compact but non-compact space in $V[\mathcal{G}]$.

The rest of this section is devoted to the proof of Theorem 1.11.10.

1.11.4. The forcing is CCC. The CCC property of P_f is crucial for us because it implies that ω_2 is preserved in the generic extension $V[\mathcal{G}]$. Indeed, properties (H1)–(H3) of definition 1.11.4 (for $\vartheta = \omega_2^V$) are easily deduced from of conditions (P1)–(P5) in 1.11.9 using straightforward density arguments. So if ω_2 is preserved then we immediately conclude that (H, i) is an ω_2-suitable pair in $V[\mathcal{G}]$.

Definition 1.11.11. Two conditions $p_0 = \langle a_0, h_0, n, i_0 \rangle$ and $p_1 = \langle a_1, h_1, n, i_1 \rangle$ from P_f are said to be *good twins* provided that

(1) p_0 and p_1 are *isomorphic*, i.e. $|a_0| = |a_1|$ and the natural order-preserving bijection e between a_0 and a_1 is an isomorphism between p_0 and p_1:
  (i) $h_1(e(\xi), j) = e[h_0(\xi, j)]$ for $\xi \in a_0$ and $j < n$,
  (ii) $i_1(e(\xi), e(\eta), j)) = e[i_0(\xi, \eta, j)]$ for $\langle \xi, \eta, j \rangle \in a_0 \otimes a_0 \otimes n$,
  (iii) $e(\xi) = \xi$ whenever $\xi \in a_0 \cap a_1$ and $j < n$;
(2) $i_1(\xi, \eta, j) = i_0(\xi, \eta, j)$ for each $\{\xi, \eta\} \in \left[a_0 \cap a_1\right]^2$;
(3) a_0 and a_1 are good for f.

The good twins p_0 and p_1 are called *very good twins* if a_0 and a_1 are very good for f.

Definition 1.11.12. If $p = \langle a, h, n, i \rangle$ and $p' = \langle a', h', n, i' \rangle$ are good twins we define the *amalgamation* $p^* = \langle a^*, h^*, n, i^* \rangle$ *of* p *and* p' as follows:

Let $a^* = a \cup a'$. For $\eta \in h[a \cap a'] \cup h'[a \cap a']$ define

$$\delta_\eta = \min\{\delta \in a \cap a' : \eta \in h(\delta, 0) \cup h'(\delta, 0)\}.$$

Now, for any $\xi \in a^*$ and $m < n$ let

$$h^*(\xi, m) = \begin{cases} h(\xi, m) \cup h'(\xi, m) & \text{if } \xi \in a \cap a', \\ h(\xi, m) \cup \{\eta \in a' \setminus a : \delta_\eta \text{ is defined and } \delta_\eta \in h(\xi, m)\} & \text{if } \xi \in a \setminus a', \\ h'(\xi, m) \cup \{\eta \in a \setminus a' : \delta_\eta \text{ is defined and } \delta_\eta \in h'(\xi, m)\} & \text{if } \xi \in a' \setminus a. \end{cases} \tag{1.108}$$

Finally for $\langle \xi, \eta, m \rangle \in a^* \otimes a^* \otimes n$ let

$$i^*(\xi, \eta, m) = \begin{cases} i(\xi, \eta, m) & \text{if } \xi, \eta \in a, \\ i'(\xi, \eta, m) & \text{if } \xi, \eta \in a', \\ f(\xi, \eta) \cap a^* & \text{otherwise.} \end{cases} \tag{1.109}$$

(Observe that i^* is well-defined because p and p' are good twins). We will write $p^* = p + p'$ for the amalgamation of p and p'.

Lemma 1.11.13. *If p and p' are good twins then their amalgamation, $p^* = p + p'$, is a common extension of p and p' in P_f.*

Proof. Lemma 1.11.13 corresponds to Lemma 1.10.14.

The proof is quite technical, but it is similar to the proof of Lemma 1.10.14. So we omit it. See in [**11**]

□

Proof of theorem 1.11.10: P_f is CCC. In every uncountable collection of conditions from P_f there are two which are good twins for f and, by Lemma 1.11.13, they are compatible. □

As was pointed out at the beginning of this section, we may now conclude that (H, i) is an ω_2-suitable pair in $V[\mathcal{G}]$. This establishes the first part of Theorem 1.11.10 up to and including (i).

1.11.5. First countability.

Proof of theorem 1.11.10: $\mathcal{X}_H$ is first countable. Since X_H is locally compact and Hausdorff it suffices to show that every point of X_H has countable pseudo-character or, in other words, every singleton is a G_δ.

To see this, fix $\langle \alpha, x\rangle \in \omega_2 \times \mathbb{C}$. We claim that there is a countable set $\Gamma \subset \alpha$ such that

$$\bigcap_{n\in\mathbb{N}} U(\alpha, n, x(n)) \subset U[\Gamma] \cup \{\langle \alpha, x\rangle\}. \tag{1.110}$$

Since every $U(\gamma)$ is clopen, this implies that

$$\{\langle \alpha, x\rangle\} = \bigcap_{n\in\mathbb{N}} U(\alpha, n, x(n)) \cap \bigcap \{\, X_H \setminus U(\gamma) : \gamma \in \Gamma\}$$

is indeed a G_δ.

Our following lemma clearly implies (1.110). To formulate it, we first fix some notation. In $V[\mathcal{G}]$, for $\alpha \in \omega_2$, $1 \le m < \omega$ and $\Gamma \subset \omega_2$ we write

$$H^1(\alpha, m) = H(\alpha, m) \setminus \{\alpha\}, \tag{1.111}$$

$$H^0(\alpha, m) = H(\alpha, 0) \setminus H(\alpha, m), \tag{1.112}$$

$$H[\Gamma] = \cup\{H(\gamma, 0) : \gamma \in \Gamma\}. \tag{1.113}$$

Recall that with this notation we have

$$U(\alpha, n, \varepsilon) = (H^\varepsilon(\alpha, n) \times \mathbb{C}) \cup (\{\alpha\} \times [n, \varepsilon])\,.$$

Lemma 1.11.14. *In $V[\mathcal{G}]$, for each $\langle \alpha, x\rangle \in \omega_2 \times \mathbb{C}$ there is a countable set $\Gamma \subset \alpha$ such that*

$$\bigcap_{n\in\mathbb{N}} H^{x(n)}(\alpha, n) \subset H[\Gamma]\,. \tag{1.114}$$

Proof. Suppose, arguing indirectly, that the lemma is false. Then, in $V[\mathcal{G}]$, for each countable set $A \subset \alpha$ there is $\gamma_A \in \alpha$ such that

$$\gamma_A \in \bigcap_{n\in\mathbb{N}} H^{x(n)}(\alpha, n) \setminus H[A]. \tag{1.115}$$

>From now on, we work in the ground model V. For every $\zeta < \omega_1$ let $A_\zeta \subseteq \alpha$ be a countable subset such that $\zeta' \le \zeta < \omega_1$ implies $A_{\zeta'} \subseteq A_\zeta$ and $\bigcup_{\zeta<\omega_1} A_\zeta = \alpha$.

Let $p_\zeta = \langle a_\zeta, h_\zeta, n_\zeta, i_\zeta\rangle \in P_f$ be a condition such that $\alpha \in a_\zeta$ and for some $\gamma_\zeta \in \alpha \cap a_\zeta$ we have

$$p_\zeta \Vdash \gamma_\zeta \in \bigcap_{n\in\mathbb{N}} H^{x(n)}(\alpha, n) \setminus H[A_\zeta]. \tag{1.116}$$

Using standard Δ-system and counting arguments and the properties of the very good Δ-function f, we may find $\zeta_1 < \zeta_2 < \omega_1$ such that

$$\alpha \cap a_{\zeta_1} \subset A_{\zeta_2}, \tag{1.117}$$

moreover p_{ζ_1}, p_{ζ_2} are very good twins for f.

Let $p = p_{\zeta_1} + p_{\zeta_2}$ with $p = \langle a, h, n, i\rangle$ be their amalgamation as in 1.11.12. We now further extend p to a condition of the form $r = \langle a, h_r, n+1, i_r\rangle$ with the following stipulations:

(r1) $h_r \supset h$,
(r2) $h_r(\xi, n) = \{\xi\}$ for $\xi \in a \setminus \{\alpha\}$,
(r3) $h_r(\alpha, n) = \{\alpha\} \cup \big(h(\alpha, 0) \cap h[\alpha \cap a_{\zeta_1}]\big)$,
(r4) $i_r \supset i$,
(r5) $i_r(\eta, \xi, n) = \emptyset$ for $\eta < \xi \in a \setminus \{\alpha\}$,
(r6) $i_r(\eta, \alpha, n) = a \cap f(\eta, \alpha)$ for $\eta < \alpha$.

It is not clear at all that r is a condition, but if it is we have reached a contradiction. Indeed, if $r \in P_f$ then $r \leq p_{\zeta_2}$, so $r \Vdash \gamma_{\zeta_2} \notin H[A_{\zeta_2}]$, hence $\gamma_{\zeta_2} \notin h[\alpha \cap a_{\zeta_1}]$ by (1.117). But then by (r3) we have

$$\gamma_{\zeta_2} \notin h_r(\alpha, n). \tag{1.118}$$

On the other hand, since $\gamma_{\zeta_1} \in \alpha \cap a_{\zeta_1} \subset h[\alpha \cap a_{\zeta_1}]$ we have

$$\gamma_{\zeta_1} \in h_r(\alpha, n) \tag{1.119}$$

by (r3). But this is a contradiction because, by (1.116), the first of these relations implies $r \Vdash x(\alpha) = 0$ while the second implies $r \Vdash x(\alpha) = 1$.

So it remains to show that $r \in P_f$. Items (P1) - (P4) of Definition 1.11.9 are clear. Also, (P5) holds if $j < n$ because $p \in P_f$. Thus we only have to check (P5) for triples of the form $\langle \eta, \xi, n \rangle$.

If $\eta < \xi \neq \alpha$ we have $\eta \notin h(\xi, n) = \{\xi\}$, and so $h_r(\eta, 0) * h_r(\xi, n) = h_r(\eta, 0) \cap h_r(\xi, n) \subseteq \eta \cap \{\xi\} = \emptyset$, hence (P5) of Definition 1.11.9 holds trivially. So assume now that $\eta < \alpha$. In view of the definition of r, our task is to show the following two assertions:

(I) if $\eta \in h_r(\alpha, n)$ then $h(\eta, 0) \setminus h_r(\alpha, n) \subset h[a \cap f(\eta, \alpha)]$,

(II) if $\eta \notin h_r(\alpha, n)$ then $h(\eta, 0) \cap h_r(\alpha, n) \subset h[a \cap f(\eta, \alpha)]$.

The fact that $p = p_{\zeta_1} + p_{\zeta_2}$ and properties $\Delta 4)$ and $\Delta 5)$ of our very good Δ-function f will play an essential role in the proofs of (I) and (II).

Proof of (I). First note that by the definition of r we have

$$h(\eta, 0) \setminus h_r(\alpha, n) = h(\eta, 0) \setminus (h(\alpha, 0) \cap h[\alpha \cap a_{\zeta_1}]) = \\ (h(\eta, 0) \setminus h(\alpha, 0)) \cup (h(\eta, 0) \setminus h[\alpha \cap a_{\zeta_1}]). \tag{1.120}$$

Since $h(\eta, 0) \setminus h(\alpha, 0) \subset h[i(\eta, \alpha, 0)] \subset h[a \cap f(\eta, \alpha)]$ is obvious, it is enough to show that

(I') if $\eta \in h_r(\alpha, n)$, then $h(\eta, 0) \setminus h[\alpha \cap a_{\zeta_1}] \subset h[a \cap f(\eta, \alpha)]$.

If $\eta \in a_{\zeta_1}$ then $h(\eta, 0) \setminus h[\alpha \cap a_{\zeta_1}] = \emptyset$ and we are done. So assume now that $\eta \notin a_{\zeta_1}$, that is $\eta \in a_{\zeta_2} \setminus a_{\zeta_1}$. Now $\eta \in h[\alpha \cap a_{\zeta_1}]$ means that there is a $\xi \in \alpha \cap a_{\zeta_1}$ with $\eta \in h(\xi, 0)$. By the definition 1.11.12 (1.108) of the amalgamation then there is $\delta \in a_{\zeta_1} \cap a_{\zeta_2}$ such that $\eta < \delta \leq \xi$ and $\eta \in h_{\zeta_2}(\delta, 0)$. Since $p_{\zeta_2} \in P_f$ this implies

$$h_{\zeta_2}(\eta, 0) \setminus h_{\zeta_2}(\delta, 0) \subseteq h_{\zeta_2}[i_{\zeta_2}(\eta, \delta, 0)]. \tag{1.121}$$

A similar argument, referring back to definition 1.11.12 (1.108), yields us that $h(\eta, 0) \setminus h_{\zeta_2}(\eta, 0) \subset h[\alpha \cap a_{\zeta_1}]$, and as $h_{\zeta_2}(\delta, 0) \subset h(\delta, 0) \subset h[\alpha \cap a_{\zeta_1}]$ we may conclude that

$$h(\eta, 0) \setminus h[\alpha \cap a_{\zeta_1}] \subset h_{\zeta_2}[i_{\zeta_2}(\eta, \delta, 0)] \subset h[i_{\zeta_2}(\eta, \delta, 0)]. \tag{1.122}$$

Since $\eta \in a_{\zeta_2} \setminus a_{\zeta_1}$ and $\delta, \alpha \in a_{\zeta_1} \cap a_{\zeta_2}$, we have $f(\eta, \delta) \subset f(\eta, \alpha)$ by $\Delta 4)$. Consequently,

$$i_{\zeta_2}(\eta, \delta, 0) \subset a_{\zeta_2} \cap f(\eta, \delta) \subset a \cap f(\eta, \alpha), \tag{1.123}$$

completing the proof of (I') and hence of (I).

Proof of (II). If $\eta \notin h_r(\alpha, n)$ then either $\eta \notin h(\alpha, 0)$ or $\eta \notin h[\alpha \cap a_{\zeta_1}]$. If $\eta \notin h(\alpha, 0)$ then $p \in P_f$ implies

$$h(\eta, 0) \cap h_r(\alpha, n) \subset h(\eta, 0) \cap h(\alpha, 0) = \\ h(\eta, 0) * h(\alpha, 0) \subset h[i(\eta, \alpha, 0)] \subset h[a \cap f(\eta, \alpha]). \tag{1.124}$$

So assume that $\eta \notin h[\alpha \cap a_{\zeta_1}]$, clearly then $\eta \notin a_{\zeta_1}$ as well. Consider any $\beta \in h(\eta, 0) \cap h_r(\alpha, n)$, we have to show that $\beta \in h[a \cap f(\eta, \alpha)]$.

Case 1. $\beta \in a_{\zeta_1}$. By using definition 1.11.12 (1.108) again, then $\beta \in h(\eta, 0)$ implies that there is a $\delta \in \eta \cap a_{\zeta_1} \cap a_{\zeta_2}$ with $\beta \in h_{\zeta_2}(\delta, 0)$. But then $\delta \in f(\eta, \alpha)$ by property $\Delta 5)$ of very good Δ-functions, hence we are done.

Case 2. $\beta \notin a_{\zeta_1}$. In this case $\beta \in h[\alpha \cap a_{\zeta_1}]$ implies that there is a $\delta \in \alpha \cap a_{\zeta_1} \cap a_{\zeta_2}$ such that $\beta \in h_{\zeta_2}(\delta, 0)$, hence $\beta \in h_{\zeta_2}(\eta, 0) \cap h_{\zeta_2}(\delta, 0)$. Moreover, $\eta \notin h[\alpha \cap a_{\zeta_1}]$ implies $\eta \notin h_{\zeta_2}(\delta, 0)$. Thus if $\eta < \delta$ then $p_{\zeta_2} \in P_f$ and $h_{\zeta_2}(\eta, 0) \cap h_{\zeta_2}(\delta, 0) = h_{\zeta_2}(\eta, 0) * h_{\zeta_2}(\delta, 0)$ imply that $\beta \in h_{\zeta_2}(\gamma, 0)$ for

some $\gamma \in i(\eta, \delta, 0) \subset f(\eta, \delta)$. But we have $f(\eta, \delta) \subset f(\eta, \alpha)$ by $\Delta 4$), so $\gamma \in a \cap f(\eta, \alpha)$ and we are done.

Finally, if $\delta < \eta$ then $\delta \in f(\eta, \alpha)$ because f satisfies $\Delta 5$), moreover we have $\beta \in h_{\zeta_2}(\delta, 0) \subset h(\delta, 0)$ and the proof of (II) is completed. □

This then completes the proof of Lemma 1.11.14 and thus of the first countability of the space X_H. □

1.11.6. ω_1-compactness. In this subsection we establish part (iii) of theorem 1.11.10. This implies that every uncountable subset of X_H has uncountable intersection with a compact set, hence every set of size ω_1 has a complete accumulation point.

Lemma 1.11.15. *If $p = \langle a, h, n, i \rangle \in P_f$ and $\beta \in \omega_2$ with $\beta > \max a$ then there is a condition $q \leq p$ such that $a \subset h_q(\beta, 0)$.*

Proof. We define the condition $q = \langle a \cup \{\beta\}, h_q, n, i_q \rangle$ with the following stipulations: $h_q \supset h$, $i_q \supset i$, $h_q(\beta, j) = a \cup \{\beta\}$ for $j < n$, $i_q(\alpha, \beta, j) = \emptyset$ for $\alpha \in a$ and $j < n$. It is straight-forward to check that $q \in P_f$ is as required. □

Lemma 1.11.16. *In $V[\mathcal{G}]$, for each set $A \in [\omega_2 \times \mathbb{C}]^{\omega_1}$ there is $\beta \in \omega_2$ such that $|A \cap U(\beta)| = \omega_1$.*

Proof. Let $\dot{A}$ be a P_f-name for A and assume that $p \in \mathcal{G}$ with

$$p \Vdash \dot{A} = \{\dot{z}_\xi : \xi < \omega_1\} \in \left[\omega_2 \times \mathbb{C}\right]^{\omega_1}.$$

We may assume that p also forces that $\{\dot{z}_\xi : \xi < \omega_1\}$ is a one-one enumeration of $\dot{A}$. For each $\xi < \omega_1$ we may pick $p_\xi \leq p$ and $\alpha_\xi \in \omega_2$ with $\alpha_\xi \in a_{p_\xi}$ such that $p_\xi \Vdash \dot{z}_\xi = \langle \alpha_\xi, \dot{x}_\xi \rangle$. Let $\sup\{\alpha_\xi : \xi < \omega_1\} < \beta < \omega_2$. By lemma 1.11.15 for each $\xi < \omega_1$ there is a condition $q_\xi \leq p_\xi$ such that $\alpha_\xi \in h_{q_\xi}(\beta, 0)$, hence $q_\xi \Vdash \dot{z}_\xi \in U(\beta)$. But P_f satisfies CCC, so there is $q \in \mathcal{G}$ such that $q \Vdash |\{\xi \in \omega_1 : q_\xi \in \mathcal{G}\}| = \omega_1$. Clearly, then $q \Vdash |\dot{A} \cap U(\beta)| = \omega_1$. □

1.11.7. Countable compactness. In this subsection we show that part (iv) of theorem 1.11.10 holds: in $V[\mathcal{G}]$, the closure of any infinite subset of $\mathcal{X}_H$ is either compact or contains a "tail" of X_H, that is $(\omega_2 \setminus \alpha) \times \mathbb{C}$ for some $\alpha < \omega_2$. Of course, this implies that X_H is countably compact and thus, together with the results of the previous section, establishes the initial ω_1-compactness of X_H. Moreover, it also implies that X_H is normal, for of any two disjoint closed sets in X_H (at least) one has to be compact.

We start by proving an extension result for conditions in P_f. We shall use the following notation that is analogous to the one that was introduced before lemma 1.11.14.

$$h^1(\alpha, m) = h(\alpha, m), \tag{1.125}$$

$$h^0(\alpha, m) = h(\alpha, 0) \setminus h(\alpha, m). \tag{1.126}$$

Lemma 1.11.17. *Assume that $p = \langle a, h, n, i \rangle \in P_f$, $\alpha \in a$, and $\varepsilon : n \longrightarrow 2$ is a function with $\varepsilon(0) = 1$. Then for every $\eta \in \alpha \setminus a$ there is a condition of the form $q = \langle a \cup \{\eta\}, h_q, n, i_q \rangle \in P_f$ such that $q \leq p$ and*

$$\eta \in \bigcap_{m<n} h_q^{\varepsilon(m)}(\alpha, m) \setminus h_q[a \cap \alpha]. \tag{1.127}$$

Proof. We define h_q and i_q with the following stipulations:

$h_q(\eta, m) = \{\eta\}$ for $m < n$,
$h_q(\alpha, m) = h(\alpha, m) \cup \{\eta\}$ if $m < n$ and $\varepsilon(m) = 1$,
$h_q(\alpha, m) = h(\alpha, m)$ if $m < n$ and $\varepsilon(m) = 0$,
$h_q(\nu, m) = h(\nu, m) \cup \{\eta\}$ if $\nu \in a \setminus \{\alpha\}$, $m < n$, and $\alpha \in h(\nu, m)$,
$h_q(\nu, m) = h(\nu, m)$ if $\nu \in a \setminus \{\alpha\}$, $m < n$, and $\alpha \notin h(\nu, m)$,
$i_q \supset i$, $i_q(\eta, \nu, m) = \emptyset$ if $\nu \in a \setminus \eta$, and $i_q(\nu, \eta, m) = \emptyset$ if $\nu \in a \cap \eta$.

To show $q \in P_f$ we need to check only (P5). But this follows from the fact that if $\eta \in h_q(\nu, 0) * h_q(\mu, m)$ then, as can be checked by examining a number of cases, we have $\nu, \mu \in a$ and $\alpha \in h(\nu, 0) * h(\mu, m)$ as well. By $p \in P_f$ then there is a $\xi \in i(\nu, \mu, m)$ with $\alpha \in h(\xi, 0)$ which implies $\eta \in h_q(\xi, 0)$ because $\varepsilon(0) = 1$, so we are done. Thus $q \in P_f$, $q \leq p$, and q clearly satisfies all our requirements. □

Lemmas 1.11.15 and 1.11.17 can be used to show that

$$D_{\alpha,n} = \{p \in P_f : \alpha \in a_p \text{ and } n < n_p\}$$

is dense in P_f for all pairs $\langle \alpha, n\rangle \in \omega_2 \times \omega$, showing that $\mathrm{dom}(H) = \omega_2 \times \omega$ and $\mathrm{dom}(i) = \omega_2 \otimes \omega_2 \otimes \omega$.

Our next lemma is a partial result on the way to what we promised to show in this section.

Lemma 1.11.18. *Assume that, in $V[\mathcal{G}]$, we have $D \in V \cap \big[\omega_2\big]^\omega$ and $Y = \{\langle \delta, x_\delta\rangle : \delta \in D\} \subset \omega_2 \times \mathbb{C}$. Then*

$$(\omega_2 \setminus \sup(D)) \times \mathbb{C} \subset \overline{Y}.$$

Proof. By lemma 1.11.5 it suffices to prove that

$$V[\mathcal{G}] \models \Big(\bigcap_{1 \leq m < n} U(\alpha, m, \varepsilon(m)) \setminus U[b] \Big) \cap Y \neq \emptyset \tag{1.128}$$

whenever $\alpha \in \omega_2 \setminus \sup D$, $n \in \mathbb{N}$, $\varepsilon : n \longrightarrow 2$ with $\varepsilon(0) = 1$, and $b \in \big[\alpha\big]^{<\omega}$. So fix these and pick a condition $p = \langle a, h, k, i\rangle \in P_f$ such that $\alpha \in a$, $b \subset a$, and $n < k$. (We know that the set E of these conditions is dense in P_f.) Let us then choose $\delta \in D \setminus a$. By lemma 1.11.17 there is a condition $q \leq p$ such that

$$\delta \in \bigcap_{1 \leq m < n} h_q^{\varepsilon(m)}(\alpha, m) \setminus h_q[b]. \tag{1.129}$$

Then

$$q \Vdash \langle \delta, x_\delta\rangle \in \bigcap_{1 \leq m < n} U(\alpha, m, \varepsilon(m)) \setminus U[b], \tag{1.130}$$

hence

$$q \Vdash \Big(\bigcap_{1 \leq m < n} U(\alpha, m, \varepsilon(m)) \setminus U[b] \Big) \cap Y \neq \emptyset\,. \tag{1.131}$$

Since $p \in E$ was arbitrary, the set of q's satisfying the last forcing relation is also dense in P_f, so we are done. □

We need a couple more, rather technical, results before we can turn to the proof of part (iv) of theorem 1.11.10. First we give a definition.

Definition 1.11.19. (1) Assume that $p = \langle a, h, n, i\rangle \in P_f$ and $a < b \in [\omega_2]^{<\omega}$ are such that $a \subset f(\gamma, \gamma')$ for any $\{\gamma, \gamma'\} \in [b]^2$. Then we define the *$b$-extension of p* to be the condition q of the form $q = \langle a \cup b, h_q, n, i_q\rangle$ with $h \subset h_q$, $i \subset i_q$, and the following stipulations:

(R1) $h_q(\gamma, \ell) = a \cup \{\gamma\}$ for $\gamma \in b$ and $\ell < n$,
(R2) $i_q(\gamma', \gamma, \ell) = a$ for $\gamma', \gamma \in b$ with $\gamma' < \gamma$ and $\ell < n$,
(R3) $i_q(\xi, \gamma, \ell) = \emptyset$ for $\xi \in a$, $\gamma \in b$, and $\ell < n$.

(2) If $q \in P_f$ and $b \subset a_q$ then $s \leq q$ is said to be a *b-fair extension of q* iff $h_s(\gamma, j) = h_s(\gamma, 0)$ holds for any $\gamma \in b$ and $n_q \leq j < n_s$.

Our following result shows that the b-extension severely restricts any further extensions.

Lemma 1.11.20. *Assume that $p = \langle a, h, n, i\rangle \in P_f$, $a < b$, and q is the b-extension of p. If $s \leq q$ is any extension of q then*

$$h_s[a] = h_s(\gamma', 0) \cap h_s(\gamma, \ell) \tag{1.132}$$

whenever $\langle \gamma', \gamma, \ell\rangle \in b \otimes b \otimes n$. If, in addition, s is a b-fair extension of q then (1.132) holds for all $\langle \gamma', \gamma, \ell\rangle \in b \otimes b \otimes n_s$.

Proof. We have $\gamma' \notin h_s(\gamma)$ by (R1) and $s \le q$, hence if $\ell < n$ then (P5) and (R2) imply

$$h_s(\gamma', 0) \cap h_s(\gamma, \ell) = h_s(\gamma', 0) * h_s(\gamma, \ell) \subset h_s[i_s(\gamma', \gamma, \ell)] = h_s[a]. \tag{1.133}$$

Similarly, for all $\xi \in a$, $\gamma'' \in b$, and $\ell'' < n$ we have

$$h_s(\xi, 0) \setminus h_s(\gamma'', \ell'') = h_s(\xi, 0) * h_s(\gamma'', \ell'') \subset h_s[i_s(\xi, \gamma'', \ell'')] = h_s[\emptyset] = \emptyset, \tag{1.134}$$

which implies $h_s[a] \subset h_s(\gamma'', \ell'')$. But then $h_s[a] \subset h_s(\gamma', 0) \cap h_s(\gamma, \ell)$ which together with (1.133) yields (1.132).

Now, if s is a b-fair extension of q and $\langle \gamma', \gamma, \ell \rangle \in b \otimes b \otimes n_s$ with $n \le \ell < n_s$ then we have (1.132) because $h_s(\gamma, 0) = h_s(\gamma, \ell)$ and $h_s[a] = h_s(\gamma', 0) \cap h_s(\gamma, 0)$. □

In our next result we are going to make use of the following simple observation.

Fact 1.11.21. *If $p = \langle a, h, n, i \rangle \in P_f$ and $X \subset a$ is an initial segment of a then $p \restriction X = \langle X, h \restriction X \times n, n, i \restriction X \otimes X \otimes n \rangle \in P_f$ as well.*

Lemma 1.11.22. *Let $p, q, s \in P_f$ be conditions and $Q \subset S < E < F$ be sets of ordinals such that*

$$a_p = Q \cup E\,,\ a_q = Q \cup E \cup F\,,\ a_s = S \cup E \cup F\,,$$

q is the F-extension of p, and s is an F-fair extension of q. Assume, moreover, that $|E| = k$ with $E = \{\gamma_i : i < k\}$ the increasing enumeration of E and $|F| = 2k$, $F = \{\gamma_{i,0}, \gamma_{i,1} : i < k\}$ with $\gamma_{i,0} < \gamma_{i,1}$ satisfying

$$\forall i < k\, \forall \xi \in S\, [\, f(\xi, \gamma_i) = f(\xi, \gamma_{i,0}) = f(\xi, \gamma_{i,1})\,]. \tag{1.135}$$

Let us now define $r = \langle a_r, h_r, n_r, i_r \rangle$ as follows:

(A) $a_r = S \cup E$, $n_r = n_s$,

(B) for $\xi \in a_r$ and $j < n_r$ let

$$h_r(\xi, j) = \begin{cases} h_s(\xi, j) \cup (S \setminus h_s[a_p]) & \text{if } \xi = \gamma_i \text{ and } \gamma_0 \in h_s(\gamma_i, j), \\ h_s(\xi, j) & \text{otherwise,} \end{cases}$$

(C) for $\langle \xi, \eta, j \rangle \in a_r \otimes a_r \otimes n_r$

$$i_r(\xi, \eta, j) = \begin{cases} i_s(\xi, \eta, j) & \text{if } \xi, \eta \in a_p \text{ or } \xi, \eta \in S, \\ f(\xi, \eta) \cap a_s & \text{otherwise.} \end{cases}$$

Then $r \in P_f$, $r \le p$, $r \le s \restriction S \in P_f$, and $S \setminus h_s[a_p] \subset h_r(\gamma_0, 0)$.

Proof. Lemma 1.11.22 corresponds to Lemma 1.10.23.

The proof is quite technical, but it is similar to the proof of Lemma 1.10.14. So we omit it. See in [**11**]

□

Proof of theorem 1.11.10: Property (iv). Our aim is to prove that the following statement holds in $V[\mathcal{G}]$:

(*iv*) *If the closure $\overline{Y}$ of a set $Y \in [X_H]^\omega$ is not compact then there is $\alpha < \omega_2$ such that $(\omega_2 \setminus \alpha) \times \mathbb{C} \subset \overline{Y}$.*

We shall make use of the following easy lemma.

Lemma 1.11.23. *A set $Z \subset X_H$ has compact closure if and only if*

$$\Gamma = \{\gamma : \exists\, x\, \langle \gamma, x \rangle \in Z\} \subset H[F]$$

for some finite set $F \subset \omega_2$.

Proof of the lemma. If $\overline{Z}$ is compact then there is a finite set $F \subset \omega_2$ such that $\overline{Z} \subset U[F]$. Clearly, then $\Gamma \subset H[F]$.

Conversely, if $\Gamma \subset H[F]$ for a finite $F \subset \omega_2$ then $Z \subset U[F]$, hence $\overline{Z} \subset U[F]$ as well. But as $U[F]$ is compact, so is $\overline{Z}$. □

Given two sets $X, E \subset \omega_2$ with $X < E$ we shall write

$$cl_f(X, E) = (\text{the } f\text{-closure of } X \cup E) \cap \sup(X). \tag{1.136}$$

Fact 1.11.24. *If $\xi \in cl_f(X,E)$ and $\eta \in cl_f(X,E) \cup E$ then $f(\xi,\eta) \subset cl_f(X,E)$.*

Let us now fix a regular cardinal ϑ that is large enough so that $\mathcal{H}_\vartheta$, the structure of sets whose transitive closure has cardinality $< \vartheta$, contains everything relevant.

Lemma 1.11.25. *Assume that*

$$V[\mathcal{G}] \models \Gamma \in \left[\omega_2\right]^\omega \text{ is not covered by finitely many } H(\xi,0) \tag{1.137}$$

and $\dot\Gamma$ is a P_f-name for Γ. If M is a σ-closed elementary submodel of $\mathcal{H}_\theta$ (in V) such that $f, \dot\Gamma \in M$, $|M| = \omega_1$, and $\delta = M \cap \omega_2 \in \omega_2$ then

$$V[\mathcal{G}] \models \Gamma \cap H(\delta,0) \setminus H[D] \neq \emptyset \text{ for each finite } D \subset \delta. \tag{1.138}$$

Proof of the lemma 1.11.25. Fix $D \in [\delta]^{<\omega}$ and a condition $p \in P_f$ with $D \cup \{\delta\} \subset a_p$ such that

$$p \Vdash \text{“}\dot\Gamma \in \left[\omega_2\right]^\omega \text{ is not covered by finitely many } H(\xi,0)\text{”}. \tag{1.139}$$

We shall be done if we can find a condition $r \leq p$ and an ordinal $\alpha \in a_r$ such that

$$r \Vdash \text{“}\alpha \in \dot\Gamma\text{” and } \alpha \in h_r(\delta,0) \setminus h_r[D]. \tag{1.140}$$

Let $Q = a_p \cap \delta$, $E = a_p \setminus \delta$, and $\{\gamma_i : i < k\}$ be the increasing enumeration of E. In particular, then we have $\gamma_0 = \delta$.

To achieve our aim, we first choose a *countable* elementary submodel N of $\mathcal{H}_\theta$ such that $M, \dot\Gamma, p \in N$ and put

$$A = \delta \cap N \text{ and } B = cl_f(A \cup Q, E)\,.$$

Note that we have $A, B \in M$ because M is σ-closed. For each $i < k$ the function $f(\,.\,,\gamma_i) \restriction B$ is in M, hence so is the set

$$T_i = \{\gamma \in \omega_2 \,:\, \forall \beta \in B \; f(\beta,\gamma) = f(\beta,\gamma_i)\}\,,$$

and $\gamma_i \in T_i \setminus M$ implies $|T_i| = \omega_2$.

By Lemma 1.11.7 there is a set of $2k$ ordinals

$$F = \{\gamma_{i,\varepsilon} \,:\, i < k, \varepsilon < 2\}$$

with $\gamma_{i,\varepsilon} \in T_i$ and $\gamma_{i,0} < \gamma_{i,1}$ for each $i < k$ such that

$$B \cup E \subset \bigcap \{f(\gamma_{i,\varepsilon}\,, \gamma_{i',\varepsilon'}) : \{\langle i,\varepsilon\rangle\,, \langle i',\varepsilon'\rangle\} \in [k \times 2]^2\}. \tag{1.141}$$

Since $a_p \subset B \cup E < F$, (1.141) implies that we can form the F-extension $q = \langle a_p \cup F, h_q, n_p, i_q\rangle \in P_f$ of p, see definition 1.11.19.

As $p \Vdash$ “$H[Q \cup E] \not\supset \dot\Gamma$”, there is a condition $t \leq q$ and an ordinal α such that

$$t \Vdash \text{“}\alpha \in \dot\Gamma \setminus H[Q \cup E]\text{”}. \tag{1.142}$$

Clearly we can assume that $\alpha \in a_t$, and then

$$t \Vdash \text{“}\alpha \in \dot\Gamma\text{” and } \alpha \in a_t \setminus h_t[Q \cup E]. \tag{1.143}$$

Since $\dot\Gamma \in N \cap M$ and P_f is CCC, we have $\alpha \in M \cap N \cap \omega_2 = N \cap \delta$. As P_f is CCC and $\alpha, \dot\Gamma \in M \cap N$ we may choose a maximal antichain $W \subset \{w \leq p : w \Vdash \alpha \in \dot\Gamma\}$ with $W \in N \cap M$ and hence $W \subset N \cap M$. By taking a further extension we can assume that $t \leq w$ for some $w \in W$.

We claim that, putting $S = B \cap a_t$, we have

$$i_t(\xi,\eta,j) \subset S \cup E \text{ for each } \langle \xi,\eta,j\rangle \in S \cup E \cup F \otimes S \cup E \cup F \otimes n_p\,. \tag{1.144}$$

Indeed, if $\xi \in S \subset B$ then fact 1.11.24 and $\gamma_{i,\varepsilon} \in T_i$ imply $f(\xi,\eta) \subset B$ and so $i_t(\xi,\eta,j) \subset S$, and if $\xi, \eta \in E \cup F$ then

$$i_t(\xi,\eta,j) = i_q(\xi,\eta,j) \subset a_p = Q \cup E \subset S \cup E$$

because q is the F-extension of p.

Let us now make the following definitions:

(s1) $a_s = S \cup E \cup F$,
(s2) $h_s(\xi,j) = h_t(\xi,j) \cap S = h_t(\xi,j) \cap a_s$ for $\xi \in S$ and $j < n_t$,
(s3) $i_s \restriction S \otimes S \otimes n_t = i_t \restriction S \otimes S \otimes n_t$,

(s4) for $\eta \in E \cup F$ and $j < n_t$ let

$$h_s(\eta, j) = \begin{cases} h_t(\eta, j) \cap a_s & \text{if } j < n_p, \\ h_t(\eta, 0) \cap a_s & \text{if } n_p \leq j < n_t, \end{cases} \tag{1.145}$$

(s5) for $\eta \in E \cup F$, $\xi \in a_s \cap \eta$ and $j < n_t$ let

$$i_s(\xi, \eta, j) = \begin{cases} i_t(\xi, \eta, j) & \text{if } j < n_p, \\ i_t(\xi, \eta, 0) & \text{if } n_p \leq j < n_t. \end{cases} \tag{1.146}$$

Then (1.144) and $t \in P_f$ imply that $s = \langle a_s, h_s, n_t, i_s \rangle \in P_f$, moreover s is an F-fair (even $E \cup F$-fair) extension of q.

Note that $t \leq w$ and $a_w \subset A \subset B$ implies $a_w \subset S$, hence by the definition of the condition s we have $s \leq w$ and even $s \restriction S \leq w$.

Things were set up in such a way that we can apply lemma 1.11.22 to the three conditions $s \leq q \leq p$ and the sets $Q \subset S < E < F$ to get a condition $r \in P_f$ such that

- $r \leq p$, $r \leq s \restriction S \leq w$,
- $\alpha \in S \setminus h_s[a_p] \subset h_s(\gamma_0)$.

Since $\delta = \gamma_0$ and $D \subset a_p$, we have $\alpha \in h_r(\delta) \setminus h_r[D]$. Moreover, $r \leq s \restriction S \leq w$ implies $r \Vdash$ "$\alpha \in \dot{\Gamma}$". So r satisfies (1.140), which completes the proof of our lemma. □

Assume now, to finish the proof of (iv), that

$$V[\mathcal{G}] \models \ Y \in \left[\omega_2 \times \mathbb{C}\right]^\omega \text{ and } \overline{Y} \text{ is not compact.} \tag{1.147}$$

Then, by lemma 1.11.23, $\Gamma = \{\gamma : \exists x \in \mathbb{C}\ \langle \gamma, x \rangle \in Y\} \in [\omega_2]^\omega$ can not be covered by finitely many $H(\xi, 0)$. Let $\dot{\Gamma}$ be a P_f-name for Γ.

Claim: *If M is a σ-closed elementary submodel of $\mathcal{H}_\theta$ with $f, \dot{\Gamma} \in M$, $|M| = \omega_1$, $\delta = M \cap \omega_2 \in \omega_2$ then $(\{\delta\} \times \mathbb{C}) \cap \overline{Y} \neq \emptyset$.*

Assume, on the contrary, that $(\{\delta\} \times \mathbb{C}) \cap \overline{Y} = \emptyset$. Then, as $U(\delta) \cap \overline{Y}$ is compact, $U(\delta) \cap Y \subset U(\delta) \cap \overline{Y} \subset U[D]$ for some finite set $D \subset \delta$ consequently we have $\Gamma \cap H(\delta, 0) \subset H[D]$. this, however, contradicts lemma 1.11.25 by which

$$\Gamma \cap H[\delta, D] \neq \emptyset \text{ for each finite } D \subset \delta. \tag{1.148}$$

This contradiction proves our claim.

Since CH holds in V, the set S of ordinals $\delta \in \omega_2$ that arise in the form $\delta = M \cap \omega_2$ for an elementary submodel $M \prec \mathcal{H}_\theta$ as in the above claim is unbounded (even stationary) in ω_2. Let D be the set of the first ω elements of S. Then $D \in V \cap [\omega_2]^\omega$ and our claim implies that, in $V[\mathcal{G}]$, for each $\delta \in D$ there is $x_\delta \in \mathbb{C}$ with $\langle \delta, x_\delta \rangle \in \overline{Y}$. But then, by lemma 1.11.18, for $\alpha = \sup D$ we have

$$(\omega_2 \setminus \ \alpha) \times \mathbb{C} \subset \overline{\{\langle \delta, x_\delta \rangle : \delta \in D\}} \subset \overline{Y}. \tag{1.149}$$

This completes the proof of theorem 1.11.10. □

CHAPTER 2

Combinatorial principles

2.1. Combinatorial principles from adding Cohen reals

(This section is based on [12] and [1])

The last 40 years have seen a furious activity in proving results that are independent of the usual axioms of set theory, that is ZFC. As the methods of these independence proofs (e.g. forcing or the fine structure theory of the constructible universe) are often rather sophisticated, while the results themselves are usually of interest to "ordinary" mathematicians (e.g. topologists or analysts), it has been natural to try to isolate a relatively small number of principles, i.e. independent statements that a) are *simple* to formulate and b) are *useful* in the sense that they have many interesting consequences. Most of these statements, we think by necessity, are of combinatorial nature, hence they have been called combinatorial principles.

In this section we present several new combinatorial principles that are all statements about $\mathcal{P}(\omega)$, the power set of the natural numbers. In fact, they all concern matrices of the form $\langle A(\alpha, n) : \langle \alpha, n\rangle \in \kappa \times \omega\rangle$, where $A(\alpha, n) \subset \omega$ for each $\langle \alpha, n\rangle \in \kappa \times \omega$, and, in the interesting cases, κ is a regular cardinal with $c = 2^\omega \geq \kappa > \omega_1$.

We show that these statements are valid in the generic extensions obtained by adding any number of Cohen reals to any ground model V, assuming that the parameter κ is a regular and ω-inaccessible cardinal in V (i.e. $\lambda < \kappa$ implies $\lambda^\omega < \kappa$).

Then we present a large number of consequences of these principles, some of them combinatorial but most of them topological, mainly concerning separable and/or countably tight topological spaces. (This, of course, is not surprising because these are objects whose structure depends basically on $\mathcal{P}(\omega)$.)

2.1.1. The combinatorial principles. The principles we formulate here are all statements on $\kappa \times \omega$ matrices of subsets of ω claiming – roughly speaking – that all these matrices contain large "submatrices" satisfying certain homogeneity properties.

To simplify the formulation of our results we introduce the following pieces of notation. If S is an arbitrary set and k is a natural number then let

$$(S)^k = \{s \in S^k : |\operatorname{ran} s| = k\}$$

and

$$(S)^{<\omega} = \bigcup_{k<\omega} (S)^k.$$

For $D_0, \dots, D_{k-1} \subset S$ we let

$$(D_0, \dots, D_{k-1}) = \{s \in (S)^k : \ \forall i \in k \ (s(i) \in D_i)\}.$$

Definition 2.1.1. If S is a set of ordinals denote by $\mathcal{M}(S)$ the family of all $S \times \omega$-matrices of subsets of ω, that is, $\mathcal{A} \in \mathcal{M}(S)$ if and only if $\mathcal{A} = \langle A(\alpha, i) : \alpha \in S, i < \omega\rangle$, where $A(\alpha, i) \subset \omega$ for each $\alpha \in S$ and $i < \omega$. If $\mathcal{A} = \langle A(\alpha, i) : \alpha \in S, i < \omega\rangle \in \mathcal{M}(S)$ and $R \subset S$ we define the restriction of $\mathcal{A}$ to R, $\mathcal{A}\restriction R$ in the straightforward way: $\mathcal{A}\restriction R = \langle A(\alpha, i) : \alpha \in R, i < \omega\rangle$. If $\mathcal{A} = \langle A(\alpha, i) : \alpha \in S, i < \omega\rangle \in \mathcal{M}(S)$, $t \in \omega^{<\omega}$ and $s \in (S)^{|t|}$ then we let

$$A(s, t) = \bigcap_{i<|t|} A(s(i), t(i)).$$

Now we formulate our first and probably most important principle that we call $C^s(\kappa)$. We also specify a weaker version of $C^s(\kappa)$ denoted by $C(\kappa)$ because in most of the applications (2.1.23, 2.1.25, 2.1.30, 2.1.33, 2.1.37, 2.1.43) we don't need the full power of $C^s(\kappa)$.

Definition 2.1.2. For $T \subset \omega^{<\omega}$ a matrix $\mathcal{A} = \langle A(\alpha, i) : \alpha \in S, i \in \omega\rangle \in \mathcal{M}(S)$ is called *T-adic* if for each $t \in T$ and $s \in (S)^{|t|}$ we have $A(s,t) \neq \emptyset$.

Definition 2.1.3. For $\kappa = \mathrm{cf}(\kappa) > \omega$ principle $C^s(\kappa)$ ($C(\kappa)$) is the following statement:
For every $T \subset \omega^{<\omega}$ and $\mathcal{A} \in \mathcal{M}(\kappa)$ we have (1) or (2) below:

(1) there is a stationary (cofinal) set $S \subset \kappa$ such that $\mathcal{A} \restriction S$ is T-adic,
(2) there are $t \in T$ and stationary (cofinal) subsets D_0, D_1, $\ldots$, $D_{|t|-1}$ of κ such that for every $s \in (D_0, \ldots, D_{|t|-1})$ we have
$$A(s,t) = \emptyset.$$

Next we formulate a dual version of principles $C^s(\kappa)$ and $C(\kappa)$. Let us remark that we don't know whether $C^s(\kappa)$ ($C(\kappa)$) implies $\hat{C}^s(\kappa)$ ($\hat{C}(\kappa)$) or vice versa.

Definition 2.1.4. If $\kappa = \mathrm{cf}(\kappa) > \omega$, then principle $\hat{C}^s(\kappa)$ ($\hat{C}(\kappa)$) is the following statement:
For every $T \subset \omega^{<\omega}$ and $\mathcal{A} \in \mathcal{M}(\kappa)$ we have (1) or (2) below:

(1) there is a stationary (cofinal) set $S \subset \kappa$ such that for each $t \in T$ and $s \in (S)^{|t|}$
$$|A(s,t)| < \omega,$$
(2) there are $t \in T$ and stationary (cofinal) subsets D_0, D_1, $\ldots$, $D_{|t|-1}$ of κ such that for every $s \in (D_0, \ldots, D_{|t|-1})$ we have
$$|A(s,t)| = \omega.$$

Let us remark that in the "plain" dual of principle $C^s(\kappa)$ we should have $|A(s,t)| = \emptyset$ in 2.1.41 and $|A(s,t)| \neq \emptyset$ in 2.1.42, but this "principle" is easily provable in ZFC.

The principles $D(\kappa)$ and $D^s(\kappa)$ that we introduce next easily follow from $C(\kappa)$ and $C^s(\kappa)$, respectively, but as their formulation is much simpler, we thought it to be worth while to have them as separate principles. We first give two auxiliary definitions.

Definition 2.1.5. If $\mathcal{A} = \langle A(\alpha, i) : \alpha < \kappa, i < \omega\rangle \in \mathcal{M}(\kappa)$, then we set
$$\hat{\mathcal{A}} = \{Y \subset \omega : |\{\alpha < \kappa : \exists i < \omega\ A(\alpha, i) \subset Y\}| = \kappa\}$$
and
$$\hat{\mathcal{A}}^s = \{Y \subset \omega : \{\alpha < \kappa : \exists i < \omega\ A(\alpha, i) \subset Y\} \text{ is stationary in } \kappa\}.$$

Now we can formulate $D^s(\kappa)$ ($D(\kappa)$) as follows.

Definition 2.1.6. For $\kappa = \mathrm{cf}(\kappa) > \omega$ principle $D^s(\kappa)$ ($D(\kappa)$) is the following statement:
If $\mathcal{A} \in \mathcal{M}(\kappa)$ and $\hat{\mathcal{A}}^s$ ($\hat{\mathcal{A}}$) is centered then there is a stationary (cofinal) set $S \subset \kappa$ such that $\mathcal{A} \restriction S$ is $\omega^{<\omega}$-adic.

Theorem 2.1.7. *$C^s(\kappa)$ ($C(\kappa)$) implies $D^s(\kappa)$ ($D(\kappa)$).*

Proof. We give the proof only for $D^s(\kappa)$ because the same argument works for $D(\kappa)$.

Let $\mathcal{A} \in \mathcal{M}(\kappa)$ so that $\hat{\mathcal{A}}^s$ is centered and put $T = \omega^{<\omega}$. By $C^s(\kappa)$ either 2.1.3(1) or 2.1.3(2) holds.

If $S \subset \kappa$ witnesses 2.1.3(1) for our T then $\mathcal{A} \restriction S$ is clearly $\omega^{<\omega}$-adic. So it is enough to show that 2.1.3(2) can not hold.

Assume, on the contrary, that there are $t \in T = \omega^{<\omega}$ and stationary subsets D_0, D_1, $\ldots$, $D_{|t|-1}$ of κ such that for each $s \in (D_0, \ldots, D_{|t|-1})$ we have
$$A(s,t) = \emptyset. \tag{+}$$
We can obviously assume that the sets D_i are pairwise disjoint. Let $X_i = \bigcup\{A(\delta, t(i)) : \delta \in D_i\}$ for every $i < |t|$. Then clearly $X_i \in \hat{\mathcal{A}}^s$ for every $i < |t|$, while (+) implies $\bigcap_{i<k} X_i = \emptyset$, contradicting that $\hat{\mathcal{A}}^s$ is centered. $\square$

Definition 2.1.8. If $\kappa = \mathrm{cf}(\kappa) > \omega$, then principle $F^s(\kappa)$ ($F(\kappa)$) is the following statement: For every $T \subset \omega^{<\omega}$ and $\mathcal{A} \in \mathcal{M}(\kappa)$ (1) or (2) below holds:

(1) there is a stationary (cofinal) set $S \subset \kappa$ such that

$$|\{A(s,t) : t \in T \text{ and } s \in (S)^{|t|}\}| \leq \omega.$$

(2) there are $t \in T$ and stationary (cofinal) subsets D_0, D_1, ..., $D_{|t|-1}$ of κ such that if $s_0, s_1 \in (D_0, \ldots, D_{|t|-1})$ with $s_0(i) \neq s_1(i)$ for each $i < |t|$ then we have

$$A(s_0, t) \neq A(s_1, t).$$

Clearly, if t, D_0, ...,$D_{|t|-1}$ satisfy (2) then

$$|\{A(s,t) : s \in (D_0, \ldots, D_{|t|-1})\}| = \kappa.$$

There is a surprising connection between these principles and the dual versions $\hat{C}^s(\kappa)$ ($\hat{C}(\kappa)$) of $C^s(\kappa)$ ($C(\kappa)$), respectively.

Theorem 2.1.9. *$F^s(\kappa)$ ($F(\kappa)$) implies $\hat{C}^s(\kappa)$ ($\hat{C}(\kappa)$).*

Proof. Let $\mathcal{A} \in \mathcal{M}(\kappa)$ and $T \subset \omega^{<\omega}$ and apply $F^s(\kappa)$ to $\mathcal{A}$ and T. Assume first that there is a stationary set $S \subset \kappa$ such that the family

$$\mathcal{I} = \{A(s,t) : t \in T \text{ and } s \in (S)^{|t|}\}$$

is countable.

Now for $t \in T$, $i < |t|$ and $I \in \mathcal{I} \cap [\omega]^\omega$ set

$$D(I,t,i) = \{\alpha \in S : A(\alpha, t(i)) \supset I\}.$$

If for some $t \in T$ and $I \in \mathcal{I} \cap [\omega]^\omega$ the set $D(I,t,i)$ is stationary for each $i < |t|$ then this t and the sets $D(I,t,0)$, ..., $D(I,t,|t|-1)$ witness 2.1.42.

So we can assume that for all $t \in T$ and $I \in \mathcal{I} \cap [\omega]^\omega$ the set

$$b(I,t) = \{i < |t| : D(I,t,i) \text{ is non-stationary in } \kappa\}$$

is not empty. Then the set

$$D = \bigcup\{D(I,t,i) : I \in \mathcal{I} \cap [\omega]^\omega, t \in T, i \in b(I,t)\}$$

is not stationary and so $S' = S \setminus D$ is stationary. We claim that S' witnesses 2.1.41. Assume on the contrary that $t \in T$, $s \in (S')^{|t|}$ and $I = A(s,t)$ is infinite. Then $I \in \mathcal{I} \cap [\omega]^\omega$ and $s(i) \in D(I,t,i)$ for each $i < |t|$. Since $s(i) \notin D$ it follows that $D(I,t,i)$ is stationary for each $i < |t|$, that is, $b(I,t) = \emptyset$, which is a contradiction.

Assume now that there are $t \in T$ and stationary subsets D_0, D_1, ..., $D_{|t|-1}$ of κ such that if $s_0, s_1 \in (D_0, \ldots, D_{|t|-1})$ with $s_0(i) \neq s_1(i)$ for each $i < |t|$ then

$$A(s_0, t) \neq A(s_1, t).$$

We show that in this case again 2.1.42 holds. Indeed, for each $I \in [\omega]^{<\omega}$ pick $s_I \in (D_0, \ldots, D_{|t|-1})$ such that $A(s_I, t) = I$ provided that there is such an s. Let $R = \bigcup\{s_I(i) : I \in [\omega]^{<\omega}, i < |t|\}$ and $D'_i = D_i \setminus R$ for $i < |t|$. Now if $s \in (D'_0, \ldots, D'_{|t|-1})$ then for any $I \in [\omega]^{<\omega}$ we have $s_I(i) \neq s(i)$ for each $i < |t|$, hence $I = A(s_I, t) \neq A(s,t)$. As I was an arbitrary element of $[\omega]^{<\omega}$ we conclude that $|A(s,t)| = \omega$. □

The following observation is almost trivial.

Proposition 2.1.10. *If $\kappa = \mathrm{cf}(\kappa) > c$ then $C^s(\kappa)$ and $F^s(\kappa)$ are valid.* □

As was mentioned, our principles are of interest only for $\kappa > \omega_1$. In fact, for $\kappa = \omega_1$, they are all false!

Theorem 2.1.11 ([12]). *$\hat{C}(\omega_1)$ (and so $F(\omega_1)$ too) and $D(\omega_1)$ are both false.* □

2.1.2. Consistency of the principles in the Cohen model. A cardinal κ is *ω-inaccessible* if $\lambda^\omega < \kappa$ holds for each $\lambda < \kappa$. Given any infinite set I we denote by $\mathcal{C}_I$ the poset $\mathrm{Fn}(I, 2, \omega)$, i.e. the standard one adding $|I|$-many Cohen reals.

In this subsection we prove that if κ is a regular ω-inaccessible cardinal in some ground model V and we add λ-many Cohen reals to V, where λ is an arbitrary cardinal, then in the extension the principles $C^s(\kappa)$, $\hat{C}^s(\kappa)$ and $F^s(\kappa)$ are all satisfied. As we remarked in subsection 2.1.1 above the case $\kappa > \lambda$ is trivial, while the case $\kappa < \lambda$ can be reduced to the case $\kappa = \lambda$.

Since the proof of the latter is long and technical, we first sketch the main idea. So let us be given a matrix $\mathcal{A} \in \mathcal{M}(\kappa)$ and a set $T \subset \omega^{<\omega}$ in $V[G]$, where G is $\mathcal{C}_\kappa$-generic over V. In the first part of the proof we find a set $I \in \left[\kappa\right]^\omega$ and a stationary set $S \subset \kappa$ such that in $V[G \restriction I]$ the sequences $\langle A(\alpha, i) : i < \omega\rangle$ for $\alpha \in S$ have also pairwise isomorphic names with disjoint supports (contained in $\kappa \setminus I$). This reduction, carried out in lemma 2.1.17, will be the place where we use that κ is regular and ω-inaccessible in V. In the second part of the proof, using slightly different arguments for $C^s(\kappa)$ and for $F^s(\kappa)$, we show that if some $\mathcal{A} \in \mathcal{M}(S)$ has names with these properties then either S witnesses 2.1.3(1) (or 2.1.8(1), respectively) or some stationary sets $D_i \subset S$ witness 2.1.3(2) (or 2.1.8(2), respectively). In this second step we don't use that κ is ω-inaccessible or regular.

In our forcing arguments we follow the notation of Kunen [**77**]. Let us first recall definition [**77**, 5.11].

Definition 2.1.12. A $\mathcal{C}_I$-name $\dot{B}$ of a subset of some ordinal μ is called *nice* if for each $\nu < \mu$ there is an antichain $B_\nu \subset \mathcal{C}_I$ such that

$$\dot{B} = \{\langle p, \hat{\nu}\rangle : \nu \in \mu \wedge p \in B_\nu\} = \bigcup\{B_\nu \times \{\hat{\nu}\} : \nu \in \mu\}.$$

We let $\mathrm{supp}(\dot{B}) = \bigcup\{\mathrm{dom}(p) : p \in \bigcup\limits_{\nu<\mu} B_\nu\}$.

It is well-known (see e.g. lemma [**77**, 5.12]) that every set of ordinals in $V[G]$ has a nice name in V.

If φ is a bijection between two sets I and J then φ lifts to a natural isomorphism between $\mathcal{C}_I$ and $\mathcal{C}_J$, which will be also denoted by φ, as follows: for $p \in \mathcal{C}_I$ let $\mathrm{dom}(\varphi(p)) = \varphi''\,\mathrm{dom}(p)$ and $\varphi(p)(\varphi(\xi)) = p(\xi)$. Moreover φ also generates a bijection between the nice $\mathcal{C}_I$-names and the nice $\mathcal{C}_J$-names (see [**77**, 7.12]): if $\dot{B}$ is a nice $\mathcal{C}_I$-name then let $\varphi(\dot{B}) = \{\left\langle \varphi(p), \hat{\xi}\right\rangle : \left\langle p, \hat{\xi}\right\rangle \in \dot{B}\}$. If I and J are sets of ordinals with the same order type then $\varphi_{I,J}$ is the natural order-preserving bijection from I onto J.

Definition 2.1.13. Assume $I, J \subset \kappa$, moreover $\dot{A}_i$ and $\dot{B}_i$ are nice $\mathcal{C}_\kappa$-names of subsets of ω for $i < \omega$, such that $\mathrm{supp}(\dot{A}_i) \subset I$ and $\mathrm{supp}(\dot{B}_i) \subset J$. We say that the structures of names $\left\langle I, \dot{A}_i : i < \omega\right\rangle$ and $\left\langle J, \dot{B}_i : i < \omega\right\rangle$ are *twins* if I and J have the same order type and for the order preserving bijection $\varphi_{I,J}$ we have

(1) $\varphi_{I,J}$ is the identity on $I \cap J$,
(2) $\varphi_{I,J}(\dot{A}_i) = \dot{B}_i$ for each $i < \omega$.

Definition 2.1.14. Assume that $I \subset \kappa$, G is a $\mathcal{C}_\kappa$-generic filter over V and $H = G \restriction I$. If $\dot{B}$ is a nice $\mathcal{C}_\kappa$-name of a subset of some ordinal μ we define in $V[H]$ the $\mathcal{C}_{\kappa\setminus I}$ name $\pi^H(\dot{B})$ as follows:

$$\pi^H(\dot{B}) = \{\langle p \restriction \kappa \setminus I, \hat{\nu}\rangle : \langle p, \hat{\nu}\rangle \in \dot{B} \wedge p \restriction I \in H\}.$$

Lemma 2.1.15. *$\pi^H(\dot{B})$ is a nice $\mathcal{C}_{\kappa\setminus I}$-name in $V[H]$ and* $\mathrm{supp}(\pi^H(\dot{B})) \subset \mathrm{supp}(\dot{B}) \setminus I$, *moreover*

$$\mathrm{val}(\pi^H(\dot{B}), G \restriction (\kappa \setminus I)) = \mathrm{val}(\dot{B}, G).$$

Proof. Straightforward from the construction. □

Definition 2.1.16. Assume that $S \subset \kappa$. A matrix $\dot{\mathcal{B}} = \left\langle \dot{B}(\alpha, i) : \alpha \in S, i < \omega\right\rangle$ of nice $\mathcal{C}_\kappa$-names of subsets of ω is called a *nice S-matrix* if conditions (i) and (ii) below hold:

(i) putting $J_\alpha = \bigcup_{i<\omega} \mathrm{supp}(\dot{B}(\alpha, i))$ the sets $\{J_\alpha : \alpha \in S\}$ are pairwise disjoint,

(ii) the structures of names $\{\left\langle J_\alpha, \dot{B}(\alpha,i) : i<\omega\right\rangle : \alpha \in S\}$ are pairwise twins.

We denote by $\mathcal{N}(S)$ the family of nice S-matrices.

Lemma 2.1.17. (Reduction lemma) *Assume that κ is a regular, ω-inaccessible cardinal, G is $\mathcal{C}_\kappa$-generic over V and $\mathcal{A} \in \mathcal{M}(\kappa)$ in $V[G]$. Then there are a countable set $I \subset \kappa$ and a stationary set $S \subset \kappa$ in V such that, in $V[G \restriction I]$, there is $\mathcal{B} \in \mathcal{N}(S)$ satisfying $V[G] \models$ "$A(\alpha,i) = \mathrm{val}(\dot{B}(\alpha,i), G \restriction (\kappa \setminus I))$" for each $\alpha \in S$ and $i \in \omega$.*

Proof. Assume that

$$1_{\mathcal{C}_\kappa} \Vdash \text{“}\dot{\mathcal{A}} = \left\langle \dot{A}(\alpha,i) : \alpha<\kappa, i<\omega\right\rangle \in \mathcal{M}(\kappa).\text{”}$$

We can assume that all the names $\dot{A}(i,\alpha)$ are nice. Let $I_\alpha = \bigcup_{i<\omega} \mathrm{supp}(\dot{A}(\alpha,i))$.

We need a strong version of Erdős-Rado Δ-system theorem saying that there is a stationary set $T \subset \kappa$ such that $\{I_\alpha : \alpha \in T\}$ forms a Δ-system with some kernel I, moreover $\sup I < \min I_\alpha \setminus I$ for each $\alpha \in T$. Although this statement is well-known, in **[12]** we presented a proof because we could not find any reference to it.

Erdős-Rado Theorem . *If κ is an ω-inaccessible regular cardinal and $\mathcal{X} = \{X_\alpha : \alpha<\kappa\}$ is a family of countable sets then there is a stationary set $I \subset \kappa$ such that $\{X_\alpha : \alpha \in I\}$ forms a Δ-system.* □

Since $2^\omega < \kappa = \mathrm{cf}(\kappa)$ and there are only 2^ω different isomorphism types of structures of names there is a stationary set $S \subset T$ such that the structures of names $\{\left\langle I_\alpha, \dot{A}(\alpha,i) : i<\omega\right\rangle : \alpha \in S\}$ are pairwise twins.

From now on we work in $V[G \restriction I]$. Let $\dot{B}(\alpha,i) = \pi^{G\restriction I}(\dot{A}(\alpha,i))$ for $\alpha \in S$ and $i \in \omega$. Then $\mathrm{supp}(\dot{B}(\alpha,i)) \subset J_\alpha = I_\alpha \setminus I$ and the structures of names $\left\langle J_\alpha, \dot{B}(\alpha,i) : i<\omega\right\rangle$ are pairwise twins by lemma 2.1.15 above. Thus $\mathcal{B} = \left\langle \dot{B}(\alpha,i) : \alpha \in S, i<\omega\right\rangle \in \mathcal{N}(S)$. □

Definition 2.1.18. Assume that $S \subset \kappa$. A sequence $\dot{\mathcal{B}} = \left\langle \left\langle J_\alpha, \dot{B}_\alpha\right\rangle : \alpha \in S\right\rangle$ is called a *nice S-sequence* if conditions (i) and (ii) below hold:

(i) $J_\alpha \in \left[\kappa\right]^\omega$, $\dot{B}_\alpha$ is a nice $\mathcal{C}_{J_\alpha}$-name, and J_α for $\alpha \in S$ are pairwise disjoint,

(ii) the structures of names $\left\langle J_\alpha, \dot{B}_\alpha\right\rangle$ for $\alpha \in S$ are pairwise twins.

We denote by $\mathcal{S}(S)$ the family of nice S-sequences.

Lemma 2.1.19. (Homogeneity lemma) *Assume that $S \subset \kappa$ and $\dot{\mathcal{B}} = \left\langle \left\langle J_\alpha, \dot{B}_\alpha\right\rangle : \alpha \in S\right\rangle$ is a nice S-sequence. If $\varphi(x_0, x_1, \dots, x_{n-1}, z)$ is a formula with free variables $x_0, x_1, \dots, x_{n-1}, z$ and Z is an element of the ground model, then* (1) *or* (2) *below holds:*

(1) $1_{\mathcal{C}_\kappa} \Vdash$ " $\varphi(\dot{B}_{s(0)}\dot{B}_{s(1)}, \dots, \dot{B}_{s(n-1)}, \hat{Z})$ for all $s \in (S)^n$ ";

(2) for some $r \in \mathcal{C}_\kappa$ we have

$r \Vdash$ " there are subsets $\dot{D}_0$, $\dot{D}_1$, …, $\dot{D}_{n-1}$ of S such that

(a) for each $i<n$ and $A \in \left[S\right]^\omega \cap V$ we have $\dot{D}_i \cap A \neq \emptyset$,

(b) $\neg\varphi(\dot{B}_{s(0)}\dot{B}_{s(1)}, \dots, \dot{B}_{s(n-1)}, \hat{Z})$ for all $s \in (\dot{D}_0, \dot{D}_1, \dots, \dot{D}_{k-1})$".

Proof. Assume that (1) fails, that is, there are $p \in \mathcal{C}_\kappa$ and $s \in (S)^k$ such that

$$p \Vdash \text{“}\neg\varphi(\dot{B}_{s(0)}, \dots, \dot{B}_{s(n-1)}, \hat{Z})\text{”}.$$

Let $J = \bigcup_{i<k} J_{s(i)}$ and $p' = p \restriction J$ and $r = p \setminus p'$. Since the sets J_α are pairwise disjoint we can assume that $\mathrm{dom}(r) \cap J_\alpha = \emptyset$ for each $\alpha \in S$.

For $\langle \alpha, \beta\rangle \in S^2$ we denote by $\varphi_{\alpha,\beta}$ the natural order preserving bijection between J_α and J_β. For $\beta \in S$ and $i<k$ let $p(\beta,i) = \varphi_{s(i),\beta}(p \restriction J_{s(i)})$. For $i<k$ define the $\mathcal{C}_\kappa$-name $\dot{D}_i$ of a subset of S as follows: $\dot{D}_i = \{\left\langle p(\beta,i), \hat{\beta}\right\rangle : \beta \in S\}$. Then

$$V[G] \models \text{“}D_i = \mathrm{val}(\dot{D}_i, G) = \{\beta \in S : p(\beta,i) \in G\}\text{”},$$

where G is $\mathcal{C}_\kappa$-generic over V. Since the supports of $p(\beta, i)$ for $\beta \in S$ are pairwise disjoint a standard density argument gives that $D_i \cap A \neq \emptyset$ whenever $A \in \left[S\right]^\omega \cap V$, hence (a) holds.

To show (b) assume that $r \in G$ and

$$V[G] \models \text{“}u \in (D_0, \dots, D_{k-1})\text{.”}$$

Since u is finite we have $u \in V$. Let $J^* = \bigcup_{i<k} J_{u(i)}$ and $\psi = \bigcup_{i<k} \varphi_{u(i),s(i)}$. Then ψ is a bijection between J^* and J and so it extends to isomorphisms between $\mathcal{C}_{J^*}$ and $\mathcal{C}_J$, and between the families of nice $\mathcal{C}_{J^*}$-names and of nice $\mathcal{C}_J$-names. Let Ψ be the natural extension of ψ to a permutation of κ:

$$\Psi(\nu) = \begin{cases} \psi(\nu) & \text{if } \nu \in J^*, \\ \psi^{-1}(\nu) & \text{if } \nu \in J, \\ \nu & \text{if } \nu \in \kappa \setminus (J \cup J^*). \end{cases}$$

Then Ψ extends to an automorphism of $\mathcal{C}_\kappa$, and also to an automorphism of nice $\mathcal{C}_\kappa$-names. Clearly if $q \in \mathcal{C}_{J^*}$ and $\dot{B}$ is a nice $\mathcal{C}_{J^*}$-name then $\psi(q) = \Psi(q)$ and $\psi(\dot{B}) = \Psi(\dot{B})$. Observe that $\Psi(r) = r$ and $\Psi(\hat{Z}) = \hat{Z}$.

Let $G^* = \Psi''G$. Then G^* is also a $\mathcal{C}_\kappa$-generic filter over V and since $\Psi(\dot{B}_{u(i)}) = \dot{B}_{s(i)}$ it follows that

$$\mathrm{val}(\dot{B}_{u(i)}, G) = \mathrm{val}(\dot{B}_{s(i)}, G^*). \tag{$\bullet$}$$

But $p(u(i), i) \in G$, so $p \restriction J_{s(i)} = \psi(p(u(i), i)) \in G^*$. Thus $p = r \cup \bigcup_{i<k} p \restriction J_{s(i)} \in G^*$ as well. Since $p \Vdash \neg\varphi(\dot{B}_{s(0)}, \dots, \dot{B}_{s(n-1)}, \hat{Z})$ and so $V[G^*] \models$ “$\neg\varphi(B_{s(0)}, \dots, B_{s(n-1)}, Z)$”, by ($\bullet$) this implies

$$V[G] \models \text{“}\neg\varphi(B_{u(0)}, \dots, B_{u(n-1)}, Z)\text{”}$$

which was to be proved. □

Theorem 2.1.20. *If κ is a regular, ω-inaccessible cardinal then for each cardinal λ we have*

$$V^{\mathcal{C}_\lambda} \models \text{“}C^s(\kappa) \textit{ and } \hat{C}^s(\kappa) \textit{ hold.”}$$

Proof. We deal only with $C^s(\kappa)$ because the same argument works for $\hat{C}^s(\kappa)$. As we observed in section 2.1.1 we can assume that $\kappa \leq \lambda$. First we investigate the case $\lambda = \kappa$.

Assume that

$$1_{\mathcal{C}_\kappa} \Vdash \text{“}\dot{\mathcal{A}} = \left\langle \dot{A}(\alpha, i) : \alpha < \kappa; i < \omega \right\rangle \in \mathcal{M}(\kappa) \text{ and } T \subset \omega^{<\omega}\text{.”}$$

Applying the reduction lemma 2.1.17 and that T is countable we can find a countable set $I \subset \kappa$ and a stationary set $S \subset \kappa$ in V and a nice S-matrix $\mathcal{B}$ in $V[G \restriction I]$ such that

$$V[G] \models \text{“}\,\mathrm{val}(\dot{A}(\alpha, i), G) = \mathrm{val}(\dot{B}(\alpha, i), G \restriction (\kappa \setminus I))\text{”}$$

for $\alpha \in S$ and $i \in \omega$, moreover $T \in V[G \restriction I]$.

We show that for each $q \in \mathcal{C}_\kappa$ there is a condition $r \leq q$ in $\mathcal{C}_\kappa$ such that $r \Vdash$ “2.1.31 or 2.1.32 holds”. Let $I' = I \cup \mathrm{dom}(q)$.

For each $t \in T$ let $\varphi_t(x_0, \dots, x_{|t|-1})$ be the following formula:

$$\varphi(\langle B_{0,k} : k < \omega\rangle, \dots, \langle B_{|t|-1,k} : k < \omega\rangle) \iff \bigcap_{i<|t|} B_{i,t(i)} \neq \emptyset.$$

Applying the homogeneity lemma 2.1.19 to $V[G \restriction I']$ as our ground model and to every φ_t we get that either $q \Vdash$ “2.1.31 holds” or $q \cup p \Vdash$ “2.1.32 holds”. Let us remark that 2.1.19(2)(a) implies that as S is stationary, so is each D_i.

Thus we have proved the theorem in the case $\kappa = \lambda$. If $\lambda > \kappa$ and $\mathcal{A} \in (\mathcal{M}(\kappa))^{V[G]}$, where G is $\mathcal{C}_\lambda$-generic over V, then there is $J \in \left[\lambda\right]^\kappa$ such that $\mathcal{A} \in V[G \restriction J]$. The stationary sets that witness 2.1.3(1) or 2.1.3(2) in $V[G \restriction J]$ remain stationary in $V[G]$, and so we are done. □

Theorem 2.1.21. *If κ is a regular, ω-inaccessible cardinal then for each cardinal λ we have*

$$V^{\mathcal{C}_\lambda} \models \text{“}F^s(\kappa) \textit{ holds.”}$$

Proof. As in 2.1.20 the important case is when $\lambda = \kappa$, because the case $\lambda < \kappa$ is trivial and the case $\kappa < \lambda$ can be reduced to the case $\kappa = \lambda$.

So assume that

$$1_{\mathcal{C}_\kappa} \Vdash \text{“}\dot{\mathcal{A}} = \left\langle \dot{A}(\alpha, i) : \alpha < \kappa, i < \omega \right\rangle \in \mathcal{M}(\kappa).\text{”}$$

Applying lemma 2.1.17 we can find a countable set $I \subset \kappa$, a stationary set $S \subset \kappa$ in V and in $V[G \restriction I]$ a nice S-matrix $\mathcal{B}$ such that $V[G] \models$ “$A(\alpha, i) = \mathrm{val}(\dot{B}(\alpha, i), G \restriction (\kappa \setminus I))$” for each $\alpha \in S$ and $i \in \omega$, moreover $T \in V[G \restriction I]$.

We need the following lemma that is probably well-known.

Lemma 2.1.22 ([**12**]). *If H is a $\mathcal{C}_\kappa$-generic filter over V and I, J are disjoint subsets of κ then*

$$V[H] \models \text{“}\mathcal{P}(\omega) \cap V[H \restriction I] \cap V[H \restriction J] = \mathcal{P}(\omega) \cap V.\text{”} \quad \square$$

To conclude the proof we show that if $q \in \mathcal{C}_\kappa$ then there is a condition $r \leq q$ in $\mathcal{C}_\kappa$ such that $r \Vdash$ “2.1.81 or 2.1.82 holds”. Let $I' = I \cup \mathrm{dom}(q)$.

For each $t \in T$ let $\varphi_t(x_0, \ldots, x_{|t|-1})$ be the following formula:

$$\varphi(\langle B_{0,k} : k < \omega \rangle, \ldots, \langle B_{|t|-1,k} : k < \omega \rangle) \Longleftrightarrow \bigcap_{i<|t|} B_{i,t(i)} \in (\mathcal{P}(\omega))^V.$$

Applying the homogeneity lemma 2.1.19 to $V[G \restriction I']$ as our ground model we get that (A) or (B) below holds:

(A) $1_{\mathcal{C}_\kappa} \Vdash$ “ $B(s,t) \in (\mathcal{P}(\omega))^V$ for each $t \in T$ and $s \in (S)^{|t|}$,”

(B) for some $t \in T$ and $p \in \mathcal{C}_\kappa$ we have

$p \Vdash$ “ there are subsets $\dot{D}_0$, $\dot{D}_1$, …, $\dot{D}_{|t|-1}$ of S such that

(a) for each $A \in [S]^\omega \cap V$ we have $A \cap \dot{D}_i \neq \emptyset$ for each $i < |t|$,

(b) If $s \in (\dot{D}_0, \dot{D}_1, \ldots, \dot{D}_{|t|-1})$ we have

$$B(\dot{s}, t) \notin (\mathcal{P}(\omega))^V.\text{”}$$

Let $J_\alpha = \bigcup_{i<\omega} \mathrm{supp}(\dot{B}(\alpha, i))$ for $\alpha \in S$ and denote by $\varphi_{\alpha,\beta}$ the natural order preserving bijection between J_α and J_β for $\langle \alpha, \beta \rangle \in S^2$.

Assume first that (A) holds. Fix $t \in T$ and $s \in (S)^{|t|}$. Write $\alpha_i = s(i)$ for $i < |t|$. Since $\mathcal{C}_\kappa$ is c.c.c , there is in V a countable set $\mathcal{I}_t \subset \mathcal{P}(\omega)$ such that $1_{\mathcal{C}_\kappa} \Vdash$ “$\bigcap_{i<k} \dot{B}(\alpha_i, n_i) \in \mathcal{I}_t$”.

Assume that $\langle \delta_0, \ldots, \delta_{|t|-1} \rangle \in (S)^k$.

Let $J^* = \bigcup_{i<k} J_{\delta_i}$, $J = \bigcup_{i<k} J_{\alpha_i}$, and $\psi = \bigcup_{i<k} \varphi_{\delta_i,\alpha_i}$. Then ψ is a bijection between J^* and J and so it lifts up to an isomorphism between $\mathcal{C}_{J^*}$ and $\mathcal{C}_J$ and between the families of nice $\mathcal{C}_{J^*}$-names and nice $\mathcal{C}_J$-names.

Let G be $\mathcal{C}_\kappa$-generic and put $G_0 = G \restriction J^*$. Since $\mathrm{supp}(\dot{B}(\delta_i, n_i)) \subset J^*$ it follows that $\mathrm{val}(\dot{B}(\delta_i, n_i), G) = \mathrm{val}(\dot{B}(\delta_i, n_i), G_0)$. Let $G_1 = \psi'' G_0$. Then G_1 is also a $\mathcal{C}_J$-generic filter and since $\psi(\dot{B}(\delta_i, n_i)) = \dot{B}(\alpha_i, n_i)$ it follows that

$$\mathrm{val}(\dot{B}(\delta_i, n_i), G_0) = \mathrm{val}(\dot{B}(\alpha_i, n_i), G_1). \tag{$\bullet$}$$

Since $1_{\mathcal{C}_\kappa} \Vdash$ “ $\bigcap_{i<k} \dot{B}(\alpha_i, n_i) \in \mathcal{I}_t$”, by ($\bullet$) we have $1_{\mathcal{C}_\kappa} \Vdash$ “ $\bigcap_{i<k} \mathrm{val}(\dot{B}(\delta_i, n_i), G) \in \mathcal{I}_t$” as well. From this it is obvious that we have

$$1_{\mathcal{C}_\kappa} \Vdash \{B(\dot{t}, s) : t \in T \wedge s \in (S)^{|t|}\} \subset \mathcal{I} = \bigcup \{\mathcal{I}_t : t \in T\}.$$

where $\mathcal{I}$ is countable as T is.

Assume now that (A) fails and so (B) holds.

Let G be $\mathcal{C}_\kappa$-generic with $p \in G$ and $\langle \gamma_0, \ldots, \gamma_{k-1} \rangle$, $\langle \delta_0, \ldots, \delta_{k-1} \rangle \in (D_0, \ldots, D_{k-1})$ such that

$$V[G] \models \text{“}\{\gamma_i, \delta_i\} \in [D_i]^2 \text{ for } i < k \text{ are pairs of distict ordinals”}.$$

Let $J^* = \bigcup_{i<k} J_{\gamma_i}$ and $J^\star = \bigcup_{i<k} J_{\delta_i}$. Then $J^* \cap J^\star = \emptyset$, hence by lemma 2.1.22 we have $\mathcal{P}(\omega) \cap V[G \restriction J^*] \cap \restriction V[G \restriction J^\star] = \mathcal{P}(\omega) \cap V$ and so $V[G] \models$ “$\bigcap_{i<k} B(\delta_i, n_i) \notin V$” implies that $V[G] \models$ “$\bigcap_{i<k} B(\delta_i, n_i) \neq \bigcap_{i<k} B(\gamma_i, n_i)$”

The theorem is proved. □

2.1.3. Applications. We start with presenting some combinatorial applications because they are quite simple and so they nicely illustrate the use of our principles.

Kunen **[76]** proved that if one adds Cohen reals to a model of CH, then in the generic extension there is no strictly $\subset^*$-increasing chain of subsets of ω of length ω_2. The first theorem we prove easily yields Kunen's above result.

Theorem 2.1.23. *If $C(\kappa)$ holds then for each $\mathcal{A} \subset [\omega]^\omega$ of size κ either*

(a) $\exists \mathcal{B} \in [\mathcal{A}]^\kappa\ \forall B \neq B' \in \mathcal{B}\ |B \setminus B'| = \omega$

or

(b) $\exists X \in [\omega]^\omega\ |\{A \in \mathcal{A} : A \subset X\}| = |\{A \in \mathcal{A} : X \subset^* A\}| = \kappa$.

Proof. Fix a 1–1 enumeration $\{A_\xi : \xi < \kappa\}$ of $\mathcal{A}$. Let $A(\xi, 2n) = A_\xi \setminus n$ and $A(\xi, 2n+1) = (\omega \setminus A_\xi) \setminus n$. Put $T = \{\langle 2i, 2i+1\rangle : i \in \omega\}$. If $S \in [\kappa]^\kappa$ witnesses 2.1.31, then $\mathcal{B} = \{A_\xi : \xi \in S\}$ satisfies (a). If on the other hand $D, E \in [\kappa]^\kappa$, $D \cap E = \emptyset$, with $\langle 2i, 2i+1\rangle \in T$ show that 2.1.32 holds, then let $X = \cup\{A_\xi : \xi \in D\}$. Then $A_\xi \subset X$ for each $\xi \in D$ and $X \setminus i \subset A_\zeta$ for each $\zeta \in E$. □

Theorem 2.1.24. *If $D(\kappa)$ or $\hat{C}(\kappa)$ holds, then κ is not embeddable into $P(\omega)/fin$.*

Remark . In the original version of the paper the statement of theorem 2.1.24 above was derived only from principle $C(\kappa)$. This strengthening of our result was pointed out by the referee.

Proof. Assume that $\{A_\alpha : \alpha < \kappa\}$ is a strictly $\subset^*$-increasing chain in $[\omega]^\omega$. For $\alpha < \kappa$ and $n < \omega$ let

$$A(\alpha, n) = \begin{cases} \omega \setminus A_\alpha & \text{if } n = 0, \\ A_\alpha \setminus n & \text{otherwise.} \end{cases}$$

We show that $\mathcal{A} = \langle A(\alpha, n) : \alpha < \kappa, n < \omega\rangle$ is a counterexample to $D(\kappa)$ and $\hat{C}(\kappa)$.

To see that $\hat{\mathcal{A}}$ is centered, observe that if $Y \in \hat{\mathcal{A}}$, then there is $\alpha_Y < \kappa$ such that $A_\beta \subset A_{\alpha_Y} \subset^* Y$ for each $\alpha_Y < \beta < \kappa$. Thus if $Y_0, \ldots, Y_{n-1} \in \hat{\mathcal{A}}$ then taking $\alpha = \max\{\alpha_{Y_i} : i < n\}$ we have $A_{\alpha+1} \subset A_\alpha \subset^* \bigcap_{i<n} Y_i$. On the other hand, if $\alpha < \beta < \kappa$ then $A_\alpha \subset^* A_\beta$, thus $A_\alpha \cap (\omega \setminus A_\beta)$ is finite and so $A(\alpha, k) \cap A(\beta, 0) = \emptyset$ for some large enough k. Thus there is no S of size κ (even of size 2) such that $\mathcal{A} \restriction S$ is $\omega^{<\omega}$-adic. Thus $D(\kappa)$ fails.

Next, let $t = \langle 0, 1\rangle$ and $T = \{t\}$. Then for any $S \subseteq \kappa$ of size κ, if $s \in S^2$ is such that $s(0) < s(1)$, then $A(s, t)$ is infinite. Hence 2.1.4.1 does not hold. Likewise if D_0, $D_1 \subseteq \kappa$ are of size κ, taking $s \in (D_0, D_1)^2$ such that $s(0) > s(1)$, $A(s, t)$ is finite. This shows that 2.1.4.2 does not hold, i.e. $\hat{C}(\kappa)$ fails. □

The next theorem can be considered as a kind of dual to 2.1.23.

Theorem 2.1.25 (**[12]**). *If $C(\kappa)$ holds then for each $\mathcal{A} \subset [\omega]^\omega$ of size κ and for each natural number k either*

(a) there is a family $\mathcal{B} \in [\mathcal{A}]^\kappa$ such that for each $\mathcal{B}' \in [\mathcal{B}]^k$ we have $|\bigcap \mathcal{B}'| = \omega$

or

(b) there are k subfamilies $\mathcal{B}_0, \ldots, \mathcal{B}_{k-1}$ of $\mathcal{B}$ of size κ such that

$$|\bigcap_{i<k} \bigcup \mathcal{B}_i| < \omega. \quad \square$$

Next we prove a consequence of theorem 2.1.25, but first we give a definition.

Definition 2.1.26. Let κ be a regular cardinal and $\mathcal{A} \subset [\omega]^\omega$ be an almost disjoint family. $\mathcal{A}$ is called a *κ-Luzin gap* if $|\mathcal{A}| = \kappa$ and there is no $X \in [\omega]^\omega$ such that both $|\{A \in \mathcal{A} : |A \setminus X| < \omega|\} = \kappa$ and $|\{A \in \mathcal{A} : |A \cap X| < \omega\}| = \kappa$. A *Luzin-gap* is an ω_1-Luzin gap.

An ω_1-Luzin gap can be constructed in ZFC and simple forcings give models in which there are 2^ω-Luzin gaps while 2^ω is as large as you wish. The next corollary of theorem 2.1.25 implies that one can not construct ω_2-Luzin gaps from the assumption $2^\omega \geq \omega_2$ alone .

Corollary 2.1.27. *If $C(\kappa)$ holds then there is no κ-Luzin gap.*

Proof. Assume that $\mathcal{A} \subset \left[\omega\right]^\omega$ is an almost disjoint family of size κ. Then we can not get a even a two element subfamily $\mathcal{B} \subset \mathcal{A}$ satisfying 2.1.25(a). So applying theorem 2.1.25 for this $\mathcal{A}$ and for $k = 2$ there are subfamilies $\mathcal{B} \subset \mathcal{A}$ and $\mathcal{D} \subset \mathcal{A}$ of size κ such that $(\bigcup \mathcal{B}) \cap (\bigcup \mathcal{D})$ is finite. Hence $X = \bigcup \mathcal{B}$ witnesses that $\mathcal{A}$ is not a κ-Luzin gap. □

Now we turn to applying our principles to topology. We start with an application of the relatively weak principle $D(\kappa)$.

A. Dow **[36]** proved that if we add ω_2 Cohen reals to a model of GCH then in the generic extension $\beta\omega$ can be embedded into every separable, compact T_2 space of size $> c = \omega_2$. Here we show that $c = \omega_2 = 2^{\omega_1}$ together with $D(\omega_2)$ suffice to imply this statement.

First we need a lemma based on the observation that large separable spaces contain many "similar" points.

Given a topological space X and a point $x \in X$ we denote by $\mathcal{V}_X(x)$ the neighbourhood filter of x in X, that is, $\mathcal{V}_X(x) = \{U \subset X : x \in \mathrm{int}_X(U)\}$. If D is a dense subset of X let $\mathcal{V}_X(x) \restriction D = \{U \cap D : D \in \mathcal{V}_X(x)\}$. We omit the subscript X if it may not cause any confusion.

In section 2.1.1 we defined the operation $\hat{\mathcal{A}}$ for $\mathcal{A} \in \mathcal{M}(\kappa)$. By an abuse of notation we define $\hat{\mathcal{A}}$ for every family $\mathcal{A}$ of subsets of ω as follow:

$$\hat{\mathcal{A}} = \{X \subset \omega : |\mathcal{A} \cap \mathcal{P}(X)| = |\mathcal{A}|\}.$$

Lemma 2.1.28. *Assume that X is a separable regular topological space of size $> c^{<c}$, where $c = 2^\omega$, $D \in \left[X\right]^\omega$, $\overline{D} = X$. Then there are a point $x \in X$ and a family $\mathcal{A} = \{A_\alpha, B_\alpha : \alpha < c\} \subset \mathcal{P}(D)$ such that*
(1) $\overline{A_\alpha} \cap \overline{B_\alpha} = \emptyset$ for each $\alpha < c$,
(2) $\hat{\mathcal{A}} \subset \mathcal{V}(x) \restriction D$.

Proof. Fix an enumeration $\{D_\xi : \xi < c\}$ of $\mathcal{P}(D)$ and let $\mathcal{D}_\alpha = \{D_\xi : \xi < \alpha\}$ for $\alpha < c$. For $x \in X$ and $\alpha < c$ let $\mathcal{V}(x, \alpha) = (\mathcal{V}(x) \restriction D) \cap \mathcal{D}_\alpha$. A point $x \in X$ is called *special* if there is an $\alpha < c$ such that $\mathcal{V}(x, \alpha) \neq \mathcal{V}(y, \alpha)$ for each $y \in X \setminus \{x\}$. Clearly there are at most $c^{<c}$ special points in X. Since $|X| > c^{<c}$ we can pick a point $x \in X$ which is not special. Then for each $\alpha < c$ we can find a point $x_\alpha \neq x$ in X such that $\mathcal{V}(x_\alpha, \alpha) = \mathcal{V}(x, \alpha)$. Since X is regular the points x and x_α have neighbourhoods U_α and W_α, respectively, with $\overline{U_\alpha} \cap \overline{W_\alpha} = \emptyset$. Let $A_\alpha = U_\alpha \cap D$ and $B_\alpha = W_\alpha \cap D$.

Now assume that $E \in \hat{\mathcal{A}}$ and pick $\xi < c$ with $E = D_\xi$. We can find $\alpha < c$ such that $\xi < \alpha$ and either $A_\alpha \subset E$ or $B_\alpha \subset E$. Hence $E \in \mathcal{V}(x, \alpha) \cup \mathcal{V}(x_\alpha, \alpha) = \mathcal{V}(x, \alpha)$. Therefore $E \in \mathcal{V}(x) \restriction D$ which was to be proved. □

Let us now recall the definition of a μ-dyadic system from **[62]**.

Definition 2.1.29. If X is a topological space a family $\{\langle A(\alpha, 0), A(\alpha, 1)\rangle : \alpha \in \mu\}$ of pairs of closed subsets of X is a *μ-dyadic system* such that

(1) $A(\alpha, 0) \cap A(\alpha, 1) = \emptyset$ for each $\alpha < \mu$,
(2) for each $\epsilon \in \mathrm{Fn}(\mu, 2, \omega)$ we have $\bigcap\limits_{\alpha \in \mathrm{dom}(\epsilon)} A(\alpha, \epsilon(\alpha)) \neq \emptyset$.

Theorem 2.1.30. *If $D(c)$ holds, X is a separable compact T_2 space of size $> c^{<c}$ then X contains a c-dyadic system, consequently X maps continuously onto $[0, 1]^c$ (and so $\beta\omega$ can be embedded into X).*

Proof. Fix a countable dense subset D of X. By lemma 2.1.28 there is a family $\mathcal{A} = \{A_\alpha, B_\alpha : \alpha < c\} \subset \mathcal{P}(D)$ such that $\overline{A_\alpha} \cap \overline{B_\alpha} = \emptyset$ for $\alpha < c$ and $\hat{\mathcal{A}}$ is centered. Let $D(\alpha, 0) = A_\alpha$, $D(\alpha, 1) = A_\alpha$ and $D(\alpha, n) = D$ for $\alpha < \kappa$ and $n \geq 2$ and consider the $\kappa \times \omega$-matrix $\mathcal{D} = \langle D(\alpha, i) : \alpha < \kappa, i < \omega\rangle$. Since $\hat{\mathcal{A}} = \hat{\mathcal{D}}$ we can apply $D(c)$ to get a cofinal $S \subset c$ such that

the family $\langle \overline{A_\alpha}, \overline{B_\alpha} : \alpha < c \rangle$ is c-dyadic. Now we can apply theorem [**62**, 3.18] to get the other consequences. □

Since the cardinality of a locally compact, scattered separable space is at most 2^ω by [**88**], the height of such a space is less then $(2^\omega)^+$. So under CH there is no such a space of height ω_2. M. Weese asked whether the existence of such a space of height ω_2 follows from $\neg CH$. This question was answered in the negative by W. Just, who proved, [**64**, theorem 2.13], that if one adds Cohen reals to a model of CH then in the generic extension there are no locally compact scattered thin spaces of height ω_2.

The next theorem is a generalization of the above mentioned result of Just.

Theorem 2.1.31. *If $C^s(\kappa)$ holds then there is no locally compact, thin scattered space of height κ.*

Proof. Assume on the contrary that there is such a space X. We can assume that $\mathrm{I}_\alpha(X) = \{\alpha\} \times \omega$ for $\alpha < \mathrm{ht}(X)$. For each $\alpha < \mathrm{ht}(X)$ fix compact open neighbourhoods $U(\alpha, n)$ of $\langle \alpha, n \rangle$ for $n \in \omega$ such that $U(\alpha, n) \subset \{\langle \alpha, n \rangle\} \cup \bigcup\{\mathrm{I}_\beta(X) : \beta < \alpha\}$ and the sets $U(\alpha, n)$ for $n < \omega$ are pairwise disjoint.

Put $A(\alpha, 2n) = U(\alpha, n) \cap \mathrm{I}_0(X)$ and $A(\alpha, 2n+1) = \mathrm{I}_0(X) \setminus \bigcup\{U(\alpha, m) : m \leq n\}$. Let

$$T = \{t \in \omega^{<\omega} : \ t(0) \text{ is even and } t(i) \text{ is odd for } i > 0 \ \}.$$

Now apply $C^s(\kappa)$ to the matrix $\langle A(\alpha, n) : \alpha < \kappa, n < \omega \rangle \in \mathcal{M}(\kappa)$ and T.

Observe that
$$A(\beta, 2n) \cap \bigcap_{i<k} A(\alpha_i, 2n_i + 1) = \emptyset \quad \text{iff}$$
$$U(\beta, n) \cap \mathrm{I}_0(X) \subset \bigcup_{i<k} U(\alpha_i, n_i) \cap \mathrm{I}_0(X) \quad \text{iff}$$
$$U(\beta, n) \subset \bigcup_{i<k} U(\alpha_i, n_i).$$

Thus if $t = \langle 2n, 2n_0 + 1, \ldots, 2n_{k-1} + 1 \rangle \in T$ and $\langle \beta, \alpha_0, \ldots, \alpha_{k-1} \rangle \in (\kappa)^{k+1}$ then $A(\beta, 2n) \cap \bigcap_{i<k} A(\alpha_i, 2n_i + 1) = \emptyset$ implies $\beta \leq \max_{i<k} \alpha_i$. This excludes 2.1.3(2). So 2.1.3(1) holds, that is we have a stationary set $S \subset \kappa$ such that if $t = \langle 2n, 2n_0 + 1, \ldots, 2n_{k-1} + 1 \rangle \in T$ and $\langle \beta, \alpha_0, \ldots, \alpha_{k-1} \rangle \in (S)^{k+1}$ then

$$A(\beta, n) \cap \bigcap_{i<k} A(\alpha_i, 2n_i + 1) \neq \emptyset,$$

that is

$$U(\beta, n) \setminus \bigcup_{i<k} \bigcup_{j \leq n_i} U(\alpha_i, j) \neq \emptyset.$$

But $U(\beta, n)$ is compact and each $U(\alpha, n)$ is open so it follows that for every $\beta \in S$ and $n \in \omega$ the set

$$D(\beta, n) = U(\beta, n) \setminus \bigcup\{U((\alpha, m) : \alpha \in S \setminus \{\beta\} \wedge m \in \omega\}$$

is not empty. For every such β and n let $\langle \gamma(\beta, n), m(\beta, n) \rangle \in D(\beta, n)$.

Since $\mathrm{I}_\beta(X)$ is dense in $X \setminus \bigcup\{\mathrm{I}_\alpha(X) : \alpha < \beta\}$ for every $\beta \in \kappa$ there is $k(\beta) \in \omega$ such that $\langle \beta, k(\beta) \rangle \in U(\beta^*, 0)$, where $\beta^* = \min S \setminus \beta + 1$. Thus $\langle \beta, k(\beta) \rangle \notin D(\beta, k(\beta))$ and so $\gamma(\beta, k(\beta)) < \beta$ for each $\beta \in S$. The set S is stationary so there are a stationary set $S' \subset S$, and ordinals $\gamma < \kappa$ and $k, m < \omega$ such that $k(\beta) = k$, $\gamma(\beta, k) = \gamma$ and $m(\beta, k) = m$ whenever $\beta \in S'$. Thus $\langle \gamma, m \rangle \in D(\beta, k)$ for each $\beta \in S'$, while $D(\beta, k) \cap D(\beta', k) = \emptyset$ for any $\{\beta, \beta'\} \in \left[S'\right]^2$ by the construction. This is a contradiction, hence the theorem is proved. □

Remark . It is easy to see that the proof of theorem 2.1.31 goes through if, instead of assuming that all levels of the space are countable, we only require that

(i) (i) all levels are of size $< \kappa$,
(ii) (ii) there are stationary many countable levels.

In [**64**] W. Just also proved that if one adds at least ω_2 Cohen reals to a model of CH then in the generic extension there is no locally compact, scattered topological space X such that

$\mathrm{ht}(X) = \omega_1 + 1$, $\mathrm{I}_0(X)$ is countable, $|\mathrm{I}_\alpha(X)| \leq \omega_1$ for $\alpha < \omega_1$ and $|\mathrm{I}_{\omega_1}(X)| = \omega_2$. The next theorem shows how to get a generalization of this result from our principles.

Theorem 2.1.32. *If* $\mathrm{cf}(\lambda) \geq \omega_1$ *and* $F(\lambda^+)$ *holds then there is no locally compact, scattered topological space* X *such that* $\mathrm{ht}(X) = \lambda + 1$, $\mathrm{I}_0(X)$ *is countable,* $|\mathrm{I}_\alpha(X)| \leq \lambda$ *for all* $\alpha < \lambda$ *and* $|\mathrm{I}_\lambda(X)| = \lambda^+$.

Proof. See in [**12**]. □

Following the terminology of [**49**] a Hausdorff space is called P_2 if it does not contain two uncountable disjoint open sets. Hajnal and Juhász in [**49**] constructed a ZFC example of a first countable, P_2 space of size ω_1 as well as consistent examples of size 2^ω with 2^ω as large as you wish . On the other hand, using a result of Z. Szentmiklóssy they proved that it is consistent with ZFC that 2^ω is as large as you wish and there are no first countable P_2 spaces of size $\geq \omega_3$. However their method was unable to replace here ω_3 with ω_2. Our next result does just this because, as is shown in [**49**], every P_2 space is separable.

Theorem 2.1.33. *If* $C(\kappa)$ *holds then every first countable, separable* T_2 *topological space* X *of size* κ *contains two disjoint open sets* U *and* V *of cardinality* κ.

Proof. Let D be a countable dense subset of X. For each $x \in X$ fix a neighborhood base $\{U(x,n) : n \in \omega\}$ of x in X. Apply $C(\kappa)$ to the matrix $\langle U(x,n) \cap D : x \in X, n < \omega\rangle$ and $T = \omega^2$. Since X is T_2, there is no $S \in \left[X\right]^\kappa$ satisfying 2.1.31. So there are $S_0, S_1 \in \left[X\right]^\kappa$ and $n, m \in \omega$ such that $U(x,n) \cap U(y,m) \cap D = \emptyset$ whenever $x \in S_0$ and $y \in S_1$. But D is dense in X, therefore $U = \bigcup\{U(x,n) : x \in S_0\}$ and $V = \bigcup\{U(y,m) : y \in S_1\}$ are disjoint open sets of size κ. □

Definition 2.1.34. Let X be a topological space and $D \subset X$. We say that D is *sequentially dense in* X iff for each $x \in X$ there is a sequence S_x from D which converges to x. A space Y is said to be *sequentially separable* if it contains a countable sequentially dense subset.

Definition 2.1.35. Given a topological space $\langle X, \tau\rangle$ and a subspace $Y \subset X$ a function f is called a *neighbourhood assignment on* Y *in* X iff $f : Y \longrightarrow \tau$ and $y \in f(y)$ for each $y \in Y$.

Our next result says that under $C(\kappa)$ if a sequentially separable space X does not contain a discrete subspace of size κ, (i.e. $\hat{\mathrm{s}}(X) \leq \kappa$ using the notation of [**62**]) then X does not contain left or right separated subspaces of size κ either. This can be written as $\hat{\mathrm{h}}(X)\,\hat{\mathrm{z}}(X) \leq \kappa$. Since in [**11**] a normal, Frechet-Urysohn, separable (hence sequentially separable) space X is forced such that $\mathrm{z}(X) \leq \omega_1$ but $\mathrm{h}(X) = \omega_2$, this result is not provable in ZFC. First, however, we need a lemma.

Lemma 2.1.36. *Assume that* $C(\kappa)$ *holds. Let* X *be a sequentially separable space with* $Y \subset X$, $|Y| = \kappa$. *If* f *is a neighbourhood assignment on* Y *in* X, *then either (a) or (b) below holds:*
(a) there is $Y' \in \left[Y\right]^\kappa$ *such that* $f(y) \cap Y' = \{y\}$ *for each* $y \in Y'$ *(hence* Y' *is discrete),*
(b) there are Y_0, $Y_1 \in \left[Y\right]^\kappa$ *such that* $y \in \overline{f(x)}$ *whenever* $x \in Y_0$ *and* $y \in Y_1$.

Proof. We can assume that $D = \omega$ is sequentially dense in X. For each $y \in Y$ choose a sequence $S_y \subset D$ converging to y. Let $A(y, 2n) = D \setminus f(y)$, $A(y, 2n+1) = S_y \setminus n$, $T = \{\langle 2n, 2m+1\rangle : n, m \in \omega\}$ and apply $C(\kappa)$. Assume first that $Y' \in \left[Y\right]^\kappa$ witnesses 2.1.31 and let $x \neq y \in Y'$. Then for each $n \in \omega$ we have $(S_y \setminus f(x)) \setminus n \neq \emptyset$, i.e. $S_y \setminus f(x)$ is infinite. But S_y converges to y, so $y \notin f(x)$, and so Y' satisfies (a). Assume now that 2.1.32 holds. Then there are $Y_0, Y_1 \in \left[\omega\right]^\omega$ and $m \in \omega$ such that $(D \setminus f(x)) \cap (S_y \setminus m) = \emptyset$ for each $x \in Y_0$ and $y \in Y_1$. But then $S_y \setminus m \subset f(x)$ hence $y \in \overline{f(x)}$ which was to be proved. □

Theorem 2.1.37. *If* $C(\kappa)$ *holds,* X *is a regular, sequentially separable space with* $\hat{\mathrm{s}}(X) \leq \kappa$ *then* $\hat{\mathrm{h}}(X)\,\hat{\mathrm{z}}(X) \leq \kappa$.

Proof. Assume on the contrary that $Y \in \left[X\right]^\kappa$ and the neighbourhood assignment $f : Y \longrightarrow \tau_X$ witnesses that Y is left (right) separated. We can assume that $Y = \kappa$ and Y is left (right) separated under the natural ordering of κ. Since X is regular we can find a neighbourhood assignment $g : Y \longrightarrow \tau$ with $\overline{g(y)} \subset f(y)$ for each $y \in Y$. Apply lemma 2.1.36 to Y and g. Now 2.1.36(a) can not hold because $\hat{\mathrm{s}}(X) \leq \kappa$, hence there are $Y_0, Y_1 \in \left[Y\right]^\kappa$ satisfying 2.1.36(b). Since

both Y_0 and Y_1 are cofinal in $Y = \kappa$ under the natural ordering of the ordinals, applying left (or right) separatedness of Y we can pick $x \in Y_0$ and $y \in Y_1$ such that $y \notin f(x)$. By the choice of g this implies $y \notin \overline{g(x)}$ which contradicts 2.1.36(b). □

The Sorgenfrey line L is weakly separated and is of size c with $\hat{\mathrm{s}}(L) = \omega_1$. This shows that theorem 2.1.37 does not remain valid if you put weakly separated subspaces instead of right or left separated ones.

As an easy consequence of 2.1.37 we get the following result in which (sequential) separability is no longer assumed. We also note that under CH the assumption of X being Frechet-Urysohn is not necessary in this result.

Theorem 2.1.38. *Assume $C(\omega_2)$. If X is regular, Frechet-Urysohn space and $\mathrm{s}(X) = \omega$ then $\mathrm{h}(X) \leq \omega_1$.*

Proof. If $C(\omega_2)$ and X is separable, then by theorem 2.1.37 even $\mathrm{s}(X) \leq \omega_1$ implies $\mathrm{h}(X)\,\mathrm{z}(X) \leq \omega_1$. Now, every uncountable space X which is both right and left separated contains an uncountable discrete subspace, hence every right separated subspace of X is (hereditarily) separable. So by the above if $Y \subset X$ is right separated then $|Y| \leq \omega_1$, i.e. $\mathrm{h}(X) \leq \omega_1$. □

In [**57**] we investigated the following question: What makes a space have weight larger than its character? To answer this question we introduced the notion of an *irreducible base of a space* and proved that any weakly separated space has such a base, moreover the weight of a space possessing an irreducible base can not be smaller than its cardinality. We asked [**57**, Problem 1] whether every first countable space of uncountable weight contains an uncountable subspace with an irreducible base? In theorem 2.1.42 and corollary 2.1.43 we will give a partial positive answer to this problem, using the principle $C(\kappa)$. First we recall some definitions from [**57**].

Definition 2.1.39. Let X be a topological space. A base $\mathcal{U}$ of X is called *irreducible* if it has an *irreducible decomposition* $\mathcal{U} = \bigcup\{\mathcal{U}_x : x \in X\}$, i.e., (i) and (ii) below hold:

(i) $\mathcal{U}_x$ is a neighbourhood base of x in X for each $x \in X$,
(ii) for each $x \in X$ the family $\mathcal{U}_x^- = \bigcup\limits_{y \neq x} \mathcal{U}_y$ is not a base of X (hence $\mathcal{U}_x^-$ does not contain a neighbourhood base of x in X).

Definition 2.1.40. Let X be a topological space with $Y \subset X$. Similarly as above, an outer base $\mathcal{U}$ of Y in X is called *irreducible* if it has an *irreducible decomposition* $\mathcal{U} = \bigcup\{\mathcal{U}_y : y \in Y\}$, i.e., (i) and (ii) below hold:

(i) $\mathcal{U}_y$ is a neighbourhood base of y in X for each $y \in Y$,
(ii) for each $y \in Y$ the family $\mathcal{U}_y^- = \bigcup\{\mathcal{U}_z : z \in Y \setminus \{y\}\}$ does not contain a neighbourhood base of y in X.

Note that in general, a subspace Y having an irreducible outer base in X does not necessarily possess an irreducible base in itself. However, if Y is dense in an open set and the irreducible outer base of Y consists of regular open sets then clearly this is the case. Moreover, by our next result, under certain conditions we can at least find another subspace of the same size as Y that does have an irreducible base.

Lemma 2.1.41. *If X is a regular, separable space and $Y \subset X$ has an irreducible outer base in X consisting of regular open sets, then there is $Z \subset X$ with $|Z| = |Y|$ such that the subspace Z has an irreducible base.*

Proof. Let $\mathcal{B} = \bigcup\{\mathcal{B}_y : y \in Y\}$ be an irreducible outer base of Y in X consisting of regular open sets and D be a countable dense subset of X. We distinguish two cases:

Case 1. $|(\operatorname{int}\overline{Y}) \cap Y| = |Y|$.

Let $Z = (\operatorname{int}\overline{Y}) \cap Y$. Since Z is dense in the open set $\operatorname{int}\overline{Y}$, by our above remark Z has an irreducible base.

Case 2. $|(\operatorname{int}\overline{Y}) \cap Y| < |Y|$.

In this case the set $Y_1 = Y \setminus \operatorname{int} \overline{Y}$ is nowhere dense, so $D_1 = D \setminus \overline{Y_1}$ is dense in X. Let $Z = D_1 \cup Y_1$, then $|Z| = |Y_1| = |Y|$. Write $D_1 = \{d_n : n < \omega\}$ and for each $d_n \in D$ let $\mathcal{B}_{d_n}$ be a neighbourhood base of d_n in X consisting of regular open sets, that are disjoint to $Y_1 \cup \{d_m : m < n\}$. Then clearly $\bigcup\{\mathcal{B}_z : z \in Z\}$ is an irreducible outer base of Z in X consisting of regular open sets and Z is dense in X, so again we are done. □

Theorem 2.1.42. *Assume $C(\kappa)$. If X is a separable, first countable, regular space with $\mathrm{w}(X) \geq \kappa$, then there is subspace $Y \subset X$ of cardinality κ that has an irreducible base.*

Proof. Let $D \subset X$ be a countable, dense subset of X. For each $x \in X$ fix a neighbourhood base $\{U(x,n) : n \in \omega\}$ consisting of regular open sets and set $V(x,n) = U(x,n) \cap D$. Since the $U(x,n)$ are regular open and D is dense, we clearly have $U(x,n) \subset U(y,m)$ iff $V(x,n) \subset V(y,m)$.

Since $\mathrm{w}(X) \geq \kappa$, by transfinite recursion on $\beta < \kappa$ we can choose points $\{x_\alpha : \alpha < \kappa\} \subset X$ such that for any $\beta < \kappa$ the family $\{U(x_\alpha, n) : \alpha < \beta, n < \omega\}$ does not contain a neighbourhood base of x_β, in other words, there is a natural number k_β such that for all $\alpha < \beta < \kappa$ and $n \in \omega$ we have

$$\neg(x_\beta \in U(x_\alpha, n) \subset U(x_\beta, k_\beta)). \tag{$*$}$$

We can assume that $k_\beta = 0$ for each $\beta < \kappa$. Let $X' = \{x_\alpha : \alpha < \kappa\}$. For $x \in X'$ and $n < \omega$ put

$$A(x, 2n) = [V(x,n) \times \{0\}] \cup [(D \setminus V(x,n)) \times \{1\}]$$

and

$$A(x, 2m+1) = [(D \setminus V(x,0)) \times \{0\}] \cup [V(x,m) \times \{1\}].$$

Note that $A(x,2n) \cap A(y, 2m+1) = \emptyset$ iff $V(y,m) \subset V(x,n) \subset V(y,0)$. Apply $C(\kappa)$ to $\langle A(x,i) : x \in X', i < \omega\rangle$ and $T = \{\langle 2n, 2m+1\rangle : n, m < \omega\}$. By $(*)$ (and $k_\beta = 0$) there are no $D, E \in \left[X'\right]^\kappa$ and $n, m \in \omega$ such that

$$V(y,m) \subset V(x,n) \subset V(y,0) \tag{$\dagger$}$$

whenever $x \in D$ and $y \in E$, because $(\dagger)$ fails if $x = x_\alpha$, $y = x_\beta$ and $\alpha < \beta$. So there is $Y \in \left[X'\right]^\kappa$ such that for all $n, m \in \omega$ and $x \neq y \in Y$ the intersection of $A(x, 2n)$ and $A(y, 2m+1)$ is not empty. This means that $\neg(V(y,m) \subset V(x,n) \subset V(y,0))$, i.e. if we set $\mathcal{B}_y = \{U(y,n) : n < \omega\}$ then it follows that $\mathcal{B} = \bigcup\{\mathcal{B}_y : y \in Y'\}$ is a an irreducible outer base of Y in X consisting of regular open sets. Now applying lemma 2.1.41 we can conclude the proof. □

Unfortunately, as $C(\omega_1)$ is false, the above result is not applicable in the perhaps most interesting case when $\mathrm{w}(X) = \omega_1$. The annoying assumption of separability, however, can be circumvented as follows.

Corollary 2.1.43. *Assume $C(\kappa)$. If X is a first countable, regular space with $\mathrm{w}(X) \geq \kappa$, then there is an uncountable subspace $Y \subset X$ that has an irreducible base.*

Proof. If X is separable, then the previous theorem can be applied. If X is not separable, then X contains an uncountable left separated subspace Y and again Y has an irreducible base. □

2.1.4. A recent application concerning Banach algebras. The following statement are proved in [**1**]

Proposition 2.1.44. *If $\hat{C}(\kappa)$ holds then the following statement denoted by $\otimes_\kappa$ holds:*

$(\otimes_\kappa)$: $\forall\{f_\alpha : \alpha < \kappa\} \subset \ell_\infty/c_0\ \forall \ell \in \omega\ \forall c_1, \ldots, c_\ell \in \mathbb{R}\ \forall x \in \mathbb{R}\ \exists S_1, \ldots S_\ell \in \left[\kappa\right]^\kappa\ \exists \rho \in \{\leq, >\}$

$$\forall a \in (S_1, \ldots, S_\ell)\ \left\| \sum_{i=1}^{\ell} c_i \cdot f_{a(i)} \right\| \rho x.$$

Proposition 2.1.45. *If $\otimes_\kappa$ holds then $C([0,\kappa]) \not\hookrightarrow \ell_\infty/c_0$.*

Proposition 2.1.46. *If $\otimes_{\omega_2}$ holds then there is a Banach space X with density $\leq \omega_2$ such that $C([0,\omega_2]) \not\hookrightarrow X$ and $X \not\hookrightarrow \ell_\infty/c_0$.*

2.2. LCS* spaces in Cohen real extension

(This section is based on [10])

W. Just proved, [**64**, theorem 2.13], that if one blows up the continuum by adding Cohen reals to a model of CH then in the resulting generic extension there is no LCS* space of height ω_2 and width ω. In this section we strengthen the result of Just by proving, in particular, that in the same Cohen real extension no LCS space may have ω_2 many countable (non-empty) levels. It seems to be an intriguing (and natural) problem if the non-existence of an LCS* space of width ω and height ω_2 implies in ZFC the above conclusion, or more generally: when is a subsequence of the cardinal sequence of an LCS* space again such a cardinal sequence? In connection with this problem let us remark that, (as is shown in [**30**] or [**63**]), in the side-by-side random real extension of a model of CH the combinatorial principle $\mathcal{C}^s(\omega_2)$ introduced in [**12**, definition 2.3] holds, consequently in such an extension there is no LCS* space X of height ω_2 and width ω. In fact, by [**12**, theorem 4.12], $\mathcal{C}^s(\omega_2)$ implies that $\{\alpha \in \omega_2 : |I_\alpha(X)| = \omega\}$ is non-stationary in ω_2. However, we do not know if our above mentioned result, namely theorem 2.2.1, is implied by $\mathcal{C}^s(\omega_2)$.

Let us formulate then the promised strengthening of Just's result. We note that no assumption (such as CH) is made on our ground model.

Theorem 2.2.1. *Let us set $\kappa = (2^\omega)^+$ and add any number of Cohen reals to our ground model. Then in the resulting extension no LCS* space contains a κ-sequence $\{E_\alpha : \alpha < \kappa\}$ of pairwise disjoint countable subspaces such that $\overline{E_\alpha} \supset E_\beta$ holds for all $\alpha < \beta < \kappa$. In particular, for any LCS* space X we have $\big|\{\alpha : |\mathrm{I}_\alpha(X)| = \omega\}\big| < \kappa$.*

In fact, we shall prove a more general statement, but to formulate that we need a definition. A family of pairs (of sets) $\mathcal{D} = \big\{\langle D_0^\alpha, D_1^\alpha\rangle : \alpha \in I\big\}$ is said to be *dyadic over a set* T iff $D_0^\alpha \cap D_1^\alpha = \emptyset$ for each $\alpha \in I$ and

$$\mathcal{D}[\varepsilon] = \bigcap\{D^\alpha_{\varepsilon(\alpha)} : \alpha \in \operatorname{dom}\varepsilon\}$$

intersects T for each $\varepsilon \in \mathrm{Fn}(I, 2)$. We simply say that $\mathcal{D}$ *is dyadic* iff it is dyadic for some T, i.e. $\mathcal{D}[\varepsilon] \neq \emptyset$ for each $\varepsilon \in \mathrm{Fn}(I, 2)$. (As usual, $\mathrm{Fn}(I, 2)$ denotes the set of all finite partial functions from I into 2.)

Now, it is obvious that in an LCS* space

- the compact open sets form a base that is closed under finite unions,
- there is no infinite dyadic system of pairs of compact sets.

Consequently, theorem 2.2.2 below immediately yields theorem 2.2.1 above.

Theorem 2.2.2. *Set $\kappa = (2^\omega)^+$ and add any number of Cohen reals to the ground model. Then in the resulting generic extension the following statement holds: If X is any T_2 space containing pairwise disjoint countable subspaces $\{E_\alpha : \alpha < \kappa\}$ such that $\overline{E_\alpha} \supset E_\beta$ for $\alpha < \beta < \kappa$ and $X = \overline{E_0}$ (i. e. E_0 is dense in X) , moreover, for each $x \in X$, we have fixed a neighbourhood base $\mathcal{B}(x)$ of x in X that is closed under finite unions then there is an infinite set $a \in \big[\kappa\big]^\omega$, for each $\alpha \in a$ there are disjoint finite subsets L^0_α and L^1_α of E_α, and for each $x \in L^0_\alpha \cup L^1_\alpha$ there is a basic neighbourhood $V(x) \in \mathcal{B}(x)$ such that the infinite family of pairs*

$$\Big\{\Big\langle \bigcup_{x\in L^0_\alpha} V(x), \bigcup_{x\in L^1_\alpha} V(x)\Big\rangle : \alpha \in a\Big\}$$

is dyadic.

This topological statement in the Cohen extension in turn will follow from a purely combinatorial one concerning certain matrices, namely theorem 2.2.7.

To formulate this theorem we again need some notation and definitions.

For an ordinal α the interval $[\omega\alpha, \omega\alpha + \omega)$ will be denoted by $\mathbb{I}_\alpha$.

Given two sets A and B we write $f : A \xrightarrow{p} B$ to denote that f is a partial function from A to B, i. e. a function from a subset of A into B. As usual, we let

$$\mathrm{Fn}(A, B) = \{f : |f| < \omega \text{ and } f : A \xrightarrow{p} B\}.$$

If $A \subset \mathrm{On}$ then for any partial function $f : A \xrightarrow{p} B$ we set

$$\gamma(f) = \begin{cases} \min \operatorname{dom} f & \text{if } \operatorname{dom} f \neq \emptyset, \\ \sup A & \text{if } \operatorname{dom} f = \emptyset. \end{cases}$$

We let

$$\Omega = \big\{\langle A, B\rangle \in \big[\omega\big]^{<\omega} \times \big[\omega\big]^{<\omega} : A \cap B = \emptyset\big\},$$

and for $\ell = \langle A, B\rangle \in \Omega$ we set $\pi_0(\ell) = A$ and $\pi_1(\ell) = B$.

If S and T are sets of ordinals, we denote by $\mathcal{M}(S,T)$ the family of all $S \times \omega$-matrices consisting of subsets of T, i. e. $\mathcal{A} \in \mathcal{M}(S,T)$ means that $\mathcal{A} = \langle A_{\alpha,i} : \alpha \in S, i \in \omega\rangle$, where $A_{\alpha,i} \subset T$ for each $\alpha \in S$ and $i < \omega$.

For $\mathcal{A} \in \mathcal{M}(S,T)$, $f : S \xrightarrow{p} S$, and $s : S \xrightarrow{p} \Omega$ the pair (f, s) is said to be *$\mathcal{A}$-dyadic (over U)* iff the family of pairs

$$\Big\{\Big\langle \cup\big\{A_{f(\alpha),n} : n \in \pi_0(s(\alpha))\big\}, \cup\big\{A_{f(\alpha),n} : n \in \pi_1(s(\alpha))\big\}\Big\rangle : \alpha \in \operatorname{dom} f \cap \operatorname{dom} s\Big\}.$$

is dyadic (over U). If the pair $\langle \mathrm{id}_S, s\rangle$ is $\mathcal{A}$-dyadic (over U) then s is simply called *$\mathcal{A}$-dyadic (over U)*. It is this latter notion of $\mathcal{A}$-dyadicity of a single partial function that is really important (that for pairs is only of technical significance). Hence we state below an alternative characterisation of it.

For $\mathcal{A} \in \mathcal{M}(S,T)$, $s : S \xrightarrow{p} \Omega$, and $\varepsilon \in \mathrm{Fn}(\operatorname{dom} s, 2)$ we write

$$\mathcal{A}[s,\varepsilon] = \bigcap_{\alpha \in \operatorname{dom} \varepsilon} \bigcup\{A_{\alpha,n} : n \in \pi_{\varepsilon(\alpha)}(s(\alpha))\}.$$

Observation 2.2.3. *If $\mathcal{A} \in \mathcal{M}(S,T)$ then $s : S \xrightarrow{p} \Omega$ is $\mathcal{A}$-dyadic over U iff $\mathcal{A}[s,\varepsilon] \cap U \neq \emptyset$ for each $\varepsilon \in \mathrm{Fn}(\operatorname{dom} s, 2)$ and*

$$\bigcup\{A_{\alpha,n} : n \in \pi_0(s(\alpha))\} \cap \bigcup\{A_{\alpha,n} : n \in \pi_1(s(\alpha))\} = \emptyset$$

for each $\alpha \in \operatorname{dom} s$.

The following easy observation will be applied later, in the proof of lemma 2.2.9:

Observation 2.2.4. *If $g : S \xrightarrow{p} S$ and $s : S \xrightarrow{p} \Omega$ satisfy $\operatorname{dom} s \subset \operatorname{ran} g$, and the pair $(g, s \circ g)$ is $\mathcal{A}$-dyadic over U then s is $\mathcal{A}$-dyadic over U, as well.*

Definition 2.2.5. Fix a cardinal κ and let $\mathcal{D} \in \mathcal{M}(\kappa,\kappa)$. For $s : \kappa \xrightarrow{p} \Omega$ we say that s is *$\mathcal{D}$-min-dyadic (m.d.)* iff s is $\mathcal{D}$-dyadic over $\mathbb{I}_{\gamma(s)}$.

Moreover, we say that the matrix $\mathcal{D}$ is *m.d.-extendible* iff for each finite $\mathcal{D}$-min-dyadic partial function $s : \kappa \xrightarrow{p} \Omega$ and for each $\gamma < \gamma(s)$ there is an $\ell \in \Omega$ such that $s \cup \{\langle \gamma, \ell\rangle\}$ is also $\mathcal{D}$-min-dyadic, i. e. $\mathcal{D}$-dyadic over $\mathbb{I}_\gamma$.

Since $\mathbb{I}_0 = \omega$, we clearly have the following.

Observation 2.2.6. *If $\mathcal{D} \in \mathcal{M}(\kappa,\kappa)$ is m.d-extendible and $s : \kappa \xrightarrow{p} \Omega$ is a finite $\mathcal{D}$-min-dyadic partial function then s is $\mathcal{D}$-dyadic over ω.*

Finally, a matrix $\mathcal{D} \in \mathcal{M}(\kappa,\kappa)$ will be called *ω-determined* iff $D_{\alpha,n} \cap D_{\alpha,m} \cap \omega = \emptyset$ implies $D_{\alpha,n} \cap D_{\alpha,m} = \emptyset$ whenever $\alpha < \kappa$ and $n < m < \omega$.

With this we now have all the necessary ingredients to formulate and prove the promised combinatorial statement that will be valid in any Cohen real extension.

Theorem 2.2.7. *Set $\kappa = (2^\omega)^+$ and add any number of Cohen reals to the ground model. Then in the resulting generic extension for every ω-determined and m.d.-extendible matrix $\mathcal{D} \in \mathcal{M}(\kappa,\kappa)$ there is an infinite $\mathcal{D}$-dyadic partial function $h : \kappa \xrightarrow{p} \Omega$.*

Before proving theorem 2.2.7, however, we show how theorem 2.2.2 can be deduced from it.

Proof of theorem 2.2.2 using theorem 2.2.7. We can assume without any loss of generality that $E_\alpha = \mathbb{I}_\alpha$ for each $\alpha < \kappa$ and then will define an appropriate matrix $\mathcal{D} \in \mathcal{M}(\kappa,\kappa)$.

To this end, for coding purposes, we first fix a bijection $\rho : \big[\omega\big]^2 \longrightarrow \omega$ and let $\eta : \omega \longrightarrow \omega$ and $\nu : \omega \longrightarrow \omega$ be the "co-ordinate" functions of its inverse, i. e. $k = \rho(\{\nu(k), \eta(k)\})$ and $\nu(k) < \eta(k)$ for each $k < \omega$.

Since X is T_2, for each $n < \omega$ we can simultaneously pick basic neighbourhoods $B^\alpha_n(m) \in \mathcal{B}(\omega\alpha + m)$ of the points $\omega \cdot \alpha + m \in E_\alpha = \mathbb{I}_\alpha$ for all $m < n$ such that the sets $\{B^\alpha_n(m)\colon m < n\}$ are pairwise disjoint.

Now we define $\mathcal{D} = \langle D_{\alpha,k} : \langle \alpha, k\rangle \in \kappa \times \omega\rangle \in \mathcal{M}(\kappa, \kappa)$ as follows:

$$D_{\alpha,k} = B^\alpha_{\eta(k)}(\nu(k)) \cap \kappa.$$

This matrix $\mathcal{D}$ is clearly ω-determined because $E_0 = \mathbb{I}_0 = \omega$ is dense in X. It is a bit less easy to establish the following

Claim . *$\mathcal{D}$ is also m.d.-extendible.*

Proof of the claim. Let $s : \kappa \xrightarrow{p} \Omega$ be a finite $\mathcal{D}$-min-dyadic partial function and let $\gamma < \gamma(s)$.

Since the sets $\{\mathcal{D}[s, \varepsilon] : \varepsilon \in {}^{\operatorname{dom} s}2\}$ are all open in the subspace κ and they all intersect $\mathbb{I}_{\gamma(s)}$, moreover every element of $\mathbb{I}_{\gamma(s)}$ is an accumulation point of $\mathbb{I}_\gamma$, it follows that $\mathcal{D}[s, \varepsilon] \cap \mathbb{I}_\gamma$ must be infinite for each $\varepsilon \in {}^{\operatorname{dom} s}2$. Thus we can easily pick two disjoint finite subsets A_0 and A_1 of $\mathbb{I}_\gamma$ such that every $\mathcal{D}[s, \varepsilon]$ intersects both A_0 and A_1. Let $n < \omega$ be chosen in such a way that $A_0 \cup A_1 \subset \{\omega\gamma + m : m < n\}$, and set $K_i = \{\rho\{m, n\} : m < n \wedge \omega\gamma + m \in A_i\}$ for $i < 2$. Since ϱ is one-to-one we have $K_0 \cap K_1 = \emptyset$, hence $\ell = \langle K_0, K_1\rangle \in \Omega$, moreover

$$\Big(\bigcup_{m\in K_0} D_{\gamma,m}\Big) \cap \Big(\bigcup_{m\in K_1} D_{\gamma,m}\Big) = \emptyset \tag{$\star$}$$

because the elements of the family $\{B^\gamma_n(m) : m < n\}$ are pairwise disjoint.

Now put $t = s \cup \{\langle \gamma, \ell\rangle\}$. Then for each $\varepsilon \in {}^{\operatorname{dom} t}2$ we clearly have

$$A_{\varepsilon(\gamma)} \cap \mathcal{D}[t, \varepsilon] \neq \emptyset, \tag{$\star\star$}$$

hence ($\star$) and ($\star\star$) together yield that the extension t of s is $\mathcal{D}$-dyadic over $\mathbb{I}_\gamma = \mathbb{I}_{\gamma(t)}$. $\square$

Thus we may apply theorem 2.2.7 to the matrix $\mathcal{D}$ to obtain an infinite $\mathcal{D}$-dyadic partial function $h : \kappa \xrightarrow{p} \Omega$. Set $a = \operatorname{dom} h$ and for each $\alpha \in a$ and $i < 2$ put $L^i_\alpha = \{\omega\alpha + \nu(k) : k \in \pi_{\mathrm{i}}(h(\alpha))\}$. For $x \in L^i_\alpha$ put

$$V(x) = \cup\{B^\alpha_{\eta(k)}(\nu(k)) : x = \omega\alpha + \nu(k) \text{ and } k \in \pi_{\mathrm{i}}(h(\alpha))\}.$$

Then $V(x) \in \mathcal{B}(x)$ because $\mathcal{B}(x)$ is closed under finite unions. Since for $i < 2$

$$\big(\cup\{V(x) : x \in L^i_\alpha\}\big) \cap \kappa = \cup\{D_{\alpha,k} : k \in \pi_{\mathrm{i}}(h(\alpha))\}$$

and

$$\cup\{D_{\alpha,k} : k \in \pi_0(h(\alpha))\} \cap \cup\{D_{\alpha,k} : k \in \pi_1(h(\alpha))\} = \emptyset,$$

we have

$$\big(\cup\{V(x) : x \in L^i_\alpha\}\big) \cap \big(\cup\{V(x) : x \in L^i_\alpha\}\big) = \emptyset$$

because the latter intersection is an open set which does not intersect the dense set $\mathbb{I}_0 \subset \kappa$. Hence the infinite family

$$\Big\{\langle \bigcup_{x\in L^0_\alpha} V(x), \bigcup_{x\in L^1_\alpha} V(x)\rangle : \alpha \in a\Big\}$$

is indeed dyadic. $\square_{2.2.2}$

Proof of theorem 2.2.7. The proof will be based on the following two lemmas, 2.2.9 and 2.2.10. For these we need some more notation and a new and rather technical notion of extendibility for set matrices.

Given a set A we set

$$\mathcal{F}(A) = \{f \in \operatorname{Fn}(A, A) : f \text{ is injective and } \operatorname{dom}(f) \cap \operatorname{ran}(f) = \emptyset\}.$$

Each function $f \in \mathcal{F}(A)$ can be extended in natural way to a bijection $f^* : A \longrightarrow A$ as follows:

$$f^*(a) = \begin{cases} f(a) & \text{if } a \in \operatorname{dom} f, \\ f^{-1}(a) & \text{if } a \in \operatorname{ran} f, \\ a & \text{otherwise.} \end{cases}$$

Definition 2.2.8. If S and T are sets of ordinals then the matrix $\mathcal{A} \in \mathcal{M}(S,T)$ is called *nicely extendible* iff for each $f \in \mathcal{F}(S)$ there are a family $N(f) \subset \operatorname{Fn}(S,\Omega)$ and a function $K^f : N(f) \longrightarrow [S]^{\leq\omega}$ such that

(1) the pair (f,s) is $\mathcal{A}$-dyadic whenever $f \in \mathcal{F}(S)$ and $s \in N(f)$,
(2) $\emptyset \in N(f)$ for each $f \in \mathcal{F}(S)$,
(3) for $f,g \in \mathcal{F}(S)$ and $s \in N(f)$ if $f^* \restriction K^f(s) = g^* \restriction K^f(s)$ then $s \in N(g)$.
(4) for any $f \in \mathcal{F}(S)$, $s \in N(f)$ and $\alpha \in S \cap \gamma(s)$ there is $\ell \in \Omega$ such that $s \cup \{\langle \alpha, \ell\rangle\} \in N(f)$.

Clearly, this last condition (4) is what explains our terminology.

Lemma 2.2.9. *If $\kappa > \omega_1$ is regular and $\mathcal{A} \in \mathcal{M}(\kappa,\omega)$ is a nicely extendible matrix then there is an infinite partial function $h : \kappa \xrightarrow{p} \omega$ that is $\mathcal{A}$-dyadic .*

Proof. By induction on $n \in \omega$ we will define functions $h_0 \subset h_1 \subset \ldots h_n \subset \ldots$ from $\operatorname{Fn}(\kappa,\Omega)$ such that $|h_n| = n$ and for each $\nu \in \kappa$

$(*)^n_\nu$ there is $g \in \mathcal{F}(\kappa)$ such that $\gamma(g) > \nu$, $\operatorname{ran} g = \operatorname{dom} h_n$ and $h_n \circ g \in N(g)$.

First observe that $h_0 = \emptyset$ satisfies our requirements because, according to (2), condition $(*)^0_\nu$ holds trivially for each $\nu < \kappa$.

Next assume that the construction has been done and the induction hypothesis has been established for n. For each $\nu < \kappa$ choose a function $g_\nu \in \mathcal{F}(\kappa)$ witnessing $(*)^n_{\nu+\omega_1}$ and then write $K_\nu = K^{g_\nu}(h_n \circ g_\nu)$ and pick $\zeta_\nu \in (\nu, \nu+\omega_1) \setminus K_\nu$. Clearly the set

$$L = \{\xi \in \kappa : |\{\nu < \kappa : \xi \notin K_\nu\}| < \kappa\}$$

is countable and so we can pick $\xi_n \in \kappa \setminus (L \cup \operatorname{dom} h_n)$; then the set

$$J = \{\nu < \kappa : \xi_n \notin K_\nu\}$$

is of size κ.

Now set $g'_\nu = g_\nu \cup \{\langle \zeta_\nu, \xi_n\rangle\}$ for every $\nu \in J$. For every such ν then $\zeta_\nu, \xi_n \notin K_\nu$ implies ${g_\nu}^* \restriction K_\nu = {g'_\nu}^* \restriction K_\nu$, hence $h_n \circ g_\nu \in N(g'_\nu)$ by (3). Since $\zeta_\nu < \nu + \omega_1 < \gamma(g_\nu) = \gamma(h_n \circ g_\nu)$, we can now apply (4) to get $\ell^\nu \in \Omega$ such that $(h_n \circ g_\nu) \cup \{\langle \zeta_\nu, \ell^\nu\rangle\} \in N(g'_\nu)$.

We can then fix $\ell_n \in \Omega$ such that $J_n = \{\nu \in J : \ell^\nu = \ell_n\}$ is of size κ and let $h_{n+1} = h_n \cup \{\langle \xi_n, \ell_n\rangle\}$.

If $\nu \in J_n$ then $h_{n+1} \circ g'_\nu = (h_n \circ g_\nu) \cup \{\langle \zeta_\nu, \ell_n\rangle\} \in N(g'_\nu)$ and $\gamma(g'_\nu) > \nu$, so g'_ν witnesses $(*)^{n+1}_\nu$. But J_n is unbounded in κ, hence the inductive step is completed.

By $(*)^n_0$, for each $n < \omega$ there is g_n such that $\operatorname{dom} h_n = \operatorname{ran} g_n$ and $h_n \circ g_n \in N(g_n)$. Hence, by (1), $(g_n, h_n \circ g_n)$ is $\mathcal{A}$-dyadic, and so h_n is $\mathcal{A}$-dyadic according to observation 2.2.4. Consequently $h = \bigcup\{h_n : n < \omega\}$ is as required: it is $\mathcal{A}$-dyadic and infinite. $\square_{2.2.9}$

Given any infinite set I we denote by $\mathcal{C}_I$ the poset $\operatorname{Fn}(I,2)$, i.e. the standard notion of forcing that adds $|I|$ many Cohen reals.

Lemma 2.2.10. *Let $\kappa = (2^\omega)^+$. Then for each λ we have*

$$V^{\mathcal{C}_\lambda} \models \textit{If } \mathcal{D} \in \mathcal{M}(\kappa,\kappa) \textit{ is both } \omega\textit{-determined and m.d.-extendible}$$
$$\textit{then there is } I \in [\kappa]^\kappa \textit{ such that}$$
$$\mathcal{D}^* = \langle D_{\alpha,n} \cap \omega : \langle \alpha, n\rangle \in I \times \omega\rangle \textit{ is nicely extendible.}$$

Proof. Assume that

$$1_{\mathcal{C}_\lambda} \Vdash \dot{\mathcal{D}} \in \mathcal{M}(\kappa,\kappa) \text{ is m.d.-extendible.}$$

Let θ be a large enough regular cardinal and consider the structure $\mathcal{H}_\theta = \left\langle H_\theta, \in, \lhd, \kappa, \lambda, \dot{\mathcal{D}}\right\rangle$, where $H_\theta = \left\{x : |\operatorname{TC}(x)| < \theta\right\}$ and $\lhd$ is a fixed well-ordering of H_θ.

Working in V, for each $\alpha < \kappa$ choose a countable elementary submodel N_α of $\mathcal{H}_\theta$ with $\alpha \in N_\alpha$. Then there is $I \in \left[\kappa\right]^\kappa$ such that the models $\{N_\alpha : \alpha \in I\}$ are not only pairwise isomorphic but, denoting by $\sigma_{\alpha,\beta}$ the unique isomorphism between N_α and N_β, we have

(i) the family $\{N_\alpha \cap \theta : \alpha \in I\}$ forms a Δ-system with kernel Λ,
(ii) $\sigma_{\alpha,\beta}(\xi) = \xi$ for each $\xi \in \Lambda$,
(iii) $\sigma_{\alpha,\beta}(\alpha) = \beta$.

For each $\alpha < \kappa$ and $n < \omega$ let $\dot{D}_{\alpha,n}$ be the $\lhd$-minimal $\mathcal{C}_\lambda$-name of the $\langle \alpha, n\rangle^{\text{th}}$ entry of $\dot{\mathcal{D}}$. Since $\lhd$ is in $\mathcal{H}_\theta$ and $\sigma_{\alpha,\beta}(\alpha) = \beta$ we have

Claim 2.2.10.1. *$\sigma_{\alpha,\beta}(\dot{D}_{\alpha,n}) = \dot{D}_{\beta,n}$ for each $\alpha, \beta \in I$ and $n \in \omega$.*

Let G be any $\mathcal{C}_\lambda$-generic filter over V. We shall show that

$$V[G] \models \text{“}\mathcal{D}^* = \langle D_{\alpha,n} \cap \omega : \langle \alpha, n\rangle \in I \times \omega\rangle \text{ is nicely extendible.”}$$

For each $f \in \mathcal{F}(I)$ define the bijection $\rho_f : \lambda \longrightarrow \lambda$ as follows:

$$\rho_f(\xi) = \begin{cases} \sigma_{\alpha,f^*(\alpha)}(\xi) & \text{if } \xi \in N_\alpha \cap \lambda \text{ for some } \alpha \in I, \\ \xi & \text{otherwise.} \end{cases}$$

In a natural way ρ_f extends to an automorphism of $\mathcal{C}_\lambda$, which will be denoted by ρ_f as well. Clearly, we have

Claim 2.2.10.2. *If $f \in \mathcal{F}(I)$, $f(\alpha) = \beta$, $p \in \mathcal{C}_\lambda \cap N_\alpha$ then $\sigma_{\alpha,\beta}(p) = \rho_f(p)$.*

For $f \in \mathcal{F}(I)$ let $G^f = \{\rho_f^{-1}(p) : p \in G\}$ and then set

$$N(f) = \{s \in \mathrm{Fn}(I, \Omega) : s \text{ is } \dot{\mathcal{D}}[G^f]\text{-min-dyadic}\} = \{s \in \mathrm{Fn}(I, \Omega) : \exists q \in G^f \ q \Vdash \text{“}s \text{ is } \dot{\mathcal{D}}\text{-min-dyadic”}\}.$$

To define K^f, for each $s \in N(f)$ pick a condition $p_s \in G$ such that

$$\rho_f^{-1}(p_s) \Vdash s \text{ is } \dot{\mathcal{D}}\text{-min-dyadic}$$

and let

$$K^f(s) = \{\alpha \in I : (N_\alpha \setminus \Lambda) \cap \operatorname{dom} p_s \neq \emptyset\}.$$

Note that $K^f(s)$ as defined above is finite, although 2.2.8.(3) only requires $K^f(s)$ to be countable.

To check property 2.2.8.(3) assume that $f, g \in \mathcal{F}(I)$ and $s \in N(f)$ with $g^* \restriction K^f(s) = f^* \restriction K^f(s)$. Then $\rho_g^{-1}(p_s) = \rho_f^{-1}(p_s)$ and so

$$\rho_g^{-1}(p_s) \Vdash s \text{ is } \dot{\mathcal{D}}\text{-min-dyadic},$$

hence s is also $\dot{\mathcal{D}}[G^g]$-min-dyadic , i.e. $s \in N(g)$.

Before checking 2.2.8.(1) we need one more observation.

Claim 2.2.10.3. *$\dot{D}_{f(\alpha),n}[G] \cap \omega = \dot{D}_{\alpha,n}[G^f] \cap \omega$ whenever $f \in \mathcal{F}(I)$, $\alpha \in \operatorname{dom} f$, and $n < \omega$.*

Proof of claim 2.2.10.3. Let $k \in \omega$. Then $k \in \dot{D}_{f(\alpha),n}[G]$ iff $\exists p \in G \ p \Vdash$ “$k \in \dot{D}_{f(\alpha),n}$” iff $\exists p \in G \cap N_{f(\alpha)} \ p \Vdash$ “$k \in \dot{D}_{f(\alpha),n}$” iff $\exists q \in G^f \cap N_\alpha \ p = \sigma_{\alpha,f(\alpha)}(q) \Vdash$ “$k \in \dot{D}_{f(\alpha),n}$” iff $\exists q \in G^f \cap N_\alpha \ q \Vdash$ “$k \in \dot{D}_{\alpha,n}$” iff $\exists q \in G^f \ q \Vdash$ “$k \in \dot{D}_{\alpha,n}$” iff $k \in \dot{D}_{\alpha,n}[G^f]$. $\square_{2.2.10}$

Now let $f \in \mathcal{F}(I)$ and $s \in N(f)$. By the definition of $N(f)$, s is $\dot{\mathcal{D}}[G^f]$-min-dyadic and so by observation 2.2.6 s is $\dot{\mathcal{D}}[G^f]$-dyadic over ω. But it follows from 2.2.10.3, that s is $\dot{\mathcal{D}}[G^f]$-dyadic over ω if and only if the pair (f, s) is $\dot{\mathcal{D}}[G]$-dyadic over ω.

2.2.8.(2) is clear because $\emptyset$ is trivially $\mathcal{A}$-min-dyadic for any $\mathcal{A} \in \mathcal{M}(\kappa, \omega)$. Finally 2.2.8.(4) follows from the definition of $N(f)$ because $\dot{\mathcal{D}}[G^f]$ is m.d.-extendible. $\square_{2.2.10}$

Now, to complete the proof of theorem 2.2.7, first apply lemma 2.2.10 to get $I \in \left[\kappa\right]^\kappa$ such that

$$\mathcal{D}^* = \langle D_{\alpha,n} \cap \omega : \langle \alpha, n\rangle \in I \times \omega\rangle$$

is nicely extendible. Then applying lemma 2.2.9 to $\mathcal{D}^*$ we obtain an infinite $\mathcal{D}^*$-dyadic function $h : \kappa \xrightarrow{p} \Omega$. Since the matrix $\mathcal{D}$ is ω-determined the function h is $\mathcal{D}$-dyadic, as well. $\square_{2.2.7}$

2.3. Weak Freeze-Nation property of posets

(This section is based on [7])

Definition 2.3.1. For a regular κ, a quasi ordering P is said to have the κ *Freese-Nation property* (the κ-*FN* for short) if there is a mapping (κ-FN mapping) $f : P \to [P]^{<\kappa}$ such that

for any p, $q \in P$ if $p \leq q$ then there is $r \in f(p) \cap f(q)$ such that $p \leq r \leq q$.

A mapping f as above is called a κ-FN mapping on P. $\aleph_1$-FN is called as *weak Freese-Nation property* (*wFN-property*, in short). A *weak Freese-Nation mapping* is a $\aleph_1$-FN mapping.

Freese and Nation [**44**] used the $\aleph_0$-FN in a characterization of projective lattices and asked if this property alone already characterizes the projectiveness. L. Heindorf gave a negative answer to the question showing that the Boolean algebras with the $\aleph_0$-FN are exactly those which are openly generated. It is known that the class of openly generated Boolean algebras contains projective Boolean algebras as a proper subclass (see [**51**] — openly generated Boolean algebras are called 'rc-filtered' there). Heindorf and Shapiro [**51**] then studied the $\aleph_1$-FN which they called the *weak Freese-Nation property* and proved some elementary properties of the Boolean algebras with this property. Partial orderings with the κ-FN for arbitrary regular κ were further studied in Fuchino, Koppelberg and Shelah [**46**]. Koppelberg [**69**] gives some nice applications of the $\aleph_1$-FN.

In the following we shall quote some elementary facts from [**46**] which we need later. First of all, it can be readily seen that every small partial ordering has the κ-FN:

Lemma* 2.3.2. ([**46**]) *Every partial ordering P of cardinality $\leq \kappa$ has the κ-FN.* $\square$

For a partial ordering P and a sub-ordering $Q \subseteq P$, we say that Q is a *κ-sub-ordering* of P and denote it with $Q \leq_\kappa P$ if, for every $p \in P$, the set $\{\, q \in Q \,:\, q \leq p \,\}$ has a cofinal subset of cardinality $< \kappa$ and the set $\{\, q \in Q \,:\, q \geq p \,\}$ has a coinitial subset of cardinality $< \kappa$.

Lemma* 2.3.3. ([**46**]) *Suppose that δ is a limit ordinal and $(P_\alpha)_{\alpha \leq \delta}$ a continuously increasing chain of partial orderings such that $P_\alpha \leq_\kappa P_\delta$ for all $\alpha < \delta$. If P_α has the κ-FN for every $\alpha < \delta$, then P_δ also has the κ-FN.* $\square$

For application of Lemma 2.3.3, it is enough to have $P_\alpha \leq_\kappa P_\delta$ and the κ-FN of P_α for every $\alpha < \delta$ such that either α is a successor or of cofinality $\geq \kappa$: $P_\alpha \leq_\kappa P_\delta$ for $\alpha < \delta$ of cofinality $< \kappa$ follows from this since such P_α can be represented as the union of $< \kappa$ many κ-sub-orderings of P_δ. Hence by inductive application of Lemma 2.3.3, we can show that P_α satisfies the κ-FN for every $\alpha \leq \delta$. Similarly, if δ is a cardinal $> \kappa$, then it is enough to have $P_\alpha <_\kappa P_\delta$ and the κ-FN of P_α for every limit $\alpha < \delta$ of cofinality $\geq \kappa$.

Proposition* 2.3.4 ([**46**]). *For a regular κ and a poset P, the following statements are equivalent:*

(1) P has the κ-FN;

(2) For some, or equivalently, any sufficiently large χ, if $M \prec \mathcal{H}_\chi = (\mathcal{H}_\chi, \in)$ is such that $P \in M$, $\kappa \subseteq M$ and $|M| = \kappa$ then $P \cap M \leq_\kappa P$ holds;

(3) $\{\, C \in [P]^\kappa \,:\, C \leq_\kappa P \,\}$ contains a club set. $\square$

Though 2.3.4,(2) is quite useful to show that a partial ordering has the κ-FN, sometimes it is quite difficult to check Proposition 2.3.4(2) as in the case of the $\aleph_1$-FN of $P(\omega)$ or $[\kappa]^{<\omega}$: in these cases it is independent if Proposition 2.3.4(2) holds. Applications like Corollary 2.3.15 in mind, we could think of another possible variant of Proposition 2.3.4, (2) in terms of the following weakening of the notion of internal approachability from [**42**]:

Definition 2.3.5. For a regular κ and a sufficiently large χ, we shall call an elementary submodel M of $\mathcal{H}_\chi$ *V_κ-like* if, either $\kappa = \aleph_0$ and M is countable, or there is an increasing sequence $(M_\alpha)_{\alpha<\kappa}$ of elementary submodels of M of cardinality less than κ such that $M_\alpha \in M_{\alpha+1}$ for all $\alpha < \kappa$ and $M = \bigcup_{\alpha<\kappa} M_\alpha$.

In [**46**], a characterization of the κ-FN using V_κ-like elementary submodels in place of elementary submodels in Proposition 2.3.4(2) was discussed. Unfortunately, in the proof an inaccuracy was found by the present author (see "Added in Proof" in [**46**]). In this section, we introduce a weakening of the very weak square principle from [**43**] — the principle $\square^{***}_{\kappa,\mu}$. In subsection 2.3.1 we show the equivalence of $\square^{***}_{\kappa,\mu}$ with the existence of a matrix $(M_{\alpha,\beta})_{\alpha<\mu^+,\beta<\mathrm{cf}(\mu)}$ — which we called (weak) (κ,μ)-dominating matrix — of elementary submodels of $\mathcal{H}(\chi)$ for sufficiently large χ with certain properties. This fact is used in subsection 2.3.2 to show that $\square^{***}_{\kappa,\lambda}$ together with a very weak version of the Singular Cardinals Hypothesis yields the characterization of partial orderings with the κ-FN in terms of V_κ-like elementary submodels (Theorem 2.3.14). ZFC or even ZFC + GCH is not enough for this characterization: in section 2.3.3, we show that, under Chang's conjecture for $\aleph_\omega$, there is a counter-example to the characterization. Together with Theorem 2.3.14, this counter-example also shows that $\square^{***}_{\aleph_1,\aleph_\omega}$ is not a theorem in ZFC + GCH.

One of the most natural questions concerning the κ-FN would be if $(\mathcal{P}(\omega),\subseteq)$ has the $\aleph_1$-FN. It is easy to see that $(\mathcal{P}(\omega),\subseteq)$ has the $\aleph_1$-FN iff $(\mathcal{P}(\omega)/fin,\subseteq^*)$ does (see [**46**]). See also Koppelberg [**69**] for some consequences of the $\aleph_1$-FN of $\mathcal{P}(\omega)/fin$. By Lemma 2.3.2, $\mathcal{P}(\omega)$ has the $\aleph_1$-FN under CH. In section 2.3.4, we show that $\mathcal{P}(\omega)$ and a lot of other ccc complete Boolean algebras still have the $\aleph_1$-FN in a model obtained by adding arbitrary number of Cohen reals to a model of, say, $V=L$. On the other hand it can happen very easily that $\mathcal{P}(\omega)$ does not have the $\aleph_1$-FN. In [**46**], it was shown that this is the case when $\mathbf{b}>\aleph_1$ or, more generally, if there is a $\subseteq^*$-sequence of elements of $\mathcal{P}(\omega)$ of order-type $\geq\omega_2$.

When κ is fixed, for $C\subseteq\kappa$, we denote with $(C)'$ the set of limit points of C other than κ.

2.3.1. Very weak square and dominating matrix. For a cardinal μ, the *weak square principle for* μ (notation: $\square^*_\mu$) is the statement: there is a sequence $(\mathcal{C}_\alpha)_{\alpha\in\mathrm{Lim}(\mu^+)}$ such that for every $\alpha\in\mathrm{Lim}(\mu^+)$

w1) $\mathcal{C}_\alpha\subseteq\mathcal{P}(\alpha)$ and $|\mathcal{C}_\alpha|\leq\mu$;

w2) every $C\in\mathcal{C}_\alpha$ is club in α and if $\mathrm{cf}(\alpha)<\mu$ then $\mathrm{otp}(C)<\mu$;

w3) for every $C\in\mathcal{C}_\alpha$ and $\delta\in(C)'$, we have $C\cap\delta\in\mathcal{C}_\delta$.

Clearly we have $\square_\mu\to\square^*_\mu$. Jensen [**53**] proved that $\square^*_\mu$ is equivalent to the existence of a special Aronszajn tree on μ^+. Ben-David and Magidor [**27**] showed that the weak square principle for a singular μ is actually weaker than the square principle: they constructed a model of $\square^*_{\aleph_\omega}$ and $\neg\square_{\aleph_\omega}$ starting from a model with a supercompact cardinal.

Foreman and Magidor considered in [**43**] the following principle which is, e.g. under GCH, a weakening of $\square^*$ principle: for a cardinal μ, *the very weak square principle for* μ holds if there is a sequence $(C_\alpha)_{\alpha<\mu^+}$ and a club $D\subseteq\mu^+$ such that for every $\alpha\in D$

v1) $C_\alpha\subseteq\alpha$, C_α is unbounded in α;

v2) for all bounded $x\in[C_\alpha]^{<\omega_1}$, there is $\beta<\alpha$ such that $x=C_\beta$.

In this paper, we shall use the following yet weaker variant of the very weak square principle. For a regular cardinal κ and $\mu>\kappa$, let $\square^{***}_{\kappa,\mu}$ be the following assertion: there exists a sequence $(C_\alpha)_{\alpha<\mu^+}$ and a club set $D\subseteq\mu^+$ such that for $\alpha\in D$ with $\mathrm{cf}(\alpha)\geq\kappa$

y1) $C_\alpha\subseteq\alpha$, C_α is unbounded in α;

y2) $[\alpha]^{<\kappa}\cap\{\,C_{\alpha'}\,:\,\alpha'<\alpha\,\}$ dominates $[C_\alpha]^{<\kappa}$ (with respect to $\subseteq$).

Since y2) remains valid when C_α's for $\alpha\in D$ are shrunk, we may replace y1) by

y1') $C_\alpha\subseteq\alpha$, C_α is unbounded in α and $\mathrm{otp}(C_\alpha)=\mathrm{cf}(\alpha)$.

A corresponding remark holds also for the sequence of the very weak square principle.

Lemma 2.3.6 ([**7**]). a) *The very weak square principle for μ implies $\square^{***}_{\omega_1,\mu}$.*

b) *For a singular cardinal μ and a regular κ such that $\mathrm{cf}([\lambda]^{<\kappa},\subseteq)\leq\mu$ for every $\lambda<\mu$, $\square^*_\mu$ implies $\square^{***}_{\kappa,\mu}$.* □

$\square^{***}_{\kappa,\mu}$ has some influence on cardinal arithmetic of cardinals below μ:

Lemma 2.3.7 ([**7**]). *Suppose that κ is regular and μ is such that $\mathrm{cf}(\mu)<\kappa$. If $\square^{***}_{\kappa,\mu}$ holds, then we have $\mathrm{cf}([\lambda]^{<\kappa},\subseteq)<\mu$ for every $\lambda<\mu$.* □

Suppose now that κ is a regular cardinal and $\mu > \kappa$ is such that $\mathrm{cf}(\mu) < \kappa$. Let $\mu^* = \mathrm{cf}(\mu)$. For a sufficiently large χ and $x \in \mathcal{H}(\chi)$, let us call a sequence $(M_{\alpha,\beta})_{\alpha<\mu^+,\beta<\mu^*}$ a (κ,μ)-*dominating matrix over* x — or just *dominating matrix over* x if it is clear from the context which κ and μ are meant — if the following conditions hold:

(j1) $M_{\alpha,\beta} \prec \mathcal{H}(\chi)$, $x \in M_{\alpha,\beta}$, $\kappa+1 \subseteq M_{\alpha,\beta}$ and $|M_{\alpha,\beta}| < \mu$ for all $\alpha < \mu^+$ and $\beta < \mu^*$
(j2) $(M_{\alpha,\beta})_{\beta<\mu^*}$ is an increasing sequence for each $\alpha < \mu^+$;
(j3) if $\alpha < \mu^+$ is such that $\mathrm{cf}(\alpha) \geq \kappa$, then there is $\beta^* < \mu^*$ such that, for every $\beta^* \leq \beta < \mu^*$, $[M_{\alpha,\beta}]^{<\kappa} \cap M_{\alpha,\beta}$ is cofinal in $([M_{\alpha,\beta}]^{<\kappa}, \subseteq)$;

j1) $M_{\alpha,\beta} \prec \mathcal{H}(\chi)$, $x \in M_{\alpha,\beta}$, $\kappa+1 \subseteq M_{\alpha,\beta}$ and $|M_{\alpha,\beta}| < \mu$ for all $\alpha < \mu^+$ and $\beta < \mu^*$;
j2)
j3) if $\alpha < \mu^+$ is such that $\mathrm{cf}(\alpha) \geq \kappa$, then there is $\beta^* < \mu^*$ such that, for every $\beta^* \leq \beta < \mu^*$, $[M_{\alpha,\beta}]^{<\kappa} \cap M_{\alpha,\beta}$ is cofinal in $([M_{\alpha,\beta}]^{<\kappa}, \subseteq)$;

For $\alpha < \mu^+$, let $M_\alpha = \bigcup_{\beta<\mu^*} M_{\alpha,\beta}$. By j1) and j2), we have $M_\alpha \prec \mathcal{H}(\chi)$.

j4) $(M_\alpha)_{\alpha<\mu^+}$ is continuously increasing and $\mu^+ \subseteq \bigcup_{\alpha<\mu^+} M_\alpha$.

Foreman and Magidor [**43**] called a sequence of subsets of μ^+ having some properties similar to those of the sequence $(\mu^+ \cap M_{\alpha,\beta})_{\alpha<\mu^+,\beta<\mu^*}$ for $(M_{\alpha,\beta})_{\alpha<\mu^+,\beta<\mu^*}$ as above Jensen matrix. Our definition of dominating matrices and some ideas in the proofs in this section are inspired by their paper. Note that, in the case of $2^{<\kappa} = \kappa$, j3) can be replaced by the following seemingly stronger property:

j3') if $\alpha < \mu^+$ is such that $\mathrm{cf}(\alpha) \geq \kappa$, then there is $\beta^* < \mu^*$ such that, for every $\beta^* \leq \beta < \mu^*$, $[M_{\alpha,\beta}]^{<\kappa} \subseteq M_{\alpha,\beta}$.

This is simple because of the following observation:

Lemma 2.3.8 ([**7**]). *Suppose that $2^{<\kappa} = \kappa$ and M is an elementary submodel of $\mathcal{H}(\chi)$ for some sufficiently large χ and $\kappa \subseteq M$. If $[M]^{<\kappa} \cap M$ is cofinal in $[M]^{<\kappa}$, then we have $[M]^{<\kappa} \subseteq M$.* $\square$

Note also that, if $M \prec \mathcal{H}(\chi)$ is V_κ-like, then $[M]^{<\kappa} \cap M$ is cofinal in $[M]^{<\kappa}$. Hence, under $2^{<\kappa} = \kappa$, $M \prec \mathcal{H}(\chi)$ is V_κ-like if and only if $|M| = \kappa$ and $[M]^{<\kappa} \subseteq M$.

In the following theorem, we show that $\square^{***}_{\kappa,\mu}$ together with a very weak version of the Singular Cardinals Hypothesis below μ implies the existence of a dominating matrix:

Theorem 2.3.9. *Suppose that κ is a regular cardinal and $\mu > \kappa$ is such that $\mathrm{cf}(\mu) < \kappa$. If we have $\mathrm{cf}([\lambda]^{<\kappa}, \subseteq) = \lambda$ for cofinally many $\lambda < \mu$ and $\square^{***}_{\kappa,\mu}$ holds, then, for any sufficiently large χ and $x \in \mathcal{H}(\chi)$, there is a (κ,μ)-dominating matrix over x.*

Proof. Let $\mu^* = \mathrm{cf}(\mu)$ and $(\mu_\beta)_{\beta<\mu^*}$ be an increasing sequence of cardinals below μ such that $\mu_0 > \mu^*$, $\sup\{\,\mu_\beta : \beta < \mu^*\,\} = \mu$ and $\mathrm{cf}([\mu_\beta]^{<\kappa}, \subseteq) = \mu_\beta$ for every $\beta < \mu^*$. Let $(C_\alpha)_{\alpha\in\mu^+}$ and $D \subseteq \mu^+$ be as in the definition of $\square^{***}_{\kappa,\mu}$. Without loss of generality, we may assume that $|C_\alpha| \leq \mathrm{cf}(\alpha)$ for all $\alpha < \mu^+$. We may also assume that $\alpha > \mu$ for every $\alpha \in D$.

In the following, we fix a well ordering $\trianglelefteq$ on $\mathcal{H}(\chi)$ and, when we talk about $\mathcal{H}(\chi)$ as a structure, we mean $\mathcal{H}(\chi) = (\mathcal{H}(\chi), \in, \trianglelefteq)$. $X \subseteq \mathcal{H}(\chi)$ as a substructure of $\mathcal{H}(\chi)$ is thus the structure $(X, \in \cap X^2, \trianglelefteq \cap X^2)$ — for notational simplicity we shall denote such a structure simply by $(X, \in, \trianglelefteq)$.

Let $N \in \mathcal{H}(\chi)$ be an elementary substructure of $\mathcal{H}(\chi)$ such that N contains every thing needed below — in particular, we let $\mu^+ \subseteq N$ and x, $(C_\alpha)_{\alpha<\mu^+}$, D, $(\mu_\alpha)_{\alpha<\mu^*} \in N$. Let $(N_\xi)_{\xi<\kappa}$ be an increasing sequence of elementary submodels of $\mathcal{H}(\chi)$ such that

(0) $N_0 = N$;
(1) $N_\xi \in \mathcal{H}(\chi)$ for every $\xi < \kappa$ and
(2) $(N_\eta)_{\eta\leq\xi} \in N_{\xi+1}$ for every $\xi < \kappa$.

Now, for each $\xi < \kappa$, let

$$\mathcal{N}_\xi = (N_\xi, \in, \trianglelefteq, R_\xi, \kappa, \mu, \mu^*, D, (C_\alpha)_{\alpha<\mu^+}, (\mu_\alpha)_{\alpha<\mu^*}, x, \eta)_{\eta<\xi}$$

where R_ξ is the relation $\{\,(\eta, N_\eta) : \eta < \xi\,\}$. For $X \subseteq \mu^+$, let us denote with $sk_\xi(X)$ the Skolem-hull of X with respect to the built-in Skolem functions of $\mathcal{N}_\xi$ (induced from $\trianglelefteq$). For $\xi < \xi' < \kappa$, $\mathcal{N}_\xi$ is an element of $\mathcal{N}_{\xi'}$ by (2) and the Skolem functions of $\mathcal{N}_\xi$ are also elements of $\mathcal{N}_{\xi'}$. In

particular, we have $sk_\xi(X) \subseteq sk_{\xi'}(X)$. It follows that $sk(X) = \bigcup_{\xi<\kappa} sk_\xi(X)$ is an elementary submodel of $\mathcal{H}(\chi)$. Note also that, if $X \subseteq \mu^+$ is an element of $sk_{\xi'}(Y)$ then, since $sk_\xi(X)$ is definable in $sk_{\xi'}(Y)$, we have $sk_\xi(X) \in sk_{\xi'}(Y)$.

For the proof of the theorem, it is clearly enough to construct $M_{\alpha,\beta}$ with j1) — j4) for every α in the club set D and for every $\beta < \mu^*$. Let

$$M_{\alpha,\beta} = sk(\mu_\beta \cup C_\alpha)$$

for $\alpha \in D$ and $\beta < \mu^*$. We show that $(M_{\alpha,\beta})_{\alpha\in D,\beta<\mu^*}$ is as desired. It is clear that j1) and j2) hold. We need the following claim to show the other properties:

Claim 2.3.9.1. *$M_\alpha = sk(\alpha)$ for every $\alpha \in D$.*

Proof of the Claim. "$\subseteq$" is clear since $\mu_\beta \cup C_\alpha \subseteq \alpha$ for every $\alpha \in D$ and $\beta < \mu^*$. For "$\supseteq$", it is enough to show that $\alpha \subseteq M_\alpha$. Let $\gamma < \alpha$. By y1), there is $\gamma_1 \in C_\alpha$ such that $\gamma < \gamma_1$. Let $f \in M_{\alpha,0}$ be a surjection from μ to γ_1, and let $\delta < \mu$ be such that $f(\delta) = \gamma$. Then we have $\gamma = f(\delta) \in M_{\alpha,\beta^*} \subseteq M_\alpha$ for $\beta^* < \mu^*$ such that $\delta < \mu_{\beta^*}$. □

For j3), suppose that $\alpha \in D$ and $\mathrm{cf}(\alpha) \geq \kappa$. Let β^* be such that $|C_\alpha| < \mu_{\beta^*}$ and $[\alpha]^{<\kappa} \cap \{ C_{\alpha'} : \alpha' < \alpha,\ \alpha' \in M_{\alpha,\beta^*} \}$ dominates $[C_\alpha]^{<\kappa}$. The last property is possible by y2), Claim 1 and $\mu^* < \kappa$. We show that this β^* is as needed in j3). Let $\beta < \mu^*$ be such that $\beta^* \leq \beta$ and suppose that $x \in [M_{\alpha,\beta}]^{<\kappa}$. Then there are $u \in [\mu_\beta]^{<\kappa}$ and $v \in [C_\alpha]^{<\kappa}$ such that $x \subseteq sk(u \cup v)$. Since $\mu_\beta \in M_{\alpha,\beta}$ and $\mathrm{cf}([\mu_\beta]^{<\kappa}, \subseteq) = \mu_\beta$, there is $X \in M_{\alpha,\beta}$ such that $X \subseteq [\mu_\beta]^{<\kappa}$, $|X| = \mu_\beta$ and X is cofinal in $([\mu_\beta]^{<\kappa}, \subseteq)$. Since $\mu_\beta \subseteq M_{\alpha,\beta}$, it follows that $X \subseteq M_{\alpha,\beta}$. Hence there is $u' \in M_{\alpha,\beta} \cap [\mu_\beta]^{<\kappa}$ such that $u \subseteq u'$. On the other hand, by definition of β^*, $\mathrm{cf}(\alpha) \geq \kappa$, there is $\alpha' \in \alpha \cap M_{\alpha,\beta}$ such that $C_{\alpha'} \in [\alpha]^{<\kappa}$ and $v \subseteq C_{\alpha'}$. Thus, we have $x \subseteq sk(u \cup v) \subseteq sk(u' \cup C_{\alpha'})$. By regularity of κ, there is $\xi < \kappa$ such that $x \subseteq sk_\xi(u' \cup C_{\alpha'})$. But $sk_\xi(u' \cup C_{\alpha'}) \in M_{\alpha,\beta}$ and $|sk_\xi(u' \cup C_{\alpha'})| < \kappa$.

j4) follows immediately from Claim 2.3.9.1. □

Note that, in the proof above, the sequence $(M_{\alpha,\beta})_{\alpha<\mu^+,\beta<\mu^*}$ satisfies also:

j5) for $\alpha < \alpha' < \mu^+$ and $\beta < \mu^*$, there is $\beta' < \mu^*$ such that $M_{\alpha,\beta} \subseteq M_{\alpha',\beta'}$.

[Suppose that α, $\alpha' \in D$ are such that $\alpha < \alpha'$ and $\beta < \mu^*$. By Claim 2.3.9.1, there is $\beta' < \mu^*$ such that $\beta < \beta'$, $\alpha \in M_{\alpha',\beta'}$ and $\mathrm{otp}(C_\alpha) \leq \mu_{\beta'}$. Then we have $C_\alpha \in M_{\alpha',\beta'}$ and $C_\alpha \subseteq M_{\alpha',\beta'}$. Also $\mu_\beta \in M_{\alpha',\beta'}$ and $\mu_\beta \subseteq \mu_{\beta'} \subseteq M_{\alpha',\beta'}$. Hence it follows that $M_{\alpha,\beta} = sk(\mu_\beta \cup C_\alpha) \subseteq M_{\alpha',\beta'}$.]

Conversely, the existence of a (κ,μ)-dominating matrix (over some/any x) implies $\square^{***}_{\kappa,\mu}$:

Theorem 2.3.10. *Suppose that κ is a regular cardinal and $\mu > \kappa$ is such that $\mathrm{cf}(\mu) < \kappa$. If there exists a (κ,μ)-dominating matrix, then $\square^{***}_{\kappa,\mu}$ holds.*

Proof. Let $\mu^* = \mathrm{cf}(\mu)$ and $(M_{\alpha,\beta})_{\alpha<\mu^+,\beta<\mu^*}$ be a (κ,μ)-dominating matrix. Let $M_\alpha = \bigcup_{\beta<\mu^*} M_{\alpha,\beta}$ for each $\alpha < \mu^+$. For

$$X = \bigcup\{ [M_{\alpha,\beta}]^{<\kappa} \cap M_{\alpha,\beta} : \alpha < \mu^+,\ \beta < \mu^* \},$$

let $\{ C_{\alpha+1} : \alpha < \mu^+ \}$ be an enumeration of X. Let

$$\begin{aligned} D = \{ \alpha < \mu^+ : {} & M_\alpha \cap \mu^+ = \alpha, \\ & \{ C_{\alpha'+1} : \alpha' < \alpha \} \supseteq [M_{\alpha^\dagger,\beta}]^{<\kappa} \cap M_{\alpha^\dagger,\beta} \\ & \text{for every } \alpha^\dagger < \alpha,\ \beta < \mu^* \ \}. \end{aligned}$$

By j4), D is a club subset of μ^+. For $\alpha \in D$ with $\mathrm{cf}(\alpha) \geq \kappa$, let $C_\alpha = M_{\alpha,\beta_\alpha} \cap \alpha$ where $\beta_\alpha < \mu^+$ be such that $M_{\alpha,\beta_\alpha} \cap \alpha$ is cofinal in α (this is possible as $M_\alpha \cap \mu^+ = \alpha$ and $\mu^* < \kappa$) and that $[M_{\alpha,\beta_\alpha}]^{<\kappa} \cap M_{\alpha,\beta_\alpha}$ is cofinal in $[M_{\alpha,\beta_\alpha}]^{<\kappa}$ (possible by j3)). For $\alpha \in \mathrm{Lim}(\mu^+) \setminus \{ \alpha \in D : \mathrm{cf}(\alpha) \geq \kappa \}$, let C_α be anything, say $C_\alpha = \emptyset$. We claim that $(C_\alpha)_{\alpha<\mu^+}$ and D as above satisfy the conditions in the definition of $\square^{***}_{\kappa,\mu}$: y1) is clear by definition of C_α's. To show y2), let $\alpha \in D$ be such that $\mathrm{cf}(\alpha) \geq \kappa$ and $x \in [C_\alpha]^{<\kappa}$. Then, by the choice of β_α, there is $y \in [\alpha]^{<\kappa} \cap M_{\alpha,\beta_\alpha}$ such that $x \subseteq y$. By j4), there are $\alpha^\dagger < \alpha$ and $\beta^\dagger < \mu^*$ such that $y \in M_{\alpha^\dagger,\beta^\dagger}$. By definition of D, it follows that $y = C_{\alpha'+1}$ for some $\alpha' < \alpha$. This shows that $[\alpha]^{<\kappa} \cap \{ C_{\alpha'} : \alpha' < \alpha \}$ dominates $[C_\alpha]^{<\kappa}$. □

Thus, by Theorem 2.3.9 and Theorem 2.3.10, if $\mathrm{cf}([\lambda]^{<\kappa}, \subseteq) = \lambda$ for cofinally many $\lambda < \mu$, $\square^{***}_{\kappa,\mu}$ is equivalent to the existence of a (κ, μ)-dominating matrix. The assumption "$\mathrm{cf}([\lambda]^{<\kappa}, \subseteq) = \lambda$ for cofinally many $\lambda < \mu$" cannot be removed from this equivalence theorem since $\square^{***}_{\kappa,\mu}$ implies this. However, using the following weakening of the notion of dominating matrix, we obtain a characterization of $\square^{***}_{\kappa,\mu}$ in ZFC: for a regular cardinal κ and $\mu > \kappa$ such that $\mu^* = \mathrm{cf}(\mu) < \kappa$, let us call a matrix $(M_{\alpha,\beta})_{\alpha<\mu^+,\beta<\mu^*}$ of elementary submodels of $\mathcal{H}(\chi)$ for a sufficiently large χ, *a weak* (κ, μ)*-dominating matrix over* x, if it satisfies j1), j2), j4) for $M_\alpha = \bigcup_{\beta<\mu^*} M_{\alpha,\beta}$, $\alpha < \mu^+$, and

j3$^-$) if $\alpha < \mu^+$ is such that $\mathrm{cf}(\alpha) \geq \kappa$, then there is $\beta^* < \mu^*$ such that, for every $\beta^* \leq \beta < \mu^*$, $[M_{\alpha,\beta}]^{<\kappa} \cap M_\alpha$ is cofinal in $([M_{\alpha,\beta}]^{<\kappa}, \subseteq)$.

Since $\mu^* < \kappa$, the condition above is equivalent to:

j3*) if $\alpha < \mu^+$ is such that $\mathrm{cf}(\alpha) \geq \kappa$, then there is $\beta^* < \mu^*$ such that, for every $\beta^* \leq \beta < \mu^*$, there is $\beta' < \mu^*$ such that $[M_{\alpha,\beta}]^{<\kappa} \cap M_{\alpha,\beta'}$ is cofinal in $([M_{\alpha,\beta}]^{<\kappa}, \subseteq)$.

Theorem 2.3.11. *Suppose that κ is a regular cardinal and $\mu > \kappa$ is such that $\mu^* = \mathrm{cf}(\mu) < \kappa$. Then $\square^{***}_{\kappa,\mu}$ holds if and only if there is a weak (κ, μ)-dominating matrix over some/any x.*

Proof. For the forward direction the proof is almost the same as that of Theorem 2.3.9. We let $(\mu_\beta)_{\beta<\mu^*}$ here merely an increasing sequence of regular cardinals with the limit μ. Then $(M_{\alpha,\beta})_{\alpha\in D,\beta<\mu^*}$ is constructed just as in the proof of Theorem 2.3.9. Lemma 2.3.7 is then used to see that j3$^-$) is satisfied by this matrix. For the converse, just the same proof as that of Theorem 2.3.10 will do. $\square$

Existence of a (weak) dominating-matrix is not a theorem in ZFC: we show in section 2.3.3 that the Chang's conjecture for $\aleph_\omega$ together with $2^{\aleph_\omega} = \aleph_{\omega+1}$ implies that there is no $(\aleph_n, \aleph_\omega)$-dominating matrix for any $n \geq 1$.

2.3.2. A characterization of the κ-Freese-Nation property. The following game over a partial ordering P was considered in [**46, 45**].

Definition 2.3.12. Let $\mathcal{G}^\kappa(P)$ be the following game played by Players I and II: in a play in $\mathcal{G}^\kappa(P)$, Players I and II choose subsets X_α and Y_α of P of cardinality less than κ alternately for $\alpha < \kappa$ such that

$$X_0 \subseteq Y_0 \subseteq X_1 \subseteq Y_1 \subseteq \cdots \subseteq X_\alpha \subseteq Y_\alpha \subseteq \cdots \subseteq X_\beta \subseteq Y_\beta \subseteq \cdots$$

for $\alpha \leq \beta < \kappa$. Thus a play in $\mathcal{G}^\kappa(P)$ looks like

$$\begin{aligned} &\textit{Player I} : X_0,\ X_1,\ \ldots,\ X_\alpha,\ \ldots \\ &\textit{Player II} : Y_0,\ Y_1,\ \ldots,\ Y_\alpha,\ \ldots \end{aligned}$$

where $\alpha < \kappa$. Player II wins the play if $\bigcup_{\alpha<\kappa} X_\alpha = \bigcup_{\alpha<\kappa} Y_\alpha$ is a κ-sub-ordering of P. Let us call a strategy τ for Player II *simple* if, in τ, each Y_α is decided from the information of the set $X_\alpha \subseteq P$ alone (i.e. also independent of α).

Another notion we need here is the following generalization of V_κ-likeness.

Definition 2.3.13. Let κ be regular and χ be sufficiently large. For $\mathcal{D} \subseteq \{ M \prec \mathcal{H}(\chi) : |M| < \kappa \}$, we say that $M \in [\mathcal{H}(\chi)]^\kappa$ is $\mathcal{D}$*-approachable* if there is an increasing sequence $(D_\alpha)_{\alpha<\kappa}$ of elements of $\mathcal{D}$ such that

a) $D_\alpha \cup \{ D_\alpha \} \subseteq D_{\alpha+1}$ for every $\alpha < \kappa$; and
b) $M = \bigcup_{\alpha<\kappa} D_\alpha$.

Clearly $M \prec \mathcal{H}(\chi)$ is V_κ-like if and only if M is $\mathcal{D}$-approachable for $\mathcal{D} = \{ M \prec \mathcal{H}(\chi) : |M| < \kappa \}$.

A slightly weaker version of the following theorem was announced in [**46**]:

Theorem 2.3.14. *Let κ be a regular uncountable cardinal and $\kappa \leq \lambda$. Suppose that*

i) $([\mu]^{<\kappa}, \subseteq)$ *has a cofinal subset of cardinality μ for every μ such that $\kappa < \mu < \lambda$ and* $\mathrm{cf}(\mu) \geq \kappa$ *; and*

ii) $\square^{***}_{\kappa,\mu}$ *holds for every* μ *such that* $\kappa \le \mu < \lambda$ *and* $\mathrm{cf}(\mu) < \kappa$.

Then, for a partial ordering P *of cardinality* $\le \lambda$, *the following are equivalent:*

1) P *has the* κ*-FN;*

2) *Player II has a simple winning strategy in* $\mathcal{G}^\kappa(P)$ *;*

3) *for some, or equivalently any sufficiently large* χ, *and any* V_κ*-like* $M \prec \mathcal{H}(\chi)$ *with* P, $\kappa \in M$, *we have* $P \cap M \le_\kappa P$ *;*

4) *for some, or equivalently any sufficiently large* χ, *there is* $\mathcal{D} \subseteq [\mathcal{H}(\chi)]^{<\kappa}$ *such that* $\mathcal{D}$ *is cofinal in* $[\mathcal{H}(\chi)]^{<\kappa}$ *and for any* $\mathcal{D}$*-approachable* $M \subseteq \mathcal{H}(\chi)$, *we have* $P \cap M \le_\kappa P$.

Note that $\neg 0^\#$ implies the conditions i) and ii). Also note that, for every $\lambda < \kappa^{+\omega}$, the condition i) holds in ZFC. Hence the characterization above holds for partial orderings of cardinality $\le \kappa^{+\omega}$ without any additional assumptions.

Proof. A proof of 1) $\Rightarrow$ 2) $\Rightarrow$ 3) is given in [**46**]. For 3) $\Rightarrow$ 4), suppose that P satisfies 3). Then P together with $\mathcal{D} = \{\, M \prec \mathcal{H}(\chi) \,:\, |M| < \kappa,\ P, \kappa \in M \,\}$ satisfies 4). The proof of 4) $\Rightarrow$ 1) is done by induction on $\nu = |P| \le \lambda$. If $\nu \le \kappa$, then P has the κ-FN by Lemma 2.3.2. For $\nu > \kappa$, let P and $\mathcal{D}$ be as in 4) and assume that 4) $\Rightarrow$ 1) holds for every partial ordering of cardinality $< \nu$. We need the following claims:

Claim 2.3.14.1. *Let* χ^* *be sufficiently large above* χ. *Suppose that* M *is an elementary submodel of* $\mathcal{H}(\chi^*)$ *such that* P, $\mathcal{H}(\chi)$, $\mathcal{D} \in M$, $\kappa + 1 \subseteq M$ *and* $[M]^{<\kappa} \cap M$ *is cofinal in* $[M]^{<\kappa}$ *with respect to* $\subseteq$. *Then we have* $P \cap M \le_\kappa P$.

Proof of the Claim. Suppose not. then there is $b \in P$ such that either

a) $(P \cap M) \restriction b$ has no cofinal subset of cardinality $< \kappa$; or
b) $(P \cap M) \uparrow b$ has no coinitial subset of cardinality $< \kappa$.

To be definite, let us assume that we have the case a) — for the case b), just the same argument will do. We can construct an increasing sequence $(N_\alpha)_{\alpha<\kappa}$ of elements of $\mathcal{D}$ such that

c) $N_\alpha \in M$ and $|N_\alpha| < \kappa$ for $\alpha < \kappa$ (since $\kappa + 1 \subseteq M$, it follows that $N_\alpha \subseteq M$) ;
d) $N_\alpha \in N_{\alpha+1}$ for every $\alpha < \kappa$;
e) $(P \cap N_\alpha) \restriction b$ is not cofinal in $(P \cap N_{\alpha+1}) \restriction b$ for every $\alpha < \kappa$.

Then $N = \bigcup_{\alpha<\kappa} N_\alpha$ is $\mathcal{D}$-approachable elementary submodel of $\mathcal{H}(\chi)$ by c) and d). Hence, by 4), we have $P \cap M \le_\kappa P$. But, by e), $(P \cap N) \restriction b$ has no cofinal subset of cardinality $< \kappa$. This is a contradiction.

To see that the construction of N_γ is possible at a limit $\gamma < \kappa$, assume that N_α, $\alpha < \gamma$ have been constructed in accordance with c),d) and e). Let $N' = \bigcup_{\alpha<\gamma} N_\alpha$. By c), we have $N' \subseteq M$ and $|N'| < \kappa$. Since $[M]^{<\kappa} \cap M$ is cofinal in $[M]^{<\kappa}$, there is some $N'' \in M$ such that $N' \subseteq N'' \subseteq \mathcal{H}(\chi)$ and $|N''| < \kappa$. Hence by elementarity of M and by $\mathcal{D} \in M$ there is $N''' \in M \cap \mathcal{D}$ such that $N'' \subseteq N'''$. Clearly, we may let $N_\gamma = N'''$. For the construction at a successor step, assume that N_α have been chosen in accordance with c),d) and e). By assumption, there is $c \in (P \cap M) \restriction b$ such that there is no $c' \in (P \cap N_\alpha) \restriction b$ with $c \le c'$. By elementarity of M there is $N^* \in M \cap \mathcal{D}$ such that $N_\alpha \cup \{\, N_\alpha, c \,\} \subseteq N^*$ and $|N^*| < \kappa$. Then $N_{\alpha+1} = N^*$ is as desired. $\square$

Claim 2.3.14.2. *If* $Q \le_\kappa P$, *then for every* $\mathcal{D}$*-approachable* $M \prec \mathcal{H}(\chi)$ *with* $Q \in M$ *we have* $Q \cap M \le_\kappa Q$. *In particular, such* Q *also satisfies the condition* 4).

Proof of the Claim. Let $M \prec \mathcal{H}(\chi)$ be $\mathcal{D}$-approachable with $Q \in M$. By assumption, we have $P \cap M \le_\kappa P$. By elementarity of M and since $Q \in M$, we have $Q \cap M \le_\kappa P \cap M$. It follows that $Q \cap M \le_\kappa P$ and hence $Q \cap M \le_\kappa Q$. Now, let $\mathcal{D}_0 = \{\, M \in \mathcal{D} \,:\, Q \in M \,\}$. Then it is clear that Q satisfies the condition 4) with $\mathcal{D}_0$ in place of $\mathcal{D}$. $\square$

Now we are ready to prove the induction steps.
Case I : ν is a limit cardinal or $\nu = \mu^+$ with $\mathrm{cf}(\mu) \ge \kappa$. Let $\nu^* = \mathrm{cf}\,\nu$. Then, by i), we can find an increasing sequence $(M_\alpha)_{\alpha<\nu^*}$ of elementary submodels of $\mathcal{H}(\chi^*)$ such that, for every $\alpha < \nu^*$,

$|M_\alpha| < \nu$ and M_α satisfies the conditions in Claim 2.3.14.1; and $P \subseteq \bigcup_{\alpha<\nu^*} M_\alpha$. By Claim 2.3.14.1, $P \cap M_\alpha \leq_\kappa P$ for every $\alpha < \nu^*$. For $\alpha < \nu^*$ let

$$P_\alpha = \begin{cases} P \cap M_\alpha, & \text{if } \alpha \text{ is a successor,} \\ P \cap (\bigcup_{\beta<\alpha} M_\beta), & \text{otherwise.} \end{cases}$$

Then $(P_\alpha)_{\alpha<\nu^*}$ is a continuously increasing sequence of sub-orderings of P such that $|P_\alpha| < \nu$ for every $\alpha < \nu^*$ and $P = \bigcup_{\alpha<\nu^*} P_\alpha$. We have also $P_\alpha \leq_\kappa P$ for every $\alpha < \nu^*$: for a successor $\alpha < \nu^*$ this is clear. If a limit $\alpha < \nu^*$ has cofinality $< \kappa$ then P_α can be represented as the union of an increasing sequence of $< \kappa$ many κ-sub-ordering of P and hence $P_\alpha \leq_\kappa P$. If $\alpha < \nu^*$ is a limit with cofinality $\geq \kappa$, then $P_\alpha = P \cap M$ where $M = \bigcup_{\beta<\alpha} M_\beta$. Now it is clear that M satisfies the conditions in Claim 2.3.14.1. Hence we again obtain that $P_\alpha = P \cap M \leq_\kappa P$. Now, by Claim 2.3.14.2, each of P_α, $\alpha < \nu^*$ satisfies the condition 4) and hence, by induction hypothesis, has the κ-FN. Thus, by Lemma 2.3.3, P also has the κ-FN.

Case II: $\nu = \mu^+$ with $\mathrm{cf}(\mu) < \kappa$. Let $\mu^* = \mathrm{cf}(\mu)$. Without loss of generality we may assume that the underlying set of P is ν. By Theorem 2.3.9, there is a (κ, μ)-dominating matrix $(M_{\alpha,\beta})_{\alpha<\nu,\beta<\mu^*}$ over $(P, \mathcal{H}(\chi))$. For $\alpha < \nu$ and $\beta < \mu^*$, let $P_{\alpha,\beta} = P \cap M_{\alpha,\beta}$ and $P_\alpha = \bigcup_{\beta<\mu^*} P_{\alpha,\beta}$. By j4), the sequence $(P_\alpha)_{\alpha<\nu}$ is continuously increasing and $\bigcup_{\alpha<\nu} P_\alpha = P$. $|P_\alpha| \leq \mu$ for every $\alpha < \nu$ by j1).

Claim 2.3.14.3. *$P_\alpha \leq_\kappa P$ for every $\alpha < \nu$.*

Proof of the Claim. For $\alpha < \nu$ such that $\mathrm{cf}(\alpha) \geq \kappa$, we have $P_{\alpha,\beta} \leq_\kappa P$ for every sufficiently large $\beta < \mu^*$ by j3) and Claim 2.3.14.1. Since $\mu^* < \kappa$, it follows that $P_\alpha \leq_\kappa P$. If $\mathrm{cf}(\alpha) < \kappa$, then, by the argument above, we have $P_{\alpha'} \leq_\kappa P$ for every $\alpha' < \alpha$ with $\mathrm{cf}(\alpha') \geq \kappa$. Since P_α can be represented as the union of $< \kappa$ many of such $P_{\alpha'}$'s, it follows again that $P_\alpha \leq_\kappa P$. □

Now, by Claim 2.3.14.2, each of P_α, $\alpha < \nu$ satisfies the condition of 4). Hence, by induction hypothesis, they have the κ-FN. By Lemma 2.3.3, it follows that P also has the κ-FN. □

In [**46**], it is shown that $\langle \mathcal{P}(\omega_1), \subseteq \rangle$ does not have the weak Freese-Nation property. If a complete Boolean algebra B does not have the c.c.c., then $\langle \mathcal{P}(\omega_1), \subseteq \rangle$ can be completely embedded into B. Hence, in this case, B can not have the weak Freese-Nation property (see Proposition 2.4.1 a) or [**46**, Lemma 2.7]).

Corollary 2.3.15. *Suppose that κ and λ satisfy* i), ii) *in Theorem 2.3.14 and $2^{<\kappa} = \kappa$. Then:*

a) *Every κ-cc complete Boolean algebra of cardinality $\leq \lambda$ has the κ-FN.*

b) *For any μ such that $\mu^{<\kappa} \leq \lambda$, the partial ordering $([\mu]^{<\kappa}, \subseteq)$ has the κ-FN.*

Proof. Let χ be sufficiently large. For a), let B be a κ-cc complete Boolean algebra. We show that B satisfies 3) in Theorem 2.3.14. Let $M \prec \mathcal{H}(\chi)$ be V_κ-like with B, $\kappa \in M$. By $2^{<\kappa} = \kappa$, Lemma 2.3.8 and by the remark after the lemma, we have $[M]^{<\kappa} \subseteq M$. Hence $B \cap M$ is a complete subalgebra of B. It follows that $B \cap M \leq_\kappa B$. For b), let $M \prec \mathcal{H}(\chi)$ be V_κ-like with λ, $\kappa \in M$. Then as above we have $[M]^{<\kappa} \subseteq M$. Hence, letting $X = \lambda \cap M$, we have $[\lambda]^{<\kappa} \cap M = [X]^{<\kappa} \leq_\kappa [\lambda]^{<\kappa}$. □

2.3.3. Chang's Conjecture for $\aleph_\omega$. Recall that $(\kappa, \lambda) \twoheadrightarrow (\mu, \nu)$ is the following assertion:

For any structure $\mathcal{A} = (A, U, \ldots)$ of countable signature with $|A| = \kappa$, $U \subseteq A$ and $|U| = \lambda$, there is an elementary substructure $\mathcal{A}' = (A', U', \ldots)$ of $\mathcal{A}$ such that $|A'| = \mu$ and $|U'| = \nu$.

In [**80**], a model of ZFC + GCH + Chang's Conjecture for $\aleph_\omega$, i.e. $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$, is constructed starting from a model with a cardinal having a property slightly stronger than huge. The following theorem together with Corollary 2.3.15 shows that the $\aleph_1$-FN of the partial ordering $([\aleph_\omega]^{\aleph_0}, \subseteq)$ is independent from ZFC (or even from ZFC + GCH).

Theorem 2.3.16. *Suppose that $(\aleph_\omega)^{\aleph_0} = \aleph_{\omega+1}$ and $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$. Then $([\aleph_\omega]^{\aleph_0}, \subseteq)$ does not have the $\aleph_1$-FN.*

Proof. Assume to the contrary that there is an $\aleph_1$-FN mapping $F : [\aleph_\omega]^{\aleph_0} \to [[\aleph_\omega]^{\aleph_0}]^{\aleph_0}$. Let us fix an enumeration $(b_\alpha)_{\alpha<\aleph_{\omega+1}}$ of $[\aleph_\omega]^{\aleph_0}$ and consider the structure:

$$\mathcal{A} = (\aleph_{\omega+1}, \aleph_\omega, \leq, E, f, g, h),$$

where

1) $\leq$ is the canonical ordering on $\aleph_{\omega+1}$;
2) $E = \{ (\alpha, \beta) : \alpha \in \aleph_\omega, \beta \in \aleph_{\omega+1}, \alpha \in b_\beta \}$;
3) $f : \aleph_{\omega+1} \times \aleph_{\omega+1} \to \aleph_{\omega+1}$ is such that, for each $\alpha \in \aleph_{\omega+1}$,

$$F(b_\alpha) = \{ b_{f(\alpha,n)} : n \in \omega \};$$

4) $g : \aleph_{\omega+1} \times \aleph_{\omega+1} \to \aleph_\omega$ is such that, for each $\alpha \in \aleph_{\omega+1}$, $g(\alpha, \cdot) \restriction \alpha$ is an injective mapping from α to $\aleph_\omega$; and
5) $h : \aleph_{\omega+1} \times \aleph_{\omega+1} \times \aleph_{\omega+1} \to \omega + 1$ is such that for each α, $\beta \in \aleph_{\omega+1}$, $h(\alpha, \beta, \cdot) \restriction (b_\alpha \cap b_\beta)$ is injective.

Note that, by 5) and since ω is definable in $\mathcal{A}$, we can express "$b_\alpha \cap b_\beta$ is finite" as a formula $\varphi(\alpha, \beta)$ in the language of $\mathcal{A}$. Now, applying the Chang's conjecture for $\mathcal{A}$ with $A = \aleph_{\omega+1}$ and $U = \aleph_\omega$, we obtain elementary substructure $\mathcal{A}' = (A', U', \leq', E', f', g', h')$ of $\mathcal{A}$ such that $|A'| = \aleph_1$ and $|U'| = \aleph_0$.

Claim 2.3.16.1. *$otp(A') = \omega_1$.*

Proof. By 4) and elementarity of $\mathcal{A}'$, every initial segment of A' can be mapped into U' injectively and hence countable. Since $|A'| = \aleph_1$, it follows that $otp(A') = \omega_1$. □

Claim 2.3.16.2. *For any $\alpha < \aleph_{\omega+1}$, there is $\gamma < \aleph_{\omega+1}$ such that $b_\beta \cap b_\gamma$ is finite for every $\beta < \alpha$.*

Proof. Since $|\alpha| \leq \aleph_\omega$, we can find a partition $(I_n)_{n\in\omega}$ of α such that $|I_n| < \aleph_\omega$ for every $n < \omega$. For $n < \omega$, let $\eta_n = \min(\aleph_\omega \setminus \bigcup\{ b_\xi : \xi \in \bigcup_{m\leq n} I_m \})$. Let $z = \{ \eta_n : n \in \omega \}$ and $\gamma < \aleph_{\omega+1}$ be such that $b_\gamma = z$. For any $\beta < \alpha$, if $\beta \in I_{m_0}$ for some $m_0 < \omega$, then we have $b_\beta \cap b_\gamma \subseteq \{ \eta_n : n < m_0 \}$. Thus this γ is as desired. □

Claim 2.3.16.3. *For any countable $I \subseteq A'$, there is $\gamma \in A'$ such that $b_\beta \cap b_\gamma$ is finite for every $\beta \in I$.*

Proof. By Claim 2.3.16.1, there is $\alpha \in A'$ such that $I \subseteq \alpha$. By elementarity of $\mathcal{A}'$, the formula with the parameter α expressing the assertion of Claim 2.3.16.2 for this α holds in $\mathcal{A}'$. Hence there is some $\gamma \in A'$ such that $b_\beta \cap b_\gamma$ is finite for every $\beta \in A' \cap \alpha$. □

Let

$$I = \{ \xi \in A' : b_\xi \in F(U') \}.$$

Then I is countable. Hence, by Claim 2.3.16.3, there is $\gamma \in A'$ such that $b_\beta \cap b_\gamma$ is finite for every $\beta \in I$. As $b_\gamma \subseteq U'$ (this holds in virtue of $h(\gamma, \gamma, \cdot)$), there is $b \in F(b_\gamma) \cap F(U')$ such that $b_\gamma \subseteq b \subseteq U'$. Let $b = b_{\xi_0}$. Then $\xi_0 \in I$ and $|b_\gamma \cap b_{\xi_0}| = |b_\gamma| = \aleph_0$. This is a contradiction to the choice of γ. □

At this point it is unclear if the assumption of Theorem 2.3.16 yields a counter example to Corollary 2.3.15, a) for $\kappa = \aleph_1$. Or, more generally:

Question 2.3.17 ([**7**, Problem 2])**.** *Is there a model of* ZFC + GCH *where some ccc complete Boolean algebra does not have the $\aleph_1$-FN?*

Of course, we need the consistency strength of some large cardinal to obtain such a model by Corollary 2.3.15.

We answer this question in the positive in Theorem 2.4.7.

The following corollary slightly improves [**43**, Theorem 4.1].

Corollary 2.3.18. *Suppose that $(\aleph_\omega)^{\aleph_0} = \aleph_{\omega+1}$ and $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$. Then the equivalence of the assertions in Theorem 2.3.14 fails. Hence we have $\neg\square^{***}_{\aleph_1,\aleph_\omega}$ under these assumptions.*

Proof. By Theorem 2.3.16 and Corollary 2.3.15, b). □

Similarly we can prove $\neg\square^{***}_{\aleph_n,\aleph_\omega}$ for every $n \in \omega$ under the assumption of $2^{\aleph_\omega} = \aleph_{\omega+1}$ and $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$.

2.3.4. Cohen models. Let V be our ground model and let G be a V-generic filter over $P = \mathrm{Fn}(\tau, 2)$ for some τ. Suppose that B is a ccc complete Boolean algebra in $V[G]$. Without loss of generality we may assume that the underlying set of B is a set X in V. B is said to be *tame* if there is a P-name $\dot{\leq}$ of partial ordering of B and a mapping $t : X \to [\tau]^{\aleph_0}$ in V such that, for every $p \in P$ and x, $y \in X$, if $p \Vdash_P$ "$x \dot{\leq} y$", then $p \restriction (t(x) \cup t(y)) \Vdash_P$ "$x \dot{\leq} y$". A lot of 'natural' ccc complete Boolean algebras in $V[G]$ are contained in the class of tame Boolean algebras:

Lemma 2.3.19. *Let G be as above. Suppose that $V[G] \models$ "B is a ccc complete Boolean algebra" and either:*

i) *there is a Boolean algebra B' in V such that B' is dense subalgebra of B in $V[G]$; or*
ii) *B is the completion of a Suslin forcing in $V[G]$.*

Then B is tame. $\square$

For Suslin forcing, see e.g. [**24**].

Theorem 2.3.20. *Let P, G be as above and λ an infinite cardinal. Assume that, in V,*

i) *$\mu^{\aleph_0} = \mu$ for every regular uncountable μ such that $\mu < \lambda$; and*
ii) *$\square^{***}_{\aleph_1,\mu}$ holds for every μ such that $\aleph_0 < \mu < \lambda$ and $\mathrm{cf}(\mu) = \omega$.*

Then, for any tame ccc complete Boolean algebra B in $V[G]$ of cardinality $\leq \lambda$ we have $V[G] \models$ "B has the $\aleph_1$-FN".

Proof. Let X, $\dot{\leq}$ and t be as in the definition of tameness for B above. We may assume $\Vdash_P$ "X is the underlying set of $\dot{B}$" and $\Vdash_P$ "$\dot{B}$ is a ccc complete Boolean algebra" where $\dot{B}$ is a P-name for B. Let χ be sufficiently large. The following is the key lemma to the proof:

Claim 2.3.20.1. *Suppose that $M \prec \mathcal{H}(\chi)$ is such that τ, X, $\dot{\leq}$, $t \in M$ and $[M]^{\aleph_0} \subseteq M$. Let $I = \tau \cap M$, $P' = \mathrm{Fn}(I, 2)$, $G' = G \cap P'$, $X' = X \cap M$ and $\dot{\leq}' = \dot{\leq} \cap M$. Then*

a) *$\dot{\leq}'$ is a P'-name, $\Vdash_P$ "$B' = (X', \dot{\leq}')$ is a ccc complete Boolean algebra", B' is tame (in $V[G']$) and the infinite sum Σ^B in $V[G]$ is an extension of the infinite sum $\Sigma^{B'}$ in $V[G']$.*

b) *$\Vdash_P$ "$(X', \dot{\leq}') \leq_\sigma (X, \dot{\leq})$".*

Proof of the Claim. a): It is easy to see that $\Vdash_{P'}$ "$(X', \dot{\leq}')$ is a Boolean algebra" and $\Vdash_P$ "$(X', \dot{\leq}')$ is a subalgebra of $(X, \dot{\leq})$". Since $\Vdash_P$ "$(X, \dot{\leq})$ has the ccc", it follows that $\Vdash_{P'}$ "$(X', \dot{\leq}')$ has the ccc". By elementarity of M, it is also easy to see that $t \restriction X'$ witnesses the tameness of B' in $V[G']$. To see that $\Vdash_{P'}$ "$(X', \dot{\leq}')$ is complete" it is enough to see that every countable subset of B' has its supremum in $V[G']$. Let $\dot{C}$ be a P'-name of countable subset of X'. Without loss of generality, we may assume that $\dot{C}$ is countable and consists of sets of the form $(p, \check{x})$ where $p \in P'$ and $x \in X'$. Since $[M]^{\aleph_0} \subseteq M$, $\dot{C} \in M$. Clearly, we have $M \models$ "$\dot{C}$ is a P-name of countable subset of X". Hence, $M \models$ "$\exists p \in P\ \exists y \in X (p \Vdash_P$ "$\Sigma\dot{C} = y$")". Let $p \in P$ and $y \in X$ be such elements of M. Then $p \in P'$ and $y \in X'$. By elementarity of M, we have $p \Vdash_P$ "$\Sigma^B\dot{C} = y$". On the other hand, from $M \models$ "$p \Vdash_P$ "$\Sigma\dot{C} = y$"" it follows that $p \Vdash_{P'}$ "$\Sigma^{B'}\dot{C} = y$".

b): By assumption, for $x \in X$ and $y \in X'$, we have $y \leq x$ in $V[G]$, if and only if there is $p \in G$, $\mathrm{dom}(p) \subseteq t(x) \cup t(y)$ such that $p \Vdash_P$ "$y \dot{\leq} x$". For $q \in G \cap \mathrm{Fn}(t(y), 2)$, the set $U_q = \{\, y \in X' : \exists p \in G' \,(p \cup q \Vdash_P$ "$y \dot{\leq} x$")$\,\}$ is in $V[G']$. Hence, by a), $\Sigma^B U_q$ is an element in B'. Since X' is closed under t, it follows that $\{\, \Sigma^B U_q : q \in G \cap \mathrm{Fn}(t(y), 2) \,\}$ is cofinal in $B' \restriction y$. $\square$

Now, let $\nu = |X|$. Without loss of generality, we may assume that $X = \nu$. We show by induction on ν that $\Vdash_P$ "$\dot{B}$ has the $\aleph_1$-FN". For $\nu \leq \aleph_1$ the assertion follows from Lemma 2.3.2 (applied in V^P). In the induction steps, we mimic the proof of Theorem 2.3.14. Let χ be sufficiently large.

Case I: ν is a limit cardinal or $\nu = \mu^+$ for some μ with $\mathrm{cf}(\mu) \geq \omega_1$. By i), we can construct a continuously increasing sequence $(M_\alpha)_{\alpha<\nu}$ of elementary submodels of $\mathcal{H}(\chi)$ such that

0) τ, ν, $\dot{\leq}$, $t \in M_0$;
1) $|M_\alpha| < \nu$ for every $\alpha < \nu$;
2) $[M_{\alpha+1}]^{\aleph_0} \subseteq M_{\alpha+1}$ for every $\alpha < \nu$ (note that it follows that the inclusion also holds for every limit $< \nu$ of cofinality $\geq \omega_1$); and
3) $\nu \subseteq \bigcup_{\alpha<\nu} M_\alpha$.

For each $\alpha < \nu$, let $X_\alpha = X \cap M_\alpha$ and $\dot{\leq}^\alpha = \dot{\leq} \cap M_\alpha$ and let $\dot{B}_\alpha$ be the P-name corresponding to $(X_\alpha, \dot{\leq}^\alpha)$. By Claim 2.3.20.1, we have $\Vdash_P$"$\dot{B}_\alpha$ is ccc complete Boolean algebra and $\dot{B}_\alpha \leq_{\aleph_1} \dot{B}$", for all $\alpha < \nu$ such that either α is a successor or of cofinality $\geq \omega_1$. By induction hypothesis, we have $\Vdash_P$"$\dot{B}_\alpha$ has the $\aleph_1$-FN" for such α's. Hence by Lemma 2.3.3 and the remark after the lemma (applied in V^P) it follows that $\Vdash_P$"$\dot{B}$ has the $\aleph_1$-FN".

Case II: $\nu = \mu^+$ for a μ with $\mathrm{cf}(\mu) = \omega$. By ii), there is an $(\aleph_1, \mu)$-dominating matrix $(M_{\alpha,n})_{\alpha<\nu, n<\omega}$ over $(\tau, \nu, \dot{\leq}, t)$. For $\alpha < \nu$, let $M_\alpha = \bigcup_{n<\omega} M_{\alpha,n}$. For $\alpha < \nu$ and $n < \omega$, let $X_{\alpha,n} = X \cap M_{\alpha,n}$, $\dot{\leq}^{\alpha,n} = \dot{\leq} \cap M_{\alpha,n}$ and $\dot{B}_{\alpha,n}$ be the P-name corresponding to $(X_{\alpha,n}, \dot{\leq}^{\alpha,n})$. Likewise, let $X_\alpha = X \cap M_\alpha$, $\dot{\leq}^\alpha = \dot{\leq} \cap M_\alpha$ and $\dot{B}_\alpha$ be the P-name corresponding to $(X_\alpha, \dot{\leq}^\alpha)$. Then we have $X_\alpha = \bigcup_{n<\omega} X_{\alpha,n}$, $\dot{\leq}^\alpha = \bigcup_{n<\omega} \dot{\leq}^{\alpha,n}$ and $\Vdash_P$"$\dot{B}_\alpha = \bigcup_{n<\omega} \dot{B}_{\alpha,n}$". By Lemma 2.3.8 and i), j3') holds for the dominating matrix. Hence, by Claim 2.3.20.1, we have $\Vdash_P$"$\dot{B}_{\alpha,n} \leq_{\aleph_1} \dot{B}_\alpha$ and $\dot{B}_{\alpha,n}$ is a ccc complete Boolean algebra" for every $\alpha < \nu$ with $\mathrm{cf}(\alpha) > \omega$ and every sufficiently large $n < \omega$. By induction hypothesis, it follows that $\Vdash_P$"$\dot{B}_{\alpha,n}$ has the $\aleph_1$-FN" for such α and n. By Lemma 2.3.3 (applied in V^P) it follows that $\Vdash_P$"$\dot{B}_\alpha$ has the $\aleph_1$-FN" for every $\alpha < \nu$ with $\mathrm{cf}(\alpha) > \omega$. Hence again by Lemma 2.3.3 and the remark after that (applied in V^P) we obtain that $\Vdash_P$"$\dot{B}$ has the $\aleph_1$-FN". □

Corollary 2.3.21. *Suppose that $V = L$ holds and let G be V-generic over $P = \mathrm{Fn}(\tau, 2)$ for some τ. Then (among others) the following ccc complete Boolean algebras have the $\aleph_1$-FN: $\mathcal{C}_\kappa$ ($\cong$ the completion of $\mathrm{Fn}(\kappa, 2)$) for any κ; $\mathcal{P}(\omega)$ (hence also $\mathcal{P}(\omega)/\mathrm{fin}$); Borel$(\mathbb{R})$/Null-sets.* □

Theorem 2.3.20 raises the following question:

Question 2.3.22 ([**7**, Problem 3])**.** *Assume that $V[G]$ is a model obtained by adding Cohen reals to a model of* ZFC + CH. *Is it true that $\mathcal{P}(\omega)$ has the $\aleph_1$-FN in $V[G]$?*

By Theorem 2.3.20, the answer to this question is positive if the number of added Cohen reals is less than $\aleph_\omega$.

In Theorem 2.4.8, we answer this question in the negative.

Question 2.3.23. *Does Theorem 2.3.20 hold also without the assumption of tameness?*

Or, more generally:

Question 2.3.24 ([**7**, Problem 5])**.** *Are the following equivalent?*

(i) $\mathcal{P}(\omega)$ has the $\aleph_1$-FN;
(ii) every ccc complete Boolean algebra has the $\aleph_1$-FN.

The answer is no, as we will see in Corollary 2.4.4.

2.4. Weak Freeze-Nation property of Boolean algebras

(This section is based on [3])

To answer Questions 2.3.17, 2.3.22, 2.3.23. and 2.3.24 we prove the following in this section.

(a) *In a model obtained by adding $\aleph_2$ Cohen reals, there is always a c.c.c. complete Boolean algebra without the weak Freese-Nation property.* (See Theorem 2.4.3)
(b) *Modulo the consistency strength of a supercompact cardinal, the existence of a c.c.c. complete Boolean algebra without the weak Freese-Nation property is consistent with* GCH. (See Theorem 2.4.7)

(c) *Modulo consistency of* $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$*, it is consistent with* GCH *that* $\mathbb{C}(\aleph_\omega)$ *does not have the weak Freese-Nation property and hence the assertion in (c) does not hold, and also that adding* $\aleph_\omega$ *Cohen reals destroys the weak Freese-Nation property of* $\langle \mathcal{P}(\omega), \subseteq \rangle$*.* (See Theorem 2.4.8)

After giving a negative answer to Question 2.3.23, the following problem still remains: *For which Boolean algebra* B*, the weak Freese-Nation property of* B *is equivalent with the weak Freese-Nation property of* $\langle \mathcal{P}(\omega), \subseteq \rangle$ *?* The following was proved in **[3]** :

(d) *If a weak form of* $\square_\mu$ *and* $\mathrm{cof}([\mu]^{\aleph_0}, \subseteq) = \mu^+$ *hold for each* $\mu > \mathrm{cf}(\mu) = \omega$*, then the weak Freese-Nation property of* $\langle \mathcal{P}(\omega), \subseteq \rangle$ *is equivalent to the weak Freese-Nation property of any of* $\mathbb{C}(\kappa)$ *or* $\mathbb{R}(\kappa)$ *for uncountable* κ*.*

The following criteria of the weak Freese-Nation property are used in the later sections. A partial ordering Q is said to be a *retract* of a partial ordering P if there are order preserving mappings $i : Q \to P$ and $j : P \to Q$ such that $j \circ i = id_Q$.

Q is said to be a *σ-subordering of* P (notation: $Q \leq_\sigma P$) if, for every $p \in P$, $Q{\restriction} p = \{q \in Q : q \leq p\}$ has a countable cofinal subset and $Q \uparrow p = \{q \in Q : q \geq p\}$ has a countable coinitial subset. Note that if C is a complete subalgebra of a complete Boolean algebra B (notation: $C \leq_c B$) or a countable union of complete subalgebras of B, then it follows that $C \leq_\sigma B$.

Proposition 2.4.1. (a) (Lemma 2.7 in **[46]**) *If* Q *is a retract of* P *and* P *has the weak Freese-Nation property then* Q *has the weak Freese-Nation property.*

(b) *Suppose that* Q *is a complete Boolean algebra and there is a strictly order-preserving embedding* f *of* Q *into* P *(i.e.* f *preserves ordering and incomparability). If* P *has the weak Freese-Nation property then* Q *also has the weak Freese-Nation property.*

(c) (Lemma 2.3 (a) in **[46]**) *If* $Q \leq_\sigma P$ *and* P *has the weak Freese-Nation property, then* Q *also has the weak Freese-Nation property.*

(d) (Lemma 2.6 in **[46]**) *If* P_α*,* $\alpha < \delta$ *is an increasing sequence of partial orderings with the weak Freese-Nation property such that* $P_\alpha \leq_\sigma P_{\alpha+1}$ *for every* $\alpha < \delta$ *and* $P_\gamma = \bigcup_{\alpha<\gamma} P_\alpha$ *for all* $\gamma < \delta$ *with* $\mathrm{cf}(\gamma) > \omega$*, then* $P = \bigcup_{\alpha<\delta} P_\alpha$ *also has the weak Freese-Nation property.* $\square$

Proposition 2.4.1 (b) follows easily from Proposition 2.4.1 (a): the mapping $g : P \to Q$ defined by $g(p) = \sum\{q \in Q : f(q) \leq p\}$ for $p \in P$ witnesses that Q is a retract of P.

2.4.1. The partial ordering $P_\mathcal{S}$. In this subsection we introduce a construction of partial orderings $P_\mathcal{S}$ and Boolean algebras $B_\mathcal{S}$ which will be used in the following Section 2.4.2 and 2.4.3. For $S \subseteq \kappa$ and an indexed family $\mathcal{S} = \langle S_\alpha : \alpha \in S \rangle$ of subsets of κ, let

$$P_\mathcal{S} = \{x_i : i \in \kappa\} \cup \{y_\alpha : \alpha \in S\}$$

where x_i's and y_α's are pairwise distinct, and let $\leq_\mathcal{S}$ be the partial ordering on $P_\mathcal{S}$ defined by

$$\begin{aligned} p \leq_\mathcal{S} q \;\Leftrightarrow\; & p = q \;\text{ or} \\ & p = x_i \text{ and } q = y_\alpha \text{ for some } i \in \kappa \text{ and } \alpha \in S \text{ with } i \in S_\alpha\ . \end{aligned}$$

Let $B_\mathcal{S}$ be the Boolean algebra generated freely from $P_\mathcal{S}$ except the relation $\leq_\mathcal{S}$. Note that the identity map on $P_\mathcal{S}$ canonically induces a strictly order-preserving embedding of $P_\mathcal{S}$ into $B_\mathcal{S}$.

Proposition 2.4.2. *Suppose that* $\mathrm{cf}(\kappa) \geq \omega_2$*,* $S \subseteq \kappa$ *is stationary such that* $S \subseteq \{\alpha < \kappa : \mathrm{cf}(\alpha) \geq \omega_1\}$ *and* $\mathcal{S} = \{S_\alpha : \alpha \in S\}$ *is such that* S_α *is a cofinal subset of* α *for each* $\alpha \in S$*. If* $P_\mathcal{S}$ *is embedded into a partial ordering* P *by a strictly order-preserving mapping then* P *does not have the weak Freese-Nation property. In particular,* $B_\mathcal{S}$ *and its completion do not have the weak Freese-Nation property.*

Proof. Without loss of generality, we may assume that $P_\mathcal{S}$ is a subordering of P. Assume to the contrary that there is a weak Freese-Nation mapping $f : P \to [P]^{\leq \aleph_0}$. Let

$$\begin{aligned} C = \{\xi < \kappa : \; & \forall \eta < \xi \, \forall p \in f(x_\eta) \\ & \big[\, \exists \alpha \in S \, [\, x_\eta \leq p \leq y_\alpha \,] \;\to\; \exists \alpha' \in S \cap \xi \, [\, x_\eta \leq p \leq y_{\alpha'} \,] \,\big] \;\}. \end{aligned}$$

Then C is a club subset of κ. Let $\alpha \in C \cap S$ and let

$$A = \{p \in f(y_\alpha) : \exists \eta \in S_\alpha \, [\, p \in F(x_\eta) \;\wedge\; x_\eta \leq p \leq y_\alpha \,] \;\}.$$

Since $\alpha \in C$, for each $p \in A$ there is $\alpha_p < \alpha$ such that $p \leq y_{\alpha_p}$. Let $\alpha^* = \sup\{\alpha_p : p \in A\}$. Since A is countable we have $\alpha^* < \alpha$. Let $\beta \in S_\alpha \setminus \alpha^*$. Since $x_\beta \leq y_\alpha$, there is a $p \in A$ such that $x_\beta \leq p \leq y_\alpha$. Hence $x_\beta \leq y_{\alpha_p}$. But this is impossible since $\alpha_p < \alpha^* \leq \beta$. □

2.4.2. Cohen models. Consider the following principle:

(∗∗) There is a sequence $\langle S_\alpha : \alpha \in Lim(\omega_2)\rangle$ such that each S_α is a cofinal subset of α and for any pairwise disjoint $\langle x_\beta : \beta < \omega_1\rangle$ with $x_\beta \in [\omega_2]^{<\aleph_0}$ for $\beta < \omega_1$, there are $\beta_0 < \beta_1 < \omega_1$ such that $x_{\beta_0} \cap S_\alpha = \emptyset$ for all $\alpha \in x_{\beta_1} \cap Lim(\omega_2)$ and that $x_{\beta_1} \cap S_\alpha = \emptyset$ for all $\alpha \in x_{\beta_0} \cap Lim(\omega_2)$.

Proposition . *Let $P = \mathrm{Fn}(\omega_2, 2)$. Then $\Vdash_P$"(∗∗)".*

Proof. Without loss of generality we may assume $P = \mathrm{Fn}(\bigcup_{\alpha\in Lim(\omega_2)} \alpha \times \{\alpha\}, 2)$. For $\alpha \in Lim(\omega_2)$, let $\dot{S}_\alpha$ be a P-name such that $\Vdash_P$"$\dot{S}_\alpha = \{\beta \in \alpha : \dot{g}(\beta, \alpha) = 1\}$" where $\dot{g}$ is the canonical name for the generic function. By genericity, $\Vdash_P$"$\dot{S}_\alpha$ is cofinal in α" for every $\alpha \in Lim(\omega_2)$. Let $\dot{S}$ be a P-name such that $\Vdash_P$"$\dot{S} = \langle \dot{S}_\alpha : \alpha \in Lim(\omega_1)\rangle$".

To show that $\dot{S}$ is forced to satisfy the property in (∗∗), let $\langle \dot{x}_\beta : \beta < \omega_1\rangle$ be a P-name of a sequence of pairwise disjoint finite subsets of ω_2. For each $\beta < \omega_1$, let p_β and $x_\beta \in [\omega_2]^{<\aleph_0}$ be such that $p_\beta \Vdash_P$"$\dot{x}_\beta = x_\beta$". By thinning out the index set ω_1, we may assume without loss of generality that $\mathrm{dom}(p_\beta)$, $\beta < \omega_1$ form a Δ-system with the root d and $p_\beta \restriction d$, $\beta < \omega_1$ are all equal to the same $p \in P$. Since p_β, $\beta < \omega_1$ are then pairwise compatible, x_β, $\beta < \omega_1$ are pairwise disjoint. Further, we may assume that s_β, $\beta < \omega_1$ form a Δ-system with the root s where $s_\beta = \{\gamma : \langle\gamma, \alpha\rangle \in \mathrm{dom}(p_\beta)$ for some $\alpha \in Lim(\omega_2)\}$. Note vthat x_β, $\beta < \omega_1$ are pairwise disjoint since p_β's are pairwise compatible.

Let $\beta_0 < \beta_1 < \omega_1$ be such that $x_{\beta_0} \cap s = \emptyset$, $x_{\beta_1} \cap s = \emptyset$, $x_{\beta_0} \cap s_{\beta_1} = \emptyset$ and $x_{\beta_1} \cap s_{\beta_0} = \emptyset$. To take such β_0 and β_1, first fix a β_0 such that $x_{\beta_0} \cap s = \emptyset$. This is possible since s is finite and x_β's are pairwise disjoint. By the same argument we can finde infinitely many β's such that $x_\beta \cap (s \cup s_{\beta_0}) = \emptyset$. Now for such β's, since $s_\beta \setminus s$ are pairwise disjoint, there are infinitely may β among them with $x_{\beta_0} \cap s_\beta = \emptyset$. Let β_1 be one of such β's.

Let

$$\begin{aligned} p^* = p_{\beta_0} \cup p_{\beta_1} \cup \{\langle\langle\beta, \alpha\rangle, 0\rangle : \beta \in x_{\beta_0}, \alpha \in x_{\beta_1} \cap Lim(\omega_2)\} \\ \cup\{\langle\langle\beta, \alpha\rangle, 0\rangle : \beta \in x_{\beta_1}, \alpha \in x_{\beta_0} \cap Lim(\omega_2)\} \end{aligned}$$

Then $p^* \in P$. Clearly $p^* \Vdash_P$"$\dot{x}_{\beta_0} \cap \dot{S}_\alpha = \emptyset$" for all $\alpha \in \dot{x}_{\beta_1} \cap Lim(\omega_2)$ and $p^* \Vdash_P$"$\dot{x}_{\beta_1} \cap \dot{S}_\alpha = \emptyset$" for all $\alpha \in \dot{x}_{\beta_0} \cap Lim(\omega_2)$. □

Proposition . *Suppose that $\langle S_\alpha : \alpha \in Lim(\omega_2)\rangle$ is as in (∗∗). Let $S = \{\alpha < \omega_2 : \mathrm{cf}(\alpha) = \omega_1\}$ and $\mathcal{S} = \langle S_\alpha : \alpha \in S\rangle$. Then $B_\mathcal{S}$ satisfies the c.c.c.*

Proof. Otherwise we can find $I_\alpha \in [\omega_2]^{<\aleph_0}$, $J_\alpha \in [S]^{<\aleph_0}$ for $\alpha < \omega_1$ and $t(\alpha, i)$, $u(\alpha, \xi) \in \{+1, -1\}$ for each $i \in I_\alpha$, $\xi \in J_\alpha$ and $\alpha < \omega_1$ such that

$$z_\alpha = \textstyle\prod_{i\in I_\alpha} t(\alpha, i)\, x_i \cdot \prod_{\xi\in J_\alpha} u(\alpha, \xi)\, y_\xi, \quad \alpha < \omega_1$$

form a pairwise disjoint family of elements of ${B_\mathcal{S}}^+$.

By Δ-system argument, we may assume that $I_\alpha \cup J_\alpha$, $\alpha < \omega_1$ are pairwise disjoint. Applying (∗∗) to $\langle I_\alpha \cup J_\alpha : \alpha < \omega_1\rangle$, we find $\beta_0 < \beta_1 < \omega_1$ such that $I_{\beta_0} \cap S_\xi = \emptyset$ for all $\xi \in J_{\beta_1}$ and that $I_{\beta_1} \cap S_\xi = \emptyset$ for all $\xi \in J_{\beta_0}$. By definition of $B_\mathcal{S}$, it follows that $z_{\beta_0} \cdot z_{\beta_1} \neq 0$. This is a contradiction. □

Theorem 2.4.3. *In a Cohen model (i.e. any model obtained by adding $\geq \aleph_2$ Cohen reals) there is a c.c.c. complete Boolean algebra B of density $\aleph_2$ without the weak Freese-Nation property.*

Proof. By Proposition , (∗∗) holds in a Cohen model. Hence $B_\mathcal{S}$ as in Proposition satisfies the c.c.c. By Proposition 2.4.2, the completion of $B_\mathcal{S}$ does not have the weak Freese-Nation property. □

Corollary 2.4.4. *The weak Freese-Nation property of $\langle\mathcal{P}(\omega), \subseteq\rangle$ does not imply the weak Freese-Nation property of all c.c.c. complete Boolean algebras.*

Proof. If we start from a model of CH and add $\aleph_2$ Cohen reals, then $\langle \mathcal{P}(\omega), \subseteq \rangle$ has the weak Freese-Nation property in the resulting model (see e.g. [**7**]). On the other hand, by Theorem 2.4.3, there is a c.c.c. complete Boolean algebra without the weak Freese-Nation property in such a model. □

Under CH, every c.c.c. complete Boolean algebra of size $\aleph_2$ has the weak Freese-Nation property ([**46**]). Hence it follows from the result above that

Proposition 2.4.5. CH *implies* $\neg(**)$.

2.4.3. The Weak Freese-Nation property of c.c.c. complete Boolean algebras under GCH**.** In [**7**] it is proved that, assuming CH and a weak form of square principle at singular cardinals of cofinality ω, every c.c.c. complete Boolean algebra has the weak Freese-Nation property. In this section we show that even GCH does not suffice for this result.

Hajnal, Juhász and Shelah [**50**] showed that, starting from a model with a supercompact cardinal, a model of GCH and the following principle can be constructed:

(∗∗∗) There are a stationary $S \subseteq \{\alpha < \omega_{\omega+1} : \mathrm{cf}(\alpha) = \omega_1\}$ and a family $\mathcal{S} = \langle S_\alpha : \alpha \in S \rangle$ such that each S_α is a cofinal subset of α of ordertype ω_1 and that, for all distinct α, $\beta \in S$, $S_\alpha \cap S_\beta$ is finite.

Proposition 2.4.6. *Suppose that* $\mathcal{S} = \langle S_\alpha : \alpha \in S \rangle$ *is as in* $(***)$. *Then* $B_{\mathcal{S}}$ *satisfies the c.c.c.*

Proof. Otherwise we can find $I_\alpha \in [\omega_{\omega+1}]^{<\aleph_0}$, $J_\alpha \in [S]^{<\aleph_0}$, $\alpha < \omega_1$ and $t(\alpha, i)$, $u(\alpha, \xi) \in \{+1, -1\}$ for each $\alpha < \omega_1$, $i \in I_\alpha$ and $\xi \in J_\alpha$ such that

$$z_\alpha = \prod_{i \in I_\alpha} t(\alpha, i)\, x_i \cdot \prod_{\xi \in J_\alpha} u(\alpha, \xi)\, y_\xi, \quad \alpha < \omega_1$$

form a pairwise disjoint family of elements of ${B_{\mathcal{S}}}^+$.

By Δ-system argument, we may assume that $I_\alpha \cup J_\alpha$, $\alpha < \omega_1$ are pairwise disjoint and each I_α has the same size, say n.

For $\alpha < \beta < \omega_1$, since $z_\alpha \cdot z_\beta = 0$, either (I) there is $\eta \in J_\alpha$ such that $I_\beta \cap S_\eta \neq \emptyset$ or else (II) there is $\xi \in J_\beta$ such that $I_\alpha \cap S_\xi \neq \emptyset$. If (I) holds then let us say that $\langle \alpha, \beta \rangle$ is of type (I).

Now, one of the following two cases should hold. We show that both of them lead to a contradiction.

Case I. There is an infinite subset S of ω such that for every $\beta \in \omega_1 \setminus \omega$, $\{k \in S : \langle k, \beta \rangle$ is of type (I)$\}$ is cofinite in S. In this case, by thinning out the index set ω_1, we may assume that, for any $k \in \omega$ and $\beta \in \omega_1 \setminus \omega$, $\langle k, \beta \rangle$ is of type (I). For all $\beta \in \omega_1 \setminus \omega$, since $|\, I_\beta \,| = n$, there are $0 \leq i^0(\beta) < i^1(\beta) < n+1$ such that $I^*_\beta = I_\beta \cap S_{\alpha^0(\beta)} \cap S_{\alpha^1(\beta)} \neq \emptyset$ for some $\alpha^k(\beta) \in J_{i^k(\beta)}$ for $k = 0, 1$ by Pigeonhole Principle. Hence we can find an infinite $X \subseteq \omega_1 \setminus \omega$, $0 \leq i^0 < i^1 < n+1$ and α^0, $\alpha^1 \in \omega_1$, $\alpha^0 \neq \alpha^1$ such that $\alpha^0(\beta) = \alpha^0$ and $\alpha^1(\beta) = \alpha^1$ for all $\beta \in X$. But then $\bigcup_{\beta \in X} I^*_\beta \subseteq S_{i^0} \cap S_{i^1}$. Since the set on the left side is infinite as a disjoint union of pairwise disjoint non-empty sets, this is a contradiction to $(***)$.

Case II. For any infinite subset $S \subseteq \omega$, there is $\beta \in \omega_1 \setminus \omega$ such that for infinitely many $k \in S$, $\langle k, \beta \rangle$ is not of type (I). In this case, by thinning out the first ω elements of the index set ω_1, we may assume that for each $k \in \omega$, there is $\xi(k) \in J_\omega$ such that $I_k \cap S_{\xi(k)} \neq \emptyset$. Note that J_ω is finite. So by thinning out further the first ω elements of the index set ω_1, we may assume that there is $\xi_0 \in J_\omega$ such that $I_k \cap S_{\xi_0} \neq \emptyset$ for every $k < \omega$. Similarly we may also assume that there are $\xi_i \in J_{\omega+i}$ for $1 \leq i \leq n$ such that $I_k \cap S_{\xi_i} \neq \emptyset$ for every $k < \omega$. Then for each $k < \omega$, we can find $i^0(k) < i^1(k) \leq n$ such that $I^*_k = I_k \cap S_{\xi_{i^0(k)}} \cap S_{\xi_{i^1(k)}} \neq \emptyset$ by Pigeonhole Principle. Since there are only $n(n-1)/2$ possibilities of $i^0(k) < i^1(k) \leq n$, there are $i^0 < i^1 \leq n$ and an infinite set $X \subseteq \omega$ such that for every $k \in X$, $i^0(k) = i^0$ and $i^1(k) = i^1$. It follows that $S_{\xi_{i^0}} \cap S_{\xi_{i^1}} \supseteq \bigcup_{k \in X} I^*_k$. As $S_{\xi_{i^0}} \cap S_{\xi_{i^1}}$ is finite, while $\bigcup_{k \in X} I^*_k$ is infinite as a union of infinitely many pairwise disjoint non-empty sets, this is a contradiction. □

Theorem 2.4.7. *It is consistent with* GCH *(modulo the consistency strength of a supercompact cardinal) that there is a c.c.c. complete Boolean algebra without the weak Freese-Nation property.*

Proof. Let $\mathcal{S}$ be a family as in $(***)$. By Proposition 2.4.6 and Proposition 2.4.2 the completion of $B_{\mathcal{S}}$ is a c.c.c. complete Boolean algebra without the weak Freese-Nation property. $\square$

2.4.4. An application of a Chang's Conjecture.

Theorem 2.4.8. *Suppose that V_0 is a transitive model of* ZFC *such that*

$$V_0 \models \mathrm{GCH} + (\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)\,.$$

Let P be a c.c.c. partial ordering in V_0 of cardinality $\aleph_1$ adding a dominating real. Let $\eta \in {}^\omega\omega$ be a dominating real over V_0 generically added by P and let $V_1 = V_0[\eta]$. Note that $V_1 \models \mathrm{GCH}$. In V_1 let $Q = \mathrm{Fn}(\aleph_\omega, \omega)$ and let $\dot{Q}$ be a corresponding P-name. Then we have:

$$V_1 \models \Vdash_Q \text{“}\langle \mathcal{P}(\omega), \subseteq\rangle \textit{ does not have the weak Freese-Nation property}\text{”}.$$

Proof. In the proofs of this and the next theorem, we shall denote by a dotted symbol a name of an element in a generic extension. By the same symbol without the dot, we denote the corresponding element in a fixed generic extension. Without further mention, we shall identify $P * \dot{Q}$ names with the corresponding Q-name in V_1 and vice versa.

Now toward a contradiction, assume that there is a Q-name $\dot{F}$ in V_1 such that

$$(\otimes) \qquad V_1 \models \Vdash_Q \text{“}\dot{F} \text{ is a weak Freese-Nation mapping on } \langle {}^\omega\omega, \leq^*\rangle\text{”}.$$

Let $\dot{\varphi}$ be a $P * \dot{Q}$-name of the function $\aleph_\omega \to \omega$ generically added by Q over V_1. Let $V_2 = V_1[\varphi]$. By GCH, we can find in V_0 a scale $\langle f_\alpha : \alpha < \aleph_{\omega+1}\rangle$ in $\langle \prod_{n\in\omega} \aleph_n, \leq^*\rangle$. Without loss of generality, we may assume that for every $\alpha < \aleph_{\omega+1}$ and $n \in \omega$, $f_\alpha(n) \in \aleph_n \setminus \aleph_{n-1}$ (where we set $\aleph_{-1} = \emptyset$). For each $\alpha \in \aleph_{\omega+1}$, let $g_\alpha : \omega \to \omega$ be defined by

$$g_\alpha = \varphi \circ f_\alpha\,.$$

Let χ be sufficiently large and let $N \prec \langle \mathcal{H}(\chi), \in\rangle$ be such that N contains everything we need in the course of the proof: in particular $\langle f_\alpha : \alpha < \aleph_{\omega+1}\rangle \in N$, $|\,\aleph_\omega \cap N\,| = \aleph_0$ and $otp(\aleph_{\omega+1} \cap N) = \omega_1$. The last two conditions are possible by the assumption of $V_0 \models (\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$.

In V_0, let $\{\xi_{n,k} : k \in \omega\}$ be an enumeration of $(\aleph_n \setminus \aleph_{n-1}) \cap N$ for each $n \in \omega$. Here again, we use the convention that $\aleph_{-1} = \emptyset$. Let $\dot{h}^*$ be a $P * \dot{Q}$-name of an element of ${}^\omega\omega$ such that

$$\Vdash_{P*\dot{Q}} \text{“}\dot{h}^*(n) = \max\{\dot{\varphi}(\xi_{n,k}) : k \leq \dot{\eta}(n)\} \text{ for all } n \in \omega\text{”}.$$

Claim 2.4.8.1. *For every $\alpha \in \aleph_{\omega+1} \cap N$, $\Vdash_{P*\dot{Q}} \text{“}\dot{g}_\alpha \leq^* \dot{h}^*\text{”}$.*

Proof of the Claim. Since $\alpha \in N$ we have $f_\alpha \in N$. Let $e_\alpha : \omega \to \omega$ be the function in V_0 such that $f_\alpha(n) = \xi_{n,e_\alpha(n)}$ for all $n \in \omega$. Since η is dominating, there is $n^* \in \omega$ such that

$$V_1 \models e_\alpha{\restriction}(\omega \setminus n^*) \;\leq\; (\eta{\restriction}\,\omega \setminus n^*).$$

By definition of $\dot{h}^*$, it follows that

$$V_2 \models g_\alpha(n) = \varphi \circ f_\alpha(n) = \varphi(\xi_{n,e_\alpha(n)}) \leq h^*(n)$$

for all $n \geq n^*$. $\square$

Let $N_0 = N$, $N_1 = N_0[\eta]$ and $N_2 = N_1[\varphi]$. Then we have $N_2 \prec \mathcal{H}(\chi)[\eta][\varphi]$.

Let $\dot{h}_l \in N_0$, $l \in \omega$ be $P * \dot{Q}$-names such that

$$\Vdash_{P*\dot{Q}} \text{“}\{\dot{h}_l : l \in \omega\} = \dot{F}(\dot{h}^*) \cap \dot{N}_2\text{”}.$$

For $l \in \omega$, Let $S_l \in [\aleph_\omega]^{\aleph_0} \cap N_0$ be such that, regarding $\dot{h}_l$ as a P-name of $\dot{Q}$-name,

$$V_1 \models \Vdash_P \text{“}\dot{h}_l \text{ is a } \mathrm{Fn}(S_l, \omega)\text{-name}\text{”}.$$

This is possible since P has the c.c.c., $\Vdash_P \text{“}\dot{Q}$ has the c.c.c.” and by the elementarity of N_0. For each $l \in \omega$, let $s_l \in \prod_{n\in\omega} \aleph_n \cap N_0$ be defined (in N_0) by $s_l(k) = \sup S_l \cap \aleph_k$ for $k \in \omega$. Since $\langle f_\alpha : \alpha < \aleph_{\omega+1}\rangle$ was taken to be a scale on $\langle \prod_{n\in\omega} \aleph_n, \leq^*\rangle$, for each $l \in \omega$ there is $\alpha_l \in \aleph_{\omega+1} \cap N_0$ such that $s_l \leq^* f_{\alpha_l}$. Since $otp(\aleph_{\omega+1} \cap N) = \omega_1$, we can find an $\alpha^* \in \aleph_{\omega+1} \cap N_0$ such that $\sup\{\alpha_l : l \in \omega\} \leq \alpha^*$.

Now, by choice of $\dot{h}_l$, $l \in \omega$, the following claim together with Claim 2.4.8.1 contradicts to $(\otimes)$ and hence proves the theorem:

Claim 2.4.8.2. *$V_1 \models \Vdash_Q$" $\dot{g}_{\alpha^*} \not\leq^* \dot{h}_l$ for all $l \in \omega$".*

Proof of the Claim. Assume to the contrary that, in V_1, we have

$$q \Vdash_Q \text{“} \dot{g}_{\alpha^*} \restriction (\omega \setminus k) \leq \dot{h}_l \restriction (\omega \setminus k) \text{”}$$

for some $q \in Q$ and k, $l \in \omega$. We may assume that

$$s_l \restriction (\omega \setminus k) < f_{\alpha^*} \restriction (\omega \setminus k)$$

and $\sup(\mathrm{dom}(p)) < f_{\alpha^*}(m)$ for all $m \in \omega \setminus k$ as well. Working further in V_1, let $m^* = k+1$ and let $q' \leq q$ be such that

$$q' \Vdash_Q \text{“} \dot{h}_l(m^*) = j^* \text{”}$$

for some $j^* \in \omega$. Then $q'' = q \cup (q' \restriction S_l)$ also forces the same statement. Since $f_{\alpha^*}(m^*) \in \aleph_{m^*} \setminus (s_l(m^*) \cup \mathrm{dom}(p))$, we have

$$f_{\alpha^*}(m^*) \notin \mathrm{dom}(q'').$$

Hence

$$q^* = q'' \cup \{\langle f_{\alpha^*}(m^*), j^*+1\rangle\}$$

is an element of Q and $q^* \leq q'' \leq q$. But

$$q^* \Vdash_Q \text{“} \dot{g}_{\alpha^*}(m^*) = \dot{\varphi} \circ f_{\alpha^*}(m^*) = j^*+1 > j^* = \dot{h}_l(m^*). \text{”}$$

This is a contradiction. □

The theorem is proved. □

2.5. Weak Freeze-Nation property of $P(\omega)$

(This section is based on [4] and [2])

In this section we study the consequences of the weak Freese-Nation property of $(\mathcal{P}(\omega), \subseteq)$. It is shown that the weak Freese-Nation property of $\mathcal{P}(\omega)$ captures many of the features of Cohen models and hence may be considered as one of the principles axiomatizing some portion of the combinatorics available in Cohen models. It is shown that the weak Freese-Nation property of $(\mathcal{P}(\omega), \subseteq)$ is one of the strongest among such principles: e.g. we prove under this assumption that most of the known cardinal invariants including all of those which appear in Cichón's diagram take the same value as in a corresponding Cohen model.

Some consequences of the weak Freese-Nation property of $\mathcal{P}(\omega)$ on algebraic behavior of $\mathcal{P}(\omega)/fin$ were studied in **[69]**.

2.5.1. Luzin gap. For a regular κ, an almost disjoint family $\mathcal{A} \subseteq [\omega]^{\aleph_0}$ is said to be a *κ-Luzin gap* if $|\mathcal{A}| = \kappa$ and for any $x \in [\omega]^{\aleph_0}$ either $|\{a \in \mathcal{A} : |a \setminus x| < \aleph_0\}| < \kappa$ or $|\{a \in \mathcal{A} : |a \cap x| < \aleph_0\}| < \kappa$.

Theorem 2.5.1. *Assume that $\mathcal{P}(\omega)$ has the $\aleph_1$-FN. Then there is no $\aleph_2$-Luzin gap.*

Proof. Let $f : \mathcal{P}(\omega) \to [\mathcal{P}(\omega)]^{\aleph_0}$ be an $\aleph_1$-FN mapping. We may assume that $f(a) = f(b) = f(\omega \setminus b)$ for all a, $b \in \mathcal{P}(\omega)$ with $a =^* b$. Thus $x \subseteq^* y$ implies that there is $z \in f(x) \cap f(y)$ such that $x \subseteq^* z \subseteq^* y$ and $|x \cap y| < \aleph_0$ implies that there is $z \in f(x) \cap f(y)$ such that $x \subseteq^* z$ and $|z \cap y| < \aleph_0$.

Suppose that $\mathcal{A} = \{a_\alpha : \alpha < \omega_2\}$ is an almost disjoint family of subsets of ω. We show that $\mathcal{A}$ is not an $\aleph_2$-Luzin gap. Let χ be sufficiently large regular cardinal and consider the model $\mathcal{H} = (\mathcal{H}(\chi), \in, \trianglelefteq)$ where $\trianglelefteq$ is any well-ordering on $\mathcal{H}$. Let N be an elementary submodel of $\mathcal{H}$ such that $\mathcal{A}$, $f \in N$, $N \cap \omega_2 \in \omega_2$ and $\mathrm{cf}(\delta) = \omega_1$ for $\delta = N \cap \omega_2$. For $\alpha \in N$ we have $|a_\alpha \cap a_\delta| < \aleph_0$ and hence $a_\alpha \subseteq^* (\omega \setminus a_\delta)$. Thus there is $b_\alpha \in f(a_\alpha) \cap f(a_\delta)$ such that $a_\alpha \subseteq^* b_\alpha \subseteq^* (\omega \setminus a_\delta)$. Since $f(a_\delta)$ is countable and $\mathrm{cf}(\delta) = \omega_1$ there is $b \in f(a_\delta)$ such that $I = \{\alpha < \delta : b_\alpha = b\}$ is cofinal in δ. We show that b witnesses that $\mathcal{A}$ is not an $\aleph_2$-Luzin gap, i.e., $J = \{\alpha < \omega_2 : a_\alpha \subseteq^* b\}$ and $K = \{\alpha < \omega_2 : |b \cap a_\alpha| < \aleph_0\}$ both have cardinality $\aleph_2$.

Claim 2.5.1.1. $|\,J\,| = \aleph_2$.

Proof of the Claim. First note that $b \in N$ since $b \in f(a_\alpha)$ for any $\alpha \in I \subseteq N$. Hence we have $J \in N$ and $I \subseteq J$. Since I is cofinal in $N \cap \omega_2$, we have $N \models J$ is cofinal in ω_2. By elementarity it follows that $\mathcal{H} \models J$ is cofinal in ω_2. Hence J is really cofinal in ω_2. □

Claim 2.5.1.2. $|\,K\,| = \aleph_2$.

Proof of the Claim. Since $b \in N$ it follows that $K \in N$. For $\beta \in N \cap \omega_2 = \delta$, we have $\mathcal{H} \models \delta \in K \wedge \beta < \delta$. Hence $\mathcal{H} \models K \not\subseteq \beta$ and $N \models K \not\subseteq \beta$ by elementarity. It follows that $N \models K$*is unbounded in* ω_2. Hence, again by elementarity, $\mathcal{H} \models K$*is unbounded in* ω_2. Thus K is really unbounded in ω_2. □

This proves the theorem. □

Corollary 2.5.2. $\mathtt{b} = \aleph_1$ *or even the statement "$\mathcal{P}(\omega)$ does not contain any strictly increasing $\subseteq^*$-chain of length ω_2" does not imply that $\mathcal{P}(\omega)$ has the $\aleph_1$-FN.*

Proof. Suppose that our ground model V satisfies the CH. Koppelberg and Shelah [**70**] proved that the forcing with $\mathrm{Fn}(\omega_2, 2)$ can be represented as a two step iteration $A * \dot{B}$ where $\Vdash_A$"$\mathcal{P}(\omega)$ contains an $\aleph_2$-Luzin gap". Thus, by Theorem 2.5.1, we have $\Vdash_A$"$\mathcal{P}(\omega)$ does not have the $\aleph_1$-WF". On the other hand, we have $\Vdash_{A*\dot{B}}$"$\mathcal{P}(\omega)$ does have the $\aleph_1$-WF" by Theorem 2.3.20. Hence $\Vdash_{A*\dot{B}}$"there is no strictly increasing $\subseteq^*$-chain in $\mathcal{P}(\omega)$ of length ω_2". It follows that $\Vdash_A$"there is no strictly increasing $\subseteq^*$-chain in $\mathcal{P}(\omega)$ of length ω_2".

□

2.5.2. Cardinal invariants. In this subsection we show that the weak Freese-Nation property of $\mathcal{P}(\omega)$ implies that all the cardinal invariants of reals appearing in [**28**] or [**65**] behave just as in a Cohen model.

We shall begin with a brief review of definitions and basic facts of some of these cardinal invariants.

$cov(meager)$ and $non(meager)$ denote the covering number and "non" of the ideal of meager sets respectively:

$$cov(meager) = \min\{|\,\mathcal{F}\,| \,:\, \forall X \in \mathcal{F}(X \subseteq \mathbb{R} \wedge X \text{ is meager}) \wedge \bigcup \mathcal{F} = \mathbb{R}\},$$
$$non(meager) = \min\{|\,X\,| \,:\, X \subseteq \mathbb{R} \wedge X \text{ is non-meager}\}.$$

The following characterization of $cov(meager)$ will be used:

Lemma 2.5.3. (for the proof see e.g. [**28**])

$$cov(meager) = \min\{|\,\mathcal{F}\,| \,:\, \mathcal{F} \subseteq {}^\omega\omega \wedge \forall f \in {}^\omega\omega \exists g \in \mathcal{F} \forall n \in \omega (f(n) \neq g(n))\}.$$

In a Cohen model, $cov(meager) = 2^{\aleph_0}$ and $non(meager) = \aleph_1$, and the values of cardinal invariants in Cichón's diagram are decided from these equations:

$$\begin{tikzcd}
cov(null) \arrow[d] & \longleftarrow & non(meager) \arrow[d, "\mathtt{b}"] & \longleftarrow & cof(meager) \arrow[d, "\mathtt{d}"] & \longleftarrow & cof(null) \arrow[d] \\
add(null) & \longleftarrow & add(meager) & \longleftarrow & cov(meager) & \longleftarrow & non(null) \\
 & \Leftarrow \aleph_1 & & & & 2^{\aleph_0} \Rightarrow &
\end{tikzcd}$$

The equations $cov(meager) = 2^{\aleph_0}$ and $non(meager) = \aleph_1$ also imply that $\mathtt{s} = \mathtt{e} = \aleph_1$ and $\mathtt{r} = \mathtt{u} = \mathtt{i} = 2^{\aleph_0}$.

The following variant of the bounding number is studied in [**39**] and [**65**].

$$\mathtt{b}^* = \min\{\kappa \,:\, \forall \mathcal{F} \subseteq {}^\omega\omega\,(\mathcal{F} \text{ is unbounded} \rightarrow \exists \mathcal{G} \subseteq \mathcal{F}\,(|\,\mathcal{G}\,| \leq \kappa \wedge \mathcal{G} \text{ is unbounded}))\}.$$

Also a similar variation of $non(meager)$ is studied in [**65**] and [**106**]:

$$shr(meager) = \min\{\kappa \,:\, \forall X \subseteq \mathtt{R}\,(\mathcal{F} \text{ is non-meager} \rightarrow \exists Y \subseteq X\,(|\,Y\,| \leq \kappa \wedge Y \text{ is non-meager}))\}.$$

Clearly $\mathtt{b} \leq \mathtt{b}^* \leq \mathtt{d}$ and $non(neager) \leq shr(meager) \leq cof(meager)$. The equation $\mathtt{b}^* \leq shr(meager)$ is proved in [**106**].

Cichón's diagram in Cohen models with these new cardinal invariants looks like this (see [**65**] and [**106**]):

$$\begin{tikzcd}
cov(null) \arrow[d] & non(meager) \arrow[l] \arrow[d, "\mathtt{b}"] & shr(meager) \arrow[l] \arrow[d, "\mathtt{b}^*"] & cof(meager) \arrow[l] \arrow[d, "\mathtt{d}"] & cof(null) \arrow[l] \arrow[d] \\
add(null) & add(meager) \arrow[l] & & cov(meager) \arrow[ll] & non(null) \arrow[l]
\end{tikzcd}$$

$$\Leftarrow \aleph_1 \qquad\qquad 2^{\aleph_0} \Rightarrow$$

$G \subseteq [\omega]^{\aleph_0}$ is said to be groupwise dense if it is downward closed with respect to $\subseteq^*$ and for every strictly increasing $f \in {}^\omega\omega$, there is an infinite $X \subseteq \omega$ such that $\bigcup_{n\in X}[f(n), f(n+1)) \in G$. The groupwise density number g is defined by

$$\mathtt{g} = \min\{|\,\mathcal{G}\,| : \forall G \in \mathcal{G}(G \subseteq [\omega]^{\aleph_0} \wedge G \text{ is groupwise dense}) \wedge \bigcap \mathcal{G} = \emptyset\}$$

(see [**29**]). $G \subseteq [\omega]^{\aleph_0}$ is groupwise dense if, and only if, G is downward closed with respect to $\subseteq^*$ and G is non-meager, under the identification of $[\omega]^{\aleph_0}$ with the subspace $\{f \in {}^\omega 2 : |\{n \in \omega : f(n) = 1\}| = \aleph_0\}$ of ${}^\omega 2$ in the canonical way (see [**28**]). In [**29**], it is shown that $\mathtt{g} = \aleph_1$ in a Cohen model.

(1) of the following proposition asserted originally the weaker equation: $non(meager) = \aleph_1$. We thank M. Kada and S. Kamo for pointing out that our proof shows actually the equation in the present form.

Proposition 2.5.4. *Assume that $\mathcal{P}(\omega)$ has the weak Freese-Nation property. Then*

(1) $shr(meager) = \aleph_1$. *Hence* $non(meager) = \mathtt{b}^* = \aleph_1$ *and* $\mathtt{s} = \mathtt{e} = \aleph_1$;
(2) $\mathtt{a} = \aleph_1$;
(3) $\mathtt{g} = \aleph_1$.

Proof. (1): Suppose that $S \subseteq \mathtt{R}$ is non-meager. We show that there is a $T \subseteq S$ of cardinality $\leq \aleph_1$ such that T is still non-meager in $\mathtt{R}$.

Let χ be large enough and $M \prec \mathcal{H}(\chi)$ be such that $|\,M\,| = \aleph_1$, $S \in M$ and that $[M]^{\aleph_0} \cap M$ is cofinal in $[M]^{\aleph_0}$ with respect to $\subseteq$. We show that $S \cap M$ is non-meager. This suffices as $|\,S \cap M\,| \leq \aleph_1$.

Let $Q = \{(q, r) \in \mathtt{Q} : q < r\}$ and let $f \in M$ be a weak Freese-Nation mapping on $(\mathcal{P}(Q), \subseteq)$. For each $x \in \mathcal{P}(Q)$, let $o(x)$ denote the union of open intervals corresponding to each element of x. Thus $o(x) = \{t \in \mathtt{R} : \exists (q, r) \in x\,(q < t < r)\}$.

Let $D \in [\mathcal{P}(Q)]^{\aleph_0}$ be such that $o(d)$ is a dense subset of $\mathtt{R}$ for all $d \in D$. We show below that $\bigcap\{o(d) : d \in D\} \cap S \cap M \neq \emptyset$. From this it follows readily that $S \cap M$ is non-meager.

For each $d \in D$, let $C(d) = f(d) \cap M \cap \{x \in \mathcal{P}(Q) : d \subseteq x\}$. Then we have $C(d) \subseteq D$ and $o(d) = \bigcap\{o(x) : x \in C(d)\}$. Let $\tilde{C}(d) \in [D]^{\aleph_0} \cap M$ be such that $C(d) \subseteq \tilde{C}(d)$. Let $\tilde{o}(d) = \bigcap\{o(x) : x \in \tilde{C}(d)\}$. Then $\tilde{o}(d) \in M$, $\tilde{o}(d)$ is co-meager and $\tilde{o}(d) \subseteq o(d)$. Let $\mathcal{F} \in [M]^{\aleph_0} \cap M$ be such that $\{\tilde{o}(d) : d \in D\} \subseteq \mathcal{F}$ and that each $X \in \mathcal{F}$ is a co-meager set $\subseteq \mathtt{R}$. Then, since $\bigcap \mathcal{F} \in M$ is co-meager, there is an $r \in S \cap M \cap \bigcap \mathcal{F}$. But $r \in \bigcap \mathcal{F} \subseteq \bigcap\{\tilde{o}(d) : d \in D\} \subseteq \bigcap\{o(d) : d \in D\}$. Hence $\bigcap\{o(d) : d \in D\} \cap S \cap M \neq \emptyset$.

(2): Let χ, $M \prec \mathcal{H}(\chi)$ be as in the proof of (1) and let $f \in M$ be a weak Freese-Nation mapping on $(\mathcal{P}(\omega), \subseteq^*)$. Let $F : \mathcal{P}(\omega) \to [\mathcal{P}(\omega)]^{\leq \aleph_0}$ be defined by $F(x) = f(x) \cup f(\omega \setminus x)$. Then F is again an element of M. Let $\langle S_\alpha : \alpha < \omega_1\rangle$ be an enumeration of $[[\omega]^{\aleph_0}]^{\aleph_0} \cap M$ and let $\langle a_\alpha : \alpha < \omega_1\rangle$ be a sequence of pairwise almost disjoint elements of $[\omega]^{\aleph_0} \cap M$ such that:

$(*)$ for any $x \in S_\alpha$, if $|\,x \setminus \bigcup_{\beta\in u} a_\beta\,| = \aleph_0$ for all $u \in [\alpha]^{<\aleph_0}$, then $|\,x \cap a_\alpha\,| = \aleph_0$.

We show that $\{a_\alpha : \alpha < \omega_1\}$ is maximal almost disjoint. Otherwise, there is a $b \in [\omega]^{\aleph_0}$ almost disjoint from all a_α, $\alpha < \omega_1$. Let $\alpha^* < \omega_1$ be such that $F(b) \cap M \subseteq S_{\alpha^*}$. Since b and a_{α^*} are almost disjoint, there is $x \in F(b) \cap F(a_{\alpha^*})$ such that (i) $b \subseteq^* x$ and (ii) $|\,x \cap a_{\alpha^*}\,| < \aleph_0$. Since $x \in F(b) \cap M \subseteq S_{\alpha^*}$, (i) implies that x satisfies the if-clause in $(*)$ for α^*. But then (ii) contradicts the choice of a_{α^*}.

(3): Let M be as before and let $f \in M$ be a weak Freese-Nation mapping on $(\mathcal{P}(\omega), \subseteq^*)$. For $C \in [[\omega]^{\aleph_0}]^{\aleph_0} \cap M$ let

$$G_C = \{x \in [\omega]^{\aleph_0} : \exists y \in [\omega]^{\aleph_0} \cap M(\forall c \in C\ (c \not\subseteq y) \wedge x \subseteq y)\}.$$

G_C is groupwise dense: Clearly G_C is downward-closed with respect to $\subseteq^*$. To show that G_C is non-meager, let $G'_C = \{y \in [\omega]^{\aleph_0} \cap M : \forall c \in C\, c \not\subseteq^* y\}$. Then $G'_C = [\omega]^{\aleph_0} \cap M \setminus \bigcup_{c \in C, n \in \omega}\{y \in [\omega]^{\aleph_0} : c \setminus n \subseteq y\}$. Since $[\omega]^{\aleph_0} \cap M$ is non-meager by the proof of (1) and each $\{y \in [\omega]^{\aleph_0} : c \setminus n \subseteq y\}$ is nowhere dense, it follows that G'_C is non-meager. By $G'_C \subseteq G_C$, G_C is non-meager as well.

Let $\mathcal{G} = \{G_C : C \in [[\omega]^{\aleph_0}]^{\aleph_0} \cap M\}$. We show that $\bigcap \mathcal{G} = \emptyset$. Since $|\,\mathcal{G}\,| = \aleph_1$, this completes the proof.

For arbitrary $x \in [\omega]^{\aleph_0}$, let $C \in [[\omega]^{\aleph_0}]^{\aleph_0} \cap M$ be such that $f(x) \cap [\omega]^{\aleph_0} \subseteq C$. Then for any $y \in [\omega]^{\aleph_0} \cap M$ with $x \subseteq^* y$, there is $c \in C$ such that $x \subseteq^* c \subseteq^* y$. It follows that such y is not in G'_C. Thus $x \notin G_C$ and hence $x \notin \cap \mathcal{G}$. □

The following proposition implies that, in the most of the cases, the right half of Cichón's diagram obtains the value $2^{\aleph_0}$ under the weak Freese-Nation property of $\mathcal{P}(\omega)$:

Proposition 2.5.5. *Suppose that χ is a sufficiently large regular cardinal and $M \prec \mathcal{H}(\chi)$ be such that*

(i) *$[M]^{\aleph_0} \cap M$ is cofinal in $[M]^{\aleph_0}$ with respect to $\subseteq$;*
(ii) *for $h \in {}^\omega\omega$, there is $f \in {}^\omega\omega \cap M$ such that $h(n) \neq f(n)$ for every $n \in \omega$;*
(iii) *$\mathtt{R} \setminus M$ is not empty.*

Then $\mathcal{P}(\omega) \cap M$ is not a σ-subordering of $\mathcal{P}(\omega)$.

Proof. Towards a contradiction, assume that $\mathcal{P}(\omega) \cap M \leq_\sigma \mathcal{P}(\omega)$. Fix $x \in \mathtt{R} \setminus M$.

Claim 2.5.5.1. *There is a countable set $C \in M$ of infinite closed subsets of $\mathtt{R}$ such that, for any closed set $c \in M$ containing x, there is $c' \in M$ such that $x \in c' \subseteq c$.*

Proof of the Claim. Let Q be as in the proof of Proposition 2.5.4, (1). Let $s = \{(q, r) \in Q : x \leq q \vee r \leq x\}$. Then, by assumption, there is a countable $D \subseteq \mathcal{P}(Q) \cap M$ cofinal in $\{y \in \mathcal{P}(Q) : y \subseteq s\}$ with respect to $\subseteq$. By (i), let $D' \in [\mathcal{P}(Q)]^{\aleph_0} \cap M$ be such that $D \subseteq D'$. Then $C = \{\mathtt{R} \setminus o(d) : d \in D', \mathtt{R} \setminus o(d) \text{ is infinite}\}$ is as desired where $o(d)$ is defined as in the proof of Proposition 2.5.4, (1). □

Let C be as in Claim 1 and let $\langle c_n : n \in \omega\rangle$ be an enumeration of C in M and let $\{u_n^m : m \in \omega, n \in \omega\} \in M$ be a family of open subsets of $\mathtt{R}$ such that for each $n \in \omega$, u_n^m, $m \in \omega$ are pairwise disjoint and, for every $m \in \omega$, $u_n^m \cap c_n \neq \emptyset$. Let $f \in {}^\omega\omega$ be such that $x \in u_n^m$ implies $f(n) = m$. This is possible as u_n^m, $m \in \omega$ are disjoint. By (ii), there is a $g \in {}^\omega\omega \cap M$ such that $f(n) \neq g(n)$ for every $n \in \omega$. Let

$$c^* = \bigcap\nolimits_{n \in \omega} \mathtt{R} \setminus u_n^{g(n)}\,.$$

Then $x \in c^*$ and by definition of c^*, $c \not\subseteq c^*$ for all $c \in C$. This is a contradiction to the choice of C. □

Using Borel coding, we can also prove the variant of Proposition above for sufficiently absolute inner models M of models of some fragment of ZFC.

The following corollary together with Proposition 2.5.4 establishes that, under the conditions as in the corollary, the weak Freese-Nation property of $\mathcal{P}(\omega)$ implies that the values of cardinal invariants of the reals are just the same as their values in a Cohen model with the same value of $2^{\aleph_0}$.

Corollary 2.5.6. *Suppose that $\{\kappa < 2^{\aleph_0} : cf([\kappa]^{\aleph_0}, \subseteq) = \kappa\}$ is cofinal in $\{\kappa < 2^{\aleph_0} : cf(\kappa) > \omega\}$ (This is the case if e.g. $2^{\aleph_0} \leq \aleph_{\omega+1}$ or if $\neg 0^\#$ holds). If $\mathcal{P}(\omega)$ has the weak Freese-Nation property then cov(meager) $= 2^{\aleph_0}$ and hence also $\mathtt{r} = \mathtt{u} = \mathtt{i} = 2^{\aleph_0}$.*

Proof. Suppose that *cov(meager)* $< 2^{\aleph_0}$. Then, by Lemma 2.5.3 and the assumption, there is an $M \prec \mathcal{H}(\chi)$ satisfying (i), (ii), (iii) of Proposition 2.5.5. By Proposition 2.5.5, $\mathcal{P}(\omega) \cap M \not\leq_\sigma \mathcal{P}(\omega)$. Hence by Theorem 2.3.4, $\mathcal{P}(\omega)$ does not have the weak Freese-Nation property. □

2.5.3. LCS* spaces. W. Just showed that there is no LCS* space X with cardinal sequence $\langle\omega\rangle_{\omega_2}$ and $\langle\omega\rangle_{\omega_1}{}^\frown\langle\omega_2\rangle$ in a Cohen model. The following theorem generalizes this non-existence theorem.

Theorem 2.5.7. *Suppose that $\mathcal{P}(\omega)$ has the weak Freese-Nation property. Then:*

(1) There are no LCS* spaceX with height ω_1+1, such that $|\,\mathrm{I}_0(X)| = \aleph_0$, $|\,\mathrm{I}_\alpha(X)| \le \aleph_1$ for all $\alpha<\omega_1$ and $|\,\mathrm{I}_{\omega_1}(X)| \ge \aleph_2$.

(2) There are no LCS* space X with height ω_2, such that $|\,\mathrm{I}_0(X)| = \aleph_0$, and $|\,\mathrm{I}_\alpha(X)| \le \aleph_1$ for all $\alpha<\omega_2$.

Proof. (1): Towards a contradiction, assume that there is an LCS* X with height ω_1+1, such that $|\,\mathrm{I}_0(X)| = \aleph_0$, $|\,\mathrm{I}_\alpha(X)| \le \aleph_1$ for all $\alpha<\omega_1$ and $|\,\mathrm{I}_{\omega_1}(X)| \ge \aleph_2$. Write X_ν for $\mathrm{I}_\nu(X)$.

Without loss of generality we may assume that $X_0=\omega$.

For each $x\in X$ let $u(x)$ be a clopen neighbor of x such that, for $\alpha=\mathrm{ht}(x)$, $u(x)\setminus X_{<\alpha}=\{x\}$. Let

$$\mathcal{S}=\{u\cap\omega \,:\, u\subseteq X \text{ is closed and covered by finitely many } u(x)\text{'s such that } \mathrm{ht}(x)<\omega_1\}$$

and

$$\mathcal{L}=\{A\subseteq\omega \,:\, \sup\{\mathrm{ht}(x) \,:\, x\in\overline{A}\}=\omega_1\}.$$

Claim 2.5.7.1. *For $A\in\mathcal{L}$, $\overline{A}\cap X_{\omega_1}\neq\emptyset$.*

Proof of the Claim. Suppose that $\overline{A}\cap X_{\omega_1}=\emptyset$. Then, by compactness there is $S\in[\omega_1]^{<\aleph_0}$, such that $\overline{A}\subseteq\bigcup_{\alpha\in S}X_{<\alpha}$. □

Claim 2.5.7.2. *For any $A\in\mathcal{L}$ there exists at most one $y\in X_{\omega_1}$ such that $A\subseteq u(y)$.*

Proof of the Claim. By the previous claim $\overline{A}\cap X_{\omega_1}\neq\emptyset$. If $A\subseteq u(y)$ then $\overline{A}\subseteq u(y)$ and hence $\overline{A}\cap X_{\omega_1}=\{y\}$. □

Now let χ be sufficiently large and let $M\prec\mathcal{H}(\chi)$ be such that $X\in M$, $|\,M\,|=\aleph_1$ and $\omega_1\subseteq M$. Note that $X_\alpha\subseteq M$ for all $\alpha<\omega_1$. Let $F\in M$ be a weak Freese-Nation mapping on $(\mathcal{P}(\omega),\subseteq)$.

By the claim above, and since $|\,M\,|=\aleph_1$ and $|\,X_{\omega_2}\,|\ge\aleph_2$, there exists an $x^*\in X_{\omega_1}$ such that, for any $A\in\mathcal{P}(\omega)\cap M$, if $A\subseteq u(x^*)\cap\omega$, then $\mathrm{ht}[\overline{A}]$ is bounded. Let $\langle A_n \,:\, n\in\omega\rangle$ be an enumeration of $\{A\in\mathcal{P}(\omega)\cap M \,:\, A\in F(u(x^*)\cap\omega),\ A\subseteq u(x^*)\cap\omega\}$. Then $\beta_n=\sup\mathrm{ht}[\overline{A_n}]$ is less than ω_1 for each $n\in\omega$. Let $\beta=\sup_{n\in\omega}\beta_n+1$ and let $y\in X_\beta\cap u(x^*)$. Then there is a $B\in[X_{<\beta}]^{<\aleph_0}$ such that $V=u(y)\setminus\bigcup_{t\in B}u(t)\subseteq u(x^*)$. Since $V\in M$ there should be an $n\in\omega$ such that $V\cap\omega\subseteq A_n$. But this is a contradiction as $\sup\mathrm{ht}[V]=\beta>\sup\mathrm{ht}[\overline{A_n}]$.

(2): Assume that there is a LCS* X with height ω_2, such that $|\,\mathrm{I}_0(X)|=\aleph_0$, and $|\,\mathrm{I}_\alpha(X)|\le\aleph_1$ for all $\alpha<\omega_2$. Write X_ν for $\mathrm{I}_\nu(X)$.

Let χ be sufficiently large and let $M\prec\mathcal{H}(\chi)$ be such that $X\in M$, $\omega_1\subseteq M$, $\omega_2\cap M\in\omega_2$ and $cof(\gamma)=\omega_1$ for $\gamma=\omega_2\cap M$.

Let $F\in M$ be a weak Freese-Nation mapping on $(\mathcal{P}(\omega),\subseteq)$. Take $x^*\in X_\gamma$ and let $\langle A_n \,:\, n\in\omega\rangle$ be an enumeration of $\{A\in\mathcal{P}(\omega)\cap M \,:\, A\in F(u(x^*)\cap\omega),\ A\subseteq u(x^*)\cap\omega\}$. By elementarity $\sup\mathrm{ht}[\overline{A_n}]<\gamma$. Hence the same argument as in the proof of (1) leads to a contradiction. □

2.5.4. Closed covers of the real reals. This is a recent application. In **[2]** we investigated the following question: Assume that $\mathbf{H}$ is a κ-fold cover of a set X. Under which assumption can you partition $\mathbf{H}$ into κ subcover?

Definition 2.5.8. If X is a set, $\mathbf{H}\subset\mathcal{P}(X)$, and $x\in X$, let $\mathbf{H}(x)=\{H\in\mathbf{H} : x\in H\}$.

Let κ be a cardinal. A partial function $c:\mathbf{H}\to On$ is a *κ-good coloring of* $\mathbf{H}$ if for each $x\in X$

$$\text{if } |\mathbf{H}(x)|\ge\kappa \text{ then } \kappa\subseteq c[\mathbf{H}(x)].$$

The coloring c is a *$[\kappa,\infty)$-good coloring of* $\mathbf{H}$ *over* Y if it is λ-good for each cardinal $\lambda\ge\kappa$.

Theorem 2.5.9 ([**2**]). *Assume that $\langle \mathcal{P}(\omega), \subseteq \rangle$ has the weak Freese-Nation property. Let (X, τ) be a topological space which has a countable base, and let* $\mathbf{H}$ *be a cover of X by closed sets. Then there exists an $[\omega_1, \infty)$-good coloring of* $\mathbf{H}$.

Corollary 2.5.10 ([**2**]). *Assume that $\langle \mathcal{P}(\omega), \subseteq \rangle$ has the weak Freese-Nation property. Then any ω_1-fold cover of the real line by closed sets can be partition into ω_1-many disjoint subcover.*

Let us remark that we also proved that under MA_{ω_1} there is an ω_1-cover of the real line by the translations of a single perfect set which can not be partitioned into two disjoint subcover.

2.6. Stick and clubs

(This section is based on [6] and [5])

In this section, we study combinatorial principles known as 'stick' and 'club', and their diverse variants which are all weakenings of $\diamondsuit$. Hence some of the consequences of $\diamondsuit$ still hold under these principles. On the other hand, they are weak enough to be consistent with the negation of the continuum hypothesis or even with a weak version of Martin's axiom in addition. We introduce a new kind of side-by-side product of partial orderings which we call pseudo-product. Using such products, we give several generic extensions where some of these principles hold together with $\neg$CH and Martin's Axiom for countable p.o.-sets. See e.g. [**31**], [**5**], [**89**] for applications of these principles.

($\spadesuit$) (read "stick") is the following principle introduced in S. Broverman, J. Ginsburg, K. Kunen and F. Tall [**31**]:

($\spadesuit$): *There exists a sequence $(x_\alpha)_{\alpha<\omega_1}$ of countable subsets of ω_1 such that for any $y \in [\omega_1]^{\aleph_1}$ there exists $\alpha < \omega_1$ such that $x_\alpha \subseteq y$.*

Note that ($\spadesuit$) follows from CH.

The principle ($\spadesuit$) suggests the following cardinal number:

$$\spadesuit = \min\{\, |\, X\, |\, :\, X \subseteq [\omega_1]^{\aleph_0},\ \forall y \in [\omega_1]^{\aleph_1}\ \exists x \in X\ x \subseteq y\,\}.$$

We have $\aleph_1 \leq \spadesuit \leq 2^{\aleph_0}$ and ($\spadesuit$) holds if and only if $\spadesuit = \aleph_1$.

The principle ($\clubsuit$) ('club'), a strengthening of ($\spadesuit$), was first formulated in Ostaszewski [**89**]. Let $Lim(\omega_1) = \{\, \gamma < \omega_1 \,:\, \gamma \text{ is a limit}\,\}$. For a stationary $E \subseteq Lim(\omega_1)$,

$\clubsuit(E)$: *There exists a sequence $(x_\gamma)_{\gamma\in E}$ of countable subsets of ω_1 such that for every $\gamma \in E$, x_γ is a cofinal subset of γ with $otp(x_\gamma) = \omega$ and for every $y \in [\omega_1]^{\aleph_1}$ there is $\gamma \in E$ such that $x_\gamma \subseteq y$.*

Let us call $(x_\gamma)_{\gamma\in E}$ as above a $\clubsuit(E)$-sequence. For $E = Lim(\omega_1)$ we shall simply write ($\clubsuit$) in place of $\clubsuit(Lim(\omega_1))$. Clearly ($\spadesuit$) follows from ($\clubsuit$). Unlike ($\spadesuit$), ($\clubsuit$) does not follow from CH since ($\clubsuit$) + CH is known to be equivalent to $\diamondsuit$ (K. Devlin, see [**89**]). This equivalence holds also in the version argumented with a stationary $E \subseteq Lim(\omega_1)$.

Fact* 2.6.1 ([**89**]). *For any stationary $E \subseteq Lim(\omega_1)$, $\clubsuit(E)$ + CH is equivalent to $\diamondsuit(E)$.* □

S. Shelah [**96**] proved the consistency of $\neg$CH + ($\clubsuit$) in a model obtained from a model of GCH by making the size of $\wp(\omega_1)$ to be $\aleph_3$ by countable conditions and then collapsing $\aleph_1$ to be countable. Soon after that, in an unpublished note, J. Baumgartner gave a model of $\neg$CH + $\clubsuit$ where collapsing of cardinals is not involved: his model was obtained from a model of $V = L$ by adding many Sacks reals by side by side product. I. Juhász then proved in an unpublished note that "$\neg$CH + MA(*countable*) + ($\clubsuit$)" is consistent. Here MA(*countable*) stands for Martin's axiom restricted to countable partial orderings. Later P. Komjáth [**67**] cited a remark by Baumgartner that Shelah's model mentioned above also satisfies $\neg$CH + MA(*countable*) + ($\clubsuit$).

These results are rather optimal in the sense that a slight strengthening of MA(*countable*) implies the negation of ($\clubsuit$). Let MA(*Cohen*) denote Martin's axiom restricted to the partial orderings of the form $\mathrm{Fn}(\kappa, 2)$ for some κ where, as in [**77**], $\mathrm{Fn}(\kappa, 2)$ is the p.o.-set for adding κ Cohen reals, i.e. the set of functions from some finite subset of κ to 2 ordered by reverse inclusion.

Fact 2.6.2. MA *for the partial ordering* $\mathrm{Fn}(\omega_1, 2)$ *implies* $\spadesuit = 2^{\aleph_0}$. □

2.6.1. Pseudo product of partial orderings. In this subsection, we introduce a new kind of side-by-side product of p.o.'s which will be used in the next section to prove various consistency results. Let X be any set and $(P_i)_{i\in X}$ be a family of partial orderings. For $p \in \Pi_{i\in X} P_i$ the support of p is defined by $supp(p) = \{\, i \in X \,:\, p(i) \neq 1_{P_i} \,\}$. For a cardinal κ, let $\Pi^*_{\kappa,i\in X} P_i$ be the set

$$\{\, p \in \Pi_{i\in X} P_i \,:\, |\, supp(p) \,| < \kappa \,\}$$

with the partial ordering

$$\begin{aligned} p \leq q \quad \Leftrightarrow \quad & p(i) \leq q(i) \text{ for all } i \in X \text{ and} \\ & \{\, i \in X \,:\, p(i) \lneqq q(i) \lneqq 1_{P_i} \,\} \text{ is finite .} \end{aligned}$$

For $\kappa = \aleph_0$ this is just a finite support product. We are mainly interested in the case where $\kappa = \aleph_1$. In this case we shall drop the subscript $\aleph_1$ and write simply $\Pi^*_{i\in X} P_i$. Further, if $P_i = P$ for some partial ordering P for every $x \in X$, we shall write $\Pi^*_{\kappa,X} P$ (or even $\Pi^*_X P$ when $\kappa = \aleph_1$) to denote this partial ordering.

For $p,\ q \in \Pi^*_{\kappa,i\in X} P_i$ the relation $p \leq q$ can be represented as a combination of the two other distinct relations which we shall call horizontal and vertical, and denote by $\leq_h$ and $\leq_v$ respectively:

$$p \leq_h q \quad \Leftrightarrow \quad supp(p) \supseteq supp(q) \text{ and } p\restriction supp(q) \subseteq q;$$

$$\begin{aligned} p \leq_v q \quad \Leftrightarrow \quad & supp(p) = supp(q),\ p(i) \leq q(i) \text{ for all } i \in X \text{ and} \\ & \{\, i \in X \,:\, p(i) \lneqq q(i) \lneqq 1_{P_i} \,\} \text{ is finite .} \end{aligned}$$

For $p \in \Pi^*_{\kappa,i\in X} P_i$ and $Y \subseteq X$ let $p\lceil Y$ denote the element of $\Pi^*_{\kappa,i\in X} P_i$ defined by $p\lceil Y(i) = 1_{P_i}$ for every $i \in X \setminus Y$ and $p\lceil Y(i) = p(i)$ for $i \in Y$.

The following is immediate from definition:

Lemma 2.6.3. *For $p,\ q \in \Pi^*_{\kappa,i\in X} P_i$, the following are equivalent:*

a) $p \leq q$;
b) *There is an $r \in \Pi^*_{\kappa,i\in X} P_i$ such that $p \leq_h r \leq_v q$;*
c) *There is an $s \in \Pi^*_{\kappa,i\in X} P_i$ such that $p \leq_v s \leq_h q$.*

Proof. b) $\Rightarrow$ a) and c) $\Rightarrow$ a) are clear. For a) $\Rightarrow$ b), let $r = p\lceil supp\, q$; for a) $\Rightarrow$ c), $s = q\restriction supp(q) \cup p\restriction (X \setminus supp(q))$. $\square$

Lemma 2.6.4.
1) *If P_i has the property K for all $i \in X$ then $P = \Pi^*_{i\in X} P_i$ preserves $\aleph_1$.*
2) *Suppose that $\lambda \leq \kappa$. If P_i has the strong λ-cc (i.e. for every $C \in [P_i]^\lambda$ there is pairwise compatible $D \in [C]^\lambda$), then $P = \Pi^*_{\kappa,i\in X} P_i$ preserves λ.*

Proof. This proof is a prototype of the arguments we are going to apply repeatedly. 1) and 2) can be proved similarly. For 1), assume that there would be $p \in P$ and a P-name $\dot{f}$ such that

*) $p \Vdash_P$ " $\dot{f} : (\omega_1)^V \to \omega$ and $\dot{f}$ is 1-1 ".

Then, let $(p_\alpha)_{\alpha<\omega_1}$ and $(q_\alpha)_{\alpha<\omega_1}$ be sequences of elements of P such that

a) $p_0 \leq p$ and $(p_\alpha)_{\alpha<\omega_1}$ is a descending sequence with respect to $\leq_h$;
b) $q_\alpha \leq_v p_\alpha$ and q_α decides $\dot{f}(\alpha)$ for all $\alpha < \omega_1$;
c) $p_\alpha\restriction S_\alpha = q_\alpha\restriction S_\alpha$ for every $\alpha < \omega_1$ where

$$S_\alpha = supp(q_\alpha) \setminus (supp(p) \cup \bigcup_{\beta<\alpha} supp(q_\beta)).$$

For $\alpha < \omega_1$ let $d_\alpha = \bigcup_{\beta<\alpha} supp(q_\beta)$. Then $(d_\alpha)_{\alpha<\omega_1}$ is a continuously increasing sequence in $[X]^{<\omega_1}$. Let $u_\alpha = \{\, \beta \in supp(q_\alpha) \,:\, q_\alpha(\beta) \neq p_\alpha(\beta) \,\}$ for $\alpha < \omega_1$. By b), u_α is finite and by c) we have $u_\alpha \subseteq d_\alpha$. Hence by Fodor's lemma, there exists an uncountable (actually even stationary) $Y \subseteq \omega_1$ such that $u_\alpha = u^*$ for all $\alpha \in Y$, for some fixed $u^* \in [X]^{<\aleph_0}$. Since $\Pi_{i\in u^*} P_i$ has the property K, there exists an uncountable $Y' \subseteq Y$ such that $\{\, q_\alpha\restriction u^* \,:\, \alpha \in Y' \,\}$ is pairwise compatible. It follows that q_α, $\alpha \in Y'$ are pairwise compatible. For each $\alpha \in Y'$ there exists an

$n_\alpha \in \omega$ such that $q_\alpha \Vdash_P$ " $n_\alpha = \dot{f}(\alpha)$ " by b). By *), n_α, $\alpha \in Y'$ must be pairwise distinct. But this is impossible as Y' is uncountable.

For 2), essentially the same proof works with sequences of elements of P of length λ, using the Δ-system lemma argument in place of Fodor's lemma.

□

Lemma 2.6.5. *If $\mid P_i \mid \leq 2^{<\kappa}$ for all $i \in X$, then $\Pi^*_{\kappa, i\in X} P_i$ has the $(2^{<\kappa})^+$-cc.*

Proof. By the usual Δ-system lemma argument. □

Corollary 2.6.6. a) *Under* CH, *if P_i satisfies the property K and $\mid P_i \mid \leq \aleph_1$ for every $i \in X$, then $P = \Pi^*_{i\in X} P_i$ preserves $\aleph_1$ and has the $\aleph_2$-cc. In particular P preserves every cardinals.*

b) *Suppose that $2^{<\kappa} = \kappa$. If P_i satisfies the strong λ-cc for every $\aleph_1 \leq \lambda \leq \kappa$ and $\mid P_i \mid \leq \kappa$ then $\Pi^*_{\kappa, i\in X} P_i$ preserves every cardinalities $\leq \kappa$ and has the κ^+-cc. In particular $\Pi^*_{\kappa, i\in X} P_i$ preserves every cardinals.*

Proof. By Lemmas 2.6.4, 2.6.5. □

Lemma 2.6.7. *For any $Y \subseteq X$ and $x \in X \setminus Y$, we have*

$$\Pi^*_{\kappa, i\in X} P_i \cong \Pi^*_{\kappa, i\in Y} P_i \times P_x \times \Pi^*_{\kappa, i\in X\setminus(Y\cup\{x\})} P_i.$$

Proof. The mapping from $\Pi^*_{\kappa, i\in X} P_i$ to $\Pi^*_{\kappa, i\in Y} P_i \times P_x \times \Pi^*_{\kappa, i\in X\setminus(Y\cup\{x\})} P_i$ defined by

$$p \mapsto (p\restriction Y, p(x), p\restriction (X \setminus (Y \cup \{x\})))$$

is an isomorphism. □

In the following we mainly use the partial orderings of the form $\mathrm{Fn}(\lambda, 2)$ for some λ as P_i in $\Pi^*_{\kappa, i\in X} P_i$. Note that $\mathrm{Fn}(\lambda, 2)$ has the property K and strong κ-cc in the sense above for every regular κ.

For a pseudo product of the form $\Pi^*_{i\in X} \mathrm{Fn}(\kappa_i, 2)$, Lemma 2.6.4 can be still improved:

Theorem* 2.6.8. (T. Miyamoto) *For any set X, and sequence $(\kappa_i)_{i\in X}$, the partial ordering $P = \Pi^*_{i\in X} \mathrm{Fn}(\kappa_i, 2)$ satisfies the Axiom A.* □

Lemma 2.6.9. *Suppose that $\mid P_i \mid \leq \kappa$ for every $i \in X$ and $P = \Pi^*_{\kappa^+, i\in X} P_i$. Then*

1) *If $\dot{x}$ is a P-name with $\Vdash_P$ " $\dot{x} \in V$ ", then for any $p \in P$ there is $q \in P$ such that $q \leq_h p$ and*

   (†) *for any $r \leq q$, if r decides $\dot{x}$ then $r\lceil supp(q)$ already decides $\dot{x}$.*

2) *Let G be P-generic. If $u \in V[G]$ is a subset of V of cardinality $< \kappa^+$, then there is a ground model set $X' \subseteq X$ of cardinality $\leq \kappa$ (in the sense of V) such that $u \in V[G \cap (\Pi^*_{\kappa^+, i\in X'} P_i)]$.*

Proof. 1): Let $\Phi : \kappa \to \kappa \times \kappa$; $\alpha \mapsto (\varphi_1(\alpha), \varphi_2(\alpha))$ be a surjection such that $\varphi_1(\alpha) \leq \alpha$ for every $\alpha < \kappa$. Let $(p_\alpha)_{\alpha<\kappa}$, $(p'_\alpha)_{\alpha<\kappa}$ and $(r_{\alpha,\beta})_{\alpha<\kappa,\beta<\kappa}$ be sequences of elements of P defined inductively by:

a) $p_0 = p$; $(p_\alpha)_{\alpha<\kappa}$ is a descending sequence with respect to $\leq_h$;

b) for a limit $\gamma < \kappa$, p_γ is such that $supp(p_\gamma) = \bigcup_{\alpha<\gamma} supp(p_\alpha)$ and, for $i \in supp(p_\gamma)$, $p_\gamma(i) = p_\alpha(i)$ for some $\alpha < \gamma$ such that $i \in supp(p_\alpha)$;

c) $(r_{\alpha,\beta})_{\beta<\kappa}$ is an enumeration of $\{\, r \in P \,:\, r \leq_v p_\alpha \,\}$;

d) let $r = r_{\varphi_1(\alpha),\varphi_2(\alpha)}$ and

$$p'_\alpha = r\restriction supp(r) \cup p_\alpha\restriction (X \setminus supp(r)).$$

If there is $s \leq_h p'_\alpha$ such that s decides $\dot{x}$, then let

$$p_{\alpha+1} = p_\alpha\restriction supp(p_\alpha) \cup s\restriction (X \setminus supp(p_\alpha)).$$

Otherwise let $p_{\alpha+1} = p_\alpha$.

Let $q \in \Pi^*_{\kappa, i\in X} P_i$ be defined by $supp(q) = \bigcup_{\alpha<\kappa} supp(p_\alpha)$ and, for $i \in supp(q)$, $q(i) = p_\alpha(i)$ for some $\alpha < \kappa$ such that $i \in supp(P_\alpha)$. We show that this q is as desired: suppose that $r \leq q$ decides $\dot{x}$. Then there is some $\alpha < \kappa$ such that

$$r\lceil supp(q) = p'_\alpha\restriction\ supp(p'_\alpha) \cup q\restriction (X \setminus supp(p'_\alpha)).$$

By d), it follows that $r\lceil supp(q) \leq r\lceil supp(p_{\alpha+1})$ decides $\dot{x}$.

2): Let $\dot{u}$ be a P-name for u and let $\dot{x}_\alpha$, $\alpha < \kappa$ be P-names such that $\Vdash_P$" $\dot{x}_\alpha \in V$" for every $\alpha < \kappa$ and $\Vdash_P$" $u = \{\, \dot{x}_\alpha \,:\, \alpha < \kappa \,\}$". By 1), for each $p \in P$, we can build a sequence $(p_\alpha)_{\alpha<\kappa}$ of elements of P decreasing with respect to $\leq_h$ such that $p_0 \leq_h p$ and

$(\dagger)_\alpha$ for any $r \leq p_\alpha$, if r decides $\dot{x}_\alpha$, then $r\lceil supp(p_\alpha)$ already decides $\dot{x}_\alpha$.

Let $q \in P$ be defined by $supp(q) = \bigcup_{\alpha<\kappa} supp(p_\alpha)$ and, for $i \in supp(q)$, $q(i) = p_\alpha(i)$ for some $\alpha < \kappa$ such that $i \in supp(p_\alpha)$. Then q satisfies:

$(\dagger\dagger)$ for any $r \leq q$, if r decides $\dot{x}_\alpha$ for some $\alpha < \kappa$, then $r\lceil supp(q)$ already decides $\dot{x}_\alpha$.

The argument above shows that q's with the property $(\dagger\dagger)$ are dense in P. Hence, by genericity, there is such $q \in G$. Clearly, $G \cap \Pi^*_{\kappa, i\in supp(q)} P_i$ contains every information needed to construct u. □

2.6.2. Some Consistency results.

Proposition 2.6.10. (CH) *For any infinite cardinal λ, let $P = \Pi^*_\lambda \mathrm{Fn}(\omega_1, 2)$. Then $\Vdash_P$" $\spadesuit = \lambda$".*

Proof.

Claim 2. $\Vdash_P$" $\spadesuit \geq \lambda$".

Proof of the Claim. If $\lambda = \aleph_1$ this is clear. So assume that $\lambda \geq \aleph_2$. For $\xi < \lambda$, let $\dot{f}_\xi$ be the P-name of the generic function from ω_1 to 2 added by the ξ-th copy of $\mathrm{Fn}(\omega_1, 2)$ in P. Let G be a P-generic filter over V. In $V[G]$ let $X \subseteq [\omega_1]^{\aleph_0}$ be such that $\mid X \mid < \lambda$. Then by $\aleph_2$-cc of P there exists $\xi < \lambda$ such that $X \in V[G']$ for $G' = G \cap \Pi^*_{\lambda\setminus\{\xi\}} \mathrm{Fn}(\omega_1, 2)$. Since $(\dot{f}_\xi)[G]$ is $\mathrm{Fn}(\omega_1, 2)$-generic over $V[G']$ by Lemma 2.6.7, we have $x \not\subseteq ((\dot{f}_\xi)[G])^{-1}\{0\}$ for every $x \in X$. □

Claim 3. $\Vdash_P$" $\spadesuit \leq \lambda$".

Proof of the Claim. For $u \in [\lambda]^{<\aleph_0}$, let $\dot{\mathcal{P}}_u$ be a P-name such that

$$\Vdash_P \text{“} \dot{\mathcal{P}}_u = ([\omega_1]^{\aleph_0})^{V[(\dot{f}_\xi)_{\xi\in u}]} \text{”}$$

where $\dot{f}_\xi$ is as in the proof of the previous claim. Let $\dot{\mathcal{P}}$ be a P-name such that

$$\Vdash_P \text{“} \dot{\mathcal{P}} = \bigcup \{\, \dot{\mathcal{P}}_u \,:\, u \in [\lambda]^{<\aleph_0} \,\} \text{”}.$$

For each $u \in [\lambda]^{<\aleph_0}$, $(\dot{f}_\xi[G])_{\xi\in u}$ corresponds to a generic filter over $\Pi^*_u \mathrm{Fn}(\omega_1, 2) \approx \mathrm{Fn}(\omega_1, 2)$. Hence, by CH, we have $\Vdash_P$" $\mid \dot{\mathcal{P}}_u \mid = \aleph_1$". It follows that $\Vdash_P$" $\mid \dot{\mathcal{P}} \mid = \lambda$". Thus it is enough to show that $\Vdash_P$" $\dot{\mathcal{P}}$ is a $\spadesuit$-set".

Let $p \in P$ and $\dot{A}$ be a P-name such that $p \Vdash_P$" $\dot{A} \in [\omega_1]^{\aleph_1}$". We show that there is an $r \leq p$ such that $r \Vdash_P$" $\exists x \in \dot{\mathcal{P}}\ x \subseteq \dot{A}$".

Now we proceed as in the proof of Lemma 2.6.4. Let $(p_\alpha)_{\alpha<\omega_1}$, $(q_\alpha)_{\alpha<\omega_1}$ be sequences of elements of P and $(\xi_\alpha)_{\alpha<\omega_1}$ be a strictly increasing sequence of ordinals $< \omega_1$ such that

a) $p_0 \leq p$ and $(p_\alpha)_{\alpha<\omega_1}$ is a descending sequence with respect to $\leq_h$;

b) $q_\alpha \leq_v p_\alpha$ and $q_\alpha \Vdash_P$" $\xi_\alpha \in \dot{A}$" for all $\alpha < \omega_1$;

c) $p_\alpha\restriction S_\alpha = q_\alpha\restriction S_\alpha$ for every $\alpha < \omega_1$ where

$$S_\alpha = supp(q_\alpha) \setminus (supp(p) \cup \bigcup_{\beta<\alpha} supp(q_\beta)).$$

For $\alpha < \omega_1$ let $u_\alpha = \{\, \beta \in supp(q_\alpha) \,:\, q_\alpha(\beta) \neq p_\alpha(\beta) \,\}$. As in the proof of Lemma 2.6.4, there exists $u^* \in [\lambda]^{<\aleph_0}$ such that $S = \{\, \alpha \in \omega_1 \,:\, u_\alpha = u^* \,\}$ is stationary. Now $(q_\alpha\restriction u)_{\alpha\in S}$

is an infinite sequence of elements of $P_{u^*} = \Pi_{u^*}\mathrm{Fn}(\omega_1, 2)$. Since P_{u^*} satisfies the ccc, there exists an $\varepsilon \in S$ and $\zeta < \omega_1$ such that $q_\varepsilon \restriction u^* \Vdash_{P_{u^*}}$ "$\{\, \xi \in S \cap \zeta \,:\, p_\xi \restriction u^* \in \dot{G} \,\}$ is infinite". Let $r = q_\varepsilon \cup p_\zeta \restriction (supp(p_\zeta) \setminus supp(p_\varepsilon))$. Let $\dot{b}$ be a P-name such that

$$r \Vdash_P \text{“}\, \dot{b} = \{\, \xi \in S \cap \zeta \,:\, q_\xi \restriction u^* \in \{\, p \restriction u^* \,:\, p \in \dot{G} \,\}\,\}\text{”}.$$

Let $\dot{x}$ be a P-name such that $r \Vdash_P$ "$\dot{x} = \{\, \xi_\alpha \,:\, \alpha \in \dot{b} \,\}$". Then $r \Vdash_P$ "$|\, \dot{x} \,| = \aleph_0$". Since $\dot{b}$ can be computed in $V[(\dot{f}_\xi[G])_{\xi \in u^*}]$ we have $r \Vdash_P$ "$\dot{x} \in \dot{\mathcal{P}}_{u^*}$". It is also clear by definition of $\dot{x}$ that $r \Vdash_P$ "$\dot{x} \subseteq \dot{A}$". □

□

Proposition above shows that $\spadesuit$ can be practically every thing. In particular we obtain:

Corollary 2.6.11. *The assertion* '$\mathrm{cof}(\spadesuit) = \omega$' *is consistent with* ZFC. □

The following Lemmas 2.6.12 and 2.6.14 show that, in spite of typographical similarity, $\Pi^*_\lambda\mathrm{Fn}(\omega_1, 2)$ and $\Pi^*_\lambda\mathrm{Fn}(\omega, 2)$ are quite different forcing notions: while the first one destroys ($\clubsuit$) or even ($\spadesuit$) by Lemma 2.6.10, the second one not only preserves a ($\clubsuit$)-sequence in the ground model but also creates such a sequence generically.

Lemma 2.6.12. *Let $S = (x_\gamma)_{\gamma \in E}$ be a $\clubsuit(E)$-sequence for a stationary $E \subseteq Lim(\omega_1)$. Let $P = \Pi^*_\kappa\mathrm{Fn}(\omega, 2)$ for arbitrary κ. Then we have $\Vdash_P$ "S is a $\clubsuit(E)$-sequence".*

Proof. Let $p \in P$ and $\dot{A}$ be a P-name such that $p \Vdash_P$ "$\dot{A} \in [\omega_1]^{\aleph_1}$". We show that there is $q \leq p$ and $\gamma \in E$ such that $q \Vdash_P$ "$x_\gamma \subseteq \dot{A}$". Let $\dot{f}$ be a P-name such that $p \Vdash_P$ "$\dot{f} : \omega_1 \to \dot{A}$ and $\dot{f}$ is 1-1". Let $(p_\alpha)_{\alpha<\omega_1}$ and $(q_\alpha)_{\alpha<\omega_1}$ be sequence of elements of P satisfying the conditions a) – c) in the proof of Lemma 2.6.4. Also, let u_α, $\alpha < \omega_1$ be as in the proof of Lemma 2.6.4. As there, we can find an uncountable $Y \subseteq \omega_1$ and $u^* \in [\kappa]^{<\aleph_0}$ such that $u_\alpha = u^*$ for all $\alpha \in Y$. Since $\Pi_{u^*}\mathrm{Fn}(\omega, 2)$ is countable we may assume that $q_\alpha \restriction u^*$ are all the same for $\alpha \in Y$. Now for each $\alpha \in Y$ let β_α be such that $q_\alpha \Vdash_P$ "$\dot{f}(\alpha) = \beta_\alpha$" and let $Z = \{\, \beta_\alpha \,:\, \alpha \in Y \,\}$. Since q_α, $\alpha \in Y$ are pairwise compatible, β_α, $\alpha \in Y$ are pairwise distinct and so Z is uncountable. Note that Z is a ground model set. Hence there exists $\gamma \in E$ such that $x_\gamma \subseteq Z$. Let $q = \bigcup_{\alpha \in Y \cap \gamma} q_\alpha$. Then $q \leq p$. Since $\sup\{\, \beta_\alpha \,:\, \alpha < \gamma \,\} \geq \gamma$ and $\Vdash_P$ "$\{\, \beta_\alpha \,:\, \alpha < \gamma \,\}$ is an initial segment of Z", we have $q \Vdash_P$ "$Z \cap \gamma \subseteq \dot{A}$". Hence $q \Vdash_P$ "$x_\gamma \subseteq \dot{A}$". □

Theorem 2.6.13. *"$\neg$CH+ MA(countable) + there exists a constructible $\clubsuit$-sequence" is consistent.*

Proof. We can obtain a model of the statement by starting from a model of $V = L$ and force with $P = \Pi^*_\kappa\mathrm{Fn}(\omega, 2)$ for a regular κ. By Corollary 2.6.6, every cardinal of V is preserved in $V[G]$. Since P adds κ many Cohen reals over V while $|\, P \,| = \kappa$ and P has the $\aleph_2$-cc, we have $V[G] \models$ "$2^{\aleph_0} = \kappa$". By Lemma 2.6.7, $V[G] \models$ "MA(*countable*)". By Lemma 2.6.12, the $\diamondsuit$-sequence in V remains a $\clubsuit$-sequence in $V[G]$. □

In fact, we do not need a $\clubsuit$-sequence in the ground model to get ($\clubsuit$) in the generic extension by $\Pi^*_\kappa\mathrm{Fn}(\omega, 2)$:

Lemma 2.6.14. *Let κ be uncountable and $P = \Pi^*_\kappa\mathrm{Fn}(\omega, 2)$. Then for any stationary $E \subseteq Lim(\omega_1)$ we have $\Vdash_P$ "$\clubsuit(E)$ holds".*

Proof. For $\gamma \in E$ let

$$f_\gamma : [\gamma,\ \gamma + \omega) \to \gamma$$

be a bijection and let

$$S_\gamma = \{\, x \subseteq \gamma \,:\, x \text{ is a cofinal subset of } \gamma,\ otp(x) = \omega \,\}.$$

For each $x \in S_\gamma$ let $p_x \in P$ be defined by

$$p_x = \{\, (\gamma + n, \{\, (0, i) \,\}) \,:\, n \in \omega,\ i \in 2,\ i = 1 \Leftrightarrow f_\gamma(\gamma + n) \in x \,\}.$$

For distinct $x, x' \in S_\gamma$, p_x and $p_{x'}$ are incompatible. Hence, for each $\gamma \in E$, we can find a P-name $\dot{x}_\gamma$ such that

$$\Vdash_P \text{“ } \dot{x}_\gamma \text{ is a cofinal subset of } \gamma \text{ and } otp(\dot{x}_\gamma) = \omega \text{ ”}$$

and

$$p_x \Vdash_P \text{“ } \dot{x}_\gamma = x \text{ ”} \quad \text{for each } x \in S_\gamma.$$

We show that $\Vdash_P$“ $(\dot{x}_\gamma)_{\gamma \in E}$ is a $\clubsuit(E)$-sequence”. For this, it is enough to show that, for any $p \in P$ and a P-name $\dot{A}$, if $p \Vdash_P$“ $\dot{A} \in [\omega_1]^{\aleph_1}$ ”, then there is $q \leq p$ and $\gamma \in E$ such that $q \Vdash_P$“ $\dot{x}_\gamma \subseteq \dot{A}$ ”. Let $\dot{f}$ be such that

$$p \Vdash_P \text{“ } \dot{f} : \omega_1 \to \dot{A} \text{ and } \dot{f} \text{ is 1-1 ”}.$$

Now let $(p_\alpha)_{\alpha<\omega_1}$, $(q_\alpha)_{\alpha<\omega_1}$, $(u_\alpha)_{\alpha\in\omega_1}$, Y and u^* be as in the proof of Lemma 2.6.4. For each $\alpha \in Y$ let β_α be such that $q_\alpha \Vdash_P$“ $\dot{f}(\alpha) = \beta_\alpha$ ” and let $Z = \{\, \beta_\alpha \,:\, \alpha \in Y \,\}$. Let

$$C = \{\, \gamma \in Lim(\omega_1) \,:\, \bigcup_{\alpha \in Y \cap \gamma} (supp(q_\alpha) \cap \omega_1) \subseteq \gamma \text{ and } Z \cap \gamma \text{ is unbounded in } \gamma \,\}.$$

Then C is closed unbounded in ω_1 and hence there exists a $\gamma^* \in C \cap E$. Let $q' = \bigcup_{\alpha \in Y \cap \gamma^*} q_\alpha$. Then we have $q' \leq q$ and $q' \Vdash_P$“ $Z \cap \gamma^* \subseteq \dot{A}$ ”. Now let $x \in S_{\gamma^*}$ be such that $x \subseteq Z \cap \gamma^*$. Finally let $q = q' \cup q_x$. Then we have $q \leq p$ and $q \Vdash_P$“ $\dot{x}_\alpha = x \subseteq Z \cap \gamma^* \subseteq \dot{A}$ ”. $\square$

Note that E's in Lemmas 2.6.12 and 2.6.14 are ground model sets. To force $\clubsuit(E)$ for every stationary $E \subseteq Lim(\omega_1)$ which may be also added generically, we need a sort of iteration described in [**6**].

2.6.3. On a theorem of Shapiro. Finally we mention one more application of our new type of product.

Theorem* 2.6.15 (L.B. Shapiro)**.** *If MA(Cohen) holds then for any compact Hausdorff space X of weight $< 2^{\aleph_0}$ and $\aleph_0 \leq \tau < 2^{\aleph_0}$ the following assertions are equivalent:*

(i) There exists a continuous surjection from X onto ${}^\tau\mathbb{I}$;
(ii) There exists a continuous injection from ${}^\tau 2$ into X;
(iii) There exists a closed subset $Y \subseteq X$ such that $\chi(y, Y) \geq \tau$ for every $y \in Y$.

In [**5**] we proved that MA(*countable*) is not enough to get the equivalence because we proved the following statement:

Theorem 2.6.16. *If principle ($\spadesuit$) holds, then there exists a Boolean algebra B of cardinality ω_1, such that $\mathrm{Fr}\,\omega_1$ is embeddable into B but there is no surjective Boolean mapping from B onto $\mathrm{Fr}\,\omega_1$.*

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