Class Numbers and Automorphic Forms

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Contents

1. Introduction

1.1. History and motivation

This dissertation deals with class number problems for quadratic number fields and with summation formulas for automorphic forms. Both subjects are important areas of number theory.

1.1.1. The class numbers of quadratic number fields were studied already by Gauss (he considered these questions in the language of quadratic forms though). Let $K = \mathbf{Q}(\sqrt{d})$, where $\mathbf{Q}$ is the rational field, and d is a fundamental discriminant. In the case of an imaginary quadratic field (i.e. $d < 0$) Gauss conjectured that if we denote by $h(d)$ the class number of K, we have $h(d) \to \infty$ as $|d| \to \infty$. This fact was first proved by Heilbronn in [Hei]. However, Heilbronn's solution was ineffective: the problem of determining all imaginary quadratic fields with class number 1 remained open for a long time. As it is well-known, it was first solved by Heegner ([Hee]), but his proof was not accepted at that time, and then it was also solved independently by Baker ([Ba]), and by Stark ([St]). Baker's solution was an immediate consequence of his famous theorem on logarithms of algebraic numbers, using earlier work of Gelfond and Linnik ([G-L]).
The situation is completely different for a general real quadratic field ($d > 0$): Gauss conjectured for this case that there are infinitely many d with class number 1. This problem is still unsolved.
However, for some special families of real quadratic fields (where the fundamental unit is very small), e.g. when $d = p^2 + 4$ with some integer p, the situation is analogous to the imaginary case: it was known for a long time that there are only finitely many fields with class number one in such a family, but the effective determination of these finitely many fields constitutes a separate problem. Chapter 2 of the present dissertation discusses the solution of Yokoi's conjecture: this conjecture stated that $h\left(p^2+4\right) > 1$ for $p > 17$.

1.1.2. In general, as it is mentioned on p. 65 of [I-K], an identity connecting one series of an arithmetic function (weighted by a test function of certain class) with another is called a summation formula. The most well-known summation formulas used in analytic number theory are the Poisson formula and the Voronoi formula. We will consider such summation formulas where the arithmetic functions are related to automorphic forms.

Automorphic forms play a central role in modern number theory. They are important both in analytic and algebraic number theory, but they are related also to many other fields of mathematics, including representation theory, ergodic theory, combinatorics, algebraic geometry.

In the analytic theory of automorphic forms several summation formulas are very important. We just mention generalizations of the classical Voronoi formula, the Selberg trace formula and the Kuznetsov formula.

In Chapter 3 of our dissertation we will present such a summation formula which is formally very similar to the classical Poisson formula, but contains triple products of automorphic forms. Roughly speaking, a triple product is the integral of a product of three automorphic forms over a fundamental domain. Such triple products are subjects of intensive research in several directions: it is enough to mention the famous Quantum Unique Ergodicity Conjecture, solved recently by Lindenstrauss and Soundararajan in the nonholomorphic case ([Li] and [So]) and by Holowinsky and Soundararajan in the holomorphic case ([H-S]), or the representation theoretic work [B-R) of Bernstein and Reznikov giving nontrivial upper bounds for triple products.

1.1.3. My interest in both subjects originates from my PhD thesis, which contained more or less the material of my papers [Bi1] and [Bi2].

The connection is more direct in the case of Chapter 3, since [Bi1] and [Bi2] dealt with automorphic forms, in particular, in [Bi1] I proved a summation formula including automorphic quantities: a generalization of the Selberg trace formula.

However, the subject of Chapter 2 is also related to automorphic forms. To see this connection in the most simple way, we note that one side of the Selberg trace formula contains a summation over conjugacy classes of a discrete subgroup Γ of $SL(2, \mathbf{R})$, see Chapter 10 of [I1]. If we choose $\Gamma = SL(2, \mathbf{Z})$, then these conjugacy classes are related to class numbers of a family of real quadratic fields with very small fundamental unit. Indeed, the subset of $\Gamma = SL(2, \mathbf{Z})$ with a given trace t, i.e.

$$\Gamma_t = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \ a, b, c, d \in \mathbf{Z}, \ ad - bc = 1, \ a + d = t \right\},$$

is obviously a union of conjugacy classes. It can be shown that there is a one-to-one correspondence between the conjugacy classes contained in Γ_t and the $SL(2, \mathbf{Z})$-equivalence classes of the integer quadratic forms with discriminant $d = t^2 - 4$. Hence for a given integer $t > 2$ the set Γ_t is a union of $h\left(t^2 - 4\right)$ conjugacy classes, and the fields $\mathbf{Q}(\sqrt{d})$ with $d = t^2 - 4$ have very small fundamental unit.

Moreover, the very first version of my proof of Yokoi's conjecture used automorphic forms: for the proof of the very important Lemma 2.1 (see Chapter 2) I expressed the function $\zeta_{P(K)}(s, \chi)$ there by integrals of Eisenstein series over certain closed geodesics of the Riemann surface obtained by factorizing the open upper half-plane by $SL(2, \mathbf{Z})$. Then, when I gave my first talk on the proof of Yokoi's conjecture in Oberwolfach in September 2001, the paper [Sh1] of Shintani was drawn to my attention by S. Egami. Using Shintani's paper I could simplify my original proof of Lemma 2.1, and the new proof (presented also here in Chapter 2) have not used already automorphic forms.

1.2. Class number problems for special real quadratic fields

Today we know that the fact (mentioned already in Subsection 1.1.1) that there are only finitely many imaginary quadratic fields with class number one is an immediate consequence of Dirichlet's class number formula and Siegel's theorem. To see this, and to analyze also the real case, we first state Dirichlet's class number formula (using [W], Chapter 3 and p. 37).

Let $K = \mathbf{Q}(\sqrt{d})$, where d is a (positive or negative) fundamental discriminant, let $h(d)$ be the class number of K, and let χ_d be the real primitive character associated to K. Then for $d < 0$ we have

$$h(d) = \frac{w\,|d|^{1/2}}{2\pi} L(1, \chi_d), \qquad (1.2.1)$$

where w is the number of roots of unity in K; for $d > 0$ we have

$$h(d) \log \epsilon_d = d^{1/2} L(1, \chi_d), \tag{1.2.2}$$

where $\epsilon_d > 1$ is the fundamental unit in K. Using Siegel's theorem for the value at 1 of a Dirichlet L-function:

$$L(1, \chi_d) \gg_\epsilon |d|^{-\epsilon}$$

(which is an ineffective estimate), we see that (1.2.1) implies indeed that there are only finitely many solutions of the imaginary class number one problem. However, for $d > 0$, we can not separate the class number and the fundamental unit. But, if we assume that the fundamental unit is small, e.g.

$$\log d \ll \log \epsilon_d \ll \log d, \tag{1.2.3}$$

then (1.2.2) implies that $h(d) > 1$ for large d. But since we used Siegel's theorem, the estimate obtained is ineffective, we cannot determine in this way all fields with class number one in a given family satisfying (1.2.3), e.g. in the family of Yokoi's discriminants $d = p^2+4$. In Chapter 2 we prove Yokoi's conjecture (formulated in [Y], and mentioned already in Subsection 1.1.1). More precisely we prove the following

THEOREM 1.1 ([Bi3]). *If d is squarefree, $h(d) = 1$ and $d = p^2 + 4$ with some integer p, then d is a square for at least one of the following moduli: $q = 5, 7, 41, 61, 1861$ (that is, $(d/q) = 0$ or 1 for at least one of the listed values of q).*

Combining this with the well-known fact that if $h(d) = 1$ then d is a quadratic nonresidue modulo any prime r with $2 < r < p$ (for the sake of completeness, we will prove it, see our Fact B stated in Section 2.2), we obtain the main result of Chapter 2:

COROLLARY 1.1 ([Bi3]). *If d is squarefree, and $d = p^2+4$ with some integer $p > 1861$, then $h(d) > 1$.*

It is easy to prove on the basis of the above-mentioned Fact B that $h(d) > 1$ if $17 < p \le 1861$, see the last part of Section 2.2 (this statement follows also from [Mi]), so we have a full solution of Yokoi's conjecture. Note that there are six exceptional fields where $h(d) = 1$, belonging to $p = 1, 3, 5, 7, 13, 17$.

The same proof with minor modifications works for Chowla's conjecture, which is a similar class number one problem (this was formulated in [C-F]). We presented that proof in the paper [Bi4]. The method was applied later to several similar cases, see e.g. [B-K-L] and [Le].

But it seems that in Yokoi's case the present proof works only for the class number one problem, the class number 2 problem (for example) remains open. But, of course, the harder problem of giving an effective lower bound tending to infinity for $h(p^2+4)$ (the similar statement in the imaginary case was proved by Goldfeld, Gross, Zagier, see [Go] and [G-Z]) is also open. We mentioned above that the fundamental unit is small (hence Siegel's theorem is applicable), however, its logarithm is as large as $\log p$, so it is large enough to cause a problem if one wants to apply the Goldfeld-Gross-Zagier method.

The starting point of our proof is an idea of the paper [Be] of J. Beck. In that paper he excluded some residue classes for p, i.e. he gave effective upper bounds for p in the class number 1 case provided p belongs to certain residue classes. He combined elementary number theory with formulas for special values of zetafunctions related to K and certain quadratic Dirichlet characters. In our proof, we use zetafunctions related to nonquadratic Dirichlet characters; this leads us to elementary algebraic number theory. Using also new elementary ingredients, we are able to exclude all residue classes modulo a given concrete modulus, hence to prove the conjecture.

Up until this proof, only quadratic characters have been used in the proof as "parameters". I mean that in the quoted paper of Beck, and also in the classical work of Gelfond-Linnik-Baker in the imaginary case, besides the quadratic Dirichlet character belonging to the given quadratic field K, there are other Dirichlet characters, and one can consider them as parameters, since one tries to choose them in a way which is most useful for the proof. Now, in the present proof these parameter characters are not quadratic. This provides a lot of new possibilities for excluding residue classes for p. The use of such characters was made possible by proving our Lemma 2.1 (see Section 2.2 for its statement), which gives a useful expression for the value at 0 of some zetafunctions. We will give a more detailed sketch of the proof in Section 2.2.

The proof requires also computer work. We emphasize that the results of the computations made by the computer program given in Section 2.5 are important for the proof of Theorem 1.1 (which is a theoretical result). So we think that this computer program belongs to the proof, consequently, for the sake of completeness it is necessary to give its details. However, if one is willing to accept the results of the computer work, one can skip Section 2.5.
As it was pointed out in [Bi5], the proof of Yokoi's conjecture can be considered to be an analogue of the Gelfond-Linnik-Baker solution of the imaginary class number one problem. But at first sight they seem to be very different, since Baker's theorem on logarithms is replaced here by elementary algebraic number theory. We return to this question in Section 2.2.

1.3. A Poisson-type formula including automorphic quantities

1.3.1. In this section we will discuss the result of Chapter 3. In order to be able to describe our formula it is unavoidable to introduce first a few notations concerning automorphic forms. Then, before actually describing the formula, we will give such an interpretation of the classical Poisson formula which will help us to show that our formula is analogous to the Poisson formula.

1.3.2. Notations. We denote by H the open upper half plane. We write

$$\Gamma_0(4) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z}) : \ c \equiv 0 \pmod 4 \right\}.$$

let D_4 be a fundamental domain of $\Gamma_0(4)$ on H, let

$$d\mu_z = \frac{dxdy}{y^2}$$

(this is the $SL(2, \mathbf{R})$-invariant measure on H), and introduce the notation

$$(f_1, f_2) = \int_{D_4} f_1(z)\overline{f_2(z)} d\mu_z.$$

Introduce the hyperbolic Laplace operator of weight l:

$$\Delta_l := y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - ily \frac{\partial}{\partial x}.$$

For a complex number $z \neq 0$ we set its argument in $(-\pi, \pi]$, and write $\log z = \log|z| + i \arg z$,where $\log|z|$ is real. We define the power z^s for any $s \in \mathbf{C}$ by $z^s = e^{s \log z}$. We write $e(x) = e^{2\pi i x}$ and $(w)_n = \frac{\Gamma(w+n)}{\Gamma(w)}$, as usual.
For $z \in H$ we write $\theta(z) = \sum_{m=-\infty}^{\infty} e(m^2 z)$, and we define

$$B_0(z) := (\mathrm{Im} z)^{\frac{1}{4}} \theta(z). \tag{1.3.1}$$

If ν is the well-known multiplier system (see e.g. [Du], (2.1) for its explicit form), we have

$$B_0(\gamma z) = \nu(\gamma) \left(\frac{j_\gamma(z)}{|j_\gamma(z)|} \right)^{1/2} B_0(z) \text{ for } \gamma \in \Gamma_0(4),$$

where for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{R})$ we write $j_\gamma(z) = cz + d$. Note that $\nu^4 = 1$.
Let $l = \frac{1}{2} + 2n$ or $l = 2n$ with some integer n. We say that a function f on H is an automorphic form of weight l for $\Gamma = SL(2, \mathbf{Z})$ or $\Gamma_0(4)$ (but, if $l = \frac{1}{2} + 2n$, we can take only $\Gamma = \Gamma_0(4)$), if it satisfies, for every $z \in H$ and $\gamma \in \Gamma$, the transformation formula

$$f(\gamma z) = \left(\frac{j_\gamma(z)}{|j_\gamma(z)|} \right)^l f(z)$$

in the case $l = 2n$,

$$f(\gamma z) = \nu(\gamma) \left(\frac{j_\gamma(z)}{|j_\gamma(z)|} \right)^l f(z)$$

in the case $l = \frac{1}{2} + 2n$, and f has at most polynomial growth in cusps. The operator Δ_l acts on smooth automorphic forms of weight l. We say that f is a Maass form of weight l for Γ, if f is an automorphic form, it is a smooth function, and it is an eigenfunction on H of the operator Δ_l. If a Maass form f has exponential decay at cusps, it is called a (Maass) cusp form.
Denote by $L_l^2(D_4)$ the space of automorphic forms of weight l for $\Gamma_0(4)$ for which we have $(f, f) < \infty$.
Take $u_{0,1/2} = c_0 B_0$, where c_0 is chosen such that $(u_{0,1/2}, u_{0,1/2}) = 1$. It is not hard to prove (using [Sa], p. 290) that the only Maass form (up to a constant factor) of weight $\frac{1}{2}$ for $\Gamma_0(4)$ with $\Delta_{1/2}$-eigenvalue $-\frac{3}{16}$ is B_0, and the other eigenvalues are smaller. Let

$u_{j,1/2}$ $(j \geq 0)$ be a Maass form orthonormal basis of the subspace of $L^2_{1/2}(D_4)$ generated by Maass forms, write

$$\Delta_{1/2} u_{j,1/2} = \Lambda_j u_{j,1/2},\ \Lambda_j = S_j(S_j - 1),\ S_j = \frac{1}{2} + iT_j,$$

then $\Lambda_0 = -\frac{3}{16}$, $\Lambda_j < -\frac{3}{16}$ for $j \geq 1$, and $\Lambda_j \to -\infty$.

For the cusps $a = 0, \infty$ denote by $E_a\left(z, s, \frac{1}{2}\right)$ the Eisenstein series of weight $\frac{1}{2}$ for the group $\Gamma_0(4)$ at the cusp a (for precise definition see Section 2). As a function of z, it is an eigenfunction of $\Delta_{1/2}$ of eigenvalue $s(s-1)$. If f is an automorphic form of weight $1/2$ and the following integral is absolutely convergent, introduce the notation

$$\zeta_a(f, r) := \int_{D_4} f(z) \overline{E_a\left(z, \frac{1}{2} + ir, \frac{1}{2}\right)} d\mu_z.$$

If $l \geq 1$ is an integer, let $S_{l+\frac{1}{2}}$ be the space of holomorphic cusp forms of weight $l + \frac{1}{2}$ with the multiplier system ν^{1+2l} for the group $\Gamma_0(4)$ (sse [I2], Section 2.7). Note that $\nu^{1+2l} = \nu$ if and only if l is even.

We will be mainly concerned with the case when l is even. If $k \geq 1$, let $f_{k,1}, f_{k,2}, ..., f_{k,s_k}$ be an orthonormal basis of $S_{2k+\frac{1}{2}}$, and write $g_{k,j}(z) = (\mathrm{Im} z)^{\frac{1}{4}+k} f_{k,j}(z)$. We note that $g_{k,j}$ is a Maass cusp form of weight $2k + \frac{1}{2}$, and $\Delta_{2k+\frac{1}{2}} g_{k,j} = \left(k + \frac{1}{4}\right)\left(k - \frac{3}{4}\right) g_{k,j}$ (see [F], formulas (4) and (7)).

We also introduce the Maass operators

$$K_k := (z - \overline{z}) \frac{\partial}{\partial z} + k = iy \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + k,$$

$$L_k := (\overline{z} - z) \frac{\partial}{\partial \overline{z}} - k = -iy \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} - k.$$

For basic properties of these operators see [F], pp. 145-146. We just mention now that if f is a Maass form of weight k, then $K_{k/2} f$ and $L_{k/2} f$ are Maass forms of weight $k+2$ and $k-2$, respectively.

1.3.3. Poisson's summation and our formula. Now, to state the Poisson formula, consider the space of smooth, 1-periodic functions on the real line $\mathbf{R}$, and let $D = \frac{d}{dx}$ be the derivation operator. Then the eigenfunctions of D in this space are the functions

$e^{2\pi inx}$, the eigenvalues are $2\pi in$, and these eigenfunctions form an orthonormal basis of the Hilbert space $L^2(\mathbf{Z}\setminus\mathbf{R})$. We parametrize the eigenvalues with the numbers n, these parameters are contained in the set $\mathbf{R}$, and the Poisson formula states that if F is a "nice" function on $\mathbf{R}$ and we write $w(n)=1$ for every n, then the expression

$$\sum_{n=-\infty}^{\infty} w(n)F(n)$$

remains unchanged if we replace F by G, where G is the Fourier transform of F. We inserted the notation $w(n)$ for the identically 1 function to emphasize the analogy, since in our case we will indeed have nontrivial weights.

In our case, instead of the smooth, 1-periodic functions on $\mathbf{R}$, consider all the smooth automorphic forms on H of any weight $\frac{1}{2}+2k$, where $k\geq 0$ is any integer. Instead of the eigenfunctions of D, we will consider the eigenfunctions of the operators $\Delta_{2k+\frac{1}{2}}$, $k\geq 0$. In fact, if $k\geq 0$ is fixed, the eigenfunctions of $\Delta_{2k+\frac{1}{2}}$ are almost in a one-to-one correspondence with the eigenfunctions of $\Delta_{2(k+1)+\frac{1}{2}}$ through the Maass operators, except that the eigenfunctions of weight $2(k+1)+\frac{1}{2}$ corresponding to holomorphic forms are annihilated by $L_{(k+1)+\frac{1}{4}}$. Hence, the essentially different eigenfunctions of the operators $\Delta_{2k+\frac{1}{2}}$ (playing a role in the spectral expansion of functions in the spaces $L^2_{2k+\frac{1}{2}}(D_4)$) are the following:

$$u_{j,1/2}\ (j\geq 0),\quad E_a\left(*,\frac{1}{2}+ir,\frac{1}{2}\right)\ (a=0,\infty,\ r\in\mathbf{R}),\quad g_{k,j}\ (k\geq 1, 1\leq j\leq s_k).$$

If u is one of these functions, we will parametrize its Laplace eigenvalue by a number T such that

$$\Delta_{2k+\frac{1}{2}}u=\left(\tfrac{1}{2}+iT\right)\left(-\tfrac{1}{2}+iT\right)u$$

with the suitable k. In particular, this parameter will be

$$T_j \text{ in case of } u_{j,1/2}\ ,\quad r \text{ in case of } E_a\left(*,\frac{1}{2}+ir,\frac{1}{2}\right),\quad i\left(\frac{1}{4}-k\right) \text{ in case of } g_{k,j}\ .$$

These numbers correspond to the numbers n in Poisson's formula. In our case these parameters are contained (at least with finitely many possible exceptions: call j exceptional, if $T_j \notin \mathbf{R}$) in the set $\mathbf{R} \cup D^+$, where

$$D^+ = \left\{ i\left(\frac{1}{4} - k\right) : \ k \geq 1 \text{ is an integer} \right\}. \qquad (1.3.2)$$

Now, in fact we prove not just one summation formula, but many formulas: to every pair u_1,u_2 of Maass cusp forms of weight 0 there will correspond a summation formula. So let us fix two such cusp forms. Our formula states that there are some weights $w_{u_1,u_2}(j)$, $w_{u_1,u_2}(a,r)$ and $w_{u_1,u_2}(k,j)$ such that if F is a "nice" function on $\mathbf{R} \cup D^+$, even on $\mathbf{R}$ (note that "nice" will mean, in particular, that the continuous part of F, i.e. the restriction of F to $\mathbf{R}$, extends as a holomorphic function to a relatively large strip containing $\mathbf{R}$, so we can speak about $F(T_j)$ even for the exceptional js), then the expression

$$\sum_{j=0}^{\infty} w_{u_1,u_2}(j)F(T_j) + \sum_{a=0,\infty} \int_{-\infty}^{\infty} w_{u_1,u_2}(a,r)F(r)\,dr + \sum_{k=1}^{\infty}\sum_{j=1}^{s_k} w_{u_1,u_2}(k,j)F\left(i\left(\frac{1}{4} - k\right)\right)$$

remains unchanged if we write $\overline{u_2}$ in place of u_1, $\overline{u_1}$ in place of u_2, and we replace F by G, where G is obtained from F by applying a certain integral transform which maps functions on $\mathbf{R} \cup D^+$, even on $\mathbf{R}$ again to such functions: this integral transform is a so-called Wilson function transform of type II, which was introduced quite recently by Groenevelt in [G1]. This integral transform plays the role what the Fourier transform played in the case of Poisson's formula. We will speak in more detail about the Wilson function transform of type II in Subsection 1.3.5 below. We just mention here that it shares some nice properties of the Fourier transform: it is an isometry on a suitably defined Hilbert space, and it is its own inverse (this last property is true at least on the even functions in the case of the Fourier transform).

The weights w_{u_1,u_2} in the above formula contain very interesting automorphic quantities. We give now only $w_{u_1,u_2}(j)$, since the other weights will be analogous, and everything will be given precisely in the theorem. So we will have for $j \geq 0$ that $w_{u_1,u_2}(j)$ equals

$$\Gamma\left(\frac{3}{4} + iT_j\right)\Gamma\left(\frac{3}{4} - iT_j\right)\int_{D_4} B_0(z)\,u_1(4z)\,\overline{u_{j,\frac{1}{2}}(z)}d\mu_z \overline{\int_{D_4} B_0(z)\,u_2(4z)\,\overline{u_{j,\frac{1}{2}}(z)}d\mu_z}.$$

1.3.4. Remarks on relations to other works and on possible future work. We have shown above that there is a strong formal analogy between our summation formula and the Poisson summation formula. I guess that this analogy may be deeper, perhaps there is a common generalization of the two formulas. I think that the explanation of this analogy and the proof of further generalization (perhaps even for groups of higher rank) may come from representation theory. Such an approach could be useful also for the understanding of the appearance of the Wilson function transform of type II in the formula, which is rather mysterious at the moment. A representation theoretic interpretation of this integral transform was given by Groenevelt himself in [G2], but it does not seem to help in the explanation of our formula. However, it is possible that the general method of [R] for proving spectral identities may be useful in better understanding of our formula.

Spectral identities having similarities to our result were proved by several authors. We mention e.g. the concrete identities proved in the above-mentioned paper [R] (as an application of the general method there), and the paper [B-M], whose method of proof based directly on the spectral structure of the space $L^2(SL(2,\mathbf{Z})\backslash SL(2,\mathbf{R}))$ may be also important in the context of our formula.

But, as far as I see, the nearest relative of our result is an identity suggested by Kuznetsov in [K] and proved by Motohashi in [Mo]. The weights are different there than in our case, but the structure of the two formulas are very similar. Indeed, on the one hand, the summation is over Laplace-eigenvalues *and* integers in both cases. On the other hand, in the case of both identities we have the same type of weights on both sides of the given identity. That formula has been successfully applied already to analytic problems (see [Iv], [J]), so perhaps our formula also may be applied along similar lines for the estimation of the weights w_{u_1,u_2}, hence the estimation of triple products, especially in view of the fact that in the case $u_1 = u_2$ the weights are nonnegative.

We mention finally that the weights $w_{u_1,u_2}(j)$ (or rather their absolute values squared) given at the end of Subsection 1.3.3 are (at least in some cases, and at least conjecturally) closely related to central values of L-functions. Indeed, let us assume that $u_{j,1/2}$ is an eigenfunction of the Hecke operator T_{p^2} (of weight 1/2) for every prime $p \neq 2$, and that $u_{j,1/2}$ is an eigenfunction of the operator L of eigenvalue 1 (see [K-S] for the definitions of the operators T_{p^2} and L). Assume also that the first Fourier coefficient at ∞ of $u_{j,1/2}$ is nonzero. Then $\mathrm{Shim}u_{j,1/2}$ (the Shimura lift of $u_{j,1/2}$) is defined in [K-S], pp 196-197. It is a Maass cusp form of weight 0 which is a simultaneous Hecke eigenform. If u_1 and u_2 are also simultaneous Hecke eigenforms, then by the Theorem of [Bi6] we see that $w_{u_1,u_2}(j)$ is closely related to

$$\int_{SL(2,\mathbf{Z})\backslash H} |u_1(z)|^2 \left(\mathrm{Shim}u_{j,1/2}\right)(z)\, d\mu_z \int_{SL(2,\mathbf{Z})\backslash H} |u_2(z)|^2 \left(\mathrm{Shim}u_{j,1/2}\right)(z)\, d\mu_z,$$

at least if we accept the unproved but likely statement that the sum in (1.4) of [Bi6] is a one-element sum (see Remark 2 of [Bi6] and Remark (a) on p 197 of [K-S]). Using the formula of Watson (see [Wat]) we finally get that $|w_{u_1,u_2}(j)|^2$ is closely related to

$$L\left(\frac{1}{2}, u_1 \times u_1 \times \mathrm{Shim}u_{j,1/2}\right) L\left(\frac{1}{2}, u_2 \times u_2 \times \mathrm{Shim}u_{j,1/2}\right).$$

1.3.5. Wilson function transform of type II. For the statement of our result the Wilson function transform of type II (introduced in [G1]) is needed. This transform will be discussed in more detail in Subsection 3.3.1, here we just give the most basic properties. Let t_1 and t_2 be two given nonzero real numbers (these numbers will come from the Laplace-eigenvalues of two cusp forms, see Theorem 1.2 below). We will define explicitly in terms of t_1 and t_2 a positive number C and a positive even function $H(x)$ on the real line in (3.3.2) and (3.3.1). Let D^+ as in (1.3.2), and for functions F on $\mathbf{R} \cup D^+$, even on $\mathbf{R}$ write

$$\int F(x)dh(x) := \frac{C}{2\pi} \int_0^\infty F(x)H(x)dx + iC \sum_{x \in D^+} F(x)\mathrm{Res}_{z=x}H(z).$$

The numbers

$$R_k = \mathrm{Res}_{z=i\left(\frac{1}{4}-k\right)} H(z)$$

will be given explicitly in (3.3.3), and it will turn out that iR_k is positive for every k. For any complex numbers λ and x the Wilson function

$$\phi_\lambda(x) = \phi_\lambda(x; a, b, c, d)$$

is defined in [G1], formula (3.2). We will use parameters a, b, c, d depending only on t_1 and t_2, and we will give them explicitly in Subsection 3.3.1. We define the Hilbert space $\mathcal{H}$ to be the space consisting of functions on $\mathbf{R} \cup D^+$, even on $\mathbf{R}$ that have finite norm with respect to the inner product

$$(f, g)_{\mathcal{H}} = \int f(x)\overline{g(x)} dh(x).$$

Then the Wilson function transform of type II is defined in [G1] as

$$(\mathcal{G}F)(\lambda) = \int F(x)\phi_\lambda(x)\, dh(x).$$

It is defined first (as in the case of the classical Fourier transform) on the dense subspace of $\mathcal{H}$ where this is absolutely convergent. Then it extends to $\mathcal{H}$, and the following nice theorem is proved in [G1], Theorem 5.10 (it will be explained in Subsection 3.3.1 that in our case Theorem 5.10 of [G1] has this form):

The operator $\mathcal{G} : \mathcal{H} \to \mathcal{H}$ is unitary, and $\mathcal{G}$ is its own inverse.

The second statement will be important for us, i.e. that $\mathcal{G}$ is its own inverse.

Since we will work separately with the continuous and discrete part of a function F on $\mathbf{R} \cup D^+$, even on $\mathbf{R}$, we introduce notations for them:

$$f(x) := F(x) \ (x \in \mathbf{R}), \qquad a_n := F\left(i\left(\frac{1}{4} - n\right)\right) \ (n \geq 1).$$

So instead of F, we will speak about a pair consisting of an even function f on $\mathbf{R}$ and a sequence $\{a_n\}_{n\geq 1}$. In this language, the Wilson function transform of type II of the pair f, $\{a_n\}_{n\geq 1}$ is the pair of the function g and the sequence $\{b_n\}_{n\geq 1}$ defined by

$$g(\lambda) = \frac{C}{2\pi}\int_0^\infty f(x)\phi_\lambda(x)\, H(x)dx + iC\sum_{k=1}^\infty a_k\phi_\lambda\left(i\left(\frac{1}{4} - k\right)\right) R_k \tag{1.3.3}$$

and

$$b_n = \frac{C}{2\pi}\int_0^\infty f(x)\phi_{i\left(\frac{1}{4}-n\right)}(x)\, H(x)dx + iC\sum_{k=1}^\infty a_k\phi_{i\left(\frac{1}{4}-n\right)}\left(i\left(\frac{1}{4} - k\right)\right) R_k \tag{1.3.4}$$

for $n \geq 1$.

1.3.6. The formula. We now state precisely the summation formula. We use the notation $\Gamma(X \pm Y) = \Gamma(X+Y)\Gamma(X-Y)$. If u is a cusp form of weight 0 for $SL(2,\mathbf{Z})$ with $\Delta_0 u = s(s-1)u$, for $n \geq 0$ define a cusp form $\kappa_n(u)$ of weight $2n$ for the group $\Gamma_0(4)$ by

$$(\kappa_n(u))(z) = \frac{(K_{n-1}K_{n-2}\dots K_1K_0u)(4z)}{(s)_n(1-s)_n}.$$

THEOREM 1.2 ([Bi7]). *Let $u_1(z)$ and $u_2(z)$ be two Maass cusp forms of weight 0 for $SL(2,\mathbf{Z})$ with Laplace-eigenvalues $s_j(s_j-1)$, where $s_j = \frac{1}{2} + it_j$ and $t_j > 0$ ($j = 1,2$). There is a positive constant K depending only on u_1 and u_2 such that proerty $P(f,\{a_n\})$ below is true, if $f(x)$ is an even holomorphic function for $|\mathrm{Im}x| < K$ satisfying that*

$$\left|f(x)e^{-2\pi|x|}(1+|x|)^K\right|$$

is bounded on the domain $|\mathrm{Im}x| < K$, and $\{a_n\}_{n\geq 1}$ is a sequence satisfying that

$$\left|n^{K+\frac{3}{2}}\left(a_n - \frac{(-1)^n}{n^{3/2}}\sum_{0\leq m<K}\frac{c_m}{n^m}\right)\right|$$

is bounded for $n \geq 1$ with some constants c_m (m runs over integers with $0 \leq m < K$).

Property $P(f,\{a_n\})$**.** *By g and b_n defined in (1.3.3) and (1.3.4) the sum of the following three lines:*

$$\sum_{j=1}^{\infty} f(T_j)\Gamma\left(\frac{3}{4} \pm iT_j\right)\left(B_0\kappa_0(u_1), u_{j,\frac{1}{2}}\right)\overline{\left(B_0\kappa_0(u_2), u_{j,\frac{1}{2}}\right)}, \tag{1.3.5}$$

$$\frac{1}{4\pi}\sum_{a=0,\infty}\int_{-\infty}^{\infty} f(r)\Gamma\left(\frac{3}{4} \pm ir\right)\zeta_a(B_0\kappa_0(u_1), r)\overline{\zeta_a(B_0\kappa_0(u_2), r)}dr, \tag{1.3.6}$$

$$\sum_{n=1}^{\infty} a_n\Gamma\left(2n+\frac{1}{2}\right)\sum_{j=1}^{s_n}(B_0\kappa_n(u_1), g_{n,j})\overline{(B_0\kappa_n(u_2), g_{n,j})} \tag{1.3.7}$$

equals the sum of the following three lines:

$$\sum_{j=1}^{\infty} g(T_j)\Gamma\left(\frac{3}{4} \pm iT_j\right)\left(B_0\kappa_0(\overline{u_2}), u_{j,\frac{1}{2}}\right)\overline{\left(B_0\kappa_0(\overline{u_1}), u_{j,\frac{1}{2}}\right)}, \tag{1.3.8}$$

$$\frac{1}{4\pi}\sum_{a=0,\infty}\int_{-\infty}^{\infty} g(r)\Gamma\left(\frac{3}{4} \pm ir\right)\zeta_a(B_0\kappa_0(\overline{u_2}), r)\overline{\zeta_a(B_0\kappa_0(\overline{u_1}), r)}dr, \tag{1.3.9}$$

$$\sum_{n=1}^{\infty} b_n \Gamma\left(2n+\frac{1}{2}\right) \sum_{j=1}^{s_n} \left(B_0 \kappa_n\left(\overline{u_2}\right), g_{n,j}\right) \overline{\left(B_0 \kappa_n\left(\overline{u_1}\right), g_{n,j}\right)}. \tag{1.3.10}$$

The sums and integrals in (1.3.3) and (1.3.4) are absolutely convergent for $|\mathrm{Im}\lambda| < \frac{3}{4}$ *and* $n \geq 1$*, and every sum and integral in (1.3.5)-(1.3.10) is absolutely convergent.*

The class of functions appearing in the theorem seems to be sufficiently general, but it may happen that the statement can be extended further for some other functions.

1.4. An expansion theorem for Wilson functions

For the proof of Theorem 1.2 it is necessary to know some properties of Wilson functions. But we prove these results only in the Appendix (i.e. in Chapter 4), since they are completely independent of automorphic forms, they belong to the area of special functions. However, we think that one of these results is interesting enough to be stated here, in the Introduction.

Let t_1, t_2, $H(x)$ and $\phi_\lambda(x)$ have the same meaning as in Subsection 1.3.5 above. So t_1 and t_2 are fixed, hence every variable and every O-constant may depend on t_1 and t_2, even if we do not denote this dependence.

The next theorem shows that a nice enough even function on $\mathbf{R}$ satisfying a vanishing property can be written as a linear combination of the functions $\phi_{i\left(\frac{1}{4}-N\right)}(x)$ $(N \geq 1)$.

THEOREM 1.3 ([Bi8]). *Assume that* K *is a positive number, and* $f(x)$ *is an even holomorphic function for* $|\mathrm{Im}x| < K$ *satisfying*

$$\int_{-\infty}^{\infty} f(\tau) H(\tau) \frac{1}{\Gamma\left(\frac{3}{4} \pm i\tau\right)} d\tau = 0 \tag{1.4.1}$$

and that

$$\left| f(x) e^{-2\pi|x|} (1+|x|)^K \right|$$

is bounded on the domain $|\mathrm{Im}x| < K$*. If* k *is a positive integer and* K *is large enough in terms of* k*, then we have a sequence* d_n *satisfying*

$$d_n = \frac{(-1)^n}{n^{5/2}} \left(\sum_{j=0}^{k} \frac{e_j}{n^j} + O\left(\frac{1}{n^{k+1}}\right) \right) \tag{1.4.2}$$

with some constants e_j *and*

$$f(x) = \sum_{n=1}^{\infty} d_n \phi_{i\left(\frac{1}{4}-n\right)}(x) \tag{1.4.3}$$

for every $|\mathrm{Im} x| < \frac{3}{4}$*, and the sum on the right-hand side of (1.4.3) is absolutely convergent for every such* x*.*

2. Yokoi's Conjecture

2.1. Structure of the chapter

In this chapter we prove Theorem 1.1. In Section 2.2 we give the plan of the proof, in Section 2.3 we prove the important Lemma 2.1 and Fact B mentioned already in the Introduction, in Section 2.4 we fix the numerical parameters, in Section 2.5 we give a BASIC program. Finally, in Section 2.6 we give the results of this computer program and conclude the proof of Theorem 1.1. This chapter is based mostly on [Bi3], but uses also [Bi5].

2.2. Outline of the proof

We use the notations of Section 1.2 and we introduce some new notations. Let R be the ring of algebraic integers of K, denote by $I(K)$ the set of nonzero ideals of R and by $P(K)$ the set of nonzero principal ideals of R. Let $N(a)$ be the norm of an $a \in I(K)$, i.e. its index in R. Let $q > 2$ be an integer with $(q, d) = 1$ (remember that $d = p^2 + 4$), and let χ be an odd (i.e we assume $\chi(-1) = -1$) primitive character with conductor q. (This will be the parameter character.) For $\Re s > 1$ define

$$\zeta_K(s) = \sum_{a \in I(K)} \frac{1}{N(a)^s}, \quad \zeta_K(s,\chi) = \sum_{a \in I(K)} \frac{\chi(N(a))}{N(a)^s},$$

and

$$\zeta_{P(K)}(s,\chi) = \sum_{a \in P(K)} \frac{\chi(N(a))}{N(a)^s}.$$

It is well-known (see e.g. [W], Theorems 4.3 and 3.11) that

$$\zeta_K(s) = \zeta(s) L(s, \chi_d), \tag{2.2.1}$$

where

$$\chi_d(n) = \left(\frac{n}{d}\right)$$

is a Jacobi symbol; moreover, if $h(d) = 1$, then d is a prime (see Fact B below), so this is a Legendre symbol. It follows easily that

$$\zeta_K(s,\chi) = L(s,\chi)L(s,\chi\chi_d).$$

It is also well-known (see e.g. [W], Theorem 4.2 and [Da], Chapter 9) that for a primitive character ψ with $\psi(-1) = -1$ and with conductor f one has

$$L(0,\psi) = -\frac{1}{f}\sum_{a=1}^{f} a\psi(a) \neq 0.$$

Consequently, since $\chi\chi_d$ is a primitive character with conductor qd by our conditions, and $\chi_d(-1) = 1$ because d is congruent to 1 modulo 4, so

$$\zeta_K(0,\chi) = \frac{1}{q^2 d}\left(\sum_{a=1}^{q} a\chi(a)\right)\left(\sum_{b=1}^{qd} b\chi(b)\chi_d(b)\right). \tag{2.2.2}$$

Now, if $h(d) = 1$, then

$$\zeta_K(s,\chi) = \zeta_{P(K)}(s,\chi) \tag{2.2.3}$$

by definition. In the next section we will prove

LEMMA 2.1. *If $d = p^2 + 4$ is squarefree, $q > 2$ is an integer with $(q,d) = 1$, and χ is a primitive character modulo q with $\chi(-1) = -1$, then $\zeta_{P(K)}(s,\chi)$ extends meromorphically in s to the whole complex plane and*

$$\zeta_{P(K)}(0,\chi) = \frac{1}{q}A_\chi(p),$$

where $\lceil t \rceil$ is the least integer not smaller than t, and for any integer a we write

$$A_\chi(a) = \sum_{0 \le C,D \le q-1} \chi(D^2 - C^2 - aCD)\lceil (aC - D)/q \rceil (C - q).$$

Note that qd divides the sum

$$\Sigma = \sum_{x=0}^{d-1} (l + xq)\chi_d(l + xq)$$

for any fixed $1 \le l \le q$. Indeed, the numbers $l + xq$ give a complete system of residues modulo d, so

$$\Sigma \equiv l \sum_{\text{y mod } d} \chi_d(y) = 0 \pmod{q}, \quad \Sigma \equiv \sum_{\text{y mod } d} y\chi_d(y) = 0 \pmod{d},$$

since χ_d is an even nonprincipal character modulo d. Now,

$$\sum_{b=1}^{qd} b\chi(b)\chi_d(b) = \sum_{l=1}^{q} \chi(l) \sum_{x=0}^{d-1} (l + xq)\chi_d(l + xq),$$

so using (2.2.2), (2.2.3), Lemma 2.1 and the last remark, we obtain the following

FACT A. *If $d = p^2 + 4$ is squarefree, $h(d) = 1$, q is an integer with $q > 2$, $(q, d) = 1$, and χ is a primitive character modulo q with $\chi(-1) = -1$, then, writing*

$$m_\chi = \sum_{a=1}^{q} a\chi(a),$$

we have that $m_\chi \ne 0$, and

$$A_\chi(p)m_\chi^{-1}$$

is an algebraic integer.

We will prove that Theorem 1.1 follows from Fact A.

First we introduce the following notation. If m is an odd positive integer, we denote by U_m the set of rational integers a satisfying that

$$\left(\frac{a^2 + 4}{r}\right) = -1$$

for every prime divisor r of m. Observe that U_m is a union of certain residue classes modulo m.

We assume that $h(d) = 1$. We will use Fact A in the following way. Denote by $\mathcal{L}_\chi$ the field generated over $\mathbf{Q}$ by the values $\chi(a)$ $(1 \le a \le q)$, and take a prime ideal I of $\mathcal{L}_\chi$ such that

$$m_\chi \in I. \tag{2.2.4}$$

Let

$$p = Pq + p_0 \text{ with } 0 \le p_0 < q, \tag{2.2.5}$$

then it is easy to see that

$$A_\chi(p) = PB_\chi(p_0) + A_\chi(p_0), \tag{2.2.6}$$

where for any integer a we write

$$B_\chi(a) = \sum_{0\le C,D\le q-1} \chi(D^2 - C^2 - aCD)C(C-q). \tag{2.2.7}$$

We then obtain by (2.2.4), (2.2.6) and Fact A that

$$PB_\chi(p_0) + A_\chi(p_0) \equiv 0 \pmod{I}. \tag{2.2.8}$$

Assume that q is odd, and that $p \in U_q$ (equivalently $p_0 \in U_q$). Observe that this already determines the ideal generated by $B_\chi(p_0)$. Indeed, if $a_1, a_2 \in U_q$, then

$$(B_\chi(a_1)) = (B_\chi(a_2)), \tag{2.2.9}$$

i.e. $B_\chi(a_1)$ and $B_\chi(a_2)$ generate the same ideal in the ring of integers of $\mathcal{L}_\chi$. We will show this statement at the end of this section. (Note that (2.2.9) is not important for the proof, but we think it is worth remarking.) Assume also that the positive integers q and r satisfy the following condition:

Condition $(*)$. *The integer q is odd, r is an odd prime, and there is an odd primitive character χ with conductor q and there is a prime ideal I of $\mathcal{L}_\chi$ lying above r such that $m_\chi \in I$, but I does not divide the ideal generated by $B_\chi(a)$ in the ring of integers of $\mathcal{L}_\chi$, if a is any rational integer with $a \in U_q$.*

Then, since $p_0 \in U_q$, we obtain by (2.2.8) that

$$P \equiv -\frac{A_\chi(p_0)}{B_\chi(p_0)} \pmod{I},$$

where we divide in the residue field of I. Combining it with (2.2.5), we see that

$$p \equiv p_0 - q\frac{A_\chi(p_0)}{B_\chi(p_0)} \pmod{I}. \tag{2.2.10}$$

Let q and p_0 be fixed. Note that in principle it may happen, if the residue field of I is not the prime field (in our concrete applications, the residue field will always be the prime field), that there is no rational integer p satisfying (2.2.10); but anyway, if there are solutions, then all the solutions belong to a unique residue class modulo r, since I lies above r. This implies that if we know q and p_0, then we can specify a congruence class modulo r such that p must belong to this class.

Summing up: let $h(d) = 1$, and let q and r satisfy Condition $(*)$. Then, if p is in a given congruence class modulo q such that $p \in U_q$, this forces p to be in a certain residue class modulo r; then we can test whether $p \in U_r$ or not. This is our key new elementary tool, and Theorem 1.1 follows by several applications of this tool. The technicalities of this are very roughly as follows.

Denote by $q \to r$ that q and r satisfy Condition $(*)$ above. We could say that we defined a directed graph (with the positive integers as vertices) in this way. We will use a certain triangle in this graph. To be concrete, we will use the arrows (more precisely, the special cases belonging to these arrows of the above-mentioned tool):

$$175 \to 61,\ 175 \to 1861,\ 61 \to 1861.$$

There are 40 residue classes modulo $175 = 5^2 \cdot 7$ contained in U_{175}, so we may assume that p belongs to one of these classes. For 20 of these classes, the arrow $175 \to 61$ forces p into a residue class modulo 61 which is not contained in U_{61}. The arrow $175 \to 1861$ similarly eliminates 10 of the remaining residue classes, so 10 possible residue classes remain for p modulo 175.

Next we apply also the arrow $61 \to 1861$, and we find that for eight of the remaining residue classes modulo 175, different residue classes modulo 1861 are prescribed for p by consecutive application of the two arrows

$$175 \to 61,\ 61 \to 1861,$$

and by the arrow $175 \to 1861$. This contradiction eliminates these classes. We are left with

$$p \equiv \pm 13 \pmod{175 \cdot 61 \cdot 1861}.$$

We then use a new arrow

$$61 \to 41,$$

and this finally forces p to residue classes modulo 41 which are not contained in U_{41}. This will prove Theorem 1.1.

We explain briefly how we found the triangle 61,175,1861. It is clear that if q and r satisfy Condition $(*)$, then there is an odd primitive character χ with conductor q such that r divides the norm of m_χ (this is a necessary, but not sufficient condition for $(*)$). Now, such divisibility relations can be found by the table on pp. 353-360. of [W]: this table lists relative class numbers of cyclotomic fields, and in view of Theorem 4.17 of [W], relative class numbers are closely related to the norms of such numbers m_χ.

To deduce Corollary 1.1 we use the following

FACT B. *If* $d = p^2 + 4$ *is squarefree and* $h(d) = 1$, *then* d *is a prime, and if* $2 < r < p$ *is also a prime, then*

$$\left(\frac{d}{r}\right) = -1$$

(Legendre symbol).

We prove it in the next section.

The small values of p, i.e. the cases $1 \le p \le 1861$, are easily handled by Fact B. In fact, it can be checked by an easy calculation that if $1 \le p \le 1861$ is an odd integer and $p \ne 1, 3, 5, 7, 13, 17$, then there is a prime $3 \le r \le 31$ such that $r < p$ and

$$\left(\frac{p^2+4}{r}\right) \ne -1.$$

Hence Yokoi's conjecture is proved.

Examining the proof, we see that Yokoi's conjecture follows from Facts A and B by elementary algebraic number theory and a finite amount of computation. I think that the present way is not the only one to prove the conjecture on the basis of these two facts.

We also see that in order to get the linear congruence (2.2.8), it was very important that once χ, its conductor q and the residue of p modulo q are fixed, then $\zeta_{P(K)}(0, \chi)$ depends linearly on p (see Lemma 2.1, (2.2.5) and (2.2.6)). In the case of quadratic characters χ, this linear dependence was proved by Beck in [Be].

We now try to explain why the proof of Yokoi's conjecture can be considered to be an analogue of the Gelfond-Linnik-Baker solution of the imaginary class number one problem, in spite of applying so different tools (elementary algebraic number theory is used here in place of Baker's theorem). Again, let d be a fundamental discriminant, and let $\chi_d(n) = \left(\frac{n}{d}\right)$. The equation

$$\zeta_K(s,\chi) = L(s,\chi)L(s,\chi\chi_d)$$

was the basis of the Gelfond-Linnik-Baker solution of the imaginary class number one problem, and this is used also here. Gelfond and Linnik considered the $s = 1$ case in the above equation (but this is equivalent to the substitution $s = 0$ because of the functional equation). If ψ is a primitive Dirichlet character modulo q, then the arithmetic nature of $L(1,\psi)$ depends on the parity of ψ: it is π times an algebraic number for odd ψ, and it is a linear combination of logarithms of algebraic numbers, if ψ is even. It is known that if $d < 0$, then χ_d is odd, and if $d > 0$, then χ_d is even. This implies that in the imaginary case ($d < 0$) it is sure, by any choice of the parameter character χ, that one of the characters χ and $\chi\chi_d$ is odd and the other one is even. Therefore, one of $L(1,\chi)$ and $L(1,\chi\chi_d)$ is a linear form of logarithms of algebraic numbers, and we are led to Baker's theorem. However, if $d > 0$, and we choose an odd χ, then both of χ and $\chi\chi_d$ are odd, $L(1,\chi)/\pi$ and $L(1,\chi\chi_d)/\pi$ are algebraic numbers, and this leads to elementary algebraic number theory.

Finally, we prove formula (2.2.9). By (2.2.7), we have

$$\frac{\chi(4)}{\chi(a_1^2+4)}B_\chi(a_1) = \sum_{0\le C,D\le q-1} \chi\Big(\frac{(2D-a_1C)^2}{a_1^2+4} - C^2\Big)C(C-q), \tag{2.2.11}$$

where dividing by a_1^2+4 means multiplying by its inverse modulo q (which exists by the assumption that $a_1 \in U_q$). Now, if C is fixed, then $(2D - a_1C)$ runs over a complete residue system modulo q. A similar formula is valid for a_2 in place of a_1. Since

$$(a_2^2+4)(a_1^2+4)^{-1}$$

is the square of a reduced residue class modulo q, if $a_1, a_2 \in U_q$, so the right-hand side of (2.2.11) remains unchanged if we replace a_1 by a_2, hence (2.2.9) is proved. In fact one can say more about the numbers $B_\chi(a)$, especially if q is a prime, but we do not need it, so we do not analyze it any further.

2.3. Proof of Lemma 2.1 and Fact B

Before proving these two important results stated in Section 2.2, we introduce some further notations.

Let α be the positive root of the equation $x^2 + px = 1$. It is easily seen that $1, \alpha^{-1}$ is an integral basis of R, and $1, \alpha$ is also an integral basis. On the other hand, α^{-1} is the fundamental unit of K, this is true because the fundamental solution of $X^2 - (p^2+4)Y^2 = -4$ is $(X, Y) = (p, 1)$. Hence the units of R are $\pm\alpha^j$ with integer j. For $\beta \in R$, denote by $\overline{\beta}$ the algebraic conjugate of β, and let

$$Q(C, D) = D^2 - C^2 - pCD.$$

It is easy to verify that for

$$\beta = C + D\alpha^{-1}$$

with integers C,D one has

$$\beta\overline{\beta} = -Q(C, D). \tag{2.3.1}$$

Proof of Lemma 2.1. Suppose that (γ) is a principal ideal of R. If $\gamma < 0$, then replace γ by $-\gamma$. If, then, $\overline{\gamma} < 0$, replace γ by $\gamma\alpha^{-1}$, which is positive, and its conjugate, $\overline{\gamma}(\overline{\alpha})^{-1}$, is also positive. Therefore, without loss of generality, we may assume that $\gamma > 0$ and $\overline{\gamma} > 0$. The units of R which are positive and whose conjugate are also positive are $(\alpha^2)^j$ with integer j. So there is a unique $\beta \in R$ such that $(\gamma) = (\beta)$ and

$$\beta > 0,\ \overline{\beta} > 0,\ 1 \le \frac{\beta}{\overline{\beta}} < \alpha^{-4}.$$

Since α^{-2} is irrational, we can write any element of K as a $\mathbf{Q}$-linear combination of 1 and α^{-2}. Say

$$\beta = X + Y\alpha^{-2}.$$

Now

$$1 \le \frac{\beta}{\overline{\beta}} \Leftrightarrow \overline{\beta} \le \beta \Leftrightarrow Y(\alpha^{-2} - \alpha^2) \ge 0 \Leftrightarrow Y \ge 0.$$

Similarly

$$\frac{\beta}{\overline{\beta}} < \alpha^{-4} \Leftrightarrow \beta < \overline{\beta}\alpha^{-4} \Leftrightarrow X(\alpha^{-4} - 1) > 0 \Leftrightarrow X > 0.$$

We deduce that every principal ideal of R can be written in a unique way in the form (β), where

$$\beta \in R,\ \beta = X + Y\alpha^{-2} \text{ with some rationals } X > 0,\ Y \geq 0.$$

Next write $X = qx + qn_1$ and $Y = qy + qn_2$ for some nonnegative integers n_1 and n_2 and real numbers $0 < x \leq 1$, $0 \leq y < 1$ which can be done in a unique way. Then $\beta \in R$ if and only if

$$q(x + y\alpha^{-2}) \in R,$$

since, evidently, $q(n_1 + n_2\alpha^{-2}) \in qR$.
Now, since $C + D\alpha^{-1}$ with integers $0 \leq C, D \leq q-1$ form a complete system of representatives of R/qR, we can uniquely select an integer pair $0 \leq C, D \leq q-1$ such that

$$q(x + y\alpha^{-2}) \in C + D\alpha^{-1} + qR.$$

Therefore

$$x + y\alpha^{-2} - \frac{C + D\alpha^{-1}}{q} \in R. \tag{2.3.2}$$

Tracing this back gives

$$X + Y\alpha^{-2} \equiv C + D\alpha^{-1} \pmod{qR},$$

and since for the principal ideal a generated by $(X + Y\alpha^{-2})$ we have

$$N(a) = (X + Y\alpha^{-2})\overline{(X + Y\alpha^{-2})}$$

because $X > 0,\ Y \geq 0$, so

$$N(a) \equiv (C + D\alpha^{-1})\overline{(C + D\alpha^{-1})} = -Q(C, D) \pmod{q},$$

where we used (2.3.1). Therefore, using also (2.3.2), if we partition the $\beta \in R$ according to the associated values for C and D we obtain the following formula of Shintani (p. 595. of [Sh2]):

$$\zeta_{P(K)}(s, \chi) = \frac{-1}{q^{2s}} \sum_{C,D=0}^{q-1} \chi(Q(C, D)) \sum_{(x,y)\in R(C,D)} \zeta\left(s, \begin{pmatrix} 1 & \alpha^{-2} \\ 1 & \alpha^{2} \end{pmatrix}, (x, y)\right) \tag{2.3.3}$$

with the following notations: $R(C,D)$ denotes the set

$$\left\{(x,y)\in \mathbf{Q}^2:\ 0<x\le 1,\ 0\le y<1,\ x+y\alpha^{-2}-\frac{C+D\alpha^{-1}}{q}\in R\right\},$$

and for a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with positive entries and $x>0$, $y\ge 0$ we write

$$\zeta\left(s,\begin{pmatrix} a & b \\ c & d \end{pmatrix},(x,y)\right)$$

for the function

$$\sum_{n_1,n_2=0}^{\infty}\left(a(n_1+x)+b(n_2+y)\right)^{-s}\left(c(n_1+x)+d(n_2+y)\right)^{-s}.$$

The key result we need to quote is easily deduced from the Corollary to Proposition 1 of [Sh1]:

Proposition (Shintani). *For any $a,b,c,d,x>0$ and $y\ge 0$ the function*

$$\zeta\left(s,\begin{pmatrix} a & b \\ c & d \end{pmatrix},(x,y)\right),$$

which is absolutely convergent for $\Re s>1$, extends meromorphically in s to the whole complex plane and the special value

$$\zeta\left(0,\begin{pmatrix} a & b \\ c & d \end{pmatrix},(x,y)\right)$$

equals

$$B_1(x)B_1(y)+\frac{1}{4}\left(B_2(x)\left(\frac{c}{d}+\frac{a}{b}\right)+B_2(y)\left(\frac{d}{c}+\frac{b}{a}\right)\right),$$

where B_1 and B_2 are the Bernoulli polynomials

$$B_1(z)=z-\frac{1}{2},\ B_2(z)=z^2-z+\frac{1}{6}.$$

We thus can substitute the result of this proposition into (2.3.3) to evaluate $\zeta_{P(K)}(0,\chi)$. Using that

$$\alpha^{-2}+\alpha^2=p^2+2,$$

we obtain

$$\zeta_{P(K)}(0,\chi) = -\sum_{0\le C,D\le q-1} \chi(Q(C,D))\Sigma_{C,D}, \tag{2.3.4}$$

where $\Sigma_{C,D}$ denotes the sum

$$\sum_{(x,y)\in R(C,D)} \left(-\frac{p^2}{2}xy - \frac{p^2+4}{4}(x+y) + \frac{p^2+2}{4}(x+y)^2 + \frac{p^2+5}{12}\right).$$

To investigate $\Sigma_{C,D}$ for a fixed pair $0 \le C, D \le q-1$, we observe that for any m, n we have

$$\frac{m\alpha^{-1}+n}{q} = \frac{(n-\frac{m}{p}) + \frac{m}{p}\alpha^{-2}}{q},$$

and so it is easy to see that the possibilities for (m,n) having $(x,y) \in R(C,D)$ with

$$(x,y) = \left(\frac{1}{q}(n-\frac{m}{p}), \frac{1}{q}\frac{m}{p}\right)$$

are

$$m_j = D + jq, \ \ n_j = C + q\left\lfloor 1 + \frac{j}{p} - \frac{(pC-D)/q}{p}\right\rfloor$$

with any integer $0 \le j \le p-1$. This is so because the possible values of m are obviously these p values, and once m is fixed, n is unique.

One has

$$0 < 1 + \frac{j}{p} - \frac{(pC-D)/q}{p} < 2,$$

so

$$n_j = C \text{ for } 0 \le j < A$$

and

$$n_j = C + q \text{ for } A \le j < p,$$

where we put

$$A = \lceil (pC-D)/q \rceil,$$

and clearly $0 \le A \le p$.

So we have

$$\Sigma_{C,D} = \sum_{j=0}^{p-1}\left(-\frac{p^2}{2q^2}(n_j - \frac{m_j}{p})\frac{m_j}{p} - \frac{p^2+4}{4q}n_j + \frac{p^2+2}{4q^2}n_j^2 + \frac{p^2+5}{12}\right).$$

By the description of n_j and m_j above, considering separately the cases $0 \le j < A$ and $A \le j < p$, using the summation formulas for $\sum_{j=0}^{N} j$ and $\sum_{j=0}^{N} j^2$ (for any integer $N \ge 0$), straightforward (but tedious) calculations give

$$\Sigma_{C,D} = A(1 - \frac{C}{q}) + \frac{p}{4q^2}\Sigma_{C,D}^{(1)} - \frac{1}{4q}\Sigma_{C,D}^{(2)}, \tag{2.3.5}$$

where

$$\Sigma_{C,D}^{(1)} = 2C^2 + D^2 + (D - pC + qA)^2,$$

and

$$\Sigma_{C,D}^{(2)} = 2pC + (p-2)D + (p+2)(D - pC + qA).$$

Remember that A depends on C and D, but for brevity we do not denote it.

We show that

$$\sum_{0 \le C,D \le q-1} \chi(Q(C,D))\Sigma_{C,D}^{(j)} = 0$$

for $j = 1, 2$. To this end we introduce the transformation

$$T\left((C,D)\right) = (\hat{C}, \hat{D})$$

with

$$\hat{C} = D - pC - q\left[(D - pC)/q\right], \quad \hat{D} = C$$

(here we used lower integer part). We will also use the notation

$$T^2\left((C,D)\right) = (\hat{\hat{C}}, \hat{\hat{D}}).$$

Note that $\hat{C}$ (similarly to $\hat{\hat{C}}$, $\hat{D}$ and $\hat{\hat{D}}$) depends on the pair (C, D). The transformation T is a permutation of the set of the pairs (C, D) with $0 \le C, D \le q - 1$.

Now, observe that

$$qA = pC - D + \hat{C}.$$

Using this relation, and

$$C = \hat{D}, \ \ \hat{C} = \hat{\hat{D}},$$

we obtain the identities

$$\Sigma^{(1)}_{C,D} = \left(D^2 + \left(\hat{D}\right)^2\right) + \left(\left(\hat{D}\right)^2 + \left(\hat{\hat{D}}\right)^2\right)$$

and

$$\Sigma^{(2)}_{C,D} = (p-2)\left(D + \hat{D}\right) + (p+2)\left(\hat{D} + \hat{\hat{D}}\right).$$

It is easy to verify that

$$Q(\hat{C}, \hat{D}) \equiv -Q(C, D) \pmod q,$$

hence

$$\chi\left(Q(\hat{C}, \hat{D})\right) = -\chi\left(Q(C, D)\right),$$

since χ is odd. Consequently, any orbit of T (where χ is not 0) has an even number of elements, and the value of $\chi\left(Q(C, D)\right)$ changes to its negative at each step by T. Our last identities then show that in fact, when we substitute (2.3.5) into (2.3.4), the terms

$$\Sigma^{(1)}_{C,D},\ \Sigma^{(2)}_{C,D}$$

give 0 after the summation over C, D (since they give 0 on each orbit). Lemma 2.1 is proved.

For the proof of Fact B, we need the following lemma.

LEMMA 2.2. *If* $0 \neq \beta \in R$, *and* $|\beta\overline{\beta}| < p$, *then* β *is associated in* R *to a rational integer.*

Proof. Let $\beta = c\alpha - d$ with integers c and d. We may assume that $\alpha \leq |\beta| \leq 1$ and $c > 0$ (since for $c = 0$ we are done). Then

$$|\overline{\beta}| = |c\frac{1}{\alpha} + d| = |c(\alpha + \frac{1}{\alpha}) - \beta| \geq c(\alpha + \frac{1}{\alpha}) - 1,$$

hence

$$p > |\beta\overline{\beta}| \geq c - \alpha.$$

The right-hand side is greater than $p-1$ for $c \geq p$, so we have $1 \leq c < p$. Then $0 < c\alpha < 1$, and by $|\beta| \leq 1$ we can assume $d = 1$, because in the case $d = 0$ the proof is complete. Then

$$p > |\beta\overline{\beta}| = 1 - c^2 + pc,$$

which is impossible for c in the given range, and the lemma is proved.

Proof of Fact B. Assume that d is not a prime (but, by our assumptions, it is odd and squarefree). Let t be the least prime divisor of d. Since $(p,d)=1$, and $(p+1)^2$ is greater than d, we have $2<t<p$. The discriminant of K is d, hence the prime t is ramified in K, so the ideal generated by t in R is a square of an ideal, say $(t)=a^2$. The class number is 1, so $a=(\beta)$ with some $0\neq\beta\in R$, and this implies that

$$|\beta\overline{\beta}|=N(a)=t,$$

hence $|\beta\overline{\beta}|<p$ and $|\beta\overline{\beta}|$ is not a square, which is a contradiction by Lemma 2.2.

So we know that d is a prime, it is obviously congruent to 1 modulo 4, and by quadratic reciprocity it is enough to prove that $\left(\frac{r}{d}\right)=-1$. Assume that $\left(\frac{r}{d}\right)=1$. It is well-known (and we can see from (2.2.1)) that the ideal (r) is then a product of two prime ideals in R; both prime ideals must have norm r. Since the class number is 1, it follows that there is a $0\neq\beta\in R$ such that $|\beta\overline{\beta}|=r$, and since $r<p$ and r is not a square, this contradicts Lemma 2.2, just as above. Fact B is proved.

2.4. Fixing the parameters

We will use the notations introduced in Section 2.2.

We will use Fact A for three concrete characters χ, denote them by χ_1, χ_2 and χ_3. The character χ_1 has conductor $175=5^2\cdot 7$, while χ_2 and χ_3 have conductor 61.

Since 2 is a primitive root modulo 25, and 3 is a primitive root modulo 7, the character χ_1 is well defined by

$$\chi_1=\chi_1^{(25)}\chi_1^{(7)},$$

where $\chi_1^{(25)}$ is a character modulo 25, $\chi_1^{(7)}$ is a character modulo 7, and

$$\chi_1^{(25)}(2)=i\xi,\ \chi_1^{(7)}(3)=\omega,$$

where ξ is a primitive fifth root of unity, i is the usual primitive fourth root of unity, and ω is a primitive third root of unity. It is easily seen that χ_1 is a primitive character modulo 175 and $\chi_1(-1)=-1$.

Since 2 is a primitive root modulo 61, the characters χ_2 and χ_3 are well defined by

$$\chi_2(2) = i\omega\xi,\ \chi_3(2) = i\xi.$$

These are obviously primitive characters modulo 61, and

$$\chi_2(-1) = \chi_3(-1) = -1.$$

Clearly

$$\mathcal{L}_\chi = \mathbf{Q}(\xi_{60}) \text{ for } \chi = \chi_1 \text{ and } \chi = \chi_2,$$

and

$$\mathcal{L}_\chi = \mathbf{Q}(\xi_{20}) \text{ for } \chi = \chi_3,$$

where ξ_n denotes a primitive nth root of unity.

Before giving the concrete examples we will work with, we quote a well-known general fact on the factorization of rational primes in cyclotomic fields. Let r be a rational prime and assume that

$$r \equiv 1 \pmod{n}. \tag{2.4.1}$$

Then, in the ring of algebraic integers of $\mathbf{Q}(\xi_n)$ the ideal (r) is a product of $\phi(n)$ distinct prime ideals, and these prime ideals have the form

$$(r, \xi_n - a), \tag{2.4.2}$$

where a runs over the rational integers $1 \le a \le r$ with

$$o_r(a) = n, \tag{2.4.3}$$

and $o_r(a)$ denotes the order of a modulo r. (See [W], pp. 14-15.) What we actually need is the fact that in the case of (2.4.1), the ideal (2.4.2) is a prime ideal for every rational integer a satisfying (2.4.3).

We now give our four examples. These examples correspond to the four arrows

$$175 \to 61,\ 175 \to 1861,\ 61 \to 1861,\ 61 \to 41,$$

respectively, mentioned in Section 2.2.

Example 1. Here

$$q = 175,\ r = 61,\ \chi = \chi_1,\ \mathcal{L}_\chi = \mathbf{Q}(\xi_{60}),$$

and we choose

$$I = (61, i\omega\xi - 10).$$

Since $o_{61}(10) = 60$, this is a prime ideal. We then have

$$\chi_1^{(25)}(2) = (i\omega\xi)^{21} \equiv 10^{21} \equiv 8 \pmod{I},$$

and

$$\chi_1^{(7)}(3) = (i\omega\xi)^{40} \equiv 10^{40} \equiv 47 \pmod{I}.$$

Consequently, for rational integers a,

$$\text{if } a \equiv 2^s \pmod{25}, \text{ then } \chi_1^{(25)}(a) \equiv 8^s \pmod{I}, \tag{2.4.4}$$

$$\text{if } a \equiv 3^t \pmod{7}, \text{ then } \chi_1^{(7)}(a) \equiv 47^t \pmod{I}. \tag{2.4.5}$$

Example 2. Here

$$q = 175,\ r = 1861,\ \chi = \chi_1,\ \mathcal{L}_\chi = \mathbf{Q}(\xi_{60}),$$

and we choose

$$I = (1861, i\omega\xi - 173).$$

Since $o_{1861}(173) = 60$, this is a prime ideal. We then have, just as above,

$$\chi_1^{(25)}(2) \equiv 173^{21} \equiv 380 \pmod{I},\ \text{and}\ \chi_1^{(7)}(3) \equiv 173^{40} \equiv 1406 \pmod{I}.$$

Consequently, for rational integers a,

$$\text{if } a \equiv 2^s \pmod{25}, \text{ then } \chi_1^{(25)}(a) \equiv 380^s \pmod{I}, \tag{2.4.6}$$

$$\text{if } a \equiv 3^t \pmod{7}, \text{ then } \chi_1^{(7)}(a) \equiv 1406^t \pmod{I}. \tag{2.4.7}$$

Example 3. Here

$$q = 61,\ r = 1861,\ \chi = \chi_2,\ \mathcal{L}_\chi = \mathbf{Q}(\xi_{60}),$$

and we choose

$$I = (1861, i\omega\xi - 1833).$$

Since $o_{1861}(1833) = 60$, this is a prime ideal. We then have, for rational integers a:

$$\text{if } a \equiv 2^s \pmod{61}, \text{ then } \chi(a) \equiv 1833^s \pmod{I}. \tag{2.4.8}$$

Example 4. Here

$$q = 61,\ r = 41,\ \chi = \chi_3,\ \mathcal{L}_\chi = \mathbf{Q}(\xi_{20}),$$

and we choose

$$I = (41, i\xi - 33).$$

Since $o_{41}(33) = 20$, this is a prime ideal. We then have, for rational integers a:

$$\text{if } a \equiv 2^s \pmod{61}, \text{ then } \chi(a) \equiv 33^s \pmod{I}. \tag{2.4.9}$$

It is clear that using formulas (2.4.4) – (2.4.9), we can verify whether Condition $(*)$ (see Section 2.2) is valid for these four (q, r) pairs or not (with the given χ and I). We will use for this the computer program of the next section, and we will find that the condition is satisfied in each case. Then we will have a possibility to apply the arguments of Section 2.2, in particular, formula (2.2.10).

2.5. The computer program

The aim of the computer program of this section is to compute m_χ modulo I, and also $A_\chi(p_0)$, $B_\chi(p_0)$ modulo I for every relevant residue class p_0 modulo q (see Section 2.2 for these notations). We will compute these quantities with the concrete parameters of the examples of Section 2.4, i.e. we compute them in four separate cases. Since I lies above r, and $|R/I| = r$, the computation modulo I is in practice a computation with rational integers modulo r.

Before giving the BASIC program itself, we say a few words about it.

We will apply the program for the four examples given in the previous section. We have to give the value of q, and then the value of r. These two values already identify the example, and the program then works with the other data (i.e. χ and I) of that example.

The program uses data from a file depending on (q, r). Each data file contains 20 numbers, we write the interesting values of p_0 followed by zeros, if there are less than 20 interesting values. See the contents of the data files below. It will turn out in Section 2.6 that indeed these are the interesting values of p_0. Firstly, the program computes the values of our characters modulo the ideal I, based on equations (3.4.4)-(3.4.9). If $q = 175$, we have

$$d(n,0) \equiv \chi_1^{(25)}(n) \pmod{I},\ d(n,1) \equiv \chi_1^{(7)}(n) \pmod{I}.$$

If $q = 61$, we have

$$d(n,2) \equiv \chi(n) \pmod{I}.$$

We use two subroutines. The first one (at line 20) is used only if $q = 175$. If $1 \le g \le 3$ is fixed, and the integers J, Z and $s(g)$ are given, this subroutine adds $\chi(J)Z$ to $s(g)$ (modulo the ideal I, of course). The second subroutine (at line 30) is the same as the previous one, but it is used when $q = 61$.

After computing the values of the characters, the program computes m_χ (we get it in result1.txt), then $A_\chi(p_0)$ (we get in result2.txt) and $B_\chi(p_0)$ (result3.txt) modulo I for every interesting value of p_0.

We now give the data files. In the first line we write the contents of data0.txt, the second line is data1.txt, the third line is data2.txt, while the fourth one is data3.txt:

3, 8, 13, 17, 18, 22, 27, 32, 38, 43, 48, 52, 53, 57, 62, 67, 73, 78, 83, 87;

8, 13, 18, 22, 32, 38, 43, 53, 67, 78, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0;

6, 10, 24, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0;

13, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0.

Here is the QBasic program:

```
DEFDBL A−Z
IF q = 175 AND r = 61 THEN OPEN "data0.txt" FOR INPUT AS #1
IF q = 175 AND r = 1861 THEN OPEN "data1.txt" FOR INPUT AS #1
IF q = 61 AND r = 1861 THEN OPEN "data2.txt" FOR INPUT AS #1
IF q = 61 AND r = 41 THEN OPEN "data3.txt" FOR INPUT AS #1
OPEN "result1.txt" FOR OUTPUT AS #2
```

```
OPEN "result2.txt" FOR OUTPUT AS #3
OPEN "result3.txt" FOR OUTPUT AS #4
DIM d(60, 2): DIM s(3)
REM ======= WE COMPUTE THE VALUES OF THE CHARACTERS
p = 1: d(1, 0) = 1: FOR J = 1 TO 19
v = p: p = (2 * p) MOD 25
IF r = 61 THEN d(p, 0) = (8 * d(v, 0)) MOD r
IF r = 1861 THEN d(p, 0) = (380 * d(v, 0)) MOD r
NEXT J
p = 1: d(1, 1) = 1: FOR J = 1 TO 5
v = p: p = (3 * p) MOD 7
IF r = 61 THEN d(p, 1) = (47 * d(v, 1)) MOD r
IF r = 1861 THEN d(p, 1) = (1406 * d(v, 1)) MOD r
NEXT J
p = 1: d(1, 2) = 1: FOR J = 1 TO 59
v = p: p = (2 * p) MOD 61
IF r = 1861 THEN d(p, 2) = (1833 * d(v, 2)) MOD r
IF r = 41 THEN d(p, 2) = (33 * d(v, 2)) MOD r
NEXT J
GOTO 40
REM ======= IF q = 175, THIS SUBROUTINE ADDS χ(J)Z 20 IF J
MOD 5 = 0 OR J MOD 7 = 0 THEN GOTO 25
s = d(((J MOD 25) + 25) MOD 25, 0): L = d(((J MOD 7) + 7) MOD 7, 1)
w = (s * L) MOD r
s(g) = (((s(g) + w * Z) MOD r) + r) MOD r
25 RETURN
REM ======= IF q = 61, THIS SUBROUTINE ADDS χ(J)Z
30 IF J MOD 61 = 0 THEN GOTO 35
s = d(((J MOD 61) + 61) MOD 61, 2)
s(g) = (((s(g) + s * Z) MOD r) + r) MOD r
35 RETURN
REM ======= WE COMPUTE m_χ (AS s(1)) 40 g = 1: FOR J = 1
TO q − 1
Z = J: IF q = 175 THEN GOSUB 20
IF q = 61 THEN GOSUB 30
NEXT J
REM ======= p(a) ARE THE POSSIBLE VALUES OF p_0
DIM p(20): FOR a = 1 TO 20: INPUT #1, p(a)
IF p(a) = 0 THEN GOTO 70
REM ======= WE COMPUTE A_χ(p_0) (AS s(2)) AND
REM ======= B_χ(p_0) (AS s(3))
FOR c = 0 TO q − 1: FOR d = 0 TO q − 1
J = d * d − c * c − p(a) * c * d
g = 2: Z = (q − c) * INT((d − p(a) * c) / q)
```

```
IF q = 175 THEN GOSUB 20
IF q = 61 THEN GOSUB 30
g = 3: Z = (c − q) ∗ c
IF q = 175 THEN GOSUB 20
IF q = 61 THEN GOSUB 30
NEXT d: NEXT c
REM ======= WE PRINT THE RESULTS
FOR g = 1 TO 3: IF a > 1 AND g = 1 THEN GOTO 60
IF g > 1 THEN PRINT #(g + 1), "for "; p(a); " we get "; s(g)
IF g = 1 THEN PRINT #(g + 1), " we get "; s(g)
s(g) = 0
60 NEXT g 70 NEXT a
CLOSE #1: CLOSE #2: CLOSE #3: CLOSE #4
```

2.6. Concluding the proof

Firstly, we show that a residue class and its negative always behave in the same way during our proof. We can spare half of the computations by this observation.

Recall the definitions of $A_\chi(j)$ and $B_\chi(j)$ from Section 2.2.

LEMMA 2.3. *Let q be a positive integer, χ a character modulo q and j an integer with $(j,q) = 1$. Then*

(i) $B_\chi(q-j) = B_\chi(j)$;

(ii) $A_\chi(q-j) + A_\chi(j) = B_\chi(j)$.

Proof. Let $(t)_q$ denote the least nonnegative residue of t modulo q. Then, replacing D by $(q-D)_q$ in the definition of $B_\chi(q-j)$, we get (i). The same reasoning gives that the left-hand side of (ii) equals

$$\sum_{C,D=0}^{q-1} \chi(D^2 - C^2 - jCD) \left(\left\lceil \frac{(q-j)C - (q-D)_q}{q} \right\rceil + \left\lceil \frac{jC - D}{q} \right\rceil \right) (C - q).$$

If $D \neq 0$, then

$$\left\lceil \frac{(q-j)C - (q-D)_q}{q} \right\rceil + \left\lceil \frac{jC - D}{q} \right\rceil = \begin{cases} C - 1, \text{ if } D \equiv jC \pmod{q} \\ C \text{ otherwise} \end{cases},$$

since the sum of the arguments of the upper integer parts is $C-1$. If $D=0$, then the sum is 1 larger. Thus, using $(j,q)=1$, the left-hand side of (ii) equals

$$B_\chi(j) - \sum_{1\le C,D\le q-1,\ D\equiv jC} \chi(-C^2)(C-q) + \sum_{1\le C\le q-1,\ D=0} \chi(-C^2)(C-q)$$

(the congruence in the first sum is meant modulo q), which proves (ii).

Proof of Theorem 1.1. Since our program in Section 2.5 applied for the four (q,r) pairs given in the examples of Section 2.4 gives 0 for m_χ (mod$I$), but gives nonzero results for $B_\chi(p_0)$ (mod I) (i.e. the results are rational integers not divisible by r) for certain values of $p_0 \in U_q$ (hence for all $p_0 \in U_q$, see (2.2.9)), we get that these four (q,r) pairs satisfy Condition $(*)$. Hence we can apply (2.2.10), and we can follow the steps outlined in Section 2.2. Note that if two rational integers are congruent modulo I, then they are congruent modulo r, so (2.2.10) gives us the value of p modulo r.

By Lemma 2.3, we have

$$j - q\frac{A_\chi(j)}{B_\chi(j)} \equiv -\left((q-j) - q\frac{A_\chi(q-j)}{B_\chi(q-j)}\right) \tag{2.6.1}$$

modulo I for every $(j,q)=1$, so (see (2.2.10)) a residue class contained in U_q and its negative determine residue classes modulo r which are again negatives of each other.

We first consider Example 1 from Section 2.4. In the first column of Table 1 we list the 20 values of p_0 (see (2.2.5) for its meaning) for which

$$0 < p_0 < \frac{175}{2},\ p_0 \equiv \pm 2 \pmod 5 \text{ and } p_0 \equiv \pm 1, \pm 3 \pmod 7.$$

These are the elements of U_{175} in the given range (for $p_0 \notin U_{175}$ we are done, p^2+4 is a square modulo 5 or 7). In the second and third columns we give $A_\chi(p_0)$ and $B_\chi(p_0)$ modulo I, respectively (obtained by the program); the fourth column gives p modulo 61, and it is computed from the first three columns, using (2.2.10). The fifth column is determined by the fourth column: if p^2+4 is a square modulo 61, then we write a number n into the fifth column such that

$$n^2 \equiv p^2 + 4 \pmod{61};$$

otherwise we leave the fifth column empty.

For the 10 values of p_0 where the fifth column of Table 1 is empty, we apply the program with the parameters of Example 2 (in particular, $q = 175$ and $r = 1861$). The results are summarized in Table 2, which is completely analogous to Table 1.

We know from (2.6.1) that if we replace a particular p_0 by $175 - p_0$ in the first column of Table 1 or Table 2, then in the fourth column we obtain the negative of the residue class belonging to p_0 in the fourth column. Consequently, $p^2 + 4$ modulo 61 (or modulo 1861 in the case of Table 2) is unchanged. Hence, if the fifth column is nonempty at the row of a p_0 in Table 1 or in Table 2, then p_0 and $175 - p_0$ are excluded in the sense that for

$$p \equiv \pm p_0 \pmod{175}$$

$p^2 + 4$ is a square modulo 61 or modulo 1861. The remaining possibilities are summarized in Table 3, where we mean that either the plus or the minus sign is valid inside a row, and one of the rows must be valid for our p.

For $p \equiv 6$, 10 or 24 modulo 61 we apply the program with the parameters of Example 3. The result is Table 4, which is completely analogous to Tables 1 and 2, but we do not need the fifth column, so we omit it. Since

$$612 \not\equiv \pm 1058,\ 881 \not\equiv \pm 1107,\ 881 \not\equiv \pm 1062 \text{ and } 460 \not\equiv \pm 1634$$

modulo 1861, so, using (2.6.1), we see by Tables 3 and 4 that the only possible values for p modulo 61 are ± 13 (since otherwise p would belong to two different residue classes modulo 1861, which is a contradiction). Hence, if we consider Example 4 ($q = 61$, $r = 41$), the only possibilities for p_0 are 13 and $61 - 13 = 48$. For $p_0 = 13$ we apply the program and we obtain

$$A_\chi(p_0) \equiv 0 \pmod{I} \text{ and } B_\chi(p_0) \equiv 13 \pmod{I}.$$

Hence (2.2.10) gives

$$p \equiv 13 \pmod{41}.$$

By (2.6.1), we know that then $p_0 = 48$ gives

$$p \equiv -13 \pmod{41}.$$

In both cases, we have

$$p^2 + 4 \equiv 173 \equiv 3^2 \pmod{41},$$

so Theorem 1.1 is proved.

TABLE 1.

We use the parameters of Example 1, in particular q=175, $r = 61$.

The second and third columns are meant modulo I.

p_0	$A_\chi(p_0)$	$B_\chi(p_0)$	$p \bmod r$	$\sqrt{p^2+4} \bmod r$
3	0	51	3	14
8	0	33	8	
13	0	24	13	
17	0	26	17	7
18	34	44	2	
22	34	53	49	
27	24	50	4	9
32	1	44	10	
38	40	30	8	
43	46	23	59	
48	20	50	39	0
52	14	32	25	18
53	13	51	6	
57	54	23	36	18
62	42	24	15	30
67	28	26	24	
73	6	32	44	7
78	27	53	51	
83	32	33	39	0
87	19	30	27	1

TABLE 2.

We use the parameters of Example 2, in particular q=175, $r = 1861$.

The second and third columns are meant modulo I.

p_0	$A_\chi(p_0)$	$B_\chi(p_0)$	p mod r	$\sqrt{p^2+4}$ mod r
8	0	1121	8	505
13	0	1498	13	
18	1254	1060	285	385
22	60	1588	1492	263
32	135	1060	1107	
38	1633	1397	321	760
43	1294	1102	1685	748
53	1275	1389	1058	
67	1773	1720	1634	
78	344	1588	1062	

TABLE 3.

p mod 175	p mod 61	p mod 1861
± 13	± 13	± 13
± 32	± 10	± 1107
± 53	± 6	± 1058
± 67	± 24	± 1634
± 78	± 51	± 1062

TABLE 4.

We use the parameters of Example 3, in particular q=61, $r = 1861$.

The second and third columns are meant modulo I.

p_0	$A_\chi(p_0)$	$B_\chi(p_0)$	p mod r
6	957	1000	612
10	1150	616	881
24	173	663	460

3. A Poisson-type summation formula

3.1. Structure of the chapter and a convention

In this chapter we prove Theorem 1.2. In Section 3.2 we give a sketch of the proof, in Section 3.3 we introduce some more notations and gather together some well-known preliminary facts. In Section 3.4 we prove our most important lemmas, then we prove a special case (the special case (3.2.1), see below) of Theorem 1.2 in Section 3.5, and the general case in Section 3.6. Some remaining lemmas on automorphic functions are proved only in Section 3.7 (but the results of Section 3.7, i.e. Lemmas 3.9-3.14 are used already in earlier sections). In this chapter we use some facts (for example Theorem 1.3, but we will quote also some other results of Chapter 4) related to the function $\phi_\lambda(x)$, but these facts will be proved only in the Appendix (i.e. in Chapter 4). The present chapter is based on [Bi7].

CONVENTION. In what follows, u_1 and u_2 (hence t_1 and t_2) will be fixed (see the statement of Theorem 1.2 for these notations). So every variable and every constant (including the constants implied in the $\ll$ and O symbols) may depend on u_1 and u_2, even if we do not denote this dependence.

3.2. Sketch of the proof of Theorem 1.2

In this sketch we ignore problems related to convergence, we just give a formal argument. Assume first that the following special case of Theorem 1.2 is already proved:

$$f(x) = 0 \quad (x \in \mathbf{R}), \qquad a_n = 0 \quad (n \neq N), \qquad a_N = 1 \tag{3.2.1}$$

with a fixed positive integer N. Using Groenevelt's result that the Wilson function transform of type II is its own inverse (see Subsection 1.3.5), we can see that this special case (reading it "in the other direction", and making the changes $u_1 \to \overline{u_2}$, $u_2 \to \overline{u_1}$) proves another case of Theorem 1.2:

$$f(x) = \phi_{i\left(\frac{1}{4}-N\right)}(x) \quad (x \in \mathbf{R}), \qquad a_n = \phi_{i\left(\frac{1}{4}-N\right)}\left(i\left(\frac{1}{4}-n\right)\right) \tag{3.2.2}$$

with a fixed positive integer N.

There is a special case of Theorem 1.2 which is easily seen to be true:

$$f(x)=\frac{1}{\Gamma\left(\frac{3}{4}\pm ix\right)}\quad (x\in\mathbf{R}),\qquad a_n=0\quad (n\geq 1). \tag{3.2.3}$$

This special case will follow trivially from the spectral theorem for weight 1/2.

It turns out that the general statement can be proved using these three special cases by purely analytical means. This will follow from Theorem 1.3 which implies that a nice enough even function on $\mathbf{R}$ can be written as a linear combination of the functions $\frac{1}{\Gamma\left(\frac{3}{4}\pm ix\right)}$ and $\phi_{i\left(\frac{1}{4}-N\right)}(x)$ $(N\geq 1)$. This will mean that if f is a given nice even function on $\mathbf{R}$, then by (3.2.2) (using it for every integer $N\geq 1$) and (3.2.3) we can prove that Theorem 1.2 is true for this f and for *some* sequence $\{a_n\}_{n\geq 1}$. But then, using (3.2.1) for every integer $N\geq 1$, we can achieve *any* sequence $\{a_n\}_{n\geq 1}$ without changing f. This will complete the proof of Theorem 1.2.

Hence, it is enough to prove the special case (3.2.1). We now give a sketch of the proof of this special case.

Observe that we have to give an expression for

$$\sum_{j=1}^{s_N}\left(B_0\kappa_N\left(u_1\right),g_{N,j}\right)\overline{\left(B_0\kappa_N\left(u_2\right),g_{N,j}\right)}, \tag{3.2.4}$$

which is the inner product of the projection of $B_0\kappa_N\left(u_1\right)$ and the projection of $B_0\kappa_N\left(u_2\right)$ to the space $(\mathrm{Im}z)^{\frac{1}{4}+N}S_{2N+\frac{1}{2}}$. This is in fact the space of Maass cusp forms of weight $2N+\frac{1}{2}$ and $\Delta_{2N+\frac{1}{2}}$-eigenvalue $\left(N+\frac{1}{4}\right)\left(N-\frac{3}{4}\right)$. We will show that this projection operator can be written as an integral operator: if U is a cusp form of weight $2N$ for $\Gamma_0(4)$, then the projection of B_0U to the above-mentioned space is

$$\int_H B_0(z)U(z)m_N(z,w)d\mu_z$$

with a suitable kernel function m_N. We can apply a theorem of Fay (see our Lemma 3.4) to determine the Fourier expansions of B_0 and U on noneuclidean circles around w. Since the behavior of $m_N(z,w)$ on such circles is well understood, we can compute this integral using geodesic polar coordinates around w, and we get that the projection equals

$$\sum_{l=0}^{\infty} C_{U,l} B_l(w) \left(U\right)_{-l}(w),$$

where the coefficients $C_{U,l}$ are explicitly known, and

$$(U)_{-l} = \frac{1}{l!} L_{N-l+1} \dots L_{N-1} L_N U, \qquad B_l = \frac{1}{l!} K_{(l-1)+\frac{1}{4}} \dots K_{\frac{5}{4}} K_{\frac{1}{4}} B_0.$$

Hence, applying it with $U = \kappa_N(u_1)$ and also with $U = \kappa_N(u_2)$ we see that for the computation of (3.2.4) we have to compute integrals of the form

$$\int_{D_4} B_{l_1}(w) \left(\kappa_N(u_1)\right)_{-l_1}(w) \overline{B_{l_2}(w) \left(\kappa_N(u_2)\right)_{-l_2}(w)} d\mu_w.$$

We will consider this integral as the inner product of $B_{l_1} \overline{\left(\kappa_N(u_2)\right)_{-l_2}}$ and $B_{l_2} \overline{\left(\kappa_N(u_1)\right)_{-l_1}}$. These are automorphic forms of weight $\frac{1}{2} + 2(l_1 + l_2 - N)$, and we will compute their inner product using the spectral theorem for this weight (in the form of Corollaries 3.1 or 3.2 below). This leads us to a sum of products of triple products of the form

$$\left(B_{l_1} \overline{\left(\kappa_N(u_2)\right)_{-l_2}}, F\right) \overline{\left(B_{l_2} \overline{\left(\kappa_N(u_1)\right)_{-l_1}}, F\right)},$$

where F is a Maass form of weight $\frac{1}{2} + 2(l_1 + l_2 - N)$. Using partial integration (in the form of Lemmas 3.1 and 3.2) it turns out in Lemma 3.7 that these triple products can be written as linear combinations of such triple products which are present in Theorem 1.2. This reasoning shows relatively easily that we can get *some* expression for (3.2.4) with the products of inner products which are present in Theorem 1.2. However, I cannot give a good explanation of the actual form of the relation, i.e. the occurrence of the Wilson function $\phi_\lambda(x)$, besides the fact that this will be the result of the computation.

3.3. Further notations and preliminaries

3.3.1. Some details on the Wilson function transform of type II. We first give explicitly the quantities C, $H(x)$ and iR_k mentioned in Subsection 1.3.5: we use the notations $\Gamma(X \pm Y) = \Gamma(X+Y)\Gamma(X-Y)$ and

$$\Gamma(X \pm Y \pm Z) = \Gamma(X+Y+Z)\Gamma(X+Y-Z)\Gamma(X-Y+Z)\Gamma(X-Y-Z),$$

and define

$$H(x)=\frac{\Gamma\left(\frac{1}{4}\pm it_1\pm ix\right)\Gamma\left(\frac{1}{4}\pm it_2\pm ix\right)\Gamma\left(\frac{1}{4}\pm ix\right)\Gamma\left(\frac{3}{4}\pm ix\right)}{\pi^2\Gamma\left(\pm 2ix\right)}, \tag{3.3.1}$$

$$C=\frac{\pi^2}{\Gamma\left(\frac{1}{2}\pm it_1\right)\Gamma\left(\frac{1}{2}\pm it_2\right)}, \tag{3.3.2}$$

and (writing $s_j=\frac{1}{2}+it_j$ for $j=1,2$, as in Theorem 1.2)

$$iR_k=\frac{2k-\frac{1}{2}}{\pi^2}\frac{\left|\Gamma\left(k+it_1\right)\right|^2\left|\Gamma\left(k+it_2\right)\right|^2}{\left|\left(s_1\right)_k\right|^2\left|\left(s_2\right)_k\right|^2}\Gamma\left(\frac{1}{2}\pm it_1\right)\Gamma\left(\frac{1}{2}\pm it_2\right). \tag{3.3.3}$$

(Note that there is a mistake in the concrete expression for this residue in Section 5.1 of [G1], the formula there should be multiplied by $4t^2$, which is $\frac{1}{4}$ in our case.)

As it was promised in Subsection 1.3.5, we now give explicitly the parameters of the Wilson function $\phi_\lambda(x)$ introduced there. Indeed, let

$$a=\frac{1}{4}+it_1, b=\frac{1}{4}+it_2, c=\frac{1}{4}-it_2, d=\frac{3}{4}+it_1. \tag{3.3.4}$$

Then this set of parameters is self-dual, i.e. for the dual parameters $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, $\tilde{d}$ defined in formula (2.6) of [G1] we have

$$\tilde{a}=a, \tilde{b}=b, \tilde{c}=c, \tilde{d}=d.$$

For the definition of the Wilson function transform of type II in Section 5.1 of [G1] one more parameter is needed, we denote it by t. We choose $t=\frac{1}{4}$ there, and then in the beginning of Section 5.1 of [G1] we see also that $\tilde{t}=t$. Then the definition of $H(x)$ in (3.3.1) above is in accordance with [G1], and since our parameters are self-dual, we quoted correctly Theorem 5.10 of [G1] in our Subsection 1.3.5.

We mention two more important facts what will be needed. The first one is $\phi_\lambda(x)=\phi_x(\lambda)$, see (3.4) of [G1] and remember that our parameters are self-dual. The second one is that $\phi_\lambda(x;a,b,c,d)$ is symmetric in $a,b,c,1-d$ (see Remark 4.5 of [G1]), hence that our Wilson function transform is symmetric in t_1 and t_2.

3.3.2. Other notations. Let D_1 be the closure of the standard fundamental domain of $SL(2,\mathbf{Z})$, hence

$$D_1=\left\{z\in H:-\frac{1}{2}\leq \mathrm{Re}z\leq \frac{1}{2},\ |z|\geq 1\right\}.$$

Then, it is easy to check that the following set is a closure of a fundamental domain of $\Gamma_0(4)$:

$$D_4=\bigcup_{j=0}^{5}\gamma_j D_1,$$

where

$$\gamma_j=\begin{pmatrix}0&-1\\1&j\end{pmatrix}\qquad (0\leq j\leq 3),$$

and

$$\gamma_4=\begin{pmatrix}1&0\\0&1\end{pmatrix},\qquad \gamma_5=\begin{pmatrix}1&0\\-2&1\end{pmatrix}.$$

In the sequel D_4 will always denote this fixed fundamental domain of $\Gamma_0(4)$.

The three cusps for $\Gamma_0(4)$ are ∞, 0 and $-\frac{1}{2}$. If a denotes one of these cusps, we take a scaling matrix $\sigma_a\in SL(2,\mathbf{R})$ as it is explained on p. 42 of [I1]. We can easily see that one can take

$$\sigma_\infty=\begin{pmatrix}1&0\\0&1\end{pmatrix},\qquad \sigma_0=\begin{pmatrix}0&\frac{-1}{2}\\2&0\end{pmatrix},\qquad \sigma_{-\frac{1}{2}}=\begin{pmatrix}-1&\frac{-1}{2}\\2&0\end{pmatrix}.$$

The only cusp for $SL(2,\mathbf{Z})$ is ∞, and, of course, we take the identity matrix σ_∞ for scaling matrix also in this case.

Let L_n^α and $F(\alpha,\beta,\gamma;z)$ be the usual notations for Laguerre polynomials and Gauss' hypergeometric functions, respectively, see [G-R], p. 990 and p. 995.

If a is a cusp for $\Gamma_0(4)$, we define χ_a by

$$\nu\left(\sigma_a\begin{pmatrix}1&1\\0&1\end{pmatrix}\sigma_a^{-1}\right)=e(-\chi_a),\qquad 0\leq\chi_a<1.$$

It is easy to check that $\chi_\infty=\chi_0=0$, and $\chi_{-\frac{1}{2}}=\frac{3}{4}$. So the cusps 0 and ∞ are said to be singular, and $-1/2$ is said to be nonsingular.

If f is a Maass form of weight l, $\Delta_l f=s(s-1)f$ with some $\mathrm{Re}s\geq\frac{1}{2}$, $s=\frac{1}{2}+it$, and a is a cusp of Γ, then $f(\sigma_a z)\left(\frac{j_{\sigma_a}(z)}{|j_{\sigma_a}(z)|}\right)^{-l}$ has the Fourier expansion

$$c_{f,a}(y)+\sum_{\substack{m\in\mathbf{Z}\\ m-\chi_a\neq 0}}\rho_{f,a}(m)W_{\frac{l}{2}\mathrm{sgn}(m-\chi_a),it}\left(4\pi\left|m-\chi_a\right|y\right)e\left(\left(m-\chi_a\right)x\right)$$

for $z = x + iy \in H$ where $W_{\alpha,\beta}$ is the Whittaker function (see [G-R], p. 1014), and $c_{f,a}(y) = 0$ if $\chi_a \neq 0$, while it is a linear combination of y^s and y^{1-s} for $s \neq \frac{1}{2}$ and of $y^{1/2}$ and $y^{1/2}\log y$ for $s = \frac{1}{2}$, if $\chi_a = 0$.

Let $P_l(D_4)$ be the space of such smooth automorphic forms of weight l for $\Gamma_0(4)$ for which we have that for any integers $B, C \geq 0$ there is an integer $A = A(B, C)$ such that

$$\left(\max_a \mathrm{Im}\sigma_a^{-1} z\right)^{-A} \left| \left(\frac{\partial^B}{\partial x^B} \frac{\partial^C}{\partial y^C} f \right)(z) \right|$$

is bounded on D_4 (i.e. every partial derivative grows at most polynomially near each cusp on the fixed fundamental domain D_4). We denote by $R_l(D_4)$ the space of such smooth automorphic forms of weight l for $\Gamma_0(4)$ for which we have that for any integers $A, B, C \geq 0$ the function

$$\left(\max_a \mathrm{Im}\sigma_a^{-1} z\right)^{A} \left| \left(\frac{\partial^B}{\partial x^B} \frac{\partial^C}{\partial y^C} f \right)(z) \right|$$

is bounded on D_4 (i.e. every partial derivative decays faster than polynomially near each cusp on the fixed fundamental domain D_4).

Let

$$\Gamma_\infty = \{\gamma \in SL(2, \mathbf{Z}) : \ \gamma\infty = \infty\}.$$

For $z, w \in H$ let

$$H(z, w) = i^{\frac{1}{2}} \left(\frac{|z - \overline{w}|}{(z - \overline{w})} \right)^{\frac{1}{2}} = \left(\frac{z - \overline{w}}{w - \overline{z}} \right)^{-\frac{1}{4}} \tag{3.3.5}$$

(the last equality holds because the fourth powers are the same, and the arguments of both sides lie in $(-\frac{\pi}{4}, \frac{\pi}{4})$), as on p. 349 of [Hej]. It is easy to see that for any $T \in SL(2, \mathbf{R})$ we have

$$\frac{H^2(Tz, Tw)}{H^2(z, w)} = \left(\frac{j_T(z)}{|j_T(z)|} \right) \left(\frac{j_T(w)}{|j_T(w)|} \right)^{-1},$$

so

$$\frac{H(Tz, Tw)}{H(z, w)} = \left(\frac{j_T(z)}{|j_T(z)|} \right)^{\frac{1}{2}} \left(\frac{j_T(w)}{|j_T(w)|} \right)^{-\frac{1}{2}}, \tag{3.3.6}$$

since both sides lie in the right half-plane. Observe also that

$$H(w, z) = \overline{H(z, w)}. \tag{3.3.7}$$

If $z \in H$ is arbitrary, let $T_z \in PSL(2, \mathbf{R})$ be such that T_z is an upper triangular matrix and $T_z i = z$. It is clear that T_z is uniquely determined by z, for $z = x + iy$ we have explicitly

$$T_z = \begin{pmatrix} y^{\frac{1}{2}} & xy^{\frac{-1}{2}} \\ 0 & y^{\frac{-1}{2}} \end{pmatrix}.$$

If $z \in H$ is fixed, the function $(\mathrm{Im} z)^{\frac{1}{4}} \theta \left(T_z \left(i \frac{1+L}{1-L} \right) \right) (1-L)^{-\frac{1}{2}}$ is holomorphic for $|L| < 1$, so it has a Taylor expansion

$$(\mathrm{Im} z)^{\frac{1}{4}} \theta \left(T_z \left(i \frac{1+L}{1-L} \right) \right) (1-L)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} B_n(z) L^n. \tag{3.3.8}$$

We defined in this way a function $B_n(z)$ $(z \in H)$ for every $n \geq 0$. For $n = 0$ this is in accordance with (1.3.1).

For $\gamma_1, \gamma_2 \in SL(2, \mathbf{R})$, we define

$$w(\gamma_1, \gamma_2) = j_{\gamma_1}(\gamma_2 z)^{1/2} j_{\gamma_2}(z)^{1/2} j_{\gamma_1 \gamma_2}(z)^{-1/2},$$

the right-hand side is indeed independent of $z \in H$. Clearly $w = \pm 1$.

For $a = 0, \infty$, $\mathrm{Re} s > 1$, $z \in H$ and any integer n define (Γ_a denotes the stability group of a in $\Gamma_0(4)$)

$$E_a\left(z, s, \frac{1}{2} + 2n\right) = \sum_{\gamma \in \Gamma_a \backslash \Gamma_0(4)} \overline{\nu(\gamma) w\left(\sigma_a^{-1}, \gamma\right)} (\mathrm{Im} \sigma_a^{-1} \gamma z)^s \left(\frac{j_{\sigma_a^{-1}\gamma}(z)}{\left| j_{\sigma_a^{-1}\gamma}(z) \right|} \right)^{-\frac{1}{2} - 2n}.$$

It follows from [F], formula (5) on p. 145 that for $n \geq 0$ we have

$$E_a\left(z, s, \frac{1}{2} + 2n\right) = c_n(s) K_{n - \frac{3}{4}} \dots K_{\frac{5}{4}} K_{\frac{1}{4}} E_a\left(z, s, \frac{1}{2}\right),$$

for $n \leq 0$ we have

$$E_a\left(z, s, \frac{1}{2} + 2n\right) = c_n(s) L_{\frac{5}{4} + n} \dots L_{-\frac{3}{4}} L_{\frac{1}{4}} E_a\left(z, s, \frac{1}{2}\right),$$

(of course s is fixed and we apply the operators in z), where

$$c_n(s) = \prod_{l=0}^{n-1} \frac{1}{s + \frac{1}{4} + l}$$

for $n \geq 0$, and

$$c_n(s) = \prod_{l=0}^{-n-1} \frac{1}{s - \frac{1}{4} + l}$$

for $n \leq 0$.

It is known that for every z the function $E_a\left(z, s, \frac{1}{2}\right)$ has a meromorphic continuation in s to the whole plane, and this function is regular at every point s with $\mathrm{Re}s = \frac{1}{2}$.

If $j \geq 0$ and $n \geq 0$ are integers, define

$$u_{j,\frac{1}{2}+2n}(z) = c_{j,n}\left(K_{n-\frac{3}{4}} \dots K_{\frac{5}{4}} K_{\frac{1}{4}} u_{j,\frac{1}{2}}\right)(z);$$

if $j \geq 1$ and $n < 0$, define

$$u_{j,\frac{1}{2}+2n}(z) = c_{j,n}\left(L_{\frac{5}{4}+n} \dots L_{-\frac{3}{4}} L_{\frac{1}{4}} u_{j,\frac{1}{2}}\right)(z),$$

where the numbers $c_{j,n}$ are chosen in such a way that $\left(u_{j,\frac{1}{2}+2n}, u_{j,\frac{1}{2}+2n}\right) = 1$, and, of course, $c_{j,0} = 1$ for every $j \geq 0$. We see by [F], pp. 145-146 that this is possible, we have $\Delta_{\frac{1}{2}+2n} u_{j,\frac{1}{2}+2n} = S_j(S_j - 1) u_{j,\frac{1}{2}+2n}$, and for a fixed n the functions $u_{j,\frac{1}{2}+2n}$ ($j \geq 0$ for $n \geq 0$, and $j \geq 1$ for $n < 0$) form an orthonormal system in $L^2_{\frac{1}{2}+2n}(D_4)$. We also see by (11) of [F] that

$$|c_{j,n}|^2 = \frac{1}{\left(S_j + \frac{1}{4}\right)_n \left(\frac{5}{4} - S_j\right)_n} \tag{3.3.9}$$

for $n \geq 0$, and

$$|c_{j,n}|^2 = \frac{1}{\left(S_j - \frac{1}{4}\right)_{-n} \left(\frac{3}{4} - S_j\right)_{-n}} \tag{3.3.10}$$

for $n \leq 0$. In this case we used also the general identity

$$\overline{K_k g} = L_{-k} \overline{g}, \tag{3.3.11}$$

and we will use frequently (and sometimes tacitly) this identity throughout this chapter.

For $n \geq k \geq 1$ and $1 \leq j \leq s_k$, let

$$g_{k,j,n} = c_{k,j,n} K_{n-\frac{3}{4}} \dots K_{k+\frac{5}{4}} K_{k+\frac{1}{4}} g_{k,j},$$

where $c_{k,j,n}$ is chosen such that $(g_{k,j,n}, g_{k,j,n}) = 1$. By [F], pp. 145-146 this is possible, $\Delta_{2n+\frac{1}{2}} g_{k,j,n} = \left(k+\frac{1}{4}\right)\left(k-\frac{3}{4}\right) g_{k,j,n}$, and for a fixed $n > 0$ the functions

$$\left\{u_{j,\frac{1}{2}+2n} : \ j \geq 0\right\} \cup \left\{g_{k,j,n} : \ 1 \leq k \leq n, 1 \leq j \leq s_k\right\}$$

form an orthonormal system in $L^2_{\frac{1}{2}+2n}(D_4)$. We also see by (11) of [F] that

$$|c_{k,j,n}|^2 = \frac{1}{\left(2k+\frac{1}{2}\right)_{n-k}(n-k)!} \tag{3.3.12}$$

for $n \geq k \geq 1$ and $1 \leq j \leq s_k$.

We will make several times a transition to geodesic polar coordinates: if $z_0 \in H$ is fixed, then for every $z \in H$ we can uniquely write

$$\frac{z-z_0}{z-\overline{z_0}} = \tanh\left(\frac{r}{2}\right)e^{i\phi} \tag{3.3.13}$$

with $r > 0$ and $0 \leq \phi < 2\pi$. The invariant measure is expressed in these new coordinates as $d\mu_z = \sinh r dr d\phi$.

3.4. Basic lemmas

3.4.1. Partial integration. We prove here two simple lemmas, but they will play an important role in the proof of Theorem 1.2, as it is mentioned in Section 3.2.

LEMMA 3.1. *Let $f_1 \in P_{2m_1}(D_4)$ and $f_2 \in P_{2m_2}(D_4)$ with $m_1 + m_2 = \frac{3}{4}$, and assume that at least one of $f_1 \in R_{2m_1}(D_4)$* and *$f_2 \in R_{2m_2}(D_4)$ is true. Then we have*

$$\int_{D_4} B_0(z)\left(L_{m_1} f_1\right)(z) f_2(z) d\mu_z = -\int_{D_4} B_0(z) f_1(z)\left(L_{m_2} f_2\right)(z) d\mu_z.$$

Proof. By (9) of [F] (we use a slight extension of that formula, because our functions are not of compact support, but the rapid decay at cusps is sufficient) and (3.3.11) we have

$$\int_{D_4} B_0(z)\left(L_{\frac{3}{4}}(f_1 f_2)\right)(z) d\mu_z = -\int_{D_4}\left(L_{\frac{1}{4}} B_0\right)(z)\left(f_1 f_2\right)(z) d\mu_z.$$

The right-hand side here is 0, since $L_{\frac{1}{4}} B_0 = 0$ by (4) of [F]. On the other hand,

$$\left(L_{m_1+m_2}(f_1 f_2)\right)(z) = \left(L_{m_1} f_1\right)(z) f_2(z) + f_1(z)\left(L_{m_2} f_2\right)(z)$$

by the definitions, and this proves the lemma.

In the next lemma we deal with the functions B_n defined in (3.3.8), the basic properties of these functions are given in Lemma 3.9 in Section 3.7.

LEMMA 3.2. *Let $l \geq 0$ be an integer, let $f \in P_{2m}(D_4)$ and $g \in P_{2n}(D_4)$ with $m+n = -\frac{1}{4} - l$, and assume that at least one of $f \in R_{2m}(D_4)$* and *$g \in R_{2n}(D_4)$ is true. Then*

$$\int_{D_4} B_l(z) f(z) g(z) d\mu_z$$

equals

$$\frac{(-1)^l}{l!} \sum_{L=0}^{l} \binom{l}{L} \int_{D_4} B_0(z) \left(K_{m+L-1} \dots K_{m+1} K_m f\right)(z) \left(K_{n+l-L-1} \dots K_{n+1} K_n g\right)(z) d\mu_z.$$

Proof. Using (3.7.2), and formula (9) of [F] (a slight extension of that formula again), we easily get that

$$\int_{D_4} B_l(z) f(z) g(z) d\mu_z = \frac{(-1)^l}{l!} \int_{D_4} B_0(z) \left(K_{-\frac{5}{4}} \dots K_{-l+\frac{3}{4}} K_{-l-\frac{1}{4}}(fg)\right)(z) d\mu_z.$$

Using the general identity

$$\left(K_{m_1+m_2}(f_1 f_2)\right)(z) = \left(K_{m_1} f_1\right)(z) f_2(z) + f_1(z) \left(K_{m_2} f_2\right)(z)$$

several times, we get the lemma.

3.4.2. Inner product of two automorphic forms of weight $\frac{1}{2} + 2n$. Here n is any integer. First we give the spectral decomposition of an $f \in R_{\frac{1}{2}+2n}(D_4)$ in Lemma 3.3: in the case $n \geq 0$ we give a complete spectral decomposition (Lemma 3.3 (i)), in the case $n < 0$ a bit less complete statement will be enough for our purposes (Lemma 3.3 (ii)). We then give two corollaries describing the inner product of two forms. We again give a complete statement in the case $n \geq 0$ (Corollary 3.1); in the case $n < 0$ (Corollary 3.2) the vanishing property (3.4.3) will suffice instead of a detailed spectral expression for the inner product.

Every statement here is more or less standard, therefore we just give brief indications of the proofs.

LEMMA 3.3. *Let n be an integer, and $f \in R_{\frac{1}{2}+2n}(D_4)$. Write*

$$\zeta_a(f,r) := \int_{D_4} f(z)\overline{E_a\left(*,\frac{1}{2}+ir,\frac{1}{2}+2n\right)}d\mu_z$$

for $a = 0, \infty$ and real r. Define

$$g_f = f - \sum_{j=j_0}^{\infty}(f,u_{j,\frac{1}{2}+2n})u_{j,\frac{1}{2}+2n} - \frac{1}{4\pi}\sum_{a=0,\infty}\int_{-\infty}^{\infty}\zeta_a(f,r)E_a\left(*,\frac{1}{2}+ir,\frac{1}{2}+2n\right)dr,$$

where $j_0 = 0$ for $n \geq 0$, and $j_0 = 1$ for $n < 0$.

(i) If $n \geq 0$, we have

$$g_f = \sum_{k=1}^{n}\sum_{j=1}^{s_k}(f,g_{k,j,n})g_{k,j,n}. \tag{3.4.1}$$

(ii) If $n < 0$, w have

$$\overline{g_f} = \sum_{k=1}^{-n}K_{-n-\frac{5}{4}}\dots K_{k+\frac{3}{4}}K_{k-\frac{1}{4}}G_{k,n}, \tag{3.4.2}$$

where $G_{k,n}(z) = (\mathrm{Im} z)^{-\frac{1}{4}+k}H_{k,n}(z)$ with some $H_{k,n} \in S_{2k-\frac{1}{2}}$.

Remarks on the proof. The case $n = 0$ (where the statement in (3.4.1) is $g_f = 0$) is well-known, and follows e.g. from [P], formula (27). For larger $|n|$ we can prove the statements by induction, applying the suitable operator L for the left-hand side of (3.4.1) and (3.4.2), and applying [F], formula (4).

COROLLARY 3.1. *If $f_1, f_2 \in R_{\frac{1}{2}+2n}(D_4)$, then for $n \geq 0$ we have that (f_1,f_2) equals the sum of*

$$\sum_{j=0}^{\infty}(f_1,u_{j,\frac{1}{2}+2n})\overline{(f_2,u_{j,\frac{1}{2}+2n})} + \sum_{k=1}^{n}\sum_{j=1}^{s_k}(f_1,g_{k,j,n})\overline{(f_2,g_{k,j,n})}$$

and

$$\frac{1}{4\pi}\sum_{a=0,\infty}\int_{-\infty}^{\infty}\zeta_a(f_1,r)\overline{\zeta_a(f_2,r)}dr.$$

Moreover, we have that the sum of

$$\sum_{j=0}^{\infty}\left|(f_1,u_{j,\frac{1}{2}+2n})\overline{(f_2,u_{j,\frac{1}{2}+2n})}\right| + \sum_{k=1}^{n}\sum_{j=1}^{s_k}\left|(f_1,g_{k,j,n})\overline{(f_2,g_{k,j,n})}\right|$$

and

$$\frac{1}{4\pi}\sum_{a=0,\infty}\int_{-\infty}^{\infty}\left|\zeta_a(f_1,r)\overline{\zeta_a(f_2,r)}\right|dr$$

is $\leq\left(\int_{D_4}\left|f_1(z)\right|^2 d\mu_z\right)^{\frac{1}{2}}\left(\int_{D_4}\left|f_2(z)\right|^2 d\mu_z\right)^{\frac{1}{2}}$.

Remarks on the proof. The expression for (f_1,f_2) follows at once from Lemma 3.3 (i). The inequality of the lemma follows by Cauchy's inequality.

COROLLARY 3.2. *If* $n<0$ *and* $f\in R_{\frac{1}{2}+2n}(D_4)$, *then we have* ($g_f$ *is defined in Lemma 3.3)*

$$K_{-\frac{3}{4}}\dots K_{n+\frac{5}{4}-r}K_{n+\frac{1}{4}-r}L_{n+\frac{5}{4}-r}\dots L_{-\frac{3}{4}+n}L_{\frac{1}{4}+n}g_f=0 \tag{3.4.3}$$

for every integer $r\geq 0$. *If* h *is another element of* $R_{\frac{1}{2}+2n}(D_4)$, *then* (f,h) *equals*

$$(g_f,h)+\sum_{j=1}^{\infty}(f,u_{j,\frac{1}{2}+2n})\overline{(h,u_{j,\frac{1}{2}+2n})}+\frac{1}{4\pi}\sum_{a=0,\infty}\int_{-\infty}^{\infty}\zeta_a(f,r)\overline{\zeta_a(h,r)}dr, \tag{3.4.4}$$

and

$$|(g_f,h)|+\sum_{j=1}^{\infty}\left|(f,u_{j,\frac{1}{2}+2n})\overline{(h,u_{j,\frac{1}{2}+2n})}\right|+\frac{1}{4\pi}\sum_{a=0,\infty}\int_{-\infty}^{\infty}\left|\zeta_a(f,r)\overline{\zeta_a(h,r)}\right|dr$$

is $\leq\left(\int_{D_4}\left|f(z)\right|^2 d\mu_z\right)^{\frac{1}{2}}\left(\int_{D_4}\left|h(z)\right|^2 d\mu_z\right)^{\frac{1}{2}}$.

Remarks on the proof. We see by (3.3.11) and Lemma 3.3 (ii) that for the proof of (3.4.3) it is enough to show that

$$L_{\frac{3}{4}}\dots L_{k-\frac{1}{4}}\left(L_{k+\frac{3}{4}}\dots L_{-n-\frac{1}{4}+r}K_{-n-\frac{5}{4}+r}\dots K_{k-\frac{1}{4}}G_{k,n}\right)=0$$

for every $1\leq k\leq -n$. This is true by (8) and (4) of [F], so (3.4.3) follows. Formula (3.4.4) follows at once from the definition of g_f. Lemma 3.3 (ii) easily implies

$$(g_f,h)=(g_f,g_h),$$

and then the inequality follows from (3.4.4) and Cauchy's inequality.

3.4.3. Fourier expansion of Laplace-eigenforms on noneuclidean circles. We reproduce here an important theorem of Fay, which will be applied several times in the sequel.

LEMMA 3.4. *Let $k \in \mathbf{R}$, $s \in \mathbf{C}$, and let f be a smooth function on H satisfying $\Delta_{2k} f = s(s-1) f$. If $z_0 \in H$ is given, then for every $z \in H$ we have the absolutely convergent expansion*

$$f(z)\left(\frac{z-\overline{z_0}}{z_0-\overline{z}}\right)^k = \sum_{n=-\infty}^{\infty} (f)_n (z_0) P^n_{s,k}(z,z_0) e^{in\phi}, \tag{3.4.5}$$

where $r = r(z,z_0) > 0$ and $0 \le \phi = \phi(z,z_0) < 2\pi$ are determined from z by (3.3.13), and

$$P^n_{s,k}(z,z_0) = \left(\tanh(\frac{r}{2})\right)^{|n|}\left(1-\tanh^2(\frac{r}{2})\right)^{k_n} F\left(s-k_n, 1-s-k_n, 1+|n|, -y\right) \tag{3.4.6}$$

with $y = \frac{\tanh^2(\frac{r}{2})}{1-\tanh^2(\frac{r}{2})}$, $k_n = k\frac{n}{|n|}$ for $n \neq 0$, $k_0 = \pm k$,

$$n!\,(f)_n(z_0) = \left(K_{k+n-1}\ldots K_{k+1}K_k f\right)(z_0) \text{ for } n \ge 0,$$

$$(-n)!\,(f)_n(z_0) = \overline{\left(K_{-k-n-1}\ldots K_{-k+1}K_{-k}\overline{f}\right)}(z_0) = \left(L_{k+n+1}\ldots L_{k-1}L_k f\right)(z_0) \text{ for } n \le 0. \tag{3.4.7}$$

Proof. This follows from Theorems 1.1 and 1.2 of [F]. Formula (3.4.6) is formally different from (13) of [F], but the right-hand side of (3.4.6) equals

$$\left(\tanh(\frac{r}{2})\right)^{|n|}\left(1-\tanh^2(\frac{r}{2})\right)^{s} F\left(s-k_n, s+|n|+k_n, 1+|n|, \tanh^2(\frac{r}{2})\right)$$

by [G-R], p. 998, 9.131.1. For the second equality in (3.4.7) we use again (3.3.11). We remark that for a fixed $r>0$ the left-hand side of (3.4.5) is a smooth 2π-periodic function of $\phi \in \mathbf{R}$ (z is determined from ϕ by (3.3.13)), and the right-hand side is its Fourier expansion, hence it is absolutely convergent. The lemma is proved.

3.5. Proof of the theorem in a special case

Let $N \ge 1$ be an integer. Our aim in this section is to prove the following special case. See Theorem 1.2 for property $P(f,\{a_n\})$.

LEMMA 3.5. *Property $P(f,\{a_n\})$ is true if f is identically zero, $a_n = 0$ for $n \neq N$, and $a_N = 1$. We have the estimates*

$$\sum_{j=1}^{\infty}\left|\phi_{T_j}\left(i\left(\frac{1}{4}-N\right)\right)\Gamma\left(\frac{3}{4}\pm iT_j\right)\left(B_0\kappa_0\left(\overline{u_2}\right), u_{j,\frac{1}{2}}\right)\overline{\left(B_0\kappa_0\left(\overline{u_1}\right), u_{j,\frac{1}{2}}\right)}\right| \le CN^D, \tag{3.5.1}$$

$$\sum_{a=0,\infty}\int_{-\infty}^{\infty}\left|\phi_r\left(i\left(\frac{1}{4}-N\right)\right)\Gamma\left(\frac{3}{4}\pm ir\right)\zeta_a\left(B_0\kappa_0\left(\overline{u_2}\right),r\right)\overline{\zeta_a\left(B_0\kappa_0\left(\overline{u_1}\right),r\right)}dr\right|\leq CN^D, \tag{3.5.2}$$

$$\sum_{k=1}^{\infty}\sum_{j=1}^{s_k}\left|\phi_{i\left(\frac{1}{4}-k\right)}\left(i\left(\frac{1}{4}-N\right)\right)\Gamma\left(2k+\frac{1}{2}\right)\left(B_0\kappa_k\left(\overline{u_2}\right),g_{k,j}\right)\overline{\left(B_0\kappa_k\left(\overline{u_1}\right),g_{k,j}\right)}\right|\leq CN^D \tag{3.5.3}$$

with positive constants C and D depending only on u_1, u_2.

In the proof of the general case of the theorem the upper bounds (3.5.1)-(3.5.3) will be important.

3.5.1. Projection to the space $S_{2N+\frac{1}{2}}$. We first construct a kernel function, then we show that the integral operator with this kernel function maps B_0U (if U is a cusp form of weight $2N$ for $\Gamma_0(4)$) into $S_{2N+\frac{1}{2}}$, finally we expand this image of B_0U in our given basis of $S_{2N+\frac{1}{2}}$.

Write

$$k_N(y)=(1+y)^{-N-\frac{1}{4}},\qquad H_N(z,w)=H(z,w)^{4N+1},$$

where $H(z,w)$ is defined in (3.3.5), and for $z,w\in H$ define

$$k_N(\mathrm{z},w)=k_N\left(\frac{|z-w|^2}{4\mathrm{Im}z\mathrm{Im}w}\right)H_N(z,w)$$

and

$$K_N(z,w)=\sum_{\gamma\in\Gamma_0(4)}k_N(\gamma z,w)\overline{\nu(\gamma)}\left(\frac{j_\gamma(z)}{|j_\gamma(z)|}\right)^{-\frac{1}{2}-2N},$$

this sum can be seen to be absolutely convergent. It is not hard to check that if $w\in H$ is fixed, then for every $\delta\in\Gamma_0(4)$ and $z\in H$ we have

$$K_N(\delta z,w)=\nu(\delta)\left(\frac{j_\delta(z)}{|j_\delta(z)|}\right)^{\frac{1}{2}+2N}K_N(z,w). \tag{3.5.4}$$

Let U be a cusp form of weight $2N$ for $\Gamma_0(4)$ with $\Delta_{2N}U=s(s-1)U$. Then we may define

$$F_U(w)=(\mathrm{Im}w)^{-N-\frac{1}{4}}\int_{D_4}B_0(z)U(z)\overline{K_N(z,w)}d\mu_z \tag{3.5.5}$$

for $w \in H$. We claim that $F_U \in S_{2N+\frac{1}{2}}$. We remark first that it is not hard to check using (3.3.6) and (3.3.7) that

$$K_N(w,z) = \overline{K_N(z,w)}. \tag{3.5.6}$$

So the required transformation property of F_U follows at once from (3.5.4). It is not hard to check that $(\mathrm{Im} w)^{-N-\frac{1}{4}} k_N(w,z)$ is holomorphic in w for every z, using the identity

$$4\mathrm{Im} z \mathrm{Im} w + |z-w|^2 = |z-\overline{w}|^2, \tag{3.5.7}$$

and then the same is true for $(\mathrm{Im} w)^{-N-\frac{1}{4}} K_N(w,z)$, using

$$\frac{\mathrm{Im} w}{|j_\gamma(w)|^2} = \mathrm{Im}\gamma w. \tag{3.5.8}$$

Hence $F_U(w)$ is holomorphic. It remains to check the behavior at cusps, i.e. that

$$\left| F_U(\sigma_a w) \left(j_{\sigma_a}(w)\right)^{-2N-\frac{1}{2}} \right| \to 0$$

as $\mathrm{Im} w \to \infty$ for each of the three cusps (in the case of $a = -\frac{1}{2}$ much less would be enough in fact, but it can be proved easily). To see this, we use the trivial estimate

$$|K_N(z,w)| \le \sum_{\gamma \in \Gamma_0(4)} k_N \left(\frac{|\gamma z - w|^2}{4\mathrm{Im}\gamma z \mathrm{Im} w} \right),$$

and the fact that $|B_0(z)U(z)|$ is bounded in z. These bounds together and the definition of $k_N(y)$ imply that the integral in (3.5.5) is bounded in w, and then the factor $(\mathrm{Im} w)^{-N-\frac{1}{4}}$ assures the required estimate (taking into account (3.5.8)). Hence indeed, $F_U \in S_{2N+\frac{1}{2}}$. Consider the inner product

$$\int_{D_4} (\mathrm{Im} w)^{2N+\frac{1}{2}} F_U(w) \overline{f_{N,j}(w)} d\mu_w \tag{3.5.9}$$

for some $1 \le j \le \mathrm{s}_N$. This is easily seen to be absolutely convergent as a double integral (see (3.5.5)). Using (3.5.6) we see by unfolding for any $z \in D_4$ that

$$\int_{D_4} \overline{K_N(z,w)} \overline{f_{N,j}(w) (\mathrm{Im} w)^{N+\frac{1}{4}}} d\mu_w = 2 \int_H k_N(w,z) \overline{f_{N,j}(w) (\mathrm{Im} w)^{N+\frac{1}{4}}} d\mu_w. \tag{3.5.10}$$

We use geodesic polar coordinates around z:

$$\frac{w-z}{w-\overline{z}} = \tanh(\frac{r}{2})e^{i\phi},$$

and since (using (3.5.7) and the definition of $k_N(y)$) we have

$$\frac{1}{1-\tanh^2(\frac{r}{2})} = \frac{|w-\overline{z}|^2}{4\mathrm{Im}z\mathrm{Im}w} \text{ and } k_N\left(\frac{|z-w|^2}{4\mathrm{Im}z\mathrm{Im}w}\right) = \left(\frac{|z-\overline{w}|^2}{4\mathrm{Im}z\mathrm{Im}w}\right)^{-N-\frac{1}{4}},$$

so (taking into account the definition of $k_N(w,z)$ and $H_N(w,z)$) we see that (3.5.10) equals

$$2i^{\frac{1}{2}+2N}\left(\frac{1}{4\mathrm{Im}z}\right)^{\frac{1}{4}+N}\int_0^\infty \left(1-\tanh^2(\frac{r}{2})\right)^{2N+\frac{1}{2}} \overline{\left(\int_0^{2\pi} F_r(\phi)d\phi\right)} \sinh r dr,$$

where we write

$$F_r(\phi) = (w-\overline{z})^{\frac{1}{2}+2N} f_{N,j}(w)$$

using the explicit expression for w in terms of r and ϕ:

$$w = \frac{z-\overline{z}\tanh(\frac{r}{2})Z}{1-\tanh(\frac{r}{2})Z} \text{ with } Z := e^{i\phi}.$$

For fixed $0<r<\infty$ and $z \in D_4$ this last expression is a regular function of Z (with values in H) in a domain containing the unit circle, hence by Cauchy‘s formula the inner integral is $2\pi\,(z-\overline{z})^{\frac{1}{2}+2N} f_{N,j}(z)$, so (3.5.10) equals (recall $g_{N,j}(z) = (\mathrm{Im}z)^{N+\frac{1}{4}} f_{N,j}(z)$)

$$4\pi\overline{g_{N,j}(z)}\int_0^\infty \left(1-\tanh^2(\frac{r}{2})\right)^{2N+\frac{1}{2}} \sinh r dr.$$

The integral can be computed, its value is $\frac{4}{4N-1}$, and so by (3.5.5) we get that (3.5.9) equals

$$\frac{16\pi}{4N-1}\int_{D_4} B_0(z)U(z)\overline{g_{N,j}(z)}d\mu_z.$$

Since the functions $f_{N,j}$ form an orthonormal basis of $S_{2N+\frac{1}{2}}$, this implies for any $w \in H$ that

$$F_U(w)\,(\mathrm{Im}w)^{N+\frac{1}{4}} = \frac{16\pi}{4N-1}\sum_{j=1}^{s_N}\left(\int_{D_4} B_0(z)U(z)\overline{g_{N,j}(z)}d\mu_z\right) g_{N,j}(w). \tag{3.5.11}$$

3.5.2. Computation in geodesic polar coordinates. We now compute the left-hand side of (3.5.11) in another way: by unfolding the right-hand side of (3.5.5). Up to some point, we continue working with a general cusp form U of weight $2N$ for $\Gamma_0(4)$, but then we will specialize to $U = \kappa_N(u)$, where u is a cusp form of weight 0 for $SL(2, \mathbf{Z})$.

By unfolding we see that

$$\int_{D_4} B_0(z)U(z)\overline{K_N(z,w)}d\mu_z = 2\int_H B_0(z)U(z)\overline{k_N(z,w)}d\mu_z \tag{3.5.12}$$

for any fixed $w \in H$. The integrand here can be written as (see (3.3.5))

$$\left(B_0(z)\left(\frac{z-\overline{w}}{w-\overline{z}}\right)^{\frac{1}{4}}\right)\left(U(z)\left(\frac{z-\overline{w}}{w-\overline{z}}\right)^{N}\right)k_N\left(\frac{|z-w|^2}{4\text{Im}z\text{Im}w}\right).$$

We now use geodesic polar coordinates around w:

$$\frac{z-w}{z-\overline{w}} = \tanh(\frac{r}{2})e^{i\phi},$$

and using the substitution $y = \frac{\tanh^2(\frac{r}{2})}{1-\tanh^2(\frac{r}{2})}$ we get that (3.5.12) equals

$$4\int_0^\infty k_N(y)\left(\int_0^{2\pi}\left(B_0(z)\left(\frac{z-\overline{w}}{w-\overline{z}}\right)^{\frac{1}{4}}\right)\left(U(z)\left(\frac{z-\overline{w}}{w-\overline{z}}\right)^{N}\right)d\phi\right)dy, \tag{3.5.13}$$

where $0 < r = r(y) < \infty$ and $z = z(y, \phi) \in H$ are determined from y and ϕ by the relations above.

For every fixed y we will now compute the inner integral by the Fourier expansions of the two functions there, and then we will integrate in y. To justify this computation remark that if

$$B_0(z)\left(\frac{z-\overline{w}}{w-\overline{z}}\right)^{\frac{1}{4}} = \sum_{l=-\infty}^{\infty} a_l(y)e^{il\phi} \text{ and } U(z)\left(\frac{z-\overline{w}}{w-\overline{z}}\right)^{N} = \sum_{l=-\infty}^{\infty} b_l(y)e^{il\phi},$$

then for any y by Cauchy's inequality in l and Parseval's formula in ϕ we have that (the implied constant in $\ll$ is absolute)

$$\sum_{l=-\infty}^{\infty} |a_l(y)b_{-l}(y)| \ll \left(\int_0^{2\pi}|B_0(z)|^2 d\phi\right)^{\frac{1}{2}}\left(\int_0^{2\pi}|U(z)|^2 d\phi\right)^{\frac{1}{2}},$$

hence by Cauchy's inequality in y we get that

$$\int_0^\infty k_N(y) \sum_{l=-\infty}^{\infty} |a_l(y)b_{-l}(y)|\, dy$$

is

$$\ll \left(\int_0^\infty k_N(y) \int_0^{2\pi} |B_0(z)|^2\, d\phi dy \right)^{\frac{1}{2}} \left(\int_0^\infty k_N(y) \int_0^{2\pi} |U(z)|^2\, d\phi dy \right)^{\frac{1}{2}},$$

which is (making backwards the steps leading from (3.5.12) to (3.5.13))

$$\ll M_U(w) := \left(\int_{D_4} K_N^*(z,w)\, |B_0(z)|^2\, d\mu_z \right)^{\frac{1}{2}} \left(\int_{D_4} K_N^*(z,w)\, |U(z)|^2\, d\mu_z \right)^{\frac{1}{2}}$$

with implied absolute constant, where

$$K_N^*(z,w) = \sum_{\gamma \in \Gamma_0(4)} k_N \left(\frac{|\gamma z - w|^2}{4 \mathrm{Im} \gamma z \mathrm{Im} w} \right).$$

We get an upper bound for this by extending the summation for $\gamma \in SL(2, \mathbf{Z})$, and then we can see by Lemma 3.11 (using (3.7.9) and (3.7.10) for fixed z_1) and the concrete form of k_N that $K_N^*(z,w)$ is bounded in z, so $M_U(w)$ is a finite number for every fixed w, hence we can compute (3.5.13) as we described above.

We now compute (3.5.13) explicitly for a given w. By Lemma 3.4 and (3.7.2), taking into account that $L_{1/4} B_0 = 0$, we get

$$B_0(z) \left(\frac{z - \overline{w}}{w - \overline{z}} \right)^{\frac{1}{4}} = \sum_{l=0}^{\infty} \left(\tanh(\frac{r}{2}) \right)^l \left(1 - \tanh^2(\frac{r}{2}) \right)^{\frac{1}{4}} B_l(w) e^{il\phi},$$

and again by Lemma 3.4 we have

$$U(z) \left(\frac{z - \overline{w}}{w - \overline{z}} \right)^N = \sum_{m=-\infty}^{\infty} (U)_m (w) P_{s,N}^m(z,w) e^{im\phi}$$

with the functions $(U)_m$ defined in Lemma 3.4, we will determine them explicitly later. Using (3.4.6) we get for any $l \geq 0$ that (recall $y = \frac{\tanh^2(\frac{r}{2})}{1 - \tanh^2(\frac{r}{2})}$)

$$\int_0^\infty k_N(y) \left(\tanh(\frac{r}{2}) \right)^l \left(1 - \tanh^2(\frac{r}{2}) \right)^{\frac{1}{4}} P_{s,N}^{-l}(z,w) dy$$

equals

$$\int_0^\infty y^l (1+y)^{-\frac{1}{2}-l} F(s+N, 1-s+N, 1+l, -y)\, dy,$$

and by [G-R], p. 807, 7.512.10 the value of this integral is

$$\frac{\Gamma(1+l)\,\Gamma\left(s-\frac{1}{2}+N\right)\Gamma\left(-s+\frac{1}{2}+N\right)}{\Gamma\left(\frac{1}{2}+l\right)\Gamma\left(\frac{1}{2}+2N\right)}.$$

So $F_U(w)\,(\mathrm{Im} w)^{N+\frac{1}{4}}$ equals (using (3.5.5), (3.5.12) and (3.5.13))

$$8\pi\frac{\Gamma\left(s-\frac{1}{2}+N\right)\Gamma\left(-s+\frac{1}{2}+N\right)}{\Gamma\left(\frac{1}{2}+2N\right)}\sum_{l=0}^{\infty}\frac{\Gamma(1+l)}{\Gamma\left(\frac{1}{2}+l\right)}B_l(w)\,(U)_{-l}(w). \tag{3.5.14}$$

It remains to determine $(U)_{-l}(w)$. By (3.4.7) for every $l \geq 0$ we have

$$\overline{(U)_{-l}(w)} = \frac{1}{l!}\left(K_{-N+l-1}\dots K_{-N+1}K_{-N}\left(\overline{U}\right)\right)(w). \tag{3.5.15}$$

We now assume that $U = \kappa_N(u)$, where u is a cusp form of weight 0 for $SL(2,\mathbf{Z})$ with $\Delta_0 u = s(s-1)u$, $s = \frac{1}{2}+it$ and $t \geq 0$. Using (3.5.15), the definition of $\kappa_n(u)$, (3.3.11) and [F], p. 145, formula (8), we get that

$$(U)_{-l}(w) = \frac{(-1)^l}{l!}\left(\kappa_{N-l}(u)\right)(w) \text{ for } 0 \leq l \leq N, \tag{3.5.16}$$

and then it follows by induction on the basis of (3.5.15) that

$$\overline{(U)_{-l}(w)} = \frac{(-1)^N}{l!}\left(K_{-N+l-1}\dots K_1K_0\left(\overline{u}\right)\right)(4w) \text{ for } l \geq N. \tag{3.5.17}$$

We remark a consequence of (3.5.16) and (3.5.17), which will be useful later: by the definition of $\kappa_n(u)$ and by [F], formula (11) we can check for every $l \geq 0$ with $v = u$ or $v = \overline{u}$ that

$$\left|(U)_{-l}(w)\right| = \left|\frac{(s-N)_l}{l!\,(s)_N}\left(\frac{1}{(s)_{|l-N|}}K_{|l-N|-1}\dots K_1K_0(v)\right)(4w)\right|. \tag{3.5.18}$$

3.5.3. The inner product of two projections. We now consider two cusp forms of weight $2N$ for $\Gamma_0(4)$, and we substitute the results of the previous two subsections. For proving convergence, we need an upper bound lemma.

Let $U_j(z) = (\kappa_N(u_j))(z)$ for $j = 1, 2$, where u_1, u_2 are as in Theorem 1.2. Then U_1 and U_2 are two cusp forms of weight $2N$ for $\Gamma_0(4)$ with $\Delta_{2N} U_j = s_j(s_j - 1)U_j$ $(j = 1, 2)$, and we have by (3.5.11), applying it for $U = U_1$ and also for $U = U_2$ that

$$\sum_{j=1}^{s_N} \int_{D_4} B_0(z) U_1(z) \overline{g_{N,j}(z)} d\mu_z \overline{\int_{D_4} B_0(z) U_2(z) \overline{g_{N,j}(z)} d\mu_z} \tag{3.5.19}$$

equals

$$\left(\frac{4N-1}{16\pi}\right)^2 \int_{D_4} F_{U_1}(w) \left(\mathrm{Im} w\right)^{N+\frac{1}{4}} \overline{F_{U_2}(w) \left(\mathrm{Im} w\right)^{N+\frac{1}{4}}} d\mu_w .$$

Using (3.5.14) twice in this last expression, we then see that (3.5.19) equals

$$\frac{\prod_{i=1}^{2} \left(\Gamma\left(s_i - \frac{1}{2} + N\right) \Gamma\left(-s_i + \frac{1}{2} + N\right)\right)}{\Gamma^2\left(-\frac{1}{2} + 2N\right)} \sum_{l_1, l_2 = 0}^{\infty} \frac{\Gamma(1 + l_1)}{\Gamma\left(\frac{1}{2} + l_1\right)} \frac{\Gamma(1 + l_2)}{\Gamma\left(\frac{1}{2} + l_2\right)} I_{l_1, l_2}, \tag{3.5.20}$$

where I_{l_1,l_2} is defined by

$$I_{l_1, l_2} = \int_{D_4} \left(B_{l_1}(w) \overline{(U_2)_{-l_2}(w)}\right) \overline{\left(B_{l_2}(w) \overline{(U_1)_{-l_1}(w)}\right)} d\mu_w \tag{3.5.21}$$

(this depends also on U_1 and U_2, of course, but we do not denote it), this computation is justified by the next lemma, which will be used also later.

LEMMA 3.6. *We have*

$$J := \sum_{l_1, l_2 = 0}^{\infty} \frac{\Gamma(1 + l_1)}{\Gamma\left(\frac{1}{2} + l_1\right)} \frac{\Gamma(1 + l_2)}{\Gamma\left(\frac{1}{2} + l_2\right)} J_{l_1, l_2} \le \frac{1}{\Gamma^2\left(\frac{1}{2} + N\right)} D_1 N^{D_2} 2^{2N}$$

with some positive constants D_1, D_2 depending only on u_1 and u_2, where

$$J_{l_1, l_2} := \left(\int_{D_4} \left|B_{l_1}(w) (U_2)_{-l_2}(w)\right|^2 d\mu_w\right)^{\frac{1}{2}} \left(\int_{D_4} \left|B_{l_2}(w) (U_1)_{-l_1}(w)\right|^2 d\mu_w\right)^{\frac{1}{2}}.$$

Proof. Let $1 < K < 3/2$ be fixed. Clearly J_{l_1,l_2} is at most

$$\frac{(1 + l_2)^K}{(1 + l_1)^K} \int_{D_4} \left|B_{l_1}(w) (U_2)_{-l_2}(w)\right|^2 d\mu_w + \frac{(1 + l_1)^K}{(1 + l_2)^K} \int_{D_4} \left|B_{l_2}(w) (U_1)_{-l_1}(w)\right|^2 d\mu_w .$$

Hence, by Lemma 3.12 and (3.5.18) we have, using $K > 1$, that

$$J \ll_{u_1, u_2} \sum_{i=1}^{2} \left|\frac{1}{(s_i)_N}\right|^2 \sum_{l=0}^{\infty} (1 + l)^{\frac{1}{2} + K} \left|\frac{(s_i - N)_l}{l!}\right|^2 \log^2 (2 + |l - N|) .$$

Using $N \geq 1$, $K < 3/2$, by simple estimates (using e.g. also the summation formula for $F(\alpha, \beta, \gamma; 1)$, see [G-R], p. 998, 9.122.1.) and Stirling's formula we obtain the lemma.

3.5.4. Inner products $\left(B_{l_1}\overline{(U_2)_{-l_2}}, F\right)$. For the computation of I_{l_1,l_2} (see (3.5.21)) using Corollaries 3.1 and 3.2, we give expressions for such inner products, mostly with Maass forms F (see (i), (ii) and (iii) of Lemma 3.7 below), but because of Corollary 3.2 we need such inner products also for some automorphic F which are not Laplace eigenfunctions (see (iv) of Lemma 3.7).

Let U_2 be as in Subsection 3.5.3. The definition of the constants $c_{j,r}$ and $c_{k,j,r}$ can be found above formulas (3.3.9) and (3.3.12), respectively. During the proof we will use several times tacitly (3.3.11) and the general fact that if $\Delta_l g = s(s-1)g$, then $\Delta_{-l}\overline{g} = \overline{s}(\overline{s}-1)\overline{g}$.

LEMMA 3.7. *Let $l_1, l_2 \geq 0$, and $m = \frac{1}{4} + (l_1 + l_2 - N)$. Introduce the notation*

$$A_{L_1}(S) = \overline{\Gamma\left(\frac{1}{4} + S\right)} \frac{(-1)^{L_1+l_2}}{l_2!} \frac{\overline{\Gamma(s_2 - N + l_2 + L_1)}}{\overline{\Gamma(s_2 + N - l_2 - L_1)}} \frac{\overline{(S-m)_{l_1-L_1}}}{\overline{\Gamma(S + m - l_1 + L_1)}}.$$

Let $F \in P_{2m}(D_4)$ satisfy the conditions of (i), (ii), (iii) or (iv) below. Then

$$\int_{D_4} B_{l_1}(w)\overline{(U_2)_{-l_2}(w)F(w)}d\mu_w = \frac{(-1)^{l_1}}{l_1!} \sum_{L_1=0}^{l_1} \binom{l_1}{L_1} J_{L_1}(F), \tag{3.5.22}$$

where $J_{L_1}(F)$ is given in the various cases as follows.

(i) If $F = u_{j,2m}$, where $j \geq 0$ for $m > 0$, and $j \geq 1$ for $m < 0$, then for every $0 \leq L_1 \leq l_1$ we have that $J_{L_1}(F)$ equals

$$A_{L_1}(S_j)\overline{c_{j,l_1+l_2-N}}\left(\overline{S_j} + \frac{1}{4}\operatorname{sgn}\left(m - \frac{1}{4}\right)\right)_{\left|m-\frac{1}{4}\right|} \int_{D_4} B_0(w)\overline{u_2(4w)u_{j,\frac{1}{2}}(w)}d\mu_w.$$

(ii) If $F = E_a(*, s, 2m)$ with $a = 0$ or ∞, $\mathrm{Re}s = \frac{1}{2}$, then for every $0 \leq L_1 \leq l_1$ we have that

$$J_{L_1}(F) = A_{L_1}(s) \int_{D_4} B_0(w)\overline{u_2(4w)E_a\left(w, s, \frac{1}{2}\right)}d\mu_w.$$

(iii) If $F = g_{k,j,l_1+l_2-N}$ with some $1 \leq k \leq l_1 + l_2 - N$, $1 \leq j \leq s_k$, then for every $0 \leq L_1 \leq l_1$ we have that

$$J_{L_1}(F) = A_{L_1}\left(k + \frac{1}{4}\right)\overline{c_{k,j,l_1+l_2-N}}\left(k + \frac{1}{2}\right)_{m-\frac{1}{4}} \int_{D_4} B_0(w)\left(\kappa_k\left(\overline{u_2}\right)\right)(w)\overline{g_{k,j}(w)}d\mu_w.$$

(iv) If $m<0$, and $F\in P_{2m}(D_4)$ is such that

$$K_{-\frac{3}{4}}\dots K_{m-r+1}K_{m-r}L_{m+1-r}\dots L_{m-1}L_mF=0$$

for every integer $r\geq 0$, then for every $0\leq L_1\leq l_1$ we have that $J_{L_1}(F)=0$.

Proof. First we assume only $F\in P_{2m}(D_4)$. By Lemma 3.2 we see that (3.5.22) holds with

$$J_{L_1}(F)=\left(B_0,\left(L_{N-l_2-L_1+1}\dots L_{N-l_2-1}L_{N-l_2}\left(U_2\right)_{-l_2}\right)\left(L_{m-l_1+L_1+1}\dots L_{m-1}L_mF\right)\right), \tag{3.5.23}$$

(the right-hand side denotes an inner product on D_4). It is clear by (3.5.15) that

$$L_{N-l_2-L_1+1}\dots L_{N-l_2-1}L_{N-l_2}\left(U_2\right)_{-l_2}=\frac{(l_2+L_1)!}{l_2!}\left(U_2\right)_{-l_2-L_1}. \tag{3.5.24}$$

For the computation of $J_{L_1}(F)$ we now distinguish between two cases.

Case I. We assume $l_2+L_1\leq N$. Then we see by (3.5.24) and (3.5.16) that

$$\overline{\left(L_{N-l_2-L_1+1}\dots L_{N-l_2-1}L_{N-l_2}\left(U_2\right)_{-l_2}\right)(w)}$$

equals

$$\frac{(-1)^N}{l_2!}\frac{\overline{\Gamma\left(s_2-N+l_2+L_1\right)}}{\overline{\Gamma\left(s_2+N-l_2-L_1\right)}}\left(L_{1-N+l_2+L_1}\dots L_{-1}L_0\overline{u_2}\right)(4w).$$

Hence, using Lemma 3.1, we see that if $l_2+L_1\leq N$, then $J_{L_1}(F)$ equals

$$\frac{(-1)^{l_2+L_1}}{l_2!}\frac{\overline{\Gamma\left(s_2-N+l_2+L_1\right)}}{\overline{\Gamma\left(s_2+N-l_2-L_1\right)}}\int_{D_4}B_0(w)\overline{\left(u_2\right)(4w)}F_{l_1,L_1}(w)d\mu_w, \tag{3.5.25}$$

where we write

$$F_{l_1,L_1}:=L_{\frac{3}{4}}\dots L_{-m+l_1-L_1-1}L_{-m+l_1-L_1}\left(\overline{L_{m-l_1+L_1+1}\dots L_{m-1}L_mF}\right). \tag{3.5.26}$$

By (3.5.25) and (3.5.26) we get (iv) of the lemma at once (since if $m<0$, then we are in Case I for every $L_1\leq l_1$).

Assume that F is a Maass form, and $\Delta_{2m}F=S(S-1)F$. Then, applying (8) of [F], we see that if $l_1+l_2\geq N\geq l_2+L_1$, then

$$F_{l_1,L_1}=\frac{\overline{\Gamma\left(S+\frac{1}{4}\right)}\,\overline{\Gamma\left(S-m+l_1-L_1\right)}}{\overline{\Gamma\left(S-\frac{1}{4}\right)}\,\overline{\Gamma\left(S+m-l_1+L_1\right)}}\overline{L_{\frac{5}{4}}\dots L_{m-1}L_mF}; \tag{3.5.27}$$

if $l_1 + l_2 < N$, then

$$F_{l_1,L_1} = \overline{\frac{\Gamma(S+m)\Gamma(S-m+l_1-L_1)}{\Gamma(S-m)\Gamma(S+m-l_1+L_1)}} L_{\frac{3}{4}} \dots L_{-m-1}L_{-m}\left(\overline{F}\right). \tag{3.5.28}$$

And, using (8) and (4) of [F], by (3.5.25), (3.5.27) and (3.5.28) we get, checking every case, that (i), (ii) and (iii) are true for the case $l_2 + L_1 \le N$. (In case (iii) we have that (3.5.27) is 0, and also $A_{L_1}\left(k+\frac{1}{4}\right) = 0$.)

Case II. Assume now that $l_2 + L_1 > N$. In this case, we need to consider F only of the following form: $F = K_{m-1}K_{m-2}\dots K_{\frac{5}{4}+t}K_{\frac{1}{4}+t}F_0$ with an integer $0 \le t \le l_1 + l_2 - N$ and a Maass form F_0 of weight $\frac{1}{2} + 2t$ for $\Gamma_0(4)$, such that we have $t = 0$ or $L_{\frac{1}{4}+t}F_0 = 0$. Let $\Delta_{\frac{1}{2}+2t}F_0 = S(S-1)F_0$. It is clear, using (4) and (8) of [F] that if $l_2 + L_1 - N < t$ (hence $m - l_1 + L_1 + 1 \le \frac{1}{4} + t \le m$ and $t > 0$), then

$$L_{m-l_1+L_1+1}\dots L_{m-1}L_mF = 0. \tag{3.5.29}$$

If $l_2 + L_1 - N \ge t$, then $L_{m-l_1+L_1+1}\dots L_{m-1}L_mF$ equals (by (8) of [F])

$$\frac{\Gamma\left(S-\frac{1}{4}-l_2-L_1+N\right)\Gamma(S+m)}{\Gamma\left(S+\frac{1}{4}+l_2+L_1-N\right)\Gamma(S-m)} K_{-\frac{3}{4}+l_2+L_1-N}\dots K_{\frac{5}{4}+t}K_{\frac{1}{4}+t}F_0,$$

and so, by (3.5.23), Lemma 3.1 and (3.5.24), $J_{L_1}(F)$ equals

$$(-1)^{l_2+L_1-N-t}\,\overline{\frac{\overline{\Gamma\left(S-\frac{1}{4}-l_2-L_1+N\right)}\,\overline{\Gamma(S+m)}}{\overline{\Gamma\left(S+\frac{1}{4}+l_2+L_1-N\right)}\,\overline{\Gamma(S-m)}}} \int_{D_4} B_0(w)V_{l_2,L_1}(w)\overline{F_0(w)}d\mu_w, \tag{3.5.30}$$

where we write

$$V_{l_2,L_1} := \frac{(l_2+L_1)!}{l_2!} L_{t+1}\dots L_{-N+l_2+L_1-1}L_{-N+l_2+L_1}\left(\overline{(U_2)_{-l_2-L_1}}\right).$$

Since $l_2 + L_1 > N$, so by (3.5.17) we get

$$\overline{(U_2)_{-l_2-L_1}}(w) = \frac{(-1)^N}{(l_2+L_1)!}\left(K_{-N+l_2+L_1-1}\dots K_1K_0\left(\overline{u_2}\right)\right)(4w),$$

hence, again by (8) of [F], for $l_2 + L_1 - N \ge t$ we get

$$V_{l_2,L_1}(w) = \frac{(-1)^N}{l_2!}\overline{\frac{\Gamma(s_2-t)\Gamma(s_2-N+l_2+L_1)}{\Gamma(s_2+t)\Gamma(s_2+N-l_2-L_1)}}\left(K_{t-1}\dots K_1K_0\left(\overline{u_2}\right)\right)(4w). \tag{3.5.31}$$

By (3.5.23), (3.5.29), (3.5.30) and (3.5.31), checking every case, we get that (i), (ii) and (iii) are true also for $l_2 + L_1 > N$. (In case (iii) and $l_2 + L_1 - N < k$ we have that (3.5.29) is 0, and also $A_{L_1}\left(k + \frac{1}{4}\right) = 0$.) The lemma is proved.

3.5.5. Expression for the sum in (3.5.20). We first compute I_{l_1,l_2} (see (3.5.21)) on the basis of the previous subsection, using Corollary 3.1 for the case $l_1 + l_2 \geq N$, and Corollary 3.2 for $l_1 + l_2 < N$. Then we substitute the obtained expressions into (3.5.20). We first note that

$$\int_{D_4} B_{l_2}(w)\overline{(U_1)_{-l_1}(w)F(w)}d\mu_w \tag{3.5.32}$$

is the same as the left-hand side of (3.5.22), if we use the substitutions $l_1 \leftrightarrow l_2$, $U_1 \leftrightarrow U_2$. Hence we can compute also (3.5.32) using Lemma 3.7.

As in Lemma 3.7, write

$$m = \frac{1}{4} + l_1 + l_2 - N.$$

In fact we should write $m = m_{l_1,l_2}$ to indicate the dependence on l_1 and l_2 (note that N is fixed), but for simplicity we use just the notation m.

In the case $l_1 + l_2 \geq N$, by Corollary 3.1 and (i), (ii) and (iii) of Lemma 3.7, using also (3.3.9) and (3.3.12) we get that I_{l_1,l_2} equals the sum of

$$\sum_{j=0}^{\infty} C_{l_1,l_2,j}(v_2, u_{j,\frac{1}{2}})\overline{(v_1, u_{j,\frac{1}{2}})} + \sum_{k=1}^{l_1+l_2-N} \sum_{j=1}^{s_k} C_{l_1,l_2}(k,j)(v_{2,k}, g_{k,j})\overline{(v_{1,k}, g_{k,j})}$$

and

$$\frac{1}{4\pi} \sum_{a=0,\infty} \int_{-\infty}^{\infty} C_{l_1,l_2}(r)\zeta_a(v_2,r)\overline{\zeta_a(v_1,r)}dr,$$

where we write

$$v_i = v_{i,0}, \quad v_{i,k} = B_0\kappa_k\left(\overline{u_i}\right) \qquad (i = 1,2 \text{ and } k = 0,1,2,...),$$

and the coefficients are defined as follows:

$$C_{l_1,l_2,j} = D_{l_1,l_2,0}(S_j), \tag{3.5.33}$$

$$C_{l_1,l_2}(k,j) = D_{l_1,l_2,k}\left(k + \frac{1}{4}\right), \tag{3.5.34}$$

$$C_{l_1,l_2}(r) = D_{l_1,l_2,0}\left(\frac{1}{2}+ir\right) \tag{3.5.35}$$

with the notations (for general S)

$$D_{l_1,l_2,k}(S) = \frac{\Gamma\left(S+\frac{1}{4}+k\right)\Gamma\left(\frac{5}{4}-S+k\right)}{\left(l_1!\right)^2\left(l_2!\right)^2}\frac{\overline{\Gamma\left(S+m\right)}}{\Gamma\left(1-S+m\right)}\Sigma_{l_1,l_2}(S), \tag{3.5.36}$$

$$\Sigma_{l_1,l_2}(S) = \sum_{L_1=0}^{l_1}\sum_{L_2=0}^{l_2}(-1)^{L_1+L_2}\binom{l_1}{L_1}\binom{l_2}{L_2}G(S,l_1,l_2,L_1,L_2), \tag{3.5.37}$$

where $G(S,l_1,l_2,L_1,L_2)$ denotes

$$\overline{\frac{\Gamma\left(s_2-N+l_2+L_1\right)}{\Gamma\left(s_2+N-l_2-L_1\right)}}\frac{\Gamma\left(s_1-N+l_1+L_2\right)}{\Gamma\left(s_1+N-l_1-L_2\right)}\overline{\frac{(S-m)_{l_1-L_1}}{\Gamma\left(S+m-l_1+L_1\right)}}\frac{(S-m)_{l_2-L_2}}{\Gamma\left(S+m-l_2+L_2\right)}. \tag{3.5.38}$$

In the case $l_1+l_2<N$, we apply Corollary 3.2 for the choices $f(w)=B_{l_2}(w)\overline{(U_1)_{-l_1}(w)}$, $h(w)=B_{l_1}(w)\overline{(U_2)_{-l_2}(w)}$. Applying (iv) of Lemma 3.7 and (3.4.3) we obtain that

$$\int_{D_4} B_{l_1}(w)\overline{(U_2)_{-l_2}(w)g_f(w)}d\mu_w = 0.$$

Then using (i) and (ii) of Lemma 3.7, after some calculations we obtain from Corollary 3.2 (using also (3.3.10) and the fact that $\mathrm{Re}S_j=\frac{1}{2}$ or S_j is real) that I_{l_1,l_2} equals

$$\sum_{j=1}^{\infty}C_{l_1,l_2,j}(v_2,u_{j,\frac{1}{2}})\overline{(v_1,u_{j,\frac{1}{2}})}+\frac{1}{4\pi}\sum_{a=0,\infty}\int_{-\infty}^{\infty}C_{l_1,l_2}(r)\zeta_a(v_2,r)\overline{\zeta_a(v_1,r)}dr$$

for $l_1+l_2<N$, with the above notations.

Then, using that $(v_1,u_{0,\frac{1}{2}})=0$ by Lemma 3.14, combining the cases $l_1+l_2\geq N$ and $l_1+l_2<N$, we get that

$$\sum_{l_1,l_2=0}^{\infty}\frac{\Gamma\left(1+l_1\right)}{\Gamma\left(\frac{1}{2}+l_1\right)}\frac{\Gamma\left(1+l_2\right)}{\Gamma\left(\frac{1}{2}+l_2\right)}I_{l_1,l_2} \tag{3.5.39}$$

equals the sum of

$$\sum_{j=1}^{\infty}C_j(v_2,u_{j,\frac{1}{2}})\overline{(v_1,u_{j,\frac{1}{2}})}+\sum_{k=1}^{\infty}\sum_{j=1}^{s_k}C(k,j)(v_{2,k},g_{k,j})\overline{(v_{1,k},g_{k,j})} \tag{3.5.40}$$

and

$$\frac{1}{4\pi}\sum_{a=0,\infty}\int_{-\infty}^{\infty}C(r)\zeta_a(v_2,r)\overline{\zeta_a(v_1,r)}dr, \tag{3.5.41}$$

where

$$C_j = \sum_{l_1,l_2=0}^{\infty} \frac{\Gamma(1+l_1)}{\Gamma\left(\frac{1}{2}+l_1\right)} \frac{\Gamma(1+l_2)}{\Gamma\left(\frac{1}{2}+l_2\right)} C_{l_1,l_2,j}, \tag{3.5.42}$$

$$C(k,j) = \sum_{l_1,l_2=0}^{\infty} \frac{\Gamma(1+l_1)}{\Gamma\left(\frac{1}{2}+l_1\right)} \frac{\Gamma(1+l_2)}{\Gamma\left(\frac{1}{2}+l_2\right)} C_{l_1,l_2}(k,j), \tag{3.5.43}$$

$$C(r) = \sum_{l_1,l_2=0}^{\infty} \frac{\Gamma(1+l_1)}{\Gamma\left(\frac{1}{2}+l_1\right)} \frac{\Gamma(1+l_2)}{\Gamma\left(\frac{1}{2}+l_2\right)} C_{l_1,l_2}(r) \tag{3.5.44}$$

(in the case of $C(k,j)$ we used that the factor $\frac{1}{\Gamma(1-S+m)}$ in (3.5.34) is 0, if $k > l_1 + l_2 - N$, since $S = k + \frac{1}{4}$). The reordering of the sum is justified by Lemma 3.6 and the inequalities in Corollaries 3.1 and 3.2, and we also see by these statements that if

$$C_j^* = \sum_{l_1,l_2=0}^{\infty} \frac{\Gamma(1+l_1)}{\Gamma\left(\frac{1}{2}+l_1\right)} \frac{\Gamma(1+l_2)}{\Gamma\left(\frac{1}{2}+l_2\right)} \left|C_{l_1,l_2,j}\right|,$$

$$C(k,j)^* = \sum_{l_1,l_2=0}^{\infty} \frac{\Gamma(1+l_1)}{\Gamma\left(\frac{1}{2}+l_1\right)} \frac{\Gamma(1+l_2)}{\Gamma\left(\frac{1}{2}+l_2\right)} \left|C_{l_1,l_2}(k,j)\right|,$$

$$C(r)^* = \sum_{l_1,l_2=0}^{\infty} \frac{\Gamma(1+l_1)}{\Gamma\left(\frac{1}{2}+l_1\right)} \frac{\Gamma(1+l_2)}{\Gamma\left(\frac{1}{2}+l_2\right)} \left|C_{l_1,l_2}(r)\right|,$$

then with a constant D_2 depending only on u_1, u_2 we have

$$\sum_{j=1}^{\infty} C_j^* \left|(v_2, u_{j,\frac{1}{2}})\overline{(v_1, u_{j,\frac{1}{2}})}\right| + \sum_{k=1}^{\infty}\sum_{j=1}^{s_k} C(k,j)^* \left|(v_{2,k}, g_{k,j})\overline{(v_{1,k}, g_{k,j})}\right| \ll_{u_1,u_2} \frac{N^{D_2} 2^{2N}}{\Gamma^2\left(\frac{1}{2}+N\right)} \tag{3.5.45}$$

and

$$\frac{1}{4\pi} \sum_{a=0,\infty} \int_{-\infty}^{\infty} C(r)^* \left|\zeta_a(v_2, r)\overline{\zeta_a(v_1, r)}\right| dr \ll_{u_1,u_2} \frac{N^{D_2} 2^{2N}}{\Gamma^2\left(\frac{1}{2}+N\right)}. \tag{3.5.46}$$

We can compute C_j, $C(k,j)$ and $C(r)$ by formulas (3.5.36)-(3.5.38) and Theorem 4.2 (proved in Chapter 4), using (3.5.42) and (3.5.33) in the case of C_j, (3.5.43) and (3.5.34) in the case of $C(k,j)$, finally (3.5.44) and (3.5.35) in the case of $C(r)$. Then, on the one hand, by (3.5.19), (3.5.20), (3.5.39)-(3.5.41) and (3.3.2), (3.3.3) we get the property $P(f, \{a_n\})$ required in Lemma 3.5; on the other hand, by (3.5.45) and (3.5.46) we obtain also the upper bounds (3.5.1)-(3.5.3), so Lemma 3.5 is proved.

3.6. Proof of the general case of the theorem

3.6.1. Some upper bounds. Formula (4.5.9) (see Chapter 4) and formula (3.5.3) with $N = 1$ imply that

$$\sum_{k=1}^{\infty} \frac{1}{k^{3/2}} \Gamma\left(2k + \frac{1}{2}\right) \sum_{j=1}^{s_k} \left| \left(B_0\kappa_k\left(\overline{u_2}\right), g_{k,j}\right) \overline{\left(B_0\kappa_k\left(\overline{u_1}\right), g_{k,j}\right)} \right| < \infty. \tag{3.6.1}$$

We now prove that there is a constant $A > 0$ depending only on u_1 and u_2 such that

$$\sum_{j=1}^{\infty} e^{\pi |T_j|} \left(1 + |T_j|\right)^{-A} \left| \left(B_0\kappa_0\left(\overline{u_2}\right), u_{j,\frac{1}{2}}\right) \overline{\left(B_0\kappa_0\left(\overline{u_1}\right), u_{j,\frac{1}{2}}\right)} \right| < \infty, \tag{3.6.2}$$

$$\sum_{a=0,\infty} \int_{-\infty}^{\infty} \left| e^{\pi |r|} \left(1 + |r|\right)^{-A} \zeta_a\left(B_0\kappa_0\left(\overline{u_2}\right), r\right) \overline{\zeta_a\left(B_0\kappa_0\left(\overline{u_1}\right), r\right)} \right| dr < \infty. \tag{3.6.3}$$

To prove this, let k be a large positive integer. It follows from Theorem 1.3 and elementary linear algebra that if $M > 0$ is large enough in terms of k, then there is a nonzero vector $(a_m)_{M \le m \le 2M}$ such that for

$$f(x) := \sum_{m=M}^{2M} \frac{a_m}{\Gamma^2\left(m \pm ix\right)}$$

formula (1.4.1) is true and the coefficients e_j in (1.4.2) are 0, so we have

$$f(x) = \sum_{N=1}^{\infty} d_N \phi_{i\left(\frac{1}{4} - N\right)}(x)$$

with some coefficients $d_N = O\left(N^{-k}\right)$. If k is large enough in terms of the constant D in (3.5.1), we get combining (3.5.1) for different integers N with coefficients d_N that

$$\sum_{j=1}^{\infty} \left| f\left(T_j\right) \Gamma\left(\frac{3}{4} \pm iT_j\right) \left(B_0\kappa_0\left(\overline{u_2}\right), u_{j,\frac{1}{2}}\right) \overline{\left(B_0\kappa_0\left(\overline{u_1}\right), u_{j,\frac{1}{2}}\right)} \right| < \infty,$$

and similarly for Eisenstein series on the basis of (3.5.2). By the definition of f and Stirling's formula this proves the estimates (3.6.2) and (3.6.3).

3.6.2. A consequence of Lemma 3.5. It is clear, in view of the upper bounds (3.5.1)-(3.5.3), that if $\{C_N\}_{N \ge 1}$ is a rapidly decreasing sequence, then we can take the linear combination of the cases of Lemma 3.5 with these coefficients, since everything is absolutely convergent. We will now show that we can take such a linear combination even in some cases when $\{C_N\}_{N\geq 1}$ is not so rapidly decreasing.

LEMMA 3.8. *For every A with $\mathrm{Re}A \geq \frac{5}{2}$ we have that*

$$\sum_{n=1}^{\infty}(-1)^n \frac{(1-A)_{n-1}}{\Gamma(n)} \frac{|(s_1)_n|^2 |(s_2)_n|^2 \Gamma\left(2n-\frac{1}{2}\right)}{|\Gamma(n+it_1)|^2 |\Gamma(n+it_2)|^2} \sum_{j=1}^{s_n} \left(B_0\kappa_n(u_1), g_{n,j}\right) \overline{\left(B_0\kappa_n(u_2), g_{n,j}\right)} \tag{3.6.4}$$

equals the sum of the following three lines (see Theorem 4.3 in Chapter 4 for the definition of $M_\lambda(A)$):

$$\sum_{j=1}^{\infty} M_{T_j}(A)\Gamma\left(\frac{3}{4}\pm iT_j\right)\left(B_0\kappa_0(\overline{u_2}), u_{j,\frac{1}{2}}\right)\overline{\left(B_0\kappa_0(\overline{u_1}), u_{j,\frac{1}{2}}\right)}, \tag{3.6.5}$$

$$\frac{1}{4\pi}\sum_{a=0,\infty}\int_{-\infty}^{\infty} M_r(A)\Gamma\left(\frac{3}{4}\pm ir\right)\zeta_a\left(B_0\kappa_0(\overline{u_2}), r\right)\overline{\zeta_a\left(B_0\kappa_0(\overline{u_1}), r\right)}dr, \tag{3.6.6}$$

$$\sum_{k=1}^{\infty} M_{i\left(\frac{1}{4}-k\right)}(A)\Gamma\left(2k+\frac{1}{2}\right)\sum_{j=1}^{s_k}\left(B_0\kappa_k(\overline{u_2}), g_{k,j}\right)\overline{\left(B_0\kappa_k(\overline{u_1}), g_{k,j}\right)}, \tag{3.6.7}$$

and every sum and integral is absolutely convergent here for every such number A.

Proof. By formulas (3.3.2), (3.3.3) and (1.3.3)-(1.3.10) we see that the identity of this lemma is obtained formally by taking a linear combination of the identities of Lemma 3.5 with coefficients $(-1)^N \frac{(1-A)_{N-1}}{iCR_N\Gamma(N)}$. It follows from (3.3.2), (3.3.3) and Lemma 3.5 that if $\mathrm{Re}A$ is large enough (depending on u_1 and u_2), then the statement of the present lemma is true (note, in particular, that (3.6.5) and (3.6.6) are absolutely convergent if $\mathrm{Re}A$ is large enough). We extend this result to $\mathrm{Re}A \geq 5/2$ by analytic continuation and continuity.

It follows from (3.6.1) (applying it with $\overline{u_1}$ in place of u_2, and $\overline{u_2}$ in place of u_1, which is possible, these are also fixed cusp forms) that (3.6.4) extends regularly to $\mathrm{Re}A > \frac{5}{2}$ and extends continuously to $\mathrm{Re}A \geq \frac{5}{2}$. The same assertions are true for (3.6.7) using Theorem 4.3 (ii) and (3.6.1).

We claim that the same assertions are true for (3.6.5) and (3.6.6) too, but the proof in this case is more complicated. Take any compact subset L of the half-plane $\mathrm{Re}A \geq \frac{5}{2}$, and let K be a large but fixed integer. Take the integer $t > 0$, complex numbers $A_1, A_2, \ldots, A_t$

and polynomials $Q_1, Q_2, \dots, Q_t$ as in Theorem 4.3 (iii). Define for Re$A \geq \frac{5}{2}$ and $|\text{Im}\lambda| < \frac{3}{4}$ (taking into account Theorem 4.3 (i))

$$S_\lambda(A) = M_\lambda(A) - \sum_{i=1}^{t} 2^{A-A_i} Q_i(A) M_\lambda(A_i). \tag{3.6.8}$$

We see by (3.6.2) and Theorem 4.3 (iii) that if K is large enough depending on u_1 and u_2, and we write $S_{T_j}(A)$ in place of $M_{T_j}(A)$ in (3.6.5), then the sum in T_j will be uniformly absolutely convergent for $A \in L$, and the resulting function of A will be regular on every open subset of L. The same is true for (3.6.6) if we write $S_r(A)$ in place of $M_r(A)$ there. We have seen in the first paragraph of the proof of the present lemma that (3.6.5) and (3.6.6) are absolutely convergent if we write any A_i in place of A (since K is large enough depending on u_1 and u_2 and Re$A_i > K$). Hence, expressing $M_{T_j}(A)$ and $M_r(A)$ from (3.6.8), we finally proved that (3.6.5) and (3.6.6) are uniformly absolutely convergent for $A \in L$, and the resulting functions are regular on every open subset of L.

By analytic continuation and continuity, these considerations prove the lemma.

3.6.3. Conclusion. We now finish the proof of Theorem 1.2, combining Lemmas 3.5, 3.8 and Theorem 1.3.

We remark first that we have to show that the statement of Theorem 1.2 is true if we fix the constant K to be large enough. We will choose K to be larger and larger several times during the proof.

The statement about the absolute convergence in (1.3.3) and (1.3.4) follows easily from the absolute convergence of the left-hand side of (4.5.11), (4.5.10) (see Corollary 4.1 in Chapter 4), (3.3.3) and Prop. 4.4 of [G1].

When f is identically 0, the statement follows at once from Lemma 3.5 and from the cases $A = \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \ldots$ of Lemma 3.8 (a finite number of them suffice). Indeed, by subtracting a suitable finite linear combination of these cases of Lemma 3.8, we can achieve that $a_n = O\left(n^{-R}\right)$ for any given $R > 0$ (we use for this Stirling's formula in the form [G-R], p. 889, 8.344), and then we can apply Lemma 3.5.

In the case when $f(x) = \frac{1}{\Gamma\left(\frac{3}{4} \pm ix\right)}$ and $a_n = 0$ for every n, we have $g(x) \equiv f(x)$ and $b_n \equiv 0$ by the formula in the proof of Theorem 6.5 of [G1] with $n = 0$ and $g = 1/4$ there. Then by Corollary 3.1 and Lemma 3.14 we see that both sides equal

$$\int_{D_4} |B_0(z)|^2 u_1(4z) \overline{u_2(4z)} d\mu_z.$$

Hence the statement is true for this case, and so we may assume that f satisfies (1.4.1) by subtracting a suitable constant multiple of $\frac{1}{\Gamma\left(\frac{3}{4} \pm ix\right)}$.

Let f be a function satisfying (1.4.1) and the conditions of Theorem 1.2, then we can apply Theorem 1.3. Define now sequences b_n and a_n $(n \geq 1)$ in the following way: $iCb_nR_n = d_n$, i.e.

$$f(x) = iC \sum_{k=1}^{\infty} b_k \phi_x \left(i\left(\frac{1}{4} - k\right)\right) R_k$$

for $|\mathrm{Im} x| < \frac{3}{4}$ on the basis of (1.4.3), and

$$a_n := iC \sum_{k=1}^{\infty} b_k \phi_{i\left(\frac{1}{4} - n\right)} \left(i\left(\frac{1}{4} - k\right)\right) R_k.$$

Observe that the pair f, $\{a_n\}$ is the Wilson function transform of type II of the pair g, $\{b_n\}$, where $g \equiv 0$. The sequences a_n and b_n satisfy the condition given for a_n in Theorem 1.2 (the constant K there may be different than the original K, but it is still large), for b_n it follows from (1.4.2) and (3.3.3), and for a_n it follows from (4.5.11). We claim that with this b_n, a_n and f formula (1.3.10) equals the sum of (1.3.5), (1.3.6) and (1.3.7). Indeed, this follows from an already proved special case of Theorem 1.2, the $P(g, \{b_n\})$ case (this is really proved already, since $g \equiv 0$), writing in this special case $\overline{u_1}$ in place of u_2, $\overline{u_2}$ in place of u_1, and taking into account that $\phi_\lambda(x; a, b, c, d)$ is symmetric in $a, b, c, 1-d$, hence that our Wilson function transform is symmetric in t_1 and t_2.

Since our Wilson function transform is its own inverse by Theorem 5.10 of [G1] (note that our functions are square integrable with respect to the measure dh of [G1]), we get that (1.3.3) and (1.3.4) are true with $g \equiv 0$ and with b_n, a_n and f above. Hence the fact (proved above) that (1.3.10) equals the sum of (1.3.5), (1.3.6) and (1.3.7) implies that our theorem is true with the given f and with this sequence a_n.

Since we proved the $f \equiv 0$ case already, Theorem 1.2 is proved.

3.7. Lemmas on automorphic functions

3.7.1. The functions B_n. We prove in Lemmas 3.9 and 3.10 basic identities and estimates for the functions B_n defined in (3.3.8). Lemma 3.11 is needed for Lemma 3.10 but it is used also at another point in this chapter. Recall that L_n^α denotes Laguerre polynomials.

Lemma 3.9. *We have*

$$B_n(z) = y^{\frac{1}{4}} \sum_{m=-\infty}^{\infty} L_n^{-\frac{1}{2}}\left(4\pi m^2 y\right) e\left(m^2 z\right) \tag{3.7.1}$$

for every $n \geq 0$ and $z = x + iy \in H$, and

$$\frac{1}{n!} K_{(n-1)+\frac{1}{4}} \dots K_{\frac{5}{4}} K_{\frac{1}{4}} B_0 = B_n \tag{3.7.2}$$

for every $n \geq 1$. We also have the following relations for every $n \geq 0$:

$$\Delta_{2n+\frac{1}{2}} B_n = \frac{1}{4}\left(\frac{1}{4} - 1\right) B_n, \tag{3.7.3}$$

$$B_n(\gamma z) = \nu(\gamma)\left(\frac{j_\gamma(z)}{|j_\gamma(z)|}\right)^{2n+\frac{1}{2}} B_n(z) \tag{3.7.4}$$

for every $\gamma \in \Gamma_0(4)$,

$$B_n\left(\frac{-1}{4z}\right) = e\left(\frac{-1}{8}\right)\left(\frac{z}{|z|}\right)^{2n+\frac{1}{2}} B_n(z), \tag{3.7.5}$$

and finally, for every $z = x+iy \in H$ and $n \geq 0$ we have that $B_n(\sigma_{-\frac{1}{2}} z)\left(\frac{j_{\sigma_{-1/2}}(z)}{\left|j_{\sigma_{-1/2}}(z)\right|}\right)^{-\frac{1}{2}-2n}$ equals

$$e(-\frac{1}{8}) y^{\frac{1}{4}} \sum_{m=-\infty}^{\infty} L_n^{-\frac{1}{2}}\left(4\pi\left(m+\frac{1}{2}\right)^2 y\right) e\left(\left(m+\frac{1}{2}\right)^2 z\right). \tag{3.7.6}$$

Proof. Using [G-R], p. 992, formula 8.975.1, we have

$$\sum_{n=0}^{\infty} L_n^{-\frac{1}{2}}\left(4\pi m^2 y\right) L^n = (1-L)^{-\frac{1}{2}} e^{\frac{4\pi m^2 y L}{L-1}}$$

for $y>0$ and $|L|<1$, from which it follows for $z=x+iy\in H$ and $|L|<1$ that

$$\sum_{n=0}^{\infty}\left(\sum_{m=-\infty}^{\infty} L_n^{-\frac{1}{2}}\left(4\pi m^2 y\right) e\left(m^2 z\right)\right) L^n=\theta\left(T_z\left(i\frac{1+L}{1-L}\right)\right)(1-L)^{-\frac{1}{2}},$$

which, together with (3.3.8), proves (3.7.1). To prove (3.7.2), it is enough to show that

$$\frac{1}{n+1}K_{n+\frac{1}{4}}B_n=B_{n+1} \tag{3.7.7}$$

for every $n\geq 0$. By the definition of the operators K and by (3.7.1) we have that $\left(K_{n+\frac{1}{4}}B_n\right)(z)$ equals (here $\left(L_n^{-\frac{1}{2}}\right)^{(1)}$ denotes the derivative of $L_n^{-\frac{1}{2}}$)

$$y^{\frac{1}{4}}\sum_{m=-\infty}^{\infty}\left(\left(-4\pi m^2 y+n+\frac{1}{2}\right)L_n^{-\frac{1}{2}}\left(4\pi m^2 y\right)+4\pi m^2 y\left(L_n^{-\frac{1}{2}}\right)^{(1)}\left(4\pi m^2 y\right)\right)e\left(m^2 z\right),$$

and applying [G-R], p. 991, 8.971.3 we get (3.7.2). Formula (3.7.3) can be checked directly for $n=0$, and then it follows for larger n from (3.7.2) and [F], p. 145, formula (6). Similarly, (3.7.4) and (3.7.5) are well-known for $n=0$, and they follow for larger n from (3.7.2) and [F], p. 145, formula (5). The case $n=0$ of (3.7.6) is known (and not hard to prove), and the general case follows by induction, using again (3.7.2), [G-R], p. 991, 8.971.3 and [F], formula (5). The lemma is proved.

LEMMA 3.10. *Let $z\in D_4$, and let $0\leq j\leq 5$ be such that $\gamma_j^{-1}z\in D_1$.*

(i) There is an absolute constant $A>0$ such that if $n\geq 0$ is an integer and $\operatorname{Im}\left(\gamma_j^{-1}z\right)\geq An$, then

$$|B_n(z)|\leq A\left(\operatorname{Im}\left(\gamma_j^{-1}z\right)\right)^{\frac{1}{4}}(n+1)^{-\frac{1}{2}}.$$

(ii) If $N\geq 0$ is an integer and $\operatorname{Im}\left(\gamma_j^{-1}z\right)\ll N+1$ with implied absolute constant, then for any $\epsilon>0$ we have

$$\sum_{n=N}^{2N}|B_n(z)|^2\ll_\epsilon (N+1)^{\frac{1}{2}+\epsilon}.$$

Proof. Part (i) follows easily from (3.7.1), (3.7.5), (3.7.6) and [G-R], p.990, formula 8.970.1, since $L_n^{-\frac{1}{2}}(0)\ll(n+1)^{-\frac{1}{2}}$.

For the proof of (ii) let $n\geq 0$, and write $h_z(L)=(\operatorname{Im}z)^{\frac{1}{4}}\theta\left(T_z\left(i\frac{1+L}{1-L}\right)\right)(1-L)^{-\frac{1}{2}}$, then

$$B_n(z)=\frac{1}{2\pi i}\int_{|L|=r}\frac{h_z(L)}{L^{n+1}}dL \tag{3.7.8}$$

for any $0 < r < 1$. Now,

$$\mathrm{Im}\left(T_z\left(i\frac{1+L}{1-L}\right)\right) = (\mathrm{Im}z)\,\frac{1-|L|^2}{|1-L|^2},$$

so

$$h_z(L) = B_0\left(T_z\left(i\frac{1+L}{1-L}\right)\right)\left(1-|L|^2\right)^{-\frac{1}{4}}\,\frac{|1-L|^{\frac{1}{2}}}{(1-L)^{\frac{1}{2}}}.$$

Hence, using Parseval's identity and (3.7.8) for a fixed r, and then averaging over $1-\frac{2}{N+2} \leq r \leq 1-\frac{1}{N+2}$, we get

$$\sum_{n=N}^{2N}|B_n(z)|^2 \ll (N+1)^{-\frac{1}{2}}\int_{1-\frac{2}{N+2}}^{1-\frac{1}{N+2}}\int_0^{2\pi}\left|B_0\left(T_z\left(i\frac{1+re^{i\phi}}{1-re^{i\phi}}\right)\right)\right|^2\frac{rd\phi dr}{(1-r^2)^2},$$

hence, using a substitution,

$$\sum_{n=N}^{2N}|B_n(z)|^2 \ll (N+1)^{-\frac{1}{2}}\int_{w\in H,\left|\frac{w-i}{w+i}\right|\leq 1-\frac{1}{N+2}}|B_0\left(T_z w\right)|^2\,d\mu_w$$

with implied absolute constant. For simplicity, instead of $|B_0|^2$, we take an $SL(2,\mathbf{Z})$-invariant majorant, write

$$F\left(Z\right) = \sum_{j=0}^{5}|B_0\left(\gamma_j Z\right)|^2.$$

Since $\frac{\left|\frac{w-i}{w+i}\right|^2}{1-\left|\frac{w-i}{w+i}\right|^2} = \frac{|w-i|^2}{4\mathrm{Im}w}$, hence

$$\sum_{n=N}^{2N}|B_n(z)|^2 \ll (N+1)^{-\frac{1}{2}}\int_{D_1}K\left(z,w;N+2\right)F(w)d\mu_w,$$

where we write

$$K\left(z,w;x\right) = \sum_{\gamma\in SL(2,\mathbf{Z}),\,\frac{|\gamma z-w|^2}{4\mathrm{Im}\gamma z\mathrm{Im}w}\leq x} 1.$$

Since we have $F(w) \ll (\mathrm{Im}w)^{\frac{1}{2}}$ for $w \in D_1$ (which follows from the $n = 0$ case of (i)), Lemma 3.11 below proves the present lemma.

LEMMA 3.11. *Let $z_1, z_2 \in D_1$, write $y_1 = \mathrm{Im}z_1$, $y_2 = \mathrm{Im}z_2$, and let $x \geq 2$. Then for every $\epsilon > 0$ we have*

$$K\left(z_1,z_2;x\right) \ll_\epsilon x^{1+\epsilon} + \left(xy_1y_2\right)^{\frac{1}{2}}, \tag{3.7.9}$$

and if E is a large enough absolute constant and $y_2 \geq Exy_1$, then

$$K(z_1, z_2; x) = 0. \tag{3.7.10}$$

Proof. It is easy to see by (1.2), (1.3) of [I1], and by the triangle inequality (for the hyperbolic distance function on H) that if $\gamma \in \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z})$ and

$$\frac{|\gamma z_1 - z_2|^2}{4\mathrm{Im}(\gamma z_1)\mathrm{Im} z_2} \leq x,$$

then

$$\frac{|\gamma(iy_1) - iy_2|^2}{4\mathrm{Im}(\gamma(iy_1))\, y_2} \leq Cx$$

with some absolute constant $C > 0$. The left-hand side here is

$$\frac{(ay_1 - dy_2)^2 + (b + cy_1y_2)^2}{4y_1y_2},$$

hence we need

$$a^2y_1^2 + d^2y_2^2 + b^2 + c^2y_1^2y_2^2 \leq (4Cx + 2)\, y_1y_2. \tag{3.7.11}$$

This implies

$$|b| \leq ((4Cx + 2)\, y_1y_2)^{\frac{1}{2}}, \qquad |c| \leq \left(\frac{4Cx + 2}{y_1y_2}\right)^{\frac{1}{2}}. \tag{3.7.12}$$

If $y_1y_2 \leq 4Cx + 2$, then the number of possible (b,c) pairs is $\ll x$. If b and c are given and $bc \neq -1$, then $ad = 1 + bc$ is also given, and $0 \neq |ad| \ll x$, hence the number of possible (a,d) pairs is $\ll_\epsilon x^\epsilon$. If $bc = -1$, then the number of possible (b,c) pairs is $\ll 1$, and $a = 0$ or $d = 0$, and we also see by (3.7.11) and the relations $y_1y_2 \leq 4Cx + 2$ and $y_1, y_2 \gg 1$ that $a^2 + d^2 \ll x^2$. This proves (3.7.9) for the case $y_1y_2 \leq 4Cx + 2$.

If $y_1y_2 > 4Cx + 2$, then (3.7.12) implies $c = 0$, hence the number of possible (a,d) pairs is $\ll 1$, and the number of possible numbers b is $\ll (xy_1y_2)^{\frac{1}{2}}$. The inequality (3.7.9) is proved. Since $d^2 + c^2y_1^2 \gg 1$, so (3.7.11) implies (3.7.10).

3.7.2. An upper bound for an integral of Maass forms.

LEMMA 3.12. *Let $C > 1/2$, and let u be a cusp form of weight 0 for $SL(2,\mathbf{Z})$ with $\Delta_0 u= s(s-1)u$, where $s = \frac{1}{2} + it$ and $t > 0$. Then for integers $n \geq 0$ we have, by the notation*

$$u_{(n)}(z) = \left(\prod_{l=0}^{n-1} \frac{1}{s+l} \right) (K_{n-1}K_{n-2}\dots K_1 K_0 u)(z),$$

the inequality

$$\int_{D_4} \left(\sum_{l=0}^{\infty} (1+l)^{-C} |B_l(z)|^2 \right) \left| u_{(n)}(4z) \right|^2 d\mu_z \ll_{u,C} \log^2(n+2).$$

Proof. We use the substitution $z \to -\frac{1}{4z}$, which normalizes $\Gamma_0(4)$. By (3.7.5) we see that

$$\left| B_l \left(-\frac{1}{4z} \right) \right|^2 = |B_l(z)|^2,$$

and $u_{(n)}\left(4\left(-\frac{1}{4z}\right)\right) = u_{(n)}(z)$ by the $SL(2,\mathbf{Z})$-invariance of u. For $z \in D_1$ we have

$$\sum_{j=0}^{5} \sum_{l=0}^{\infty} (1+l)^{-C} |B_l(\gamma_j z)|^2 \ll_C (\mathrm{Im} z)^{1/2}$$

by Lemma 3.10, so it is enough to prove that

$$\int_{D_1} (\mathrm{Im} z)^{1/2} \left| u_{(n)}(z) \right|^2 d\mu_z \ll_u \log^2(n+2).$$

We will give an upper bound by extending the integration to $\mathrm{Im} z \geq \sqrt{3}/2$, $|\mathrm{Re} z| \leq 1/2$, and using Parseval's formula. Consider the Fourier expansion

$$u_{(n)}(z) = \sum_{m \neq 0} b_{u,n}(m) W_{n\mathrm{sgn}(m),it}(4\pi |m| y) e(mx).$$

It is well-known (see [Du], formulas (2.4) and (2.6), and take into account our formula (3.3.11)) that for $m > 0$ we have

$$b_{u,n}(m) = (-1)^n \left(\prod_{l=0}^{n-1} \frac{1}{s+l} \right) b_{u,0}(m),$$

and for $m < 0$ we have

$$b_{u,n}(m) = (-1)^n \left(\prod_{l=1}^{n} (s-l) \right) b_{u,0}(m).$$

By [G-R], p. 814, 7.611.4 and p. 893, 8.362.1 we see for real $t \neq 0$ and any integer m (note that $W_{m,it}(y)$ is real) that

$$\int_0^\infty \left| W_{m,it}(y)\Gamma\left(\frac{1}{2}-m+it\right)\right|^2 \frac{dy}{y} = \frac{\pi}{\sin 2\pi t}\sum_{k=0}^{\infty}\left(\frac{1}{\frac{1}{2}-it-m+k}-\frac{1}{\frac{1}{2}+it-m+k}\right),$$

which is $\ll_t \log(|m|+2)$. By these relations, Lemma 3.13 below and formulas (8.17) and (8.5) of [I1] we easily get the lemma.

LEMMA 3.13. *There are positive absolute constants* C_1, C_2, C_3 *such that if* $n \in \mathbf{Z}$, $t \geq 0$, *then*

$$\left| W_{n,it}(y)\Gamma\left(\frac{1}{2}-n+it\right)\right| \leq C_1 e^{-C_2 y} \text{ for } y \geq C_3 \max(1+t, n). \tag{3.7.13}$$

Proof. By [G-R], p.1015, formula 9.223 we have for $y > 0$ and $t \geq 0$ that

$$W_{n,it}(y)\Gamma\left(\frac{1}{2}-n+it\right) - \frac{e^{-\frac{y}{2}}}{2\pi i}\int_{(1/4)} \frac{\Gamma(u-n)\Gamma\left(\frac{1}{2}-u-it\right)\Gamma\left(\frac{1}{2}-u+it\right)}{\Gamma\left(\frac{1}{2}-n-it\right)} y^u du$$

is $\ll$ than

$$e^{-\frac{y}{2}}\sum_{j=1}^{n}\left|\mathrm{Res}_{u=j}\frac{\Gamma(u-n)\Gamma\left(\frac{1}{2}-u-it\right)\Gamma\left(\frac{1}{2}-u+it\right)}{\Gamma\left(\frac{1}{2}-n-it\right)} y^u\right|$$

(this is of course 0 for $n \leq 0$). We use the well-known statement that if σ is a real number which is not a nonpositive integer, then $\max_{\tau\in\mathbf{R}}|\Gamma(\sigma+i\tau)| = |\Gamma(\sigma)|$. We apply this statement to estimate $\Gamma(u-n)$ if $\mathrm{Re}u = 1/4$, and $\Gamma\left(\frac{1}{2}-u-it\right)$ if $u=j$. Then Stirling's formula easily implies (3.7.13), the lemma is proved.

3.7.3. An orthogonality relation.

LEMMA 3.14. *If* u *is a cusp form of weight* 0 *for* $SL(2,\mathbf{Z})$, *then*

$$\int_{D_4} |B_0(z)|^2 u(4z) d\mu_z = 0. \tag{3.7.14}$$

Proof. By the substitution $z \to -\frac{1}{4z}$ and by (3.7.5) with $n=0$ we get (as in the proof of Lemma 3.12) that the left-hand side of (3.7.14) equals

$$\int_{D_4} |B_0(z)|^2 u(z) d\mu_z = \int_{D_1}\left(\sum_{j=0}^{5}|B_0(\gamma_j z)|^2\right) u(z) d\mu_z. \tag{3.7.15}$$

We now determine the Fourier expansion of $F(z) := \sum_{j=0}^{5} |B_0(\gamma_j z)|^2$. We use that

$$\begin{pmatrix} 0 & -1 \\ 1 & j \end{pmatrix} = \begin{pmatrix} 0 & -1/2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1/2 & j/2 \\ 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -1/2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1/2 \\ 0 & -1 \end{pmatrix}.$$

It is also not hard to see for any integer n and $y > 0$ that

$$\int_0^1 \left(\sum_{j=0}^{3} \left| B_0\left(\frac{x+iy+j}{4} \right) \right|^2 \right) e(-nx)dx = 4 \int_0^1 \left| B_0\left(x + \frac{iy}{4} \right) \right|^2 e(-4nx)dx,$$

hence, using (3.7.5) and (3.7.6), we get that $\int_0^1 F(x+iy)\, e(-nx)dx$ equals

$$y^{\frac{1}{2}} \sum_{\substack{m_1, m_2 \in \frac{1}{2}\mathrm{Z} \\ m_1^2 - m_2^2 = n}} e^{-2\pi\left(m_1^2+m_2^2\right)y} + 2y^{\frac{1}{2}} \sum_{\substack{m_1, m_2 \in \mathrm{Z} \\ m_1^2 - m_2^2 = 4n}} e^{-2\pi\left(m_1^2+m_2^2\right)\frac{y}{4}}.$$

One easily checks that the two sums above have the same value, and, writing $a = m_1 - m_2$, $d = m_1 + m_2$ in the first sum, we finally get for any $y > 0$ and integer n that

$$\int_0^1 F(x+iy)\, e(-nx)dx = 3y^{\frac{1}{2}} \sum_{\substack{a, d \in \mathrm{Z} \\ ad = n}} e^{-\pi\left(a^2+d^2\right)y}. \tag{3.7.16}$$

We now show that a certain incomplete Eisenstein series has the same Fourier coefficients. Indeed, for $z \in H$ let

$$G(z) := E(z, \psi) = \sum_{\gamma \in \Gamma_\infty \backslash SL_2(\mathrm{Z})} \psi\left(\mathrm{Im}(\gamma z)\right),$$

where

$$\psi(y) = \sum_{m=1}^{\infty} e^{-\pi \frac{m^2}{y}}.$$

Then by [I1], (3.17) we have for $y > 0$ and $n \neq 0$ that

$$\int_0^1 G(x+iy)\, e(-nx)dx = \sum_{c=1}^{\infty} S(0,n,c) \int_{-\infty}^{\infty} \psi\left(\frac{yc^{-2}}{t^2+y^2} \right) e(-nt)dt,$$

where $S(0,n,c)$ is given by [I1], (2.26). We can compute easily that

$$\int_{-\infty}^{\infty} \psi\left(\frac{yc^{-2}}{t^2+y^2} \right) e(-nt)dt = d\sqrt{y} \sum_{m=1}^{\infty} \frac{1}{cm} e^{-\pi y\left(m^2c^2 + \frac{n^2}{m^2c^2}\right)}$$

with a nonzero absolute constant d. Since for any positive integer a we have

$$\sum_{c|a} S(0,n,c) = \begin{cases} a, \text{ if } a\,|n, \\ 0 \text{ otherwise,} \end{cases}$$

so we get finally for any $y > 0$ and nonzero integer n that

$$\int_0^1 G\left(x+iy\right) e(-nx)dx = d\sqrt{y}\sum_{a|n} e^{-\pi y\left(a^2+\frac{n^2}{a^2}\right)}.$$

This and (3.7.16) imply that there is a nonzero absolute constant D such that $F(z)-DG(z)$ depends only on Imz. Since $F(z) - DG(z)$ is $SL(2,\mathrm{Z})$-invariant, so it is a constant. This implies that (3.7.15) is 0 (since cusp forms are orthogonal to incomplete Eisenstein series and constants), the lemma is proved.

4. Appendix: some properties of Wilson functions

4.1. Statement of the results

In this chapter we prove some results applied in Chapter 3. These results are independent of automophic forms, we deal only with special functions here.

We first give the necessary notations. Let t_1 and t_2 be two nonzero real numbers and let $s_1 = \frac{1}{2} + it_1, s_2 = \frac{1}{2} + it_2$. As before, we write $(w)_n = \Gamma(w+n)/\Gamma(w)$, $\Gamma(X \pm Y) = \Gamma(X+Y)\Gamma(X-Y)$, and

$$\Gamma(X \pm Y \pm Z) = \Gamma(X+Y+Z)\Gamma(X+Y-Z)\Gamma(X-Y+Z)\Gamma(X-Y-Z).$$

We will use the notations (3.3.1), (3.3.2), (3.3.4). We use the Wilson function

$$\phi_\lambda(x) = \phi_\lambda(x; a, b, c, d)$$

as in Chapter 3, i.e. with the parameters given in (3.3.4).

See [S] for the definition of the generalized hypergeometric functions $F_{A,B}$. Sometimes we will write $F(\alpha, \beta, \gamma; z)$ in place of $F_{2,1}(\alpha, \beta, \gamma; z)$. Introduce the notation

$$F_N(x) = \frac{1}{-N\left(N-\frac{1}{2}\right)} \frac{F_{5,4}\left(\begin{matrix} -N, N-\frac{1}{2}, \frac{1}{4}+ix, \frac{1}{4}-ix, 1 \\ \frac{1}{2}+it_1, \frac{1}{2}-it_1, \frac{1}{2}+it_2, \frac{1}{2}-it_2 \end{matrix}; 1\right) - 1}{\Gamma\left(\frac{3}{4} \pm ix\right)}$$

for integers $N \geq 1$. We make a similar convention as in Chapter 3.

CONVENTION. In what follows, t_1 and t_2 will be fixed. So every variable and every constant (including the constants implied in the $\ll$ and O symbols) may depend on t_1 and t_2, even if we do not denote this dependence.

We can now state our three theorems. Theorem 4.1 is a more precise form of Theorem 1.3 (i.e. it is enough to prove Theorem 4.1 for the proof of Theorem 1.3). Theorem 4.2 shows that the result of a complicated summation is a Wilson function. Theorem 4.3 gives estimates for a certain series of Wilson functions.

THEOREM 4.1 ([Bi8]). *Assume that K is a positive number, and $f(x)$ is an even holomorphic function for $|\mathrm{Im}x| < K$ satisfying*

$$\int_{-\infty}^{\infty} f(\tau) H(\tau) \frac{1}{\Gamma\left(\frac{3}{4} \pm i\tau\right)} d\tau = 0 \tag{4.1.1}$$

and that

$$\left| f(x) e^{-2\pi|x|} (1+|x|)^K \right|$$

is bounded on the domain $|\mathrm{Im}x| < K$. If k is a positive integer and K is large enough in terms of k, then

$$\int_0^{\infty} f(x) F_N(x) H(x) dx = \sum_{j=4}^{k} \frac{c_{f,j}}{N^j} + O\left(\frac{1}{N^{k+1}}\right) \tag{4.1.2}$$

with some constants $c_{f,j}$ as $N \to \infty$. If K is large enough (depending only on t_1 ad t_2), we have

$$f(x) = \frac{C}{2\pi} \sum_{N=1}^{\infty} \frac{\Gamma\left(\frac{1}{2}+2N\right)}{(1-N)_{N-1}\left(N+\frac{1}{2}\right)_{N-1}} \left(\int_0^{\infty} f(\xi) F_N(\xi) H(\xi) d\xi\right) \phi_{i\left(\frac{1}{4}-N\right)}(x) \tag{4.1.3}$$

for every $|\mathrm{Im}x| < \frac{3}{4}$, and the sum on the right-hand side of (4.1.3) is absolutely convergent for every such x.

THEOREM 4.2 ([Bi8]). *Let n be a fixed positive integer, and write*

$$m_{l_1,l_2} = \frac{1}{4} + l_1 + l_2 - n.$$

For nonnegative integers l_1, l_2 and complex S define

$$\Sigma_{l_1,l_2}(S) = \sum_{L_1=0}^{l_1} \sum_{L_2=0}^{l_2} (-1)^{L_1+L_2} \binom{l_1}{L_1} \binom{l_2}{L_2} G(S, l_1, l_2, L_1, L_2),$$

where $G(S, l_1, l_2, L_1, L_2)$ denotes

$$\frac{\overline{\Gamma(s_2-n+l_2+L_1)}}{\overline{\Gamma(s_2+n-l_2-L_1)}} \frac{\Gamma(s_1-n+l_1+L_2)}{\Gamma(s_1+n-l_1-L_2)} \frac{\overline{(S-m_{l_1,l_2})_{l_1-L_1}}}{\overline{\Gamma(S+m_{l_1,l_2}-l_1+L_1)}} \frac{(S-m_{l_1,l_2})_{l_2-L_2}}{\Gamma(S+m_{l_1,l_2}-l_2+L_2)}.$$

Note that since n is given. we have not denoted the dependence on n in m_{l_1,l_2}, $\Sigma_{l_1,l_2}(S)$ and $G(S, l_1, l_2, L_1, L_2)$.

Then, if $S = \frac{1}{2} + i\tau$*, where* τ *is either real or purely imaginary, we have that the sum*

$$\sum_{l_1,l_2=0}^{\infty} \frac{\overline{\Gamma\left(S+m_{l_1,l_2}\right)}}{\Gamma\left(1-S+m_{l_1,l_2}\right)} \frac{\Sigma_{l_1,l_2}(S)}{\Gamma\left(\frac{1}{2}+l_1\right)\Gamma\left(1+l_1\right)\Gamma\left(\frac{1}{2}+l_2\right)\Gamma\left(1+l_2\right)}$$

is absolutely convergent, and equals

$$\frac{\left|\Gamma\left(s_1\right)\right|^2\left|\Gamma\left(s_2\right)\right|^2\Gamma\left(-\frac{1}{2}+2n\right)}{\left|\Gamma\left(s_1+n\right)\right|^2\left|\Gamma\left(s_2+n\right)\right|^2}\phi_\tau\left(i\left(\frac{1}{4}-n\right)\right).$$

THEOREM 4.3 ([Bi8]). *Write*

$$M_\lambda(A) = \frac{1}{\Gamma\left(1-A\right)}\sum_{k=1}^{\infty}(-1)^k\frac{\Gamma\left(k-A\right)}{\Gamma\left(k\right)}\phi_\lambda\left(i\left(\frac{1}{4}-k\right)\right).$$

(i) $M_\lambda(A)$ *is absolutely convergent if* $\mathrm{Re}A > 1 + 2\left|\mathrm{Im}\lambda\right|$*, or if* $\lambda = i\left(\frac{1}{4}-k\right)$ *with a positive integer* k *and* $\mathrm{Re}A \geq 2$*.*
(ii) If a compact set L *on the complex plane is given such that* $2 \leq \mathrm{Re}A$ *for every* $A \in L$*, then*

$$M_{i\left(\frac{1}{4}-k\right)}(A) = O_L\left(k^{-3/2}\right)$$

for any $A \in L$ *and positive integer* k*. The left-hand side here is regular in* A *on every open subset of* L *for every fixed positive integer* k*.*
(iii) If a compact set L *on the complex plane and an integer* $K \geq 2$ *are given such that* $2 \leq \mathrm{Re}A$ *for every* $A \in L$*, then we can find an integer* $t > 0$*, complex numbers* $A_1, A_2, \ldots, A_t$ *with* $\mathrm{Re}A_i > K$ $(i = 1, 2, \ldots, t)$ *and polynomials* $Q_1, Q_2, \ldots, Q_t$ *such that*

$$M_\lambda(A) - \sum_{i=1}^{t} 2^{A-A_i}Q_i\left(A\right)M_\lambda(A_i) = O_{L,K}\left(e^{2\pi|\lambda|}\left(1+|\lambda|\right)^{-K}\right)$$

for any $A \in L$ *and real* λ*. The left-hand side here is regular in* A *on every open subset of* L *for every fixed real* λ*.*

REMARK. In Chapter 3 we applied the following results on Wilson functions: Theorem 1.3, Theorem 4.2, Theorem 4.3, and one more result, namely Corollary 4.1 below. Hence (since Theorem 1.3 follows from Theorem 4.1), to complete the proof of Theorem 1.2 it will be enough to prove the three theorems stated above and Corollary 4.1.

Theorems 4.1., 4.2 and 4.3 are proved in Sections 4.2, 4.3 and 4.4, respectively, using Lemmas 4.5-4.9 and Corollaries 4.1 and 4.2 proved only in Section 4.5.

4.2. Expansion in Wilson functions

4.2.1. A biorthogonal system. We will consider now the inner product

$$\int_0^\infty f_1(x)\overline{f_2(x)}H(x)dx$$

for two even functions f_1,f_2 on $\mathbf{R}$. We will show that the system of functions F_N $(N\geq 1)$ is biorthogonal to the system $\phi_{i\left(\frac{1}{4}-N\right)}$ $(N\geq 1)$ with respect to this inner product. Then we will consider for a given function f satisfying (4.1.1) and some additional properties the asymptotic behaviour of the sequence of inner products of f with F_N. The reason of imposing the condition (4.1.1) is that it is true for every $f=\phi_{i\left(\frac{1}{4}-N\right)}$, as we will show shortly.

Recall the symmetry property $\phi_\lambda(x)=\phi_x(\lambda)$ from (3.4) of [G1] (we use here that our parameters a,b,c,d are self-dual, see (2.6) of [G1]). By the formula in the proof of Theorem 6.5 of [G1] with $n=0$ and $g=\frac{1}{4}+k$ we have (in general, $(X\pm Y)_n$ means $(X+Y)_n(X-Y)_n$)

$$\frac{C}{2\pi}\int_0^\infty \phi_\lambda(x)\,H(x)\frac{\left(\frac{1}{4}\pm ix\right)_k}{\Gamma\left(\frac{3}{4}\pm ix\right)}dx=\frac{\left(\frac{1}{2}\pm it_1\right)_k\left(\frac{1}{2}\pm it_2\right)_k}{\Gamma\left(\frac{3}{4}+k\pm i\lambda\right)} \tag{4.2.1}$$

for any integer $k\geq 0$ and any λ (see Prop. 4.4 of [G1] to see the absolute convergence). So for any integer $N\geq 1$ we have that

$$\frac{C}{2\pi}\int_0^\infty \phi_\lambda(x)\,H(x)\frac{F_{5,4}\left(\begin{matrix}-N,N-\frac{1}{2},\frac{1}{4}+ix,\frac{1}{4}-ix,1\\ \frac{1}{2}+it_1,\frac{1}{2}-it_1,\frac{1}{2}+it_2,\frac{1}{2}-it_2\end{matrix};1\right)-1}{\Gamma\left(\frac{3}{4}\pm ix\right)}dx$$

equals

$$\frac{-N\left(N-\frac{1}{2}\right)}{\Gamma\left(\frac{7}{4}\pm i\lambda\right)}F_{3,2}\left(\begin{matrix}1-N,N+\frac{1}{2},1\\ \frac{7}{4}+i\lambda,\frac{7}{4}-i\lambda\end{matrix};1\right),$$

and by [S], (2.3.1.4) this equals

$$\frac{-N\left(N-\frac{1}{2}\right)}{\left(N-\frac{1}{4}+i\lambda\right)\left(N-\frac{1}{4}-i\lambda\right)\Gamma\left(\frac{3}{4}\pm i\lambda\right)}=-N\left(N-\frac{1}{2}\right)\frac{\left(\frac{3}{4}\pm i\lambda\right)_{N-1}}{\Gamma\left(\frac{3}{4}\pm i\lambda+N\right)}.$$

This implies that we have for every pair of integers $k, N \geq 1$ that

$$\frac{C}{2\pi}\int_0^\infty \phi_{i\left(\frac{1}{4}-k\right)}(x)\, F_N(x)\, H(x)dx = 0 \text{ for } k \neq N, \tag{4.2.2}$$

$$\frac{C}{2\pi}\int_0^\infty \phi_{i\left(\frac{1}{4}-k\right)}(x)\, F_N(x)\, H(x)dx = \frac{(1-N)_{N-1}\left(N+\frac{1}{2}\right)_{N-1}}{\Gamma\left(\frac{1}{2}+2N\right)} \text{ for } k = N, \tag{4.2.3}$$

i.e. these two systems ($\phi_{i\left(\frac{1}{4}-N\right)}(x)$ and $F_N(x)$) are biorthogonal in this sense.

Note that (4.2.1) with $k = 0$ and $\lambda = i\left(\frac{1}{4}-N\right)$ implies indeed that (4.2.1) holds for any $N \geq 1$ with $f = \phi_{i\left(\frac{1}{4}-N\right)}$.

In the proof of the next lemma we will use Lemma 4.6 which will be proved later. For the function f we impose the condition of Lemma 4.6. Some of these conditions will be released later.

LEMMA 4.1. *Let f be a function satisfying (4.2.1) and the conditions of Lemma 4.6. If k is a positive integer and K in Lemma 4.6 is large enough in terms of k, then we have*

$$\int_0^\infty f(x)\, F_N(x)\, H(x)dx = \sum_{j=4}^{k}\frac{c_{f,j}}{N^j} + O\left(\frac{1}{N^{k+1}}\right)$$

with some constants $c_{f,j}$ as $N \to \infty$.

Proof. By shifting the path of integration to the right, we see that if $-\frac{1}{4} < \alpha < 0$, τ is real and $m \geq 0$ is an integer, then

$$\frac{1}{2\pi i}\int_{(\alpha)} \frac{\Gamma(-A \pm i\tau)\,\Gamma\left(\frac{1}{4}+A\pm it_1\right)\Gamma\left(\frac{1}{4}+A+m\right)}{\Gamma\left(\frac{1}{4}\pm A\right)\Gamma\left(\frac{3}{4}\pm A\right)\Gamma\left(\frac{3}{4}+A+m\right)}dA \tag{4.2.4}$$

equals the sum of

$$\frac{\Gamma(-2i\tau)\,\Gamma\left(\frac{1}{4}+i\tau\pm it_1\right)\Gamma\left(\frac{1}{4}+i\tau+m\right)}{\Gamma\left(\frac{1}{4}\pm i\tau\right)\Gamma\left(\frac{3}{4}\pm i\tau\right)\Gamma\left(\frac{3}{4}+i\tau+m\right)}F_{3,2}^{+}$$

and

$$\frac{\Gamma(2i\tau)\,\Gamma\left(\frac{1}{4}-i\tau\pm it_1\right)\Gamma\left(\frac{1}{4}-i\tau+m\right)}{\Gamma\left(\frac{1}{4}\pm i\tau\right)\Gamma\left(\frac{3}{4}\pm i\tau\right)\Gamma\left(\frac{3}{4}-i\tau+m\right)}F_{3,2}^{-},$$

where

$$F_{3,2}^{+} = F_{3,2}\left(\begin{matrix}\frac{1}{4}+i\tau+it_1, \frac{1}{4}+i\tau-it_1, \frac{1}{4}+i\tau+m \\ 1+2i\tau, \frac{3}{4}+i\tau+m\end{matrix};1\right),$$

and $F^-_{3,2}$ is obtained from $F^+_{3,2}$ by writing $-\tau$ in place of τ. Using [S], (2.4.4.4) we then get that (4.2.4) equals

$$\frac{\Gamma\left(\frac{1}{4}\pm i\tau\pm it_1\right)\Gamma\left(\frac{1}{4}\pm i\tau+m\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\pm i\tau\right)\Gamma\left(\frac{3}{4}\pm i\tau\right)\Gamma\left(\frac{1}{2}\pm it_1+m\right)}. \tag{4.2.5}$$

This implies that if $-\frac{1}{4}<\alpha<0$ and τ is real, then for every integer $N\geq 1$ we have that

$$\frac{1}{2\pi i}\int_{(\alpha)}\frac{\Gamma\left(-A\pm i\tau\right)\Gamma\left(\frac{1}{4}+A\pm it_1\right)}{\Gamma\left(\frac{1}{4}-A\right)\Gamma\left(\frac{3}{4}-A\right)\Gamma^2\left(\frac{3}{4}+A\right)}\left(F_{4,3}\left(\begin{matrix}-N,N-\frac{1}{2},\frac{1}{4}+A,1\\ \frac{1}{2}+it_2,\frac{1}{2}-it_2,\frac{3}{4}+A\end{matrix};1\right)-1\right)dA \tag{4.2.6}$$

equals

$$\frac{\Gamma\left(\frac{1}{4}\pm i\tau\pm it_1\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{4}\pm i\tau\right)\Gamma\left(\frac{1}{2}\pm it_1\right)}\left(F_{5,4}\left(\begin{matrix}-N,N-\frac{1}{2},\frac{1}{4}+i\tau,\frac{1}{4}-i\tau,1\\ \frac{1}{2}+it_1,\frac{1}{2}-it_1,\frac{1}{2}+it_2,\frac{1}{2}-it_2\end{matrix};1\right)-1\right). \tag{4.2.7}$$

We claim that if $N\geq 1$ is an integer and $\mathrm{Re}A>-\frac{5}{4}$, then

$$F_{4,3}\left(\begin{matrix}1-N,N+\frac{1}{2},\frac{5}{4}+A,1\\ \frac{3}{2}+it_2,\frac{3}{2}-it_2,\frac{7}{4}+A\end{matrix};1\right)=\frac{\Gamma\left(\frac{7}{4}+A\right)\Gamma\left(\frac{3}{2}-it_2\right)}{\Gamma\left(\frac{5}{4}+A\right)\Gamma\left(1-it_2\right)}I \tag{4.2.8}$$

with

$$I=\int_0^1 z^{-\frac{1}{2}}\left(1-z\right)^{-it_2}F_{2,1}\left(\begin{matrix}1-N,N+\frac{1}{2}\\ \frac{3}{2}+it_2\end{matrix};z\right)F_{2,1}\left(\begin{matrix}\frac{3}{4}+A,\frac{1}{2}\\ 1-it_2\end{matrix};\frac{z-1}{z}\right)dz. \tag{4.2.9}$$

Indeed, this follows by writing the first $F_{2,1}$ function as a polynomial in z, then applying (6.6) of [G1] (note that there is a misprint there, $\frac{y}{1-y}$ should be replaced by $\frac{-y}{1-y}$) after the substitution $y=1-z$. This proves the equality of (4.2.8) and (4.2.9). We remark also that (by [G-R], p. 998, 9.131.1) the second $F_{2,1}$ function in (4.2.9) equals

$$z^{\frac{1}{2}}F_{2,1}\left(\begin{matrix}\frac{1}{4}-A-it_2,\frac{1}{2}\\ 1-it_2\end{matrix};1-z\right). \tag{4.2.10}$$

It follows then that if f is a function satisfying the conditions of Lemma 4.6, then for every integer $N\geq 1$ we have that

$$\int_{-\infty}^{\infty}f\left(\tau\right)F_N\left(\tau\right)\frac{\Gamma\left(\frac{1}{4}\pm it_1\pm i\tau\right)\Gamma\left(\frac{1}{4}\pm it_2\pm i\tau\right)\Gamma\left(\frac{1}{4}\pm i\tau\right)\Gamma\left(\frac{3}{4}\pm i\tau\right)}{\Gamma\left(\pm 2i\tau\right)}d\tau \tag{4.2.11}$$

equals

$$c\int_0^1\left(1-z\right)^{-it_2}F_{2,1}\left(\begin{matrix}1-N,N+\frac{1}{2}\\ \frac{3}{2}+it_2\end{matrix};z\right)K(z)dz \tag{4.2.12}$$

with a constant $c \neq 0$ (c may depend on t_1 and t_2, since these are fixed numbers), where (for $G(A)$ see Lemma 4.6)

$$K(z) = \frac{1}{2\pi i} \int_{(\alpha)} \frac{\Gamma\left(\frac{1}{4} + A \pm it_1\right) G(A)}{\Gamma\left(\frac{1}{4} \pm A\right) \Gamma\left(\frac{3}{4} \pm A\right)} F_{2,1}\left(\begin{matrix} \frac{1}{4} - A - it_2, \frac{1}{2} \\ 1 - it_2 \end{matrix}; 1 - z \right) dA \tag{4.2.13}$$

with $-\frac{1}{4} < \alpha < 0$. Indeed, by Lemma 4.6 and [G-R], p 995, 9.111 we see that the double integral in A, z is absolutely convergent here, we first compute the integral in z by (4.2.8), (4.2.9), (4.2.10), then we insert the definition (4.5.12) of $G(A)$, the resulting integral in A and τ is again absolutely convergent, and we compute the integral in A by the equality of (4.2.6) and (4.2.7), we obtain in this way the equality of (4.2.11) and (4.2.12). We also see by Lemma 4.6 and [G-R], p 995, 9.111 that if k is a positive integer and the number K in Lemma 4.6 is large enough in terms of k, then the function $\left(\frac{d}{dz}\right)^j K$ is bounded on the closed interval $[\frac{1}{2}, 1]$ for every $0 \leq j \leq k$. For the estimation of K and its derivatives on $[0, \frac{1}{2}]$ we use that the $F_{2,1}$ function in (4.2.13) equals, by [G-R], p. 998, 9.131.2, the sum of

$$\frac{\Gamma(1 - it_2)\Gamma\left(\frac{1}{4} + A\right)}{\Gamma\left(\frac{1}{2} - it_2\right)\Gamma\left(\frac{3}{4} + A\right)} F_{2,1}\left(\begin{matrix} \frac{1}{4} - A - it_2, \frac{1}{2} \\ \frac{3}{4} - A \end{matrix}; z \right) \tag{4.2.14}$$

and

$$z^{\frac{1}{4} + A} \frac{\Gamma(1 - it_2)\Gamma\left(-\frac{1}{4} - A\right)}{\Gamma\left(\frac{1}{4} - A - it_2\right)\Gamma\left(\frac{1}{2}\right)} F_{2,1}\left(\begin{matrix} \frac{1}{2} - it_2, \frac{3}{4} + A \\ \frac{5}{4} + A \end{matrix}; z \right). \tag{4.2.15}$$

We estimate here the $F_{2,1}$ functions by their power series expansions. Observe first that the zeroth term of the $F_{2,1}$ function in (4.2.14) gives 0 in (4.2.13), since (4.1.1) holds. This follows from the definition of $G(A)$ in (4.5.12) and from the equality of (4.2.4) and (4.2.5) for $m = 0$, see also (3.3.1). Now, using Lemma 4.6 and shifting the line of A-integration to the right in (4.2.13) when we substitute (4.2.15) we see that, if the number K is large enough in terms of k, then on the closed interval $[0, \frac{1}{2}]$ we have

$$K(z) = \sum_{j=1}^{k} a_j z^j + z^{\frac{1}{2} + it_2} L(z) \tag{4.2.16}$$

with some constants a_j and a function L such that $\left(\frac{d}{dz}\right)^j L$ is bounded on the closed interval $[0, \frac{1}{2}]$ for every $0 \leq j \leq k$. By our investigation above on the behavior of K on $[\frac{1}{2}, 1]$, we finally see that in fact (4.2.16) is valid and $\left(\frac{d}{dz}\right)^j L$ is bounded on the whole interval $[0,1]$ for every $0 \le j \le k$.

Observe that by [G-R], p 990, 8.962.1 we have ($P_n^{(\alpha,\beta)}$ is Jacobi's polynomial)

$$F_{2,1}\left(\begin{matrix}1-N, N+\frac{1}{2}\\ \frac{3}{2}+it_2\end{matrix}; z\right) = \frac{(-1)^{N-1}(N-1)!}{\left(\frac{3}{2}+it_2\right)_{N-1}} P_{N-1}^{\left(-it_2, \frac{1}{2}+it_2\right)}(2z-1),$$

so

$$\int_0^1 (1-z)^{-it_2} F_{2,1}\left(\begin{matrix}1-N, N+\frac{1}{2}\\ \frac{3}{2}+it_2\end{matrix}; z\right) z^{\frac{1}{2}+it_2} L(z)dz \tag{4.2.17}$$

equals $d\frac{(-1)^{N-1}(N-1)!}{\left(\frac{3}{2}+it_2\right)_{N-1}}$ times

$$\int_{-1}^1 (1-x)^{-it_2} (1+x)^{\frac{1}{2}+it_2} P_{N-1}^{\left(-it_2, \frac{1}{2}+it_2\right)}(x) L\left(\frac{1+x}{2}\right) dx \tag{4.2.18}$$

with a nonzero constant d. By [G-R], p 989, 8.960.1 we see that

$$(1-x)^{-it_2} (1+x)^{\frac{1}{2}+it_2} P_{N-1}^{\left(-it_2, \frac{1}{2}+it_2\right)}(x)$$

equals

$$\frac{(-1)^k (N-k-1)!}{2^k (N-1)!} \left(\frac{d}{dx}\right)^k \left((1-x)^{-it_2+k} (1+x)^{\frac{1}{2}+it_2+k} P_{N-1-k}^{\left(-it_2+k, \frac{1}{2}+it_2+k\right)}(x)\right),$$

hence by repeated partial integration we see by the property of L that (4.2.18) is $\ll$ than

$$N^{-k} \int_{-1}^1 \left|(1-x)^{-it_2+k} (1+x)^{\frac{1}{2}+it_2+k} P_{N-1-k}^{\left(-it_2+k, \frac{1}{2}+it_2+k\right)}(x)\right| dx.$$

In order to get a Jacobi polynomial with real parameters, remark that by [G-R], p. 807, 7.512.12 we have that

$$F_{2,1}\left(\begin{matrix}1+k-N, N+k+\frac{1}{2}\\ \frac{3}{2}+it_2+k\end{matrix}; \frac{1+x}{2}\right)$$

equals

$$\frac{\Gamma\left(\frac{3}{2}+it_2+k\right)}{\Gamma(1+it_2)\Gamma\left(k+\frac{1}{2}\right)} \int_0^1 y^{k-\frac{1}{2}} (1-y)^{it_2} F_{2,1}\left(\begin{matrix}1+k-N, N+k+\frac{1}{2}\\ \frac{1}{2}+k\end{matrix}; y\frac{1+x}{2}\right) dy.$$

This, together with [G-R], p 990, 8.962.1 (we use this formula twice) and substituting $a = y(1+x) - 1$ in place of y, implies that (4.2.18) is $\ll$ than

$$N^{1-k} \int_{-1}^{1} \left| (1+a)^{k-\frac{1}{2}} P_{N-1-k}^{\left(1+k,k-\frac{1}{2}\right)}(a) \right| \int_{a}^{1} (1-x)^k \, dx da.$$

Computing the inner integral, and using Cauchy's inequality and [G-R], p 800, 7.391.1 we finally see that (4.2.18) is $\ll$ than N^{1-k}, so the same is true for (4.2.17).

Noting that by [G-R], p. 807, 7.512.12

$$\int_0^1 (1-z)^{-it_2} F_{2,1}\left(\begin{matrix} 1-N, N+\frac{1}{2} \\ \frac{3}{2}+it_2 \end{matrix} ; z \right) z^j dz = f_j F_{3,2}\left(\begin{matrix} 1-N, N+\frac{1}{2}, j+1 \\ \frac{3}{2}+it_2, j-it_2+2 \end{matrix} ; 1 \right)$$

with a constant f_j, and this equals

$$g_j \frac{\Gamma(-j+it_2-N)\,\Gamma\left(-\frac{1}{2}-j+it_2+N\right)}{\Gamma(1+it_2-N)\,\Gamma\left(\frac{1}{2}+it_2+N\right)}$$

with a constant g_j by [S], (2.3.1.4), so by (3.3.1), (4.2.11), (4.2.12), (4.2.16) and the above estimate for (4.2.17) we proved the lemma.

4.2.2. A nonvanishing result. In order to eliminate some conditions imposed on f in Lemma 4.1 (namely the vanishing of the integral for $0 \leq j \leq K-10$ in the statement of Lemma 4.6), we prove the next lemma.

LEMMA 4.2. *If P is a nonzero polynomial, then there is an integer $n \geq 1$ such that*

$$\int_{-\infty}^{\infty} \frac{\Gamma\left(\frac{1}{4} \pm ix\right)\Gamma\left(\frac{3}{4} \pm ix\right)\Gamma\left(\frac{1}{4} \pm it_1 \pm ix\right)}{\Gamma(\pm 2ix)} P\left(x^2\right) \phi_{i\left(\frac{1}{4}-n\right)}(x)\, dx \neq 0.$$

Proof. If $k \geq 0$ and $n \geq 1$ are integers, define

$$I_{n,k} = \int_{-\infty}^{\infty} \frac{\Gamma\left(\frac{1}{4} \pm ix\right)\Gamma\left(\frac{3}{4} \pm ix\right)\Gamma\left(\frac{1}{4} \pm it_1 \pm ix\right)}{\Gamma(\pm 2ix)} \left(x^2 + \left(\frac{1}{4}+it_1\right)^2 \right)^k \phi_{i\left(\frac{1}{4}-n\right)}(x)\, dx.$$

We first prove that

$$I_{n,0} = (D + o(1))\, n \tag{4.2.19}$$

with a nonzero constant D.

By Proposition 4.4 of [G1] we see that $I_{n,0}$ equals the sum of

$$\frac{\Gamma\left(\frac{1}{2}-2n\right)}{\Gamma\left(\frac{1}{2}\pm it_1-n\right)\Gamma\left(\frac{1}{2}\pm it_2-n\right)}\sum_{m_1=0}^{\infty}\frac{(n\pm it_1)_{m_1}(n\pm it_2)_{m_1}}{m_1!\left(\frac{1}{2}+2n\right)_{m_1}}J\left(\frac{1}{4}+n+m_1\right) \tag{4.2.20}$$

and

$$\frac{\Gamma\left(2n-\frac{1}{2}\right)}{\Gamma\left(n\pm it_1\right)\Gamma\left(n\pm it_2\right)}\sum_{m_2=0}^{\infty}\frac{\left(\frac{1}{2}\pm it_1-n\right)_{m_2}\left(\frac{1}{2}\pm it_2-n\right)_{m_2}}{m_2!\left(\frac{3}{2}-2n\right)_{m_2}}J\left(\frac{3}{4}-n+m_2\right) \tag{4.2.21}$$

with the abbreviation

$$J(a)=\int_{-\infty}^{\infty}\frac{\Gamma\left(\frac{1}{4}\pm ix\right)\Gamma\left(\frac{3}{4}\pm ix\right)\Gamma\left(\frac{1}{4}\pm it_1\pm ix\right)}{\Gamma\left(\pm 2ix\right)\Gamma\left(a\pm ix\right)}dx.$$

Using that for $m_1\geq 0$ and real x we have

$$\frac{1}{\Gamma\left(\frac{1}{4}+n+m_1\pm ix\right)}\ll\frac{1}{\Gamma\left(\frac{1}{4}\pm ix\right)}\frac{1}{\Gamma^2\left(\frac{1}{4}+n+m_1\right)},$$

and for $m_2\geq n$ and real x we have

$$\frac{1}{\Gamma\left(\frac{3}{4}-n+m_2\pm ix\right)}\ll\frac{1}{\Gamma\left(\frac{3}{4}\pm ix\right)}\frac{1}{\Gamma^2\left(\frac{3}{4}-n+m_2\right)},$$

it is not hard to see (estimating every term separately) that for any fixed $\epsilon>0$ the whole sum (4.2.20) and the $m_2>n$ part of (4.2.21) give $o(n)$. (To see this estimate it helps to consider separately the cases $m_1,m_2\geq n^{2-\epsilon}$ and $m_1,m_2\leq n^{2-\epsilon}$.) We thus see that the difference of $I_{n,0}$ and the $0\leq m_2\leq n$ part of (4.2.21) is $o(n)$. In this case we have

$$J\left(\frac{3}{4}-n+m_2\right)=c\Gamma\left(\frac{1}{2}+n-m_2\pm it_1\right)$$

by (3.6.1) of [A-A-R] with a nonzero constant c. So we proved that

$$I_{n,0}=o(n)+c^*n^2\left(1+o(1)\right)\sum_{m=0}^{n}\frac{\Gamma\left(\frac{1}{2}-n+m\pm it_2\right)}{m!\Gamma\left(\frac{3}{2}-2n+m\right)}. \tag{4.2.22}$$

with a nonzero constant c^*. By the relations

$$\sum_{n+1\leq m\leq n^{2-\epsilon}}\left|\frac{\Gamma\left(\frac{1}{2}-n+m\pm it_2\right)}{m!\Gamma\left(\frac{3}{2}-2n+m\right)}\right|=o\left(\frac{1}{n}\right),$$

$$\sum_{m>n^{2-\epsilon}} \left| \frac{\Gamma\left(\frac{1}{2}-n+m\pm it_2\right)}{m!\Gamma\left(\frac{3}{2}-2n+m\right)} - \frac{\Gamma\left(-n+m\pm it_2\right)}{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(1-2n+m\right)} \right| = o\left(\frac{1}{n}\right),$$

$$\sum_{2n\le m\le n^{2-\epsilon}} \left| \frac{\Gamma\left(-n+m\pm it_2\right)}{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(1-2n+m\right)} \right| = o\left(\frac{1}{n}\right),$$

$$\sum_{m=2n}^{\infty} \frac{\Gamma\left(-n+m\pm it_2\right)}{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(1-2n+m\right)} = \frac{\Gamma\left(n\pm it_2\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(n+\frac{1}{2}\pm it_2\right)},$$

$$\sum_{m=0}^{\infty} \frac{\Gamma\left(\frac{1}{2}-n+m\pm it_2\right)}{m!\Gamma\left(\frac{3}{2}-2n+m\right)} = \frac{\Gamma\left(\frac{1}{2}-n\pm it_2\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(1-n\pm it_2\right)}$$

we get (the last two relations follow from [S], (1.7.6)) that

$$\sum_{m=0}^{n} \frac{\Gamma\left(\frac{1}{2}-n+m\pm it_2\right)}{m!\Gamma\left(\frac{3}{2}-2n+m\right)} = \frac{\Gamma\left(\frac{1}{2}-n\pm it_2\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(1-n\pm it_2\right)} - \frac{\Gamma\left(n\pm it_2\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(n+\frac{1}{2}\pm it_2\right)} + o\left(\frac{1}{n}\right).$$

Since

$$\frac{\Gamma^2\left(\frac{1}{2}\pm it_2\right)}{\Gamma\left(1\pm it_2\right)\Gamma\left(\pm it_2\right)} \neq 1,$$

hence, together with (4.2.22) this implies (4.2.19).

By Proposition 3.1 and (3.4) of [G1] we have the recursion relation ($a = \frac{1}{4} + it_1$)

$$\left(x^2+a^2\right)\phi_{i\left(\frac{1}{4}-n\right)}(x) = a_n\phi_{i\left(\frac{1}{4}-(n-1)\right)}(x) + b_n\phi_{i\left(\frac{1}{4}-n\right)}(x) + c_n\phi_{i\left(\frac{1}{4}-(n+1)\right)}(x)$$

with (for the functions A and B see [G1])

$$a_n = B\left(-i\left(\frac{1}{4}-n\right)\right), b_n = A\left(i\left(\frac{1}{4}-n\right)\right) + A\left(-i\left(\frac{1}{4}-n\right)\right), c_n = B\left(i\left(\frac{1}{4}-n\right)\right),$$

and since

$$A\left(\pm i\left(\frac{1}{4}-n\right)\right) = \frac{n^2}{4}\left(1+o(1)\right),\ B\left(\pm i\left(\frac{1}{4}-n\right)\right) = \frac{n^2}{4}\left(1+o(1)\right),$$

we get by induction on the basis of (4.2.19) that for any fixed integer $k \geq 0$ we have

$$I_{n,k} = \left(D_k + o(1)\right)n^{1+2k}$$

as $n \to \infty$ with a nonzero constant D_k. This proves the lemma.

4.2.3. Proof of Theorem 4.1. The first statement in Theorem 4.1 is a strengthening of Lemma 4.1, since we prove the same conclusion from weaker conditions. The second statement is the promised theorem on the expression of a general function in the system $\phi_{i\left(\frac{1}{4}-N\right)}$.

It follows from Lemma 4.2 (applying it writing t_2 in place of t_1, which is possible, since $\phi_\lambda(x)$ is symmetric in t_1 and t_2, see Remark 4.5 (iii) of [G1]) by elementary linear algebra that there is a finite linear combination

$$g(x) := \sum_{n=1}^{N_0} d_n \phi_{i\left(\frac{1}{4}-n\right)}(x)$$

such that for $h(x) := f(x) - g(x)$ we have that

$$\int_{-\infty}^{\infty} h(x) \frac{\Gamma\left(\frac{1}{4} \pm ix\right)\Gamma\left(\frac{3}{4} \pm ix\right)\Gamma\left(\frac{1}{4} \pm it_2 \pm ix\right)}{\Gamma(\pm 2ix)} x^j dx = 0$$

for every integer $0 \leq j \leq K - 10$. We see by (4.2.1) with $k = 0$ that (4.1.1) is still satisfied if we write h in place of f there. Since g is an entire function and satisfies a much better upper bound by Proposition 4.4 of [G1] than the one needed here, we finally see that the conditions of Lemma 4.1 are satisfied by writing h in place of f there. Since

$$\int_0^{\infty} g(x) F_N(x) H(x) dx = 0$$

for $N > N_0$ by (4.2.2), so Lemma 4.1 applied for h implies (4.1.2).

Denote by $d(x)$ the difference of the left-hand side and the right-hand side of (4.1.3). By analytic continuation, using (4.1.2) and (4.5.7) we see the statement about the absolute convergence and that it is enough to prove that $d(x) = 0$ for every real x. By (4.1.2), (4.5.6), (4.2.2), (4.2.3), (4.1.1) and (4.2.1) (with $k = 0$ and $\lambda = i\left(\frac{1}{4} - N\right)$) we see that $d(x)$ is an even continuous function on the real line satisfying

$$d(x) \ll e^{2\pi x} (1 + |x|)^2$$

and

$$\int_0^{\infty} d(x) F_N(x) H(x) dx = 0, \quad \int_{-\infty}^{\infty} d(x) \frac{1}{\Gamma\left(\frac{3}{4} \pm ix\right)} H(x) dx = 0$$

for every $N \geq 1$. Hence

$$\int_{-\infty}^{\infty} d\left(x\right) \frac{H(x)}{\Gamma\left(\frac{3}{4} \pm ix\right)} P(x)dx = 0$$

for every polynomial P. In view of the definition of H and these properties of d, applying [A-A-R], Theorem 6.5.2 we get that d is identically 0. This proves (4.1.3), hence Theorem 4.1.

4.3. An expression for the Wilson function

4.3.1. Transforming the double sum into a double integral. We first construct a function $P_Z(S)$ such that when $\mathrm{Re}S = \frac{1}{2}$ or S is real, then

$$\sum_{l_1,l_2=0}^{\infty} \frac{\overline{\Gamma\left(S+m_{l_1,l_2}\right)}}{\Gamma\left(1-S+m_{l_1,l_2}\right)} \frac{\Sigma_{l_1,l_2}(S)}{\Gamma\left(\frac{1}{2}+l_1\right)\Gamma\left(1+l_1\right)\Gamma\left(\frac{1}{2}+l_2\right)\Gamma\left(1+l_2\right)} = P_1\left(S\right), \qquad (4.3.1)$$

and $P_Z(S)$ has the properties that for fixed $|Z| \leq 1$ it is entire in S, and for a fixed S it is holomorphic on $|Z| < 1$ and continuous on $|Z| \leq 1$. Hence it is enough to determine $P_1(S)$, and by analyticity in S it is enough to consider $\mathrm{Re}S = \frac{1}{2}$. And for a fixed $\mathrm{Re}S = \frac{1}{2}$, for $|Z| < 1/3$ we can apply the lemmas of Subsection 4.5.3 to get an expression for $P_Z(S)$, which will then be extended also to $Z = 1$.

For simplicity, we write $m = m_{l_1 l_2}$ in the sequel. Now, it is not hard to see for any S that

$$\frac{\overline{\Gamma\left(S+m\right)}}{\Gamma\left(1-S+m\right)} \sum_{L_1=0}^{l_1} (-1)^{L_1} \binom{l_1}{L_1} \frac{\overline{\Gamma\left(s_2-n+l_2+L_1\right)}}{\overline{\Gamma\left(s_2+n-l_2-L_1\right)}} \frac{\overline{\left(S-m\right)_{l_1-L_1}}}{\overline{\Gamma\left(S+m-l_1+L_1\right)}}$$

equals

$$\frac{(-1)^{l_1}\overline{\Gamma\left(s_2-n+l_1+l_2\right)}}{\overline{\Gamma\left(s_2+n-l_1-l_2\right)}\Gamma\left(1-S+m\right)} \overline{F_{3,2}\left(\begin{matrix} -l_1, S-m, 1-S-m \\ s_2+n-l_1-l_2, 1-s_2+n-l_1-l_2 \end{matrix}; 1\right)}. \qquad (4.3.2)$$

If $\mathrm{Re}S = \frac{1}{2}$ or S is real, then (using that $\{S, 1-S\}$ =$\{\overline{S}, 1-\overline{S}\}$) this equals

$$\frac{(-1)^{l_1}\overline{\Gamma\left(s_2-n+l_1+l_2\right)}}{\overline{\Gamma\left(s_2+n-l_1-l_2\right)}\Gamma\left(1-S+m\right)} F_{3,2}\left(\begin{matrix} -l_1, S-m, 1-S-m \\ \overline{s_2}+n-l_1-l_2, 1-\overline{s_2}+n-l_1-l_2 \end{matrix}; 1\right). \qquad (4.3.3)$$

Again for any S, we have that

$$\sum_{L_2=0}^{l_2} (-1)^{L_2} \binom{l_2}{L_2} \frac{\Gamma\left(s_1-n+l_1+L_2\right)}{\Gamma\left(s_1+n-l_1-L_2\right)} \frac{\left(S-m\right)_{l_2-L_2}}{\Gamma\left(S+m-l_2+L_2\right)}$$

equals

$$\frac{(-1)^{l_2}\Gamma\left(s_1-n+l_1+l_2\right)}{\Gamma\left(s_1+n-l_1-l_2\right)\Gamma\left(S+m\right)}F_{3,2}\left(\begin{matrix}-l_2, S-m, 1-S-m\\ s_1+n-l_1-l_2, 1-s_1+n-l_1-l_2\end{matrix};1\right). \tag{4.3.4}$$

For any complex S, denote the product of (4.3.3) and (4.3.4) by $P_{l_1,l_2}(S)$. We just proved that

$$\frac{\overline{\Gamma\left(S+m\right)}}{\Gamma\left(1-S+m\right)}\Sigma_{l_1,l_2}(S)=P_{l_1,l_2}(S), \tag{4.3.5}$$

if $\mathrm{Re}S=\frac{1}{2}$ or S is real.

Let Z be a fixed complex number with $|Z|\leq 1$. We can see using Lemma 4.7 (ii) (by writing $S=\frac{1}{2}+i\tau$, $a=n-l_1-l_2=\frac{1}{4}-m$, and estimating trivially, term by term the new $F_{3,2}$ functions) that the series

$$P_Z(S):=\sum_{l_1,l_2=0}^{\infty}\frac{Z^{l_1+l_2}}{\Gamma\left(\frac{1}{2}+l_1\right)\Gamma\left(1+l_1\right)\Gamma\left(\frac{1}{2}+l_2\right)\Gamma\left(1+l_2\right)}P_{l_1,l_2}(S) \tag{4.3.6}$$

is locally uniformly absolutely convergent on the whole plane. Hence, for any given $|Z|\leq 1$ the function $P_Z(S)$ is an entire function. We also see by the same estimates that for a given complex S the function $P_Z(S)$ is holomorphic on the open disc $|Z|<1$ and extends continuously to $|Z|\leq 1$. We also see by (4.3.5) and (4.3.6) that (4.3.1) is valid if $\mathrm{Re}S=\frac{1}{2}$ or S is real. Hence for the proof of Theorem 4.2 it is enough to show for every complex S that

$$P_1\left(S\right)=\frac{\left|\Gamma\left(s_1\right)\right|^2\left|\Gamma\left(s_2\right)\right|^2\Gamma\left(-\frac{1}{2}+2n\right)}{\left|\Gamma\left(s_1+n\right)\right|^2\left|\Gamma\left(s_2+n\right)\right|^2}\phi_\tau\left(i\left(\frac{1}{4}-n\right)\right), \tag{4.3.7}$$

if we write $S=\frac{1}{2}+i\tau$.

For any complex S, denote the product of (4.3.2), (4.3.4) and $\frac{\Gamma(1-S+m)}{\overline{\Gamma(S+m)}}$ by $R_{l_1,l_2}(S)$. Let Z be a fixed complex number with $|Z|<1$, and write

$$R_Z(S):=\sum_{l_1,l_2=0}^{\infty}\frac{Z^{l_1+l_2}}{\Gamma\left(\frac{1}{2}+l_1\right)\Gamma\left(1+l_1\right)\Gamma\left(\frac{1}{2}+l_2\right)\Gamma\left(1+l_2\right)}R_{l_1,l_2}(S).$$

Observe that for $\mathrm{Re}S=\frac{1}{2}$ we have

$$R_Z(S)=P_Z(S). \tag{4.3.8}$$

Applying Corollary 4.2 and Lemma 4.9 we see that this series is locally uniformly absolutely convergent for $\frac{1}{2} \leq \mathrm{Re}S < \frac{3}{4}$, hence $R_Z(S)$ is a continuous function of S there (remark that it is not holomorphic).

And by Lemma 4.7 (i) for $\frac{1}{2} < \mathrm{Re}S < \frac{3}{4}$, if $S = \frac{1}{2} + i\tau$, we see that $R_{l_1,l_2}(S)$ equals the product of

$$\frac{2^{4\mathrm{Re}S\,-\,2}(-1)^{l_1+l_2} l_1! l_2! \left|\Gamma\left(s_1\right)\right|^2 \left|\Gamma\left(s_2\right)\right|^2}{4\pi^2 \overline{\Gamma\left(\frac{1}{4}+it_2 \pm i\tau\right)} \Gamma\left(\frac{1}{4}+it_1 \pm i\tau\right) \overline{\Gamma\left(\frac{1}{2}+n-l_2-it_2\right)} \Gamma\left(\frac{1}{2}+n-l_1-it_1\right)},$$

$$\overline{\int_0^\infty y_1^{\frac{5}{4}+it_2-i\tau} L_{l_1}^{-\frac{1}{2}}\left(y_1\right) J\left(n-l_1-l_2, y_1, \tau\right) e^{-\frac{y_1}{2}} \frac{dy_1}{y_1^2}}$$

and

$$\int_0^\infty y_2^{\frac{5}{4}+it_1-i\tau} L_{l_2}^{-\frac{1}{2}}\left(y_2\right) J\left(n-l_1-l_2, y_2, \tau\right) e^{-\frac{y_2}{2}} \frac{dy_2}{y_2^2},$$

where we write

$$J\left(a, y, \tau\right) = \int_{-\infty}^{\infty} \left(1+T^2\right)^{-\frac{1}{2}-i\tau} \left(\frac{1+iT}{1-iT}\right)^{\frac{1}{4}-a} e^{-y\frac{iT}{2}} dT.$$

For $\frac{1}{2} < \mathrm{Re}S < \frac{3}{4}$ and $|Z| < \frac{1}{3}$ we then get, using also Lemma 4.8 that $R_Z(S)$ equals the product of

$$\frac{2^{4\mathrm{Re}S\,-\,\frac{3}{2}+i(t_1-t_2+\overline{\tau}-\tau)} \left|\Gamma\left(s_1\right)\right|^2 \left|\Gamma\left(s_2\right)\right|^2}{4\pi^3 \Gamma\left(\frac{1}{4}-it_2-i\overline{\tau}\right) \Gamma\left(\frac{1}{4}+it_1+i\tau\right) \Gamma\left(\frac{1}{2}+n+it_2\right) \Gamma\left(\frac{1}{2}+n-it_1\right)} \tag{4.3.9}$$

and

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F\left(T_1, T_2\right) G_Z\left(T_1, T_2\right) H_Z\left(T_1, T_2\right) dT_1 dT_2, \tag{4.3.10}$$

where $F\left(T_1, T_2\right)$ denotes

$$\left(1+iT_1\right)^{-\frac{3}{4}+n+i\overline{\tau}} \left(1-iT_1\right)^{-\frac{1}{2}-n+it_2} \left(1+iT_2\right)^{-\frac{1}{2}-n-it_1} \left(1-iT_2\right)^{-\frac{3}{4}+n-i\tau},$$

$G_Z\left(T_1, T_2\right)$ denotes

$$\left(1+Z\frac{1+iT_2}{1-iT_2}\right)^{-\frac{1}{2}+n-it_1} F\left(\frac{1}{2}-n+it_1, \frac{1}{4}+it_2-i\overline{\tau}, \frac{1}{2}; \frac{2Z\frac{1+iT_2}{(1+iT_1)(1-iT_2)}}{1+Z\frac{1+iT_2}{1-iT_2}}\right),$$

and $H_Z(T_1, T_2)$ denotes

$$\left(1 + Z\frac{1-iT_1}{1+iT_1}\right)^{-\frac{1}{2}+n+it_2} F\left(\frac{1}{2} - n - it_2, \frac{1}{4} - it_1 + i\tau, \frac{1}{2}; \frac{2Z\frac{1-iT_1}{(1+iT_1)(1-iT_2)}}{1 + Z\frac{1-iT_1}{1+iT_1}}\right).$$

By continuity of $R_Z(S)$ in S this is also valid for $|Z| < \frac{1}{3}$ and $\mathrm{Re}S = \frac{1}{2}$. By (4.3.8) we then see that $P_Z(S)$ equals the product of (4.3.9) and (4.3.10), if $|Z| < \frac{1}{3}$, $S = \frac{1}{2} + i\tau$ and τ is real. But since by using [S], (1.8.1.11) (for the case when the absolute value of the argument of the hypergeometric function is greater than 1) it can be checked that for a fixed S with $\mathrm{Re}S = \frac{1}{2}$ the product of (4.3.9) and (4.3.10) is holomorphic on the open disc $|Z| < 1$ and tends to the same expression with $Z = 1$ when Z tends to 1, we finally get that if $S = \frac{1}{2} + i\tau$ and τ is real, then

$$P_1(S) = \frac{2^{2n-\frac{1}{2}} |\Gamma(s_1)|^2 |\Gamma(s_2)|^2}{4\pi^3 \Gamma\left(\frac{1}{4} - it_2 - i\tau\right) \Gamma\left(\frac{1}{4} + it_1 + i\tau\right) \Gamma\left(\frac{1}{2} + n + it_2\right) \Gamma\left(\frac{1}{2} + n - it_1\right)} M(n) \tag{4.3.11}$$

with the definition

$$M(\nu) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} M(T_1, T_2, \nu)\, dT_1 dT_2,$$

where $M(T_1, T_2, \nu)$ denotes the product of

$$(1 + iT_1)^{-\frac{1}{4}-it_2+i\tau} (1 - iT_1)^{-\frac{1}{2}-\nu+it_2} (1 + iT_2)^{-\frac{1}{2}-\nu-it_1} (1 - iT_2)^{-\frac{1}{4}+it_1-i\tau}$$

and

$$F\left(\frac{1}{2} - \nu + it_1, \frac{1}{4} + it_2 - i\tau, \frac{1}{2}; \frac{1+iT_2}{1+iT_1}\right) F\left(\frac{1}{2} - \nu - it_2, \frac{1}{4} - it_1 + i\tau, \frac{1}{2}; \frac{1-iT_1}{1-iT_2}\right).$$

4.3.2. Identifying $M(\nu)$ with a Wilson function. In the above formula (4.3.11) for $P_1(S)$ we have only the value $M(n)$, but we will determine it by analytic continuation in ν.

It can be checked (using again [S], (1.8.1.11)) that $M(\nu)$ is a holomorphic function for $\mathrm{Re}\nu > \frac{1}{4}$, and in the case $\frac{1}{4} < \mathrm{Re}\nu < \frac{1}{2}$ we can deform the path of integration in the following way. For a fixed T_2 instead of the line segment [-R,R] (where R is a large positive number) we integrate in T_1 on the following route:

$$[-R, a] \cup [a, b] \cup [b, c] \cup [c, d] \cup [d, R],$$

where

$$a=-\frac{1}{R}-iR, b=-\frac{1}{R}-i+\frac{i}{R}, c=\frac{1}{R}-i+\frac{i}{R}, d=\frac{1}{R}-iR.$$

Letting $R\to\infty$, the integral on $[-R,a],[b,c]$ and $[d,R]$ tends to 0, and we are left with two integrals: $1-iT_1$ runs once on the "upper side" and once on the "lower side" of the negative real axis. Writing $x=\frac{iT_1-1}{2}$ we get in this way that

$$\int_{-\infty}^{\infty} M\left(T_1,T_2,\nu\right)dT_1$$

equals

$$4\sin\left(\pi\left(\frac{1}{2}+\nu-it_2\right)\right)\left(1+iT_2\right)^{-\frac{1}{2}-\nu-it_1}\left(1-iT_2\right)^{-\frac{1}{4}+it_1-i\tau}\int_0^{\infty} M^*\left(x,T_2,\nu\right)dx,$$

where $M^*\left(x,T_2,\nu\right)$ denotes the product of

$$(2x+2)^{-\frac{1}{4}-it_2+i\tau}\left(2x\right)^{-\frac{1}{2}-\nu+it_2}$$

and

$$F\left(\frac{1}{2}-\nu+it_1,\frac{1}{4}+it_2-i\tau,\frac{1}{2};\frac{1+iT_2}{2(1+x)}\right)F\left(\frac{1}{2}-\nu-it_2,\frac{1}{4}-it_1+i\tau,\frac{1}{2};\frac{-2x}{1-iT_2}\right).$$

Then for a given x we compute the integral in T_2 in the same way, but this time $1+iT_2$ goes to the negative real axis, and we write $y=\frac{-iT_2-1}{2}$. We finally obtain in this way in the case $\frac{1}{4}<\mathrm{Re}\nu<\frac{1}{2}$ that

$$M(\nu)=16\sin\left(\pi\left(\frac{1}{2}+\nu+it_1\right)\right)\sin\left(\pi\left(\frac{1}{2}+\nu-it_2\right)\right)\int_0^{\infty}\int_0^{\infty} M^{**}\left(x,y,\nu\right)dxdy,$$

where $M^{**}\left(x,y,\nu\right)$ denotes the product of

$$(2+2x)^{-\frac{1}{4}-it_2+i\tau}\left(2x\right)^{-\frac{1}{2}-\nu+it_2}\left(2y\right)^{-\frac{1}{2}-\nu-it_1}\left(2+2y\right)^{-\frac{1}{4}+it_1-i\tau}$$

and

$$F\left(\frac{1}{2}-\nu+it_1,\frac{1}{4}+it_2-i\tau,\frac{1}{2};\frac{-y}{1+x}\right)F\left(\frac{1}{2}-\nu-it_2,\frac{1}{4}-it_1+i\tau,\frac{1}{2};\frac{-x}{1+y}\right).$$

We apply now [S], (1.6.1.6) for both hypergeometric functions, and we compute the resulting integral in x and in y by [G-R], p 312, 3.191.2. We get in this way for $\frac{1}{4} < \mathrm{Re}\nu < \frac{1}{2}$ that $M(\nu)$ equals for any $\mathrm{Re}\nu - \frac{1}{2} < \sigma < 0$ the product of

$$16 \frac{2^{-\frac{3}{2}-2\nu} \sin\left(\pi\left(\frac{1}{2}+\nu+it_1\right)\right) \sin\left(\pi\left(\frac{1}{2}+\nu-it_2\right)\right) \Gamma^2\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}-\nu+it_1\right)\Gamma\left(\frac{1}{2}-\nu-it_2\right)\Gamma\left(\frac{1}{4}+it_2-i\tau\right)\Gamma\left(\frac{1}{4}-it_1+i\tau\right)}$$

and

$$\frac{1}{(2\pi i)^2}\int_{(\sigma)}\int_{(\sigma)} f(S_1)\, g(S_2)\, h(S_1,S_2)\, dS_1 dS_2,$$

where

$$f(S_1) = \frac{\Gamma\left(\frac{1}{2}-\nu+it_1+S_1\right)\Gamma\left(\frac{1}{2}-\nu-it_1+S_1\right)\Gamma(-S_1)}{\Gamma\left(\frac{1}{2}+S_1\right)},$$

$$g(S_2) = \frac{\Gamma\left(\frac{1}{2}-\nu+it_2+S_2\right)\Gamma\left(\frac{1}{2}-\nu-it_2+S_2\right)\Gamma(-S_2)}{\Gamma\left(\frac{1}{2}+S_2\right)},$$

$$h(S_1,S_2) = \Gamma\left(-\frac{1}{4}+\nu+i\tau+S_2-S_1\right)\Gamma\left(-\frac{1}{4}+\nu-i\tau-S_2+S_1\right).$$

For any fixed S_2 we have by the Second Barnes Lemma ([S], (4.2.2.2)) that

$$\frac{1}{2\pi i}\int_{(\sigma)} f(S_1)\, h(S_1,S_2)\, dS_1 = \frac{\Gamma\left(\frac{1}{2}-\nu+it_1\right)\Gamma\left(\frac{1}{2}-\nu-it_1\right)\Gamma\left(-\frac{1}{2}+2\nu\right)}{\Gamma(\nu-it_1)\,\Gamma(\nu+it_1)} G(S_2)$$

with

$$G(S_2) = \frac{\Gamma\left(-\frac{1}{4}+\nu-i\tau-S_2\right)\Gamma\left(\frac{1}{4}+i\tau+S_2+it_1\right)\Gamma\left(\frac{1}{4}+i\tau+S_2-it_1\right)}{\Gamma\left(\frac{3}{4}-\nu+i\tau+S_2\right)}.$$

By shifting the line of integration to the right, we have that

$$\frac{1}{2\pi i}\int_{(\sigma)} g(S_2)\, G(S_2)\, dS_2$$

equals the sum of

$$\frac{\Gamma\left(\frac{1}{2}-\nu+it_2\right)\Gamma\left(\frac{1}{2}-\nu-it_2\right)\Gamma\left(-\frac{1}{4}+\nu-i\tau\right)\Gamma\left(\frac{1}{4}+i\tau+it_1\right)\Gamma\left(\frac{1}{4}+i\tau-it_1\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{4}-\nu+i\tau\right)} F_1$$

and

$$\frac{\Gamma(\nu+it_1)\,\Gamma(\nu-it_1)\,\Gamma\left(\frac{1}{4}-\nu+i\tau\right)\Gamma\left(\frac{1}{4}-i\tau+it_2\right)\Gamma\left(\frac{1}{4}-i\tau-it_2\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}+\nu-i\tau\right)} F_2$$

with

$$F_1 = F_{4,3}\left(\begin{matrix} \frac{1}{2}-\nu+it_2, \frac{1}{2}-\nu-it_2, \frac{1}{4}+i\tau+it_1, \frac{1}{4}+i\tau-it_1 \\ \frac{1}{2}, \frac{3}{4}-\nu+i\tau, \frac{5}{4}-\nu+i\tau \end{matrix};1\right),$$

$$F_2 = F_{4,3}\left(\begin{matrix} \nu+it_1, \nu-it_1, \frac{1}{4}-i\tau+it_2, \frac{1}{4}-i\tau-it_2 \\ \frac{1}{2}, \frac{1}{4}+\nu-i\tau, \frac{3}{4}+\nu-i\tau \end{matrix};1\right).$$

By formula (3.3) of [G1] this sum equals the product of

$$\Gamma\left(\frac{1}{2}-\nu\pm it_2\right)\Gamma\left(\frac{1}{4}+i\tau\pm it_1\right)\Gamma\left(\nu\pm it_1\right)\Gamma\left(\frac{1}{4}-i\tau\pm it_2\right)$$

and

$$\phi_{t_1}\left(t_2; \frac{1}{2}-\nu, \nu, \frac{1}{4}+i\tau, \frac{3}{4}+i\tau\right). \tag{4.3.12}$$

By [G2], Lemma 5.3 (i) and (ii) we see that (4.3.12) equals

$$\phi_\tau\left(i\left(\frac{1}{4}-\nu\right); \frac{1}{4}+it_1, \frac{1}{4}+it_2, \frac{1}{4}-it_2, \frac{3}{4}+it_1\right). \tag{4.3.13}$$

Finally, we get for $\frac{1}{4} < \mathrm{Re}\nu < \frac{1}{2}$ that $M(\nu)$ equals the product of

$$2^{-\frac{3}{2}-2\nu}16\pi^3\frac{\Gamma\left(\frac{1}{4}-i\tau-it_2\right)\Gamma\left(\frac{1}{4}+i\tau+it_1\right)\Gamma\left(-\frac{1}{2}+2\nu\right)}{\Gamma\left(\frac{1}{2}+\nu+it_1\right)\Gamma\left(\frac{1}{2}+\nu-it_2\right)}$$

and (4.3.13).

We now use the fact that the Wilson function $\phi_\lambda(x)$ is analytic in both variables, which is mentioned in [G1], and can be seen at once from (3.3) of [G1]. Then, by (4.3.11) and analytic continuation in ν, if $S=\frac{1}{2}+i\tau$ and τ is real, we see that $P_1(S)$ equals the product of

$$\frac{|\Gamma(s_1)|^2|\Gamma(s_2)|^2\Gamma\left(-\frac{1}{2}+2n\right)}{|\Gamma(s_1+n)|^2|\Gamma(s_2+n)|^2} \tag{4.3.14}$$

and

$$\phi_\tau\left(i\left(\frac{1}{4}-n\right); \frac{1}{4}+it_1, \frac{1}{4}+it_2, \frac{1}{4}-it_2, \frac{3}{4}+it_1\right). \tag{4.3.15}$$

By analytic continuation this is valid for every complex S. Hence we proved (4.3.7), so the proof of Theorem 4.2 is complete.

4.4. Proof of Theorem 4.3

4.4.1. The easy statements. We will apply Lemma 4.5 several times. Note that we have

$$M_\lambda(A) = \frac{\Sigma_\lambda(A)}{\Gamma(1-A)},$$

where M_λ is defined in Theorem 4.3, and Σ_λ is defined in Lemma 4.5.

Theorem 4.3 (i) follows from Lemma 4.5 (i) and (ii). Theorem 4.3 (ii) follows from Lemma 4.5 (ii), taking there $\lambda = i\left(\frac{1}{4} - k\right)$ and shifting the integration in formula (4.5.1) there to $\sigma = \frac{5}{4} + \epsilon$ with a small $\epsilon > 0$.

It remains to prove Theorem 4.3 (iii). This will follow from Lemma 4.4 below and formula (4.5.2).

4.4.2. Estimation of a Fourier transform. Our aim is to prove that taking a suitable linear combination of functions G_A (see (4.5.3)) for different A, the Fourier transform in (4.5.2) will be rapidly decreasing, see Lemma 4.4. We first need an elementary claim.

LEMMA 4.3. *If a positive integer K is given, then we can find an integer $t > 0$, complex (non-integer) numbers $A_1, A_2, \dots, A_t$ with $\mathrm{Re}A_i > K$ $(i = 1, 2, \dots, t)$ and polynomials $Q_1, Q_2, \dots, Q_t$ with the following property. If a compact set L on the complex plane and a real σ are given such that*

$$\frac{1}{2} < \sigma + \frac{K}{2}, \qquad \frac{1}{2} < \sigma + \frac{\mathrm{Re}A}{2}$$

for every $A \in L$, then

$$\frac{\Gamma\left(S + \frac{A-1}{2}\right)\Gamma\left(S + \frac{A}{2}\right)}{\Gamma\left(S - \frac{3}{4}\right)\Gamma\left(S + \frac{1}{4} + A\right)} - \sum_{i=1}^{t} Q_i(A)\,\frac{\Gamma\left(S + \frac{A_i-1}{2}\right)\Gamma\left(S + \frac{A_i}{2}\right)}{\Gamma\left(S - \frac{3}{4}\right)\Gamma\left(S + \frac{1}{4} + A_i\right)} = O\left(\frac{1}{(1+|S|)^K}\right)$$

if $A \in L$ and $\mathrm{Re}S \geq \sigma$.

Proof. Let us remark that since

$$\frac{\Gamma\left(S + \frac{A-1}{2}\right)\Gamma\left(S + \frac{A}{2}\right)}{\Gamma\left(S - \frac{3}{4}\right)\Gamma\left(S + \frac{1}{4} + A\right)} = F\left(-\frac{1}{4} - \frac{A}{2}, \frac{3}{4} + \frac{A}{2}, S + \frac{A}{2}; 1\right)$$

for $\frac{1}{2} < \mathrm{Re}S + \frac{\mathrm{Re}A}{2}$ by [S], (1.7.6), so there are polynomials $P_0, P_1, \ldots, P_{K-1}$ such that

$$\frac{\Gamma\left(S+\frac{A-1}{2}\right)\Gamma\left(S+\frac{A}{2}\right)}{\Gamma\left(S-\frac{3}{4}\right)\Gamma\left(S+\frac{1}{4}+A\right)} = \sum_{k=0}^{K-1}\frac{P_k\left(A\right)}{S^k} + O\left(\frac{1}{\left(1+|S|\right)^K}\right)$$

if σ is a fixed real number, and A is in a fixed compact set such that $\frac{1}{2} < \sigma + \frac{\mathrm{Re}A}{2}$ for every element A of this compact set, and $\mathrm{Re}S \geq \sigma$.

It is clear that P_0 is identically 1. Let $t > 0$ be maximal with the property that there are integers $0 \leq k_1 < k_2 < \ldots < k_t \leq K-1$ such that the t-variable polynomial

$$\det\left(P_{k_i}\left(X_j\right)\right)_{1\leq i,j\leq t}$$

is not identically 0, and fix such integers k_i. Then it is clear that we can find non-integer numbers $A_1, A_2, \ldots, A_t$ with arbitrarily large real part such that the number

$$\det\left(P_{k_i}\left(A_j\right)\right)_{1\leq i,j\leq t}$$

is nonzero, fix such numbers $A_1, A_2, \ldots, A_t$. Then we see by Cramer's rule that there are polynomials $Q_1, Q_2, \ldots, Q_t$ such that

$$\sum_{j=1}^{t} P_k\left(A_j\right) Q_j\left(A\right) = P_k\left(A\right)$$

for every $k = k_i$ $(1 \leq i \leq t)$, but it follows easily by the maximality property above that it is true also for every $0 \leq k \leq K-1$. The lemma is proved.

LEMMA 4.4. *If a compact set L on the complex plane and an integer $K \geq 2$ are given such that $1 < \mathrm{Re}A$ for every $A \in L$, then we can find an integer $t > 0$, complex (non-integer) numbers $A_1, A_2, \ldots, A_t$ with $\mathrm{Re}A_i > K$ $(i = 1, 2, \ldots, t)$ and polynomials $Q_1, Q_2, \ldots, Q_t$ such that (see (4.5.3) for the definition of the function $G_A(x)$)*

$$\int_{-\infty}^{\infty} e^{i\lambda x}\left(G_A\left(x\right) - \sum_{i=1}^{t} 2^{A-A_i} Q_i\left(A\right) G_{A_i}\left(x\right)\right) dx = O\left(\frac{1}{\left(1+|\lambda|\right)^K}\right)$$

for any $A \in L$ and real λ. The left-hand side here is regular in A on every open subset of L for every fixed real λ.

Proof. We choose t, $A_1, A_2, \ldots, A_t$ and $Q_1, Q_2, \ldots, Q_t$ for this K as in Lemma 4.3, and choose a fixed $-\frac{1}{4} < \sigma < 0$ in (4.5.3) such that $\frac{1}{2} < \sigma + \frac{\mathrm{Re}A}{2}$ for every $A \in L$. Using the identity

$$\Gamma(2S + A - 1) = \frac{1}{2\sqrt{\pi}} 2^{2S+A-1} \Gamma\left(S + \frac{A-1}{2}\right) \Gamma\left(S + \frac{A}{2}\right)$$

the claim follows by repeated partial integration. Indeed, because of the rapid decay in $|S|$ (assured by Lemma 4.3), we can compute for $x > 0$ and also for $x < 0$ the first few derivatives in x of

$$G_A(x) - \sum_{i=1}^{t} 2^{A-A_i} Q_i(A) G_{A_i}(x) \tag{4.4.1}$$

by taking the derivatives of $\left(e^{\frac{x}{2}} + e^{-\frac{x}{2}}\right)^{2S}$ in x (in the integral representation of (4.4.1) derived from (4.5.3), using the above σ there), and we can see that these derivatives of (4.4.1) are continuous at 0 (we see this fact by shifting the S-integration to the right using again Lemma 4.3, since $\Gamma(-2S)\left(e^{2\pi i S} - e^{-2\pi i S}\right)$ is regular everywhere). The last statement is obvious by (4.5.3), the lemma is proved.

In view of our remarks at the beginning of this section, this completes the proof of Theorem 4.3.

4.5. Remaining lemmas

4.5.1. Basic properties of the functions $\phi_\lambda\left(i\left(\frac{1}{4} - k\right)\right)$.

LEMMA 4.5. *Write*

$$\Sigma_\lambda(A) = \sum_{k=1}^{\infty} (-1)^k \frac{\Gamma(k-A)}{\Gamma(k)} \phi_\lambda\left(i\left(\frac{1}{4} - k\right)\right).$$

(i) $\Sigma_\lambda(A)$ *is absolutely convergent, if* $\mathrm{Re}A > 1 + 2|\mathrm{Im}\lambda|$, *and* A *is not an integer.*

(ii) If $\mathrm{Re}A$ *is large enough (*$\mathrm{Re}A \geq 2$*, say),* A *is not an integer, and* λ *is either a real number or* $\lambda = i\left(\frac{1}{4} - n\right)$ *with a positive integer* n*, then* $\Sigma_\lambda(A)$ *is absolutely convergent and equals the product of*

$$-\frac{\Gamma(1-A)}{\Gamma\left(\frac{1}{4} \pm it_1 \pm i\lambda\right)\Gamma\left(\frac{1}{4} \pm it_2 \pm i\lambda\right)}$$

and

$$\frac{1}{2\pi i}\int_{(\sigma)}\frac{\Gamma\left(\pm i\lambda-S\right)\Gamma\left(\frac{1}{4}+S\pm it_1\right)\Gamma\left(\frac{1}{4}+S\pm it_2\right)\Gamma\left(2S+A-1\right)}{\Gamma\left(S-\frac{1}{4}\right)\Gamma\left(S+\frac{1}{4}+A\right)\Gamma\left(2S\right)}dS \tag{4.5.1}$$

with $-\frac{1}{4}<\sigma<0$*. In the case* $\lambda>0$ *we have that*

$$\Sigma_\lambda(A)=-\frac{\Gamma\left(1-A\right)e^{-2\pi\lambda}}{\Gamma\left(\frac{1}{4}\pm it_1\pm i\lambda\right)\Gamma\left(\frac{1}{4}\pm it_2\pm i\lambda\right)}\int_{-\infty}^{\infty}e^{i\lambda x}G_A\left(x\right)dx, \tag{4.5.2}$$

where $G_A\left(x\right)$ *denotes*

$$\frac{1}{2\pi i}\int_{(\sigma)}\frac{\Gamma\left(-2S\right)\Gamma\left(\frac{1}{4}+S\pm it_1\right)\Gamma\left(\frac{1}{4}+S\pm it_2\right)\Gamma\left(2S+A-1\right)}{\Gamma\left(S-\frac{1}{4}\right)\Gamma\left(S+\frac{1}{4}+A\right)\Gamma\left(2S\right)}\gamma\left(x,S\right)dS \tag{4.5.3}$$

with $-\frac{1}{4}<\sigma<0$ *and*

$$\gamma\left(x,S\right)=\left(e^{\frac{x}{2}}+e^{-\frac{x}{2}}\right)^{2S}e^{2\pi iS\frac{x}{|x|}}.$$

(iii) If k *is a positive integer and* λ *is any complex number such that* $\frac{1}{4}\pm it_1\pm i\lambda$ *and* $\frac{1}{4}\pm it_2\pm i\lambda$ *are not integers, then* $\phi_\lambda\left(i\left(\frac{1}{4}-k\right)\right)$ *equals the product of*

$$\frac{1}{\Gamma\left(\frac{1}{4}\pm it_1\pm i\lambda\right)\Gamma\left(\frac{1}{4}\pm it_2\pm i\lambda\right)}$$

and

$$\frac{1}{2\pi i}\int\frac{\Gamma\left(\pm i\lambda-S\right)\Gamma\left(\frac{1}{4}+S\pm it_1\right)\Gamma\left(\frac{1}{4}+S\pm it_2\right)}{\Gamma\left(\frac{1}{2}+S\pm\left(\frac{1}{4}-k\right)\right)}dS \tag{4.5.4}$$

with an integration route from $-i\infty$ *to* $i\infty$ *such that the poles of* $\Gamma\left(\frac{1}{4}+S\pm it_1\right)$ *and* $\Gamma\left(\frac{1}{4}+S\pm it_2\right)$ *are on the left of the route, and the poles of* $\Gamma\left(\pm i\lambda-S\right)$ *are on the right of it. In the case* $\lambda>0$ *we have that (4.5.4) equals*

$$e^{-2\pi\lambda}\int_{-\infty}^{\infty}e^{i\lambda x}\left(\frac{1}{2\pi i}\int_{(\sigma)}\frac{\Gamma\left(-2S\right)\Gamma\left(\frac{1}{4}+S\pm it_1\right)\Gamma\left(\frac{1}{4}+S\pm it_2\right)}{\Gamma\left(\frac{1}{2}+S\pm\left(\frac{1}{4}-k\right)\right)}\gamma\left(x,S\right)dS\right)dx \tag{4.5.5}$$

with $\gamma\left(x,S\right)$ *as above and* $-\frac{1}{4}<\sigma<0$*.*

(iv) If k *is a positive integer and* λ *is real, then for any* $\epsilon>0$ *we have that*

$$\left|\phi_\lambda\left(i\left(\frac{1}{4}-k\right)\right)\right|\ll_\epsilon e^{2\pi|\lambda|}\left(1+|\lambda|\right)^2k^\epsilon. \tag{4.5.6}$$

For any given compact set L on the complex plane and any $\epsilon > 0$ we have

$$\left|\phi_\lambda\left(i\left(\frac{1}{4}-k\right)\right)\right| \ll_{\epsilon,L} k^{\epsilon+2|\mathrm{Im}\lambda|} \tag{4.5.7}$$

for positive integers k and $\lambda \in L$.

Proof. Note first that (i) will follow from (4.5.7). Note also that by continuity we may assume $\lambda \neq 0$ in (ii).

By Proposition 5.3 of [G1] we see that if $\mathrm{Re}A \geq 2$ and A is not an integer, and λ is either a nonzero real number or $\lambda = i\left(\frac{1}{4}-n\right)$ with a positive integer n, then $\Sigma_\lambda(A)$ equals

$$\frac{\Gamma(-2i\lambda)}{\Gamma\left(\frac{1}{4}\pm it_1 - i\lambda\right)\Gamma\left(\frac{1}{4}\pm it_2 - i\lambda\right)}S(\lambda) + \frac{\Gamma(2i\lambda)}{\Gamma\left(\frac{1}{4}\pm it_1 + i\lambda\right)\Gamma\left(\frac{1}{4}\pm it_2 + i\lambda\right)}S(-\lambda),$$

where $S(L)$ denotes

$$\sum_{m=0}^{\infty}\frac{\left(\frac{1}{4}\pm it_1+iL\right)_m\left(\frac{1}{4}\pm it_2+iL\right)_m}{m!\,(1+2iL)_m}\sum_{k=1}^{\infty}(-1)^k\frac{\Gamma(k-A)}{\Gamma(k)\,\Gamma\left(\frac{1}{2}+iL+m\pm\left(\frac{1}{4}-k\right)\right)}.$$

By the given conditions the double sum here is absolutely convergent for $L = \pm\lambda$. To see this, we first remark that if $iL = \frac{1}{4}-n$ with some positive integer n, then we have a factor $\frac{1}{\Gamma(1+m-n-k)}$, hence we can take $m > n$, since the other terms are 0. It is not hard to check then that we have $\mathrm{Re}(iL+m) \geq 0$ in every case, and this implies that the sequence $\left|\frac{1}{\Gamma\left(\frac{1}{2}+iL+m\pm\left(\frac{1}{4}-k\right)\right)}\right|$ is monotonically decreasing for $k \geq 1$. This proves the absolute convergence of the double sum. Hence $\Sigma_\lambda(A)$ is absolutely convergent, and the inner sum in $S(L)$ equals

$$-\frac{\Gamma(1-A)\,\Gamma(2iL+2m+A-1)}{\Gamma\left(-\frac{1}{4}+iL+m\right)\Gamma\left(\frac{1}{4}+iL+m+A\right)\Gamma(2iL+2m)},$$

which follows from [S], (1.7.6) in the case when L is real or $L = i\left(\frac{1}{4}-n\right)$ with a positive integer n, and follows from [S], (1.7.7) in the case when $L = i\left(n-\frac{1}{4}\right)$ with a positive integer n. The expression for $\Sigma_\lambda(A)$ involving (4.5.1) in (ii) then follows easily by shifting the line of integration to the right in (4.5.1).

Let us now assume that λ is a nonzero real number. Since for $\mathrm{Re}S < 0$ we have

$$\Gamma(\pm i\lambda - S) = \Gamma(-2S)\int_{-\infty}^{\infty} e^{i\lambda x}\left(e^{\frac{x}{2}}+e^{-\frac{x}{2}}\right)^{2S}dx \tag{4.5.8}$$

by [G-R], p. 332, 3.313.2, hence we see in this case that

$$\Sigma_\lambda(A) = -\frac{\Gamma(1-A)}{\Gamma\left(\frac{1}{4} \pm it_1 \pm i\lambda\right)\Gamma\left(\frac{1}{4} \pm it_2 \pm i\lambda\right)} \int_{-\infty}^{\infty} e^{i\lambda x} g_A(x)\, dx,$$

where $g_A(x)$ denotes $(-\frac{1}{4} < \sigma < 0)$

$$\frac{1}{2\pi i}\int_{(\sigma)} \frac{\Gamma(-2S)\Gamma\left(\frac{1}{4}+S \pm it_1\right)\Gamma\left(\frac{1}{4}+S \pm it_2\right)\Gamma(2S+A-1)}{\Gamma\left(S-\frac{1}{4}\right)\Gamma\left(S+\frac{1}{4}+A\right)\Gamma(2S)} \left(e^{\frac{x}{2}}+e^{-\frac{x}{2}}\right)^{2S} dS.$$

We now extend analytically the function $g_A(x)$. To this end, let

$$D = \{z \in \mathbf{C}:\ 0 < \mathrm{Im} z < 2\pi,\ z \notin [\pi i, 2\pi i)\},$$

and observe that $\log\left(e^{\frac{z}{2}}+e^{-\frac{z}{2}}\right)$ can be defined holomorphically on the domain D in such a way that this function is real on $(0, i\pi)$. Denote this unique holomorphic function by $h(z)$. It is easy to see that

$$|\mathrm{Im} h(z)| < \pi$$

for $z \in D$, hence $(-\frac{1}{4} < \sigma < 0)$

$$g_A(z) := \frac{1}{2\pi i}\int_{(\sigma)} \frac{\Gamma(-2S)\Gamma\left(\frac{1}{4}+S \pm it_1\right)\Gamma\left(\frac{1}{4}+S \pm it_2\right)\Gamma(2S+A-1)}{\Gamma\left(S-\frac{1}{4}\right)\Gamma\left(S+\frac{1}{4}+A\right)\Gamma(2S)} e^{2Sh(z)} dS$$

is a holomorphic function on D. We claim that $g_A(z)$ extends holomorphically to the open strip $0 < \mathrm{Im} z < 2\pi$. Indeed, if $\epsilon > 0$ is fixed, then we can take a small open neighborhood G of the closed line segment $[i\epsilon, i(2\pi - \epsilon)]$ such that

$$\left|e^{\frac{z}{2}}+e^{-\frac{z}{2}}\right| < 2$$

for $z \in G$. Then we can compute $g_A(z)$ for $z \in G$ by shifting the path of integration to the right, and since $2S$ is a nonnegative integer at the poles, we get in this way that $g_A(z)$ extends holomorphically to G, hence to the whole strip $0 < \mathrm{Im} z < 2\pi$. Then we can see that

$$\int_{-\infty}^{\infty} e^{i\lambda(x+ih)} g_A(x+ih)\, dx$$

is independent of $0 < h < 2\pi$, and taking the limits as $h \to 0+0$ and $h \to 2\pi - 0$, using the dominated convergence theorem, we finally complete the proof of (ii).

Completely similarly as in the case of $\Sigma_\lambda(A)$ in (ii), we can prove the statements of (iii). Since we can take σ arbitrarily close to 0 in (4.5.5), and $\phi_\lambda\left(i\left(\frac{1}{4}-k\right)\right)$ is even in λ, we easily get (4.5.6) from (iii). Formula (4.5.7) can be seen from (4.5.4) using that $\phi_\lambda\left(i\left(\frac{1}{4}-k\right)\right)$ is entire in λ. The lemma is proved.

COROLLARY 4.1. *(i) We have*

$$\phi_{-\frac{3}{4}i}\left(i\left(\frac{1}{4}-k\right)\right)=c\frac{(-1)^k}{k^{3/2}}\left(1+o(1)\right) \tag{4.5.9}$$

with a nonzero constant c as $k\to\infty$.

(ii) For any given compact set L on the complex plane and any $\epsilon>0$ we have

$$\left|\phi_\lambda\left(i\left(\frac{1}{4}-k\right)\right)\right| \ll_{\epsilon,L} k^{\epsilon+2|\mathrm{Im}\lambda|} \tag{4.5.10}$$

for positive integers k and $\lambda\in L$.

(iii) If a_n $(n\geq 1)$ is any given sequence satisfying $a_n=O\left(n^d\right)$ with a number $d<\frac{1}{2}$, then for any positive integer M there are constant coefficients b_m such that

$$\sum_{n=1}^{\infty} a_n\phi_{i\left(\frac{1}{4}-n\right)}\left(i\left(\frac{1}{4}-k\right)\right)=\frac{(-1)^k}{k^{3/2}}\left(\sum_{m=0}^{M-1}\frac{b_m}{k^m}+O\left(k^{-M}\right)\right) \tag{4.5.11}$$

as $k\to\infty$ over positive integers, and the left-hand side here is absolutely convergent for every integer $k\geq 1$.

Proof. Part (i) follows from (4.5.4). Indeed, we shift the route of integration to the right of $\mathrm{Re}S=\frac{3}{4}$ in (4.5.4), and we get the first pole at $S=\frac{3}{4}$. In the same way (but this time shifting the route of integration a bit further, to a large but fixed $\mathrm{Re}S$) we get part (iii). Part (ii) is contained in Lemma 4.5 (iv).

4.5.2. Properties of a function transform. The next lemma is used in the proof of Lemma 4.1.

LEMMA 4.6. *Assume that K is a positive number, and $f(\tau)$ is an even holomorphic function for $|\mathrm{Im}\tau|<K$, it satisfies that*

$$\left|f(\tau)e^{-2\pi|\tau|}\left(1+|\tau|\right)^K\right|$$

is bounded on the domain $|\mathrm{Im}\tau| < K$*, and*

$$\int_{-\infty}^{\infty} f(\tau) \frac{\Gamma\left(\frac{1}{4} \pm i\tau\right) \Gamma\left(\frac{3}{4} \pm i\tau\right) \Gamma\left(\frac{1}{4} \pm it_2 \pm i\tau\right)}{\Gamma(\pm 2i\tau)} \tau^j d\tau = 0$$

for every integer $0 \leq j \leq K - 10$ *(say). For* $\mathrm{Re}A < 0$ *define*

$$G(A) = \int_{-\infty}^{\infty} f(\tau) \frac{\Gamma\left(\frac{1}{4} \pm i\tau\right) \Gamma\left(\frac{3}{4} \pm i\tau\right) \Gamma\left(\frac{1}{4} \pm it_2 \pm i\tau\right)}{\Gamma(\pm 2i\tau)} \Gamma(-A \pm i\tau)\, d\tau. \tag{4.5.12}$$

Then, if k *is a positive integer and* K *is large enough in terms of* k*, then* $G(A)$ *extends meromorphically to the domain* $\mathrm{Re}A < k$ *with possible singularities only at the points*

$$\frac{1}{4} + j, \frac{3}{4} + j, \frac{1}{4} + it_2 + j, \frac{1}{4} - it_2 + j$$

with integers $0 \leq j \leq k - 1$*, it has at most simple poles at these points, and*

$$\left| G(A) e^{\pi|A|} (1 + |A|)^k \right|$$

is bounded on the set $-\frac{1}{4} \leq \mathrm{Re}A < k$*,* $|\mathrm{Im}A| > 1 + |t_2|$*.*

Proof. We see by (4.5.8) that for $\mathrm{Re}A < 0$ we have

$$G(A) = \Gamma(-2A) \int_{-\infty}^{\infty} F(x) \left(e^{\frac{x}{2}} + e^{-\frac{x}{2}}\right)^{2A} dx \tag{4.5.13}$$

with the definition

$$F(x) = \int_{-\infty}^{\infty} f(\tau) \frac{\Gamma\left(\frac{1}{4} \pm i\tau\right) \Gamma\left(\frac{3}{4} \pm i\tau\right) \Gamma\left(\frac{1}{4} \pm it_2 \pm i\tau\right)}{\Gamma(\pm 2i\tau)} e^{i\tau x} d\tau.$$

It is not hard to see that if k is a positive integer, and K is large enough in terms of k, then, on the one hand, F is k times continuously differentiable on the real line and $\left(\left(\frac{d}{dx}\right)^l F\right)(0) = 0$ for every $0 \leq l \leq 2k$, on the other hand $\left(\frac{d}{dx}\right)^l F$ is even for even l and odd for odd l, and (shifting the line of τ-integration upwards, using that $t_2 \neq 0$)

$$\left(\left(\frac{d}{dx}\right)^l F\right)(x) = \sum_{j=0}^{k-1} e^{-\left(\frac{1}{4}+j\right)x} \left(A_{j,l} + B_{j,l} e^{-\frac{x}{2}} + C_{j,l} e^{-it_2 x} + D_{j,l} e^{it_2 x}\right) + O\left(e^{-kx}\right)$$

as $x \to +\infty$ with some constant coefficients for every $0 \leq l \leq k$.

Define now $F_0 = F$, and

$$F_{j+1}(x) = -\left(\frac{d}{dx}\left(\frac{2F_j(x)}{e^{\frac{x}{2}} - e^{-\frac{x}{2}}}\right)\right)(x)$$

for $j \geq 0$. It is not hard to see that for any $0 \leq j \leq k$ the function $F_j(x)$ is continuous at 0, and the behaviour of the function

$$F_j(x)\left(e^{\frac{x}{2}} + e^{-\frac{x}{2}}\right)^j$$

as $x \to +\infty$ is the same as we saw above for the derivatives of F. Then by repeated partial integration, we get from (4.5.13) for $-\frac{1}{4} \leq \mathrm{Re}A<0$ that

$$G(A) = \Gamma(-2A)\frac{1}{(2A+1)_k}\int_{-\infty}^{\infty} F_k(x)\left(e^{\frac{x}{2}} + e^{-\frac{x}{2}}\right)^{2A+k} dx.$$

By the above-mentioned properties of F_k this almost proves the lemma, using also the easy fact that if w is a given complex number and $M > 0$ is an integer, then there are constants $\gamma_{w,1}, \gamma_{w,2}, \ldots, \gamma_{w,M-1}$ such that

$$e^{wx} = \left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right)\sum_{m=0}^{M-1}\gamma_{w,m}\left(e^{\frac{x}{2}} + e^{-\frac{x}{2}}\right)^{2\left(w-\frac{1}{2}-m\right)} + O\left(e^{(w-M)x}\right)$$

for real x as $x \to \infty$.

The only fact which still requires a proof is that $G(A)$ is regular at the poles of $\Gamma(-2A)$. For the proof of this fact we return to (4.5.12).

Let b be a large positive integer (we will fix it later). Since we have

$$\frac{\Gamma(z)}{\Gamma(z+b)} = \sum_{a=0}^{b-1} c_{a,b}(z+a)^{-1}$$

with some constants $c_{a,b}$, so, applying it for $z = -A \pm i\tau$, we see that

$$\Gamma(-A \pm i\tau) = \Gamma(-A \pm i\tau + b)\sum_{0 \leq a_1, a_2 \leq b-1}\frac{c_{a_1,b}c_{a_2,b}}{(-A+i\tau+a_1)(-A-i\tau+a_2)}.$$

We use the identity

$$\frac{-2i\tau + a_2 - a_1}{(-A+i\tau+a_1)(-A-i\tau+a_2)} = \frac{1}{-A+i\tau+a_1} - \frac{1}{-A-i\tau+a_2}, \qquad (4.5.14)$$

and because of the presence of the factor $\frac{1}{\Gamma(\pm 2i\tau)}$, shifting the line of integration in (4.5.12) in the case Re$A<0$ to Im$\tau=\pm\frac{b}{2}$ (the minus sign is used in the case of the first term on the right-hand side of (4.5.14), and the plus sign is used in the case of the second term), since b can be arbitrarily large, we get such an expression for $G(A)$ which proves that it is regular at the poles of $\Gamma(-2A)$. The lemma is proved.

4.5.3. Lemmas needed for Theorem 4.2

LEMMA 4.7. *Let $n\geq 0$ be an integer, and let t and a be real numbers. Then we have that*

(i) if $-\frac{1}{4}<\mathrm{Im}\tau<0$, then

$$\frac{2\pi\Gamma\left(\frac{1}{4}+it\pm i\tau\right)\Gamma\left(\frac{1}{2}+a-it+n\right)}{2^{2i\tau}n!\Gamma\left(\frac{1}{2}+a\pm it\right)\Gamma\left(\frac{3}{4}-a+i\tau\right)}F_{3,2}\left(\begin{matrix}-n,\frac{1}{4}+i\tau+a,\frac{1}{4}-i\tau+a\\ \frac{1}{2}+it+a,\frac{1}{2}-it+a\end{matrix};1\right) \tag{4.5.15}$$

equals

$$\int_{-\infty}^{\infty}\left(1+T^2\right)^{-\frac{1}{2}-i\tau}\left(\frac{1+iT}{1-iT}\right)^{\frac{1}{4}-a}\left(\int_0^{\infty}y^{\frac{5}{4}+it-i\tau}L_n^{-\frac{1}{2}}(y)\,e^{-y\frac{1+iT}{2}}\frac{dy}{y^2}\right)dT; \tag{4.5.16}$$

(ii) for any τ, (4.5.15) equals

$$\frac{2\pi\Gamma\left(\frac{1}{4}+it\pm i\tau\right)\left(\frac{1}{2}\right)_n}{2^{2i\tau}n!\Gamma\left(\frac{1}{2}+a+it\right)\Gamma\left(\frac{3}{4}-a+i\tau\right)}F_{3,2}\left(\begin{matrix}-n,\frac{1}{4}+i\tau+it,\frac{1}{4}-i\tau+it\\ \frac{1}{2}+it+a,\frac{1}{2}\end{matrix};1\right). \tag{4.5.17}$$

Proof. By Corollary 3.3.5 of [A-A-R] we see (ii). Since for $k\geq 0$ and real T we have

$$\int_0^{\infty}y^{\frac{5}{4}+it-i\tau+k}e^{-y\frac{1+iT}{2}}\frac{dy}{y^2}=\left(\frac{1+iT}{2}\right)^{-\left(\frac{1}{4}+it-i\tau+k\right)}\Gamma\left(\frac{1}{4}+it-i\tau+k\right)$$

by [G-R], p 884, 8.312.2, hence by [G-R], p 990, 8.970.1 and p 899, 8.381.1 we get that (4.5.16) equals (4.5.17) for $-\frac{1}{4}<Im\tau<0$, the lemma is proved.

COROLLARY 4.2. *Let $n\geq 0$ be an integer, let t and a be real numbers, and let $-\frac{1}{4}<\mathrm{Im}\tau\leq 0$. Then we have that (4.5.15) equals*

$$\int_0^{\infty}y^{\frac{5}{4}+it-i\tau}L_n^{-\frac{1}{2}}(y)\,I(a,y,\tau)\,e^{-\frac{y}{2}}\frac{dy}{y^2},$$

where $I(a,y,\tau)$ denotes the sum of

$$I_1(a,y,\tau)=\int_{-T_{a,y}}^{T_{a,y}}\left(1+T^2\right)^{-\frac{1}{2}-i\tau}\left(\frac{1+iT}{1-iT}\right)^{\frac{1}{4}-a}e^{-y\frac{iT}{2}}dT,$$

$$I_2\left(a,y,\tau\right)=-\frac{2i}{y}\int_{T>\left|T_{a,y}\right|}\frac{\partial}{\partial T}\left(\left(1+T^2\right)^{-\frac{1}{2}-i\tau}\left(\frac{1+iT}{1-iT}\right)^{\frac{1}{4}-a}\right)e^{-y\frac{iT}{2}}dT,$$

and

$$\frac{2i}{y}\left(1+T_{a,y}^2\right)^{-\frac{1}{2}-i\tau}\left(\left(\frac{1-iT_{a,y}}{1+iT_{a,y}}\right)^{\frac{1}{4}-a}e^{y\frac{iT_{a,y}}{2}}-\left(\frac{1+iT_{a,y}}{1-iT_{a,y}}\right)^{\frac{1}{4}-a}e^{-y\frac{iT_{a,y}}{2}}\right),$$

and here $T_{a,y}=\left(y+\frac{1}{y}+|a|\right)^{10}$.

Proof. For $-\frac{1}{4}<Im\tau<0$ this follows at once from Lemma 4.7 (i) by partial integration in T. Since everything is absolutely convergent in this new expression even for $Im\tau=0$, by continuity we get the result.

LEMMA 4.8. *If* $y>0$ *and* z,w *are complex numbers, write*

$$S_{y,z}(w)=\sum_{l=0}^{\infty}\frac{(-1)^l}{\left(\frac{1}{2}\right)_l\Gamma\left(z-l\right)}L_l^{-\frac{1}{2}}\left(y\right)w^l.$$

If $\mathrm{Re}b>0$, $\mathrm{Re}s>\left|\frac{w}{w-1}\right|$ *and* $|w|<\frac{1}{2}$, *then*

$$\int_0^{\infty}e^{-sy}y^{b-1}S_{y,z}(w)dy=\frac{\Gamma\left(b\right)\left(1-w+\frac{w}{s}\right)^{z-1}s^{-b}}{\Gamma\left(z\right)}F\left(1-z,\frac{1}{2}-b,\frac{1}{2};\frac{w}{s+w-sw}\right).$$

Proof. By [G-R], p 990, 8.970.1, for $|w|<\frac{1}{2}$ we have for any $y>0$ that $S_{y,z}(w)$ equals

$$\sum_{k=0}^{\infty}\frac{y^k}{k!\left(\frac{1}{2}\right)_k}\sum_{l=k}^{\infty}\frac{(-1)^{l-k}}{(l-k)!\Gamma\left(z-l\right)}w^l$$

(for $|w|<\frac{1}{2}$ the double sum is absolutely convergent, since $\left|\frac{(-l)_k}{k!}\right|\leq 2^l$). By the binomial theorem, the inner sum here is $w^k\frac{(1-w)^{z-k-1}}{\Gamma(z-k)}$. For a given k the y-integral can be computed by [G-R], p 884, 8.381.1, and since

$$F\left(1-z,b,\frac{1}{2};\frac{w}{s(w-1)}\right)=F\left(1-z,\frac{1}{2}-b,\frac{1}{2};\frac{w}{s+w-sw}\right)\left(1+\frac{w}{s(1-w)}\right)^{z-1}$$

by [G-R], p. 998, 9.131.1, we get the lemma.

LEMMA 4.9. *(i) Let* $n\geq 0$ *be an integer, and* $y>0$. *If* $y\geq 100n$, *then*

$$\left|L_n^{-\frac{1}{2}}\left(y\right)\right|\leq Ce^{\frac{y}{4}}$$

with some absolute constant C.

(ii) If $N \geq 0$ is an integer and $y > 0$, then

$$\sum_{n=N}^{2N} \left| L_n^{-\frac{1}{2}}(y)\, e^{-\frac{y}{2}} \right|^2 \leq C \log(N+2)$$

with some absolute constant C.

Proof. It is easy to see that

$$\binom{n-\frac{1}{2}}{t} \leq C 2^n$$

for $0 \leq t \leq n$ with some absolute constant C, so by [G-R], p 990, 8.970.1 we have for any $M \geq 1$ that

$$\left| L_n^{-\frac{1}{2}}(y) \right| \leq C 2^n M^n \sum_{m=0}^{n} \frac{\left(\frac{y}{M}\right)^m}{m!} \leq C (2M)^n e^{\frac{y}{M}}.$$

Taking $M = 100$ we get (i).

For the proof of (ii), remark that by [G-R], p 992, 8.975.1 we have for any $0 < r < 1$ that

$$L_n^{-\frac{1}{2}}(y)\, e^{-\frac{y}{2}} = \frac{1}{2\pi i} \int_{|z|=r} (1-z)^{-\frac{1}{2}} e^{y\left(\frac{z}{z-1} - \frac{1}{2}\right)} \frac{dz}{z^{n+1}}.$$

Since $\mathrm{Re}\left(\frac{z}{z-1} - \frac{1}{2}\right) \leq 0$ for $|z| < 1$, hence taking $r = 1 - \frac{1}{N+2}$, from Parseval's identity we obtain (ii).

References

[A-A-R] G.E. Andrews, R. Askey, R. Roy, Special Functions, *Cambridge Univ. Press,* 1999

[Ba] A. Baker, *Linear forms in the logarithms of algebraic numbers,* Mathematika, 13 (1966), 204-216.

[Be] J. Beck, *Diophantine approximation and quadratic fields,* 55-93., in: Number Theory, eds. Győry, Pethő, Sós; Walter de Gruyter, 1998

[Bi1] A. Biró, *On a generalization of the Selberg trace formula*, Acta Arithmetica, 87 (1999), 319-338.

[Bi2] A. Biró, *Cycle integrals of Maass forms of weight 0 and Fourier coefficients of Maass forms of weight 1/2*, Acta Arithmetica, 94 (2000), 103-152.

[Bi3] A. Biró, *Yokoi's conjecture,* Acta Arithmetica, 106 (2003), 85-104.

[Bi4] A. Biró, *Chowla's conjecture,* Acta Arithmetica, 107 (2003), 178-194.

[Bi5] A. Biró, *On the class number one problem for some special real quadratic fields,* Proc. of the 2003 Nagoya Conference "On Yokoi-Chowla Conjecture and Related Problems", Furukawa Total Pr. Co., 2004, 1-9.

[Bi6] A. Biró, *A relation between triple products of weight* 0 *and weight* $\frac{1}{2}$ *cusp forms,* Israel J. of Math., 182 (2011), no. 1, 61-101.

[Bi7] A. Biró, *A duality relation for certain triple products of automorphic forms*, to appear in Israel J. of Math.

[Bi8] A. Biró, *Some properties of Wilson functions*, prepint, 2011

[B-K-L] D. Byeon, M. Kim, J. Lee, *Mollin's conjecture,* Acta Arithmetica, 126 (2007), 99-114.

[B-M] R.W. Bruggeman, Y. Motohashi, *A new approach to the spectral theory of the fourth moment of the Riemann zetafunction*, J. Reine Angew. Math., 579 (2005), 75-114.

[B-R] J. Bernstein and A. Reznikov, *Periods, subconvexity of L-functions and representation theory*, J. of Diff. Geom., 70 (2005), 129-142.

[C-F] S. Chowla and J. Friedlander, *Class numbers and quadratic residues,* Glasgow Math. J. 17 (1976), 47-52.

[Da] H. Davenport, Multiplicative Number Theory, 2nd edition, *Springer*, 1980

[Du] W. Duke, *Hyperbolic distribution problems and half-integral weight Maass forms*, Invent Math., 92, 73-90 (1988)

[F] J.D. Fay, *Fourier coefficients of the resolvent for a Fuchsian group*, J. Reine Angew. Math., 294 (1977), 143-203.

[Go] D. Goldfeld, *The class number of quadratic fields and the conjectures of Birch and Swinnerton-Dyer,* Ann. Scuola Norm. Sup. Pisa (4) 3 (1976), 623-663.

[G1] W. Groenevelt, *The Wilson function transform*, Int. Math. Res. Not. 2003, no. 52, 2779–2817

[G2] W. Groenevelt, *Wilson function transforms related to Racah coefficients*, Acta Appl. Math. 91 (2006), no. 2, 133–191.

[G-L] A.O. Gelfond and Yu.V. Linnik, *On Thue's method and the effectiveness problem in quadratic fields (in Russian),* Dokl. Akad. Nauk SSSR, 61 (1948), 773-776.

[G-Z] B. Gross and J. Zagier, *Points de Heegner et derivees de fonctions L,* C.R. Acad. Sci. Paris, 297 (1983), 85-87.

[G-R] I.S. Gradshteyn, I.M. Ryzhik, Table of integrals, series and products, 6th edition, *Academic Press*, 2000

[Hee] K. Heeegner, *Diophantische Analysis und Modulfunktionen,* Math. Z., 56 (1952), 227-253.

[Hei] H. Heilbronn, *On the class number in imaginary quadratic fields,* Quart. J. Math. Oxford Ser., 25 (1934), 150-160.

[Hej] D.A. Hejhal, The Selberg Trace Formula for $PSL(2, \mathbf{R})$, vol. 2, Springer, 1983

[H-S] R. Holowinski, K. Soundararajan, *Mass equidistribution for Hecke eignforms*, Ann. of Math. (2) 172 (2010), no 2, 1517-1528.

[I1] H. Iwaniec, Introduction to the spectral theory of automorphic forms, *Rev. Mat. Iberoamericana*, 1995

[I2] H. Iwaniec, Topics in Classical Automorphic Forms, *American Mathematical Society*, 1997

[Iv] A. Ivic, *On the moments of Hecke series at central points*, Functiones et Approximatio, 30 (2002), 49-82.

[I-K] H. Iwaniec, E. Kowalski, Analytic Number Theory, *Providence, RI*, 2004

[J] M. Jutila, *The twelfth moment of central values of Hecke series*, J. of Number Theory, 108 (2004), 157-168.

[K] N.V. Kuznetsov, *Sums of Kloosterman sums and the eighth power moment of the Riemann zetafunction*, T.I.F.R. Stud. Math., 12, 57-117, (1989)

[K-S] S. Katok, P. Sarnak, *Heegner points, cycles and Maass forms,* Israel J. of Math., 84 (1993), 193-227.

[Le] J. Lee, *The complete determination of wide Richaud-Degert types which are not 5 modulo 8 with class number one,* Acta Arithmetica, 140 (2009), 1-29.

[Li] E. Lindenstrauss, *Invariant measures and arithmetic quantum unique ergodicity,* Ann. of Math., 163 (2006), 165-219.

[Mi] Ming-yao Zhang, *On Yokoi's conjecture*, Math. Comp. 64 (212) (1995), 1675-85.

[Mo] Y. Motohashi, *A functional equation for the spectral fourth moment of modular Hecke L-functions,* in: D.R. Heath-Brown, B.Z. Moroz (eds), Proceedings of the session in analytic number theory and Diophantine equations (Bonn, January-June, 2002), Bonner Math. Schriften Nr. 360, Math. Inst. Univ. Bonn, Bonn, 2003

[P] N.V. Proskurin, *On general Klosterman sums (in Russian)*, Zap. Naucn. Sem. POMI 302 (2003), 107-134.

[R] A. Reznikov, *Rankin-Selberg without unfolding and bounds for spherical Fourier coefficients of Maass forms*, J. of the Amer. Math. Soc., 21 (2) 2008, 439-477.

[Sa] P. Sarnak, *Additive number theory and Maass forms,* in: Chudnovsky, D.V., Chudnovsky, G.V., Cohn, H., Nathatnson, M.B., (eds) Number Theory. Proceedings, New York 1982 (Lecture Notes Math. vol. 1052., pp. 286-309.), *Berlin, Heidelberg, New York, Springer*, 1982

[Sh1] T. Shintani, *On evaluation of zeta functions of totally real algebraic number fields at non-positive integers*, J. Fac. Sci. Univ. Tokyo 23 (1976), 393-417.

[Sh2] T. Shintani, *On special values of zeta functions of totally real algebraic number fields*, in: Proc. Internat. Cong. of Math., Helsinki, 1978, 591-597.

[S] L.J. Slater, Generalized hypergeometric functions, *Cambridge Univ. Press*, 1966

[So] K. Soundararajan, *Quantum unique ergodicity for* $SL(2,\mathbf{Z})\setminus H$, Ann. of Math. (2) 172 (2010), no 2, 1529-1538.

[St] H.M. Stark, *A complete determination of the complex quadratic fields of class-number one,* Michigan Math. J., 14 (1967), 1-27.

[Y] H.Yokoi, *Class number one problem for certain kind of real quadratic fields*, in: Proc. Internat. Conference, Katata, Japan (1986), 125-137., Nagoya Univ., Nagoya, 1986

[W] L.C. Washington, Introduction to Cyclotomic Fields, *Springer*, 1982

[Wat] T. Watson, Rankin triple products and quantum chaos, *Thesis, Princeton University, 2002*