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D.SC. THESIS

REPRESENTATION THEORY

BASED ON RELATIVIZED SET ALGEBRAS ORIGINATING FROM LOGIC

by

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Contents

Introduction

Representation theorems In this Thesis a new representation theory for algebraic logic is analised in which the representative structures are relativized set algebras instead of "ordinary" (square) set algebras. With this kind of representation we can associate Henkin-style semantics and completeness theorems in Logic, but the results in the Thesis are essential generalizations and extensions of Henkin's classical results. We deal with cylindric-type- and polyadic-type algebras, as algebraizations.

In the first Part of the Thesis we formulate representation theorems. As is known, in contrast with Boolean algebras, cylindric algebras are not representable *in the classical sense* in general (as isomorphic copies of cylindric set algebras in Gs_α or as subdirect products of cylindric set algebras). However, the celebrated Resek-Thompson-Andréka theorem states that if the system of cylindric axioms is extended by a new axiom schema, the merry-go-round property (MGR, for short, see Definition 1.8), and axiom (C_4) (the commutativity property of the cylindrifications) is weakened (see $(C_4)^*$), then the *cylindric–type* algebra obtained is representable by a *cylindric relativized set algebra* and, in particular, by a set algebra in D_α (instead of Gs_α). By an *r-representation* of a cylindric- or polyadic-type algebra we mean a representation by a cylindric- or polyadic-type *relativized* set algebra.

Upon analyzis of the merry-go-round property, it turns out, that in the background of this property the existence of a kind of transposition operator is. As is known, the general transposition operator cannot be introduced in every cylindric algebra (see [Fe07b]). These facts led to research into the representability of *transposition algebras* (TA_α). Transposition algebras are cylindric algebras extended by abstract transposition operators (p_{ij}) and single substitutions (s^i_j), $i, j < \alpha$ (Definition 2.3). They are weakening of the (so-called) finitary polyadic equality algebras introduced in [Sa-Th]. Furthermore, TA_α is definitionally equivalent to the non-commutative quasi-polyadic equality algebras (Theorem 3.6). Transposition algebras are not necessarily representable in the classical sense. However, it is proven that they are *r-representable* by relativized set algebras in Gwt_α (Theorem 2.8), where the unit V of a Gwt_α is of the form $\bigcup\limits_{k\in K} {}^{\alpha}U_k^{(p_k)}$ (see Definition 2.2). Approaching our topic from the starting point of the representative set algebras, this theorem says that the class Gwt_α is *first-order axiomatizable by a finite schema of equations and the axioms can be the* TA_α *axioms.* As is known, if the disjointness of the members ${}^{\alpha}U_k^{(p_k)}$ is assumed in the above decompositions of V, then the classical class Gws_α is obtained and this class is no longer first order finite schema axiomatizable (for classical representability, some additional non-first order conditions are needed, for example, the condition of local finiteness).

A next question is whether or not *polyadic equality algebras* are r- representable. Recall that polyadic algebras are essentially different from the quasi-polyadic algebras mentioned above: in the case of polyadic algebras the substitution operations are defined for real *infinite* transformations. The problem of r-representability of polyadic equality algebras is answered affirmatively here for a large class: polyadic equality algebras having *single* cylindrifications, called *cylindric polyadic equality algebras* (class CPE_α, Definition 3.17). Our representation theorem says that this class is r-representable by algebras in $\mathsf{Gp}_\alpha^{\mathrm{reg}}$ (Theorem 2.8). This is a kind of answer for the problem asked in [An-Go-Ne]: is the class G_α (the cylindric version of Gp_α) is a variety for infinite α? Furthermore, we prove that Halmos's result on the representability of locally finite, quasi-polyadic algebras ([Ha56]) can be generalized to m-quasi, locally–m cylindric polyadic algebras and r-representability, where m is infinite (Theorem 3.24).

The representant set algebras Gwt_α and $\mathsf{Gp}_\alpha^{\mathrm{reg}}$ related to the above r-representations are attractive and simple. The only difference between these kinds of set algebras and the classical Gws_α and Gs_α is the disjointness of the subunit components of the unit V.

These r-representation theorems may be regarded as immediate generalizations of the Stone representation theorem for Boolean algebras.

To briefly state *the techniques* used in the proofs of the r-representation theorems: three methods are used, all of them are known from the literature, but new ideas are needed for their applications here. The first one is the step-by-step method (this technique is closely related to the technique "games", see [Hi-Ho]). This technique is applied to prove the main r-representation theorems for cylindric-type algebras and transposition algebras. The other technique is the neat embedding technique. This method is applied to prove the representation theorems for cylindric polyadic equality algebras (when the previous technique cannot be used because of the infinite substitutions s_τ). This technique is based on the so-called "neat embedding theorems". Inside the neat embedding technique, we use the *ultrafilter technique* due to Tarski. Further, we use the technique of the translation from Algebra to Logic.

The concept of neatly embeddability is interesting in itself of course, discussion is dedicated to this concept here.

Neat embedding theorems In the second Part of the Thesis we deal with neat embedding theorems and their applications. Neat embedding is a concept of (universal) algebra (see Definition 4.2). The classical *neat embedding theorem* for cylindric algebras says that (classical) representability is equivalent to (classical) neat embeddability (see [He-Mo-Ta II.], 3.2.10). Neat embeddability may be considered as the *abstract algebraic characterization* of representability. On considering *cylindric-type* algebras, the question arises: is it possible to characterize the concept of r-representability of a cylindric-type algebra in terms of neat embeddability? The answer is affirmative (see Theorems 4.5 and 4.6). Of course, the concept of neat embeddability obtained in this way is different from the standard one. This is a neat embedding into a *many sorted* structure, where the axioms (C_4) and (C_6) are weakened, i.e., as an embedding class, a *larger class* than in the case of classical neat embedding is allowed.

The next question is whether this new kind of neat embedding theorem concerning cylindric-type algebras can be transfered to other structures, to transposition algebras, quasi-polyadic equality algebras or cylindric polyadic equality algebras. The answer depends on which particular class we are considering. The answer is obviously affirmative for transposition algebras and for quasi-polyadic equality algebras ([FePrepr]), but in the case of cylindric polyadic equality algebras, in the presence of infinite substitutions, the situation is essentially different. For polyadic equality algebras, as is known, there is *no* classical neat embedding theorem (neatly embeddability does not imply classical representability, see [He-Mo-Ta II.]). The question whether some kind of neat embedding theorem for polyadic equality algebras exists is a long standing problem. We prove a neat embedding theorem here for these kind of algebras and, in particular, for m-quasi, locally-m cylindric polyadic equality algebras (Theorem 6.2).

In order to apply neat embedding theorems to prove representability, we need *neatly embeddable classes* of algebras, of course. To meet this need the Daigneault-Monk-Keisler theorem ([Da-Mo]) and its variants is used.

There are remarkable connections between our subject and Logic. We mentioned that there is a close connection between r-representation theorems and Henkin-style completeness theorems in Logic, as well as between relativized set algebras and Henkin-style semantics. In terms of neat embeddability we prove a theorem concerning *conservative extensions* of provability relations (see Theorem 5.1). On considering the proof of the classical representation theorem of cylindric algebras in terms of neat embeddability and the resultant weakenings of the axioms (C_4) and (C_6), we can conclude that at proving the completeness of the respective Logic, we need only a part of the usual calculus (Theorem 4.19).

History The pioneer of the research discussed here is Leon Henkin. He introduced the concept of cylindric relativized set algebra (Crs_α), developed the merry-go-round properties, he was Resek's the doctoral advisor (Resek formulated her representation theorem concerning cylindric relativized set algebras in her PhD Thesis [Res]) and, he developed the famous completeness theorem in mathematical logic based on Henkin-style semantics.

The detailed research of the class Crs_α was initianed by István Németi. An extensive paper on Crs_α was published in [HMTAN] by Andréka and Németi. It was proved ([Nem81], [Nem86]) that Crs is a decidable variety, but it is not finite schema axiomatizable. In Crs the commutativity of the cylindrifications (axiom (C_4)) fails to be true. It was Németi who called attention to the importance of the commutativity of the cylindrification in the cylindric algebra theory, and proved that for the lack of this property implies decidability ([Nem86]). Some remarkable subclasses of Crs were investigated in a detailed way, e.g., the "locally square" set algebras G_α (see [Nem86], [Nem92], [An-Ne-Be]). There are many interesting applications of Crs. Crs may be considered as the algebraization of semantics of several non-classical logics, e.g., many-sorted, higher order, modal, etc. logics ([An-Ge-Ne], [An-Ne-Be]). Amongs of these, the most important is the so-called "guarded segment" ([An-Ne-Be], [Ben12], [Ben97], [Ben05]). It is a part of first order logic which corresponds to a kind of decidable, first order modal logic. This logic has remarkable applications in Computer Science.

Resek was the first to prove a *representation theorem* concerning *cylindric relativized set algebras*. She proved it for simple, complete and atomic cylindric algebras satisfying infinitely many merry-go-round equalities. This result was improved, in a sense, by Thompson and Andréka who reduced the infinitely many merry-go-round equalities to just two and replaced the cylindric axiom (C_4) by a weaker axiom. The theorem thus improved is called Resek-Thompson-Andréka theorem (RTA theorem, for short). Though the theorem was announced in [He-Mo-Ta II.], in Remark 3.2.88, a proof was only published in 1986 ([An-Th]). That proof is relatively short (in contrast with Resek's long proof) and it is based on the step-by-step method. Later, some variants of the RTA theorem have also found their way into the literature. Maddux has proved a somewhat stronger version of the theorem (see [Mad89]). He also investigated the problem of representation by relativized set algebras for relation algebras. The present author has published a simplification of the RTA theorem, replacing the axiom (C_4) with the commutativity of single substitutions (see

[Fe07a]). For some classes of relativized set algebras, the existence of axiomatizability and the fact of decidability was investigated (see [Sai],[An-Go-Ne], [Nem92] and [An-Ne-Be]). Andréka in [And] constructed a *concrete* axiom system, for finite dimensional G_α.

The results concerning r-representation of *polyadic-type* algebras are due to the present author. In [Fe11a] the connection between the merry-go-round properties and the operator transposition is investigated. In [Fe11a], the r-representation theorem is proven for transposition algebras (also for quasi-polyadic equality algebras). In [An-Fe-Ne] and [Fe11b], the r-representation theorem is proved for cylindric polyadic equality algebras. As regards neat embeddability of cylindric-type algebras, the present author has published neat embedding theorems for r-representation ([Fe10], [Fe00]). In [Fe09b] and [Fe09a], the logical applications of the topic are investigated.

Conclusions The question can be asked: considering the Resek -Thompson- Andréka theorem, which new ideas and aspects were developed after the publication of the theorem? This Thesis answers the question as follows:

1. An interesting aspect is that in the new representation theorems the representant classes are more attractive and simpler than the class D_α included in the RTA theorem. These classes (for example, Gwt_α or Gp_α) can easily be described and visualized geometrically. Furthermore, because of their simplicity, these representative classes are expected to be applied in different areas of mathematics (in set theory, measure theory, topology, etc.), similarly to the Stone theorem.

2. The cost of the r-representability of a cylindric-type or polyadic-type equality algebra by relativized set algebras is that *certain restrictions* of the classical structures of algebraic logic comes into the focus of research, for example, the MGR axioms for cylindric algebras or the assumption of the existence of a transposition operator. However, in order to obtain an elegant representation, certain axioms must be modified (weakened) a little in addition. A common feature of the algebras occurring in these theorems is that the commutativity of cylindrifications is *not required.* Instead of this, a weakening of it is assumed, for example,

the commutativity of single substitutions. Similarly, in the polyadic case, instead of the last two non diagonal axioms, certain weakenings are assumed.

3. The concept of r-representability (representability by relativized set algebras) can be characterized by a kind of neat embeddability, i.e., a kind of neat embedding theorem holds for r-representation. As is known, it is remarkable that no classical neat embedding theorem previously existed for polyadic equality algebra, i.e., classical representability could not be characterized by neat embeddability.

There are interesting applications of this new kind of neat embeddability, too. In terms of the new neat embedding theorems, we can prove r-representation theorems, e.g., Henkin's classical theorem on the representability of locally finite, quasi-polyadic algebras can be generalized to locally-m, m-quasi-algebras, where m is infinite or a new proof can be given for the RTA theorem.

4. There are remarkable logical aspects of the subject. For example, on proving the completeness of the logical calculus corresponding to cylindric algebras, it was realized that it is enough to use a part of the usual logical calculus. Neat embeddability has remarkable applications at conservative extensions of provability relations.

In the first Part, representation theorems concerning relativized set algebras are formulated. In the Chapters 1, 2 and 3 we deal with cylindric-type, transposition and cylindric polyadic-type algebras respectively. In the second Part, neat embeddability theorems are stated and their applications are investigated. In Chapter 4 we deal with the neat embeddability of cylindric-type algebras, in Chapter 5 the logical applications are considered and in Chapter 6 the cylindric polyadic-type case is discussed.

Part I

Representation theorems for cylindric and polyadic-type algebras, based on relativized set algebras

Chapter 1

Representation theorems for cylindric-type algebras

In this Chapter the celebrated Resek-Thompson-Andréka theorem is analysed, and, a variant of the theorem is claimed.

First, we recall the concepts of cylindric *relativized set* algebras:

Definition 1.1 (Crs_α)$\mathfrak{A}$ is a *cylindric relativized set algebra* of dimension α ($\alpha \geq 2$) with unit V if $\mathfrak{A}$ is of the form:

$$\langle A,\ \cup,\ \cap,\ \sim_V,\ 0,\ V,\ C_i^V,\ D_{ij}^V\ \rangle_{\ i,j<\alpha}$$

where the unit V is a set of α–termed sequences, such that $V \subseteq {}^\alpha U$ for some base set U, A is a non-empty set of subsets of V, closed under the Boolean operations $\cup$, $\cap$, $\sim_V$ and under the cylindrifications

$$C_i^V X = \{y \in V\ :\ y_u^i \in X \ \text{ for some } u\}$$

where $i < \alpha$, $X \in A$, and A contains the sets $\emptyset$, V and the diagonals

$$D_{ij}^V = \{y \in V \,:\, y_i = y_j\}$$

(see [He-Mo-Ta II.] 3.1.1).

Here the definition of y_u^i is $(y_u^i)_j = y_j$ if $j \neq i$, and $(y_u^i)_j = u$ if $j = i$. Another notation for y_u^i is $y(i \,/\, u)$. If y is the sequence of ordinals, then y_u^i is denoted by $[i \,/\, u]$ and it is called *elementary substitution*. The superscript, V is often omitted from the notations C_i^V and D_{ij}^V. We note that an algebra in Crs_α satisfies all the cylindric axioms, with the possible exception of the axioms (C_4) and (C_6) (see [He-Mo-Ta II.] 3.1.19).

Let us denote $C_i^V(D_{ij}^V \cap X)$ $(i \neq j)$ by ${}^V S_j^i X$. Notice that ${}^V S_j^i X =$ $\{y \in V : y \circ [i \,/\, j] \in X\}$, where $X \in A$. Here $y \circ [i \,/\, j] = y_{y_j}^i$, by definition. In this sense, if $\{y\} \in A$, then the elementary substitution $y_{y_j}^i$ can be defined in Crs_α in terms of ${}^V \mathsf{S}_j^i$.

Definition 1.2 (D_α) D_α is the subclass of Crs_α such that ${}^V S_j^i V = V$ for every $i, j \in \alpha$, where V is the unit of the algebra (see [An-Th]).

It is easy to check that in Crs_α the equality ${}^V S_j^i V = V$ and (C_6) are equivalent, thus D_α satisfies all the CA axioms with the possible exception of (C_4).

Definition 1.3 (G_α) G_α is a subclass of D_α, called the class of "*locally square*" cylindric set algebras , such that the unit V is of the form $\bigcup\limits_{k \in K} {}^\alpha U_k$ for some sets U_k, $k \in K$ (G_α was introduced in [Nem86]).

Recall that given a set U and a mapping $p \in {}^\alpha U$, the set

${}^\alpha U^{(p)} = \{x \in {}^\alpha U \,:\, x$ and p are different only in finitely many members$\}$

is called the *weak space* determined by p and U.

Definition 1.4 (Gw_α) It is a subclass of D_α such that the unit V is of the form $\bigcup_{k\in K} {}^\alpha U_k^{(p_k)}$ for some sets U_k, $k \in K$, and sequences $p_k \in {}^\alpha U_k$.

The difference between the classical class Gs_α ([He-Mo-Ta II.], 3.1.1) and G_α is that the disjointness for U_k's in G_α is not assumed. The difference between the classes Gws_α ([He-Mo-Ta II.], 3.1.1) and Gw_α is analogous.

Now, we define some *abstract* classes of algebras.

Definition 1.5 (CA_α) A Boolean algebra $\langle A, +, \cdot, -, 0, 1\rangle$ enriched with a set of additional unary operations c_i $(i < \alpha)$ and constants d_{ij} $(i, j < \alpha)$ is said to be a *cylindric algebra* $(\alpha \geq 2)$ of dimension α, if it satisfies the following axioms for every $i, j < \alpha$:

(C_1) $c_i 0 = 0$

(C_2) $x \leq c_i x$

(C_3) $c_i(x \cdot c_i y) = c_i x \cdot c_i y$

(C_4) $c_i c_j x = c_j c_i x$

(C_5) $d_{ii} = 1$

(C_6) $c_j(d_{ji} \cdot d_{jk}) = d_{ik} \qquad j \notin \{i, k\}$

(C_7) $d_{ij} \cdot c_i(d_{ij} \cdot x) = d_{ij} x \qquad i \neq j.$

An algebra is a *cylindric-type algebra* if its type is that of cylindric algebras.

If K is a class of algebras, then $\mathbf{I}\mathsf{K}$ denotes the class of the *isomorphic copies* of the members of K.

Definition 1.6 A cylindric–type algebra $\mathfrak{A}$ is *r-representable* if $\mathfrak{A} \in \mathbf{I}\mathsf{Crs}_\alpha$.

As is known, axiom (C_6) is equivalent to the set of the following four properties:

$$\begin{array}{llll} a. & d_{ij} = d_{ji} & b. & d_{ij} \cdot d_{jk} \le d_{ik} \\ c. & c_k d_{ij} = d_{ij} \quad k \notin \{i, j\} & d. & c_i d_{ij} = 1. \end{array} \tag{1.1}$$

Lemma 1.7 *The following propositions* (i) *and* (ii) *hold for every* $i, j < \alpha$:

(i) *If* $\mathfrak{A} \in \mathsf{Crs}_\alpha$, *then* $\mathfrak{A} \in \mathsf{D}_\alpha$ *if and only if* $x \in V$ *implies* $x \circ [i \ / \ j] \in V$.

(ii) *If* $\mathfrak{B}$ *is a cylindric–type algebra such that* $s^i_j 1 = 1$ *and* $\mathfrak{B}$ *is r-representable, then* $\mathfrak{B} \in \mathbf{I}\mathsf{D}_\alpha$.

Proof.

(i) The statement that $x \in V$ implies $x \circ [i \ / \ j] \in V$ means that $V \subseteq {}^V S^i_j V$. But ${}^V S^i_j V \subseteq V$ is always satisfied, thus ${}^V S^i_j V = V$. The latter together with $\mathfrak{A} \in \mathsf{Crs}_\alpha$ are equivalent to $\mathfrak{A} \in \mathsf{D}_\alpha$, by definition.

(ii) If h denotes an isomorphism between $\mathfrak{B}$ and an algebra in Crs_α, then $hs^i_j 1 = S^i_j h1$, where S^i_j is the abbreviation of ${}^V S^i_j$. But, in the previous equality, $hs^i_j 1 = h1 = V$, and $S^i_j h1 = S^i_j V$, i.e., $S^i_j V = V$.

qed.

The operator s^i_j (single substitution operator) is defined for the element x as $c_i(d_{ij} \cdot x)$ if $i \neq j$, and x if $i = j$.

Definition 1.8 The *merry-go-round properties* are:

$$s^k_i s^i_j s^j_k \, c_k x = s^k_j s^j_i s^i_k \, c_k x$$

$$s^k_i s^i_j s^j_m s^m_k \, c_k x = s^k_j s^j_m s^m_i s^i_k \, c_k x$$

for distinct ordinals i, j, k and n (see [He-Mo-Ta II.] 3.2.88). The two properties *together* are denoted by MGR (for an equivalent form of MGR, see (1.9)).

Definition 1.9 (CNA_α^-) The axioms of CNA_α^- ($\alpha \geq 4$) are obtained from the cylindric axioms so that the axiom (C_4) is replaced by the property

$$^-(C_4) : \; s_k^i s_m^j x = s_m^j s_k^i x \tag{1.2}$$

$i, k \notin \{j, m\}$.

Definition 1.10 (CNA_α^+) If the CNA_α^- axioms are extended by the MGR property, then the axioms of CNA_α^+ are obtained ([Fe07a]).

Definition 1.11 (NA_α^+) The axioms of NA_α^+ are obtained from those of the class CNA_α^+ ($\alpha \geq 4$) if the axiom $^-(C_4)$ is replaced by the axiom

$$(C_4)^* : \; d_{ik} \cdot c_i c_j x \;\leq\; c_j c_i x \tag{1.3}$$

([Fe07a] and Lemma 1.14 below).

Definition 1.12 (NA_α) The axioms of NA_α are obtained from those of CNA_α^- ($\alpha \geq 2$) if the axiom $^-(C_4)$ is replaced by $(C_4)^*$.

The following theorem is the main r-representation theorem for cylindric-type algebras in NA_α^+:

Theorem (Resek-Thompson-Andréka):

$$\mathfrak{A} \in \mathsf{NA}_\alpha^+ \;\; \textit{if and only if} \;\; \mathfrak{A} \in \; \mathbf{I}\mathsf{D}_\alpha.$$

where $\alpha \geq 4$ ([An-Th]).

In other words, the theorem says that the class D_α is *first-order axiomatizable by a finite schema of equations and the axioms can be the* NA_α^+ *axioms.* We note that, on modifying $(\mathrm{C}_4)^*$ and MGR a little, the theorem also remains true for $\alpha = 2$ and $\alpha = 3$ too.

If Σ is a set of formulas, let Mod Σ denote the *class of models satisfying* Σ.

Let CA_α^+ denote the class of cylindric algebras satisfying the MGR property. D_α satisfies (C_6), thus the following holds:

Corollary 1.13 $\mathfrak{A} \in \mathsf{CA}_\alpha^+$ *if and only if* $\mathfrak{A} \in \mathbf{I}(\mathsf{D}_\alpha \cap \text{Mod } (\mathrm{C}_4))$, $\alpha \geq 4$.

The lemma below lists some equivalents of $^-(\mathrm{C}_4)$. Let us denote by Σ the set of cylindric axioms except for (C_4) and let us assume that $\alpha \geq 4$.

Lemma 1.14 *Under* Σ *the following properties are equivalent*:

(i) $s_k^i s_m^j x = s_m^j s_k^i x$ (*property* $^-(\mathrm{C}_4)$)

(ii) $c_i s_m^j x \leq s_m^j c_i x$

(iii) $d_{ik} \cdot d_{jm} \cdot c_i c_j x = d_{jm} \cdot d_{ik} \cdot c_j c_i x$

(iv) $d_{ik} \cdot c_i c_j x \leq c_j c_i x$ (*property* $(\mathrm{C}_4)^*$)

where i, j, k *and* m *are different, except for* $k = m$ *maybe (see* [Fe07a] *and* [Tho]).

Proof.

A little more is proven than necessary, some pairwise equivalences are proven.

First, we prove the equivalences of (i) and (ii).

(ii)⇒(i). Substitute $x = d_{ik} \cdot y$ in (ii), we obtain: $c_i s_m^j (d_{ik} \cdot y) \leq s_m^j c_i (d_{ik} \cdot y)$. But (C_3) and (C_6)c. imply that $c_i s_m^j (d_{ik} \cdot y) = c_i (d_{ik} \cdot s_m^j y)$. This latter is $s_k^i s_m^j y$. Therefore $s_k^i s_m^j y = s_m^j s_k^i y$. By symmetry, we obtain (i).

(i)⇒(ii)

First $c_i\ c_i x = c_i x$ is proven.

$c_i\ c_i x = c_i\ (c_i x \cdot c_i x) = c_i\ c_i x \cdot c_i x \leq c_i x \leq c_i\ c_i x$ by (C_3) and (C_2).

On one hand,

$$s^j_m c_i x = s^j_m s^i_k c_i x = s^i_k s^j_m c_i x = c_i(d_{ik} \cdot s^j_m c_i x) \tag{1.4}$$

using (C$_6$)d. and condition (i). Applying c_i to both sides of (1.4) we obtain: $c_i s^j_m c_i x = c_i(c_i(d_{ik} \cdot s^j_m c_i x))$. Because of $c_i c_i x = c_i x$ and (1.4), we obtain:
$c_i s^j_m c_i x = c_i(c_i(d_{ik} \cdot s^j_m c_i x)) = c_i(d_{ik} \cdot s^j_m c_i x) = s^j_m c_i x$.

On the other hand, $c_i s^j_m x \leq c_i s^j_m c_i x$ is true (by monotonicity of c_i). Using that $c_i s^j_m c_i x = s^j_m c_i x$, we obtain (ii), i.e., $c_i s^j_m x \leq s^j_m c_i x$ is true in fact.

Then the equivalence of (i) and (iii) are proven.

Here the well-known operator t^i_j defined in cylindric algebras where $t^i_j x = d_{ij} \cdot c_i x$ if $i \neq j$ (and $t^i_j x = x$, if $i = j$) is used. Obviously, (iii) is equivalent to the property (iii)' below:

(iii)' $\qquad t^i_k t^j_m x = t^j_m t^i_k x.$

We prove the equivalence of (i) and (iii)'.

(i)⇒(iii)'. Under Σ the operators s^j_m and t^j_m are conjugates of each other in the Boolean algebraic sense, consequently if $\mathfrak{A} \vDash \Sigma$ then

a. $t^j_m s^j_m y \leq y$

b. $y \leq s^j_m t^j_m y$

for all $y \in A$ and $j, m \in \alpha$.

On one hand, it can be stated:

$$t^j_m t^i_k s^i_k (s^j_m t^i_k t^j_m x) \overset{a.}{\leq} (t^j_m s^j_m) t^i_k t^j_m x \overset{a.}{\leq} t^i_k t^j_m x. \tag{1.5}$$

That is, $t^j_m t^i_k s^i_k s^j_m y \leq t^j_m s^j_m y \leq y$. Let y be $t^i_k t^j_m x$, we obtain (1.5).

On the other hand,

$$t^j_m t^i_k s^j_m s^i_k t^i_k t^j_m x \geq t^j_m t^i_k x. \tag{1.6}$$

Namely, $s^j_m s^i_k t^i_k t^j_m y \overset{b.}{\geq} s^j_m t^j_m y \overset{b.}{\geq} y$. Let us apply the transformation $t^j_m t^i_k$ to this inequality and replace $y = x$, we obtain (1.6).

(i) implies that the left-hand sides of (1.5) and (1.6) coincide. Comparing (1.5) and (1.6) we obtain that

$$t^j_m t^i_k x \leq t^i_k t^j_m x.$$

By symmetry, $t^i_k t^j_m x = t^i_k t^j_m x$ follows.

The proof of (iii)'⇒(i) is completely similar: we swap s, t and swap $\leq$, $\geq$ throughout the proof.

The proof of equivalence of (iii) and (iv):

Instead of (iv) we use the property (iv)' below:

(iv)' $\quad d_{ik} \cdot c_i c_j x \leq d_{ik} \cdot c_j c_i x.$

Multiplying (iv) by d_{ik} we can see that (iv) is really equivalent to the property (iv)'.

(iv)' implies (iii), because by multiplying (iv)' by d_{jm} we obtain the one direction of (iii). By symmetry, the opposite inequality follows, too.

(iii) implies (iv)'. Apply the operation c_j to both sides of (iii). We obtain by (C_6)c., (C_6)d. and (C_3) that

$$d_{ik} \cdot s^j_m c_i c_j x = d_{ik} \cdot c_j c_i x. \tag{1.7}$$

Now we state that

$$d_{ik} \cdot c_i c_j x \leq d_{ik} \cdot s^j_m c_i c_j x. \tag{1.8}$$

We can use the property (ii) because the equivalence of (i) and (ii), and the equivalence of (i) and (iii) are proven above. Therefore by (ii), $c_i s^j_m(c_j x) \leq s^j_m c_i(c_j x)$. But $c_i s^j_m(c_j x) = c_i c_j x$ because (C_3) and $c_j d_{jm} = 1$. So $c_i c_j x \leq s^j_m c_i(c_j x)$. Multiplying this inequality by d_{ik}, (1.8) is obtained.

Comparing (1.7) and (1.8) we really obtain (iv)'.

qed.

Taking into consideration the previous lemma, the Resek-Thompson-Andéka theorem can be reformulated as follows (due to the present author, see [Fe07a], Corollary 3.2):

Theorem 1.15

$$\mathfrak{A} \in \mathsf{CNA}^+_\alpha \quad \textit{if and only if} \ \mathfrak{A} \in \ \mathbf{I}\mathsf{D}_\alpha.$$

where $\alpha \geq 4$.

***

We can ask the question: *what is the intuitive background of the merry-go-round properties* playing a key role in the Resek-Thompson-Andréka theorem?

Recall that by the *elementary transposition operator* $[i,\ j]$ we mean the operator changing i and j (in the sequence of ordinals).

Let us consider the operator ${}_k s(i,j)$ in CA_α, where ${}_k s(i,j)y = s^k_i s^i_j s^j_k y$ and i,j,k are different. The properties of ${}_k s(i,j)$ are investigated in detail in [He-Mo-Ta I.]1.5. Andréka and Thompson proved that the following property is equivalent to the two merry-go-round properties:

$${}_k s(i,j)_k s(j,m) c_k x = \ {}_k s(j,m)_k s(m,i) c_k x \tag{1.9}$$

under the other NA_α axioms if $k \notin \{i, j, m\}$, $j \notin \{m, i\}$ (Proposition 3 in [An-Th]). Elementary transposition, of course, satisfies (1.9).

(1.9) means that the cylindric algebra has a kind of "weak" abstract transposition operator (for the meaning of "weak", see [Fe11a]). Thus, the Resek-Thompson-Andréka theorem says that the existence of such an operator implies r-representability.

It is known that, in general, abstract transposition operators cannot be introduced in arbitrary cylindric algebra ([Fe07b]), and, likewise, the substitution operator s_τ for finite τ. For example, a sufficient condition for this is that the α-dimensional cylindric algebra is a "neat subreduct" of some $\alpha+2$-dimensional cylindric algebra (see in [Fe07b]).

***

In the first published proof of the Resek-Thompson-Andréka theorem due to Andréka (see [An-Th]), the so-called *step-by-step method* (or iteration method, see [Hi-Ho]) is applied to construct the suitable representation. We will refer to this proof in the next Chapter, therefore Andréka's proof is outlined below.

The proof of the non-trivial part of the theorem is decomposed into parts (Parts 1–4) so that the beginning of the original proof is cited almost word for word (Parts 1–3), while the remainder is only outlined (Part 4).

The sketch of the proof of the non-trivial part of the RTA *Theorem:*

Part 1 *About the framework of the proof.*

$\mathfrak{A}$ can be assumed to be atomic. Namely, by [He-Mo-Ta I.], 2.7.5, 2.7.13, every Boolean algebra $\mathfrak{B}$ with operators can be embedded into an atomic one such that all the equations valid in $\mathfrak{B}$, and in which "$-$" does not occur, continue to hold in the atomic one. This latter condition is satisfied in $\mathfrak{B}$ because it is easy to eliminate the "$-$" from the axioms. As a consequence, from now on $\mathfrak{A}$ is assumed to be atomic, satisfying the axioms.

Let At$\mathfrak{A}$ denote the set of all atoms of $\mathfrak{A}$. We want to "build" an isomorphism rep: $\mathfrak{A} \twoheadrightarrow \mathfrak{B}$, for some $\mathfrak{B} \in \mathrm{Crs}_\alpha$, such that the equality below holds:

$$\mathrm{rep}(x) = \bigcup \{\mathrm{rep}(a) : a \in \mathrm{At}\mathfrak{A},\ \mathrm{a} \leq x\} \text{ for every } x \in A. \tag{1.10}$$

Let V be a set of α-sequences and for every $X \subseteq V$ and $i, j < \alpha$ let $C_i X \stackrel{d}{=} \{f \in V : (\exists u)\ f(i \,/\, u) \in X\}$, $D_{ij} \stackrel{d}{=} \{f \in V : f_i = f_j\}$. Assume that rep: $A \to \{X : X \subseteq V\}$ is a function such that (1.10) holds. Then it is easy to check that rep is an isomorphism onto a $\mathfrak{B} \in \mathrm{Crs}_\alpha$ with $1^{\mathfrak{B}} \subseteq V$ if and only if conditions (i)$-$(v) below hold for every $a, b \in$ At$\mathfrak{A}$ and $i, j < \alpha$:

(i) $\mathrm{rep}(a) \cap \mathrm{rep}(b) = \emptyset$ if $a \neq b$

(ii) $\mathrm{rep}(a) \subseteq D_{ij}$ if $a \leq d_{ij}^{\mathfrak{A}}$ and $\mathrm{rep}(a) \cap D_{ij} = \emptyset$ if $a \cdot d_{ij}^{\mathfrak{A}} = 0$

(iii) $\mathrm{rep}(a) \subseteq C_i \mathrm{rep}(b)$ if $a \leq c_i^{\mathfrak{A}} b$,

(iv) $\mathrm{rep}(a) \cap C_i \mathrm{rep}(b) = \emptyset$ if $a \cdot c_i^{\mathfrak{A}} b = 0$

(v) $\mathrm{rep}(a) \neq \emptyset$.

$$. \tag{1.11}$$

A set V of α-sequences and a function rep with the above properties will be constructed, step by step.

Part 2 *About the 0th step.*

For every α-sequence f let ker $(f) \stackrel{d}{=} \left\{(i, j) \in {}^2\alpha : f_i = f_j\right\}$.

For every $a \in \mathrm{At}\mathfrak{A}$ let $\mathrm{Ker}(a) \stackrel{d}{=} \left\{(i, j) \in {}^2\alpha : a \leq d_{ij}^{\mathfrak{A}}\right\}$.

Then Ker(a) is an equivalence relation on α by the axioms (C_5)–(C_7). For every $a \in$ At$\mathfrak{A}$ let f_a be an α–sequence such that for every $a, b \in \mathrm{At}\mathfrak{A}$

$$\text{a) } \ker(f_a) = \mathrm{Ker}(a),\ \text{ b) } \mathrm{Rg}\ (f_a) \cap \mathrm{Rg}\ (f_b) = \emptyset \text{ if } a \neq b. \tag{1.12}$$

Such a system $\{f_a : a \in \mathrm{At}\mathfrak{A}\}$ of α-sequences does exist. Define

$$\mathrm{rep}_0(a) \stackrel{d}{=} \{f_a\}\,,\ \text{ for every } a \in \mathrm{At}\mathfrak{A}.$$

Then the function rep_0 satisfies conditions (i),(ii),(iv) and (v) but it does not satisfy condition (iii). Below, we shall make condition (iii) become true step by step, and later we shall check that conditions (i),(ii),(iv) and (v) remain true in each step.

Part 3 *About the (n+1)th step, i.e., about the definition of the function* rep_{n+1}.

Let $R = \mathrm{At}\mathfrak{A} \times \mathrm{At}\mathfrak{A} \times \alpha$, ρ be an ordinal and let $r : \rho \to R$ be an enumeration of R such that for all $n \in \rho$ and $(a, b, i) \in R$ there is $m \in \rho, m > n$ such that $r(m) = (a, b, i)$. Such ρ and r clearly exists.

Assume that $n \in \rho$ and $\mathrm{rep}_n : \mathrm{At}\mathfrak{A} \to \{X : X \subseteq V'\}$ is already defined where V' is a set of α-sequences. We define $\mathrm{rep}_{n+1} : \mathrm{At}\mathfrak{A} \to \{X : X \subseteq V\,"\}$, where $V\,"$ is a set of α-sequences. Let $r(n) = (a, b, i)$. If $a \nleq c_i b$, then

$$\mathrm{rep}_{n+1} \stackrel{d}{=} \mathrm{rep}_n. \tag{1.13}$$

Assume $a \leq c_i b$. Then $\mathrm{rep}_{n+1}(e) \stackrel{d}{=} \mathrm{rep}_n(e)$ for all $e \in \mathrm{At}\mathfrak{A}$, $e \neq b$.

Furthermore,

case 1. $b \leq d_{ij}$ for some $j < \alpha, j \neq i$. Then

$$\mathrm{rep}_{n+1}(b) = \mathrm{rep}_n(b) \cup \{f\ (\ i\ /\ f_j) : f \in \mathrm{rep}_n(a)\}\,. \tag{1.14}$$

case 2. $b \nleq d_{ij}$ for all $j < \alpha,\ j \neq i$. For every $f \in \mathrm{rep}_n(a)$ let u_f be such that

(i) $u_f \notin \bigcup\{\mathrm{Rg}\ (h) : h \in \bigcup\{\mathrm{rep}_n(e) : e \in \mathrm{At}\mathfrak{A}\}\}$

(ii) $u_f \neq u_h$ if $f \neq h,\ f, h \in \mathrm{rep}_n(a)$.

Now

$$\mathrm{rep}_{n+1}(b) = \mathrm{rep}_n(b) \cup \{f\ (i\ /\ u_f) : f \in \mathrm{rep}_n(a)\}\,. \tag{1.15}$$

Let $n \in \rho$ be a limit ordinal and assume that rep_m is defined for all $m < n$. Then

$$\mathrm{rep}_n(e) \stackrel{d}{=} \bigcup \{\mathrm{rep}_m(e) : m < n\} \tag{1.16}$$

for all $e \in \mathrm{At}\mathfrak{A}$.

By this, $\langle \mathrm{rep}_n : n \in \rho \rangle$ is defined. Now we define

$$\mathrm{rep}(a) \stackrel{d}{=} \bigcup \{\mathrm{rep}_n(a) : n \in \rho\} \tag{1.17}$$

for all $a \in \mathrm{At}\mathfrak{A}$. Let

$$V \stackrel{d}{=} \bigcup \{\mathrm{rep}(a) : a \in \mathrm{At}\mathfrak{A}\}\,. \tag{1.18}$$

We will check that conditions (i)–(v) hold for the above rep and V.

Part 4. *On the proof of the properties* (i)–(v).

They are proven by induction. The proof of the properties (ii), (iii) and (v) are relatively easy. Instead of (i) and (iv) a stronger property, denoted by (iv)', is proven such that it implies both (i) and (iv). In the proof of (iv)' Jónsson's famous theorem plays a key role ([He-Mo-Ta II.], 3.2.17, p. 68). It concerns the extension of a mapping, having certain fixed properties, from the elementary transformations $[i\ /\ j]$ and $[i,\ j]$, to arbitrary finite transformations.

End of the sketch of the proof.

Main references in this Chapter are: [An-Th], [And], [Fe07a], [An-Ne-Be], [Hi-Ho97, Hi-Ho97], [Ben12], [Nem86] and [Fe07b].

Chapter 2

Representation theorems for transposition algebras

In this Chapter the concept of transposition algebra is introduced. In the previous Chapter we noted that if a cylindric algebra has at least a weak transposition operator, then the algebra is r-representable. In accordance with this, the cylindric reduct of transposition algebras will be r-representable. Next, we investigate the problem whether or not the transposition algebras themselves are r-representable.

Definition 2.1 (Trs_α) The structure

$$\langle A,\ \cup,\ \cap,\ \sim_V,\ \emptyset,\ V,\ C_i^V,\ [i,\ j]^V,\ D_{ij}^V\ \rangle_{\ i,j<\alpha}$$

is a *transposition relativized set algebra,* if its cylindric reduct is in Crs_α, and $\mathfrak{A}$ is closed under $[i,\ j]^V$, where

$$[i,\ j]^V X = \{y \in V : y \circ [i,\ j] \in X\}\,.$$

Here $[i,\ j]$ denotes the elementary transposition.

The upper index V is often omitted from $[i,\ j]^V$ and, in this case we can disambiguate $[i,\ j]$ taking the context into consideration.

Notice that $[i,\ j]^V V = V$ in Trs_α. To see this, recall that $[i,\ j]^V V \subseteq V$, by definition. Now, let us apply $[i,\ j]^V$ to this inclusion. Then the equality $y \circ [i,\ j] \circ [i,\ j] = y$ implies that, for the left-hand side, $[i,\ j]^V\ [i,\ j]^V V = V$, and thus we obtain the opposite inclusion $V \subseteq [i,\ j]^V V$.

Definition 2.2 (Gwt_α) A set algebra $\mathfrak{A}$ in Trs_α is called a *generalized weak transposition relativized set algebra* ($\mathfrak{A} \in \mathsf{Gwt}_\alpha$) if there are sets $U_k,\ k \in K$ and sequences $p_k \in {}^\alpha U_k$ such that $V = \bigcup\limits_{k \in K} {}^\alpha U_k^{(p_k)}$, where V is the unit.

We can associate the cylindric set algebra class Gws_α with the class Gwt_α (see [He-Mo-Ta II.] 3.1.1). Besides their different types, a further difference between these classes is that the disjointness of the sets ${}^\alpha U_k^{(p_k)}$ is not assumed in Gwt_α. The subclass of Gwt_α in which this disjointness is assumed is denoted by $\dot{\mathsf{Gwt}}_\alpha$.

Now, we define some *abstract* classes of algebras.

Definition 2.3 (TA_α) A *transposition algebra of dimension* α ($\alpha \geq 3$) is the algebra

$$\mathfrak{A} = \langle A,\ +,\ \cdot,\ -,\ 0,\ 1,\ c_i,\ \ s_j^i,\ p_{ij},\ \ d_{ij}\rangle_{i,j<\alpha}$$

where $+$ and $\cdot$ are binary operations, $-$, c_i, s_j^i, p_{ij} are unary operations, d_{ij} are constants, and the axioms (F0–F11) below are assumed for every $i, j, k < \alpha$:

(F0) $\langle A,\ +,\ \cdot,\ -,\ 0,\ 1\rangle$ is a Boolean algebra, $\mathsf{s}_i^i = p_{ii} = \mathsf{d}_{ii} = Id \restriction A$ and $\mathsf{p}_{ij} = p_{ji}$

(F1) $x \leq c_i x$

(F2) $c_i(x + y) = c_i x +\ c_i y$

(F3) $s_j^i c_i x = c_i x$

(F4) $c_i s^i_j x = s^i_j x \;\; i \neq j$

(F5)* $s^i_j s^k_m x = s^k_m s^i_j x, \;\; if \;\; i, j \notin \{k, \, m\}$

(F6) s^i_j and p_{ij} are Boolean endomorphisms

(i.e., $s^i_j(-x) = -s^i_j x$, etc.)

(F7) $p_{ij} p_{ij} x = x$

(F8) $p_{ij} p_{ik} x = p_{jk} p_{ij} x$, if i, j, k are distinct

(F9) $p_{ij} s^i_j x = s^j_i x$

(F10) $s^i_j d_{ij} = 1$

(F11) $x \cdot d_{ij} \leq s^i_j x$.

Notice that axiom (F5)*is the same as $^-(\mathrm{C}_4)$ for cylindric algebras.

Definition 2.4 (TAS_α) The concept of *strong transposition algebra* can be obtained from that of transposition algebra TA_α, if the axiom (F5)* is changed by the stronger axiom

$$\text{(F5)}: \;\; s^i_j c_k x = c_k s^i_j x \;\; k \notin \{i, j\} \, .$$

The class TAS_α is the same as the class of *finitary polyadic equality algebras* (FPEA_α) introduced in [Sa-Th]. We preserve the notation of the axioms in [Sa-Th], but it seems expedient to change the terminology of FPEA_α, especially in the case of TA_α.

Definition 2.5 A transformation τ defined on α is called *finite* if $\tau i = i$ with finitely many exceptions ($i \in \alpha$). The notation of the set of finite transformations on α is FT_α.

By [Sa-Th] Theorem 1 (i), a substitution operator s_τ can be introduced in every FPEA_α so that the extended algebra is a quasi-polyadic equality algebra (see Definition 3.3). The existence of such a substitution operator s_τ holds for TA_α, too (instead of FPEA_α) namely, it is easy to check that the proof in [Sa-Th] works supposing (F5)* instead of (F5) (e.g., the inequality $s^k_j p_{ij} x \leq s^k_j p_{ij} s^i_k x$ in (16) on p. 553 there, follows from the TA_α axioms). Therefore throughout this Chapter we assume that the transposition algebras occurring here *are equipped with the operator* s_τ, where τ is finite. Further, s_τ is assumed to have the following properties for arbitrary finite transformations τ and λ and ordinals $i, j < \alpha$ (by [Sa-Th], p. 547):

$$s_{\tau\circ\lambda} = s_\tau \circ s_\lambda$$

$$p_{ij} = s_{[i,\, j]}$$

$$s^i_j = s_{[i\,/\,j]}$$

$$s_\tau d_{ij} = d_{\tau i\ \tau j}$$

$c_i s_\tau \leq s_\tau c_{\tau^{-1} i}$, where τ is finite permutation.

$$. \tag{2.1}$$

Definition 2.6 An algebra $\mathfrak{A}$ with the type of TA_α is *r-representable* if $\mathfrak{A} \in \mathbf{I}\mathsf{Trs}_\alpha$.

Lemma 2.7 *The following propositions* (i) *and* (ii) *hold*:

(i) *If* $\mathfrak{A} \in \mathsf{Trs}_\alpha$, *then* $\mathfrak{A} \in \mathsf{Gwt}_\alpha$ *if and only if* $x \in V$ *implies both* $x \circ [i,\, j] \in V$ *and* $x \circ [i\,/\,j] \in V$, *for every* $i, j < \alpha$.

(ii) *If* $\mathfrak{B} \in \mathsf{TA}_\alpha$ *and* $\mathfrak{B}$ *is r-representable, then* $\mathfrak{B} \in \mathbf{I}\mathsf{Gwt}_\alpha$.

Proof.

(i) If $\mathfrak{A} \in \mathsf{Gwt}_\alpha$, then, by the definition of V, V is closed under the operators $[i,\, j]$ and $[i\,/\,j]$. Conversely, we need to prove that V is of the form $\bigcup\limits_{k\in K} {}^\alpha U_k^{(p_k)}$. The condition implies that V is closed under the finite transformations of α, i.e., $x \in V$ implies $x \circ \tau \in V$ if τ is finite, since, as is known, finite transformations can be composed by finitely many applications of elementary transpositions and replacements. It can now be shown that V is of the form $\bigcup\limits_{x\in V} {}^\alpha(\mathrm{Rg}\ x)^{(x)}$ (this latter is really a Gwt_α unit). $V \subseteq \bigcup\limits_{x\in V} {}^\alpha(\mathrm{Rg}\ x)^{(x)}$ obviously holds by definition.

Conversely, if $y \in \bigcup_{x\in V} {}^{\alpha}(\mathrm{Rg}\ x)^{(x)}$, then $y = x \circ \tau$ for some $x \in V$ and finite τ, by the definition of the weak space ${}^{\alpha}(\mathrm{Rg}\ x)^{(x)}$. But, $x \circ \tau \in V$, by assumption. Thus $\bigcup_{x\in V} {}^{\alpha}(\mathrm{Rg}\ x)^{(x)} \subseteq V$ and, consequently, $V = \bigcup_{x\in V} {}^{\alpha}(\mathrm{Rg}\ x)^{(x)}$, as we claimed.

(ii) The proof is similar to that of Lemma 1.7 (ii), making use of the above part (i) and the fact that the isomorphism h, in question, preserves the operators s^i_j and p_{ij}.

qed.

The following main r-representation theorem holds for TA_α ([Fe11a], Theorem 3.1):

Theorem 2.8 (*Ferenczi*):

$$\mathfrak{A} \in \mathsf{TA}_\alpha \ \ \textit{if and only if} \ \ \mathfrak{A} \in \mathbf{I}\mathsf{Gwt}_\alpha$$

where $\alpha \geq 3$.

If we set out from the problem of the axiomatizability of the class Gwt_α of set algebras, then the reformulation of the theorem is the following one: *The class* Gwt_α *is first-order axiomatizable by a finite schema of equations and the axioms can be the* TA_α *axioms.*

Notice that Gwt_α is a canonical variety (see [HHGames], 2.69). Notice that he theorem above is valid also for finite α's, while, in general, the classical representation theorems are not.

By Definition 2.4, the class TAS_α is obtained from TA_α so that axiom (F5)* is replaced by the stronger (F5). Thus, the following is obtained:

Corollary 2.9 $\mathfrak{A} \in \mathsf{TAS}_\alpha$ *if and only if* $\mathfrak{A} \in \mathbf{I}(\mathsf{Gwt}_\alpha \cap$ Mod (F5)) ($\alpha \geq 3$).

As is known, TAS_α is not representable in the classical sense (see [Sa-Th]), thus Gwt_α cannot be replaced by $\overset{\bullet}{\mathsf{Gwt}}_\alpha$ in the Corollary and in Theorem 2.8.

The proof of Theorem 2.8 follows Andréka's proof (step-by-step method) for the Resek-Thompson-Andréka theorem (from now on, AP or the *cylindric case*), assuming some modifications in accordance with the transposition type of the algebras and some additional requirements. But, the proof is a non-trivial modification of Andréka's proof. Among others, a difference between the cylindric and transposition cases is that the definition of the function rep_0 is more complex in the transposition case. Here only the differences between the two proofs are emphasized, discussing the proof in accordance with the Parts 1–4 of the AP.

The proof of Theorem 2.8:

The following lemma states the easy part of the theorem:

Lemma 2.10 *If* $\mathfrak{A} \in \mathsf{Gwt}_\alpha$*, then* $\mathfrak{A} \in \mathsf{TA}_\alpha$*, where* $\alpha \geq 4$.

Proof.

We assume that $\mathfrak{A} \in \mathsf{Gwt}_\alpha$ and we need to check the axioms (F1)–(F11). As examples we check the axioms (F4), (F9) and (F10):

Axiom (F4): $c_i s^i_j x = s^i_j x \;\; i \neq j$.

$z \in C_i S^i_j X \Leftrightarrow z^i_u \in S^i_j X$ for some $u \Leftrightarrow z^i_{z_j} \in X$.

$z \in S^i_j X \Leftrightarrow z^i_{z_j} \in X$.

Axiom (F9): $p_{ij} s^i_j x = s^j_i x$.

$z \in [i,\ j]\ S^i_j X \Leftrightarrow z \circ [i,\ j] \in S^i_j X \Leftrightarrow z^j_{z_i} \in X$.

$z \in S^j_i X \Leftrightarrow z^j_{z_i} \in X$.

Axiom (F10): $s^i_j d_{ij} = 1$.

We show that $z \in V$ implies $z \in S^i_j D_{ij}$. Namely if $z \in V$, then $z^i_{z_j} \in V$ by the definition of a Gwt unit V. But this implies that $z^i_{z_j} \in D_{ij}$, i.e., $z \in S^i_j D_{ij}$.

qed.

First, let us consider the *framework* (Part 1) of Andréka and Thompson's proof in Chapter 1. *On the modification of that framework:*

The only necessary change is that a property (vi) is needed which states the preservation of the operator p_{ij}. By (2.1) p_{ij} may be considered as $s_{[i,\ j]}$. We will use $s_{[i,\ j]}$ rather than p_{ij}. So we need to prove:

$$\text{(vi) rep}(s_{[i,\ j]}\, a) = [i,\ j]\,\text{rep}(a).$$

We will prove the following more general property

$$\text{(vi)' rep}(s_\sigma a) = S_\sigma \text{rep}(a) \tag{2.2}$$

where σ is an arbitrary finite permutation on α.

We note that the original representation is *complete* (see (1.10)), and this will also be transmitted to our construction.

The next part (Part 2) of the original proof is the *definition of the 0th step*, i.e., the definition of the function rep_0..

We need to essentially change the definition of rep_0 to handle property (vi)'.

First, as a preparation, we introduce two equivalence relations:

1. Let a be an arbitrary *fixed* atom. The definition of the relation $\equiv_a$ ($\equiv$, for short) on α is:

$$i \equiv j \text{ if and only if } s_{[i,\ j]}a = a. \tag{2.3}$$

$\equiv$ is an equivalence relation. For example, if $i \equiv j$ and $j \equiv k$, then $i \equiv k$, because $s_{[i,\,j]}a = a$ and $s_{[j,\,k]}a = a$ imply $s_{[i,\,k]}a = a$. Namely, by (2.1*),* $[i,\ k] = [i,\ j] \circ [j,\ k] \circ [i,\ j]$ implies that $s_{[i,\,k]}a = (s_{[i,\,j]} \circ s_{[j,\,k]} \circ s_{[i,\,j]})a$.

Notice that

$$(i, j) \in \text{ Ker}(a) \text{ implies that } i \equiv j. \tag{2.4}$$

Namely, $a \le d_{ij}$ implies that $a = s_{[i,\,j]}a$. (C_7) is equivalent to $(\mathrm{C}_7)^* : d_{ij} \cdot c_i(d_{ij} \cdot x) = d_{ij} \cdot x$. If $x = a$, then $a \le d_{ij}$ implies that $d_{ij} \cdot c_i a = a$. Applying $s_{[i,\ j]}$ to this equality we obtain that $s_{[i,\ j]}(d_{ij} \cdot c_i a) = s_{[i,\ j]}a$, i.e., $d_{ji} \cdot s_i^j(c_i a) = s_{[i,\ j]}a$.

Replacing $c_i a$ for x in $(\mathrm{C}_7)^*$ and changing i and j we obtain that $d_{ji} \cdot c_j(d_{ij} \cdot c_i a) = d_{ji} \cdot c_i a$, i.e., $d_{ji} \cdot s_i^j c_i a = d_{ji} \cdot c_i a$. Comparing this equality with $d_{ij} \cdot c_i a = a$ and with $d_{ji} \cdot s_i^j(c_i a) = s_{[i,\ j]}a$ we obtain that $a = s_{[i,\,j]}a$.

2. Let us consider the following equivalence relation $\sim$ on $\mathrm{At}\mathfrak{A}$:

$$a \sim b \text{ if and only if } b = s_\tau a \text{ for some finite permutation } \tau \tag{2.5}$$

$a, b \in \mathrm{At}\mathfrak{A}$.

In fact, the relation $\sim$ is an *equivalence relation*: it is reflexive because $a = s_I a$. It is symmetrical because $b = s_\tau a$ implies $s_{\tau^{-1}}b = a$. It is transitive because $b = s_\tau a$ and $c = s_\sigma b$ imply that $c = s_\sigma(s_\tau a) = s_{\sigma\circ\tau}a$, where $\sigma \circ \tau$ is also a finite permutation.

Let us choose and fix representative points for the equivalence classes concerning $\sim$ and let Rp denote this fixed set of representative points.

We define the function rep_0 :

Definition 2.11 If $c \in Rp$, then let

$$\mathrm{rep}_0(c) = \{S_\tau f_c : s_\tau c = c\} \tag{2.6}$$

where f_c is the sequence defined in the original proof and τ is a finite permutation on α. If $b = s_\sigma c$, then let

$$\mathrm{rep}_0(b) = S_\sigma \ \mathrm{rep}_0(c). \tag{2.7}$$

Lemma 2.12 *The above definition is unique.*

Proof.

It must be proved that if

$$s_\tau c = s_\sigma c \tag{2.8}$$

for some $c \in Rp$ and finite permutations τ and σ, then

$$\mathrm{rep}_0(s_\tau c) = \mathrm{rep}_0(s_\sigma c). \tag{2.9}$$

(2.8) is equivalent to $c = (s_{\tau^{-1}} \circ s_\sigma)c = s_{\tau^{-1} \circ \sigma} c$, so is equivalent to

$$c = s_\beta c \tag{2.10}$$

where $\beta = \tau^{-1} \circ \sigma$. Similarly, using (2.7), (2.9) is equivalent to

$$\mathrm{rep}_0(c) = S_\beta\ \mathrm{rep}_0(c). \tag{2.11}$$

By (2.6), (2.11) is equivalent to $\{S_{\tau_1} f_c : s_{\tau_1} c = c\} = S_\beta \{S_{\tau_2} f_c : s_{\tau_2} c = c\}$.

But $S_\beta \{S_{\tau_2} f_c : s_{\tau_2} c = c\} = \{(S_\beta\ S_{\tau_2}) f_c : s_{\tau_2} c = c\}$. So it must be proved that

$$\{S_{\tau_1} f_c : s_{\tau_1} c = c\} = \{(S_\beta S_{\tau_2}) f_c : s_{\tau_2} c = c\}. \tag{2.12}$$

We show that the left-hand side of (2.12) is a subset of the right-hand side and conversely. Assume that $S_\tau f_c \in \{S_{\tau_1} f_c : s_{\tau_1} c = c\}$ for some fixed $\tau_1 = \tau$. Then let us choose $\beta^{-1} \circ \tau$ on the right-hand side for τ_2. We need to prove that $s_{\beta^{-1}\circ\tau} c = c$. But $s_{\beta^{-1}\circ\tau} c = (s_{\beta^{-1}} s_\tau) c = s_{\beta^{-1}}(s_\tau c)$. $s_\tau c = c$ by condition and $s_{\beta^{-1}} c = c$ by (2.10). So, really $s_{\beta^{-1}\circ\tau} c = c$. The proof of the converse inclusion in (2.12) is completely similar.

qed.

Lemma 2.13 $\mathrm{rep}_0(s_\sigma a) = S_\sigma\ \mathrm{rep}_0(a)$, *where σ is an arbitrary finite permutation on α and a is an arbitrary atom, i.e., the property* (vi)*' in (2.2) is satisfied.*

Proof.

We need to prove that (2.7) is true for arbitrary atoms b and a with $b = s_\sigma a$, not only for representative points c, i.e., we need to prove that

$$\mathrm{rep}_0(b) = S_\sigma \mathrm{rep}_0(a). \tag{2.13}$$

Namely if the representative point representing a is c and $a = s_\tau c$ for τ, then $\mathrm{rep}_0(b) = \mathrm{rep}_0(s_\sigma a) = \mathrm{rep}_0(s_\sigma s_\tau c) = \mathrm{rep}_0(s_{\sigma\circ\tau} c)$. But $\mathrm{rep}_0(s_{\sigma\circ\tau} c) =$
$= S_{\sigma\circ\tau} \mathrm{rep}_0(c)$ by (2.7). $S_{\sigma\circ\tau} \mathrm{rep}_0(c) = (S_\sigma S_\tau) \mathrm{rep}_0(c) = S_\sigma \mathrm{rep}_0(s_\tau c) = S_\sigma \mathrm{rep}_0(a)$ by (2.7). So, really $\mathrm{rep}_0(b) = S_\sigma \mathrm{rep}_0(a)$ and the proof is complete.

qed.

Similarly to the original proof, we show that rep_0 satisfies the conditions (i), (ii) and (iv) in (1.11). The proof requires a bit more complex consideration than the original proof.

Lemma 2.14 $\mathrm{rep}_0(a) \cap \mathrm{rep}_0(b) = \emptyset$ *if* $a \neq b \;\; a, b \in \mathrm{At}\mathfrak{A}$, *i.e., the property* (i) *in (1.11) is true.*

Proof.

If $a \nsim b$ and $a = s_\sigma c,\ b = s_\lambda d$ for some $c, d \in Rp$ and finite permutations σ and λ, then the condition b) $\mathrm{Rg}(f_c) \cap \mathrm{Rg}(f_d) = \emptyset$ in (1.12) and (2.6) imply that $\mathrm{rep}(a) \cap \mathrm{rep}(b) = \emptyset$.

Assume that $a \sim b$ and $a = s_\sigma c,\ b = s_\eta c$ for some $c \in Rp$ and finite permutations σ and η, $a \neq b$, i.e., $s_\sigma c \neq s_\eta c$. We need to prove that
$\mathrm{rep}_0(s_\sigma c) \cap \mathrm{rep}_0(s_\eta c) = \emptyset$.

Indirectly, assume that $\mathrm{rep}_0(s_\sigma c) \cap \mathrm{rep}_0(s_\eta c) \neq \emptyset$. We show that $a = b$, i.e., $s_\sigma c = s_\eta c$ and this contradicts the condition $a = b$.

Taking into consideration (2.7) we obtain that

$S_\sigma \left\{S_{\tau_1} f_c : s_{\tau_1} c = c\right\} \cap S_\eta \left\{S_{\tau_2} f_c : s_{\tau_2} c = c\right\} \neq \emptyset$,

i.e., $S_\sigma S_{\tau_1} f_c = S_\eta S_{\tau_2} f_c$ for some finite permutations τ_1 and τ_2. This latter equality is equivalent to $f_c = S_{\tau_1^{-1}} S_{\sigma^{-1}} S_\eta S_{\tau_2} f_c$, i.e., to $f_c = S_{\tau_1^{-1} \circ \sigma^{-1} \circ \eta \circ \tau_2}\ f_c$. Let γ denote the permutation $\tau_1^{-1} \circ \sigma^{-1} \circ \eta \circ \tau_2$, then $f_c = S_\gamma f_c$.

Using that $s_{\tau_1} c = c$ and $s_{\tau_2} c = c$, $a = b$ (i.e., $s_\sigma c = s_\eta c$) is equivalent to $s_\sigma s_{\tau_1} c = s_\eta s_{\tau_2} c$. Similarly to the equivalences above, this latter equality is equivalent to $c = s_{\tau_1^{-1} \circ \sigma^{-1} \circ \eta \circ \tau_2} c$, i.e., to $c = s_\gamma c$.

Generally, we prove that

$$f_c = S_\beta f_c \text{ implies } c = s_\beta c, \tag{2.14}$$

where β is an arbitrary finite permutation on α.

We remind the reader that every finite permutation β can be formulated as a composition of finitely many cyclic permutations. Further, a cyclic permutation δ of length n can be formulated as a composition $\left[\delta^{n-1}i,\ \delta^{n}i\right] \circ ... \circ \left[\delta i,\ \delta^{2}i\right] \circ [i,\ \delta i]$ of transpositions, where $\delta^n i = i$. This obviously implies that every finite permutation β can be formulated as a finitely many compositions of transpositions of the form $[j,\ \beta j]$. To prove (2.14) let us decompose β in this form:

$$\beta = [j_m,\ \beta j_m] \circ ... \circ [j_2,\ \beta j_2] \circ [j_1,\ \beta j_1]\,. \tag{2.15}$$

$f_c = S_\beta f_c$ implies that $(f_c)_j = (S_\beta f_c)_j$ for every $j < \alpha$. Therefore $(S_\beta f_c)_j = (f_c)_{\beta^{-1}j}$. Therefore $(f_c)_j = (f_c)_{\beta^{-1}j}$ for every $j < \alpha$. Here β is an arbitrary finite permutation, so it can be written that $(f_c)_{\beta i} = (f_c)_i$ for arbitrary $j < \alpha$ and permutation β. This latter is equivalent to $(f_c)_{\beta j} = (f_c)_j$. This means that $(j, \beta j) \in \ker(f)$. The property a) in (1.12), i.e., Ker(c)=ker(f_c) implies that $(j, \beta j)$ $\in$Ker(c). (2.4) implies that $j \equiv \beta j$, i.e.,

$$s_{[j,\ \beta j]}c = c \tag{2.16}$$

for every $j < \alpha$.

Applying (2.15) we obtain that $s_\beta c = s_{[j_m,\ \beta j_m]\circ ... \circ[j_2,\ \beta j_2]\circ[j_1,\ \beta j_1]}c =$
$= s_{[j_m,\ \beta j_m]} ... s_{[j_2,\ \beta j_2]} s_{[j_1,\ \beta j_1]} c$. Using (2.16) step by step, we obtain that

$$s_\beta c = c \tag{2.17}$$

and (2.14) is proven.

Applying (2.14) to the transformation γ we obtain a contradiction and the proof is complete.

qed.

Lemma 2.15 $\mathrm{rep}_0(a) \subseteq D_{ij}$ *if* $a \leq d_{ij}$ *and* $\mathrm{rep}_0(a) \cap D_{ij} = \emptyset$ *if* $a \cdot d_{ij} = 0$ *for every* $i, j < \alpha$, *i.e., the property* (ii) *in (1.11) is true.*

Proof.

First we prove that $\mathrm{rep}_0(a) \subseteq D_{ij}$ if $a \leq d_{ij}$.

Assume that $a = s_\sigma c$ for some $c \in Rp$. Then the condition $a \leq d_{ij}$ is of the form $s_\sigma c \leq d_{ij}$. By (2.7) we need to prove that

$$S_\sigma \{S_\tau f_c : s_\tau c = c\} \subseteq D_{ij} \tag{2.18}$$

i.e.,

$$(S_\sigma S_\tau f_c)_i = (S_\sigma S_\tau f_c)_j. \tag{2.19}$$

Let us consider the following equivalences:

$(S_\sigma S_\tau f_c)_i = (S_\sigma S_\tau f_c)_j \Leftrightarrow (S_{\sigma\circ\tau} f_c)_i =$

$= (S_{\sigma\circ\tau} f_c)_j \Leftrightarrow (f_c)_{(\sigma\circ\tau)^{-1}\, i} = (f_c)_{(\sigma\circ\tau)^{-1}\, j} \Leftrightarrow (\lambda^{-1} i, \lambda^{-1} j) \in \ker(f_c)$=$\mathrm{Ker}(c)$ by (1.12) , where λ denotes the permutation $\sigma \circ \tau$.

$(\lambda^{-1} i, \lambda^{-1} j) \in \mathrm{Ker}(c)$ means that $c \leq d_{\lambda^{-1} i\, \lambda^{-1} j}$. Applying s_λ to this inequality, we obtain that $s_\lambda c \leq d_{ij}$. So (2.19) is equivalent to

$$s_\lambda c \leq d_{ij}. \tag{2.20}$$

But $s_\lambda c = s_{\sigma\circ\tau} c = s_\sigma s_\tau c = s_\sigma c = a$ using the fact that $s_\tau c = c$ in (2.18) and $a = s_\sigma c$. Therefore the condition $s_\sigma c \leq d_{ij}$ is equivalent to $s_\lambda c \leq d_{ij}$.

Applying the equivalences above we obtain that, $a = s_\sigma c \leq d_{ij}$ implies (2.18).

The other case: we need to prove that $\mathrm{rep}_0(a) \cap D_{ij} = \emptyset$ if $a \cdot d_{ij} = 0$. Indirectly, assume that $\mathrm{rep}(a) \cap D_{ij} \neq \emptyset$ for some $i, j < \alpha$. Similarly to the first case, (2.19) is true for some $i, j < \alpha$. $(S_\sigma S_\tau f_c)_i = (S_\sigma S_\tau f_c)_j$ for *some* finite permutation τ and $i, j < \alpha$. By the argument above, $a \le d_{ij}$ follows and this contradicts the condition $a \cdot d_{ij} = 0$.

qed.

Lemma 2.16 $\mathrm{rep}_0(a) \cap C_i\mathrm{rep}_0(b) = \emptyset$ *if* $a \cdot c_i^{\mathfrak{A}} b = 0$, *i.e., the property* (iv) *in* (*1.11*) *is true.*

Proof.

If $a \nsim b$, then (2.7) and (1.12) b) imply that $\mathrm{rep}_0(a) \cap C_i\mathrm{rep}_0(b) = \emptyset$.

Assume that $a \sim b$ ($a \ne b$) and the representative point is c, so $a = s_\sigma c$ and $b = s_\eta c$ for some permutations σ and η.

Indirectly, assume that

$$\mathrm{rep}_0(a) \cap C_i\mathrm{rep}_0(b) \ne \emptyset. \tag{2.21}$$

This means that there exists a $g \in \mathrm{rep}_0(b)$ such that $g_u^i \in \mathrm{rep}_0(a)$ for some u. By (2.7), this means that $g = S_\eta S_{\tau_2} f_c$ for some τ_2 and $g_u^i = S_\sigma S_{\tau_1} f_c$ for some τ_1, that is, $(S_\eta S_{\tau_2} f_c)_u^i = S_\sigma S_{\tau_1} f_c$. Therefore

$$f_c = S_{\tau_1^{-1}} S_{\sigma^{-1}} (S_\eta S_{\tau_2} f_c)_u^i = (S_{\tau_1^{-1}} S_{\sigma^{-1}} S_\eta S_{\tau_2} f_c)_u^{\delta i}, \tag{2.22}$$

where $\delta = \tau_1^{-1} \circ \sigma^{-1}$.

Let β denote the permutation $\tau_1^{-1} \circ \sigma^{-1} \circ \eta \circ \tau_2$. Then $f_c = (S_\beta f_c)_u^{\delta i}$ implies that $(f_c)_j = (S_\beta f_c)_j$ for every $j \ne \delta i$, i.e.,

$$(f_c)_j = (f_c)_{\beta^{-1} j}, \text{ if } j \ne \delta i. \tag{2.23}$$

Let us consider the decomposition of β^{-1} being analogous with (2.15): $\beta^{-1} = \left[k_m,\ \beta^{-1}k_m\right] \circ ... \circ \left[k_2,\ \beta^{-1}k_2\right] \circ \left[k_1,\ \beta^{-1}k_1\right]$, where $k_m = \delta i$ can be assumed without loss of generality. Similarly to (2.17) we obtain that

$$s_{\beta^{-1}}c = s_{[\delta i,\ \beta^{-1}(\delta i)]}c. \tag{2.24}$$

By definition of β, $\beta^{-1} = \tau_2^{-1} \circ\ \eta^{-1} \circ\ \sigma \circ\ \tau_1$, therefore $s_{\beta^{-1}}c =$

$= s_{\tau_2^{-1} \circ \eta^{-1}}(s_{\sigma \circ \tau_1}c) = s_{\tau_2^{-1} \circ \eta^{-1}}a$ because $s_{\tau_1}c = c$ and $s_\sigma = a$.

(2.24) implies that $s_{\tau_2^{-1} \circ \eta^{-1}}a = s_{[\delta i,\ \beta^{-1}(\delta i)]}c$. Applying $s_{\eta \circ \tau_2}$ to this equality we obtain that

$$a = s_{\eta \circ \tau_2} s_{[\delta i,\ \beta^{-1}(\delta i)]}c = s_{[(\eta \circ \tau_2 \circ \delta)i\ ,\ (\eta \circ \tau_2 \circ \beta^{-1} \circ \delta)i]} s_{\eta \circ \tau_2}c \tag{2.25}$$

The second equality follows from $(\eta \circ \tau_2) \circ \left[\delta,\ \beta^{-1} \circ \delta\right] = [\eta \circ \tau_2 \circ \delta,\ \eta \circ \tau_2 \circ \beta^{-1} \circ \delta] \circ$ $(\ \eta \circ \tau_2)$ on α and from the property $s_{\tau \circ \lambda} = s_\tau \circ s_\lambda$ in (2.1) applying it to both sides.

The transformation $\eta \circ \tau_2 \circ \beta^{-1} \circ \delta$ is the identity, namely $\eta \circ \tau_2 \circ \beta^{-1} \circ \delta = \eta \circ \tau_2 \circ \tau_2^{-1} \circ \eta^{-1} \circ \sigma \circ \tau_1 \circ \tau_1^{-1} \circ \sigma^{-1} = I$. So (2.25), $c = s_{\tau_2}$ and $b = s_\eta c$ imply that

$$a = s_{[m,\ i]}b \tag{2.26}$$

where m denotes $(\eta \circ \tau_2 \circ \delta)\ i$.

(2.13) and (2.26) imply that $\text{rep}_0(a) = [m,\ i]\ \text{rep}_0(b)$.

Notice that $a = s_{[m,\ i]}b$, $a \neq b$ and Lemma 2.14 imply that

$$[m,\ i]\,\mathrm{rep}_0(b) \cap \mathrm{rep}_0(b) = \emptyset. \tag{2.27}$$

Further, the (indirect) condition in (2.21) is of the form

$$[m,\ i]\,\mathrm{rep}_0(b) \cap C_i \mathrm{rep}_0(b) \neq \emptyset.$$

Then, on one hand, there exists a $g \in \mathrm{rep}_0(b)$ such that $g \circ [m,\ i] \in \mathrm{C}_i\ \mathrm{rep}_0(b)$, i.e., $(g \circ [m,\ i])^i_w \in \mathrm{rep}_0(b)$, for some w. Let h denote the sequence $(g \circ [m,\ i])^i_w$.

On the other hand, both g and h are elements of rep(b) so (2.6) implies that both of them are finite permutations of the representative sequence f_c. Therefore they are finite permutations of each others too, for example, let $h = S_\tau g$ for some finite permutation τ. If $\tau k = i$, then $g_k = w$. For the sake of simplicity let us consider here the finite permutation τ to be defined on some finite subset of α.

Let us denote g_i and g_m by u and v, so $g_i = u$, $g_m = v$ and $g_k = w$. Then $h = (g \circ [m,\ i])^i_w$, $h_i = w, h_m = u$ and

$$g_j = h_j \tag{2.28}$$

for every $j \notin \{i, m\}$. We state that the expected finite permutation τ between the sequences g and h, having the above properties, cannot exist.

The problem, in question, will now be discussed. First notice that $u \neq v$. Namely $u = v$ implies $g = g \circ [m,\ i]$, so $g \circ [m,\ i] \in \mathrm{rep}_0(b)$ and this contradicts (2.27). For similar reason, $u \neq w$.

First, let us consider the case $v \neq w$. We show that this case is impossible. Assume that $\tau m = t$ for some $t < \alpha$. Then $h_t = v$, $v \notin \{u, w\}$ imply that $t \notin \{i, m\}$. $h_t = v$ and

$g_t = h_t$ imply that $g_t = v$ by (2.28). Assume that $\tau t = p$ for some $p < \alpha$. $u \neq v$ and the τ is finite permutation, therefore $p \notin \{i, m, t\}$. Similarly to the previous step, we obtain that $g_p = h_p = v$. τ is a *finite* permutation so there are only finitely many n_j and q_j such that $\tau n_j = q_j$ and $g_{n_j} = h_{q_j} = g_{q_j} = v$. Let q_n be the last q_j with this property in this sequence. τ is a permutation, so $\tau q_n = i$ or $\tau q_n = m$. Therefore $g_{q_n} = v$ implies that $h_i = v$ or $h_m = v$ which contradicts the conditions $h_i = w$, $h_m = u$ and $v \notin \{u, w\}$.

If $v = w$, then $h \circ [m, i] = h$, i.e., $g \circ [m, i] = g$. This contradicts (2.27), so the original proposition is true.

qed.

Notice that in the 0th step, similarly to the original proof, the condition $a \cdot c_i^{\mathfrak{A}} b = 0$ is not used.

As regards the (n+1)*th step* of the proof**,** i.e., the definition of the function rep_{n+1}, let us consider Andréka's proof again (see Part 3 in the proof). *The modified construction is:*

In order to assure the validity of the property (vi)' in (2.2), the original construction is modified. Here equivalence classes of triples are considered instead of single triples. From the point of view of the original proof, this means that the single triples are classified according to an equivalence relation to be introduced.

The original construction uses an *arbitrary fixed* free transfinite enumeration of the (a, b, i) triples, where $a, b \in \mathrm{At}\mathfrak{A}$, $i < \alpha$. In contrast with this, certain restrictions for this enumeration will be assumed, and the triples will be classified in a sense. The function rep_n will be defined in accordance with this classification. Beyond this small change, the original procedure is *not* changed, so the original proof works. We shall prove that property (vi)' in (2.2) is *preserved* in every step.

Let us consider the following relation $\approx$ on R :

$$(a_1, b_1, i_1) \approx (a_2, b_2, i_2) \text{ if and only if } a_2 = s_\sigma a_1, b_2 = s_\sigma b_1, i_2 = \sigma i_1 \qquad (2.29)$$

for some permutation σ. $\approx$ is obviously an equivalence relation. Let us fix representative points in the equivalence classes and denote by R' the class of the representative points.

We note that the relation $\approx$ preserves the inequalities $a \leq c_i b$ and $b \leq d_{ij}$ in the following sense: if $(a_1, b_1, i_1) \approx (a_2, b_2, i_2)$, then

$$a_1 \leq c_{i_1} b_1 \text{ if and only if } a_2 \leq c_{i_2} b_2 \qquad (2.30)$$

and

$$b \leq d_{ij} \text{ if and only if } s_\sigma b \leq d_{\sigma i\ \sigma j}. \qquad (2.31)$$

Namely, if $a_1 \leq c_{i_1} b_1$, i.e., $s_{\sigma^{-1}} s_\sigma a_1 \leq c_{i_1} s_{\sigma^{-1}} s_\sigma b_1$, then by the last property in (2.1) $c_{i_1} s_{\sigma^{-1}} s_\sigma b_1 \leq s_{\sigma^{-1}} c_{\sigma i_1} s_\sigma b_1$, therefore applying s_σ, $s_\sigma a_1 \leq c_{\sigma i_1} s_\sigma b_1$ so $a_2 \leq c_{i_2} b_2$. This argument is symmetrical. The second property is trivial.

Now it is possible to define a special enumeration of R. If $p \in R'$, then let R_p be the members of the $\approx$ -equivalence class with representative point p. So R equals the union of the sets R_p $(r \in R')$, obviously.

Let us fix an ordering $\leq^*$ of R' and fix the following lexicographic extension of $\leq^*$ to R :

$$\text{if } q \in R_p \text{, then set } p \leq^* q$$

$$\text{if } p_1 \leq^* p_2 \ (p_1, p_2 \in R') \text{ and } p_1 \approx \gamma,\ p_2 \approx \lambda \ (\gamma, \lambda \in R),\ \text{then let } \gamma \leq^* \lambda. \tag{2.32}$$

Let ρ be an ordinal and let $r : \rho \to R$ be an enumeration of R such that r preserves the lexicographic ordering $\leq^*$ and for all $n \in \rho$ and $(a, b, i) \in R$ there is a $m \in \rho, m > n$ such that $r(m) = (a, b, i)$. Such ρ and r clearly exist.

Now the *definition of the function* rep_{n+1} is:

We will define rep_{n+1} for this case. In the case of the limit ordinal and the general definition of the function rep, let rep_{n+1} be the same as the originals in (1.16) and (1.17). Assume that n is a successor ordinal.

For the *representative point* $p = (a, b, i)$ $(p \in R')$ let the definition of rep_{n+1} be the same as the original one, so be the same as the one included in (1.13), (1.14) or (1.15), depending on the cases discussed there.

Then we extend the definition of rep_{n+1} for the members of the equivalence class including the respective representative points depending on the cases included in the original definition. The motivation of these definitions is that $\approx$ preserves the respective inequalities (see (2.30) and (2.31)).

If $a \not\leq c_i b$, let

$$\mathrm{rep}_{n+1} = \mathrm{rep}_n \tag{2.33}$$

for all the members of the equivalence class containing (a, b, i).

If $a \leq c_i b$, we define the function rep_{n+1} simultaneously for all the triples (a_1, b_1, i_1) such that $(a_1, b_1, i_1) \approx (a, b, i)$.

Assume that $a_1 = s_\tau a, b_1 = s_\tau b, i_1 = \tau i$ for some permutation τ.

If $b \leq d_{ij}$ for some $j < \alpha, j \neq i$, then let

$$\mathrm{rep}_{n+1}(s_\tau b) = \mathrm{rep}_n(s_\tau b) \cup \{g\ (\tau i\ /\ g_{\tau j}) : g \in \mathrm{rep}_n(s_\tau a)\}\,. \tag{2.34}$$

If $b \nleq d_{ij}$ for all $j < \alpha, j \neq i$, then let

$$\mathrm{rep}_{n+1}(s_\tau b) = \mathrm{rep}_n(s_\tau b) \cup \{g\ (\tau i\ /\ u_h) : g \in \mathrm{rep}_n(s_\tau a)\}\,, \tag{2.35}$$

where u_h is the constant in (1.15) and h denotes the sequence $S_{\tau^{-1}}g$.

That is, $\mathrm{rep}_{n+1}(s_\tau b) = S_\tau\ \mathrm{rep}_{n+1}(b)$ in (2.34) and (2.35) by definition. In both cases infinitely many steps of the original proof can be reduced into one schema. The definition of rep_{n+1} for the representative triple (a, b, i) assures the desired properties of rep_{n+1} for the members of the equivalence class determined by (a, b, i). For example, such a property is the one denoted by (iv)' in the original proof.

The definition of the function rep should be the original (1.17).

We check property (vi)' in (2.2) for the function rep:

Lemma 2.17 $\mathrm{rep}(s_\sigma e) = S_\sigma \mathrm{rep}(e)$ *for every* $e \in \mathrm{At}\mathfrak{A}$*, where* S_σ *is the substitution on the unit* V *and* σ *is a permutation on* α *– i.e., the property* (vi)' *in (2.2) is true for the function rep.*

Proof.

It is proven that if $\mathrm{rep}_n(s_\sigma e) = S_\sigma \mathrm{rep}_n(e)$ for every $e \in \mathrm{At}\mathfrak{A}$, then

$$\mathrm{rep}_{n+1}(s_\sigma e) = S_\sigma\ \mathrm{rep}_{n+1}(e) \tag{2.36}$$

for every $e \in \mathrm{At}\mathfrak{A}$ and successor ordinal n.

If this implication is proven, then by the definition in (1.16) and the induction condition, (2.36) is true for every ordinal n. From this and from (2.7) we obtain that $\mathrm{rep}(s_\sigma e) = S_\sigma \mathrm{rep}(e)$ for every $e \in \mathrm{At}\mathfrak{A}$, i.e., the proposition of the lemma is true.

To prove (2.36), the definition of rep_{n+1} will be used. Let us consider the representative point (a, b, i) for the equivalence relation $\approx$ and consider an arbitrary point $(s_\tau a, s_\tau b, \tau i)$ being $\approx$ -equivalent to (a, b, i). Let us consider the cases listed in the definition of rep_{n+1}:

Case 1.

If $a \nleq c_i b$ (i.e., $s_\tau a \nleq c_{\tau i} s_\tau b$), then by the definition in (2.33), $\mathrm{rep}_{n+1} = \mathrm{rep}_n$ for all the members of the class containing (a, b, i), therefore the property (2.36) is transmitted from n to $n + 1$.

Case 2.

$a \leq c_i b$ (i.e., $s_\tau a \leq c_{\tau i} s_\tau b$) and $b \leq d_{ij}$ for some j and for every $i \neq j$. We need to prove (2.36) for $e = s_\tau b$, i.e., that

$$\mathrm{rep}_{n+1}(s_\sigma(s_\tau b)) = S_\sigma \mathrm{rep}_{n+1}(s_\tau b) \tag{2.37}$$

for any permutation σ.

Let us consider the left-hand side of (2.37):

$\mathrm{rep}_{n+1}(s_\sigma(s_\tau b)) = \mathrm{rep}_{n+1}(s_{\sigma\circ\tau} b) = \mathrm{rep}_n(s_\alpha b) \cup \{g\ (\alpha i\ /\ g_{\alpha j}) : g \in \mathrm{rep}_n(s_\alpha a)\}$ by (2.34), where $\alpha = \sigma \circ \tau$. Here $\mathrm{rep}_n(s_\alpha b) = \mathrm{rep}_n(s_\sigma(s_\tau b)) = S_\sigma \mathrm{rep}_n(s_\tau b)$ by induction.

For the right-hand side of (2.37):

$S_\sigma\ \mathrm{rep}_{n+1}(s_\tau b) = S_\sigma(\mathrm{rep}_n(s_\tau b) \cup \{g\ (\tau i\ /\ g_{\tau j}) : g \in \mathrm{rep}_n(s_\tau a)\}) =$

$= S_\sigma\ \mathrm{rep}_n(s_\tau b) \cup S_\sigma \{g\ (\tau i\ /\ g_{\tau j}) : g \in \mathrm{rep}_n(s_\tau a)\}$.

Comparing the above reformulations of the left and right-hand sides, it is sufficient to prove that

$$\{g\ (\alpha i\ /\ g_{\alpha j}) : g \in \mathrm{rep}_n(s_\alpha a)\} = S_\sigma \{g\ (\tau i\ /\ g_{\tau j}) : g \in \mathrm{rep}_n(s_\tau a)\}. \tag{2.38}$$

To prove (2.38), first let us consider the left-hand side. We show that for any finite permutation β the following is true:

$$\{g\ (\beta i\ /\ g_{\beta j}) : g \in \text{rep}_n(s_\beta a)\} = S_\beta \{f\ (i\ /\ f_j) : f \in \text{rep}_n(a)\}. \tag{2.39}$$

But $S_\beta \{f\ (i\ /\ f_j) : f \in \text{rep}_n(a)\} = \{(S_\beta\ f)\ (\beta i\ /\ f_j) : f \in \text{rep}_n(a)\}$. Denoting $S_\beta f$ by g we obtain that $f = S_{\beta^{-1}} g$. Further,

$$(S_\beta f)\ (\beta i\ /\ f_j) = g\ (\beta i\ / g_{\beta j}). \tag{2.40}$$

So

$$S_\beta\ f\ (i\ /\ f_j) = \ g\ (\beta i\ / g_{\beta j}). \tag{2.41}$$

Considering (2.39) if $f = S_{\beta^{-1}} g$, then $f \in \text{rep}_n(a)$ is equivalent to $g \in S_\beta\ \text{rep}_n(a) = \text{rep}_n(s_\beta a)$. So (2.39) is true.

Now let us consider (2.38). On one hand, $\{g\ (\alpha i\ /\ g_{\alpha j}) : g \in \text{rep}_n(s_\alpha a)\} = S_\alpha \{f\ (i\ /\ f_j) : f \in \text{rep}_n(a)\} = S_\sigma S_\tau \{f\ (i\ /\ f_j) : f \in \text{rep}_n(a)\}$ applying (2.39) for $\beta = \alpha$. On the other hand,

$S_\sigma \{g\ (\tau i\ /\ g_{\tau j}) : g \in \text{rep}_n(s_\tau a)\} = S_\sigma(S_\tau(\{f\ (i\ /\ f_j) : f \in \text{rep}_n(a)\})$ applying (2.39) for $\beta = \tau$. Therefore (2.38), so (2.37) is proven.

Case 3.

$a \leq c_i b$ (i.e., $s_\tau a \leq c_{\tau i} s_\tau b$) and $b \nleq d_{ij}$ (i.e., $s_\tau b \nleq d_{\tau i\ \tau j}$) for all $j < \alpha$, $j \neq i$.

Similarly to the above arguments, considering the definition in (2.35) instead of (2.34), we need to prove the following equality rather than (2.38):

$$\{g\ (\alpha i\ /\ u_{h_1}) : g \in \text{rep}_n(s_\alpha a)\} = S_\sigma \{g\ (\tau i\ /\ u_{h_2}) : g \in \text{rep}_n(s_\tau a)\}, \tag{2.42}$$

where $\alpha = \tau \circ \tau_1$, $h_1 = S_{\alpha^{-1}}\, g$ and $h_2 = S_{\tau^{-1}}\, g$.

We can prove the following equality by being analogous with (2.39) for an arbitrary finite permutation β :

$$\{g\ (\beta i\ /\ u_h) : g \in \mathrm{rep}_n(s_\beta a)\} = S_\beta\, \{f\ (i\ /\ u_f) : f \in \mathrm{rep}_n(a)\}\,, \tag{2.43}$$

where u_h is the constant in (1.15) and h denotes $S_{\beta^{-1}} g$.

Namely let us apply the same argument as in the proof of (2.39), but in (2.41) let us use $S_\beta f\ (i\ /\ u_f) = g\ (\beta i\ /\ u_h)$ instead of $S_\beta f\ (i\ /\ f_j) = g\ (\beta i\ /\ g_{\beta j})$, where $h = S_{\beta^{-1}}\, g$.

The proof of (2.42):

$\{g\ (\alpha i\ /\ u_{h_1}) : g \in \mathrm{rep}_n(s_\alpha a)\} = S_\alpha\, \{f\ (i\ /\ u_f) : f \in \mathrm{rep}_n(a)\} =$

$= S_\sigma S_\tau\, \{f\ (i\ /\ u_f) : f \in \mathrm{rep}_n(a)\}$ applying (2.43) for β and $\alpha = \sigma \circ \tau$. Further,

$S_\sigma\, \{g\ (\tau i\ /\ u_{h_2}) : g \in \mathrm{rep}_n(s_\tau a)\} \ = \ S_\sigma(S_\tau(\{f\ (i\ /\ u_f) : f \in \mathrm{rep}_n(a)\})$ applying (2.43) for $\beta = \tau$. Therefore (2.42), so (2.37) is proven.

qed.

Let Dp_α denote the polyadic version of the cylindric class D_α.

Lemma 2.18 $\mathsf{Dp}_\alpha = \mathsf{Gwt}_\alpha$.

Proof.

The notation introduced in Chapter 1 is used. $\mathsf{Gwt}_\alpha \subseteq \mathsf{Dp}_\alpha$ is trivial. To prove the converse inclusion, we use the following characterization of Gwt_α : $y \in V$ implies $y \circ \tau \in V$ for every finite transformation τ. But τ can be composed in terms of finitely many elementary transformations substitution $[i\ /\ j]$ and transposition $[i,\ j]$. It is sufficient to prove that V is closed under these transformations. But V is closed under $[i\ /\ j]$ because

V is a Dp_α unit. Furthermore, V is a Trs_α unit, therefore it is closed under $[i,\ j]$ too. qed.

The completion of the proof of Theorem 2.8 is:

In [Fe07a] it is proven that (F5)* and (C$_4$)* are equivalent under the other F_α axioms. Andréka and Thompson proved that there is an isomorphism, denoted by rep', between the algebra $\mathfrak{Rd}_{ca}\mathfrak{A}$ and some algebra $\mathfrak{B}$'$\in \mathsf{D}_\alpha$. We proved in Lemma 2.17 that this isomorphism preserves the operators s_σ for any finite permutations σ on α. Therefore $\mathfrak{B}$' may be considered as an algebra $\mathfrak{B}$ in Dp_α. Lemma 2.18 implies that $\mathfrak{B} \in \mathsf{Gwt}_\alpha$. So rep' is an isomorphism between $\mathfrak{A}$ and a $\mathfrak{B} \in \mathsf{Gwt}_\alpha$.

qed.

Gw_α denotes the class $\{\mathrm{Rd}_{ca}\mathfrak{B}\colon \mathfrak{B} \in \mathsf{Gwt}_\alpha\}$ by definition, where $\mathfrak{Rd}_{ca}\mathfrak{B}$ denotes the cylindric reduct of $\mathfrak{B}$ (see [He-Mo-Ta I.], p. 226). The following claim obviously follows from Theorem 2.8:

Corollary 2.19 *If* $\mathfrak{A} \in \mathsf{TA}_\alpha$, *then* $\mathfrak{Rd}_{ca}\mathfrak{A} \in \mathbf{I}\mathsf{Gw}_\alpha$, $\alpha \geq 4$.

Main references in this Chapter are: [Fe12a], [Sa-Th] and [Fe11a].

Chapter 3

Representation theorems for polyadic-type equality algebras

In this Chapter we deal with "polyadic-type" algebras other than transposition algebras. We assume that these algebras have only single cylindrifications c_i, because the non-commutativity of cylindrifications (for quasi-polyadic algebras this is only a formal restriction, but for polyadic algebras, in general, not). This is the reason for the terminology *cylindric* polyadic algebras. While the type of cylindric-type algebras is unique, the type of polyadic-type algebras depends on the definite subset Q of ${}^{\alpha}\alpha$, where the transformation τ of s_τ runs. The following concrete classes of "polyadic-type" algebras will be investigated: cylindric quasi-polyadic equality, cylindric polyadic equality, cylindric m-quasi-polyadic equality algebras.

3.1 Cylindric quasi-polyadic equality algebras

The concept of quasi-polyadic algebra was introduced in Halmos [Ha56] (here Definition 3.3). Sain and Thompson proved ([Sa-Th]) that quasi-polyadic equality algebras and algebras in FPEA_α (or strong transposition algebras) are definitionally equivalent. Nevertheless, it is worth investigating quasi-polyadic algebras in themselves because quasi-polyadic algebra is a well-known class and can be considered as a bridge to the polyadic algebras having infinite substitution operators.

The following two definitions are closely related to the Definitions 2.1 and 2.2 concerning transposition algebras.

Definition 3.1 (Cqrs_α) The structure

$$\langle A,\ \cup,\ \cap,\ \sim_V,\ 0,\ V,\ C_i^V,\ S_\tau^V,\ D_{ij}^V\ \rangle_{\tau\in \mathrm{FT}_\alpha,\ i,j<\alpha}$$

is a *cylindric quasi-polyadic relativized set algebra* if its cylindric reduct is in Crs_α, and A is closed under the substitutions

$$S_\tau^V X = \{y \in V : y \circ \tau \in X,\ \tau \in \mathrm{FT}_\alpha\}$$

([He-Mo-Ta II.], 5.4.22).

Definition 3.2 (Gwq_α) A set algebra in Cqrs_α is called a *generalized weak quasi-polyadic relativized set algebra* if there are sets $U_k,\ k \in K$ and sequences $p_k \in {}^\alpha U_k$ such that $V = \bigcup\limits_{k\in K} {}^\alpha U_k^{(p_k)}$, where V is the unit.

Recall the *classical definition* of quasi-polyadic equality algebra (containing general cylindrification $c_{(\Gamma)}$, $\Gamma \subset \alpha$):

Definition 3.3 (QPEA_α) By a *quasi-polyadic equality algebra* of dimension α, we mean an algebra $\mathfrak{A}=\left\langle \mathfrak{B},\ c_{(\Gamma)}, s_\tau,\ d_{ij}\right\rangle_{i,j<\alpha}$ such that $c_{(\Gamma)}$ and s_τ are unary operations, d_{ij} are constants and the following equations (Q_0)-(Q_9),(E_1)-(E_3) are valid in $\mathfrak{A}$ for every finite $\Gamma, \Delta\ (\Gamma, \Delta \subset \alpha), \tau, \sigma \in \mathrm{FT}_\alpha$ and $i, j < \alpha$:

(Q_0) $\mathfrak{B} = \langle A; +, \cdot, -, 0, 1\rangle$ is a Boolean algebra

(Q_1) $x \leq c_{(\Gamma)}x$

(Q_2) $c_{(\Gamma)}(x \cdot c_{(\Gamma)}(y)) = c_{(\Gamma)}(x) \cdot c_{(\Gamma)}(y)$

(Q_3) $c_{(\emptyset)}x = x$

(Q_4) $c_{(\Gamma)}c_{(\Delta)}x = c_{(\Gamma\cup\Delta)}x$

(Q_5) $s_{Id}x = x$

(Q_6) $s_{\sigma\circ\tau}x = s_\sigma s_\tau x$

(Q_7) $s_\sigma(x + y) = s_\sigma x + s_\sigma y$ and $s_\sigma(-x) = -s_\sigma x$

(Q_8) *if* $\sigma|_{\alpha\sim\Gamma} = \tau|_{\alpha\sim\Gamma}$ then $s_\sigma c_{(\Gamma)}x = s_\tau c_{(\Gamma)}x$

(Q_9) $c_{(\Gamma)}s_\tau x = s_\tau c_{(\Delta)}x$, where $\Delta = \tau^{-1}[\Gamma]$ and $\tau|_\Delta$ is one-one

(E_1) $d_{ii} = 1$

(E_2) $x \cdot d_{ij} \leq s_{[i/j]}x$

(E_3) $s_\tau d_{ij} = d_{\tau(i)\tau(j)}$.

(see Halmos [Ha57], [Sa-Th], Def. 5, or [He-Mo-Ta II.]).

It is obvious that replacing the general cylindrifications $c_{(\Gamma)}$ by single cylindrifications c_i, this does not mean any essential change due to the finiteness of the sets Γ. In [Fe13] it is proven that this usual axiom system is redundant, because axiom (Q_8) can be omitted.

.

The following definition is closely related to that of quasi-polyadic equality algebra, but, as it was mentioned above, this latter is adapted to the non-commutative case of cylindrifications (the polyadic axiom (Q_4) is missing and (Q_9) has changed, see [He-Mo-Ta II.], 5.4.1.

Definition 3.4 (CQE_α) A *cylindric quasi-polyadic equality algebra* of dimension α ($\alpha \geq 2$) is a structure

$$\mathfrak{A} = \langle A,\ +,\ \cdot,\ -,\ 0,\ 1,\ c_i,\ s_\tau,\ d_{ij}\rangle_{\tau\in\, \mathrm{FT}_\alpha,\, i.j<\alpha} \tag{3.1}$$

where $+$, and $\cdot$ are binary operations, $-$, c_i and s_τ are unary operations, $0, 1$ and d_{ij} are constants in $\mathfrak{A}$ such that for every $i, j \in \alpha$, $x, y \in A$, $\sigma, \tau \in \mathrm{FT}_\alpha$, the following postulates are satisfied:

(CP0) $\langle A,\ +,\ \cdot,\ -,\ 0,\ 1\rangle$ is a Boolean algebra

(CP1) $c_i 0 = 0$

(CP2) $x \leq c_i x$

(CP3) $c_i(x \cdot c_i y) = c_i x \cdot c_i y$

(CP4) $s_{Id} x = x$

(CP5) $s_{\sigma\circ\tau} x = s_\sigma s_\tau x$

(CP6) $s_\sigma(x + y) = s_\sigma x + s_\sigma y$

(CP7) $s_\sigma(-x) = \sim s_\sigma x$

(CP8) $s_\sigma x = s_\tau x$, assuming that $\sigma i = \tau i$ if $i \notin \Gamma$ and Γ is such that $c_i x = x$ if $i \in \Gamma$ $(\Gamma \subset \alpha)$

(CP9)* $c_i s_\sigma x \leq s_\sigma c_j x$ if $\sigma^{-1*}\{i\}$ equals $\{j\}$ or the empty set (in this latter case c_i is the identity operator), and the equality holds instead of $\leq$ if σ is a permutation of α

(E1) $d_{ii} = 1$

(E2) $x \cdot d_{ij} \leq s_{[i\,/\,j]} x$

(E3) $s_\tau d_{ij} = d_{\tau i\, \tau j}$.

Definition 3.5 (CQES_α) A *strong cylindric quasi-polyadic equality algebra* is such a CQE_α that, instead of (CP9)*, the axiom

$$\text{(CP9)} : c_i s_\sigma x = s_\sigma c_j x$$

is assumed, where $\sigma^{-1*}\{i\}$ equals $\{j\}$ or the empty set (in this latter case c_i is the identity operator) and, in addition, the cylindric axiom (C_4), i.e., the commutativity of cylindrifications

$$c_i c_j x = c_j c_i x \quad \text{if } i, j \in \alpha$$

is assumed.

As a consequence of Sain and Thompson's result (Theorem 1 in [Sa-Th]) TAS_α (also FPEA_α), CQES_α and quasi-polyadic equality algebras are definitionally equivalent ($\alpha \geq 3$). The question arises: which class is the quasi-polyadic counterpart of the class TA_α?

The following theorem answers this question ([Fe13, Fe13]):

Theorem 3.6 *The axiomatizations of* TA_α *and* CQE_α *are definitionally equivalent* ($\alpha \geq 3$).

Proof.

If $\mathfrak{A} \in \mathsf{CQE}_\alpha$, then checking the FPEA_α (TAS_α) axioms, the commutativity of the cylindrifications is used only in the proof of (F5). So, now we only need to prove (F5)*. The property $s^i_j s^k_m x = s^k_m s^i_j x$ $(i, j \notin \{k,\ m\})$ is equivalent to the special case of (CP9)*:

$$c_i s^j_m x \leq s^j_m c_i x \quad (i \notin \{j, m\}) \tag{3.2}$$

supposing that both properties hold for every possible ordinal in the conditions (see [Fe07b], Theorem 1). Here we need the direction that (3.2) implies axiom (F5)* (in this proof only the polyadic axiom (Q_2) is used in [Fe07b]). Originally, the polyadic axiom (Q_9) is applied in proving axiom (F3). But, (F3) follows from (CP9)*.

Conversely, assume that $\mathfrak{A} \in \mathsf{TA}_\alpha$. We refer to the proof of Theorem 1 in [Sa-Th], following the applications of axiom (F5) in that proof, and investigating whether (F5) can be replaced by axiom (F5)*.

The first occurrence of (F5) is in the proof of the commutativity of the cylindrifications (Claim 1.1). In CQE_α, this latter property fails to be true, therefore we must not use Claim 1.1.

The next occurrences of (F5) are in the Claims 1.2 and 1.3 which state that the operator s_τ can be introduced in the algebra for an arbitrary $\tau \in \mathrm{FT}_\alpha$. These claims are based on Jonsson's famous theorem which requires the validity of certain conditions (J1)–(J7). These properties can obviously be proven in CQE_α without (F5) or they are axioms (e.g., (J6) is exactly (F5)*). The only critical property is (J4): $p_{ij}s^k_i x = s^k_j p_{ij}x$, because the proof of this property uses axiom (F5) in proving the inequality $s^k_j p_{ij}x \leq s^k_j p_{ij} s^i_k x$ (row (16) there). We show that this property can be proven without (F5):

$s^k_j x = c_k(x \cdot d_{kj})$ holds in TA_α (the proof is similar to that of [He-Mo-Ta II.] Thm. 5.4.3). Then

$s^k_j p_{ij}x = c_k(p_{ij}x \cdot d_{kj}) = c_k(d_{kj} \cdot p_{ij}x \cdot d_{kj})$. But $d_{kj} = p_{ij}d_{ki}$ ($d_{kj} = s^i_j d_{kj} = p_{ij}s^j_i d_{kj} = p_{ij}d_{ki}$ by (F9)). Thus, $c_k(d_{kj} \cdot p_{ij}x \cdot d_{kj}) = c_k(d_{kj} \cdot p_{ij}x \cdot p_{ij}d_{ki}) = c_k(d_{kj} \cdot p_{ij}(x \cdot d_{ki})) \leq c_k(d_{kj} \cdot p_{ij}c_i(x \cdot d_{ki})) = s^k_j p_{ij} s^i_k x$.

Thus, the existence of the operator s_τ is proven.

The next part of the proof in Theorem 1 in [Sa-Th] is the proof of the CQE_α axioms. The only non-trivial case is the proof of the polyadic axiom (Q_9), namely (F5) occurs in Lemma 1.5 (iii). This part (iii) states that $c_i s_\tau x = s_\tau c_i x$ if $\tau i = i$. The proof uses that $c_i s^j_m x = s^j_m c_i x$ $(i \notin \{j, m\})$. But, instead of this, we can use property (3.2) above. As it is mentioned above, (F5)* implies this property (see [Fe07a]) and the proof uses only (CP3) and $\mathrm{c}_i d_{ij} = 1$ (this latter is trivially true in CQE_α). Therefore in Lemma 1.5 (iii) only the inequality $c_i s_\tau x \leq s_\tau c_i x$ (where $\tau i = i$) holds instead of equality. Using this inequality in the remainder of the proof of (Q_9), we obtain exactly (CP9)* instead of (Q_9).

qed.

Definition 3.7 An algebra $\mathfrak{A}$ with the type of CQE_α is *r-representable,* if $\mathfrak{A} \in \mathbf{I}\mathsf{Cqrs}_\alpha$.

Lemma 3.8 *The following propositions* (i) *and* (ii) *hold:*

(i) *If* $\mathfrak{A} \in \mathsf{Cqrs}_\alpha$, *then* $\mathfrak{A} \in \mathsf{Gwq}_\alpha$ *if and only if* $x \in V$ *implies* $x \circ \tau \in V$ *for every finite* τ *on* α.

(ii) *If* $\mathfrak{B} \in \mathsf{CQE}_\alpha$ *and* $\mathfrak{B}$ *is r-representable, then* $\mathfrak{B} \in \mathbf{I}\mathsf{Gwq}_\alpha$.

The proposition can be reduced to Lemma 2.7, noticing that for finite τ on α, the condition $x \circ \tau \in V$ is equivalent to the pair of conditions

$$x \circ [i, j] \in V \text{ and } x \circ [i/j] \in V.$$

The following basic representation theorem follows from Theorem 2.8, Theorem 3.6 and from the proof of Lemma 3.8.

Theorem 3.9 (Main r-representation theorem for algebras in CQE_α):

$$\mathfrak{A} \in \mathsf{CQE}_\alpha \;\; \textit{if and only if} \;\; \mathfrak{A} \in \mathbf{I}\mathsf{Gwq}_\alpha$$

where $\alpha \geq 3$.

The reformulation of the theorem is:

The class Gwq_α *is first-order axiomatizable by a finite schema of equations and the axioms can be the* CQE_α *axioms.*

By definition, the class CQES_α is obtained from CQE_α if axiom (CP9) is replaced by (CP9)*, and, (C_4) is assumed. Thus, we obtain the following:

Corollary 3.10 $\mathfrak{A} \in \mathsf{CQES}_\alpha$ *if and only if*

$$\mathfrak{A} \in \mathbf{I}(\mathsf{Gwq}_\alpha \cap \mathrm{Mod}\,\{(\mathrm{CP9}), (\mathrm{C}_4)\})$$

$(\alpha \geq 3)$.

Let us denote by $\overset{\bullet}{\mathsf{Gwq}}_\alpha$ the subclass of Gwq_α such that the disjointness of subunit is assumed. CQES_α is not representable in the classical sense (see[Sa-Th]), thus Gwq_α in the Corollary cannot be replaced by $\overset{\bullet}{\mathsf{Gwq}}_\alpha$. But, recall that the locally finite algebras in CQES_α are already representable in the classical sense ([Ha56], [Ha57]).

3.2 Cylindric polyadic and m-quasi polyadic equality algebras

In this Section we study α-dimensional "polyadic-type" equality algebras having *infinite* substitution operators (s_τ or S_τ). Here "polyadic" is used in the classical, Halmos polyadic sense, except for the fact that the algebra contains only single cylindrifications. From now on, the dimension α is assumed to be infinite (because the finite dimensional case is closely connected to the quasi-polyadic case). The other ordinals included later in the chapter (e.g., m) are infinite, as well. These investigations focus on the analysis of the substitution operators with *infinite* transformations and equalities (in another terminology, on *transformation systems* with equalities, see [Da-Mo]). The techniques needed for these investigations are different from the case of finite transformations.

First, some classes of *set algebras* are introduced: the classes Cprs_α, Gp_α, $\mathsf{Gp}_\alpha^{\mathrm{reg}}$, ${}_m\mathsf{Cprs}_\alpha$, Gpw_α and $\mathsf{Gpw}_\alpha^{\mathrm{reg}}$.

The following definition is a variant of that of Cqrs_α. It includes ${}^\alpha\alpha$ instead of FT_α, where α is infinite.

Definition 3.11 (Cprs_α) The structure

$$\langle A,\ \cup,\ \cap,\ \sim_V,\ 0,\ V,\ C_i^V,\ S_\tau^V,\ D_{ij}^V\ \rangle_{\tau\,\in\,{}^\alpha\alpha,\ i,j<\alpha}$$

is a *cylindric polyadic relativized set algebra* if its cylindric reduct is in Crs_α, and A is closed under the substitutions

$$S_\tau^V X = \{y \in V : y \circ \tau \in X,\ \tau \in\ {}^\alpha\alpha\}$$

(see [He-Mo-Ta II.], Definition 5.4.22).

Obviously, the cylindric reduct of a Cprs_α is a Crs_α.

A dimension set Δx of an element x of a cylindric or polyadic-type algebra is the set $(i : c_i x \neq x, i < \alpha)$.

Definition 3.12 (Gp_α and $\mathsf{Gp}_\alpha^{\mathrm{reg}}$) A set algebra $\mathfrak{A}$ in Cprs_α is called a *generalized polyadic relativized set algebra* ($\mathfrak{A} \in \mathsf{Gp}_\alpha$) if there are sets U_k and $k \in K$, such that $V = \bigcup_{k\in K} {}^\alpha U_k$, where V is the unit. An algebra $\mathfrak{A}$ in Gp_α is called *regular* ($\mathfrak{A} \in \mathsf{Gp}_\alpha^{\mathrm{reg}}$) if, for each $X \in A$, $x \in X$ and $y \in V$, the condition $(\Delta X \cup 1) \upharpoonright x \subseteq y$ implies $y \in X$.

Remarks

a) One of the differences between the classical cylindric class Gs_α (generalized cylindric set algebras, see [He-Mo-Ta II.], Definition 3.1.2) and Gp_α is that in Gp_α the pairwise disjointness of the U_k's is not required.

b) The cylindric reduct of a Gp_α is the "locally square" cylindric set algebra G_α, introduced by Németi (see [Nem86], [An-Go-Ne] and [And]).

c) The concept of *regularity* (see [He-Mo-Ta II.], Definition 3.1.1 (viii)) compensates, in a sense, for the lack of general cylindrification C_Γ $(\Gamma \subset \alpha)$ because if such a cylindrification exists, then $(\Delta X \cup 1) \upharpoonright x \subseteq y$ implies that $y \in C_{(\alpha \sim (\Delta X \cup 1))} X = X$.

d) The subclass of Gp_α such that the pairwise disjointness of the U_k's is assumed is denoted by $\dot{\mathsf{Gp}}_\alpha$.

Assume that $m < \alpha$ is infinite and fixed. Given a set U and a fixed sequence $p \in {}^\alpha U$, the set

$$ {}^\alpha_m U^{(p)} = \{x \in {}^\alpha U \ : \ x \text{ and } p \text{ are different at most in } m\text{-many members}\} $$

is called the *m-weak space* (or m-weak Cartesian space) determined by p and U. Here p is called a *support* of the m-weak space and U is called the *base*.

Recall that the definition of the *weak space*, in notation ${}^\alpha U^{(p)}$ (see Chapter 1 here, and [He-Mo-Ta II.], 3.1.2) is the ω-version of the above definition if the term "at most in" is replaced by "less than" in it.

Definition 3.13 A transformation τ defined on α is said to be an *m-transformation* ($m \leq \alpha$ is infinite and fixed) if $\tau i = i$ except for m-many $i \in \alpha$. The class of m-transformations is denoted by ${}_m\mathsf{T}_\alpha$.

Definition 3.14 (${}_m\mathsf{Cprs}_\alpha$) If, in the definition of Cprs_α, ${}^\alpha\alpha$ is changed by ${}_m\mathsf{T}_\alpha$ ($m < \alpha$ infinite and fixed), then the definition of the class ${}_m\mathsf{Cprs}_\alpha$ is obtained.

Obviously, ${}_\alpha\mathsf{Cprs}_\alpha$ is Cprs_α. We note that there exists a *generalized definition* of Cprs_α such that, instead of ${}^\alpha\alpha$, the domain of the τ's is a fixed subset Q of ${}^\alpha\alpha$. In this case it is necessary to assume certain compatibility conditions for Q (see [Sai]).

Now, we can summarize the types of polyadic-type algebras included in the Thesis: the types of Trs_α, Cqrs_α, Cprs_α and ${}_m\mathsf{Cprs}_\alpha$.

Definition 3.15 (${}_m\mathsf{Gwp}_\alpha$ and ${}_m\mathsf{Gwp}^{\mathrm{reg}}$) A set algebra $\mathfrak{A}$ in ${}_m\mathsf{Cprs}_\alpha$ ($m < \alpha$ infinite and fixed) is called a *generalized m-quasi* ($m < \alpha$) *polyadic relativized set algebra* ($\mathfrak{A} \in_m \mathsf{Gwp}_\alpha$) if there are sets U_k, $k \in K$ and sequences $p_k \in {}^\alpha U_k$ such that $V = \bigcup_{k\in K} {}^\alpha_m U$, where V is the unit. The relation of ${}_m\mathsf{Gwp}^{\mathrm{reg}}_\alpha$ and ${}_m\mathsf{Gwp}_\alpha$ is similar to that of ${}_m\mathsf{Gp}_\alpha$ and ${}_m\mathsf{Gp}^{\mathrm{reg}}_\alpha$.

The characterizations of the classes ${}_m\mathsf{Gwp}_\alpha$ and Gp_α are the following ones:

Lemma 3.16

(i) *If* $\mathfrak{A} \in {}_m\mathsf{Cprs}_\alpha$, *then* $\mathfrak{A} \in {}_m\mathsf{Gwp}_\alpha$ *if and only if* $x \in V$ *implies* $x \circ \tau \in V$ *for every transformation* τ, $\tau \in {}_m\mathsf{T}_\alpha$. *Another equivalent condition for* $\mathfrak{A} \in {}_m\mathsf{Gwp}_\alpha$ *is:* $\mathsf{S}_\tau V = V$ *for every transformation* τ, $\tau \in {}_m\mathrm{T}_\alpha$.

(ii) *If* $\mathfrak{A} \in \mathsf{Cprs}_\alpha$, *then* $\mathfrak{A} \in \mathsf{Gp}_\alpha$ *if and only if* $x \in V$ *implies* $x \circ \tau \in V$ *for every transformation* τ, $\tau \in {}^\alpha\alpha$. *Another equivalent condition for* $\mathfrak{A} \in \mathsf{Gp}_\alpha$ *is:* $S_\tau V = V$ *for every* τ, $\tau \in {}^\alpha\alpha$ (see [And]).

This lemma is analogous with the Lemma 3.8. As regards the equivalency of the first property and $S_\tau V = V$ in (i), for example, the condition $x \in V$ *implies* $x \circ \tau \in V$ *for every transformation* $\tau, \tau \in {}_m\mathsf{T}_\alpha$ means that $V \subseteq S_\tau V$. Conversely, $S_\tau V \subseteq V$ is always holds in ${}_m\mathsf{Gwp}_\alpha$.

Now, some classes of *abstract algebras* are introduced: the classes CPE_α, CPES_α and ${}_m\mathsf{CPE}_\alpha$.

Definition 3.17 (CPE_α) If, in the definition of CQE_α, FT_α is changed by ${}^\alpha\alpha$ (α is infinite), and, instead of (CP8) the axiom

(CP8)* : $d \cdot s_\sigma x = d \cdot s_\tau x$ if the product d of the elements $d_{\tau i\ \sigma i}$ $(i \in \triangle x)$ exists.

is assumed, then the concept of *cylindric polyadic equality algebra* of dimension α is obtained.

Definition 3.18 (CPES_α) A *strong cylindric polyadic equality algebra* of dimension α is an algebra in CPE_α such that instead of (CP9)* the axiom

$$\text{(CP9)} : c_i s_\sigma x = \ s_\sigma c_j x$$

is required if $\sigma^{-1*}\{i\}$ equals $\{j\}$ or the empty set (in the latter case c_j is the identity) and, in addition, the axiom

$$(\text{C}_4) : c_i c_j x = c_j c_i x$$

is assumed, where α is infinite, $i, j \in \alpha$, $\sigma \in {}^\alpha\alpha$.

Definition 3.19 (${}_m\mathsf{CPE}_\alpha$) If, in the definition of CPE_α the transformations τ and σ are assumed to be m-transformations ($m < \alpha$ infinite and fixed), i.e., $\tau, \sigma \in {}_m\mathsf{T}_\alpha$, then the concept of *cylindric m-quasi-polyadic equality algebra* of dimension α (${}_m\mathsf{CPE}_\alpha$) is obtained.

Lemma 3.20 ${}_m\mathsf{Gwp}_\alpha^{\mathrm{reg}} \cup \mathsf{Gp}_\alpha^{\mathrm{reg}} \subset \mathsf{CPE}_\alpha$

Proof.

As examples, we check the validity of (CP8)* and (CP9)* for an algebra $\mathfrak{A} \in \mathsf{Gp}_\alpha^{\mathrm{reg}}$.

Axiom (CP8)*. Assume that $z \in d \cap S_\sigma X$, where $X \in A$. Then, $S_\sigma z \in X$, by definition. $z \in d$ implies $z_{\tau i} = z_{\sigma i}$ if $i \in \Delta X$, i.e., $(S_\sigma z)_i = (S_\tau z)_i$ if $i \in \Delta X$. The regularity of $\mathfrak{A}$ implies that $S_\tau z \in X$, as well. Thus, $z \in S_\tau X$. Therefore $z \in S_\sigma X$ implies $z \in S_\tau X$, i.e., $S_\sigma X \subseteq S_\tau X$. By symmetry, $S_\sigma X = S_\tau X$.

Axiom (CP9)*. Assume that $z \in C_i S_\sigma X$. Then, $z_u^i \in S_\sigma X$ for some u. By definition, $S_\sigma z_u^i \in X$. Notice that $S_\sigma z_u^i = (S_\sigma z)_u^j$, where $\{j\} = \sigma^{-1*}\{i\}$ and $S_\sigma z \in V$ (the latter follows from the facts that $\mathrm{Rg} S_\sigma z = \mathrm{Rg} z$ and the definition of a Gp_α unit). Thus, $(S_\sigma z)_u^j \in X$, as well. $(S_\sigma z)_u^j \in X$ means that $z \in S_\sigma C_j X$. Therefore $C_i S_\sigma X \subseteq S_\sigma C_j X$.

We check the converse inclusion, assuming that σ is a permutation of α. Assume that $z \in S_\sigma C_j X$, where $\{j\} = \sigma^{-1*}\{i\}$. This means that $(S_\sigma z)_u^j \in X$ for some u. If σ is a permutation, then $\mathrm{Rg}(S_\sigma z)_u^j = \mathrm{Rg}\ z_u^i$, therefore by definition of a Gp_α unit, $z_u^i \in V$. In this case, the argument above can be repeated, i.e., $(S_\sigma z)_u^j = S_\sigma z_u^i$ implies $z \in C_i S_\sigma X$. Thus, $S_\sigma C_j X \subseteq C_i\ S_\sigma X$.

qed.

Remarks

a) An algebra in Cprs_α satisfies all the CPES_α axioms, with the possible exceptions of the axioms (C_4), (CP5), (CP7), (CP8)*, (CP9) and (E3) (see [He-Mo-Ta II.], Theorem 5.4.15). ${}_m\mathsf{Gwp}_\alpha^{\mathrm{reg}} \cup \mathsf{Gp}_\alpha^{\mathrm{reg}} \nsubseteq \mathsf{CPES}_\alpha$, because the CPES_α axioms (C_4) and (CP9) fail to hold for the union on the left-hand side. But, ${}_m\dot{\mathsf{Gwp}}_\alpha^{\mathrm{reg}} \cup \dot{\mathsf{Gp}}_\alpha^{\mathrm{reg}} \subseteq \mathsf{CPES}_\alpha$. Notice that ${}_m\mathsf{Gwp}_\alpha \cup \mathsf{Gp}_\alpha$ satisfies all the CPE_α axioms except for (CP8)*.

b) We note that CPE_α and CPES_α can be conceived of as so-called *transformation systems* equipped by diagonals and cylindrifications (see [Da-Mo], 3§ and 4§).

Definition 3.21 An algebra $\mathfrak{A}$ with the type of CPE_α is *r-representable if* $\mathfrak{A} \in \mathbf{I}\mathsf{Cprs}_\alpha$. An algebra $\mathfrak{A}$ with the type of ${}_m\mathsf{CPE}_\alpha$ is *r-representable if* $\mathfrak{A} \in \mathbf{I}\ {}_m\mathsf{Cprs}_\alpha$.

The next lemma *motivates* the representation theorems. For r-representable algebras it gives *necessary* conditions for the representants.

Lemma 3.22 *The following propositions* (i) *and* (ii) *hold:*

(i) *If* $\mathfrak{B}$ *is r-representable and* $\mathfrak{B} \in {}_m\mathsf{CPE}_\alpha$, *then* $\mathfrak{B} \in \mathbf{I}\ {}_m\mathsf{Gwp}_\alpha$

(ii) *If* $\mathfrak{B}$ *is r-representable and* $\mathfrak{B} \in \mathsf{CPE}_\alpha \cup \mathsf{CPES}_\alpha$, *then* $\mathfrak{B} \in \mathbf{I}\mathsf{Gp}_\alpha$.

Proof.

(i) By r-representability, $\mathfrak{B} \in \mathbf{I}\mathfrak{A}$ for some $\mathfrak{A} \in {}_m\mathsf{Cprs}_\alpha$ implies that $f(\mathsf{s}_\lambda 1) = S_\lambda f1$, where f is an isomorphism between $\mathfrak{B}$ and $\mathfrak{A}$, and λ is an arbitrary m-transformation (i.e., $\lambda \in {}_m\mathsf{T}_\alpha$). But $\mathsf{s}_\lambda 1 = 1$ and $f1 = V$, and therefore $f1 = S_\lambda V$, i.e., $V = S_\lambda V$. By Lemma 3.16 (i), $\mathfrak{B} \in \mathbf{I}\ {}_m\mathsf{Gwp}_\alpha$.

(ii) The proof is similar to the previous one, but we have to use Lemma 3.16 (ii) instead of (i).

qed.

Definition 3.23 Assume that m is infinite and $m < \alpha$. An algebra $\mathfrak{A} \in {}_m\mathsf{CPE}_\alpha$ is *locally-m dimensional* (locally-m, for short), if $|\Delta b| \leq m$ for each $b \in A$. The class of α-dimensional locally-m algebras is denoted by $\mathsf{L}m_\alpha$.

The main r-representation theorems concerning *cylindric polyadic equality algebras* are the following ones (see [Fe12b, Fe12b], [Fe11b, Fe11b]):

Theorem 3.24 (Representation theorem for ${}_m\mathsf{CPE}_\alpha \cap \mathsf{L}m_\alpha$)
$\mathfrak{A} \in {}_m\mathsf{CPE}_\alpha \cap \mathsf{L}m_\alpha$ *if and only if* $\mathfrak{A} \in \mathbf{I}({}_m\mathsf{Gwp}_\alpha^{\mathrm{reg}} \cap \mathsf{L}m_\alpha)$, *where m is infinite, $m < \alpha$.*

This theorem generalizes Halmos's classical theorem that locally finite, infinite dimensional, quasi-polyadic algebras are representable (see [Ha56]). Similarly to Halmos's theorem, where the local finiteness condition implies that the quasi-polyadic condition can be omitted, in the theorem above the (implicit) condition m-quasi can be omitted.

Let α be infinit.

Theorem 3.25 (Representation theorem for CPE_α and CPES_α)

(i) $\mathfrak{A} \in \mathsf{CPE}_\alpha$ *if and only if* $\mathfrak{A} \in \mathbf{I}\mathsf{Gp}_\alpha^{\mathrm{reg}}$.

(ii) $\mathfrak{A} \in \mathsf{CPES}_\alpha$ *if and only if* $\mathfrak{A} \in \mathbf{I}(\mathsf{Gp}_\alpha^{\mathrm{reg}} \cap \mathrm{Mod}\{(\mathrm{C}_4), (\mathrm{CP9})\})$.

We return to the proofs of the above theorems in Part 2 dealing with neat embedding theorems.

This result, in a sense, generalizes Andréka's result ([And]) concerning the finite scheme axiomatizability of the class G_α of *finite* dimensional locally square cylindric algebras (α is infinit).

Theorem 3.25 gives a kind of answer for the problem asked in [An-Go-Ne] and [And] whether G_α is a variety. And, Theorems 3.24 and 3.25 answer the other problem, whether transformation systems equipped with equalities and cylindrifications are representable (see [Kei] and [Slo]).

We do not know whether r-representation theorem exists for classical polyadic equality algebras (having infinite cylindrifications).

Remarks

a) The classes CPES_α and CPE_α are not representable in the classical sense (see [Da-Mo], [Slo]), therefore the class $\mathsf{Gp}_\alpha^{\mathrm{reg}}$ cannot be replaced by $\dot{\mathsf{Gp}}_\alpha^{\mathrm{reg}}$ in the above representation theorems. Similarly, ${}_m\mathsf{Gwp}_\alpha^{\mathrm{reg}}$ cannot be replaced by ${}_m\dot{\mathsf{Gwp}}_\alpha$ in Theorem 3.24.

b) With the second proposition of Theorem 3.25, the following cylindric algebraic theorem can be associated: cylindric algebras satisfying the merry-go-round axioms are representable by set algebras in $\mathsf{Crs}_\alpha \cap$ Mod$\{(\mathrm{C}_4), (\mathrm{C}_6)\}$ (or in $\mathsf{Crs}_\alpha \cap \mathsf{CA}_\alpha$, see [He-Mo-Ta II.], 3.2.88).

Finally, we state a consequence of Theorem 3.25 for cylindric algebras.

Corollary 3.26 *If* $\mathfrak{B}$ *is the cylindric reduct of some* $\mathfrak{A} \in \mathsf{CPE}_\alpha$*, where* α *is infinite,* then $\mathfrak{B}$ *is r-representable and* $\mathfrak{B} \in \mathbf{I}\mathsf{G}_\alpha^{\mathrm{reg}}$.

Concerning the concept of *cylindric reduct*, see below Definition 4.1.

Main references in this Chapter are: [Fe12a], [And], [Sa-Th], [Ha57], [Da-Mo], [Fe12b], [Nem86], [An-Go-Ne] and [Sai] .

Part II

Neat embedding theorems and their applications

Chapter 4

Neat embedding theorems for cylindric-type algebras

The classical neat embedding theorem of cylindric algebras says: $\mathfrak{A}$ is representable if and only if $\mathfrak{A} \in \mathrm{SNr}_\alpha \mathsf{CA}_{\alpha+\varepsilon}$, where $\varepsilon \geq \omega$ is an arbitrary but fixed ordinal, $\alpha \geq 2$, and $\mathsf{SNr}_\alpha \mathsf{CA}_{\alpha+\varepsilon}$ is the class of CA_α's that have the neat embedding property. The following question arises: *can this theorem be generalized from classical representability to r-representability*? In this Chapter, this question is investigated. At the end of the Chapter, some conclusions are drawn about the *classical* neat embedding theorem with respect to itself.

Definition 4.1 The *α-reduct* of a β-dimensional ($\alpha < \beta$) cylindric algebra

$$\mathfrak{C} = \langle A, +, \cdot, -, 0, 1, c_i, d_{ij} \rangle_{i,j<\beta}$$

is the cylindric algebra $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, c_i, d_{ij} \rangle_{i,j<\alpha}$, in notation $\mathfrak{A} = \mathfrak{Rd}_\alpha \mathfrak{C}$. The *neat α-reduct* of $\mathfrak{C}$ is the algebra $\mathfrak{D} = \langle D, +, \cdot, -, 0, 1, c_i, d_{ij} \rangle_{i,j<\alpha}$, where $D = \{b \in A : c_i b = b$ for every $\alpha \leq i < \beta\}$, in notation $\mathfrak{D} = \mathfrak{Nr}_\alpha \mathfrak{C}$.

Definition 4.2 An $\mathfrak{A} \in \mathsf{CA}_\alpha$ is *neatly embeddable into a* $\mathfrak{C} \in \mathsf{CA}_\beta$ $(\alpha < \beta)$ if there is an embedding e of $\mathfrak{A}$ into $\mathfrak{Rd}_\alpha\mathfrak{C}$ such that we have $c_i ea = ea$ for every $a \in A$ and for every $\alpha \leq i < \beta$. So $\mathfrak{A}$ is *neatly embeddable* into $\mathfrak{C}$, if it is isomorphic to a subalgebra of $\mathfrak{Nr}_\alpha\mathfrak{C}$, i.e., $\mathfrak{A} \in \mathrm{S}\mathfrak{Nr}_\alpha\mathfrak{C}$.

If K is a fixed subclass of CA_β, then $\mathrm{SNr}_\alpha\mathsf{K}$ denotes the class of the algebras neatly embeddable into some member of K, where S denotes forming subalgebra. These definitions can be reformulated analogously for cylindric-type algebras.

Recall that CNA^+_α denotes the class of cylindric-type algebras where the commutativity of the single substitutions is assumed instead of that of the cylindrifications, furthermore, the MGR is supposed (Definition 1.9). In this Chapter we use the short notation F_α for CNA^+_α.

Two unusual classes of algebras are introduced, denoted by $\mathsf{F}^\alpha_{\alpha+\varepsilon}$ and $\mathsf{M}^\alpha_{\alpha+\varepsilon}$. These classes are obtained from $\mathsf{F}_{\alpha+\varepsilon}$ and $\mathsf{CA}_{\alpha+\varepsilon}$, respectively. Instead of the axioms (C_4) and (C_6) certain *consequences* of them are postulated moreover, the schemas of these consequences are *restricted* to certain ordinals depending on α and ε. These axiom schemas may be considered as many sorted schemas.

Definition 4.3 ($\mathsf{F}^\alpha_{\alpha+\varepsilon}$) The axioms of $\mathsf{F}^\alpha_{\alpha+\varepsilon}$ are obtained from those of $\mathsf{F}_{\alpha+\varepsilon}$ if axioms (C_4), (C_6) and MGR are replaced by the axioms (C_4^-), (C_6^-) and MGR$^-$ below, where $\alpha \geq 3$, $\varepsilon \geq 1$ and $\alpha + \varepsilon$ is denoted by β:

(C_4^-) is the set of the following four properties:

(C_4^-)a) $s^i_m s^j_n x = s^j_n s^i_m x$ if $i, j, m, n \in \beta$, $i \neq j$ except for two cases: $i, j \in \alpha$, $m \notin \alpha$ and $i, j \in \alpha$, $n \notin \alpha$

(C_4^-)b) $s^i_m s^j_n x \leq s^j_n s^i_m x$ if $i, j, n \in \alpha$, $m \notin \alpha$, (i, j, n, m are different)

(C_4^-)c) $d_{ik} \cdot s^i_m s^j_n x \leq s^j_n s^i_m x$ if $i, j, k \in \alpha$, $n \notin \alpha$, (i, j, k, n, m are different)

(C_4^-)d) $c_i c_m x = c_m c_i x$, $m \notin \alpha$

(C_6^-) is the set of the diagonal properties in (1.1) with the following restriction for property d., denoted by (C_6^-)d.:

$c_i d_{ij} = 1$ if $i, j \in \beta$, except for the case $i \in \alpha$, $j \notin \alpha$

MGR$^-$: MGR restricted to α.

Another notation for $\mathsf{F}^{\alpha}_{\alpha+\varepsilon}$ is $\mathsf{F}_{\alpha,\ \alpha+\varepsilon}$.

Definition 4.4 ($M^{\varepsilon}_{\alpha+\varepsilon}$) This class is obtained from $\mathsf{F}^{\alpha}_{\alpha+\varepsilon}$ if we assume axiom (C_4) and (C_6) for the α-reduct of $\mathsf{F}^{\alpha}_{\alpha+\varepsilon}$ ($\alpha \geq 3, \varepsilon \geq 1$).

Obviously $\mathsf{M}^{\alpha}_{\alpha+\varepsilon} \subseteq \mathsf{F}^{\alpha}_{\alpha+\varepsilon}$ and, the α-reducts of the algebras in $\mathsf{F}^{\alpha}_{\alpha+\varepsilon}$ and $\mathsf{M}^{\alpha}_{\alpha+\varepsilon}$ are algebras in F_{α} and CA_{α} respectively. A *generic example* for an algebra in $\mathsf{F}^{\alpha}_{\alpha+\varepsilon}$ will be shown in the proof of Theorem 4.6.

Remark

The class $\mathsf{F}^{\alpha}_{\alpha+\varepsilon}$ is essentially different from the class $\mathsf{CA}_{\alpha+\varepsilon}$. For example, the equation $c_i d_{im} = 1$ is not necessarily true in $\mathsf{F}^{\alpha}_{\alpha+\varepsilon}$ if $i \in \alpha$, $m \notin \alpha$. Also the equations $c_j c_i d_{im} = c_i d_{im}$ or $c_j c_i c_j d_{jm} = c_i c_j d_{jm}$ are *not* necessarily true in $\mathsf{F}^{\alpha}_{\alpha+\varepsilon}$ (see the proof of Theorem 4.6). For example, the latter equation may be considered as a special case of $c_j c_i c_j b = c_i c_j b$ which is *not true* in $\mathsf{F}^{\alpha}_{\alpha+\varepsilon}$ in general.

Recall that an $\mathfrak{A} \in \mathsf{F}_{\alpha}$ is called r-representable if $\mathfrak{A} \in \mathbf{I}\mathsf{D}_{\alpha}$. The following two theorems are necessary and sufficient parts of a Main neat embedding theorem concerning r-representability (Corollary 4.7 due to the present author):

Theorem 4.5 *If* $\mathfrak{A} \in \mathrm{SNr}_{\alpha}\mathsf{F}^{\alpha}_{\alpha+\varepsilon}$, *then* $\mathfrak{A} \in \mathbf{I}\mathsf{D}_{\alpha}$, *where* $\alpha \geq 4$, ε *is any fixed infinite ordinal.*

Theorem 4.6 *If* $\mathfrak{A} \in \mathsf{D}_{\alpha}$, *then* $\mathfrak{A} \in \mathrm{SNr}_{\alpha}\mathsf{F}^{\alpha}_{\alpha+\varepsilon}$ *for any fixed* $\alpha \geq 4$, $\varepsilon \geq 2$.

These theorems will be proven below.

Theorem 4.5 and Theorem 4.6 together imply the following neat embedding theorem for r-representability:

Corollary 4.7 *Let* $\mathfrak{A} \in \mathsf{F}_\alpha$ $(\alpha \geq 4)$ *and let* ε *be any fixed infinite ordinal. Then the following properties* (i) *and* (ii) *are equivalent:*

(i) $\mathfrak{A}$ is r-representable (i.e., $\mathfrak{A} \in \mathbf{I}\mathsf{D}_\alpha$)

(ii) $\mathfrak{A} \in \mathrm{S}\mathrm{Nr}_\alpha \mathsf{F}^\alpha_{\alpha+\varepsilon}$.

The following proposition is an easy consequence of Theorem 4.6 and the RTA theorem:

Corallary 4.8 $\mathsf{F}_\alpha \in \mathrm{S}\mathrm{Nr}_\alpha \mathsf{F}^\alpha_{\alpha+\varepsilon}$ for $\varepsilon \geq 2$, $\alpha \geq 4$ (see [Fe07a]).

The following theorems are variants of Theorems 4.5 and 4.6, they are necessary and sufficient parts of a neat embedding theorem concerning *r-representation* and *cylindric algebras*. Since, recall that an $\mathfrak{A} \in \mathsf{CA}_\alpha$ is r-representable if $\mathfrak{A} \in \mathbf{I}\mathsf{Crs}_\alpha$, i.e., $\mathfrak{A} \in \mathbf{I}\mathsf{Crs}_\alpha \cap \mathsf{CA}_\alpha$.

Theorem 4.9 *If* $\mathfrak{A} \in \mathrm{S}\mathrm{Nr}_\alpha \mathsf{M}^\alpha_{\alpha+\varepsilon}$, $\alpha \geq 4$, *then* $\mathfrak{A} \in \mathbf{I}\mathsf{Crs}_\alpha \cap \mathsf{CA}_\alpha$, *where* ε *is any fixed infinite ordinal.*

Theorem 4.10 *If* $\mathfrak{A} \in \mathbf{I}\mathrm{Crs}_\alpha \cap \mathsf{CA}_\alpha$, $\alpha \geq 4$, *then* $\mathfrak{A} \in \mathrm{S}\mathrm{Nr}_\alpha \mathsf{M}^\alpha_{\alpha+\varepsilon}$ *for any fixed* $\varepsilon \geq 2$.

* * *

Now we come to the proofs of Theorem 4.5 and Theorem 4.6. First Theorem 4.5 is proved.

The outline of the proof is: *We define a* D_α*-unit, denoted by* V*, then we define an embedding of* $\mathfrak{A}$ *into the full set algebra in* D_α *with unit* V. To perform this, some lemmas are needed.

Let us fix an algebra $\mathfrak{B} \in \mathsf{F}^{\alpha}_{\alpha+\varepsilon}$ such that $\mathfrak{A}$ is a subalgebra of the neat α-reduct of $\mathfrak{B}$.

First, we introduce some concepts needed in the proof:

Let τ be a transformation on $\alpha+\varepsilon$ such that $\tau_{i_0} = m_0$, $\tau_{i_1} = m_1, \ldots, \tau_{i_{n-1}} = m_{n-1}$, else $\tau_i = i$ if $i \notin \{i_0, \ldots, i_{n-1}\}$. We refer to $\{i_0, \ldots, i_{n-1}\}$ as the domain of τ (Dom τ), and to $\{m_0, \ldots, m_{n-1}\}$ as the range of τ (Rg τ). τ defines a unary operator s_τ on $\mathfrak{B}$ as follows: $s_\tau = s^{i_0}_{m_i} \ldots s^{i_{n-1}}_{m_{n-1}}$.

Such a transformation s_τ is called an *admitted* transformation if Dom $\tau \subseteq \alpha$ and Rg $\tau \cap \alpha \neq \emptyset$ (this latter is equivalent to Rg $\tau \subseteq \beta \sim \alpha$ if $\beta = \alpha + \varepsilon$).

We refer to the single substitutions s^i_n contained in s_τ as the *members* of s_τ. Let R be the set of the admitted transformations on $\mathfrak{B}$.

A Boolean ultrafilter F in $\mathfrak{B}$ is perfect *if, for any element of the form $s_\tau c_j x$ included in F, where $j \in \alpha$, $x \in A$ and s_τ is any admitted transformation, there exists an $m \notin \alpha \cup$* Rg τ *such that $s_\tau s^j_m x \in F$.*

As is known, neat embeddability into ω extra dimensions implies neat embeddability into any infinite number of infinitely many extra dimensions, i.e., $\mathrm{SNr}_\alpha \mathsf{F}^{\alpha}_{\alpha+\varepsilon} = \mathrm{SNr}_\alpha \mathsf{F}^{\alpha}_{\alpha+\omega}$. Therefore from now on, we can assume that $\varepsilon > \max(\alpha, |A|)$ and ε is infinite.

Lemma 4.11 *Let a be an arbitrary, but fixed non zero element of A and $\varepsilon > \max(\alpha, |A|)$. Then there exists a perfect ultrafilter F in the algebra $\mathfrak{B} \in \mathsf{F}^{\alpha}_{\alpha+\varepsilon}$ such that $a \in F$.*

Proof.

Henkin's proof for completeness is adapted to the axioms of $\mathsf{F}^{\alpha}_{\alpha+\varepsilon}$.

Let

$$X = \{s_\tau c_j y \,:\, \tau \in R,\ j \in \alpha,\ y \in A\}.$$

Let β denote the ordinal $\alpha + \varepsilon$. By condition, $\varepsilon > \max(|A|, \alpha)$, ε is infinite, therefore $\alpha + \varepsilon = \varepsilon$ ($\beta = \varepsilon$) and $|X| = \beta$. Let $\rho : \beta \to X$ be a fixed enumeration of X.

Let F_0 be the Boolean (BA) filter of $\mathfrak{B}$ generated by a. Now we define *recursively* an increasing sequence $\langle F_i : i < \beta\rangle$ of proper BA filters of $\mathfrak{B}$.

Let n be a fixed ordinal ($n < \beta$). Assume that F_i $(0 \le i < n-1)$ has been defined.

Let $\rho_n = s_\tau c_j y$, where $\tau \in R$, $j \in \alpha$ and $y \in A$.

If n is *a limit ordinal*, then let $F_n = \bigcup\limits_{i<n} F_i$.

And, let $\langle m_n : n < \beta,\ m_n < \beta,\ \alpha < m_n\rangle$ be a sequence of ordinals such that $m_n \notin \alpha \cup \bigcup\limits_{i<n}(\text{ dim } \rho_i) \cup \text{Rg } \tau$.

Such a sequence m_n exists because $n < \beta$ $(= \varepsilon)$ and the finiteness of (dim ρ_k) $\sim \alpha$ imply that $\bigcup\limits_{i<n}(\text{ dim } \rho_i) \sim \alpha < \varepsilon$. Further Rg τ is finite and ε is infinite.

If n is a *successor ordinal*, let F_n be the filter generated by the set

$$F_{n-1} \cup \{s_\tau c_j y \to s_\tau s^j_{m_n} y\}$$

where $y \in A$. Obviously $F_i \subseteq F_n$ if $i < n$.

We show that F_n is a *proper* filter. The only case worthwhile considering is the case when n is a successor ordinal. So assume that F_{n-1} is proper and assume, seeking a contradiction, that F_n is not.

Let m denote now m_n. Suppose, on the contrary, that $-(s_\tau c_j y \to s_\tau s^j_m y)$ belongs to F_n. The property of generating filters in Boolean algebras implies that there are finitely many elements in F_{n-1} such that

$$a\ (s_{\tau_1} c_{j_1} y_1 \to s_{\tau_1} s^{j_1}_{m_1} y_1) \ldots (s_{\tau_k} c_{j_k} y_k \to s_{\tau_k} s^{j_k}_{m_k} y_k) \le -(s_\tau c_j y \to s_\tau s^j_m y) \qquad (4.1)$$

where $y_1, y_2, \ldots, y_k, y$ are in A. Let us apply c^∂_m to both sides of this inequality (where c^∂_m denotes the operator $-c_m-$).

If x is any factor of the left-hand side, then the condition $m \notin$dim ρ_k, $x \in F_{n-1}$ in the construction imply that

$$c_m(s_{\tau_i} c_{j_i} y_i \to s_{\tau_i} s^{j_i}_{m_i} y) = s_{\tau_i} c_{j_i} y_i \to s_{\tau_i} s^{j_i}_{m_i} y. \tag{4.2}$$

But (4.2) is true for c^{∂}_m instead for c_m, using that $c_m(-c_m x) = -c_m x$, $x \in B$. Thus applying c^{∂}_m to the left-hand side of (4.1), it does not change and it must be different from 0 because F_{n-1} is a proper one. Here we used that $c^{\partial}_m(u+v) = c^{\partial}_m u + c^{\partial}_m v$, which is a consequence of (C_3), and therefore it is true in $\mathsf{F}^{\alpha}_{\alpha+\varepsilon}$.

Applying c^{∂}_m to the right-hand side of (4.1), we show that it is zero. We have

$$c^{\partial}_m(-(s_\tau c_j y \quad \to \quad s_\tau s^j_m y)) = -[c_m(-s_\tau c_j y) + s_\tau s^j_m y)] = \tag{4.3}$$

$$-[c_m(-s_\tau c_j y) + c_m s_\tau s^j_m y] \tag{4.4}$$

because $c_m(u+v) = c_m u + c_m v$.

On one hand, by $m \notin \text{Dom}\ \tau$, ($C^-_4$)d) and (4.16)

$$c_m(-s_\tau c_j y) = c_m(-c_m s_\tau c_j y)$$

and here

$$c_m s_\tau c_j y = s_\tau c_j c_m y = s_\tau c_m s^j_m c_m y$$

therefore

$$c_m(-s_\tau c_j y) = -s_\tau c_m s^j_m c_m y \tag{4.5}$$

by $c_m(-c_m x) = -c_m x$.

On the other hand, similarly, for $c_m s_\tau s^j_m y$, i.e., for $c_m s_\tau s^j_m c_m y$

$$c_m s_\tau s^j_m c_m y = s_\tau c_m s^j_m c_m y. \tag{4.6}$$

From (4.5) and (4.6) we obtain that (4.3) is zero. Therefore applying c_m^{∂} to the right-hand side of (4.1), we show that it is zero. It is a contradiction, because the left-hand side is different from zero. It has been shown that F_n is a proper filter, in fact.

Now we have a sequence $G_0 = F_0 \subset F_1 \subset F_2 \subset \ldots \subset F_n \subset \ldots$ of proper filters. Now let $D = \cup\{F_n \,:\, n < \beta\}$. Then D is a proper filter, too. Let F be the ultrafilter generated by this filter. It is easily seen that F is a desired ultrafilter including the element a.

qed.

Let F be any fixed perfect ultrafilter. Let us consider the following equivalence relation $\equiv$ on $\beta \sim \alpha$:

$$m \equiv n \quad (m, n \in \beta \sim \alpha) \text{ if and only if } d_{mn} \in F. \tag{4.7}$$

The axioms (C_5), (C_6)a. and (C_6)b. ensure that $\equiv$ is an equivalence relation. Denote by Π the set of the equivalence classes and let us denote by M, N, L, $\ldots$ the classes $m/\equiv, n/\equiv, l/\equiv \ldots$, respectively.

First we prove *three useful properties*:

Lemma 4.12 *Assume that s_ν, s_σ and s_τ are admitted substitutions. The following properties* (i), (ii) *and* (iii) *are true:*

(i)

$$s_\nu s_m^j s_\sigma z = s_\nu s_m^j s_\sigma s_m^j z, \tag{4.8}$$

where $j \notin Dom\ \sigma$, i.e., supplying $s_\nu s_m^j s_\sigma$ by s_m^j on the right-hand side, "nothing changes",

(ii)

$$s_\tau z \in F \ \textit{ if and only if } \ s_\tau s_m^j z \in F \tag{4.9}$$

if $j \notin \text{Dom}\ \tau$, $j \in \alpha$ and $d_{jm} \in F$, $m \in \beta \sim \alpha$,

(iii)

$$\text{For every } i \in \alpha \text{ there exists a unique } m/\equiv,\ m \notin \alpha \text{ such that } d_{im} \in F. \tag{4.10}$$

Proof.

(i)

$$s_\nu s^j_m s_\sigma z \stackrel{\text{def. of } s^j_m}{=} s_\nu s^j_m (d_{jm} \cdot s_\sigma z) \stackrel{(\mathrm{C}_6)\mathrm{c. \ and \ }(\mathrm{C}_3)}{=} s_\nu s^j_m s_\sigma (d_{jm} \cdot z) \tag{4.11}$$

because $j \notin \operatorname{Dom} \sigma$. But really $s_\nu s^j_m s_\sigma (d_{jm} \cdot z) \stackrel{(\mathrm{C}_7)}{=} s_\nu s^j_m s_\sigma (d_{jm} \cdot s^j_m z) \stackrel{(4.11)}{=} s_\nu s^j_m s_\sigma s^j_m z$.

(ii) Similarly to the proof of (i), $d_{jm} \cdot s_\tau u = s_\tau (d_{jm} \cdot u)$. Therefore

$$d_{jm} \cdot s_\tau s^j_m z = s_\tau (d_{jm} \cdot s^j_m z) \stackrel{(\mathrm{C}_7)}{=} s_\tau (d_{jm} \cdot z) = d_{jm} \cdot s_\tau z.$$

So $d_{jm} \in F$ implies that $s_\tau s^j_m z \in F$ if and only if $s_\tau z \in F$.

(iii) Namely, $1 \stackrel{(\mathrm{C}_6)\mathrm{d.}}{=} c_j d_{ji} \in F$ $(j \neq i)$ and the perfect ultrafilter property imply that $s^j_m d_{ji} \in F$ for some $m \notin \alpha$. But $s^j_m d_{ij} = c_j (d_{jm} \cdot d_{ji}) \stackrel{(\mathrm{C}_6)\mathrm{b.}}{\leq} c_j d_{mi} \stackrel{(\mathrm{C}_6)\mathrm{c.}}{=} d_{mi}$ implies that $d_{mi} \in F$.

If $d_{im} \in F$ and $d_{in} \in F$ for different n and m, then $n \equiv m$. Namely $d_{im} \cdot d_{in} \in F$ and $d_{im} \cdot d_{in} \leq d_{mn}$, so really $d_{mn} \in F$. So by (4.7), with every $i \in \alpha$ an equivalence class with respect $\equiv$ can be uniquely associated.

qed.

Now, we define a D_α-unit V, as we indicated in the outline of the proof. The members of the *alpha*-sequences in V will be equivalence classes with respect to $\equiv$. We define V by "subunits".

For the fixed y, $y \in A$, $y \neq 0$, let us consider a fixed ultrafilter F_y containing y. Such a filter exists by Lemma 4.11. With y and F_y we associate a subset W_y of V in the following way (we omit the index y if misunderstanding is excluded):

Let the *support* of W_y be an α-sequence Y such that Y_i is the equivalence class in Π associated with i by (4.10). Let

$$W_y = \{f_\tau Y \,:\, s_\tau 1 \in F_y, s_\tau \text{ is admitted}\}\,, \tag{4.12}$$

where $f_\tau Y$ is defined in the following way: with the admitted substitution $s_\tau = s^i_n s^j_m \dots s^k_p$ and let us associate the α-sequence

$$(((Y^i_N)^j_M)\dots)^k_P,$$

where $N, M, \dots, P$ are the classes in Π containing $n, m, \dots, p$, respectively, i.e., the classes $n/\equiv, m/\equiv, \dots, p/\equiv$. Here the meaning of Y^i_N is the usual, i.e., $(Y^i_N)_i = n/\equiv$ and $(Y^i_N)_j = Y_j$ if $j \neq i$. We denote the sequence Y^i_M by $f^i_M Y$ too, so the sequence $(((Y^i_N)^j_M)\dots)^k_P$ can be denoted by $f^k_P \dots f^j_M f^i_N Y$ or by $f_\tau Y$, for short.

Lemma 4.13 *The definition* (4.12) *of* W_y *is sound and* W_y *is a subunit with support* Y*.*

Proof.

We show that the definition (4.12) does not depend on the choice of the representative points, i.e.,

$$s^i_{n_1} s^j_{m_1} \dots s^k_{p_1} 1 \in F_y \text{ if and only if } s^i_{n_2} s^j_{m_2} \dots s^k_{p_2} 1 \in F_y \tag{4.13}$$

where $n_1 \equiv n_2,\ m_1 \equiv m_2, \dots, p_1 \equiv p_2$ (i.e., $d_{n_1 n_2}, d_{m_1 m_2}, \dots, d_{p_1 p_2} \in F_y$). Namely, for example,

$$d_{n_1 n_2} \cdot s^i_{n_1} s^j_{m_1} \dots s^k_{p_1} 1 \overset{(C_3)}{=} s^i_{n_1}(d_{n_1 n_2} \cdot s^j_{m_1} \dots s^k_{p_1} 1) \overset{def.\ of\ s^i_{n_1}}{=}$$
$$= c_i(d_{i n_1} \cdot d_{n_1 n_2} \cdot s^j_{m_1} \dots s^k_{p_1} 1) \overset{(C_6)\text{b.}}{\leq} c_i(d_{i n_2} \cdot s^j_{m_1} \dots s^k_{p_1} 1) = s^i_{n_2} s^j_{m_2} \dots s^k_{p_2} 1.$$

But $d_{n_1n_2} \in F_y$ and $s^i_{n_1}s^j_{m_1}\dots s^k_{p_1}1 \in F_y$ imply $s^i_{n_2}s^j_{m_2}\dots s^k_{p_2}1 \in F_y$. Repeating this procedure, multiplying by the elements $d_{m_1m_2},\dots,d_{p_1p_2}$ step by step, we obtain that $s^i_{n_2}s^j_{m_2}\dots s^k_{p_2}1 \in F_y$. The proof of the other implication in (4.13) is similar.

Now it is shown that $Y \in W_y$. Namely, we choose the relativized identity for τ in (4.12), i.e., let $s_\tau = s^i_n s^j_m \dots s^k_p$ such that $d_{in}, d_{jm},\dots,d_{kp} \in F_y$. Then $s_\tau 1 = s^i_n s^j_m \dots s^k_p 1 = c_i c_j \dots c_k(d_{in}\cdot d_{jm}\cdot\ldots\cdot d_{kp}) \geq d_{in}\cdot d_{jm}\cdot\ldots\cdot d_{kp} \in F_y$, so really $s_\tau 1 \in F_y$. It is obvious that W_y is a subunit of a Crs_α, with support Y.

qed.

It will be proved in Lemma 4.17 that V is a D_α unit.

We continue to realize our plan for the proof. Let the definition of the expected embedding h' of $\mathfrak{A}$ into the full Crs_α with unit V be

$$hz = \{f_\tau Y \,:\, s_\tau z \in F,\ s_\tau \text{ is admitted}\}, \tag{4.14}$$

where $z \in A$ and h denotes the restriction of h' to the subunit W_y.

Two remarks concerning the definitions (4.12) and (4.14) are:

a) $W_z = h1$, by definition.

b) Notice that $hz \subseteq W_y$ because $s_\tau z \leq s_\tau 1$, therefore $s_\tau z \in F_y$ imply that $s_\tau 1 \in F_y$. Therefore really $f_\tau Y \in W_y$, by definition.

It will be shown in Lemma 4.16 that h' is really an embedding of $\mathfrak{A}$. But first, in Lemma 4.14 below we check that the definition in (4.14) is sound. That is, we prove that the definition does not depend on the choice of τ (especially from the choice of the representatives concerning $\equiv$).

Lemma 4.14 *Assume that $\mathcal{E} \in \mathsf{F}^\alpha_{\alpha+\varepsilon}$. The following two properties are true:*

$$s^{j_0}_t s^{j_1}_t \dots s^{j_{n-1}}_t \; s^{j_0}_{m_0} s^{j_1}_{m_1} \dots s^{j_{n-1}}_{m_{n-1}} z = s^{j_0}_t s^{j_1}_t \dots s^{j_{n-1}}_t \; s^{k_0}_{p_0} s^{k_1}_{p_1} \dots s^{k_{n-1}}_{p_{n-1}} z, \tag{4.15}$$

where $z \in \mathfrak{C}$, $j_0, j_1, \ldots, j_{n-1} \in \alpha$ *are distinct,* $t \notin \{j_0, j_1, \ldots, j_{n-1}\}$, $m_0, m_1, \ \ldots, m_{n-1} \in \beta \sim \alpha$, *and the sequence* $s^{k_0}_{p_0} s^{k_1}_{p_1} \ldots s^{k_{n-1}}_{p_{n-1}}$ *is a permutation of the sequence* $s^{j_0}_{m_0} s^{j_1}_{m_1} \ldots s^{j_{n-1}}_{m_{n-1}}$, *furthermore*

$$c_m s^j_m c_m z = c_j c_m z \tag{4.16}$$

$j \in \alpha$, $m \notin \alpha$.

Proof of (4.15): Consider the case $n = 2$. We prove that $s^j_t s^i_t s^j_m s^i_n z = s^j_t s^i_t s^i_n s^j_m z$. But

$$s^j_t s^i_t s^j_m s^i_n z \overset{\text{def. of } s^j_t}{=} s^j_t (d_{jt} \cdot s^i_t s^j_m s^i_n z) \overset{(\mathrm{C}_6)\mathrm{c.}}{=} s^j_t s^i_t (d_{jt} \cdot s^j_m s^i_n z) \overset{(\mathrm{C}^-_4)\ \mathrm{c})}{\leq} s^j_t s^i_t s^i_n s^j_m z).$$

The proof of the converse is the same.

The proof of the general case is similar because we can change any neighboring members of $s^{j_0}_{m_0} s^{j_1}_{m_1} \ldots s^{j_{n-1}}_{m_{n-1}}$ making use of (C_6)c. and (C^-_4) c) and the fact that $j_0, j_1, \ldots, j_{n-1}$ and t are different.

Proof of (4.16):

$$\begin{aligned} & c_m s^j_m c_m z = \\ & = \ c_m c_j (d_{jm} \cdot c_m z) \overset{(\mathrm{C}^-_4)\ \mathrm{d})}{=} c_j c_m (d_{jm} \cdot c_m z) \overset{(\mathrm{C}_3)}{=} c_j (c_m z \cdot c_m d_{jm}) \\ & \quad \overset{(\mathrm{C}^-_6)\mathrm{d.}}{=} c_j c_m z. \end{aligned}$$

qed.

Let the admitted substitutions s_τ *with the property* $s_\tau 1 \in F_y$ *be called* realized *substitutions.*

The origin of this definition is that W_y is obtained in (4.12) in terms of this kind of substitutions only. The substitutions in (4.14) are also this kind of substitutions.

Fix the subunit W_y (W, for short) corresponding to the perfect ultrafilter F_y (F, for short).

Lemma 4.15 *If $f_\tau y = f_\sigma y$ for some realized substitutions s_τ and s_σ, then*

$$s_\tau z \in F \quad \text{if and only if} \quad s_\sigma z \in F \tag{4.17}$$

for every $z \in \mathfrak{B}$.

Proof.

First, consider the case when the upper indices are different in s_τ and the single members of s_σ are a *permutation* of that of s_τ.

So let s_τ be of the form: $s^{j_0}_{m_0} s^{j_1}_{m_1} \dots s^{j_{n-1}}_{m_{n-1}}$, where $j_0, j_1, \dots, j_{n-1}$ are different and let $s^{k_0}_{p_0} s^{k_1}_{p_1} \dots s^{k_{n-1}}_{p_{n-1}}$ be a permutation of the members $s^{j_i}_{m_i}$ in s_τ.

Let s_H denote the transformation $s^{j_0}_t s^{j_1}_t \dots s^{j_{n-1}}_t$, where $t \in \alpha$ is an arbitrary fixed and $t \notin \mathrm{Dom}\{j_0, j_1, \dots, j_{n-1}\}$.

Consider the transformation $s_H s_\tau$.

One one hand,

$$s_\tau z \leq s_H s_\tau z. \tag{4.18}$$

To prove this, consider the special case $n = 3$. If

$$s_\tau = s^j_m s^i_p s^l_r \tag{4.19}$$

then

$$s_H s_\tau z = s^j_t s^i_t s^l_t s^j_m s^i_p s^l_r z \geq s^j_t s^i_t s^j_m s^l_t s^i_p s^l_r z \geq s^j_t s^i_t s^j_m s^i_p s^l_t s^l_r z \tag{4.20}$$

by (C_4^-) b).

But $s^l_t s^l_r\, z = c_l(d_{lt} \cdot (c_l(d_{lr} \cdot z))) = (c_l d_{lt}) \cdot c_l(d_{lr} \cdot z) = s^l_r z$ in $\mathfrak{A}$ by (C_3) and by $c_l d_{lt} = 1$. So we can eliminate s^l_t from the right-hand side in (4.20). Repeating this procedure for the element $s^j_t s^i_t s^j_m s^i_p s^l_t s^l_r z$ obtained, for s^i_t and s^j_t we obtain (4.18).

The proof of the general case of (4.18) is completely similar. On the other hand,

$$s_H s_\tau z = s_H s_\sigma z \tag{4.21}$$

by (4.15).

Comparing (4.18) and (4.21)

$$s_\tau z \leq s_H s_\sigma z. \tag{4.22}$$

To go on with the proof of (4.17) we prove the following inequality:

$$s_H s_\sigma z \cdot s_\sigma 1 \leq s_\sigma z. \tag{4.23}$$

We consider again the case $n = 3$. Assume that s_τ is of the form as in (4.19) and let s_σ (a permutation of the members in s_τ) be $s_r^l s_m^j s_n^i$. Then the inequality (4.23) is:

$$s_t^j s_t^i s_t^l s_r^l s_m^j s_n^i z \cdot s_r^l s_m^j s_n^i 1 \leq s_r^l s_m^j s_n^i z. \tag{4.24}$$

By (C_4^-) a).

$$s_t^j s_t^i s_t^l s_r^l s_m^j s_n^i z = s_t^l s_t^j s_t^i s_r^l s_m^j s_n^i z \tag{4.25}$$

so the right-hand side begins with s_t^l. Then we can move the first factor $s_t^l s_t^j s_t^i s_r^l s_m^j s_n^i z$ of the left-hand side of (4.24) into $s_r^l s_m^j s_n^i 1$ behind s_r^l, i.e.,

$$s_t^l s_t^j s_t^i s_r^l s_m^j s_n^i z \cdot s_r^l s_m^j s_n^i 1 = s_r^l ((s_t^l s_t^j s_t^i s_r^l s_m^j s_n^i z) \cdot s_m^j s_n^i 1) \tag{4.26}$$

using (C_3).

Applying the argument above twice, we obtain that the left-hand side of (4.24) is equal to

$$s_r^l s_m^j s_n^i (s_t^i s_t^j s_t^l s_r^l s_m^j s_n^i z). \tag{4.27}$$

Notice here that the order of the upper indices in $s_t^i s_t^j s_t^l s_r^l s_m^j s_n^i$ is symmetrical: i, j, l, l, j, i.

The element in (4.27) is less than the element

$$s_r^l s_m^j s_n^i s_t^i s_t^j c_l s_r^l s_m^j s_n^i z = s_r^l s_m^j s_n^i s_t^i s_t^j s_r^l s_r^j s_m^i s_n^i z \tag{4.28}$$

so we can eliminate s_t^l in (4.27).

But on the right-hand side in (4.28) the single substitution s_r^l is repeated and l does not occur in the "upper indices" between these members. Therefore similarly to (4.8) in Lemma 4.12 (i), the right-hand side of (4.28) is

$$s_r^l s_m^j s_n^i s_t^i s_t^j s_m^j s_n^i z. \tag{4.29}$$

So, by increasing the element in (4.27) we eliminated s_t^l, then the second s_r^l. Similarly, increasing the element in (4.29), we can eliminate s_t^j and the second s_m^j, then s_t^i and the second s_n^i. So really we obtain (4.24).

The proof of the general case in (4.23) is similar.

Comparing the relations in (4.22) and (4.23), using that s_σ is realized, i.e., $s_\sigma 1 \in F$, we obtain that $s_\tau z \in F$ implies $s_\sigma z \in F$. Using symmetry we obtain in the same way that $s_\sigma z \in F$ implies $s_\tau z \in F$, therefore Lemma 4.15 is proven for the case of permutations.

Proof of the general case:

The general case is reduced to the case of permutation.

So let s_τ be of the form $s_{m_0}^{j_0} s_{m_1}^{j_1} \ldots s_{m_{n-1}}^{j_{n-1}}$, where repetitions *are allowed* in the sequence $j_0, j_1, \ldots, j_{n-1}$.

First the "multiple upper indices" are eliminated from s_τ, i.e., we achieve that the upper indices should be different.

Choose a $t \in \alpha$ such that $t \notin \{j_0, j_1, \dots, j_{n-1}\}$. We know that for example, $c_{j0}d_{j0t} = 1$. Then by (C$_3$)

$$(c_{j0}d_{j0t}) \cdot s^{j_0}_{m_0} s^{j_1}_{m_1} \dots s^{j_{n-1}}_{m_{n-1}} z = c_{j0}(d_{j0t} \cdot s^{j_0}_{m_0} s^{j_1}_{m_1} \dots s^{j_{n-1}}_{m_{n-1}} z). \tag{4.30}$$

If $j_0 \neq j_1$, then by (C$_4^-$) c), (4.30) is less than $c_{j0}(d_{j0t} \cdot s^{j_1}_{m_1} s^{j_0}_{m_0} \dots s^{j_{n-1}}_{m_{n-1}} z)$.

If $j_0 = j_1$, then for (4.30), by (C$_3$)

$$\begin{aligned} & c_{j0}(d_{j0t} \cdot c_{j0}(d_{j0m0} \cdot s^{j_1}_{m_1} \dots s^{j_{n-1}}_{m_{n-1}} z)) \leq \\ \leq \quad & c_{j0}(d_{j0t} \cdot s^{j_1}_{m_1} \dots s^{j_{n-1}}_{m_{n-1}} z) \cdot c_{j0}d_{j0m0} \leq c_{j0}(d_{j0t} \cdot s^{j_1}_{m_1} \dots s^{j_{n-1}}_{m_{n-1}} z). \end{aligned}$$

Repeating this procedure we can eliminate the single substitutions with upper index $j0$ except for the last one in s_τ. Moreover, this procedure can be applied for the indices $j_1, \dots, j_{n-1}$ too, and we obtain that

$$s_\tau z \leq s_{\tau_1} z \tag{4.31}$$

where the upper indices in $s_{\tau_1} z$ are already different.

Now among the upper indices of s_{τ_1} there is no repetition, but in s_{τ_1} members of type s^i_n can occur, where $i \equiv n$, i.e., $d_{in} \in F$. Let us omit these members from s_{τ_1} and denote by s_{τ_2} the substitution obtained. We state that

$$s_{\tau_1} z \in F \text{ if and only if } s_{\tau_2} z \in F. \tag{4.32}$$

If s_{τ_1} is of the form $s^j_m s^i_n s^l_r$ e.g., then multiplying it by d_{in} we obtain that $d_{in} \cdot s^j_m s^i_n s^l_r z = s^j_m(d_{in} \cdot s^i_n s^l_r z) = s^j_m(d_{in} \cdot s^l_r z) = d_{in} \cdot s^j_m s^l_r z$ by (C$_6$)c. and (C$_7$). Since $d_{in} \in F$, really $s^j_m s^i_n s^l_r z \in F$ if and only if $s^j_m s^l_r z \in F$. The proof of the general case in (4.32) is similar.

Let us associate the transformations s_{σ_1} and s_{σ_2} with s_σ in the same way as we associated s_{τ_1} and s_{τ_2} with s_τ. The condition $f_\tau y = f_\sigma y$ of Lemma 4.15 and the constructions of s_{τ_2} and s_{σ_2} imply that s_{σ_2} is a permutation of the single substitutions in s_{τ_2} and $f_{\tau_2} y = f_{\sigma_2} y$.

Therefore applying (4.17) for the case of permutation τ

$$s_{\tau_2} z \in F \text{ if and only if } s_{\sigma_2} z \in F. \tag{4.33}$$

Further, similarly to (4.32) and (4.18) we obtain

$$s_{\sigma_1} z \in F \text{ if and only if } s_{\sigma_2} z \in F \tag{4.34}$$

and

$$s_{\sigma_1} z \leq s_K s_{\sigma_2} z, \tag{4.35}$$

where the set K for σ_1 is analogous with the set H for τ.

We state the following inequality:

$$s_K s_{\sigma_1} z \cdot s_\sigma s_{\sigma_1} 1 \leq s_\sigma s_{\sigma_1} z. \tag{4.36}$$

The proof is similar to that of (4.23):

First, by (C_4^-) a) changing the order of the members in s_K, we can move $s_K s_{\sigma_1} z$ behind $s_\sigma s_{\sigma_1}$ in $s_\sigma s_{\sigma_1} 1$ and similarly to (4.27) we obtain that the left-hand side of (4.36) equals

$$s_\sigma s_{\sigma_1} s_K s_{\sigma_1} z. \tag{4.37}$$

As at the proof of (4.23), increasing (4.37), $s_K s_{\sigma_1}$ can be eliminated from (4.37), so $s_\sigma s_{\sigma_1} s_K s_{\sigma_1} z \leq s_\sigma s_{\sigma_1} z$, so really (4.36) is true.

Given that s_σ is realized ($s_\sigma 1 \in F$), and using the definition of s_{σ_1}, by (4.8)

$$s_\sigma 1 \in F \text{ implies } s_\sigma s_{\sigma_1} 1 \in F.$$

Moreover, by (4.8) and the definitions of s_σ and s_{σ_1}

$$s_\sigma s_{\sigma_1} z \in F \text{ if and only if } s_\sigma z \in F. \tag{4.38}$$

Finally, let us compare the following relations:

$$\begin{aligned} s_\tau z &\overset{(4.31)}{\leq} s_{\tau_1} z \overset{(4.32)}{\sim} s_{\tau_2} z \overset{(4.33)}{\sim} s_{\sigma_2} z \overset{(4.34)}{\sim} s_{\sigma_1} z \\ &\overset{(4.35)}{\leq} s_K s_{\sigma_1} z \overset{(4.36)}{\sim} s_\sigma s_{\sigma_1} z \overset{(4.38)}{\sim} s_\sigma z, \end{aligned}$$

where the meaning of $a \sim b$ is: $a \in F$ is equivalent to $b \in F$.

Therefore $s_\tau z \in F$ implies $s_\sigma z \in F$.

Using the symmetry of the argument, this relation can be reversed. The proof is complete.

qed.

Now we will prove that the mapping h' defined in (4.14) is an embedding of $\mathfrak{A}$.

Lemma 4.16 *h' is a homomorphism on $\mathfrak{A}$ and $h'z \neq \emptyset$* if $z \neq 0$ (i.e., *h' is an embedding of* $\mathfrak{A}$).

Proof.

First we check that $h'z \neq \emptyset$ if $z \neq 0$. It can be proved in the same way as checking in Lemma 4.13 that $Y \in W_y$ because specially $z \in F_z$, by the definition of F_z. We state that $s^i_m z \in F_z$, where m is the ordinal associated with i in (4.10). Namely, $d_{im} \in F$ and $z \in F_z$ imply that $s^i_m z = c_i(d_{im} \cdot z) \in F$. Then $z^i_M \in hz$ by (4.14), so really $h'z \neq \emptyset$.

We prove the homomorphism property *by subunits.* Let us fix a subunit W_y corresponding to the ultrafilter F_y. Let us denote W_y and F_y by W and F, for short and, further, let h denote the restriction of h' to W_y.

First we prove that

$$hc_iz = C_ihz$$

where C_i is relativized to W, so C_i is $C_i^{[W]}$.

The left-hand side is

$$hc_iz = \{f_\nu Y \ : \ s_\nu c_iz \in F\}, \tag{4.39}$$

the right-hand side is $C_ihz = C_i\{f_\tau Y \ : \ s_\tau z \in F\}$. $hc_iz \subseteq C_ihz$. If $f_\nu Y \in hc_iz$, i.e., $s_\nu c_iz \in F$, then $s_\nu s_n^i z \in F$ for some $n \notin \alpha$ because F is perfect. Therefore by definition of hz, $f_N^i f_\nu Y \in hz$, so $f_\nu Y \in C_ihz$.

$C_ihz \subseteq hc_iz$. If $f_\nu Y \in C_ihz$, then $f_N^i f_\nu Y \in hz$ for some $N \in \Pi$, so $s_\nu s_n^i z \in F$.

But $s_\nu s_n^i z = s_\nu c_i d_{in} z \leq s_\nu c_i z$, therefore $s_\nu c_i z \in F$. Thus by (4.39), really $f_\nu Y \in hc_iz$.

We state that $h(u+v) = hu \cup hv$. Here $h(u+v) = \{f_\tau Y \ : \ s_\tau(u+v) \in F\}$, $hu = \{f_\tau Y \ : \ s_\tau u \in F\}$, $hv = \{f_\tau Y \ : \ s_\tau v \in F\}$. $c_i(u+v) = c_iu + c_iv$ implies that $s_\tau(u+v) = s_\tau u + s_\tau v$.

If $f_\tau Y \in hu \cup hv$ then, for example, $f_\tau Y \in hv$, i.e., $s_\tau v \in F$. But $s_\tau v \in F$ and the ultrafilter property imply that

$$s_\tau u + s_\tau v \in F. \tag{4.40}$$

So $s_\tau(u+v) \in F$, consequently $f_\tau Y \in h(u+v)$.

If $f_\tau Y \in h(u+v)$, then $s_\tau(u+v) = s_\tau u + s_\tau v \in F$. F is an ultrafilter, therefore $s_\tau u \in F$ or $s_\tau v \in F$. Therefore $f_\tau Y \in hu$ or $f_\tau Y \in hv$, so $f_\tau Y \in hu \cup hv$.

Then we prove that

$$h(-z) = \ \sim hz$$

where $\sim$ concerns W, so $\sim$ is $\sim_W$.

Here $hz = \{f_\sigma Y \ : \ s_\sigma z \in F\}$ and $h(-z) = \{f_\tau Y \ : \ s_\tau(-z) \in F\}$, therefore using the ultrafilter property of F

$$\sim hz = \ \sim \{f_\sigma Y \,:\, s_\sigma z \in F\} = \{f_\sigma Y \,:\, s_\sigma z \notin F\} = \{f_\sigma Y \,:\, -s_\sigma z \in F\}$$

where s_σ is a realized substitution.

We note that

$$s_\tau z + s_\tau(-z) = s_\tau(z + (-z)) = s_\tau 1 \in F. \tag{4.41}$$

We state that $\sim hz \subseteq h(-z)$. Assume that $f_\sigma Y \in \sim hz$, i.e., $s_\sigma z \notin F$. It must be proved that $f_\sigma Y \in h(-z)$, i.e., $s_\sigma(-z) \in F$. By (4.41) we know that $s_\sigma z + s_\sigma(-z) \in F$. So $s_\sigma z \notin F$ and the ultrafilter property imply that $s_\sigma(-z) \in F$.

Conversely $h(-z) \subseteq \sim hz$.

We note that $s_\tau(-z) = -s_\tau z$ is not true in an $\mathsf{F}^\alpha_{\alpha+\varepsilon}$ algebra, in general. But we prove the following two properties of s_τ (where s_τ is a realized substitution):

$$s_\tau z \cdot s_\tau(-z) = 0 \tag{4.42}$$

$$s_\tau(-z) = s_\tau 1 \cdot (-s_\tau z). \tag{4.43}$$

We prove them simultaneously, by induction, by the number k of the single substitutions in s_τ.

Assume that $k = 1$. Then for (4.42)

$$s^i_n z \cdot s^i_n(-z) = 0$$

is true by (C_7).

Adding $-s^i_n z \cdot s^i_n(-z)$ to both sides we obtain

$$s^i_n(-z) = -s^i_n z \cdot s^i_n(-z). \tag{4.44}$$

By (4.41), $s_n^i z + s_n^i(-z) = s_n^i 1$. Multiplying this equation by $-s_n^i z$ and using (4.44), we obtain $s_n^i(-z) = s_n^i 1 \cdot -s_n^i z$, and this is really (4.43).

Assume that the properties (4.42) and (4.43) are true if $k \leq m$.

They are proved if $k = m+1$. Let s_τ be of the form $s_n^i s_\sigma$, where the number of the single substitutions in s_σ is m.

$(s_n^i s_\sigma z) \cdot (s_n^i s_\sigma(-z)) = s_n^i s_\sigma z \cdot s_n^i(s_\sigma 1 \cdot (-s_\sigma z))$ using (4.43) for $s_\sigma(-z)$. But $s_\sigma 1 \leq 1$, therefore $s_n^i s_\sigma z \cdot s_n^i(s_\sigma 1 \cdot (-s_\sigma z)) \leq s_n^i s_\sigma z \cdot s_n^i(-s_\sigma z) = 0$ by (C_7). So

$$(s_n^i s_\sigma z) \cdot (s_n^i s_\sigma(-z)) = 0$$

i.e., (4.42) follows. From this, similarly to (4.44), we obtain

$$(s_n^i s_\sigma(-z)) = (-s_n^i s_\sigma z) \cdot (s_n^i s_\sigma(-z)). \tag{4.45}$$

To prove (4.43), if $k = m+1$, (4.41) is used, i.e., $(s_n^i s_\sigma z) + (s_n^i s_\sigma(-z)) = s_n^i s_\sigma 1$. Multiplying the equation by $-s_n^i s_\sigma z$ and using (4.45) we obtain (4.43).

Coming to the proof of $h(-z) \subseteq \sim hz$, assume that $f_\tau Y \in h(-z)$, i.e., $s_\tau(-z) \in F$. We prove that $f_\tau Y \in \sim hz$, i.e., $-s_\tau z \in F$. Indirectly if $-s_\tau z \notin F$, then $s_\tau z \in F$. (4.43) and the ultrafilter property imply that $s_\tau(-z) \notin F$. This is a contradiction. So really $-s_\tau z \in F$.

h' preserves 0 and 1 by definition. Now we prove that h preserves the diagonals.

We fix a subunit W with support element Y. D_{ij}^W will denote the restriction of the diagonal element D_{ij} to W.

First it is shown that we can assume, without the loss of generality, that D_{ij} restricted to W is of the form:

$$D_{ij} = \left\{ f_L^i f_L^j f_\nu Y \ : \ s_\nu s_m^j s_n^i 1 \in F, \ d_{mn} \in F \right\} \tag{4.46}$$

where L is the equivalence class containing m and n.

Namely, assume that $X \in D_{ij}^W$ and X is of the form $f_\tau Y$ ($f_\tau Y \in V$), so $X_i = X_j$. By definition of V, s_τ is realized, so $s_\tau 1 \in F$.

First case: $i, j \in$Dom τ. Then f_τ can be composed into the form

$$f_\alpha f_L^i f_\beta f_L^j f_\gamma \tag{4.47}$$

where α and β are such that $i, j \notin$ Dom α, $j \notin$ Dom β. So $s_\tau = s_\gamma s_m^j s_\beta s_n^i s_\alpha$ is a realized substitution (where $m \equiv n$ and L is the equivalence class containing them).

But by (4.8), $s_\tau 1 = s_\gamma s_m^j s_\beta s_n^i s_\alpha 1 = s_\gamma s_m^j s_\beta s_n^i s_\alpha s_m^j s_n^i 1 = s_\tau s_m^j s_n^i 1$ so $s_\tau s_m^j s_n^i$ is also realized. Further $f_L^i f_L^j f_\tau Y = f_\tau Y$ because f_τ is of the form in (4.47), so D_{ij}^W is of the form in (4.46).

Second case: $i, j \notin$ Dom τ. Then obviously $f_\tau Y = f_L^i f_L^j f_\tau Y$, where L is the class containing m and n, and m, n are such that $d_{jm} \in F$ and $d_{jn} \in F$. Such m and n exist by (4.10). By (4.9), $s_\tau 1 \in F$ if and only if $s_\tau s_m^j 1 \in F$ if and only if $s_\tau s_m^j s_n^i 1 \in F$. Further, $d_{jm} \cdot d_{jn} \leq d_{mn}$ implies that $d_{mn} \in F$. So $s_\tau s_m^j s_n^i$ is realized too. So D_{ij}^W is of the form (4.46).

Third case: exactly one of i and j is an element of Dom τ. This case can be reduced to the first and second case.

So it can be assumed that D_{ij}^W is of the form in (4.46).

We prove that $hd_{ij} = D_{ij}^W$.

$$hd_{ij} = \{f_\nu Y \, : \, s_\nu d_{ij} \in F\} \, .$$

By (4.8) $s_\nu d_{ij} = s_\nu s_m^j s_n^i d_{ij}$, where s_m^j and s_n^i are the last members of s_ν respectively, with upper indices j and i, supposing that there are such members in s_ν. If i and j are not included in Dom ν then, by (4.8) and (4.9) $s_\nu d_{ij} \in F$ if and only if $s_\nu s_m^j s_n^i d_{ij} \in F$, where $d_{jm} \in F$, $d_{in} \in F$. The case $i \in$ Dom ν, $j \notin$ Dom ν is similar. Therefore hd_{ij} may be considered to be of the form

$$\Big\{ f_N^i f_M^j f_\nu Y \,:\, s_\nu s_m^j s_n^i d_{ij} \in F \Big\}. \tag{4.48}$$

We prove that $hd_{ij} \subseteq D_{ij}^W$. Assume that $f_N^i f_M^j f_\nu Y \in hd_{ij}$, i.e., $s_\nu s_m^j s_n^i d_{ij} \in F$.

First we show that

$$s_\nu s_m^j s_n^i d_{ij} \in F \text{ implies that } d_{mn} \in F. \tag{4.49}$$

We check that

$$d_{jm} \cdot d_{in} \cdot d_{ij} = d_{mn} \cdot d_{jm} \cdot d_{in}. \tag{4.50}$$

Namely, $d_{jm} \cdot d_{in} \cdot d_{ij} \leq d_{in} \cdot d_{mi} \leq d_{mn}$ by (C_6)b., so by multiplying the inequality by $d_{jm} \cdot d_{in}$ we obtain the one direction of (4.50)

$$d_{jm} \cdot d_{in} \cdot d_{ij} \leq d_{jm} \cdot d_{in} \cdot d_{mn}. \tag{4.51}$$

As regards the other direction: $d_{in} \cdot d_{jm} \cdot d_{mn} \leq d_{in} \cdot d_{jn} \leq d_{ij}$ by (C_6)b. and multiplying the inequality by $d_{in} \cdot d_{jm}$ we obtain

$$d_{in} \cdot d_{jm} \cdot d_{mn} \leq d_{in} \cdot d_{jm} \cdot d_{ij}.$$

Further

$$\begin{aligned} s_\nu s_m^j s_n^i d_{ij} &= s_\nu c_j (d_{jm} \cdot c_i (d_{in} \cdot d_{ij})) = s_\nu c_j c_i (d_{jm} \cdot d_{in} \cdot d_{ij}) \\ &= s_\nu c_j c_i (d_{mn} \cdot d_{jm} \cdot d_{in}) = d_{mn} \cdot s_\nu s_m^j s_n^i 1 \end{aligned} \tag{4.52}$$

by (4.50) and (C_6)c.

$s_\nu s_m^j s_n^i 1 \in F$ since $s_\nu s_m^j s_n^i d_{ij} \in F$. So by (4.52), $s_\nu s_m^j s_n^i d_{ij} \in F$ implies that really

$d_{mn} \in F$. If L is the equivalence class containing m and n, then $f_N^i f_M^j f_\nu Y = f_L^i f_L^j f_\nu Y$, where $s_\nu s_m^j s_n^i 1 \in F$ and $d_{mn} \in F$. So considering the form of D_{ij}^W in (4.46), $hd_{ij} \subseteq D_{ij}^W$ is proven.

The proof of the inequality $D_{ij}^W \subseteq hd_{ij}$ is similar comparing the forms in (4.46) and (4.48) and using (4.52) in the other direction.

qed.

Lemma 4.17 *V is a D_α unit, i.e., $C_i D_{ij} = V$ for any fixed $i, j \in \alpha$.*

Proof.

It must be proved that $V \subseteq C_i D_{ij}$. Assume that $X \in W_y \subseteq V$, where W_y is a fixed subunit defined by a perfect ultrafilter F_y with support Y. Then $X = f_\tau Y$ for a transformation τ such that $s_\tau 1 \in F_y$.

By (4.8) (for the case $j \in \tau$) and by (4.9) (for the case $j \notin \tau$)

$$s_\tau 1 \in F_y \text{ if and only if } s_\tau s_m^j 1 \in F_y$$

for the fixed j and some $m \in \beta \sim \alpha$ such that $j \equiv m$ (i.e., $d_{jm} \in F_y$). $s_\tau s_m^j 1 \in F_y$, so

$$f_M^j f_\tau Y \in W_y \subset V.$$

By definition and by $j \equiv m$, $X = f_M^j f_\tau Y$. We know that $c_i d_{ij} = 1$ and $s_\tau 1 \in F_y$, therefore $s_\tau s_m^j 1 = s_\tau s_m^j c_i d_{ij} \in F_y$ if and only if $s_\tau s_m^j s_n^i d_{ij} \in F_y$ for some $n \notin \alpha$. Therefore $s_\tau s_m^j s_n^i d_{ij} \in F_y$. $s_\tau s_m^j s_n^i d_{ij} \leq s_\tau s_m^j s_n^i 1$ implies that $s_\tau s_m^j s_n^i$ is realized, so

$$f_N^i f_M^j f_\tau Y \in W_y \subset V. \tag{4.53}$$

Using (4.52) for $\nu = \tau$ we obtain that $d_{mn} \in F_y$, therefore $M = N$ in (4.53), so

$$f_M^i f_M^j f_\tau Y \in W_y \subset V.$$

Considering that $X = f_M^j f_\tau Y$ we obtain that $X \in C_i D_{ij}$.

qed.

* * *

Summing up the above lemmas, the proof of Theorem 4.5 can be completed:

By Lemma 4.17 a D_α unit V is constructed, and by Lemma 4.16 h' is an embedding of $\mathfrak{A}$ into the full set algebra in D_α with unit V. Therefore really $\mathfrak{A} \in \mathbf{I}\mathsf{D}_\alpha$.

Now, we come to the proof of Theorem 4.6:

We refer to the following *Proposition* concerning neat reducts of algebras in Crs_α:

Proposition. Let us assume that $\mathfrak{A}$ is in Crs_α with base U and unit element V. Assume that $\alpha \le \beta$, $W \subseteq {}^\beta U$ and the next two hypotheses are satisfied:

$$V = \{x : x = \alpha \upharpoonright y \text{ for some } y \in W\} \tag{4.54}$$

$$\text{for every } y \in W,\ i \in \alpha \text{ and } u \in U \quad \text{if } \ (\alpha \upharpoonright y)_u^i \in V, \text{ then } \ y_u^i \in W. \tag{4.55}$$

Let $\Theta X = \{y \in W : \alpha \upharpoonright y \in X\}$, $X \in A$. Then there exists an algebra $\mathfrak{B}$ with unit W in Crs_β such that $\Theta \in \mathbf{I}(\mathfrak{A}, \mathfrak{Rd}_\alpha \mathfrak{B})$ and $C_i^{\mathfrak{B}} \Theta X = \Theta X$ for every $X \in A$ and $i \in \beta \sim \alpha$ (see [He-Mo-Ta II.] Lemma 3.1.120). So $\mathfrak{A}$ is neatly embeddable into $\mathfrak{B}$.

We extend $\mathfrak{A}$ by subunits to an algebra in $\mathsf{F}^{\alpha}_{\beta}$, where β denotes $\alpha+\varepsilon$. Let an arbitrary, fixed subunit of $\mathfrak{A}$ be Q_k and its subbase be U_k. Let us extend Q_k to a β-dimensional subunit W_k to let $W_k = Q_k \times U^{\varepsilon}$. Let $\mathfrak{B}$ denote the full Crs_{β} algebra with subunit W_k.

Then the conditions (4.54) and (4.55) are obviously satisfied. Therefore by the above *Proposition,* $\mathfrak{A}$ is neatly embeddable into an algebra $\mathfrak{B}$ in Crs_{β}, with subunits W_k respectively.

We state that $\mathfrak{B}$ *is a member of* $\mathsf{F}^{\alpha}_{\beta}$.

As is known, algebras in Crs_{β} with D_{α} α-reduct satisfy all the axioms of $\mathsf{F}^{\alpha}_{\beta}$ except for (C_4^-) maybe. We check (C_4^-).

Consider (C_4^-) b). Assume that $x \in S^i_m S^j_n X$, $i, j, n \in \alpha$, $m \notin \alpha$. Then $\left(x^i_{x_m}\right)^j_{x_n} \in X$. But $\left(x^i_{x_m}\right)^j_{x_n} = \left(x^j_{x_n}\right)^i_{x_m}$. $x^j_{x_n} \in W$ by construction, because $\alpha \upharpoonright x^j_{x_n} \in Q$, namely $C_j D_{jn} \supseteq W$ is true in $\mathfrak{A}$. So $x^j_{x_n} \in W_k$ and $\left(x^j_{x_n}\right)^i_{x_m} \in X$ imply that $x \in S^j_n S^i_m X$.

The proof of (C_4^-) a) is similar.

Consider (C_4^-) c). Assume that $x \in D_{ik} \cap S^i_m S^j_n X$, $i, j, k \in \alpha$, $n \notin \alpha$. Then $x_i = x_k$ and $\left(x^i_{x_m}\right)^j_{x_n} \in X$. But $\left(x^i_{x_m}\right)^j_{x_n} = \left(x^j_{x_n}\right)^i_{x_m}$. We state that $x^j_{x_n} \in W$. By construction of W, it is sufficient to show that $\alpha \upharpoonright x^j_{x_n} \in Q$. We have $\left(x^i_{x_m}\right)^j_{x_n} \in W$ so $\alpha \upharpoonright \left(x^i_{x_m}\right)^j_{x_n} \in Q$. $(\alpha \upharpoonright \left(x^i_{x_m}\right)^j_{x_n})^i_{x_k} \in Q$ by $C_i D_{ik} \supseteq W$. But $(\left(x^i_{x_m}\right)^j_{x_n})^i_{x_k} = (x^j_{x_n})^i_{x_k} = x^j_{x_n}$ by $x_i = x_k$, so $(\alpha \upharpoonright \left(x^i_{x_m}\right)^j_{x_n})^i_{x_k} = \alpha \upharpoonright x^j_{x_n}$.

Therefore $\alpha \upharpoonright x^j_{x_n} \in Q$. So $x^j_{x_n} \in W$ and $\left(x^j_{x_n}\right)^i_{x_m} \in X$ imply that $x \in S^j_n S^i_m X$.

Consider (C_4^-) d). Assume that $x \in C_m C_i X$, $m \notin \alpha$. Then $(x^m_v)^i_u \in X$ for some $u, v \in U$. We prove that $x \in C_i C_m X$, i.e., $\left(x^i_u\right)^m_v \in X$. But $\left(x^i_u\right)^m_v = (x^m_v)^i_u$. It merely needs to be shown that $x^i_u \in W$. But $(x^m_v)^i_u \in X$, therefore $\alpha \upharpoonright (x^m_v)^i_u \in Q$. But $\alpha \upharpoonright (x^m_v)^i_u = \alpha \upharpoonright x^i_u$ by definition of W, therefore $x^i_u \in W$ indeed. So $C_m C_i X \subseteq C_i C_m X$. The proof of the opposite inclusion is similar.

qed.

***

Now we return to the *classical* neat embedding theorem. We apply the methods used previously in the Chapter.

Recall the theorem:

$\mathfrak{A}$ is representable, $\alpha \geq 2$ (i.e., $\mathfrak{A} \in \mathsf{Gws}_\alpha$) if and only if $\mathfrak{A} \in \mathrm{SNr}_\alpha \mathsf{CA}_{\alpha+\varepsilon}$, where $\varepsilon \geq \omega$ is an arbitrary but fixed ordinal, Gws_α is the class of generalized cylindric set algebras of dimension α, CA_α is the class of cylindric algebras of dimension α and $\mathsf{SNr}_\alpha \mathsf{CA}_{\alpha+\varepsilon}$ is the class of CA_α's that have the neat embedding property.

The part "only if" is trivial. Regarding the other part "if $\mathfrak{A} \in \mathsf{SNr}_\alpha \mathsf{CA}_{\alpha+\varepsilon}$ then $\mathfrak{A}$ is r-representable" the following question arises: *is it possible to replace the class* CA *in the hypothesis* $\mathfrak{A} \in \mathsf{SNr}_\alpha \mathsf{CA}_{\alpha+\varepsilon}$ *by a larger class so that the theorem still holds?*

The answer is affirmative. Such a larger class K^α_β will be defined below where $\beta = \alpha + \varepsilon$, $\varepsilon \geq \omega$. The character of K^α_β is similar to the class F^α_β included in Theorem 4.5.

The weakenings $(\mathrm{C}_4)-$ and $(\mathrm{C}_6)-$ of the cylindric axioms (C_4) and (C_6) are introduced and it is shown that the class satisfying these axioms, together with the other cylindric axioms is suitable to replace the class CA in the hypothesis above.

Assume that $\omega \leq \alpha < \beta$. We now introduce the class K^α_β indexed by the ordinals α and β, with the similarity type of CA_β. K^α_β is defined as follows:

Definition 4.18 (K^α_β) K^α_β is a class such that

$$\mathsf{K}^\alpha_\beta \models \{(\mathrm{C}_0), (\mathrm{C}_1), (\mathrm{C}_2), (\mathrm{C}_3), (\mathrm{C}_5), (\mathrm{C}_7), (\mathrm{C}_4)-, (\mathrm{C}_6)-\}$$

where $(\mathrm{C}_0), \ldots, (\mathrm{C}_7)$ denote the usual cylindric axioms, $(\mathrm{C}_4)-$ denotes the following instances $(\mathrm{C}_4)-$ a) and $(\mathrm{C}_4)-$ b):

$(\mathrm{C}_4)-$ a) $\quad c_m s^j_n c_m x = s^j_n c_m x$ if $j \in \alpha,\ n, m \in \beta,\ n \neq m$

$(\mathrm{C}_4)-$ b) $\quad c_m s^j_m c_m x = c_j c_m x$ if $j \in \alpha,\ m \in \beta$

and $(\mathrm{C}_6)-$ denotes the properties $a., b., c.$, in (1.1).

Another notation for K^α_β is $\mathsf{K}_{\alpha,\beta}$.

Suppose that $\mathfrak{A} \in \mathsf{CA}_\alpha$, $\alpha \geq \omega$. Now, the neat embedding theorem in question is formulated:

Theorem 4.19 *$\mathfrak{A}$ is representable (i.e., $\mathfrak{A} \in \mathbf{I}\mathsf{Gws}_\alpha$) if and only if $\mathfrak{A} \in \mathrm{SNr}_\alpha \mathsf{K}^\alpha_{\alpha+\varepsilon}$, where $\varepsilon \geq \omega$, $\alpha \geq 2$ are fixed.*

We omit the proof (see [Fe00]).

Remark

Notice that if $\mathfrak{B} \in \mathsf{K}^\alpha_\beta$, $\mathfrak{A} \subseteq \mathfrak{Nr}_\alpha \mathfrak{B}$ and $\mathfrak{A}$ happens to be a CA_α, then part d. in (1.1), follows only partially in $\mathfrak{B}$, that is,

$$c_m d_{im} = 1 \text{ if } i \in \alpha,\ m \in \beta\,. \tag{4.56}$$

Indeed, in $(C_4)-$ b) let us choose $a = d_{ij}$, $i,j \in \alpha$, $m \neq i$, where i, j, m are all distinct. Then

$$1 = c_j d_{ij} = c_j c_m d_{ij} = c_m s^j_m c_m(d_{ij}) = c_m s^j_m d_{ij} =$$

$$= c_m c_j (d_{mj} d_{ij}) \leq c_m c_j d_{mi} = c_m d_{mi}\,.$$

It is easy to show that $i \in \alpha$ is necessary in (4.56).

Let us onsider $\mathfrak{A} \in \mathsf{Crs}_\alpha \cap \mathsf{CA}_\alpha$ with unit $V = \cup P^{(r)}$, where $P^{(r)}$'s are the subunits. Let T_r be the subbase of $P^{(r)}$. Let us take any sets U_r's and form the extended unit $W = \cup P^{(r)} \times U^\varepsilon_r$ of $\alpha + \varepsilon$ dimension and consider the full $\mathfrak{B} \in \mathsf{Crs}_{\alpha+\varepsilon}$ with unit W. W satisfies 4.54 and 4.55 obviously. Let us consider the following special cases (i) and (ii):

(i) Let $\mathfrak{A}$ be a non-representable $\mathsf{Crs}_\alpha \cap \mathsf{CA}_\alpha$ (see [He-Mo-Ta II.], p. 85). $(\mathrm{C}_4)-$a) is false in $\mathfrak{B}$. Namely if $a = d_{ij}$, $m = i$, then $c_i c_j d_{jn} = c_j d_{jn}$ fails to be true in $\mathfrak{B}$. $\mathfrak{B}$ is in K^α_β obviously. This example shows that $(\mathrm{C}_4)-$ *cannot be rejected* in Theorem 4.19. It is also an *example* for an algebra $\mathfrak{A}$ such that $\mathfrak{A} \in \mathsf{SNrK}^\alpha_{\alpha+\omega} \sim \mathsf{RCA}_\alpha$, where RCA is the class of the representable cylindric algebras.

(ii) Let $\mathfrak{A}$ be a Gws_α and choose the sets U_r's so that T_r should be a proper subset of U_r. It is easy to check that $\mathfrak{B} \in \mathsf{K}^\alpha_\beta$. But $c_i d_{im} = 1$, $i \in \alpha$, $m \notin \alpha$ is not satisfied in $\mathfrak{B}$, i.e. $\mathfrak{B} \notin \mathsf{CA}_\beta$. This example shows that CA_β is a *proper subclass* of K^α_β, furthermore Theorem 4.19 is *stronger* than the classical representation theorem because it states that it is enough to embed an algebra neatly into $\mathsf{K}^\alpha_{\alpha+\varepsilon}$ instead of CA, to be representable.

Main references in this Chapter are: [Fe10], [Fe07a] and [Fe00].

Chapter 5

Logical applications

An obvious logical application of our results (e.g., that of representation theorems) is that they can be translated to the Logics corresponding to the respective cylindric-type- or polyadic-type algebras ([He-Mo-Ta II.], [Kei]). In this way we obtain new Henkin-style completeness theorems. In this Chapter we deal with a logical application of our topic, with conservative extensions of provability relations. Mainly, the concept “neat embeddability” and the logical calculus corresponding to cylindric algebras are used to obtain these results. There are also many other logical aspects of our subject. For example, considering the weakenings of the axioms (C_4) and (C_6) and the results at the end of the previous Chapter (Theorem 4.19), their logical background can be summarized as follows: thinking of the logical calculus corresponding to cylindric algebras (see [He-Mo-Ta II.]) and the proof of its completeness, only a fragment of the calculus is needed to construct a model for a consistent set of sentences. Another logical connection of our subject is that Crs_α occurs in the algebraizations of the semantics of many non-classical logics (e.g., many-sorted, higher-order and modal logics). Among these logics, one of the most important is the so-called *guarded segment* which corresponds to a kind of first order modal logic (see van Benthem, Andréka, Németi [An-Ne-Be]). Crs_α apply to Stochastics as well ([Fe09a]).

We come to conservative extensions of provability relations. Let us consider the standard first order logic with a usual deduction system. If the language is extended by any set of new individual variables preserving the other components of the original deduction system, then the provability relation $\overset{+}{\vdash}$ obtained is a *conservative extension* of the original one $\vdash$. That is, if φ *is any formula of the original language, then* $\overset{+}{\vdash} \varphi$ *implies* $\vdash \varphi$. Namely, at the deduction of φ by $\overset{+}{\vdash}$, the new individual variables *can be changed* to old ones and in this way a deduction of φ by $\vdash$ is obtained. This method works if we set out from a first order logic with predicates of ranks being at most β, where $\beta < \alpha$ and α is a limit ordinal, where α is associated with the sequence of the individual variables in the original language.

Now, we deal with first order logic with *infinitary predicates* (i.e., with relations of *arbitrary* infinite ranks). This logic was investigated in [Kei], [He-Mo-Ta II.] e.g., it can be associated with cylindric algebras and quasi -polyadic algebras, among others. If we set out from such a logic and we extend the original deduction system so that the language is extended by new individual variables, then the respective extension *fails* to be conservative, as counterexamples show.

We present conditions for these logics to have a conservative extension of the kind above (Theorem 5.1). On one hand, a slightly stronger deduction system is chosen for the *basic logic* than usual, namely, we suppose an additional axiom, the merry-go-round axiom (this property is always satisfied in classical first order logic). On the other hand, instead of the extended deduction system above, a *restricted* deduction system is assumed: the usual commutativity of quantifiers and the equality axioms are weakened. We can show that these latter restrictions are crucial: if the extended deduction system is not a restricted one, i.e., it is of the same kind as in the classical case, then the extension is not conservative, even if the merry-go-round axiom is supposed in the basic system.

Next, we briefly review the basic notions to be used.

Let $\mathcal{L}$ be the type-free first-order language described in [He-Mo-Ta II.] Sect.4.3. So $\mathcal{L}$ has the logical constants $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$, $\neg$, $\exists$, $\forall$, the equality symbol $=$, a sequence of α-many individual variables $\langle v_j : j \in \alpha\rangle$ and a sequence of relation symbols $\langle \mathrm{R}_i : i \in Q\rangle$, where the rank ρ_i of R_i is allowed to be infinite ($\rho_i \leq \alpha$). By the type-free property, the formulas in $\mathcal{L}$ are restricted, i.e., the atomic subformulas are of the forms $v_k = v_j$ $(k, j \in \alpha)$ or $\mathrm{R}_i(v_0, v_{1,} v_2 \ldots.)$. Let Z denote the set of individual variables.

We suppose the following Hilbert type system of axioms (see [He-Mo-Ta II.] 4.3 and [Mon76] p.196).

(0) φ is a propositional tautology

(1) $\forall v_i(\varphi \to \psi) \to (\forall v_i\varphi \to \forall v_i\psi)$

(2) $\forall v_i\varphi \to \varphi$

(3) $\varphi \to \forall v_i\varphi$ if v_i does not occur freely in φ

(4) $\exists v_i \exists v_j \varphi \leftrightarrow \exists v_j \exists v_i \varphi$

(5) $v_i = v_i$

(6) $\exists v_i(v_i = v_j)$

(7) $v_i = v_j \to (v_i = v_k \to v_j = v_k) \quad j \notin \{i, k\}$

(8) $v_i = v_j \to (\varphi \to \forall v_i(v_i = v_j \to \varphi)) \quad i \neq j$

(9) $\exists v_i\varphi \leftrightarrow \neg\forall v_i \neg\varphi$

where φ and ψ are arbitrary restricted formulas, i, j and k are ordinals $(i, j, k < \alpha)$.

Let Ax_0^Z *(or* Ax_0*, for short) denote this system of axioms.*

Inferences rules are the *modus ponens* and the *generalization.*

Let us suppose a *fixed set* Σ *of non-logical axioms* in $\mathcal{L}$ and let $\overset{r}{\vdash}$ denote the *provability relation* obtained above. Thus $\overset{r}{\vdash} \varphi$ denotes $\Sigma \overset{r}{\vdash} \varphi$, for short.

We obtain an extended system of axioms (see [He-Mo-Ta II.] 3.2.88, and [An-Th]) if the system Ax_0 is extended by the *merry-go-round axiom*

$$\exists u(u = v_i \wedge \exists v_i(v_i = v_j \wedge \exists v_j(v_j = v_n \wedge \exists v_n(v_n = u \wedge \exists u\varphi)))) \leftrightarrow$$

$$\leftrightarrow \exists u(u = v_j \wedge \exists v_j(v_j = v_n \wedge \exists v_n(v_n = v_i \wedge \exists v_i(v_i = u \wedge \exists u\varphi))))$$

where $u \notin \{v_j, v_n\}$ and $v_i \notin \{v_j, u, v_n\}$.

(5.1)

Denote this extended system of axioms by Ax^Z (*or just by* Ax, *for short)*, and denote the resulting *provability relation* by $\overset{q}{\vdash}$ (the set of non-logical axioms remains the same).

We note that the system Ax_0 has some redundancy because axiom (3) implies axiom (4), but this form of the system of axioms will be more adequate for our investigations (see [Mon76] p.193).

If the language $\mathcal{L}$ is extended by a set of new individual variables (where the extended set is denoted by Z^+), while *the set of relation symbols remains the same*, then the new language is denoted by $\mathcal{L}^+$ and the extensions of the axiom systems Ax_0^Z and Ax^Z are denoted by $Ax_0^{Z^+}$ and Ax^{Z^+} respectively. Here the original language, system of axioms and provability relation will be referred to as the *basic* language, *basic* system of axioms and *basic* provability relation respectively.

In $\mathcal{L}^+$ we can speak about the conservative extension of the provability relation defined on the formulas of the basic language, too:

As is known, if $\overset{r}{\vdash}$ *is a provability relation defined on the formulas in* $\mathcal{L}$ *and* $\overset{r^+}{\vdash}$ *is a provability relation defined on the formulas in* $\mathcal{L}^+$ *extending* $\overset{r}{\vdash}$*, then* $\overset{r^+}{\vdash}$ *is said to be a* conservative extension *of* $\overset{r}{\vdash}$ *if* $\overset{r^+}{\vdash} \gamma$ *implies* $\overset{r}{\vdash} \gamma$ *for any formula* γ *in* $\mathcal{L}$.

It is known that, with the language $\mathcal{L}$, with the provability relation $\overset{r}{\vdash}$, a *formula alge-*

bra can be associated as an α-dimensional *cylindric algebra* – it is denoted by $\mathrm{Fm}_r^{\mathcal{L}}$ (see [He-Mo-Ta II.] 4.3.1.). Conversely if $\mathfrak{A}$ is an α-dimensional cylindric algebra, then $\mathfrak{A} \simeq \mathrm{Fm}_r^{\mathcal{L}}$ for a suitable language $\mathcal{L}$ and provability relation $\overset{r}{\vdash}$ of the kind above, so cylindric algebras can be representable by formula algebras (see [He-Mo-Ta II.] Theorem 4.3.28). The element of a formula algebra corresponding to the formula φ is denoted by $|\varphi|$.

In general, if $\mathcal{L}$' is a language and $\overset{r'}{\vdash}$ is a provability relation on the formulas in $\mathcal{L}$', then $\mathrm{Fm}_{r'}^{\mathcal{L}'}$ will denote the formula algebra associated with $\mathcal{L}$' and $\overset{r'}{\vdash}$. So, in particular if $\overset{r'}{\vdash}$ is specially the relation $\overset{q}{\vdash}$ (so the merry-go-round axiom is supposed), then the formula algebra is denoted by $\mathrm{Fm}_q^{\mathcal{L}}$.

Let us take $\mathcal{L}$ as *basic* language, take the system Ax as *basic* logical axioms and the provability relation $\overset{q}{\vdash}$ as *basic* provability relation – so the merry-go-round axiom and a fixed set Σ of non-logical axioms are assumed.

Let us extend the language $\mathcal{L}$ by $\beta{\sim}\alpha$ -many new individual variables v_i's ($\alpha \leq i < \beta$), where β is any fixed ordinal, $\beta > \alpha$. Let us denote by $\mathcal{L}^+$ the extended language and denote by Z^+ the set of individual variables in $\mathcal{L}^+$. We will show that if a *restricted* version of the system Ax^{Z^+} is assumed (Ax^{Z^+} is the system Ax with the set Z^+ of individual variables), then the provability relation obtained in this way will be a *conservative extension* of $\overset{q}{\vdash}$.

Definition of the restricted axioms in $\mathcal{L}^+$:

Consider the system Ax^{Z^+} in $\mathcal{L}^+$. This system is modified so that the schemas of axioms are restricted, i.e., we restrict the possibilities for the choice of the formulas and the individual variables occurring in the schemas (3), (4), (6) and 5.1.

The schemas $(3)^-$, $(4)^-$, $(6)^-$ and MGR$^-$ rather than (3), (4), (6) and the merry-go-round axioms are:

(3)$^-$ $\varphi \to \forall v_i \varphi$ *if* φ is *in* $\mathcal{L}$ and v_i is not free in φ, $i \in \beta$

(4)$^-$ $\exists v_i \exists v_j \varphi \leftrightarrow \exists v_j \exists v_i \varphi$ *except for the case* if φ is *not* in $\mathcal{L}$ and $i,\ j \in \alpha$

(6)$^-$ $\exists v_i (v_i = v_j)$ *except for* $i \in \alpha$ *and* $j \notin \alpha$

$^-$MGR is the merry-go-round formula in (5.1) if φ, u, v_i, v_j and v_n are in $\mathcal{L}$.

The other axioms in Ax^{Z^+} are the same.

Let us denote by Ax^+ *the system of axioms* obtained in this way.

In $\mathcal{L}^+$, assume the system Ax^+, suppose the set Σ of non-logical axioms (the same as in $\mathcal{L}$) and denote the provability relation obtained by $\overset{r_1}{\vdash}$.

The following theorem due to the present author holds (see [Fe09b]):

Theorem 5.1 *The provability relation* $\overset{r_1}{\vdash}$ *is a conservative extension of the provability relation* $\overset{q}{\vdash}$.

Proof.

Obviously $\overset{r_1}{\vdash}$ is an extension of $\overset{q}{\vdash}$. It must be proved that if $\overset{r_1}{\vdash} \varphi$ holds, then also $\overset{q}{\vdash} \varphi$ holds for any formula φ in $\mathcal{L}$.

Let us consider the formula algebra $\mathrm{Fm}_q^{\mathcal{L}}$ and a *representation* $\mathfrak{A}$ of this algebra by a set algebra in $\mathbf{I}\mathsf{Crs}_\alpha \cap \mathsf{CA}_\alpha$ (such a representation exists). Let g denote an isomorphism from $\mathrm{Fm}_q^{\mathcal{L}}$ onto $\mathfrak{A}$. First, we show that $\mathfrak{A}$ is *neatly embeddable* into a β-dimensional set algebra $\mathfrak{B}$ in Crs_β.

We need the *Proposition* concerning neat reducts of algebras in Crs, cited in the proof of Theorem 4.6 (see [He-Mo-Ta II.] Lemma 3.1.120). The notation introduced there used.

To apply the *Proposition* we will extend $\mathfrak{A}$ to an algebra in $\mathsf{Crs}_{\alpha+\varepsilon}$, where $\beta = \alpha+\varepsilon,\ \varepsilon \geq 1$.

Let us extend V to a β-dimensional subunit W to let $W = V \times {}^{\varepsilon}U$. Then the conditions (4.54) and (4.55) above are obviously satisfied. Therefore by the *Proposition* above, $\mathfrak{A}$ is neatly embeddable into an algebra $\mathfrak{B}$ in Crs_β with unit W.

With every *atomic formula* in $\mathcal{L}^+$ an element (a set) can be associated in the β-dimensional set algebra $\mathfrak{B}$ defined above. Namely with the formula $v_i = v_j \quad i, j \in \beta$ we can associate the diagonal element D_{ij} of $\mathfrak{B}$, and with any other atomic formula R we can associate the element in $\mathfrak{B}$ which corresponds to the image of the equivalence class $|R|$ in $\mathrm{Fm}_q^{\mathcal{L}}$, under the composition of the isomorphism $\mathrm{Fm}_q^{\mathcal{L}} \simeq \mathfrak{A}$ and the neat embedding of $\mathfrak{A}$ into $\mathfrak{B}$. Because R is included in $\mathcal{L}$ by definition and the type-free property of $\mathcal{L}^+$, R does not include new variables. Further $\mathfrak{A}$ is neatly embeddable into $\mathfrak{B}$.

Therefore by formula induction, with every *formula* ψ in $\mathcal{L}^+$ a unique element, denoted by $[\psi]$, can be associated in the algebra $\mathfrak{B}$ (here, using axiom (9), $\forall v_i \varphi$ is considered as $\neg \exists v_i \neg \varphi$, so we can use only the quantifier $\exists$ in the language $\mathcal{L}$). Denote by h this assignment from the formulas of $\mathcal{L}^+$ into $\mathfrak{B}$, so let

$$h\psi = [\psi] . \tag{5.2}$$

We note that if ψ is in $\mathcal{L}$, then $V \upharpoonright [\psi]$ is in $\mathfrak{A}$ and $V \upharpoonright [\psi] = g\,|\psi|$ because of the definition of h, the homomorphism property of g and the embeddability of $\mathfrak{A}$ into $\mathfrak{B}$.

First, we state that if ψ is an axiom in Ax^+, then

$$[\psi] = W \tag{5.3}$$

where W is the unit of $\mathfrak{B}$.

On evaluating $[\psi]$, i.e., $h\psi$, we may consider the cylindric algebraic expression corresponding to ψ because the type of $\mathcal{L}^+$ and that of cylindric algebras coincide ($\varphi \to \psi$ and $\forall v_i \varphi$ are defined in $\mathcal{L}^+$ as $\neg \varphi \vee \psi$ and $\neg \exists v_i \neg \varphi$). So h may be considered to be defined on cylindric algebraic expressions (for example, the "translation" of axiom (2) is $-c_i(-y) \leq y$, where $\leq$ is the usual defined concept in Boolean algebras or the translation of axiom (5) is $d_{ii} = 1$. So it is sufficient to prove that the value of the cylindric algebraic expressions corresponding to the axioms in Ax^+ is W in $\mathfrak{B}$.

The cylindric expressions corresponding to the axioms (0), (1), (2), (5), (7), (8) and $^-$MGR are the cylindric axioms (C_0), (C_1), (C_2), (C_3), (C_5), (C_7) and the merry-go-round axiom respectively, or known consequences of these axioms (see [He-Mo-Ta II.] proof of Lemma 4.3.25). Therefore the interpretation of these expressions is exactly the set W in $\mathfrak{B}$, because $\mathfrak{B} \in \mathsf{Crs}_\beta$, and $\mathfrak{B}$ satisfies the cylindric axioms except for (C_4) and (C_6)d.

(5.3) is also true for those instances of the axioms (3), (4) and (6) which include individual variables only from $\mathcal{L}$. Namely, the cylindric expressions corresponding to these axioms are cylindric axioms or simple consequences of cylindric axioms. Further, h associates an element in $\mathfrak{A}$ with these expressions apart from isomorphism and $\mathfrak{A}$ is a cylindric algebra. It remains to check the other instances of the axioms (3), (4) and (6).

We start with $(4)^-$. With $(4)^-$ and the case $i \in \alpha$, $j \notin \alpha$ we can associate the cylindric expression $c_i c_j y = c_j c_i y$, i.e., it must be proved that

$$C_i C_j b = C_j C_i b \tag{5.4}$$

in $\mathfrak{B}$ where $b \in B$.

Suppose that $x \in C_i C_j b$. Then $(x^i_u)^j_v \in b$ for some $u, v \in U$ but $(x^i_u)^j_v = (x^j_v)^i_u$. $x^j_v \in W$ by $j \notin \alpha$ and the definition of W, therefore $(x^j_v)^i_u \in b$ implies that $x \in C_j C_i b$. Conversely, suppose that $x \in C_j C_i b$, then $(x^j_v)^i_u \in b$ for some $u, v \in U$. It is sufficient to prove that $(x^i_u)^j_v \in b$. Because $(x^i_u)^j_v = (x^j_v)^i_u$ it is sufficient to prove that $x^i_u \in W$. But $x^i_u = ((x^i_u)^j_v)^j_{x_j} = ((x^j_v)^i_u)^j_{x_j}$. From the definition of W, it follows that $(x^j_v)^i_u \in W$ implies that $((x^j_v)^i_u)^j_{x_j} \in W$.

The proof for $(4)^-$ is trivial in the case $i \notin \alpha$, $j \notin \alpha$.

If ψ is the axiom $(3)^-$ in (5.3), first we show that

$$C_i\,[\varphi] = [\varphi] \tag{5.5}$$

whenever φ is in $\mathcal{L}$, $i<\beta$ and v_i is not free in φ.

If $i<\alpha$ and v_i is not free in φ, then $v_i \to \forall v_i\varphi$ is an axiom of $\overset{q}{\vdash}$, so its equivalence class in $\mathrm{Fm}_q^{\mathcal{L}}$ is 1 in $\mathfrak{B}$ by neat embeddability. If $i \geq \alpha$, since φ is in $\mathcal{L}$, we have $[\varphi] = \Theta g\,|\varphi|$. By the *Proposition* [He-Mo-Ta II.] Lemma 3.1.120 (here, after the conditions (4.54) and (4.55)), $C_i\Theta(a) = \Theta(a)$ for all $a \in A$ and $i \geq \alpha$. Since $[\varphi]$ has the form $\Theta(a)$, where $a = g\,|\varphi|$, we have $C_i\,[\varphi] = [\varphi]$.

Having the relation (5.5) we can translate axiom $(3)^-$ in this way: $y \leq -c_i(-y)\ i \notin \alpha$ if $c_iy = y$ is true. So we need to prove that $b \leq\ {\sim}C_i({\sim}b)\ i \notin \alpha$ holds in $\mathfrak{B}$, if $b = C_ib$. But $C_ib \leq\ {\sim}C_i({\sim}C_ib)$ or equivalently $C_i({\sim}C_ib) \leq\ {\sim}C_ib$ holds in $\mathfrak{B}$. Namely, if $x \in C_i({\sim}C_ib)$, then $x^i_u \in {\sim}C_ib$ for some u, where $x^i_u \in W$, so $x^i_u \notin C_ib$. This implies the relation $x \in {\sim}C_ib$, i.e., $x \notin C_ib$ because $x \in C_ib$ implies $x^i_u \in C_ib$ if $x^i_u \in W$.

With the axiom $(6)^-$ and for example, with the cases $i \notin \alpha,\ j \in \alpha$ the expression $c_id_{ij} = 1\ i \notin \alpha,\ j \in \alpha$ can be associated so we need to check that $C_iD_{ij} = W$ if $i \in \alpha,\ j \notin \alpha$ in $\mathfrak{B}$. It is sufficient to check that $W \subseteq C_iD_{ij}$. We need to prove that if $x \in W$, then $x \in C_iD_{ij}$, that is, $x^i_u \in D_{ij}$ for some $u \in U$. But, by the definition of W, for every $v \in U$, $x^i_v \in W$, so $x^i_u \in W$ too.

The other instances of $(6)^-$ are obvious. So (5.3) is proven.

Then we prove that

$$\text{if} \overset{r_1}{\vdash} \varphi, \text{ then } [\varphi] = W \tag{5.6}$$

for an arbitrary φ in $\mathcal{L}$.

Suppose that $\varphi_1, \varphi_2, \varphi_3, ..., \varphi_n = \varphi$ is a deduction of φ by $\overset{r_1}{\vdash}$, where φ is a formula in $\mathcal{L}$. We state that

$$[\varphi_i] = W \tag{5.7}$$

$i = 1, 2, ...n$. We prove it by induction.

$[\varphi_1] = W$. Namely, if φ_1 is a logical axiom, then (5.7) is true by (5.3). If $\varphi_1 \in \Sigma$, then $|\varphi_1| = 1$ in $\mathrm{Fm}_q^{\mathcal{L}}$ by definition, further $[\varphi_1] = g\,|\varphi_1|$ because of the homomorphism property of g, where g is the isomorphism from $\mathrm{Fm}_q^{\mathcal{L}}$ into $\mathfrak{A}$. Further, $g\,|\varphi_1| = W$ because of the embeddability of $\mathfrak{A}$ into $\mathfrak{B}$. Assume that (5.7) is true if $i \leq k$ ($1 \leq k < n$). We prove (5.7) for $k + 1$.

If φ_{k+1} is a logical axiom in Ax^+ or non-logical axiom in Σ, then the proof is completely similar to the case $k = 1$.

If we obtain φ_{k+1} by generalization from a formula φ_i, that is, $\varphi_{k+1} = \forall v \varphi_i$ for some $i \leq k$, then by definition (5.2) we obtain $[\forall v \varphi_i] = [\neg \exists v \neg \varphi_i] = \sim C_i \sim [\varphi_i]$. The induction condition $[\varphi_i] = W$ and $C_i 0 = 0$ imply that $\sim C_i \sim [\varphi_i] = W$.

If we obtain φ_{k+1} by modus ponens from the formulas φ_i and φ_j and $\varphi_j = \varphi_i \to \varphi_{k+1}$, $i, j \leq k$, then $\varphi_i \to \varphi_{k+1} \overset{r_2}{\leftrightarrow} \neg \varphi_i \vee \varphi_{k+1}$ by axiom (0). But $[\varphi_i \to \varphi_{k+1}] = [\neg \varphi_i \vee \varphi_{k+1}] = \sim [\varphi_i] \cup [\varphi_{k+1}]$. By the induction condition, $[\varphi_i] = W$ and by $\sim [\varphi_i] \cup [\varphi_{k+1}] = W$, we obtain $[\varphi_{k+1}] = W$.

By the remark after (5.2), if φ is in $\mathcal{L}$, then $V \upharpoonright [\varphi] = g\,|\varphi|$, but if $\overset{r_1}{\vdash} \varphi$, then $[\varphi] = W$ by (5.6). But $V \upharpoonright W = V$ so $g\,|\varphi| = V$.

Therefore with φ we associate the unit element V at the isomorphism $\mathrm{Fm}_q^{\mathcal{L}} \simeq \mathfrak{A}$, i.e., $|\varphi| = 1$ in $\mathrm{Fm}_q^{\mathcal{L}}$. By definition of the formula algebra, this means that $\overset{q}{\vdash} \varphi$ is true.

qed.

We may ask whether there are other restrictions of the system Ax^{Z^+}such that the Theorem 5.1 should remain true. The answer is affirmative.

Analysing the proof, a given weakening φ of axiom (4) could be a new axiom (as a part of the restriction for Ax^{Z^+}) if the corresponding cylindric algebraic expression equals 1 in the embedding algebra $\mathfrak{B}$. For example, Ax^{Z^+} can be restricted also by the following additional weakening $(4)^{--}$ of (4)

$$\exists v_i(v_i = v_m \wedge \exists v_j(v_j = v_n \wedge \varphi)) \rightarrow \exists v_j(v_j = v_n \wedge \exists v_i(v_i = v_m \wedge \varphi))$$

where $i,\ j,\ n \in \alpha$ and $m \notin \alpha$. Because the respective cylindric algebraic (defined) expression $s^i_m s^j_n x \ \leq s^j_n s^i_m x \ \ i, j, n \in \alpha,\ m \notin \alpha$ is true in $\mathfrak{B}$ – we assume here that i, j and n are distinct. Since, if $t \in S^i_m S^j_n b$, $b \in B$ then $(t^i_u)^j_{t_n} \in x$. And $(t^i_u)^j_{t_n}$ $=(t^j_{t_n})^i_u$. But $C_j D_{jn} = W$ implies that $t^j_{t_n} \in W$. Therefore $t \in S^j_n S^i_m x$, in fact.

The next question is: Does a distinguished *restriction* of axiom (4) exist in $\mathcal{L}^+$ among the possible ones? The following is true: there is such a restriction of axiom (4) in $\mathcal{L}^+$ that the conservative extensibility of $\overset{q}{\vdash}$ into this restricted system of axioms already implies the completeness of $\overset{q}{\vdash}$ (we do not prove this proposition).

Main references in this Chapter are: [Fe09b], [Kei], [He-Mo-Ta II.], [Fe07a], [Fe10] and [Fe07b].

Chapter 6

Neat embedding theorem for polyadic-type algebras and its applications

For transposition algebras and quasi-polyadic algebras similar neat embedding theorems hold, as was proved for cylindric-type algebras in Chapter 4 (see [FePrepr]). But the case of polyadic-type algebras having substitution with infinite τ's is essentially different.

In this Chapter, first we prove a neat embedding theorem for cylindric m-quasi-polyadic equality, locally-m algebras (algebras in ${}_m\mathsf{CPE}_\alpha \cap \mathrm{L}m_\alpha$). Let $m < \alpha < \beta$ be fixed, infinite ordinals and let K_β be a class of algebras with the type of ${}_m\mathsf{CPE}_\beta$.

The definition of neat embeddability of an algebra $\mathfrak{A}$ in ${}_m\mathsf{CPE}_\alpha$ into an algebra $\mathfrak{B}$ in ${}_m\mathsf{CPE}_\beta$ is specified as follows (see [He-Mo-Ta II.], Def. 5.4.16 and [Say12]):

Definition 6.1 Let

$$\mathfrak{Nr}_\alpha\mathfrak{B} = \left\langle \mathfrak{B}^0,\ +,\ \cdot,\ -,\ 0,\ 1,\ c_i,\ s'_\tau,\ d_{ij} \right\rangle_{\tau\in\ {}_m\mathrm{T}_\alpha\ ,\ i,.j<\alpha}$$

where $\mathfrak{B}^0 = \{b \in B : c_i b = b, \text{for every } i \in \beta \sim \alpha\}$, and $s'_\tau = s_\sigma$ with $\sigma = \tau \cup \{i : i \in \beta \sim \alpha\}$ for each $\tau \in {}_m\mathrm{T}_\alpha$. An algebra $\mathfrak{A} \in {}_m\mathsf{CPE}_\alpha$ is *neatly embeddable* into $\mathfrak{B}$ ($\mathfrak{B} \in \mathrm{K}_\beta$) if $\mathfrak{A} \in \mathrm{S}\mathfrak{Nr}_\alpha\mathfrak{B}$.

Let ${}_m\mathsf{CPE}^-_{\alpha+\varepsilon}$ denote the class such that the ${}_m\mathsf{CPE}_{\alpha+\varepsilon}$ axioms hold in it, except for the axiom (CP9)* in which the part "the equality holds if σ is a permutation" is replaced by the following two instances of (CP9)*:

$$c_i x = c_m s_{[i\ /\ m]} x \quad \text{if } i \in \alpha, m \notin \alpha, x \in A \tag{6.1}$$

$$c_m s_\tau z = s_\tau c_m z \text{ if } \tau m = m,\ m \notin \alpha,\ \tau \in {}_m\mathrm{T}_\beta,\ z \in B. \tag{6.2}$$

The following theorem holds (see [Fe11b], [Fe12b]):

Theorem 6.2 (Neat embedding theorem for ${}_m\mathsf{CPE}_\alpha \cap \mathrm{L}m_\alpha$) *Assume that* $\mathfrak{A} \in {}_m\mathsf{CPE}_\alpha \cap \mathrm{L}m_\alpha$, *$m$ is infinite, $m < \alpha$. Then* $\mathfrak{A} \in \mathbf{I}\,{}_m\mathsf{Gwp}^{\mathrm{reg}}_\alpha$ (*i.e.,* $\mathfrak{A}$ *is r-representable*) *if and only if* $\mathfrak{A} \in \mathrm{S}\mathfrak{Nr}_\alpha\mathfrak{B}$ *for some* $\mathfrak{B} \in {}_m\mathsf{CPE}^-_{\alpha+\varepsilon}$, *where ε is infinite.*

Let us consider the direction in which $\mathfrak{A} \in \mathrm{S}\mathfrak{Nr}_\alpha\mathfrak{B}$ implies $\mathfrak{A} \in \mathbf{I}\,{}_m\mathsf{Gwp}^{\mathrm{reg}}_\alpha$. The proof follows a classical line of thought, it is analogous with that of Theorem 4.5. In addition to the necessary adaptation to the ${}_m\mathsf{CPE}_\alpha$ axioms, a further unusual aspect of the proof is the *simulation of the relativization* in algebraic syntax (see the definition of the set M below, i.e., that of the subunit W_y in (4.12)).

The outline of the proof of Theorem 6.2 is: *A* Cprs_α*-unit* V *will be defined, next, an embedding of* $\mathfrak{A}$ *into the full set algebra with unit* V *is constructed. Finally, it will be shown that* V *is a* ${}_m\mathsf{Gwp}_\alpha$ *unit and the set algebra is regular.*

To implement this plan some concepts and lemmas are needed.

Assume that $\mathfrak{A} \in \mathsf{Cprs}_\alpha$, V is the unit of $\mathfrak{A}$. Let us consider the following equivalence relation $\sim$ on V :

$$x \sim y \text{ if and only if } x \text{ and } y \text{ are different at most in } m\text{-ary members.} \tag{6.3}$$

Definition 6.3 The equivalence classes concerning $\sim$, regarding them as subsets of V, are called the *m-subunits* of A. If W is an m-subunit, then $\bigcup\limits_{x\in W} \mathrm{Rg}x$ is called the m-base of W, and any $x \in W$ is called a *support* of W.

If, at the definition (6.3) of the equivalence relation $\sim$, "m-ary" is replaced by "finitely many", then the concept of m-subunit means *subunit* ([HMTAN] Def. 0.1). Notice that a subunit is a subset of an m-weak space with the same m-base and support. The subunits are disjoint, by definition. If $\mathfrak{A} \in {}_m\mathsf{Gwp}_\alpha$, then an m-subunit, in particular, is a union of some ${}^{\alpha}_{m}U^{(p_k)}_k$.

A preparation for the first lemma is needed. Let us fix an algebra $\mathfrak{B}$ occuring in the theorem. Let us denote by *adm* the class of m-transformations $\tau \in {}^\alpha\beta$, i.e., $\tau \in {}_m\mathrm{T}_\alpha \cap {}^\alpha\beta$, where $\alpha + \varepsilon$ is denoted by β.

We formulate a version of the concept perfect ultrafilter introduced in Chapter 4:

A Boolean ultrafilter F *in* $\mathfrak{B}$ *is a* regular perfect *ultrafilter if for any element of the form* $s_\tau c_j x$ *included in* F*, where* $j \in \alpha$*,* $x \in A$ *and* $\tau \in adm$*, there exists an* m*,* $m \notin \alpha$*,* $\tau m = m$ *such that* $s_\tau s_{[j\,/\,m]} x \in F$*.*

Lemma 6.4 *Let a be an arbitrary, but fixed non-zero element of A and* let $m < \alpha$ *be fixed ordinal, let $\varepsilon >$ max $(\alpha, |A|)$, where $\varepsilon + \alpha$ is regular, and assume that $\mathfrak{A} \in \mathrm{S}\mathfrak{Nr}_\alpha \mathfrak{B}$ for some $\mathfrak{B} \in {}_m\mathsf{CPE}^{-}_{\alpha+\varepsilon}$. Then, there exists a proper Boolean filter $\mathfrak{D}$ in $\mathfrak{B}$, such that $a \in D$ and an arbitrary ultrafilter containing $\mathfrak{D}$ is a regular perfect ultrafilter in $\mathfrak{B}$.*

The proof is similar to that of Lemma 4.11. Among others, the properties (6.1) and (6.2) need to be used. We omit the proof.

We prefix a regular perfect ultrafilter F in $\mathfrak{B}$, extending the filter $\mathfrak{D}$ guaranteed in the lemma, letting it be defined as follows: let us take the cylindric algebraic completion $\mathfrak{B}$' of $\mathfrak{B}$ (see [He-Mo-Ta I.], 2.7.21). Let us consider the filter F' in $\mathfrak{B}$', generated by the generators of $\mathfrak{D}$ – such a filter F' exists. Let us consider any fixed ultrafilter $(F')^+$ in $\mathfrak{B}$', which extends F'. The restriction F of $(F')^+$ with respect to $\mathfrak{B}$ is an ultrafilter in $\mathfrak{B}$. Let us choose such an ultrafilter F for the extension of the filter $\mathfrak{D}$ in $\mathfrak{B}$.

Let us consider the following relation $\equiv$ on β, where β denotes the ordinal $\alpha + \varepsilon$:

$$m \equiv n \ (m, n \in \beta) \text{ if and only if } d_{mn} \in F. \tag{6.4}$$

Lemma 6.5 *$\equiv$ is an equivalence relation on β and, furthermore, for every $i \in \alpha$ there exists an $m \notin \alpha$ such that $d_{im} \in F$.*

Proof.

The (E1), (E2) and (E3) axioms ensure that $\equiv$ is an equivalence relation on β. Let us denote by Π the set of the equivalence classes.

$1 = c_j d_{ji} \in F$ $(i, j \in \alpha,\ j \neq i)$. The regular perfect ultrafilter property implies that $s_{[j\ /\ m]} d_{ji} \in F$ for some $m \notin \alpha$. By (E3), $s_{[j\ /\ m]} d_{ji} = d_{mi}$, therefore $d_{mi} = d_{im} \in F$ follows.

qed.

As it was mentioned at the outline of the proof, we define a set M of m-transformations in ${}^{\alpha}\beta$ (in some steps). Let us assume that m, α are infinite and $m < \alpha$.

Let R be the set $\{m : m \in \beta, \exists i \in \alpha$ such that $d_{im} \in F\}$.

An m-transformation $\tau \in {}^{\alpha}\beta$ is called a *basic transformation* on α if for τ, there is a set N ($N \subset \alpha$) such that $|N| \leq m$ and $d_{i\,\tau i} \in F$ if $i \in N$ and $\tau i = i$ if $i \notin N$, and, in addition, $\prod_{i \in N} d_{i\,\tau i} \in F$.

Let M_0 be the set $\{\tau : \tau$ is basic transformation on $\alpha\}$.

Let M_1 be the set

$\{\eta : \eta \in {}^{\alpha}\beta,\ \eta i = \tau i$ for some $\tau \in M_0$ except for finitely many $i \in \alpha\}$.

Let M denote the set $\{\eta \circ \lambda : \eta \in M_1, \lambda \in {}^{\alpha}\alpha, \lambda$ is an m-transformation$\}$ of m- transformations.

Remarks

a) $M_0 \neq \emptyset$. For example, if N is finite, then let τ be such that $\tau i = n_i$, where $i \equiv n_i$ ($d_{in_i} \in F$), $n_i \notin \alpha$, and let τ be the identity otherwise. Lemma 6.5 implies that such a τ exists and $\tau \in M_0$.

b) In general, $M \neq {}^{\alpha}\beta \cap {}_m\mathrm{T}_{\alpha}$. Furthermore, R is infinite by Lemma 6.5, $\alpha \subset R$.

c) Notice that if the set N ($|N| \leq m$) occuring in the definition of the *basic* transformation is replaced by a set having cardinality α (for example, by the set α), and $\mathfrak{B}$' is locally-m (this may be assumed, too), then $\prod_{i \in \alpha} d_{i\,\tau i} \in F$ cannot hold, because $\prod_{i \in \alpha} d_{i\,\tau i} = c_0^{\partial} d_{01}$ (by [He-Mo-Ta I.] 1.11.6). Indeed, $c_0^{\partial} d_{01} \notin F$, apart from trivial cases.

Lemma 6.6 *The following propositions* (i), (ii) and (iii) *hold:*

(i) $Id_{\alpha} \in M_0$

(ii) $\tau \in M$ *implies that* $\tau \circ [i\ /\ m] \in M$ *for each fixed* $i \in \alpha$ *and* $m \in \beta$

(iii) $\tau \in M$ *implies that* $\tau \circ \lambda_1 \in M$ *for each* m*-transformation* λ_1, $\lambda_1 \in {}^{\alpha}\alpha$.

Proof.

(i) It follows from $d_{ii} = 1 \in F$, $i \in \alpha$, and the definition of M_0.

(ii) Assume that τ is of the form $\eta \circ \lambda$, where $\eta \in M_1$. For a fixed $i \in \alpha$, let us fix a $j \in \alpha$ such that $\lambda j = j$, $\tau j = j$ and there is no $k \in \alpha, k \neq j$ such that $\lambda k = j$. λ and τ are m-transformations, thus such a j exists. Let $\lambda' \in {}_m\mathrm{T}_\alpha$ such that $\lambda' i = j$ and $\lambda' k = \lambda k$ if $k \neq i$ and, furthermore, let $\tau' \in {}^\alpha\beta \cap {}_m\mathrm{T}_\alpha$ be such that $\tau' j = m$ and $\tau' l = \tau l$ if $l \neq j$. It is easy to see that $\tau \circ [i \ / \ m] = \tau' \circ \lambda'$ and $\tau' \circ \lambda' \in M$. That is, $\tau \circ [i \ / \ m] \in M$.

(iii) It follows from the fact that the composition of m-transformations is an m-transformation and from the definition of M.

qed.

Remark

Of course, part (ii) is true for finitely many compositions, too, i.e., for $\tau \circ [i_1 \ / \ m_1] \circ [i_2 \ / \ m_2] \circ \ldots \circ [i_n \ / \ m_n]$. (iii) fails to be true for compositions by an arbitrary m-transformation $\eta \in {}^\alpha\beta$, i.e., for $\tau \circ \eta$. This will be the reason why the proof of Theorem 6.2 *does not work* for polyadic equality algebras, i.e., for infinite cylindrifications.

Now, we define a Cprs_α-unit V, as we indicated in the outline of the proof. The members of the α-sequences in V will be equivalence classes with respect to $\equiv$. V will be defined by *m-subunits*.

For the fixed y ($y \in A$, $y \neq 0$), let us consider the fixed ultrafilter F_y containing y, defined after Lemma 6.4, and let Π_y denote the set of equivalence classes corresponding to F_y, defined in (6.4). Let Z_y be a β-sequence such that

$$(Z_y)_n = n/\equiv \text{ if } n \in \beta. \tag{6.5}$$

With y, Z_y and F_y we can associate an m-subunit W_y in the following way (we omit the index y if misunderstanding is excluded):

$$W_y = \{S_\tau Z_y \,:\, \tau \in M\}\,. \tag{6.6}$$

Let the definition of the expected embedding h' of $\mathfrak{A}$ into the full Cprs_α with unit V be

$$hx = \{S_\tau Z_y \,:\, s_\tau x \in F_y,\ \tau \in M\} \tag{6.7}$$

where $x \in A$ and h denotes the restriction of h' to the m-subunit W_y.

Remarks

a) By Lemma 6.6 (i), τ may be Id_α in (6.6). Then we obtain a *support* of W_y, i.e., we obtain the α-sequence Z_y^0 such that $(Z_y^0)_i$ is the equivalence class in Π_y associated with i by Lemma 6.5 ($Z_y^0 \in W_y$). W_y is a subset of the m-weak space by support Z_y^0 and m-base Π_y. By Lemma 6.6 (iii), W_y is really an m-subunit, because $\tau \in M$ implies $\tau \circ \lambda \in M$ for each m-transformation λ.

b) $W_y = h1$ because $s_\tau 1 = 1$, by the neat embedding property. Notice that $hx \subseteq W_y$, by definition.

In the lemma below, we check that the definition in (6.7) is sound. Next, it is shown in Lemma 6.8 that h' is indeed an embedding of $\mathfrak{A}$.

Lemma 6.7 *$S_\tau Z_y = S_\sigma Z_y$ implies that $s_\tau x \in F$ if and only if $s_\sigma x \in F$, where $\tau, \sigma \in M$, $x \in A$.*

Proof.

Indirectly. Assume that

$$S_\tau Z_y = S_\sigma Z_y,\ s_\tau x \in F,\ \text{but } s_\sigma x \notin F \tag{6.8}$$

for some $\tau, \sigma \in M$, $x \in A$. By (6.5), $S_\tau Z_y = S_\sigma Z_y$ means that $\tau i \equiv \sigma i$, i.e., $d_{\tau i\ \sigma i} \in F$ if $i \in \alpha$. This implies that $d_{\tau i\ \sigma i} \in F$ if $i \in \Delta x$, of course. (Here $|\Delta x| \leq m$, by condition).

Let us consider the product $\prod_{i \in \Delta x} d_{\tau i\ \sigma i}$. This product does not necessarily exists or, if it exists, does not necessarily belongs to F.

Let us consider the completion $\mathfrak{B}$' of $\mathfrak{B}$ and recall the definition of F (after Lemma 6.4) in $\mathfrak{B}'$. From now on, we *identify* the elements in $\mathfrak{B}$ and their images at the embedding.

$\prod_{i \in \Delta x} d_{\tau i\ \sigma i}$ exists in $\mathfrak{B}$', by the completion property. It is shown that

$$\prod_{i \in \Delta x} d_{\tau i\ \sigma i} \in F \tag{6.9}$$

where $d_{\tau i\ \sigma i} \in F$ if $i \in \Delta x$.

Let us take the definition of the transformations $\tau, \sigma \in M$. Assume that $\tau = \overline{\tau}\lambda_1$ and $\sigma = \overline{\sigma}\lambda_2$ for some $\overline{\tau}, \overline{\sigma} \in M_1$ and m-transformations $\lambda_1, \lambda_2 \in {}^\alpha\alpha$ (where $\overline{\tau}\lambda_1$ abbreviates $\overline{\tau}$ $\circ\lambda_1$, e.g.,). Then $\prod_{i \in \Delta x} d_{\tau i\ \sigma i}$ is of the form $\prod_{i \in \Delta x} d_{(\overline{\tau}\lambda_1)i\ (\overline{\sigma}\lambda_2)i}$. In this product, let us separate the diagonal elements such that at least one of their indices is not in R. By the definition of M_1, there are only *finitely many diagonals* in $\prod_{i \in \Delta x} d_{(\overline{\tau}\lambda_1)i\ (\overline{\sigma}\lambda_2)i}$ having this property, so let us assume that this property is satisfied for $i \in P$, for example, (P may be infinite). Thus we obtain:

$$\prod_{i \in \Delta x} d_{(\overline{\tau}\lambda_1)i\ (\overline{\sigma}\lambda_2)i} = \prod_{i \in \Delta x \cap P} d_{(\overline{\tau}\lambda_1)i\ (\overline{\sigma}\lambda_2)i} \cdot \prod_{i \in \Delta x \sim P} d_{(\overline{\tau}\lambda_1)i\ (\overline{\sigma}\lambda_2)i}.$$

The first member of this product is an element of F because it contains finitely many diagonals and the diagonals are elements of F, by assumption. As regards the second member of the product, let us consider the following inequality:

$$\prod_{i \in \Delta x \sim P} d_{(\overline{\tau}\lambda_1)i\ (\overline{\sigma}\lambda_2)i} \geq \prod_{i \in \Delta x \sim P} d_{(\overline{\tau}\lambda_1)i\ \lambda_1 i}$$

This follows from the known property $d_{nm} \geq d_{ni} \cdot d_{im}$ of diagonals.

After the above separation, $\overline{\tau}$ and $\overline{\sigma}$ may already be considered as basic transformations in M_0 by the definition of M_1, considering these transformations to be the identity if $i \in P$.

Therefore the following inequality is true:

$$\prod_{i\in\Delta x\sim P} d_{(\overline{\tau}\lambda_1)i\ \lambda_1 i}\cdot \prod_{i\in\Delta x\sim P} d_{\lambda_2 i\ (\overline{\sigma}\lambda_2)i} \geq \prod_{k\in N_1} d_{\overline{\tau}k\ k}\cdot \prod_{k\in N_2} d_{k\ \overline{\sigma}k}$$

where N_1 and N_2 are the sets occuring in the definition of basic transformation. This inequality follows from the fact that the set of the members on the right-hand side is a subset of those on the left-hand side, by the definition of M and M_0.

But, by the definition of M_0, the two products on the right-hand side are elements of F. Therefore, using the filter properties, we obtain that (6.9) is true.

Then, let us consider the inequality $\prod_{i\in\Delta x} d_{\tau i\ \sigma i}\cdot s_\tau x \leq s_\sigma x$ (i.e., $\prod_{i\in\Delta x} d_{\tau i\ \sigma i}\cdot\ x(\tau 1,\tau 2,\ldots,\tau k,\ldots) \leq x(\sigma 1,\sigma 2,\ldots,\sigma k,\ldots)$). This holds, by (CP8)*. This inequality implies a contradiction because $\prod_{i\in\Delta x} d_{\tau i\ \sigma i}\in F$, $s_\tau x\in F$ and the properties of filters imply $s_\sigma x\in F$, contradicting (6.8). Thus the lemma is proven.

qed.

Now, we will prove that the mapping h' defined in (6.7) is an embedding of $\mathfrak{A}$.

Lemma 6.8 *h' is a homomorphism defined on $\mathfrak{A}$ and $h'y\neq\emptyset$ if $y\neq 0$ (i.e., h' is an embedding of $\mathfrak{A}$).*

Proof.

First, we check that $h'y\neq\emptyset$ if $y\neq 0$.

Since $\tau = Id_\alpha\in M$, by Lemma 6.6 (i), therefore by Remark a) before Lemma 6.7, $Z_y^0\in hy$, where Z_y^0 is a support.

We prove the homomorphism property by m-subunits. Let us fix an m-subunit W_y corresponding to the ultrafilter F_y. Let us denote W_y, Z_y and F_y by W, Z and F, for short, and let h denote the restriction of h' to W_y. We need to show that h preserves the operations c_i, s_λ, $+$, $-$ and the diagonals.

1. *h preserves the cylindrifications* $c_i, i \in \alpha$, i.e.,

$$hc_i x = C_i hx \tag{6.10}$$

where $x \in A$ *and* C_i *abbreviates* $C_i^{[W]}$.

We use axiom (CP5) several times. By definition, (6.10) means that

$$\{S_\tau Z \,:\, s_\tau c_i x \in F, \tau \in M\} = C_i \{S_\eta Z \,:\, s_\eta x \in F, \eta \in M\}. \tag{6.11}$$

For the right-hand side of (6.11), $C_i \{S_\eta Z \,:\, s_\eta x \in F, \eta \in M\} =$

$=\big\{S_{[i\ /\ n]} S_\eta Z : s_\eta x \in F,\ \eta \circ [i\ /\ n] \in M \text{ for some } n \in \beta\big\} =$

$$= \big\{S_{\eta\circ[i\ /\ n]}\ Z : s_\eta x \in F,\ \eta \circ [i\ /\ n] \in M \text{ for some } n \in \beta\big\} \tag{6.12}$$

by $S_{[i\ /\ n]} S_\eta Z = S_{\eta\circ[i\ /\ n]}\ Z$.

First, we prove that the left-hand side is a subset of the right-hand side in (6.11). Assume that $S_\tau Z$ is an element of the left-hand side in (6.11).

By the regular perfect ultrafilter property, $s_\tau c_i x \in F$ implies $s_\tau s_{[i\ /\ m]} x \in F$ for some $m \notin (\alpha \cup \mathrm{Rg}\ \tau)$. And,

$$s_\tau s_{[i\ /\ m]} x = s_{\tau\circ[i\ /\ m]} x. \tag{6.13}$$

$\tau \in M$ implies that $\tau \circ [i\ /\ m] \in M$, by Lemma 6.6 (ii).

Let us choose $\tau \circ [i\ /\ m]$ for η in (6.12). So, $\eta \in M$ holds. $s_\eta x \in F$, by (6.13). $\eta = \tau \circ [i\ /\ m]$ implies that τ is of the form $\eta \circ [i\ /\ n]$ for some $n \in \beta$. $\eta \in M$ implies that $\eta \circ [i\ /\ n] \in M$ by Lemma 6.6 (ii). Hence $S_\tau Z$, i.e., $S_{\eta\circ[i\ /\ n]} Z$ is indeed in the set in (6.12).

Next, we check that the right-hand side in (6.11) (i.e., the set in (6.12)) is a subset of the left-hand side. Assume that $S_{\eta\circ[i\ /\ n]}Z$ is an element of (6.12)

$s_\eta x \in F$ implies that $s_\eta c_i x \in F$. Considering the left-hand side of (6.11), let τ be $\eta \circ [i\ /\ n]$. Then $\tau \in M$ holds. But, by (CP5),

$$s_{\eta\circ[i\ /\ n]}c_i x = (s_\eta \circ s_{[i\ /\ n]})c_i x. \tag{6.14}$$

Here $(s_\eta \circ s_{[i\ /\ n]})c_i x = s_\eta c_i x$, hence $(s_\eta \circ s_{[i\ /\ n]})c_i x \in F$. This latter together with (6.14) imply $s_{\eta\circ[i\ /\ n]}c_i x \in F$, i.e., $s_\tau c_i x \in F$. Hence, $S_{\eta\circ[i\ /\ n]}Z$ is in $\{S_\tau Z : s_\tau c_i x \in F, \tau \in M\}$.

2. *h preserves the transformations s_λ for every m-transformation $\lambda \in {}^\alpha\alpha$, i.e.,*

$$h(s_\lambda x) = S_\lambda hx, \tag{6.15}$$

where S_λ abbreviates S_λ^W.

(6.15) means that

$$\{S_\tau Z\ :\ s_\tau(s_\lambda x) \in F, \tau \in M\} = S_\lambda \{S_\eta Z\ :\ s_\eta x \in F, \eta \in M\} \tag{6.16}$$

where $x \in A$, $\lambda \in {}^\alpha\alpha$ is m-transformation.

We use (CP5) again. Let us denote the set $\{S_\eta Z\ :\ s_\eta x \in F, \eta \in M\}$ by X. For the right-hand side of (6.16), by the definition (6.6) of W_y,

$S_\lambda X = \{S_\delta Z\ :\ S_\lambda S_\delta Z \in X,\ \delta \in M\} = \{S_\delta Z\ :\ S_{\delta\,\circ\,\lambda} Z \in X,\ \delta \in M\}$. And,

$$\{S_\delta Z\ :\ S_{\delta\,\circ\,\lambda} Z \in X,\ \delta \in M\} = \{S_\delta Z\ :\ s_{\delta\,\circ\,\lambda} x \in F, \delta \in M,\ \delta \circ \lambda \in M\ \} \tag{6.17}$$

by the definition of the set X.

For the left-hand side of (6.16)

$$\{S_\tau Z \,:\, s_\tau(s_\lambda x) \in F, \tau \in M\} = \{S_\tau Z \,:\, s_{\tau\,\circ\,\lambda} x \in F,\ \tau \in M\ \}\,. \tag{6.18}$$

Comparing (6.17) and (6.18), choosing $\tau = \delta$, and recalling that $\delta \in M$ implies $\delta \circ \lambda \in M$ by Lemma 6.6 (iii), we obtain that these sets coincide.

3. *h preserves the diagonals, i.e.,*

$$hd_{ij} = D_{ij},$$

where $i, j \in \alpha$ and D_{ij} abbreviates D^W_{ij}.

$hd_{ij} = D_{ij}$ means that

$$\{S_\tau Z \,:\, s_\tau d_{ij} \in F, \tau \in M\} = \{S_\tau Z \,: (S_\tau Z)_i = (S_\tau Z)_j, \tau \in M\,\} \tag{6.19}$$

where $i, j \in \alpha$.

The left-hand side of (6.19) is a subset of the right-hand side. Indeed, by (E3), $s_\tau d_{ij} = d_{\tau i\, \tau j}$, hence $d_{\tau i\,\tau j} \in F$. But $(S_\tau Z)_i = (S_\tau Z)_j$, i.e., $\tau i \;/\equiv\; = \tau j \;/\equiv$ means, by definition of $\equiv$, that $d_{\tau i\,\tau j} \in F$. Conversely, the right-hand side of (6.19) is a subset of the left-hand side. Similarly to the previous line of reasoning, $(S_\tau Z)_i = (S_\tau Z)_j$ means that $d_{\tau i\,\tau j} \in F$. From this, by (E3), $s_\tau d_{ij} \in F$ obviously follows.

4. *h preserves the operation $+$, i.e.,*

$$h(x + z) = hx \cup hz$$

if $x, z \in A$.

Here $h(x+z) = \{S_\tau Z : s_\tau(x+z) \in F, \tau \in M\}$, $hx =$
$\{S_\tau Z : s_\tau x \in F, \tau \in M\}$, $hz = \{S_\tau Z : s_\tau z \in F, \tau \in M\}$, where $x, z \in A$. By (CP6), $s_\tau(x+z) = s_\tau x + s_\tau z$.

If $S_\tau Z \in hx \cup hz$, then, for example, $S_\tau Z \in hz$, which means by the definition of h that $s_\tau z \in F$. But $s_\tau z \in F$ and the ultrafilter properties imply that

$$s_\tau x + s_\tau z \in F. \tag{6.20}$$

By (CP6), $s_\tau(x+z) \in F$, consequently, $S_\tau Z \in h(x+z)$, by the definition of h.

The converse is similar. If $S_\tau Z \in h(x+z)$, then $s_\tau(x+z) = s_\tau x + s_\tau z \in F$. F is a filter, therefore $s_\tau x \in F$ or $s_\tau z \in F$. Thus, $S_\tau Z \in hx$ or $S_\tau Z \in hz$, so, $S_\tau Z \in hx \cup hz$.

5. *h preserves the operation* $-$, *i.e.,*

$$h(-x) = \ \sim hx$$

where $\sim$ *abbreviates* $\sim_W$.

Here $hx = \{S_\sigma Z : s_\sigma x \in F, \sigma \in M\}$ and $h(-x) =$
$=\{S_\tau Z : s_\tau(-x) \in F, \tau \in M\}$. Using the ultrafilter properties and (CP7)

$$\begin{aligned} \sim \ \ hx = \ & W \sim \{S_\sigma Z : s_\sigma x \in F, \sigma \in M\} = \{S_\sigma Z : s_\sigma x \notin F, \sigma \in M\} = \\ = \ & \{S_\sigma Z : -s_\sigma x \in F, \sigma \in M\} = \{S_\sigma Z : s_\sigma(-x) \in F, \sigma \in M\}. \end{aligned}$$

Comparing $h(-x)$ and $\sim hx$, choosing $\tau = \sigma$, we obtain the proposition.

So, h' preserves the operations restricted to the m-subunits. Notice that the preservation is true for the unit V as well, instead of the m-subunits W's. Here, the only non-trivial case is the operation minus. But, the disjointness of the m-subunits assures that h' preserves the minus, too.

qed.

The proofs of the preservation of $+$, $-$ and the diagonals are similar. They are not detailed.

Finally, using the Lemmas 6.4–6.8, we obtain

The proof of Theorem 6.2:

By Lemma 6.8, h' is an isomorphism between $\mathfrak{A}$ and a Cprs_α with unit V. We need to prove that V is a ${}_m\mathsf{Gwp}_\alpha$ unit and the representant algebra is regular. The ${}_m\mathsf{Gwp}_\alpha$ unit property follows from Lemma 3.16 (i), i.e., from the preservation of the operator s_λ, where $\lambda \in {}^\alpha\alpha$ and λ is m-transformation (Lemma 6.8, part 2). In particular, we know that $h'(s_\lambda x) = S_\lambda h'x$. Let us choose 1 for x. On one hand, $h'(s_\lambda 1) = h'1 = V$. On the other hand, $S_\lambda h'1 = S_\lambda V$. Comparing these equalities, we obtain that $S_\lambda V = V$, i.e., $\mathfrak{A} \in {}_m\mathsf{Gwp}_\alpha$.

To prove the regularity property, let us consider an arbitrary element hx in the representant algebra (see 6.7). Assume that $t \in h'x$. By definition, t is an element of a subunit W_y for some y. By the definition of regularity of ${}_m\mathsf{Gwp}_\alpha$, assume that $q \in W_y$ such that $(\Delta h'x \cup 1) \upharpoonright t \subseteq q$ ($q \in W_y$ may be assumed).

Using (6.6) and (6.7), t is of the form $S_\tau Z_y$ for some $\tau \in M$, where τ is such that $s_\tau x \in F_y$, and q is of the form $S_\sigma Z_y$ for some $\sigma \in M$. It must be proved that $q \in h'x$, i.e., $s_\sigma x \in F_y$. h' is an isomorphism, therefore $\Delta h'x = \Delta x$. By condition, $(S_\tau Z_y)_i = (S_\sigma Z_y)_i$ if $i \in (\Delta x \cup 1)$, i.e., $\tau i \equiv \sigma i$ if $i \in (\Delta x \cup 1)$. But, by the proof of Lemma 6.7, $s_\sigma x \in F_y$ follows.

As regards the proof of the other part of the Theorem 6.2, $\mathfrak{A} \in \mathbf{I}\, {}_m\mathsf{Gwp}_\alpha$ implies $\mathfrak{A} \in {}_m\mathsf{CPE}_\alpha$ (by Lemma 3.20). Then we can refer to the respective version of Daigneault-Monk-Keisler theorem (see also the proof of Theorem 3.24 below).

qed.

* * *

We come to the applications of the above neat embedding theorem. It was mentioned that neat embedding theorems, together with theorems about neatly embeddable algebras, imply representation theorems. In terms of our neat embedding theorem and the Daigneault-Monk-Keisler theorem below (and its variants), we prove two representation theorems.

Let us recall the definitions of polyadic and polyadic equality algebras (PA_α and PEA_α, [He-Mo-Ta II.], 5.4.1) and the following important result, closely related to our subject:

Theorem (Daigneault–Monk–Keisler) *If* $\mathfrak{A} \in \mathsf{PA}_\alpha$, *then* $\mathfrak{A} \in \mathrm{S}\mathfrak{Nr}_\alpha\mathfrak{B}$ *for some* $\mathfrak{B} \in \mathsf{PA}_{\alpha+\varepsilon}$*, where* α *is a fixed infinite ordinal and* $\varepsilon > 1$ (see [Da-Mo], [Kei] and [He-Mo-Ta II.] Thm. 5.4.17).

This form of the theorem (apart from terminology) is due to Daigneault and Monk ([Da-Mo], Theorem 4.3). Keisler published the proof theoretical variant of the theorem in the same issue ([Kei]). Here we will refer to the proof of Theorem 4.3 in [Da-Mo] and its variant for polyadic *equality* algebras ([He-Mo-Ta II.] 5.4.17).

The Daigneault–Monk–Keisler theorem holds if the class PA_α is replaced by ${}_m\mathsf{CPE}_\alpha$ and $\mathsf{PA}_{\alpha+\varepsilon}$ is replaced by the class ${}_m\mathsf{CPE}^-_{\alpha+\varepsilon}$. We return to these versions below.

The proof of Theorem 3.24:

Assume that $\mathfrak{A} \in {}_m\mathsf{CPE}_\alpha \cap \mathrm{L}m_\alpha$. By Theorem 6.2, it is enough to prove that $\mathfrak{A} \in \mathrm{S}\mathfrak{Nr}_\alpha\mathfrak{B}$, for some $\mathfrak{B} \in {}_m\mathsf{CPE}^-_{\alpha+\varepsilon}$, where ε is infinite. We refer to the proof of Daigneault-Monk-Keisler's theorem, specifically to the proof of Theorem 4.3 in [Da-Mo] and its variant for algebras with equality ([He-Mo-Ta II.] 5.4.17).

A special case of the proof is when only single cylindrifications are defined. Omitting the axiom of the commutativity of cylindrifications (axiom (P5) there), the proof also works.

If the transformations in $\mathfrak{A}$ are supposed to be m-transformations, where m is infinite, i.e., $\mathfrak{A}$ is an m-quasi-polyadic algebra, then it is easy to check that each transformation occuring in the proof is m-transformation. Thus we obtain only m-transformations in the embedding algebra $\mathfrak{B}$, i.e., $\mathfrak{B}$ also is an m-quasi one. Thus, all ${}_m\mathsf{CPE}_\alpha$ axioms are satisfied in $\mathfrak{B}$, except for (CP9)* maybe. An important special case is when $\mathfrak{A}$ is locally-m, m is infinite, then, as the proof implies, $\mathfrak{B}$ can be assumed to be locally-m, too.

It must be checked that the properties (6.1) and (6.2) are satisfied in $\mathfrak{B}$. These equations follow from the construction included in the proof of Theorem 4.3 in [Da-Mo]. We refer to the notation used there. (6.1) means the equation in (16) there if $K = \{m\}, \tau = [j \ / \ m]$ and $\rho = [m \ / \ j]$. This holds, obviously. If $K = \{m\}$ and τ is such that $\tau m = m$, then (16) means $c_m s_\tau c_m x = c_m s_\tau x$, which is equivalent to (6.2). In this case, in the next equation (following (16)) instead of equality, the inequality $\leq$ holds by the original (CP9)*. But the right-hand side of this inequality equals that of the equation in (16).
The other direction of the theorem follows by Lemma 3.20.

qed.

The proof of Theorem 3.25:

First, assume that $\mathfrak{A} \in \mathsf{CPE}_\alpha$. Similarly to the proof of Theorem 4.3 in [Da-Mo], we can obtain that $\mathfrak{A}$ is neatly embeddable into a β-dimensional algebra $\mathfrak{B}$ satisfying all the CPE_β axioms, except for (CP9)* maybe, where $\alpha < \beta$. The embedding of $\mathfrak{A}$ in $\mathfrak{B}$ may be considered as a β-dimensional algebra. Let us denote this algebra by $\mathfrak{A}$'. This algebra is a locally-α and α-quasi β-dimensional algebra for each β ($\beta < \alpha$), i.e., $\mathfrak{A}$'$\in {}_\alpha\mathsf{CPE}^-_\beta \cap \mathrm{L}\alpha_\beta$. Now, applying to $\mathfrak{A}$', as to β-dimensional algebra, the same argument as in the proof of Theorem 3.24, we obtain that there exists a $\beta+\varepsilon$-dimensional algebra $\mathfrak{C} \in {}_\alpha\mathsf{CPE}^-_{\beta+\varepsilon}$ (ε is infinite) such that $\mathfrak{A}$'$\in \mathrm{S}\mathfrak{Nr}_\beta\mathfrak{C}$. Thus, the conditions of Theorem 6.2 are satisfied with the following choices: $\mathfrak{A}$' for $\mathfrak{A}$, α for m and β for α. By Theorem 6.2, $\mathfrak{A}$'$\in \mathbf{I}\,{}_\alpha\mathsf{Gwp}^{\mathrm{reg}}_\beta$. But, as is known, the α-reduct of an algebra in ${}_\alpha\mathsf{Gwp}^{\mathrm{reg}}_\beta$ ($\alpha < \beta$) is a set algebra in Gp_α, and the regularity is preserved as well. This set algebra in $\mathsf{Gp}_\alpha^{\mathrm{reg}}$ is obviously isomorphic to $\mathfrak{A}$. If $\mathfrak{A} \in \mathbf{I}\mathsf{Gp}_\alpha^{\mathrm{reg}}$, then the proposition follows by Lemma 3.20.

The second proposition of the theorem follows immediately from the first proposition and the definition of the class CPES_α.

qed.

Main references in this Chapter: [Ha57], [Da-Mo], [Fe12b], [Fe10] and [Fe07a].

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