Multicriteria Decision Models with Imprecise Information

DSC Dissertation

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Contents

Chapter 1

Introduction

Many decision-making tasks are too complex to be understood quantitatively, however, humans succeed by using knowledge that is imprecise rather than precise. Fuzzy logic resembles human reasoning in its use of imprecise information to generate decisions. This work summarizes my main results in multiple criteria decision making with imprecise information, where the imprecision is modelled by possibility distributions. It is organized as follows. It begins, in Chapter 'Preliminaries', with some basic principles and definitions.

The process of information aggregation appears in many applications related to the development of intelligent systems. In 1988 Yager introduced a new aggregation technique based on the ordered weighted averaging operators (OWA) [142]. The determination of ordered weighted averaging (OWA) operator weights is a very important issue of applying the OWA operator for decision making. One of the first approaches, suggested by O'Hagan, determines a special class of OWA operators having maximal entropy of the OWA weights for a given level of *orness*; algorithmically it is based on the solution of a constrained optimization problem. In 2001, using the method of Lagrange multipliers, Fullér and Majlender solved this constrained optimization problem analytically and determined the optimal weighting vector [84], and in 2003 they computed the exact minimal variability weighting vector for any level of orness [86]. 313 independent citations show that the scientific community has accepted these two approaches to obtain OWA operator weights. In 1994 Yager [145] discussed the issue of weighted $\min$ and $\max$ aggregations and provided for a formalization of the process of importance weighted transformation. In 2000 Carlsson and Fullér [24] discussed the issue of weighted aggregations and provide a possibilistic approach to the process of importance weighted transformation when both the importances (interpreted as *benchmarks*) and the ratings are given by symmetric triangular fuzzy numbers. Furthermore, we show that using the possibilistic approach (i) small changes in the membership function of the importances can cause only small variations in the weighted aggregate; (ii) the weighted aggregate of fuzzy ratings remains stable under small changes in the crisp importances; (iii) the weighted aggregate of crisp ratings still remains stable under small changes in the crisp importances whenever we use a continuous implication operator for the importance weighted transformation. 52 independent citations show that the scientific community has accepted our approach to importance weighted aggregations. In 2000 and 2001 Carlsson and Fullér [25, 30] introduced a novel statement of fuzzy mathematical programming problems and provided a method for finding a fair solution to these problems. Suppose we are given a mathematical programming problem in which the functional relationship between the decision variables and the objective function is not completely known. Our knowledge-base consists of a block of fuzzy if-then rules, where the antecedent part of the rules contains some linguistic values of the decision variables, and the consequence part consists of a linguistic value of the objective function. We suggested the use of Tsukamoto's fuzzy reasoning method to determine the crisp functional relationship between the objective function and the decision variables, and solve the resulting (usually nonlinear) programming problem to find a fair optimal solution to the original fuzzy problem. 60 independent citations show that the scientific community has accepted our statement and solution approach to fuzzy mathematical programming problems.

Typically, in complex, real-life problems, there are some unidentified factors which effect the values of the objective functions. We do not know them or can not control them; i.e. they have an impact we can not control. The only thing we can observe is the values of the objective functions at certain points. And from this information and from our knowledge about the problem we may be able to formulate the impacts of unknown factors (through the observed values of the objectives). In 1994 Carlsson and Fullér [13] stated the multiple objective decision problem with independent objectives and then adjusted their model to reality by introducing interdependences among the objectives. Interdependences among the objectives exist whenever the computed value of an objective function is not equal to its observed value. We claimed that the real values of an objective function can be identified by the help of feed-backs from the values of other objective functions, and showed the effect of various kinds (linear, nonlinear and compound) of additive feed-backs on the compromise solution. 35 independent citations show that the scientific community has accepted this statement of multiple objective decision problems.

Even if the objective functions of a multiple objective decision problem are exactly known, we can still measure the *complexity* of the problem, which is derived from the *grades of conflict* between the objectives. In 1995 Carlsson and Fullér [15] introduced the measure the *complexity* of multiple objective decision problems and to find a good compromise solution to these problems they employed the following heuristic: increase the value of those objectives that support the majority of the objectives, because the gains on their (concave) utility functions surpass the losses on the (convex) utility functions of those objectives that are in conflict with the majority of the objectives. 59 independent citations show that the scientific community has accepted this heuristic.

In Chapter "OWA Operators in Multiple Criteria Decisions" we first discuss Fullér and Majlender [84, 86] papers on obtaining OWA operator weights and survey some later works that extend and develop these models. Then following Carlsson and Fullér [24] we show a possibilistic approach to importance weighted aggregations. Finally, following Carlsson and Fullér [25, 30] we show a solution approach to fuzzy mathematical programming problems in which the functional relationship between the decision variables and the objective function is not completely known (given by fuzzy if-then rules).

Possibilisitic linear equality systems are linear equality systems with fuzzy coefficients, defined by the Zadeh's extension principle. In 1988 Kovács [108] showed that the fuzzy solution to possibilisitic linear equality systems with symmetric triangular fuzzy numbers is stable with respect to small changes of centres of fuzzy parameters. In Chapter "Stability in Fuzzy Systems" first we generalize Kovács's results to possibilisitic linear equality systems with Lipschitzian fuzzy numbers (Fullér, [74]) and to fuzzy linear programs (Fullér, [73]). Then we consider linear (Fedrizzi and Fullér, [72]) and quadratic (Canestrelli, Giove and Fullér, [12]) possibilistic programs and show that the possibility distribution of their objective function remains stable under small changes in the membership function of the fuzzy number coefficients. Furthermore, we present similar results for multiobjective possibilistic linear programs (Fullér and Fedrizzi, [82]).

In 1973 Zadeh [154] introduced the compositional rule of inference and six years later [156] the theory of approximate reasoning. This theory provides a powerful framework for reasoning in the face of imprecise and uncertain information. Central to this theory is the representation of propositions as statements assigning fuzzy sets as values to variables. In 1993 Fullér and Zimmermann [81] showed two very important features of the compositional rule of inference under triangular norms. Namely, they proved that (i) if the t-norm defining the composition and the membership function of the observation are continuous, then the conclusion depends continuously on the observation; (ii) if the t-norm and the membership function of the relation are continuous, then the observation has a continuous membership function. The stability property of the conclusion under small changes of the membership function of the observation and rules guarantees that small rounding errors of digital computation and small errors of measurement of the input data can cause only a small deviation in the conclusion, i.e. every successive approximation method can be applied to the computation of the linguistic approximation of the exact conclusion in control systems. In 1992 Fullér and Werners [80] extended the stability theorems of [81] to the compositional rule of inference with several relations. These stability properties in fuzzy inference systems were used by a research team - headed by Professor Hans-Jürgen Zimmermann - when developing a fuzzy control system for a "fuzzy controlled model car" [5] during my DAAD Scholarship at RWTH Aachen between 1990 and 1992.

In possibility theory we can use the principle of expected value of functions on fuzzy sets to define variance, covariance and correlation of possibility distributions. Marginal probability distributions are determined from the joint one by the principle of 'falling integrals' and marginal possibility distributions are determined from the joint possibility distribution by the principle of 'falling shadows'. Probability distributions can be interpreted as carriers of *incomplete information* [106], and possibility distributions can be interpreted as carriers of *imprecise information*. A function $f\colon [0,1] \to \mathbb{R}$ is said to be a weighting function if f is non-negative, monotone increasing and satisfies the following normalization condition $\int_0^1 f(\gamma)d\gamma = 1$. Different weighting functions can give different (case-dependent) importances to level-sets of possibility distributions. In Chapter "A Normative View on Possibility Distributions" we will discuss the weighted lower possibilistic and upper possibilistic mean values, crisp possibilistic mean value and variance of fuzzy numbers, which are consistent with the extension principle. We can define the mean value (variance) of a possibility distribution as the f-weighted average of the probabilistic mean values (variances) of the respective uniform distributions defined on the γ-level sets of that possibility distribution. A measure of possibilistic covariance (correlation) between marginal possibility distributions of a joint possibility distribution can be defined as the f-weighted average of probabilistic covariances (correlations) between marginal probability distributions whose joint probability distribution is defined to be uniform on the γ-level sets of their joint possibility distribution [88]. We should note here that the choice of uniform probability distribution on the level sets of possibility distributions is not without reason. Namely, these possibility distributions are used to represent imprecise human judgments and they carry non-statistical uncertainties. Therefore we will suppose that each point of a given level set is equally possible. Then we apply Laplace's principle of Insufficient Reason: if elementary events are equally possible, they should be equally probable (for more details and generalization of principle of Insufficient Reason see [71], page 59). The main new idea here is to equip the alpha-cuts of joint possibility distributions with uniform probability distributions. In Chapter "A Normative View on Possibility Distributions" we will introduce the concepts of possibilistic mean value, variance, covariance and correlation. The related publications are the following: Carlsson and Fullér [26] Carlsson, Fullér and Majlender [45], Fullér and Majlender [88] and Fullér, Mezei and Várlaki [96], 941 independent citations show that the scientific community has accepted these principles.

Properties of operations on interactive fuzzy numbers, when their joint possibility distribution is defined by a t-norm have been extensively studied in the literature. In Chapter "Operations on Interactive Fuzzy Numbers", following Fullér [76, 77] we will compute the exact membership function of product-sum and Hamacher-sum of triangular fuzzy numbers, and following Fullér and Keresztfalvi [79] we will compute the exact membership function of t-norm-based sum of L-R fuzzy numbers. We will consider the extension principle with interactive fuzzy numbers, where the interactivity relation between fuzzy numbers is defined by their joint possibility distribution. Following Fullér and Keresztfalvi [75] and Carlsson, Fullér and Majlender [41] we will show that Nguyen's theorem remains valid for interactive fuzzy numbers.

In Chapter "Selected Industrial Applications" I will describe 6 industrial research projects in which I participated as a researcher at Institute for Advanced Management Systems Research (IAMSR), Åbo Akademi University, Åbo, Finland between 1992 and 2011. In the majority of these projects our research team implemented computerized decision support systems, where all input data and information were imprecise (obtained from human judgments) and, therefore, possessed non-statistical uncertainties. Longer descriptions of these projects can be found in our three monographs: Carlsson and Fullér [33], Carlsson, Fedrizzi and Fullér [44], and Carlsson and Fullér [63]. In many cases I developed the mathematical models and algorithms for the decision problems arised in these projects. My doctoral students (and later colleagues at IAMSR, Åbo Akademi University) Péter Majlender and József Mezei (both graduated from Eötvös Loránd University) also participated in the development and verification of mathematical models and algorithms.

"The Knowledge Mobilization project" has been a joint effort by IAMSR, Åbo Akademi University and VTT Technical Research Centre of Finland. Its goal was to better "mobilize" knowledge stored in heterogeneous databases for users with various backgrounds, geographical locations and situations. The working hypothesis of the project was that fuzzy mathematics combined with domain-specific data models, in other words, fuzzy ontologies, would help manage the uncertainty in finding information that matches the user's needs. In this way, Knowledge Mobilization places itself in the domain of knowledge management. I will describe an industrial demonstration of fuzzy ontologies in information retrieval in the paper industry where problem solving reports are annotated with keywords and then stored in a database for later use.

In the Woodstrat project we built a support system for strategy formation and show that the effectiveness and usefulness of hyperknowledge support systems for strategy formation can be further advanced using adaptive fuzzy cognitive maps.

In the Waeno project we implemented fuzzy real options theory as a series of models, which were built on Excel platforms. The models were tested on a number of real life investments, i.e. real (so-called) giga-investment decisions were made on the basis of the results. The methods were thoroughly tested and validated in 2001. The new series of models, for fuzzy real option valuation (ROV), have been tested with real life data and the impact of the innovations have been traced and evaluated against both the traditional ROV-models and the classical net present value (NPV) models. The fuzzy real options were found to offer more flexibility than the traditional models; both versions of real option valuation were found to give better guidance than the classical NPV models. The models are being run from a platform built by standard Excel components, but the platform was enhanced with an adapted user interface to guide the users to both a proper use of the tools and better insight. A total of 8 actual giga-investment decisions were studied and worked out with the real options models.

In the AssessGrid project we developed a hybrid probabilistic and possibilistic model to assess the success of computing tasks in a Grid. Using the predictive probabilistic approach we developed a framework for resource management in grid computing, and by introducing an upper limit for the number of possible failures, we approximated the probability that a particular computing task can be executed. We also showed a lower limit for the probability of success of a computing task in a grid. In the possibilistic model we estimated the possibility distribution defined over the set of node failures using a fuzzy nonparametric regression technique.

In the OptionsPort project we developed a model for valuing options on R&D projects, when future cash flows and expected costs are estimated by trapezoidal fuzzy numbers. Furthermore, we represented the optimal R&D portfolio selection problem as a fuzzy mathematical programming problem, where the optimal solutions defined the optimal portfolios of R&D projects with the largest (aggregate) possibilistic deferral flexibilities.

In the EM-S Bullwhip project we worked out a fuzzy approach to reduce the bullwhip effect in supply chains. The research work focused on the demand fluctuations in paper mills caused by the frictions of information handling in the supply chain and worked out means to reduce or eliminate the fluctuations with the help of information technology. The program enhanced existing theoretical frameworks with fuzzy logic modelling and built a hyperknowledge platform for fast implementation of the theoretical results.

Chapter 2

Preliminaries

Fuzzy sets were introduced by Zadeh [153] in 1965 to represent/manipulate data and information possessing nonstatistical uncertainties. It was specifically designed to mathematically represent uncertainty and vagueness and to provide formalized tools for dealing with the imprecision intrinsic to many problems. Fuzzy sets serve as a means of representing and manipulating data that was not precise, but rather fuzzy. Some of the essential characteristics of fuzzy logic relate to the following [157]: (i) In fuzzy logic, exact reasoning is viewed as a limiting case of approximate reasoning; (ii) In fuzzy logic, everything is a matter of degree; (iii) In fuzzy logic, knowledge is interpreted a collection of elastic or, equivalently, fuzzy constraint on a collection of variables; (iv) Inference is viewed as a process of propagation of elastic constraints; and (v) Any logical system can be fuzzified. There are two main characteristics of fuzzy systems that give them better performance for specific applications: (i) Fuzzy systems are suitable for uncertain or approximate reasoning, especially for systems with mathematical models that are difficult to derive; and (ii) Fuzzy logic allows decision making with estimated values under incomplete or uncertain information.

Definition 2.1. *[153] Let X be a nonempty set. A fuzzy set A in X is characterized by its membership function $\mu_A\colon X \to [0,1]$, and $\mu_A(x)$ is interpreted as the degree of membership of element x in fuzzy set A for each $x \in X$.*

It should be noted that the terms *membership function* and *fuzzy subset* get used interchangeably and frequently we will write simply $A(x)$ instead of $\mu_A(x)$. The family of all fuzzy (sub)sets in X is denoted by $\mathcal{F}(X)$. Fuzzy subsets of the real line are called *fuzzy quantities*. Let A be a fuzzy subset of X; the *support* of A, denoted $\mathrm{s}upp(A)$, is the crisp subset of X whose elements all have nonzero membership grades in A. A fuzzy subset A of a classical set X is called *normal* if there exists an $x \in X$ such that $A(x) = 1$. Otherwise A is subnormal. An α-level set (or α-cut) of a fuzzy set A of X is a non-fuzzy set denoted by $[A]^\alpha$ and defined by $[A]^\alpha = \{t \in X | A(t) \geq \alpha\}$, if $\alpha > 0$ and $\mathrm{cl}(\mathrm{supp}A)$ if $\alpha = 0$, where $\mathrm{cl}(\mathrm{supp}A)$ denotes the closure of the support of A. A fuzzy set A of X is called *convex* if $[A]^\alpha$ is a convex subset of X for all $\alpha \in [0,1]$.

Definition 2.2. *A fuzzy number A is a fuzzy set of the real line with a normal, (fuzzy) convex and upper semi-continuous membership function of bounded support. The family of fuzzy numbers will be denoted by $\mathcal{F}$.*

Let A be a fuzzy number. Then $[A]^\gamma$ is a closed convex (compact) subset of $\mathbb{R}$ for all $\gamma \in [0,1]$. Let us introduce the notations

$$a_1(\gamma) = \min[A]^\gamma \text{ and } a_2(\gamma) = \max[A]^\gamma.$$

In other words, $a_1(\gamma)$ denotes the left-hand side and $a_2(\gamma)$ denotes the right-hand side of the γ-cut. It is easy to see that if $\alpha \leq \beta$ then $[A]^\alpha \supset [A]^\beta$. Furthermore, the left-hand side function $a_1\colon [0,1] \to \mathbb{R}$ is monotone increasing and lower semi-continuous, and the right-hand side function $a_2\colon [0,1] \to \mathbb{R}$ is monoton decreasing and upper semi-continuous. we will use the notation

$$[A]^\gamma = [a_1(\gamma), a_2(\gamma)].$$

The support of A is the open interval $(a_1(0), a_2(0))$. If A is not a fuzzy number then there exists an $\gamma \in [0,1]$ such that $[A]^\gamma$ is not a convex subset of $\mathbb{R}$.

Definition 2.3. *A fuzzy set A is called triangular fuzzy number with peak (or center) a, left width $\alpha > 0$ and right width $\beta > 0$ if its membership function has the following form*

$$A(t) = \begin{cases} 1 - \dfrac{a-t}{\alpha} & \text{if } a - \alpha \leq t \leq a \\ 1 - \dfrac{t-a}{\beta} & \text{if } a \leq t \leq a + \beta \\ 0 & \text{otherwise} \end{cases}$$

and we use the notation $A = (a, \alpha, \beta)$. It can easily be verified that

$$[A]^\gamma = [a - (1-\gamma)\alpha, a + (1-\gamma)\beta], \ \forall \gamma \in [0,1].$$

The support of A is $(a - \alpha, b + \beta)$. A triangular fuzzy number with center a may be seen as a fuzzy quantity "x is approximately equal to a".

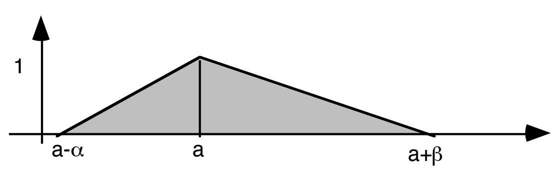

Figure 2.1: Triangular fuzzy number.

Definition 2.4. *A fuzzy set A is called trapezoidal fuzzy number with tolerance interval $[a,b]$, left width α and right width β if its membership function has the following form*

$$A(t) = \begin{cases} 1 - \dfrac{a-t}{\alpha} & \text{if } a - \alpha \leq t \leq a \\ 1 & \text{if } a \leq t \leq b \\ 1 - \dfrac{t-b}{\beta} & \text{if } a \leq t \leq b + \beta \\ 0 & \text{otherwise} \end{cases}$$

and we use the notation

$$A = (a, b, \alpha, \beta). \tag{2.1}$$

It can easily be shown that $[A]^\gamma = [a - (1-\gamma)\alpha, b + (1-\gamma)\beta]$ for all $\gamma \in [0,1]$. The support of A is $(a - \alpha, b + \beta)$.

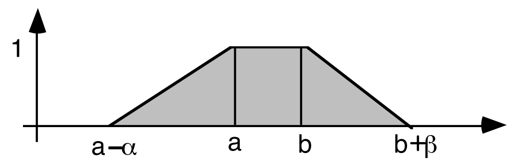

Figure 2.2: Trapezoidal fuzzy number.

A trapezoidal fuzzy number may be seen as a fuzzy quantity "x is approximately in the interval $[a, b]$".

Definition 2.5. *Any fuzzy number $A \in \mathcal{F}$ can be described as*

$$A(t) = \begin{cases} L\left(\dfrac{a-t}{\alpha}\right) & \textit{if } t \in [a-\alpha, a] \\ 1 & \textit{if } t \in [a, b] \\ R\left(\dfrac{t-b)}{\beta}\right) & \textit{if } t \in [b, b+\beta] \\ 0 & \textit{otherwise} \end{cases}$$

where $[a, b]$ is the peak *or* core *of A, $L\colon [0,1] \to [0,1]$ and $R\colon [0,1] \to [0,1]$ are continuous and non-increasing shape functions with $L(0) = R(0) = 1$ and $R(1) = L(1) = 0$. We call this fuzzy interval of LR-type and refer to it by $A = (a, b, \alpha, \beta)_{LR}$. The support of A is $(a - \alpha, b + \beta)$.*

Definition 2.6. *Let $A = (a, b, \alpha, \beta)_{LR}$ be a fuzzy number of type LR. If $a = b$ then we use the notation*

$$A = (a, \alpha, \beta)_{LR} \tag{2.2}$$

and say that A is a quasi-triangular fuzzy number. Furthermore if $L(x) = R(x) = 1 - x$, then instead of $A = (a, b, \alpha, \beta)_{LR}$ we write $A = (a, b, \alpha, \beta)$.

Let A and B are fuzzy subsets of a classical set $X \neq \emptyset$. We say that A is a subset of B if $A(t) \leq B(t)$ for all $t \in X$. Furthermore, A and B are said to be equal, denoted $A = B$, if $A \subset B$ and $B \subset A$. We note that $A = B$ if and only if $A(x) = B(x)$ for all $x \in X$. The intersection of A and B is defined as

$$(A \cap B)(t) = \min\{A(t), B(t)\} = A(t) \wedge B(t),\ \forall t \in X.$$

The union of A and B is defined as

$$(A \cup B)(t) = \max\{A(t), B(t)\} = A(t) \vee B(t),\ \forall t \in X.$$

The complement of a fuzzy set A is defined as $(\neg A)(t) = 1 - A(t), \forall t \in X$.

A fuzzy set $\bar{r}$ in the real line is said to be a fuzzy point, if its membership function is defined by

$$\bar{r}(z) = \begin{cases} 1 & \text{if } z = r, \\ 0 & \text{if } z \neq r. \end{cases}$$

That is, $\bar{r}$ is nothing else but the characteristic function of the singleton $\{r\}$.

Triangular norms were introduced by Schweizer and Sklar [131] to model distances in probabilistic metric spaces. In fuzzy sets theory triangular norms are extensively used to model logical connective *and*.

Definition 2.7. *A mapping* $T\colon [0,1] \times [0,1] \to [0,1]$ *is said to be a triangular norm (t-norm for short) iff it is symmetric, associative, non-decreasing in each argument and* $T(a,1) = a$*, for all* $a \in [0,1]$*. In other words, any t-norm* T *satisfies the properties:*

$T(x,y) = T(y,x),\ \forall x, y \in [0,1]$ *(symmetricity)*

$T(x, T(y,z)) = T(T(x,y), z),\ \forall x, y, z \in [0,1]$ *(associativity)*

$T(x,y) \le T(x',y')$ *if* $x \le x'$ *and* $y \le y'$ *(monotonicity)*

$T(x,1) = x,\ \forall x \in [0,1]$ *(one identy)*

These axioms attempt to capture the basic properties of set intersection. The basic t-norms are:

- minimum (or Mamdani [126]): $T_M(a,b) = \min\{a,b\}$,
- Łukasiewicz: $T_L(a,b) = \max\{a+b-1, 0\}$
- product (or Larsen [110]): $T_P(a,b) = ab$
- weak:

$$T_W(a,b) = \begin{cases} \min\{a,b\} & \text{if } \max\{a,b\} = 1 \\ 0 & \text{otherwise} \end{cases}$$

- Hamacher [99]:

$$H_\gamma(a,b) = \frac{ab}{\gamma + (1-\gamma)(a+b-ab)},\ \gamma \ge 0 \tag{2.3}$$

All t-norms may be extended, through associativity, to $n > 2$ arguments. A t-norm T is called strict if T is strictly increasing in each argument. A t-norm T is said to be Archimedean iff T is continuous and $T(x,x) < x$ for all $x \in (0,1)$. Every Archimedean t-norm T is representable by a continuous and decreasing function $f\colon [0,1] \to [0,\infty]$ with $f(1) = 0$ and $T(x,y) = f^{-1}(\min\{f(x) + f(y), f(0)\})$. The function f is the additive generator of T. A t-norm T is said to be nilpotent if $T(x,y) = 0$ holds for some $x, y \in (0,1)$. The operation *intersection* can be defined by the help of triangular norms.

Definition 2.8. *Let* T *be a t-conorm. The* T*-intersection of* A *and* B *is defined as*

$$(A \cap B)(t) = T(A(t), B(t)),\ \forall t \in X.$$

Triangular conorms are extensively used to model logical connective *or*.

Definition 2.9. *A mapping* $S\colon [0,1] \times [0,1] \to [0,1]$ *is said to be a triangular co-norm (t-conorm) if it is symmetric, associative, non-decreasing in each argument and* $S(a,0) = a$*, for all* $a \in [0,1]$*. In other words, any t-conorm* S *satisfies the properties:*

$S(x,y) = S(y,x)$ *(symmetricity)*

$S(x, S(y, z)) = S(S(x, y), z)$ *(associativity)*

$S(x, y) \leq S(x', y')$ *if* $x \leq x'$ *and* $y \leq y'$ *(monotonicity)*

$S(x, 0) = x, \ \forall x \in [0, 1]$ *(zero identy)*

If T is a t-norm then the equality $S(a, b) := 1 - T(1 - a, 1 - b)$, defines a t-conorm and we say that S is derived from T. The basic t-conorms are:

- maximum: $S_M(a, b) = \max\{a, b\}$
- Łukasiewicz: $S_L(a, b) = \min\{a + b, 1\}$
- probabilistic: $S_P(a, b) = a + b - ab$
- strong:

$$STRONG(a, b) = \begin{cases} \max\{a, b\} & \text{if } \min\{a, b\} = 0 \\ 1 & \text{otherwise} \end{cases}$$

- Hamacher:

$$HOR_\gamma(a, b) = \frac{a + b - (2 - \gamma)ab}{1 - (1 - \gamma)ab}, \ \gamma \geq 0$$

The operation *union* can be defined by the help of triangular conorms.

Definition 2.10. *Let S be a t-conorm. The S-union of A and B is defined as*

$$(A \cup B)(t) = S(A(t), B(t)), \ \forall t \in X.$$

2.1 The extension principle

In order to use fuzzy numbers and relations in any intelligent system we must be able to perform arithmetic operations with these fuzzy quantities. In particular, we must be able to to *add*, *subtract*, *multiply* and *divide* with fuzzy quantities. The process of doing these operations is called *fuzzy arithmetic*. we will first introduce an important concept from fuzzy set theory called the *extension principle*. We then use it to provide for these arithmetic operations on fuzzy numbers. In general the extension principle pays a fundamental role in enabling us to extend any point operations to operations involving fuzzy sets. In the following we define this principle.

Definition 2.11 (Zadeh's extension principle, [153])**.** *Assume X and Y are crisp sets and let f be a mapping from X to Y, $f\colon X \to Y$, such that for each $x \in X, f(x) = y \in Y$. Assume A is a fuzzy subset of X, using the extension principle, we can define $f(A)$ as a fuzzy subset of Y such that*

$$f(A)(y) = \begin{cases} \sup_{x \in f^{-1}(y)} A(x) & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases} \tag{2.4}$$

where $f^{-1}(y) = \{x \in X \mid f(x) = y\}$.

If $f(x) = \lambda x$ and $A \in \mathcal{F}$ then we will write $f(A) = \lambda A$. Especially, if $\lambda = -1$ then we have

$$(-1A)(x) = (-A)(x) = A(-x),\ x \in \mathbb{R}.$$

It should be noted that Zadeh's extension principle is nothing else but a straightforward generalization of set-valued functions (see [114] for details).

The extension principle can be generalized to n-place functions using the sup-min operator.

Definition 2.12 (Zadeh's extension principle for n-place functions, [153])**.** *Let $X_1, X_2, \dots, X_n$ and Y be a family of sets. Assume f is a mapping*

$$f\colon X_1 \times X_2 \times \cdots \times X_n \to Y,$$

that is, for each n-tuple $(x_1, \dots, x_n)$ such that $x_i \in X_i$, we have

$$f(x_1, x_2, \dots, x_n) = y \in Y.$$

Let $A_1, \dots, A_n$ be fuzzy subsets of $X_1, \dots, X_n$, respectively; then the (sup-min) extension principle allows for the evaluation of $f(A_1, \dots, A_n)$. In particular, $f(A_1, \dots, A_n) = B$, where B is a fuzzy subset of Y such that

$$f(A_1, \dots, A_n)(y) = \begin{cases} \sup\{\min\{A_1(x_1), \dots, A_n(x_n)\} \mid x \in f^{-1}(y)\} & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise.} \end{cases} \tag{2.5}$$

For $n = 2$ then the sup-min extension principle reads

$$f(A_1, A_2)(y) = \sup_{f(x_1,x_2)=y} \{A_1(x_1), A_2(x_2)\}.$$

Example 2.1. *Let $f\colon X \times X \to X$ be defined as $f(x_1, x_2) = x_1 + x_2$, i.e. f is the addition operator. Suppose A_1 and A_2 are fuzzy subsets of X. Then using the sup-min extension principle we get*

$$f(A_1, A_2)(y) = \sup_{x_1+x_2=y} \min\{A_1(x_1), A_2(x_2)\} \tag{2.6}$$

and we use the notation $f(A_1, A_2) = A_1 + A_2$.

Example 2.2. *Let $f\colon X \times X \to X$ be defined as $f(x_1, x_2) = x_1 - x_2$, i.e. f is the subtraction operator. Suppose A_1 and A_2 are fuzzy subsets of X. Then using the sup-min extension principle we get*

$$f(A_1, A_2)(y) = \sup_{x_1-x_2=y} \min\{A_1(x_1), A_2(x_2)\},$$

and we use the notation $f(A_1, A_2) = A_1 - A_2$.

The sup-min extension principle for n-place functions is also a straightforward generalization of set-valued functions. Namely, let $f\colon X_1 \times X_2 \to Y$ be a function. Then the image of a (crisp) subset $(A_1, A_2) \subset X_1 \times X_2$ by f is defined by

$$f(A_1, A_2) = \{f(x_1, x_2) \mid x_1 \in A \text{ and } x_2 \in A_2\}$$

and the characteristic function of $f(A_1, A_2)$ is

$$\chi_{f(A_1,A_2)}(y) = \sup\{\min\{\chi_{A_1}(x), \chi_{A_2}(x)\} \mid x \in f^{-1}(y)\}.$$

Then replacing the characteristic functions by fuzzy sets we get Zadeh's sup-min extension principle for n-place functions (2.5).

Let $A = (a_1, a_2, \alpha_1, \alpha_2)_{LR}$, and $B = (b_1, b_2, \beta_1, \beta_2)_{LR}$, be fuzzy numbers of LR-type. Using the sup-min extension principle we can verify the following rules for addition and subtraction of fuzzy numbers of LR-type.

$$A + B = (a_1 + b_1, a_2 + b_2, \alpha_1 + \beta_1, \alpha_2 + \beta_2)_{LR}$$

$$A - B = (a_1 - b_2, a_2 - b_1, \alpha_1 + \beta_2, \alpha_2 + \beta_1)_{LR}.$$

In particular if $A = (a_1, a_2, \alpha_1, \alpha_2)$ and $B = (b_1, b_2, \beta_1, \beta_2)$ are fuzzy numbers of trapezoidal form then

$$A + B = (a_1 + b_1, a_2 + b_2, \alpha_1 + \beta_1, \alpha_2 + \beta_2) \tag{2.7}$$

$$A - B = (a_1 - b_2, a_2 - b_1, \alpha_1 + \beta_2, \alpha_2 + \beta_1). \tag{2.8}$$

If $A = (a, \alpha_1, \alpha_2)$ and $B = (b, \beta_1, \beta_2)$ are fuzzy numbers of triangular form then

$$A + B = (a + b, \alpha_1 + \beta_1, \alpha_2 + \beta_2), \quad A - B = (a - b, \alpha_1 + \beta_2, \alpha_2 + \beta_1)$$

and if $A = (a, \alpha)$ and $B = (b, \beta)$ are fuzzy numbers of symmetric triangular form then

$$A + B = (a + b,\ \alpha + \beta), \quad A - B = (a - b,\ \alpha + \beta), \quad \lambda A = (\lambda a,\ |\lambda|\alpha).$$

Let A and B be fuzzy numbers with $[A]^\alpha = [a_1(\alpha), a_2(\alpha)]$ and $[B]^\alpha = [b_1(\alpha), b_2(\alpha)]$. Then it can easily be shown that

$$\begin{aligned} [A + B]^\alpha &= [a_1(\alpha) + b_1(\alpha), a_2(\alpha) + b_2(\alpha)], \\ [A - B]^\alpha &= [a_1(\alpha) - b_2(\alpha), a_2(\alpha) - b_1(\alpha)], \\ [\lambda A]^\alpha &= \lambda [A]^\alpha, \end{aligned}$$

where $[\lambda A]^\alpha = [\lambda a_1(\alpha), \lambda a_2(\alpha)]$ if $\lambda \geq 0$ and $[\lambda A]^\alpha = [\lambda a_2(\alpha), \lambda a_1(\alpha)]$ if $\lambda < 0$ for all $\alpha \in [0, 1]$, i.e. any α-level set of the extended sum of two fuzzy numbers is equal to the sum of their α-level sets. We note here that from

$$\sup_{x_1 - x_2 = y} \min\{A_1(x_1), A_2(x_2)\} = \sup_{x_1 + x_2 = y} \min\{A_1(x_1), A_2(-x_2)\},$$

it follows that the equality $A_1 - A_2 = A_1 + (-A_2)$ holds. However $A - A$ is defined by the sup-min extension principle as

$$(A - A)(y) = \sup_{x_1 - x_2 = y} \min\{A(x_1), A(x_2)\},\ y \in \mathbb{R}$$

which turns into

$$[A - A]^\alpha = [a_1(\alpha) - a_2(\alpha), a_2(\alpha) - a_1(\alpha)],$$

which is generally not a fuzzy point.

Theorem 2.1 (Nguyen, [129]). *Let $f\colon X \to X$ be a continuous function and let A be fuzzy numbers. Then*

$$[f(A)]^\alpha = f([A]^\alpha)$$

where $f(A)$ is defined by the extension principle (2.4) and $f([A]^\alpha) = \{f(x) \mid x \in [A]^\alpha\}$.

If $[A]^\alpha = [a_1(\alpha), a_2(\alpha)]$ and f is monoton increasing then from the above theorem we get

$$[f(A)]^\alpha = f([A]^\alpha) = f([a_1(\alpha), a_2(\alpha)]) = [f(a_1(\alpha)), f(a_2(\alpha))].$$

Theorem 2.2 (Nguyen, [129]). *Let $f\colon X \times X \to X$ be a continuous function and let A and B be fuzzy numbers. Then*

$$[f(A, B)]^\alpha = f([A]^\alpha, [B]^\alpha),$$

where

$$f([A]^\alpha, [B]^\alpha) = \{f(x_1, x_2) \,|\, x_1 \in [A]^\alpha, x_2 \in [B]^\alpha\}.$$

Let $f(x, y) = xy$ and let $[A]^\alpha = [a_1(\alpha), a_2(\alpha)]$ and $[B]^\alpha = [b_1(\alpha), b_2(\alpha)]$ be two fuzzy numbers. Applying Theorem 2.2 we get

$$[f(A, B)]^\alpha = f([A]^\alpha, [B]^\alpha) = [A]^\alpha [B]^\alpha.$$

However the equation

$$[AB]^\alpha = [A]^\alpha [B]^\alpha = [a_1(\alpha)b_1(\alpha), a_2(\alpha)b_2(\alpha)]$$

holds if and only if A and B are both nonnegative, i.e. $A(x) = B(x) = 0$ for $x \le 0$.

2.2 Fuzzy implications

If p is a proposition of the form "x is A" where A is a fuzzy set, for example, "big pressure" and q is a proposition of the form "y is B" for example, "small volume" then one encounters the following problem: *How to define the membership function of the fuzzy implication $A \to B$?* It is clear that $(A \to B)(x, y)$ should be defined *pointwise* i.e. $(A \to B)(x, y)$ should be a function of $A(x)$ and $B(y)$. That is $(A \to B)(u, v) = I(A(u), B(v))$. We shall use the notation $(A \to B)(u, v) = A(u) \to B(v)$. In our interpretation $A(u)$ is considered as the truth value of the proposition "u is big pressure", and $B(v)$ is considered as the truth value of the proposition "v is small volume".

$$u \text{ is big pressure} \to v \text{ is small volume} \equiv A(u) \to B(v)$$

One possible extension of material implication to implications with intermediate truth values is

$$A(u) \to B(v) = \begin{cases} 1 & \text{if } A(u) \le B(v) \\ 0 & \text{otherwise} \end{cases}$$

This implication operator is called *Standard Strict*.

$$\text{"4 is big pressure"} \to \text{"1 is small volume"} = A(4) \to B(1) = 0.75 \to 1 = 1.$$

However, it is easy to see that this fuzzy implication operator is not appropriate for real-life applications. Namely, let $A(u) = 0.8$ and $B(v) = 0.8$. Then we have

$$A(u) \to B(v) = 0.8 \to 0.8 = 1.$$

Let us suppose that there is a small error of measurement or small rounding error of digital computation in the value of $B(v)$, and instead 0.8 we have to proceed with 0.7999. Then from the definition of Standard Strict implication operator it follows that

$$A(u) \to B(v) = 0.8 \to 0.7999 = 0.$$

This example shows that small changes in the input can cause a big deviation in the output, i.e. our system is very sensitive to rounding errors of digital computation and small errors of measurement.

A smoother extension of material implication operator can be derived from the equation

$$X \to Y = \sup\{Z | X \cap Z \subset Y\},$$

where X, Y and Z are classical sets. Using the above principle we can define the following fuzzy implication operator

$$A(u) \to B(v) = \sup\{z | \min\{A(u), z\} \leq B(v)\}$$

that is,

$$A(u) \to B(v) = \begin{cases} 1 & \text{if } A(u) \leq B(v) \\ B(v) & \text{otherwise} \end{cases}$$

This operator is called *Gödel* implication. Using the definitions of negation and union of fuzzy subsets the material implication $p \to q = \neg p \vee q$ can be extended by

$$A(u) \to B(v) = \max\{1 - A(u), B(v)\}$$

This operator is called *Kleene-Dienes* implication.

In many practical applications one uses Mamdani's implication operator to model causal relationship between fuzzy variables. This operator simply takes the minimum of truth values of fuzzy predicates

$$A(u) \to B(v) = \min\{A(u), B(v)\}$$

It is easy to see this is not a correct extension of material implications, because $0 \to 0$ yields zero. However, in knowledge-based systems, we are usually not interested in rules, in which the antecedent part is false. There are three important classes of fuzzy implication operators:

- S**-implications**: defined by

  $$x \to y = S(n(x), y)$$

  where S is a t-conorm and n is a negation on $[0, 1]$. These implications arise from the Boolean formalism

  $$p \to q = \neg p \vee q.$$

  Typical examples of S-implications are the Łukasiewicz and Kleene-Dienes implications.

- R**-implications**: obtained by residuation of continuous t-norm T, i.e.

  $$x \to y = \sup\{z \in [0, 1] \,|\, T(x, z) \leq y\}$$

  These implications arise from the *Intutionistic Logic* formalism. Typical examples of R-implications are the Gödel and Gaines implications.

- **t-norm implications**: if T is a t-norm then

  $$x \to y = T(x, y)$$

  Although these implications do not verify the properties of material implication they are used as model of implication in many applications of fuzzy logic. Typical examples of t-norm implications are the Mamdani ($x \to y = \min\{x, y\}$) and Larsen ($x \to y = xy$) implications.

Name	Definition
Early Zadeh	$x \to y = \max\{1-x, \min(x,y)\}$
Łukasiewicz	$x \to y = \min\{1, 1-x+y\}$
Mamdani	$x \to y = \min\{x, y\}$
Larsen	$x \to y = xy$
Standard Strict	$x \to y = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{otherwise} \end{cases}$
Gödel	$x \to y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$
Gaines	$x \to y = \begin{cases} 1 & \text{if } x \leq y \\ y/x & \text{otherwise} \end{cases}$
Kleene-Dienes	$x \to y = \max\{1-x, y\}$
Kleene-Dienes-Łukasiewicz	$x \to y = 1 - x + xy$
Yager	$x \to y = y^x$

Table 2.1: Fuzzy implication operators.

The use of fuzzy sets provides a basis for a systematic way for the manipulation of vague and imprecise concepts. In particular, we can employ fuzzy sets to represent linguistic variables. A linguistic variable can be regarded either as a variable whose value is a fuzzy number or as a variable whose values are defined in linguistic terms.

Definition 2.13. *A linguistic variable is characterized by a quintuple*

$$(x, T(x), U, G, M)$$

in which x is the name of variable; $T(x)$ is the term set of x, that is, the set of names of linguistic values of x with each value being a fuzzy number defined on U; G is a syntactic rule for generating the names of values of x; and M is a semantic rule for associating with each value its meaning.

For example, if *speed* is interpreted as a linguistic variable, then its term set T (speed) could be

$$T = \{\text{slow, moderate, fast, very slow, more or less fast, sligthly slow, } \dots\}$$

where each term in T (speed) is characterized by a fuzzy set in a universe of discourse $U = [0, 100]$. We might interpret

- *slow* as ”a speed below about 40 mph”
- *moderate* as ”a speed close to 55 mph”
- *fast* as ”a speed above about 70 mph”

These terms can be characterized as fuzzy sets whose membership functions are

$$\text{slow}(v) = \begin{cases} 1 & \text{if } v \le 40 \\ 1 - (v-40)/15 & \text{if } 40 \le v \le 55 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{moderate}(v) = \begin{cases} 1 - |v-55|/30 & \text{if } 40 \le v \le 70 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{fast}(v) = \begin{cases} 1 & \text{if } v \ge 70 \\ 1 - \dfrac{70-v}{15} & \text{if } 55 \le v \le 70 \\ 0 & \text{otherwise} \end{cases}$$

In many practical applications we normalize the domain of inputs and use the following type of fuzzy partition: NVB (Negative Very Big), NB (Negative Big), NM (Negative Medium), NS (Negative Small), ZE (Zero), PS (Positive Small), PM (Positive Medium), PB (Positive Big), PVB (Positive Very Big). We will use the following parametrized standard fuzzy partition of the unit inteval. Suppose that $U = [0,1]$ and $\mathcal{T}(x)$ consists of $K+1$, $K \ge 2$, terms,

$$\mathcal{T} = \{\text{small}_1, \text{around 1/K, around 2/K}, \dots, \text{around (K-1)/K}, \text{big}_K\}$$

which are represented by triangular membership functions $\{A_1, \dots, A_{K+1}\}$ of the form

$$A_1(u) = [\text{small}_1](u) = \begin{cases} 1 - Ku & \text{if } 0 \le u \le 1/K \\ 0 & \text{otherwise} \end{cases} \tag{2.9}$$

$$A_k(u) = [\text{around } k/K](u) = \begin{cases} Ku - k + 1 & \text{if } (k-1)/K \le u \le k/K \\ k + 1 - Ku & \text{if } k/K \le u \le (k+1)/K \\ 0 & \text{otherwise} \end{cases} \tag{2.10}$$

for $1 \le k \le (K-1)$, and

$$A_{K+1}(u) = [\text{big}_K](u) = \begin{cases} Ku - K + 1 & \text{if } (K-1)/K \le u \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{2.11}$$

If $K = 1$ then the fuzzy partition for the [0,1] interval consists of two linguistic terms $\{\text{small}, \text{big}\}$ which are defined by

$$\text{small}(t) = 1 - t, \quad \text{big}(t) = t, \ t \in [0,1]. \tag{2.12}$$

Suppose that $U = [0,1]$ and $\mathcal{T}(x)$ consists of $2K+1$, $K \ge 2$, terms,

$$\mathcal{T} = \{\text{small}_1, \dots, \text{small}_K = \text{small}, \text{big}_0 = \text{big}, \text{big}_1, \dots, \text{big}_K\}$$

which are represented by triangular membership functions as

$$\text{small}_k(u) = \begin{cases} 1 - \dfrac{K}{k}u & \text{if } 0 \le u \le k/K \\ 0 & \text{otherwise} \end{cases} \tag{2.13}$$

for $k \le k \le K$,

$$\text{big}_k(u) = \begin{cases} \dfrac{u - k/K}{1 - k/K} & \text{if } k/K \le u \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{2.14}$$

for $0 \le k \le K - 1$.

2.3 The theory of approximate reasoning

In 1979 *Zadeh* introduced the theory of approximate reasoning [156]. This theory provides a powerful framework for reasoning in the face of imprecise and uncertain information. Central to this theory is the representation of propositions as statements assigning fuzzy sets as values to variables. Suppose we have two interactive variables $x \in X$ and $y \in Y$ and the causal relationship between x and y is completely known. Namely, we know that y is a function of x, that is $y = f(x)$. Then we can make inferences easily

$$"y = f(x)" \ \& \ "x = x_1" \longrightarrow "y = f(x_1)".$$

This inference rule says that if we have $y = f(x)$, for all $x \in X$ and we observe that $x = x_1$ then y takes the value $f(x_1)$. More often than not we do not know the complete causal link f between x and y, only we now the values of $f(x)$ for some particular values of x, that is

$$\begin{array}{ll} \Re_1: & \text{If } x = x_1 \text{ then } y = y_1 \\ \Re_2: & \text{If } x = x_2 \text{ then } y = y_2 \\ & \qquad \dots \\ \Re_n: & \text{If } x = x_n \text{ then } y = y_n \end{array}$$

If we are given an $x' \in X$ and want to find an $y' \in Y$ which correponds to x' under the rule-base $\Re = \{\Re_1, \dots, \Re_m\}$ then we have an interpolation problem.

Let x and y be linguistic variables, e.g. "x is high" and "y is small". The basic problem of approximate reasoning is to find the membership function of the consequence C from the rule-base $\{\Re_1, \dots, \Re_n\}$ and the fact A.

$$\begin{array}{lll} \Re_1: & \text{if } x \text{ is } A_1 \text{ then } y \text{ is } C_1, & \\ \Re_2: & \text{if } x \text{ is } A_2 \text{ then } y \text{ is } C_2, & \\ & \dots\dots\dots\dots & \\ \Re_n: & \text{if } x \text{ is } A_n \text{ then } y \text{ is } C_n & \\ \text{fact:} & \quad x \text{ is } A & \\ \hline \text{consequence:} & & y \text{ is } C \end{array}$$

In fuzzy logic and approximate reasoning, the most important fuzzy inference rule is the *Generalized Modus Ponens* (GMP). The classical *Modus Ponens* inference rule says:

$$\begin{array}{lll} \text{premise} & \text{if } p \text{ then} & q \\ \text{fact} & \quad p & \\ \hline \text{consequence} & & q \end{array}$$

This inference rule can be interpreted as: If p is true and $p \to q$ is true then q is true. If we have fuzzy sets, $A \in \mathcal{F}(U)$ and $B \in \mathcal{F}(V)$, and a fuzzy implication operator in the premise, and the fact is also a fuzzy set, $A' \in \mathcal{F}(U)$, (usually $A \neq A'$) then the consequnce, $B' \in \mathcal{F}(V)$, can be derived from the premise and the fact using the compositional rule of inference suggested by Zadeh [154]. The *Generalized Modus Ponens* inference rule says

$$\begin{array}{lll} \text{premise} & \text{if } x \text{ is } A \text{ then} & y \text{ is } B \\ \text{fact} & \quad x \text{ is } A' & \\ \hline \text{consequence:} & & y \text{ is } B' \end{array}$$

where the consequence B' is determined as a composition of the fact and the fuzzy implication operator $B' = A' \circ (A \to B)$, that is,

$$B'(v) = \sup_{u \in U} \min\{A'(u), (A \to B)(u, v)\}, \ v \in V.$$

The consequence B' is nothing else but the shadow of $A \to B$ on A'. The Generalized Modus Ponens, which reduces to classical modus ponens when $A' = A$ and $B' = B$, is closely related to the forward data-driven inference which is particularly useful in the Fuzzy Logic Control. In many practical cases instead of sup-min composition we use sup-t-norm composition.

Definition 2.14. *Let T be a t-norm. Then the sup-T compositional rule of inference rule can be written as,*

$$\begin{array}{lll} \textit{premise} & \textit{if } x \textit{ is } A \textit{ then} & y \textit{ is } B \\ \textit{fact} & x \textit{ is } A' & \\ \hline \textit{consequence:} & & y \textit{ is } B' \end{array}$$

where the consequence B' is determined as a composition of the fact and the fuzzy implication operator $B' = A' \circ (A \to B)$, that is,

$$B'(v) = \sup\{T(A'(u), (A \to B)(u, v)) \mid u \in U\}, \ v \in V.$$

It is clear that T can not be chosen independently of the implication operator.

Suppose that A, B and A' are fuzzy numbers. The GMP should satisfy some rational properties

Property 2.1. *Basic property:*

$$\begin{array}{lll} \textit{if} & x \textit{ is } A \textit{ then} & y \textit{ is } B \\ & x \textit{ is } A & \\ \hline & & y \textit{ is } B \end{array}$$

Property 2.2. *Total indeterminance:*

$$\begin{array}{lll} \textit{if} & x \textit{ is } A \textit{ then} & y \textit{ is } B \\ & x \textit{ is } \neg A & \\ \hline & & y \textit{ is } unknown \end{array}$$

Property 2.3. *Subset:*

$$\begin{array}{lll} \textit{if} & x \textit{ is } A \textit{ then} & y \textit{ is } B \\ & x \textit{ is } A' \subset A & \\ \hline & & y \textit{ is } B \end{array}$$

Property 2.4. *Superset:*

$$\begin{array}{lll} \textit{if} & x \textit{ is } A \textit{ then} & y \textit{ is } B \\ & x \textit{ is } A' & \\ \hline & & y \textit{ is } B' \supset B \end{array}$$

Suppose that A, B and A' are fuzzy numbers. The GMP with Mamdani implication inference rule says

$$\begin{array}{lll} \text{if} & x \text{ is } A \text{ then} & y \text{ is } B \\ & x \text{ is } A' & \\ \hline & & y \text{ is } B' \end{array}$$

where the membership function of the consequence B' is defined by

$$B'(y) = \sup\{A'(x) \wedge A(x) \wedge B(y) | x \in \mathbb{R}\},\ y \in \mathbb{R}.$$

It can be shown that the Generalized Modus Ponens inference rule with Mamdani implication operator does not satisfy all the four properties listed above. However, it does satisfy all the four properties with Gödel implication.

Chapter 3

OWA Operators in Multiple Criteria Decisions

The process of information aggregation appears in many applications related to the development of intelligent systems. In 1988 Yager introduced a new aggregation technique based on the ordered weighted averaging operators (OWA) [142]. The determination of ordered weighted averaging (OWA) operator weights is a very important issue of applying the OWA operator for decision making. One of the first approaches, suggested by O'Hagan, determines a special class of OWA operators having maximal entropy of the OWA weights for a given level of *orness*; algorithmically it is based on the solution of a constrained optimization problem. In 2001, using the method of Lagrange multipliers, Fullér and Majlender [84] solved this constrained optimization problem analytically and determined the optimal weighting vector. In 2003 using the Karush-Kuhn-Tucker second-order sufficiency conditions for optimality, Fullér and Majlender [86] computed the exact minimal variability weighting vector for any level of orness.

In 1994 Yager [145] discussed the issue of weighted $\min$ and $\max$ aggregations and provided for a formalization of the process of importance weighted transformation. In 2000 Carlsson and Fullér [24] discussed the issue of weighted aggregations and provide a possibilistic approach to the process of importance weighted transformation when both the importances (interpreted as *benchmarks*) and the ratings are given by symmetric triangular fuzzy numbers. Furthermore, we show that using the possibilistic approach (i) small changes in the membership function of the importances can cause only small variations in the weighted aggregate; (ii) the weighted aggregate of fuzzy ratings remains stable under small changes in the *nonfuzzy* importances; (iii) the weighted aggregate of crisp ratings still remains stable under small changes in the crisp importances whenever we use a continuous implication operator for the importance weighted transformation.

In 2000 and 2001 Carlsson and Fullér [25, 30] introduced a novel statement of fuzzy mathematical programming problems and provided a method for finding a fair solution to these problems. Suppose we are given a mathematical programming problem in which the functional relationship between the decision variables and the objective function is not completely known. Our knowledge-base consists of a block of fuzzy if-then rules, where the antecedent part of the rules contains some linguistic values of the decision variables, and the consequence part consists of a linguistic value of the objective function. We suggested the use of Tsukamoto's fuzzy reasoning method to determine the crisp functional relationship between the objective function and the decision variables, and solve the resulting (usually nonlinear) programming problem to find a fair optimal solution to the original fuzzy problem.

In this Chapter we first discuss Fullér and Majlender [84, 86] papers on obtaining OWA operator weights and survey some later works that extend and develop these models. Then following Carlsson and Fullér [24] we show a possibilistic approach to importance weighted aggregations. Finally, following Carlsson and Fullér [25, 30] we show a solution approach to fuzzy mathematical programming problems in which the functional relationship between the decision variables and the objective function is not completely known (given by fuzzy if-then rules).

3.1 Averaging operators

In a decision process the idea of *trade-offs* corresponds to viewing the global evaluation of an action as lying between the *worst* and the *best* local ratings. This occurs in the presence of conflicting goals, when a compensation between the corresponding compatibilities is allowed. Averaging operators realize trade-offs between objectives, by allowing a positive compensation between ratings. An averaging (or mean) operator M is a function $M\colon [0,1]\times[0,1]\to[0,1]$ satisfying the following properties

- $M(x,x)=x,\ \forall x\in[0,1],$ (idempotency)
- $M(x,y)=M(y,x),\ \forall x,y\in[0,1],$ (commutativity)
- $M(0,0)=0$, $M(1,1)=1,$ (extremal conditions)
- $M(x,y)\leq M(x',y')$ if $x\leq x'$ and $y\leq y'$ (monotonicity)
- M is continuous

It is easy to see that if M is an averaging operator then

$$\min\{x,y\}\leq M(x,y)\leq\max\{x,y\},\ \forall x,y\in[0,1]$$

An important family of averaging operators is formed by quasi-arithmetic means

$$M(a_1,\dots,a_n)=f^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}f(a_i)\right)$$

This family has been characterized by Kolmogorov as being the class of all decomposable continuous averaging operators. For example, the quasi-arithmetic mean of a_1 and a_2 is defined by

$$M(a_1,a_2)=f^{-1}\left[\frac{f(a_1)+f(a_2)}{2}\right].$$

The concept of *ordered weighted averaging* (OWA) operators was introduced by Yager in 1988 [142] as a way for providing aggregations which lie between the maximum and minimums operators. The structure of this operator involves a nonlinearity in the form of an ordering operation on the elements to be aggregated. The OWA operator provides a new information aggregation technique and has already aroused considerable research interest [149].

Definition 3.1 ([142]). *An OWA operator of dimension n is a mapping $F\colon \mathbb{R}^n \to \mathbb{R}$, that has an associated weighting vector $W = (w_1, w_2, \dots, w_n)^T$ such as $w_i \in [0,1],\ 1 \leq i \leq n$, and $w_1 + \dots + w_n = 1$. Furthermore*

$$F(a_1, \dots, a_n) = w_1 b_1 + \dots + w_n b_n = \sum_{j=1}^{n} w_j b_j,$$

where b_j is the j-th largest element of the bag $\langle a_1, \dots, a_n \rangle$.

A fundamental aspect of this operator is the re-ordering step, in particular an aggregate a_i is not associated with a particular weight w_i but rather a weight is associated with a particular ordered position of aggregate. When we view the OWA weights as a column vector we will find it convenient to refer to the weights with the low indices as weights at the top and those with the higher indices with weights at the bottom. It is noted that different OWA operators are distinguished by their weighting function. In [142] Yager pointed out three important special cases of OWA aggregations:

- F^*: In this case $W = W^* = (1, 0 \dots, 0)^T$ and $F^*(a_1, \dots, a_n) = \max\{a_1, \dots, a_n\}$,
- F_*: In this case $W = W_* = (0, 0 \dots, 1)^T$ and $F_*(a_1, \dots, a_n) = \min\{a_1, \dots, a_n\}$,
- F_A: In this case $W = W_A = (1/n, \dots, 1/n)^T$ and $F_A(a_1, \dots, a_n) = \dfrac{a_1 + \dots + a_n}{n}$.

A number of important properties can be associated with the OWA operators. we will now discuss some of these. For any OWA operator F holds

$$F_*(a_1, \dots, a_n) \leq F(a_1, \dots, a_n) \leq F^*(a_1, \dots, a_n).$$

Thus the upper an lower star OWA operator are its boundaries. From the above it becomes clear that for any F

$$\min\{a_1, \dots, a_n\} \leq F(a_1, \dots, a_n) \leq \max\{a_1, \dots, a_n\}.$$

The OWA operator can be seen to be *commutative*. Let $\langle a_1, \dots, a_n \rangle$ be a bag of aggregates and let $\{d_1, \dots, d_n\}$ be any *permutation* of the a_i. Then for any OWA operator $F(a_1, \dots, a_n) = F(d_1, \dots, d_n)$. A third characteristic associated with these operators is *monotonicity*. Assume a_i and c_i are a collection of aggregates, $i = 1, \dots, n$ such that for each i, $a_i \geq c_i$. Then $F(a_1, \dots, a_n) \geq F(c_1, c_2, \dots, c_n)$, where F is some fixed weight OWA operator. Another characteristic associated with these operators is *idempotency*. If $a_i = a$ for all i then for any OWA operator $F(a_1, \dots, a_n) = a$. From the above we can see the OWA operators have the basic properties associated with an *averaging operator*.

Example 3.1. *A window type OWA operator takes the average of the m arguments around the center. For this class of operators we have*

$$w_i = \begin{cases} 0 & \text{if } i < k \\ \dfrac{1}{m} & \text{if } k \leq i < k+m \\ 0 & \text{if } i \geq k+m \end{cases} \tag{3.1}$$

In order to classify OWA operators in regard to their location between *and* and *or*, a measure of *orness*, associated with any vector W is introduced by Yager [142] as follows

$$\text{orness}(W) = \frac{1}{n-1}\sum_{i=1}^{n}(n-i)w_i.$$

It is easy to see that for any W the $\text{orness}(W)$ is always in the unit interval. Furthermore, note that the nearer W is to an *or*, the closer its measure is to one; while the nearer it is to an *and*, the closer is to zero. It can easily be shown that $\text{orness}(W^*) = 1$, $\text{orness}(W_*) = 0$ and $\text{orness}(W_A) = 0.5$. A measure of *andness* is defined as $\text{andness}(W) = 1 - \text{orness}(W)$. Generally, an OWA operator with much of nonzero weights near the top will be an *orlike* operator, that is, $\text{orness}(W) \geq 0.5$, and when much of the weights are nonzero near the bottom, the OWA operator will be *andlike*, that is, $\text{andness}(W) \geq 0.5$. In [142] Yager defined the measure of dispersion (or entropy) of an OWA vector by

$$\text{disp}(W) = -\sum_{i=1}^{n} w_i \ln w_i.$$

We can see when using the OWA operator as an averaging operator $\text{disp}(W)$ measures the degree to which we use all the aggregates equally.

3.2 Obtaining OWA operator weights

One important issue in the theory of OWA operators is the determination of the associated weights. One of the first approaches, suggested by O'Hagan, determines a special class of OWA operators having maximal entropy of the OWA weights for a given level of *orness*; algorithmically it is based on the solution of a constrained optimization problem. Another consideration that may be of interest to a decision maker involves the variability associated with a weighting vector. In particular, a decision maker may desire low variability associated with a chosen weighting vector. It is clear that the actual type of aggregation performed by an OWA operator depends upon the form of the weighting vector [144]. A number of approaches have been suggested for obtaining the associated weights, i.e., quantifier guided aggregation [142, 144], exponential smoothing and learning [150]. O'Hagan [98] determined a special class of OWA operators having maximal entropy of the OWA weights for a given level of *orness*. His approach is based on the solution of he following mathematical programming problem,

$$\begin{array}{ll} \text{maximize} & \text{disp}(W) = -\displaystyle\sum_{i=1}^{n} w_i \ln w_i \\ \text{subject to} & \text{orness}(W) = \displaystyle\sum_{i=1}^{n} \frac{n-i}{n-1}\cdot w_i = \alpha,\ 0 \leq \alpha \leq 1 \\ & w_1 + \cdots + w_n = 1,\ 0 \leq w_i,\ i = 1, \dots, n. \end{array} \tag{3.2}$$

In 2001, using the method of Lagrange multipliers, Fullér and Majlender [84] transformed constrained optimization problem (3.2) into a polynomial equation which is then was solved to determine the maximal entropy OWA operator weights. By their method, the associated weighting vector is easily obtained by

$$\ln w_j = \frac{j-1}{n-1}\ln w_n + \frac{n-j}{n-1}\ln w_1 \Longrightarrow w_j = \sqrt[n-1]{w_1^{n-j} w_n^{j-1}}$$

and

$$w_n = \frac{((n-1)\alpha - n)w_1 + 1}{(n-1)\alpha + 1 - nw_1}$$

then

$$w_1[(n-1)\alpha + 1 - nw_1]^n = ((n-1)\alpha)^{n-1}[((n-1)\alpha - n)w_1 + 1]$$

where $n \geq 3$. For $n = 2$ then from $\text{orness}(w_1, w_2) = \alpha$ the optimal weights are uniquely defined as $w_1^* = \alpha$ and $w_2^* = 1 - \alpha$. Furthemore, if $\alpha = 0$ or $\alpha = 1$ then the associated weighting vectors are uniquely defined as $(0, 0, \dots, 0, 1)^T$ and $(1, 0, \dots, 0, 0)^T$, respectively.

An interesting question is to determine the minimal variability weighting vector under given level of orness [148]. The variance of a given weighting vector is computed as follows

$$D^2(W) = \sum_{i=1}^{n} \frac{1}{n}(w_i - E(W))^2 = \frac{1}{n}\sum_{i=1}^{n} w_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} w_i\right)^2 = \frac{1}{n}\sum_{i=1}^{n} w_i^2 - \frac{1}{n^2}.$$

where $E(W) = (w_1 + \dots + w_n)/n = 1/n$ stands for the arithmetic mean of weights.

In 2003 Fullér and Majlender [86] suggested a minimum variance method to obtain the minimal variability OWA operator weights. A set of OWA operator weights with minimal variability could then be generated. Their approach requires the solution of the following mathematical programming problem:

$$\begin{aligned} &\text{minimize} \qquad D^2(W) = \frac{1}{n} \cdot \sum_{i=1}^{n} w_i^2 - \frac{1}{n^2} \\ &\text{subject to} \quad \text{orness}(w) = \sum_{i=1}^{n} \frac{n-i}{n-1} \cdot w_i = \alpha, \ 0 \leq \alpha \leq 1, \\ &\qquad\qquad w_1 + \dots + w_n = 1, \ 0 \leq w_i, \ i = 1, \dots, n. \end{aligned} \tag{3.3}$$

Fullér and Majlender [86] computed the exact minimal variability weighting vector for any level of orness using the Karush-Kuhn-Tucker second-order sufficiency conditions for optimality:

Let us consider the constrained optimization problem (3.3). First we note that if $n = 2$ then from $\text{orness}(w_1, w_2) = \alpha$ the optimal weights are uniquely defined as $w_1^* = \alpha$ and $w_2^* = 1 - \alpha$. Furthemore, if $\alpha = 0$ or $\alpha = 1$ then the associated weighting vectors are uniquely defined as $(0, 0, \dots, 0, 1)^T$ and $(1, 0, \dots, 0, 0)^T$, respectively, with variability

$$D^2(1, 0, \dots, 0, 0) = D^2(0, 0, \dots, 0, 1) = \frac{1}{n} - \frac{1}{n^2}.$$

Suppose now that $n \geq 3$ and $0 < \alpha < 1$. Let us

$$L(W, \lambda_1, \lambda_2) = \frac{1}{n}\sum_{i=1}^{n} w_i^2 - \frac{1}{n^2} + \lambda_1\left(\sum_{i=1}^{n} w_i - 1\right) + \lambda_2\left(\sum_{i=1}^{n} \frac{n-i}{n-1} w_i - \alpha\right).$$

denote the Lagrange function of constrained optimization problem (3.3), where λ_1 and λ_2 are real numbers. Then the partial derivatives of L are computed as

$$\begin{aligned}\frac{\partial L}{\partial w_j} &= \frac{2w_j}{n} + \lambda_1 + \frac{n-j}{n-1}\cdot\lambda_2 = 0,\ 1 \le j \le n, \\ \frac{\partial L}{\partial \lambda_1} &= \sum_{i=1}^{n} w_i - 1 = 0, \\ \frac{\partial L}{\partial \lambda_2} &= \sum_{i=1}^{n} \frac{n-i}{n-1}\cdot w_i - \alpha = 0.\end{aligned} \tag{3.4}$$

We shall suppose that the optimal weighting vector has the following form

$$W = (0, \ldots, 0, w_p, \ldots, w_q, 0 \ldots, 0)^T \tag{3.5}$$

where $1 \le p < q \le n$ and use the notation

$$I_{\{p,q\}} = \{p, p+1, \ldots, q-1, q\},$$

for the indexes from p to q. So, $w_j = 0$ if $j \notin I_{\{p,q\}}$ and $w_j \ge 0$ if $j \in I_{\{p,q\}}$.

For $j = p$ we find that

$$\frac{\partial L}{\partial w_p} = \frac{2w_p}{n} + \lambda_1 + \frac{n-p}{n-1}\cdot\lambda_2 = 0,$$

and for $j = q$ we get

$$\frac{\partial L}{\partial w_q} = \frac{2w_q}{n} + \lambda_1 + \frac{n-q}{n-1}\cdot\lambda_2 = 0.$$

That is,

$$\frac{2(w_p - w_q)}{n} + \frac{q-p}{n-1}\cdot\lambda_2 = 0$$

and therefore, the optimal values of λ_1 and λ_2 (denoted by λ_1^* and λ_2^*) should satisfy the following equations

$$\lambda_1^* = \frac{2}{n}\left[\frac{n-q}{q-p}\cdot w_p - \frac{n-p}{q-p}\cdot w_q\right] \quad \text{and} \quad \lambda_2^* = \frac{n-1}{q-p}\cdot\frac{2}{n}\cdot(w_q - w_p). \tag{3.6}$$

Substituting λ_1^* for λ_1 and λ_2^* for λ_2 in (3.4) we get

$$\frac{2}{n}\cdot w_j + \frac{2}{n}\left[\frac{n-q}{q-p}\cdot w_p - \frac{n-p}{q-p}\cdot w_q\right] + \frac{n-j}{n-1}\cdot\frac{n-1}{q-p}\cdot\frac{2}{n}\cdot(w_q - w_p) = 0.$$

That is the jth optimal weight should satisfy the equation

$$w_j^* = \frac{q-j}{q-p}\cdot w_p + \frac{j-p}{q-p}\cdot w_q,\ j \in I_{\{p,q\}}. \tag{3.7}$$

From representation (3.5) we get

$$\sum_{i=p}^{q} w_i^* = 1,$$

that is,

$$\sum_{i=p}^{q}\left(\frac{q-i}{q-p}\cdot w_p+\frac{i-p}{q-p}\cdot w_q\right)=1,$$

i.e.

$$w_p+w_q=\frac{2}{q-p+1}.$$

From the constraint $\mathrm{orness}(w)=\alpha$ we find

$$\sum_{i=p}^{q}\frac{n-i}{n-1}\cdot w_i=\sum_{i=p}^{q}\frac{n-i}{n-1}\cdot\frac{q-i}{q-p}\cdot w_p+\sum_{i=p}^{q}\frac{n-i}{n-1}\cdot\frac{i-p}{q-p}\cdot w_q=\alpha,$$

that is,

$$w_p^*=\frac{2(2q+p-2)-6(n-1)(1-\alpha)}{(q-p+1)(q-p+2)}, \tag{3.8}$$

and

$$w_q^*=\frac{2}{q-p+1}-w_p^*=\frac{6(n-1)(1-\alpha)-2(q+2p-4)}{(q-p+1)(q-p+2)}. \tag{3.9}$$

The optimal weighting vector

$$W^*=(0,\dots,0,w_p^*,\dots,w_q^*,0\dots,0)^T$$

is feasible if and only if $w_p^*,w_q^*\in[0,1]$, because according to (3.7) any other w_j^*, $j\in I_{\{p,q\}}$ is computed as their convex linear combination.

Using formulas (3.8) and (3.9) we find

$$w_p^*,w_q^*\in[0,1]\iff\alpha\in\left[1-\frac{1}{3}\cdot\frac{2q+p-2}{n-1},1-\frac{1}{3}\cdot\frac{q+2p-4}{n-1}\right]$$

The following (disjunctive) partition of the unit interval $(0,1)$ will be crucial in finding an optimal solution to problem (3.3):

$$(0,1)=\bigcup_{r=2}^{n-1}J_{r,n}\cup J_{1,n}\cup\bigcup_{s=2}^{n-1}J_{1,s}. \tag{3.10}$$

where

$$J_{r,n}=\left(1-\frac{1}{3}\cdot\frac{2n+r-2}{n-1},\,1-\frac{1}{3}\cdot\frac{2n+r-3}{n-1}\right],\ r=2,\dots,n-1,$$
$$J_{1,n}=\left(1-\frac{1}{3}\cdot\frac{2n-1}{n-1},\,1-\frac{1}{3}\cdot\frac{n-2}{n-1}\right),$$
$$J_{1,s}=\left[1-\frac{1}{3}\cdot\frac{s-1}{n-1},\,1-\frac{1}{3}\cdot\frac{s-2}{n-1}\right),\ s=2,\dots,n-1.$$

Consider again problem (3.3) and suppose that $\alpha\in J_{r,s}$ for some r and s from partition (3.10). Such r and s always exist for any $\alpha\in(0,1)$, furthermore, $r=1$ or $s=n$ should hold.

Then

$$W^*=(0,\dots,0,w_r^*,\dots,w_s^*,0\dots,0)^T, \tag{3.11}$$

where

$$
\begin{aligned}
w_j^* &= 0, \text{ if } j \notin I_{\{r,s\}}, \\
w_r^* &= \frac{2(2s+r-2) - 6(n-1)(1-\alpha)}{(s-r+1)(s-r+2)}, \\
w_s^* &= \frac{6(n-1)(1-\alpha) - 2(s+2r-4)}{(s-r+1)(s-r+2)}, \\
w_j^* &= \frac{s-j}{s-r} \cdot w_r + \frac{j-r}{s-r} \cdot w_s, \text{ if } j \in I_{\{r+1,s-1\}}.
\end{aligned}
\tag{3.12}
$$

and $I_{\{r+1,s-1\}} = \{r+1, \dots, s-1\}$. We note that if $r = 1$ and $s = n$ then we have

$$
\alpha \in J_{1,n} = \left(1 - \frac{1}{3} \cdot \frac{2n-1}{n-1},\ 1 - \frac{1}{3} \cdot \frac{n-2}{n-1}\right),
$$

and

$$
W^* = (w_1^*, \dots, w_n^*)^T,
$$

where

$$
\begin{aligned}
w_1^* &= \frac{2(2n-1) - 6(n-1)(1-\alpha)}{n(n+1)}, \\
w_n^* &= \frac{6(n-1)(1-\alpha) - 2(n-2)}{n(n+1)}, \\
w_j^* &= \frac{n-j}{n-1} \cdot w_1 + \frac{j-1}{n-1} \cdot w_n, \text{ if } j \in \{2, \dots, n-1\}.
\end{aligned}
$$

Furthermore, from the construction of W^* it is clear that

$$
\sum_{i=1}^{n} w_i^* = \sum_{i=r}^{s} w_i^* = 1,\ w_i^* \geq 0,\ i = 1, 2, \dots, n,
$$

and $\text{orness}(W^*) = \alpha$, that is, W^* is feasible for problem (3.3).

We will show now that W^*, defined by (3.11), satisfies the Kuhn-Tucker second-order sufficiency conditions for optimality ([66], page 58). Namely,

(i) There exist $\lambda_1^*, \lambda_2^* \in \mathbb{R}$ and $\mu_1^* \geq 0, \dots, \mu_n^* \geq 0$ such that,

$$
\begin{aligned}
\frac{\partial}{\partial w_k}\Bigg(D^2(W) + \lambda_1^*\Bigg[\sum_{i=1}^{n} w_i - 1\Bigg] + \lambda_2^*\Bigg[\sum_{i=1}^{n-1} \frac{n-i}{n-1} \cdot w_i - \alpha\Bigg] \\
+ \sum_{j=1}^{n} \mu_j^*(-w_j)\Bigg)\Bigg|_{W=W^*} = 0
\end{aligned}
$$

for $1 \leq k \leq n$ and $\mu_j^* w_j^* = 0$, $j = 1, \dots, n$.

(ii) W^* is a regular point of the constraints,

(iii) The Hessian matrix,

$$\frac{\partial^2}{\partial W^2}\Bigg(D^2(W) + \lambda_1^*\Big[\sum_{i=1}^{n} w_i - 1\Big] + \lambda_2^*\Big[\sum_{i=1}^{n-1}\frac{n-i}{n-1}\cdot w_i - \alpha\Big] + \sum_{j=1}^{n}\mu_j^*(-w_j)\Bigg)\Bigg|_{W=W^*}$$

is positive definite on

$$\hat{X} = \Big\{ y \Big| h_1 y^T = 0, h_2 y^T = 0 \text{ and } g_j y^T = 0 \text{ for all } j \text{ with } \mu_j > 0 \Big\}, \tag{3.13}$$

where

$$h_1 = \left(\frac{n-1}{n-1}, \frac{n-2}{n-1}, \dots, \frac{1}{n-1}, 0\right)^T, \tag{3.14}$$

and

$$h_2 = (1, 1, \dots, 1, 1)^T. \tag{3.15}$$

are the gradients of linear equality constraints, and

$$g_j = (0, 0, \dots, 0, \overbrace{-1}^{j\text{th}}, 0, 0, \dots, 0)^T \tag{3.16}$$

is the gradient of the jth linear inequality constraint of problem (3.3).

Proof

(i) According to (3.6) we get

$$\lambda_1^* = \frac{2}{n}\cdot\left[\frac{n-s}{s-r}\cdot w_r^* - \frac{n-r}{s-r}\cdot w_s^*\right] \quad \text{and} \quad \lambda_2^* = \frac{n-1}{s-r}\cdot\frac{2}{n}\cdot(w_s^* - w_r^*)$$

and

$$\frac{2}{n}\cdot w_k^* + \lambda_1^* + \frac{n-k}{n-1}\cdot\lambda_2^* - \mu_k = 0.$$

for $k = 1, \dots, n$. If $k \in I_{\{r,s\}}$ then

$$\begin{aligned}
\mu_k^* &= \frac{2}{n}\cdot\left[\frac{s-k}{s-r}\cdot w_r^* + \frac{k-r}{s-r}\cdot w_s^*\right] + \frac{2}{n}\cdot\left[\frac{n-s}{s-r}\cdot w_r^* - \frac{n-r}{s-r}\cdot w_s^*\right] \\
&\quad + \frac{n-k}{n-1}\cdot\frac{n-1}{s-r}\cdot\frac{2}{n}\cdot(w_s^* - w_r^*) \\
&= \frac{2}{n}\cdot\frac{1}{s-r}\big[(s-k+n-s-n+k)w_r^* + (k-r-n+r+n-k)w_s^*\big] \\
&= 0.
\end{aligned}$$

If $k \notin I_{\{r,s\}}$ then $w_k^* = 0$. Then from the equality

$$\lambda_1^* + \frac{n-k}{n-1}\cdot\lambda_2^* - \mu_k = 0,$$

we find

$$\begin{aligned}
\mu_k^* &= \lambda_1^* + \frac{n-k}{n-1} \cdot \lambda_2^* \\
&= \frac{2}{n} \cdot \left[\frac{n-s}{s-r} \cdot w_r^* - \frac{n-r}{s-r} \cdot w_s^* \right] + \frac{n-k}{n-1} \cdot \frac{n-1}{s-r} \cdot \frac{2}{n} \cdot (w_s^* - w_r^*) \\
&= \frac{2}{n} \cdot \frac{1}{s-r} \cdot \left[(k-s) w_r^* + (r-k) w_s^* \right].
\end{aligned}$$

We need to show that $\mu_k^* \geq 0$ for $k \notin I_{\{r,s\}}$. That is,

$$\begin{aligned}
(k-s)w_r^* + (r-k)w_s^* = (k-s) \cdot \frac{2(2s+r-2) - 6(n-1)(1-\alpha)}{(s-r+1)(s-r+2)} \\
+(r-k) \cdot \frac{6(n-1)(1-\alpha) - 2(s+2r-4)}{(s-r+1)(s-r+2)} \geq 0.
\end{aligned} \tag{3.17}$$

If $r = 1$ and $s = n$ then we get that $\mu_k^* = 0$ for $k = 1, \ldots, n$. Suppose now that $r = 1$ and $s < n$. In this case the inequality $k > s > 1$ should hold and (3.17) leads to the following requirement for α,

$$\alpha \geq 1 - \frac{(s-1)(3k-2s-2)}{3(n-1)(2k-s-1)}.$$

On the other hand, from $\alpha \in J_{1,s}$ and $s < n$ we have

$$\alpha \in \left[1 - \frac{1}{3} \cdot \frac{s-1}{n-1}, 1 - \frac{1}{3} \cdot \frac{s-2}{n-1} \right),$$

and, therefore,

$$\alpha \geq 1 - \frac{1}{3} \cdot \frac{s-1}{n-1}$$

Finally, from the inequality

$$1 - \frac{1}{3} \cdot \frac{s-1}{n-1} \geq 1 - \frac{(s-1)(3k-2s-2)}{3(n-1)(2k-s-1)}$$

we get that (3.17) holds. The proof of the remaining case ($r > 1$ and $s = n$) is carried out analogously.

(ii) The gradient vectors of linear equality and inequality constraints are computed by (3.14), (3.15) and (3.16), respectively. If $r = 1$ and $s = n$ then $w_j^* \neq 0$ for all $j = 1, \ldots, n$. Then it is easy to see that h_1 and h_2 are linearly independent. If $r = 1$ and $s < n$ then $w_j^* = 0$ for $j = s+1, \ldots, n$, and in this case

$$g_j = (0, 0, \ldots, 0, \overbrace{-1}^{j\text{th}}, 0, 0, \ldots, 0)^T,$$

for $j = s+1, \ldots, n$. Consider the matrix

$$G = [h_1^T, h_2^T, g_{s+1}^T, \ldots, g_n^T] \in \mathbb{R}^{n \times (n-s+2)}.$$

Then the determinant of the lower-left submatrix of dimension $(n-s+2)\times(n-s+2)$ of G is

$$\begin{vmatrix} \frac{n-s+1}{n-1} & 1 & 0 & \dots & 0 \\ \frac{n-s}{n-1} & 1 & 0 & \dots & 0 \\ \frac{n-s-1}{n-1} & 1 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 1 & 0 & \dots & -1 \end{vmatrix} = (-1)^{n-s} \begin{vmatrix} \frac{n-s+1}{n-1} & 1 \\ \frac{n-s}{n-1} & 1 \end{vmatrix} = \frac{1}{n-1}(-1)^{n-s}$$

which means that the columns of G are linearly independent, and therefore, the system

$$\{h_1, h_2, g_{s+1}, \dots, g_n\},$$

is linearly independent.

If $r > 1$ and $= n$ then $w_j^* = 0$ for $j = 1, \dots, r-1$, and in this case

$$g_j = (0, 0, \dots, 0, \overbrace{-1}^{j\text{th}}, 0, 0, \dots, 0)^T,$$

for $j = 1, \dots, r-1$. Consider the matrix

$$F = [h_1^T, h_2^T, g_1^T, \dots, g_{r-1}^T] \in \mathbb{R}^{n\times(r+1)}.$$

Then the determinant of the upper-left submatrix of dimension $(r+1)\times(r+1)$ of F is

$$\begin{vmatrix} \frac{n-1}{n-1} & 1 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ \frac{n-r+1}{n-1} & 1 & 0 & \dots & -1 \\ \frac{n-r}{n-1} & 1 & 0 & \dots & 0 \\ \frac{n-r-1}{n-1} & 1 & 0 & \dots & 0 \end{vmatrix} = (-1)^{r-1} \begin{vmatrix} \frac{n-r}{n-1} & 1 \\ \frac{n-r-1}{n-1} & 1 \end{vmatrix} = \frac{1}{n-1}(-1)^{r-1}$$

which means that the columns of F are linearly independent, and therefore, the system

$$\{h_1, h_2, g_1, \dots, g_{r-1}\},$$

is linearly independent. So W^* is a regular point for problem (3.3).

(iii) Let us introduce the notation

$$K(W) = D^2(W) + \lambda_1^*\left[\sum_{i=1}^n w_i - 1\right] + \lambda_2^*\left[\sum_{i=1}^{n-1} \frac{n-i}{n-1}\cdot w_i - \alpha\right] + \sum_{j=1}^n \mu_j^*(-w_j).$$

The Hessian matrix of K at W^* is

$$\frac{\partial^2}{\partial w_k \partial w_j} K(W)\bigg|_{W=W^*} = \frac{\partial^2}{\partial w_k \partial w_j} D^2(W)\bigg|_{W=W^*} = \frac{2}{n}\cdot\delta_{kj},$$

where

$$\delta_{jk} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{otherwise.} \end{cases}$$

That is,

$$K''(W^*) = \frac{2}{n} \cdot \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

which is a positive definite matrix on $\mathbb{R}^n$.

So, the objective function $D^2(W)$ has a local minimum at point $W = W^*$ on

$$X = \left\{ W \in \mathbb{R}^n \middle| W \geq 0, \sum_{i=1}^{n} w_i = 1, \sum_{i=1}^{n} \frac{n-i}{n-1} \cdot w_i = \alpha \right\} \tag{3.18}$$

where X is the set of feasible solutions of problem (3.3). Taking into consideration that $D^2\colon \mathbb{R}^n \to \mathbb{R}$ is a strictly convex, bounded and continuous function, and X is a convex and compact subset of $\mathbb{R}^n$, we can conclude that D^2 attains its (unique) global minimum on X at point W^*.

Following Fullér and Majlender [86] we show an example for obtaining the minimal variability five-dimensional weighting vector under orness levels $\alpha = 0, 0.1, \dots, 0.9$ and 1.0. First, we construct the corresponding partition as

$$(0,1) = \bigcup_{r=2}^{4} J_{r,5} \cup J_{1,5} \cup \bigcup_{s=2}^{4} J_{1,s}.$$

where

$$J_{r,5} = \left(\frac{1}{3} \cdot \frac{5-r-1}{5-1}, \frac{1}{3} \cdot \frac{5-r}{5-1} \right] = \left(\frac{4-r}{12}, \frac{5-r}{12} \right],$$

for $r = 2, 3, 4$ and

$$J_{1,5} = \left(\frac{1}{3} \cdot \frac{5-2}{5-1}, \frac{1}{3} \cdot \frac{10-1}{5-1} \right) = \left(\frac{3}{12}, \frac{9}{12} \right),$$

and

$$J_{1,s} = \left[1 - \frac{1}{3} \cdot \frac{s-1}{5-1}, 1 - \frac{1}{3} \cdot \frac{s-2}{5-1} \right) = \left[\frac{13-s}{12}, \frac{14-s}{12} \right),$$

for $s = 2, 3, 4$, and, therefore we get,

$$\begin{aligned} (0,1) = \left(0, \frac{1}{12}\right] \cup \left(\frac{1}{12}, \frac{2}{12}\right] \cup \left(\frac{2}{12}, \frac{3}{12}\right] \cup \left(\frac{3}{12}, \frac{9}{12}\right) \\ \cup \left[\frac{9}{12}, \frac{10}{12}\right) \cup \left[\frac{10}{12}, \frac{11}{12}\right) \cup \left[\frac{11}{12}, \frac{12}{12}\right). \end{aligned}$$

Without loss of generality we can assume that $\alpha < 0.5$, because if a weighting vector W is optimal for problem (3.3) under some given degree of orness, $\alpha < 0.5$, then its reverse, denoted by W^R, and defined as

$$w_i^R = w_{n-i+1}$$

is also optimal for problem (3.3) under degree of orness $(1-\alpha)$. Really, as was shown by Yager [144], we find that

$$D^2(W^R) = D^2(W) \text{ and } \mathrm{orness}(W^R) = 1 - \mathrm{orness}(W).$$

Therefore, for any $\alpha > 0.5$, we can solve problem (3.3) by solving it for level of orness $(1-\alpha)$ and then taking the reverse of that solution.

Then we obtain the optimal weights from (3.12) as follows

- if $\alpha = 0$ then $W^*(\alpha) = W^*(0) = (0, 0, \dots, 0, 1)^T$ and, therefore,

$$W^*(1) = (W^*(0))^R = (1, 0, \dots, 0, 0)^T.$$

- if $\alpha = 0.1$ then

$$\alpha \in J_{3,5} = \left(\frac{1}{12}, \frac{2}{12}\right],$$

and the associated minimal variablity weights are

$$\begin{aligned}
w_1^*(0.1) &= 0, \\
w_2^*(0.1) &= 0, \\
w_3^*(0.1) &= \frac{2(10+3-2) - 6(5-1)(1-0.1)}{(5-3+1)(5-3+2)} = \frac{0.4}{12} = 0.0333, \\
w_5^*(0.1) &= \frac{2}{5-3+1} - w_3^*(0.1) = 0.6334, \\
w_4^*(0.1) &= \frac{1}{2} \cdot w_3^*(0.1) + \frac{1}{2} \cdot w_5^*(0.1) = 0.3333,
\end{aligned}$$

So,

$$W^*(\alpha) = W^*(0.1) = (0, 0, 0.033, 0.333, 0.633)^T,$$

and, consequently,

$$W^*(0.9) = (W^*(0.1))^R = (0.633, 0.333, 0.033, 0, 0)^T.$$

with variance $D^2(W^*(0.1)) = 0.0625$.

- if $\alpha = 0.2$ then

$$\alpha \in J_{2,5} = \left(\frac{2}{12}, \frac{3}{12}\right]$$

and in a similar manner we find that the associated minimal variablity weighting vector is

$$W^*(0.2) = (0.0, 0.04, 0.18, 0.32, 0.46)^T,$$

and, therefore,

$$W^*(0.8) = (0.46, 0.32, 0.18, 0.04, 0.0)^T,$$

with variance $D^2(W^*(0.2)) = 0.0296$.

- if $\alpha = 0.3$ then
$$\alpha \in J_{1,5} = \left(\frac{3}{12}, \frac{9}{12}\right]$$
and in a similar manner we find that the associated minimal variablity weighting vector is
$$W^*(0.3) = (0.04, 0.12, 0.20, 0.28, 0.36)^T,$$
and, therefore,
$$W^*(0.7) = (0.36, 0.28, 0.20, 0.12, 0.04)^T,$$
with variance $D^2(W^*(0.3)) = 0.0128$.
- if $\alpha = 0.4$ then
$$\alpha \in J_{1,5} = \left(\frac{3}{12}, \frac{9}{12}\right]$$
and in a similar manner we find that the associated minimal variablity weighting vector is
$$W^*(0.4) = (0.12, 0.16, 0.20, 0.24, 0.28)^T,$$
and, therefore,
$$W^*(0.6) = (0.28, 0.24, 0.20, 0.16, 0.12)^T,$$
with variance $D^2(W^*(0.4)) = 0.0032$.
- if $\alpha = 0.5$ then
$$W^*(0.5) = (0.2, 0.2, 0.2, 0.2, 0.2)^T.$$
with variance $D^2(W^*(0.5)) = 0$.

3.3 Constrained OWA aggregations

Yager [147] considered the problem of maximizing an OWA aggregation of a group of variables that are interrelated and constrained by a collection of linear inequalities and he showed how this problem can be modeled as a mixed integer linear programming problem. The constrained OWA aggregation problem [147] can be expressed as the following mathematical programming problem
$$\max F(x_1, \dots, x_n)$$
$$\text{subject to } Ax \leq b, x \geq 0,$$
where $F(x_1, \dots, x_n) = w^T y = w_1 y_1 + \dots + w_n y_n$ and y_j denotes the jth largest element of the bag $< x_1, \dots, x_n >$.

Following Carlsson, Fullér and Majlender [37] we shall show an algorithm for solving the following (nonlinear) constrained OWA aggregation problem
$$\max w^T y; \text{ subject to } \{x_1 + \dots + x_n \leq 1, x \geq 0\}. \tag{3.19}$$

Note 1. *As an illustration of the general constrained OWA aggregation problem, Yager [147] considered problem (3.19) for $n = 3$ and showed how it can be modelled as a mixed integer linear programming problem. Then he used the Storm software to solve it. In fact, our work has been motivated by the observation that the dual of problem (3.19) can be solved by a simple inspection.*

First using the relations $y_1 \geq y_2 \geq \cdots \geq y_n \geq 0$, we rewrite (3.19) in the form

$$\max\, w^T y;\ \text{subject to } \hat{G}y \leq q, \tag{3.20}$$

where

$$\hat{G} = \begin{bmatrix} e^T \\ G \end{bmatrix},$$

and $q = (1, 0, 0, \ldots, 0)^T \in \mathbb{R}^{n+1}$, $e = (1, 1, \ldots, 1)^T \in \mathbb{R}^n$, and $G = (g_{ij})$ with $g_{ij} = 1$ if $i = j - 1$, $g_{ij} = -1$ if $i = j$, and $g_{ij} = 0$ otherwise, for $i, j = 1, \ldots, n$.

We note here that the condition $y \geq 0$ is implicitly included in problem (3.20). The dual problem of (3.20) can be formulated as

$$\min q^T \hat{z};\ \text{subject to } \{\hat{z}^T \hat{G} = w^T,\ \hat{z} \geq 0\}, \tag{3.21}$$

where $\hat{z} = [t, z_1, \ldots, z_n]^T \in \mathbb{R}^{n+1}$ and $t \in \mathbb{R}$ is a real number. It is easy to see that problem (3.21) can be written as

$$\min t;\ \text{subject to } \{t - z_1 = w_1, t - z_2 + z_1 = w_2, \ldots, t - z_n + z_{n-1} = w_n\}, \tag{3.22}$$

where $t \geq 0$ and $z \geq 0$. Summing up the first k conditions of (3.22) for $k = 1, \ldots, n$, we get $kt - z_k = w_1 + \cdots + w_k$, that is,

$$t = \frac{w_1 + \cdots + w_k}{k} + \frac{z_k}{k},\ k = 1, \ldots, n. \tag{3.23}$$

So problem (3.21) is equivalent to the problem

$$\min t;\ \text{subject to } \left\{ t = \frac{w_1 + \cdots + w_k}{k} + \frac{z_k}{k}, k = 1, \ldots, n \right\}, \tag{3.24}$$

where $z_1, \ldots, z_n \geq 0$. The optimal solution $\hat{z}^* = [t^*, z_1^*, \ldots, z_n^*]$ to (3.24) can be obtained immediately by inspection. We merely have,

$$t^* = \frac{w_1 + \cdots + w_{k^*}}{k^*},$$

and

$$\frac{z_k^*}{k} = \frac{w_1 + \cdots + w_{k^*}}{k^*} - \frac{w_1 + \cdots + w_k}{k}, \quad k = 1, \ldots, n,$$

where $k^* \in \{1, \ldots, n\}$ is such that

$$\frac{w_1 + \cdots + w_{k^*}}{k^*} = \max_{k=1,\ldots,n} \frac{w_1 + \cdots + w_k}{k}.$$

Let us introduce the notations

$$y^k = (\overbrace{1/k}^{\text{1-st}}, \ldots \overbrace{1/k}^{k\text{-th}}, 0, \ldots, 0)^T \in \mathbb{R}^n,\ k = 1, \ldots, n. \tag{3.25}$$

It can easily be checked that each y^k satisfies all conditions of problem (3.20). Using the duality theorem we have

$$\max_{k=1,\dots,n} \frac{w_1 + \dots + w_k}{k} = \max_{k=1,\dots,n} w^T y^k \le \max\{w^T y | \hat{G} y \le q\} \le$$

$$\min\{q^T \hat{z} | \hat{z}^T \hat{G} = w^T, \hat{z} \ge 0\} = \max_{k=1,\dots,n} \frac{w_1 + \dots + w_k}{k},$$

which means that the optimal value

$$t^* = \frac{w_1 + \dots + w_{k^*}}{k^*}$$

can be reached with y^{k^*}.

Summary 1. *To find an optimal solution to (3.20) we should proceed as follows: select the maximal element of the set*

$$\max\left\{ w_1, \frac{w_1 + w_2}{2}, \dots, \frac{w_1 + \dots + w_n}{n} \right\},$$

and then choose the corresponding element from (3.25).

Note 2. *Let $d > 0$ be a real number. Then the constraint $x_1 + \dots + x_n \le 1$ can be replaced by $x_1 + \dots + x_n \le d$ without modifying the solution algorithm. Thus, the problem studied in this paper is nothing else but a nonlinear version of the well-known continuous knapsack problem.*

Following Carlsson, Fullér and Majlender [37] we shall show an example. Consider the following 4-dimensional constrained OWA aggregation problem

$$\max\ F(x_1, x_2, x_3, x_4);\ \text{subject to } \{x_1 + x_2 + x_3 + x_4 \le 1, x \ge 0\}. \tag{3.26}$$

Then the set of all conceivable optimal values is constructed as

$$H = \left\{ w_1, \frac{w_1 + w_2}{2}, \frac{w_1 + w_2 + w_3}{3}, \frac{w_1 + w_2 + w_3 + w_4}{4} \right\}$$

and, the correspending optimal solutions are

1. If $\max H = w_1$ then an optimal solution to problem (3.26) will be $x_1^* = 1, x_2^* = x_3^* = x_4^* = 0$ with $F(x^*) = w_1$.
2. If $\max H = (w_1 + w_2)/2$ an optimal solution to problem (3.26) will be $x_1^* = x_2^* = 1/2, x_3^* = x_4^* = 0$ with $F(x^*) = (w_1 + w_2)/2$.
3. If $\max H = (w_1 + w_2 + w_3)/3$ an optimal solution to problem (3.26) will be $x_1^* = x_2^* = x_3^* = 1/3, x_4^* = 0$ with $F(x^*) = (w_1 + w_2 + w_3)/3$.
4. If $\max H = (w_1 + w_2 + w_3 + w_4)/4$ an optimal solution to problem (3.26) will be $x_1^* = x_2^* = x_3^* = x_4^* = 1/4$ with $F(x^*) = (w_1 + w_2 + w_3 + w_4)/4$.

Note 3. *From the commutativity of OWA operators it follows that all permutations of the coordinates of an optimal solution are also optimal solutions to constrained OWA aggregation problems.*

3.4 Recent advances

In this Section we will give a short chronological survey of some later works that extend and develop the maximal entropy and the minimal variability OWA operator weights models. We will mention only those works in which the authors extended, improved or used the findings of our original papers Fullér and Majlender [84, 85].

In 2004 Liu and Chen [116] introduced the concept of parametric geometric OWA operator (PGOWA) and a parametric maximum entropy OWA operator (PMEOWA) and showed the equivalence of parametric geometric OWA operator and parametric maximum entropy OWA operator weights.

In 2005 Wang and Parkan [137] presented a minimax disparity approach, which minimizes the maximum disparity between two adjacent weights under a given level of orness. Their approach was formulated as

$$\begin{aligned} \text{minimize} \max_{i=1,2,\dots,n-1} \mid w_i - w_{i+1} \mid \\ \text{subject to orness}(w) = \sum_{i=1}^{n} \frac{n-i}{n-1} w_i = \alpha,\ 0 \leq \alpha \leq 1, \\ w_1 + \cdots + w_n = 1,\ 0 \leq w_i \leq 1,\ i = 1, \dots, n. \end{aligned}$$

Majlender [124] developed a *maximal Rényi entropy* method for generating a parametric class of OWA operators and the maximal Rényi entropy OWA weights. His approach was formulated as

$$\begin{aligned} \text{maximize } H_\beta(w) = \frac{1}{1-\beta} \log_2 \sum_{i=1}^{n} w_i^\beta \\ \text{subject to orness}(w) = \sum_{i=1}^{n} \frac{n-i}{n-1} w_i = \alpha,\ 0 \leq \alpha \leq 1, \\ w_1 + \cdots + w_n = 1,\ 0 \leq w_i \leq 1,\ i = 1, \dots, n. \end{aligned}$$

where $\beta \in \mathbb{R}$ and $H_1(w) = -\sum_{i=1}^{n} w_i \log_2 w_i$. Liu [117] extended the the properties of OWA operator to the RIM (regular increasing monotone) quantifier which is represented with a monotone function instead of the OWA weighting vector. He also introduced a class of parameterized equidifferent RIM quantifier which has minimum variance generating function. This equidifferent RIM quantifier is consistent with its orness level for any aggregated elements, which can be used to represent the decision maker's preference. Troiano and Yager [133] pointed out that OWA weighting vector and the fuzzy quantifiers are strongly related. An intuitive way for shaping a monotonic quantifier, is by means of the threshold that makes a separation between the regions of what is satisfactory and what is not. Therefore, the characteristics of a threshold can be directly related to the OWA weighting vector and to its metrics: the attitudinal character and the entropy. Usually these two metrics are supposed to be independent, although some limitations in their value come when they are considered jointly. They argued that these two metrics are strongly related by the definition of quantifier threshold, and they showed how they can be used jointly to verify and validate a quantifier and its threshold.

In 2006 Xu [141] investigated the dependent OWA operators, and developed a new argument-dependent approach to determining the OWA weights, which can relieve the influence of unfair arguments on the aggregated results. Zadrozny and Kacprzyk [158] discussed the use of the Yager's OWA operators within a flexible querying interface. Their key issue is the adaptation of an OWA operator to the specifics of a user's query. They considered some well-known approaches to the manipulation of the weights vector and proposed a new one that is simple and efficient. They discussed the tuning (selection of weights) of the OWA operators, and proposed an algorithm that is effective and efficient in the context of their FQUERY for Access package. Wang, Chang and Cheng [138] developed the query system of practical hemodialysis database for a regional hospital in Taiwan, which can help the doctors to make more accurate decision in hemodialysis. They built the fuzzy membership function of hemodialysis indices based on experts' interviews. They proposed a fuzzy OWA query method, and let the decision makers (doctors) just need to change the weights of attributes dynamical, then the proposed method can revise the weight of each attributes based on aggregation situation and the system will provide synthetic suggestions to the decision makers. Chang et al [65] proposed a dynamic fuzzy OWA model to deal with problems of group multiple criteria decision making. Their proposed model can help users to solve MCDM problems under the situation of fuzzy or incomplete information. Amin and Emrouznejad [4] introduced an extended minimax disparity model to determine the OWA operator weights as follows,

$$\begin{aligned} &\text{minimize } \delta \\ &\text{subject to } \text{orness}(w) = \sum_{i=1}^{n} \frac{n-i}{n-1} w_i = \alpha,\ 0 \leq \alpha \leq 1, \\ &w_j - w_i + \delta \geq 0,\ i = 1, \dots, n-1, j = i+1, \dots, n \\ &w_i - w_j + \delta \geq 0,\ i = 1, \dots, n-1, j = i+1, \dots, n \\ &w_1 + \dots + w_n = 1,\ 0 \leq w_i \leq 1,\ i = 1, \dots, n. \end{aligned}$$

In this model it is assumed that the deviation $|w_i - w_j|$ is always equal to δ, $i \neq j$.

In 2007 Liu [118] proved that the solutions of the minimum variance OWA operator problem under given orness level and the minimax disparity problem for OWA operator are equivalent, both of them have the same form of maximum spread equidifferent OWA operator. He also introduced the concept of maximum spread equidifferent OWA operator and proved its equivalence to the minimum variance OWA operator. Llamazares [123] proposed determining OWA operator weights regarding the class of majority rule that one should want to obtain when individuals do not grade their preferences between the alternatives. Wang, Luo and Liu [139] introduced two models determining as equally important OWA operator weights as possible for a given orness degree. Their models can be written as

$$\begin{aligned} &\text{minimize } J_1 = \sum_{i=1}^{n-1} (w_i - w_{i+1})^2 \\ &\text{subject to } \text{orness}(w) = \sum_{i=1}^{n} \frac{n-i}{n-1} w_i = \alpha,\ 0 \leq \alpha \leq 1, \\ &w_1 + \dots + w_n = 1,\ 0 \leq w_i \leq 1,\ i = 1, \dots, n. \end{aligned}$$

and

$$\begin{aligned} &\text{minimize } J_2 = \sum_{i=1}^{n-1} \left(\frac{w_i}{w_{i+1}} - \frac{w_{i+1}}{w_i} \right)^2 \\ &\text{subject to } \text{orness}(w) = \sum_{i=1}^{n} \frac{n-i}{n-1} w_i = \alpha,\ 0 \leq \alpha \leq 1, \\ &w_1 + \dots + w_n = 1,\ 0 \leq w_i \leq 1,\ i = 1, \dots, n. \end{aligned}$$

Yager [151] used stress functions to obtain OWA operator weights. With this stress function, a user can "stress" which argument values they want to give more weight in the aggregation. An important feature of this stress function is that it is only required to be nonnegative function on the unit interval. This allows a user to completely focus on the issue of where to put the stress in the aggregation without having to consider satisfaction of any other requirements.

In 2008 Liu [119] proposed a *general optimization model with strictly convex objective function* to obtain the OWA operator under given orness level,

$$\text{minimize } \sum_{i=1}^{n} F(w_i)$$
$$\text{subject to } \text{orness}(w) = \sum_{i=1}^{n} \frac{n-i}{n-1} w_i = \alpha,\ 0 \le \alpha \le 1,$$
$$w_1 + \cdots + w_n = 1,\ 0 \le w_i \le 1,\ i = 1, \dots, n.$$

and where F is a strictly convex function on $[0, 1]$, and it is at least two order differentiable. His approach includes the *maximum entropy* (for $F(x) = x \ln x$) and the *minimum variance* (for $F(x) = x^2$ problems as special cases. More generally, when $F(x) = x^\alpha, \alpha > 0$ it becomes the OWA problem of Rényi entropy [124], which includes the maximum entropy and the minimum variance OWA problem as special cases. Liu also included into this general model the solution methods and the properties of maximum entropy and minimum variance problems that were studied separately earlier. The consistent property that the aggregation value for any aggregated set monotonically increases with the given orness value is still kept, which gives more alternatives to represent the preference information in the aggregation of decision making. Then, with the conclusion that the RIM quantifier can be seen as the continuous case of OWA operator with infinite dimension, Liu [120] further suggested a general RIM quantifier determination model, and analytically solved it with the optimal control technique. Ahn [1] developed some new quantifier functions for aiding the quantifier-guided aggregation. They are related to the weighting functions that show properties such that the weights are strictly ranked and that a *value of orness is constant independently of the number of criteria considered*. These new quantifiers show the same properties that the weighting functions do and they can be used for the quantifier-guided aggregation of a multiple-criteria input. The proposed RIM and regular decreasing monotone (RDM) quantifiers produce the same orness as the weighting functions from which each quantifier function originates. the quantifier orness rapidly converges into the value of orness of the weighting functions having a constant value of orness. This result indicates that a quantifier-guided OWA aggregation will result in a similar aggregate in case the number of criteria is not too small.

In 2009 Wu et al [140] used a linear programming model for determining ordered weighted averaging operator weights with maximal Yager's entropy [146]. By analyzing the desirable properties with this measure of entropy, they proposed a novel approach to determine the weights of the OWA operator. Ahn [2] showed that a closed form of weights, obtained by the least-squared OWA (LSOWA) method, is equivalent to the minimax disparity approach solution when a condition ensuring all positive weights is added into the formulation of minimax disparity approach. Liu [121] presented some methods of OWA determination with different dimension instantiations, that is to get an OWA operator series that can be used to the different dimensional application cases of the same type. He also showed some OWA determination methods that can make the elements distributed in monotonic, symmetric or any function shape cases with different dimensions. Using Yager's stress function method [151] he managed to extend an OWA operator to another dimensional case with the same aggregation properties.

In 2010 Ahn [3] presented a general method for obtaining OWA operator weights via an extreme point approach. The extreme points are identified by the intersection of an attitudinal character constraint and a fundamental ordered weight simplex that is defined as

$$K = \{w \in \mathbb{R}^n \mid w_1 + w_2 + \cdots + w_n = 1, w_j \geq 0, \, j = 1, \dots, n\}.$$

The parameterized OWA operator weights, which are located in a convex hull of the identified extreme points, can then be specifically determined by selecting an appropriate parameter. Vergara and Xia [136] proposed a new method to find the weights of an OWA for uncertain information sources. Given a set of uncertainty data, the proposed method finds the combination of weights that reduces aggregated uncertainty for a predetermined orness level. Their approach assures best information quality and precision by reducing uncertainty. Yager [152] introduced a measure of diversity related to the problem of selecting of selecting n objects from a pool of candidates lying in q categories.

In 2011 Liu [122] summarizing the main OWA determination methods (the optimization criteria methods, the sample learning methods, the function based methods, the argument dependent methods and the preference methods) showed some relationships between the methods in the same kind and the relationships between different kinds. Hon [105] proved the extended minimax disparity OWA problem.

3.5 Benchmarking in linguistic importance weighted aggregations

In this Section we concentrate on the issue of weighted aggregations and provide a possibilistic approach to the process of importance weighted transformation when both the importances (interpreted as *benchmarks*) and the ratings are given by symmetric triangular fuzzy numbers. Following Carlsson and Fullér [18, 24] we will show that using the possibilistic approach

(i) small changes in the membership function of the importances can cause only small variations in the weighted aggregate;

(ii) the weighted aggregate of fuzzy ratings remains stable under small changes in the *nonfuzzy* importances;

(iii) the weighted aggregate of crisp ratings still remains stable under small changes in the crisp importances whenever we use a continuous implication operator for the importance weighted transformation.

In many applications of fuzzy sets such as multi-criteria decision making, pattern recognition, diagnosis and fuzzy logic control one faces the problem of weighted aggregation. Unlike Herrera and Herrera-Viedma [101] who perform direct computation on a finite and totally ordered term set, we use the membership functions to aggregate the values of the linguistic variables *rate* and *importance*. The main problem with finite term sets is that the impact of small changes in the weighting vector can be disproportionately large on the weighted aggregate (because the set of possible output values is finite, but the set of possible weight vectors is a subset of $\mathbb{R}^n$). For example, the *rounding* operator in the *convex combination of linguistic labels*, defined by Delgado et al. [67], is very sensitive to the values around 0.5 ($round(0.499) = 0$ and $round(0.501) = 1$).

Following Carlsson and Fullér [24] we consider the process of importance weighted aggregation when both the aggregates and the importances are given by an infinite term set, namely by the values of the linguistic variables "rate" and "importance". In this approach the importances are considered as benchmark levels for the performances, i.e. an alternative performs well on all criteria if the degree of satisfaction to each of the criteria is at least as big as the associated benchmark. The proposed ”stable” method in [24] ranks the alternatives by measuring the degree to which they satisfy the proposition: ”All ratings are larger than or equal to their importance”. We will also use OWA operators to measure the degree to which an alternative satisfies the proposition: ”Most ratings are larger than or equal to their importance”, where the OWA weights are derived from a well-chosen linguistic quantifier.

Recall that a fuzzy set A is called a symmetric triangular fuzzy number with center a and width $\alpha > 0$ if its membership function has the following form

$$A(t) = \begin{cases} 1 - \dfrac{|a-t|}{\alpha} & \text{if } |a-t| \leq \alpha \\ 0 & \text{otherwise} \end{cases}$$

and we use the notation $A = (a, \alpha)$. If $\alpha = 0$ then A collapses to the characteristic function of $\{a\} \subset \mathbb{R}$ and we will use the notation $A = \bar{a}$. We will use symmetric triangular fuzzy numbers to represent the values of linguistic variables *rate* and *importance* in the universe of discourse $I = [0, 1]$. The set of all symmetric triangular fuzzy numbers in the unit interval will be denoted by $\mathcal{F}(I)$. Let $A = (a, \alpha)$ and $B = (b, \beta)$. The degree of possibility that the proposition ”A is less than or equal to B” is true, denoted by $\text{Pos}[A \leq B]$, is computed by

$$\text{Pos}[A \leq B] = \begin{cases} 1 & \text{if } a \leq b \\ 1 - \dfrac{a-b}{\alpha+\beta} & \text{if } 0 < a-b < \alpha+\beta \\ 0 & \text{otherwise} \end{cases} \tag{3.27}$$

Let A be an alternative with ratings $(A_1, A_2, \dots, A_n)$, where $A_i = (a_i, \alpha_i) \in \mathcal{F}(I)$, $i = 1, \dots, n$. For example, the symmetric triangular fuzzy number $A_j = (0.8, \alpha)$ when $0 < \alpha \leq 0.2$ can represent the property ”the rating on the j-th criterion is around 0.8” and if $\alpha = 0$ then $A_j = (0.8, \alpha)$ is interpreted as ”the rating on the j-th criterion is equal to 0.8” and finally, the value of α can not be bigger than 0.2 because the domain of A_j is the unit interval.

Assume that associated with each criterion is a weight $W_i = (w_i, \gamma_i)$ indicating its importance in the aggregation procedure, $i = 1, \dots, n$. For example, the symmetric triangular fuzzy number $W_j = (0.5, \gamma) \in \mathcal{F}(I)$ when $0 < \gamma \leq 0.5$ can represent the property ”the importance of the j-th criterion is approximately 0.5” and if $\gamma = 0$ then $W_j = (0.5, \gamma)$ is interpreted as ”the importance of the j-th criterion is equal to 0.5” and finally, the value of γ can not be bigger than 0.5 because the domain of W_j is the unit interval. The general process for the inclusion of importance in the aggregation involves the transformation of the ratings under the importance. Following Carlsson and Fullér [24] we suggest the use of the transformation function $g\colon \mathcal{F}(I) \times \mathcal{F}(I) \to [0, 1]$, where, $g(W_i, A_i) = \text{Pos}[W_i \leq A_i]$, for $i = 1, \dots, n$, and then obtain the weighted aggregate,

$$\phi(A, W) = \mathbf{Agg}\langle \text{Pos}[W_1 \leq A_1], \dots, \text{Pos}[W_n \leq A_n] \rangle. \tag{3.28}$$

where **Agg** denotes an aggregation operator.

For example if we use the min function for the aggregation in (3.28), that is,

$$\phi(A, W) = \min\{\text{Pos}[W_1 \leq A_1], \dots, \text{Pos}[W_n \leq A_n]\} \tag{3.29}$$

then the equality $\phi(A, W) = 1$ holds iff $w_i \leq a_i$ for all i, i.e. when the mean value of each performance rating is at least as large as the mean value of its associated weight. In other words, if a performance rating with respect to a criterion exceeds the importance of this criterion with possibility one, then this rating does not matter in the overall rating. However, ratings which are well below the corresponding importances (in possibilistic sense) play a significant role in the overall rating. In this sense the importance can be considered as *benchmark* or *reference level* for the performance. Thus, formula (3.28) with the min operator can be seen as a measure of the degree to which an alternative satisfies the following proposition: "All ratings are larger than or equal to their importance". It should be noted that the $\min$ aggregation operator does not allow any compensation, i.e. a higher degree of satisfaction of one of the criteria can not compensate for a lower degree of satisfaction of another criterion. Averaging operators realize *trade-offs* between criteria, by allowing a positive compensation between ratings. We can use an *andlike* or an *orlike* OWA-operator to aggregate the elements of the bag

$$\langle \text{Pos}[W_1 \leq A_1], \dots, \text{Pos}[W_n \leq A_n] \rangle.$$

In this case (3.28) becomes,

$$\phi(A, W) = \mathbf{OWA}\langle \text{Pos}[W_1 \leq A_1], \dots, \text{Pos}[W_n \leq A_n] \rangle,$$

where **OWA** denotes an Ordered Weighted Averaging Operator. Formula (3.28) does not make any difference among alternatives whose performance ratings exceed the value of their importance with respect to all criteria with possibility one: the overall rating will always be equal to one. Penalizing ratings that are "larger than the associated importance, but not large enough" (that is, their intersection is not empty) we can modify formula (3.28) to measure the degree to which an alternative satisfies the following proposition: "All ratings are essentially larger than their importance". In this case the transformation function can be defined as

$$g(W_i, A_i) = \text{Nes}[W_i \leq A_i] = 1 - \text{Pos}[W_i > A_i],$$

for $i = 1, \dots, n$, and then obtain the weighted aggregate,

$$\phi(A, W) = \min\{\text{Nes}[W_1 \leq A_1], \dots, \text{Nes}[W_n \leq A_n]\}. \tag{3.30}$$

If we do allow a positive compensation between ratings then we can use OWA-operators in (3.30). That is,

$$\phi(A, W) = \mathbf{OWA}\langle \text{Nes}[W_1 \leq A_1], \dots, \text{Nes}[W_n \leq A_n] \rangle.$$

The following theorem shows that if we choose the $\min$ operator for **Agg** in (3.28) then small changes in the membership functions of the weights can cause only a small change in the weighted aggregate, i.e. the weighted aggregate depends continuously on the weights.

Theorem 3.1 (Carlsson and Fullér, [24]). *Let $A_i = (a_i, \alpha) \in \mathcal{F}(I)$, $\alpha_i > 0$, $i = 1, \dots, n$ and let $\delta > 0$ such that*

$$\delta < \alpha := \min\{\alpha_1, \dots, \alpha_n\}$$

If $W_i = (w_i, \gamma_i)$ and $W_i^\delta = (w_i^\delta, \gamma^\delta) \in \mathcal{F}(I),\ i = 1, \dots, n$, satisfy the relationship

$$\max_i D(W_i, W_i^\delta) \leq \delta \tag{3.31}$$

then the following inequality holds,

$$|\phi(A, W) - \phi(A, W^\delta)| \leq \frac{\delta}{\alpha} \tag{3.32}$$

where $\phi(A, W)$ is defined by (3.29) and

$$\phi(A, W^\delta) = \min\{Pos[W_1^\delta \leq A_1], \dots, Pos[W_n^\delta \leq A_n]\}.$$

From (3.31) and (3.32) it follows that

$$\lim_{\delta \to 0} \phi(A, W^\delta) = \phi(A, W)$$

for any A, which means that if δ is small enough then $\phi(A, W^\delta)$ can be made arbitrarily close to $\phi(A, W)$.

As an immediate consequence of (3.32) we can see that Theorem 3.1 remains valid for the case of crisp weighting vectors, i.e. when $\gamma_i = 0,\ i = 1, \dots, n$. In this case

$$\text{Pos}[\bar{w}_i \leq A_i] = \begin{cases} 1 & \text{if } w_i \leq a_i \\ A(w_i) & \text{if } 0 < w_i - a_i < \alpha_i \\ 0 & \text{otherwise} \end{cases}$$

where $\bar{w}_i$ denotes the characteristic function of $w_i \in [0,1]$; and the weighted aggregate, denoted by $\phi(A, w)$, is computed as

$$\phi(A, w) = \textbf{Agg}\{\text{Pos}[\bar{w}_1 \leq A_1], \dots, \text{Pos}[\bar{w}_n \leq A_n]\}$$

If **Agg** is the minimum operator then we get

$$\phi(A, w) = \min\{\text{Pos}[\bar{w}_1 \leq A_1], \dots, \text{Pos}[\bar{w}_n \leq A_n]\} \tag{3.33}$$

If both the ratings and the importances are given by crisp numbers (i.e. when $\gamma_i = \alpha_i = 0,\ i = 1, \dots, n$) then $\text{Pos}[\bar{w}_i \leq \bar{a}_i]$ implements the *standard strict* implication operator, i.e.,

$$\text{Pos}[\bar{w}_i \leq \bar{a}_i] = w_i \to a_i = \begin{cases} 1 & \text{if } w_i \leq a_i \\ 0 & \text{otherwise} \end{cases}$$

It is clear that whatever is the aggregation operator in

$$\phi(a, w) = \textbf{Agg}\{\text{Pos}[\bar{w}_1 \leq \bar{a}_1], \dots, \text{Pos}[\bar{w}_n \leq \bar{a}_n]\},$$

the weighted aggregate, $\phi(a, w)$, can be very sensitive to small changes in the weighting vector w. However, we can still sustain the *benchmarking character* of the weighted aggregation if we use an R-implication operator to transform the ratings under importance [15, 17]. For example, for the operator

$$\phi(a, w) = \min\{w_1 \to a_1, \dots, w_n \to a_n\}. \tag{3.34}$$

where $\to$ is an R-implication operator, the equation $\phi(a, w) = 1$, holds iff $w_i \leq a_i$ for all i, i.e. when the value of each performance rating is at least as big as the value of its associated weight. However, the crucial question here is: Does the

$$\lim_{w^\delta \to w} \phi(a, w^\delta) = \phi(a, w),\ \forall a \in I,$$

relationship still remain valid for any R-implication?

The answer is negative. ϕ will be continuous in w if and only if the implication operator is continuous. For example, if we choose the Gödel implication in then ϕ will not be continuous in w, because the Gödel implication is not continuous.

To illustrate the sensitivity of ϕ defined by the Gödel implication consider (3.34) with $n = 1$, $a_1 = w_1 = 0.6$ and $w_1^\delta = w_1 + \delta$. In this case

$$\phi(a_1, w_1) = \phi(w_1, w_1) = \phi(0.6, 0.6) = 1,$$

but

$$\phi(a, w_1^\delta) = \phi(w_1, w_1 + \delta) = \phi(0.6, 0.6 + \delta) = (0.6 + \delta) \rightarrow 0.6 = 0.6,$$

that is,

$$\lim_{\delta \to 0} \phi(a_1, w_1^\delta) = 0.6 \neq \phi(a_1, w_1) = 1.$$

But if we choose the (continuous) Łukasiewicz implication in (3.34) then ϕ will be continuous in w, and therefore, small changes in the importance can cause only small changes in the weighted aggregate. Thus, the following formula

$$\phi(a, w) = \min\{(1 - w_1 + a_1) \wedge 1, \dots, (1 - w_n + a_n) \wedge 1\}. \tag{3.35}$$

not only keeps up the benchmarking character of ϕ, but also implements a stable approach to importance weighted aggregation in the nonfuzzy case.

If we do allow a positive compensation between ratings then we can use an OWA-operator for aggregation in (3.35). That is,

$$\phi(a, w) = \mathbf{OWA}\ \langle (1 - w_1 + a_1) \wedge 1, \dots, (1 - w_n + a_n) \wedge 1 \rangle. \tag{3.36}$$

Taking into consideration that OWA-operators are usually continuous, equation (3.36) also implements a stable approach to importance weighted aggregation in the nonfuzzy case.

We illustrate our approach by an example. Consider the aggregation problem,

$$A = \begin{pmatrix} (0.7, 0.2) \\ (0.5, 0.3) \\ (0.8, 0.2) \\ (0.9, 0.1) \end{pmatrix} \quad \text{and} \quad W = \begin{pmatrix} (0.8, 0.2) \\ (0.7, 0.3) \\ (0.9, 0.1) \\ (0.6, 0.2) \end{pmatrix}.$$

Using formula (3.29) for the weighted aggregate we find

$$\phi(A, W) = \min\{3/4, 2/3, 2/3, 1\} = 2/3.$$

The reason for the relatively high performance of this object is that, even though it performed low on the second criterion which has a high importance, the second importance has a relatively large tolerance level, 0.3.

In this Section we have introduced a possibilistic approach to the process of importance weighted transformation when both the importances and the aggregates are given by triangular fuzzy numbers. In this approach the importances have been considered as benchmark levels for the performances, i.e. an alternative performs well on all criteria if the degree of satisfaction to each of the criteria is at least as big as the associated benchmark. We have suggested the use of measure of necessity to be able to distinguish alternatives with overall rating one (whose performance ratings exceed the value of their importance with respect to all criteria with possibility one). We have shown that using the possibilistic approach (i) small changes in the membership function of the importances can cause only small variations in the weighted aggregate; (ii) the weighted aggregate of fuzzy ratings remains stable under small changes in the *nonfuzzy* importances; (iii) the weighted aggregate of crisp ratings still remains stable under small changes in the crisp importances whenever we use a continuous implication operator for the importance weighted transformation. These results have further implications in several classes of multiple criteria decision making problems, in which the aggregation procedures are rough enough to make the finely tuned formal selection of an optimal alternative meaningless.

3.6 Optimization with linguistic variables

In 2000 and 2001 Carlsson and Fullér [25, 30] introduced a novel statement of fuzzy mathematical programming problems and provided a method for finding a fair solution to these problems. Suppose we are given a mathematical programming problem in which the functional relationship between the decision variables and the objective function is not completely known. Our knowledge-base consists of a block of fuzzy if-then rules, where the antecedent part of the rules contains some linguistic values of the decision variables, and the consequence part consists of a linguistic value of the objective function. We suggest the use of Tsukamoto's fuzzy reasoning method to determine the crisp functional relationship between the objective function and the decision variables, and solve the resulting (usually nonlinear) programming problem to find a fair optimal solution to the original fuzzy problem. When Bellman and Zadeh [6], and a few years later Zimmermann [159], introduced fuzzy sets into optimization problems, they cleared the way for a new family of methods to deal with problems which had been inaccessible to and unsolvable with standard mathematical programming techniques. Fuzzy optimization problems can be stated and solved in many different ways. Usually the authors consider optimization problems of the form

$$\max/\min\ f(x);\ \text{subject to } x \in X,$$

where f or/and X are defined by fuzzy terms. Then they are searching for a crisp x^* which (in certain) sense maximizes f under the (fuzzy) constraints X. For example, fuzzy linear programming (FLP) problems are stated as [130]

$$\begin{array}{ll} \max/\min & f(x) := \tilde{c}_1 x_1 + \cdots + \tilde{c}_n x_n \\ \text{subject to} & \tilde{a}_{i1} x_1 + \cdots + \tilde{a}_{in} x_n \lesssim \tilde{b}_i,\ i = 1, \dots, m, \end{array} \tag{3.37}$$

where $x \in \mathbb{R}^n$ is the vector of crisp decision variables, $\tilde{a}_{ij}$, $\tilde{b}_i$ and $\tilde{c}_j$ are fuzzy quantities, the operations addition and multiplication by a real number of fuzzy quantities are defined by Zadeh's extension principle, the inequality relation, $\lesssim$, is given by a certain fuzzy relation, f is to be maximized in the sense of a given crisp inequality relation between fuzzy quantities, and the (implicite) X is a fuzzy set describing the concept "x satisfies all the constraints".

Unlike in (3.37) the fuzzy value of the objective function $f(x)$ may not be known for any $x \in \mathbb{R}^n$. In many cases we are able to describe the causal link between x and $f(x)$ linguistically using fuzzy if-then rules. Following Carlsson and Fullér [30] we consider a new statement of constrained fuzzy optimization problems, namely

$$\max/\min\ f(x); \text{subject to } \{\Re(x) \mid x \in X\}, \tag{3.38}$$

where $x_1, \dots, x_n$ are linguistic variables, $X \subset \mathbb{R}^n$ is a (crisp or fuzzy) set of constrains on the domains of $x_1, \dots, x_n$, and $\Re(x) = \{\Re_1(x), \dots, \Re_m(x)\}$ is a fuzzy rule base, and

$$\Re_i(x) : \text{if } x_1 \text{ is } A_{i1} \text{ and} \dots \text{and } x_n \text{ is } A_{in} \text{ then } f(x) \text{ is } C_i,$$

constitutes the only knowledge available about the (linguistic) values of $f(x)$, and A_{ij} and C_i are fuzzy numbers.

Generalizing the fuzzy reasoning approach introduced by Carlsson and Fullér [14] we shall determine the crisp value of f at $y \in X$ by Tsukamoto's fuzzy reasoning method, and obtain an optimal solution to (3.38) by solving the resulting (usually nonlinear) optimization problem max/min $f(y)$, subject to $y \in X$.

The use of fuzzy sets provides a basis for a systematic way for the manipulation of vague and imprecise concepts. In particular, we can employ fuzzy sets to represent linguistic variables. A linguistic variable can be regarded either as a variable whose value is a fuzzy number or as a variable whose values are defined in linguistic terms. Fuzzy points are used to represent crisp values of linguistic variables. If x is a linguistic variable in the universe of discourse X and $y \in X$ then we simple write "$x = y$" or "x is $\bar{y}$" to indicate that y is a crisp value of the linguistic variable x.

Recall the three basic t-norms: (i) minimum: $T(a, b) = \min\{a, b\}$, (ii) Łukasiewicz: $T(a, b) = \max\{a + b - 1, 0\}$, and (iii) product (or probabilistic): $T(a, b) = ab$. We briefly describe Tsukamoto's fuzzy reasoning method [134]. Consider the following fuzzy inference system,

$$\begin{array}{llll}
\Re_1: & \text{if} & x_1 \text{ is } A_{11} \text{ and} \dots \text{and } x_n \text{ is } A_{1n} \text{ then} & z \text{ is } C_1 \\
\dots & & & \\
\Re_m: & \text{if} & x_1 \text{ is } A_{m1} \text{ and} \dots \text{and } x_n \text{ is } A_{mn} \text{ then} & z \text{ is } C_m \\
\text{Input:} & & x_1 \text{ is } \bar{y}_1 \quad \text{and} \dots \text{and } x_n \text{ is } \bar{y}_n & \\
\hline
\text{Output:} & & & z_0
\end{array}$$

where $A_{ij} \in \mathcal{F}(U_j)$ is a value of linguistic variable x_j defined in the universe of discourse $U_j \subset \mathbb{R}$, and $C_i \in \mathcal{F}(W)$ is a value of linguistic variable z defined in the universe $W \subset \mathbb{R}$ for $i = 1, \dots, m$ and $j = 1, \dots, n$. We also suppose that W is bounded and each C_i has strictly monotone (increasing or decreasing) membership function on W. The procedure for obtaining the crisp output, z_0, from the crisp input vector $y = \{y_1, \dots, y_n\}$ and fuzzy rule-base $\Re = \{\Re_1, \dots, \Re_m\}$ consists of the following three steps:

- We find the firing level of the i-th rule as
$$\alpha_i = T(A_{i1}(y_1), \dots, A_{in}(y_n)),\ i = 1, \dots, m, \tag{3.39}$$
where T usually is the minimum or the product t-norm.

- We determine the (crisp) output of the i-th rule, denoted by z_i, from the equation $\alpha_i = C_i(z_i)$, that is,
$$z_i = C_i^{-1}(\alpha_i),\ i = 1, \dots, m,$$
where the inverse of C_i is well-defined because of its strict monotonicity.

- The overall system output is defined as the weighted average of the individual outputs, where associated weights are the firing levels. That is,
$$z_0 = \frac{\alpha_1 z_1 + \cdots + \alpha_m z_m}{\alpha_1 + \cdots + \alpha_m} = \frac{\alpha_1 C_1^{-1}(\alpha_1) + \cdots + \alpha_m C_m^{-1}(\alpha_m)}{\alpha_1 + \cdots + \alpha_m}$$
i.e. z_0 is computed by the discrete Center-of-Gravity method.

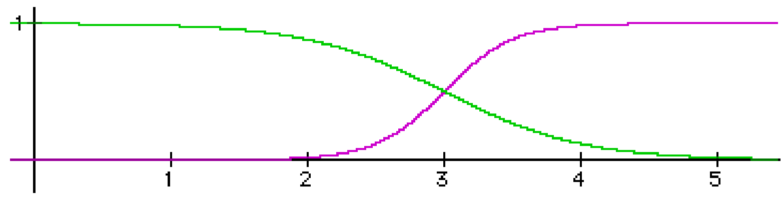

Figure 3.1: Sigmoid membership functions for "z is small" and "z is big".

If $W = \mathbb{R}$ then all linguistic values of $x_1, \dots, x_n$ also should have strictly monotone membership functions on $\mathbb{R}$ (that is, $0 < A_{ij}(x) < 1$ for all $x \in \mathbb{R}$), because $C_i^{-1}(1)$ and $C_i^{-1}(0)$ do not exist. In this case A_{ij} and C_i usually have sigmoid membership functions of the form

$$\mathrm{big}(t) = \frac{1}{1+\exp(-b(t-c))}, \quad \mathrm{small}(t) = \frac{1}{1+\exp(b'(t-c'))}$$

where $b, b' > 0$ and $c, c' > 0$.

Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a function and let $X \subset \mathbb{R}^n$. A constrained optimization problem can be stated as

$$\min f(x); \text{ subject to } x \in X.$$

In many practical cases the function f is not known exactly. In this Section we consider the following fuzzy optimization problem

$$\min f(x); \text{ subject to } \{\Re_1(x), \dots, \Re_m(x) \mid x \in X\}, \tag{3.40}$$

where $x_1, \dots, x_n$ are linguistic variables, $X \subset \mathbb{R}^n$ is a (crisp or fuzzy) set of constrains on the domain of $x_1, \dots, x_n$, and the only available knowledge about the values of f is given as a block fuzzy if-then rules of the form

$$\Re_i(x) : \text{if } x_1 \text{ is } A_{i1} \text{ and } \dots \text{and } x_n \text{ is } A_{in} \text{ then } f(x) \text{ is } C_i,$$

here A_{ij} are fuzzy numbers (with continuous membership function) representing the linguistic values of x_i defined in the universe of discourse $U_j \subset \mathbb{R}$; and C_i, $i = 1, \dots, m$, are linguistic values (with strictly monotone and continuous membership functions) of the objective function f defined in the universe $W \subset \mathbb{R}$. To find a fair solution to the fuzzy optimization problem (3.40) we first determine the crisp value of the objective function f at $y \in X$ from the fuzzy rule-base $\Re$ using Tsukamoto's fuzzy reasoning method as

$$f(y) := \frac{\alpha_1 C_1^{-1}(\alpha_1) + \cdots + \alpha_m C_m^{-1}(\alpha_m)}{\alpha_1 + \cdots + \alpha_m}$$

where the firing levels,

$$\alpha_i = T(A_{i1}(y_1), \dots, A_{in}(y_n)),$$

for $i = 1, \dots, m$, are computed according to (3.39). To determine the firing level of the rules, we suggest the use of the product t-norm (to have a smooth output function).

In this manner our constrained optimization problem (3.40) turns into the following crisp (usually nonlinear) mathematical programming problem

$$\min f(y); \text{ subject to } y \in X.$$

The same principle is applied to constrained maximization problems

$$\max f(x);\ \text{subject to } \{\Re_1(x), \dots \Re_m(x) \mid x \in X\}. \tag{3.41}$$

If X is a fuzzy set in $U_1 \times \cdots \times U_n \subset \mathbb{R}^n$ with membership function μ_X (e.g. given by soft constraints as in [159]) and $W = [0,1]$ then following Bellman and Zadeh [6] we define the fuzzy solution to problem (3.41) as

$$D(y) = \min\{\mu_X(y), f(y)\},$$

for $y \in U_1 \times \cdots \times U_n$, and an optimal (or maximizing) solution, y^*, is determined from the relationship

$$D(y^*) = \sup_{y \in U_1 \times \cdots \times U_n} D(y). \tag{3.42}$$

Example 3.2. *Consider the optimization problem*

$$\min f(x);\ \{x_1 + x_2 = 1/2,\ 0 \le x_1, x_2 \le 1\}, \tag{3.43}$$

and $f(x)$ is given linguistically as

$$\begin{aligned} \Re_1 :\quad & \textit{if } x_1 \textit{ is small and } x_2 \textit{ is small then } f(x) \textit{ is small},\\ \Re_2 :\quad & \textit{if } x_1 \textit{ is small and } x_2 \textit{ is big then } f(x) \textit{ is big}, \end{aligned}$$

and the universe of discourse for the linguistic values of f is also the unit interval $[0,1]$.

We will compute the firing levels of the rules by the product t-norm. Let the membership functions in the rule-base $\Re$ be defined by (2.12) and let $[y_1, y_2] \in [0,1] \times [0,1]$ be an input vector to the fuzzy system. Then the firing levels of the rules are

$$\begin{aligned} \alpha_1 &= (1 - y_1)(1 - y_2),\\ \alpha_2 &= (1 - y_1)y_2, \end{aligned}$$

It is clear that if $y_1 = 1$ then no rule applies because $\alpha_1 = \alpha_2 = 0$. So we can exclude the value $y_1 = 1$ from the set of feasible solutions. The individual rule outputs are

$$z_1 = 1 - (1 - y_1)(1 - y_2), \quad z_2 = (1 - y_1)y_2,$$

and, therefore, the overall system output, interpreted as the crisp value of f at y, is

$$f(y) := \frac{(1 - y_1)(1 - y_2)(1 - (1 - y_1)(1 - y_2)) + (1 - y_1)y_2(1 - y_1)y_2}{(1 - y_1)(1 - y_2) + (1 - y_1)y_2} =$$

$$y_1 + y_2 - 2y_1y_2$$

Thus our original fuzzy problem

$$\min f(x);\ \textit{subject to } \{\Re_1(x), \Re_2(x) \mid x \in X\},$$

turns into the following crisp nonlinear mathematical programming problem

$$(y_1 + y_2 - 2y_1y_2) \to \min$$

$$y_1 + y_2 = 1/2,$$

$$0 \le y_1 < 1,\ 0 \le y_2 \le 1.$$

which has the optimal solution

$$y_1^* = y_2^* = 1/4$$

and its optimal value is

$$f(y^*) = 3/8.$$

It is clear that if there were no other constraints on the crisp values of x_1 and x_2 then the optimal solution to (3.43) would be $y_1^* = y_2^* = 0$ with $f(y^*) = 0$.

This example clearly shows that we can not just choose the rule with the smallest consequence part (the first first rule) and fire it with the maximal firing level ($\alpha_1 = 1$) at $y^* \in [0, 1]$, and take $y^* = (0, 0)$ as an optimal solution to (3.40). The rules represent our knowledge-base for the fuzzy optimization problem. The fuzzy partitions for linguistic variables will not ususally satisfy ε-completeness, normality and convexity. In many cases we have only a few (and contradictory) rules. Therefore, we can not make any preselection procedure to remove the rules which *do not play any role* in the optimization problem. All rules should be considered when we derive the crisp values of the objective function. We have chosen Tsukamoto's fuzzy reasoning scheme, because the individual rule outputs are crisp numbers, and therefore, the functional relationship between the input vector y and the system output $f(y)$ can be relatively easily identified (the only thing we have to do is to perform inversion operations).

Consider the problem

$$\max_X f(x) \tag{3.44}$$

where X is a fuzzy subset of the unit interval with membership function

$$\mu_X(y) = \frac{1}{1+y},\ y \in [0, 1],$$

and the fuzzy rules are

$$\begin{aligned} \Re_1 :&\quad \text{if } x \text{ is } small \text{ then } f(x) \text{ is } small, \\ \Re_2 :&\quad \text{if } x \text{ is } big \text{ then } f(x) \text{ is } big, \end{aligned}$$

Let $y \in [0, 1]$ be an input to the fuzzy system $\{\Re_1, \Re_2\}$. Then the firing leveles of the rules are

$$\begin{aligned} \alpha_1 &= 1 - y, \\ \alpha_2 &= y. \end{aligned}$$

the individual rule outputs are computed by

$$\begin{aligned} z_1 &= (1 - y)y, \\ z_2 &= y^2, \end{aligned}$$

and, therefore, the overall system output is

$$f(y) = (1 - y)y + y^2 = y.$$

Then according to (3.42) our original fuzzy problem (3.44) turns into the following crisp biobjective mathematical programming problem

$$\max \min\{y, \frac{1}{1+y}\};\ \text{subject to } y \in [0, 1],$$

which has the optimal solution

$$y^* = \frac{\sqrt{5}-1}{2}$$

and its optimal value is $f(y^*) = y^*$.

Consider the following one-dimensional problem

$$\max f(x); \text{ subject to } \{\Re_1(x), \dots, \Re_{K+1}(x) \mid x \in X\}, \tag{3.45}$$

where $U = W = [0, 1]$,

$$\Re_i(x): \quad \text{if } x \text{ is } A_i \text{ then } f(x) \text{ is } C_i.$$

and A_i is defined by equations (2.9, 2.10, 2.11), the linguistic values of f are selected from (2.13, 2.14), $i = 1, \dots, K+1$. It is clear that exactly two rules fire with nonzero degree for any input $y \in [0, 1]$. Namely, if

$$y \in I_k := \left[\frac{k-1}{K}, \frac{k}{K}\right],$$

then $\Re_k$ and $\Re_{k+1}$ are applicable, and therefore we get

$$f(y) = (k - Ky)C_k^{-1}(k - Ky) + (Ky - k + 1)C_{k+1}^{-1}(Ky - k + 1)$$

for any $k \in \{1, \dots, K\}$. In this way the fuzzy maximization problem (3.45) turns into K independent maximization problem

$$\max_{k=1,\dots,K} \{\max_{X \cap I_k} (k - Ky)C_k^{-1}(k - Ky) + (Ky - k + 1)C_{k+1}^{-1}(Ky - k + 1)\}$$

If $x \in \mathbb{R}^n$, with $n \geq 2$ then a similar reasoning holds, with the difference that we use the same fuzzy partition for all the linguistic variables, $x_1, \dots, x_n$, and the number of applicable rules grows to 2^n. It should be noted that we can refine the fuzzy rule-base by introducing new lingusitic variables modeling the linguistic dependencies between the variables and the objectives [15].

The principles presented above can be extended to multiple objective optimization problems under fuzzy if-then rules. Namely, following Carlsson and Fullér [25], we consider the following statement of multiple objective optimization problem

$$\max/\min \{f_1(x), \dots, f_K(x)\}; \text{subject to } \{\Re_1(x), \dots, \Re_m(x) \mid x \in X\}, \tag{3.46}$$

where $x_1, \dots, x_n$ are linguistic variables, and

$$\Re_i(x): \text{if } x_1 \text{ is } A_{i1} \text{ and} \dots \text{and } x_n \text{ is } A_{in} \text{ then} f_1(x) \text{ is } C_{i1} \text{ and} \dots \text{and } f_K(x) \text{ is } C_{iK},$$

constitutes the only knowledge available about the values of $f_1, \dots, f_K$, and A_{ij} and C_{ik} are fuzzy numbers. To find a fair solution to the fuzzy optimization problem (3.46) with continuous A_{ij} and with strictly monotone and continuous C_{ik}, representing the linguistic values of f_k, we first determine the crisp value of the k-th objective function f_k at $y \in \mathbb{R}^n$ from the fuzzy rule-base $\Re$ using Tsukamoto's fuzzy reasoning method as

$$f_k(y) := \frac{\alpha_1 C_{1k}^{-1}(\alpha_1) + \dots + \alpha_m C_{mk}^{-1}(\alpha_m)}{\alpha_1 + \dots + \alpha_m}$$

where

$$\alpha_i = T(A_{i1}(y_1), \dots, A_{in}(y_n))$$

denotes the firing level of the i-th rule, $\Re_i$ and T is a t-norm. To determine the firing level of the rules, we suggest the use of the product t-norm (to have a smooth output function). In this manner the constrained optimization problem (3.46) turns into the crisp (usually nonlinear) multiobjective mathematical programming problem

$$\text{max/min}\ \{f_1(y), \dots, f_K(y)\};\ \text{subject to}\ y \in X. \tag{3.47}$$

Example 3.3. *Consider the optimization problem*

$$\max\ \{f_1(x), f_2(x)\};\ \{x_1 + x_2 = 3/4,\ 0 \le x_1, x_2 \le 1\}, \tag{3.48}$$

where $f_1(x)$ and $f_2(x)$ are given linguistically by

$$\begin{aligned} \Re_1(x):&\quad \textit{if } x_1 \textit{ is } small \textit{ and } x_2 \textit{ is } small \textit{ then } f_1(x) \textit{ is } small \textit{ and } f_2(x) \textit{ is } big, \\ \Re_2(x):&\quad \textit{if } x_1 \textit{ is } small \textit{ and } x_2 \textit{ is } big \textit{ then } f_1(x) \textit{ is } big \textit{ and } f_2(x) \textit{ is } small, \end{aligned}$$

and the universe of discourse for the linguistic values of f_1 and f_2 is also the unit interval $[0, 1]$. We will compute the firing levels of the rules by the product t-norm. Let the membership functions in the rule-base $\Re = \{\Re_1, \Re_2\}$ be defined by $\text{small}(t) = 1 - t$ *and* $\text{big}(t) = t$*. Let $0 \le y_1, y_2 \le 1$ be an input to the fuzzy system. Then the firing leveles of the rules are*

$$\alpha_1 = (1 - y_1)(1 - y_2), \quad \alpha_2 = (1 - y_1)y_2.$$

It is clear that if $y_1 = 1$ then no rule applies because $\alpha_1 = \alpha_2 = 0$. So we can exclude the value $y_1 = 1$ from the set of feasible solutions. The individual rule outputs are

$$\begin{aligned} z_{11} &= 1 - (1 - y_1)(1 - y_2), \\ z_{21} &= (1 - y_1)y_2, \\ z_{12} &= (1 - y_1)(1 - y_2), \\ z_{22} &= 1 - (1 - y_1)y_2, \end{aligned}$$

and, therefore, the overall system outputs are

$$f_1(y) = \frac{(1 - y_1)(1 - y_2)(1 - (1 - y_1)(1 - y_2)) + (1 - y_1)y_2(1 - y_1)y_2}{(1 - y_1)(1 - y_2) + (1 - y_1)y_2} = y_1 + y_2 - 2y_1y_2,$$

and

$$f_2(y) = \frac{(1 - y_1)(1 - y_2)(1 - y_1)(1 - y_2) + (1 - y_1)y_2(1 - (1 - y_1)y_2)}{(1 - y_1)(1 - y_2) + (1 - y_1)y_2} = 1 - (y_1 + y_2 - 2y_1y_2).$$

Modeling the anding of the objective functions by the minimum t-norm our original fuzzy problem (3.48) turns into the following crisp nonlinear mathematical programming problem

$$\max \min\{y_1 + y_2 - 2y_1y_2, 1 - (y_1 + y_2 - 2y_1y_2)\}$$

$$\text{subject to}\ \{y_1 + y_2 = 3/4, 0 \le y_1 < 1,\ 0 \le y_2 \le 1\}.$$

which has the following optimal solutions

$$y^* = \begin{pmatrix} y_1^* \\ y_2^* \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/4 \end{pmatrix},$$

and

$$\begin{pmatrix} 1/4 \\ 1/2 \end{pmatrix},$$

from symmetry, and its optimal value is

$$(f_1(y^*), f_2(y^*)) = (1/2, 1/2).$$

We can introduce trade-offs among the objectives function by using an OWA-operator in (3.47). However, as Yager has pointed out in [147], constrained OWA-aggregations are not easy to solve, because the usually lead to a mixed integer mathematical programming problem of very big dimension.

Typically, in complex, real-life problems, there are some unidentified factors which effect the values of the objective functions. We do not know them or can not control them; i.e. they have an impact we can not control. The only thing we can observe is the values of the objective functions at certain points. And from this information and from our knowledge about the problem we may be able to formulate the impacts of unknown factors (through the observed values of the objectives). In 1994 Carlsson and Fullér [13] stated the multiobjective decision problem with independent objectives and then adjusted their model to reality by introducing interdependences among the objectives. Interdependences among the objectives exist whenever the computed value of an objective function is not equal to its observed value. We claimed that the real values of an objective function can be identified by the help of feed-backs from the values of other objective functions, and show the effect of various kinds (linear, nonlinear and compound) of additive feed-backs on the compromise solution. 35 independent citations show that the scientific community has accepted this statement of multiobjective decision problems.

Even if the objective functions of a multiobjective decision problem are exactly known, we can still measure the *complexity* of the problem, which is derived from the *grades of conflict* between the objectives. In 1995 Carlsson and Fullér [15] introduced the measure the *complexity* of multi objective decision problems and to find a good compromise solution to these problems they employd the following heuristic: increase the value of those objectives that support the majority of the objectives, because the gains on their (concave) utility functions surpass the losses on the (convex) utility functions of those objectives that are in conflict with the majority of the objectives. 59 independent citations show that the scientific community has accepted this heuristic.

Chapter 4

Stability in Fuzzy Systems

Possibilisitic linear equality systems are linear equality systems with fuzzy coefficients, defined by the Zadeh's extension principle. In 1988 Kovács [108] showed that the fuzzy solution to possibilisitic linear equality systems with symmetric triangular fuzzy numbers is stable with respect to small changes of centres of fuzzy parameters. First we generalize Kovács's results to possibilisitic linear equality systems with Lipschitzian fuzzy numbers (Fullér, [74]) and to fuzzy linear programs (Fullér, [73]). Then we consider linear (Fedrizzi and Fullér, [72]) and quadratic (Canestrelli, Giove and Fullér, [12]) possibilistic programs and show that the possibility distribution of their objective function remains stable under small changes in the membership function of the fuzzy number coefficients. Furthermore, we present similar results for multiobjective possibilistic linear programs (Fullér and Fedrizzi, [82]).

In 1973 Zadeh [154] introduced the compositional rule of inference and six years later [156] the theory of approximate reasoning. This theory provides a powerful framework for reasoning in the face of imprecise and uncertain information. Central to this theory is the representation of propositions as statements assigning fuzzy sets as values to variables. In 1993 Fullér and Zimmermann [81] showed two very important features of the compositional rule of inference under triangular norms. Namely, they proved that (i) if the t-norm defining the composition and the membership function of the observation are continuous, then the conclusion depends continuously on the observation; (ii) if the t-norm and the membership function of the relation are continuous, then the observation has a continuous membership function. The stability property of the conclusion under small changes of the membership function of the observation and rules guarantees that small rounding errors of digital computation and small errors of measurement of the input data can cause only a small deviation in the conclusion, i.e. every successive approximation method can be applied to the computation of the linguistic approximation of the exact conclusion in control systems. In 1992 Fullér and Werners [80] extended the stability theorems of [81] to the compositional rule of inference with several relations. These stability properties in fuzzy inference systems were used by a research team - headed by Professor Hans-Jürgen Zimmermann - when developing a fuzzy control system for a "fuzzy controlled model car" [5] during my DAAD Scholarship at RWTH Aachen between 1990 and 1992.

4.1 Stability in possibilistic linear equality systems

Modelling real world problems mathematically we often have to find a solution to a linear equality system

$$a_{i1}x_1 + \cdots + a_{in}x_n = b_i,\ i = 1, \ldots, m, \tag{4.1}$$

or shortly, $Ax = b$, where a_{ij}, b_i and x_j, $j = 1, \dots, n$ are real numbers. It is known that system (4.1) generally belongs to the class of ill-posed problems, so a small perturbation of the parameters a_{ij} and b_i may cause a large deviation in the solution. A possibilistic linear equality system is

$$\tilde{a}_{i1}x_1 + \cdots + \tilde{a}_{in}x_n \;=\; \tilde{b}_i, \; i = 1, \dots, m, \tag{4.2}$$

or shortly, $\tilde{A}x = \tilde{b}$, where $\tilde{a}_{ij}$, $\tilde{b}_i \in \mathcal{F}(\mathbb{R})$ are fuzzy quantities, $x \in \mathbb{R}^n$, the operations addition and multiplication by a real number of fuzzy quantities are defined by Zadeh's extension principle and the equation is understood in possibilistic sense. Recall the truth value of the assertion "$\tilde{a}$ is equal to $\tilde{b}$", written as $\tilde{a} = \tilde{b}$, denoted by $\mathrm{Pos}(\tilde{a} = \tilde{b})$, is defined as

$$\mathrm{Pos}(\tilde{a} = \tilde{b}) = \sup_t \{\tilde{a}(t) \wedge \tilde{b}(t)\} = (\tilde{a} - \tilde{b})(0). \tag{4.3}$$

We denote by $\mu_i(x)$ the degree of satisfaction of the i-th equation in (4.2) at the point $x \in \mathbb{R}^n$, i.e.

$$\mu_i(x) = \mathrm{Pos}(\tilde{a}_{i1}x_1 + \cdots + \tilde{a}_{in}x_n = \tilde{b}_i).$$

Following Bellman and Zadeh [6] the fuzzy solution (or the fuzzy set of feasible solutions) of system (4.2) can be viewed as the intersection of the μ_i's such that

$$\mu(x) = \min\{\mu_1(x), \dots, \mu_m(x)\}. \tag{4.4}$$

A measure of consistency for the possibilistic equality system (4.2) is defined as

$$\mu^* = \sup\{\mu(x) \mid x \in \mathbb{R}^n\}. \tag{4.5}$$

Let X^* be the set of points $x \in \mathbb{R}^n$ for which $\mu(x)$ attains its maximum, if it exists. That is

$$X^* = \{x^* \in \mathbb{R}^n \mid \mu(x^*) = \mu^*\}$$

If $X^* \neq \emptyset$ and $x^* \in X^*$, then x^* is called a maximizing (or best) solution of (4.2).

If $\tilde{a}$ and $\tilde{b}$ are fuzzy numbers with $[a]^\alpha = [a_1(\alpha), a_2(\alpha)]$ and $[b]^\alpha = [b_1(\alpha), b_2(\alpha)]$ then their Hausdorff distance is defined as

$$D(\tilde{a}, \tilde{b}) = \sup_{\alpha \in [0,1]} \max\{|a_1(\alpha) - b_1(\alpha)|, |a_2(\alpha) - b_2(\alpha)|\}.$$

i.e. $D(\tilde{a}, \tilde{b})$ is the maximal distance between the α-level sets of $\tilde{a}$ and $\tilde{b}$.

Let $L > 0$ be a real number. By $\mathcal{F}(L)$ we denote the set of all fuzzy numbers $\tilde{a} \in \mathcal{F}$ with membership function satisfying the Lipschitz condition with constant L , i.e.

$$|\tilde{a}(t) - \tilde{a}(t')| \leq L|t - t'|, \; \forall t, t' \in \mathbb{R}.$$

In many important cases the fuzzy parameters $\tilde{a}_{ij}$, $\tilde{b}_i$ of the system (4.2) are not known exactly and we have to work with their approximations $\tilde{a}^\delta_{ij}$, $\tilde{b}^\delta_i$ such that

$$\max_{i,j} D(\tilde{a}_{ij}, \tilde{a}^\delta_{ij}) \leq \delta, \quad \max_i D(\tilde{b}_i, \tilde{b}^\delta_i) \leq \delta, \tag{4.6}$$

where $\delta \geq 0$ is a real number. Then we get the following system with perturbed fuzzy parameters

$$\tilde{a}^\delta_{i1}x_1 + \cdots + \tilde{a}^\delta_{in}x_n = \tilde{b}^\delta_i, \; i = 1, \dots, m \tag{4.7}$$

or shortly, $\tilde{A}^\delta x = \tilde{b}^\delta$. In a similar manner we define the solution

$$\mu^\delta(x) = \min\{\mu_1^\delta(x), \dots \mu_m^\delta(x)\},$$

and the measure of consistency

$$\mu^*(\delta) = \sup\{\mu^\delta(x) \mid x \in \mathbb{R}^n\},$$

of perturbed system (4.7), where

$$\mu_i^\delta(x) = \mathrm{Pos}(\tilde{a}_{i1}^\delta x_1 + \cdots + \tilde{a}_{in}^\delta x_n = \tilde{b}_i^\delta)$$

denotes the degree of satisfaction of the i-th equation at $x \in \mathbb{R}^n$. Let $X^*(\delta)$ denote the set of maximizing solutions of the perturbed system (4.7).

The following lemmas build up connections between C_∞ and D distances of fuzzy numbers.

Lemma 4.1.1 (Kaleva, [107])**.** *Let $\tilde{a}$, $\tilde{b}$, $\tilde{c}$ and $\tilde{d}$ be fuzzy numbers. Then*

$$D(\tilde{a} + \tilde{c}, \tilde{b} + \tilde{d}) \leq D(\tilde{a}, \tilde{b}) + D(\tilde{c}, \tilde{d}), \quad D(\tilde{a} - \tilde{c}, \tilde{b} - \tilde{d}) \leq D(\tilde{a}, \tilde{b}) + D(\tilde{c}, \tilde{d})$$

and $D(\lambda\tilde{a}, \lambda\tilde{b}) = |\lambda| D(\tilde{a}, \tilde{b})$ for any $\lambda \in \mathbb{R}$.

Let $\tilde{a} \in \mathcal{F}$ be a fuzzy number. Then for any $\theta \geq 0$ we define $\omega(\tilde{a}, \theta)$, the modulus of continuity of $\tilde{a}$ as

$$\omega(\tilde{a}, \theta) = \max_{|u-v| \leq \theta} |\tilde{a}(u) - \tilde{a}(v)|.$$

The following statements hold [100]:

$$\text{If } 0 \leq \theta \leq \theta' \text{ then } \omega(\tilde{a}, \theta) \leq \omega(\tilde{a}, \theta') \tag{4.8}$$

$$\text{If } \alpha > 0, \beta > 0, \text{ then } \omega(\tilde{a}, \alpha + \beta) \leq \omega(\tilde{a}, \alpha) + \omega(\tilde{a}, \beta). \tag{4.9}$$

$$\lim_{\theta \to 0} \omega(\tilde{a}, \theta) = 0 \tag{4.10}$$

Recall, if $\tilde{a}$ and $\tilde{b}$ are fuzzy numbers with $[\tilde{a}]^\alpha = [a_1(\alpha), a_2(\alpha)]$ and $[\tilde{b}]^\alpha = [b_1(\alpha), b_2(\alpha)]$ then

$$[\tilde{a} + \tilde{b}]^\alpha = [a_1(\alpha) + b_1(\alpha), a_2(\alpha) + b_2(\alpha)]. \tag{4.11}$$

Lemma 4.1.2 (Fullér, [78])**.** *Let $\lambda \neq 0, \mu \neq 0$ be real numbers and let $\tilde{a}$ and $\tilde{b}$ be fuzzy numbers. Then*

$$\omega(\lambda\tilde{a}, \theta) = \omega\left(\tilde{a}, \frac{\theta}{|\lambda|}\right), \tag{4.12}$$

$$\omega(\lambda\tilde{a} + \lambda\tilde{b}, \theta) \leq \omega\left(\frac{\theta}{|\lambda| + |\mu|}\right), \tag{4.13}$$

where

$$\omega(\theta) := \max\{\omega(\tilde{a}, \theta), \omega(\tilde{b}, \theta)\},$$

for $\theta \geq 0$.

Lemma 4.1.3 (Fullér, [78]). *Let $\tilde{a} \in \mathcal{F}$ be a fuzzy number and. Then $a_1\colon [0,1] \to \mathbb{R}$ is strictly increasing and*

$$a_1(\tilde{a}(t)) \leq t,$$

for $t \in cl(\text{supp}\tilde{a})$, furthemore $\tilde{a}(a_1(\alpha)) = \alpha$, for $\alpha \in [0,1]$ and

$$a_1(\tilde{a}(t)) \leq t \leq a_1(\tilde{a}(t) + 0),$$

for $a_1(0) \leq t < a_1(1)$, where

$$a_1(\tilde{a}(t) + 0) = \lim_{\epsilon \to +0} a_1(\tilde{a}(t) + \epsilon). \tag{4.14}$$

Lemma 4.1.4 (Fullér, [78]). *Let $\tilde{a}$ and $\tilde{b}$ be fuzzy numbers.Then*

(i) $D(\tilde{a}, \tilde{b}) \geq |a_1(\alpha + 0) - b_1(\alpha + 0)|$, *for* $0 \leq \alpha < 1$,

(ii) $\tilde{a}(a_1(\alpha + 0)) = \alpha$, *for* $0 \leq \alpha < 1$,

(iii) $a_1(\alpha) \leq a_1(\alpha + 0) < a_1(\beta)$, *for* $0 \leq \alpha < \beta \leq 1$.

Proof. (i) From the definition of the metric D we have

$$\begin{aligned} |a_1(\alpha + 0) - b_1(\alpha + 0)| &= \lim_{\epsilon \to +0} |a_1(\alpha + \epsilon) - \lim_{\epsilon \to +0} b_1(\alpha + \epsilon)| \\ &= \lim_{\epsilon \to +0} |a_1(\alpha + \epsilon) - b_1(\alpha + \epsilon)| \\ &\leq \sup_{\gamma \in [0,1]} |a_1(\gamma) - b_1(\gamma)| \leq D(\tilde{a}, \tilde{b}). \end{aligned}$$

(ii) Since $\tilde{a}(a_1(\alpha + \epsilon)) = \alpha + \epsilon$, for $\epsilon \leq 1 - \alpha$, we have

$$\tilde{a}(a_1(\alpha + 0)) = \lim_{\epsilon \to +0} A(a_1(\alpha + \epsilon)) = \lim_{\epsilon \to +0} (\alpha + \epsilon) = \alpha.$$

(iii) From strictly monotonity of a_1 it follows that $a_1(\alpha + \epsilon) < a_1(\beta)$, for $\epsilon < \beta - \alpha$. Therefore,

$$a_1(\alpha) \leq a_1(\alpha + 0) = \lim_{\epsilon \to +0} a_1(\alpha + \epsilon) < a_1(\beta),$$

which completes the proof. □

The following lemma shows that if all the α-level sets of two (continuous) fuzzy numbers are close to each other, then there can be only a small deviation between their membership grades.

Lemma 4.1.5 (Fullér, [78]). *Let $\delta \geq 0$ and let $\tilde{a}, \tilde{b}$ be fuzzy numbers. If $D(\tilde{a}, \tilde{b}) \leq \delta$, then*

$$\sup_{t \in \mathbb{R}} |\tilde{a}(t) - \tilde{b}(t)| \leq \max\{\omega(\tilde{a}, \delta), \omega(\tilde{b}, \delta)\}. \tag{4.15}$$

Proof. Let $t \in \mathbb{R}$ be arbitrarily fixed. It will be sufficient to show that

$$|\tilde{a}(t) - \tilde{b}(t)| \leq \max\{\omega(\tilde{a}, \delta), \omega(\tilde{b}, \delta)\}.$$

If $t \notin \text{supp}\tilde{a} \cup \text{supp}\tilde{b}$ then we obtain (4.15) trivially. Suppose that $t \in \text{supp}\tilde{a} \cup \text{supp}\tilde{b}$. With no loss of generality we will assume $0 \leq \tilde{b}(t) < \tilde{a}(t)$. Then either of the following must occur:

$$\begin{array}{ll} (a) & t \in (b_1(0), b_1(1)), \\ (b) & t \leq b_1(0), \\ (c) & t \in (b_2(1), b_2(0)) \\ (d) & t \geq b_2(0). \end{array}$$

In this case of (a) from Lemma 4.1.4 (with $\alpha = \tilde{b}(t), \beta = \tilde{a}(t)$) and Lemma 4.1.3(iii) it follows that

$$\tilde{a}(a_1(\tilde{b}(t)+0)) = \tilde{b}(t), \quad t \geq a_1(\tilde{a}(t)) \geq a_1(\tilde{b}(t)+0)$$

and

$$D(\tilde{a}, \tilde{b}) \geq |a_1(\tilde{b}(t)+0) - a_1(\tilde{b}(t)+0))|.$$

Therefore from continuity of $\tilde{a}$ we get

$$\begin{aligned} |\tilde{a}(t) - \tilde{b}(t)| &= |\tilde{a}(t) - \tilde{a}(a_1(\tilde{b}(t)+0))| \\ &= \omega(\tilde{a}, |t - a_1(\tilde{b}(t)+0)|) \\ &= \omega(\tilde{a}, t - a_1(\tilde{b}(t)+0)) \\ &\leq \omega(\tilde{a}, b_1(\tilde{b}(t)+0) - a_1(\tilde{b}(t)+0)) \leq \omega(\tilde{a}, \delta). \end{aligned}$$

In this case of (b) we have $\tilde{b}(t) = 0$; therefore from Lemma 4.1.3(i) it follows that

$$\begin{aligned} |\tilde{a}(t) - \tilde{b}(t)| &= |\tilde{a}(t) - 0| \\ &= |\tilde{a}(t) - \tilde{a}(a_1(0))| \\ &\leq \omega(\tilde{a}, |t - a_1(0)|) \\ &\leq \omega(\tilde{a}, |b_1(0) - a_1(0)|) \leq \omega(\tilde{a}, \delta). \end{aligned}$$

A similar reasoning yields in the cases of (c) and (d); instead of properties a_1 we use the properties of a_2. □

Let $L > 0$ be a real number. By $\mathcal{F}(L)$ we denote the set of all fuzzy numbers $\tilde{a} \in \mathcal{F}$ with membership function satisfying the Lipschitz condition with constant L , i.e.

$$|\tilde{a}(t) - \tilde{a}(t')| \leq L|t - t'|, \; \forall t, t' \in \mathbb{R}.$$

In the following lemma (which is a direct consequence of Lemma 4.1.2 and Lemma 4.1.5) we see that (i) linear combinations of Lipschitzian fuzzy numbers are also Lipschitzian ones, and (ii) if all the α-level sets of two Lipschitzian fuzzy numbers are closer to each other than δ, then there can be maximum $L\delta$ difference between their membership grades.

Lemma 4.1.6 (Fullér, [74]). *Let $L > 0$, $\lambda \neq 0$, $\mu \neq 0$ be real numbers and let $\tilde{a}$, $\tilde{b} \in \mathcal{F}(L)$ be fuzzy numbers. Then*

$$\lambda\tilde{a} \in \mathcal{F}\left(\frac{L}{|\lambda|}\right),$$

$$\lambda\tilde{a} + \mu\tilde{b} \in \mathcal{F}\left(\frac{L}{|\lambda| + |\mu|}\right).$$

Furthermore, if $D(\tilde{a}, \tilde{b}) \leq \delta$, then

$$\sup_t |\tilde{a}(t) - \tilde{b}(t)| \leq L\delta.$$

If the fuzzy $\tilde{a}$ and $\tilde{a}$ are of symmetric triangular form then Lemma 4.1.6 reads

Lemma 4.1.7 (Fullér, [74]). *Let $\delta > 0$ be a real number and let $\tilde{a} = (a, \alpha)$ and $\tilde{b} = (b, \beta)$ be symmetric triangular fuzzy numbers. Then*

$$\lambda\tilde{a} \in \mathcal{F}\left[\frac{1}{\alpha|\lambda|}\right],$$

$$\lambda\tilde{a} + \mu\tilde{b} \in \mathcal{F}\left(\frac{\max\{1/\alpha, 1/\beta\}}{|\lambda| + |\mu|}\right).$$

Furthermore, from the inequality $D(\tilde{a}, \tilde{b}) \leq \delta$ it follows that

$$\sup_t |\tilde{a}(t) - \tilde{b}(t)| \leq \max\left\{\frac{\delta}{\alpha}, \frac{\delta}{\beta}\right\}.$$

Kovács [108] showed that the fuzzy solution to system (4.2) with symmetric triangular fuzzy numbers is a stable with respect to small changes of centres of fuzzy parameters. Following Fullér [74] in the next theorem we establish a stability property (with respect to perturbations (4.6)) of the solution of system (4.2).

Theorem 4.1 (Fullér, [74]). *Let $L > 0$ and $\tilde{a}_{ij}$, $\tilde{a}^\delta_{ij}$, $\tilde{b}_i$, $\tilde{b}^\delta_i \in \mathcal{F}(L)$. If (4.6) holds, then*

$$||\mu - \mu^\delta||_\infty = \sup_{x \in \mathbb{R}^n} |\mu(x) - \mu^\delta(x)| \leq L\delta, \tag{4.16}$$

where $\mu(x)$ and $\mu^\delta(x)$ are the (fuzzy) solutions to systems (4.2) and (4.7), respectively.

Proof. It is sufficient to show that

$$|\mu_i(x) - \mu^\delta_i(x)| \leq L\delta$$

for each $x \in \mathbb{R}^n$ and $i = 1, \ldots, m$. Let $x \in \mathbb{R}^n$ and $i \in \{1, \ldots, m\}$ be arbitrarily fixed. From (4.3) it follows that

$$\mu_i(x) = \left(\sum_{j=1}^n \tilde{a}_{ij}x_j - \tilde{b}_i\right)(0) \quad \text{and} \quad \mu^\delta_i(x) = \left(\sum_{j=1}^n \tilde{a}^\delta_{ij}x_j - \tilde{b}^\delta_i\right)(0).$$

Applying Lemma 4.1.1 we have

$$D\left(\sum_{j=1}^n \tilde{a}_{ij}x_j - \tilde{b}_i, \sum_{j=1}^n \tilde{a}^\delta_{ij}x_j - \tilde{b}^\delta_i\right) \leq$$

$$\sum_{j=1}^n |x_j| D(\tilde{a}_{ij}, \tilde{a}^\delta_{ij}) + D(\tilde{b}_i, \tilde{b}^\delta_i) \leq \delta(|x|_1 + 1),$$

where $|x|_1 = |x_1| + \cdots + |x_n|$. Finally, by Lemma 4.1.6 we have

$$\sum_{j=1}^{n} \tilde{a}_{ij} x_j - \tilde{b}_i \in \mathcal{F}\left(\frac{L}{|x|_1 + 1}\right) \quad \text{and} \quad \sum_{j=1}^{n} \tilde{a}_{ij}^{\delta} x_j - \tilde{b}_i^{\delta} \in \mathcal{F}\left(\frac{L}{|x|_1 + 1}\right)$$

therefore,

$$\begin{aligned} |\mu_i(x) - \mu_i^{\delta}(x)| &= \left| \left(\sum_{j=1}^{n} \tilde{a}_{ij} x_j - \tilde{b}_i\right)(0) - \left(\sum_{j=1}^{n} \tilde{a}_{ij}^{\delta} x_j - \tilde{b}_i^{\delta}\right)(0) \right| \\ &\leq \sup_{t \in \mathbb{R}} \left| \left(\sum_{j=1}^{n} \tilde{a}_{ij} x_j - \tilde{b}_i\right)(t) - \left(\sum_{j=1}^{n} \tilde{a}_{ij}^{\delta} x_j - \tilde{b}_i^{\delta}\right)(t) \right| \\ &\leq \frac{L}{|x|_1 + 1} \times \delta(|x|_1 + 1) = L\delta. \end{aligned}$$

Which proves the theorem. □

From (4.16) it follows that

$$|\mu^* - \mu^*(\delta)| \leq L\delta,$$

where μ^*, $\mu^*(\delta)$ are the measures of consistency for the systems (4.2) and (4.7), respectively. It is easily checked that in the general case $\tilde{a}_{ij}, \tilde{b}_i \in \mathcal{F}(\mathbb{R})$ the solution to possibilistic linear equality system (4.2) may be unstable (in metric C_∞) under small variations in the membership function of fuzzy parameters (in metric D). When the problem is to find a maximizing solution to a possibilistic linear equality system (4.2), then according to Negoita [128], we are led to solve the following optimization problem

$$\begin{aligned} \text{maximize} \quad & \lambda \\ \mu_1(x_1, \ldots, x_n) \quad & \geq \lambda, \\ & \cdots \\ \mu_m(x_1, \ldots, x_n) \quad & \geq \lambda, \\ x \in \mathbb{R}^n,\ 0 \leq & \lambda \leq 1. \end{aligned} \tag{4.17}$$

Finding the solutions of problem (4.17) generally requires the use of nonlinear programming techniques, and could be tricky. However, if the fuzzy numbers in (4.2) are of trapezoidal form, then the problem (4.17) turns into a quadratically constrained programming problem. Even though the fuzzy solution and the measure of consistency of system (4.2) have a stability property with respect to changes of the fuzzy parameters, the behavior of the maximizing solution towards small perturbations of the fuzzy parameters can be very fortuitous, i.e. supposing that, X^*, the set of maximizing solutions to system (4.2) is not empty, the distance between $x^*(\delta)$ and X^* can be very big, where $x^*(\delta)$ is a maximizing solution of the perturbed possibilistic equality system (4.7).

Consider now the possiblistic equality system (4.2) with fuzzy numbers of symmetric triangular form

$$(a_{i1}, \alpha)x_1 + \cdots + (a_{in}, \alpha)x_n \ = \ (b_i, \alpha),\ i = 1, \ldots, m,$$

or shortly,

$$(A, \alpha)x = (b, \alpha) \tag{4.18}$$

Then the fuzzy solution of (4.18) can be written in a compact form

$$\mu(x) = \begin{cases} 1 & \text{if } Ax = b \\ 1 - \dfrac{||Ax-b||_\infty}{\alpha(|x|_1+1)} & \text{if } 0 < ||Ax-b||_\infty \leq \alpha(|x|_1+1) \\ 0 & \text{if } ||Ax-b||_\infty > \alpha(|x|_1+1) \end{cases}$$

where

$$||Ax-b||_\infty = \max\{|\langle a_1, x\rangle - b_1|, \dots, |\langle a_m, x\rangle - b_m|\}.$$

If

$$[\mu]^1 = \{x \in \mathbb{R}^n \mid \mu(x) = 1\} \neq \emptyset$$

then the set of maximizing solutions, $X^* = [\mu]^1$, of (4.18) coincides with the solution set, denoted by X^{**}, of the crisp system $Ax = b$. The stability theorem for system (4.18) reads

Theorem 4.2 (Kovács, [108])**.** *If* $D(\tilde{A}, \tilde{A}^\delta) = \max_{i,j} |a_{ij} - a_{ij}^\delta| \leq \delta,\ D(\tilde{b}, \tilde{b}^\delta) = \max_i |b_i - b_i^\delta| \leq \delta$ *hold, then*

$$||\mu - \mu^\delta||_\infty = \sup_x |\mu(x) - \mu^\delta(x)| \leq \frac{\delta}{\alpha},$$

where $\mu(x)$ *and* $\mu^\delta(x)$ *are the fuzzy solutions to possibilistic equality systems* $(A, \alpha)x = (b, \alpha)$ *and* $(A^\delta, \alpha)x = (b^\delta, \alpha)$*, respectively.*

Theorem 4.1 can be extended to possibilistic linear equality systems with (continuous) fuzzy numbers.

Theorem 4.3 (Fullér, [78])**.** *Let* $\tilde{a}_{ij}$, $\tilde{a}_{ij}^\delta$, $\tilde{b}_i$, $\tilde{b}_i^\delta \in \mathcal{F}$ *be fuzzy numbers. If (4.6) holds, then* $||\mu - \mu^\delta||_\infty \leq \omega(\delta)$*, where* $\omega(\delta)$ *denotes the maximum of modulus of continuity of all fuzzy coefficients at* δ *in (4.2) and (4.7).*

In 1992 Kovács [109] showed a wide class of fuzzified systems that are well-posed extensions of ill-posed linear equality and inequality systems.

Consider the following two-dimensional possibilistic equality system

$$(1, \alpha)x_1 + (1, \alpha)x_2 = (0, \alpha) \tag{4.19}$$

$$(1, \alpha)x_1 - (1, \alpha)x_2 = (0, \alpha)$$

Then its fuzzy solution is

$$\mu(x) = \begin{cases} 1 & \text{if } x = 0 \\ \tau_2(x) & \text{if } 0 < \max\{|x_1 - x_2|, |x_1 + x_2|\} \leq \alpha(|x_1| + |x_2| + 1) \\ 0 & \text{if } \max\{|x_1 - x_2|, |x_1 + x_2|\} > \alpha(|x_1| + |x_2| + 1) \end{cases}$$

where

$$\tau_2(x) = 1 - \frac{\max\{|x_1 - x_2|, |x_1 + x_2|\}}{\alpha(|x_1| + |x_2| + 1)},$$

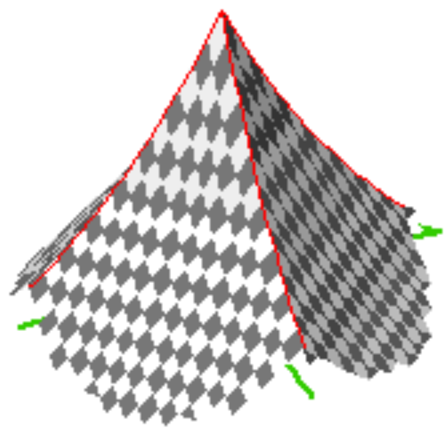

Figure 4.1: The graph of fuzzy solution of system (4.19) with $\alpha = 0.4$.

and the only maximizing solution of system (4.19) is $x^* = (0, 0)$. There is no problem with stability of the solution even for the crisp system

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

because $\det(A) \neq 0$.

The fuzzy solution of possibilistic equality system

$$(1, \alpha)x_1 + (1, \alpha)x_2 = (0, \alpha) \tag{4.20}$$

$$(1, \alpha)x_1 + (1, \alpha)x_2 = (0, \alpha)$$

is

$$\mu(x) = \begin{cases} 1 & \text{if } |x_1 + x_2| = 0 \\ 1 - \dfrac{|x_1 + x_2|}{\alpha(|x_1| + |x_2| + 1)} & \text{if } 0 < |x_1 + x_2| \leq \alpha(|x_1| + |x_2| + 1) \\ 0 & \text{if } |x_1 + x_2| > \alpha(|x_1| + |x_2| + 1) \end{cases}$$

and the set of its maximizing solutions is

$$X^* = \{x \in \mathbb{R}^2 \mid x_1 + x_2 = 0\}.$$

In this case we have

$$X^* = X^{**} = \{x \in \mathbb{R}^2 \mid Ax = b\}.$$

We might experience problems with the stability of the solution of the crisp system

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

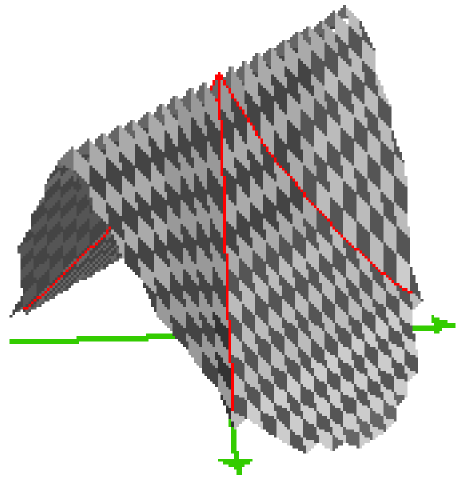

Figure 4.2: The graph of fuzzy solution of system (4.20) with $\alpha = 0.4$.

because $\det(A) = 0$.

Really, the fuzzy solution of possibilistic equality system

$$(1, \alpha)x_1 + (1, \alpha)x_2 = (\delta_1, \alpha) \tag{4.21}$$

$$(1, \alpha)x_1 + (1, \alpha)x_2 = (\delta_2, \alpha)$$

where $\delta_1 = 0.3$ and $\delta_2 = -0.3$, is

$$\mu(x) =$$

$$\begin{cases} \tau_1(x) & \text{if } 0 < \max\{|x_1 + x_2 - 0.3|, |x_1 + x_2 + 0.3|\} \leq \alpha(|x_1| + |x_2| + 1) \\ 0 & \text{if } \max\{|x_1 + x_2 - 0.3|, |x_1 + x_2 + 0.3|\} > \alpha(|x_1| + |x_2| + 1) \end{cases}$$

where

$$\tau_1(x) = 1 - \frac{\max\{|x_1 + x_2 - 0.3|, |x_1 + x_2 + 0.3|\}}{\alpha(|x_1| + |x_2| + 1)}$$

and the set of the maximizing solutions of (4.21) is empty, and X^{**} is also empty. Even though the set of maximizing solution of systems (4.20) and (4.21) varies a lot under small changes of the centers of fuzzy numbers of the right-hand side, δ_1 and δ_2, their fuzzy solutions can be made arbitrary close to each other by letting

$$\frac{\max\{\delta_1, \delta_2\}}{\alpha}$$

to tend to zero.

4.2 Stability in fuzzy linear programming problems

In this Section, following Fullér [73] we investigate the stability of the solution in FLP problems (with symmetric triangular fuzzy numbers and extended operations and inequalities) with respect to changes

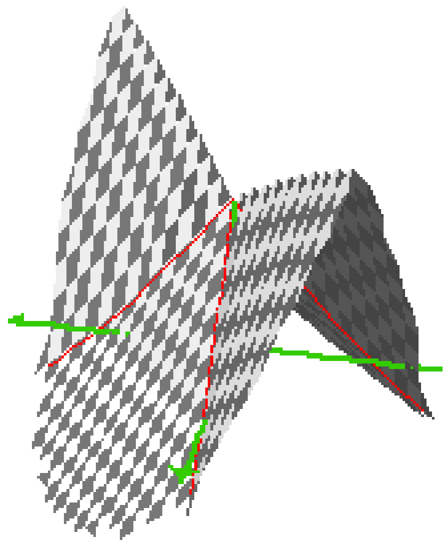

Figure 4.3: The graph of fuzzy solution of system (4.21) with $\alpha = 0.4$.

of fuzzy parameters and show that the solution to these problems is stable (in metric C_∞) under small variations in the membership functions of the fuzzy coefficients.

The conventional model of linear programming (LP) can be stated as

$$\langle a_0, x\rangle \to \min$$

$$\text{subject to } Ax \le b.$$

In many real-world problems instead of minimization of the objective function $\langle a_0, x\rangle$ it may be sufficient to determine an x such that

$$a_{01}x_1 + \cdots + a_{0n}x_n \le b_0; \text{ subject to } Ax \le b. \tag{4.22}$$

where b_0 is a predetermined aspiration level.

Assume that all parameters in (4.22) are fuzzy quantities and are described by symmetric triangular fuzzy numbers. Then the following flexible (or fuzzy) linear programming (FLP) problem can be obtained by replacing crisp parameters a_{ij}, b_i with symmetric triangular fuzzy numbers $\tilde{a}_{ij} = (a_{ij}, \alpha)$ and $\tilde{b}_i = (b_i, d_i)$ respectively,

$$(a_{i1}, \alpha)x_1 + \cdots + (a_{in}, \alpha)x_n \le (b_i, d_i), \quad i = 0, \dots, m. \tag{4.23}$$

Here d_0 and d_i are interpreted as the tolerance levels for the objective function and the i-th constraint, respectively. The parameter $\alpha > 0$ will guarantee the stability property of the solution of (4.23) under small changes in the coefficients a_{ij} and b_i. We denote by $\mu_i(x)$ the degree of satisfaction of the i-th restriction at the point $x \in \mathbb{R}^n$ in (4.23), i.e.

$$\mu_i(x) = \mathrm{Pos}(\tilde{a}_{i1}x_1 + \cdots + \tilde{a}_{in}x_n \le \tilde{b}_i).$$

Then the (fuzzy) solution of the FLP problem (4.23) is defined as a fuzzy set on R^n whose membership function is given by

$$\mu(x) = \min\{\mu_0(x), \mu_1(x), \dots, \mu_m(x)\},$$

and the maximizing solution x^* of the FLP problem (4.23) satisfies the equation

$$\mu(x^*) = \mu^* = \max_x \mu(x).$$

The degree of satisfaction of the i-th restriction at x in (4.23) is the following:

$$\mu_i(x) = \begin{cases} 1 & \text{if } \langle a_i, x\rangle \le b_i, \\ 1 - \dfrac{\langle a_i, x\rangle - b_i}{\alpha |x|_1 + d_i} & \text{otherwise}, \\ 0 & \text{if } \langle a_i, x\rangle > b_i + \alpha|x|_1 + d_i, \end{cases} \tag{4.24}$$

where $|x|_1 = |x_1| + \cdots + |x_n|$ and $\langle a_i, x\rangle = a_{i1}x_1 + \cdots + a_{in}x_n,\ i = 0, 1, \ldots, m.$

In the extremal case $\alpha = 0$ but $d_i > 0$ in (4.24), we get a linear membership function for μ_i, i.e. Zimmermann's principle [160]. Really, for $\alpha = 0$ we get

$$(a_{i1}, 0)x_1 + \cdots + (a_{in}, 0)x_n \le (b_i, d_i), \tag{4.25}$$

and the μ_i's have a very simple form

$$\mu_i(x) = \begin{cases} 1 & \text{if } \langle a_i, x\rangle \le b_i, \\ 1 - \dfrac{\langle a_i, x\rangle - b_i}{d_i} & \text{if } b_i < \langle a_i, x\rangle \le b_i + d_i, \\ 0 & \text{if } \langle a_i, x\rangle > b_i + d_i, \end{cases}$$

for $i = 0, 1, \ldots, m$.

If $\alpha = 0$ then μ_i has an easy interpretation: If for an $x \in \mathbb{R}^n$ the value of $\langle a_i, x\rangle$ is less or equal than b_i then x satisfies the i-th constraint with the maximal conceivable degree one; if $b_i < \langle a_i, x\rangle < b_i + d_i$ then x is not feasible in classical sense, but the decision maker can still tolerate the violation of the crisp constraint, and accept x as a solution with a positive degree, however, the bigger the violation the less is the degree of acceptance; and if $\langle a_i, x\rangle > b_i + d_i$ then the violation of the i-th costraint is untolerable by the decision maker, that is, $\mu_i(x) = 0$. Following Fullér [73] we investigate the stability of the solution in FLP problems (with symmetric triangular fuzzy numbers and extended operations and inequalities) with respect to changes of fuzzy parameters and show that the solution to these problems is stable (in metric C_∞) under small variations in the membership functions of the fuzzy coefficients. Consider now the perturbed FLP problem,

$$(a^\delta_{i1}, \alpha)x_1 + \cdots + (a^\delta_{in}, \alpha)x_n \le (b^\delta_i, d_i),\ i = 0, \ldots, m. \tag{4.26}$$

where a^δ_{ij} and b^δ_i satisfy the inequalities

$$\max_{i,j} |a_{ij} - a^\delta_{ij}| \le \delta, \quad \max_i |b_i - b^\delta_i| \le \delta. \tag{4.27}$$

In a similar manner we can define the solution of FLP problem (4.26) by

$$\mu^\delta(x) = \min\{\mu^\delta_0(x), \mu^\delta_1(x), \ldots, \mu^\delta_m(x)\},\ x \in \mathbb{R}^n,$$

where $\mu^\delta_i(x)$ denotes the degree of satisfaction of the i-th restriction at $x \in \mathbb{R}^n$ and the maximizing solution $x^*(\delta)$ of FLP problem (4.26) satisfies the equation

$$\mu^\delta(x^*(\delta)) = \mu^*(\delta) = \sup_x \mu^\delta(x). \tag{4.28}$$

In the following theorem we establish a stability property of the fuzzy solution of FLP problem (4.23).

Theorem 4.4 (Fullér, [73]). *Let $\mu(x)$ and $\mu^\delta(x)$ be solution of FLP problems (4.23) and (4.26) respectively. Then*

$$||\mu - \mu^\delta||_\infty = \sup_{x\in R^n} |\mu(x) - \mu^\delta(x)| \le \delta\left[\frac{1}{\alpha} + \frac{1}{d}\right] \tag{4.29}$$

where $d = \min\{d_0, d_1, \dots, d_m\}$.

Proof. First let $\delta \ge \min\{\alpha, d\}$. Then from $|\mu(x) - \mu^\delta(x)| \le 1, \forall x \in \mathbb{R}^n$ and

$$\frac{\delta}{\alpha + d} \ge 1,$$

we obtain (4.29). Suppose that

$$0 < \delta < \min\{\alpha, d\}.$$

It will be sufficient to show that

$$|\mu_i(x) - \mu_i^\delta(x)| \le \delta\left[\frac{1}{\alpha} + \frac{1}{d}\right],\ \forall x \in \mathbb{R}^n,\ i = 0, \dots, m, \tag{4.30}$$

because from (4.30) follows (4.29). Let $x \in \mathbb{R}^n$ and $i \in \{0, \dots, m\}$ be arbitrarily fixed.

Consider the following cases:

(1) $\mu_i(x) = \mu_i^\delta(x)$. In this case (4.30) is trivially obtained.

(2) $0 < \mu_i(x) < 1$ and $0 < \mu_i^\delta(x) < 1$. In this case from (4.24), (4.27) we have

$$\begin{aligned}
|\mu_i(x) - \mu_i^\delta(x)| &= \left|1 - \frac{\langle a_i, x\rangle - b_i}{\alpha|x|_1 + d_i} - \left(1 - \frac{\langle a_i^\delta, x\rangle - b_i^\delta}{\alpha|x|_1 + d_i}\right)\right| \\
&= \frac{|b_i - b_i^\delta| + \langle a_i^\delta, x\rangle - \langle a_i, x\rangle|}{\alpha|x|_1 + d_i} \\
&\le \frac{|b_i - b_i^\delta| + |\langle a_i^\delta - a_i, x\rangle|}{\alpha|x|_1 + d_i} \\
&\le \frac{\delta + |a_i^\delta - a_i|_\infty |x|_1}{\alpha|x|_1 + d_i} \\
&\le \frac{\delta + \delta|x|_1}{\alpha|x|_1 + d_i} \\
&\le \delta\left[\frac{1}{\alpha} + \frac{1}{d_i}\right] \le \delta\left[\frac{1}{\alpha} + \frac{1}{d}\right],
\end{aligned}$$

where $a_i^\delta = (a_{i1}^\delta, \dots, a_{in}^\delta)$ and $|a_i^\delta - a_i|_\infty = \max_j |a_{ij}^\delta - a_{ij}|$.

(3) $\mu_i(x) = 1$ and $0 < \mu_i^\delta(x) < 1$. In this case we have $\langle a_i, x\rangle \leq b_i$. Hence

$$\begin{aligned}|\mu_i(x) - \mu_i^\delta(x)| &= \left|1 - \left[1 - \frac{\langle a_i^\delta, x\rangle - b_i^\delta}{\alpha|x|_1 + d_i}\right]\right| \\ &= \frac{\langle a_i^\delta, x\rangle - b_i^\delta}{\alpha|x|_1 + d_i} \\ &\leq \frac{(\langle a_i^\delta, x\rangle - b_i^\delta) - (\langle a_i, x\rangle - b_i)}{\alpha|x|_1 + d_i} \\ &\leq \delta\left[\frac{1}{\alpha} + \frac{1}{d}\right].\end{aligned}$$

(4) $0 < \mu_i(x) < 1$ and $\mu_i^\delta(x) = 1$. In this case the proof is carried out analogously to the proof of the preceding case.

(5) $0 < \mu_i(x) < 1$ and $\mu_i^\delta(x) = 0$. In this case from

$$\langle a_i^\delta, x\rangle - b_i^\delta > \alpha|x|_1 + d_i$$

it follows that

$$\begin{aligned}|\mu_i(x) - \mu_i^\delta(x)| &= \left|1 - \frac{\langle a_i, x\rangle - b_i}{\alpha|x|_1 + d_i}\right| \\ &= \frac{1}{\alpha|x|_1 + d_i} \times \left|\alpha|x|_1 + d_i - (\langle a_i, x\rangle - b_i)\right| \\ &\leq \frac{|\langle a_i(\delta), x\rangle - b_i(\delta) - (\langle a_i, x\rangle - b_i)|}{\alpha|x|_1 + d_i} \\ &\leq \delta\left[\frac{1}{\alpha} + \frac{1}{d}\right].\end{aligned}$$

(6) $\mu_i(x) = 0$ and $0 < \mu_i^\delta(x) < 1$. In this case the proof is carried out analogously to the proof of the preceding case.

(7) $\mu_i(x) = 1$ $\mu_i^\delta(x) = 0$, or $\mu_i(x) = 0$, $\mu_i^\delta(x) = 1$. These cases are not reasonable. For instance suppose that case $\mu_i(x) = 1$, $\mu_i^\delta(x) = 0$ is conceivable. Then from (4.27) it follows that

$$\begin{aligned}|\langle a_i, x\rangle - b_i - (\langle a_i(\delta), x\rangle - b_i(\delta))| &\leq |b_i - b_i^\delta| + |a_i^\delta - a_i|_\infty |x|_1 \\ &\leq \delta(|x|_1 + 1).\end{aligned}$$

On the other hand we have

$$\begin{aligned}|\langle a_i, x\rangle - b_i - (\langle a_i^\delta, x\rangle - b_i^\delta)| &\geq |\langle a_i^\delta, x\rangle - b_i^\delta| \geq \alpha|x|_1 + d_i \\ &> \delta|x|_1 + \delta = \delta(|x|_1 + 1).\end{aligned}$$

So we arrived at a contradiction, which ends the proof.

□

From (4.29) it follows that

$$|\mu^* - \mu^*(\delta)| \leq \delta \left[\frac{1}{\alpha} + \frac{1}{d} \right]$$

and

$$||\mu - \mu^\delta||_C \to 0 \text{ if } \delta/\alpha \to 0 \text{ and } \delta/d \to 0,$$

which means stability with respect to perturbations (4.27) of the solution and the measure of consistency in FLP problem (4.23). To find a maximizing solution to FLP problem (4.23) we have to solve the following nonlinear programming problem

$$\max \lambda$$

$$\lambda(\alpha|x|_1 + d_0) - \alpha|x|_1 + \langle a_0, x\rangle \leq b_0 + d_0,$$

$$\lambda(\alpha|x|_1 + d_1) - \alpha|x|_1 + \langle a_1, x\rangle \leq b_1 + d_1,$$

$$\dots\dots$$

$$\lambda(\alpha|x|_1 + d_m) - \alpha|x|_1 + \langle a_m, x\rangle \leq b_m + d_m,$$

$$0 \leq \lambda \leq 1,\ x \in \mathbb{R}^n.$$

It is easily checked that in the extremal case $\alpha = 0$ but $d_i > 0$, the solution of FLP problem (4.23) may be unstable with respect to changes of the crisp parameters a_{ij}, b_i.

4.3 Stability in possibilistic linear programming problems

Following Fedrizzi and Fullér [72] we show that the possibility distribution of the objective function of a possibilistic linear program with continuous fuzzy number parameters is stable under small perturbations of the parameters. First, we will briefly review possibilistic linear programming and set up notations. A possibilitic linear program is (see Buckley in [8])

$$\max/\min\ Z = x_1\tilde{c}_1 + \cdots + x_n\tilde{c}_n, \tag{4.31}$$

$$\text{subject to} \quad x_1\tilde{a}_{i1} + \cdots + x_n\tilde{a}_{in} * \tilde{b}_i,\ 1 \leq i \leq m,\ x \geq 0.$$

where $\tilde{a}_{ij}$, $\tilde{b}_i$, $\tilde{c}_j$ are fuzzy numbers, $x = (x_1, \dots, x_n)$ is a vector of (nonfuzzy) decision variables, and $*$ denotes $<$, $\leq$, $=$, $\geq$ or $>$ for each i. We will assume that all fuzzy numbers $\tilde{a}_{ij}$, $\tilde{b}_i$ and $\tilde{c}_j$ are non-interactive. Non-interactivity means that we can find the joint possibility distribution of all the fuzzy variables by calculating the min-intersection of their possibility distributions. Following Buckley [8], we define $\mathrm{Pos}[Z = z]$, the possibility distribution of the objective function Z. We first specify the possibility that x satisfies the i-th constraints. Let $\Pi(a_i, b_i) = \min\{\tilde{a}_{i1}(a_{i1}), \dots, \tilde{a}_{in}(a_{in}), \tilde{b}_i(b_i\}$, where $a_i = (a_{i1}, \dots, a_{in})$, which is the joint distribution of $\tilde{a}_{ij}$, $j = 1, \dots, n$, and $\tilde{b}_i$. Then

$$\mathrm{Pos}[x \in \mathcal{F}_i] = \sup_{a_i, b_i} \{\, \Pi(a_i, b_i) \mid a_{i1}x_1 + \cdots + a_{in}x_n * b_i \,\},$$

which is the possibility that x is feasible with respect to the i-th constraint. Therefore, for $x \geq 0$,

$$\mathrm{Pos}[x \in \mathcal{F}] = \min_{1 \leq i \leq m} \mathrm{Pos}[x \in \mathcal{F}_i],$$

which is the possibility that x is feasible. We next construct $\mathrm{Pos}[Z = z|x]$ which is the conditional possibility that Z equals z given x. The joint distribution of the $\tilde{c}_j$ is

$$\Pi(c) = \min\{\tilde{c}_1(c_1), \dots, \tilde{c}_n(c_n)\}$$

where $c = (c_1, \dots, c_n)$. Therefore,

$$\mathrm{Pos}[Z = z|x] = \sup_c\{\Pi(c)|c_1x_1 + \cdots + c_nx_n = z\}.$$

Finally, applying Bellman and Zadeh's method for fuzzy decision making [6], the possibility distribution of the objective function is defined as

$$\mathrm{Pos}[Z = z] = \sup_{x\geq 0}\min\{\mathrm{Pos}[Z = z|x], \mathrm{Pos}[x \in \mathcal{F}]\}.$$

It should be noted that Buckley [9] showed that the solution to an appropriate linear program gives the correct z values in $\mathrm{Pos}[Z = z] = \alpha$ for each $\alpha \in [0, 1]$.

An important question is the influence of the perturbations of the fuzzy parameters to the possibility distribution of the objective function. We will assume that there is a collection of fuzzy parameters $\tilde{a}^\delta_{ij}$, $\tilde{b}^\delta_i$ and $\tilde{c}^\delta_j$ available with the property

$$D(\tilde{A}, \tilde{A}^\delta) \leq \delta,\ D(\tilde{b}, \tilde{b}^\delta) \leq \delta,\ D(\tilde{c}, \tilde{c}^\delta) \leq \delta, \tag{4.32}$$

where $D(\tilde{A}, \tilde{A}^\delta) = \max_{i,j} D(\tilde{a}_{ij}, \tilde{a}^\delta_{ij}), D(\tilde{b}, \tilde{b}^\delta) = \max_i D(\tilde{b}_i, \tilde{b}^\delta_i) and D(\tilde{c}, \tilde{c}^\delta) = \max_j D(\tilde{c}_j, \tilde{c}^\delta_j)$. Then we have to solve the following perturbed problem:

$$\max/\min\ Z^\delta = x_1\tilde{c}^\delta_1 + \cdots + x_n\tilde{c}^\delta_n \tag{4.33}$$
$$\text{subject to}\quad x_1\tilde{a}^\delta_{i1} + \cdots + x_n\tilde{a}^\delta_{in} * \tilde{b}^\delta_i,\ 1 \leq i \leq m,\ x \geq 0.$$

Let us denote by $\mathrm{Pos}[x \in \mathcal{F}^\delta_i]$ the possibility that x is feasible with respect to the i-th constraint in (4.33). Then the possibility distribution of the objective function Z^δ is defined as follows:

$$\mathrm{Pos}[Z^\delta = z] = \sup_{x\geq 0}(\min\{\mathrm{Pos}[Z^\delta = z \mid x], \mathrm{Pos}[x \in \mathcal{F}^\delta]\}).$$

The next theorem shows a stability property (with respect to perturbations (4.32) of the possibility dostribution of the objective function of the possibilistic linear programming problems (4.31) and (4.33).

Theorem 4.5 (Fedrizzi and Fullér, [72]). *Let $\delta \geq 0$ be a real number and let $\tilde{a}_{ij}$, $\tilde{b}_i$, $\tilde{a}^\delta_{ij}$, $\tilde{c}_j$, $\tilde{c}^\delta_j$ be (continuous) fuzzy numbers. If (4.32) hold, then*

$$\sup_{z\in\mathbb{R}} \mid \mathrm{Pos}[Z^\delta = z] - \mathrm{Pos}[Z = z] \mid\leq \omega(\delta) \tag{4.34}$$

where $\omega(\delta) = \max_{i,j}\{\omega(\tilde{a}_{ij}, \delta), \omega(\tilde{a}^\delta_{ij}, \delta), \omega(\tilde{b}_i, \delta), \omega(b^\delta_i, \delta), \omega(\tilde{c}_j, \delta), \omega(\tilde{c}^\delta_j, \delta)\}$.

From (4.34) follows that $\sup_z |\mathrm{Pos}[Z^\delta = z] - \mathrm{Pos}[Z = z]| \to 0$ as $\delta \to 0$, which means the stability of the possiibility distribution of the objective function with respect to perturbations (4.32). As an immediate consequence of this theorem we obtain the following result: If the fuzzy numbers in (4.31) and (4.33) satisfy the Lipschitz condition with constant $L > 0$, then

$$\sup_{z\in\mathbb{R}} \mid \mathrm{Pos}[Z^\delta = z] - \mathrm{Pos}[Z = z] \mid\leq L\delta$$

It is easy to see that in the case of non-continuous fuzzy parameters the possibility distribution of the objective function may be unstable under small changes of the parameters.

4.4 Stability in possibilistic quadratic programming problems

In this Section, following Canastrelli, Giove and Fullér [12] we show that possibilistic quadratic programs with crisp decision variables and continuous fuzzy number coefficients are well-posed, i.e. small changes in the membership function of the coefficients may cause only a small deviation in the possibility distribution of the objective function.

A possibilistic quadratic program is

$$\begin{array}{ll} \text{maximize} & Z := x^T\tilde{C}x + \langle \tilde{d}, x\rangle \\ \text{subject to} & \langle \tilde{a}_i, x\rangle \leq \tilde{b}_i,\ 1 \leq i \leq m,\ x \geq 0 \end{array} \tag{4.35}$$

where $\tilde{C} = (\tilde{c}_{kj})$ is a matrix of fuzzy numbers, $\tilde{a}_i = (\tilde{a}_{ij})$ and $\tilde{d} = (\tilde{d}_j)$ are vectors of fuzzy numbers, $\tilde{b}_i$ is a fuzzy number and

$$\langle \tilde{d}, x\rangle = \tilde{d}_1 x_1 + \cdots + \tilde{d}_n x_n.$$

We will assume that all fuzzy numbers are non-interactive. We define, $\mathrm{Pos}[Z = z]$, the possibility distribution of the objective function Z. We first specify the possibility that x satisfies the i-th constraint. Let

$$\Pi(a_i, b_i) = \min\{\tilde{a}_{i1}(a_{i1}), \ldots, \tilde{a}_{in}(a_{in}), \tilde{b}_i(b_i)\}$$

where $a_i = (a_{i1}, \ldots, a_{in})$, which is the joint possibility distribution of $\tilde{a}_i$, $1 \leq j \leq n$ and $\tilde{b}_i$. Then

$$\mathrm{Pos}[x \in \mathcal{F}_i] = \sup_{a_i, b_i}\{\Pi(a_i, b_i) \mid a_{i1}x_1 + \cdots + a_{in}x_n \leq b_i\}$$

which is the possibility that x is feasible with respect to th i-th constraint. Therefore, for $x \geq 0$,

$$\mathrm{Pos}[x \in \mathcal{F}] = \min\{\mathrm{Pos}[x \in \mathcal{F}_1], \ldots, \mathrm{Pos}[x \in \mathcal{F}_m]\}.$$

We next construct $\mathrm{Pos}[Z = z|x]$ which is the conditional possibility that Z equals z given x. The joint possibility distribution of $\tilde{C}$ and $\tilde{d}$ is

$$\Pi(C, d) = \min_{k,j}\{\tilde{C}_{kj}(c_{kj}), \tilde{d}_j(d_j)\}$$

where $C = (c_{kj})$ is a crisp matrix and $d = (d_j)$ a crisp vector. Therefore,

$$\mathrm{Pos}[Z = z|x] = \sup_{C,d}\{\Pi(C, d) \mid x^T C x + \langle d, x\rangle = z\}.$$

Finally, the possibility distribution of the objective function is defined as

$$\mathrm{Pos}[Z = z] = \sup_{x \geq 0}\min\{\mathrm{Pos}[Z = z|x], \mathrm{Pos}[x \in \mathcal{F}]\}.$$

We show that possibilistic quadratic programs with crisp decision variables and continuous fuzzy number coefficients are well-posed, i.e. small changes in the membership function of the coefficients may cause only a small deviation in the possibility distribution of the objective function. We will assume that there is a collection of fuzzy parameters $\tilde{A}^\delta$, $\tilde{b}^\delta$, $\tilde{C}^\delta$ and $\tilde{d}^\delta$ are available with the property

$$D(\tilde{A}, \tilde{A}^\delta) \leq \delta,\ D(\tilde{C}, \tilde{C}^\delta) \leq \delta,\ D(\tilde{b}, \tilde{b}^\delta) \leq \delta,\ D(\tilde{d}, \tilde{d}^\delta) \leq \delta, \tag{4.36}$$

Then we have to solve the following perturbed problem:

$$\begin{aligned} \text{maximize} \quad & x^T \tilde{C}^\delta x + \langle \tilde{d}^\delta, x \rangle \\ \text{subject to} \quad & \tilde{A}^\delta x \leq \tilde{b}^\delta, \ x \geq 0 \end{aligned} \tag{4.37}$$

Let us denote by $\text{Pos}[x \in \mathcal{F}_i^\delta]$ that x is feasible with respect to the i-th constraint in (4.37). Then the possibility distribution of the objective function Z^δ is defined as follows

$$\text{Pos}[Z^\delta = z] = \sup_{x \geq 0} \min\{\text{Pos}[Z^\delta = z|x], \text{Pos}[x \in \mathcal{F}^\delta]\}.$$

The next theorem shows a stability property of the possibility distribution of the objective function of the possibilistic quadratic programs (4.35) and (4.37).

Theorem 4.6 (Canastrelli, Giove and Fullér, [12])**.** *Let $\delta > 0$ be a real number and let $\tilde{c}_{kj}$, $\tilde{a}_{ij}$, $\tilde{d}_j$, $\tilde{b}_i$, $\tilde{c}_{kj}^\delta$, $\tilde{a}_{ij}^\delta$, $\tilde{d}_j^\delta$, $\tilde{b}_i^\delta \in \mathcal{F}$ be fuzzy numbers. If (4.36) hold then*

$$\sup_{z \in \mathbb{R}} |\text{Pos}[Z^\delta = z] - \text{Pos}[Z = z]| \leq \omega(\delta)$$

where $\omega(\delta)$ denotes the maximum of modulus of continuity of all fuzzy number coefficients at δ in (4.35) and (4.37).

From Theorem 4.6 it follows that $\sup_z |\text{Pos}[Z^\delta = z] - \text{Pos}[Z = z]| \to$ as $\delta \to 0$ which means the stability of the possibility distribution of the objective function with respect to perturbations (4.36).

4.5 Stability in multiobjective possibilistic linear programming problems

In this Section, following Fullér and Fedrizzi [82], we show that the possibility distribution of the objectives of an multiobjective possibilistic linear program (MPLP) with (continuous) fuzzy number coefficients is stable under small changes in the membership function of the fuzzy parameters.

A multiobjective possibilistic linear program (MPLP) is

$$\max/\min Z = (\tilde{c}_{11}x_1 + \cdots + \tilde{c}_{1n}x_n, \ldots, \tilde{c}_{k1}x_1 + \cdots + \tilde{c}_{kn}x_n) \tag{4.38}$$

$$\text{subject to} \quad \tilde{a}_{i1}x_1 + \cdots \tilde{a}_{in}x_n * \tilde{b}_i, \ i = 1, \ldots, m, \ x \geq 0,$$

where $\tilde{a}_{ij}$, $\tilde{b}_i$, and $\tilde{c}_{lj}$ are fuzzy quantities, $x = (x_1, \ldots, x_n)$ is a vector of (non-fuzzy) decision variables and $|ast$ denotes $<$, $\leq$, =, $\geq$ or $>$ for each $i, \ i = 1, \ldots, m$.

Even though $*$ may vary from row to row in the constraints, we will rewrite the MPLP (4.38) as

$$\max/\min Z = (\tilde{c}_1 x, \ldots, \tilde{c}_k x)$$

$$\text{subject to} \quad \tilde{A}x * \tilde{b}, x \geq 0,$$

where $\tilde{a} = \{\tilde{a}_{ij}\}$ is an $m \times n$ matrix of fuzzy numbers and $\tilde{b} = (\tilde{b}_1, ..., \tilde{b}_m)$ is a vector of fuzzy numbers. The fuzzy numbers are the possibility distributions associated with the fuzzy variables and hence place a restriction on the possible values the variable may assume. For example, $\text{Pos}[\tilde{a}_{ij} = t] = \tilde{a}_{ij}(t)$. We will assume that all fuzzy numbers $\tilde{a}_{ij}$, $\tilde{b}_i$, $\tilde{c}_l$ are non-interactive. Following Buckley [10], we define

$\mathrm{Pos}[Z = z]$, the possibility distribution of the objective function Z. We first specify the possibility that x satisfies the i-th constraints. Let

$$\Pi(a_i, b_i) = \min\{\tilde{a}_{i1}(a_{i1}), \dots, \tilde{a}_{in}(a_{in}), \tilde{b}_i(b_i\},$$

where $a_i = (a_{i1}, \dots, a_{in})$, which is the joint distribution of $\tilde{a}_{ij}, j = 1, \dots, n$, and $\tilde{b}_i$. Then

$$\mathrm{Pos}[x \in \mathcal{F}_i] = \sup_{a_i, b_i}\{\Pi(a_i, b_i) \mid a_{i1}x_1 + \cdots + a_{in}x_n * b_i \},$$

which is the possibility that x is feasible with respect to the i-th constraint. Therefore, for $x \geq 0$,

$$\mathrm{Pos}[x \in \mathcal{F}] = \min\{\mathrm{Pos}[x \in \mathcal{F}_1], \dots, \mathrm{Pos}[x \in \mathcal{F}_m]\}.$$

which is the possibility that x is feasible. We next construct $\mathrm{Pos}[Z = z|x]$ which is the conditional possibility that Z equals z given x. The joint distribution of the $\tilde{c}_{lj},\ j = 1, \dots, n$, is

$$\Pi(c_l) = \min\{\tilde{c}_{l1}(c_{l1}), \dots, \tilde{c}_{ln}(c_{ln})\}$$

where $c_l = (c_{l1}, \dots, c_{ln}),\ l = 1, \dots, k$. Therefore,

$$\mathrm{Pos}[Z = z|x] = \mathrm{Pos}[\tilde{c}_1 x = z_1, \dots, \tilde{c}_k x = z_k] = \min_{1 \leq l \leq k} \mathrm{Pos}[\tilde{c}_l x = z_l] =$$

$$\min_{1 \leq l \leq k} \sup_{c_{l1}, \dots, c_{lk}} \{\Pi(c_l) \mid c_{l1}x_1 + \cdots + c_{ln}x_n = z_l\}.$$

Finally, the possibility distribution of the objective function is defined as

$$\mathrm{Pos}[Z = z] = \sup_{x \geq 0} \min\{\mathrm{Pos}[Z = z|x], \mathrm{Pos}[x \in \mathcal{F}]\}$$

We will assume that there is a collection of fuzzy parameters $\tilde{a}^{\delta}_{ij}$, $\tilde{b}^{\delta}_i$, $\tilde{c}^{\delta}_{lj}$ available with the property

$$\max_{i,j} D(\tilde{a}_{ij}, \tilde{a}^{\delta}_{ij}) \leq \delta, \quad \max_i D(\tilde{b}_i, \tilde{b}^{\delta}_i) \leq \delta, \quad \max_{l,j} D(\tilde{c}_{lj}, \tilde{c}^{\delta}_{lj}) \leq \delta. \tag{4.39}$$

Then we have to solve the following problem:

$$\text{max/min } Z^{\delta} = (\tilde{c}^{\delta}_1 x, \dots, \tilde{c}^{\delta}_k x) \tag{4.40}$$

$$\text{subject to} \quad \tilde{A}^{\delta} x * \tilde{b}^{\delta},\ x \geq 0.$$

Let us denote by $\mathrm{Pos}[x \in \mathcal{F}^{\delta}_i]$ the possibility that x is feasible with respect to the i-th constraint in (4.40). Then the possibility distribution of the objective function Z^{δ} in (4.40) is defined as:

$$\mathrm{Pos}[Z^{\delta} = z] = \sup_{x \geq 0}(\min\{\mathrm{Pos}[Z^{\delta} = z \mid x], \mathrm{Pos}[x \in \mathcal{F}^{\delta}]\}).$$

The next theorem shows a stability property (with respect to perturbations (4.39) of the possibility distribution of the objective function, Z, of multiobjective possibilistic linear programming problems (4.38) and (4.40).

Theorem 4.7 (Fullér and Fedrizzi, [82]). *Let $\delta \geq 0$ be a real number and let $\tilde{a}_{ij}$, $\tilde{b}_i$, $\tilde{a}^{\delta}_{ij}$, $\tilde{c}_{lj}$, $\tilde{c}^{\delta}_{lj}$ be (continuous) fuzzy numbers. If (4.39) hold, then*

$$\sup_{z \in \mathbb{R}^k} \mid \mathrm{Pos}[Z^{\delta} = z] - \mathrm{Pos}[Z = z] \mid \leq \omega(\delta)$$

where $\omega(\delta)$ is the maximum of moduli of continuity of all fuzzy numbers at δ.

From Theorem 4.7 it follows that

$$\sup_{z \in \mathbb{R}^k} \mid \mathrm{Pos}[Z^{\delta} = z] - \mathrm{Pos}[Z = z] \mid \rightarrow 0 \text{ as } \delta \rightarrow 0$$

which means the stability of the possibility distribution of the objective function with respect to perturbations (4.39). It is easy to see that in the case of non-continuous fuzzy parameters the possibility distribution of the objective function may be unstable under small changes of the parameters.

Example 4.1 (Fullér and Fedrizzi, [82]). *As an example, consider the following biobjective possibilistic linear program*

$$\textit{max/min}\ (\tilde{c}x, \tilde{c}x) \tag{4.41}$$

$$\textit{subject to} \quad \tilde{a}x \leq \tilde{b},\ x \geq 0.$$

where $\tilde{a} = (1, 1)$, $\tilde{b} = (2, 1)$ and $\tilde{c} = (3, 1)$ are fuzzy numbers of symmetric triangular form. Here x is one-dimensional $(n = 1)$ and there is only one constraint $(m = 1)$. We find

$$\mathrm{Pos}[x \in \mathcal{F}] = \begin{cases} 1 & \textit{if } x \leq 2, \\ \dfrac{3}{x+1} & \textit{if } x > 2. \end{cases}$$

and $\mathrm{Pos}[Z = (z_1, z_2)|x] = \min\{\mathrm{Pos}[\tilde{c}x = z_1], \mathrm{Pos}[\tilde{c}x = z_2]\}$*, where*

$$\mathrm{Pos}[\tilde{c}x = z_i] = \begin{cases} 4 - \dfrac{z_i}{x} & \textit{if } z_i/x \in [3, 4], \\ \dfrac{z_i}{x} - 2 & \textit{if } z_i/x \in [2, 3], \\ 0 & \textit{otherwise}, \end{cases}$$

for $i = 1, 2$ and $x \neq 0$, and

$$\mathrm{Pos}[Z = (z_1, z_2)|0] = \mathrm{Pos}[0 \times \tilde{c} = z] = \begin{cases} 1 & \textit{if } z = 0, \\ 0 & \textit{otherwise}. \end{cases}$$

Both possibilities are nonlinear functions of x, however the calculation of $\mathrm{Pos}[Z = (z_1, z_2)]$ *is easily performed and we obtain*

$$\mathrm{Pos}[Z = (z_1, z_2)] = \begin{cases} \theta_1 & \textit{if } z \in M_1, \\ \min\{\theta_1, \theta_2, \theta_3\} & \textit{if } z \in M_2, \\ 0 & \textit{otherwise}, \end{cases}$$

where

$$M_1 = \{z \in \mathbb{R}^2 \mid |z_1 - z_2| \le \min\{z_1, z_2\}, z_1 + z_2 \le 12\},$$
$$M_2 = \{z \in \mathbb{R}^2 \mid |z_1 - z_2| \le \min\{z_1, z_2\}, z_1 + z_2 > 12\},$$

and

$$\theta_i = \frac{24}{z_i + 7 + \sqrt{z_i^2 + 14z_i + 1}}.$$

for $i = 1, 2$ *and*

$$\theta_3 = \frac{4\min\{z_1, z_2\} - 2\max\{z_1, z_2\}}{z_1 + z_2}.$$

Consider now a perturbed biobjective problem with two different objectives (derived from (4.41) by a simple δ*-shifting of the centres of* $\tilde{a}$ *and* $\tilde{c}$*):*

$$\textit{max/min}\ (\tilde{c}x, \tilde{c}^\delta x) \tag{4.42}$$
$$\textit{subject to}\quad \tilde{a}^\delta x \le \tilde{b},\ x \ge 0.$$

where $\tilde{a} = (1 + \delta, 1)$*,* $\tilde{b} = (2, 1)$*,* $\tilde{c} = (3, 1)$*,* $\tilde{c}^\delta = (3 - \delta, 1)$ *and* $\delta \ge 0$ *is the error of measurement. Then*

$$\mathrm{Pos}[x \in \mathcal{F}^\delta] = \begin{cases} 1 & \textit{if } x \le \dfrac{2}{1+\delta}, \\ \dfrac{3 - \delta x}{x + 1} & \textit{if } x > \dfrac{2}{1+\delta}. \end{cases}$$

and

$$\mathrm{Pos}[Z^\delta = (z_1, z_2)|x] = \min\{\mathrm{Pos}[\tilde{c}x = z_1], \mathrm{Pos}[\tilde{c}^\delta x = z_2]\}$$

where

$$\mathrm{Pos}[\tilde{c}x = z_1] = \begin{cases} 4 - \dfrac{z_1}{x} & \textit{if } z_1/x \in [3, 4], \\ \dfrac{z_1}{x} - 2 & \textit{if } z_1 i/x \in [2, 3], \\ 0 & \textit{otherwise}, \end{cases}$$

$$\mathrm{Pos}[\tilde{c}^\delta x = z_2] = \begin{cases} 4 - \delta - \dfrac{z_2}{x} & \textit{if } z_2/x \in [3 - \delta, 4 - \delta], \\ \dfrac{z_2}{x} - 2 + \delta & \textit{if } z_2/x \in [2 - \delta, 3 - \delta], \\ 0 & \textit{otherwise}, \end{cases}$$

$x \ne 0$*, and*

$$\mathrm{Pos}[Z^\delta = (z_1, z_2)|0] = \mathrm{Pos}[0 \times \tilde{c} = z] = \begin{cases} 1 & \textit{if } z = 0, \\ 0 & \textit{otherwise}. \end{cases}$$

So,

$$\mathrm{Pos}[Z^\delta = (z_1, z_2)] = \begin{cases} \theta_1(\delta) & \textit{if } z \in M_1(\delta), \\ \min\{\theta_1(\delta), \theta_2(\delta), \theta_3(\delta)\} & \textit{if } z \in M_2(\delta), \\ 0 & \textit{otherwise}, \end{cases}$$

where

$$M_1(\delta) = \left\{ z \in \mathbb{R}^2 \mid |z_1 - z_2| \le (1 - 0.5\delta)\min\{z_1, z_2\}, z_1 + z_2 \le \frac{2(6-\delta)}{1+\delta} \right\},$$

$$M_2(\delta) = \left\{ z \in \mathbb{R}^2 \mid |z_1 - z_2| \le (1 - 0.5\delta)\min\{z_1, z_2\}, z_1 + z_2 > \frac{2(6-\delta)}{1+\delta} \right\},$$

$$\theta_1(\delta) = \frac{24 + \delta\left(7 - z_1 - \sqrt{z_1^2 + 14z_1 + 1 + 4z_1\delta}\right)}{z_1 + 7 + \sqrt{z_1^2 + 14z_1 + 1 + 4z_1\delta} + 2\delta}$$

$$\theta_2(\delta) = \frac{24 - \delta\left(\delta + z_2 - 1 + \sqrt{(1-\delta-z_2)^2 + 16z_2}\right)}{z_2 + 7 + \sqrt{(1-\delta-z_2)^2 + 16z_2} + \delta}$$

and

$$\theta_3(\delta) = \frac{(4-\delta)\min\{z_1, z_2\} - 2\max\{z_1, z_2\}}{z_1 + z_2}.$$

It is easy to check that

$$\sup_{x \ge 0} |\mathrm{Pos}[x \in \mathcal{F}] - \mathrm{Pos}[x \in \mathcal{F}^\delta]| \le \delta,$$

$$\sup_{z} |\mathrm{Pos}[Z = z|x] - \mathrm{Pos}[Z^\delta = z|x]| \le \delta, \ \forall x \ge 0,$$

$$\sup_{z} |\mathrm{Pos}[Z = z] - \mathrm{Pos}[Z^\delta = z]| \le \delta.$$

On the other hand, from the definition of metric D the modulus of continuity and Theorem 4.7 it follows that

$$D(\tilde{a}, \tilde{a}^\delta) = \delta, D(\tilde{c}, \tilde{c}^\delta) = \delta, D(\tilde{c}, \tilde{c}) = 0, D(\tilde{b}, \tilde{b}) = 0, \omega(\delta) = \delta,$$

and, therefore $\sup_z |\mathrm{Pos}[Z = z] - \mathrm{Pos}[Z^\delta = z]| \le \delta$.

4.6 Stability in fuzzy inference systems

In this Section following Fullér and Zimmermann [81], and Fullér and Werners [80] we show two very important features of the compositional rule of inference under triangular norms. Namely, we prove that (i) if the t-norm defining the composition and the membership function of the observation are continuous, then the conclusion depends continuously on the observation; (ii) if the t-norm and the membership function of the relation are continuous, then the observation has a continuous membership function. We consider the compositional rule of inference with different observations P and P',

Observation:	X has property P		Observation:	X has property P'
Relation:	X and Y are in relation R		Relation m:	X and Y are in relation R
Conclusion:	Y has property Q		Conclusion:	Y has property Q'

According to Zadeh's compositional rule of inference, Q and Q' are computed as $Q = P \circ R$ and $Q' = P' \circ R$ i.e.,

$$\mu_Q(y) = \sup_{x \in \mathbb{R}} T(\mu_P(x), \mu_R(x,y)), \quad \mu_{Q'}(y) = \sup_{x \in \mathbb{R}} T(\mu_{P'}(x), \mu_R(x,y)).$$

The following theorem shows that when the observations are close to each other in the metric D, then there can be only a small deviation in the membership functions of the conclusions.

Theorem 4.8 (Fullér and Zimmermann, [81]). *Let $\delta \geq 0$ and T be a continuous triangular norm, and let P, P' be fuzzy intervals. If $D(P, P') \leq \delta$ then*

$$\sup_{y \in \mathbb{R}} |\mu_Q(y) - \mu_{Q'}(y)| \leq \omega_T(\max\{\omega_P(\delta), \omega_{P'}(\delta)\}).$$

where $\omega_P(\delta)$ and $\omega_{P'}(\delta)$ denotes the modulus of continuity of P and P' at δ.

It should be noted that the stability property of the conclusion Q with respect to small changes in the membership function of the observation P in the compositional rule of inference scheme is independent from the relation R (it's membership function can be discontinuous). Since the membership function of the conclusion in the compositional rule of inference can have unbounded support, it is possible that the maximal distance between the α-level sets of Q and Q' is infinite, but their membership grades are arbitrarily close to each other. The following theorem establishes the continuity property of the conclusion in the compositional rule of inference scheme.

Theorem 4.9 (Fullér and Zimmermann, [81]). *Let R be continuous fuzzy relation, and let T be a continuous t-norm. Then Q is continuous and $\omega_Q(\delta) \leq \omega_T(\omega_R(\delta))$, for each $\delta \geq 0$.*

From Theorem 4.9 it follows that the continuity property of the membership function of the conclusion Q in the compositional rule of inference scheme is independent from the observation P (it's membership function can be discontinuous).

Theorems 4.8 and 4.9 can be easily extended to the compositional rule of inference with several relations:

Observation:	X has property P	Observation:	X has property P'
Relation 1:	X and Y are in relation W_1	Relation 1:	X and Y are in relation W_1
...		...	
Relation m:	X and Y are in relation W_m	Relation m:	X and Y are in relation W_m
Conclusion:	Y has property Q	Conclusion:	Y has property Q'.

According to Zadeh's compositional rule of inference, Q and Q' are computed by sup-T composition as follows

$$Q = \bigcap_{i=1}^{m} P \circ W_i \quad \text{and} \quad Q' = \bigcap_{i=1}^{m} P' \circ W_i. \tag{4.43}$$

Generalizing Theorems 4.8 and 4.9 about the case of single relation, we show that when the observations are close to each other in the metric D, then there can be only a small deviation in the membership function of the conclusions even if we have several relations.

Theorem 4.10 (Fullér and Werners, [80]). *Let $\delta \geq 0$ and T be a continuous triangular norm, and let P, P' be continuous fuzzy intervals. If $D(P, P') \leq \delta$ then*

$$\sup_{y \in \mathbb{R}} |\mu_Q(y) - \mu_{Q'}(y)| \leq \omega_T(\max\{\omega_P(\delta), \omega_{P'}(\delta)\})$$

where Q and Q' are computed by (4.43).

In the following theorem we establish the continuity property of the conclusion under continuous fuzzy relations W_i and continuous t-norm T.

Theorem 4.11 (Fullér and Werners, [80]). *Let W_i be continuous fuzzy relation, i=1,...,m and let T be a continuous t-norm. Then Q is continuous and $\omega_Q(\delta) \leq \omega_T(\omega(\delta))$ for each $\delta \geq 0$ where $\omega(\delta) = \max\{\omega_{W_1}(\delta), \dots, \omega_{W_m}(\delta)\}$.*

The above theorems are also valid for Multiple Fuzzy Reasoning (MFR) schemes:

$$\begin{array}{lll}
\text{Observation:} & P & P' \\
\text{Implication 1:} & P_1 \to Q_1 & P'_1 \to Q'_1 \\
 & \dots & \dots \\
\text{Implication } m\text{:} & P_m \to Q_m & P'_m \to Q'_m \\
\hline
\text{Conclusion:} & Q & Q'
\end{array}$$

where Q and Q' are computed by sup-T composition as follows

$$Q = P \circ \bigcap_{i=1}^{m} P_i \to Q_i, \quad Q' = P' \circ \bigcap_{i=1}^{m} P'_i \to Q'_i,$$

i.e.,

$$\mu_Q(y) = \sup_{x \in \mathbb{R}} T(\mu_P(x), \min_{i=1,\dots,m} \mu_{P_i}(x) \to \mu_{Q_i}(y)),$$

$$\mu_{Q'}(y) = \sup_{x \in \mathbb{R}} T(\mu_{P'}(x), \min_{i=1,\dots,m} \mu_{P'_i(x)} \to \mu_{Q'_i}(y)).$$

Then the following theorems hold.

Theorem 4.12 (Fullér and Werners, [80]). *Let $\delta \geq 0$, let T be a continuous triangular norm, let P, P', P_i, P'_i, Q_i, Q'_i, $i = 1, \dots, m$, be fuzzy intervals and let $\to$ be a continuous fuzzy implication operator. If*

$$\max\{D(P, P'), \max_{i=1,\dots,m} D(P_i, P'_i), \max_{i=1,\dots,m} D(Q_i, Q'_i)\} \leq \delta,$$

then

$$\sup_{y \in \mathbb{R}} |\mu_Q(y) - \mu_{Q'}(y)| \leq \omega_T(\max\{\omega(\delta), \omega_{\to}(\omega(\delta))\}),$$

where $\omega(\delta) = max\{\omega_{P_i}(\delta), \omega_{P'_i}(\delta), \omega_{Q_i}(\delta), \omega_{Q'_i}(\delta)\}$, and $\omega_{\to}$ denotes the modulus of continuity of the fuzzy implication operator.

Theorem 4.13 (Fullér and Werners, [80]). *Let $\to$ be a continuous fuzzy implication operator, let P, P',P_i, P'_i, Q_i, Q'_i, $i = 1, \dots, m$, be fuzzy intervals and let T be a continuous t-norm. Then Q is continuous and*

$$\omega_Q(\delta) \leq \omega_T(\omega_{\to}(\omega(\delta)) \quad \textit{for each } \delta \geq 0,$$

where $\omega(\delta) = \max\{\omega_{P_i}(\delta), \omega_{P'_i}(\delta), \omega_{Q_i}(\delta), \omega_{Q'_i}(\delta)\}$ and $\omega_{\to}$ denotes the modulus of continuity of the fuzzy implication operator.

From $\lim_{\delta\to 0}\omega(\delta) = 0$ and Theorem 4.12 it follows that

$$\|\mu_Q - \mu_{Q'}\|_\infty = \sup_y |\mu_Q(y) - \mu_{Q'}(y)| \to 0$$

whenever $D(P, P') \to 0,\ D(P_i, P_i') \to 0$ and $D(Q_i, Q_i') \to 0,\ i = 1, \dots, m$, which means the stability of the conclusion under small changes of the observation and rules.

The stability property of the conclusion under small changes of the membership function of the observation and rules guarantees that small rounding errors of digital computation and small errors of measurement of the input data can cause only a small deviation in the conclusion, i.e. every successive approximation method can be applied to the computation of the linguistic approximation of the exact conclusion.

These stability properties in fuzzy inference systems were used by a research team - headed by Professor Hans-Jürgen Zimmermann - when developing a fuzzy control system for a ”fuzzy controlled model car” [5] during my DAAD Scholarship at RWTH Aachen between 1990 and 1992.

Chapter 5

A Normative View on Possibility Distributions

In possibility theory we can use the principle of expected value of functions on fuzzy sets to define variance, covariance and correlation of possibility distributions. Marginal probability distributions are determined from the joint one by the principle of 'falling integrals' and marginal possibility distributions are determined from the joint possibility distribution by the principle of 'falling shadows'. Probability distributions can be interpreted as carriers of *incomplete information* [106], and possibility distributions can be interpreted as carriers of *imprecise information*. A function $f\colon [0,1] \to \mathbb{R}$ is said to be a weighting function if f is non-negative, monotone increasing and satisfies the following normalization condition $\int_0^1 f(\gamma)d\gamma = 1$. Different weighting functions can give different (case-dependent) importances to level-sets of possibility distributions. In Chapter "A Normative View on Possibility Distributions" we will discuss the weighted lower possibilistic and upper possibilistic mean values, crisp possibilistic mean value and variance of fuzzy numbers, which are consistent with the extension principle. We can define the mean value (variance) of a possibility distribution as the f-weighted average of the probabilistic mean values (variances) of the respective uniform distributions defined on the γ-level sets of that possibility distribution. A measure of possibilistic covariance (correlation) between marginal possibility distributions of a joint possibility distribution can be defined as the f-weighted average of probabilistic covariances (correlations) between marginal probability distributions whose joint probability distribution is defined to be uniform on the γ-level sets of their joint possibility distribution [88]. We should note here that the choice of uniform probability distribution on the level sets of possibility distributions is not without reason. Namely, these possibility distributions are used to represent imprecise human judgments and they carry non-statistical uncertainties. Therefore we will suppose that each point of a given level set is equally possible. Then we apply Laplace's principle of Insufficient Reason: if elementary events are equally possible, they should be equally probable (for more details and generalization of principle of Insufficient Reason see [71], page 59). The main new idea here is to equip the alpha-cuts of joint possibility distributions with uniform probability distributions and to derive possibilistic mean value, variance, covariance and correlation of possibility distributions, in such a way that they would be consistent with the extension principle. The idea of equipping the alpha-cuts of fuzzy numbers with a uniform probability refers to early ideas of simulation of fuzzy sets by Yager [143], and possibility/probability transforms by Dubois et al [70] as well as the pignistic transform of Smets [132]. In this Chapter, following Carlsson and Fullér [26] Carlsson, Fullér and Majlender [45], Fullér and Majlender [88] and Fullér, Mezei and Várlaki [96], we will introduce the concepts of possibilistic mean value, variance, covariance and correlation. 941 independent citations show that the scientific community has accepted these principles.

5.1 Possibilistic mean value, variance, covariance and correlation

Fuzzy numbers can be considered as possibility distributions [153, 155]. Possibility distributions are used to represent imprecise human judgments and therefore they carry non-statistical uncertainties. If $A \in \mathcal{F}$ is a fuzzy number and $x \in \mathbb{R}$ a real number then $A(x)$ can be interpreted as the degree of possiblity of the statement "x is A". Let $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$ with $a \leq b$, then the degree of possibility that $A \in \mathcal{F}$ takes its value from interval $[a, b]$ is defined by [155]

$$\mathrm{Pos}(A \in [a, b]) = \max_{x \in [a,b]} A(x).$$

We should note here that if $[a, b]$ and $[c, d]$ are two disjoint intervals such that they both belong to the support of fuzzy number A then

$$\mathrm{Pos}(A \in [a, b] \cup [c, d]) < \mathrm{Pos}(A \in [a, b]) + \mathrm{Pos}(A \in [c, d]).$$

since

$$\max_{x \in [a,b] \cup [c,d]} A(x) < \max_{x \in [a,b]} A(x) + \max_{x \in [c,d]} A(x)$$

That is, Pos is a sub-additive set function and there is no way that it can be considered as a (probability) measure. The degree of necessity that $A \in \mathcal{F}$ takes its value from $[a, b]$ is defined by $\mathrm{Nec}(A \in [a, b]) = 1 - \mathrm{Pos}(A \notin [a, b])$.

Definition 5.1. *Let $n \geq 2$ an integer. A fuzzy set C in $\mathbb{R}^n$ is said to be a joint possibility distribution of fuzzy numbers $A_1, \dots, A_n$ if its projection on the i-th axis is A_i, that is,*

$$A_i(x_i) = \max_{x_j \in \mathbb{R},\ j \neq i} C(x_1, \dots, x_n),\ \forall x_i \in \mathbb{R}, i = 1, \dots, n. \tag{5.1}$$

Then A_i is called the i-th marginal possibility distribution of C.

For example, if $n = 2$ then C is a joint possibility distribution of fuzzy numbers $A, B \in \mathcal{F}$ if

$$A(x) = \max_{y \in \mathbb{R}} C(x, y),\ \forall x \in \mathbb{R}, \quad B(y) = \max_{x \in \mathbb{R}} C(x, y),\ \forall y \in \mathbb{R}.$$

We should note here that there exists a large family of joint possibility distributions that can not be defined directly from the membership values of its marginal possibility distributions by any aggregation operator. On the other hand, if A and B are fuzzy numbers and T is a t-norm, then

$$C(x, y) = T(A(x), B(y))$$

always defines a joint possibility distribution with marginal possibility distributions A and B.

Definition 5.2. *Fuzzy numbers $A_i \in \mathcal{F}$, $i = 1, \dots, n$ are said to be non-interactive if their joint possibility distribution C satisfies the relationship*

$$C(x_1, \dots, x_n) = \min\{A_1(x_1), \dots, A_n(x_n)\},$$

or, equivalently,

$$[C]^\gamma = [A_1]^\gamma \times \cdots \times [A_n]^\gamma$$

hold for all $x_1, \dots, x_n \in \mathbb{R}$ and $\gamma \in [0, 1]$.

If $A_i \in \mathcal{F}$, $i = 1, \dots, n$ and C is their joint possibility distribution then the relationships

$$C(x_1, \dots, x_n) \leq \min\{A_1(x_1), \dots, A_n(x_n)\},$$

or, equivalently,

$$[C]^\gamma \subseteq [A_1]^\gamma \times \cdots \times [A_n]^\gamma$$

hold for all $x_1, \dots, x_n \in \mathbb{R}$ and $\gamma \in [0, 1]$.

If $A, B \in \mathcal{F}$ are non-interactive then their joint membership function is defined by $A \times B$, where $(A \times B)(x, y) = \min\{A(x), B(y)\}$ for any $x, y \in \mathbb{R}$. It is clear that in this case any α-level set of their joint possibility distribution is a rectangular. On the other hand, A and B are said to be interactive if they can not take their values independently of each other [69].

The possibilistic mean (or expected value), variance, covariance and correlation were originally defined from the measure of possibilistic interactivity (as shown in [45, 88]) but for simplicity, we will present the concept of possibilistic mean value, variance, covariance and possibilistic correlation in a probabilistic setting and point out the fundamental difference between the standard probabilistic approach and the possibilistic one. Let $A \in \mathcal{F}$ be fuzzy number with $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$ and let U_γ denote a uniform probability distribution on $[A]^\gamma$, $\gamma \in [0, 1]$. Recall that the probabilistic mean value of U_γ is equal to

$$M(U_\gamma) = \frac{a_1(\gamma) + a_2(\gamma)}{2},$$

and its probabilistic variance is computed by

$$\operatorname{var}(U_\gamma) = \frac{(a_2(\gamma) - a_1(\gamma))^2}{12}.$$

In 2001 Carlsson and Fullér [26] defined the *possibilistic mean (or expected) value* of fuzzy number A as

$$E(A) = \int_0^1 M(U_\gamma) 2\gamma \, \mathrm{d}\gamma = \int_0^1 \frac{a_1(\gamma) + a_2(\gamma)}{2} 2\gamma \, \mathrm{d}\gamma = \int_0^1 (a_1(\gamma) + a_2(\gamma))\gamma \, \mathrm{d}\gamma,$$

where U_γ is a uniform probability distribution on $[A]^\gamma$ for all $\gamma \in [0, 1]$.

In [26] we named $E(A)$ as the ”possibilistic mean value” of A since it can be defined by using possibilities. Really, following [26] we can rewrite $E(A)$ as

$$E(A) = \int_0^1 \gamma(a_1(\gamma) + a_2(\gamma)) d\gamma = \frac{2 \cdot \displaystyle\int_0^1 \gamma a_1(\gamma) d\gamma + 2 \cdot \displaystyle\int_0^1 \gamma a_2(\gamma) d\gamma}{2}$$

$$= \frac{1}{2}\left(\frac{\displaystyle\int_0^1 \gamma a_1(\gamma) d\gamma}{\dfrac{1}{2}} + \frac{\displaystyle\int_0^1 \gamma a_2(\gamma) d\gamma}{\dfrac{1}{2}} \right) = \frac{1}{2}\left(\frac{\displaystyle\int_0^1 \gamma a_1(\gamma) d\gamma}{\displaystyle\int_0^1 \gamma d\gamma} + \frac{\displaystyle\int_0^1 \gamma a_2(\gamma) d\gamma}{\displaystyle\int_0^1 \gamma d\gamma} \right).$$

Let us take a closer look at the right-hand side of the equation for $E(A)$. The first quantity, denoted by $E_*(A)$ can be reformulated as

$$E_*(A) = 2\int_0^1 \gamma a_1(\gamma) d\gamma = \frac{\displaystyle\int_0^1 \gamma a_1(\gamma) d\gamma}{\displaystyle\int_0^1 \gamma d\gamma}$$

$$= \frac{\displaystyle\int_0^1 \mathrm{Pos}[A \leq a_1(\gamma)] a_1(\gamma) d\gamma}{\displaystyle\int_0^1 \mathrm{Pos}[A \leq a_1(\gamma)] d\gamma} = \frac{\displaystyle\int_0^1 \mathrm{Pos}[A \leq a_1(\gamma)] \times \min[A]^\gamma d\gamma}{\displaystyle\int_0^1 \mathrm{Pos}[A \leq a_1(\gamma)] d\gamma},$$

where Pos denotes possibility, i.e.

$$\mathrm{Pos}[A \leq a_1(\gamma)] = \sup_{u \leq a_1(\gamma)} A(u) = \gamma.$$

since A is upper-semicontinuous. So $E_*(A)$ is nothing else but the lower possibility-weighted average of the minima of the γ-sets, and it is why we call it the lower possibilistic mean value of A. In a similar manner we introduce $E^*(A)$, the upper possibilistic mean value of A, as

$$E^*(A) = 2\int_0^1 \gamma a_2(\gamma) d\gamma = \frac{\displaystyle\int_0^1 \gamma a_2(\gamma) d\gamma}{\displaystyle\int_0^1 \gamma d\gamma}$$

$$= \frac{\displaystyle\int_0^1 \mathrm{Pos}[A \geq a_2(\gamma)] a_2(\gamma) d\gamma}{\displaystyle\int_0^1 \mathrm{Pos}[A \geq a_2(\gamma)] d\gamma} = \frac{\displaystyle\int_0^1 \mathrm{Pos}[A \geq a_2(\gamma)] \times \max[A]^\gamma d\gamma}{\displaystyle\int_0^1 \mathrm{Pos}[A \leq a_2(\gamma)] d\gamma},$$

where we have used the equality

$$\mathrm{Pos}[A \geq a_2(\gamma)] = \sup_{u \geq a_2(\gamma)} A(u) = \gamma.$$

In [26] we introduced the *crisp possibilistic mean value* of A as the arithemetic mean of its lower possibilistic and upper possibilistic mean values, i.e.

$$\bar{E}(A) = \frac{E_*(A) + E^*(A)}{2}.$$

In 1986 Goetschel and Voxman [97] introduced a method for ranking fuzzy numbers $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$ and $[B]^\gamma = [b_1(\gamma), b_2(\gamma)]$ as

$$A \leq B \iff \int_0^1 \gamma(a_1(\gamma) + a_2(\gamma))\, d\gamma \leq \int_0^1 \gamma(b_1(\gamma) + b_2(\gamma))\, d\gamma$$

As was pointed out by Goetschel and Voxman this definition of ordering was motivated in part by the desire to give less importance to the lower levels of fuzzy numbers. In this terminology, the ordering by Goetschel and Voxman can be written as $A \leq B \iff E(A) \leq E(B)$. We note further that from the equality

$$E(A) = \int_0^1 \gamma(a_1(\gamma) + a_2(\gamma))d\gamma = \frac{\displaystyle\int_0^1 2\gamma \cdot \frac{a_1(\gamma) + a_2(\gamma)}{2}\, d\gamma}{\displaystyle\int_0^1 2\gamma\, d\gamma},$$

it follows that $E(A)$ is nothing else but the level-weighted average of the arithmetic means of all γ-level sets, that is, the weight of the arithmetic mean of $a_1(\gamma)$ and $a_2(\gamma)$ is just 2γ.

Example 5.1. *If $A = (a, \alpha, \beta)$ is a triangular fuzzy number with center a, left-width $\alpha > 0$ and right-width $\beta > 0$ then a γ-level of A is computed by*

$$[A]^\gamma = [a - (1-\gamma)\alpha, a + (1-\gamma)\beta],\ \forall \gamma \in [0,1],$$

Then,

$$E(A) = \int_0^1 \gamma[a - (1-\gamma)\alpha + a + (1-\gamma)\beta]d\gamma = a + \frac{\beta - \alpha}{6}.$$

When $A = (a, \alpha)$ is a symmetric triangular fuzzy number we get $E(A) = a$.

Let $A \in \mathcal{F}$ be fuzzy number with $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$, $\gamma \in [0,1]$. A function $f\colon [0,1] \to \mathbb{R}$ is said to be a weighting function if f is non-negative, monoton increasing and satisfies the following normalization condition

$$\int_0^1 f(\gamma)d\gamma = 1. \tag{5.2}$$

Definition 5.3 (Fullér and Majlender, [85])**.** *We define the f-weighted possibilistic mean (or expected) value of fuzzy number A as*

$$E_f(A) = \int_0^1 \frac{a_1(\gamma) + a_2(\gamma)}{2} f(\gamma)d\gamma. \tag{5.3}$$

It should be noted that if $f(\gamma) = 2\gamma$, $\gamma \in [0,1]$ then

$$E_f(A) = \int_0^1 \frac{a_1(\gamma) + a_2(\gamma)}{2} 2\gamma d\gamma = \int_0^1 [a_1(\gamma) + a_2(\gamma)]\, \gamma d\gamma = E(A).$$

That is the f-weighted possibilistic mean value defined by (5.3) can be considered as a generalization of possibilistic mean value introduced earlier by Carlsson and Fullér [26]. From the definition of a weighting function it can be seen that $f(\gamma)$ might be zero for certain (unimportant) γ-level sets of A. So by introducing different weighting functions we can give different (case-dependent) importances to γ-levels sets of fuzzy numbers. Let us introduce a family of weighting function (which stands for the principle "all level sets are equally important") defined by

$$\mathrm{one}(\gamma) = \begin{cases} 1 & \text{if } \gamma \in (0,1] \\ a & \text{if } \gamma = 0 \end{cases}$$

where $a \in [0,1]$ is an arbitrary real number. Then,

$$E_{\mathrm{one}}(A) = \int_0^1 \frac{a_1(\gamma) + a_2(\gamma)}{2} \times \mathrm{one}(\gamma)d\gamma = \int_0^1 \frac{a_1(\gamma) + a_2(\gamma)}{2} d\gamma. \tag{5.4}$$

Definition 5.4 (Fullér and Majlender, [85]). *Let f be a weighting function and let $A \in \mathcal{F}$ be fuzzy number with $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$, $\gamma \in [0,1]$. Then we define the f-weighted interval-valued possibilistic mean of A as*

$$M_f(A) = [M_f^-(A), M_f^+(A)],$$

where

$$M_f^-(A) = \int_0^1 a_1(\gamma) f(\gamma) d\gamma, \quad M_f^+(A) = \int_0^1 a_2(\gamma) f(\gamma) d\gamma.$$

The following two theorems can directly be proved using the definition of f-weighted interval-valued possibilistic mean.

Theorem 5.1 (Fullér and Majlender, [85]). *Let $A, B \in \mathcal{F}$ two non-interactive fuzzy numbers and let f be a weighting function, and let λ be a real number. Then*

$$M_f(A + B) = M_f(A) + M_f(B), \quad M_f(\lambda A) = \lambda M_f(A),$$

where the non-interactive sum of fuzzy numbers A and B is defined by the sup-min extension principle 2.6.

Note 4. *The f-weighted possibilistic mean of A, defined by (5.3), is the arithmetic mean of its f-weighted lower and upper possibilistic mean values, i.e.*

$$E_f(A) = \frac{M_f{}^-(A) + M_f^+(A)}{2}. \tag{5.5}$$

Theorem 5.2 (Fullér and Majlender, [85]). *Let A and B be two non-interactive fuzzy numbers, and let $\lambda \in \mathbb{R}$. Then we have*

$$E_f(A + B) = E_f(A) + E_f(B), \quad E_f(\lambda A) = \lambda E_f(A),$$

where the non-interactive sum of fuzzy numbers A and B is defined by the sup-min extension principle 2.6

We will show an important relationship between the interval-valued probabilistic mean $D(A) = [D_*(A), D^*(A)]$ introduced by Dubois and Prade in [68] and the f-weighted interval-valued possibilistic mean $M_f(A) = [M_f^-(A), M_f^+(A)]$ for any fuzzy number with strictly decreasing shape functions.

An LR-type fuzzy number A can be described with the following membership function:

$$A(u) = \begin{cases} L\left(\dfrac{q_- - u}{\alpha}\right) & \text{if } q_- - \alpha \le u \le q_- \\ 1 & \text{if } u \in [q_-, q_+] \\ R\left(\dfrac{u - q_+}{\beta}\right) & \text{if } q_+ \le u \le q_+ + \beta \\ 0 & \text{otherwise} \end{cases}$$

where $[q_-, q_+]$ is the peak of fuzzy number A; q_- and q_+ are the lower and upper modal values; $L, R\colon [0,1] \to [0,1]$ with $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$ are non-increasing, continuous functions. We will use the notation $A = (q_-, q_+, \alpha, \beta)_{LR}$. Hence, the closure of the support of A is exactly $[q_- - \alpha, q_+ + \beta]$. If L and R are strictly decreasing functions then the γ-level sets of A can easily be computed as

$$[A]^\gamma = [q_- - \alpha L^{-1}(\gamma), q_+ + \beta R^{-1}(\gamma)], \gamma \in [0,1].$$

The *lower and upper probability mean values of the fuzzy number A are computed by* Dubois and Prade [68] as

$$D_*(A) = q_- - \alpha \int_0^1 L(u)du, \quad D^*(A) = q_+ + \beta \int_0^1 R(u)du. \tag{5.6}$$

and we will use the notation

$$\bar{D}(A) = \frac{D_*(A) + D^*(A)}{2}.$$

The f-weighted lower and upper possibilistic mean values are computed by

$$\begin{aligned} M_f^-(A) &= \int_0^1 \big(q_- - \alpha L^{-1}(\gamma)\big) f(\gamma)d\gamma = \int_0^1 q_- f(\gamma)d\gamma - \int_0^1 \alpha L^{-1}(\gamma)f(\gamma)d\gamma \\ &= q_- - \alpha \int_0^1 L^{-1}(\gamma)f(\gamma)d\gamma, \\ M_f^+(A) &= \int_0^1 \big(q_+ + \beta R^{-1}(\gamma)\big) f(\gamma)d\gamma = \int_0^1 q_+ f(\gamma)d\gamma + \int_0^1 \beta R^{-1}(\gamma)f(\gamma)d\gamma \\ &= q_+ + \beta \int_0^1 R^{-1}(\gamma)f(\gamma)d\gamma. \end{aligned} \tag{5.7}$$

We can state the following theorem.

Theorem 5.3 (Fullér and Majlender, [85]). *Let f be a weighting function and let A be a fuzzy number of type LR with strictly decreasing and continuous shape functions. Then, the f-weighted interval-valued possibilistic mean value of A is a subset of the interval-valued probabilistic mean value, i.e. $M_f(A) \subseteq D(A)$.*

Example 5.2. *Let $f(\gamma) = (n+1)\gamma^n$ and let $A = (a, \alpha, \beta)$ be a triangular fuzzy number with center a, left-width $\alpha > 0$ and right-width $\beta > 0$ then a γ-level of A is computed by*

$$[A]^\gamma = [a - (1-\gamma)\alpha, a + (1-\gamma)\beta], \ \forall \gamma \in [0,1].$$

Then the power-weighted lower and upper possibilistic mean values of A are computed by

$$\begin{aligned} M_f^-(A) &= \int_0^1 [a - (1-\gamma)\alpha](n+1)\gamma^n d\gamma \\ &= a(n+1)\int_0^1 \gamma^n d\gamma - \alpha(n+1)\int_0^1 (1-\gamma)\gamma^n d\gamma = a - \frac{\alpha}{n+2}, \end{aligned}$$

and,

$$\begin{aligned} M_f^+(A) &= \int_0^1 [a + (1-\gamma)\beta](n+1)\gamma^n d\gamma \\ &= a(n+1)\int_0^1 \gamma^n d\gamma + \beta(n+1)\int_0^1 (1-\gamma)\gamma^n d\gamma = a + \frac{\beta}{n+2}, \end{aligned}$$

and therefore,

$$M_f(A) = \left[a - \frac{\alpha}{n+2}, a + \frac{\beta}{n+2}\right].$$

That is,

$$E_f(A) = \frac{1}{2}\left(a - \frac{\alpha}{n+2} + a + \frac{\beta}{n+2}\right) = a + \frac{\beta - \alpha}{2(n+2)}.$$

So,

$$\lim_{n\to\infty} E_f(A) = \lim_{n\to\infty}\left(a + \frac{\beta-\alpha}{2(n+2)}\right) = a.$$

Example 5.3. *Let $A = (a, b, \alpha, \beta)$ be a fuzzy number of trapezoidal form with peak $[a, b]$, left-width $\alpha > 0$ and right-width $\beta > 0$, and let $f(\gamma) = (n+1)\gamma^n$, $n \geq 0$. A γ-level of A is computed by*

$$[A]^\gamma = [a - (1-\gamma)\alpha, b + (1-\gamma)\beta],\ \forall \gamma \in [0,1],$$

then the power-weighted lower and upper possibilistic mean values of A are computed by

$$\begin{aligned} M_f^-(A) &= \int_0^1 [a - (1-\gamma)\alpha](n+1)\gamma^n d\gamma \\ &= a(n+1)\int_0^1 \gamma^n d\gamma - \alpha(n+1)\int_0^1 (1-\gamma)\gamma^n d\gamma = a - \frac{\alpha}{n+2}, \end{aligned}$$

and,

$$\begin{aligned} M_f^+(A) &= \int_0^1 [b + (1-\gamma)\beta](n+1)\gamma^n d\gamma \\ &= b(n+1)\int_0^1 \gamma^n d\gamma + \beta(n+1)\int_0^1 (1-\gamma)\gamma^n d\gamma = b + \frac{\beta}{n+2}, \end{aligned}$$

and therefore,

$$M_f(A) = \left[a - \frac{\alpha}{n+2}, b + \frac{\beta}{n+2}\right]$$

That is,

$$E_f(A) = \frac{1}{2}\left(a - \frac{\alpha}{n+2} + b + \frac{\beta}{n+2}\right) = \frac{a+b}{2} + \frac{\beta-\alpha}{2(n+2)}.$$

So,

$$\lim_{n\to\infty} E_f(A) = \lim_{n\to\infty}\left(\frac{a+b}{2} + \frac{\beta-\alpha}{2(n+2)}\right) = \frac{a+b}{2}.$$

Example 5.4. *Let $f(\gamma) = (n+1)\gamma^n$, $n \geq 0$ and let $A = (a, \alpha, \beta)$ be a triangular fuzzy number with center a, left-width $\alpha > 0$ and right-width $\beta > 0$ then*

$$M_f(A) = \left[a - \frac{\alpha}{n+2}, a + \frac{\beta}{n+2}\right] \subset D(A) = \left[a - \frac{\alpha}{2}, a + \frac{\beta}{2}\right]$$

and for $n > 0$ we have

$$E_f(A) = a + \frac{\beta-\alpha}{2(n+2)} \neq \bar{D}(A) = a + \frac{\beta-\alpha}{4}.$$

Note 5. *When A is a symmetric fuzzy number then the equation $E_f(A) = \bar{D}(A)$ holds for any weighting function f. In the limit case, when $A = (a, b, 0, 0)$ is the characteristic function of interval $[a, b]$, the f-weighted possibilistic and probabilistic interval-valued means are equal, $D(A) = M_f(A) = [a, b]$.*

Definition 5.5 (Fullér and Majlender, [88])**.** *The f-weighted* possibilistic variance *of $A \in \mathcal{F}$ can be written as*

$$\mathrm{Var}_f(A) = \int_0^1 \mathrm{var}(U_\gamma) f(\gamma) d\gamma = \int_0^1 \frac{(a_2(\gamma) - a_1(\gamma))^2}{12} f(\gamma) d\gamma.$$

where U_γ is a uniform probability distribution on $[A]^\gamma$ and $\mathrm{var}(U_\gamma)$ denotes the variance of U_γ.

If $f(\gamma) = 2\gamma$ then the f-weighted *possibilistic variance* is said to be a *possibilistic variance* of A, denoted by $\mathrm{Var}(A)$, and is defined by

$$\mathrm{Var}(A) = \int_0^1 \mathrm{var}(U_\gamma) 2\gamma \,\mathrm{d}\gamma = \frac{1}{6}\int_0^1 (a_2(\gamma) - a_1(\gamma))^2 \gamma \,\mathrm{d}\gamma,$$

where U_γ is a uniform probability distribution on $[A]^\gamma$ and $\mathrm{var}(U_\gamma)$ denotes the variance of U_γ.

Example 5.5. *If $A = (a, \alpha, \beta)$ is a triangular fuzzy number then*

$$\mathrm{Var}(A) = \frac{1}{6}\int_0^1 \gamma\big(a + \beta(1-\gamma) - (a - \alpha(1-\gamma))\big)^2 d\gamma = \frac{(\alpha+\beta)^2}{72}.$$

Example 5.6. *Let $A = (a, b, \alpha, \beta)$ be a trapezoidal fuzzy number and let $f(\gamma) = (n+1)\gamma^n$ be a weighting function. Then,*

$$\mathrm{Var}_f(A) = (n+1)\int_0^1 \left[\frac{a_2(\gamma) - a_1(\gamma)}{2}\right]^2 \gamma^n d\gamma = \frac{n+1}{4}\int_0^1 \left[(b-a) + (\alpha+\beta)(1-\gamma)\right]^2 \gamma^n d\gamma$$
$$= \frac{n+1}{4}\left[(b-a)^2 \int_0^1 \gamma^n d\gamma + 2(b-a)(\alpha+\beta)\int_0^1 (1-\gamma)\gamma^n d\gamma + (\alpha+\beta)^2 \int_0^1 (1-\gamma)^2 \gamma^n d\gamma\right]$$
$$= \frac{n+1}{4}\left[\frac{(b-a)^2}{n+1} + \frac{2(b-a)(\alpha+\beta)}{(n+1)(n+2)} + \frac{2(\alpha+\beta)^2}{(n+1)(n+2)(n+3)}\right]$$
$$= \frac{(b-a)^2}{4} + \frac{(b-a)(\alpha+\beta)}{2(n+2)} + \frac{(\alpha+\beta)^2}{2(n+2)(n+3)} = \left[\frac{b-a}{2} + \frac{\alpha+\beta}{2(n+2)}\right]^2 + \frac{(n+1)(\alpha+\beta)^2}{4(n+2)^2(n+3)}.$$

So,

$$\lim_{n\to\infty} \mathrm{Var}_f(A) = \lim_{n\to\infty}\left(\left[\frac{b-a}{2} + \frac{\alpha+\beta}{2(n+2)}\right]^2 + \frac{(n+1)(\alpha+\beta)^2}{4(n+2)^2(n+3)}\right) = \frac{b-a}{2}.$$

In 2001 Carlsson and Fullér [26] originally introduced the possibilistic variance of fuzzy numbers as

$$\mathrm{Var}(A) = \frac{1}{2}\int_0^1 (a_2(\gamma) - a_1(\gamma))^2 \gamma \,\mathrm{d}\gamma,$$

and in 2003 Fullér and Majlender [85] introduced the f-weighted possibilistic variance of A by

$$\mathrm{Var}_f(A) = \frac{1}{4}\int_0^1 (a_2(\gamma) - a_1(\gamma))^2 f(\gamma) d\gamma.$$

In 2004 Fullér and Majlender [88] introduced a measure of possibilistic covariance between marginal distributions of a joint possibility distribution C as the expected value of the interactivity relation between the γ-level sets of its marginal distributions. In 2005 Carlsson, Fullér and Majlender [45] showed that the possibilistic covariance between fuzzy numbers A and B can be written as the weighted average of the probabilistic covariances between random variables with uniform joint distribution on the level sets of their joint possibility distribution C.

Definition 5.6 (Fullér and Majlender, [88]; Carlsson, Fullér and Majlender [45])**.** *The f-weighted* measure of possibilistic covariance *between $A, B \in \mathcal{F}$, (with respect to their joint distribution C), can be written as*

$$\mathrm{Cov}_f(A, B) = \int_0^1 \mathrm{cov}(X_\gamma, Y_\gamma) f(\gamma) d\gamma,$$

where X_γ and Y_γ are random variables whose joint distribution is uniform on $[C]^\gamma$ and $\mathrm{cov}(X_\gamma, Y_\gamma)$ denotes their covariance, for all $\gamma \in [0, 1]$.

Now we show how the possibilistic variance can be derived from possibilistic covariance. Let $A \in \mathcal{F}$ be fuzzy number with $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$ and let U_γ denote a uniform probability distribution on $[A]^\gamma$, $\gamma \in [0, 1]$. First we compute the level-wise covariances by

$$\begin{aligned}
\mathrm{cov}(U_\gamma, U_\gamma) &= M(U_\gamma^2) - (M(U_\gamma))^2 \\
&= \frac{1}{a_2(\gamma) - a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} x^2 dx - \left(\frac{1}{a_2(\gamma) - a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} x dx \right)^2 \\
&= \frac{a_1^2(\gamma) + a_1(\gamma)a_2(\gamma) + a_2^2(\gamma)}{3} - \left(\frac{a_1(\gamma) + a_2(\gamma)}{2} \right)^2 \\
&= \frac{a_1^2(\gamma) - 2a_1(\gamma)a_2(\gamma) + a_2^2(\gamma)}{12} = \frac{(a_2(\gamma) - a_1(\gamma))^2}{12},
\end{aligned}$$

and we get

$$\mathrm{Var}_f(A) = \mathrm{Cov}_f(A, A) = \int_0^1 \mathrm{cov}(U_\gamma, U_\gamma) f(\gamma) d\gamma = \int_0^1 \frac{(a_2(\gamma) - a_1(\gamma))^2}{12} f(\gamma) d\gamma.$$

If A and B are non-interactive, i.e. $C = A \times B$. Then $[C]^\gamma = [A]^\gamma \times [B]^\gamma$, that is, $[C]^\gamma$ is rectangular subset of $\mathbb{R}^2$ for any $\gamma \in [0, 1]$. Then X_γ, the first marginal probability distribution of a uniform distribution on $[C]^\gamma = [A]^\gamma \times [B]^\gamma$, is a uniform probability distribution on $[A]^\gamma$ (denoted by U_γ) and Y_γ, the second marginal probability distribution of a uniform distribution on $[C]^\gamma = [A]^\gamma \times [B]^\gamma$, is a uniform probability distribution on $[B]^\gamma$ (denoted by V_γ) that is X_γ and Y_γ are independent. So,

$$\mathrm{cov}(X_\gamma, Y_\gamma) = \mathrm{cov}(U_\gamma, V_\gamma) = 0,$$

for all $\gamma \in [0, 1]$, and, therefore, we have

$$\mathrm{Cov}_f(A, B) = \int_0^1 \mathrm{cov}(X_\gamma, Y_\gamma) f(\gamma) \mathrm{d}\gamma = \int_0^1 \mathrm{cov}(U_\gamma, V_\gamma) f(\gamma) \mathrm{d}\gamma = 0.$$

If A and B are non-interactive then $\mathrm{Cov}_f(A, B) = 0$ for any weighting function f.

We should emphasize here that the inclusion of the weighting function f does not play any crucial role in our theory, since by setting $f(\gamma) = 1$ for all $\gamma \in [0, 1]$, f could be eliminated from the definition.

Example 5.7. *Now consider the case when $A(x) = B(x) = (1 - x) \cdot \chi_{[0,1]}(x)$, for $x \in \mathbb{R}$, that is, $[A]^\gamma = [B]^\gamma = [0, 1 - \gamma]$ for $\gamma \in [0, 1]$. Suppose that their joint possibility distribution is given by*

$$F(x, y) = (1 - x - y) \cdot \chi_T(x, y),$$

where

$$T = \{(x, y) \in \mathbb{R}^2 | x \geq 0, y \geq 0, x + y \leq 1\}.$$

This situation is depicted on Fig. 5.7, where we have shifted the fuzzy sets to get a better view of the situation.

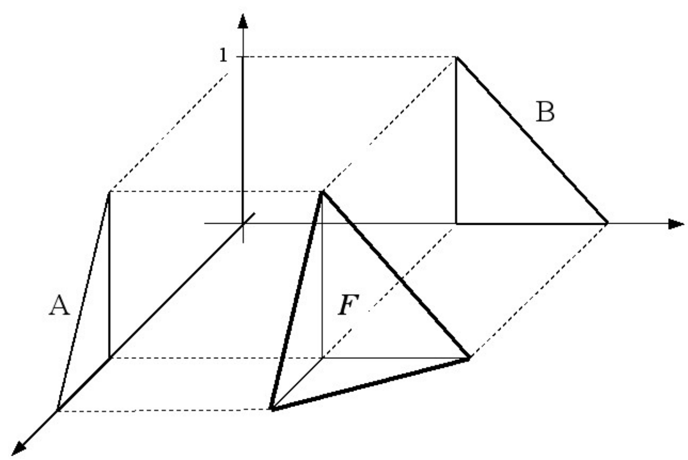

Figure 5.1: Illustration of joint possibility distribution F.

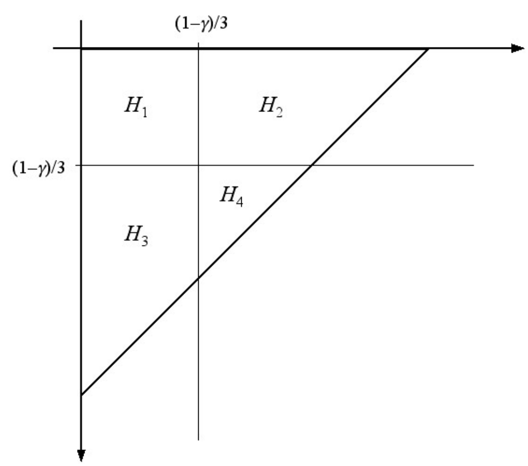

Figure 5.2: Partition of $[F]^\gamma$.

It is easy to check that A and B are really the marginal distributions of F. A γ-level set of F is computed by

$$[F]^\gamma = \{(x, y) \in \mathbb{R}^2 | x \geq 0, y \geq 0, x + y \leq 1 - \gamma\}.$$

The density function of a uniform distribution on $[F]^\gamma$ can be written as

$$f(x,y)=\begin{cases}\dfrac{1}{\displaystyle\int_{[F]^\gamma}dxdy} & \textit{if } (x,y)\in[F]^\gamma\\ 0 & \textit{otherwise}\end{cases}$$

in details,

$$f(x,y)=\begin{cases}\dfrac{1}{\displaystyle\int_0^{1-\gamma}\int_0^{1-\gamma-x}dxdy} & \textit{if } (x,y)\in[F]^\gamma\\ 0 & \textit{otherwise}\end{cases}$$

that is,

$$f(x,y)=\begin{cases}\dfrac{2}{(1-\gamma)^2} & \textit{if } (x,y)\in[F]^\gamma\\ 0 & \textit{otherwise}\end{cases}$$

The marginal functions are obtained as

$$f_1(x)=\frac{1}{\displaystyle\int_{[F]^\gamma}dxdy}\int_0^{1-\gamma-x}dy=\begin{cases}\dfrac{2(1-\gamma-x)}{(1-\gamma)^2} & \textit{if } 0\le x\le 1-\gamma\\ 0 & \textit{otherwise}\end{cases}$$

and

$$f_2(y)=\frac{1}{\displaystyle\int_{[F]^\gamma}dxdy}\int_0^{1-\gamma-y}dx=\begin{cases}\dfrac{2(1-\gamma-y)}{(1-\gamma)^2} & \textit{if } 0\le y\le 1-\gamma\\ 0 & \textit{otherwise}\end{cases}$$

Furthermore, the probabilistic expected values of marginal distributions of X_γ and Y_γ are equal to $(1-\gamma)/3$ see (Fig. 5.2). Really,

$$M(X_\gamma)=\frac{2}{(1-\gamma)^2}\int_0^{1-\gamma}x(1-\gamma-x)dx=(1-\gamma)/3.$$

$$M(Y_\gamma)=\frac{2}{(1-\gamma)^2}\int_0^{1-\gamma}y(1-\gamma-y)dy=(1-\gamma)/3.$$

And the covariance between X_γ and Y_γ is positive on H_1 and H_4 and negative on H_2 and H_3. In this case we get (see Fig. 5.2 for a geometrical interpretation),

$$\operatorname{cov}(X_\gamma,Y_\gamma)=M(X_\gamma Y_\gamma)-M(X_\gamma)M(Y_\gamma)=\frac{1}{\displaystyle\int_{[F]^\gamma}dxdy}\int_{[F]^\gamma}xydxdy$$

$$-\frac{1}{\displaystyle\int_{[F]^\gamma}dxdy}\int_{[F]^\gamma}xdxdy\times\frac{1}{\displaystyle\int_{[F]^\gamma}dxdy}\int_{[F]^\gamma}ydxdy.$$

That is,

$$\mathrm{cov}(X_\gamma, Y_\gamma) = \frac{2}{(1-\gamma)^2} \times \int_0^{1-\gamma} \int_0^{1-\gamma-x} xy dx dy - \frac{(1-\gamma)^2}{9}.$$

$$\mathrm{cov}(X_\gamma, Y_\gamma) = \frac{2}{(1-\gamma)^2} \times \int_0^{1-\gamma} x(1-\gamma-x) dx - \frac{(1-\gamma)^2}{9}.$$

$$\mathrm{cov}(X_\gamma, Y_\gamma) = \frac{(1-\gamma)^2}{12} - \frac{(1-\gamma)^2}{9} = -\frac{(1-\gamma)^2}{36}.$$

Therefore we get

$$\mathrm{Cov}_f(A,B) = -\frac{1}{36}\int_0^1 (1-\gamma)^2 f(\gamma) d\gamma,$$

and

$$\mathrm{Var}_f(A) = \mathrm{Var}_f(B) = \frac{1}{12}\int_0^1 (1-\gamma)^2 f(\gamma) d\gamma.$$

Definition 5.7 (Fullér, Mezei and Várlaki, [96])**.** *The f-weighted* possibilistic correlation coefficient *of $A, B \in \mathcal{F}$ (with respect to their joint distribution C) is defined by*

$$\rho_f(A,B) = \int_0^1 \rho(X_\gamma, Y_\gamma) f(\gamma) \mathrm{d}\gamma \tag{5.8}$$

where

$$\rho(X_\gamma, Y_\gamma) = \frac{\mathrm{cov}(X_\gamma, Y_\gamma)}{\sqrt{\mathrm{var}(X_\gamma)}\sqrt{\mathrm{var}(Y_\gamma)}}$$

and, where X_γ and Y_γ are random variables whose joint distribution is uniform on $[C]^\gamma$ for all $\gamma \in [0,1]$.

For any joint distribution C and for any f we have, $-1 \leq \rho_f(A,B) \leq 1$. In other words, the f-weighted is nothing else, but the f-weighted average of the probabilistic correlation coefficients $\rho(X_\gamma, Y_\gamma)$ for all $\gamma \in [0,1]$. Since $\rho_f(A,B)$ measures an average index of interactivity between the level sets of A and B, we sometimes call this measure as the index of interactivity between A and B.

Note 6. *There exist several other ways to define correlation coefficient for fuzzy numbers, e.g. Liu and Kao [115] used fuzzy measures to define a fuzzy correlation coefficient of fuzzy numbers and they formulated a pair of nonlinear programs to find the α-cut of this fuzzy correlation coefficient, then, in a special case, Hong [104] showed an exact calculation formula for this fuzzy correlation coefficient. Vaidyanathan [135] introduced a new measure for the correlation coefficient between triangular fuzzy variables called credibilistic correlation coefficient.*

In 2005 Carlsson, Fullér and Majlender [45] defined the f-weighted *possibilistic correlation* of $A, B \in \mathcal{F}$, (with respect to their joint distribution C) as

$$\rho_f^{old}(A,B) = \frac{\mathrm{Cov}_f(A,B)}{\sqrt{\mathrm{Var}_f(A)\,\mathrm{Var}_f(B)}}. \tag{5.9}$$

where U_γ is a uniform probability distribution on $[A]^\gamma$ and V_γ is a uniform probability distribution on $[B]^\gamma$, and X_γ and Y_γ are random variables whose joint probability distribution is uniform on $[C]^\gamma$ for all $\gamma \in [0,1]$. If $[C]^\gamma$ is convex for all $\gamma \in [0,1]$ then $-1 \leq \rho_f^{\mathrm{old}}(A,B) \leq 1$ for any f.

The main drawback of the definition of the former index of interactivity (5.9) is that it does not necessarily take its values from $[-1, 1]$ if some level-sets of the joint possibility distribution are not convex. For example, consider a joint possibility distribution defined by

$$C(x, y) = 4x \cdot \chi_T(x, y) + \ 4/3(1 - x) \cdot \chi_S(x, y), \tag{5.10}$$

where,

$$T = \left\{(x, y) \in \mathbb{R}^2 \mid 0 \le x \le 1/4, 0 \le y \le 1/4, x \le y\right\},$$

and,

$$S = \left\{(x, y) \in \mathbb{R}^2 \mid 1/4 \le x \le 1, 1/4 \le y \le 1, y \le x\right\}.$$

Furthermore, we have,

$$[C]^\gamma = \left\{(x, y) \in \mathbb{R}^2 \mid \gamma/4 \le x \le 1/4, x \le y \le \ 1/4\right\} \bigcup \\ \left\{(x, y) \in \mathbb{R}^2 \mid 1/4 \le x \le 1 - 3/4\gamma, 1/4 \le y \le x\right\}.$$

We can see that $[C]^\gamma$ is not a convex set for any $\gamma \in [0, 1)$ (see Fig. 5.3).

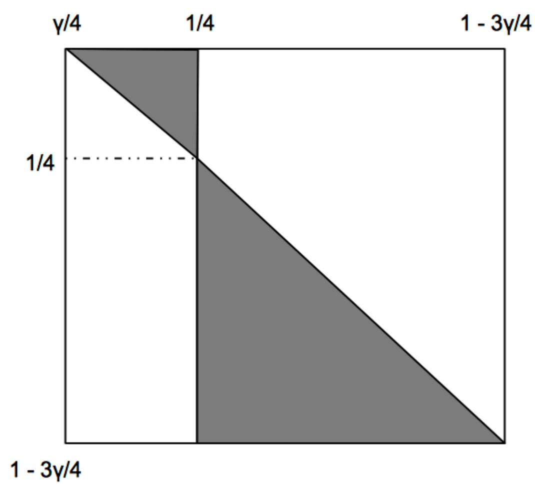

Figure 5.3: Not convex γ-level set.

Then the marginal possibility distributions of (5.10) are computed by (see Fig. 5.4),

$$A(x) = B(x) = \begin{cases} 4x, & \text{if } 0 \le x \le 1/4 \\ \dfrac{4}{3}(1 - x), & \text{if } 1/4 \le x \le 1 \\ 0, & \quad\text{otherwise} \end{cases}$$

After some computations we get $\rho_f^{old}(A, B) \approx 1.562$ for the weighting function $f(\gamma) = 2\gamma$. We get here a value bigger than one since the variance of the first marginal distributions, X_γ, exceeds the variance of the uniform distribution on the same support.

We will show five important examples for the possibilistic correlation coefficient. If A and B are non-interactive then their joint possibility distribution is defined by $C = A \times B$. Since all $[C]^\gamma$ are

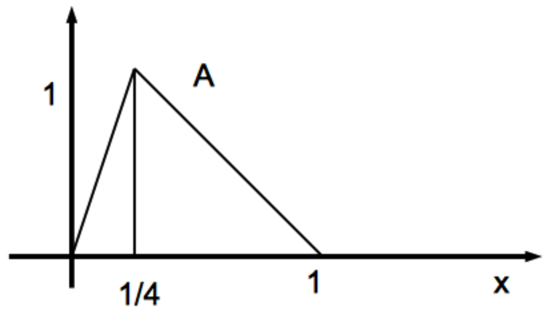

Figure 5.4: Marginal distribution A.

rectangular and the probability distribution on $[C]^\gamma$ is defined to be uniform we get $\mathrm{cov}(X_\gamma, Y_\gamma) = 0$, for all $\gamma \in [0,1]$. So $\mathrm{Cov}_f(A,B) = 0$ and $\rho_f(A,B) = 0$ for any weighting function f.

Fuzzy numbers A and B are said to be in perfect correlation, if there exist $q, r \in \mathbb{R}$, $q \neq 0$ such that their joint possibility distribution is defined by [45]

$$C(x_1,x_2) = A(x_1) \cdot \chi_{\{qx_1+r=x_2\}}(x_1,x_2) = B(x_2) \cdot \chi_{\{qx_1+r=x_2\}}(x_1,x_2), \tag{5.11}$$

where $\chi_{\{qx_1+r=x_2\}}$, stands for the characteristic function of the line

$$\{(x_1,x_2) \in \mathbb{R}^2 | qx_1 + r = x_2\}.$$

In this case we have

$$[C]^\gamma = \Big\{(x, qx+r) \in \mathbb{R}^2 \Big| x = (1-t)a_1(\gamma) + ta_2(\gamma), t \in [0,1]\Big\}$$

where $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$; and $[B]^\gamma = q[A]^\gamma + r$, for any $\gamma \in [0,1]$, and, finally,

$$B(x) = A\left(\frac{x-r}{q}\right),$$

for all $x \in \mathbb{R}$. Furthermore, A and B are in a perfect positive [see Fig. 5.6] (negative [see Fig. 5.7]) correlation if q is positive (negative) in (5.11).

If A and B have a perfect positive (negative) correlation then from $\rho(X_\gamma, Y_\gamma) = 1$ ($\rho(X_\gamma, Y_\gamma) = -1$) [see [45] for details], for all $\gamma \in [0,1]$, we get $\rho_f(A,B) = 1$ ($\rho_f(A,B) = -1$) for any weighting function f.

Consider the case, when $A(x) = B(x) = (1-x) \cdot \chi_{[0,1]}(x)$, for $x \in \mathbb{R}$, that is $[A]^\gamma = [B]^\gamma = [0, 1-\gamma]$, for $\gamma \in [0,1]$. Suppose that their joint possibility distribution is given by $F(x,y) = (1-x-y) \cdot \chi_T(x,y)$, where

$$T = \left\{(x,y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x+y \leq 1\right\}.$$

A γ-level set of F is computed by

$$[F]^\gamma = \left\{(x,y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x+y \leq 1-\gamma\right\}.$$

This situation is depicted on Fig. 5.8, where we have shifted the fuzzy sets to get a better view of the situation.

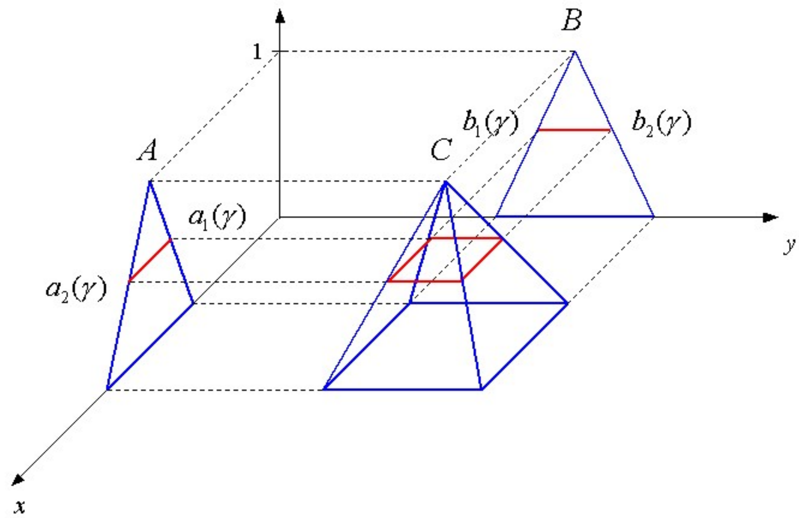

Figure 5.5: The case of non-interactive marginal distributions.

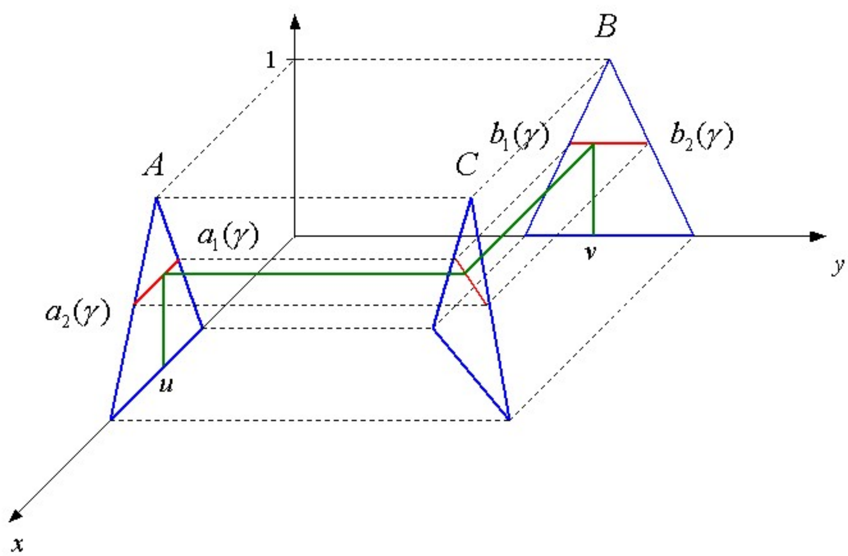

Figure 5.6: Perfect positive correlation.

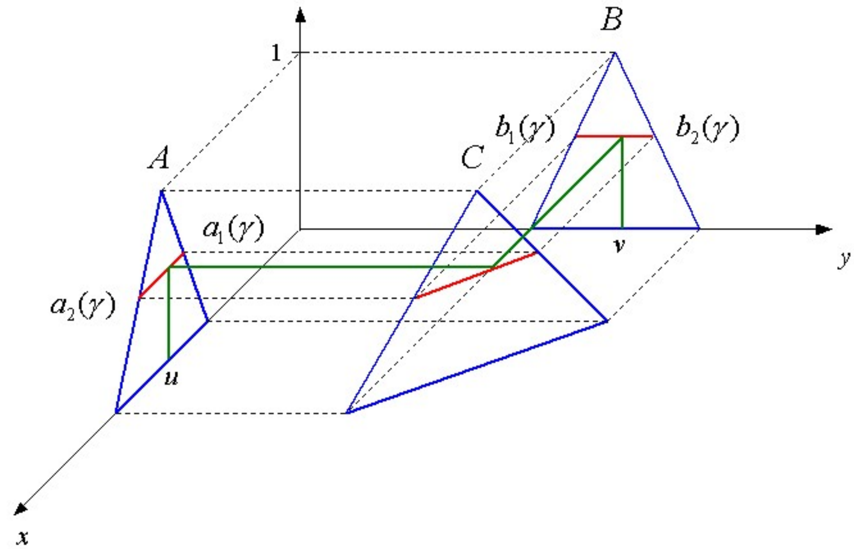

Figure 5.7: Perfect negative correlation.

The density function of a uniform distribution on $[F]^\gamma$ can be written as

$$f(x,y) = \begin{cases} \dfrac{1}{\int_{[F]^\gamma} dxdy}, & \text{if } (x,y) \in [F]^\gamma \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \dfrac{2}{(1-\gamma)^2}, & \text{if } (x,y) \in [F]^\gamma \\ 0 & \text{otherwise} \end{cases}$$

The marginal functions are obtained as

$$f_1(x) = \begin{cases} \dfrac{2(1-\gamma-x)}{(1-\gamma)^2}, & \text{if } 0 \le x \le 1-\gamma \\ 0 & \text{otherwise} \end{cases} \qquad f_2(y) = \begin{cases} \dfrac{2(1-\gamma-y)}{(1-\gamma)^2}, & \text{if } 0 \le y \le 1-\gamma \\ 0 & \text{otherwise} \end{cases}$$

We can calculate the probabilistic expected values of the random variables X_γ and Y_γ, whose joint distribution is uniform on $[F]^\gamma$ for all $\gamma \in [0,1]$:

$$M(X_\gamma) = \frac{2}{(1-\gamma)^2} \int_0^{1-\gamma} x(1-\gamma-x)dx = \frac{1-\gamma}{3}$$

and,

$$M(Y_\gamma) = \frac{2}{(1-\gamma)^2} \int_0^{1-\gamma} y(1-\gamma-y)dy = \frac{1-\gamma}{3}.$$

We calculate the variations of X_γ and Y_γ with the formula $\text{var}(X) = M(X^2) - M(X)^2$:

$$M(X_\gamma^2) = \frac{2}{(1-\gamma)^2} \int_0^{1-\gamma} x^2(1-\gamma-x)dx = \frac{(1-\gamma)^2}{6}$$

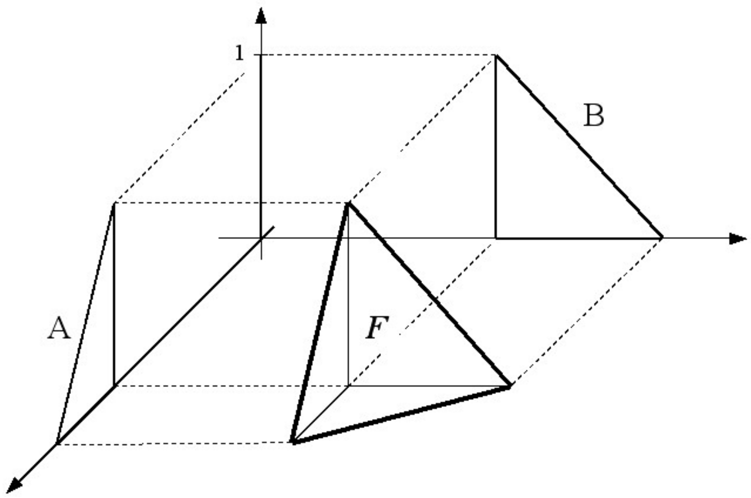

Figure 5.8: Illustration of joint possibility distribution F.

and,

$$\mathrm{var}(X_\gamma) = M(X_\gamma^2) - M(X_\gamma)^2 = \frac{(1-\gamma)^2}{6} - \frac{(1-\gamma)^2}{9} = \frac{(1-\gamma)^2}{18}.$$

And similarly we obtain

$$\mathrm{var}(Y_\gamma) = \frac{(1-\gamma)^2}{18}.$$

Using that

$$M(X_\gamma Y_\gamma) = \frac{2}{(1-\gamma)^2}\int_0^{1-\gamma}\int_0^{1-\gamma-x} xydydx = \frac{(1-\gamma)^2}{12},$$

$$\mathrm{cov}(X_\gamma, Y_\gamma) = M(X_\gamma Y_\gamma) - M(X_\gamma)M(Y_\gamma) = -\frac{(1-\gamma)^2}{36},$$

we can calculate the probabilistic correlation of the random variables:

$$\rho(X_\gamma, Y_\gamma) = \frac{\mathrm{cov}(X_\gamma, Y_\gamma)}{\sqrt{\mathrm{var}(X_\gamma)}\sqrt{\mathrm{var}(Y_\gamma)}} = -\frac{1}{2}.$$

And finally the f-weighted possibilistic correlation of A and B:

$$\rho_f(A, B) = \int_0^1 -\frac{1}{2}f(\gamma)d\gamma = -\frac{1}{2}.$$

We note here that using the former definition (5.9) we would obtain $\rho_f^{\mathrm{old}}(A, B) = -1/3$ for the correlation coefficient (see [45] for details).

Now consider the case when $A(1-x) = B(x) = x \cdot \chi_{[0,1]}(x)$ for $x \in \mathbb{R}$, that is, $[A]^\gamma = [0, 1-\gamma]$ and $[B]^\gamma = [\gamma, 1]$, for $\gamma \in [0, 1]$. Let $E(x, y) = (y - x) \cdot \chi_S(x, y)$, where

$$S = \{(x, y) \in \mathbb{R}^2 | x \geq 0, y \leq 1, y - x \geq 0\}.$$

This situation is depicted on Fig. 5.9, where we have shifted the fuzzy sets to get a better view of the situation. A γ-level set of E is computed by

$$[E]^\gamma = \{(x, y) \in \mathbb{R}^2 | x \geq 0, y \leq 1, y - x \geq \gamma\}.$$

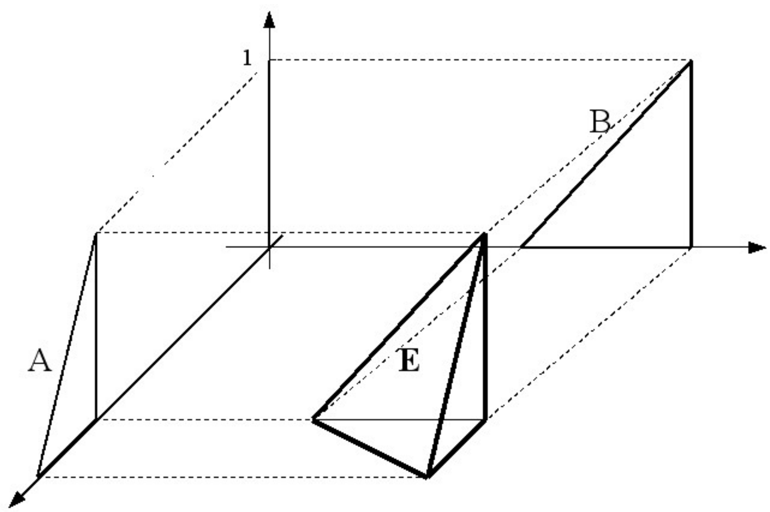

Figure 5.9: $\rho_f(A, B) = 1/2$.

In this case, the probabilistic expected value of marginal distribution X_γ is equal to $(1-\gamma)/3$ and the probabilistic expected value of marginal distribution of Y_γ is equal to $2(1-\gamma)/3$ see (Fig. 5.10). And the covariance between X_γ and Y_γ is positive on H_1 and H_4 and negative on H_2 and H_3. After some calculations (see Fig. 5.10) we get $\rho_f(A, B) = 1/2$, for any weighting function f.

Consider the case, when $A(x) = B(x) = (1 - x^2) \cdot \chi_{[0,1]}(x)$, for $x \in \mathbb{R}$, that is $[A]^\gamma = [B]^\gamma = [0, \sqrt{1-\gamma}]$, for $\gamma \in [0, 1]$. Suppose that their joint possibility distribution is given by:

$$C(x, y) = (1 - x^2 - y^2) \cdot \chi_T(x, y),$$

where

$$T = \left\{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x^2 + y^2 \leq 1\right\}.$$

A γ-level set of C is computed by

$$[C]^\gamma = \left\{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x^2 + y^2 \leq 1 - \gamma\right\}.$$

The density function of a uniform distribution on $[F]^\gamma$ can be written as

$$f(x, y) = \begin{cases} \dfrac{1}{\int_{[C]^\gamma} dxdy}, & \text{if } (x, y) \in [C]^\gamma \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \dfrac{4}{(1-\gamma)\pi}, & \text{if } (x, y) \in [C]^\gamma \\ 0 & \text{otherwise} \end{cases}$$

The marginal functions are obtained as

$$f_1(x) = \begin{cases} \dfrac{4\sqrt{1-\gamma-x^2}}{(1-\gamma)\pi}, & \text{if } 0 \leq x \leq 1-\gamma \\ 0 & \text{otherwise} \end{cases} \quad f_2(y) = \begin{cases} \dfrac{4\sqrt{1-\gamma-y^2}}{(1-\gamma)\pi}, & \text{if } 0 \leq y \leq 1-\gamma \\ 0 & \text{otherwise} \end{cases}$$

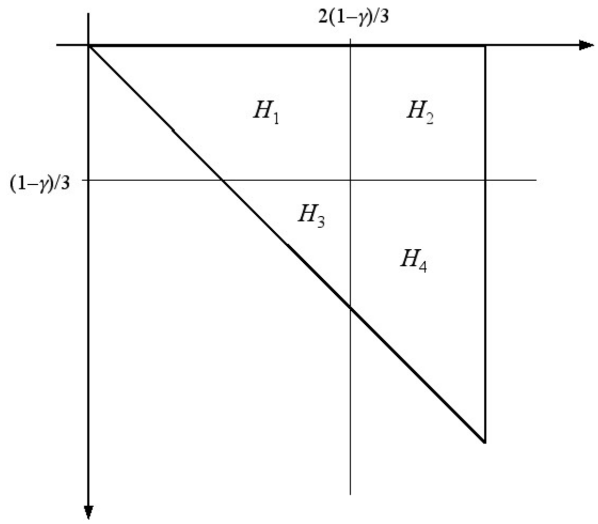

Figure 5.10: Partition of $[E]^\gamma$.

We can calculate the probabilistic expected values of the random variables X_γ and Y_γ, whose joint distribution is uniform on $[C]^\gamma$ for all $\gamma \in [0,1]$:

$$M(X_\gamma) = \frac{4}{(1-\gamma)\pi} \int_0^{\sqrt{1-\gamma}} x\sqrt{1-\gamma-x^2}dx = \frac{4\sqrt{1-\gamma}}{3\pi}$$

$$M(Y_\gamma) = \frac{4}{(1-\gamma)\pi} \int_0^{\sqrt{1-\gamma}} y\sqrt{1-\gamma-y^2}dx = \frac{4\sqrt{1-\gamma}}{3\pi}.$$

We calculate the variations of X_γ and Y_γ with the formula $var(X) = M(X^2) - M(X)^2$:

$$M(X_\gamma^2) = \frac{4}{(1-\gamma)\pi} \int_0^{\sqrt{1-\gamma}} x^2\sqrt{1-\gamma-x^2}dx = \frac{1-\gamma}{4}$$

$$\text{var}(X_\gamma) = M(X_\gamma^2) - M(X_\gamma)^2 = \frac{1-\gamma}{4} - \frac{16(1-\gamma)}{9\pi^2} = \frac{(1-\gamma)(9\pi^2-64)}{36\pi^2}.$$

And similarly we obtain

$$\text{var}(Y_\gamma) = \frac{(1-\gamma)(9\pi^2-64)}{36\pi^2}.$$

Using that

$$\text{cov}(X_\gamma, Y_\gamma) = M(X_\gamma Y_\gamma) - M(X_\gamma)M(Y_\gamma) = \frac{(1-\gamma)(9\pi-32)}{18\pi^2},$$

we can calculate the probabilisctic correlation of the reandom variables:

$$\rho(X_\gamma, Y_\gamma) = \frac{\text{cov}(X_\gamma, Y_\gamma)}{\sqrt{\text{var}(X_\gamma)}\sqrt{\text{var}(Y_\gamma)}} = \frac{2(9\pi-32)}{(9\pi^2-64)} \approx -0.302.$$

And finally the f-weighted possibilistic correlation of A and B:

$$\rho_f(A,B) = \int_0^1 \frac{2(9\pi-32)}{(9\pi^2-64)} f(\gamma)d\gamma = \frac{2(9\pi-32)}{(9\pi^2-64)}.$$

Suppose that the joint possibility distribution of A and B is defined by,

$$C(x,y) = \begin{cases} A(x) & \text{if } y = 0 \\ B(y) & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

where,

$$A(x) = B(x) = (1-x) \cdot \chi_{[0,1]}(x),$$

for $x \in \mathbb{R}$. Then a γ-level set of C is computed by

$$[C]^\gamma = \left\{(x,0) \in \mathbb{R}^2 \mid 0 \le x \le 1-\gamma\right\} \bigcup \left\{(0,y) \in \mathbb{R}^2 \mid 0 \le y \le 1-\gamma\right\}.$$

Since all γ-level sets of C are degenerated, i.e. their integrals vanish, we calculate everything as a limit of integrals. We calculate all the quantities with the γ-level sets:

$$[C]^\gamma_\delta = \left\{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1-\gamma, 0 \le y \le \delta\right\} \bigcup \left\{(x,y) \in \mathbb{R}^2 \mid 0 \le y \le 1-\gamma, 0 \le x \le \delta\right\}.$$

First we calculate the expected value and variance of X_γ and Y_γ:

$$M(X_\gamma) = \lim_{\delta \to 0} \frac{1}{\int_{[C]^\gamma_\delta} dxdy} \int_{[C]^\gamma_\delta} xdx = \frac{1-\gamma}{4},$$

$$M(X^2_\gamma) = \lim_{\delta \to 0} \frac{1}{\int_{[C]^\gamma_\delta} dxdy} \int_{[C]^\gamma_\delta} x^2 dx = \frac{(1-\gamma)^2}{6},$$

$$\mathrm{var}(X_\gamma) = M(X^2_\gamma) - M(X_\gamma)^2 = \frac{(1-\gamma)^2}{6} - \frac{(1-\gamma)^2}{16} = \frac{5(1-\gamma)^2}{48}.$$

Because of the symmetry, the results are the same for Y_γ. We need to calculate their covariance,

$$M(X_\gamma Y_\gamma) = \lim_{\delta \to 0} \frac{1}{\int_{[C]^\gamma_\delta} dxdy} \int_{[C]^\gamma_\delta} xydydx = 0,$$

Using this we obtain,

$$\mathrm{cov}(X_\gamma, Y_\gamma) = -\frac{(1-\gamma)^2}{16},$$

and for the correlation,

$$\rho(X_\gamma, Y_\gamma) = \frac{\mathrm{cov}(X_\gamma, Y_\gamma)}{\sqrt{\mathrm{var}(X_\gamma)}\sqrt{\mathrm{var}(Y_\gamma)}} = -\frac{3}{5}.$$

Finally we obtain the f-weighted possibilistic correlation:

$$\rho_f(A,B) = \int_0^1 -\frac{3}{5} f(\gamma) d\gamma = -\frac{3}{5}.$$

In this extremal case, the joint distribution is unequivocally constructed from the knowledge that $C(x,y) = 0$ for positive x, y.

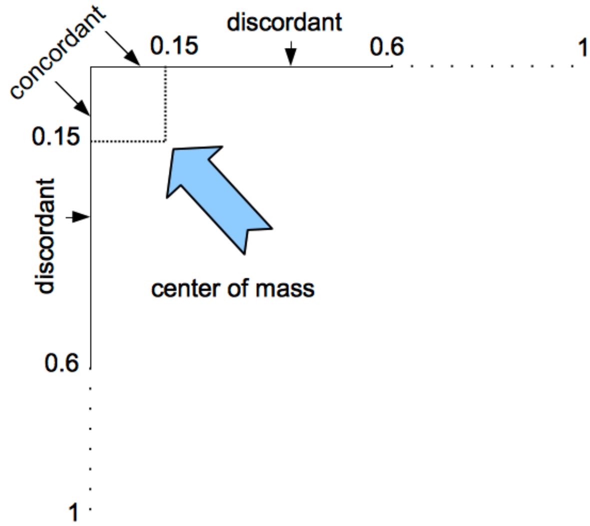

Figure 5.11: Illustration of $[C]^{0.4}$.

We emphasize here that zero correlation does not always imply non-interactivity. Let $A, B \in \mathcal{F}$ be fuzzy numbers, let C be their joint possibility distribution, and let $\gamma \in [0, 1]$. Suppose that $[C]^\gamma$ is symmetrical, i.e. there exists $a \in \mathbb{R}$ such that

$$C(x, y) = C(2a - x, y),$$

for all $x, y \in [C]^\gamma$ *(the line defined by* $\{(a, t)|t \in \mathbb{R}\}$ *is the axis of symmetry of* $[C]^\gamma$*)*. In this case $\mathrm{cov}(X_\gamma, Y_\gamma) = 0$. Indeed, let

$$H = \{(x, y) \in [C]^\gamma | x \le a\},$$

then

$$\int_{[C]^\gamma} xydxdy = \int_H \big(xy + (2a - x)y\big)dxdy = 2a\int_H ydxdy,$$

$$\int_{[C]^\gamma} xdxdy = \int_H \big(x + (2a - x)\big)dxdy = 2a\int_H dxdy,$$

$$\int_{[C]^\gamma} ydxdy = 2\int_H ydxdy, \quad \int_{[C]^\gamma} dxdy = 2\int_H dxdy,$$

therefore, we obtain

$$\mathrm{cov}(X_\gamma, Y_\gamma) = \frac{1}{\displaystyle\int_{[C]^\gamma} dxdy}\int_{[C]^\gamma} xydxdy - \frac{1}{\displaystyle\int_{[C]^\gamma} dxdy}\int_{[C]^\gamma} xdxdy\frac{1}{\displaystyle\int_{[C]^\gamma} dxdy}\int_{[C]^\gamma} ydxdy = 0.$$

For example, let G be a joint possibility distribution with a symmetrical γ-level set, i.e., there exist $a, b \in \mathbb{R}$ such that

$$G(x, y) = G(2a - x, y) = G(x, 2b - y) = G(2a - x, 2b - y),$$

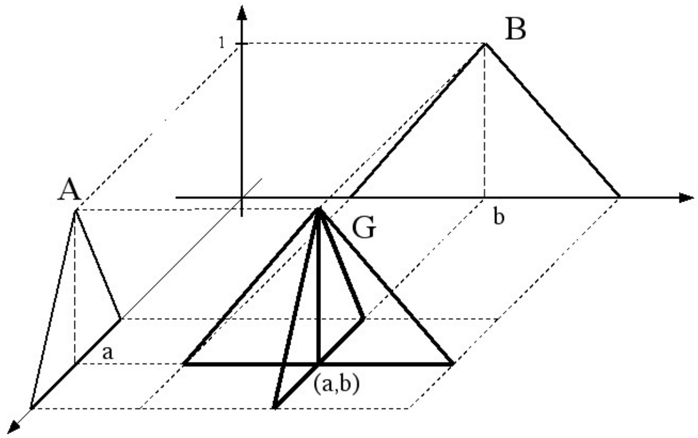

Figure 5.12: A case of $\rho_f(A, B) = 0$ for interactive fuzzy numbers.

for all $x, y \in [G]^\gamma$, where (a, b) is the center of the set $[G]^\gamma$, In Fig. 5.12, the joint possibility distribution is defined from *symmetrical marginal distributions* as $G(x, y) = T_W(A(x), B(y))$, where T_W denotes the weak t-norm.

Consider now joint possibility distributions that are derived from given marginal distributions by aggregating their membership values. Namely, let $A, B \in \mathcal{F}$. We will say that their joint possibility distribution C is *directly defined* from its marginal distributions if $C(x, y) = T(A(x), B(y)),\ x, y \in \mathbb{R}$, where $T\colon [0, 1] \times [0, 1] \to [0, 1]$ is a function satisfying the properties

$$\max_y T(A(x), B(y)) = A(x), \forall x \in \mathbb{R}, \tag{5.12}$$

and

$$\max_x T(A(x), B(y)) = B(y), \forall y \in \mathbb{R}, \tag{5.13}$$

for example a triangular norm. In this case the joint distribution depends barely on the membership values of its marginal distributions, and the covariance (and, consequently, the correlation) between its marginal distributions will be zero whenever at least one of its marginal distributions is symmetrical.

Theorem 5.4 (Carlsson, Fullér and Majlender, [42])**.** *Let $A, B \in \mathcal{F}$ and let their joint possibility distribution C be defined by $C(x, y) = T(A(x), B(y))$, for $x, y \in \mathbb{R}$, where T is a function satisfying conditions (5.12) and (5.13). If A is a symmetrical fuzzy number then* $\mathrm{Cov}_f(A, B) = 0$*, for any fuzzy number B, aggregator T, and weighting function f.*

Really, if A is a symmetrical fuzzy number with center a such that $A(x) = A(2a - x)$ for all $x \in \mathbb{R}$ then, $C(x, y) = T(A(x), B(y)) = T(A(2a - x), B(y)) = C(2a - x, y)$, that is, C is symmetrical. Hence, considering the results obtained above we have $\mathrm{cov}(X_\gamma, Y_\gamma) = 0$, and, therefore, $\mathrm{Cov}_f(A, B) = 0$, for any weighting function f. In 2010 Fullér, Mezei and Várlaki [93] introduced a possibilistic correlation ratio (for marginal possibility distributions of joint possibility distributions).

Chapter 6

Operations on Interactive Fuzzy Numbers

Properties of operations on interactive fuzzy numbers, when their joint possibility distribution is defined by a t-norm have been extensively studied in the literature [33]. In this Chapter, following Fullér [76, 77] we will compute the exact membership function of product-sum and Hamacher-sum of triangular fuzzy numbers, and following Fullér and Keresztfalvi [79] we will compute the exact membership function of t-norm-based sum of L-R fuzzy numbers. We will consider the extension principle with interactive fuzzy numbers, where the interactivity relation between fuzzy numbers is defined by their joint possibility distribution. Following Fullér and Keresztfalvi [75] and Carlsson, Fullér and Majlender [41] we will show that Nguyen's theorem remains valid for interactive fuzzy numbers.

In the definition of the extension principle (2.5) one can use any t-norm for modeling the conjunction operator.

Definition 6.1. *Let T be a t-norm and let f be a mapping from $X_1 \times X_2 \times \cdots \times X_n$ to Y, Assume $(A_1, \dots, A_n)$ is a fuzzy subset of $X_1 \times X_2 \times \cdots \times X_n$, using the extension principle, we can define $f(A_1, A_2, \dots, A_n)$ as a fuzzy subset of Y such that*

$$f(A_1, A_2, \dots, A_n)(y) \quad = \quad \begin{cases} \sup\{T(A_1(x), \dots, A_n(x)) \,|\, x \in f^{-1}(y)\} & \textit{if } f^{-1}(y) \neq \emptyset \\ 0 & \textit{otherwise} \end{cases} \tag{6.1}$$

This is called the sup-T (or generally sup-t-norm) extension principle.

Specially, if T is a t-norm and $*$ is a binary operation on $\mathbb{R}$ then $*$ can be extended to fuzzy quantities in the sense of the sup-T extension principle as

$$\big(A_1 * A_2\big)(z) = \sup_{x_1 * x_2 = z} T\big(A_1(x_1), A_2(x_2)\big), \; z \in \mathbb{R}.$$

For example, if A and B are fuzzy numbers, $T_P(u, v) = uv$ is the product t-norm and $f(x_1, x_2) = x_1 + x_2$ is the addition operation on the real line then the sup-product extended sum of A and B, called product-sum and denoted by $A + B$, is defined by

$$f(A_1, A_2)(y) = (A_1 + A_2)(y) = \sup_{x_1 + x_2 = y} T(A_1(x_1), A_2(x_2)) = \sup_{x_1 + x_2 = y} A_1(x_1)A_2(x_2).$$

The sup-T extension principle is a very important in fuzzy arithmetic. For example, if we have a sequence of symmetric triangular fuzzy numbers $\tilde{a}_i, \; i \in \mathbb{N}$ then their sup-min extended sum $\tilde{a}_1 + \tilde{a}_2 + \cdots + \tilde{a}_n + \cdots$ is always the universal fuzzy set in $\mathbb{R}$ independently of α. This means that the minimum norm, because it is too big, might be inappropriate in situations where we have to manipulate with many fuzzy quantities (for example, fuzzy time series analysis, fuzzy linear programming problems, fuzzy control with a large number of rules, etc.). To compute the exact membership function of the interactive sum of fuzzy numbers is not an easy task. However if all the fuzzy numbers are of symmetrical triangular form then it is possible to compute the exact membership function of their product-sum. Namely, 1991 Fullér [76] proved the following theorem.

Theorem 6.1 (Fullér, [76]). *Let $\tilde{a}_i = (a_i, \alpha),\ i \in \mathbb{N}$ be symmetrical triangular fuzzy numbers. If*

$$A := a_1 + a_2 + \cdots + a_n + \cdots = \sum_{i=1}^{\infty} a_i,$$

exists and is finite, then with the notations $\tilde{A}_n := \tilde{a}_1 + \cdots + \tilde{a}_n$ and $A_n := a_1 + \cdots + a_n,\ n \in \mathbb{N}$ we have

$$\left(\lim_{n\to\infty} \tilde{A}_n\right)(z) = \exp\left(-\frac{|A-z|}{\alpha}\right),\ z \in \mathbb{R}.$$

Proof. It will be sufficient to show that

$$\tilde{A}_n(z) = \begin{cases} \left(1 - \dfrac{|A_n - z|}{n\alpha}\right)^n & \text{if } |A_n - z| \le n\alpha \\ 0 & \text{otherwise} \end{cases} \tag{6.2}$$

for each $n \ge 2$, because from (6.2) it follows that

$$\left(\lim_{n\to\infty} \tilde{A}_n\right)(z) = \lim_{n\to\infty}\left(1 - \frac{|A_n - z|}{n\alpha}\right)^n = \exp\left(-\frac{|\lim_{n\to\infty} A_n - z|}{\alpha}\right) = \exp\left(-\frac{|A - z|}{\alpha}\right)$$

for $z \in \mathbb{R}$. From the definition of product-sum of fuzzy numbers it follows that

$$\text{supp}\tilde{A}_n = \text{supp}(\tilde{a}_1 + \cdots + \tilde{a}_n) = \text{supp}\tilde{a}_1 + \cdots + \text{supp}\tilde{a}_n =$$

$$[a_1 - \alpha, a_1 + \alpha] + \cdots + [a_n - \alpha, a_n + \alpha] = [A_n - n\alpha, A_n + n\alpha],\ n \in \mathbb{N}.$$

We prove (6.2) by making an induction argument on n. Let $n = 2$. In order to determine $\tilde{A}_2(z),\ z \in [A_2 - 2\alpha, A_2 + 2\alpha]$ we need to solve the following mathematical programming problem:

$$\left(1 - \frac{|a_1 - x|}{\alpha}\right)\left(1 - \frac{|a_2 - y|}{\alpha}\right) \to \max$$

$$\text{subject to } |a_1 - x| \le \alpha,$$

$$|a_2 - y| \le \alpha,\ \ x + y = z.$$

By using Lagrange's multipliers method and decomposition rule of fuzzy numbers into two separate parts it is easy to see that $\tilde{A}_2(z),\ z \in [A_2 - 2\alpha, A_2 + 2\alpha]$ is equal to the optimal value of the following mathematical programming problem:

$$\left(1 - \frac{a_1 - x}{\alpha}\right)\left(1 - \frac{a_2 - z + x}{\alpha}\right) \to \max \tag{6.3}$$

$$\text{subject to } a_1 - \alpha \le x \le a_1,$$

$$a_2 - \alpha \le z - x \le a_2,\ x + y = z.$$

Using Lagrange's multipliers method for the solution of (6.3) we get that its optimal value is

$$\left(1 - \frac{|A_2 - z|}{2\alpha}\right)^2$$

and its unique solution is

$$x = \frac{a_1 - a_2 + z}{2},$$

where the derivative vanishes. Indeed, it can be easily checked that the inequality

$$\left(1 - \frac{|A_2 - z|}{2\alpha}\right)^2 \ge 1 - \frac{A_2 - z}{\alpha}$$

holds for each $z \in [A_2 - 2\alpha, A_2]$.

In order to determine $\tilde{A}_2(z),\ z \in [A_2, A_2 + 2\alpha]$ we need to solve the following mathematical programming problem:

$$\left(1 + \frac{a_1 - x}{\alpha}\right)\left(1 + \frac{a_2 - z + x}{\alpha}\right) \to \max \tag{6.4}$$

$$\text{subject to } a_1 \le x \le a_1 + \alpha,$$

$$a_2 \le z - x \le a_2 + \alpha.$$

In a similar manner we get that the optimal value of (6.4) is

$$\left(1 - \frac{|z - A_2|}{2\alpha}\right)^2.$$

Let us assume that (6.2) holds for some $n \in \mathbb{N}$. By similar arguments we obtain

$$\begin{aligned}\tilde{A}_{n+1}(z) = (\tilde{A}_n + \tilde{a}_{n+1})(z) = \sup_{x+y=z} \tilde{A}_n(x) \cdot \tilde{a}_{n+1}(y) &= \sup_{x+y=z} \left(1 - \frac{|A_n - x|}{n\alpha}\right)\left(1 - \frac{|a_{n+1} - y|}{\alpha}\right) \\ &= \left(1 - \frac{|A_{n+1} - z|}{(n+1)\alpha}\right)^{n+1},\end{aligned}$$

for $z \in [A_{n+1} - (n+1)\alpha, A_{n+1} + (n+1)\alpha]$, and $\tilde{A}_{n+1}(z) = 0$, for $z \notin [A_{n+1} - (n+1)\alpha, A_{n+1} + (n+1)\alpha]$. This ends the proof. □

If $\tilde{a}$ and $\tilde{b}$ are fuzzy numbers and $\gamma \ge 0$ a real number, then their Hamacher-sum (H_γ-sum for short) is defined as

$$(\tilde{a} + \tilde{b})(z) = \sup_{x+y=z} H_\gamma(\tilde{a}(x), \tilde{b}(y)) = \sup_{x+y=z} \frac{\tilde{a}(x)\tilde{b}(y)}{\gamma + (1-\gamma)(\tilde{a}(x) + \tilde{b}(y) - \tilde{a}(x)\tilde{b}(y))},$$

for $x, y, z \in \mathbb{R}$, where H_γ the Hamacher t-norm (2.3) with parameter γ.

If all the fuzzy numbers are of symmetrical triangular form then it is possible to compute the exact membership function of their Hamacher-sum.

Theorem 6.2 (Fullér, [77]). *Let $\gamma = 0$ and $\tilde{a}_i = (a_i, \alpha)$, $i \in \mathbb{N}$. Suppose that $A := \sum_{i=1}^{\infty} a_i$ exists and is finite, then with the notation $\tilde{A}_n = \tilde{a}_1 + \cdots + \tilde{a}_n$ and $A_n = a_1 + \cdots + a_n$ we have*

$$\left(\lim_{n\to\infty} \tilde{A}_n\right)(z) = \frac{1}{1 + |A - z|/\alpha},\ z \in \mathbb{R}.$$

Theorem 6.3 (Fullér, [77]). *Let $\gamma = 2$ and $\tilde{a}_i = (a_i, \alpha)$, $i \in \mathbb{N}$. If $A := \sum_{i=1}^{\infty} a_i$ exists and is finite, then we have*

$$\left(\lim_{n\to\infty} \tilde{A}_n\right)(z) = \frac{2}{1 + \exp\left[\dfrac{-2|A - z|}{\alpha}\right]},\ z \in \mathbb{R}.$$

In 1992 Fullér and Keresztfalvi [79] determined a class of t-norms in which the addition of fuzzy numbers is very simple.

Theorem 6.4 (Fullér and Keresztfalvi, [79]). *Let T be an Archimedean t-norm with additive generator f and let $\tilde{a}_i = (a_i, b_i, \alpha, \beta)_{LR}$, $i = 1, \dots, n$, be fuzzy numbers of LR-type. If L and R are twice differentiable, concave functions, and f is twice differentiable, strictly convex function then the membership function of the T-sum $\tilde{A}_n = \tilde{a}_1 + \cdots + \tilde{a}_n$ is*

$$\tilde{A}_n(z) = \begin{cases} 1 & \text{if } A_n \le z \le B_n \\ f^{[-1]}\left(n \times f\left(L\left(\dfrac{A_n - z}{n\alpha}\right)\right)\right) & \text{if } A_n - n\alpha \le z \le A_n \\ f^{[-1]}\left(n \times f\left(R\left(\dfrac{z - B_n}{n\beta}\right)\right)\right) & \text{if } B_n \le z \le B_n + n\beta \\ 0 & \text{otherwise} \end{cases}$$

where $A_n = a_1 + \cdots + a_n$ and $B_n = b_1 + \cdots + b_n$.

In 1991 Fullér and Keresztfalvi [75] generalized Theorems 2.1 and 2.2 to sup-t-norm extended functions.

Theorem 6.5 (Fullér and Keresztfalvi [75]). *Let $X \neq \emptyset$, $Y \neq \emptyset$, $Z \neq \emptyset$ be sets and let T be a t-norm. If $f\colon X \times Y \to Z$ is a two-place function and $A \in \mathcal{F}(X)$, $B \in \mathcal{F}(Y)$ then a necessary and sufficient condition for the equality*

$$[f(A, B)]^{\alpha} = \bigcup_{T(\xi,\eta)\ge\alpha} f([A]^{\xi}, [B]^{\eta}),\ \alpha \in (0, 1], \tag{6.5}$$

is, that for each $z \in Z$,

$$\sup_{f(x,y)=z} T(A(x), B(y))$$

is attained.

The next theorem shows that the equality (6.5) holds for all upper semi-continuous triangular norm T and continuous function f in the class of upper semi-continuous fuzzy sets of compact support. When X is a topological space, we denote by $\mathcal{F}(X, \mathcal{K})$ the set of all fuzzy sets of X having upper semi-continuous, membership function of compact support.

Theorem 6.6 (Fullér and Keresztfalvi, [75]). *If $f : X \times Y \to Z$ is continuous and the t-norm T is upper semi-continuous, then*

$$[f(A,B)]^\alpha = \bigcup_{T(\xi,\eta)\geq\alpha} f([A]^\xi, [B]^\eta),\ \alpha \in (0,1], \tag{6.6}$$

holds for each $A \in \mathcal{F}(X,\mathcal{K})$ and $B \in \mathcal{F}(Y,\mathcal{K})$.

Equation (6.6) is known in the literature as Nguyen-Fullér-Keresztfalvi (NFK) formula [11].

Example 6.1. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as $f(x,y) = x + y$. Suppose $[A]^\alpha = [a_1(\alpha), a_2(\alpha)]$ and $[B]^\alpha = [b_1(\alpha), b_2(\alpha)]$ are fuzzy numbers. Then using the sup-product-t-norm extension principle we get*

$$(A+B)(z) = \sup_{x+y=z} A(x)B(y)$$

Then,

$$[A+B]^\alpha = \bigcup_{\xi\eta\geq\alpha} \left([A]^\xi + [B]^\eta\right)$$

for all $\alpha \in [0,1]$.

The interactivity relation between fuzzy numbers may be given by a more general joint possibility distribution, which can not be directly defined from the membership values of its marginal possibility distributions by t-norms. In 2004 using the concept of joint possibility distribution Carlsson, Fullér and Majlender, [41] introduced the following extension principle.

Definition 6.2 (Carlsson, Fullér and Majlender, [41]). *Let C be the joint possibility distribution with marginal possibility distributions $A_1, \dots, A_n \in \mathcal{F}$, and let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. Then $f_C(A_1, \dots, A_n)$ is defined by*

$$f_C(A_1, \dots, A_n)(y) = \sup_{y=f(x_1,\dots,x_n)} C(x_1, \dots, x_n). \tag{6.7}$$

if $f^{-1}(y) \neq \emptyset$ and $f_C(A_1, \dots, A_n)(y) = 0$ if $f^{-1}(y) = \emptyset$ (the supremum is set to zero).

We should note here that if $A_1, \dots, A_n$ are non-interactive, that is, their joint possibility distribution is defined by $C(x_1, \dots, x_n) = \min\{A_1(x_1), \dots, A_n(x_n)\}$, then (6.7) turns into the extension principle (2.4) introduced by Zadeh in 1965 [153],

$$f(A_1, \dots, A_n)(y) = \sup_{y=f(x_1,\dots,x_n)} \min\{A_1(x_1), \dots, A_n(x_n)\}.$$

Furthermore, if $C(x_1, \dots, x_n) = T(A_1(x_1), \dots, A_n(x_n))$, where T is a t-norm then we get the t-norm-based extension principle,

$$f_C(A_1, \dots, A_n)(y) = \sup_{y=f(x_1,\dots,x_n)} T(A_1(x_1), \dots, A_n(x_n)).$$

Carlsson, Fullér and Majlender [41] showed that Nguyen's theorem remains valid in this environment.

Theorem 6.7 (Carlsson, Fullér and Majlender, [41])**.** *Let $A_1, \dots, A_n \in \mathcal{F}$ be fuzzy numbers, let C be their joint possibility distribution, and let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. Then,*

$$[f_C(A_1, \dots, A_n)]^\gamma = f([C]^\gamma),$$

for all $\gamma \in [0, 1]$.

Following [41] we will give explicit formulas for the γ-level sets of the extended sum of two interactive fuzzy numbers in perfect correlation. We will show that (i) the interactive sum $A + B$ of two fuzzy numbers A and B having a correlation coefficient minus one, where $B(x) = (-A)(x) = A(-x)$ for all $x \in \mathbb{R}$, is equal to fuzzy zero; (ii) the interactive difference $A - B$, of two fuzzy numbers A and B having a correlation coefficient one and having identical membership functions, is equal to fuzzy zero. Recall that fuzzy numbers A and B are said to be in perfect correlation, if there exist $q, r \in \mathbb{R}$, $q \neq 0$ such that their joint possibility distribution is defined by

$$C(x_1, x_2) = A(x_1) \cdot \chi_{\{qx_1 + r = x_2\}}(x_1, x_2) = B(x_2) \cdot \chi_{\{qx_1 + r = x_2\}}(x_1, x_2), \tag{6.8}$$

where $\chi_{\{qx_1+r=x_2\}}$, stands for the characteristic function of the line $\{(x_1, x_2) \in \mathbb{R}^2 | qx_1 + r = x_2\}$. In this case we have,

$$[C]^\gamma = \big\{(x, qx + r) \in \mathbb{R}^2 \big| x = (1 - t)a_1(\gamma) + ta_2(\gamma), t \in [0, 1]\big\}$$

where $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$, and $[B]^\gamma = q[A]^\gamma + r$, for any $\gamma \in [0, 1]$, and, finally, their membership functions satisfy the following property,

$$B(x) = A\left(\frac{x - r}{q}\right),$$

for all $x \in \mathbb{R}$.

Now let us consider the extended addition of interactive fuzzy numbers A and B that are in perfect correlation,

$$(A + B)(y) = \sup_{y = x_1 + x_2} C(x_1, x_2).$$

That is,

$$(A + B)(y) = \sup_{y = x_1 + x_2} A(x_1) \cdot \chi_{\{qx_1 + r = x_2\}}(x_1, x_2).$$

Then from (6.7) and (6.8) we find,

$$\begin{aligned} [A + B]^\gamma &= \mathrm{cl}\{x_1 + x_2 \in \mathbb{R} | A(x_1) > \gamma, qx_1 + r = x_2\} \\ &= \mathrm{cl}\{(q + 1)x_1 + r \in \mathbb{R} | A(x_1) > \gamma\} \\ &= (q + 1)\mathrm{cl}\{x_1 \in \mathbb{R} | A(x_1) > \gamma\} + r = (q + 1)[A]^\gamma + r, \end{aligned}$$

that is,

$$[A + B]^\gamma = (q + 1)[A]^\gamma + r, \tag{6.9}$$

for all $\gamma \in [0, 1]$. If $q = -1$ then (see Fig. 6.1) $[B]^\gamma = -[A]^\gamma + r$ for all $\gamma \in [0, 1]$, then $A + B$ will be a crisp number. Really, from (6.9) we get $[A + B]^\gamma = 0 \times [A]^\gamma + r = [r, r]^\gamma = \{r\}$, for all $\gamma \in [0, 1]$.

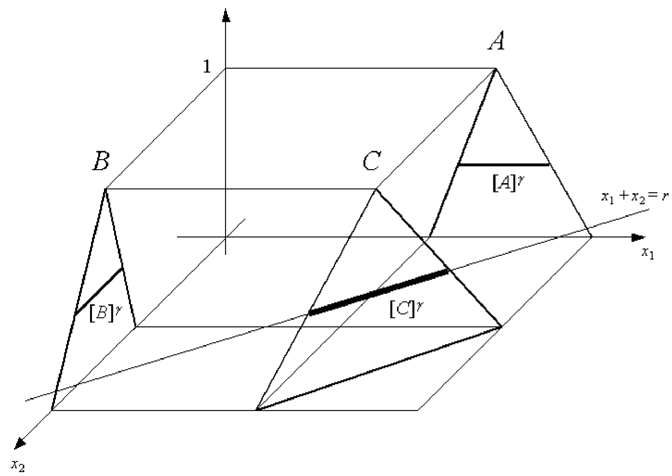

Figure 6.1: The correlation coefficient between A and B is -1.

If $q = -1$ and $r = 0$, i.e. $A(x) = (-B)(x) = B(-x), \forall x \in \mathbb{R}$, then from (6.9) we get

$$(A + B)(z) = \begin{cases} 0 & \text{if } z \neq 0 \\ 1 & \text{if } z = 0 \end{cases}$$

that is, $A + B = \bar{0}$, where $\bar{0}$ denotes a fuzzy point with support $\{0\}$. The interactive sum $A + B$, of two fuzzy numbers A and B having a correlation coefficient -1 and with $A(x) = (-B)(x) = B(-x), \forall x \in \mathbb{R}$, is equal to $\bar{0}$.

Let us consider now the subtraction operator for interactive fuzzy numbers A and B, where their joint possibility distribution is defined by (6.8).

$$(A - B)(y) = \sup_{y = x_1 - x_2} C(x_1, x_2).$$

That is,

$$(A - B)(y) = \sup_{y = x_1 - x_2} A(x_1) \cdot \chi_{\{qx_1 + r = x_2\}}(x_1, x_2).$$

Then for a γ-level set of $A - B$ we get,

$$[A - B]^\gamma = \text{cl}\{x_1 - x_2 \in \mathbb{R} | A(x_1) > \gamma, qx_1 + r = x_2\} = (1 - q)[A]^\gamma - r$$

for all $\gamma \in [0, 1]$. In particular if $q = 1$, i.e. $[B]^\gamma = [A]^\gamma + r$, $\forall \gamma \in [0, 1]$ then $[A - B]^\gamma = -[r, r]^\gamma = -\{r\}$, that is, the fuzziness of $A - B$ vanishes.

If $q = 1$ and $r = 0$ we have $A(x) = B(x)$, for $x \in \mathbb{R}$ and

$$C(x, y) = A(x)\chi_{\{x=y\}}(x, y) = B(y)\chi_{\{x=y\}}(x, y)$$

for all $x, y \in \mathbb{R}$ and from (6.9) we get $A - B = \bar{0}$, for all $z \in \mathbb{R}$. The interactive difference $A - B$, of two fuzzy numbers A and B having a correlation coefficient 1 and having identical membership functions, is equal to $\bar{0}$.

Let A and B be fuzzy numbers, where the membership function of B is defined by

$$B(x) = A\left(\frac{x-r}{q}\right),$$

for any $x \in \mathbb{R}$, then for any $q > 0$ we find

$$[A+B]^{\gamma} = [A]^{\gamma} + [B]^{\gamma} = [A]^{\gamma} + q[A]^{\gamma} + r = (q+1)[A]^{\gamma} + r = [A +_C B]^{\gamma}.$$

for all $\gamma \in [0,1]$. So, $A +_C B = A + B$. that is, the membership function of the interactive sum of fuzzy numbers with correlation coefficient one (defined by (6.7) and (6.8)) is equal to the membership function of their non-interactive sum (defined by their sup-min convolution).

Chapter 7

Selected Industrial Applications

In Chapter "Selected Industrial Applications" I will describe 6 industrial research projects in which I participated as a researcher at Institute for Advanced Management Systems Research (IAMSR), Åbo Akademi University, Åbo, Finland between 1992 and 2011. In the majority of these projects our research team implemented computerized decision support systems, where all input data and information were imprecise (obtained from human judgments) and, therefore, possessed non-statistical uncertainties.

"The Knowledge Mobilization project" has been a joint effort by Institute for Advanced Management Systems Research, Åbo Akademi University and VTT Technical Research Centre of Finland. Its goal was to better "mobilize" knowledge stored in heterogeneous databases for users with various backgrounds, geographical locations and situations. The working hypothesis of the project was that fuzzy mathematics combined with domain-specific data models, in other words, fuzzy ontologies, would help manage the uncertainty in finding information that matches the user's needs. In this way, Knowledge Mobilization places itself in the domain of knowledge management. We will describe an industrial demonstration of fuzzy ontologies in information retrieval in the paper industry where problem solving reports are annotated with keywords and then stored in a database for later use.

In the Woodstrat project we implemented a support system for strategy formation and show that the effectiveness and usefulness of hyperknowledge support systems for strategy formation can be further advanced using adaptive fuzzy cognitive maps.

In the Waeno research project we implemented fuzzy real options theory as a series of models, which were built on Excel platforms. The models were tested on a number of real life investments, i.e. real (so-called) giga-investment decisions were made on the basis of the results. The new series of models, for fuzzy real option valuation (ROV), have been tested with real life data and the impact of the innovations have been traced and evaluated against both the traditional ROV-models and the classical net present value (NPV) models. The fuzzy real options were found to offer more flexibility than the traditional models; both versions of real option valuation were found to give better guidance than the classical NPV models. A total of 8 actual giga-investment decisions were studied and worked out with the real options models.

In the EM-S Bullwhip project we suggested a fuzzy approach to reduce the bullwhip effect in supply chains. The research work focused on the demand fluctuations in paper mills caused by the frictions of information handling in the supply chain and worked out means to reduce or eliminate the fluctuations with the help of information technology. The program enhanced existing theoretical frameworks with fuzzy logic modelling and built a hyperknowledge platform for fast implementation of the theoretical results.

In the Assessgrid project we developed a hybrid probabilistic and possibilistic model to assess the success of computing tasks in a Grid. Using the predictive probabilistic approach we developed a framework for resource management in grid computing, and by introducing an upper limit for the number of possible failures, we approximated the probability that a particular computing task can be executed. We also showed a lower limit for the probability of success of a computing task in a grid. In the possibilistic model we estimated the possibility distribution defined over the set of node failures using a fuzzy nonparametric regression technique.

In the OptionsPort project we developed a model for valuing options on R&D projects, when future cash flows and expected costs are estimated by trapezoidal fuzzy numbers. Furthermore, we represented the optimal R&D portfolio selection problem as a fuzzy mathematical programming problem, where the optimal solutions defined the optimal portfolios of R&D projects with the largest (aggregate) possibilistic deferral flexibilities.

7.1 The Knowledge Mobilisation project

Knowledge Mobilisation - KNOWMOBILE, TEKES [40211/08] project (2008-2011). Partners: Institute for Advanced Management Systems Research, Åbo University, VTT Technical Research Centre of Finland, UC Berkeley. Industrial partners: Metso Automation, Kemira, Ruukki, UPM-Kymmene. Our publications in this project: Carlsson, Fullér and Mezei [59], Carlsson, Fullér and Fedrizzi [61], Juhani Hirvonen, Tommila, Pakonen, Carlsson, Fedrizzi and Fullér, [102]. A longer description of this project can be found in Carlsson and Fullér [63]. The key research question of the Knowledge Mobilisation project was [61, 102]: How to build fuzzy ontologies for the process industry domain to enhance knowledge retrieval? My contribution to this project: Carlsson, Fedrizzi and Fullér [61] showed an algorithm for approximating keyword dependencies in the keyword ontology, then computed the degrees of dependency between keywords on the immediate upper level using the max-min approach. Then repeated this procedure until the top layer.

In the Knowledge Mobilisation project, we have developed a concept of a tool for searching plant knowledge with a search engine based on a fuzzy ontology. The usage scenario for the tool was that a process expert, dealing with a problem in the process chemistry of a paper machine, wishes to find past problem solving cases of a similar setting in order to find possible solutions to a current issue. This setting is a universal one: pieces of knowledge, called ”nuggets”, are written and stored by companies on different domains in the form of incident reports.

In the Knowledge Mobilisation project we have focused on the chemistry of the ”wet end” in order to limit the work effort needed to construct the domain ontology and concentrate on a subject on which domain expertise and actual data were available. Nuggets are documents than can contain all kinds of raw data or multimedia extracted from different information systems. An expert author annotates the nuggets with suitable keywords, and it is these keywords that the search is then based on. In addition to providing exact results to queries, the tool uses a fuzzy domain ontology to extend the query to related keywords (see Figure 7.1). As a result, the search results include nuggets that may not necessary deal with exactly the same process equipment, variable, function or chemical, but nuggets that may still provide valuable insight to solving the problem at hand. The key research question of the Knowledge Mobilisation project was [61, 102]: How to build fuzzy ontologies for the process industry domain to enhance knowledge retrieval?

Our demonstration works with both engineering and operational knowledge of an industrial plant. Therefore, the fuzzy ontology should not be developed in separation from existing engineering tools

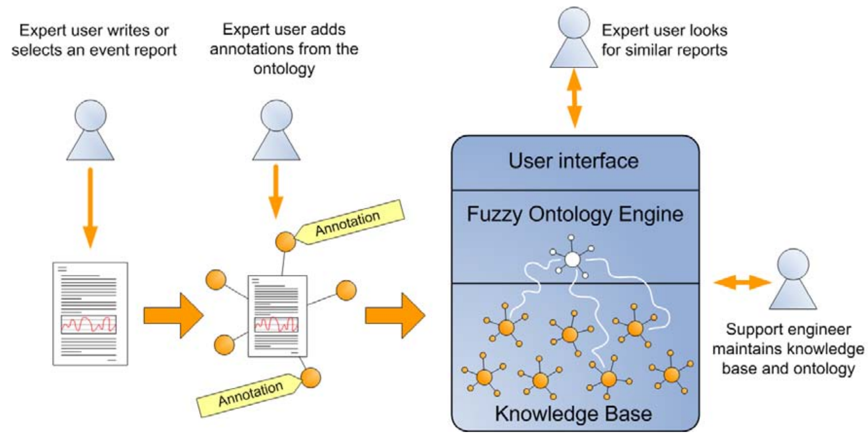

Figure 7.1: System concept-a knowledge base of event reports [61, 102].

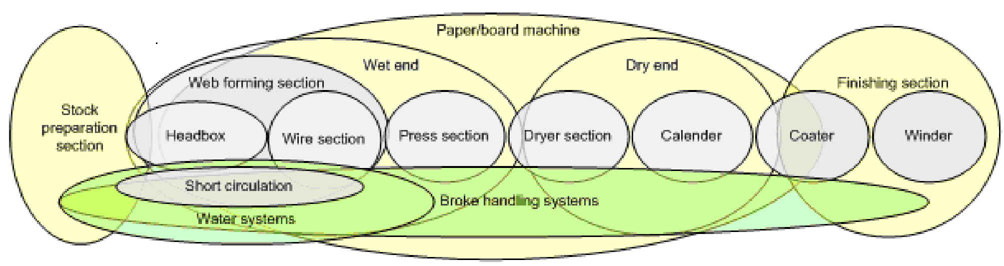

Figure 7.2: System example - a fuzzy decomposition of a paper making line. [61, 102].

and knowledge repositories, but existing terminologies, taxonomies and data models should be used if possible. This leads to a taxonomic system consisting of several layers,

- Top layer: general concepts (i.e. based on international standards) that apply to several industries.
- Middle layer: vocabulary defined and shared by business partners (within a certain industry, again based on standards) to share knowledge of, e.g. the type and structure of process equipment. This layer extends the top layer with domain-specific keywords.
- Bottom layer: custom, company-specific concepts, e.g. specific products and component types, or even individual process plants.

In order to speak about an ontology, our system of keywords should represent concepts, properties, relationships, axioms, and reasoning schemes relevant for the application area. On the basis of various upper ontologies and industrial data models we identified that the following keyword categories are needed to characterize event reports:

- Systems: types of real-world components of a process plant, e.g. machines, buildings, software and people.

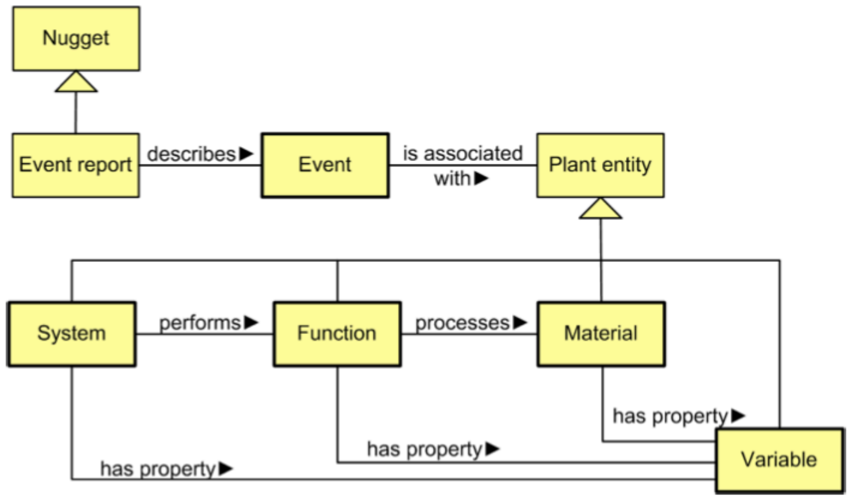

Figure 7.3: Overall domain concepts [61, 102].

- Functions: phenomena and activities carried out at an industrial plant in order to fulfill its purpose.
- Variables: properties and state variables of various entities, e.g. temperature.
- Events: types of interesting periods of plant life described in event reports, e.g. test runs or equipment failures.
- Materials: raw materials, products, consumables etc. handled in a process plant.

Our basic approach to conceptualize our application is shown in the informal UML class diagram below (Figure 7.1).

Event reports describe events that are related to various entities of a process plant, e.g. to equipment, processing functions and materials. Nothing is assumed about the internal structure of event reports. Instead, they are characterized by an expert with keywords selected from a fuzzy ontology. The expert can select the keywords from five categories: event, system, function, material and variable. All keywords represent an entity type and can have subtypes and smaller parts. Therefore, keywords can be understood as representatives of populations of real-world entities that overlap and are related in many ways. For example, the keyword ”paper machine” might represent the set of all paper machines in the world. Classification (is-a) and decomposition (part-of) can be found in most ontologies and data models. They are important in the industrial context as well. So, the keywords in each category are linked by is-a and part-of relationships. Furthermore, the ontology should model functional and other kinds of dependencies between keywords in various keyword categories. As an example, systems can be or are used for some purposes, i.e. they play various roles in carrying out one or more functions. This creates a link between the keywords ”wire section (a part of paper machine)” and ”formation (a quality measure of the produced paper)”. Modeling classifications, decompositions and various dependencies leads to a situation where we have a taxonomy tree for each keyword category and a set of partonomy (part-of relationships) trees describing the decomposition to various domain entities.

For developing a software tool the ontology should be expressed and stored in a more formal way. The basic approach for representing a fuzzy ontology is illustrated in Figure 7.4 with a combination of

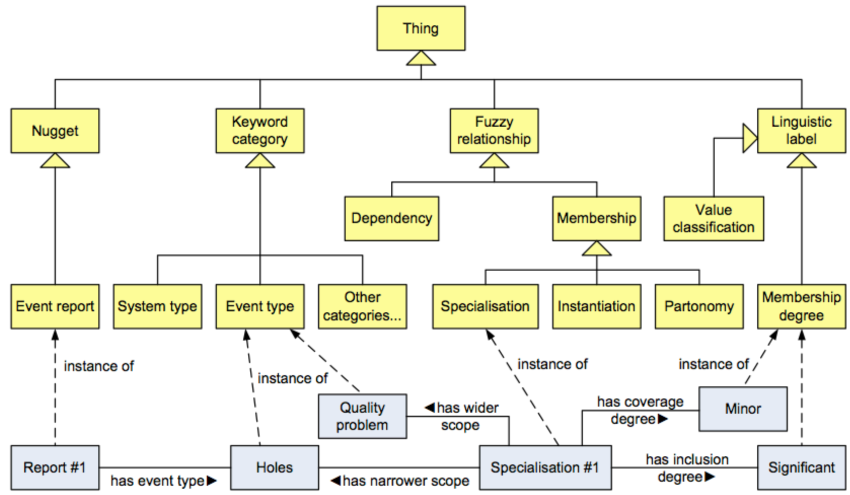

Figure 7.4: Representing fuzzy keyword ontology with object classes and their instances [61, 102].

UML class and instance diagrams. Fuzzy dependencies between keyword instances are described by a few fundamental relationship types like is-a (specialization), part-of and, as an option, instantiation. In the first version we focus on ”specializations”, i.e. fuzzy classification of keywords. The degree of overlapping (or inclusion) of the sets represented by the keywords is described by linguistic labels, i.e. natural language words like ”moderate” or ”significant”. So, the instance named ”Specialization #1” in Figure 7.4 tells us that ”Holes” is ”to a large extent” understood as a subclass of ”Quality problem but only represents a minor part of its scope”. In addition, the keyword ”Holes” may also specialize other problem types.

Carlsson, Fedrizzi and Fullér [61] showed a method for approximating keyword dependencies in the keyword ontology. Their method uses Bellman-Zadeh's principle to fuzzy decision making [6]. An event type fuzzy taxonomy is shown in Figure 7.5. For example, consider the second column of event classification matrix ”Problem”. All ”Technical problems” are ”Problems” and they represent around 80% of all possible problems. That is ”Technical problem” covers ”Problem” with degree 0.8. Similarly, ”Human errors” are ”Problems ” and they represent around 30% of all possible problems. That is ”Human error” covers ”Problem” with degree 0.3.

We will assume that if A and B are two keywords in the keyword taxonomic tree then

$$\text{coverage}(A, B) = \text{fuzzy inclusion}(A, B). \tag{7.1}$$

For example, ”System fault = {Device fault, Design flaw}” that is, ”System fault” is a union of these two events. Furthermore, ”Function failure = {Design flaw, Drift, Oscillation}” that is ”Function failure” is the union of these three events. ”Design flaw” covers ”System fault” with degree 0.6 and at the same time ”Design flaw” covers ”Function failure” with degree 0.4 (see Figure 7.5). Moreover, ”System fault” and ”Function failure” do not have any more component in common. We compute the degree of dependency between ”System fault” (SF) and ”Function failure” (FF) as their joint coverage by ”Design flaw” (DF)

$$\text{dependency}(SF, FF) = \min\{\text{coverage}(DF, SF), \text{coverage}(DF, FF)\},$$

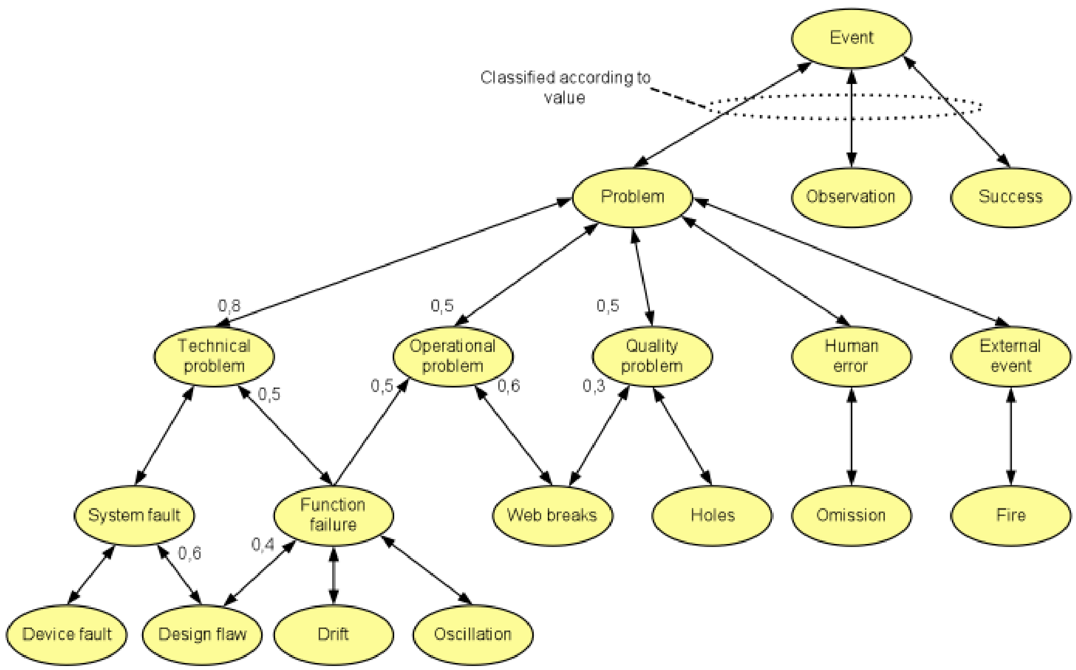

Figure 7.5: A fragment of the event type fuzzy taxonomy [61, 102].

that is,

$$\text{dependency}(SF, FF) = \min\{0.6, 0.4\} = 0.4 \tag{7.2}$$

It is easy to see in Figure 7.5 that keywords ”Function failure” and ”Fire” are independent since they do not have any component in common. In this case we have,

$$\text{dependency}(\text{"Function failure"}, \text{"Fire"}) = 0. \tag{7.3}$$

Zero means independence, one means full dependence, and values between zero and one denote intermediate degrees of dependency between keywords. It can happen that two keywords have more than one joint component. Then we apply Bellman-Zadeh’s principle (max-min approach) to fuzzy decision making to measure their dependency. For example, suppose that ”System fault” and ”Function failure” wrere to have two joint components, where the first one is ”Design flaw” and the second one ”Fluctuation” that has a coverage values 0.7 and 0.5, respectively. Then we measure the degree of dependency between ”System fault” and ”Function failure” according to Bellman-Zadeh’s principle to fuzzy decision making as

$$\text{dependency}\{SF, FF\} = \max\{\min\{0.6, 0.4\}, \min\{0.7, 0.5\}\} = \max\{0.4, 0.5\} = 0.5.$$

Supposing that all the coverage degrees are given by experts. Then we can summarize our algorithm as follows: *Compute the degrees of dependency between keywords on the immediate upper level using the max-min approach. Then repeat this procedure until the top layer.* For example, consider keywords ”Technical problem” (TP) and ”Operational problem” (OP). Then we find (Figure 7.5),

$$\text{dependency}\{TP, OP\} = \min\{\text{coverage}(FF, TP), \text{coverage}(FF, OP)\} = 0.5.$$

One can further improve this model by introducing degrees of inclusion and coverage between concepts as suggested by Holi and Hyvönen [103].

Demo architecture and implementation

The component-based demo architecture has been implemented by VTT Technical Research Centre of Finland, using the Protégé ontology editor to maintain the fuzzy ontology in OWL format. The GUI component (Graphical User Interface) guides the user in specifying the information query, and presents the results. Tools for browsing and evaluating the fuzzy reasoner component directly were also provided. A database adapter is used to access report data, which in this case was stored locally in XML files. Similarly, an ontology adapter is used to provide access to the fuzzy ontology, in this case stored in OWL files. The adapters help hide the different interfaces and protocols of different data sources (e.g. SQL, HTTP) and provide transparent access via an agreed interface. The fuzzy ontology reasoner component is used to process ontology-based information. Its main function in the demo is to extend a list of query keywords to a list of their closest neighbours in terms of fuzzy ontology relationships. For maintenance and evaluation purposes, the component interface also provides methods for directly accessing the ontology concepts and relationships. Finally, the application logic component binds all the functionality together by taking the query, using the reasoner component to extend it, passing the extended query to the report database and then combining and ordering the results for the GUI. The fuzzy ontology with fuzzy concepts, relations, and instances was defined using Protégé version 3.4.

7.2 The Woodstrat project

Woodstrat/Tekes-konsortium(1992-1994), ERUDIT-WOOD/Tekes-konsortium (1995-1996). Ipari partnerek: Metsä-Serla (coordinator), L M Ericsson, Sampo és Valmet. Our publications in this project: Carlsson and Fullér [16]. My contribution to the project: In [16] we suggested an adaptive fuzzy cognitive map for modelling the strategy formation process and and implemented an error correction learning algorithm for fine-tuning the cause-effect relationships among the elements of the strategy building process.

Strategic Management is defined as a system of action programs which form sustainable competitive advantages for a corporation, its divisions and its business units in a strategic planning period. A research team of the IAMSR institute has developed a support system for strategic management, called the *Woodstrat*, in two major Finnish forest industry corporations in 1992-96. The system is modular and is built around the actual business logic of strategic management in the two corporations, i.e. the main modules cover the *market position* (MP), the *competitive position* (CP), the *productivity position* (PROD), the *profitability* (PROF) , the investments (INV) and the *financing of investments* (FIN).

The innovation in *Woodstrat* is that these modules are linked together in a hyperknowledge fashion, i.e. when a strong market position is built in some market segment it will have an immediate impact on profitability through links running from key assumptions on expected developments to the projected income statement. There are similar links making the competitive position interact with the market position, and the productivity position interact with both the market and the competitive positions, and with the profitability and financing positions. The basis for this is rather unusual: the *Woodstrat* system was built with Visual Basic in which the objects to create a hyperknowledge environment were built. The *Woodstrat* offers an intuitive and effective strategic planning support with object-oriented expert systems elements and a hyperknowledge user interface.

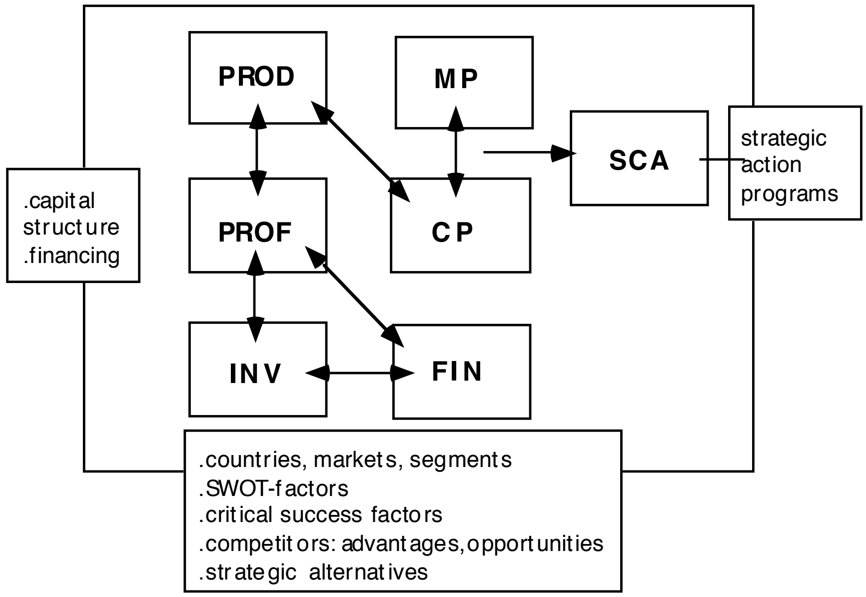

Figure 7.6: The framework of Woodstrat.

Carlsson and Fullér [16] showed that the effectiveness and usefulness of a hyperknowledge support system can be further advanced using adaptive fuzzy cognitive maps (FCM).

It is relatively easy to create cause-effect relationships among the elements of the strategy building process, however it is time-consuming and difficult to fine-tune them. Neural nets give a shortcut to tuning fuzzy cognitive maps. The trick is to let the fuzzy causal edges change as if they were synapses (weights) in a neural net. Each arrow in Fig. 7.7 defines a fuzzy rule. We weigh these rules or arrows with a number from the interval $[-1, 1]$, or alternatively we could use *word weights* like *little*, or *somewhat*, or *more or less*. The states or nodes are fuzzy too. Each state can fire to some degree from 0% to 100%. In the crisp case the nodes of the network are *on* or *off*. In a real FCM the nodes are fuzzy and fire more as more causal juice flows into them. Adaptive fuzzy cognitive maps can learn the weights from historical data. Once the FCM is trained it lets us play what-if games (e.g. *What if demand goes up and prices remain stable? - i.e. we improve our MP*)

Carlsson and Fullér [16] described a learning mechanism for the fuzzy cognitive maps of the strategy building process, and illustrated the effectiveness of the map by a simple training set.

Inputs of states are computed as the weighted sum of the outputs of its causing states

$$net = Wo$$

where W denotes the matrix of weights, o is the vector of computed outputs, and net is the vector of

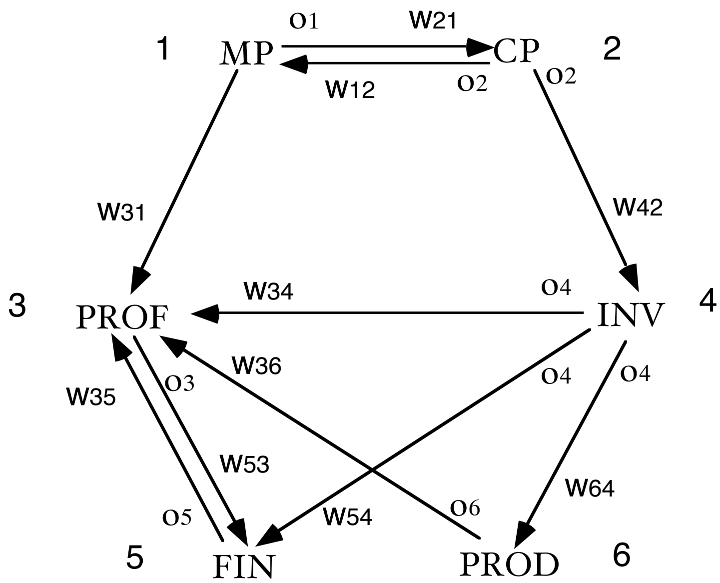

Figure 7.7: Adaptive fuzzy cognitive map for the strategy formation process (see Carlsson and Fullér [16]).

inputs to the states. In our case the weight matrix is given by

$$W = \begin{pmatrix} 0 & w_{12} & 0 & 0 & 0 & 0 \\ w_{21} & 0 & 0 & 0 & 0 & 0 \\ w_{31} & 0 & 0 & w_{34} & w_{35} & w_{36} \\ 0 & w_{42} & 0 & 0 & 0 & 0 \\ 0 & 0 & w_{53} & w_{54} & 0 & 0 \\ 0 & 0 & 0 & w_{64} & 0 & 0 \end{pmatrix}$$

where the zero elements denote no causal link between the states, and

$$net = \begin{bmatrix} net_1 \\ net_2 \\ net_3 \\ net_4 \\ net_5 \\ net_6 \end{bmatrix} = \begin{bmatrix} net(MP) \\ net(CP) \\ net(PROF) \\ net(INV) \\ net(FIN) \\ net(PROD) \end{bmatrix} \qquad o = \begin{bmatrix} o_1 \\ o_2 \\ o_3 \\ o_4 \\ o_5 \\ o_6 \end{bmatrix} = \begin{bmatrix} o(MP) \\ o(CP) \\ o(PROF) \\ o(INV) \\ o(FIN) \\ o(PROD) \end{bmatrix}$$

That is,

$$net_1 = net(MP) = w_{12}o_2,\ net_2 = net(CP) = w_{21}o_1,$$

$$net_3 = net(PROF) = w_{31}o_1 + w_{34}o_4 + w_{35}o_5 + w_{36}o_6,\ net_4 = net(INV) = w_{42}o_2,$$

$$net_5 = net(FIN) = w_{54}o_4 + w_{53}o_3,\ net_6 = net(PROD) = w_{64}o_4$$

The output of state i is is computed by a squashing function

$$o_i = \frac{1}{1 + \exp(-net_i)}$$

Suppose we are given a set of historical training data

$$(MP(t), CP(t), PROF(t), INV(t), FIN(t), PROD(t))$$

where $t = 1, \dots, K$. Here $MP(t)$ is the observed value of the market position, $CP(t)$ is the value of the competitive position at time t, and so on. Using an error correction learning procedure we find the weights by minimizing the overall error

$$E(W) = \frac{1}{2} \sum_{t=1}^{K} \{ (MP(t) - o_1(t))^2 + (CP(t) - o_2(t))^2 +$$

$$(PROF(t) - o_3(t))^2 + (INV(t) - o_4(t))^2 + (FIN(t) - o_5(t))^2 + (PROD(t) - o_6(t))^2 \}$$

where $o_i(t)$, the computed value of the i-th state at time t, is determined as

$$o_i(t) = \frac{1}{1 + \exp[-net_i(t-1)]} = \frac{1}{1 + \exp[-\sum_j w_{ij} o_j(t-1)]}$$

where j is a causing state for state i. The weights are initialized at small random values. The rule for changing the weights of the states is derived from he gradient descent method.

In the *Woodstrat* project the development work was done interactively with the strategic business units management teams and involved more than 60 managers in 14 strategic business units. In terms of technology, the *Woodstrat* is a hybrid of an object-oriented expert system and a hyperknowledge system. It is built with Visual Basic (version 3.0) in which the expert and hyperknowledge properties of the previous prototypes in Lisp and Toolbook were reconstructed.

7.3 The AssessGrid project

AssessGrid - Advanced Risk Assessment & Management for Trustable Grids, EU Sixth Framework, Programme acronym: FP6-IST, Contract number: IST-2005-031772. Partners: CETIC, University of Leeds, Wincor-Nixdorf, Paderborn Center for Parallel Computing, Atos Origin, Technical University of Berlin, Åbo Akademi University. Publications in this project: Carlsson and Fullér [57, 60, 64], Carlsson, Fullér and Mezei [53, 56]. A longer description of this project can be found in Carlsson and Fullér [63]. My contribution to this project: Carlsson and Fullér [64] developed a hybrid probabilistic and possibilistic model to assess the success of computing tasks in a Grid.

The AssessGrid project aimed to satisfy the demands for transparent and understandable risk evaluation by extending the Grid technology with methods for risk assessment and management as core services of future Grids. Since end-users, brokers, and providers have different perspectives on risk enhanced Grid services, they define the three major AssessGrid objectives: the end-user seeks a reliable and trustworthy provider, the broker looks for the best offer for its customer, and the provider aims to reduce the risk of Service Level Agreement (SLA) violation. Furthermore, providers need objective measures to lower the execution risk and to analyse their infrastructure in order to remove bottlenecks. Therefore, the main objectives of the AssessGrid project were: (i) Mechanisms for risk identification, risk assessment, risk treatment, and risk monitoring on all Grid layers as decisive components for negotiation and enforcement of SLAs as well as the definition of business models. (ii) Risk-based support for decision making in quality and capacity planning leading to higher productivity and cost-effective usage of virtualized resources. (iii) Risk measures as parameters for self-organizing fault-tolerant systems Improved transparency, usability, and trustiness by customized presentation of confidence and risk information to end-users, brokers, and providers. (iv) Aggregated quality and reliability information of performance as fundament for provider evaluation and competition (v) A consistent realization based on existing Grid developments and standards as well as the evaluation of risk management methods in real-world environments. (see http://pc2.uni-paderborn.de/research-projects/project/assessgrid/)

Our research team at IAMSR, Åbo Akademi University, developed and integrated methods for risk assessment and management for Grids. Namely, Carlsson and Fullér [64] developed a hybrid probabilistic and possibilistic model to assess the success of computing tasks in a Grid. Using the predictive probabilistic approach we developed a framework for resource management in grid computing, and by introducing an upper limit for the number of possible failures, we approximated the probability that a particular computing task can be executed. We also showed a lower limit for the probability of success of a computing task in a grid [56]. In the possibilistic model we estimated the possibility distribution defined over the set of node failures using a fuzzy nonparametric regression technique. The probabilistic models scale from 10 nodes to 100 nodes (and then on to any number of nodes); while the possibilistic models scale to 100 nodes. The resource Provider can use both models to get two alternative risk assessments. In the AssessGrid project we carried out a number of validation tests in order to find out (i) how well the predictive possibilistic models can be fitted to the Los Alamos National Laboratory dataset, (ii) what differences can be found between the probabilistic and possibilistic predictions and (iii) if these differences can be given reasonable explanations. In the testing we worked with short and long duration computing tasks scheduled on a varying number of nodes and the Service Level Agreement probabilities of failure estimates remained reasonable throughout the testing.

7.4 The Waeno project

Waeno research project on giga-investments (TEKES [40682/99; 40470/00], industrial partners: Fortum, M-Real, Outokumpu, Rautaruukki). Publications in this project: Carlsson and Fullér [21, 27, 28, 32, 36]. My contribution to this project: Carlsson and Fullér [36] developed a hybrid heuristic fuzzy real option valuation method which was used in assessing the productivity and profitability of the original giga-investment.

Giga-investments made in the paper- and pulp industry, in the heavy metal industry and in other base industries, today face scenarios of slow (or even negative) growth (2-3 % p.a.) in their key markets and a growing over-capacity in Europe. The energy sector faces growing competition with lower prices and cyclic variations of demand. There is also some statistics, which shows that productivity improvements in these industries have slowed down to 1-2 % p.a., which opens the way for effective competitors to gain footholds in their main markets. Giga-investments compete for major portions of the risk-taking capital, and as their life is long, compromises are made on their short-term productivity. The short-term productivity may not be high, as the life-long return of the investment may be calculated as very good. Another way of motivating a giga-investment is to point to strategic advantages, which would not be possible without the investment and thus will offer some indirect returns. The core products and services produced by giga-investments are enhanced with life-time service, with gradually more advanced maintenance and financial add-on services. These make it difficult to actually assess the productivity and profitability of the original giga-investment, especially if the products and services are repositioned to serve other or emerging markets. New technology and enhanced technological innovations will change the life cycle of a giga-investment. The challenge is to find the right time and the right innovation to modify the life cycle in an optimal way.

Decision trees are excellent tools for making financial decisions where a lot of vague information needs to be taken into account. They provide an effective structure in which alternative decisions and the implications of taking those decisions can be laid down and evaluated. They also help us to form an accurate, balanced picture of the risks and rewards that can result from a particular choice. In our empirical cases we have represented strategic planning problems by dynamic decision trees, in which the nodes are projects that can be deferred or postponed for a certain period of time. Using the theory of real options we have been able to identify the optimal path of the tree, i.e. the path with the biggest real option value in the end of the planning period.

In 1973 Black and Scholes [7] made a major breakthrough by deriving a differential equation that must be satisfied by the price of any derivative security dependent on a non-dividend paying stock. For risk-neutral investors the *Black-Scholes pricing formula* for a call option is

$$C_0 = S_0 N(d_1) - Xe^{-rT} N(d_2),$$

where

$$d_1 = \frac{\ln(S_0/X) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T},$$

and where, C_0 is the option price, S_0 is the current stock price, $N(d)$ is the probability that a random draw from a standard normal distribution will be less than d, X is the exercise price, r is the annualized continuously compounded rate on a safe asset with the same maturity as the expiration of the option, T is the time to maturity of the option (in years) and σ denotes the standard deviation of the annualized continuously compounded rate of return of the stock. In 1973 Merton [127] extended the Black-Scholes option pricing formula to dividends-paying stocks as

$$C_0 = S_0 e^{-\delta T} N(d_1) - Xe^{-rT} N(d_2) \tag{7.4}$$

where,

$$d_1 = \frac{\ln(S_0/X) + (r - \delta + \sigma^2/2)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}$$

where δ denotes the dividends payed out during the life-time of the option. *Real options* in option thinking are based on the same principles as financial options. In real options, the options involve ”real” assets as opposed to financial ones. To have a ”real option” means to have the possibility for a certain period to either choose for or against making an investment decision, without binding oneself up front. For example, owning a power plant gives a utility the opportunity, but not the obligation, to produce electricity at some later date.

Real options can be valued using the analogue option theories that have been developed for financial options, which is quite different from traditional discounted cash flow investment approaches. Leslie and Michaels [113] suggested the following rule for computing the value of a real option,

$$\mathrm{ROV} = S_0 e^{-\delta T} N(d_1) - Xe^{-rT} N(d_2) \tag{7.5}$$

where,

$$d_1 = \frac{\ln(S_0/X) + (r - \delta + \sigma^2/2)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}$$

and where ROV denotes the current real option value, S_0 is the present value of expected cash flows, X is the (nominal) value of fixed costs, σ quantifies the uncertainty of expected cash flows, and δ denotes the value lost over the duration of the option.

Usually, the present value of expected cash flows can not be be characterized by a single number. However, our experiences with the Waeno research project on giga-investments show that managers are able to estimate the present value of expected cash flows by using a trapezoidal possibility distribution of the form $\tilde{S}_0 = (s_1, s_2, \alpha, \beta)$, i.e. the most possible values of the present value of expected cash flows lie in the interval $[s_1, s_2]$ (which is the core of the trapezoidal fuzzy number $\tilde{S}_0$), and $(s_2 + \beta)$ is the upward potential and $(s_1 - \alpha)$ is the downward potential for the present value of expected cash flows. In a similar manner one can estimate the expected costs by using a trapezoidal possibility distribution of the form $\tilde{X} = (x_1, x_2, \alpha', \beta')$, i.e. the most possible values of expected cost lie in the interval $[x_1, x_2]$ (which is the core of the trapezoidal fuzzy number $\tilde{X}$), and $(x_2 + \beta')$ is the upward potential and $(x_1 - \alpha')$ is the downward potential for expected costs.

Following Carlsson and Fullér [36] we suggest the use of the following (heuristic) formula for computing fuzzy real option values

$$\text{FROV} = \tilde{S}_0 e^{-\delta T} N(d_1) - \tilde{X} e^{-rT} N(d_2), \tag{7.6}$$

where,

$$d_1 = \frac{\ln(E(\tilde{S}_0)/E(\tilde{X})) + (r - \delta + \sigma^2/2)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}, \tag{7.7}$$

and where, $E(\tilde{S}_0)$ denotes the possibilistic mean value [26] of the present value of expected cash flows, $E(\tilde{X})$ stands for the the possibilistic mean value of expected costs and $\sigma := \sigma(\tilde{S}_0)$ is the possibilistic variance [26] of the present value expected cash flows. Using formulas (2.7 - 2.8) for arithmetic operations on trapezoidal fuzzy numbers we find

$$\begin{aligned}
\text{FROV} = (s_1, s_2, \alpha, \beta)e^{-\delta T} N(d_1) - (x_1, x_2, \alpha', \beta')e^{-rT} N(d_2) = \\
(s_1 e^{-\delta T} N(d_1) - x_2 e^{-rT} N(d_2), s_2 e^{-\delta T} N(d_1) - x_1 e^{-rT} N(d_2), \\
\alpha e^{-\delta T} N(d_1) + \beta' e^{-rT} N(d_2), \beta e^{-\delta T} N(d_1) + \alpha' e^{-rT} N(d_2)).
\end{aligned} \tag{7.8}$$

We have a specific context for the use of the real option valuation method with fuzzy numbers, which is the main motivation for our approach. Giga-investments require a basic investment exceeding 300 million euros and they normally have a life length of 15-25 years. The standard approach with the NPV or DCF methods is to assume that uncertain revenues and costs associated with the investment can be estimated as probabilistic values, which in turn are based on historic time series and observations of past revenues and costs. We have discovered that giga-investments actually influence the end-user markets in *non-stochastic ways* and that they are normally significant enough to have an impact on market strategies, on technology strategies, on competitive positions and on business models. Thus, the use of assumptions on purely stochastic phenomena is not well-founded.

We will show now a simple example for computing FROV. Suppose we want to find a fuzzy real option value under the following assumptions,

$$\tilde{S}_0 = (\$400 \text{ million}, \$600 \text{ million}, \$150 \text{ million}, \$150 \text{ million}),$$

$r = 5\%$ per year, $T = 5$ years, $\delta = 0.03$ per year and

$$\tilde{X} = (\$550 \text{ million}, \$650 \text{ million}, \$50 \text{ million}, \$50 \text{ million}),$$

First calculate

$$\sigma(\tilde{S}_0) = \sqrt{\frac{(s_2 - s_1)^2}{4} + \frac{(s_2 - s_1)(\alpha + \beta)}{6} + \frac{(\alpha + \beta)^2}{24}} = \$154.11 \text{ million},$$

i.e. $\sigma(\tilde{S}_0) = 30.8\%$,

$$E(\tilde{S}_0) = \frac{s_1 + s_2}{2} + \frac{\beta - \alpha}{6} = \$500 \text{ million},$$

and

$$E(\tilde{X}) = \frac{x_1 + x_2}{2} + \frac{\beta' - \alpha'}{6} = \$600 \text{ million},$$

furthermore,

$$N(d_1) = N\left(\frac{\ln(600/500) + (0.05 - 0.03 + 0.308^2/2) \times 5}{0.308 \times \sqrt{5}}\right) = 0.589, \quad N(d_2) = 0.321.$$

Thus, from (7.6) we obtain the fuzzy value of the real option as

$$\text{FROV} = (\$40.15 \text{ million}, \$166.58 \text{ million}, \$88.56 \text{ million}, \$88.56 \text{ million}).$$

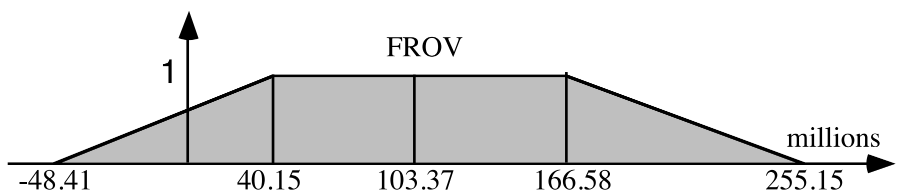

Figure 7.8: The possibility distribution of real option values.

The expected value of FROV is \$103.37 million and its most possible values are bracketed by the interval $[\$40.15 \text{ million}, \$166.58 \text{ million}]$, the downward potential (i.e. the maximal possible loss) is \$48.41 million, and the upward potential (i.e. the maximal possible gain) is \$255.15 million. From Fig. 7.4 we can see that the set of most possible values of fuzzy real option $[40.15, 166.58]$ is quite big. It follows from the huge uncertainties associated with cash inflows and outflows.

Following Carlsson and Fullér [21, 27, 28, 32] we shall generalize the probabilistic decision rule for optimal investment strategy to a fuzzy setting: Where the maximum deferral time is T, make the investment (exercise the option) at time t^*, $0 \le t^* \le T$, for which the option, $\tilde{C}_{t^*}$, attends its maximum value,

$$\tilde{C}_{t^*} = \max_{t=0,1,\dots,T} \tilde{C}_t = \tilde{V}_t e^{-\delta t} N(d_1) - \tilde{X} e^{-rt} N(d_2), \tag{7.9}$$

where

$$\tilde{V}_t = \text{PV}(\tilde{\text{cf}}_0, \dots, \tilde{\text{cf}}_T, \beta_P) - \text{PV}(\tilde{\text{cf}}_0, \dots, \tilde{\text{cf}}_t, \beta_P) = \text{PV}(\tilde{\text{cf}}_{t+1}, \dots, \tilde{\text{cf}}_T, \beta_P),$$

that is,

$$\tilde{V}_t = \tilde{\text{cf}}_0 + \sum_{j=1}^{T} \frac{\tilde{\text{cf}}_j}{(1+\beta_P)^j} - \tilde{\text{cf}}_0 - \sum_{j=1}^{t} \frac{\tilde{\text{cf}}_j}{(1+\beta_P)^j} = \sum_{j=t+1}^{T} \frac{\tilde{\text{cf}}_j}{(1+\beta_P)^j},$$

where $\tilde{\mathrm{cf}}_t$ denotes the expected (fuzzy) cash flow at time t, β_P is the risk-adjusted discount rate (or required rate of return on the project). However, to find a maximizing element from the set

$$\{\tilde{C}_0, \tilde{C}_1, \ldots, \tilde{C}_T\},$$

is not an easy task because it involves ranking of trapezoidal fuzzy numbers. In our computerized implementation we have employed the following value function to order fuzzy real option values, $\tilde{C}_t = (c_t^L, c_t^R, \alpha_t, \beta_t)$, of trapezoidal form:

$$v(\tilde{C}_t) = \frac{c_t^L + c_t^R}{2} + r_A \cdot \frac{\beta_t - \alpha_t}{6},$$

where $r_A \geq 0$ denotes the degree of the investor's risk aversion. If $r_A = 0$ then the (risk neutral) investor compares trapezoidal fuzzy numbers by comparing their possibilistic expected values, i.e. he does not care about their downward and upward potentials.

In 2003 Carlsson and Fullér [36] outlined the following methodology used in the Waeno project (to keep confidentiality we have modified the real setup).

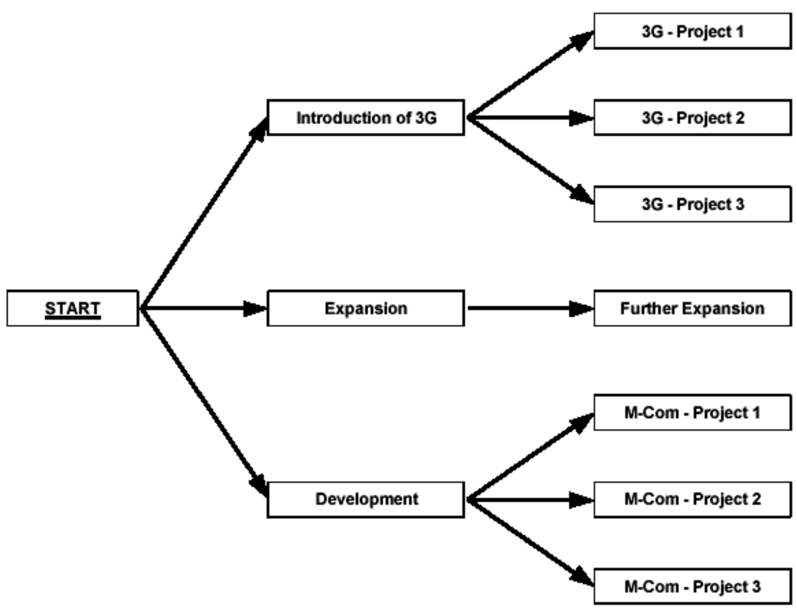

Figure 7.9: A simplified decision tree for Nordic telecom Inc. (Carlsson and Fullér [36]).

The World's telecommunications markets are undergoing a revolution. In the next few years mobile phones may become the World's most common means of communication, opening up new opportunities for systems and services. Characterized by large capital investment requirements under conditions of high regulatory, market, and technical uncertainty, the telecommunications industry faces many situations where strategic initiatives would benefit from real options analysis. As the FROV method is applied to the telecom markets context and to the strategic decisions of a telecom corporation we will have to understand in more detail how the real option values are formed. The FROV will increase with an increasing volatility of cash flow estimates. The corporate management can be proactive and find (i) ways to expand to new markets, (ii) product innovations and (iii) (innovative) product combinations as end results of their strategic decisions. If the current value of expected cash flows will increase, then the FROV will increase. A proactive management can influence this by (for instance) developing market strategies or developing subcontractor relations. The FROV will decrease if value is lost during the postponement of the investment, but this can be countered by either creating business barriers for competitors or by better managing key resources. An increase in risk-less returns will increase the FROV, and this can be further enhanced by closely monitoring changes in the interest rates. If the expected value of fixed costs goes up, the FROV will decrease as opportunities of operating with less cost are lost. This can be countered by using the postponement period to explore and implement production scalability benefits and/or to utilise learning benefits. The longer the time to maturity, the greater will be the FROV. A proactive management can make sure of this development by (i) maintaining protective barriers, (ii) communicating implementation possibilities and (iii) maintaining a technological lead.

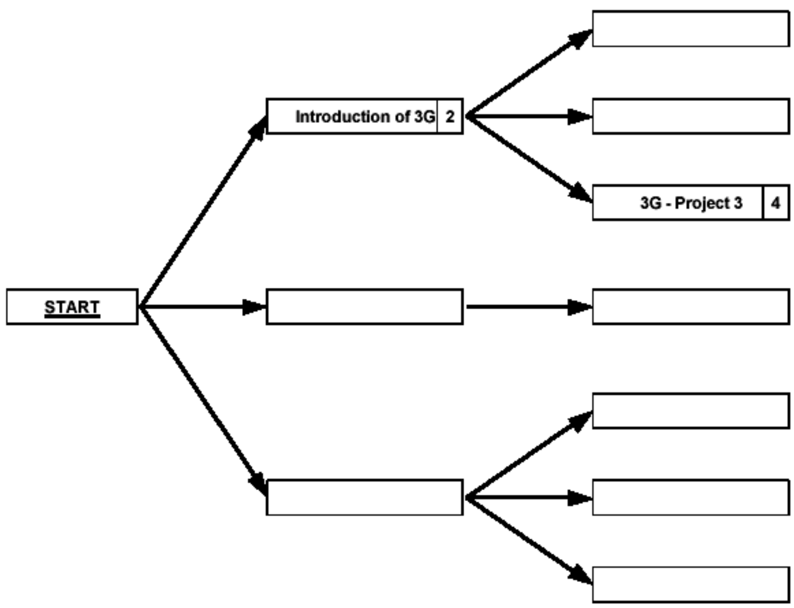

Figure 7.10: The optimal path (Carlsson and Fullér [36]).

The following example outlines the methodology used (to keep confidentiality we have modified the real setup) in the the Nordic Telekom Inc. (NTI) case: Nordic Telekom Inc. is one of the most successful mobile communications operators in Europe [NTI is a fictional corporation, but the dynamic tree model of strategic decisions has been successfully implemented for the 4 Finnish companies which participate in the Waeno project on giga-investments.] and has gained a reputation among its competitors as a leader in quality, innovations in wireless technology and in building long-term customer relationships.

Still it does not have a dominating position in any of its customer segments, which is not even advisable in the European Common market, as there are always 4-8 competitors with sizeable market shares. NTI would, nevertheless, like to have a position which would be dominant against any chosen competitor when defined for all the markets in which NTI operates. NTI has associated companies that provide GSM services in five countries and one region: Finland, Norway, Sweden, Denmark, Estonia and the St. Petersburg region. We consider strategic decisions for the planning period 2004-2012. There are three possible alternatives for NTI: (i) introduction of third generation mobile solutions (3G); (ii) expanding its operations to other countries; and (iii) developing new m-commerce solutions. The introduction of a 3G system can be postponed by a maximum of two years, the expansion may be delayed by maximum of one year and the project on introduction of new m-commerce solutions should start immediately.

In 2003 our goal was to maximize the company's cash flow at the end of the planning period (year 2012). In our computerized implementation we have represented NTI's strategic planning problem by a dynamic decision tree, in which the future expected cash flows and costs are estimated by trapezoidal fuzzy numbers. Then using the theory of fuzzy real options we have computed the real option values for all nodes of the dynamic decision tree. Then we have selected the path with the biggest real option value in the end of the planning period. The imprecision we encounter when judging or estimating future cash flows is genuine, i.e. we simply do not know the exact levels of future cash flows. The proposed model that incorporates subjective judgments and statistical uncertainties may give investors a better understanding of the problem when making investment decisions.

7.5 The OptionsPort project

OptionsPort - Real Option Valuation and Optimal Portfolio Strategies, TEKES [662/04]. Industrial partners: Kemira Oyj, Cargotec Oyj, UPM-Kymmene Oyj, Kuntarahoitus Oyj. Publications in this project: Carlsson, Fullér, Heikkilä and Majlender [51] and Carlsson, Fullér and Heikkilä [58]. My contribution to the project: Using the possibilistic mean value and variance for ranking projects with imprecise future cash flows.

A major advance in development of project selection tools came with the application of options reasoning to R&D. The options approach to project valuation seeks to correct the deficiencies of traditional methods of valuation through the recognition that managerial flexibility can bring significant value to a project. The main concern is how to deal with non-statistical imprecision we encounter when judging or estimating future cash flows. In our OptionsPort project we developed a model for valuing options on R&D projects, when future cash flows and expected costs are estimated by trapezoidal fuzzy numbers. Furthermore, we represented the optimal R&D portfolio selection problem as a fuzzy mathematical programming problem, where the optimal solutions defined the optimal portfolios of R&D projects with the largest (aggregate) possibilistic deferral flexibilities. Carlsson, Fullér, Heikkilä and Majlender [51] suggested the following algorithm for ordering R&D projects. This paper was Number 1 in Top 25 Hottest Articles Computer Science, International Journal of Approximate Reasoning April to June 2007. (see http://top25.sciencedirect.com/subject/computer-science/7/journal/international-journal-of-approximate-reasoning/0888613X/archive/12/)

Facing a set of project opportunities of R&D type, the company is usually able to estimate the expected investment costs, denoted by X, of the projects with a high degree of certainty. Thus, in the following we will assume that the X is a crisp number. However, the cash flows received from the projects do involve uncertainty, and they are modelled by trapezoidal possibility distributions. Let us fix a particular project of length L and maximum deferral time T with cash flows

$$\tilde{\mathrm{cf}}_i = (A_i, B_i, \Phi_i, \Psi_i).$$

Now, instead of the absolute values of the cash flows, we shall consider their *fuzzy returns on investment*

(FROI) by computing the return that we receive on investment X at year i of the project as

$$\mathrm{FROI}_i = \tilde{R}_i = \left(\frac{A_i}{X}, \frac{B_i}{X}, \frac{\Phi_i}{X}, \frac{\Psi_i}{X}\right) = (a_i, b_i, \alpha_i, \beta_i).$$

We compute the fuzzy net present value of project by

$$\mathrm{FNPV} = \left(\sum_{i=0}^{L} \frac{\tilde{R}_i}{(1+r)^i} - 1\right) \times X.$$

where r is the project specific risk-adjusted discount rate. If a project with fuzzy returns on investments $\{\tilde{R}_0, \tilde{R}_1, \dots, \tilde{R}_L\}$ can be postponed by maximum of T years then we will define the value of its possibilistic deferral flexibility by

$$\mathcal{D}_T = (1 + \sigma(\tilde{R}_0)) \times (1 + \sigma(\tilde{R}_1)) \times \cdots \times (1 + \sigma(\tilde{R}_{T-1})) \times \mathrm{FNPV},$$

where $1 \leq t \leq L$. If a project cannot be postponed then its possibilistic flexibility equals to its fuzzy net present value. That is, if $T = 0$ then $\mathcal{D}_T = FNPV$. The basic optimal R&D project portfolio selection problem can be formulated as the following fuzzy mixed integer programming problem

$$\begin{aligned} \text{maximize} \qquad & \mathcal{D} = \sum_{i=1}^{N} u_i \mathcal{D}_i \\ \text{subject to} \quad & \sum_{i=1}^{N} u_i X_i + \sum_{i=1}^{N} (1 - u_i) c_i \leq B \\ & u_i \in \{0, 1\}, i = 1, \dots, N. \end{aligned} \tag{7.10}$$

where N is the number of R&D projects; B is the whole investment budget; u_i is the decision variable associated with project i, which takes value one if project i starts now (i.e. at time zero) and takes value zero if it is postponed and is going to start at a later time; c_i denotes the cost of postponing project i (i.e. the capital expenditure required to keep the associated real option alive); finally, X_i and $\mathcal{D}_i$ stand for the investment cost and the possibilistic deferral flexibility of project i, respectively, $i = 1, \dots, N$. In our approach to fuzzy mathematical programming problem (7.10) , we have used the following defuzzifier operator for $\mathcal{D}$,

$$\nu(\mathcal{D}) = (E(\mathcal{D}) - \tau\sigma(\mathcal{D})) \times X$$

where $0 \leq \tau \leq 1$ denotes the decision makers risk aversion parameter.

I presented the following example at *Seminar on New Trends in Intelligent Systems and Soft Computing*, February 8-9, 2007, Granada, Spain. Let us assume that we have 5 different types of R&D projects with the following characteristics:

Project 1 has a large negative estimated NPV (which is due to the huge uncertainty it involves), and it can be deferred up to 2 years ($\nu(FNPV) < 0, T = 2$).

Project 2 includes positive NPV with low risks, and has no deferral flexibility ($\nu(FNPV) > 0, T = 0$).

Project 3 has revenues with large upward potentials and managerial flexibility, but its reserve costs (c) are very high.

Deferral time (*T*)	Project 1	Project 2	Project 3	Project 4	Project 5
0	*FNPV* = ((75%),17%,15%,126%) *M* = (10.5%) σ = 71.5%	*FNPV* = (12%,20%,45%,56%) *M* = 17.8% σ = 24%	*FNPV* = (5%,24%,17%,218%) *M* = 48.0% σ = 56.0%	*FNPV* = ((12%),85%,71%,6%) *M* = 25.7% σ = 62.0%	*FNPV* = ((5%),12%,4%,358%) *M* = 62.5% σ = 81.0%
1	$FROV_1$ = ((90%),20%,18%,151%) *M* = (12.6%) σ = 85.8%		$FROV_1$ = (6%,26%,19%,240%) *M* = 52.8% σ = 61.6%	$FROV_1$ = ((15%),106%,89%,8%) *M* = 32.1% σ = 77.5%	$FROV_1$ = ((6%),13%,4%,394%) *M* = 68.8% σ = 89.1%
2	$FROV_2$ = ((104%),23%,21%,174%) *M* = (14.5%) σ = 98.7%		$FROV_2$ = (7%,31%,23%,288%) *M* = 63.4% σ = 73.9%		$FROV_2$ = ((7%),14%,5%,433%) *M* = 75.7% σ = 98.0%

Figure 7.11: Expected cash-flows from projects.

Project 4 requires a large capital expenditure once it has been undertaken, and has a deferral flexibility of a maximum of 1 year.

Project 5 represents a small flexible project with low revenues, but it opens the possibility of further projects that are much more profitable.

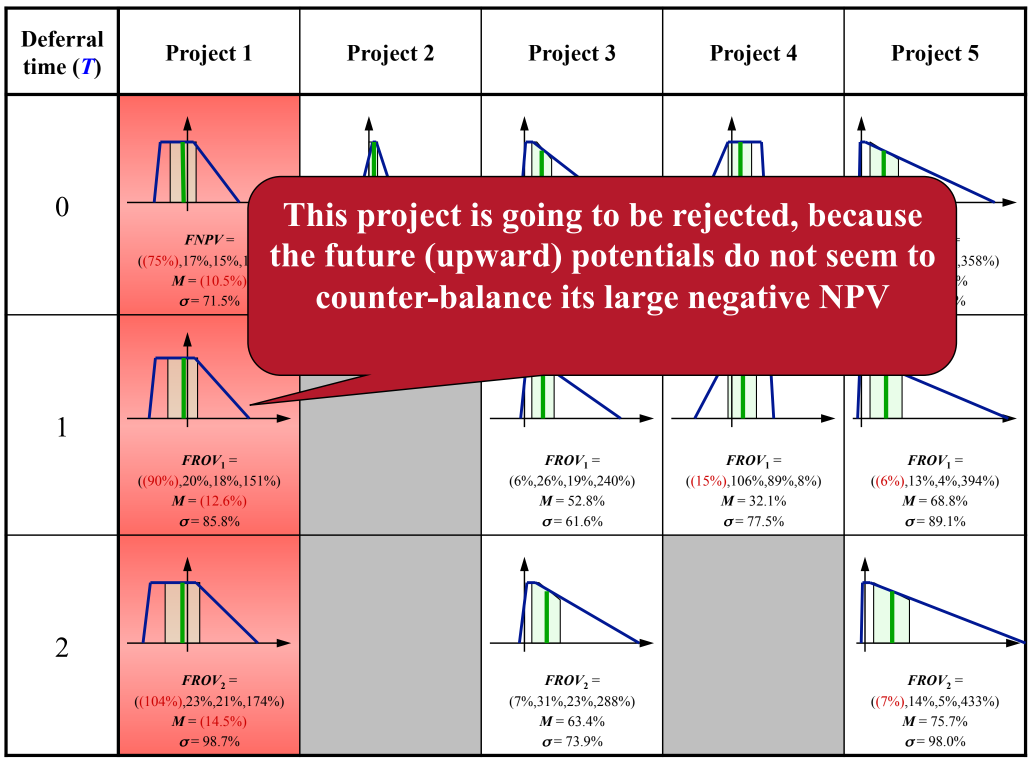

Figure 7.12: Analysis of Project 1.

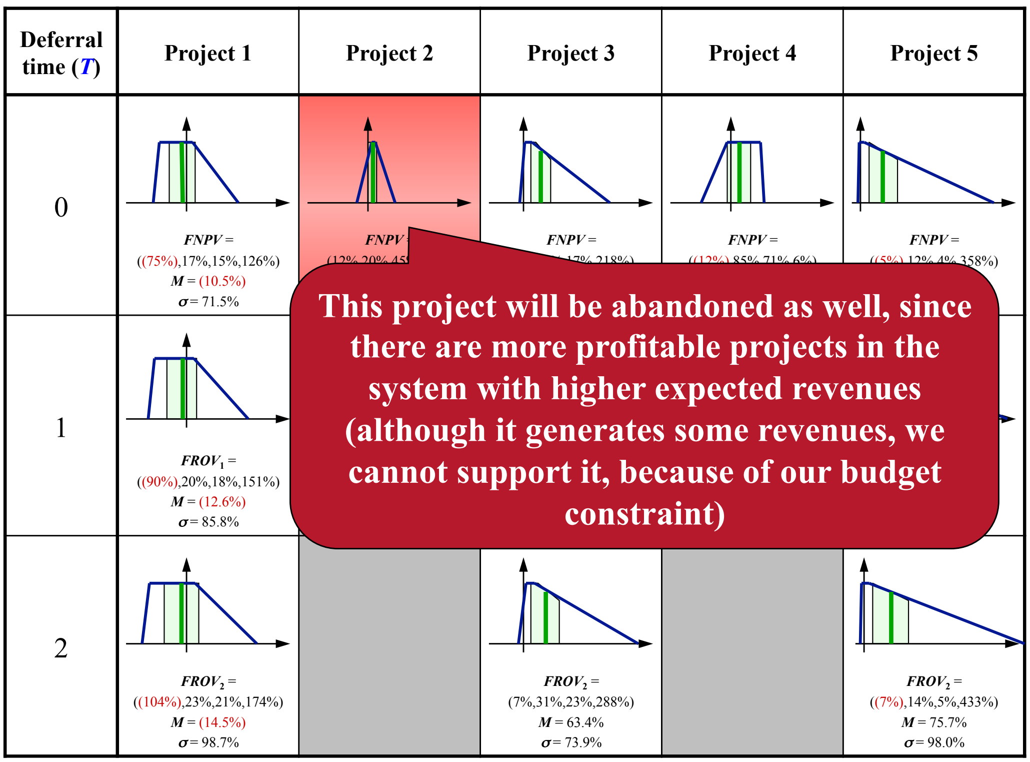

Figure 7.13: Analysis of Project 2.

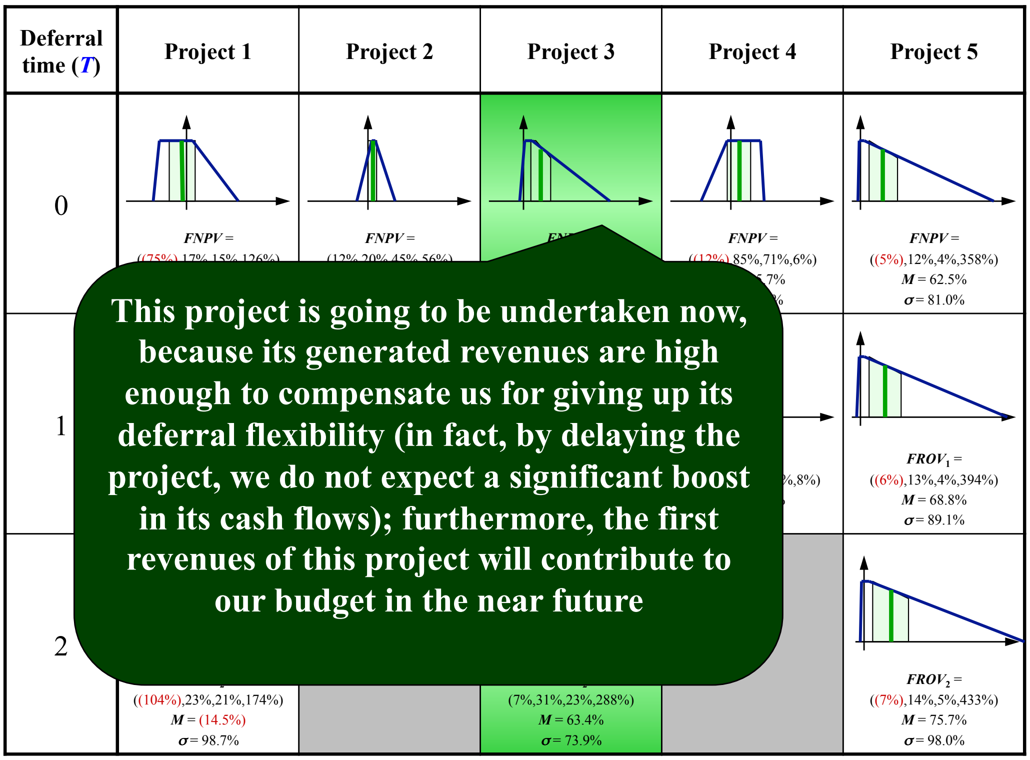

Figure 7.14: Analysis of Project 3.

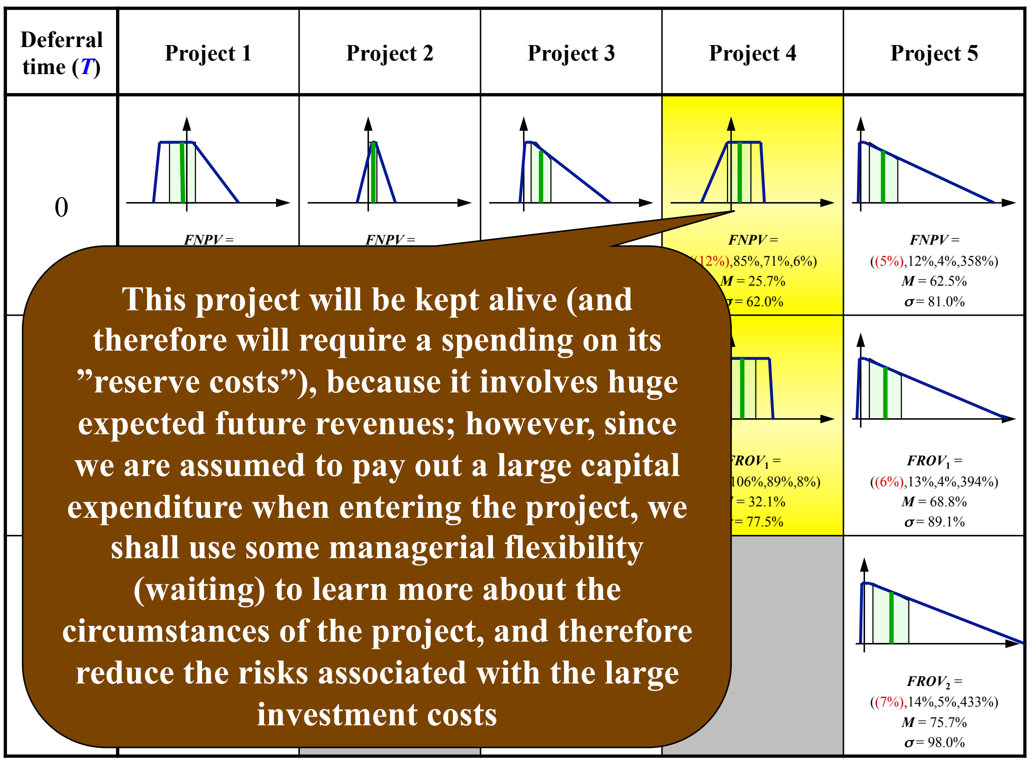

Figure 7.15: Analysis of Project 4.

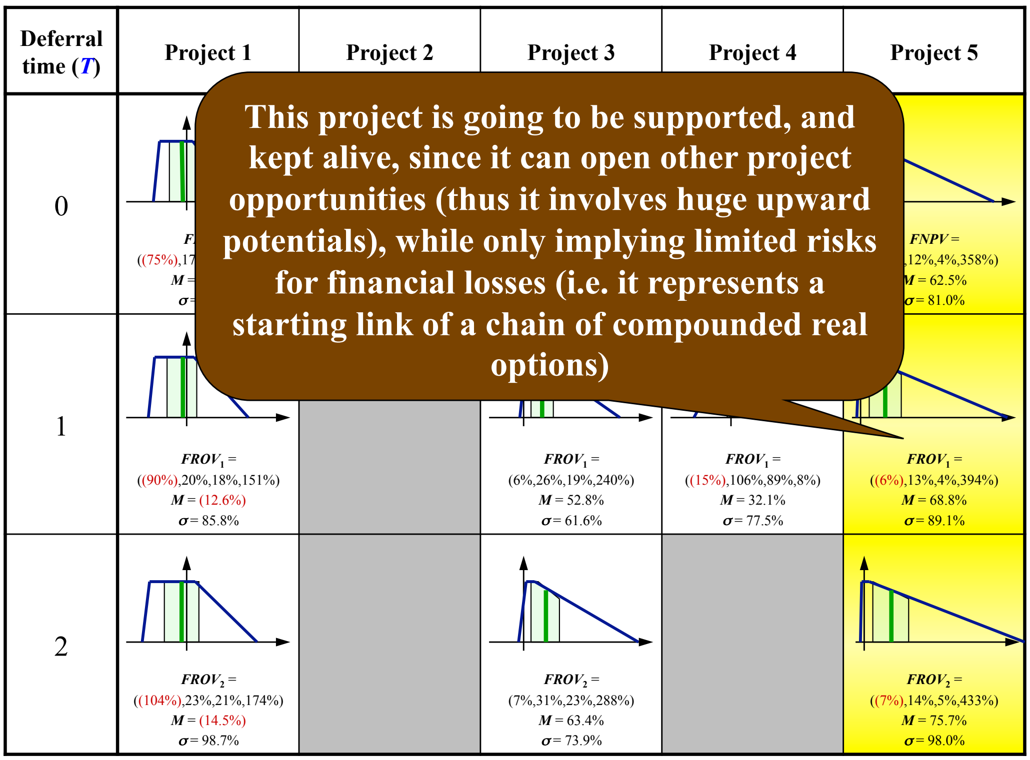

Figure 7.16: Analysis of Project 5.

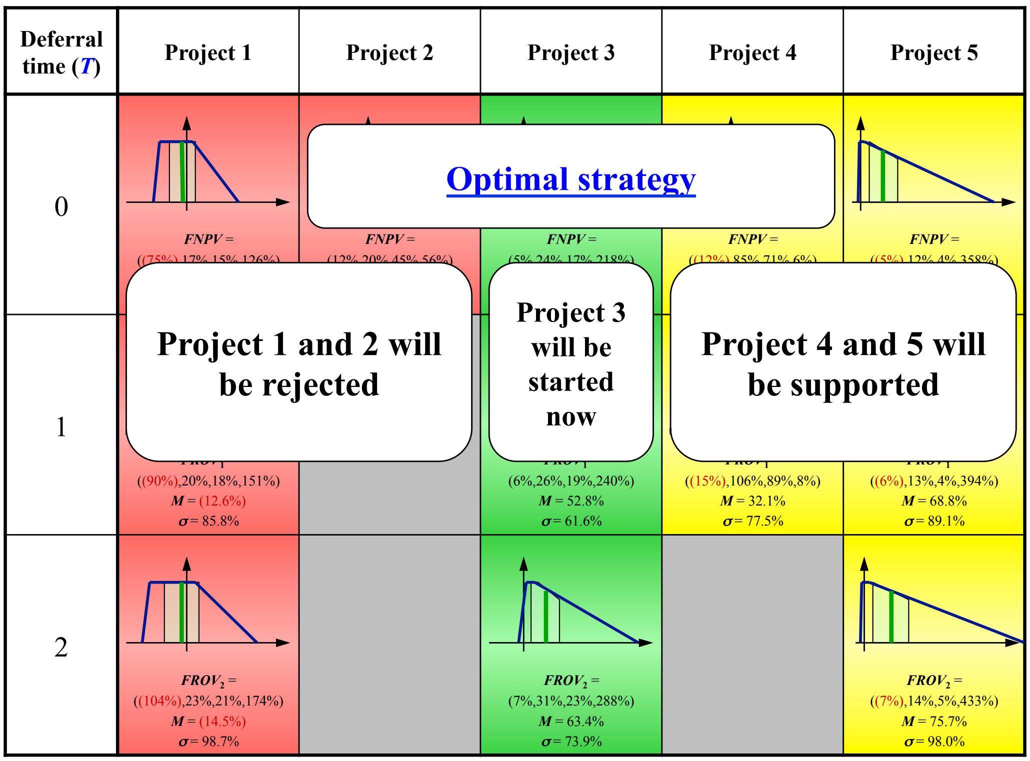

Figure 7.17: The optimal strategy.

7.6 The EM-S Bullwhip project

EM-S Bullwhip [TEKES 40965/98], industrial partners: Metsä-Serla and Stora-Enso. Our publications in this project: Carlsson and Fullér [20, 22, 23, 29, 35]. A longer description of this project can be found in Carlsson and Fullér [33] and Carlsson, Fedrizzi and Fullér [44]. My contribution to this project: As the optimal, crisp ordering policy drives the bullwhip effect we decided to try a policy in which orders are imprecise. This means that orders can be fuzzy intervals, and we will allow the actors in the supply chain to make their orders more precise as the (time) point of delivery gets closer. I suggested a neural fuzzy system for reducing the bullwhip effect in demand signal processing (possibilistic variance of orders).

We will consider a series of companies in a supply chain, each of which orders from its immediate upstream collaborators. Usually, the retailer's order do not coincide with the actual retail sales. The *bullwhip effect* refers to the phenomenon where orders to the supplier tend to have larger variance than sales to the buyer (i.e. demand distortion), and the distortion propagates upstream in an amplified form (i.e. variance amplification). The factors driving the bullwhip effect appear to form a hyper-complex, i.e. a system where factors show complex interactive patterns. The theoretical challenges posed by a hyper-complex merit study, even if significant economic consequences would not have been involved. The costs incurred by the consequences of the bullwhip effect (estimated at 200-300 Million Finnish Marks annually for a 300 kiloton paper mill) offer a few more reasons for carrying out serious work on the mechanisms driving the bullwhip. Thus, we have built a theory to explain at least some of the factors and their interactions, and we have created a support system to come to terms with them and to find effective means to either reduce or eliminate the bullwhip effect. With a little simplification there appears to be three possible approaches to counteract the bullwhip effect:

1. Find some means to share information from downstream the supply chain with all the preceding actors.

2. Build channel alignment with the help of some co-ordination of pricing, transportation, inventory planning and ownership - when this is not made illegal by anti-trust legislation.

3. Improve operational efficiency by reducing cost and by improving on lead times.

In 1998-2000 we carried out a research program on the bullwhip effect with two major fine paper producers: Metsä-Serla and Stora-Enso. The project, known as EM-S Bullwhip, worked with actual data and in interaction with senior decision makers. The two corporate members of the EM-S Bullwhip consortium had observed the bullwhip effects in their own markets and in their own supply chains for fine paper products. They also readily agreed that the bullwhip effect is causing problems and significant costs, and that any good theory or model, which could give some insight into dealing with the bullwhip effect, would be a worthwhile effort in terms of both time and resources. Besides the generic reasons we introduced above, there are a few practical reasons why we get the bullwhip effect in the fine paper markets.

The first reason is to be found in the structure of the market (see Fig. 7.18).

The paper mills do not deal directly with their end-customers, the printing houses, but fine paper products are distributed through wholesalers, merchants and retailers. The paper mills may (i) own some of the operators in the market supply chain, (ii) they may share some of them with competitors or (iii) the operators may be completely independent and bound to play the market game with the paper producers. The operators in the market supply chain do not willingly share their customer and market data, information and knowledge with the paper mills. Thus, the paper producers do not get *neither precise nor updated information* on the real customer demand, but get it in a filtered and/or manipulated way from the market supply chain operators. Market data is collected and summarized by independent data providers, and market forecasts are produced by professional forest products consultants and market study agencies, but it still appears that these macro level studies and forecasts do not apply exactly to the markets of a single paper producer.

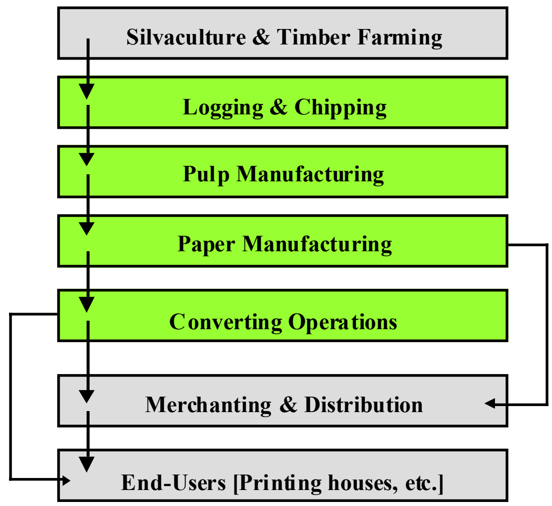

Figure 7.18: The supply chain of the market for fine paper products.

The second, more practical, reason for the bullwhip effect to occur is found earlier in the supply chain. The demand and price fluctuations of the pulp markets dominate also the demand and price patterns of the paper products markets, even to such an extent, that the customers for paper products anticipate the expectations on changes in the pulp markets and act accordingly. If pulp prices decline, or are expected to decline, demand for paper products will decline, or stop in anticipation of price reductions. Then, eventually, prices will in fact go down as the demand has disappeared and the paper producers get nervous. The initial reason for fluctuations in the pulp market may be purely speculative, or may have no reason at all. Thus, the construction of any reasonable, explanatory cause-effect relationships to find out the market mechanisms that drive the bullwhip may be futile.

The third practical reason for the bullwhip effect is specialized form of order batching. The logistics systems for paper products favour shiploads of paper products, the building of inventories in the supply chain to meet demand fluctuations and push ordering to meet end-of-quarter or end-of-year financial needs. The logistics operators are quite often independent of both the paper mills and the wholesalers and/or retailers, which will make them want to operate with optimal programs in order to meet their financial goals. Thus they decide their own tariffs in such a way that their operations are effective and profitable, which will - in turn - affect the decisions of the market supply chain operators, including the paper producers.

There is a fourth practical reason, which is caused by the paper producers themselves. There are attempts at influencing or controlling the paper products markets by having occasional low price campaigns or special offers. The market supply chain operators react by speculating in the timing and the level of low price offers and will use the (rational) policy of buying only at low prices for a while. This normally triggers the bullwhip effect.

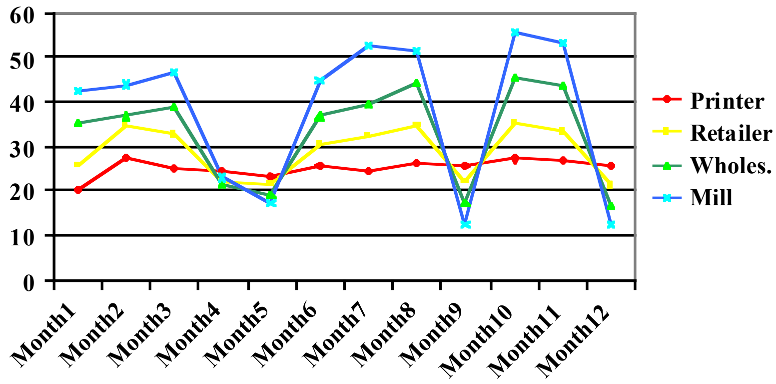

Figure 7.19: The bullwhip effect in the fine paper products market.

The bullwhip effect may be illustrated as in Fig. 7.19 The variations shown in Fig. 7.19 are simplifications, but the following patterns appear: (i) the printer (an end-customer) orders once per quarter according to the real market demand he has or is estimating; (ii) the dealer meets this demand and anticipates that the printer may need more (or less) than he orders; the dealer acts somewhat later than his customer; (iii) the paper mill reacts to the dealer's orders in the same fashion and somewhat later than the dealer. The resulting overall effect is the bullwhip effect.

Lee et al [111, 112] focus their study on the demand information flow and worked out a theoretical framework for studying the effects of systematic information distortion as information works its way through the supply chain. They simplify the context for their theoretical work by defining an idealised situation. They start with a multiple period inventory system, which is operated under a periodic review policy. They include the following assumptions: (i) past demands are not used for forecasting, (ii) resupply is infinite with a fixed lead time, (iii) there is no fixed order cost, and (iv) purchase cost of the product is stationary over time. If the demand is stationary, the standard optimal result for this type of inventory system is to order up to S, where S is a constant. The optimal order quantity in each period is exactly equal to the demand of the previous period, which means that orders and demand have the same variance (and there is no bullwhip effect).

This idealized situation is useful as a starting point, as is gives a good basis for working out the consequences of distortion of information in terms of the variance, which is the indicator of the bullwhip effect. By relaxing the assumptions (i)-(iv), one at a time, it is possible to produce the bullwhip effect.

Let us focus on the retailer-wholesaler relationship in the fine paper products market (the framework applies also to a wholesaler-distributor or distributor-producer relationship). Now we consider a multiple period inventory model where demand is non-stationary over time and demand forecasts are updated from observed demand. Lets assume that the retailer gets a much higher demand in one period. This will be interpreted as a signal for higher demand in the future, the demand forecasts for future periods get adjusted, and the retailer reacts by placing a larger order with the wholesaler. As the demand is non-stationary, the optimal policy of ordering up to S also gets non-stationary. A further consequence is that the variance of the orders grows, which is starting the bullwhip effect. If the lead-time between ordering point and the point of delivery is long, uncertainty increases and the retailer adds a "safety margin" to S, which will further increase the variance - and add to the bullwhip effect.

Lee et al simplify the context even further by focusing on a single-item, multiple period inventory, in order to be able to work out the exact bullwhip model.

The timing of the events is as follows: At the beginning of period t, a decision to order a quantity z_t is made. This time point is called the "decision point" for period t. Next the goods ordered ν periods ago arrive. Lastly, demand is realized, and the available inventory is used to meet the demand. Excess demand is backlogged. Let S_t denote the amount in stock plus on order (including those in transit) after decision z_t has been made for period t. Lee at al [111] assume that the retailer faces serially correlated demands which follow the process

$$D_t = d + \rho D_{t-1} + u_t$$

where D_t is the demand in period t, ρ is a constant satisfying $-1 < \rho < 1$, and u_t is independent and identically normally distibuted with zero mean and variance σ^2. Here σ^2 is assumed to be significantly smaller than d, so that the probability of a negative demand is very small. The existence of d, which is some constant, basic demand, is doubtful; in the forest products markets a producer cannot expect to have any "granted demand". The use of d is technical, to avoid negative demand, which will destroy the model, and it does not appear in the optimal order quantity. Lee et al proved the following theorem,

Theorem 7.1 (Lee, Padmanabhan and Whang, [111]). *In the above setting, we have,*

1. *If* $0 < \rho < 1$, *the variance of retails orders is strictly larger than that of retail sales; that is,* $\mathrm{Var}(z_1) > \mathrm{Var}(D_0)$.
2. *If* $0 < \rho < 1$, *the larger the replenishment lead time, the larger the variance of orders; i.e.* $\mathrm{Var}(z_1)$ *is strictly increasing in* ν.

This theorem has been proved from the relationships

$$z_1^* = S_1 - S_0 + D_0 = \frac{\rho(1-\rho^{\nu+1})}{1-\rho}(D_0 - D_{-1}) + D_0, \tag{7.11}$$

and

$$\mathrm{Var}(z_1^*) = \mathrm{Var}(D_0) + \frac{2\rho(1-\rho^{\nu+1})(1-\rho^{\nu+2})}{(1+\rho)(1-\rho)^2} > \mathrm{Var}(D_0),$$

where z_1^* denotes the optimal amount of order. Which collapses into $\mathrm{Var}(z_1^*) = \mathrm{Var}(D_0) + 2\rho$, for $\nu = 0$.

The optimal order quantity is an optimal ordering policy, which sheds some new light on the bullwhip effect. The effect gets started by rational decision making, i.e. by decision makers doing the best they can. In other words, there is no hope to avoid the bullwhip effect by changing the ordering policy, as it is difficult to motivate people to act in an irrational way. Other means will be necessary.

It appears obvious that the paper mill could counteract the bullwhip effect by forming an alliance with either the retailers or the end-customers. The paper mill could, for instance, provide them with forecasting tools and build a network in order to continuously update market demand forecasts. This is, however, not allowed by the wholesalers.

As the optimal, crisp ordering policy drives the bullwhip effect we decided to try a policy in which orders are imprecise. This means that orders can be intervals, and we will allow the actors in the supply chain to make their orders more precise as the (time) point of delivery gets closer. We can work out such a policy by replacing the crisp orders by fuzzy numbers. Following Carlsson and Fullér

[20, 22, 23, 29, 35] we will carry this out only for the demand signal processing case. It should be noted, however, that the proposed procedure can be applied also to the price variations module and - with some more modeling efforts - to the cases with the rationing game and order batching.

Let us consider equation (7.11) with trapezoidal fuzzy numbers

$$z_1^* = S_1 - S_0 + D_0 = \frac{\rho(1-\rho^{\nu+1})}{1-\rho}(D_0 - D_{-1}) + D_0. \tag{7.12}$$

Then from the definition of possibilistic mean value [26] we get,

$$\mathrm{Var}(z_1^*) > \mathrm{Var}(D_0),$$

so the simple adaptation of the probabilistic model (i.e. the replacement of probabilistic distributions by possibilistic ones) does not reduce the bullwhip effect.

We will show, however that by including better and better estimates of future sales in period one, D_1, we can reduce the variance of z_1 by replacing the old rule for ordering (7.12) with an adjusted rule. If the participants of the supply chain do not share information, or they do not agree on the value of D_1 then we can apply a neural fuzzy system that uses an error correction learning procedure to predict z_1. This system should include historical data, and a supervisor who is in the position to derive some initial linguistic rules from past situations which would have reduced the bullwhip effect. A typical fuzzy logic controller (FLC) describes the relationship between the change of the control $\Delta u(t) = u(t) - u(t-1)$ on the one hand, and the error $e(t)$ (the difference between the desired and computed system output) and its change

$$\Delta e(t) = e(t) - e(t-1).$$

on the other hand. The actual output of the controller $u(t)$ is obtained from the previous value of control $u(t-1)$ that is updated by $\Delta u(t)$. This type of controller was suggested originally by Mamdani and Assilian in 1975 and is called the *Mamdani-type* FLC [125].

A prototype rule-base of a simple FLC, which is realized with three linguistic values {N: negative, ZE: zero, P: positive} is listed in Table 7.1. To reduce the bullwhip effect we suggest the use of a fufCarzzy logic controller. Demand realizations D_{t-1} and D_{t-2} denote the volumes of retail sales in periods $t-1$ and $t-2$, respectively. We use a FLC to determine the change in *order*, denoted by Δz_1, in order to reduce the bullwhip effect, that is, the variance of z_1.

$\Delta e(t) \mid e(t) \rightarrow$ $\downarrow$	N	ZE	P
N	N	N	ZE
ZE	N	ZE	P
P	ZE	P	P

Table 7.1: A Mamdani-type FLC in a tabular form.

We shall derive z_1 from D_0, D_{-1} (sales data in the last two periods) and from the last order z_0 as

$$z_1 = z_0 + \Delta z_1$$

where the crisp value of Δz_1 is derived from the rule base $\{\Re_1, \ldots, \Re_5\}$, where $e = D_0 - z_0$ is the difference between the past realized demand (sales), D_0 and order z_0, and the change of error

$$\Delta e := e - e_{-1} = (D_0 - z_0) - (D_{-1} - z_{-1}),$$

is the change between $(D_0 - z_0)$ and $(D_{-1} - z_{-1})$.

To improve the performance (approximation ability) we can include more historical data D_{t-3}, $D_{t-4} \ldots$, in the antecedent part of the rules. The problem is that the fuzzy system itself can not learn the membership function of Δz_1, so we could include a neural network to approximate the crisp value of z_1, which is the most typical value of $z_0 + \Delta z_1$. It is here, that the supervisor should provide crisp historical learning patterns for the concrete problem, for example, $\{5, 30, 20\}$ which tells us that if at some past situations $(D_{k-2} - z_{k-2})$ was 5 and $(D_{k-1} - z_{k-1})$ was 30 then then the value of z_k should have been $(z_{k-1} + 20)$ in order to reduce the bullwhip effect. The meaning of this pattern can be interpreted as: if the preceding chain member ordered a little bit less than he sold in period $(k - 2)$ and much less in period $(k - 1)$ then his order for period k should have been enlarged by 20 in order to reduce the bullwhip effect (otherwise - at a later time - the order from this member would unexpectedly jump in order to meet his customers' demand - and that is the bullwhip effect). Then the parameters of the fuzzy system (i.e. the shape functions of the error, change in error and change in order) can be learned by a neural network (see Fullér [83]).

Demo architecture and implementation

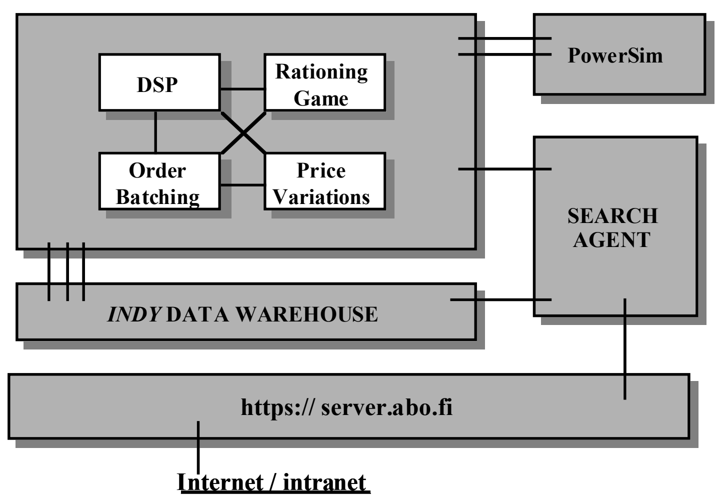

Figure 7.20: A soft computing platform for reducing the bullwhip effect.

The platform shown in Fig. 7.20 is a prototype, which was built in 2000 mainly to validate and to verify the theory we have developed for coping with the bullwhip effect. The platform is built in Java 2.0 and it was designed to operate over the Internet or through a corporate intranet. This makes it possible for a user to work with the bullwhip effect as (i) part of a corporate strategic planning session, as (ii) part of a negotiation program with retailers and/or wholesalers, as (iii) part of finding better solutions when dealing with end customers, as (iv) support for negotiating with transport companies and logistics subcontractors, and as (v) a basis for finding new solutions when organizing the supply chain for the end customers.

The platform is operated on a secure server, which was built at IAMSR in order to include some non-standard safety features. There are four models operated on the platform: (i) *DSP* for demand signal processing, (ii) *Rationing Game* for handling the optimal strategies as demand exceeds supply and the deliveries have to be rationed, (iii) *Order Batching* for working out optimal delivery schemes when there are constraints like *full shipload*, and (iv) *Price Variations* for working out the best pricing policies when the paper mill wants to shift between low and high prices. The hyperknowledge features allow the models to be interconnected, which means that the effects of the DSP can be taken as input when working out either Order Batching or Price Variations effects. Thus, models can be operated either individually or as cause-effect chains. Data for the models is collected with search agents, which operate on either databases in the corporate intranet or on data sources in the Internet. Also the search agents have been designed, built in Java and implemented for corporate partners by IAMSR as part of a series of research programs.

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