Appraisal and Development of Transportation Systems Using Multiple Criteria Decision Making Methodology

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Abbreviations

AFV	=	alternative-fuel vehicle
AHP	=	analytic hierarchy process
CBA	=	cost-benefit analysis
CD	=	conventional diesel engine
CI	=	consistency index
CNG	=	compressed natural gas
CR	=	consistency ratio
CTP	=	Common Transport Policy
DM	=	decision maker
DSS	=	decision support system
ECMT	=	European Conference of Ministers of Transport
EV	=	electric vehicle
FCV	=	fuel-cell vehicle
GIS	=	Geographic Information System
GPS	=	Global Positioning System
GSM	=	Global System for Mobile
HEV	=	hybrid electric vehicle
HSPC	=	high strength prestressed concrete
HTP	=	Hungarian Transport Policy
ICEV	=	internal combustion engine vehicle
ILWIS	=	Integrated Land and Water Information System
ITS	=	Intelligent Transportation System
KSIM	=	Kane simulation
LPG	=	liquefied propane gas
LS	=	least-squares
LSR	=	least-squares recursion
MADA	=	multi-attribute decision analysis
MAROM	=	multi attribute object measurement
MCDM	=	multi-criteria decision making
MOO	=	multi-objective optimization
MSU	=	mean spatial utility
NHDP	=	National Hungarian Development Policy
N-K	=	Newton-Kantorovich
PC	=	prestressed concrete
PCM	=	pairwise comparison matrix
PM	=	particulate matter
RC	=	reinforced concrete
SDM	=	spectral density matrix
SMCE	=	spatial multiple criteria evaluation
SPI	=	symmetric permutation invariant
SR	=	symmetrically reciprocal
STP	=	Sequential Transport Policy
TEN-T	=	Trans-European Network - Transportation
TOPSIS	=	technique for order preference by similarity to ideal solution
Triple R-I	=	recursive rank-one residue iteration
UTDS	=	Unified Transport Development Strategy
UTM	=	Universal Transverse Mercator

Preface

This work summarizes the results of the scholarly research I have conducted in the fields of transportation systems engineering and multi-criteria decision making (MCDM) during the last twenty years. My major goals were to reveal and to give exact explanations of a mathematical type on some known shortcomings of existing scaling methods and developing new techniques while applying them to a variety of transportation problems. Central to my interest was to look at these methods with a healthy skepticism. Another objective of this thesis was to stimulate new applications of MCDM by specialists and non-specialists through providing many uses. MCDM methods have become a tool that must be understood not only by the research engineer, operations research analyst and mathematician, but also by the transport planner, geographer, regional expert, civil engineer, public servant and other people without extensive mathematical backgrounds.

On the other hand, a pragmatic approach is not enough to support the analysis of such complicated systems. In my view, findings which are presented as a mixture of theoretically based propositions (with proofs) and propositions based on numerical calculations, or only on speculations cannot be sufficient. Therefore, when I was writing the text, I followed solid mathematical rigor, as well as precision in the numerical computations. An implicit goal was to move the practitioners to the computer and to an intensive use of the huge opportunities provided by the internet. Additionally, I tried to motivate the users to apply the ever-growing capabilities of intelligent transportation systems such as, for example, the geographic information systems (GIS). Similarly, I have placed a large emphasis on the question of sustainable transport, as well as the inclusion of the stakeholders (transport providers, services and users) in the decision making processes and the preparation of new projects using multi-criteria decision analysis (MCDA) methods in such complex areas like transport policy, transport planning, transport design and evaluation of new project plans.

Several people have contributed to this work. My distinguished appreciation goes to my dear colleague and friend, the late professor Pál Rózsa, whom I had been working very hard with on these exciting topics over the past twenty years. He has made great achievements concerning some complex mathematical derivations. His intellectual prowess was matched by his genuine character and humanity. I felt privileged to have had this long opportunity to work with him. His valuable remarks have improved the clarity of presentation of our joint papers in a large extent.

I wish to express my deepest gratitude to my international collaborators also, to professor Peter Lancaster of University of Calgary, holder of the Hans Schneider Prize, ILAS, who has made significant contributions to the development of both the linear and the non-linear models presented in Chapter 4, and to professor Richard Wendell of University of Pittsburgh, for his inspiring and challenging advices during the early stage of my research.

My warmest thanks and appreciations are expressed to my esteemed co-authors in this subject, to professors Pál Rózsa, Peter Lancaster, Etelka Stubnya, Géza Tassi and András György. Special acknowledgement goes to professors Katalin Tánczos, Pál Michelberger and István Zobory following my research and encouraging the preparation of this dissertation.

Finally, I wish to thank my family for the patience and unwaivering support.

Chapter 1

1 On the Development of Transportation Systems

In this Chapter, the basic concepts, notions and the characterization of the transportation systems are discussed. Special focus is devoted to transportation systems design, project planning and transport policy, emphasizing the multidisciplinary nature of these issues. A sequential transport policy model for sustainable transport is developed that can be useful for global, regional and urban transportation projects. This framework aims to satisfy the increased needs of the transport infrastructure- and service providers, the transport users and the urban communities. To support effective decision making a hierarchy composed of a structured set of multiple objectives and criteria, for choosing the most favorable option for transport projects, is also proposed along with a comprehensive application to planning a metro-rail network with a combined GIS-MCDA approach.

1.1 Basic Concepts, Notions and Characteristics of Transportation Systems

In this section the commonly accepted concepts, notions and main characteristics related to transportation systems are presented in line with the contemporary literature [8], [9], [45] and [89]. Some definitions and interpretations described here were drawn from the excellent book of Cascetta [9].

A *transportation system* can be defined as a combination of elements and their interactions, which produce the *demand* for travel within a given area and the *supply* of transportation services to satisfy this demand. Usually, transportation systems are very complex, sometimes they involve nonlinear interactions, a number of feedback cycles and have stochastic nature (e.g. travel times). Therefore, the designer or the analyst cannot consider all the interacting elements to solve such a sophisticated problem. Newer technologies like *intelligent transportation systems* (ITS) include advanced traveler information systems and traffic control systems. ITSs aim to provide innovative services relating to various modes of transport and traffic management, where information and communication technologies are applied in the field of road transport, including infrastructure, vehicles and human users (e.g. car navigation traffic signal, space recognition, users' guidance and sensing technologies). *Transportation engineering* utilizes a set of appropriate technologies and applied scientific principals to the planning, design, operation and maintenance of facilities of any mode of transportation in order to promote a safe, efficient, rapid, comfortable, convenient, economically sound and environmentally compatible transport through the movement of people and goods. Using an integrated logistics framework, Farkas and Koltai (1988) have introduced three dimensions which are referring to the spatial constitution, the position in time of the units and the completion state of the construction of the objects. The use of these dimensions enabled them to define transportation systems in a more abstract sense, according to which it is a series of *primary movements* of people and goods taking place in a three-dimensional space determined by the *space-time-state variables* within specified *systems boundaries*.

As a part of the physical and naturally build environment, transportation systems are tightly connected to *civil engineering* works and structures, like roads, railways, bridges, buildings, etc. This field comprises, among others, the geotechnical, structural, environmental and construction engineering disciplines to accomplish residential, commercial, industrial and public projects of all sizes and levels of construction. *Vehicle engineering* plays a significant role in the operation of transportation systems as well, by designing, manufacturing and in the maintenance of different transportation means, i.e., motorcycles, automobiles, buses, trucks, railway and subway vehicles, ships and aircrafts and their respective engineering subsystems.

The common engineering approach is to isolate the parts of the system under study, which are most relevant to the solution of the problem. These parts, and the relationships among them, constitute the *internal* system. The *external* environment is taken into account only through their mutual *interactions* with the internal system. *Transportation systems engineering* is mainly focusing on the planning and evaluation of transportation supply projects. A transportation system with its well-defined internal elements within its boundaries and beyond has interactions with the external environment called the *analysis system* [9]. It follows, that there is a strict interdependence between the identification of the analysis system and the problem to be solved. A transportation system of a given area is usually regarded as a sub-system of a wider territorial system with which it strongly interacts. For example, a town may be described by a set of households, institutions, recreation places, industrial plants, services, transit facilities, municipalities, regulations, etc. Several sub-systems can also be identified including transit and activity systems. What has become known as *activity-based* approaches, a basic goal is to account for decisions concerning activities which affect the demand for travel [2]. We extended the original flow-chart of Cascetta [9, p.2] aiming to describe the interconnections that exist between the transportation systems and activity systems by including civil engineering components and some additional activities also, as is exhibited in Figure 1.

The *activity system* encompasses social and economic behavior and interactions of the people that give rise to travel demand. Transportation activity surveys investigate, when, where and how individuals change their spatial positions and provide information necessary for transportation planning. Most recently, smartphone-based systems have used to replace the traditional household surveys [94]. They are equipped with various sensors to collect GPS, GSM, accelerometer and other sources of information for place detection, trajectory tracking and transportation mode exploration by using different algorithms. The activity system of an urban area can be decomposed into three sub-systems involving [9]: (*i*) the households which are divided into categories (by income level, life-cycle, etc.) living in each zone; (*ii*) the economic activities that are assigned to zones and divided into different industrial and service sectors and into economic (expenses) and physical (number of employees) indicators, and, (*iii*) the floorspace which is available in each zone for various uses (industrial production, offices, residences, shops, buildings) and relative market prices (real-estate system). Transportation performances strongly influence the relative accessibility of different zones of an urban area. To measure performance, a common way is to assess the "cost" of reaching other zones, called *active accessibility*, or being reached from other zones, called *passive accessibility* [9].

One of the major goals of transportation systems engineering is to design *transportation supply projects* using appropriate quantitative methods. In general, transportation systems engineering focuses on the planning, analysis and control of the single elements and the relationships, assuming the activity system as is given exogenously. The *identification* of the transportation system includes the definition of the elements and the relationships making up the analysis system. Due to Cascetta [9], such identification may be done in three phases: (*i*) identification of relevant spatial and supply characteristics; (*ii*) definition of relevant components of transport demand, and, (*iii*) identification of relevant temporal dimensions.

Relevant components of transportation demand are the *transport* and the *travel demand flows* which can formally be defined as the number of users (set of households activities) with their given characteristics consuming the services offered by a transport system in a given time period [9]. A *trip* is defined as the act of moving from one place (origin) to another (destination) using one or more means of transport mode, in order to carry out one or several activities. A sequence of trips is defined as a *journey* or *trip-chain*. An urban transportation system may contain many feedback *cycles*. The trips between the various zones made with a given mode (e.g. by car) are using different paths and result in traffic flows on the different supply elements. (e.g., for road sections) [89].

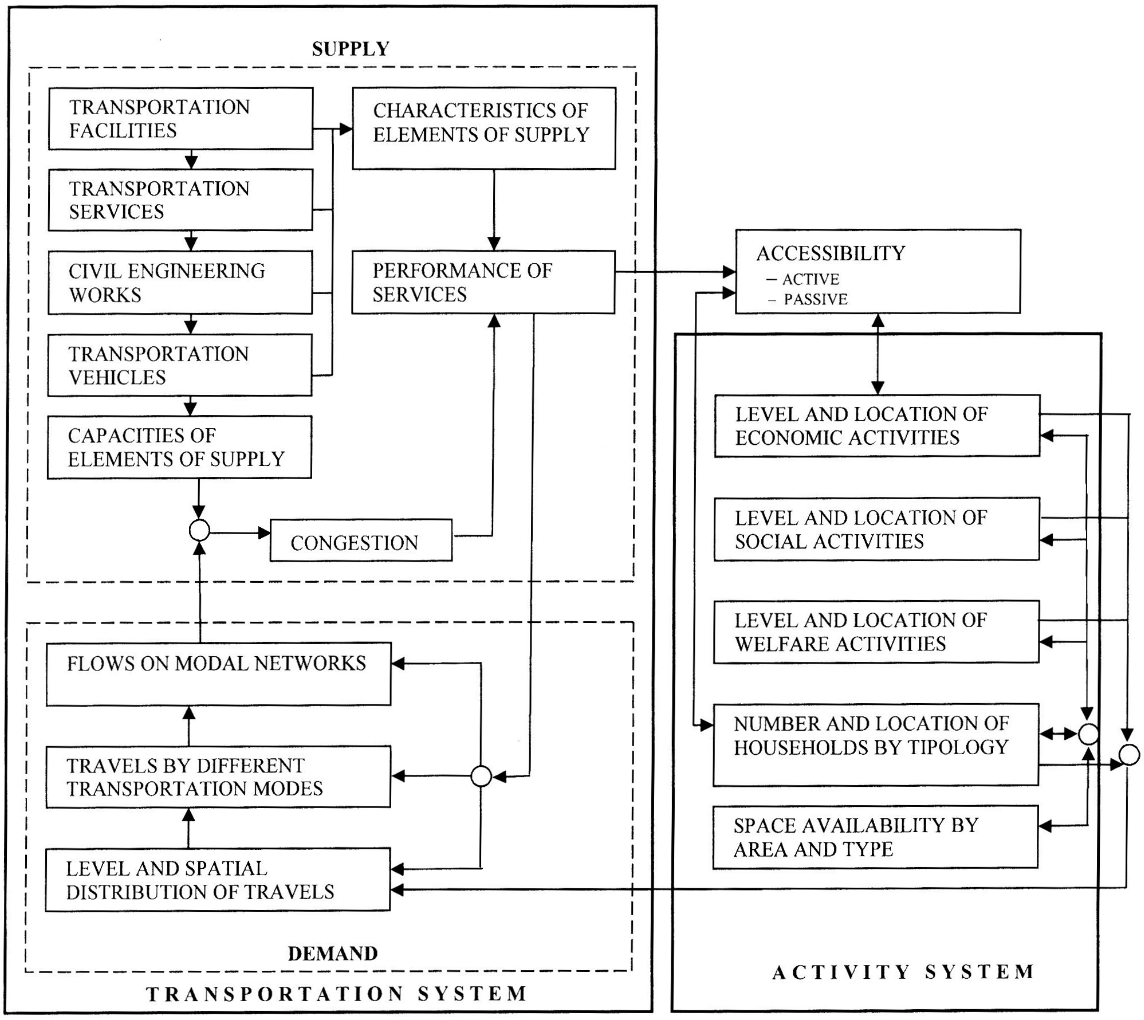


Figure 1. Mutual interdependencies between the transportation system and the activity system.

Spatial characterization is of utmost importance in transportation studies. Household activities are characterized by the number of travels, trip frequency, destination, mode, paths, etc. in an area, since they determine *transportation demand*. Household members are the users of the *transportation supply* system and make *mobility* and *travel choices* in order to undertake different activities (work, school, shopping, etc.) into different locations. These choices result in the transportation demand, e.g., the number of trips made among the different zones of a city, for different purposes, in different periods, by means of available different transportation modes. Similarly, economic activities transport goods that are needed for different consumers. The movements of goods make up the freight transportation demand.

Both mobility and travel choices are influenced by some characteristics of the transportation services, offered by the different travel modes and transit [45]. These features are termed *level of service* or *performance attributes* (traveling times, costs, service reliability, clearness, riding comfort, etc.). The characteristic of transportation services depend on the transportation supply. The available range of the transportation facilities (roads, parking, railway lines, etc.), regulations (rules) and prices (transit fares, parking prices, road tolls, etc.) provide the *travel opportunities*. Construction facilities in transportation supply systems have finite *capacities*.

The interconnected physical and human elements of the transportation *infrastructure* with quantitative measures on the arcs may form a single, or a multi-modal *capacitated network*. Trips and journeys can be modeled by *network flows*. When a flow approaches its upper capacity limit, interactions among users will increase and *congestion* effects will be triggered. Congestion can significantly deteriorate the performance of transportation services for the users, e.g., travel times, service delays, fuel consumptions will each increase with congestion. Any congestion has *negative external effects* (such as noise, air pollution and visual impacts in road and highway traffic) and *cross-modal effects*, e.g. it can influence the surface of transit services. *Transportation systems modeling* have physical and functional delimitations, since it is usually assumed that it is a part of a larger territory or region. This is said *study area*. The external parts are considered only through their links to the analysis system. These are valid to both demand (exchange and crossing demand) and supply (transportation infrastructures and services connecting the external area with the analysis system) and are true for both regional and urban transportation projects [9].

Mathematical models and *algorithms* applied to real transportation systems are fundamental tools for designing, appraising, evaluating, ranking the physical elements and their alternative solutions (e.g. a new railway line) and managing (traffic control) transportation systems. It is usually necessary to divide the study area into a number of discrete geographical units called traffic analysis *zones*, represented by fictitious nodes termed *centroids*. Latter points are located near to the geographic ‘center of gravity’ of a particular zone. Natural geographic separators (e.g. rivers, railway lines, etc.) are conventionally used as zone boundaries to present prohibited traffic access. Transportation networks are complex, large-scale systems that appear in a variety of forms, such as road, rail, air, and waterway networks. To handle such problems, specific mathematical methods and models, known from graph theory and network flow analysis might be employed. In this work, we will not use these approaches. The reader may find a collection of these methods with applications, e.g. in Farkas (1986).

An effective *transportation strategy*, based on a continuous forecasting, should protect the distinctive lifestyle of the regions and cities under development while meeting the cost-effective transport infrastructure needs of a growing and aging population. Sustainability requires the designers to devote special attention to ecological aspects and reduce greenhouse gas emissions since transport is a major contributor of it. Such a strategy expect to contribute an increase in job opportunities, quality levels of goods and services and provide transit oriented developments through *integrated* land use and transport planning.

Transportation projects relate to construction of transportation infrastructure, facilities and services. They should be designed and evaluated from the perspective of the community. In the prepared feasibility studies infrastructure and service providers should include project alternatives with performance measures and enhance timely and safe delivery of people and freights by indicating proper choices on vehicles and technologies in coordination with the public needs. *Decision making* must consider the effects of proposed actions on the overall community. Reorganizations often imply external impacts which strongly influence final decisions. A planned transportation project should consider the general objectives directed to the transport sector through the statements of the C*ommon Transport Policy* (CTP) of the European Union. The latest statements appeared in the so-called White Paper [91]. Present targets, set by the European Conference of Ministers of Transport (ECMT) state that the developments of coherent, integrated and internally consistent transport systems using the best available technology, reducing disparities between regions, e.g., by infrastructure construction, sustainable patterns of deployments by respecting environment, encouraging social cohesion and actions to promote safety should be accomplished [24].

1.2 Directions of Transport Policy Developments in the EU and in Hungary

Setting targets is a rather difficult and politically sensitive task. Transportation can predominantly be regarded as a means that implies very different transport policy strategies and transport policy targets being derived from more general targets. We will focus here on both scientific and political terms and will discuss target setting for environmental, regional developments and efficiency issues as well. Investment in new transport infrastructure development or reorganizing an old one is very capital-intensive and sometimes may be irreversible in nature. Therefore, such investments should maintain, preserve and extend the life and the utility of the communities, to provide for and improve the safety and security of transportation customers, promote energy conservation, protect environment and continuously improve effectiveness and efficiency of the transportation system.

We first give a brief summary of the past history of the common European transport policy based on the excellent study of Fleischer [27]. According to this work, such an idea has already originated in the Treaty of Rome. The first official document in this field was published under the title Future Development of a Common Transport Policy (CTP, 1992), also known as the Union White Paper. In this document, the EU laid down the guidelines and the key elements of the Trans-European Networks (TEN) the system of the pan-European transportation corridors (TEN-T). A revised version of the EU transport policy came out in the century turn in the White Paper, 2001 [91], in which, itemized transportation duties were assigned to four designated areas with a total of 60 measures applied. Then, in the year 2004, a rethinking of the earlier TEN-T concept has resulted in the extension of this network toward Eastern-Europe in order to enhance competitiveness and achieve better regional transit links. Another change in the year 2006 diverged strongly from the progressive line taken in earlier White Paper publications. The main direction of EU's transport policy turned to primarily focus on sustainability questions like the issues of assuring a friendly environment, higher energy efficiency and providing better balances concerning the regional developments.

Taking into consideration the EU's common transport policy concepts as guiding principles for Hungary, the currently valid strategy over 2003-2015 has been determined in the Hungarian Transport Policy (HTP) [34]. The general objectives, approved by the Parliament were as follows: (*i*) improvement of the quality of life, preservation of health, reduction of regional disparities, increasing safety and protection of the built and natural environment, (*ii*) supporting successful integration into the EU, (*iii*) improvement and extension of connections to the neighboring countries, (*iv*) promotion and implementation of regional development objectives and (*v*) creation the conditions for efficient operation and maintenance by regulated competition. As it marks out from these itemized goals and objectives, they are not defined sharply, their grouping is not entirely coherent, they are not really well-structured and they lack of priority setting [27]. A more focused transport policy description was contained by the documents of the Unified Transport Development Strategy (UTDS) [85] that spans the years 2007-2020. This policy concept imposes priorities on (*i*) the development of the passenger transport and the transport of goods by improving the international accessibility of the country and its region-centers as well as the regional accessibility (within and between the regions), (*ii*) the development of inter-modal logistic centers (in order to create efficient distribution functions toward Eastern and South Europe) and the transport infrastructure of urban and sub-urban communities and prevent the highway capacity overload originating from public road vehicles (freight traffic hubs), (*iii*) the development of public transport in cities and their agglomeration (personal traffic hubs) and, finally, (*iv*) the introduction or more environmentally friendly and energy efficient regional and urban transport systems and vehicle usage policies including an increased number of ITS applications. From these settings, we lay down that instead of the earlier planned construction of the TEN-T corridors (No. 4: from NW to SE; No. 5: from SW to NE; and No. 7: Danube and No. 10. Highway #6 to Croatia, which has partially accomplished) [54], the major emphasis was shifted to building other motorways and pavement enforcements on major roads and turned to paying much larger attention to railroad and waterways developments.

1.3 A Sequential Transport Policy (STP) Model for Sustainable Transport

To elaborate robust transport policies for the future bears many challenges. Some of them can be raised as follows: What kind of a balance could be found among engineering, economic, social, institutional and environmental aspects and among the conflicting interests of the different stakeholders? How to reconcile socio-economic and public acceptance of the project due to their unique preferences which are usually different? What extent transport policy objectives, like high level technical quality, sustainability, economic efficiency, preservation of environment, energy conservation, public service, safety etc. could be met as a result of the actions that have been made? A rich collection of transport policy objectives and a wide range of policy instruments with their measures (land use, attitudinal and behavioral, infrastructure, influential and pricing) have been described in [81] with reference to the comprehensive study that appeared in [42]. Also, when we were making efforts to shaping a new transport policy framework, we have consulted and drawn upon from the works of [46], [60], [61] and [79].

Farkas (2014) has developed a new transport policy framework what he called a *Sequential Transport Policy* (STP) model for *sustainable transport*, aiming to conform to both the EU and the domestic goals and objectives of the same kind. STP enables the prioritization of multiple measures using multi-criteria decision making techniques. This framework sets primary goals that ensure resilience and adaptability in the transport requirements of a given region or a city with special focus on satisfying societal and environmental needs. The model seems to be useful in a global and a local sense as well. STP was planned by keeping in mind to create a powerful tool that promotes sustainable transport. Its structure encompasses four consecutive phases (stages) representing the main goals. As follows, a successful accomplishment of a particular phase is a prerequisite for a start-up of the succeeding phase. This model seems to be useful for governmental departments, transport and infrastructure providers and local authorities when they are faced different alternatives of transport projects and also for transportation users, communities. Now, we describe our STP model in detail below.

Stage 1
Policy Goal #1: REFORMULATE DEMAND PATTERNS OF TRANSPORT

Glossary: *Demand* is interpreted as the amount of motorized road transport associated with the amount of street/road/highway use to access a service/activity/household item. *Demand patterns* describe the set of behavioral habits that are characteristic to the users of transport modes.

Objectives: To achieve a change in the current practice of users by reducing their needs for gasoline powered transport, i.e., lessening gasoline powered vehicle km passenger journeys and gasoline powered vehicle km freight distribution per year, traveled to deliver people, goods and services. (The term gasoline powered is meant internal combustion engines here.)

Benefits: Reduction of traveling distances; depletion of fossil fuels and thus decreasing harmful air and noise impacts on environment; motivating other transport modes; implement sustainable mobility; inspiring more effective use of existing transportation facilities and resources; to ensure lower motorized mobility, i.e., fewer journeys, shorter distances; switch to public transportation/biking/walking.

Drawbacks: New demand may arise which would neutralize the results by displacing one demand with another.

Tools: Planning and implementing decentralized new global, regional and local commercial/institutional/social infrastructure for communities in order to improve access to goods/services/activities in short distances; expanding the supply and accessibility of favorable traveling destinations; improving pedestrian-oriented design establishments, e.g., short crossings, wide sidewalks, gardens, and existing public transportation infrastructure, e.g. underground entrances, bus stops; subsidizing transit costs for employees and students; e.g. instead of providing 'commute allowances' pay for employees for parking to enjoy free parking opportunities at firms and institutions, give more incentives for car-pool to work, or especially for biking or walking; utilizing flexible time work schedules; applying road pricing tariffs during peak-hours; developing workplace travel plans; introducing time-, distance- and place road pricing for automobile users depending upon when, where and how much they drive; developing ITSs to achieve an effective and wider ranged traveler information service, e.g. about current traffic conditions, apply public notice about congestions and choices for alternate routes; introducing congestion pricing to reduce traffic jams and thus vehicle carbon emissions and heavy gasoline consumption due to idle engines; do not construct new freeways which encourages strongly sub-urban sprawl, instead build sub-urban trains leading to city centers; employing new zoning strategies, i.e. build more compact new neighborhoods with transit and shopping centers possibly within walking distance; letting new apartment houses locate around transit modes and near to corridors.

Measuring demand minimization: Achieved distance reductions in journeys with gasoline powered vehicles [passenger km/year], [ton km/year]; investment costs of new regional/local infrastructure [million $], specific measures related to the effect of reduction of gasoline powered vehicle use in urban public transport, e.g. air pollution measures etc.

Stage 2
Policy Goal #2: TRANSITION TO OTHER TRANSPORTATION MODES

Glossary: A *transition* from one transportation mode to another is called a *modal shift*. A *modal share* (*modal split*) represents the percentage of travelers or the number of trips using a particular type of transportation. In freight transportation this is usually measured in mass. *Inter-modal* passenger transport (also called *mixed-mode commuting*) involves using two or more modes of transportation in a journey.

Objectives: The purpose of mixed-mode commuting is to combine the strengths (and offset the weaknesses) of the various transportation options. The major objectives are: to reduce dependence on the automobile as the major mode of current ground transportation and increase the use of public transport and, similarly, a considerable amount of freight delivery happens on highways should transfer to railways and/or waterways.

Benefits: Comparative advantages have many forms, such as reducing pollution of environment, cost savings, since additional congestions would carry economic-cost, capacity extensions, traveling time reduction, extensions of existing flexibility and achieving a higher reliability: depending on what, where and when is being transported the worth of the above factors can significantly vary: the higher the gain is the more incentives are to switch from one mode to another; decreasing bottlenecks would produce large benefits as congestion adds to journey times and makes logistics less predictable which complicates supply chain management routines.

Drawbacks: A significant drop in comparative advantages would contribute to an undesired stopping of this phase, as the new mode gets increasingly crowded, furthermore, opportunity loss may emerge since the previous mode loses traffic, e.g. when some routes have to closed, price cutting must be employed, etc.

Tools: Supporting transitions into modal shift, since these actions take freight off the roads and transfer it to rail and/or to waterway transport; setting modal share targets for transport modes in urban transport (e.g. let 30 % of non-motorized and 30 % public transport), since modal share is an important component in developing sustainable transport within a city or a region; keeping road journeys as short as possible; imposing restrictions on moving freight by road over the weekends; making cars less attractive and in parallel, walking and cycling more attractive in urban transport; enhancing the quality of the waiting facilities at bus stops and rail stations; improving security with the use of ITS devices and reducing vandalism; a general use of electronic information at bus/tram stops and rail/subway stations; building more and larger parking lots at rail/subway stations and also for trucks at the sub-urban areas of the cities; diminishing bus travel times so that to build new bus lanes; reallocating road space to give more priority to pedestrians; creating better integration among modes covering physical interchanges, time-tables, information and ticketing.

Measures of transition: Average speed and/or average time to reach target destinations, traveling convenience and comfort, measures of environmental impacts (emission and noise), proximity to mass transit, frequency of congestion occurrences, changes in scheduling issues (time-table coordination).

Stage 3
Policy Goal #3: IMPROVE TRANSPORT EFFICIENCY

Glossary: *Transport efficiency* is a measure of transportation system performance that shows how well a transportation system and its constituting elements consume resources in a given time period. It is a ratio of the effective (useful) outputs to the total input. Outputs are typically equal to the total supply of transportation services during that period, while inputs are equal to the cost of transportation resources required to produce that output.

Objectives: To improve transportation sustainability and achieve a continuous reduction in transportation costs in order to increase global competitiveness. In other words, to get better outputs from given inputs.

Benefits: Fuel-efficient vehicles require less gas to take a given distance; to burn less gas requires less fuel use (oil), therefore the cost per journey would become lower; fuel-efficient vehicles contribute to reducing global warming, harmful materials' emission, noise impacts and protect public health; for both freight and passenger transportation there could be less cost per journeys; better capacity utilization of the vehicles; land-use improvements.

Drawbacks: There are some challenges concerning efficiency, e.g., the 'rebound' effect which means that improved efficiency will not reduce the need for gasoline powered transport and lead to more frequent travel, which would increase both energy and transportation demand (through the generated traffic or by the induced demand).

Tools: Promoting behavioral changes in the driving habits, e.g., environment- and economic friendly driving style; utilizing longer vehicle combinations to reduce the number of trucks on the roads and highways; permitting use of long vehicles on highways with greater load capacities as opposed to idle running; letting vehicles drive in columns by keeping short distances between them to reduce air drug and improve utilization of the highway network; implementing the so called 'FreightBus' concept in urban areas which carries both passengers and goods; introducing the use of purpose-designed load modules which can be transferred to smaller delivery vehicles assigned specifically to urban conditions; extending the use of ITS for helping the drivers to avoid road congestions, places of accidents, etc.

Measures of transport efficiency: Transportation efficiency is a compound term. It implies fuel-efficiency, inter-modal conditions, land-use, vehicle occupancy and a set of trip and routing data. Therefore, traditional measures, like [kWh/tkm] for freight, and [kWh/pkm] for passenger transport are usually not enough to make a thorough analysis. There are a variety of input measures, e.g. volume/mass for materials; labor hour for human resources; navigation prescriptions; terminal operations for services; physical and monetary units for investment capital; weight, power, etc. data for planning and cargo trips, number of vehicle trips, vehicle distances and capacity data for transportation. Similarly, there are a great number of common output measures, e.g., ton-kilometers, passenger-kilometers, special dimension with system boundaries, time dimensions as transit time, peak hours etc., quality of service like speed, reliability, dependability, flexibility, etc. Transport efficiency for the different transportation modes can be expressed as fuel consumption per unit distance per vehicle [l/100 km] or fuel consumption per unit distance per passenger [l/pkm], or fuel consumption per unit distance per unit mass of cargo transported. [l/tkm].

Stage 4
Policy Goal #4: ENHANCE AVAILABILITY AND USE OF RENEWABLE ENERGY POWERED TRANSPORTS

Glossary: *Renewable fuels* are those derived from renewable biomass energy sources in contrast to fossil fuels (petrol and diesel). Renewable energy powered transport includes *alternative-fuel vehicles* including electric, hybrid electric, biomass-fuel, hydrogen, ethanol, methanol, compressed natural gas (CNG), liquefied propane gas (LPG) and other ecologically preferred power sources.

Objectives: To have a sustainable transportation system by improving quality of life for individuals of a society with human and ecosystem health and with efficient operations; offering a choice of transport mode and supporting economy as well by implementing a balanced regional and urban transport. Due to factors, such as environmental concerns, high oil prices, dusty operations, the development of advanced power systems for vehicles must gain one the highest priorities for governments, municipalities, engineering firms, industrial and transportation engineers all around the world.

Benefits: Saving significant amount of costs, since many of these alternative-fuels have high energy efficiency, hence a potential for excellent fuel economy (electric, hybrids, hydrogen); considerable reduction in harmful carbon dioxide, nitrous oxide and particulate matter emissions (electric, ethanol, bio-diesel, hydrogen); less noise impacts on the environment (electric, hydrogen); cost much less than gasoline (CNG, LPG, ethanol, methanol); ensuring environment-friendly operation, providing opportunities to increase choice of transport mode and to fit everyone to meet different life styles; acting as a means of a broad area of engineering and manufacturing developments together with new business opportunities for companies and entrepreneurships and increasing employment especially by inducing need for hiring more highly educated graduates and skilled workers.

Drawbacks: The different types of alternative-fuel vehicles have dissimilar benefits and drawbacks, e.g., purchasing costs of alternative fuel vehicles are very high related to the conventional (diesel and gasoline) vehicles (bio-diesel, electric, hybrid, hydrogen); Huge gas tanks/batteries trunk spaces are needed with less storing capacities of fuel due to their low upper limits (CNG, electric and hydrogen); short cruising distances, long recharging times, frequent need for recharging, low speed (electric); inefficient fuel-economy (ethanol).

Tools: The growing financial support (EU/EBRD/regional) for funding innovation, energy, environmental, societal projects in transportation research and, additionally, the introduction of such new productive technologies with promoting start-ups for the manufacturing of these alternative-fuel vehicles should be fully utilized.

Measures: There are a broad variety of different performance measures, technical, economic, social and environmental related to different engineering characteristics, implementation costs, exhaust fumes emissions (CO_x, NO_x, particulates), energy efficiency measures, then a great number of qualitative indicators for measuring achieved quality of life improvements for the transport users and communities (see a detailed study about alternative-fuel vehicles in Application 8, Chapter 6).

Suppose now, that we wish to implement our STP model in practice and assume that the project alternatives of a planned transportation development project are known (e.g, to build new infrastructure in an urban area). At this point, the following question can be raised. How can the decision maker(s) select a feasible option from the available set of projects if he/she wants to follow *optimal transport policy* through each the four stages. For this purpose, a systematic tool, termed *multistage dynamic programming technique* is proposed as displayed in Figure 2.

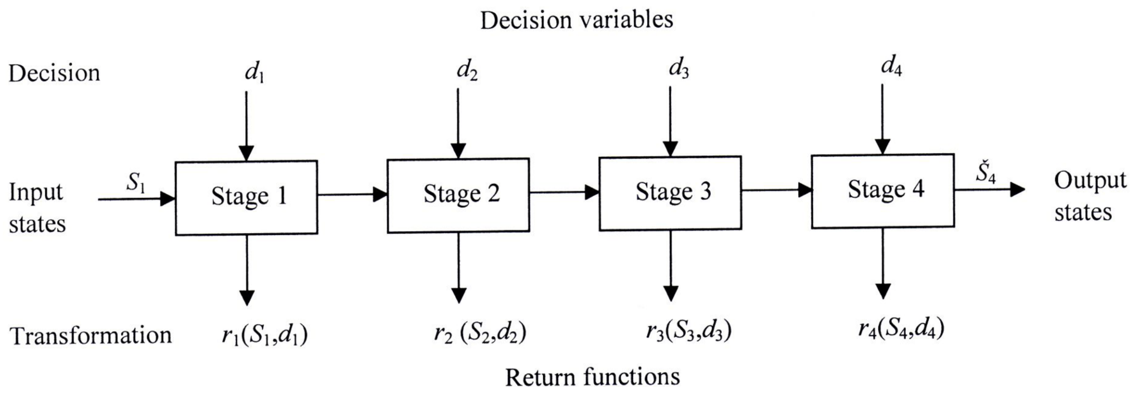


Figure 2. A four-stage dynamic programming approach for the STP model.

In this four-stage model, we denote a point at which a policymaker makes a decision as the nth stage, and its corresponding input parameters as the *state*, S_n. A decision itself is governed by some sort of rules, called a *transformation*. At each stage, regarding a given policy goal, the decision maker should make a *decision*. Every decision has a relative worth. Let these worth (benefit or loss) be represented by a *return function*, $r_n(S_n, d_n)$, since for every set of decision one makes, he/she gets a return on each decision. This return function will, in general, depend on both the state variable S_n, and a decision variable d_n, chosen from the set of feasible decision variables at stage n, $n = 1, \ldots, N$. In our case, $N = 4$. An optimal decision at stage n would be that decision which yields the most favorable (maximum or minimum) outcome for a given value of the state variable S_n. Each of these stages (decision points) are related by a *transition function*, i.e. $S_n = S_{n-1} \circledast d_n$, where the symbol $\circledast$ denotes an appropriate mathematical operation of the stage transformation that depends upon the problem under study. The units of S_{n-1}, d_n and S_n must be homogeneous. The designations of these units are determined by the particular problem being solved. Since a state variable is both the output from one stage and an input to another, it is sometimes represented by more than one symbol. Such a dynamic programming approach lands itself best to suit to our transport policy model, since such a multivariable optimization problem can be solved sequentially, one stage at a time. Hence, it is necessary to keep track of all the returns accumulated in this process as one proceeds from stage to stage. Denote by $f_n(S_n, d_n)$ the accumulated total return calculated over the four-stages given a particular state variable. Similarly, denote by $f_n^*(S_n)$ the optimal four-stage total return for a particular input state S_n. That is, a particular value of S_n might give rise to many possible decisions, d_n, among which is a decision, d_n^*, which produces an optimal n-stage total return $[f_n^*\,(S_n)]$. It is now apparent, that our STP model can be represented as the following optimization problem for determining an *optimal transport policy*, which might be achieved by using the *forward recursion*:

$$f_N^*\left(S_n\right)=\underset{d_1,d_2,\ldots,d_N}{\text{optimize}}\left\{r_1\left(d_1,S_1\right)\circledast r_2\left(d_2,S_2\right)\circledast\cdots\circledast r_N\left(d_N,S_N\right)\right\}. \tag{1.1}$$

In this general expression (1.1), the symbol represents any operand dictated within the context of the transport policy problem at hand and, in addition, might change from one stage to the next. As a matter of fact, designing transport projects are usually very complex tasks. Therefore, in many cases, putting them into practice would impose large difficulties for the participants. It appears especially difficult to find the appropriate variables and transform them into homogeneous units. For that, the use of multi-criteria analysis techniques (discussed in Chapter 2 and in Appendix A in detail) are recommended including standardization and normalization of the performance measures, which are very often given in different units of measurement originally, so it is recommended that they are represented by *utility values*.

Concerning the one-way linear structure of the STP model, however, one may raise proper criticisms. Indeed, there is room for its further improvement. For example, incorporating feedback opportunities and/or amplifying the model to have iterative features mainly between stages 2 and 4, seem to be necessary and are subject of future research.

1.4 Transportation Systems Design and Related Decision-Making

To establish and maintain transport infrastructure through planning, design, construction and traffic management have considerable economic, social, environmental consequences. These are spatially distributed across areas. The choice of the most appropriate transport policy involves balancing engineering, economic and environmental considerations, as well as their spatial distribution. It also implies balancing the demands of the stakeholders. The planning aspects of transport engineering involve intercity transportation problems, such as route/site selection, vehicle assignment and routing, fleet sizing for passenger and freight deliveries and urban transportation planning problems, such as trip generation and distribution, mode choice, route assignment, land-use forecasting and selection of residential or business locations [9]. The complexities of transport policies, for example transport land use and transport environment interactions prompt the need for sophisticated assessment methods and comprehensive decision making processes. Multi-criteria analysis methods can serve to systematically identifying and structuring objectives making trade-offs and balancing risks. They may be used to analyze the system and to choose proper options to account for a proposed policy and to test the appropriateness of a certain policy.

Changes in transportation systems may affect a community and its members in a variety of ways [9]. Building a new facility, for example, may not only change the service experienced by the network users but also produce economic, financial, social and environmental impacts on groups of individuals, land owners, businesses and institutions. The rational approach to decision-making involves a thorough evaluation of the impacts of these projects on the various affected parties. The natural dynamics of society and changes in civil attitudes have resulted in the recognition of the relevance of such decisions. Planning is no longer seen as an activity and preparing a single master plan. Rather, it is now viewed as a complex process with a sequence of decisions concerning a lot of external and internal factors. In this framework, the use of quantitative/qualitative methods for ranking, sorting, evaluating these projects is inevitable. Thus, the identification and the use of proper decision-making methodology are extremely important for transportation systems, indicating the clear identification of objectives, the range of constrains and account for the effects of planned actions. These might be explicitly or implicitly defined. Transportation systems design and evaluation can be made from different points of view (many conflicting interests of the stakeholders). Objectives would typically dictate profit-, effectiveness- and efficiency maximization, subject to constraints, like existing regulations, standards, available budget and technical limits. Public decision makers may argue for environmental aspects, enhancing safety, improving accessibility to economic and social activities, fostering land development, protecting resources, saving energy consumption. Objectives and constraints synthesize the values and attitudes of the firm or of society.

From the modeling perspective these factors have an impact on the definition of the analysis system. That is, identification of the relevant elements and their relationships included in the models of the systems in order to evaluate correctly the effects of the planned actions. A new underground line, for example, requires reorganization of the surface transit lines to increase the catchments area of the stations (complementary action). Restricting the car access to part of an urban area requires the design of appropriate parking lots, transit lines, etc. (integrated actions) [9]. A critical issue is the collection of reliable and appropriate data associated with the model and its mathematical description concerning the present and the planned system, which provides inputs for the models formulated and also some performance indicators that would be too costly to measure directly.

Conventional decision making techniques have largely been *non-spatial*. In transportation engineering problems, the assumption that the study area is spatially homogenous is rather unrealistic, because in many cases evaluation criteria vary across space. The most significant difference between spatial decision support methods and the conventional decision support methods is the explicit presence of a spatial component. The presence of the spatial component in transportation projects implies that problem solving highly depends on the geographic pattern of the area under study. Recent advances in geo-information technology through various remote sensing techniques has offered appropriate technology for data collection from the earth's surface, georeferencing, information extraction, data management, and visualization. Spatial Multiple Criteria Evaluation (SMCE) is based on multiple attribute decision analysis (MCDA) combines these evaluation methods and spatio-temporal analysis performed in a GIS environment [50]. The spatial decision problem can be visualized as a two or three dimensional table of maps, or map of tables. In the SMCE, the decision alternatives, a_i, are the three series of maps, and the criteria, c_j, are the pixels (basic units for which information is explicitly recorded) or polygons in the maps. The model in Figure 3 shows that not only an aggregation of effects (function f), but also a spatial aggregation (function g) is necessary to arrive at a ranking of alternatives.

Different paths lead to different results in the ranking of the alternatives. The distinguishing feature of Path 1 and Path 2 is the order in which aggregation takes place. Most computer applications of SMCE follow the aggregation of effects of Path 2 (the first step is aggregation across criteria, the second step is aggregation across spatial units) [72]. Thus, SMCA is a process that combines and transforms geographical data (the input) into a decision (the output). This process, called an *integrated GIS-MCDA* approach consists of procedures that involve the utilization of geographical data, the decision maker's preferences and the manipulation of data and preferences according to specified decision rules [49]. As a decision making support, the AHP method (see in Chapter 2) is built into most computer softwares.

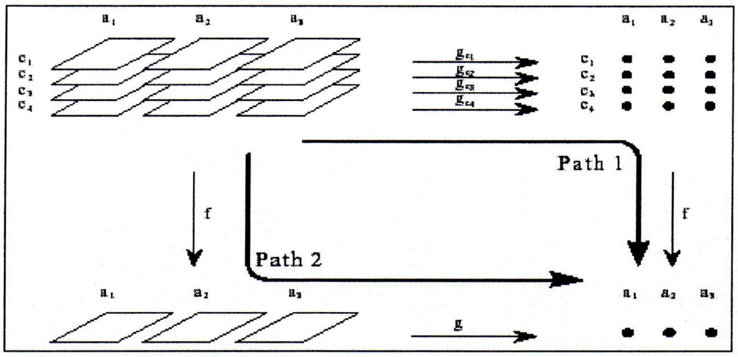


Figure 3. Two possible pathways of spatial multi-criteria evaluation [72, p.2]

An effective and coherent decision-making in the areas of transportation engineering and civil engineering assumes a scientifically based, well-structured, systems-oriented evaluation of the feasible choices for a given real-world problem. In these fields of interest, proper decision making methodologies must be capable of handling the usually multi-disciplinary nature of the problems emerging. In addition, very often, both quantitative and qualitative attributes with different units are incorporated in the decision problem when one attempts to find a solution. Apparently, this fact complicates evaluations in the decision-making processes. In the traditional approach, the selection of the best solution was found by the economic valorization of the transportation/civil engineering projects, i.e. the by the static of dynamic analysis of the costs that might incur and the profits that might be generated. Such a world-widely used framework for transportation and civil engineering project appraisal and for investments decisions is the well-known cost-benefit analysis (CBA); see e.g. in [80]. Due to the recognized inherent limitations of CBA in many transport applications, they have gradually lost their popularity. The complexities of the problems like the inclusion of a great number of non-quantifiable attributes into the decision making process, e.g. land use, environmental and social aspects, gave rise to the need for more comprehensive approaches to making coherent assessments. Fortunately, over the last two decades, the so-called multi-criteria decision making methods have gone through a considerable progress. These multi-criteria evaluation methods may be used for the purpose of [88]: (*i*) a descriptive analysis of the spatial system (*ii*) selecting the most favorable option from a predefined given set of feasible alternatives, (*iii*) accounting for a proposed line of action of policy, (*iv*) testing the likelihood of the appropriateness of a certain policy.

According to Keeney [40], two major approaches can be distinguished in MCDM: (*i*) the alternative-focused and (*ii*) the value-focused approach. The alternative-focused approach starts with development of alternative options, specification of values and criteria followed by the evaluation and recommendation of an option. The value-focused approach considers the values as the fundamental component in decision analysis. Therefore, first, it concentrates on the specification of values (value structure), then, it develops the values feasible options and evaluates them with respect to the predefined value and criteria structure. This implies that the decision alternatives should be generated in a way that values specified for a decision situation are best met. Hence, the order of thinking is focused on what is desired, rather than the evaluation. For example, in the context of route/site selection problems of urban transportation, the value-focused approach has many advantages over the other [72].

Farkas (2009b) proposed a top-down decision analysis process to determine and structure the goal, the objectives and their related indicators suited to the facility selection problem of planning a metro-rail system with three different alternative route options. This hierarchical decision tree model is presented in Figure 4 prepared for a route/site selection problem (see Application 1). In the decision making phase, a consulting team, technical committee members, designers, investors, local authority officials and public representatives are involved as the basis for development and evaluation of the project. The various elements of this criteria structure are now described.

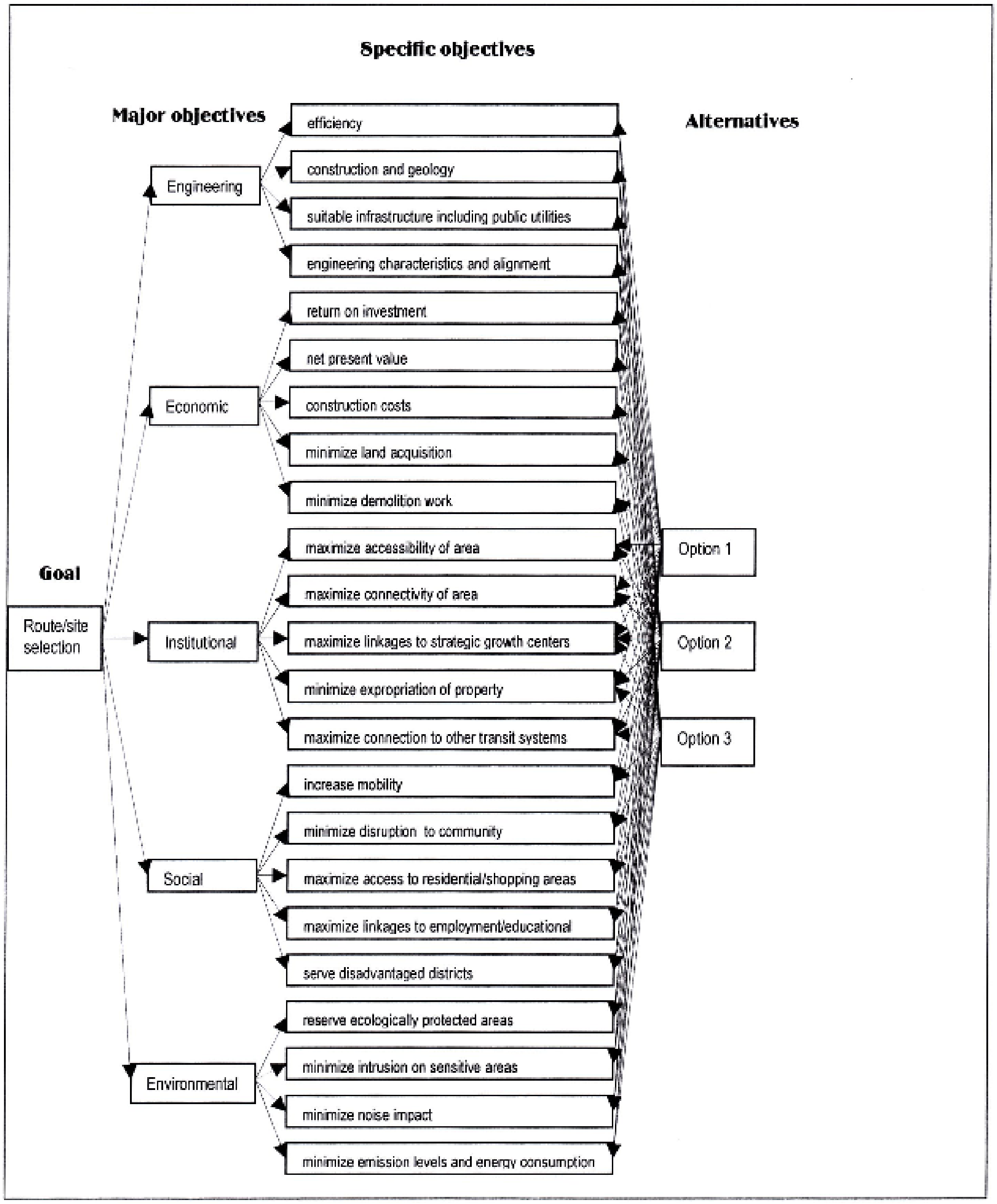


Figure 4. Hierarchy of goals, objectives, criteria and indicators for urban transport planning [Farkas (2009b)]

Goal and Objectives: The goal of this framework is to identify an effective public mass transportation system for a metropolitan area integrated with an efficient land-use so that it meets the present and long-term socio-economic and environmental requirements of the residents of the marked territory. This goal can be achieved if the following objectives are met:

Economic Objective: The economic objective seeks to maximize feasible economic return on investment from the system. A number of criterion is used to measure how well an option performs on each indicator, e.g., benefit/cost ratio, first year return, internal rate of return, net present value, construction cost and operation cost, as well as minimizing land/real estate acquisition (expropriation of property), intensification of existing land-use and maximizing the potential of the location.

Engineering Objective: This objective looks at three main concerns that are the efficiency of the system, the construction issues and the effective use of the network for work and non-work travels. The criteria which measure the extent to which such achievements are met by the transit route or facility options are the following:

- Efficiency is measured by examining the minimum number of transfer, (whereby an alternative with excessive transfer will score low for this criteria) A transit option, which contributes to a reduction in travel time compared to time spent on the roads and provides a close to optimal convenience for pedestrian access and links to other local and commuter transportation modes, further, an effective connection of housing, jobs, retail centers, recreation areas would be beneficial and will score high.
- From the construction perspective, alternatives that have rail routes passing through high demand areas like high density built-up areas, commercial, industrial and institutional areas, will score high for this criterion. This aspect, however, comes to the front, particularly when it is accompanied by poor geological conditions at a route/site option, conflicts with a low construction cost requirement. To build metro-line stations, the commonly used construction modes are: open-cast construction (just below grade, building pit is beveled or secured by walls, requires large construction areas, more flexibility in design); bored-piled and cover-slab construction with or without inner shell (bored-piled wall, generates column free space, reduces surface interruption); diaphragm wall and cover-slab construction (excavation after diaphragm and cover-slab are constructed, multi-story basement structure, structure growths from top to downwards); mine tunneling construction (extremely deep situation, use of shot crete but cracks and leakages are not avoidable).
- Engineering characteristics and alignment in this plan are evaluated with respect to the measures, attributes constituting the geological environment (including soil mechanics, intrusive rock structure, stratification); hydro-geological conditions (including underground water-level, chances of inrush, perviousness, locations of permeable or impermeable layers, chemical and physical characteristics of underground water and their effects on the built-in architectural structures) and geotechnics (rock boundaries, response surfaces, geographic configuration). Special focus should be given to safety. Therefore, the recognition and control of risk factors are of utmost importance (a water intrusion, a gas explosion, a chance of an earthquake).
- Infrastructure involves the careful examination/analysis of the overground building up, the suitability of the existing public utility network and the required overground organization to be made before the construction works are started.

Institutional Objective: This objective measures the match between the transit system and spatial policies of the government/urban municipality, e.g. to maximize interconnectivity to existing public transport systems; to maximize linkages to strategic growth centers (as designated/proposed in local plans), to provide good linkages among urban centers and sub-urban railway networks, airports, long-distance bus stations, park and ride lots as well as to minimize land acquisition.

Social Objective: Establishment of a transit system should increase social mobility by way of easy access to existing and future settlements. This can be measured by forecasting the passenger/km reduction from residential to employment areas, and from residential areas to educational institutions. Based on plans and ideas of future settlements, employment and educational institutions, efficiency of the land use objective should be achieved by maximizing access between residential areas and shopping, service and recreational centers. Such systems would serve highly populated areas and particularly disadvantaged areas (low cost settlements); would increase access to tourism attraction areas; minimize disruption to neighborhood communities; and maximize linkages to major employment areas/centers.

Environmental Objective: The designed transit project should minimize intrusion and damage to the environment. Protected areas must be excluded from the set of the potential options. The expected accomplishments are: a reduction in energy consumption and minimizing emission levels, the intrusion into environmentally sensitive and reserved areas and the noise impact to sensitive land-use (such as hospitals, residential buildings and schools) during site construction.

Criteria and Indicators: To further support the design and evaluation of a metro-rail network, the major objectives are further broken down into specific objectives with their corresponding indicators (sub-criteria). These indicators are then used to measure the performance of each alternative route/site option on each objective.

We conclude that, as opposed to the conventional planning methodology where the route/site alternatives (location selection of the stations and tracing the track) are designated in advance followed by the analysis of the impacts on the global and local environment, in our approach, first, the suitability of each area (which can be chosen as arbitrarily small units) are evaluated with respect to multiple criteria, then, the most favorable locations of the stations (and thus the track of the line) are determined. We propose an integrated GIS-MCDA tool with the hierarchy of Figure 4, which would enable transportation engineers to select the best option from among a given set of feasible alternatives as is demonstrated in Application 1.

Application 1: An Intelligent GIS-based planning of a metro-rail network

In Farkas (2009a and 2009b), the planning of a new urban transportation infrastructure project using an integrated GIS-MCDA approach was presented. In this study, we showed how a combined GIS-SMCE system (as a Path 2 analysis in Figure 3) can assist the design of alternative solutions for urban transit zone locations in a given metropolitan area. Unfortunately, spatially referenced data (with geometric positions and attribute data) cannot be accessed in direct ways. Therefore, the author has chosen a built-in database taken from the ILWIS (Integrated Land and Water Information System) library [37], which has been developed by the International Institute for Aerospace Survey and Earth Sciences (ITC), Enschede, The Netherlands. ILWIS is a Windows-based remote sensing and GIS software which integrates image, vector and thematic data in one powerful package available on the desktop. In this study, Release 3.4 is applied (as an open source software as of July 1, 2007) which contains a strong SMCA module as well [36].

Study area

The study area is Cochabamba city, a fast growing center located in the Andean region of Bolivia with a fast growing population of approximately 550 000. The city is located at an elevation of about 2 600 meters above sea level in a large valley on the alluvial fans at the foot of steep mountains. The city's northeastern side area is occasionally subjected to landslides, soil erosion and heavy flashfloods. Hence, from a perspective of urban transportation development, the improvement of its transport infrastructure would be of large importance, however, topographical and geological attributes do form quite serious considerations in building a metro-rail system.

Geographic data

Spatial data includes field collected data and GIS datasets (which consist of data derived by remote sensing from satellite imagery and/or field measurements). Attribute data are partly based on actual measurements, but, for the most part, are elicited from judgments, and, thus, they are fictive. To display geographic data (spatial and attribute data) on screen or in a printout, digitized vector maps (point, segment and polygon maps) and raster maps are used as visual representation forms chosen in a convenient way. Each map must contain the same coordinate system and georeference. In a raster map, spatial data are organized in pixels (grid cells). Pixels in a raster map all have the same dimensions. A particular pixel is uniquely determined by its geographic coordinates expressed in Latitudes (parallels) and Longitudes (meridians). With the help of a map projection, geographic coordinates are then converted into a metric coordinate system, measuring the X and Y directions in meters (UTM). This way, a very high degree of accuracy is reached.

Description of data sets

The geographic area of the planned metro-rail project (network system) is given by the polygon map "Cityblock" and is shown in Figure 5. (The skewness of the chart is due to the north-pole orientation of the map.) This map has a total of 1408 blocks (polygons). A code is assigned to each of these polygons for unique identification. Block attributes are the geometric area in square meters; the prevailing land use type, i.e., residential (city blocks used primarily for housing), commercial (city blocks containing banks, hotels, malls, supermarkets, shops), institutional (such as governmental offices, universities, schools, hospitals, museums), industrial (buildings dedicated to industrial activities, storages), recreational (including protected areas, parks, sport fields), existing transport facilities (railway stations, bus stations, taxi services, public parking lots), water (including lakes and rivers), airport and vacant (blocks that are not used for any urban activity); the codes of city districts; and population (number of persons living or using a city block).

Identifying assessment objectives/criteria

As a simplified illustration of the site selection problem, that is to find the potential locations for metro-rail stations, consider the central part of the city. This dependent polygon map "Center" has 137 blocks and its location is shown by the shaded area that is added to the layer "Cityblock" as is shown in Figure 6. Its block attributes include the following specific objectives (with their computed or estimated numerical data) for each polygon:

Figure 5. Polygon map "Cityblock"

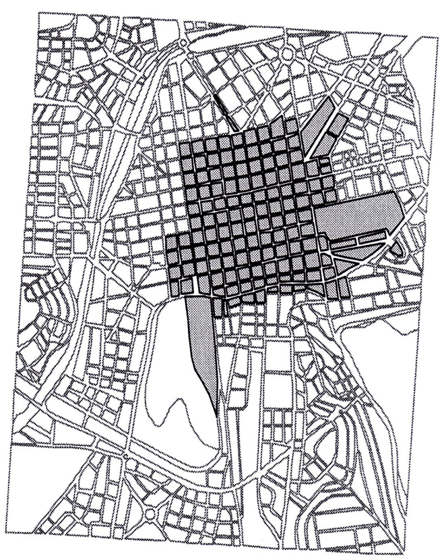


Figure 6. The embedded polygon map "Center"

C_1 = engineering characteristics and geological soil structure (rocks) [% scale],
C_2 = ecological suitability [% scale],
C_3 = connectivity index [m] (converted to an inverse interval scale),
C_4 = population density [number of people/area-hectare], and
C_5 = projected construction costs [mi$].

In the course of the aggregation to calculate the values of the composite attributes, among these criteria, C_2 represents a spatial constraint that determines areas which are not at all suitable (these areas will get a value of 0 for that pixel in the final output); C_1, C_3 and C_4 are criteria representing spatial benefits that contribute positively to the output (the higher their values are, the better they are with respect to those criteria) and C_5 represents a spatial cost factor that contributes negatively to the output (the lower its value is, the better it is with respect to that criterion).

Processing of raster datasets

Raster layers were derived by applying an appropriate GIS raster processing method for the vector maps. Vector maps contain the data sets required for the SMCE. ILWIS requires all raster overlays to have the same pixel size. In this study, a pixel size of 20.00 meter was chosen to rasterize all vector layers.

Weighting of criteria

Weights of the major objectives seen on the hierarchical decision model in Figure 4 were determined by a group of experts formed of five transportation engineers, three mechanical engineers and two economists using pairwise comparison matrices (PCMs) of the AHP. In real-life problems, apparently, more groups of stakeholders supposed to be requested. Our results, therefore, will not represent the positions involved organizations and civil members take and are only indicative. Yet we attempted to demonstrate the deviations in the views coming to the surface represented by the different stakeholders' groups in the course of the evaluation process. The consistency measures μ_i, of the PCMs (see Definition 2.3) generated by the committee' members varied between 0.023 and 0.042, which are fairly good, and thus acceptable.

Spatial multi-criteria assessment

A criteria tree was constracted for the major objectives, their associated factors and constraints and attached importance weights by ILWIS considering three different project policies (equal vision, engineering vision, economic vision). In a SMCA, each criterion is represented by a map. Due to the different units of measurement, standardization of all criteria had to carry out using an appropriate setting ("Attribute", "Goal", or "Maximum") depending on the given factor and data characteristics. As a result, all the input maps were normalized and then, utility values were computed using a closed interval between 0 (not suitable) and 1 (highly suitable). The prepared criteria tree in the ILWIS format is exhibited in Figure 7 for the engineering vision. In this simplified study we selected only one specific objective from each set of the five sets of the major objectives as depicted in Figure 4. Observe in this snapshot the weighting numbers, standardization conditions and the associated raster maps indicated in the ILWIS window.

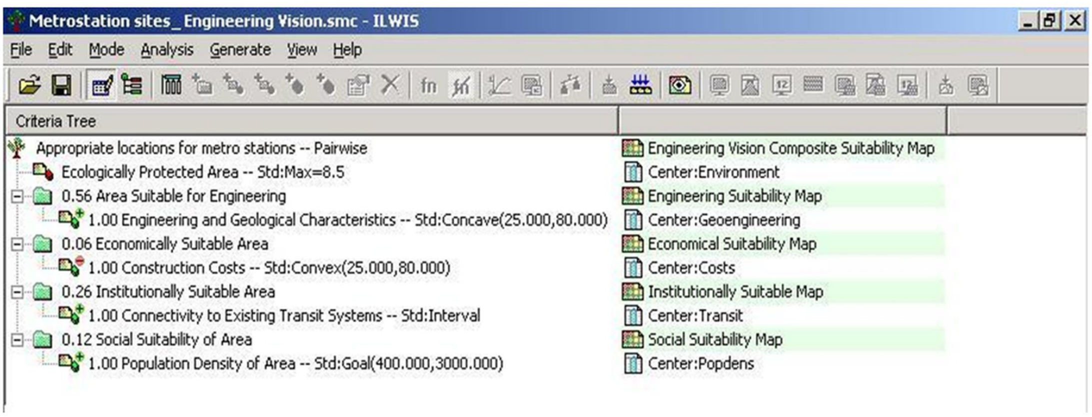


Figure 7. ILWIS screenshot of the criteria tree for identifying suitable locations

The assessment process resulted in the output maps for the policy visions showing the suitable locations of metro-rail stations in the inner part of the city. As an example, the suitability maps of the single objectives (criteria) and the composite suitability map for selecting appropriate locations for the stations of the metro-rail are shown in Figure 8 for the engineering vision. In these raster maps, areas of low suitability (valued 0 or close to 0) are symbolized by the color red, while areas of highest suitability (valued 1 or close to 1) by the color green. For color interpretation the reader is referred to the electronic version of this dissertation on the web. For every pixel, the pixel information catalog contains the utility values in quantitative manner. We remark that the pixel information is invariant within a particular polygon (e.g. city block), since the functionality of these blocks can be regarded homogenous.

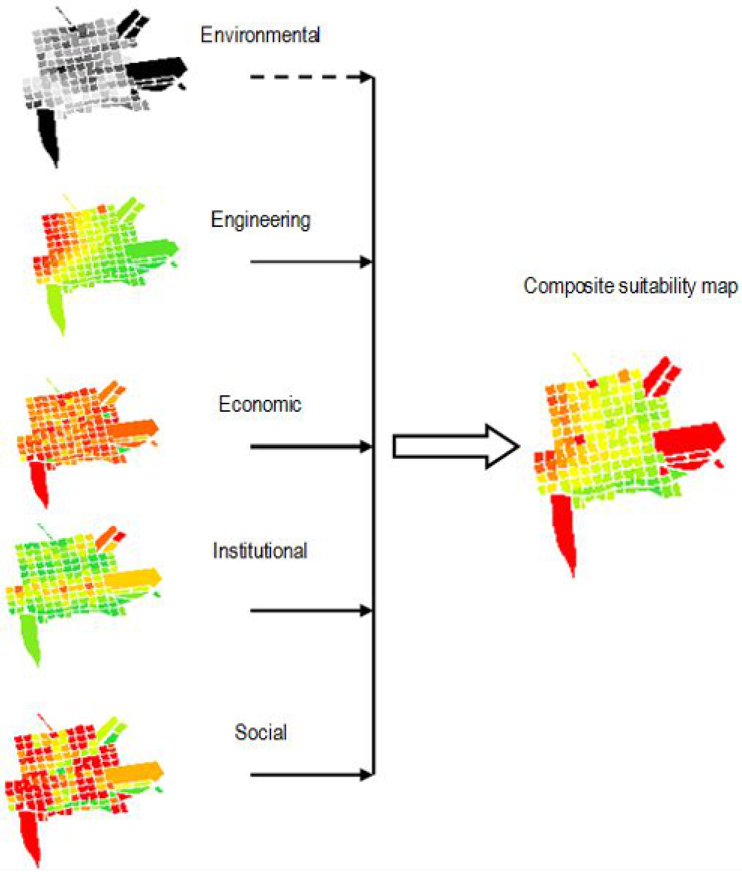


Figure 8. Aggregation of suitability maps of the objectives to an overall composite map

Designing alternative metro-rail paths (tracks)

In this step of the planning process the selection of appropriate metro-rail routes are performed. We first extended the processing of our raster datasets to all other city blocks (beyond the blocks contained by the "Center" raster map) then generated the output suitability maps for the polygon map "Cityblock". A careful analysis of the maps obtained for suitable locations of metro-rail stations enabled us to design proper tracks (pathways) leading between the two major transit zones of the city, i.e., from the origin node (South Railway Station) to the destination node (North Railway Station). These corridors, whose width span more than one block in the polygon map of the city, are indicated by the shaded areas in Figure 9.

We also had to consider the technical requirements, like rail-track geometry, vehicle engineering standards and specifications (e.g., feasible length and radius of transition curves, possible slope of the tracks, etc.), when such a corridor was mapped out. As is displayed by gray color in Figure 10, three metro-rail routes for potential metro line alternatives have been established (Blue Line, Red Line and Green Line). By a thorough investigation of the suitability values of the multiple factors at different pixels along these three corridors, ultimately, the locations for the metro-rail stations were fixed. Thus, a rough feasibility plan of this metro network project was completed as it is shown in Figure 10.

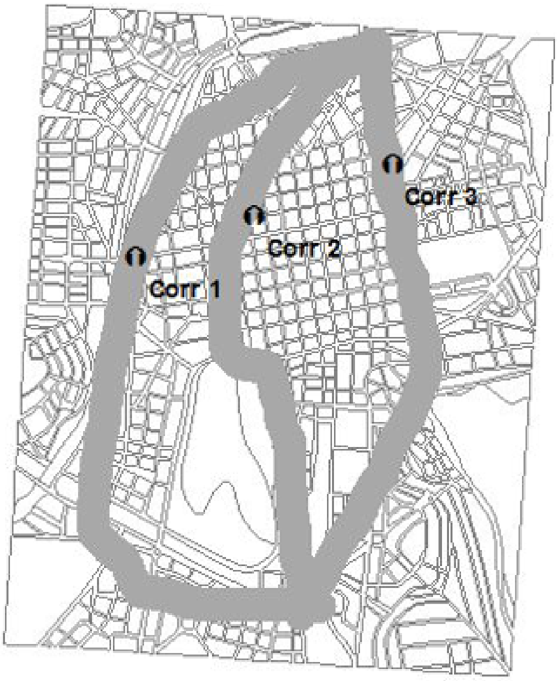


Figure 9. Corridors for the metro-rail tracks

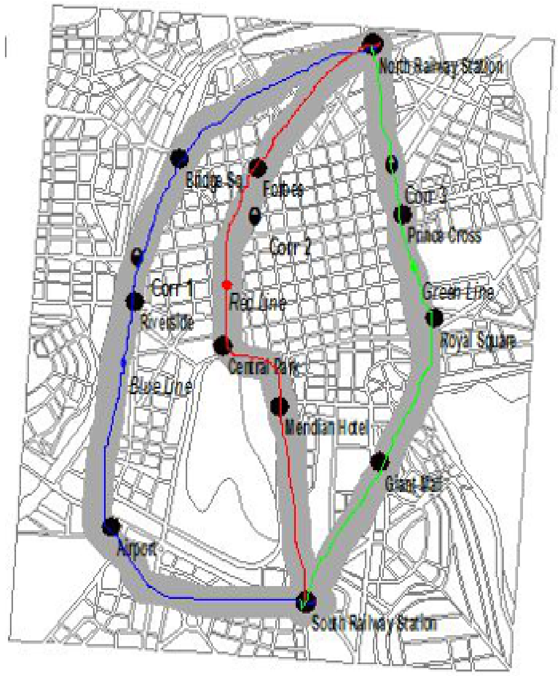


Figure 10. Feasibility plan of the metro-rail network

Network analysis via evaluating alternative metro-rail routes

Effectiveness and efficiency of both construction and operation of a particular route are mostly determined by the characteristics of the stations along that route. Therefore, it is reasonable to measure the extent to which an average suitability of the metro stations along a given route contributes to these characteristics. Introducing the mean spatial utility measure of a given metro-rail route as

$$\mathrm{MSU}_i = \frac{\sum_{j=1}^{N} u_j}{N}, \qquad i = 1, 2, \ldots, M, \tag{1.2}$$

where in (1.2), u_j is the utility (suitability index) of the pixel (raster cell) underlying the jth site (metro station) along the ith route, N is the number of the selected sites along the ith route and M is the number of the alternative route options. In order to form a commonly used measure in similar transportation problems called impedance, we compute the complementary of the value of MSU_i and multiplying it by the total length of the routes. Hence, the value for impedance of the ith route of the metro-rail network system yield

$$\Omega_i = (1 - \mathrm{MSU}_i) \cdot L_i, \qquad i = 1, 2, \ldots, M, \tag{1.3}$$

where in (1.3), L_i is the length of the ith route option (the length of the ith polyline). The higher the value of the impedance Ω_i is, the greater the costs associated with that route and the lower the benefits attained by it. Thereby, the best route option can be obtained by

$$\Omega^* = \min_i \{\Omega_i\}, \qquad i = 1, 2, \ldots, M. \tag{1.4}$$

A multiple criteria evaluation of the planned metro-rail network was done based on the performance of each route with respect to the total impedance accumulated by that route. The result of this process for the three competitive metro-rail routes is presented in Table 1 for the engineering vision. This table contains, the route options defined by the respective sequences of nodes (the raster cell codes and the names of the metro-rail stations with their utility values/suitability indexes (composite index scores), the length of these lines (obtained by the distance calculation module of ILWIS) and the total impedance of the routes.

Table 1. Effect table of the three potential metro-rail tracks (routes)

Route 1 (Blue Line)	**Route 2 (Red Line)**	**Route 3 (Green Line)**
(463) South Railway Station 0.75	(463) South Railway Station 0.75	(463) South Railway Station 0.75
(400) Airport 0.78	(341) Meridian Hotel 0.70	(508) Giant Mall 0.65
(355) Riverside 0.74	(349) Central Park 0.61	(295) Royal Square 0.87
(147) Bridge Square 0.55	(118) Forbes 0.31	(265) Prince Cross 0.80
(181) North Railway Station 0.83	(181) North Railway Station 0.83	(181) North Railway Station 0.83
$L_1 = 5801$ m	$L_2 = 4443$ m	$L_3 = 4146$ m
$\mathrm{MSU}_1 = 0.73$	$\mathrm{MSU}_2 = 0.64$	$\mathrm{MSU}_3 = 0.78$
$\Omega_1 = 1566.27$	$\Omega_2 = 1599.48$	$\Omega_3 = 912.12$
		$\boldsymbol{\Omega}^*$

Results in Table 1 demonstrate that there exists no route option that would entirely dominate over the other options. It is evident here that, if a route is shorter than another, this fact not necessarily means that it represents a better route option. The best option, Route 3 (Green Line), however, outperforms the other two routes both in terms of total impedance and length of line. Hence, considering the enormous construction costs of the whole metro-rail project, implementation of the Green Line might be proposed. Perhaps the best conceivable proposal could be to lengthen the track of the Green Line to the airport.

We conclude that a GIS combined with the value-focused approach of MCDM is a viable tool in supporting decision makers in the design, evaluation and implementation of spatial decision making processes. The analytical capabilities and the computational functionality of GIS promote to produce policy relevant information to decision makers. Although different stakeholders usually have different priorities to highest level objectives, utilizing this approach provides a considerable help in reaching a satisfactory compromise ranking of the objectives under conflicting interests. To find the appropriate route/site locations of facilities in urban transportation problems is one of the most promising areas of application for such an integrated GIS and MCDM methodology as was demonstrated in this metro-rail network system study.

Chapter 2

2 Multiple Criteria Decision Making (MCDM) Methods

In this Chapter, the classification of the multi-criteria decision making (MCDM) methods is done. A comprehensive summary of its classes, i.e., the set of the multi-objective optimization (MOO) methods and the set of the multi-criteria decision analysis (MCDA) methods together with several real-world problem solving applications to transportation and civil engineering cases are presented in Appendix A (due to the page limit of D.Sc. dissertations). The analytic hierarchy process (AHP) method is discussed in this part with necessary detail, since it is directly related to author's research. Two civil engineering applications are discussed at the end of this Chapter.

2.1 Taxonomy of MCDM Methods and their Applications to Transportation and Civil Engineering Projects

A large variety of different problems emerging in transportation/civil engineering projects can effectively be solved using the multiple-criteria decision making (MCDM) methodology and its related techniques. MCDM is a sub-discipline of operations research and considers multiple criteria in decision-making environments. This field of interest has a long history and has shown an extensively grown popularity over the recent decades both in terms of theoretical developments and applications. This phenomenon can be explained by the rapid growth of the complexity of real-world problems, their ill-defined structures and the presence of divergent multiple criteria.

We present a comprehensive summary of MCDM techniques in Appendix A and, in parallel, we cite and describe briefly some of their thousands of real-world applications to certain transportation/civil engineering problems. MCDM methods are commonly divided into two distinct types depending upon their formal statement. These two main classes, distinguished by the properties of their feasible solutions, are the *multi-criteria decision analysis* (MCDA) also termed as *multi-attribute decision making* (MADM), and the *multi-objective optimization* (MOO) methods. In MCDA, the set of feasible alternatives is discrete, pre-specified and finite. Examples of such explicitly given alternatives are, for instance, the location selection of civil engineering objects or vehicle procurement for urban transport. In MOO problems, the feasible alternatives are not known explicitly in advance. In this latter class of MCDM a finite number of explicit constraints given in forms of mathematical functions describe the restrictions formally and are constructed for an infinite number of feasible alternatives. These problems are called continuous MCDM problems, where one has to generate the alternatives before they can be evaluated.

In the MOO and also in some MCDA problems, several criteria can simultaneously be optimized in the feasible set of alternatives. However, one particular alternative does not exist, which can optimize all criteria one at a time, where we usually see an improvement in the value of one criterion so that it produces an opposite effect in the value of at least one other criterion. This set of alternatives (solutions) is called a set of the *non-dominating* or *Pareto optimal* solutions (see Appendix A). Each alternative in this set could be a solution of the multi-criteria problem. In order to select one of them, it is necessary to have additional information from the human decision maker.

The MOO models are appropriate for multi-criteria choice problems which are *well-structured*, and where the present state and the desired future state of the decision problem are both known together with the way to achieve the desired goal. The most favorable solution can be attained by solving a proper mathematical model and then, from this solution, the decision maker selects the 'best' one. The MCDA models are appropriate for *ill-structured* problems, which have very complex objectives often vaguely formulated, contain uncertainties and the originally observed state of the decision problem may change during the problem solving process [17]. These features impede to find any unique solution for them. MCDA methods allow a finite number of alternate solutions that are known at the beginning. These solutions, however, cannot be optimal solutions in a mathematical sense. Additionally, the criteria (attributes) identified by the decision makers at the beginning of the decision process are usually associated with different quantitative and qualitative scales of measurement, and are usually weighted (different relative importance of the attributes). As many transportation engineering problems have ill-defined structures, mostly MCDA methods are used for them in practice. Characteristic features of the MOO and MCDA methods are summarized in Table 2 taken from [51].

Table 2. Characteristic features of MOO and MCDA methods [51]

Characteristic feature	MOO	MCDA
Criteria defined by	Objectives	Attributes
Objectives defined	Explicitly	Implicitly
Attributes defined	Implicitly	Explicitly
Constraints	Active	Not active
Alternatives defined	Implicitly	Explicitly
Number of alternatives	Infinite (large number)	Finite (small number)
Decision maker's control	Significant	Limited
Application for	Design, choice (finding the solution and selection)	Choice, evaluation, ranking, sorting (solutions are known in advance)

It should be mentioned here that the first general work on MCDA methods appeared in Hungary was the book of Kindler and Papp [41].

2.2 The Analytic Hierarchy Process (AHP) with Applications

As several results of the author of this dissertation presented in the following chapters are strongly related to the analytic hierarchy process (AHP) method, in this section, this MCDA method is described with necessary detail. The AHP was founded by Thomas L. Saaty and appeared first in the seminal paper of Saaty [63]. There, three basic principles are introduced for problem solving purposes: decomposition, comparison, and synthesis of priorities.

DECOMPOSITION OF THE DECISION PROBLEM

In the AHP, the decision problem is broken into three components: goal, alternatives and criteria. The decomposition principle is applied to construct a hierarchy for the given decision making problem with elements in a level independent from those in succeeding levels, working downward from the goal at the top, to criteria bearing on the goal in the second level, to sub-criteria in the third level, etc., from the general (and sometimes uncertain) to the particular alternatives at the bottom level. A simple three-level hierarchy is illustrated in Figure 11. The aim of this structure is to make it possible to judge the importance of the elements in a given level with respect to some or all of the elements in the adjacent level above. At this phase, one must include enough relevant detail to represent the problem as thoroughly as possible, identify the issues, attributes, parameters, etc., that the individual decision maker (or a participant's group) feels should contribute to the solution. The benefits for structuring a decision problem as a hierarchy are that the complex problem is laid out in a much clearer fashion.

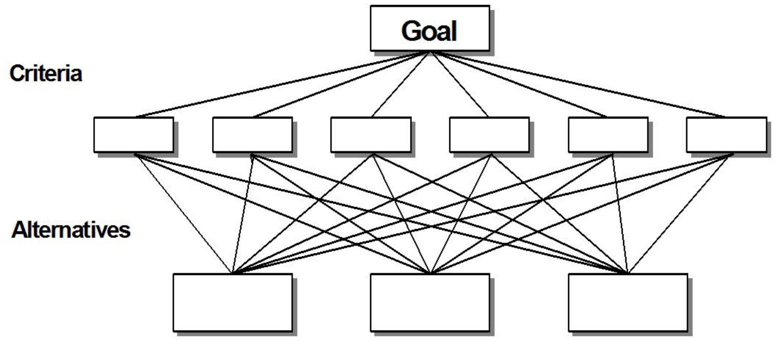


Figure 11. Model of a simple three-level decision hierarchy

The hierarchy does not need to be complete; that is, an element in a given level does not have to function as a criterion for *all* elements in the level below. The elements being compared, however, should be homogeneous. The task of setting priorities requires that the criteria, the sub-criteria, the properties or features of the alternatives be compared among themselves in relation to the elements of the next higher level.

COMPARATIVE JUDGEMENTS

The principle of comparative judgments is applied to construct a mapping of notions, rankings, and objects to numerical values [68]. In the AHP, utilizing pairwise comparisons between different options, a relative *ratio scale* is created**.** The method can be used with both absolute and relative types of comparisons to derive ratio scales of measurement. In absolute comparisons, alternatives are compared with a standard. Usually, relative measurement is employed, when a ratio scale value, w_i, i=1,…,n, of each n elements should be derived by comparing it in pairs with the others. In paired comparisons two elements, i and j are compared with respect to a property they have in common. Thus, such a matrix of these ratio comparisons, denoted by **A**, may be given in the following form:

$$\mathbf{A} = (a_{ij}) = \begin{bmatrix} 1 & \dfrac{w_1}{w_2} & \dots & \dfrac{w_1}{w_n} \\ \dfrac{w_2}{w_1} & 1 & \dots & \dfrac{w_2}{w_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{w_n}{w_1} & \dfrac{w_n}{w_2} & \dots & 1 \end{bmatrix}, \qquad i,j = 1,2,\dots,n.$$

This $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ with all entries positive numbers, where $n \geq 3$, is called a *pairwise comparison matrix* (PCM) and is usually being constructed by eliciting decision makers' (experts) judgements. An entry a_{ij} from R^n of $\mathbf{A}$ represents the strength or the relative importance ratio of A_i over alternative A_j with respect to a common criterion C_k. Note that the ratio in entry a_{ij} is the ratio of w_i to w_j, hence, this matrix is an example of a *consistent matrix* [68]:

Definition 2.1 A consistent matrix $\mathbf{A}$ is one in which for each entry a_{ij} (the entry in the ith row and jth column), $a_{ij} = a_{ik}/a_{jk}$. Otherwise the matrix is called inconsistent.

Saaty called $\mathbf{A}$ a *reciprocal matrix*, if [68]:

Definition 2.2 A reciprocal matrix $\mathbf{A}$ is one in which for each entry $a_{ji} = 1/a_{ij}$ and $a_{ii} = 1$.

The basic objective is to derive implicit positive *weights* (priority scores), $w_1, w_2, \ldots, w_n$, with respect to each criterion C_k. There is an infinite number of ways to derive a vector of the *weights*, $\mathbf{w} = [w_i]$, $w_i > 0$, $i = 1, \ldots, n$, from the matrix $\mathbf{A}$. Saaty showed that the weight w_i of an alternative A_i, what he called the *relative dominance* of the ith alternative A_i, up through k-dominance (as the sum of the ratio of the intensity of a k-walk in a directed graph [65, p. 161]) can be given through formulating the corresponding *eigenvalue-eigenvector* problem as: $\mathbf{Aw} = \lambda_{\max}\mathbf{w}$. Accordingly, it is the ith component of the principal right eigenvector, u_i, of $\mathbf{A}$ provided that $\mathbf{A}$ is consistent [64, p. 848]. This solution for the weights is *unique* up to a multiplicative constant. The *principal* right eigenvector belongs to the eigenvalue of largest modulus. Let us call the eigenvalue of largest modulus as the *maximal* eigenvalue. By Perron's theorem, for matrices with positive elements, the maximal eigenvalue is always *positive*, *simple* and the components of its associated eigenvectors are *positive*, see e.g. in [93]. Saaty [64, p. 853] claimed to prove that his above result holds also for a reciprocal matrix that is *not* necessarily consistent. Thus, this process produces a ratio scale score for each alternative. These scores (weights) obtained for the alternatives are usually normalized so that their sum is equal to unity.

Saaty gave a proof that for each case occurring in the AHP the principal eigenvalue, $\lambda_{\max}$ will be greater than or equal to n [63]. That is, $\lambda_{\max} \geq n$, which suggests using $\lambda_{\max}$ as an index of departure from consistency. He introduced the idea of measuring inconsistency as [63]:

Definition 2.3 The consistency index (CI) is the value of $\mu = (\lambda_{\max} - n)/(n-1)$.

It is interesting to note here that $2(\lambda_{\max} - n)/(n-1)$ is the variance of the error incurred in estimating an entry a_{ij}. Results might be accepted if $\mu \leq 0.08$. Otherwise the problem should be reconsidered and the associated PCM must be revised [66]. Obviously, for a consistent PCM: $\mu = 0.00$, since this follows apparently from the above considerations. An alternative concept to measure inconsistency of matrix $\mathbf{A}$ is the *consistency ratio (CR)* which is the ratio of *CI*($\mathbf{A}$)/*RI*($\mathbf{A}$), where the random index, *RI*($\mathbf{A}$) is the average *CI* calculated from a large number of randomly generated reciprocal matrices of size n [66]. In Table 3, the fundamental scale of numerical values proposed by Saaty to represent the intensities of judgments is shown.

Since the different criteria in a real-world decision problem are usually not of equal importance, therefore, a vector of the *weighting factors* of the criteria, $\mathbf{s} = [s_k]$, where s_k, $k = 1, 2, \ldots, m$ should be determined which is often normalized so that $0 < s_k < 1$. For this procedure, in most applications, the AHP technique is used as well.

SYNTHESIS OF PRIORITIES

The third principle is to synthesize the priorities downward the hierarchy by weighting their local priorities by the priority of their corresponding criterion in the level above, and adding for each element in a level according to the criteria it affects. This gives the composite or global priority score of that element, which is then used to weight the local priorities of the elements in the level below compared to each other with it as the criterion, and so on to the bottom level. To compute the components of the *overall priority scores*, $\pi_1, \pi_2, \ldots, \pi_n$, (overall weights) for

Table 3. The fundamental scale [66]

Intensity of importance Strength of preference	Definition	Explanation
1	**Equal importance**	Two activities contribute equally to the objective
2	Weak	
3	**Moderate importance**	Judgment slightly favor one activity over another
4	Moderate plus	
5	**Strong importance**	Judgement strongly favor one activity over another
6	Strong plus	
7	**Very strong importance**	An activity is favored very strongly over another
8	Very, very strong	
9	**Extreme importance**	Favoring one activity over another is of the highest affirmation
Reciprocals of above	If activity i has one of the above nonzero numbers assigned to it when compared with activity j, then j has the reciprocal value when compared with i	

We remark that any positive numbers can also be used, e.g. 4.1 or 6.87, or even beyond the lower and upper boundaries of the proposed scale, e.g. 23.6 or 0.05.

the set of the alternatives, i.e. when taking into account the weighting factors of each of the criteria, the AHP utilizes an *additive* type aggregation function: $\pi_i = \sum_{k=1}^{m} s_{ik} w_i$, $i = 1, 2, \ldots, n$. We note that there are alternative ways of computing the overall priorities, e.g. a *multiplicative* weighted geometric-mean aggregation is proposed in [3].

APPLICATIONS OF AHP TO TRANSPORTATION/CIVIL ENGINEERING PROBLEMS

There are numerous applications of this method for many problems occurring in different areas of transportation/civil engineering. A great number of such applications are referenced in the related survey literature as for example Saaty and Forman [69], Ehrgott et al. [23], Zak [95] and for combined GIS-AHP applications Malczewski [48]. Based on our literature research, we now accentuate some recent applications of AHP.

In the paper of Kumru and Kumru [44], the AHP was deployed for a logistics company to select the most suitable transportation mode between two given locations in Turkey. The criteria used were identified: cost, speed, safety, accessibility, reliability, environmental friendliness, and flexibility. Several cost parameters (transportation, storage, handling, bosphorus crossover) were incorporated in the decision-making process and the AHP was successfully applied. In the study of Piantanakulchai and Saengkhao [57], social interest groups were modeled in a decision process to reflect social preference in a transport problem. Using AHP, the relative importance of each attribute was elicited by combining an engineering model with the decision model. An investigation of alternative motorway alignments in Thailand was conducted. Impacts were estimated by the aid of a GIS system. Composite weighted AHP scores were then used to generate a decision surface for the problem. Finally, the best alignment was proposed by determining the socially preferred least cost path. In their interesting study Thanesuen et al. [78] have shown that AHP can help to reveal an appropriate speed limit from the viewpoints of the road users. Safety, driving comfort and travel time were the criteria. The results showed that safety was the most important criterion, followed by travel time and driving comfort, respectively. How they concluded, according to the road and traffic conditions, their research verified that the preferred speed limits are convenient, therefore they can be regarded as guidelines for the new limits on Hokkaido roads in Japan.

In the paper of Kopytov and Abramov [43] a multimodal freight transportation system with finite number of given alternatives (routes and modes) was considered. Their objective was to suggest a method to evaluate and make choice from the alternatives of cargo transportation. They wished to find proper indices that characterize the efficiency of multimodal transportations, optimization criteria for multimodal freight transportation, a right construction to modeling a multimodal transportation system and calculate the performance criteria of cargo transportation. The study claimed the AHP method as the most suitable approach for comparative evaluation of different routes and modes of cargo transportation. Dalala et al. [15] presented a systematic methodology under the consideration of multiple factors and objectives that are witnessed to be crucial in construction operations in the process of construction works. The model included an AHP hierarchy with a criteria tree and the alternatives in order to select the most favorable tower crane from among three types of cranes.

To get more insight in the appliance of MCDA for transport projects several articles reported a fast growing attention for MCDA due to its favourable features as opposed to other DSS systems, see e.g. Grant-Muller [29] and Morisugi [52]. Several researchers, who used computerized automated search and commercial search robots on the web have focused on the frequency of use of particular MCDA techniques for different transportation engineering problems in the past two decades. Macharis and Ampo [47] have found that 40.2% of the publications they analyzed the AHP was used and was considered to be trustworthy and robust. A similar result was reported by Deluka-Tibljas et al. [17] whose survey in the process of planning, design, maintanance and construction of transport infrastructure in urban areas have shown that the most often used MCDA method was the AHP with an approximately 65% frequency of occurrance in their sample of publications $(n = 46)$. In the area of spatial decision making problems, Malczewski [48, p. 710] investigated 259 articles that used a MCDA method in combination with a GIS technique to visualize and capture geographic data. Out of a total of 259 works 34 applied the AHP for structuring, evaluating and prioritizing the decision alternatives. In the light of these findings, even if they have statistical nature only, the present author is truly convinced that he made the right decision 25 years ago, when he started to carry out intensive research related to the AHP methodology and to other scaling methods of similar kind.

Application 2: Evaluation and selection of a bridge design using the AHP method

We now present a civil engineering application of the use of the AHP for selecting the most appropriate bridge design which has appeared in Farkas (2010). We demonstrate here that the AHP is able to link hard measurement to human values in the physical and the engineering sciences. The following study concerns an actual construction project to provide an alternative route across the Monongahela River in the city of Pittsburgh, USA. The author took part in one of the seven decision making groups of this project. A detailed report of this study has been documented in [56]. The three types of bridges $(n = 3)$ considered by The Port Authority of Allegheny County were as follows:

A $=$ A **Cable-stayed bridge** (Figure 12); it belongs to the group of the longest bridges called suspension bridges. The deck is hung from suspenders of wire rope, eyebars or other materials. Materials for the other parts also vary: piers may be steel or masonry; the deck may be made of girders or trussed. This type of bridge is usually applied with very high tensile strength, which minimizes beam deflection as the span is increased significantly. Moreover, adding several stay cables allows the use of more slender deck beams, which require less flexural stiffness. By decreasing the cable spacing supports, local bending moments in the girders are also reduced. Simple double-edge girders supporting transverse floor beams and top slabs provide a synergistic reinforcing action. The economic viability and aesthetic appeal make this type of bridge to be very popular.

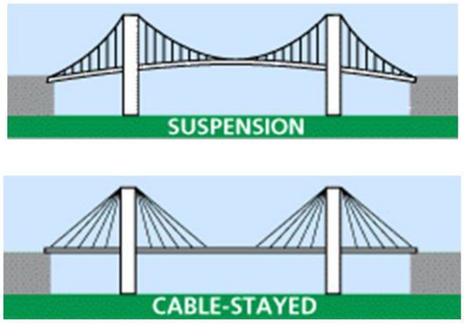


Figure 12. Suspension bridges including their cousin the cable-stayed bridge [33]

B $=$ A **Truss bridge** (Figure 13); which allows applied loads to be resisted primarily by axial forces in its straight truss members. Its open web system permits the use of a greater overall depth than for an equivalent solid web girder. These factors lead to an economy in material and a reduced dead weight. Deflection is reduced and the structure is more rigid. However, fabrication and maintenance costs are increased. In addition, a truss bridge rarely possesses aesthetic beauty.

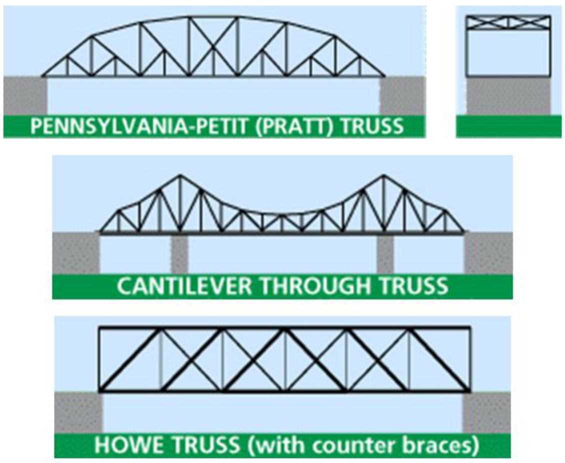


Figure 13. Bridges of Truss type [33]

C $=$ A **Tied-Arch bridge** (Figure 14); which has been used for its architectural beauty and outstanding strength for centuries. With the aid of its inward-acting horizontal components, the arch is capable of distributing loads both above and below its structure. In a tied-arch design the horizontal reactions to the arch rib are supplied by a tie at deck level. It reduces bending moments in the superstructure and is fairly economical. Aesthetically, the arch has been perhaps the most appealing of all bridge types. It has, however, high relative fabrication and building costs.

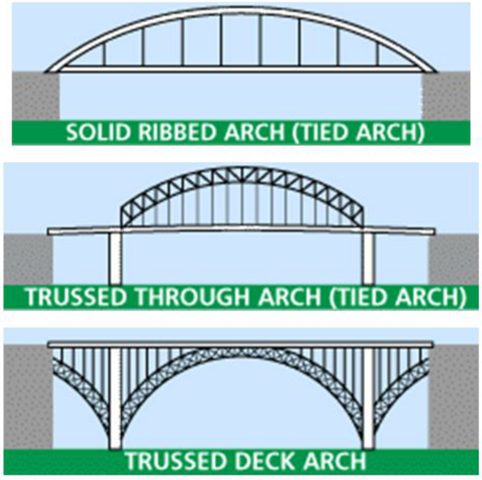


Figure 14. Arch bridges with different configurations including the tied-arch type bridges [33]

The most desirable bridge type would conceivably be the one that brings the most satisfaction to the greatest number of stakeholders. Keeping an eye on this goal, a hierarchy was developed with major stakeholders at the second level, the driving criteria at the third level and the three alternative bridge types at the fourth level. The major stakeholders were then arranged into seven groups with a number of 8-15 people in each:

FWHA = A Federal Agency; which represents an array of federal departments. It is a key financier of the project and will have dictates with respect to the engineering integrity of any bridge type.

CBD = The Commercial Business District; which broadly represents the businesses in the downtown of Pittsburgh. Its interest implies to maintain the historical appearance of the building site as well.

PUB = The Public; which represents the population of the city that would use the new bridge.

DOT = The Pennsylvania Department of Transportation; which represents the complex interest of the state. These interests are financial (as the state provides part of the capital), political, technical and environmental.

DES = The Designers; who represent engineers, architects and planners and their professional organizations. They provide crucial technical input and so, they have a great influence.

SIG = Special Interest Groups; this means a very broad category with diverse and possibly conflicting interests. They are the concrete suppliers, the steel manufacturers and the environmentalists. Steel industry has declined in size and influence in this region, however, the concrete industry remained strong. Environmentalists are active and vocal.

PAT = The Port Authority Transit; it is the ultimate project owner. This premier stakeholder cares of all management issues from conception to construction, as well as maintenance.

In the level below the stakeholders identified the following six criteria with respect to which the bridge types were evaluated $(m = 6)$:

$C1$ = Engineering Feasibility (**EF**): The technical knowledge and experience of both the designers and contractors in regard to the bridge type.

$C2$ = Capital Cost (**CC**): Necessary funding. Because the costs were committed, low costs are included in the overall benefits hierarchy as one of the criteria.

$C3$ = Maintenance (**MA**): Cleaning, painting, repair, inspection vary dramatically with bridge type.

$C4$ = Aesthetics (**AE**): Architectural attractiveness.

$C5$ = Environmental Impact (**EI**): The ecological and historical adjustments that must be compromised.

$C6$ = Durability (**DU**): The lifetime of the bridge and the potential major repairs over and above the routine maintenance.

Tangible data supporting the engineering characteristics ($C1$, $C2$, $C3$, $C6$) have been derived from measurements, while the ratios for the intangible attributes ($C4$, $C5$) were judged by the groups of stakeholders.

Computations were done by the software Expert Choice [25]. First, the actors were compared in order to determine their relative importance (weighting factors). The 7×7 sized pairwise comparison matrix $\mathbf{A}$ is displayed below. We note that matrix $\mathbf{A}$ is near consistent. Its consistency measure yielded: $\mu = 0.03$.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1/5 & 1 & 1/2 & 1/3 & 3 \\ 1/2 & 1 & 1/6 & 1/2 & 1/3 & 1/4 & 2 \\ 5 & 6 & 1 & 5 & 4 & 3 & 7 \\ 1 & 2 & 1/5 & 1 & 1/2 & 1/3 & 3 \\ 2 & 3 & 1/4 & 2 & 1 & 1/2 & 4 \\ 3 & 4 & 1/3 & 3 & 2 & 1 & 5 \\ 1/3 & 1/2 & 1/7 & 1/3 & 1/4 & 1/5 & 1 \end{bmatrix}.$$

The criteria were then compared according to each factor and the composite priorities calculated (see Table 4.).

Table 4. Weighting factors and weights (priorities) of the criteria

Weighting factor, s_k	0.135	0.221	0.029	0.136	0.085	0.056	0.337	
Stakeholder Criterion C_k	FHWA	CBD	PUB	DOT	DES	SIG	PAT	Weight w_i
$C1 = \mathrm{EF}$	0.117	0.048	0.037	0.216	0.313	0.033	0.260	0.173
$C2 = \mathrm{CC}$	0.340	0.048	0.297	0.082	0.197	0.357	0.100	0.147
$C3 = \mathrm{MA}$	0.069	0.116	0.297	0.052	0.118	0.097	0.260	0.154
$C4 = \mathrm{AE}$	0.069	0.401	0.074	0.216	0.136	0.224	0.061	0.174
$C5 = \mathrm{EI}$	0.202	0.270	0.114	0.352	0.117	0.224	0.061	0.181
$C6 = \mathrm{DU}$	0.202	0.116	0.182	0.082	0.118	0.064	0.260	0.171
Consistency μ	0.02	0.07	0.05	0.04	0.08	0.06	0.05	

In the last step of the evaluation process, the alternatives were compared with respect to each criterion and the composite priority scores computed. This information was synthesized to yield the overall priority ranking and the overall priorities of the bridges:

Overall ranking and the overall priorities π_i
B (0.371)
|
C (0.320)
|
A (0.309)

Thus, in this project, the most desirable bridge is of a **Truss** type. It is quite interesting to note that a couple of months later this result was reconsidered. The major difference in the duplicated decision making process was the addition of a new stakeholder, the US Coast Guard (USCG), the responsible authority for the river traffic, and the deletion of the Public (PUB). On the effect of the USCG concerning the reinforcement of the safety aspects of river transportation and the further ecological claims of the environmentalists the final ranking of the types of bridges has been changed in favor of a **Tied-arch** type bridge. Since then, the new bridge has been built to the Wabash Tunnel, consisting of three high occupancy vehicle lanes and a lane for pedestrian traffic.

Finally, we remark that further applications of the AHP applied to complex evaluations of bridge constructions have been published in Farkas (2011b) and a multi-criteria urban transportation analysis of four bridges of the city of Budapest leading across the river Danube in Farkas (2011a).

Application 3: Effect of prestressing on the appearance of concrete structures

In Tassi, Szlivka and Farkas (2005), the authors investigated the effect of prestressing on the shape of concrete elements. The AHP method was used to judge the aesthetic appeal of concrete structures.

By prestressing, we can influence the forces acting in structures, affecting the crack formation, the stiffness, the applicability of high strength material and reaching many other benefits which influence advantageously the appearance of the structure.

The cross section shape is influenced by the intensity of pre-stressing. The main features of the phenomenon are represented by the depth, cross section area and the thickness of the web. The beneficial effect of prestress on the thickness of the web plays an important role in the shape of the concrete member. Even in the case if only longitudinal tendons are present, prestressing significantly reduces the principal tensile stresses. This enables us to apply very thin webs. It is well known, that the web is not only possible to be thin but it is advisable (see the paradox of pre-stressing).

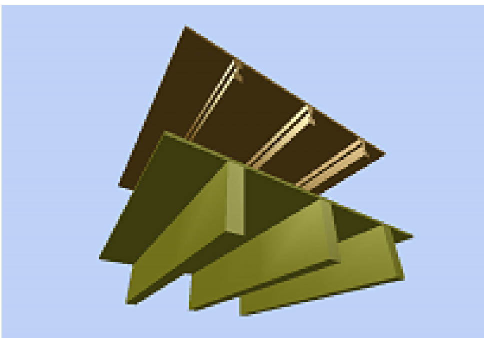


Figure 15. Difference between the appearances of RC and PC members

The dead load is decreased by the thin web, and the member can be more slender. It is obvious that the prestressed concrete beam is lighter anyway because the full or the approximately full cross section can work, and materials of higher strength, even HSC can be used. The appearance of an I-beam with its broken surface seems to be better than that of an even one (see on Figure 15). The relative value of dead and live load plays a role in shape of PC elements, too. It will not be discussed here but also the ratio of the cross section of the bottom flange to the top flange's is a relevant factor which is influenced by the percentage of the self weight to the full load.

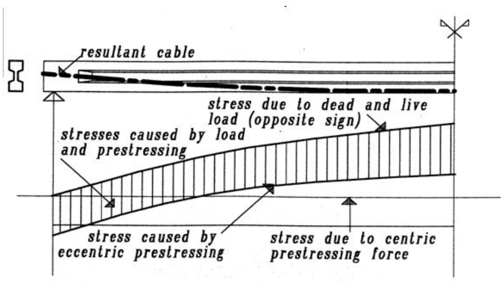
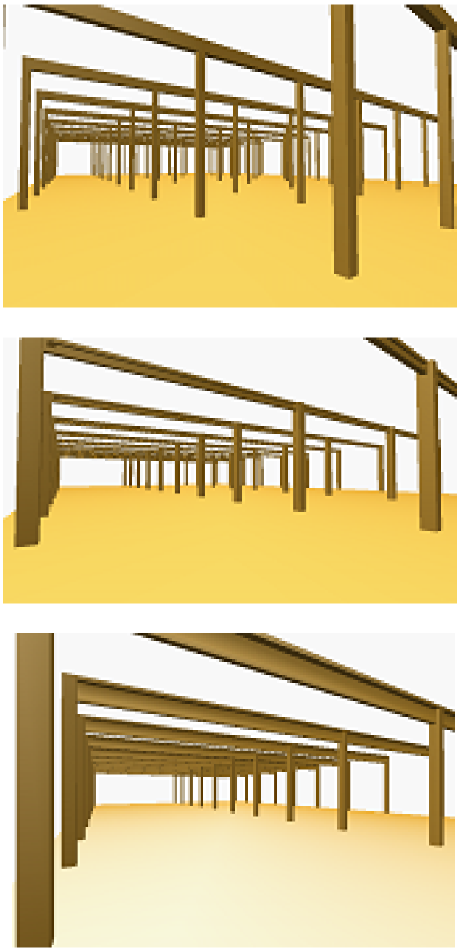


Figure 16. The effect of prestressing on the top fibre stresses along a simple supported beam

The uniform cross section of a simply supported beam (except some roof girders) and the parallel flange are practically needed and aesthetically convenient. The so called uniform strength beam in case of a steel or reinforced concrete member would result in a "fish-belly" shape side view which results in material saving but it is neither nice nor practical in production. Well designed post-tensioning produces in a beam having uniform cross section an approximately constant compression along the top extreme fibre, as it is shown in Figure 16.

Suitable arrangement of pre-tensioning can result in almost similar stress distribution. It should not be discussed, that generally, a simply supported beam having uniform cross section provides a more favorable view than the one with a depth increasing towards midspan. The depth/span ratio can be significantly reduced by prestressing. As it was pointed out, this statement follows from multiple based reasons. The use of HSC gives one of the opportunities in the near for future developments.



top ***A*** (RC, 6×6 m)
middle ***B*** (PC, 6×12 m)
bottom ***C*** (HSPC, 6×18m)

Figure 17. Variants for single storey concrete skeleton buidings

Let us now compare the RC, PC and HSPC structures when they are used to single storey skeleton buildings. These three relative similar building constructions, e.g. commercial or industrial buildings, are $(n = 3)$:
A = reinforced concrete (RC), ***B*** = prestressed concrete (PC),
C = high strength prestressed concrete (HSPC) load bearing structure.
Figure 17 shows only the skeletons with simply supported beams. No suspended ceiling is supposed.
In a real-world situation, the number of criteria is rather high, but in this study it was limited to eight by drawing some of them together. The eight criteria (m=8) with respect to which these concrete structures were evaluated are listed below:
$C1$ = Mass,
$C2$ = Costs of the structure,
$C3$ = Difficulty of production and assembly,
$C4$ = Variety of the structure, multipurpose use, service,
$C5$ = Deformation, cracking,
$C6$ = Load capacity, durability, fire resistance, maintenance,
$C7$ = Functional usability of the building,
$C8$ = Appearance, psychological impression.

The data concerning the engineering characteristics ($C1$, $C2$, $C5$, $C6$) have been derived from calculations and for the intangible attributes ($C3$, $C4$, $C7$, $C8$) by subjective judgements elicited from a group of ten members (engineering experts, constructors, owners, users and laymen). Computations were made by [25]. Results from the analysis of the eight 3×3 PCMs are presented in Table 5. (Here the weights were generated by solving an 8×8 PCM, where the consistency measure yielded: $\mu \leq 0.04$, which is fairly good, and therefore acceptable).The overall priority ranking and the overall priority scores of the structures are also given in Table 5.

Table 5. Weights, priority rankings and the normalized scores of the structures with respect to each criterion and the overall priority ranking and priority scores for the single storey skeleton buildings

Criterion, C_k	$C1$	$C2$	$C3$	$C4$	$C5$	$C6$	$C7$	$C8$
Weight, w_k [%]	4	7	5	11	17	21	12	23
Priority, p_{ik} and ranking	***C*** (0.52) | ***B*** (0.32) | ***A*** (0.16)	***B*** (0.37) | ***C*** (0.33) | ***A*** (0.30)	***C*** (0.43) | ***B*** (0.32) | ***A*** (0.26)	***A*** (0.50) | ***B_C*** (0.25) (0.25)	***C*** (0.51) | ***B*** (0.39) | ***A*** (0.10)	***A_C*** (0.36) | (0.36) ***B*** (0.28)	***C*** (0.54) | ***B*** (0.38) | ***A*** (0.08)	***C*** (0.50) | ***B*** (0.38) | ***A*** (0.12)
Inconsistency	μ_1=0.00	μ_2=0.00	μ_3=0.00	μ_4=0.00	μ_5=0.00	μ_6=0.00	μ_7=0.00	μ_8=0.00

C (0.43)
|
B (0.34)
|
A (0.23)

Regarding the main goal of this study, we conclude as: since criterion $C8$ (Aesthetics appeal) has received the highest weight, this fact strongly contributed to the overall priority ranking of the single storey skeleton buildings as of ***C-B-A***.

Chapter 3

3 On the Development of the Analytic Hierarchy Process (AHP)

In this Chapter, symmetrically reciprocal (SR) and transitive matrices are defined and spectral properties of certain SR perturbations of transitive matrices are developed. Matrix theory is used to derive the principal eigenvector components of pairwise comparison matrices (PCMs) in an explicit form. In the AHP, such matrices may have only positive entries. Special focus is devoted to the most controversial phenomenon of this method, the issue of rank reversal. Proofs are given for the existence of rank reversals of the priority rankings generated by the AHP method. Exact intervals are established for the range of values over which such reversals occur as function of a continuous perturbation parameter. It is also shown that such matrices occur in macroeconomics, where negative elements may also be contained by the respective PCMs, and in vehicle dynamics, where the entries of these matrices are complex numbers. Applications are presented for each of the discussed theoretical cases.

3.1 Preliminaries

Cardinal utility is a class of preference measurement based on the presumption that utility is a quantifiable characteristic of human activity that can be measured with numerical values on at least an interval scale (see Appendix A.1.1). Although this subject has ancient roots mainly originated in the economic theory of consumer behavior, unfortunately, it has never been successfully achieved. Its modern forms have only taken shape in the contemporaneous developments of the latest era of economics, operations research and management science. This period has been host to significant advances in the theory of cardinal measurement of decision makers' preferences, most notably in the works of Debreu [16], Allais [1], Churchman and Ackoff [13], Tversky and Kahnemann [83], Fishburn [26] and Chew [10]. One of the most remarkable evolutions was the formation and axiomatization of an original idea, a ratio scale framework called the analytic hierarchy process (AHP). This is a historically and theoretically different and independent theory of decision making, therefore, it cannot be compared with utility theory as its author argued [67].

The exciting questions of *cardinal preference measurement* aroused authors' interest around the mid nineteenth. As with every achievement, AHP has inevitable benefits yet giving rise to some deficiencies also. At this point, author declares his standpoint for that he is willing to accept AHP methodology, however, he has focused on some of its controversial issues as the main directions of his research with a strong intention of achieving refinement. Initially, the investigations aimed the properties of the pairwise comparison matrices (PCM), denoted by $\mathbf{A}$ and defined in sub-chapter 2.2. Certain preliminary findings concerning the spectral properties of matrix $\mathbf{A}$ have been appeared in Farkas and Rózsa (1996b), Farkas and Rózsa (1996a) and Farkas, Rózsa and Stubnya (1998).

3.2 Transitive and Symmetrically Reciprocal Matrices

In order to extend the scope and thus the usage opportunities in other fields of interest as well, we generalized the original interpretation of a PCM. For this purpose, in Farkas, Rózsa and Stubnya (1999a), the authors proposed the notion of a *symmetrically reciprocal* (SR) matrix and delineated some potential engineering applications of such matrices. In Farkas, Rózsa and Stubnya (1999b), a more restrictive class of matrices has been introduced called *transitive* matrices. We now give the definitions of these matrices as follows:

Let $\mathbf{A} = (a_{ij})$ denote an n by n matrix whose entries are all nonzero. Although our main interest is the real or complex numbers, the entries may come from any field.

Definition 3.1 We call $\mathbf{A}$ *transitive* if

$$a_{ik}a_{kj} = a_{ij}, \quad \text{for all } i, j, k \text{ in } [1, \ldots, n]. \tag{3.1}$$

Definition 3.2 We call $\mathbf{A}$ *symmetrically reciprocal* (SR) if

$$a_{ij}a_{ji} = 1, \quad \text{for all } i, j \text{ in } [1, \ldots, n] \quad \text{and} \quad a_{ii} = 1, \quad \text{for all } i \text{ in } [1, \ldots, 1]. \tag{3.2}$$

Proposition 3.1 *Let $\mathbf{A} = (a_{ij})$ denote an n by n matrix with entries from a field. Let $\mathbf{P}$ denote an n by n permutation matrix.*

(a) *If $\mathbf{A}$ is transitive,*

(i) $\mathbf{B} = \mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P}$ *is transitive,*

(ii) $\mathbf{A}$ *is* SR*,*

(iii) $\mathbf{A} = \mathbf{u}\mathbf{v}^{\mathrm{T}}$*, where $\mathbf{u}$ (resp. $\mathbf{v}^{\mathrm{T}}$) is the first column (resp. row) of $\mathbf{A}$. The ith entries of $\mathbf{u}$ and $\boldsymbol{v}^{\mathrm{T}}$ are related by $u_i v_i = 1$ and $u_1 = v_1 = 1$.*

(iv) $\mathbf{x}\mathbf{y}^{\mathrm{T}}$ *is transitive if*

$$\mathbf{x}^{\mathrm{T}} = \left[1, \frac{1}{y_2}, \frac{1}{y_3}, \ldots, \frac{1}{y_n}\right] \qquad \textit{and} \qquad \mathbf{y}^{\mathrm{T}} = [1, y_2, y_3, \ldots, y_n]\,. \tag{3.3}$$

(v) $\{n = \mathbf{v}^{\mathrm{T}}\mathbf{u},\ 0, \ldots, 0\}$ *is the spectrum of $\mathbf{A}$ and $\mathbf{u}$ and nonzero vectors $\mathbf{w}$ perpendicular to $\mathbf{v}^{\mathrm{T}}$ are respective eigenvectors of $\mathbf{A}$, and*

(vi) $\mathbf{A}^2 = n\mathbf{A}$*, so $(1/n)\mathbf{A}$ is a projection.*

(b) *If $\mathbf{A}$ is* SR*,*

(i) $\mathbf{B} = \mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P}$ *is* SR*, and*

(ii) if the rank of $\mathbf{A}$ is 1, $\mathbf{A}$ is transitive.

Proof.
(a)*(i)*, (b)*(i)* If $\mathbf{B} = (b_{ij})$, $b_{ij} = a_{\delta(i)\delta(j)}$ for some permutation δ. Then the b_{ij} satisfy the identities defining transitivity (resp. SR) because the a_{ij} do.

(a)*(ii)* Set $k = j = i$ to get $a_{ii} = 1$ and $k = j$ and $j = i$ to get $a_{ij}a_{ji} = 1$.

(a)*(iii)* $a_{ij} = a_{i1}a_{1j}$. $u_1 = v_1 = a_{11} = 1$ and $u_i v_i = a_{i1}a_{1i} = 1$, since $\mathbf{A}$ is SR.

(a)*(iv)*

$$\left(\frac{1}{y_i}y_k\right)\left(\frac{1}{y_k}y_j\right) = \frac{y_j}{y_i}.$$

(a)*(v)* From a(iii).

(a)*(vi)* $\sum_{k=1}^{n} a_{ik}a_{kj} = na_{ij}$.

(b)*(ii)* Since $\mathbf{A}$'s rank is 1, $\mathbf{A} = \mathbf{f}\mathbf{g}^{\mathrm{T}}$, where $\mathbf{f}$ and $\mathbf{g}^{\mathrm{T}}$ are column and row vectors, respectively. Since $\mathbf{A}$ has no zero entry, $\mathbf{A} = \mathbf{x}\mathbf{y}^{\mathrm{T}}$, where $\mathbf{x} = g_1\mathbf{f}$ and $\mathbf{y} = (1/g_1)\mathbf{g}$. Then $1 = a_{ii} = x_i y_i$, so by (a)*(iv)* $\mathbf{A}$ is transitive. □

Definition 3.3 A matrix $\mathbf{A}$ with positive entries is called a *specific* PCM if it is transitive.

According to (a)*(iii)* in Proposition 1, any transitive matrix can be expressed as the product of a column vector $\mathbf{u}$ and a row vector $\mathbf{v}^{\mathrm{T}}$, i.e. as a dyadic (outer product):

$$\mathbf{A} = \mathbf{u}\mathbf{v}^{\mathrm{T}}. \tag{3.4}$$

Let

$$\mathbf{v}^{\mathrm{T}} = [1, x_1, x_2, \ldots, x_{n-1}] \qquad \text{and} \qquad \mathbf{y}^{\mathrm{T}} = \left[1, \frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_{n-1}}\right]. \tag{3.5}$$

Introducing the diagonal matrix $\mathbf{D} = \operatorname{diag}\langle 1, 1/x_1, 1/x_2, \ldots, 1/x_{n-1}\rangle$ and the vector $\mathbf{e}^{\mathrm{T}} = [1, 1, \ldots, 1]$, obviously $\mathbf{D}^{-1}\mathbf{A}\mathbf{D} = \mathbf{e}\mathbf{e}^{\mathrm{T}}$. It is easy to see that the characteristic polynomial of $\mathbf{A}$, $p_n(\lambda)$, can be obtained in the following form:

$$p_n(\lambda) \equiv \det[\lambda\mathbf{I}_n - \mathbf{A}] = \det\left[\lambda\mathbf{I}_n - \mathbf{e}\mathbf{e}^{\mathrm{T}}\right] = \lambda^{n-1}(\lambda - n), \tag{3.6}$$

where $\mathbf{I}_n$ is the identity matrix of order n. From (3.6) it is readily apparent that $\mathbf{A}$ has a zero eigenvalue with multiplicity $n-1$ and one simple positive eigenvalue, $\lambda = n$, with the corresponding right and left eigenvectors, $\mathbf{u}$ and $\mathbf{v}^{\mathrm{T}}$, respectively.

3.3 The Spectrum of Certain SR Perturbations of Transitive Matrices

In problems, emerging in practice, these SR matrices are usually non-transitive due to among others subjective judgements, measurement errors and the presence of random components which cause perturbations in the a_{ij} entries of $\mathbf{A}$. Therefore, it is reasonable to detect how the maximal eigenvalue and its associated eigenvector vary when matrix $\mathbf{A}$ is perturbed such that it remains in SR, however, its transitivity is lost. The derivations in this sub-chapter aim to determine the spectral properties of perturbed PCMs. These results have appeared in Farkas and Rózsa (2001).

Definition 3.4 A square matrix with positive entries is called a *perturbed* PCM and denoted by $\mathbf{A}_{\mathrm{p}}$, if the matrix is in SR but it is not transitive.

Consider the transitive matrix $\mathbf{A} = \mathbf{D}\mathbf{e}\mathbf{e}^{\mathrm{T}}\mathbf{D}^{-1}$ with the elements $a_{ij} = 1/a_{ji} = x_j/x_i$, $i, j = 0, 1, 2, \ldots, n-1$. Let the elements of matrix $\mathbf{A}$ be perturbed in its first row and in its first column. This perturbed matrix $\mathbf{A}_{\mathrm{p}}$ can now be written as

$$\mathbf{A}_{\mathrm{p}} = \begin{bmatrix} 1 & x_1\delta_1 & x_2\delta_2 & \ldots & x_{n-1}\delta_{n-1} \\ \dfrac{1}{x_1\delta_1} & 1 & \dfrac{x_2}{x_1} & \ldots & \dfrac{x_{n-1}}{x_1} \\ \dfrac{1}{x_2\delta_2} & \dfrac{x_1}{x_2} & 1 & \ldots & \dfrac{x_{n-1}}{x_2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \dfrac{1}{x_{n-1}\delta_{n-1}} & \dfrac{x_1}{x_{n-1}} & \dfrac{x_2}{x_{n-1}} & \ldots & 1 \end{bmatrix}, \tag{3.7}$$

where $\delta_1, \delta_2, \ldots, \delta_{n-1}$ are arbitrary positive numbers with $\delta_i \neq 1$, $i = 1, 2, \ldots, n-1$. Performing a similarity transformation, the characteristic polynomial of $\mathbf{A}_\mathrm{p}$, $p_n^\mathrm{P}(\lambda)$, is obtained as

$$p_n^\mathrm{P}(\lambda) \equiv \det\left[\lambda \mathbf{I}_n - \mathbf{A}_\mathrm{P}\right] = \det\left[\lambda \mathbf{I}_n - \mathbf{D}^{-1}\mathbf{A}_\mathrm{P}\mathbf{D}\right] = \det \mathbf{K}_\mathrm{P}(\lambda), \tag{3.8}$$

where

$$\det \mathbf{K}_\mathrm{P}(\lambda) = \begin{vmatrix} \lambda - 1 & -\delta_1 & -\delta_2 & \ldots & -\delta_{n-1} \\ -\dfrac{1}{\delta_1} & \lambda - 1 & -1 & \ldots & -1 \\ -\dfrac{1}{\delta_2} & -1 & \lambda - 1 & \ldots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\dfrac{1}{\delta_{n-1}} & -1 & -1 & \ldots & \lambda - 1 \end{vmatrix}.$$

The matrix $\mathbf{K}_\mathrm{P}(\lambda) = \lambda \mathbf{I}_n - \mathbf{D}^{-1}\mathbf{A}_\mathrm{P}\mathbf{D}$ may be interpreted in the form of a modified matrix:

$$\mathbf{K}_\mathrm{P}(\lambda) = \lambda \mathbf{I}_n + \mathbf{U}_\mathrm{P}\mathbf{V}_\mathrm{P}^\mathrm{T} - \mathbf{e}\mathbf{e}^\mathrm{T}, \tag{3.9}$$

with the notation

$$\mathbf{U}_\mathrm{P} = \begin{bmatrix} 0 & 1 \\ 1 - \dfrac{1}{\delta_1} & 0 \\ \vdots & \vdots \\ 1 - \dfrac{1}{\delta_{n-1}} & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{V}_\mathrm{P}^\mathrm{T} = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 - \delta_1 & \ldots & 1 - \delta_{n-1} \end{bmatrix}.$$

In order to find the inverse of a matrix that is modified by a rank-one matrix [see the determinant (B1) in Appendix B] through applying the Sherman-Morrison formula [73, p. 126], introduce the matrix $\mathbf{T}_\mathrm{P}(\lambda)$ as

$$\mathbf{T}_\mathrm{P}(\lambda) = \lambda \mathbf{I}_n + \mathbf{U}_\mathrm{P}\mathbf{V}_\mathrm{P}^\mathrm{T}. \tag{3.10}$$

Thus, the modified matrix $\mathbf{K}_\mathrm{P}(\lambda)$ can now be described as

$$\mathbf{K}_\mathrm{P}(\lambda) = \mathbf{T}_\mathrm{P}(\lambda) - \mathbf{e}\mathbf{e}^\mathrm{T}. \tag{3.11}$$

In Appendix B it is shown that the determinant of $\mathbf{K}_\mathrm{P}(\lambda)$, i.e. the characteristic polynomial, $p_n^\mathrm{P}(\lambda)$, yields

$$\begin{aligned} p_n^\mathrm{P}(\lambda) \equiv \det \mathbf{K}_\mathrm{P}(\lambda) = \lambda^{n-3} \Bigg\{ \lambda^3 - n\lambda^2 + (n-1) \sum_{i=1}^{n-1} (1 - \delta_i)\left(1 - \frac{1}{\delta_i}\right) - \\ - \sum_{i=1}^{n-1} \left(1 - \frac{1}{\delta_i}\right) \sum_{i=1}^{n-1} (1 - \delta_i) \Bigg\}. \end{aligned} \tag{3.12}$$

Expression (3.12) clearly shows that even if all elements are perturbed in one (arbitrary) row and in its corresponding column of matrix $\mathbf{A}$, then, $\mathbf{A}_{\mathrm{P}}$ has a zero eigenvalue with multiplicity $\geq n-3$, if $n>2$ and a trinomial equation is obtained for the nonzero eigenvalues. Writing the characteristic polynomial, $p_n^{\mathrm{P}}(\lambda)$, defined as (3.12) in a simplified form, we get

$$p_n^{\mathrm{P}}(\lambda) \equiv \lambda^{n-3}\left\{\lambda^3 - n\lambda^2 - C\right\}, \tag{3.13}$$

where the constant term C with the perturbation factors $\delta_i \neq 1$, $i=1,2,\ldots,n-1$, has the form

$$C = -(n-1)\sum_{i=1}^{n-1}(1-\delta_i)\left(1-\frac{1}{\delta_i}\right) + \sum_{i=1}^{n-1}\left(1-\frac{1}{\delta_i}\right)\sum_{i=1}^{n-1}(1-\delta_i) = \sum_{i=1}^{n-1}\delta_i\sum_{i=1}^{n-1}\frac{1}{\delta_i} - (n-1)^2.$$

In the sequel, we restrict our investigations to PCMs with *one* perturbed pair of elements only, say $\delta_1 = \delta \neq 1$, while $\delta_i = 1$ for $i \neq 1$.

Definition 3.5 If one pair of elements, a_{12} and a_{21} of a specific PCM has the form $a_{12} = x_1\delta$, $a_{21} = 1/x_1\delta$, and $\delta > 0$, then it is called a *simple perturbed* PCM and denoted by $\mathbf{A}_{\mathrm{S}}$.

In this special case, we have the simple perturbed matrix $\mathbf{A}_{\mathrm{S}}$ as

$$\mathbf{A}_{\mathrm{S}} = \begin{bmatrix} 1 & x_1\delta & x_2 & \ldots & x_{n-1} \\ \dfrac{1}{x_1\delta} & 1 & \dfrac{x_2}{x_1} & \ldots & \dfrac{x_{n-1}}{x_1} \\ \dfrac{1}{x_2} & \dfrac{x_1}{x_2} & 1 & \ldots & \dfrac{x_{n-1}}{x_2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \dfrac{1}{x_{n-1}} & \dfrac{x_1}{x_{n-1}} & \dfrac{x_2}{x_{n-1}} & \ldots & 1 \end{bmatrix}. \tag{3.14}$$

Performing a similarity transformation [see (3.6) and (3.8)], the characteristic polynomial of $\mathbf{A}_{\mathrm{S}}$, $p_n^{\mathrm{P}}(\lambda)$, can be written as

$$p_n^{\mathrm{S}}(\lambda) \equiv \det\left[\lambda\mathbf{I}_n - \mathbf{A}_{\mathrm{S}}\right] = \det\left[\lambda\mathbf{I}_n - \mathbf{D}^{-1}\mathbf{A}_{\mathrm{S}}\mathbf{D}\right] = \det\mathbf{K}_{\mathrm{S}}(\lambda), \tag{3.15}$$

where

$$\det\mathbf{K}_{\mathrm{S}}(\lambda) = \begin{vmatrix} \lambda-1 & 1-\delta & -1 & \ldots & -1 \\ 1-\dfrac{1}{\delta} & \lambda-1 & -1 & \ldots & -1 \\ -1 & -1 & \lambda-1 & \ldots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \ldots & \lambda-1 \end{vmatrix}.$$

Similarly to (3.9), the matrix $\mathbf{K}_{\mathrm{S}}(\lambda)$ in (3.15) given in the form of

$$\mathbf{K}_{\mathrm{S}}(\lambda) = \lambda\mathbf{I}_n - \mathbf{D}^{-1}\mathbf{A}_{\mathrm{S}}\mathbf{D}, \tag{3.16}$$

may also be interpreted as a modified matrix

$$\mathbf{K}_{\mathrm{S}}(\lambda) = \lambda\mathbf{I}_n + \mathbf{U}_{\mathrm{S}}\mathbf{V}_{\mathrm{S}}^{\mathrm{T}} - \mathbf{e}\mathbf{e}^{\mathrm{T}}, \tag{3.17}$$

where we used the notations:

$$\mathbf{U}_{\mathrm{P}} = \begin{bmatrix} 0 & 1 \\ 1-\dfrac{1}{\delta} & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{V}_{\mathrm{S}}^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1-\delta_1 & \dots & 0 \end{bmatrix}.$$

Introducing

$$\mathbf{T}_{\mathrm{S}}(\lambda) = \lambda\mathbf{I}_n + \mathbf{U}_{\mathrm{S}}\mathbf{V}_{\mathrm{S}}^{\mathrm{T}} = \begin{bmatrix} \lambda & 1-\delta & 0 & \dots & 0 \\ 1-\dfrac{1}{\delta} & \lambda & 0 & \dots & 0 \\ 0 & 0 & \lambda & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda \end{bmatrix}, \tag{3.18}$$

the modified matrix $\mathbf{K}_{\mathrm{S}}(\lambda)$ can be written as

$$\mathbf{K}_{\mathrm{S}}(\lambda) = \mathbf{T}_{\mathrm{S}}(\lambda) - \mathbf{e}\mathbf{e}^{\mathrm{T}}. \tag{3.19}$$

In this special case, the characteristic polynomial (3.12) has the form

$$p_n^{\mathrm{S}}(\lambda) \equiv \lambda^{n-3}\left[\lambda^3 - n\lambda^2 - C_{\mathrm{S}}\right] \tag{3.20}$$

where, the constant term, C_{S}, now becomes

$$C_{\mathrm{S}} = -(n-2)(1-\delta)\left(1-\frac{1}{\delta}\right) = (n-2)Q,$$

and Q is expressed as a function of the perturbation δ as

$$Q = \delta + \frac{1}{\delta} - 2, \qquad \delta > 0 \quad (\delta \neq 1). \tag{3.21}$$

Let r denote the maximal eigenvalue of a simple perturbed PCM, $\mathbf{A}_{\mathrm{S}}$. Then, r can be obtained from the equation [cf. (3.20)]:

$$r^3 - nr^2 - (n-2)Q = 0, \tag{3.22}$$

where Q is given by (3.21). Since $Q > 0$, from (3.22) it is easy to see that $r > n$. The proof can be found in Farkas, Rózsa and Stubnya (1999b). The components of the principal eigenvector can be obtained from the rank-one matrix

$$\text{adj}\,(r\mathbf{I}_n - \mathbf{A}_\text{S}) = \left[u_{ij}^\text{S}(r)\right], \tag{3.23}$$

since any column of the adjoint gives the elements of the principal eigenvector. In Appendix C, we show that the elements, $u_{ij}^\text{S}(r)$, of the principal eigenvector for the simple perturbed case are as follows:

$$\begin{bmatrix} u_{11}^\text{S}(r) \\ u_{21}^\text{S}(r) \\ \dots \\ u_{i1}^\text{S}(r) \\ \dots \end{bmatrix} = \begin{bmatrix} r^{n-2}\left[r-(n-1)\right] \\ \dfrac{1}{x_1} r^{n-3}\left\{r - \left(1-\dfrac{1}{\delta}\right)\left[r-(n-2)\right]\right\} \\ \dots \\ \dfrac{1}{x_{i-1}} r^{n-3}\left\{r - \left(1-\dfrac{1}{\delta}\right)\right\} \\ \dots \end{bmatrix}; \qquad i = 3,4,\dots,n, \tag{3.24}$$

$$\begin{bmatrix} u_{12}^\text{S}(r) \\ u_{22}^\text{S}(r) \\ \dots \\ u_{i2}^\text{S}(r) \\ \dots \end{bmatrix} = \begin{bmatrix} r^{n-3}\left\{r+(\delta-1)\left[r-(n-2)\right]\right\} \\ \dfrac{1}{x_1} r^{n-2}\left[r-(n-1)\right] \\ \dots \\ \dfrac{1}{x_{i-1}} r^{n-3}\left\{r+(\delta-1)\right\} \\ \dots \end{bmatrix} x_1; \qquad i = 3,4,\dots,n, \tag{3.25}$$

and

$$\begin{bmatrix} u_{1j}^\text{S}(r) \\ u_{2j}^\text{S}(r) \\ \dots \\ u_{ij}^\text{S}(r) \\ \dots \end{bmatrix} = \begin{bmatrix} r^{n-3}\left[r+(\delta-1)\right] \\ \dfrac{1}{x_1} r^{n-3}\left\{r - \left(1-\dfrac{1}{\delta}\right)\right\} \\ \dots \\ \dfrac{1}{x_{i-1}} r^{n-2}\left\{\dfrac{r-2}{n-2}\right\} \\ \dots \end{bmatrix} x_{j-1}; \qquad i,j = 3,4,\dots,n. \tag{3.26}$$

3.4 The Issue of Rank Reversal

It is well known that one of its most controversial aspects of the AHP is the phenomenon of rank reversal. Both proponents and opponents of AHP agree that rank reversal may occur, but disagree on its legitimacy. The problem has been considered by many authors and a persistent debate has followed; see in [90], [71], [5], [86], [30], [19], [67] and [31].

It has been shown that a rank reversal *may* occur in the AHP: (*i*) by introducing continuous SR perturbation(s) at one or more pairs of elements of a transitive PCM see e.g., Watson and Freeling [90] and Dyer and Wendell [21], or, (*ii*) by adding a new alternative to a SR perturbed PCM that is a replica (copy) of any of the old alternatives see e.g., Belton and Gear [4] and Dyer and Wendell [21] and (*iii*) using the additive aggregation rule when synthesizing priorities see e.g. Barzilai and Golany [3]. In this sub-section and in Appendix D, exact intervals are determined for cases (*i*) and (*ii*), respectively, over which such rank reversals of the rankings appear for cases where even slightest departures from perfect consistency occur in a PCM. We consider here PCM's with a single criterion only. The following result has published by Farkas (2007).

The concept of rank reversal is now introduced. Consider the simple perturbed matrix $\mathbf{A}_\mathrm{S}$ defined by (3.14). In the specific versus the simple perturbed case, the maximal eigenvalue r of matrix $\mathbf{A}_\mathrm{S}$ can be determined from (3.22), where $r > n$ $(n \geq 3)$ always holds [Farkas, Rózsa and Stubnya (1999)]. The components of the principal eigenvector can be obtained from the rank-one matrix (3.23). Since any column of this matrix gives the elements of the principal eigenvector, say, let $j = n$. Suppose that for two consecutive elements, u_i and u_{i+1} of the principal eigenvector of a *specific* PCM

$$u_i < u_{i+1} \tag{3.27}$$

holds. Furthermore, suppose that for the corresponding two elements, $u^\mathrm{S}_{in}(r)$ and $u^\mathrm{S}_{i+1,n}(r)$, of the adjoint matrix (3.23), i.e. for those of the principal eigenvector of a *simple perturbed* PCM

$$u^\mathrm{S}_{in}(r) > u^\mathrm{S}_{i+1,n}(r) \tag{3.28}$$

holds. If this case occurs, then, the rank order of the alternatives A_i and A_{i+1} has been reversed. This phenomenon is called a *rank reversal* of the alternatives under question.

According to the cardinal theory of preferences an opposite order of the corresponding components of the principal eigenvector *cannot* be yielded. Yet we give proofs for the occurrence of rank reversals in the AHP between the alternatives A_1 and A_2. To see that such a reversal is inherent in the AHP, it is sufficient to compare the order of the first two components of the principal eigenvectors.

For the specific case, the maximal eigenvalue of $\mathbf{A}$ equals n. The first two components of the principal eigenvector of $\mathbf{A}$ are as follows [cf. (3.5)]

$$1 \quad ; \quad \frac{1}{x_1} \tag{3.29}$$

i.e., the components of the principal eigenvector are monotonously increasing for $x_1 < 1$, whereas they are monotonously decreasing for $x_1 > 1$. In Theorem 3.1, necessary and sufficient condition is given for the occurrence of a rank reversal in the specific versus the simple perturbed case.

Theorem 3.1 *Let $\mathbf{A} = [a_{ij}]$ be a transitive (consistent) pairwise comparison matrix of order n, $n \geq 3$. Between the alternatives A_1 and A_2 when the elements a_{12} and a_{21} of $\mathbf{A}$ are perturbed, a rank reversal occurs if and only if*

$$1 > x_1 > \frac{r-1+\dfrac{1}{\delta}}{r-1+\delta} = 1 - \frac{\delta - \dfrac{1}{\delta}}{\delta + (r-1)}, \quad \textit{for} \quad \delta > 1, \tag{3.30}$$

or

$$1 < x_1 < \frac{r-1+\dfrac{1}{\delta}}{r-1+\delta} = 1 + \frac{\dfrac{1}{\delta} - \delta}{\delta + (r-1)}, \quad \textit{for} \quad 0 < \delta < 1. \tag{3.31}$$

Proof. Using (C5) from Appendix C, after performing the necessary algebraic manipulations the first two elements of the nth column of adj $(r\mathbf{I}_n - \mathbf{D}^{-1}\mathbf{A}_{\mathrm{S}}\mathbf{D})$, i.e., the cofactors corresponding to the first two elements of the nth row of $(r\mathbf{I}_n - \mathbf{D}^{-1}\mathbf{A}_{\mathrm{S}}\mathbf{D})$ are obtained as [cf. (3.26)]

$$\begin{aligned} \left\{\text{adj}\left(r\mathbf{I}_n - \mathbf{D}^{-1}\mathbf{A}_{\mathrm{S}}\mathbf{D}\right)\right\}_{1n} &= r^{n-3}\left[r - (1-\delta)\right]; \\ \left\{\text{adj}\left(r\mathbf{I}_n - \mathbf{D}^{-1}\mathbf{A}_{\mathrm{S}}\mathbf{D}\right)\right\}_{2n} &= r^{n-3}\left[r - (1 - \frac{1}{\delta})\right]. \end{aligned} \tag{3.32}$$

Taking into account (C4) in Appendix C, the first two components of the principal right eigenvector of the simple perturbed PCM, $\mathbf{A}_{\mathrm{S}}$, are proportional to

$$r - 1 + \delta \quad ; \quad \frac{1}{x_1}\left(r - 1 + \frac{1}{\delta}\right). \tag{3.33}$$

A rank reversal occurs if the elements in (3.33) are monotonously decreasing for $x_1 < 1$, or they are monotonously increasing for $x_1 > 1$ [cf. (3.29)]. Depending on whether δ is greater than unity, or δ is less than unity, two cases are distinguished:

(i) if $\delta > 1$ and $x_1 < 1$, then the elements in (3.33) are monotonously *decreasing* if x_1 resides in the interval given by (3.30), and

(ii) if $0 < \delta < 1$ and $x_1 > 1$, then the elements in (3.33) are monotonously *increasing* if x_1 resides in the interval given by (3.31).

This means that the condition is *necessary*. Furthermore, since all operations in the proof can be performed in the opposite direction, the condition is *sufficient* as well.

We note that according to (3.21) and (3.22), r is dependent on the value of δ. This fact, however, has no impact on the *existence* of the intervals (3.30) and (3.31), where rank reversal occurs. □

As concerns the other elements of the principal eigenvector, they can be obtained by making similar considerations. As a result, for these elements we have

$$u_{in}^{\mathrm{S}} = \frac{1}{x_{i-1}} r \frac{r-2}{n-2}, \quad i = 3, 4, \ldots, n. \tag{3.34}$$

From (3.34), it is obvious that rank reversal cannot occur between any pair of the alternatives $A_3, A_4, \ldots, A_n$. The occurrence of a rank reversal between alternatives A_1 and A_i, $i = 3, 4, \ldots, n$, or between A_2 and A_i, $i = 3, 4, \ldots, n$, could be analyzed in a similar way as was shown above.

Farkas (2008) also derived the spectral properties of augmented PCMs in explicit form. Such a PCM is constructed so that a new alternative is added to a simple perturbed PCM which is a replica (copy) of any of the old alternatives. Any column and the respective row can be selected as a replica. The principal eigenvector of this bordered matrix with such an extension was determined in an explicit form. A proof was given for the existence of a rank reversal between the specific and the augmented perturbed case and exact intervals were established for such cases. This analysis is presented in Appendix D. Latter results have also demonstrated that this phenomenon is inherent in the AHP.

Consider now a perturbed PCM $\mathbf{A}_{\mathrm{P}}$ as defined in Definition 3.4 and displayed by (3.7). Now, let the elements of $\mathbf{A}_{\mathrm{P}}$ be nonzero complex numbers. The multiplicative type perturbations δ_i, $i = 1, \ldots, n-1$, $\delta_i \neq 1$ at the entries may be arbitrary complex numbers. Dividing the characteristic polynomial (3.12) by n^3 and introducing the normalized eigenvalue $\mu = \lambda/n$, we obtain the general form of the trinomial equation as function of μ and the constant C_{P} as follows:

$$L(\mu) = \mu^3 - \mu^2 - C_{\mathrm{P}} = 0. \tag{3.35}$$

In Farkas, Rózsa and Stubnya (1998) we showed how the roots of equation (3.35), i.e. the nonzero eigenvalues of $\mathbf{A}_{\mathrm{P}}$ vary with a continuous change in the constant term C_{P}. Without loss of generality we may restrict our investigations to SR matrices with *one* perturbed pair of elements only. Let $\delta_1 = \delta \neq 1$. Thus, we obtain matrix $\mathbf{A}_{\mathrm{S}}$ defined by (3.14). Let us write the perturbation factor δ in the form: $\delta = r^2 e^{2\mathrm{i}t}$. We now get the constant term

$$C_{\mathrm{S}} = \frac{n-2}{n^3}\left(re^{\mathrm{i}t} - \frac{1}{r}e^{-\mathrm{i}t}\right)^2, \tag{3.36}$$

where

$$r = \sqrt{|\delta|}, \quad \text{and} \quad t = \mathrm{arc}\sqrt{\delta}.$$

From an application oriented point of view, those cases are of importance where expression (3.36) is real valued. Depending on the real parameters r and t, three cases can be distinguished:

Case (a). If $t = 0$, i.e., $\delta > 0$, and $r > 1$, then

$$L(\mu, r) = \mu^3 - \mu^2 - \frac{n-2}{n^3}\left(r - \frac{1}{r}\right)^2, \tag{3.37}$$

where the maximal eigenvalue of $\mathbf{A}_{\mathrm{S}}$ is given by (3.22). In this case the constant term C_{S} has the form:

$$C_{\mathrm{S}} = -(n-2)(1-\delta)\left(1 - \frac{1}{\delta}\right) = (n-2)Q,$$

where Q is given by (3.21) as function of the perturbation factor δ.

Case (b). If $t = \pi/2$, i.e., $\delta < 0$, and $r > 1$, then

$$L(\mu, r) = \mu^3 - \mu^2 + \frac{n-2}{n^3}\left(r + \frac{1}{r}\right)^2. \tag{3.38}$$

Case (c). If $r = 1$, then the modified matrix $\mathbf{K}_{\mathrm{P}}(\lambda)$ in (3.9) becomes a Hermitian matrix and so

$$L(\mu, t) = \mu^3 - \mu^2 + \frac{n-2}{n^3} 4\sin^2 t, \tag{3.39}$$

where t is an arbitrary parameter. Notice that in cases (a) and (b) it is sufficient to consider $r > 1$, since $L(\mu, r) = L(\mu, 1/r)$. Figure 18 displays the characteristic polynomial as function of μ, for various values of r and t, respectively, when $n = 4$, where each of the three regions corresponds to one of the cases (a), (b) or (c).

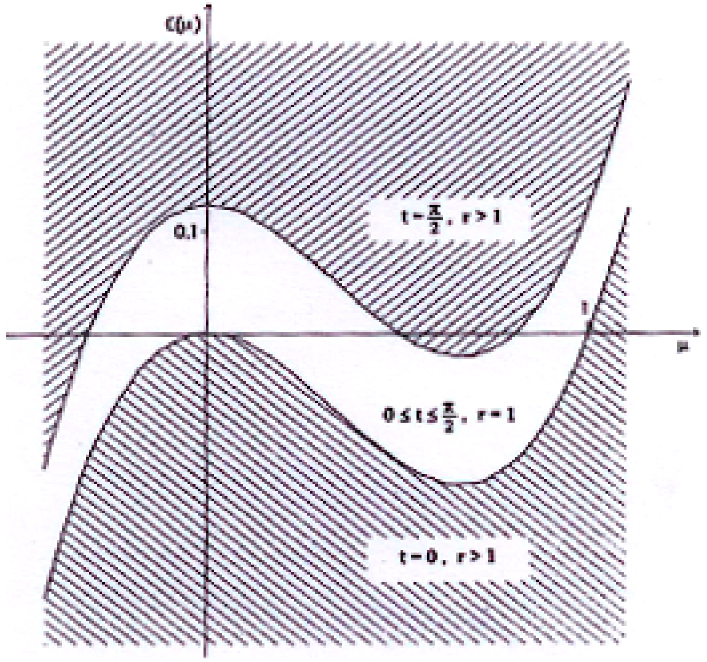


Figure 18. The characteristic polynomial for various values of r and t, when $r = 4$ [Farkas, Rózsa and Stubnya (1999b)]

Next, applications are shown for each of the above cases [Farkas and Rózsa (2001)].

Application 4: On the priority ranking problem of the AHP

Case (a) is applicable for positive SR matrices, e.g. for the problem of extracting the weights from a PCM in the AHP. This application demonstrates that the method cannot give the true ranking of the alternatives if the PCM is inconsistent even in the slightest degree, since then the principal eigenvector components do not give the true relative dominance of the alternatives. Obviously, this result can be extended to PCMs with arbitrary number of perturbed pairs of elements, regarding the fact that in the practical applications of the AHP, neither the cardinal consistency, nor the ordinal consistency of the judgements cannot be guaranteed in advance. In case of small perturbations, i.e., when the PCM is near consistent, the error does not affect the order of magnitude of the alternatives and hence, the relative dominance remains approximately the same as that of in a case of a perfect consistency. However, if there appears a serious violation of the transitivity condition, the problem should be studied again and the original PCM must be revised.

Given n, and specifying a value for Saaty's consistency index $CI = r - n/n - 1$ (see Definition 2.3), the maximal eigenvalue r, of a simple perturbed PCM $\mathbf{A}_{\mathrm{S}}$, given by (3.14) can be obtained as

$$r = n + CI(n-1), \tag{3.40}$$

then, from (3.22), for the term Q we have

$$Q = \frac{n-1}{n-2} r^2 CI. \tag{3.41}$$

Next, using (3.21), the roots of the following equation can be calculated from

$$\delta^2 - (2+Q)\delta + 1 = 0. \tag{3.42}$$

Finally, making use of (3.30) and (3.31) and with a chosen value for CI, the intervals of the range of values of x_1 over which a rank reversal occurs are

$$1 > x_1 > \frac{(n-1)(1+CI)+\dfrac{1}{\delta}}{(n-1)(1+CI)+\delta} = \frac{r-1+\dfrac{1}{\delta}}{r-1+\delta}, \quad \text{for } \delta > 1, \tag{3.43}$$

and

$$1 < x_1 < \frac{(n-1)(1+CI)+\dfrac{1}{\delta}}{(n-1)(1+CI)+\delta} = \frac{r-1+\dfrac{1}{\delta}}{r-1+\delta}, \quad \text{for } 0 < \delta < 1. \tag{3.44}$$

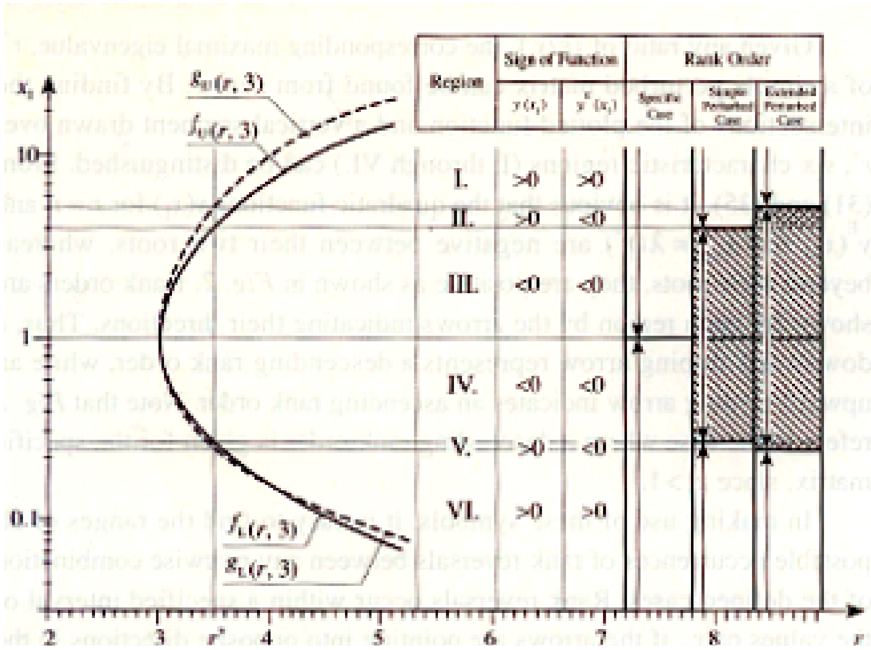


Figure 19. Characteristic zones of rank reversals for a perturbed SR matrix $\mathbf{A}_\mathrm{S}$ for $n = 3$ [Farkas and Rózsa (1996)]

Consider a simple perturbed PCM of order $n = 3$, that departs from consistency arbitrarily small. Let $CI = 0.01$. Using the appropriate table in Saaty [66], the corresponding $RI = 0.58$. Thus, $CR = 0.017$. From (3.40), (3.41) and (3.42) the computed parameters are, $r = 3.02$, $Q = 0.1824$, $\delta = 1.5279$, $1/\delta = 0.6545$, respectively. Using (3.43) and (3.44), the range of values of x_1, over which a rank reversal occurs lie in the interval of 0.7538 to 1.3266. This result verifies that if a PCM is perturbed even in the slightest degree, yet there exists a relatively large interval, over which a rank reversal occurs, i.e. in our case between alternatives A_1 and A_2. This example demonstrates well that any violation of transitivity might be serious in practice, because from a decision maker's perspective an undesired alternative may eventually be chosen as the best one. In summary, when the PCM is near consistent, the error does not affect the order of magnitude of the alternatives and hence, the relative dominance remains approximately the one as that of for a perfect consistency. However, if there is a serious violation of the transitivity condition, the original PCM should be revised. As a graphical illustration, this is displayed impressively in Figure 19, where characteristic zones of rank reversals are determined for an $\mathbf{A}_\mathrm{S}$ subject to a fairly large perturbation entered into the pair of its entries a_{12} and a_{21} (for this case the principal eigenvalue is: $r^* = 3.5$). We note that such a PCM must be revisited and the respective ratio estimate be corrected.

Application 5: Spectrum of the matrix of economic growth in the dynamic input-output analysis of macroeconomics

In case (b) the SR matrix may contain negative elements also. Such a case occurs in the static input-output analysis of the classic Leontief framework in macroeconomics. Steenge [77] has shown that the matrix of direct input coefficients can easily be transformed to an SR matrix of rank-one. Similarly, a specific case of the dynamic input-output analysis, the matrix of economic growth, can also be reformulated into a SR matrix form, provided that the economy produces a balanced growth (von-Neumann path) with growth rates of equal size. In reality, however, an economy never produces identical growth rates in its constituted sectors. Therefore, a question of utmost importance can now be raised. What amount of departures from equilibrium (in terms of perturbations in the associated SR matrix of growth) may occur so that the system still remains relative stable?

In response to this challenging issue we developed a system of perturbations suited to this problem. Making use of (3.10) and denoting by k_j the order of the blocks for the pattern of perturbations of these blocks located along the main diagonal of the matrix displayed in (3.7), the perturbed characteristic matrix is introduced as follows:

$$\mathbf{T}_{k_j}(\lambda) = \lambda \mathbf{I}_{k_j} + \mathbf{U}_j \mathbf{V}_j^{\mathrm{T}}, \qquad j = 1, 2, \ldots, p; \quad \sum_{j=1}^{p} k_j = n.$$

Let the block matrix, perturbed in the above described manner, be denoted by $\mathbf{A}_{\mathrm{Z}}$. By (3.11), the characteristic polynomial $p_n^{\mathrm{Z}}(\lambda)$ of this more complex matrix $\mathbf{A}_{\mathrm{Z}}$ is equal to the determinant:

$$p_n^{\mathrm{Z}}(\lambda) \equiv \det\left(\langle \mathbf{T}_{k_j}(\lambda) \rangle - \mathbf{e}\mathbf{e}^{\mathrm{T}}\right). \tag{3.45}$$

Following the procedure presented in Appendix B and applying it to matrix $\mathbf{A}_{\mathrm{Z}}$, a general expression in a closed form is obtained for the characteristic polynomial of $\mathbf{A}_{\mathrm{Z}}$:

$$\begin{aligned} p_n^{\mathrm{Z}}(\lambda) \equiv \lambda^{n-2p-1} \prod_{j=1}^{p} \left(\lambda^2 - \sum_{\mu=1}^{k_j-1} \left(1 - \frac{1}{\delta_{\mu j}}\right)(1 - \delta_{\mu j}) \right) & \\ \left\{ \lambda - n + \sum_{v=1}^{p} \frac{(\lambda - 1) \sum\limits_{\mu=1}^{k_v-1} \left(1 - \frac{1}{\delta_{\mu v}}\right)(1 - \delta_{\mu v}) - \sum\limits_{\mu=1}^{k_v-1} \left(1 - \frac{1}{\delta_{\mu v}}\right) \sum\limits_{\mu=1}^{k_v-1} (1 - \delta_{\mu v})}{\lambda^2 - \sum\limits_{\mu=1}^{k_v-1} \left(1 - \frac{1}{\delta_{\mu v}}\right)(1 - \delta_{\mu v})} \right\} & . \end{aligned} \tag{3.46}$$

By virtue of Tsukui's theorem, there exists a balanced growth solution that is relatively stable, if and only if there exists a positive eigenvalue λ_1 of $\mathbf{A}_{\mathrm{Z}}$, such that $\lambda_1 > |\lambda_i|$, where the λ_i's, $i = 2, 3, \ldots, n$, are the other eigenvalues [82]. Hence, using (3.46), the eigenvalues of the perturbed matrix of economic growth $\mathbf{A}_{\mathrm{Z}}$ can explicitly be determined. Then by comparing them among each other, one can model as to whether an economy, being initially in a balanced growth state, remained to be relatively stable on the effect of introduced arbitrary economic disturbances.

Application 6: A vehicle system dynamics problem considering n-axle railway carriages

Case (c) represents a Hermitian matrix with nonzero complex elements. Such a case occurs in vehicle system dynamics, where the rail/road track or surface unevenness affects the vibration characteristics of the bogies and the vehicle body through this multiple input excitation. In Farkas, Rózsa and Stubnya (2000), an original approach was proposed to describe and handle these vehicle engineering problems.

We consider an n-axle railway vehicle (railway carriage) running along a track. The vehicle is subject to n vertically imposed displacements, one at each wheel set. The excitation of the vehicle is represented by the generalized vertical displacement, $\tilde{x}(l)$ given as a function of the longitudinal coordinate of the track. The expression, $\tilde{x}(vt)$ provides a time-dependent excitation function at a constant translation speed, v, from which its values related to the particular wheel sets are yielded with delays. The excitation of the entire vehicle system is described by a vector-valued time function, $\mathbf{x}(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^{\mathrm{T}}$ which can be developed from the track unevenness variables, $x_i(t)$.

If the excitation environment is restricted to random undulations the input variables are representative members of a stationary, stochastic Gaussian process with a mean zero, attained by field-response data measurements. Therefore, the variables $x_i(t)$, $t = 1, 2, \ldots, N-1$, are given as discrete sequences of N points, each sampled at equal increments, Δt. For such a digitally sampled time series, $t = 1, 2, \ldots, N-1$, the discrete finite Fourier transform, X_k, can be written as

$$X_k = \Delta t \sum_{t=0}^{N-1} \hat{x}(t) e^{-i\frac{2\pi kt}{n}}, \qquad k = 0, 1, 2, \ldots, N-1.$$

The spectral density of the time series as function of the spatial frequency, f_k, is defined in terms of the series X_k:

$$G(f_k) = 2E\left\{\frac{|X_k|^2}{T}\right\}, \qquad k = 0, 1, 2, \ldots, \frac{N}{2},$$

where $f_k = k\Delta f$, and $\Delta f = 1/T$. Thus, a stochastic track unevenness, $\tilde{x}(l)$, can be described by the spectral density function, $g_{\tilde{x}\tilde{x}}(f)$. By introducing the angular frequency, ω, the excitation spectrum related to a steady speed ν, can be obtained through using the following scalar-valued formula:

$$g_{x_i x_i}(\omega) = \frac{1}{\nu} g_{\tilde{x}_i \tilde{x}_i}\left(\frac{\omega}{\nu}\right).$$

For a multi-wheel railway vehicle, it may be assumed that each wheel set rolls on identical track unevenness retarded in time. Hence, the spectral density at each wheel set becomes

$$g_{x_1 x_1}(\omega) = g_{x_2 x_2}(\omega) = \cdots = g_{x_n x_n}(\omega) = g_{xx}(\omega).$$

Further simplification can be achieved by considering that for zero mean stationary processes the elements of the cross spectral densities differ in the phase information only:

$$g_{x_j x_k}(\omega) = e^{-i\omega\frac{a_h}{\nu}} g_{xx}(\omega), \qquad j = 1, \ldots, n-1, \quad k = 2, \ldots, n,$$

and

$$g_{x_k x_j}(\omega) = e^{i\omega\frac{a_h}{\nu}} g_{xx}(\omega),$$

where a_h, $h = 1, \ldots, n-1$, is the axle-base between the front wheel set and the second, third,..., nth wheel set, respectively. Thus, for a vertical plane model of a 4-axle railway vehicle as seen in Figure 20, the input *spectral density matrix* (SDM) can be defined as

$$\mathbf{G}_x(\omega) = g_{xx}(\omega) \begin{bmatrix} 1 & e^{-i\omega\frac{a_1}{\nu}} & e^{-i\omega\frac{a_2}{\nu}} & e^{-i\omega\frac{a_3}{\nu}} \\ e^{i\omega\frac{a_1}{\nu}} & 1 & e^{-i\omega\frac{a_2-a_1}{\nu}} & e^{-i\omega\frac{a_3-a_1}{\nu}} \\ e^{i\omega\frac{a_2}{\nu}} & e^{i\omega\frac{a_2-a_1}{\nu}} & 1 & e^{-i\omega\frac{a_3-a_2}{\nu}} \\ e^{i\omega\frac{a_3}{\nu}} & e^{i\omega\frac{a_3-a_1}{\nu}} & e^{i\omega\frac{a_3-a_2}{\nu}} & 1 \end{bmatrix} \tag{3.47}$$

or writing in a brief form

$$\mathbf{G}_x(\omega) = g_{xx}(\omega)\mathbf{S}_x(\omega). \tag{3.48}$$

Such matrices were used by Dodds and Robson [18], who described the excitation process caused by road surface roughness, and also by Zobory [97], who analyzed the impacts of the track unevenness on the drive system of a railway traction vehicle. We consider 4-axle vehicles here such as displayed in Figure 20, i.e., the order of $\mathbf{S}_x(\omega)$, and thus, also of $\mathbf{G}_x(\omega)$, is $n = 4$. We remark that our approach may be generalized for any n. In railway vehicle applications, usually, for $n = 6$, and $n = 8$.

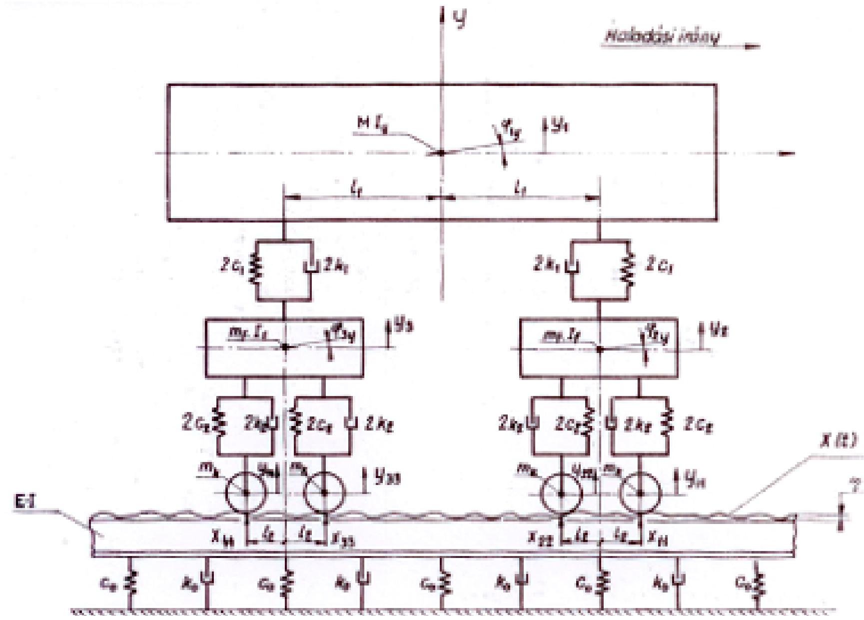


Figure 20. A vertical plane vibration model of a 4-axle railway carriage

Applying (3.4), (3.5) and (3.6) for an input SDM, it is easy to see that $\mathbf{S}_x(\omega)$ conforms to $\mathbf{A}$. Therefore, we can write that $\mathbf{S}_x(\omega) = \mathbf{u}\mathbf{v}^{\mathrm{T}}$. Thus, $\lambda_{\max} = n$, and for all other eigenvalues of $\mathbf{S}_x(\omega)$, $\lambda_i = 0$, $i = 2, \ldots, n$. The only one nonzero eigenvalue n, of $\mathbf{S}_x(\omega)$ should be multiplied by the scalar $g_{xx}(\omega)$ and evaluated at each relevant frequency band. This result indicates that a perfect linear relationship exists between the input excitation signals, i.e. each excitation source is linearly dependent on the others. In other words, the multiple-input system is in a complete coherence.

Definition 3.6 The *multiple coherence function*, γ_{ix}, $[0 < \gamma_{ix} < 1]$, for the set of inputs x_i; alone, i.e., between x_i, and all the other inputs $x_1, x_2, \ldots, x_n$, excluding x_i; is a real-valued statistical measure of the fraction of power in the x_i, accounted for by a simultaneous linear filter relationship with all other inputs.

Stationary random data from a number of measured input points (the excitations) are assumed to pass through the linear vehicle system whose effects are then summed to produce an output response. The multiple-input linear system is characterized by the matrix of its frequency response (transfer) function, $\mathbf{H}(\omega)$, which is computed as

$$\mathbf{H}(\omega) = \mathbf{G}_x^{-1}(\omega)\mathbf{G}_{xy}(\omega), \tag{3.49}$$

where $\mathbf{G}_{xy}(\omega)$ is the spectrum matrix of the output with the inputs. Eq. (3.49) is often referred to as the base formula for computing the transfer characteristics of the bogies and the vehicle body. The computational procedure involves a dynamic model of finite degree of freedom and strives to determine the vertical and the horizontal displacements for the lumped inertial, elastic and dissipative elements. It can immediately be seen from (3.47) that $\mathbf{G}_x(\omega)$, and thus $\mathbf{S}_x(\omega)$, is *singular*, therefore, this form of input that seems to be realistic, cannot be used to evaluate the frequency response functions. The matrix $\mathbf{S}_x(\omega)$ is *Hermitian*. This implies that the eigenvalues of $\mathbf{S}_x(\omega)$ are real numbers. Furthermore, $\mathbf{G}_x(\omega)$ is *positive semidefinite*, which is *not* a sufficient condition for computing (3.49), as $\mathbf{G}_x(\omega)$ must also be positive definite. Thus, these inputs may be correlated, but they may not be completely coherent. Latter case commands interest from vehicle dynamics practitioners.

Symmetrically reciprocal (SR) perturbations of input spectral density matrices

Such cases occur in vehicle dynamics, when an SR perturbation enters a transitive input SDM. Let this perturbation represent the change in the multiple-input excitation of the vehicle system on the effect of a bias in the axle-base between the front and the second wheel sets, and be denoted by Δa_1. That is, let $a_1^{\mathrm{P}} = a_1 + \Delta a_1$. Such a bias may occur due to manufacturing deficiencies which result in geometric inaccuracy, or frequent overstrain causing undesired structural wear, or a local track unevenness which impacts the bogie, or due to an irregular movement of the vehicle in curves, etc. Ordinarily, these perturbations are available for the engineer as statistical estimates of the unknown true parameters with expected values and variances acquired from an assumed type of probability distribution. Let δ_1 denote such a perturbation which is introduced in matrix $\mathbf{S}_x(\omega)$ given by (3.47). The respective perturbation factor δ_1 becomes

$$\delta_1 = e^{-i\omega\frac{\Delta a_1}{\nu}}, \quad \text{since } e^{-i\omega\frac{a_1+\Delta a_1}{\nu}} = e^{-i\omega\frac{a_1}{\nu}} e^{-i\omega\frac{\Delta a_1}{\nu}} = e^{-i\omega\frac{a_1}{\nu}}\delta_1.$$

Hence, the SR perturbed input SDM has now the form

$$\mathbf{G}_x^{\mathrm{S}}(\omega) = g_{xx}(\omega) \begin{bmatrix} 1 & e^{-i\omega\frac{a_1}{\nu}}\delta_1 & e^{-i\omega\frac{a_2}{\nu}} & e^{-i\omega\frac{a_3}{\nu}} \\ e^{i\omega\frac{a_1}{\nu}}\frac{1}{\delta_1} & 1 & e^{-i\omega\frac{a_2-a_1}{\nu}} & e^{-i\omega\frac{a_3-a_1}{\nu}} \\ e^{i\omega\frac{a_2}{\nu}} & e^{i\omega\frac{a_2-a_1}{\nu}} & 1 & e^{-i\omega\frac{a_3-a_2}{\nu}} \\ e^{i\omega\frac{a_3}{\nu}} & e^{i\omega\frac{a_3-a_1}{\nu}} & e^{i\omega\frac{a_3-a_2}{\nu}} & 1 \end{bmatrix} \tag{3.50}$$

or in brief

$$\mathbf{G}_x^{\mathrm{S}}(\omega) = g_{xx}(\omega)\mathbf{S}_x^{\mathrm{S}}(\omega). \tag{3.51}$$

It is apparent from (3.50) that $\mathbf{S}_x^{\mathrm{S}}(\omega)$ remained in SR, but its *transitivity* has been lost. Furthermore, $\mathbf{S}_x^{\mathrm{S}}(\omega)$ is still *singular*. However, $\mathbf{S}_x^{\mathrm{S}}(\omega)$ has preserved its *Hermitian* property. Given these properties, there are some limitations for the vehicle system analyst because Eq. (3.49) is still not applicable with $\mathbf{S}_x^{\mathrm{S}}(\omega)$. Nevertheless, the degree of the nonlinear undulations caused by a known type of perturbation may be analyzed by studying the change of the eigenvalue of largest modulus of the perturbed SDM on the complex plane.

Symmetrical perturbations of input spectral density matrices

In addition to the Hermitian property, input SDMs should be nonsingular and positive definite. To attain this goal, the multiple coherence function can be applied. This application illustrates the coherence functions in terms of a multivariate statistical analysis on complex variables. Power spectra (see Glossary at the end of this Chapter) may be interpreted as variances, thus, the multiple coherence is the fraction of the variance of the dependent variable accounted for by the input variables. An SDM may be viewed as a *covariance* matrix of complex random variables, one particular matrix at each frequency band in the course of the spectral analysis, provided that they are both *Hermitian* and *positive definite*. These properties imply that the eigenvalues of such SDMs are real, positive numbers.

In the following we will utilize the correspondence between the interpretation of the *variance* (*power*) and the *multiple coherence function*. Principal component analysis refers to the problem of determining a few linear combinations of the originally observed data that account for most of the variance in the dependent variable. The first principal component will be a variable (input to a multiple linear system) that has maximum multiple coherence with the output. The second component will be an other variable that is incoherent with the first and has the next largest multiple coherence with the output, etc. If $\mathbf{G}_x(\omega)$ is a spectral density matrix, then the solution to the principal component problem results in the equation [11]:

$$[\lambda \mathbf{I} - \mathbf{G}_x(\omega)]\, \mathbf{u} = 0.$$

Above equation conforms to the conventional eigenvalue-eigenvector problem. The principal components are

$$C_i = \mathbf{u}_i^{\mathrm{T}} \mathbf{x}(\omega), \quad i = 1, \ldots, n,$$

where $\mathbf{x}(\omega)$ is the Fourier transform vector of the original n-dimensional data vector. As is well known, the variance of C_i is equal to the eigenvalue λ_i of $\mathbf{G}_x(\omega)$, so we can write that

$$\mathrm{Var}[C_i] = \lambda_i, \quad i = 1, \ldots, n. \tag{3.52}$$

According to (3.52), the eigenvalues represent the fraction of power accounted for by the principal components as a result of certain minimum means square transformations applied to data.

Let us now enter symmetrical coherence coefficients in a transitive input SDM, $\mathbf{S}_x(\omega)$. Let $\gamma_1 \neq \gamma_2 \neq$ $\neq \gamma_3 \neq \gamma_4$, and $0 < \gamma_i < 1$, for all i. We assume here that there is identical coherence between the first and the third and between the second and the fourth wheel sets (γ_2), and also, between the first and the fourth and between the second and the third wheel sets (γ_4). As a matter of fact, another set-up could have been established, as well. The *symmetrically* perturbed input SDM now becomes

$$\mathbf{G}_x^{\mathrm{C}}(\omega) = g_{xx}(\omega) \begin{bmatrix} 1 & \gamma_1 e^{-i\omega \frac{a_1}{\nu}} & \gamma_2 e^{-i\omega \frac{a_2}{\nu}} & \gamma_4 e^{-i\omega \frac{a_3}{\nu}} \\ \gamma_1 e^{i\omega \frac{a_1}{\nu}} & 1 & \gamma_4 e^{-i\omega \frac{a_2 - a_1}{\nu}} & \gamma_2 e^{-i\omega \frac{a_3 - a_1}{\nu}} \\ \gamma_2 e^{i\omega \frac{a_1}{\nu}} & \gamma_4 e^{i\omega \frac{a_2 - a_1}{\nu}} & 1 & \gamma_3 e^{-i\omega \frac{a_3 - a_2}{\nu}} \\ \gamma_4 e^{i\omega \frac{a_1}{\nu}} & \gamma_2 e^{i\omega \frac{a_3 - a_1}{\nu}} & \gamma_3 e^{i\omega \frac{a_3 - a_2}{\nu}} & 1 \end{bmatrix} \tag{3.53}$$

or in short

$$\mathbf{G}_x^{\mathrm{C}}(\omega) = g_{xx}(\omega) \mathbf{S}_x^{\mathrm{C}}(\omega). \tag{3.54}$$

With these coherence coefficients, it is apparent from (3.53) that the matrix $\mathbf{S}_x^{\mathrm{C}}(\omega)$ will *no longer* be neither SR nor *transitive*. However, the *Hermitian* property has been preserved. Furthermore, $\mathbf{S}_x^{\mathrm{C}}(\omega)$ has become *nonsingular*. Performing a similarity transformation [cf. (3.8)] and introducing the matrices $\mathbf{F}$ and $\mathbf{E}$ as

$$\mathbf{F} = \begin{bmatrix} 1 & \gamma_1 & \gamma_2 & \gamma_4 \\ \gamma_1 & 1 & \gamma_4 & \gamma_2 \\ \gamma_2 & \gamma_4 & 1 & \gamma_3 \\ \gamma_4 & \gamma_2 & \gamma_3 & 1 \end{bmatrix}, \qquad \mathbf{E} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

it can easily be shown that the characteristic matrix, $[\lambda \mathbf{I} - \mathbf{F}]$, contains four commutative blocks of order 2, since each of these blocks are linear functions of matrix $\mathbf{E}$. Therefore, its determinant can be calculated as the determinant of a block determinant:

$$\det \mathbf{S}_x^{\mathrm{C}}(\omega) = \det \left\{ \left[(\lambda - 1)^2 + \gamma_1 \gamma_2 - \gamma_2^2 - \gamma_4^2 \right] \mathbf{I} - \left[(\lambda - 1)\left(\gamma_1 + \gamma_3 \right) + 2\gamma_2 \gamma_4 \right] \mathbf{E} \right\}. \tag{3.55}$$

Then a short calculation produces the eigenvalues of $\mathbf{S}_x^{\mathrm{C}}(\omega)$ as

$$\lambda_{1,2} = 1 - \frac{\gamma_1 + \gamma_3}{2} \pm \sqrt{\left(\frac{\gamma_1 - \gamma_3}{2}\right)^2 + (\gamma_2 - \gamma_4)^2}, \tag{3.56}$$

and

$$\lambda_{3,4} = 1 + \frac{\gamma_1 + \gamma_3}{2} \pm \sqrt{\left(\frac{\gamma_1 - \gamma_3}{2}\right)^2 + (\gamma_2 + \gamma_4)^2}. \tag{3.57}$$

From (3.56) and (3.57), the *necessary* and *sufficient* conditions, required the matrix $\mathbf{S}_x^{\mathrm{C}}(\omega)$ be *positive definite* are as follows

$$\gamma_2 + \gamma_4 < \sqrt{(1+\gamma_1)(1+\gamma_3)} \quad \text{and} \quad |\gamma_2 - \gamma_4| < \sqrt{(1-\gamma_1)(1-\gamma_3)}. \tag{3.58}$$

If conditions (3.58) hold for a matrix $\mathbf{S}_x^{\mathrm{C}}(\omega)$, then it may be viewed as a *covariance matrix* of complex random variables that should be evaluated at each relevant frequency band. These properties of a matrix $\mathbf{S}_x^{\mathrm{C}}(\omega)$ ensure that the frequency response function (3.49) can be computed. Furthermore, all eigenvalues of $\mathbf{S}_x^{\mathrm{C}}(\omega)$ will be positive and they can be interpreted as variances (fraction of power). From (3.55), it is easy to derive that for the special case of $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = \gamma$, the eigenvalues of the perturbed matrix $\mathbf{S}_x^{\mathrm{C}}(\omega)$ yield

$$\lambda_{1,2,4} = 1 - \gamma \quad \text{and} \quad \lambda_3 = 1 + 3\gamma. \tag{3.59}$$

Therefore, given the coherence coefficients γ_i, $i = 1,2,3,4$, the variance that is transferred to the output (response function of the vehicle body) can be determined, since by (3.52), each eigenvalue, λ_i, $i = 1,2,3,4$, of $\mathbf{S}_x^{\mathrm{C}}(\omega)$ represents a particular fraction of power (i.e. the $\ddot{x}(t)$ process) caused by the multiple-input excitation.

Glossary:

Power spectrum [92]. The well known expression of the mean square of a stationary and ergodic random process is given by

$$\Psi_x^2 = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} x^2(t)\, dt.$$

Dimensionally, Ψ_x^2 is proportional to the mean square energy per unit time, which is by definition, *power*. The *power spectral density* of the function x, written $G_x(f)$ is an extension of this concept. The interpretation of $G_x(f)$ is that the integral

$$\Psi_x^2(f_1, f_2) = \int_{f_1}^{f_2} G_x(f)\, df, \quad 0 \le f_1 \le f_2,$$

is the power between the frequences f_1 and f_2. Thus

$$\Psi_x^2 = \int_0^{\infty} G_x(f)\, df.$$

Chapter 4

4 Consistency Adjustments to Pairwise Comparison Matrices

In this Chapter, a "best" rank one transitive approximation to a general paired comparison matrix in a least-squares sense is developed. A direct attack on the nonlinear problem may be replaced by a sub-optimal problem and, here, the best procedure of this kind is obtained. The Newton-Kantorovich (N-K) method for the solution of the nonlinear problem developed in the form of an inhomogeneous system of n equations is presented. Convergence of the process to (at least) a local minimum is discussed. It is shown that multiple solutions may occur, provided that the SR matrix has a particular structure. Sufficient conditions for the non-uniqueness problem of the solution are derived and proved. A specific, user defined, flexible multiplicative SR perturbation of transitive matrices of exponential type is proposed. An application of latter result is shown, which is an extension of the vehicle excitation problem of Application 6 to nonlinear cases.

4.1 Preliminaries and Problem Statement

The results presented in the next sections 4.1, 4.2 and 4.3 have appeared in the paper of Farkas, Lancaster and Rózsa (2003). Consider $n \times n$ matrices with positive entries. We call $\mathbf{A} = [a_{ij}]$ a *symmetrically reciprocal* (SR) matrix and $\mathbf{B} = [b_{ij}]$ a transitive matrix as defined in sub-chapter 3.2. We wish to assess weights, $w_1, w_2, \ldots, w_n$ associated with the n decision alternatives. Recall that an SR matrix has rank one if and only if it is transitive (see Proposition 3.1). Saaty argued in [71, p. 853] that the eigenvector method applied to a particular SR matrix provides an optimal choice, even if the matrix is *not* transitive. Here a different argument is proposed in which a "best" rank-one transitive approximation to a general SR matrix is the objective, and "best" is assessed in a least-squares sense.

There are various possible ways to generate approximations for $\mathbf{A}$ (see Appendix A, in A.2.3). The most notable least-squares (LS) approaches were the direct least-squares method (Jensen [38]), and the weighted least-squares method (Chu et al. [12] and Blankmeyer [6]). Bozóki and Lewis [7] proposed the resultant method for the LS problem as a general solution procedure, while Fülöp [28] applied a logarithmic transformation to this problem, first converting it into a separable programming form, then he used a branch and bound technique.

It will be convenient to have two related notations: $\mathbf{W}$ is the diagonal matrix with diagonal entries $w_1, w_2, \ldots, w_n$ and the (column) vector from R^n with entries $w_1, w_2, \ldots, w_n$ denoted by $\mathbf{w}$. Thus $\mathbf{W}$ is a positive definite diagonal matrix if and only if vector $\mathbf{w}$ is an element-wise positive column vector. Introduce the $n \times n$ matrix $\mathbf{E}$ whose entries are all equal to one and the (row) vector, $\mathbf{e}^{\mathrm{T}} = [1, 1, \ldots, 1]$. It is easy to see that any transitive matrix can now be written in the form

$$\mathbf{B} = \mathbf{W}^{-1}\mathbf{E}\mathbf{W} = \mathbf{W}^{-1}\mathbf{e}\mathbf{e}^{\mathrm{T}}\mathbf{W} = \left[\frac{w_j}{w_i}\right], \quad i, j = 1, \ldots, n. \tag{4.1}$$

Mathematically, the problem is to find a transitive matrix $\mathbf{B}$ which is the best approximation to $\mathbf{A}$ in some sense. From (4.1), it can immediately be shown that

$$\mathbf{B}\mathbf{W}^{-1}\mathbf{e} = n\mathbf{W}^{-1}\mathbf{e}. \tag{4.2}$$

According to (4.2) it is readily apparent that the only nonzero (dominant) eigenvalue of $\mathbf{B}$ is n and its associated principal eigenvector is $\mathbf{u} = \mathbf{W}^{-1}\mathbf{e}$, i.e., whose components are the weights. This vector is usually normalized so that its entries sum to unity.

A transitive matrix $\mathbf{B}$ is to be found which is, in a suitable sense, the best approximation to an SR matrix $\mathbf{A}$ which is *not* transitive. A natural way to do this is to find the $\mathbf{B}$ (or $\mathbf{W}$) which will minimize the expression

$$S^3(\mathbf{w}) = \|\mathbf{A} - \mathbf{B}\|_\mathrm{F}^2 = \min \sum_{i=1}^{n} \sum_{j=1}^{n} \left(a_{ij} - \frac{w_j}{w_i} \right)^2 . \tag{4.3}$$

Here, the subscript F denotes the Frobenius norm, the square root of the sum of squares of the elements. As this problem is nonlinear in the w_j, $j = 1, \ldots, n$, one may observe that

$$\|\mathbf{A} - \mathbf{B}\|_\mathrm{F} = \left\|\mathbf{A} - \mathbf{W}^{-1}\mathbf{E}\mathbf{W}\right\|_\mathrm{F} \leq \left\|\mathbf{W}^{-1}\right\|_\mathrm{S} \|\mathbf{W}\mathbf{A} - \mathbf{E}\mathbf{W}\|_\mathrm{F} . \tag{4.4}$$

In (4.4) the subscript S denotes the spectral norm. Consider the sub-optimal, but linear problem of minimizing $\|\mathbf{W}\mathbf{A} - \mathbf{E}\mathbf{W}\|_\mathrm{F}^2$. To avoid the trivial (and useless) solution $\mathbf{W} = 0$, an inhomogeneous linear constraint is added. Thus, for a given elementwise positive vector $\boldsymbol{\Phi}^\mathrm{T} = [\Phi_1, \Phi_2, \ldots, \Phi_n]$, one must find the matrix $\mathbf{W}_0(\boldsymbol{\Phi})$ for which

$$S_0^2(\phi) := \inf_{\Phi^\mathrm{T}\mathbf{w}=1} \|\mathbf{W}\mathbf{A} - \mathbf{E}\mathbf{W}\|_\mathrm{F}^2 = \|\mathbf{W}_0\mathbf{A} - \mathbf{E}\mathbf{W}_0\|_\mathrm{F}^2 . \tag{4.5}$$

A strategy of similar kind was adopted by Chu et al. in [12] and discussed further by Blankmayer [6]. Here it will be shown that there is an optimal choice of $\boldsymbol{\Phi}$, i.e. there is an easily determined $\boldsymbol{\Phi}_\mathrm{opt}$ and associated $\mathbf{W}_0(\boldsymbol{\Phi}_\mathrm{opt})$ for which $S_0(\boldsymbol{\Phi}_\mathrm{opt}) \leq S_0(\boldsymbol{\Phi})$ for all $\boldsymbol{\Phi} > 0$ (see Theorem 4.4). This clearly provides an improved estimate $\mathbf{W}_0$ for the solution of the nonlinear problem. Observe that, if $\mathbf{A}$ happens to be transitive, then $\mathbf{A} = \mathbf{U}^{-1}\mathbf{E}\mathbf{U}$ for some positive definite diagonal matrix $\mathbf{U}$. Then the choice $\mathbf{W} = \mathbf{U}$ gives $S^2 = 0$. Furthermore, the sub-optimal strategy also yields $S_0^2(\boldsymbol{\Phi}) = 0$ with the choice $\mathbf{W} = \mathbf{U}$ and any $\boldsymbol{\Phi} > 0$ satisfying $\boldsymbol{\Phi}^\mathrm{T}\mathbf{u} = 1$.

In sub-chapter 4.3, it will be shown that the necessary conditions for a stationary value of $S^2(\mathbf{w})$ have a tractable form. This facilitates the use of the Newton-Kantorovich (N-K) method to the solution of these nonlinear equations and (when successful) provides a solution to the original nonlinear problem. In contrast to the claims of some earlier papers, and as *n* is usually no larger than ten or twenty, direct solution of the nonlinear problem in this way is not formidable *provided* a good initial approximation to the solution is known. It is claimed that $\mathbf{W}_0(\boldsymbol{\Phi}_\mathrm{opt})$ is a good candidate. The author's experience with this strategy is very favorable. To assist in the analysis of the N-K method, closed forms are found for all the first and second derivatives of S^2 in sub-chapter 4.3 and it is shown, using an example in 4.3.1 how this can be used in numerical analysis of the nonlinear optimization problem. Solution of sub-optimal, but *linear* problem is first discussed in the next section.

4.2 The Linear Approximation

First, we show that the objective function associated with equation (4.5) can be written in quadratic form (P. Lancaster):

Lemma 4.1 *If a quadratic matrix of* n *order* $\mathbf{P}_n$ *is defined by*

$$\mathbf{P}_n = (n\mathbf{I}_n + \mathbf{G}_n) - \left(\mathbf{A} + \mathbf{A}^\mathrm{T}\right) , \tag{4.6}$$

where $\mathbf{I}_n$ *is the identity matrix of n order, and*

$$\mathbf{G}_n = \mathrm{diag}\left[\sum_{j=1}^{n} a_{1j}^2, \sum_{j=1}^{n} a_{2j}^2, \ldots, \sum_{j=1}^{n} a_{nj}^2,\right],$$

then

$$\|\mathbf{WA} - \mathbf{EW}\|_{\mathrm{F}}^2 = \mathbf{w}^{\mathrm{T}}\mathbf{P}_n\mathbf{w}. \tag{4.7}$$

Proof. Defining $\mathbf{e}$ to be the column vector of R^n with all entries equal to 1, observe that $\mathbf{We} = \mathbf{w}$ and, writing tr for the trace of the matrix,

$$\begin{aligned}\|\mathbf{WA} - \mathbf{EW}\|_{\mathrm{F}}^2 &= \mathrm{tr}\left[(\mathbf{WA} - \mathbf{EW})^{\mathrm{T}}(\mathbf{WA} - \mathbf{EW})\right] = \\ &= \mathrm{tr}\left(\mathbf{A}^{\mathrm{T}}\mathbf{W}^2\mathbf{A}\right) - \mathrm{tr}\left(\mathbf{w}\mathbf{w}^{\mathrm{T}}\mathbf{A}\right) - \mathrm{tr}\left(\mathbf{A}^{\mathrm{T}}\mathbf{w}\mathbf{w}^{\mathrm{T}}\right) + n\mathrm{tr}\left(\mathbf{w}\mathbf{w}^{\mathrm{T}}\right).\end{aligned} \tag{4.8}$$

Using the simple fact that, for any $\mathbf{a}, \mathbf{b} \in R^n$, $\mathrm{tr}\left(\mathbf{a}\mathbf{b}^T\right) = \mathbf{b}^T\mathbf{a}$, the lemma follows readily from the last displayed equation. □

It is apparent from Lemma 1 that $\mathbf{P}_n$ is positive semidefinite; $\mathbf{P}_n \geq 0$. Furthermore, it can be shown that $\mathbf{P}_n$ is *not* positive definite if $\mathbf{A}$ is transitive. But a stronger statement can be made (P. Lancaster):

Lemma 4.2 *If* $\mathbf{A}$ *is not transitive and* $n \geq 3$*, then the matrix* $\mathbf{P}_n$ *is positive definite.*

Proof. The proof is by induction on n. When $n = 3$ a computation shows that the first two leading minors of $\mathbf{P}_3$ are positive and that

$$\det \mathbf{P}_3 = (1 - a_{31}a_{12}a_{23})^2 \geq 0.$$

Since $a_{13} = a_{31}^{-1}$, it follows that $\det(\mathbf{P}_3) = 0$ if and only if $a_{13} = a_{12}a_{23}$. As it is assumed that $\mathbf{A}$ is not transitive, it follows that $\mathbf{P}_3 > 0$. Suppose that $n > 3$, $\mathbf{P}_{n-1} > 0$, and observe that (since $a_{ii} = 1$) the diagonal elements of $\mathbf{P}_n$ have the form

$$n + \sum_{j=1}^{n} a_{ij}^2 - 2 = (n-1) + \sum_{j=1,\, j\neq i}^{n} a_{ij}^2, \quad i = 1, \ldots, n.$$

It follows that

$$\mathbf{P}_n = \begin{bmatrix} \mathbf{P}_{n-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \mathbf{D} & \alpha \\ \alpha^{\mathrm{T}} & (n-1) + \sum_{j=1}^{n-1} a_{nj}^2 \end{bmatrix}, \tag{4.9}$$

where $\mathbf{D} = \mathrm{diag}\left[1 + a_{1,n}^2, \ldots, 1 + a_{n-1,n}^2\right]$ and, since $\mathbf{A}$ is SR,

$$\alpha^{\mathrm{T}} = \left[-\left(a_{1n} + \frac{1}{a_{1n}}\right), \ldots, -\left(a_{n-1,n} + \frac{1}{a_{n-1,n}}\right)\right].$$

The determinant of the last matrix in (4.9) is

$$\Delta := [\det \mathbf{D}]\left\{(n-1) + \sum_{j=1}^{n-1} a_{nj}^2 - \alpha^{\mathrm{T}}\mathbf{D}^{-1}\alpha\right\}.$$

It is found that

$$\alpha^{\mathrm{T}}\mathbf{D}^{-1}\alpha = (n-1) + \sum_{j=1}^{n-1} \frac{1}{a_{jn}^2},$$

and, since $a_{nj} = 1/a_{jn}$, it follows that $\Delta = 0$ and the last matrix of (4.9) (like the first) is positive semidefinite. If $\mathbf{P}_n$ is *not* positive definite there is a nonzero vector $[\ \mathbf{x}^{\mathrm{T}} \quad \xi\ ]$, where ξ is real, such that

$$\begin{bmatrix} \mathbf{x}^{\mathrm{T}} & \xi \end{bmatrix} \mathbf{P}_n \begin{bmatrix} \mathbf{x} \\ \xi \end{bmatrix} = 0.$$

But this implies that both

$$\begin{bmatrix} \mathbf{x}^{\mathrm{T}} & \xi \end{bmatrix} \begin{bmatrix} \mathbf{P}_{n-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \xi \end{bmatrix} = 0$$

and

$$\begin{bmatrix} \mathbf{x}^{\mathrm{T}} & \xi \end{bmatrix} \begin{bmatrix} \mathbf{D} & \alpha \\ \alpha^{\mathrm{T}} & (n-1) + \sum_{j=1}^{n-1} a_{nj}^2 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \xi \end{bmatrix} = 0,$$

and, from $\mathbf{P}_{n-1} > 0$ it follows that both $\mathbf{x} = 0$ and $\xi = 0$. Hence $\mathbf{P}_n > 0$. □

The next Lemma is proposed to find a positive optimal solution directly (P. Lancaster):

Lemma 4.3 *If* $\mathbf{A}$ *is a non-transitive SR matrix of size* $n \geq 3$*, and* $\boldsymbol{\Phi} > 0$ *is given in* R^n*, then*
(a) there exists a unique positive definite diagonal matrix $\mathbf{W}_0(\boldsymbol{\Phi})$ *such that*

$$\inf_{\Phi^{\mathrm{T}}\mathbf{w}=1} \|\mathbf{WA} - \mathbf{EW}\|_{\mathrm{F}}^2 = \|\mathbf{W}_0\mathbf{A} - \mathbf{EW}_0\|_{\mathrm{F}}^2. \tag{4.10}$$

(b) $\mathbf{P}_n^{-1}$ *is elementwise positive.*

Proof. The Lagrange multiplier technique is used to solve this linearly constrained minimization problem. Using Lemma 4.1, the Lagrangian is written in the form:

$$\mathcal{L}(\mathbf{W}) = \mathbf{w}^{\mathrm{T}}\mathbf{P}_n\mathbf{w} + 2\lambda\left(\phi^{\mathrm{T}}\mathbf{w} - 1\right).$$

Then the normal equations are

$$\begin{bmatrix} \mathbf{P}_n & \phi \\ \phi^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{4.11}$$

By Lemma 4.2, $\mathbf{P}_n$ is positive definite, and the determinant of the coefficient matrix is

$$-[\det(\mathbf{P}_n)]\phi^{\mathrm{T}}\mathbf{P}_n^{-1}\phi \neq 0.$$

So these equations have a unique solution. Also, from the definition (4.6) it follows that $\mathbf{P}_n$ is irreducible and, therefore, a Stieljes matrix. Hence $\mathbf{P}_n^{-1}$ is elementwise positive. Then it follows from definition (4.6) that

$$\mathbf{w} = (\phi^{\mathrm{T}}\mathbf{P}_n^{-1}\phi)^{-1}\mathbf{P}_n^{-1}\phi. \tag{4.12}$$

Since $\boldsymbol{\Phi}$ is elementwise positive, it follows that the same is true of $\mathbf{w}$. This completes the proof. □

Theorem 4.1 *If* $\mathbf{A}$ *is a non-transitive SR matrix of size* $n \geq 3$ *and* $\mathbf{W}_0(\boldsymbol{\Phi})$ *is defined as in* Lemma 3*, then there is a unique positive definite diagonal matrix* $\hat{\mathbf{W}}$ *such that*

$$\inf_{\Phi} \|\mathbf{W}_0(\phi)\mathbf{A} - \mathbf{E}\mathbf{W}_0(\phi)\|_{\mathrm{F}}^2 = \left\|\hat{\mathbf{W}}\mathbf{A} - \mathbf{E}\hat{\mathbf{W}}\right\|_{\mathrm{F}}^2. \tag{4.13}$$

Furthermore, $\hat{\mathbf{w}}$ *is proportional to the Perron-eigenvector of* $\mathbf{P}_n^{-1}$.

Proof. Given a $\boldsymbol{\Phi} > 0$, the error associated with the corresponding optimal $\mathbf{w}$ satisfying $\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{w} = 1$ is, from Lemma 1 and equation (4.12),

$$S_0^2 = \mathbf{w}^{\mathrm{T}}\mathbf{P}_n\mathbf{w} = \frac{\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{P}_n^{-1}}{\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{P}_n^{-1}\boldsymbol{\Phi}}\mathbf{P}_n\frac{\mathbf{P}_n^{-1}\boldsymbol{\Phi}}{\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{P}_n^{-1}\boldsymbol{\Phi}} = \frac{1}{\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{P}_n^{-1}\boldsymbol{\Phi}} \geq \frac{\|\boldsymbol{\Phi}\|^2}{\lambda_{\max}}, \tag{4.14}$$

where $\lambda_{\max}$ is the maximal (Perron) eigenvalue of $\mathbf{P}_n^{-1}$. The required infimum with respect to $\boldsymbol{\Phi}$ now occurs when equality occurs above, i.e. when $\boldsymbol{\Phi} = \boldsymbol{\Phi}_{\mathrm{opt}}$, the Perron-eigenvector of $\mathbf{P}_n^{-1}$. Then (from (4.12))

$$\hat{\mathbf{w}} = \frac{\mathbf{P}_n^{-1}\boldsymbol{\Phi}_{opt}}{\left(\boldsymbol{\Phi}_{opt}^{\mathrm{T}}\mathbf{P}_n^{-1}\boldsymbol{\Phi}_{opt}\right)} = \frac{\boldsymbol{\Phi}_{opt}}{\|\boldsymbol{\Phi}_{opt}\|^2}. \tag{4.15}$$

□

Note that the Perron-eigenvector for $\mathbf{P}_n^{-1}$ is also an eigenvector associated with the minimal eigenvalue of $\mathbf{P}_n$. This can be used to avoid computation of $\mathbf{P}_n^{-1}$, if desired. In formula (4.15), the vector $\hat{\mathbf{w}}$ represents the optimal solution of the linear problem associated with the minimization problem (4.3).

4.3 The Nonlinear Problem

First a new and convenient reformulation of the objective function (4.3) is obtained.

Proposition 4.1 *For any set of positive weights,* $w_1, w_2, \ldots, w_n$*, with* $\mathbf{w}^{-\mathrm{T}} = \left[w_1^{-1}, \ldots, w_n^{-1}\right]$*, the error* $S^2(\mathbf{w})$ *of equation (4.3) has the form*

$$S^2(\mathbf{w}) = \|\mathbf{A}\|_{\mathrm{F}}^2 - 2\left(\mathbf{w}^{-\mathrm{T}}\mathbf{A}\mathbf{w}\right) + \left\|\mathbf{w}^{-\mathrm{T}}\right\|_{\mathrm{F}}^2 \|\mathbf{w}\|_{\mathrm{F}}^2. \tag{4.16}$$

Proof. Write

$$\begin{aligned} S^2(\mathbf{w}) &= \operatorname{tr}\left\{\left(\mathbf{A} - \mathbf{W}^{-1}\mathbf{E}\mathbf{W}\right)^{\mathrm{T}}\left(\mathbf{A} - \mathbf{W}^{-1}\mathbf{E}\mathbf{W}\right)\right\}, \\ &= \operatorname{tr}\left(\mathbf{A}^{\mathrm{T}}\mathbf{A}\right) - \operatorname{tr}\left(\mathbf{W}^{-1}\mathbf{E}\mathbf{W}\mathbf{A}^{\mathrm{T}}\right) - \operatorname{tr}\left(\mathbf{A}\mathbf{W}\mathbf{E}\mathbf{W}^{-1}\right) + \\ &\qquad + \operatorname{tr}\left(\mathbf{W}^{-1}\mathbf{E}\mathbf{W}^2\mathbf{E}\mathbf{W}^{-1}\right), \end{aligned} \tag{4.17}$$

and apply the technique of proof used for Lemma 1. □

The necessary conditions for a stationary value of the error functional S^2 require the first derivatives of S^2 with respect to the w_j. A direct computation with (4.3) produces the following result:

Proposition 4.2

$$\begin{bmatrix} \dfrac{\partial\left(S^2\right)}{\partial w_1} \\ \vdots \\ \dfrac{\partial\left(S^2\right)}{\partial w_n} \end{bmatrix} = 2\mathbf{R}(\mathbf{w})\mathbf{w}, \tag{4.18}$$

where

$$\mathbf{R}(\mathbf{w}) = \mathbf{W}^{-2}\left(\mathbf{A} - \mathbf{W}^{-1}\mathbf{E}\mathbf{W}\right) - \left(\mathbf{A} - \mathbf{W}^{-1}\mathbf{E}\mathbf{W}\right)^{\mathrm{T}}\mathbf{W}^{-2}. \tag{4.19}$$

In Farkas, Lancaster and Rózsa (2005) we have shown that expression (4.19) is more generally useful: it can be used in the approximation of *merely positive* $\mathbf{A} > 0$ matrices by transitive matrices.

It is apparent from Eq. (4.19) that matrix $\mathbf{R}(\mathbf{w})$ is skew-symmetric. Thus:

Corollary 4.1 *A stationary value,* $\mathbf{w}$*, of* S^2 *satisfies the homogeneous nonlinear equation*

$$\mathbf{R}(\mathbf{w})\mathbf{w} = 0, \tag{4.20}$$

and $\mathbf{R}(\mathbf{w})$ *is a real skew-symmetric matrix.*

In its usual form, the N-K method cannot be applied directly to find the solution of equation (4.20), because it is homogeneous. This apparent difficulty is easily circumvented, since the following proposition shows that any one of the n scalar equations represented by Eq. (4.20) can be dropped without affecting the solution set. Here, it is reasonable to denote the jth row of any matrix $\mathbf{M}$ by $\mathbf{M}_{j*}$.

Proposition 4.3 *Let* $\mathbf{c} \in R^n$ *be nonzero and let* (4.20) *have a positive solution* $\mathbf{w}$ *normalized so that* $\mathbf{c}^{\mathrm{T}}\mathbf{w} = 1$*. Then, for any* j*,* $1 \le j \le n$*,* $\mathbf{w}$ *is a solution of the inhomogeneous system of* n *equations:*

$$\mathbf{c}^{\mathrm{T}}\mathbf{w} = 1, \qquad \mathbf{R}_{k*}(\mathbf{w})\mathbf{w} = 0, \quad k \ne j, \quad 1 \le k \le n. \tag{4.21}$$

Proof. Since $\mathbf{R}^{\mathrm{T}} = -\mathbf{R}$, it follows from (4.20) that $\mathbf{w}^{\mathrm{T}}\mathbf{R}(\mathbf{w}) = 0$ and hence

$$\mathbf{w}_j\mathbf{R}_{j*}(\mathbf{w}) = -\sum_{k \ne j}\mathbf{w}_k\mathbf{R}_{k*}(\mathbf{w}). \tag{4.22}$$

Since $\mathbf{w}$ is element-wise positive, $\mathbf{R}_{k*}(\mathbf{w})\mathbf{w} = 0$ for all $k \ne j$ together with $w_j \ne 0$ yields $\mathbf{R}_{j*}(\mathbf{w})\mathbf{w} = 0$. □

In the light of the last result the minimization problem may be solved by applying the N-K procedure to any inhomogeneous system of form (4.21). In the applications with pairwise comparison matrices having positive elements it is convenient that $j = 1$ (the first nonlinear equation is dropped), and $\mathbf{c}^{\mathrm{T}} = [1, 0, \ldots, 0]$, i.e. the normalization condition is $w_1 = 1$. We remark that authors' computations with different choices of $\mathbf{c}$ and j have always led to scalar multiples of the same solution of the stationary vector $\mathbf{w}^*$. Consequently, the same rank-one approximating matrix is determined. This experience supports the conjecture that, in these cases, $\mathbf{B}$ is associated with a global minimum of S^2.

Application of the N-K method now requires to determine the Jacobian matrix, say $\mathbf{M}$, of the system (4.21). Most of the entries are, of course, second derivatives of S^2. The following proposition can be established by direct computation:

Proposition 4.4 *If $j = 1$ in Proposition* (4.3)*, then:*

- *For $k = 1, 2, \ldots, n$, $\quad m_{1k} = c_k$.*
- *For $i = 2, \ldots, n$ $\quad$ and $\quad k = 1, 2, \ldots, n$,*

$$m_{ii} = \frac{\partial^2 (S^2)}{\partial w_i^2} = -\frac{2}{w_i^3}\sum_{j=1}^{n} a_{ij} w_j - \frac{2}{w_i^2} + \sum_{j=1}^{n} \frac{1}{w_j^2} + \frac{3}{w_i^4}\sum_{j=1}^{n} w_j^2, \tag{4.23}$$

- *and, when $i \neq k$,*

$$m_{ik} = \frac{\partial^2 (S^2)}{\partial w_i \partial w_k} = \frac{a_{ik}}{w_i^2} + \frac{a_{ki}}{w_k^2} - 2\left(\frac{w_i}{w_k^3} + \frac{w_k}{w_i^3}\right). \tag{4.24}$$

The sufficient condition of the existence of a stationary value requires to compute the Hessian matrix, $\mathbf{H}$, whose elements are the derivatives of (4.23) and (4.24). Our computational experience has shown that, in every case $\mathbf{H}$ was found to be positive definite, thus ensuring that each stationary value computed was a *local minimum*. It is well-known that $S^2(\mathbf{w})$ does not necessarily have a unique (positive) global minimum. Due to the non-convex nature of the least-squares optimization problems, the question of *non-uniqueness* should be raised. In our approach, non-uniqueness would be reflected in multiple linearly independent solutions of (4.20). By assigning different vectors $\mathbf{c}$ and different values of j in Propositions 4.3 and 4.4, (as well as different starting vectors) it may be possible to search for multiple solutions, if this thought to be necessary. It should be noted that we always started the N-K iteration with the optimal linear approximation $\hat{\mathbf{w}}$ of Theorem 4.1 as given in (4.15), and *convergence* of the process necessarily ensued. Such a proof would, of course, require the results of this section. However, our numerical experiments have shown that this strategy always determined a convergent process.

4.3.1 Numerical Illustration

This example comes from Saaty [70, p. 227] and concerns five primary criteria for the major economic indicators of national welfare: inflation, unemployment, growth, domestic stability and foreign relations. Here, Saaty's nine-point scale is used (see Table 3, Chapter 2). The response matrix, $\mathbf{A}$ is:

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 5 & 4 & 6 \\ 1/3 & 1 & 4 & 4 & 6 \\ 1/5 & 1/4 & 1 & 2 & 2 \\ 1/4 & 1/4 & 1/2 & 1 & 2 \\ 1/6 & 1/6 & 1/2 & 1/2 & 1 \end{bmatrix}.$$

We first compare the sub-optimal $\hat{\mathbf{w}}$ of Theorem 4.1 with the optimal $\mathbf{w}^*$ (local minimum) and the right Perron-eigenvector of $\mathbf{A}$. Eigenvector components are normalized so that their sums equal one.

$$\hat{\mathbf{w}}^{-1} = [0.4048, 0.3391, 0.1164, 0.0879, 0.0519].$$
$$(\mathbf{w}^*)^{-1} = [0.4027, 0.3531, 0.0895, 0.0929, 0.0617].$$
$$\mathbf{u}^{\mathrm{T}} = [0.4767, 0.2865, 0.1029, 0.0819, 0.0520].$$

The three methods predict the same ordering of alternatives (indicators), but they produce different values for the weights. Compare now the errors $S(\mathbf{w}) = \|\mathbf{A} - \mathbf{W}^{-1}\mathbf{E}\mathbf{W}\|_{\mathrm{F}}$ obtained with four different approximations to $\mathbf{A}$ with transitive rank-one matrices; which are associated with the right Perron-eigenvector of $\mathbf{A}$, with the linear problem set up here (Theorem 4.1 and Eq. (4.15)), by Chu et al. [12] and with the nonlinear problem (Propositions (4.3) and (4.4)). They are: 4.2628, 3.4117, 3.3199 and 2.5470, respectively. The largest error is obtained with the eigenvector estimate (this is not surprising since its fundamental principle is *not* related to an optimization criterion) while our approach produces the least error.

The best transitive rank-one approximation to $\mathbf{A}$ itself in a least-square sense, is the matrix:

$$\widetilde{\mathbf{B}} = \begin{bmatrix} 1 & 1.1405 & 4.4981 & 4.3356 & 6.5211 \\ 0.8768 & 1 & 3.9438 & 3.8013 & 5.7176 \\ 0.2223 & 0.2536 & 1 & 0.9639 & 1.4497 \\ 0.2306 & 0.2631 & 1.0375 & 1 & 1.5041 \\ 0.1533 & 0.1749 & 0.6898 & 0.6649 & 1 \end{bmatrix}.$$

4.4 On the Non-uniqueness of the Solution to the Nonlinear Problem

In the paper of Farkas and Rózsa (2004), *sufficient* conditions for the occurrence of multiple solutions to the inhomogeneous system of n equations (4.21) were given. The following matrices play an important role in this subject matter:

Definition 4.1 An $n \times n$ matrix $\mathbf{Z} = [z_{ij}]$, $i, j = 1, 2, \ldots, n$, is said to be *persymmetric* if its entries satisfy $z_{ij} = z_{n+1-j,n+1-i}$, i.e., if its entries are symmetric about the counter (secondary) diagonal.

Definition 4.2 An $n \times n$ matrix $\mathbf{P}_n$ is called a *permutation* matrix and is described by $\mathbf{P}_n = [\mathbf{e}_{j_1} \quad \mathbf{e}_{j_2} \ldots \mathbf{e}_{j_n}]$ where the n numbers in the indices, $p = (j_1\ j_2 \ldots j_n)$, indicate a particular permutation from the standard order of the numbers $1, 2, \ldots, n$ and $\mathbf{P}_n^{\mathrm{T}}\mathbf{P}_n = \mathbf{I}_n$.

Definition 4.3 An $n \times n$ matrix $\mathbf{M} = [m_{ij}]$, $i, j = 1, 2, \ldots, n$, is called a *symmetric permutation invariant* (SPI) matrix if there exists an $n \times n$ permutation matrix $\mathbf{P}_n$ such that $\mathbf{P}_n^{\mathrm{T}}\mathbf{M}\mathbf{P}_n = \mathbf{M}$ is satisfied.

Definition 4.4 An $n \times n$ matrix $\mathbf{C} = [c_{jk}]$, $j, k = 1, 2, \ldots, n$, is said a *circulant matrix*, or *circulant* for short, $\mathbf{C} = \mathrm{circ}\,[c_{11}, c_{12}, \ldots, c_{1n}]$ and

$$c_{jk} = \begin{cases} c_{1,k+1-j}, & \text{if} \quad j \le k, \\ c_{1,n+k+1-j}, & \text{if} \quad j > k. \end{cases}$$

Definition 4.5 The special $n \times n$ permutation matrix $\mathbf{\Omega}_1$ of the form

$$\mathbf{\Omega}_1 = \mathrm{circ}\,[\mathbf{e}_n \quad \mathbf{e}_1 \quad \mathbf{e}_2 \ldots \mathbf{e}_{n-1}]$$

is said to be the *elementary (primitive) circulant* matrix, i.e. $\mathbf{\Omega}_1 = \mathrm{circ}\,[0, 1, 0, \ldots, 0]$. The other $n \times n$ *circulant* permutation matrices $\mathbf{\Omega}_k$ of the form $\mathbf{\Omega}_k = [\mathbf{e}_{n-k+1} \quad \mathbf{e}_{n-k+2} \ldots \mathbf{e}_n \quad \mathbf{e}_1 \ldots \mathbf{e}_{n-k}]$, $k = 1, 2, \ldots, n$, are composed of the powers of matrix $\mathbf{\Omega}_1$.

Notice here that the relation $\mathbf{\Omega}_k = \mathbf{\Omega}_1^k$, holds for all $k = 1, \ldots, n-1$, and, obviously, $\mathbf{\Omega}_1^n = \boldsymbol{I}_n$. Furthermore, a circulant $\mathbf{C}$ is invariant to a cyclic (simultaneous) permutation of the rows and the columns, hence $\mathbf{\Omega}_k^{\mathrm{T}} \mathbf{C} \mathbf{\Omega}_k = \mathbf{C}$, $k = 1, \ldots, n-1$. Thus, by Definition 4.3, any circulant matrix is an SPI matrix. Also, it can be readily shown that a circulant $\mathbf{C}$ may be expressed as a polynomial of the matrix $\mathbf{\Omega}_1$ in the form

$$\mathbf{C} = c_{11}\mathbf{I}_n + c_{12}\mathbf{\Omega}1 + c_{13}\mathbf{\Omega}_1^2 + \cdots + c_{1n}\mathbf{\Omega}_1^{n-1}.$$

Definition 4.6 The special $n \times n$ permutation matrix $\mathbf{K}_n$, which has 1's on the main counter-diagonal and 0's elsewhere is called a *counteridentity* matrix.

It can be easily shown that $\mathbf{K}_n \mathbf{A} \mathbf{K}_n = \mathbf{A}^{\mathrm{T}}$ holds. In addition, $\mathbf{\Omega}_k$ and $\mathbf{K}_n$ are persymmetric matrices.

Proposition 4.5 *Let $\mathbf{A} = [a_{ij}]$ be an $n \times n$ SR matrix with positive entries. Let a (positive) stationary vector of the error functional* (4.3) *be derived and denoted by $\mathbf{w}^*$. If $\mathbf{A}$ is a symmetric permutation invariant (SPI) matrix to a certain permutation matrix $\mathbf{P}_n$, then $\mathbf{P}_n^{\mathrm{T}}\mathbf{w}^*$ produces an alternate stationary vector, provided that $\mathbf{P}_n^{\mathrm{T}}\mathbf{w}^*$ and $\mathbf{w}^*$ are linearly independent. If this permutation is consecutively repeated (not more than n times over), then the vectors, $\mathbf{P}_n^{\mathrm{T}}\mathbf{w}^*, \mathbf{P}_n^{\mathrm{T}^2}\mathbf{w}^*, \mathbf{P}_n^{\mathrm{T}^3}\mathbf{w}^*, \ldots$ represent alternate stationary vectors provided that they are linearly independent.*

Proof. Write the Frobenius norm of the nonlinear LS optimization problem (4.3) in the form

$$S^2(\mathbf{w}) = \left\|\mathbf{A} - \mathbf{W}^{-1}\mathbf{e}\mathbf{e}^{\mathrm{T}}\mathbf{W}\right\|_{\mathrm{F}}^2. \tag{4.25}$$

Let $\mathbf{P}_n$ be an arbitrary $n \times n$ permutation matrix. Considering the fact that the sum of squares of the elements of a matrix is not affected by any permutation of the rows and the columns, the Frobenius norm does not vary by postmultiplying the matrix $\left(\mathbf{A} - \mathbf{W}^{-1}\mathbf{e}\mathbf{e}^{\mathrm{T}}\mathbf{W}\right)$ by $\mathbf{P}_n$, and then by premultiplying it by its transpose $\mathbf{P}_n^{\mathrm{T}}$. Therefore,

$$\begin{aligned} S^2(\mathbf{w}) &= \left\|\mathbf{P}_n^{\mathrm{T}}\left(\mathbf{A} - \mathbf{W}^{-1}\mathbf{e}\mathbf{e}^{\mathrm{T}}\mathbf{W}\right)\mathbf{P}_n\right\|_{\mathrm{F}}^2 = \\ &= \left\|\mathbf{P}_n^{\mathrm{T}}\mathbf{A}\mathbf{P}_n - \mathbf{P}_n^{\mathrm{T}}\mathbf{W}^{-1}\mathbf{P}_n\mathbf{P}_n^{\mathrm{T}}\mathbf{e}\mathbf{e}^{\mathrm{T}}\mathbf{P}_n\mathbf{P}_n^{\mathrm{T}}\mathbf{W}\mathbf{P}_n\right\|_{\mathrm{F}}^2. \end{aligned} \tag{4.26}$$

Observe that in (4.26), $\mathbf{P}_n^{\mathrm{T}}\mathbf{e} = \mathbf{e}$ and $\mathbf{e}^{\mathrm{T}}\mathbf{P}_n = \mathbf{e}^{\mathrm{T}}$. Since, for an SPI matrix $\mathbf{A}$, $\mathbf{P}_n^{\mathrm{T}}\mathbf{A}\mathbf{P}_n = \mathbf{A}$ holds, hence

$$S^2(\mathbf{w}) = \left\|\mathbf{A} - \mathbf{P}_n^{\mathrm{T}}\mathbf{W}^{-1}\mathbf{P}_n\mathbf{e}\mathbf{e}^{\mathrm{T}}\mathbf{P}_n^{\mathrm{T}}\mathbf{W}\mathbf{P}_n\right\|_{\mathrm{F}}^2. \tag{4.27}$$

In (4.27), the terms $\mathbf{P}_n^{\mathrm{T}}\mathbf{W}\mathbf{P}_n$ and $\mathbf{P}_n^{\mathrm{T}}\mathbf{W}^{-1}\mathbf{P}_n$ represent the permutations of the elements of $\mathbf{W}$ and $\mathbf{W}^{-1}$, respectively. After they have been permuted by the permutation matrix $\mathbf{P}_n = [\mathbf{e}_{j_1} \quad \mathbf{e}_{j_2} \ldots \mathbf{e}_{j_n}]$, the elements of $\mathbf{P}_n^{\mathrm{T}}\mathbf{W}\mathbf{P}_n$ (and the elements of $\mathbf{P}_n^{\mathrm{T}}\mathbf{W}^{-1}\mathbf{P}_n$) are: $w_{j_1}, w_{j_2}, \ldots, w_{j_n}$ (and their inverses). If the derived stationary vector, $\mathbf{w}^*$ is linearly independent of the vector $\mathbf{P}_n^{\mathrm{T}}\mathbf{w}^*$, i.e., if $\mathbf{P}_n^{\mathrm{T}}\mathbf{w}^* \neq c\mathbf{w}^*$, where c is an arbitrary constant, then

$$\mathbf{P}_n\mathbf{W}\mathbf{P}_n\mathbf{e} = \mathbf{P}_n^{\mathrm{T}}\mathbf{W}\mathbf{e}$$

becomes an alternate stationary vector. By repeating this procedure we may get

$$\mathbf{P}_n^{\mathrm{T}^2}\mathbf{W}\mathbf{e},$$

which constitutes an other stationary vector, provided that this solution is linearly independent of both of the previous solutions. This way, the process can be continued as long as new linearly independent solutions are obtained. This completes the proof. □

Corollary 4.2 *If an $n \times n$ SR matrix $\mathbf{A}$ is a circulant matrix as defined in Definitions* 4.4 *and* 4.5*, then its factorization consists of one cycle of full length by the circulant permutations, $\mathbf{\Omega}_k\mathbf{w}^*$, $k = 1, 2, \ldots, n$, (i.e. if $\mathbf{P}_n^{\mathrm{T}}$ is an elementary circulant matrix that can be written in the form of a block diagonal matrix whose entries are the powers, s_i, $i = 1, 2, \ldots, r$, so that $\sum_{i=1}^{r} s_i = n$, of $\mathbf{\Omega}_k$) and the total number of alternate stationary vectors of the nonlinear inhomogeneous system of n equations given by* (4.21) *is n.*

Corollary 4.3 *Let $\tilde{\mathbf{A}}$ be an $n \times n$ positive SR matrix whose rows and columns have been appropriately rearranged to be an SPI matrix. Let a (positive) stationary vector of the error functional* (4.3) *be determined and denoted by $\mathbf{w}^*$. Then the permutations $\mathbf{P}_n^{\mathrm{T}}\mathbf{w}^*, \mathbf{P}_n^{\mathrm{T}^2}\mathbf{w}^*, \mathbf{P}_n^{\mathrm{T}^3}\mathbf{w}^*, \ldots$ are also solutions, where $\mathbf{P}_n^{\mathrm{T}}$ is defined by Definition* 4.2*. The total number of the alternate stationary vectors as solutions to problem* (4.3) *cannot exceed the least common multiple of $s_1, s_2, \ldots, s_r$.*

Proofs of Corollaries 4.2 and 4.3 are presented in Farkas, Lancaster and Rózsa (2005, p. 1038).

Proposition 4.6 *Let $\mathbf{A} = [a_{ij}]$ be an $n \times n$ SR matrix with positive entries. Let a (positive) stationary vector of the error functional* (4.3) *be determined and let this solution of Eq.* (4.21) *be denoted by $\mathbf{w}^{*\mathrm{T}(1)} = \left[1, w_2^{*(1)}, \ldots, w_n^{*(1)}\right]$. If $\mathbf{A}$ is a persymmetric matrix as of in Definition* 4.1*, then*

$$\mathbf{w}^{*\mathrm{T}(2)} = \left[\frac{1}{w_{n-1}^{*(1)}}, \frac{1}{w_{n-2}^{*(1)}}, \ldots, 1\right] \tag{4.28}$$

is an alternate stationary vector as an other solution of equation (4.21)*, provided that the latter solution $\mathbf{w}^{*(2)}$ is linearly independent of $\mathbf{w}^{*(1)}$, i.e., if*

$$w_n^{*(1)} \neq \left(w_i^{*(1)}\right)\left(w_{n+1-i}^{*(1)}\right), \qquad i = 1, 2, \ldots, n. \tag{4.29}$$

Proof. Write the Frobenius norm of the nonlinear LS optimization problem (4.3) in the form

$$S^2(\mathbf{w}) = \left\|\mathbf{A} - \mathbf{W}^{-1}\mathbf{e}\mathbf{e}^{\mathrm{T}}\mathbf{W}\right\|_{\mathrm{F}}^2. \tag{4.30}$$

Consider the $n \times n$ counteridentity (permutation) matrix, $\mathbf{K}_n$ as defined in Definition 4.6. Since $\mathbf{K}_n$ is an involutory matrix, therefore, $\mathbf{K}_n^2 = \mathbf{I}_n$. Let $\mathbf{P}_n$ be an arbitrary $n \times n$ permutation matrix. Recognize that $\mathbf{I}_n = \mathbf{P}_n\mathbf{P}_n^{\mathrm{T}} = \mathbf{P}_n\mathbf{K}_n\mathbf{K}_n\mathbf{P}_n^{\mathrm{T}}$. Now apply the same technique that was used for the proof of Proposition 4.5. Thus, one may write that

$$\begin{aligned} S^2(\mathbf{w}) &= \left\|\mathbf{K}_n\mathbf{A}\mathbf{K}_n - \mathbf{K}_n\mathbf{W}^{-1}\mathbf{K}_n\mathbf{e}\mathbf{e}^{\mathrm{T}}\mathbf{K}_n\mathbf{W}\mathbf{K}_n\right\|_{\mathrm{F}}^2 = \\ &= \left\|\mathbf{A}^{\mathrm{T}} - \mathbf{K}_n\mathbf{W}^{-1}\mathbf{K}_n\mathbf{e}\mathbf{e}^{\mathrm{T}}\mathbf{K}_n\mathbf{W}\mathbf{K}_n\right\|_{\mathrm{F}}^2. \end{aligned} \tag{4.31}$$

Since $\mathbf{K}_n\mathbf{A}\mathbf{K}_n = \mathbf{A}^{\mathrm{T}}$, the transpose of the matrix in the right hand side of (4.31) is, therefore,

$$S^2(\mathbf{w}) = \left\|\mathbf{A} - \mathbf{K}_n\mathbf{W}\mathbf{K}_n\mathbf{e}\mathbf{e}^{\mathrm{T}}\mathbf{K}_n\mathbf{W}^{-1}\mathbf{K}_n\right\|_{\mathrm{F}}^2. \tag{4.32}$$

It is obvious from (4.32) that the elements of the matrix $\mathbf{K}_n\mathbf{W}^{-1}\mathbf{K}_n$ are composed of the elements of a vector $\mathbf{w}^{*(2)}$, which also constitutes a stationary vector. If this solution is linearly independent of $\mathbf{w}^{*(1)}$, then it must represent an alternate stationary vector as the entries of $\mathbf{K}_n\mathbf{W}^{-1}\mathbf{K}_n$ are: $\frac{1}{w_{n-1}^{*(1)}}, \frac{1}{w_{n-2}^{*(1)}}, \ldots, 1$. If (4.29) is satisfied, then these entries are linearly independent. □

Corollary 4.4 Suppose that for the stationary vector $\mathbf{w}^{*(1)}$ the equality

$$\left[1, \frac{w_n^{*(1)}}{w_{n-1}^{*(1)}}, \frac{w_n^{*(1)}}{w_{n-2}^{*(1)}}, \ldots, w_n^{*(1)}\right] = \left[1, w_2^{*(1)}, w_3^{*(1)}, \ldots, w_n^{*(1)}\right] \tag{4.33}$$

is satisfied, i.e., the relation

$$w_n^{*(1)} = \left(w_i^{*(1)}\right)\left(w_{n+1-i}^{*(1)}\right), \qquad i = 1, 2, \ldots, n, \tag{4.34}$$

holds. Then, (4.33) provides *one* solution to the nonlinear optimization problem (4.3). In this case no trivial alternate stationary vector can be found. It should be noted however, that one might not call this solution a *unique* solution until the *necessary* conditions for the non-uniqueness problem of the solution of equation (4.21) have not been found, because, at this point, the existence of an other stationary value cannot be excluded.

Summarizing the results obtained in this section the next theorem gives *sufficient* conditions for the occurrence of a *non-unique* stationary vector of $S^2(\mathbf{w})$ given by (4.3) as a solution to equation (4.21).

Theorem 4.2 *Let* $\mathbf{A}$ *be an* $n \times n$ *SR matrix with positive entries, then*

(i) *If* $\mathbf{A}$ *is a circulant matrix, or*

(ii) *if* $\mathbf{A} = [\mathbf{A}_{ij}]$ *is a block SR matrix with* $s_i \times s_j$ *blocks, where the* $\mathbf{A}_{ii}$*'s are circulant SR matrices, furthermore, for* $i \neq j$*, and* $s_i = s_j$ *the* $\tilde{\mathbf{A}}_{ij}$ *are circulant matrices with equal entries for* $i \neq j$, $s_i \neq s_j$ *whose rows and columns were simultaneously rearranged such that for each block satisfies* $\tilde{\mathbf{A}}_{ij} = \tilde{\mathbf{A}}_{ij}^{\mathrm{T}}$ *and* $\mathbf{\Omega}^{(s_i)\mathrm{T}}\,\tilde{\mathbf{A}}_{ij}\mathbf{\Omega}^{(s_i)} = \tilde{\mathbf{A}}_{ij}$*, where* $i, j = 1, \ldots, r$*, or*

(iii) *if* $\mathbf{A}$ *is a persymmetric matrix, and for a given solution the relation*

$$w_i^* \neq \frac{w_n^*}{w_{n+1-i}^*}, \qquad i = 1, 2, \ldots, n, \tag{4.35}$$

is satisfied, and

(iv) *if a (positive) solution to equation* (4.21)*, under the condition (i), or (ii), or (iii), represents a stationary vector* $\mathbf{w}^* = [1, w_2^*, \ldots, w_n^*]$ *(a local minimum),*

then this solution $\mathbf{w}^*$ *of the nonlinear optimization problem* (4.3) *is a non-unique stationary point.*

We remark that the likelihood of occurrences of the cases (i)-(iv) is infinitesimally small in practice.

4.5 SR Matrices with perturbations of Exponential Type

Utilizing the results obtained for the nonlinear problem discussed in section 4.2, Farkas, György and Rózsa (2004) proposed a flexible tool to circumvent or at least moderate the inherent difficulties in the AHP. For this purpose, a two parameter exponential type multiplicative perturbation factor was introduced at each element of a so constructed positive SR matrix $\hat{\mathbf{A}} = [\hat{a}_{pq}]$ with the entries:

$$\hat{a}_{pq} = \frac{\grave{w}_q^*}{\grave{w}_p^*} \exp\{\alpha\delta_{pq}\}, \qquad p, q = 0, 1, 2, \ldots, n-1, \tag{4.36}$$

where

(i) The $\grave{w}_q^{*-1}$'s represent the inverses of the stationary values of the weights (as solutions to the LS minimization problem (4.3)) arranged in a descending order with respect to their magnitudes, i.e., $\grave{w}_0^{*-1} \geq \grave{w}_1^{*-1} \geq \cdots \geq \grave{w}_{n-1}^{*-1}$. The generated rank-one matrix $\grave{\mathbf{B}} = \left[\grave{w}_q^*/\grave{w}_p^*\right]$ is the 'best' transitive matrix approximation to $\grave{\mathbf{A}} = \left[\grave{a}_{pq}\right]$ in a least-squares sense matrix, where $\grave{\mathbf{A}}$ is obtained from columns (and rows) of the original SR matrix $\mathbf{A}$ by rearranging them in descending array with respect to the magnitudes of the initial Perron-eigenvector components w_{p0} in order to get the components $\grave{w}_{p0}$.

(ii) The δ_{pq}'s, $p, q = 0, 1, \ldots, n-1$, are the entries of an $n \times n$ perturbation matrix $\boldsymbol{\Delta} = [\delta_{pq}]$ and represent monotonic perturbations with the notation

$$\delta_{pq} = \begin{cases} -(m-q+p), & \text{for } q-p>0, \\ m+q-p, & \text{for } q-p<0, \quad p,q=0,1,2,\ldots,n-1, \\ 0, & \text{for } q=p, \end{cases} \tag{4.37}$$

where parameter $m \geq 1$ is an arbitrary positive integer of adjusting $\grave{\mathbf{w}}^* = \left[\grave{w}_q^*\right]$, $q = 0, 1, 2, \ldots, n-1$, through a transformation of the weights (i.e. the priority ranking) of the alternatives in terms of direction and intensity. The properties of the exponential function with the chosen values for m in (4.37) and thus in (4.36), i.e. the data perturbation of the transitive matrix $\grave{\mathbf{B}}$, affects the magnitude of the components of the Perron-eigenvector (modified weights) of the resulted matrix $\hat{\mathbf{A}}$ (and of $\grave{\mathbf{A}}$). If $n > 2m$, then each pairwise ratio of the modified weights increases as compared to those of that can be formed from the $\grave{\mathbf{w}}^*$'s. As a result, the scores of the alternatives in the priority ranking are getting farther from each other. Therefore, we call this transformation of the weights $\grave{w}_q^*$ a *stretching*. If $n < 2m$, then each possible pairwise ratio of the modified weights decreases as compared to those that can be formed from the $\grave{\mathbf{w}}^*$'s. As a result, the scores of the alternatives in the priority ranking are getting closer to each other. Therefore, we call such a transformation of the weights $\grave{w}_q^*$ a *shrinking*. If $n = 2m$, then model (4.36) leaves the weights $\grave{w}_q^{*-1}$ unchanged. Latter case may only occur if n is even.

(iii) The scalar α is called factor of magnification. Proper settings for α are $0 < \alpha \leq 1$ in the AHP. The choice for α determines the strength of the transformation for adjusting the magnitude of the weights of the alternatives. A greater value of α selected by the user, results in the rise of inconsistency of matrix $\hat{\mathbf{A}}$ and also a larger rate of increase of the pairwise ratios. A smaller value of α can be used to investigate the sensitivity of the Perron-eigenvector of $\grave{\mathbf{B}}$ subject to slight or moderate SR perturbations. The values of α may assume imaginary numbers as well (see Application 7).

Using (4.36) and (4.37) matrix $\hat{\mathbf{A}}$ can be written in the form of a Hadamard product as

$$\hat{\mathbf{A}} = \left[\frac{\grave{w}_q^*}{\grave{w}_p^*}\right] \circ \left[\frac{\exp\{q\alpha\}}{\exp\{p\alpha\}}\right] \circ \exp\{-m\,\mathrm{sign}\,(q-p)\alpha\} = \grave{\mathbf{B}} \circ \mathbf{Z} \circ \mathbf{S},\ p,q=0,1,\ldots,n-1, \tag{4.38}$$

where $\grave{\hat{\mathbf{B}}} = \left[\dfrac{\grave{w}_q^*}{\grave{w}_p^*}\right]$ and $\mathbf{Z} = [z_{pq}] = \left[\dfrac{\exp\{q\alpha\}}{\exp\{p\alpha\}}\right]$ are transitive and $\mathbf{S} = \left[s^{-\mathrm{sign}\,(q-p)}\right]$, with $s = \exp\{m\alpha\}$.

Let $\phi = \varphi + \mathrm{i}\psi$ denote an arbitirary complex number. We now utilize pseudo-circulant matrices $\mathbf{\Omega}(\phi)$ which are extensions of the elementary (primitive) circulant matrices $\mathbf{\Omega}_1$ defined by Definition 4.5.

In order to decompose $\mathbf{S}$, we write it as a polynomial of a primitive pseudo-circulant matrix, introduced as $\mathbf{\Omega}(s) = [s^2\mathbf{e}_n \quad \mathbf{e}_1 \quad \mathbf{e}_2 \ \ldots \ \mathbf{e}_{n-1}]$, where $s^2 = \exp\boldsymbol{\phi}$, $\boldsymbol{\phi} \neq 0$. Thus, we get

$$\mathbf{S} = \mathbf{I}_n + \frac{1}{s}\sum_{\nu=1}^{n-1}\mathbf{\Omega}^{\nu}(s). \tag{4.39}$$

Observe that the spectral decomposition of $\hat{\mathbf{A}}$ can be traced back to the spectral decomposition of $\mathbf{S}$. Hence, from the eigenvalues $\lambda_k(\mathbf{\Omega})$, the eigenvalues $\lambda_k(\mathbf{S})$ can be obtained as

$$\lambda_k(\mathbf{S}) = 1 + \frac{1}{s}\frac{\lambda_k(\mathbf{\Omega}) - \lambda_k\left(\mathbf{\Omega}^n\right)}{1 - \lambda_k(\mathbf{\Omega})}, \qquad k = 0, 1, 2, \ldots, n-1,$$

where, since $s^2 = \exp\boldsymbol{\phi}$, $\boldsymbol{\phi} \neq 0$, $\lambda_k(\mathbf{\Omega}) \neq 1$. Namely, in the trivial case of $\boldsymbol{\phi} = 0$, i.e. whenever $s^2 = 1$, then from (4.39), $\mathbf{S} = \mathbf{e}\mathbf{e}^{\mathrm{T}}$. Thus, by (4.38), in this trivial case $\hat{\mathbf{A}} = {}`\mathbf{B}$. Substituting $s^2 = \exp\boldsymbol{\phi}$, some algebraic manipulations yield the eigenvalues $\lambda_k(\mathbf{S})$ in the form:

$$\lambda_k(\mathbf{S}) = 1 - \frac{\exp\left\{-\alpha\frac{2m-1}{2} + \mathrm{i}\frac{k}{2}\frac{\pi}{m}\right\} - \exp\left\{\alpha\frac{2m-1}{2}\mathrm{i}\frac{k}{2}\frac{\pi}{m}\right\}}{\exp\left\{\frac{\alpha}{2} + \mathrm{i}\frac{k}{2}\frac{\pi}{m}\right\} - \exp\left\{-\frac{\alpha}{2} - \mathrm{i}\frac{k}{2}\frac{\pi}{m}\right\}}, \tag{4.40}$$

where $k = 0, 1, 2, \ldots, n-1$. The spectral decomposition of $\hat{\mathbf{A}}$ is obtained as

$$\begin{aligned}\hat{\mathbf{A}} = \left\langle\frac{1}{`w_p^*\exp\{p\alpha\}}\right\rangle\left\langle\exp\left\{\frac{2}{n}pm\alpha\right\}\right\rangle\left[\frac{1}{\sqrt{n}}\varepsilon_n^{pk}\right]\langle\lambda_k(\mathbf{S})\rangle\left[\frac{1}{\sqrt{n}}\varepsilon_n^{-qk}\right]\\ \left\langle\frac{1}{\exp\left\{\frac{2}{n}qm\alpha\right\}}\right\rangle\left\langle`w_q^*\exp\{q\alpha\}\right\rangle,\end{aligned} \tag{4.41}$$

where $p, q, k = 0, 1, 2, \ldots, n-1$ and $\varepsilon_n^k = \exp\left\{\mathrm{i}\frac{2k\pi}{n}\right\}$ denotes the nth roots of unity. By (4.38), (4.40) and (4.41), with $s = \exp\{m\alpha\}$, a direct computation produces the next result:

Proposition 4.7 *The eigenvalues* $\lambda_k(\hat{\mathbf{A}})$ *of matrix* $\hat{\mathbf{A}}$ *defined by* (4.36) (*being equal to the eigenvalues* $\lambda_k(\mathbf{S})$ *of the pseudo-circulant matrix* (4.39)) *are*

$$\lambda_k(\hat{\mathbf{A}}) = \lambda_k(\mathbf{S}) = 1 + \frac{\operatorname{sh}\left[\frac{m(n-1)}{n}\alpha - \mathrm{i}\frac{k\pi}{n}\right]}{\operatorname{sh}\left[\frac{m}{n}\alpha + \mathrm{i}\frac{k\pi}{n}\right]}, \qquad k = 0, 1, 2, \ldots, n-1. \tag{4.42}$$

Application 7: Generating the spectrum of perturbed input spectral density matrices

As an extension of the results in Appendix 6, suppose that there are inhomogeneities along the track (e.g. between the successive supports on a bridge, track irregularities, etc.). Then the balanced motion of the wheels is locally disturbed due to a sudden vertical displacement of the track system. Let such a bias be denoted by

Δa. Let the complex entries δ_{pq} of the perturbation matrix $\mathbf{\Delta} = [\delta_{pq}]$ be the same at all axle-bases, since we may assume that this bias produces the same geometric changes affecting identically all axle-bases, as each wheel is subject to the same impact from an external source. Thus the following SR matrix of perturbations can be constructed:

$$\mathbf{\Delta}_x(\omega) = \begin{bmatrix} 1 & e^{-i\frac{\omega}{\nu}\mathbf{\Delta}a} & e^{-i\frac{\omega}{\nu}\mathbf{\Delta}a} & e^{-i\frac{\omega}{\nu}\mathbf{\Delta}a} \\ e^{i\frac{\omega}{\nu}\mathbf{\Delta}a} & 1 & e^{-i\frac{\omega}{\nu}\mathbf{\Delta}a} & e^{-i\frac{\omega}{\nu}\mathbf{\Delta}a} \\ e^{i\frac{\omega}{\nu}\mathbf{\Delta}a} & e^{i\frac{\omega}{\nu}\mathbf{\Delta}a} & 1 & e^{-i\frac{\omega}{\nu}\mathbf{\Delta}a} \\ e^{i\frac{\omega}{\nu}\mathbf{\Delta}a} & e^{i\frac{\omega}{\nu}\mathbf{\Delta}a} & e^{i\frac{\omega}{\nu}\mathbf{\Delta}a} & 1 \end{bmatrix} \tag{4.43}$$

The effect of this type of perturbation is that the multiple-input excitation of the vehicle will no longer be linear. As it is well-known in systems dynamics the degree of the nonlinear undulations caused by *any* type of perturbation is determined by the dominant eigenvalue of the SDM on a complex plane. This issue is of interest for the designers who determine the dynamic characteristics of the bogie.

Recognize that the structure of matrix $\hat{\mathbf{A}}$ conforms well to the above problem. However, transitivity of matrix $\mathbf{S}_x(\omega)$ in (3.47) will be lost, due to the introduced perturbation. Observe that the structure of the complex perturbation matrix $\mathbf{\Delta}_x(\omega)$ in (4.43), is the same as that of the matrix $\mathbf{S}_x^{\mathrm{S}}$ given in (3.50). Form the Hadamard product for the perturbed SDM as $\mathbf{G'}_x(\omega) = \mathbf{G}_x(\omega) \circ \mathbf{\Delta}_x(\omega) = \mathbf{S}_x(\omega) \circ g_{xx}(\omega)\mathbf{\Delta}_x(\omega)$ and write s for $\boldsymbol{\phi}$ in the form of

$$s = \exp \mathrm{i} \left\{ \frac{\omega}{\nu} \Delta a \right\} = \exp \mathrm{i} \left\{ \frac{1}{2} \psi \right\}, \tag{4.44}$$

since now $\varphi = 0$. We remark that if $\varphi = 0$ and $\psi \neq 0$ and $\psi \neq \pi$, then the pseudo-circulant matrix, $\mathbf{\Omega}(\phi)$, corresponds to the class of the so-called $\{k\}$-circulant matrices, $k = 0, 1, 2, \ldots, n-1$, where $|k| = |\phi| = 1$. Substituting (4.44) into (4.42) we obtain an explicit form for the eigenvalue, $\lambda_k(\mathbf{G'}_x(\omega))$, of the input spectral density matrix as follows:

$$\lambda_k(\mathbf{G'}_x(\omega)) = g_{xx}(\omega) \left[1 + \frac{\sin\left(\frac{(n-1)}{n} \frac{\omega}{\nu} \mathbf{\Delta} a - \frac{k\pi}{n} \right)}{\sin\left(\frac{1}{n} \frac{\omega}{\nu} \mathbf{\Delta} a + \frac{k\pi}{n} \right)} \right], \qquad k = 0, 1, 2, \ldots, n-1. \tag{4.45}$$

Since under the effect of the perturbation (4.43), $\mathbf{S}_x(\omega)$ remains Hermitian, the eigenvalues of the matrix $\mathbf{G'}_x(\omega)$ (and of $\mathbf{S}_x(\omega)$) are real as it can also be seen from (4.45). For $n = 4$, the dominant eigenvalue of the input spectral density matrix is

$$\lambda_0(\mathbf{G'}_x(\omega)) = g_{xx}(\omega) \left[4 \cos^2 \frac{\omega}{4\nu} \mathbf{\Delta} a \right] \cong g_{xx}(\omega) \left[4 - \left(\frac{1}{2} \frac{\omega}{\nu} \mathbf{\Delta} a \right)^2 \right]. \tag{4.46}$$

In the right hand side of expression (4.46) the term $4g_{xx}(\omega)$ would be the only nonzero eigenvalue if the multiple-input excitation system were completely coherent. The second term gives the decay of this nonzero eigenvalue caused by the perturbation (4.43). This way, using (4.46), and by altering the magnitude of the bias Δa [cf. (4.43)], the change in the eigenvalue of largest modulus $\lambda_0(\mathbf{G'}_x(\omega))$ can be analyzed at each relevant frequency band.

For the perturbed SDM in this vehicle dynamics problem the parameters of the model (4.36), m and α, are inversely related. To see this, compare (4.45) with (4.42) and observe that

$$m\alpha = \mathrm{i} \frac{\omega}{\nu} \mathbf{\Delta} a. \tag{4.47}$$

Expression (4.47) clearly indicates that now α is a pure imaginary number consisting of continuous variables (the engineering characteristics of the dynamic vibrations of the vehicle) and m can be any positive integer. It is interesting to note that with such a choice for $m\alpha$ as is given by (4.47) the spectra of the matrices for two entirely *different* problems (occurring in decision theory and in vehicle dynamics) are equivalent with each other.

We conclude that the spectral decomposition of specific SR perturbations of exponential type of transitive matrices together with the introduction of a complex parameter in a pseudo-circulant matrix have been proven useful in the analysis for a pure imaginary case as well. In particular, we have developed the spectra (eigenvalues) of such SR matrices with nonzero complex numbers in closed forms suited to the dynamic excitation problem of running n-axle railway carriage in the frequency domain.

Chapter 5

5 Balancing SR Matrices by Transitive Matrices

In this Chapter, a recursive rank-one residue iteration (triple R-I) to balancing positive SR matrices is developed. Furthermore, it is found that this algorithm seems to be more generally useful and can be applied to merely positive matrices also. It is proven that the procedure is equivalent with a line-sum-symmetric diagonal similarity scaling. It is shown that mutual correspondence might be established in a tractable form between two different scaling methods: the eigenvector method and the least-squares method for any positive SR matrix. Convergence proof for the algorithm is given. Numerical analysis, demonstrating the theoretical results, is included.

5.1 A Recursive Rank-one Residue Iteration

Farkas and Rózsa (2013) has developed a particular scaling method to balancing SR matrices. In this sub-chapter first, a least-squares (LS) optimization algorithm called a *recursive rank-one residue iteration* (triple R-I) is presented.

Let the set Ω denote the feasible region for minimization problem (4.3):

$$\Omega = \left\{ \mathbf{w} \in \mathbb{R}^n \;\middle|\; \mathbf{c}^{\mathrm{T}}\mathbf{w} = 1, \mathbf{w} > 0 \right\}.$$

The triple R-I starts by using the N-K method for solving equation (4.21) to find a stationary vector $\mathbf{w}^{*(0)}$ (and thus the diagonal matrix $\mathbf{W}_0^*$) at the initial step, $k = 0$. The normalization condition $\mathbf{c}^{\mathrm{T}}\mathbf{w} = 1$ is imposed in order to hold $\left\{\mathbf{w}^{*(k)}\right\}$, $k = 0, 1, 2, \ldots$, in a bounded set throughout the entire process. By $\mathbf{W}_0^*$ and with expression (4.1), the 'best' transitive matrix approximation $\mathbf{B}_0$ to the original SR matrix $\mathbf{A}$ in a LS sense can thus be determined.

A strategy to design an iterative procedure by establishing a successively adjusted sequence of rank-one matrices is the following. It is clear that the 'best' approximation of an entry a_{ij} of matrix $\mathbf{A}$ is $w_j^{*(0)}/w_i^{*(0)}$ $i, j = 1, 2, \ldots, n$. Since we may reasonably expect that $\left(w_i^{*(0)}/w_j^{*(0)}\right) a_{ij}$ produces a 'good' approximation of 1, it is readily apparent that

$$\left[\frac{w_i^{*(0)}}{w_j^{*(0)}} a_{ij}\right] = \mathbf{W}_0^* \mathbf{A} \mathbf{W}_0^{*(-1)} \approx \mathbf{E}, \qquad i, j = 1, 2, \ldots, n. \tag{5.1}$$

The main idea is to achieve continuous improvement in further approximating $\mathbf{E}$. For this purpose, let a positive $n \times n$ matrix $\mathbf{H}_k = \left(h_{ij}^{(k)}\right)$, $k = 0, 1, 2, \ldots$, called a *residue* be defined. It is convenient to set $\mathbf{H}_0 = \mathbf{A}$, at $k = 0$. Hence, necessarily, $\mathbf{H}_k$ is also in SR. Next, at the consecutive steps of the iteration process, each entries of $\mathbf{H}_k$ will simultaneously be updated by performing a similarity transformation (diagonal similarity scaling) of the previous update $\mathbf{H}_{k-1}$ with the generating diagonal matrices $\mathbf{W}_{k-1}$ and $\mathbf{W}_{k-1}^{-1}$. This yields the *updating rule*:

$$\mathbf{H}_k = \mathbf{W}_{k-1} \mathbf{H}_{k-1} \mathbf{W}_{k-1}^{-1} = \left[\frac{w_i^{(k-1)}}{w_j^{(k-1)}}\right] \circ \left[h_{ij}^{(k-1)}\right], \qquad k = 1, 2, \ldots. \tag{5.2}$$

Note here that (5.2) can be written in the form of a Hadamard product. For updating matrix $\mathbf{H}_{k-1}$, formula (5.2) is referred to as the *step-operator*, $\mathbf{S}_k(\mathbf{H}_k)$. It is desired that the rank-one matrix $\tilde{\mathbf{B}}_k$ be recursively adjusted to the original matrix $\mathbf{A}$ at the consecutive iteration steps, so that

$$\tilde{\mathbf{B}}_k{=}\tilde{\mathbf{W}}_k^{-1}\mathbf{E}\tilde{\mathbf{W}}_k, \quad k=0,1,\ldots, \quad \text{where } \tilde{\mathbf{W}}_k{=}\prod_{i=0}^{k-1}\mathbf{W}_i \text{ and } \tilde{\mathbf{W}}_k^{-1}{=}\prod_{i=0}^{k-1}\mathbf{W}_i^{-1}. \tag{5.3}$$

It can be readily seen that each of the adjustment errors, $S\left(\tilde{\mathbf{w}}^{(k)}\right) = \left\|\mathbf{A}-\tilde{\mathbf{B}}_k\right\|_{\mathrm{F}}$, will be greater for $k=1,2,\ldots$, than that of for $k=0$. An other transitive matrix,

$$\mathbf{B}_k^P{=}\mathbf{W}_k^{-1}\mathbf{E}\mathbf{W}_k,=\left[\frac{w_j^{(k)}}{w_i^{(k)}}\right]\circ\left[e_{ij}^{(k)}\right], \qquad k=1,2,\ldots, \tag{5.4}$$

called a *pattern* will represent the 'best' transitive matrix approximation to its corresponding residue $\mathbf{H}_k$. Its approximation error is: $S\left(\mathbf{w}^{(k)}\right)=\left\|\mathbf{H}_k-\mathbf{B}_k^P\right\|_{\mathrm{F}}$. Obviously, $\mathbf{B}_0{=}\mathbf{B}_0^P{=}\tilde{\mathbf{B}}_0$. It is evident that the updating rule (5.2) will force all entries of $\tilde{\mathbf{B}}_k$ to be set to one, while the elements of the Perron-eigenvectors of the pattern $\mathbf{B}_k^P$ will successively approach to those of associated with the original matrix $\mathbf{A}$.

The process is repeated until some convergence criterion is met. The stopping rule is to halt the algorithm at iteration step $k=q$, once the numerical error falls below a predefined tolerance (a reasonably small positive number, $\varepsilon>0$) yielding the "stabilized" matrices $\tilde{\mathbf{W}}_q, \mathbf{H}_q, \tilde{\mathbf{B}}_q$ and $\mathbf{B}_q^P$.

The formal description of the algorithm is presented below:

Triple R-I Algorithm

Input module. Enter the SR matrix $\mathbf{A}$. Calculate its Perron-eigenvalue, $\lambda_{\max}(\mathbf{A})$, and its normalized right and left Perron-eigenvectors, $\mathbf{u}_{\max}(\mathbf{A})$ and $\mathbf{v}_{\max}(\mathbf{A})$.

Initial module. For $k=0$. Given a positive initial value ϕ_0 and a reasonably small $\varepsilon>0$. An appropriate choice for the starting vector ϕ_0 is the solution to the 'best' linear approximation as given by (4.15). Using the N-K method find the stationary vector $\mathbf{w}^{*(0)}$ (and thus the diagonal matrix $\mathbf{W}_0^*$) by solving the following system of nonlinear equations [cf. (4.19]:

$$\left\{\mathbf{W}_0^{-2}\left(\mathbf{A}-\mathbf{W}_0^{-1}\mathbf{E}\mathbf{W}_0\right)-\left(\mathbf{A}-\mathbf{W}_0^{-1}\mathbf{E}\mathbf{W}_0\right)^{\mathrm{T}}\mathbf{W}_0^{-2}\right\}\mathbf{W}_0\mathbf{e}=0, \tag{5.5}$$

where in (5.5), $\mathbf{W}_0\mathbf{e}=\mathbf{w}^{(0)}=\left[w_i^{(0)}\right]$, $i=1,2,\ldots,n$, is normalized so that $\mathbf{c}^{\mathrm{T}}\mathbf{w}^{(0)}=1$.

a) If $\mathbf{w}^{(0)}$ is stationary, compute

$$\tilde{\mathbf{B}}_0=\mathbf{B}_0^P=\mathbf{W}_0^{*(-1)}\mathbf{E}\mathbf{W}_0^*=\left[\frac{w_j^{*(0)}}{w_i^{*(0)}}\right], \qquad i,j=1,2,\ldots,n. \tag{5.6}$$

b) Else choose an other promising positive initial value and repeat the N-K procedure until $\mathbf{w}^{(0)}$ is stationary, then compute $\tilde{\mathbf{B}}_0=\mathbf{B}_0^P$ according to (5.6).

c) Calculate the error of the 'best' transitive matrix approximation $\tilde{\mathbf{B}}_0$ to the original matrix $\mathbf{A}$ as: $S\left(\mathbf{w}^{(0)}\right)=S\left(\tilde{\mathbf{w}}^{(0)}\right)=\left\|\mathbf{A}-\tilde{\mathbf{B}}_0\right\|_{\mathrm{F}}$.

Set $\mathbf{H}_0=\mathbf{A}$.

Recursion module. For $k=1,2,\ldots$. Using the N-K method find the stationary vector $\mathbf{w}^{*(k)}$ (and $\mathbf{W}_k^*$) by solving the following system of nonlinear equations:

$$\left\{\mathbf{W}_k^{-2}\left(\mathbf{H}_k-\mathbf{W}_k^{-1}\mathbf{E}\mathbf{W}_k\right)-\left(\mathbf{H}_k-\mathbf{W}_k^{-1}\mathbf{E}\mathbf{W}_k\right)^{\mathrm{T}}\mathbf{W}_k^{-2}\right\}\mathbf{W}_k\mathbf{e}=0, \tag{5.7}$$

where in (5.7), the vector $\mathbf{W}_k\mathbf{e} = \mathbf{w}^{(k)} = \left[w_i^{(k)}\right]$, $i = 1, 2, \ldots, n$, is normalized so that $\mathbf{c}^{\mathrm{T}}\mathbf{w}^{(k)} = 1$ and the residue (updating rule) is given by the formula (5.2).

a) If $\|\mathbf{W}_k - \mathbf{I}_n\| < \varepsilon$, for $k > N(\varepsilon)$, set $k = q$, then compute $\tilde{\mathbf{W}}_q$, $\mathbf{H}_q$, $\tilde{\mathbf{B}}_q$ and $\mathbf{B}_q^P$, next calculate the Perron-eigenvalue, $\lambda_{\max}(\mathbf{H}_q)$, and its normalized right and left Perron-eigenvectors, $\mathbf{u}_{\max}(\mathbf{H}_q)$ and $\mathbf{v}_{\max}(\mathbf{H}_q)$ and stop.

b) Else compute $\tilde{\mathbf{W}}_k$, $\mathbf{H}_k$, $\tilde{\mathbf{B}}_k$ and $\mathbf{B}_k^P$.

c) Calculate the adjustment error of the rank-one matrix $\tilde{\mathbf{B}}_k$ to the original matrix $\mathbf{A}$ as: $S(\tilde{\mathbf{w}}^{(k)}) = \|\mathbf{A} - \tilde{\mathbf{B}}_k\|_{\mathrm{F}}$.

d) Continue the iteration for $k + 1$.

Steps of the triple R-I algorithm and a computer program written in 'Mathematica' code was presented in Farkas (2012). Find it in Appendix E.

5.2 Diagonal Similarity Scaling of Pairwise Comparison Matrices

This section discusses the matrix balancing problem to be done through successive adjustments of the residue and the pattern matrices. It will be shown that matrices $\mathbf{A}$ and $\mathbf{B}_0$ are balanceable in the sense of (5.9) and can be balanced by virtue of (5.8). The balanced matrices have useful properties which provide some novel contributions to the theory of the AHP as well. In particular, we will give proofs that our triple R-I algorithm with a user specified termination criterion, $\varepsilon > 0$ results in the similarity scalings, $\mathbf{B}_q^P$ and $\mathbf{H}_q$.

Definition 5.1 An $n \times n$ matrix $\mathbf{A}$ with nonnegative entries is said to be *balanced* if for each $i = 1, 2, \ldots, n$, the sum of the elements in the ith row of $\mathbf{A}$ equals the sum of the elements in the ith column of $\mathbf{A}$, i.e., if $\mathbf{A}$ is *line-sum-symmetric* so that

$$\mathbf{A}\mathbf{e} = \mathbf{A}^{\mathrm{T}}\mathbf{e}. \tag{5.8}$$

Definition 5.2 A matrix $\mathbf{A}$ is said to be *balanceable via diagonal similarity-scaling* if there exists a nonsingular diagonal matrix $\mathbf{W}$ with positive diagonal elements such that $\mathbf{W}\mathbf{A}\mathbf{W}^{-1}$ is balanced, i.e., if

$$\mathbf{W}\mathbf{A}\mathbf{W}^{-1}\mathbf{e} = \mathbf{W}^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{W}\mathbf{e}. \tag{5.9}$$

Consider now the residue matrix $\mathbf{H}_k$. We will show that the updating rule of the triple R-I algorithm is essentially analogous to the *fixed point iteration*

$$\mathbf{H}_{k+1} = \mathbf{S}_k(\mathbf{H}_k) = \mathbf{W}_k\mathbf{H}_k\mathbf{W}_k^{-1}, \quad k = 0, 1, \ldots, \tag{5.10}$$

where $\mathbf{S}_k(\mathbf{H}_k)$ is the step-operator of the triple R-I. The objective is to minimize the Frobenius norm:

$$\|\mathbf{H}_{k+1} - \mathbf{H}_k\|_{\mathrm{F}} \Rightarrow \text{minimum}, \qquad k = 0, 1, \ldots. \tag{5.11}$$

The convergence theorem for the sequence $\{\mathbf{H}_k\}$ is stated below.

Theorem 5.1 *Let* $\mathbf{H} = (h_{ij})$, $i, j = 1, 2, \ldots, n$, *be an SR matrix with positive entries and called a residue. The sequence* $\{\mathbf{H}_k\}$, $k = 1, 2, \ldots$, *generated by the fixed point iteration* (5.10) *using the step operator* $\mathbf{S}_k$ *converges to some* $\mathbf{H}_q^* \in \mathcal{H}^*$, *where* $\mathcal{H}^*$ *is the set of stationary points of problem* (5.11) *over the feasible set* Ω, *and* q *indicates the step of the termination of the triple R-I for a prescribed reasonably small tolerance,* $\varepsilon > 0$.

Proof. As follows from its construct, $\mathbf{S}_k(\mathbf{H}_k)$ is non-expansive, therefore $\{\mathbf{H}_k\}$ lies in a compact set and must have a limit point, say $\hat{\mathbf{H}} = \lim_{j\to\infty} \mathbf{H}_{k_j}$. Additionally, for any $\mathbf{H}_q^* \in \mathcal{H}^*$,

$$\left\|\mathbf{H}_{k+1} - \mathbf{H}_q^*\right\|_{\mathrm{F}} = \left\|(\mathbf{S}_k(\mathbf{H}_k) - \mathbf{S}_{k_j}(\mathbf{H}_q^*)\right\|_{\mathrm{F}} \le \left\|\mathbf{H}_k - \mathbf{H}_q^*\right\|_{\mathrm{F}},$$

which implies that the sequence $\{\|\mathbf{H}_k - \mathbf{H}_q^*\|_{\mathrm{F}}\}$ is monotonically non-increasing under the updating rule (5.2). Hence,

$$\lim_{k\to\infty} \left\|\mathbf{H}_k - \mathbf{H}_q^*\right\|_{\mathrm{F}} = \left\|\hat{\mathbf{H}} - \mathbf{H}_q^*\right\|_{\mathrm{F}}, \tag{5.12}$$

where $\hat{\mathbf{H}}$ can be any limit point of $\{\mathbf{H}_k\}$. Considering that $\mathbf{S}_k(\mathbf{H}_k)$ is continuous, the step operator for $\hat{\mathbf{H}}$,

$$\mathbf{S}_{k_j}(\hat{\mathbf{H}}) = \lim_{j\to\infty} \mathbf{S}_{k_j}\left(\mathbf{H}_{k_j}\right) = \lim_{j\to\infty} \mathbf{H}_{k_j+1},$$

produces also a limit point of $\{\mathbf{H}_k\}$. Therefore, we have

$$\left\|\mathbf{S}_{k_j}(\hat{\mathbf{H}}) - \mathbf{S}_q(\mathbf{H}_q^*)\right\|_{\mathrm{F}} = \left\|\mathbf{S}_{k_j}(\hat{\mathbf{H}}) - \mathbf{H}_q^*\right\|_{\mathrm{F}} = \left\|\hat{\mathbf{H}} - \mathbf{H}_q^*\right\|_{\mathrm{F}}$$

which shows that $\hat{\mathbf{H}}$ is a stationary point of problem (5.11). Finally, by setting $\mathbf{H}_q^* = \hat{\mathbf{H}} \in \mathcal{H}^*$ in (5.12), we obtain

$$\lim_{k\to\infty} \left\|\mathbf{H}_k - \hat{\mathbf{H}}\right\|_{\mathrm{F}} = \lim_{j\to\infty} \left\|\mathbf{H}_{k_j} - \hat{\mathbf{H}}\right\|_{\mathrm{F}} = 0,$$

i.e. $\{\mathbf{H}_k\}$ converges to its limit point $\hat{\mathbf{H}}$. In each step k of the recursive algorithm the N-K method is used to solve the system of nonlinear equations (5.7). Therefore, at step $k = q$, when the iteration has converged to any limit point $\mathbf{H}_q^*$ in the interior of the feasible region Ω, this point is necessarily a stationary point (see Proposition 4.4). This completes the proof. $\square$

We now show that matrix $\mathbf{H}_q^*$ is in line-sum-symmetry.

Corollary 5.1 *For the limit matrix* $\mathbf{H}_q^*$, *of the triple R-I the right and the left eigenvectors associated with the zero eigenvalue of the skew-symmetric matrix* $\left(\mathbf{H}_q^* - \mathbf{H}_q^{*\mathrm{T}}\right)$, *are the vectors* $\mathbf{e}$ *and* $\mathbf{e}^{\mathrm{T}}$, *respectively.*

Proof. Using the diagonal matrix $\tilde{\mathbf{W}}$ defined as of (5.3) we can write the product of the diagonal matrices $\mathbf{W}_k$ in a limiting sense as

$$\lim_{k\to\infty} (\mathbf{W}_{k-1}\mathbf{W}_{k-2}\ldots\mathbf{W}_2\mathbf{W}_1\mathbf{W}_0) = \tilde{\mathbf{W}}. \tag{5.13}$$

By taking the limit of (5.2) we have

$$\lim_{k\to\infty} \mathbf{H}_k = \tilde{\mathbf{W}}_q \mathbf{A} \tilde{\mathbf{W}}_q^{-1}. \tag{5.14}$$

Applying (5.14) for $k > N$, the system of nonlinear equations (4.21) leads to the following equation:

$$\left(\tilde{\mathbf{W}}_q \mathbf{A} \tilde{\mathbf{W}}_q{}^{-1} - \tilde{\mathbf{W}}_q{}^{-1} \mathbf{A}^{\mathrm{T}} \tilde{\mathbf{W}}_q\right) \mathbf{e} = \left(\mathbf{H}_q^* - \mathbf{H}_q^{*\mathrm{T}}\right) \mathbf{e} = 0, \tag{5.15}$$

where it is apparent that the right eigenvector associated with the zero eigenvalue of the skew-symmetric matrix $\left(\mathbf{H}_q^* - \mathbf{H}_q^{*\mathrm{T}}\right)$ is $\mathbf{e}$, while the left eigenvector is $\mathbf{e}^{\mathrm{T}}$. As it is readily seen from equation (5.15) the matrix $\mathbf{H}_q^*$ is balanced, since it is in line-sum-symmetry in the sense of (5.8). □

It is possible to find an alternate way to obtain the same result as given in (5.15). Consider $\tilde{\mathbf{W}}_k$ defined as a product in (5.3). We have the recurrence

$$\tilde{\mathbf{W}}_k \text{ minimizes } \left\| \tilde{\mathbf{W}}_{k-1} \left(\mathbf{A} - \tilde{\mathbf{W}}_k^{-1} \mathbf{E} \tilde{\mathbf{W}}_k\right) \tilde{\mathbf{W}}_{k-1} \right\|_{\mathrm{F}}^2,$$

and a fixed point of the iteration satisfies (5.15). Observe that if $\mathbf{A}$ happens to be transitive, then we can satisfy the fixed point equation by setting $\tilde{\mathbf{W}}$ to diagonal scaling by the Perron-eigenvector $\left(\text{since then } \tilde{\mathbf{W}} \mathbf{A} \tilde{\mathbf{W}}^{-1} = \mathbf{E}\right)$. This is equivalent to saying that there is a diagonal similarity $\tilde{\mathbf{W}}$ such that the row and column sums of $\tilde{\mathbf{W}} \mathbf{A} \tilde{\mathbf{W}}^{-1}$ are equal.

As is readily seen from (5.15), the triple R-I algorithm produces a stabilized matrix $\mathbf{H}_q$ which is in sum-symmetry, exactly the condition that characterizes a fixed point for this iteration. This means, that in this case, the SR matrix $\mathbf{A}$ has a unique line-sum-symmetric similarity-scaling. Hence, our recursive least-squares method enables to obtain a solution and is equivalent to balancing a matrix through diagonal similarity-scaling. Latter problem has a solution if and only if the underlying matrix is completely reducible, see the characterization theorems on nonnegative balanceable matrices in Osborne [55, see Theorem 1 and Lemma 2] and in Eaves et al. [22, see Theorem 3]. The above recognition gives confidence in the viability of both approaches. We remark that this issue is discussed in author's most recent paper [Farkas (2014c)].

Computational experience with many different choices for SR perturbations of transitive matrices has shown that depending upon the degree of consistency of matrix $\mathbf{A}$, four mutually exclusive cases, denoted by $l = \mathrm{i}, \mathrm{ii}, \mathrm{iii}, \mathrm{iv}$, may occur in the iteration process:

Case (i). The iteration terminates at step q yielding *one* matrix, denoted by $\mathbf{H}_q^{(\mathrm{i})}$. Continuing the run of the algorithm, this stabilized matrix repeats itself with the same pattern in consecutive steps. The cardinality (cycle length) of this recurrent sequence is thus: $l = 1$. In this case, we call matrix $\mathbf{A}$ a *weakly perturbed* PCM.

Case (ii). The iteration terminates at steps q and $q + 1$ yielding *two* matrices, denoted by $\mathbf{H}_q^{(\mathrm{ii})}$ and $\mathbf{H}_{q+1}^{(\mathrm{ii})}$. Continuing the run of the algorithm, these stabilized matrices repeat themselves with the same pattern in consecutive steps. The cardinality of this recurrent sequence is thus: $l = 2$. In this case, we call matrix $\mathbf{A}$ a *moderately perturbed* PCM.

Case (iii). The iteration terminates at steps q, $q + 1$ and $q + 2$ yielding *three* matrices, denoted by $\mathbf{H}_q^{(\mathrm{iii})}$, $\mathbf{H}_{q+1}^{(\mathrm{iii})}$ and $\mathbf{H}_{q+2}^{(\mathrm{iii})}$. Continuing the run of the algorithm, these stabilized matrices repeat themselves with the same pattern in consecutive steps. The cardinality of this recurrent sequence is thus: $l = 3$. In this case, we call matrix $\mathbf{A}$ a *strongly perturbed* PCM.

Case (iv). The iteration terminates at steps q, $q + 1$, $q + 2$ and $q + 3$ yielding *four* matrices, denoted by $\mathbf{H}_q^{(\mathrm{iv})}$, $\mathbf{H}_{q+1}^{(\mathrm{iv})}$, $\mathbf{H}_{q+2}^{(\mathrm{iv})}$ and $\mathbf{H}_{q+3}^{(\mathrm{iv})}$. Continuing the run of the algorithm, these stabilized matrices repeat themselves with the same pattern in consecutive steps. The cardinality of this recurrent sequence is thus: $l = 4$. However, this case occurs within a narrow perturbation zone of $\mathbf{A}$ only and may be regarded as a 'subcase' of (ii).

Next, we show that in case (i), the sequence of $\{\mathbf{W}_k\}$ converges to the identity matrix $\mathbf{I}_n$.

Proposition 5.1 *Let a positive SR matrix $\mathbf{A}$ be given and $\{\mathbf{W}_k\}$, $k = 0, 1, 2, \ldots$, be a sequence of diagonal matrices composed of the positive stationary vectors $\{\mathbf{w}^{(k)}\}$ produced by the triple R-I. Then, in case* (i), *the iterate $\mathbf{W}_k$ converges to $\mathbf{I}_n$ in a limiting sense:*

$$\lim_{k\to\infty} \mathbf{W}_k = \lim_{k\to\infty} \mathbf{W}_k^{-1} = \mathbf{I}_n, \qquad k = 0, 1, 2, \ldots. \tag{5.16}$$

For a stipulated arbitrarily small $\varepsilon > 0$ the algorithm stops, if a certain N can be found such that

$$\|\mathbf{W}_q - \mathbf{I}_n\| < \varepsilon, \qquad \textit{for } q > N(\varepsilon).$$

Proof. According to equation (5.7), for step q we have

$$\left\{\mathbf{W}_q^{-2}\left(\mathbf{H}_q - \mathbf{W}_q^{-1}\mathbf{E}\mathbf{W}_q\right) - \left(\mathbf{H}_q - \mathbf{W}_q^{-1}\mathbf{E}\mathbf{W}_q\right)^{\mathrm{T}}\mathbf{W}_q^{-2}\right\}\mathbf{W}_q\mathbf{e} = 0. \tag{5.17}$$

For step $q + 1$, we can write that

$$\begin{aligned}&\left\{\mathbf{W}_{q+1}^{-2}\left(\mathbf{W}_q\mathbf{H}_q\mathbf{W}_q^{-1} - \mathbf{W}_{q+1}^{-1}\mathbf{E}\mathbf{W}_{q+1}\right) - \right.\\ &\qquad \left. - \left(\mathbf{W}_q\mathbf{H}_q\mathbf{W}_q^{-1} - \mathbf{W}_{q+1}^{-1}\mathbf{E}\mathbf{W}_{q+1}\right)^{\mathrm{T}}\mathbf{W}_{q+1}^{-2}\right\}\mathbf{W}_{q+1}\mathbf{e} = 0.\end{aligned} \tag{5.18}$$

In case (i), one matrix $\mathbf{H}_{q+(r-1)}^{(l)}$, $l = \mathrm{i}$, $r = 1$, has been stabilized at step q, which repeats itself in the succeeding steps. Therefore, $\mathbf{H}_q^{(\mathrm{i})} = \mathbf{H}_{q+1}^{(\mathrm{i})}$. Observe that equation (5.17) corresponds to equation (5.18). Thus, $S^2\left(\mathbf{w}^{(q+1)}\right) = S^2\left(\mathbf{w}^{(q)}\right)$ holds, which implies that

$$\mathbf{W}_q = \mathbf{W}_{q+1} = \mathbf{I}_n.$$

This completes the proof. □

We remark that proofs for the relevant properties of the other cases that may occur in the triple R-I are given in Farkas and Rózsa (2013).

In this paper we introduced the the geometric Hadamard mean matrix as being the geometric means of the Hadamard products of the stabilized residue matrices,

$$\mathbf{H}_g^{(l)} = \left(\prod_{r=1}^{N} \circ \mathbf{H}_{q+(r-1)}^{(l)}\right)^{1/N},$$

and showed that the *entries* of the matrices $\mathbf{H}_{q+(r-1)}^{(l)} = (h_{ij})$, $i, j = 1, 2, \ldots, n$, for $l = \mathrm{i}$, $r = 1$ or $l = \mathrm{ii}$, $r = 1, 2$ or $l = \mathrm{iii}$, $r = 1, 2, 3$ or $l = \mathrm{iv}$, $r = 1, 2, 3, 4$, and of their associated geometric Hadamard mean matrices $\mathbf{H}_g^{(l)} = (h_{ij})$, represent the single consistency errors in the responses which are, in fact, the perturbation factors δ_{ij}. We assumed that any perturbation is identified as a continuous log-normally distributed random variable, justified theoretically by Ramsay [59] as a distribution of errors in the judgments. The following propositions provides us two appropriate measures for the *inconsistency* of the SR matrices $\mathbf{A}$:

Proposition 5.2 *If* $\mathbf{A}$ *is an SR matrix, then the average magnitude of the inconsistency of matrix* $\mathbf{A}$ *in an LS sense is the geometric mean of the elements* h_{ij}, $i, j = 1, 2, \ldots, m$, *of the matrices* $\mathbf{H}^{(l)}_{q+(r-1)}$, *for* $l = \text{i}$, $r = 1$ *or* $l = \text{ii}$, $r = 1, 2$ *or* $l = \text{iii}$, $r = 1, 2, 3$ *or* $l = \text{iv}$, $r = 1, 2, 3, 4$, *or of* $\mathbf{H}^{(l)}_g$ *and it is given by the formula (we omit the codes for* h_{ij}*)*

$$g(\mathbf{A}) = \frac{1}{m}\left(\prod_{i<j}^{m} h_{ij}\right)^{1/2}, \qquad i, j = 1, 2, \ldots, m. \tag{5.19}$$

Proposition 5.3 *If* $\mathbf{A}$ *is an SR matrix, then the variability of the inconsistency of matrix* $\mathbf{A}$ *in an LS sense is the geometric standard deviation of the elements* h_{ij}, $i, j = 1, 2, \ldots, m$, *of the matrices* $\mathbf{H}^{(l)}_{q+(r-1)}$, *for* $l = \text{i}$, $r = 1$ *or* $l = \text{ii}$, $r = 1, 2$ *or* $l = \text{iii}$, $r = 1, 2, 3$ *or* $l = \text{iv}$, $r = 1, 2, 3, 4$, *or of* $\mathbf{H}^{(l)}_g$ *and it is given by the antilogarithm of (we omit the codes for* h_{ij}*)*

$$\ln s_g(\mathbf{A}) = \frac{1}{m}\left[\sum_{i<j}(\ln h_{ij} - \ln g(\mathbf{A}))^2\right]^{1/2}, \qquad i, j = 1, 2, \ldots, m. \tag{5.20}$$

5.3 Numerical Analysis

Illustrations reported here are selected to demonstrate the results discussed in the previous sections. Computations are made by 'Mathematica'. The prescribed accuracy for a run of the triple R-I was: $\varepsilon = 10^{-6}$.

Example. This example concerns data of the same 5×5 sized SR matrix $\mathbf{A}$ which was considered in section 4.3.1. The principal eigenvalue of $\mathbf{A}$ is: $\lambda_{\max} = 5.2247$ and the respective right and left Perron-eigenvectors of $\mathbf{A}$ are:

$$\mathbf{u}^{\mathrm{T}}(\mathbf{A}) = [0.4767, 0.2865, 0.1029, 0.0820, 0.0520],$$
$$\text{and} \quad \mathbf{v}^{\mathrm{T}}(\mathbf{A}) = [0.0475, 0.0776, 0.2085, 0.2647, 0.4017].$$

The output of the initial module of the algorithm gives the stationary vector $\mathbf{w}^{*(0)}$ that constitutes the first row of the 'best' transitive matrix approximation $\mathbf{B}^P_0$ to matrix $\mathbf{A}$ in an LS sense (given in a non-normalized form):

$$\mathbf{w}^{*\mathrm{T}(0)} = [1.0000, 1.1405, 4.4981, 4.3356, 6.5211].$$

The algorithm terminates at step $q = 9$, producing the stationary vector:

$$\mathbf{w}^{*\mathrm{T}(\mathrm{q})} = [1.0000, 1.0000, 1.0000, 1.0000, 1.0000],$$

which would repeat itself if we continue the iteration. Recognize that case (i) has occurred, yielding one stabilized matrix $\mathbf{H}^*_q$ which is clearly in SR:

$$\mathbf{H}^*_q = \begin{bmatrix} 1 & 1.8308 & 1.1135 & 0.7040 & 0.6822 \\ 0.5462 & 1 & 1.4596 & 1.1536 & 1.1179 \\ 0.8981 & 0.6851 & 1 & 1.5806 & 1.0212 \\ 1.4205 & 0.8669 & 0.6327 & 1 & 1.2921 \\ 1.4658 & 0.8945 & 0.9793 & 0.7739 & 1 \end{bmatrix}.$$

The principal eigenvalue of $\mathbf{H}_q^*$ is: $\lambda_{\max} = 5.2247$. The right and the left Perron-eigenvectors of $\mathbf{H}_q^*$, are now computed. Notice their asymmetry, since they are not element-wise reciprocal:

$$\mathbf{u}_q^{\mathrm{T}}(\mathbf{H}_q^*) = [0.2045, 0.2014, 0.1983, 0.1997, 0.1961], \quad \text{and}$$
$$\mathbf{v}_q^{\mathrm{T}}(\mathbf{H}_q^*) = [0.2034, 0.2027, 0.1988, 0.1995, 0.1956].$$

Up to step $q = 9$, the inverse matrix $\tilde{\mathbf{W}}_q^{-1}$ of the product of the diagonal matrices $\mathbf{W}_k^{-1}$, $k = 0, 1, 2, \ldots, q$, is obtained as

$$\tilde{\mathbf{W}}_q^{-1} = \begin{bmatrix} 1 & & & & 0 \\ & 0.6103 & & & \\ & & 0.2227 & & \\ & & & 0.1760 & \\ 0 & & & & 0.1137 \end{bmatrix}.$$

It is easy to check that $\tilde{\mathbf{W}}_q^{-1}\mathbf{u}_q(\mathbf{H}_q^*) = \mathbf{u}(\mathbf{A})$ and $\mathbf{v}_q^{\mathrm{T}}(\mathbf{H}_q^*)\tilde{\mathbf{W}}_q = \mathbf{v}^{\mathrm{T}}(\mathbf{A})$ holds. To obtain the 'best' adjustment of the Perron-eigenvectors to those of the original SR matrix $\mathbf{A}$, compute $\tilde{\mathbf{B}}_q$, as defined in (5.3). Notice also that the Perron-eigenvectors of matrix $\tilde{\mathbf{B}}_q$:

$$\tilde{\mathbf{w}}^{(q)-1\mathrm{T}} = [0.4711, 0.2875, 0.1049, 0.0829, 0.0536], \quad \text{and}$$
$$\tilde{\mathbf{w}}^{(q)\mathrm{T}} = [0.0463, 0.0759, 0.2079, 0.2631, 0.4069],$$

provide very good adjustments to $\mathbf{u}(\mathbf{A})$ and $\mathbf{v}^{\mathrm{T}}(\mathbf{A})$. For the steps $k = 0$, $k = 1$ and $k = q$, the consistency adjustment errors to matrix $\mathbf{A}$ are

$$S\left(\mathbf{w}^{(0)}\right) = S\left(\tilde{\mathbf{w}}^{(0)}\right) = 2.5470, \quad S\left(\tilde{\mathbf{w}}^{(1)}\right) = 4.5463 \quad \text{and} \quad S\left(\tilde{\mathbf{w}}^{(1)}\right) = 3.9968.$$

In accordance with the characteristic features of the algorithm, indeed, $S\left(\mathbf{w}^{(0)}\right)$ produces the smallest consistency adjustment error, whereas $S\left(\tilde{\mathbf{w}}^{(1)}\right)$ produces the largest error. Above results verify that $\mathbf{H}_q^*$ is in line-sum-symmetry and thus, $\mathbf{A}$ is a weakly perturbed SR matrix. By formulae (5.19) and (5.20), respectively, the average magnitude of the inconsistency of $\mathbf{A}$ is: $g(\mathbf{A}) = 1.1441$ and the variability of the judgmental errors committed by the respondent is: $s_g(\mathbf{A}) = 1.3537$.

In order to examine the sensitivity of SR matrices to increased perturbations and study their effects on the triple R-I, let one particular entry of $\mathbf{A}$, $a_{15} = 6$ (and thus $a_{51} = 1/6$) be continuously changed over the interval a_{15}: [6.00–0.001] while the values of the other entries are held fixed. In Figure 21, we display the change in the values of g, i.e. the behavior of the least-squares recursion (LSR) as function of the increasing perturbation. It can be seen well how the graph pulls apart into two, four and three branches over the interval a_{15}: [6.00–0.001]. In Figure 22, we display the ranges of the weakly perturbed matrices, denoted by $\mathbf{A}^{(\mathrm{w})}$, of the moderately perturbed matrices, denoted by $\mathbf{A}^{(\mathrm{m})}$ and of the strongly perturbed matrices, denoted by $\mathbf{A}^{(\mathrm{s})}$. For these regions, both the average magnitude (geometric mean) and the variability (standard deviation) of the inconsistency of matrix $\mathbf{A}$ are graphed. The convexity of these curves is well discernible in Figure 22, as well as a change in the gradients at the transition points.

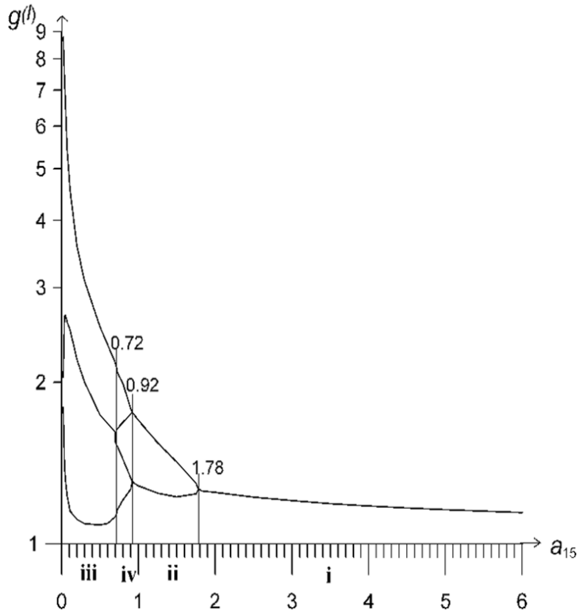


Figure 21. The behavior of the LSR for matrix $\mathbf{A}$ subject to perturbations of $a_{15} = [0.001 - 6]$

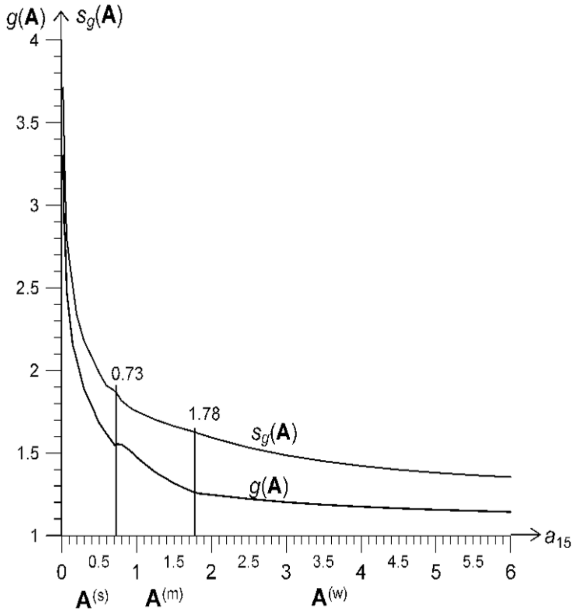


Figure 22. The average magnitude and variability of the inconsistency of $\mathbf{A}$ for $a_{15} = [0.001 - 6]$

Starting from $a_{15} = 1.78$ and moving towards the opposite direction now (i.e. when the values of a_{15} are continuously increasing), the graph $g^{(l)}$, $l = 1, 2, 3, 4$, is plotted in Figure 23 for all cases, where the set of the perturbed values varies a_{15}: [1.78–500]. As shown in Figure 24, the average magnitude of perturbations across $\mathbf{A}$, measured by $g(\mathbf{A})$, changes at a very slow rate despite the fact that most values within this region, say $a_{15} > 100$, represent a 'colossal' perturbation for $\mathbf{A}$ in practice. This phenomenon clearly demonstrates the necessity to make cautious interpretations based on only averages. As it can be determined from formulas (5.19) and (5.20) and also seen from Figure 24, for a value of $a_{15} \cong 50$, $g(\mathbf{A}) = 1$.

This would suggest that the corresponding $\mathbf{A}$ is transitive. Obviously, however, this is not the case, since the variability of the perturbations of $\mathbf{A}$ should, simultaneously, be taken into consideration, i.e. at $a_{15} \cong 50$, $g(\mathbf{A}) \cong 1.5$. At $a_{15} \cong 11$, the value of the statistic $s_g(\mathbf{A})$ attains a minimum. It is interesting to note that the principal eigenvalue of $\mathbf{A}$ has a minimum at $\lambda_{\max} = 5.1759$, just at the same value of $a_{15} \cong 11$. This result suggests the conjecture that the $s_g(\mathbf{A})$ measure seems to share global properties with the AHP's consistency measure (see μ in Definition 2.3, in Chapter 2).

We conclude that the recursive iteration when applied to a specific class of positive matrices called pairwise comparison matrices is equivalent with a line-sum-symmetric diagonal similarity scaling. We have proven that a series of transitive matrix approximations to the original SR matrix produces a convergent process, which, with a prescribed accuracy, yields stabilized matrices. In particular, we tied together two different approaches, the eigenvector method (EM) and the least-squares method (LS) and showed that there exists a tractable *relationship* between them when they are applied for matrices with such a characteristic structure. Furthermore, our triple R-I algorithm seems to be more generally useful as it can be applied to merely positive matrices also.

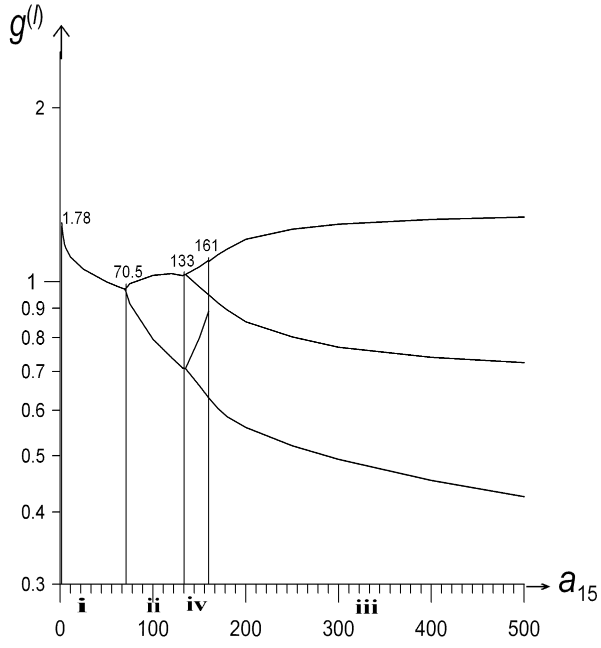


Figure 23. The behavior of the LSR for matrix $\mathbf{A}$ subject to perturbations of $a_{15} = [1.78 - 500]$

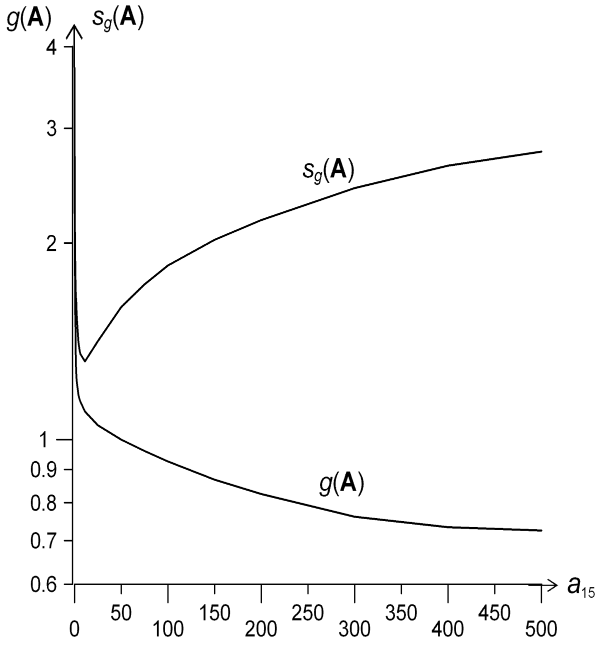


Figure 24. The average magnitude and variability of the inconsistency of $\mathbf{A}$ for $a_{15} = [0.001 - 500]$

Chapter 6

6 Development of a Combined MOO/MCDA Scaling Method

In this Chapter, a scaling method as a combination of the multi-criteria optimization (MOO) and the multi-criteria decision analysis (MCDA) approaches is developed. Although, there are a great number of scaling methods used in practice, however, most of them have been designed to measure the preferences of the decision makers among the alternatives on one particular scale of measurement. Here, a more versatile method based on multi-attribute utility theory is presented, which enables each attribute to be measured on that scale where the attribute belongs to de facto. A unique feature of this method is the deployment of distinct metric functions on each scale of measurement, even on the two qualitative (nominal and ordinal) scales. Preferences, termed relative standings, are derived as weighted sums of the composite scores for each alternative and are measured on interval scale. This method, called MAROM, is applied to an up-to-date transportation and environmental problem of utmost interest, namely the problem of alternative-fuel modes of buses used for public transportation in urban areas is considered. MAROM is compared to a widely used classic MCDA procedure called TOPSIS. Evaluation procedures and a scenario of the alternative-fuel modes of twelve buses and their rankings are analyzed for an international project taken from the literature.

6.1 Preliminaries

As we discussed in detail in Chapter 2 and in Appendix A, both the multi-objective optimization (MOO) techniques and the multi-criteria decision analysis (MCDA) methods may be applied to many decision making processes. They are most applicable to solving complex evaluation problems and to making a choice among alternatives with respect to a set of criteria. When used for group decision making, these MCDM methods allow the respondents to consider the values that each views as being important.

The seminal models of preference measurement based on multi-attribute utility theory postulate that the preference of an individual decision maker (DM) towards a choice object is related to its “distance” from his/her ideal object which may well be a hypothetical object (see e.g. Dyer and Sarin [20], Zeleny [96], Horsky and Rao [32] and Hwang et al. [35]). The closer the object is to the ideal one, the greater the preference for it. The distance is a compound measure which takes into account the location of each alternative (systems, projects, products, brands, persons) on several attitudes (criteria) which characterize them.

Recently, there are a great number of such methods used in practice but most of them have been designed to evaluate alternatives on *one* particular scale of measurement only. In contrast to this, since the alternatives are characterized by many attributes which may have totally different properties, they should be assigned to different types of measurement scales. A particular scale is homogeneous and, thus, only those transformations are allowed which let the inherent structure of the scale remain to be invariant. To overcome these shortcomings, we developed a combined MOO/MCDA scaling method designed to be capable of incorporating tangible and intangible attributes simultaneously. Our fundamental concept requires that non-quantifiable and quantifiable criteria be treated in different manners. Therefore, its data matrix is partitioned into four blocks. Every criterion is then assigned to that block which represents its associated scale of measurement. Thus, the entries of this matrix are mixed, i.e., they appear in forms of binary variables, rank numbers and quantitative data with different units of measurement. A weighting number is also assigned to each criterion as its relative importance.

The first description of the method was published in Farkas (1994). Later an improved version of the procedure has appeared in Farkas (2006).

6.2 Formal Description of the Method MAROM

We now turn to the description of the methodology of the multi-criteria scaling method called *MultiAttRibute Object Measurement* (MAROM) according to the recent paper of Farkas (2014a). Relevant definitions and interpretations of the terms used in this sub-chapter can be found in Appendix A, section A.2.1.

MAROM perfectly conforms to the theory of measurement, since its basic concept is that each criterion is to be assigned to their corresponding scale of measurement. The alternatives have to be evaluated with respect to each criterion (attribute). After the evaluation process has been completed, the scores (ratings) what the objects received, are available in a data table in the form similar to that of displayed in Figure A.2. Then, a reference object (alternative) is defined that is arbitrarily chosen by the decision making group. It either represents the target (desired) values of an object with respect to each criterion, or it can be composed of the best values of each criterion as an "ideal" one. This object may also be referred to as a benchmark, i.e. a standard or a reference point by which something can be measured or judged. Next, each alternative is compared to the reference object. As a matter of fact, usually, everyone wishes his/her favorable alternative to be located to the "ideal" object as close as possible. In other words, the best object is the one which is closest to the ideal object. MAROM measures "closeness" in terms of distances, by defining *distance functions* for each scale of measurement.

Individual preferences appear as differences between the particular alternatives and the reference object and are quantified by attaching a *composite score* to them termed *relative standing* of the alternatives on an *interval* scale. The ordinal ranking of the objects is given by the order of the magnitude of the relative standings of the objects. For each alternative, by adding up the committee members' scores, then normalizing them within the highest order scales, the resultant vector produces the *compromise ranking* of the decision making group. Intensity of preference between the alternatives indicates the difference between the aggregated single relative standings of the alternatives on interval scale. Thus, MAROM is a method of *absolute measurement* and may be used for competitive benchmarking as well.

Hereafter we present the formal description of MAROM:

We are given the data matrix:

$$\mathbf{A} = [a_{ik}], \qquad i = 1, 2, \ldots, m, \quad k = 1, \ldots, n, \tag{6.1}$$

representing n distinguishable *alternatives*. The n columns give for every option the values of the m specified *variables* (rows) denoting various characteristics (attributes, criteria) of these alternatives. In (6.1), a value (number) is assigned to each entry a_{ik} which is either elicited from respondents' judgments or arisen from physical measurements. Thereby, the nature of a particular data may be of a subjective type (qualitative) or of an objective type (quantitative). Every column vector $\mathbf{a}_k$ of matrix $\mathbf{A}$ is decomposed, therefore it represents a *composite vector*

$$\mathbf{a}_k = \left(\mathbf{a}_k^{(N)}, \mathbf{a}_k^{(O)}, \mathbf{a}_k^{(I)}, \mathbf{a}_k^{(R)}\right),$$

partitioned into four blocks. Thus, $\mathbf{A}$ includes variables of *mixed* type, where N refers to *nominal* (usually binary), O to *ordinal*, I to *interval* and R to *ratio* variables. Of course, in a concrete case, variables of any type may be missing.

A column vector $\mathbf{b} = [b_i]$, $i = 1, \ldots, m$, called a *reference vector*, is to be constructed and added to $\mathbf{A}$ as its $n + 1$th column. Vector $\mathbf{b}$ represents an ideal (hypothetical) option, entries of which receive the "best" values of each criterion given to any alternative(s) in the course of the evaluation process and it has the same element-wise structure as that of vector $\mathbf{a}_k$. Numerical scales, that generally used are: $[0,1]$ on a nominal scale; $[1,\ldots,5]$ on an ordinal scale, $[0,\ldots,1]$ or $[1,\ldots,100]$ on an interval scale, and, [actual values, i.e. row data from measurements] on a *ratio* scale.

Because the ratio (and sometimes the interval) variables have different units of measurements the row vectors $\mathbf{a}_i^{(R)\mathrm{T}}$ $\left(\text{and } \mathbf{a}_i^{(I)\mathrm{T}}\right)$ are standardized so that let their means be equal to zero and their standard deviations be equal to one. For instance, the standard deviations for the ratio variables can be computed in the following way

$$s_i^{(R)} = \sqrt{\frac{1}{n-1}\left[\sum_{i=1^{(R)}}^{k^{(R)}} a_{ik}^{(R)2} - \frac{1}{n}\left(\sum_{i=1^{(R)}}^{m^{(R)}} a_{ik}^{(R)}\right)^2\right]},\ i = 1^{(R)},\ldots,m^{(R)},\ k = 1,\ldots,n. \quad (6.2)$$

With (6.2), the standardized elements can be obtained as

$$a_{ik}^{\prime(R)} = \frac{1}{s_i^{(R)}}\left(a_{ik}^{(R)} - \overline{a}_{i.}^{(R)}\right), \qquad i = 1,\ldots,m, \quad k = 1,\ldots,n. \quad (6.3)$$

A representative group of respondents (experts, consumers, users, etc.) is then formed. Every committee member evaluates each alternative by supplying his/her judgments on each qualitative variable on the nominal and ordinal scale criteria. It is recommended that the number of voters l, $l = 1,\ldots,q$, be at least $q = 10$.

The general form of the preference measurement model used in MAROM is as follows:

$$\overline{d}_k^{\,l} = \sum_{i=1}^{m} w_i^l d_{ik}^l + \varepsilon_k^l, \qquad k = 1,\ldots,n, \quad l = 1,\ldots,q, \quad (6.4)$$

where $\overline{d}_k^{\,l}$ is the overall distance of alternative k from the "ideal" alternative for the lth voter; w_i^l is the weight of attribute i; d_{ik}^l is the distance of the kth alternative from the reference point on attribute i, i.e. for the lth voter: $d_{ik}^l = b_i - a_{ik}^{\prime l(\cdot)}$; ε_k is the value of an error random variable which may include model misspecification, measurement errors, and respondents' uncertainties. Assumptions underlying the use of model (6.4) are: $Exp(\varepsilon_k) = 0$ and $Var(\varepsilon_k) = \sigma^2$ and constant, for all k. To determine the weights of the attributes w_i^l the analytic hierarchy process (AHP) method is proposed; see in Chapter 2. These weights are then normalized so that $\sum_{k=1}^{n} w_i^l = 1$.

The main goal is to obtain the *relative standings* of the alternatives (and thus their priority ranking), denoting them by s_k, $k = 1,2,\ldots,n$. The relative standing of an alternative becomes the composite rating of that particular alternative, which can be determined by the distance functions over the set of all composite vectors. The relative standings express the closeness (similarity) of the alternatives to the 'ideal' object and are determined as: $s_k^l = 1 - \overline{d}_k^{\,l}$. In other words, the relative standing is a measure of the relative 'goodness' of an alternative yielded on an *interval* scale that varies from 1 to 100 or from 0 to 1. The reference vector has a value as its relative standing equal to 100 or 1, since it represents an ideal object. Thus, using MAROM, the best alternative, A^*, is determined as the one having the largest relative standing on an interval scale (for the lth voter):

$$A^{*l} = \max_k s_k^l, \qquad k = 1,2,\ldots,n \quad l = 1,2,\ldots,q. \quad (6.5)$$

In real-world applications, the set of the criteria usually consists of variables that can only be measured on nominal or ordinal scales. In such cases, the respective distance functions should conform to the requirements of *metric* distance functions, since otherwise, the corresponding block vectors cannot be transformed up to an interval scale. The conditions that a metric distance function must satisfy are; see e.g. in Späth [76]:

Axiom 6.1 (Metric Requirements). *For any three composite vectors x, y, z*

(i) $d(x,y) \geq 0$,

(ii) $d(x,y) = 0$ *when* $x = y$,

(iii) $d(x,y) = d(y,x)$,

(iv) $d(x,z) \leq d(x,y) + d(y,z)$.

Axiom 6.2 (Proportionality). *The distance between any two composite vectors is proportional to the degree of intensity.*

In MAROM, the distance measure d_{ik}, defined by the model (6.4) takes on different functional forms for an alternative k.

a) For two arbitrary nominal vectors $d_{ik}^{(N)}\left(a_{ik}^{(N)}, c_{ik}^{(N)}\right)$, denoting them simply as $x, y \in N$, the distance measure is the Tanimoto (also called Jaccard) coefficient [75]:

$$d_{ik}^{(N)}(x,y) = 1 - \frac{\alpha}{\alpha+\beta+\gamma} = \frac{\beta+\gamma}{\alpha+\beta+\gamma}, \tag{6.6}$$

where

$$\alpha = \sum_i \min(x_i, y_i), \quad \beta = \sum_i x_i - \alpha, \quad \gamma = \sum_i y_i - \alpha, \quad i \in N.$$

b) For two arbitrary ordinal vectors $d_{ik}^{(O)}\left(a_{ik}^{(O)}, c_{ik}^{(O)}\right)$, denoted them simply as $x, y \in O$, the distance measure is the Soergel number [74]:

$$d_{ik}^{(O)}(x,y) = \frac{\sum_i x_i + \sum_i y_i - 2\sum_i \min(x_i,y_i)}{\sum_i x_i + \sum_i y_i - \sum_i \min(x_i,y_i)}, \qquad i \in O. \tag{6.7}$$

c) For two arbitrary interval and ratio vectors, $d_{ik}^{(I,R)}\left(a_{ik}^{(I,R)}, c_{ik}^{(I,R)}\right)$, denoted them either $x, y \in I$, or $x, y \in R$ and introducing the L_2 norm of a vector x, we have

$$\|x\|_2 = \sqrt{\sum_i x_i^2} = \sqrt{x^{\mathrm{T}}x}, \qquad i \in I, \quad \text{or} \quad i \in R.$$

The distance measure then is the well-known Euclidean-metric:

$$d_{ik}^{(I,R)}(x,y) = \|x-y\|_2 = \sqrt{(x-y)^{\mathrm{T}}(x-y)}. \tag{6.8}$$

In Farkas (2004), proofs were given to show that the metric properties hold for (6.6) and (6.7).

It is convenient to define a distance function of the composite vectors d_C in an additive fashion, since for MAROM, the metric properties hold for its incorporated component distance functions and the scales are *linear*. Therefore, such a composite vector is also metric. Furthermore, it is *unique* and for each row, $0 \leq d_{ik}\big(\mathbf{a}_k^{(\cdot)\mathrm{T}}, b_i\big) \leq 1$ holds. The distance between any two $\overline{d}_i^{\,l}$ adjacent rankings of the alternatives is *proportional* to the degree of adjacency. The proportionality unit is taken to be one.

Theorem 6.1 *If the metric properties hold for the distance functions of the nominal, ordinal, interval and ratio vectors of a data matrix* $\mathbf{A}$ *defined in* (6.1)*, the composite distance vector* d_C *is also metric and it is unique and measures the true distance between any two composite vectors of* $\mathbf{A}$ *on an interval scale.*

Proof. The proof immediately follows from the earlier considerations. $\square$

Once the pairwise overall distances between each composite vector and the reference vector have been determined, a Pareto optimal solution (see in Appendix A, Definition A.1), the (column) vector of the relative standings (ratings) of the alternatives is obtained, $\mathbf{s} = [s_k]$, $k = 1, 2, \ldots, n$. Hence, the overall priority ranking of the alternatives simply yields from: $s_k^l = 1 - \overline{d}_k^{\,l}$, respectively. To establish either a $[0-1]$ or a $[1-100]$ interval scale, a normalization procedure is performed. For aggregating the single individual preferences into a compromise ranking the minimum variance method (Cook and Seiford [14]) is proposed. MAROM, with its accompanied software program written in FORTRAN 77, was applied to various fields of interest. One of these applications is described in the next section in detail.

Finally, we note that Farkas (2004) established a direct relationship between MAROM and the AHP. For that, he utilized the SR property and the structure of the entries (pairwise ratios) of a PCM. He used the relative standings, the output of MAROM as an input to AHP. This manner, a transformed matrix $\mathbf{S} = [s_{ij}]$, where $s_{ij} = s_i/s_j$, was constructed which is obviously in SR. Remarkable implications of this synthesis were that $\mathbf{S}$ turned to be a transitive matrix and, by this simple way, an interval scale outcome could readily be transformed onto a ratio scale. Thereby, applying the AHP methodology for a perfectly consistent $\mathbf{S}$, *true* priority scores may be gained.

Application 8: Evaluation, ranking and a scenario of alternative-fuel modes of buses

A characteristic feature of the modern age is the issue of ever growing urbanization, i.e. the physical growth of urban areas as a result of rural migration as well as the ever growing suburban concentration into many cities, particularly the very largest ones. The sustainable development of cities largely depends upon a sound urban transportation policy that is capable of drastically reducing air and noise pollution in the urban world in order to preserve human health and the environment. Total transport energy use and carbon emission are projected to be approximately 80% higher than current levels by 2030. Road transport accounts for by 23% of world energy-related CO_2 emissions (in 2006 approx. 6.3 Giga tons).

In the paper of Farkas (2013), the author focused on the modern technology and its applicability to mass transit systems as major contributors to sustainable urbanization. The main parameter in defining alternative-fuel solutions is the fuel mode. Worldwide efforts have made for developments and use of alternative fuels which possess different characteristics than the traditional ones. This issue has attracted immense interest in recent years, see e.g. in Romm [62]. Of the various options available for public transportation, efficient bus systems appears to be effective and affordable.

Tzeng et al. [84] reviewed the most promising developments of alternative-fuel buses suitable for urban areas and compared them to the characteristics of the conventional internal combustion diesel engine bus. They presented a comprehensive multi-attribute investigation of these alternative-fuel modes with a set of data provided by different groups of Taiwanese experts using the method called TOPSIS (Technique for

Order Preference by Similarity to Ideal Solution), which is one of the most popular advanced procedures for evaluating and ranking different alternatives, see Appendix A, section A.2.2.

In this study we compared the priority rankings and the performance scores of the alternative-fuel buses resulting from the use of the two methods MAROM and TOPSIS when they are applied to the same data set which was provided in the seminal paper of Tzeng et al. [84]. Furthermore, we intended to reveal deterministic interactions between the alternative-fuel modes, i.e., between their constituting attributes. Utilizing these so-called cross-impacts, a dynamic simulation model called KSIM [39] was used to make trend projections for the alternative-fuel modes in order to estimate the changes in their characteristics and in the resulting new ranking over a long-term perspective.

In the paper of Morita [53], an excellent overview was presented about the characteristics and the main directions of the engineering developments of automotive power sources to be expected over the next two decades. Morita suggested and set up four categories for the fuel-modes [53]: (i) *internal combustion engine vehicles* (ICEVs), (ii) *electric vehicles* (EVs), (iii) *hybrid electric vehicle*s (HEVs) and (iv) *fuel cell vehicles* (FCVs). Tzeng and his co-authors considered 12 alternative-fuel modes of buses for public transportation in their seminal paper [84]. We will utilize these twelve choices for the types of alternative-fuel vehicles denoting them as AFV$_k$, $k = 1, \dots, 12$.

AFV 1: Conventional diesel engine — CD
The diesel engine still is one of the major contenders as a power source in the 21st century. Its main advantages are low purchasing costs, flexibility to the speed of traffic and low sensitivity to road facility. However, it has very high exhaust emission rates (PM, NO_x, CO, CO_2). This vehicle is introduced in the set of alternatives in order to compare it with the new fuel modes.

AFV 2: Compressed natural gas — CNG
Interest for natural gas as an alternative fuel arises from its clean-burning qualities and its wide resource base. Natural gas has numerous benefits in terms of pollutants, comfort, and general abundance. CNG vehicles emit only slight amounts of CO and CO_2, they have high-octane value and they cost less than diesel buses. Meanwhile, natural gas vehicles are saddled with problems in many countries such as supply, distribution and especially risk of explosion.

AFV 3: Liquefied propane gas — LPG
There are countries that use this mode of fuel for public transportation. In Japan, Italy and Canada, 7% of transit buses are powered by LPG, and several European countries are planning to employ LPG vehicles, due to pollution considerations.

AFV 4: Fuel cell (hydrogen) — FC (H)
Research on a fuel cell-hydrogen bus has already been concluded with success. Test results with the experimental vehicle operating on hydrogen fuel indicate that this vehicle has a broad surface in the burning chamber, low burning temperature, and the fuel is easily inflammable. No detrimental substance is produced and only pure water, in the form of vapor, is emitted. A fully loaded fuel tank can last as far as 250 km.

AFV 5: Methanol — MET
The fuel of methanol is related to vehicles with gasoline engines. The combination rate of methanol in the fuel is 85% (so-called M85). The engine can run smoothly with any combination rate of gas with methanol, and methanol will act as an alternative fuel and help to reduce the emission of black smoke and NO_2 as well as pollutants and greenhouse gases. Fuel stations providing methanol are already available in several countries. The thermal energy of methanol is lower than that of gasoline, and the capability of continuous travel by this vehicle is inferior to that of conventional vehicles.

AFV 6: Electric vehicle - opportunity charging — E-OC
The source of power for the opportunity charging electric vehicle is a combination of a loaded battery and fast opportunity charging during the time the bus is idle. Whenever the bus starts from the depot, its battery will be fully charged. During the 10–20 sec when the bus is stopped, the power reception sensor on the electric bus (installed under the bus) will be lowered to the charging supply plate installed in front of the bus

stop to charge the battery. Within 10 sec of a stop, the power supply is done, so that the battery is charged with 0.15 kWh, which is adequate for it to move to the next bus stop.

AFV 7: Direct electric charging — E-DEC
The big appeal of electricity is a clean and quiet operating system. This is to be contrasted to its high costs and short cruising distance. The power for this vehicle comes from a loaded battery. Once the battery power is insufficient, the vehicle will have to return to the plant to conduct recharging. The development of a suitable battery is critical for this mode of vehicle.

AFV 8: Electric bus with exchangeable batteries — E-EB
Here, the goals are to accomplish a fast battery charge and achieve a longer cruising distance. The bus is modified to create more on-board battery space, and the number of on-board batteries is adjusted to meet the needs of different routes. The fast exchanging facility has to be ready to conduct a rapid battery exchange.

AFV 9: Hybrid electric bus with gasoline engine — HE-G
The electric-gasoline vehicle has an electric motor as its major source of power and a small-sized gasoline engine. When electric power fails, the gasoline engine can take over the drive and continue the trip. The kinetic energy rendered during the drive will be turned into electric power to increase the cruising distance of these vehicles.

AFV 10: Hybrid electric bus with diesel engine — HE-D
The electric–diesel vehicle has an electric motor and small-sized diesel engine as its major source of power. When electric power fails, the diesel engine can take over the drive and continue the trip, while the kinetic energy rendered during the drive will be turned into electric power to increase the cruising distance of these vehicles.

AFV 11: Hybrid electric bus with CNG engine — HE-CNG
The hybrid electric-CNG vehicle has an electric motor and a small-sized CNG engine as its major source of power. When electric power fails, the CNG engine takes over the drive and provides the power, with the kinetic energy produced converted to electric power to permit continuous travel.

AFV 12: Hybrid electric bus with LPG engine — HE-LPG
The hybrid electric-LPG vehicle has an electric motor and a small-sized LPG engine as its major source of power. When electric power fails, the LPG engine takes over the drive and provides the power, with the kinetic energy produced converted to electric power to permit continuous travel.

Tzeng et al. [84] used the following 11 single criteria to evaluate the alternative-fuel modes:

C 1: Energy supply — Annual costs of supply, storage and fuel
C 2: Energy efficiency — Energy consumption related to fuel heating value
C 3: Air pollution — Chemical substance harmful to health
C 4: Noise pollution — Noise produced during operation
C 5: Industrial relationship — Impact on other locomotive industry branches
C 6: Costs of implementation — Costs of production, purchase and implementation
C 7: Costs of maintenance — Annual costs of maintenance
C 8: Vehicle capacity — Cruising distance, gradeability, speed of vehicle, etc.
C 9: Road facility — Necessary features of road for the vehicle, e.g. pavement, slope, etc.
C 10: Speed of traffic flow — Conformity to traffic flow
C 11: Sense of comfort — Traveling comfort and aesthetic appeal

In Table 6, the normalized average weights (relative importance of each criterion) are indicated according to Tzeng [84, p. 1377]. These weights were determined by groups of Taiwanese experts using the AHP method. In Table 6, the averages of the assessed values for the performance of each of the alternative-fuel modes with respect to every criterion are also presented. These values, denoted them by u_{ij}, $0 \le u_{ij} \le 1$, are taken from Tzeng [84, p. 1378]. They were derived through conducting a survey by applying a Delphi procedure and using experts' judgments that was repeated twice. The experts represented academic faculties, governmental departments, manufacturing industries, energy committees and research institutes.

Table 6. Criteria weights and results of the value assessment for the alternative-fuel vehicles [84]

	C 1	C 2	C 3	C 4	C 5	C 6	C 7	C 8	C 9	C 10	C 11
Weight	0.0313	0.0938	0.1661	0.0554	0.0629	0.0829	0.0276	0.1239	0.0805	0.1994	0.0761
AFV 1	0.82	0.59	0.18	0.42	0.58	0.36	0.49	0.79	0.81	0.82	0.56
AFV 2	0.77	0.70	0.73	0.55	0.55	0.52	0.53	0.73	0.78	0.66	0.67
AFV 3	0.79	0.70	0.73	0.55	0.55	0.52	0.53	0.73	0.78	0.66	0.67
AFV 4	0.36	0.63	0.86	0.58	0.51	0.59	0.74	0.56	0.63	0.53	0.70
AFV 5	0.40	0.54	0.69	0.58	0.51	0.52	0.68	0.52	0.63	0.60	0.70
AFV 6	0.69	0.76	0.89	0.60	0.72	0.80	0.72	0.54	0.35	0.79	0.73
AFV 7	0.77	0.79	0.89	0.59	0.73	0.80	0.72	0.47	0.44	0.87	0.75
AFV 8	0.77	0.79	0.89	0.59	0.73	0.80	0.72	0.51	0.48	0.87	0.75
AFV 9	0.77	0.63	0.63	0.52	0.66	0.63	0.65	0.67	0.70	0.80	0.74
AFV 10	0.77	0.63	0.51	0.58	0.66	0.63	0.65	0.67	0.70	0.80	0.74
AFV 11	0.77	0.73	0.80	0.48	0.63	0.66	0.65	0.67	0.71	0.62	0.78
AFV 12	0.77	0.73	0.80	0.48	0.63	0.66	0.65	0.67	0.71	0.62	0.78

The MAROM procedure requires that nature of the data for each criterion be adequate with the properties of the type of the scale of measurement to which these data correspond. Therefore, as the first step, each criterion should be assigned to the appropriate scale of measurement. In addition, the number of criteria was extended from 11 to 15, because in the article of Tzeng et al. [84] some additional information was presented which were not directly captured by their analysis. These supplementary data related to a number of relevant engineering and chemical characteristics of alternative fuels originated from reliable sources (physical measurements) were presented in [84, pp. 1382–1383] with their associated units of measurement. To preserve the uniformity of the two data sets as much as possible (which were used by Tzeng et al. [84] and the present author; see Table 6 and Table 7) only minimal changes have been made. This way, criteria C4, C5, C9, C10 and C11 of the original data set have been retained, but they were assigned to ordinal scales so that their original performance values, u_{ij}, see them in Table 6, were converted to rank numbers by using a nine-grade ordinal scale $[1, 1.5, 2, 2.5, \ldots, 5]$, where an ideally best performance, if there exists any, received 5.

Utilizing the technical data collected by Tzeng et al [84], several new criteria were introduced. As seen in Table 7, these are: 'Depot', which can be small or large characterizing the depositary needs of the buses, as a nominal variable [0 or 1], while 'Cruising distance', 'Number of passengers', 'Maximum speed' for urban/sub-urban services and 'Recharge time' are ratio scale variables with specific units of measurements. They constitute the extended form of the "old" criterion 'Vehicle capability' whose weight has been uniformly allocated to them. The "old" criterion 'Energy efficiency' is a dimensionless variable, since it gives the ratio of the alternative-fuel efficiency/fuel heating value related to that of the diesel bus, and hence, it is reasonable to assign it to an interval scale. We hoped that establishing this new data base for the same problem provides us more powerful and reliable evaluation outcomes. Table 7 presents this reformulation of the original data set that meets the requirements of the theory of measurement. Here, the characteristic values for the 12 alternative-fuel buses, the 15 criteria weights and the aggregated weights for the scales of measurement are indicated.

The results of the multi-criteria evaluation of the 12 alternative-fuel buses are shown in Table 8. Here, both the ranks and the evaluation indices called relative standings or scores yielded by TOPSIS (basic and compromise solutions) and MAROM (for the individual and the aggregate weighting cases) are indicated on interval scales. As it does not come as a surprise, the two methods have produced rather different rankings and scores. Comparisons of the findings, however, should be made very carefully.

As a remarkable outcome, observe the big differences in the ranks of the conventional diesel engine bus. The last position of the diesel engine in the TOPSIS rankings seems to be rather strange regarding the fact that Tzeng et al.'s investigations refer to the year 2005. It is also striking that there are significant differences in the priority scores of the alternative-fuel modes produced by the two methods. We intend not to go into detailed technical explanations. We believe the MAROM ranking reflects better the situation at around 2005 than that of the TOPSIS. The relative high positions of the conventional diesel engine bus in the MAROM rankings as opposed to those obtained for the buses of alternative-fuel modes follows mainly from the tardiness of the required engineering developments and the limited bus manufacturing capabilities, as well as the weak achievements of the civil initiatives concerning environmental protection. However, there is no doubt as urban mass transit technology gets stronger and improves, more buses will be powered by alternative means in the search for more efficient energy use, cleaner air, quieter operation and more traveling convenience, especially, if they could efficiently serve in sub-urban areas as well.

Table 7. Input data of the alternative-fuel vehicles for MAROM proposed by Farkas (2013)

		AFV1	AFV2	AFV3	AFV4	AFV5	AFV6	AFV7	AFV8	AFV9	AFV10	AFV11	AFV12
Nominal scale	Aggregated weight of nominal scale				0.0666								
	Criterion weight				0.0248								
	Best value on nominal scale for criterion				C1	1							
	1. Depot	1	0	0	1	1	0	0	0	0	0	0	0
Ordinal scale	Aggregated weight of ordinal scale				0.3333								
	Criteria weights				0.0805	0.0554	0.0629	0.1994	0.0761				
	Best values on ordinal scale for criteria				C2-C6	4.0	3.0	3.5	4.5	4..0			
	2. Road facility	4.0	4.0	4.0	3.0	3.0	1.5	2.0	2.5	3.5	3.5	3.5	3.5
	3. Noise pollution	2.0	3.0	3.0	3.0	3.0	3.0	3.0	3.0	2.5	3.0	2.5	2.5
	4. Indust. rel. ship	3.0	3.0	3.0	2.5	2.5	3.5	3.5	3.5	3.5	3.5	3.0	3.0
	5. Speed of traffic	4.0	3.5	3.5	2.5	3.0	4.0	4.5	4.5	4.0	4.0	3.0	3.0
	6. Sense of comfort	3.0	3.5	3.5	3.5	3.5	3.5	4.0	4.0	3.5	3.5	4.0	4.0
Interval scale	Aggregated weight of interval and ratio scales				0.6000								
	Criterion weight				0.0938								
	Best value on interval scale for criterion				C7	10.9							
	Worst value on interval scale for criterion				C7	0.7							
	7. Energy efficiency [dim.less]	1.0	0.8	0.7	1.9	0.8	10.9	5.5	3.2	1.5	1.5	1.5	1.5
Ratio scale	Criteria weights				0.0313	0.1661	0.0248	0.0248	0.0248	0.0248	0.0248	0.0829	0.0276
	Best values for criteria				C8-C15	3875	0.30	500	80	120	10	100000	10410
	Worst values for criteria				C8-C15	46600	30.15	80	40	60	360	624000	30720
	8. Fuel costs [1000 NT$]	14000	11450	15000	46600	14495	3875	4000	8000	7880	7450	7250	7300
	9. Exhaust emission (PM+NOx+HC+COx) [%]	30.15	19.27	8.2	6.78	12.71	0.30	0.35	0.38	9.97	11.25	10.30	10.55
	10. Cruising distance [km]	450	500	400	325	225	100	80	220	250	250	350	350
	11. Number of passengers [No]	80	70	70	60	60	50	40	40	50	50	55	55
	12. Max speed [km/h]	120	80	90	75	110	60	65	65	70	70	75	75
	13. Recharge time [min]	10	200	100	10	10	360	300	300	360	360	360	360
	14. Costs of implementation [1000 NT$]	100000	420000	300000	624000	144000	340000	360000	380000	400000	420000	440000	450000
	15. Costs of maintenance [1000 NT$]	11400	10410	12500	30720	14700	18495	19600	19200	22200	22400	23500	23800

Hereafter we considered each alternative-fuel mode as being a quantity Q (a compound construct, as each AFV constitutes 15 variables). These attributes are listed in Table 7. Furthermore, the set of the alternative-fuel buses were regarded an interrelated complex system as it is apparent that in the course of their evolution they interact with one another. At this point, a question of vital importance can be raised. Namely, how the characteristics of these alternative-fuel modes will change and what will their spread in public transportation means look like over the successive two decades. To make technology assessments and study the dynamic behavior of this system, we employed a dynamic model of Kane [39] which is based on computer simulation.

Table 8. Comparison of the rankings and the evaluation scores for TOPSIS [84] and MAROM

	TOPSIS				MAROM			
	Rank Basic	Score	Rank Compr.	Score	Rank Indiv.	Score	Rank Aggreg.	Score
Electric bus with exchangeable batteries	1	0.945	1	0.975	5	0.514	4	0.675
Electric bus with opportunity charging	2	0.933	3	0.964	7	0.498	3	0.677
Electric bus with direct charging	3	0.931	2	0.967	4	0.514	2	0.681
Hybrid electric with gasoline engine	4	0.749	9	0.756	9	0.482	7	0.630
Hybrid electric with CNG engine	5	0.700	4	0.889	11	0.449	11	0.599
Hybrid electric with LPG engine	6	0.700	5	0.889	12	0.448	12	0.599
Hybrid electric with diesel engine	7	0.700	11	0.488	8	0.484	8	0.629
Fuel cell (hydrogen)	8	0.563	6	0.865	3	0.733	10	0.601
Methanol	9	0.527	10	0.698	1	0.791	1	0.691
Compressed natural gas engine (CNG)	10	0.399	7	0.830	10	0.467	9	0.611
Liquidate propane gas engine (LPG)	11	0.345	8	0.830	6	0.499	5	0.670
Conventional diesel engine bus	12	0.301	12	0.097	2	0.785	6	0.650

Kane procedures for modeling such systems require that we specify a set of quantities Q (AFVs in our case); a set of binary interactions C, between any two pairs (q_i, q_j), $i, j = 1, \ldots, n$, including possible self-interactions, defined on $Q \times Q$; and a set of initial values for each of the quantities q_i, denoted as q_{i0} (individual scores from MAROM ranking as given in the third column of Table 8). This model conforms well to our problem, since all variables q_i are bounded $0 < q_i(t) < 1$ for all $i = 1, \ldots, n$, and for all $t > 0$, and, thus, no rescaling is needed. The projected trends of the variables (AFVs) will be of sigmoidal type as the solution of the following differential equation (for small Δt time increments, i.e., for one iteration in the simulation run):

$$\frac{dq_i}{dt} = -\sum_{j=1}^{n} c_{ij} q_i q_j \ln q_i, \qquad i = 1, \ldots, n,$$

where c_{ij} is a binary interaction coefficient of q_j upon q_i. From here, it is clear that q_i accumulates the effect of q_j since it is easy to see that:

$$q_i(t) = q_{i0} + \int_{j=1}^{n} f[q_j(\tau)]d\tau, \qquad i = 1, \ldots, n.$$

Observe here that the structure of the KSIM model doesn't imply that the interaction coefficients are constants.

To gain insight into the conjectures of pairwise causal relationships of deterministic type between the AFVs, the actual interaction coefficients were revealed following an extensive research of the related literature and juries of executive opinion. For this purpose, interviews with different experts' groups (formed from researchers and faculties of transportation engineering from the Technical University, Budapest) were conducted and repeated three times to achieve a compromise decision to confirm structure; see the cross-impact matrix **C**=$[c_{ij}]$ representing a coherent pattern of causality assertions in Farkas (2014a, p.218). To simplify our system we have merged AFVs which had identical individual scores: Direct electric charges – Electric bus with exchangeable batteries (DEL); Hybrid electric bus with gasoline – diesel engines (HGD) and Hybrid electric bus with CNG – LPG engines (HCL).

In Figure 25 the projected trends, as a result of the interactions among the different alternative-fuel modes over a two decade time horizon after completing 50 simulation runs, are exhibited. Tendency and the changes in the alternative-fuel modes could be analyzed from this scenario by keeping in mind that any change in the behaviour of an impacted AFV is a result in the common effect of its self-development and the changes in its constituting variables caused by the total impact of the changes in the impacting AFVs as well as different external factors, i.e., new international regulations, users' concerns, etc. From the new priority ranking in Figure 25, it turns out that the Fuel cell (hydrogen) bus will take over the "lead" before the Hybrid

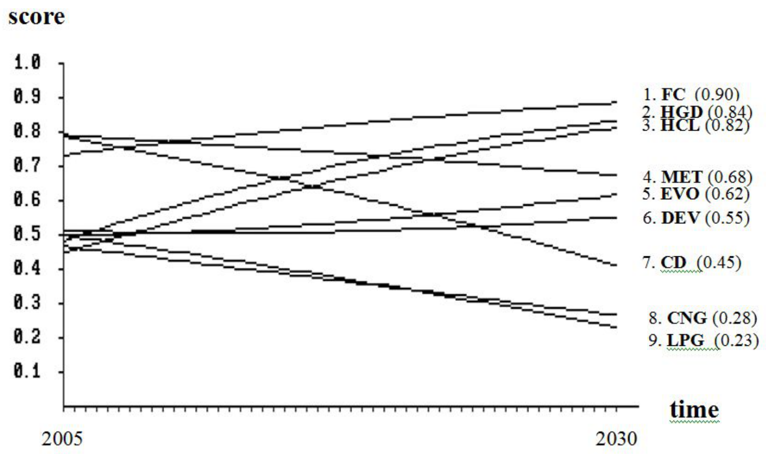


Figure 25. Projected trends and scores of the alternative-fuel modes of buses [Farkas (2014a)]

electric buses operating with gasoline/diesel engines and the Hybrids with CNG/LPG engines by 2030. The performance score of the conventional diesel bus will decline significantly. Similar decays can be observed for the CNG and LPG buses.

The projected trends of this model depicted in Figure 25 reflect the international directions in terms of both technological developments and environmental protection fairly well. Japanese manufacturers are seriously investing in hybrids, what they see as a promising market segment. US manufacturers are starting to use hybrids to disguise the environmental impacts of vehicles that consumers want. European manufacturers respond to the market's need for high performance and less polluting vehicles by investing in diesel technologies and tend to ignore or dismiss hybrid technology as an overly complicated half-solution that introduces excess weight and hampers performance. Instead, they intend to favour fuel cell research as the way forward. In their view, hybrid buses are only a medium-term interim solution filling the gap until a more efficient technology, ideally fuel cell buses mature and become available. It should be mentioned that, most recently, the struggle to reduce vehicle emission has speeded up strongly. As an illustration for this, the bus emission standards for NO_x and PM in the US and in the EU have become much more rigorous between 2000 and 2010, i.e., the NO_x emission in [g/kWh] should be reduced from 5.8 down to 0.16 and from 5.0 down to 2.0, respectively and the PM emission from 0.075 down to 0.0075 and from 0.1 to 0.02, respectively [87]. Technical limitations of electric and hybrid vehicles are mainly related to capacity, durability and price conditions of the batteries. In summary, our projections for the AFVs are fairly close to those scenarios reported by the European Commission [58], with the one exception that the CNG vehicles performance and popularity of their use in public transport seem to be a little bit under-estimated by 2030.

Summary and Theses

The primary objective of this dissertation was to provide the academic community with an appreciation of the power, as well as the limitations of a number of advanced mathematical approaches and specialized techniques, which can be used in transportation and civil engineering requiring quantitative decisions. The emphasis was upon developing problem solving methods in the field of multiple criteria decision making, by providing a careful discussion of the preliminaries, problem formulation, mathematical analysis with proofs, solution procedures including, where it was necessary, the development of appropriate algorithms, and utilizing profound numerical examples. Applications were also integral parts of this work.

The first part of the text started with the characterization of transportation systems followed by the development of a novel transport policy model for sustainable transport that conforms fairly well to the contemporary directions laid down by the decision makers of present EU and domestic authorities. The next part showed that how can the stakeholders, i.e., transport providers and users, be involved in the decision making processes of transportation projects, and an efficient treatment of the multidisciplinary nature of these problems was proposed. Special focus was devoted to economic, social and environmental issues beyond that of the engineering type. The next chapter aimed to detect and clarify the explicit mathematical background of a world widely used technique, the analytic hierarchy process, especially concentrating on one of its most controversial issues the phenomenon of rank reversal. An extension of the original construction of the pairwise comparison matrix to complex numbers enabled a more generalized use of these matrices in some important fields in economics (static and dynamic input-output analyses) and in engineering (vehicle dynamics) also. The next chapter covered the development of a new scaling method by developing a rank-one transitive approximation to a general SR matrix in a least-squares sense. In formulating an adequate optimization model, solutions to both the linear and the nonlinear problems were presented with proofs and a numerical analysis of the findings. Additionally, sufficient conditions for possible non-uniqueness of the solutions were given. A flexible, multiplicative, two parameter perturbation of exponential type was proposed and successfully applied to a relevant nonlinear vibration problem of railway vehicles running along the track in the form of an adequate input spectral density matrix. In the next chapter a recursive rank-one residue algorithm was developed whose properties were proven to be valuable in supporting the decision making methodology of the AHP. It was shown that there exists a mutual correspondence between two entirely different approaches, the eigenvector method and the least-squares technique. Based on the results obtained by the use of this algorithm, new measures of inconsistency for perturbed SR matrices were derived. In the last chapter, the formulation of a combined MOO/MCDA technique was discussed that perfectly suits to the requirements of the theory of measurement as concerns the nature of the rough (input) data. A comprehensive application to an up-to-date problem, in particular, an interactive scenario and evaluation of alternative-fuel modes of buses used in urban transportation demonstrated the viability of this compensatory multi-attribute decision making method called MAROM. The results generated from MAROM were in a remarkable matching to those of reported by some respective European Transport and Environmental Committee documents.

Major results of my scholarly research are laid into the next six theses following the order of their discussion in the text and are conceived as follows.

Thesis 1

Effective transportation in the European context is a sine qua non to integration. In our modern age, transport policy faces many, although difficult, but fascinating challenges, among others, the rapid increase in the amount of travel and its concomitant impact on global warming; the environmental externalities mainly the effects of the ever growing air pollution to human health; the regional and local imbalances between transportation supply and demand; the shortages of transshipments across countries and the required logistical services and the lack of a proper inter-modal freight transport. Recognizing these challenges and some shortcomings, I prepared a novel framework called a sequential transport policy (STP) model for sustainable transport. Responding to these challenges, I recommended four subsequent stages with distinct transport policy goals, which are, in turn

Stage #1. Policy goal: REFORMULATE DEMAND PATTERNS OF TRANSPORT

Stage #2. Policy goal: TRANSITION TO OTHER TRANSPORTATION MODES

Stage #3. Policy goal: IMPROVE TRANSPORT EFFICIENCY

Stage #4. Policy goal: ENHANCE AVAILABILITY AND USE OF RENEWABLE ENERGY POWERED TRANSPORT

Every stage has the same unified internal structure and for each, I indicated the particular objectives, benefits and drawbacks. I designated the tools and instruments needed to achieve these objectives and proposed certain indicators to measure as to whether the targeted goals have been met. In a strong conformity with the STP model, a multistage, forward recursion, dynamic programming model was proposed to attain optimal transport policy through the accomplishment of the single stages in a consecutive manner and assisting its implementation for practical uses:

$$f_N^*(S_n) = \underset{d_1,d_2,\ldots,d_N}{\text{optimize}} \{r_1(d_1,S_1) \circledast r_2(d_2,S_2) \circledast \cdots \circledast r_N(d_n,S_N)\}\,.$$

The STP model seems to be useful in both global and local sense for governmental departments of transportation, transport infrastructure and service providers, local authorities, social experts, transportation engineers as well as for users and urban communities. The model appears to outperform many of its counterparts in the subject in terms of its definite and radical goals, coherent and logical structure and the huge variety of its assigned tools and instruments. Possible forthcoming directions of its improvement are the addition of negative feedbacks built-in between the stages which would further increase its effectiveness and efficiency, but this effort is subject of future research.

Related publication to Thesis 1 has appeared in Farkas (2014b).

Thesis 2

In case of the majority of transportation and civil engineering projects the involvement of the representatives of the stakeholders and public users is not at all sufficient, or at most to only a lesser extent, as it would be necessary. Another pitfall of these projects is that, usually, all the relevant characteristic features of their typically multidisciplinary nature cannot be taken into consideration simultaneously, due to the limitations of the existing conventional planning, design and evaluation methods and technical devices that are used nowadays in practice. These facts motivated me to create an adequate tool that suits to the specific requirements of transport and civil engineering projects as much as possible. For these purposes, I have made the following achievements:

(i) I have conducted a comprehensive survey of the literature and discussed the multi-objective optimization and multi-criteria decision analysis methods, which have widely been accepted by the academic community, and then, I analyzed their similarities as well as their distinguishing features (Chapter 2 and Appendix A). I collected and briefly described a great number of applications to different transportation and civil engineering problems. Finally, I gave a solid reasoning in regard to their theoretical, structural and implementation characteristics in order to further enhance their excessive use in these areas of interest.

(ii) Contrary to the traditional approach which starts with the development of alternative options, specification of values and criteria and ends by the evaluation and recommendation of a particular option, I proposed just a reversed procedure, the value-focused approach that first focusing on the specification of the values (value-structure) as the fundamental elements of decision analysis, then considers and develops the values' feasible options which then are evaluated based on the predefined value and criteria structure. This implies that the decision alternatives should be generated in such a way that the desired values specified for a decision problem are best achieved. To implement the value-focused approach, a top-down structure in the form of a four-level hierarchy was created to define the goal, the objectives and the associated multidisciplinary indicators of the required transportation network.

(iii) Using an adequate intelligent transportation system to designing transport infrastructure, I implemented the proposed approach on the example of a real-world project of planning a metro-rail network system where I utilized an integrated GIS/MCDA methodology. I used an advanced remote sensing and geographic information system (ILWIS) which also contains a strong spatial multi-criteria analysis module (SMCA). Selection of the locations for the stations and the best option for the track were shown and explained in detail (Application 1).

Related publications to Thesis 2 have appeared in Farkas (2009a), (2009b), (2010), (2011b).

Thesis 3

Ever since its first publication in the late 1970's, numerous criticisms of the AHP method have also appeared in the literature, mainly attacking its controversial phenomenon the rank reversal of its generated priority ranking. As strange as it may sound, but the opponents have enumerated arguments based on numerical analysis (sometimes even on verbal speculations) only. I have conjectured that this undesired issue must have a clear mathematical reasoning. Since that time, we have had the following pioneering contributions to this topic:

(i) We have introduced and defined symmetrically reciprocal (SR) matrices $\mathbf{A}$, and transitive matrices $\mathbf{B}$, and we have developed their spectral properties (Proposition 3.1).

(ii) We have provided the principal (Perron) eigenvectors for certain SR perturbations of transitive matrices in explicit forms and established exact intervals over which a rank reversal occurs for any non-transitive SR matrix (Theorem 3.1 and Theorem D.1 in Appendix D).

(iii) We have derived the characteristic polynomial of an SR perturbed matrix $\mathbf{A}_\mathrm{p}$, where the perturbation factors $\delta_i \neq 1$, $i = 1, 2, \ldots, n-1$, are arbitrary positive numbers (Proof in Appendix B):

$$p_n^\mathrm{P}(\lambda) \equiv \det \mathbf{K}_\mathrm{p}(\lambda) = \lambda^{n-3} \left\{ \lambda^3 - n\lambda^2 + (n-1) \sum_{i=1}^{n-1} (1-\delta_i)\left(1 - \frac{1}{\delta_i}\right) - \right. \\ \left. - \sum_{i=1}^{n-1} \left(1 - \frac{1}{\delta_i}\right) \sum_{i=1}^{n-1} (1-\delta_i) \right\}.$$

(iv) We extended the scope of interpretation of matrices $\mathbf{A}$ and $\mathbf{B}$ to complex numbers. Then, by dividing the characteristic polynomial p_n^P by n^3 and normalizing its principal eigenvalue as $\mu = \lambda/n$, a general form of the trinomial equation as function of μ is obtained as follows:

$$L(\mu) = \mu^3 - \mu^2 - C_\mathrm{p} = 0.$$

For the simple perturbed matrix $\mathbf{A}_\mathrm{S}$, the constant term C_S, with $r = \sqrt{|\delta|}$ and $t = \mathrm{arc}\sqrt{\delta}$, yielded:

$$C_\mathrm{S} = \frac{n-2}{n^3} \left(re^{\mathrm{i}t} - \frac{1}{r} e^{-\mathrm{i}t} \right)^2.$$

Depending upon the values of the real parameters r and t, three cases were distinguished and applications for each one, emerging from the very different three fields of interest: decision theory, macroeconomics and vehicle dynamics, have been presented (Applications 4,5,6).

Related publications to Thesis 3 have appeared in Farkas (2007), (2008), in Farkas and Rózsa (1996b), (2001), and in Farkas, Rózsa and Stubnya (1998), (1999a), (1999b), (2000).

Thesis 4

This thesis summarizes the results of the research I have conducted with my co-authors to determine the "best" rank-one transitive matrix approximation $\mathbf{B}$ to a general SR matrix $\mathbf{A}$ in a least-squares sense. Major results were as follows:

(i) First, we considered the sub-optimal, but linear problem of minimizing the Frobenius-norm $\|\mathbf{WA} - \mathbf{EW}\|_\mathrm{F}^2$, where $\mathbf{W}$ is a positive definite diagonal matrix if and only if the unknown vector of the weights $\mathbf{w}$ is an element-wise positive column vector. We introduced the $n \times n$ matrix $\mathbf{E}$ whose entries are all equal to one. To avoid the trivial solution $\mathbf{W} = 0$, an inhomogeneous linear constraint was added. Thus, for a given element-wise positive vector $\boldsymbol{\Phi}^\mathrm{T} = [\Phi_1, \Phi_2, \ldots, \Phi_n]$, one must find the matrix $\mathbf{W}_0(\boldsymbol{\Phi})$ for which

$$S_0^2(\boldsymbol{\phi}) := \inf_{\boldsymbol{\phi}^\mathrm{T}\mathbf{w}=1} \|\mathbf{WA} - \mathbf{EW}\|_\mathrm{F}^2 = \|\mathbf{W}_0\mathbf{A} - \mathbf{EW}_0\|_\mathrm{F}^2.$$

We showed that there is an optimal choice of $\boldsymbol{\Phi}$, i.e. there is a $\boldsymbol{\Phi}_\mathrm{opt}$ and associated $\mathbf{W}_0(\boldsymbol{\Phi}_\mathrm{opt})$ for which $S_0(\boldsymbol{\Phi}_\mathrm{opt}) \leq S_0(\boldsymbol{\Phi})$ for all $\boldsymbol{\Phi} > 0$. With $\boldsymbol{\Phi}_\mathrm{opt}$, the optimal solution $\hat{\mathbf{w}}$ to the linear problem may be produced that is superior to the hitherto reported result. (Theorem 4.1)

(ii) To find a solution to the nonlinear problem we considered the minimization of the Frobenius-norm:

$$S^2(\mathbf{w}) := \|\mathbf{A}-\mathbf{B}\|_{\mathrm{F}}^2 = \min \sum_{i=1}^{n}\sum_{j=1}^{n}\left(a_{ij}-\frac{w_j}{w_i}\right)^2 .$$

To describe the necessary conditions we formulated the system of n nonlinear equations as: $\mathbf{R}(\mathbf{w})\mathbf{w} = 0$, where the skew-symmetric matrix has the following closed form:

$$\mathbf{R}(\mathbf{w}) = \mathbf{W}^{-2}\left(\mathbf{A}-\mathbf{W}^{-1}\mathbf{E}\mathbf{W}\right) - \left(\mathbf{A}-\mathbf{W}^{-1}\mathbf{E}\mathbf{W}\right)^{\mathrm{T}}\mathbf{W}^{-2}.$$

This expression was proven to be more generally useful in the approximation of merely positive matrices by transitive matrices. In order to be able to employ the Newton-Kantorovich procedure, a linear equality constraint, $\mathbf{c}^{\mathrm{T}}\mathbf{w} = 0$, was added to the nonlinear equations, where $\mathbf{c}^{\mathrm{T}} = [1, 0, \ldots, 0]$ stands for holding $\mathbf{w}$ in a bounded set in the steps of the iteration. For an SR matrix $\mathbf{A}$, a convenient normalization condition is $w_1 = 1$. Computational experience with the formulated system of n inhomogeneous nonlinear equations has shown that the iteration process was always convergent and produced a local minimum, $\mathbf{w}^*$. The Hessian matrices were found to be positive definite. (Propositions 4.1, 4.2, 4.3 and 4.4)

(iii) Due to the well-known non-convex nature of the least-squares optimization problems, we investigated the non-uniqueness problem of the solution to the inhomogeneous system of n equations generated in (ii). As a result, we have given sufficient conditions for the occurrence of multiple solutions. (Propositions 4.5, 4.6 and Theorem 4.2)

(iv) Utilizing the rank-one transitive matrices generated in (ii), a flexible, user defined, two parameter, multiplicative perturbation structure of exponential type was proposed and successfully applied to a nonlinear vibration problem of railway vehicles using an adequately constructed input spectral density matrix. (Proposition 4.7 and Application 7)

Related publications to Thesis 4 have appeared in Farkas, Lancaster and Rózsa (2003), (2005), in Farkas, György and Rózsa (2004) and in Farkas and Rózsa (2004).

Thesis 5

A least-squares recursion algorithm was proposed for balancing positive SR matrices. The following results have been achieved:

(i) A recursive rank-one residue iteration called triple R-I was developed by establishing a successively adjusted sequence of rank-one matrices, based on the reasonable conjecture that the 'best' approximation of an entry a_{ij} of an SR matrix $\mathbf{A}$ is $w_j^{*(0)}/w_i^{*(0)}$, thus

$$\left[\frac{w_i^{*(0)}}{w_j^{*(0)}}a_{ij}\right] = \mathbf{W}_0^*\mathbf{A}\mathbf{W}_0^{*(-1)} \approx \mathbf{E}, \qquad ij = 1, 2, \ldots, n.$$

The main idea was to achieve continuous improvement in approximating $\mathbf{E}$ consecutively, in the further steps of the iteration. For this purpose, a positive $n \times n$ matrix $\mathbf{H}_k$ and an updating rule: $\mathbf{H}_k = \mathbf{W}_{k-1}\mathbf{H}_{k-1}\mathbf{W}_{k-1}$, were introduced. Next, using the optimal solution of the linear problem, $\hat{\mathbf{w}}$, as a starting vector and is given by Thesis 4 (i), the following system of nonlinear equations were solved in each iteration step k:

$$\left\{\mathbf{W}_k^{-2}\left(\mathbf{H}_k-\mathbf{W}_k^{-1}\mathbf{E}\mathbf{W}_k\right)-\left(\mathbf{H}_k-\mathbf{W}_k^{-1}\mathbf{E}\mathbf{W}_k\right)^{\mathrm{T}}\mathbf{W}_k^{-2}\right\}\mathbf{W}_k\mathbf{e}=0,\quad k=1,2,\ldots\,.$$

The algorithm terminates at step $k=q$, once the numerical error falls below a prescribed tolerance, a reasonable small $\varepsilon>0$. Convergence proofs were given that the sequences $\{\mathbf{H}_k\}$ and $\{\mathbf{W}_k\}$ converge to a limit point $\mathbf{H}_q^*$ and to the identity matrix $\mathbf{I}_n$, respectively. (Theorem 5.1 and Proposition 5.1), [Algorithm is given in Appendix E]

(ii) We have shown that there exists a direct relationship between the eigenvector method and the least-squares technique for matrices with such a structure.

(iii) We have shown that the triple R-I algorithm is analogous to a diagonal similarity scaling of nonnegative matrices which is a well-known procedure in numerical linear algebra. Therefore, both stabilized matrices, the limit matrix $\mathbf{H}_q^*$ and (obviously) $\mathbf{I}_n$ are balanced since they are, as we have shown, in line-sum-symmetry.

(iv) Utilizing the probabilistic nature of the entries of pairwise comparison matrices, we assumed that they represent log-normally distributed random variables. Two new measures of inconsistency for perturbed SR matrices were derived, the average error and the variability of the errors due to perturbations of an SR matrix $\mathbf{A}$. A numerical analysis has suggested the conjecture that the measure of variability of the perturbation errors of an SR matrix i.e., the geometric standard deviation of the elements h_{ij} of the stabilized matrix $\mathbf{H}_q^*$ seems to share global properties with the AHP's consistency measure μ. (Propositions 5.1, 5.2 and 5.3)

Related publications to Thesis 5 have appeared in Farkas (2012) and in Farkas and Rózsa (2013).

Thesis 6

The challenges that inspired me to elaborate a combined MOO/MCDA multi-criteria scaling method called MultiAttRibute Object Measurement (MAROM) were twofold. The first one concerns my observations which showed that although there exist a great number of such methods most of them have been designed to evaluate alternatives on *one* particular scale of measurement only, yet the alternatives are usually characterized by many attributes which may have totally different properties. Thus, they should be assigned to different scales of measurement. The second indication has come from my long application experience with human decision makers. I have found that the respondents are unable to subscribe the requirements of rational decision making (von-Neumann Morgenstern axioms) when they are confronted with the obligation to provide judgements on the basis of relative measurements. My responses on these issues were the following:

(i) This multiple technique for systems evaluation applies to absolute measurement and works with the original (raw) data, a_{ik}, arranged in a data matrix, disregarding as to whether they have emerged from subjective estimates or physical measurements. Additionally, the method MAROM conforms perfectly to the theory of measurement, because according to its basic concept each criterion is to be assigned to its corresponding scale of measurement.

(ii) As a novel contribution, I have employed metric distance functions, $d_{ik}^{l} = b_i - a_{ik}^{\prime l(\cdot)}$, for the kth alternative from a reference point b_i on attribute i, on each scale of measurement (nominal, ordinal, interval and ratio); the importance weights, w_i^l, on each attribute i; and a random error term, ε_k, for the measurement errors. Then, I have constructed the following model for preference measurement which is based on the overall distance, $\overline{d}_k^{\,l}$, of alternative k from the "ideal" alternative, for the lth voter:

$$\overline{d}_k^{\,l} = \sum_{i=1}^{m} w_i^l d_{ik}^l + \varepsilon_k^l, \qquad k = 1, \ldots, n; \quad l = 1, \ldots, q.$$

Next, determination of the relative standings of the alternatives followed by a normalization and aggregation procedure of the single scores (ratings) were taken place to obtain the aggregated overall composite scores of the alternatives which have been evaluated by the respondents.

(iii) A proof was given that the overall performance indices (relative standings) appear on an interval scale. (Theorem 6.1)

(iv) I have shown that the output of MAROM may easily be transformed up to a ratio scale. This way, the transformed data as pairwise ratios matches to the AHP methodology directly. The huge benefit is that the pairwise comparison matrix will be perfectly consistent.

(v) Although my statement is based on numerical evidence only, yet the method seems to produce more realistic outcomes than those of generated by other scaling methods as it was demonstrated by an up-to-date transport and environmental application in which I evaluated, ranked and projected trends from a generated scenario of alternative-fuel modes of buses used in urban transportation. (Application 8)

Related publications to Thesis 6 have appeared in Farkas (2004), (2006), (2013) and (2014a).

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Appendix A

Summary of Multi-Criteria Decision Making (MCDM) Methods with Applications to Transportation/Civil Engineering Problems

In this Appendix, a comprehensive summary of scholarly recognized multi-criteria decision making methods (MCDM) with some robust applications to transportation/civil engineering problems is presented. As it was classified in Chapter 2, there are two types of multi-criteria methods, the multi-objective optimization (MOO) and the multi-criteria decision analysis (MCDA) methods. Differences between these two fundamental groups were also discussed there. In order to prepare this overview, we deployed an extensive search for the related literature and consulted the works of Figueira et al. [28], Malczewski [55], Marler and Arora [57], Srdjevic [87], Triantaphyllou [92] and Vassilev et al. [97].

A.1 Multi-Objective Optimization (MOO) Methods with Some Applications to Transportation/Civil Engineering Problems

Throughout our discussion in this section, we follow the outstanding survey of Marler and Arora [57] as a major source-material. Some parts of this work were literally adopted. First, we give the notion of MOO. The process of optimizing simultaneously a collection of objective functions is called *multi-objective optimization* (MOO) or *vector optimization*. The general multi-objective optimization problem is formulated as follows

$$\underset{\mathbf{x}}{\text{minimize}}\, \mathbf{f}(\mathbf{x}) = [f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_k(\mathbf{x})]^{\mathrm{T}}, \tag{A.1}$$

$$\begin{aligned} \text{subject to } g_j(\mathbf{x}) &\le 0, \quad j = 1, 2, \ldots, m. \\ h_l(\mathbf{x}) &= 0, \quad l = 1, 2, \ldots, e, \end{aligned}$$

where k is the number of objective functions, m is the number of inequality constraints, and e is the number of equality constraints The vector $\mathbf{x} \in R^n$ is a vector of *design variables*, (also called *decision variables*), where n is the number of independent variables, and $\mathbf{f}(\mathbf{x})$ is a vector of objective functions $f_i(\mathbf{x})$. The functions $f_i(\mathbf{x})$ are also called *objectives*, *criteria*, *pay-off functions*, *cost functions*, or *value functions*. The gradient of $f_i(\mathbf{x})$ with respect to $\mathbf{x}$ is written as $\boldsymbol{\nabla}_{\mathbf{x}} f_i(\mathbf{x})$. The stationary point, denoted by $\mathbf{x}^*$, is the point that minimizes the objective function $f_i(\mathbf{x})$.

The *feasible design space* $\mathbf{X}$ (often called the feasible *decision space* or *constraint set*) is defined as the set $\{\mathbf{x}|g_j(\mathbf{x}) < 0, j = 1, 2, \ldots, m,$ and $h_i(\mathbf{x}) = 0, i = 1, 2, \ldots, e\}$. The *feasible criterion space* $\mathbf{Z}$ (also called the feasible *cost space* or the *attainable set*) is defined as the set $\{\mathbf{f}(\mathbf{x})|(\mathbf{x}) \in \mathbf{X}\}$. Feasibility implies that no constraint is violated. Attainability implies that a point in the criterion space maps to a point in the design space. Each point in the design space maps to a point in the criterion space, but the reverse may not be true. Consequently, even with an unconstrained problem, only certain points in the criterion space are *attainable*.

Since the primary goal of MOO is to model a decision maker's preferences (ordering or relative importance of objectives and goals), methods can be categorized depending on how the DM specifies these preferences. Three categories may be distinguished: a'priori specification of preferences (user indicates the relative importance of the objective functions before running the algorithm), a'posteriori specification of preferences (selecting a single solution from a set of mathematically equivalent solutions) and no specification of preferences are addressed [57].

Preferences refer to a decision maker's opinions concerning points in the criterion space. With a'posteriori specification of preferences, the decision maker imposes preferences directly on the set of potential solution points. With a' priori specification of preferences, one must quantify options before actually viewing points in the criterion space. In this sense, the term *preference* is often used in relation to the relative importance of different objective functions. A *preference function* is an abstract function (of points in the criterion space) in the mind of the decision maker, which perfectly incorporates his/her preferences.

In the context of economics, *utility*, denoted by *u,* is modeled with a *utility function*. It represents an individual's or a group's degree of contentment. This is slightly different from the meaning of *usefulness* or *worth* which are meant satisfaction. The utility function *U*, is a mathematical expression that attempts to model the decision maker's preferences to approximate the preference function, which, typically cannot be expressed in mathematical form. We describe here the related main result of the famous von-Neumann-Morgenstern *cardinal utility theory* [101]. Their *preference axioms* are stated as [101]:

Let X be a nonempty set and let P be a convex set of probability distributions on X, such that if $p, q \in P$ and $0 \leq \lambda \leq 1$ then $\lambda p + (1-\lambda)q \in P$. Also let $>$ be an individual's *is preferred to* relation on *P*, with indifference defined by

$$p \sim q \quad \text{if neither} \quad p > q \quad \text{nor} \quad q > p.$$

We say that $>$ on *P* is a *weak order* if it is asymmetric ($p > q \Longrightarrow$ not $q > p$) and both $>$ and $\sim$ are transitive, i.e., $(p > q, q > r) \Longrightarrow p > r$ and $(p \sim q, q \sim r) \Longrightarrow p \sim r$.

The *von-Neumann and Morgenstern axioms* are, for all $p, q, r \in P$ and for all $0 < \lambda < 1$:

1. *order* : $>$ on P is a weak order;
2. *independence* : $p > q \Longrightarrow \lambda p + (1-\lambda)r > \lambda q + (1-\lambda)r$;
3. *continuity* : $p > q > r \Longrightarrow \alpha p + (1-\alpha)r > q > \beta p + (1-\beta)r$, for some α and β strictly between 0 and 1.

Note that the axioms apply solely to preference comparisons between distributions in P. They say that preferences are ordered; that similar convex combinations with a third distribution preserve preference; and that if a distribution is between two others in preference then it is also between nontrivial convex combinations of those others. The von-Neumann-Morgenstern linear utility theorem asserts that the preceding axioms hold if and only if there is a linear functional u on P such that, for all $p, q \in P$,

$$p > q \Longleftrightarrow u(p) > u(q). \tag{A.2}$$

Linearity for u means that, for all $p, q \in P$ and all $0 \leq \lambda \leq 1$,

$$u\left[\lambda p + (1-\lambda)q\right] = \lambda u(p) + (1-\lambda)u(q). \tag{A.3}$$

The theorem goes on to assert that such a u is unique up to *a positive affine transformation*, which is to say that a linear functional v on P also satisfies the representation if and only if there are numbers $a > 0$ and b such that $v = au + b$.

The familiar *expected utility* form follows from linearity. Assume all degenerate (one-point) distributions are in P, and define $u(x)$ as $u(p)$ when $x \in X$ and $p(x) = 1$. Then for any *simple* distribution p on X, $u(p) = \sum_x p(x)u(x)$. Thus, for simple p and q in P,

$$p > q \Longleftrightarrow \sum_{\mathbf{x}} p(x)u(x) > \sum_{\mathbf{x}} q(x)u(x), \tag{A.4}$$

is the expected utility representation for preference between risky prospects.

A *global criterion* is a scalar function that mathematically combines multiple objective functions; it does not necessarily involve utility or preference. A predominant classification of multi-objective approaches is that of *scalarization* methods and *vector optimization* methods in order to form a single scalar objective function. The major concept in defining an optimal point in this framework is that of *Pareto optimality* [73]:

Definition A.1 Pareto optimality: A point $\mathbf{x}^* \in \mathbf{x}$, is Pareto optimal iff there does not exist another point $\mathbf{x} \in \mathbf{X}$, such that $\mathbf{f}(\mathbf{x}) \leq \mathbf{f}(\mathbf{x}^*)$ and $f_i(\mathbf{x}) < f_i(\mathbf{x}^*)$ for at least one function [weak Pareto optimality exists iff $\mathbf{f}(\mathbf{x}) < \mathbf{f}(\mathbf{x}^*)$].

All Pareto optimal points lie on the boundary of the feasible criterion space **Z** [1], [18]. With regard to a global criterion f_g, Stadler [88] presented the following sufficiency condition for a Pareto optimal point:

Theorem A.1 *Let* $\mathbf{f} \in \mathbf{Z}$*,* $\mathbf{x}^* \in \mathbf{x}$*, and* $\mathbf{f}^* = \mathbf{f}(\mathbf{x}^*)$*. Let a scalar global criterion* $f_g(\mathbf{f}) : \mathbf{Z} \longrightarrow R^1$ *be differentable with* $\nabla_f f_g(\mathbf{f}) > 0$ *for all* $\mathbf{f} \in \mathbf{Z}$*. Assume* $f_g(\mathbf{f}^*) = \min\{f_g(\mathbf{f}) < \mathbf{f} \in \mathbf{Z}\}$*. Then* $\mathbf{x}^*$ *is Pareto optimal.*

Theorem A.1 suggests that minimization of a global function $f_g(\mathbf{f})$ is sufficient for Pareto optimality if $f_g(\mathbf{f})$ increases monotonically with respect to each objective function. If minimizing $f_g(\mathbf{f})$ is to provide a necessary condition for Pareto optimality, the Hessian of $f_g(\mathbf{f})$ with respect to $\mathbf{f}$ must be negative definite [1]. A Pareto optimal point in the criterion space is often called a *non-dominated* point (Yu [106]; Yu and Leitmann [107]):

Definition A.2 Non-dominated and dominated points: A vector of objective functions, $\mathbf{f}(\mathbf{x}^*) \in \mathbf{Z}$ is non-dominated iff there does not exist another vector, $\mathbf{f}(\mathbf{x}) \in \mathbf{Z}$, such that $\mathbf{f}(\mathbf{x}) \leq \mathbf{f}(\mathbf{x}^*)$ with at least one $f_i(\mathbf{x}) < f_i(\mathbf{x}^*)$. Otherwise, $\mathbf{f}(\mathbf{x}^*)$ is dominated.

An alternative idea of Pareto optimality which yields a single solution point is the idea of a *compromise solution* (Salukvadze [82],[83]). It entails minimizing the difference between the potential optimal point and a *utopia point* (also called *ideal point*) (Vincent and Grantham [99]):

Definition A.3 Utopia point: A point, $\mathbf{f}^\circ \in \mathbf{Z}^k$, is an utopia point iff for each $i = 1, 2, \ldots, k$, $f_i^\circ = \text{minimum}_x\{f_i(\mathbf{x})|\mathbf{x} \in \mathbf{X}\}$.

In general, $\mathbf{f}$ is unattainable. The next best thing is a solution that is as close as possible to the utopia point. Such a solution is called a compromise solution and is Pareto optimal. The term *close* usually implies that one minimizes the Euclidean distance:

$$D(\mathbf{x}) = \underset{\mathbf{x}}{\text{minimize}}\, |\mathbf{f}(\mathbf{x}) - \mathbf{f}^\circ| = \left\{ \sum_{i=1}^{k} [f_i(\mathbf{x}) - f_i^\circ]^2 \right\}^{\frac{1}{2}}. \tag{A.5}$$

In many cases, especially with scalarization methods that involve a'priori specification of preferences, it is advantageous to transform the original objective function. The most robust approach, regardless of their original range, is referred to as *normalization* (Koski [50]; Rao and Freiheit [77]):

$$f_i^{\text{trans}} = \frac{f_i(\mathbf{x}) - f_i^{\circ}}{f_i^{\max} - f_i^{\circ}}. \tag{A.6}$$

In this case, $f_i^{\text{trans}}(\mathbf{x})$ generally has values between zero and one.

A.1.1 MOO methods with a'priori specification of preferences and applications

Most of these methods incorporate parameters which are coefficients, exponents, etc. that either can be set to reflect decision makers' preferences, or be continuously altered in an effort to represent the complete Pareto optimal set. They usually develop a kind of a utility function. The most common general scalarization method is the *single objective global criterion* method. Its simplest form is the *weighted exponential sum*:

$$U = \sum_{i=1}^{k} w_i \left[f_i(\mathbf{x})\right]^p, \quad \text{or} \quad U = \sum_{i=1}^{k} \left[w_i f_i(\mathbf{x})\right]^p, \quad f_i(\mathbf{x}) > 0, \quad \text{for all } i, \tag{A.7}$$

and its extension (Yu and Leitmann [107], Zeleny [110]):

$$U = \left\{ \sum_{i=1}^{k} w_i \left[f_i(\mathbf{x}) - f_i^{\circ}\right]^p \right\}^{\frac{1}{p}} \quad \text{or} \quad U = \left\{ \sum_{i=1}^{k} w_i^p \left[f_i(\mathbf{x}) - f_i^{\circ}\right]^p \right\}^{\frac{1}{p}}. \tag{A.8}$$

In (A.7) and (A.8), $\mathbf{w}$ is a vector of weights, typically set by the DM such that $\sum_{i=1}^{k} w_i = 1$ and $\mathbf{w} > \mathbf{0}$. These methods are often called *compromise programming* methods, as the DM usually has to compromise between the final solution and the *utopia point* (Miettinen [65]), or its approximated *reference point* (Wierzbicki [103]), or the *target point* (Hallefjord and Jornsten [39]). The solutions of these approaches depend on the value of p. The most common approach of form (A.7) is when $p = 1$, what is called the *weighted sum method*:

$$U = \sum_{i=1}^{k} w_i f_i(\mathbf{x}). \tag{A.9}$$

With positive weights, the minimum of (A.9) is Pareto optimal (Zadeh [108]), however this formulation sometimes does not provide a necessary condition as well (Zionts [112]). Systematic approaches to selecting weights have been proposed by many authors, most notably Hwang and Yoon [44] and Voogd [102]. With *ranking* and *categorization* methods, the different objective functions are ordered by importance (Yoon and Hwang [105]). Weights are ranged from the least important objective to the higher evaluated ones and they have integer values (rank numbers). With *rating methods*, where there is a more than ordinal significance, decision makers assign consistent increments to the objectives as values of relative importance usually on a $[1 - 100]$ interval scale. *Ratio questioning* provides systematic means to rate objective functions by comparing two objectives at a time (Saaty [81]). Rao and Roy [75] provided a method for determining weights based on *fuzzy set* theory.

The *lexicographic method* arranges the objective functions in order of importance (Stadler [88]). Then, the following optimization problems are solved one at a time:

$$\begin{aligned} &\underset{\mathbf{x}\in\mathbf{X}}{\text{minimize}}\ f_i(\mathbf{x}) \\ &\text{subject to } f_j(\mathbf{x}) \le f_j(\mathbf{x}_j^*), \quad j = 1, 2, \ldots, i-1,\ i > 1,\ i = 1, 2, \ldots, k. \end{aligned} \tag{A.10}$$

The *weighted min-max method* employs the utility function:

$$U = \max_i \{w_i [f_i(\mathbf{x}) - f_i^\circ]\}. \tag{A.11}$$

Then, for treating (A.11), the method introduces an additional unknown parameter: λ so as to minimize$_{\mathbf{x},\lambda}$ λ, subject to $w_i[f_i(\mathbf{x}) - f_i^\circ] - \lambda \leq 0$, i=1,2,...,k. As $p \longrightarrow \infty$, (A.10) is the limit of (A.8), therefore (A.10) can provide the complete Pareto optimal set with variation in the weights. It provides a necessary condition for Pareto optimality (Miettinen [65]). In addition, it is sufficient for weak Pareto optimality (Koski and Silvennoinen [52]).

In response to the inability of the weighted sum method to capture points on non-convex portions of the Pareto optimal surface, Athan and Papalambros [1] proposed the exponential weighted criterion, as follows:

$$U = \sum_{i=1}^{k} (e^{pw_i} - 1)\, e^{pf_i(\mathbf{x})}, \tag{A.12}$$

where the argument of the summation represents an individual utility function for $f_i(\mathbf{x})$. Minimizing (A.12) provides a necessary and sufficient condition for Pareto optimality.

Bridgman [8] proposed the following approach:

$$U = \prod_{i=1}^{k} [f_i(\mathbf{x})]^{w_i}, \tag{A.13}$$

where w_i are weights indicating the relative significance of the objective functions. Gerasimov and Repko [32] successfully applied this method to the multi-objective optimization of a truss. They minimized the weight, displacement and difficulty of a construction. The cross-sectional areas of the rods were the design variables and constraints were on strength and stability.

Charnes et al. [16], and Charnes and Cooper [15] developed the *goal programming method*, in which goals b_j, are specified for each objective function $f_j(\mathbf{x})$. Then, the total deviation from the goals $\sum_{j=1}^{k} |d_j|$ is minimized, where d_j is the deviation from the goal b_j for the jth objective. To model the absolute values d_j is split into positive and negative parts such that $d_j = d_j^+ - d_j^-$, with nonnegative members and $d_j^+ d_j^- = 0$. Consequently, $|d_j| = d_j^+ + d_j^-$. d_i^+ and d_i^- represent underachievement and overachievement, respectively, where achievement implies that a goal has been reached. The optimization problem is formulated as follows:

$$\underset{\mathbf{x}\in\mathbf{X},\ \mathbf{d}^-,\ \mathbf{d}^+}{\text{minimize}} \sum_{i=1}^{k} (d_i^+ + d_i^-) \tag{A.14}$$

$$\begin{aligned} \text{subject to} \quad & f_j(\mathbf{x}) + d_j^+ - d_j^- = b_j, \quad j = 1, 2, \ldots, k, \\ & d_j^+,\ d_j^- \geq 0, \qquad j = 1, 2, \ldots, k, \\ & d_j^+ d_j^- = 0, \qquad j = 1, 2, \ldots, k. \end{aligned}$$

The *bounded objective function method* minimizes the single most important objective function $f_s(\mathbf{x})$. All other objective functions are used to form additional constraints (Hwang and Md. Musad [43]). A general description of the so-called ε-constraint approach is provided by Haimes et al. [38] and Carmichel [12]:

$$f_i(x_i^*) \leq \varepsilon_i \leq f_s(x_i^*). \tag{A.15}$$

Carmichel [12] applied this approach to a five-bar two-objective truss problem. Weight was minimized with an ε-constraint approach on nodal displacement, where ε was varied to yield a series of Pareto optimal solutions for the four design variables representing different areas of the truss members.

The *physical programming method* (Messac [59]) maps the general classifications of the goals and the objectives. In general, the DM customizes individual utility functions called class functions as non-dimensional unimodal transformations. These are then combined into a utility function

$$f_a(\mathbf{x}) = \log\left\{\frac{1}{dm}\sum_{i}^{dm}\overline{f_i}\left[f_i(\mathbf{x})\right]\right\}, \tag{A.16}$$

where dm represents the number of design metrics being considered. This method was applied, among others, to high-speed transport planes (Messac and Hattis [61]). The design metrics were the tank-volume ratio, recurring cost per passenger seat, initial cost per passenger seat, propellant mass ratio, fuselage-length/wing-root-length ratio, engine inlet area, wing sweep-back angle, and number of passengers. The design parameters were the engine inlet area, wingspan, wing sweep-back angle, number of passengers, and propellant-tank volume. Messac and Wilson [63] applied this method to the design of a robust controller for a two degree-of-freedom spring-and-mass system. There were five design metrics: settling time, stability, noise amplification, control effort (output of controller), and controller complexity (indicated by controller order). The nine design variables were mathematical parameters used in the development of the controller. Messac [60] modeled unconstrained simple beams with three objectives (mass, displacement and width) and two design variables (beam height and width). Physical programming has also been used with complex problems such as finite element sizing optimization involving inflatable thin-walled structural members for housing (Messac et al. [64]).

A.1.2 MOO methods with a'posteriori specification of preferences and applications

The core feature of the multi-objective methods with a'posteriori specification of preferences is that they allow the decision maker to choose from a set of (or subset of) the computed Pareto optimal solutions. Some algorithms are designed specifically to produce a set of Pareto optimal points that accurately represents the complete Pareto set.

In response to deficiencies in the weighted sum approach Das [23] and Das and Dennis [24] have presented the *normal boundary intersection* (NBI) *method.* This method provides a means for obtaining an even distribution of Pareto optimal points for a consistent variation in the user-supplied parameter vector $\mathbf{w}$, even for a non-convex Pareto optimal problem. The basic mathematical model is formulated as follows

$$\begin{aligned} &\underset{\mathbf{x}\in\mathbf{X},\ \lambda}{\text{minimize}} \\ &\text{subject to} \quad \Omega\mathbf{w} + \lambda\mathbf{n} = \mathbf{f}(\mathbf{x}) - \mathbf{f}^\circ. \end{aligned} \tag{A.17}$$

In the constraint of (A.17), $\mathbf{\Omega}$ is a $k \times k$ pay-off matrix in which the ith column is composed of the vector $\mathbf{f}(\mathbf{x}_i^*) - \mathbf{f}^\circ$, where $\mathbf{f}(\mathbf{x}_i^*)$ is the vector of objective functions evaluated at the minimum of the ith objective function. The diagonal elements of $\mathbf{\Omega}$ are all zeros. The sum of the non-negative vector $\mathbf{w}$ is normalized, so that the sum of its components is unity. The vector $\mathbf{n}$ can be written in the form: $\mathbf{n} = -\mathbf{\Omega}\mathbf{e}$, where $\mathbf{e} \in R^k$ is a (column) vector with all entries ones called a *quasi-normal* vector. Since each nonzero component of $\mathbf{\Omega}$ is positive, the negative sign ensures that $\mathbf{n}$ points towards the origin of the criterion space and gives the property that for any $\mathbf{w}$, a solution point is independent of how the objective functions are scaled. The method may also yield non-Pareto optimal points i.e. it does not provide a sufficient condition for Pareto optimality. Das and Dennis [24] applied the NBI method to a three-bar truss design problem where five objective functions were used to represent the total volume, the nodal displacement and the absolute value of the stress in each bar. The four design variables were the cross-sectional area of each bar and the position of the vertical bar, which had a fixed length. The constraints consisted of limits on the stresses.

The *normal constrained method* (NC) provided an alternative to the NBI method with some improvements (Messac et al. [62]) when used with normalized objective functions and with a *Pareto filter*, which eliminates non-Pareto optimal solutions. This approach provides a set of Pareto optimal points in the criterion optimal space. The method first determines the utopia point and then the individual minima of the normalized objective functions for the vertices of what is called the *utopia hyperplan* (in the criterion space). This method requires the formulation of additional inequality constraints. Messac at al. [62] applied this approach to a three-bar truss problem emerging from Koski [51], where the cross-sectional areas of the bars were the design variables. The linear combination of nodal displacement and the volume were minimized. Limits were placed on the design variables and on the stresses in each bar.

A.1.3 MOO methods with no specification of preferences and applications

Often the DM cannot concretely define what he/she prefers. This section describes methods that do not implicitly require any articulation of preferences. Most of the methods are simplifications of the methods discussed in sub-section A.1.1, typically with exclusion of method parameters. The fundamental idea behind most global criterion methods is the use of an *exponential sum* which is formed by setting all of the weights in (A.8) to one. This yields a single function $f_g(\mathbf{f})$. The primary general global criteria formulation which can be reduced to many other formulations is given by (A.8) with all of the weights equal to one (Yoon [104]; Hwang et al. [42]). However there are a lot of variations of the basic global criterion method. When forming a measure of distance, it is possible and often necessary to seek a point that not only is as close as possible to the utopia point but also is as far away as possible from some detrimental point.

The *technique for order preference by similarity to ideal solution* (TOPSIS) *method* takes this approach and is a form of compromise programming (Hwang et al. [42]). The utopia point is the *positive ideal* solution, and the vector in the criterion space that is composed of the worst or most undesirable solutions for the objective functions is called the *negative ideal*. *Similarity* is developed as a function that is inversely proportional to the distance from the positive ideal and directly proportional to the distance from the negative ideal. Then, the similarity is maximized.

When (A.7) is used with $p = 1$ and $\mathbf{w} = 1$, the result is simply the sum of the objective functions. Not only is this special case of a global criterion method, it is a special case of the weighted sum method discussed earlier. The *objective sum method*, thus, highlights a fundamental approach that always provides a Pareto optimal solution.

A basic min-max formulation is derived by excluding the weights in (A.7) and using $p = \infty$. Then, the basic *min-max method* is formulated as follows

$$\underset{\mathbf{x}\in\mathbf{X}}{\text{minimize}} \max_i \left[f_i(\mathbf{x})\right]. \tag{A.18}$$

Osyczka [72] treats (A.18) as a standard single objective function, where $\max_i[f_i(\mathbf{x})]$ provides the objective function values at point $\mathbf{x}$. Tseng and Lu [93] have utilized this approach for a ten-member cantilever truss, where there were four objectives: minimize the weight, minimize the maximum member-stress, minimize the maximum nodal displacement, and maximize the natural frequency. The cross-sectional areas of the members represented the design variables and the constraints were on member-stress and areas. Vásárhelyi and Lógó [96] used a similar approach to design a steel frame. Volume and shear stress were minimized using ten design variables representing cross-sectional dimensions.

In terms of a mathematical formulation, in which individual objective functions are minimized, the *Nash arbitration* (derived from game theory) and *objective product method* entails maximizing the following global criterion (Straffin [89]):

$$f_g(\mathbf{x}) = \prod_{i=1}^{k} \left[s_i - f_i(\mathbf{x}) \right], \tag{A.19}$$

where $s_i \geq f_i(\mathbf{x})$. If s_i is selected as an upper limit on each function, guaranteeing that $\mathbf{f}(\mathbf{x}) < \mathbf{s}$, then (A.19) yields a Pareto optimal point. Mazumdar et al. [58] have used (A.19) to solve an optimal network flow problem which has two objective functions (general performance indices for each network user), two design variables (the throughput for each user, associated with a specific objective function), and four basic constraints.

Rao [76] introduced the *Rao's method* that is based on the use of a product-type global criterion shown in (A.19). In this method, first, the following '*super criterion*' is minimized:

$$SU = \prod_{i=1}^{k} \left[1 - f_i^{\mathrm{norm}}(\mathbf{x}) \right], \tag{A.20}$$

where $f_i^{\mathrm{norm}}(\mathbf{x})$ is a normalized objective function, with values between zero and one, such that $f_i^{\mathrm{norm}} = 1$ is the worst possible value. Rao and Hati [78] applied this method to a three-degree-of-freedom spring-and-dumper system. The relative displacement and transmitted force were minimized subject to limits on the design variables which were the mass, spring constant, and damping coefficient for each degree-of-freedom. Rao and Freiheit [77] applied this approach to the probabilistic design of an eighteen-speed gear train. Reliability in bending and in wear was maximized while weight was minimized. The width of each gear was used as a design variable.

Data envelopment analysis (DEA) is a linear programming methodology (a data oriented approach) to measure the efficiency of multiple decision-making units (DMUs) when a process presents a structure of multiple inputs and outputs. In their originating study, Charnes et al. [17] described *DEA* as a mathematical programming model applied to observational data which provides a new way of obtaining empirical estimates of relations. A drawback of this technique is that model specification and inclusion/exclusion of variables can affect the results. In spite of this fact, the popularity of this method shows a rapid increase. For a particular DMU the ratio of this single virtual output to single virtual input provides a measure of *efficiency* that is a function of the multipliers. In mathematical programming language, this ratio, which is to be maximized, forms the objective function for the particular DMU being evaluated, so that symbolically [17]:

$$\max h_0(u,v) = \sum_r u_r y_{ro} / \sum_r v_i x_{ro}. \tag{A.21}$$

where the variables are the u_r's and the v_i's and the y_{ro}'s and x_{io}'s are the observed output and input values, respectively, of $DMU_\circ$, the DMU to be evaluated. The work of Caulfiel et al. [14] used DEA as a new project investment appraisal tool to identify the most efficient solution for a city centre-airport route and to establish the reasons for inefficiency. This research extended the DEA applications by implementing the method in the field of public transport investment analysis. Fülöp and Markovits-Somogyi [30] showed that DEA as an optimization based method can also be considered for eliciting the ranking values (priorities of the options) from a nonreciprocal pairwise comparison matrix.

In a specific sub-class of the MOO methods the emphasis is put on the active participation of the DM during the whole process. These procedures are called *interactive algorithms*, such as e.g. the *VIMDA method* (Korhonen [49]), the *aspiration-level method* (Lotfi et al. [54]), the *InterQuad method* (Sun and Steuer [91]) and the *classification oriented interactive methods* (Miettinen [65]). In these methods, usually, only a subset of the Pareto optimal solutions are generated and evaluated by the DM. The optimization and the evaluation phases of the decision-making process are cyclically repeated so that the DM can change his/her preferences, until a satisfying solution is found. These interactive algorithms are especially appropriate for solving linear and convex non-linear MOO problems in which the expected time duration for scalarization does not play an important role as opposed to integer, combinatorial-type and non-convex nonlinear problems (Vassilev et al. [97]).

Other approaches such as the *genetic algorithm* (Holland [40]) can also be tailored to solve multi-objective problems directly. These algorithms do not require gradient information so they can be effective regardless of the nature of the objective functions and constraints. Genetic algorithms are *global optimization techniques* which means they converge to a global solution rather than to a local solution. For an overview of the fundamentals of genetic algorithms the reader is referred to Goldberg [35]. Some techniques which address the development of generic algorithms for multi-objective problems are the *vector evaluated genetic algorithm* (VEGA) (Schaffer [84]), the *niche technique* (Deb [25]) and the *tournament selection technique* (Horn et al. [41]). Two applications of the genetic algorithms to a complex civil engineering and a transportation planning problem are presented by Schauman et al. [85], who applied genetic algorithms to the optimization of a reinforced concrete structure (using 112 design variables to represent the dimensions of 217 structural members) and to an urban planning problem. The urban planning problem involved minimizing the traffic travel time, the cost and the change in land use. Constraints were used to limit housing capacity and to ensure housing for five income brackets. 155 discrete design variables were used to represent different land zones and street characteristics.

A.2 Multi-Criteria Decision Analysis (MCDA) Methods with Some Applications to Transportation/Civil Engineering Problems

The multi-criteria decision analysis (MCDA) problems can be divided into three types: problems of multi-criteria choice, problems of multi-criteria ranking and problems of multi-criteria sorting. A great number of methods have been developed to solve these kinds of problems. In the literature, there are different classification schemes. For instance, Vincke [100] assigned these methods to three classes, multi-attribute utility (value) theory methods together with some weighting methods, outranking methods and interactive algorithms. In addition, the reader will recognize that some methods discussed in the MOO framework may also be grouped as being an MCDA method as well (e.g. TOPSIS, lexicographic method, weighting sum method or the AHP in some sense). This phenomenon indicates that there are several overlapping in the grouping principles. In this sub-section, we follow another categorization that is due to Hwang and Yoon [44], who distinguished non-compensatory and compensatory MCDA methods. In addition, we discuss here the so-called prioritization methods (also called scaling methods).

A.2.1 Definitions of the basic terms used in the MCDA framework

In this section, we begin by presenting the most relevant definitions commonly used in the MCDA framework:

Definition A.4 A *decision-maker* (DM) is an *individual* who makes a possibly logical choice from among finite number of available alternatives (options); so that he/she is solely responsible for the kind of decision achieved; takes full accountability for the outcome and its consequences; cannot delegate the task; he/she may be influenced by subjectivity and sometimes uses intuition. A decision worked out more than one people by forming a *decision-making group* has potentially a greater chance to be more effective than that of an individual effort, because it is an outcome of a collective choice and cohesive minds and, generally, more creative solutions may be generated.

Definition A.5 A *decision problem* exists, when a DM perceives a discrepancy between the current and the desired states of a system, and (*i*) the DM has alternative courses of action available; (*ii*) the choice of an action can have a significant effect on the perceived difference; and (*iii*) the DM is motivated to make a decision, but he is uncertain as to which option should be selected.

Definition A.6 A rational *decision-making* is a multi-step process; with formulating goals (objectives), identifying criteria and alternatives (options), making evaluations of the alternatives with respect to each criterion and choosing the 'best', or prioritizing, sorting the alternatives based on a formal *decision model*. In theory, it is required that a rational DM subscribes the von-Neumann-Morgenstern axioms (see these axioms in Appendix A.1).

Definition A.7 The *goal* of the decision problem is the overarching purpose that drives the decision. A goal is the state of affairs that a plan is intended to achieve. The goal of the problem, sometimes also called objective, determines the DM's efforts or actions which are intended to attain or accomplish. It indicates the desired direction of change.

Definition A.8 A *criterion* is a standard of judgment to test the desirability of an option. In a real-world decision problem the criteria, denoted by C_k, $k = 1, 2, \ldots, m$, are the factors that are used to evaluate the alternatives to see how well they meet the goal of the decision problem. Thus, a criterion is a measure of performance for the evaluation of an alternative. In MCDM, the concept of a criterion includes both *attributes* and *objectives*. The term *attribute*, which is used mainly (but not exclusively) in the MCDA framework, is a qualitative or quantitative property. It is an inherent characteristic of an alternative that can be measured on different scales of measurement. An *objective* is a statement about the desired state and mainly used in the MOO framework which indicate the directions of improvement (minimizing or maximizing an attribute).

Above defined terms should be understood clearly. The distinctions and the relationships among criteria, attributes, and objectives are illustrated in Figure A.1 [86]. As shown here, criteria are emerging in either an attribute or an objective form, and attributes with desired directions are regarded objectives [44]. Quoted the expressive example of Sun [90], the level of comfort is a criterion when evaluating an aircraft; cabin volume and noise are attributes of the aircraft which can be used to measure the level of comfort; while the maximization of cabin volume and the minimization of noise are objectives in the aircraft design process.

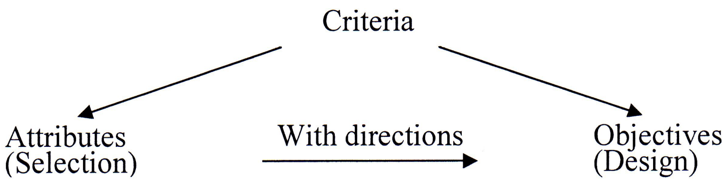

Figure A.1. The Relationship among criteria, attributes, and objectives [86]

Definition A.9 The *alternatives* or *options* represent the different choices of action available for the DM. In a real-world decision problem, an alternative, denoted by A_i, $i = 1, 2, \dots, n$, is a well-defined entity of a finite set of things, e.g., opportunities, objects, persons. Alternatives will be prioritized or sorted and/or one of them will be chosen as the 'best' one at the end of the decision process. They are usually being weighted with respect to each criterion (relative importance). *Feasible options* must meet the satisfaction level (constrains) specified by the DM maker for a set of criteria. A *non-dominated alternative* or *option* refers to the one that is at least equal in all criterion scores and at least is better in one criterion than other (dominated) options.

Definition A.10 A *decision table* of a size $m \times n$, as displayed in Figure A.2, is an arrangement where a *value* (*score*), a_{ki}, in the body of the table indicates the performance of alternative A_i evaluated in terms of criterion C_k. These values are either elicited from people as subjective judgments or given by physical measurements. The "raw" performances are expressed usually in different non-comparable units, therefore, the corresponding transformed values, as a result of an appropriate standardization, should also be generated.

$$
\begin{array}{cc|ccc|}
 & & u_1 & \dots & u_n \\
 & & A_1 & \dots & A_n \\
\hline
s_1 & C_1 & a_{11} & \dots & a_{n1} \\
\vdots & \vdots & \vdots & & \vdots \\
s_m & C_m & a_{m1} & \dots & a_{mn} \\
\hline
\end{array}
$$

Figure A.2. The form of a decision table

The transformed values are exhibited in a similar arrangement in a decision table. A higher score represents a better performance. If any goal of minimization occurs with respect to a given criterion, this can easily be converted into a goal of maximization. As shown in the decision table, *weighting factors*, $s_1, \dots, s_m$, which are assumed to be positive real numbers (and which are usually normalized on a $[0-1]$ scale) are assigned to the criteria reflecting the relative importance of them. A *utility value* or a *utility value function*, u_i, associated with the alternatives in the decision table, is a mathematical representation of the human judgments in terms of the expected behavior of a criterion (e.g. linear, Gaussian-shaped, user defined) over a specified range of values and is constituted by a preference mapping. This latter approach is used in the methods of multi-attribute utility theory (MAUT).

A.2.2 Non-compensatory MCDA methods with applications

Non-compensatory multi-criteria decision analysis methods do not permit trade-offs among criteria, which means that a disadvantage in one criterion cannot be offset by an advantage in other criterion.

The *conjunctive method* is probably the simplest MCDA technique. The DM sets up the criteria values as acceptable minima. Any alternative which has a criterion value less than the standard level will be rejected [44]. This method does not require the criteria to be in numerical form, and the relative importance of the criteria is not needed. It is usually employed for dichotomizing alternatives into acceptable and unacceptable categories. E.g., Benyoucef et al. [4] have shown an application of this method for a supplier selection problem.

In the *disjunctive method*, an alternative is evaluated on its greatest value with respect to a criterion (Hwang-Yoon [44]). When larger criteria values are preferred, the ith alternative A_i, is classified as an acceptable alternative, only if it is greater than c_k^0 the desirable level of the jth criterion C_k. As with the *conjunctive method*, the *disjunctive method* does not require the criteria to be in numerical form, and it does not need information on the relative importance of the criteria. Khademi et al. [48] considered an intelligent transportation system, where 33 user services were analyzed with respect to 38 criteria through a hybrid model of the non-compensatory disjunctive satisfying model (DSM) and the compensatory analytic network process (ANP) to prioritize user services.

The *dominance method* can be used to screen the alternatives in order to obtain a set of *non-dominated* solutions before the final choice has been made (Calpine and Golding [11]). The *dominance method* does not require any assumption or any transformation of criteria. The non-dominated set usually has multiple alternatives, hence, this method is mainly used for initial filtering. The so-called *scoring methods* may also be grouped into this class (e.g. Kesselring's method). Norris and Marshal [69] applied this method for a problem of an effective choice among feasible alternatives of buildings.

The *ELECTRE* (Elimination and Choice Translation Reality) *methods* utilize the concept of the outranking relation introduced by Benayoun et al. [3]. Suppose there are n alternatives based on m evaluation criteria, with weighing factors $[s_1, s_2, \ldots, s_m]$, and a_{ki} or u_{ki} is the performance score of alternative A_i with respect to criterion C_k. An *outranking relation* between, say alternative A_k and alternative A_l $(k, l = 1, 2, \ldots, n;\ k \neq l)$ is defined as: A_k is preferred to A_l when A_k is at least as good as A_l with respect to a majority of criteria and when A_k is not significantly poor regarding any other criteria. After the assessment of the outranking relations for each pair of alternatives, dominated alternatives can be eliminated and non-dominated alternatives can be obtained for further consideration. There are several different versions of *ELECTRE* methods, including the basic *ELECTRE I*, and its extensions concerning the definition of the *outranking relations* and the construction of the *concordance dominance matrix* and the *discordance dominance matrix* and whether or not they require criteria weights in the calculation procedures: *ELECTRE IS, II, III, IV* and *TRI* (Roy [80]). The main characteristics of all existing versions of *ELECTRE* methods were summarized by Roy [80].

In the *lexicographic method*, the DM compares the alternatives with respect to the most important criterion. If one alternative has a better criterion value than all the other alternatives, the alternative is chosen and the decision process ends. However, if some alternatives are tied on the most important criterion, the subset of tied alternatives is then compared to the second most important criterion. The process continues sequentially until a single alternative is chosen or until all the criteria have been considered. The *lexicographic method* does not require comparability across criteria, and the preference information on the criteria is not necessarily given in the form of crisp values. It only utilizes a small part of available information in making the final decision.

A.2.3 Compensatory MCDA methods with applications

Compensatory multi-criteria decision analysis methods permit trade-offs among criteria, that is, small changes in one criterion can be offset by opposing changes in any other criteria.

The large group of the *multi-attribute utility theory* (MAUT) methods utilizes the concept of a utility function. This fact requires to perform a preference mapping of the non-formal preference of the DM into a mathematical function (Keeney and Raiffa [47]). The most widely used form is the additive multi-attribute utility method given by equation (A.22), with two assumptions; stating that the utility functions of all attributes are independent and that the weighing factor of an attribute can be determined regardless of the weighing factors of other attributes:

$$U = (x_1, x_2, \ldots, x_m) = \sum_{k}^{n} s_k u_k(x_k), \tag{A.22}$$

where s_k is the weighting factor of the kth attribute (criterion) C_k, and $u_k(C_k)$ is its corresponding utility function. The additive multi-attribute utility theory provides a formal utility function to represent the DM's preference information. However, the two assumptions including the independence of utility functions and the weighting factors do not hold true for many practical decision making problems, which limits the use of this method. The SMART method is the simplest form of the MAUT methods. The ranking score of an alternative is simply obtained as the weighted algebraic mean of the utility values associated with it. We can also mention a very popular version of the MAUT methods called MACBETH (Bana e Costa and Vansnick [2]). In Zietsman et al.'s paper [111] the authors illustrateted how a multi-attribute utility theory approach can be used to promote a decision making problem, related to transportation corridors and how these decisions would differ from the ones that can be achieved through conventional single-objective techniques. The performance measures used in this MAUT approach included monetary benefits as well as negative externalities such as fuel consumption, vehicular emissions, mobility, and traffic safety. The MAUT approach made it possible to include a broad range of negative externalities even though these criteria cannot be expressed in monetary terms. In Reed et al. [79], the application of a MAUT-based framework for the process of transit system design is described and illustrated. Specifically, MAUT provided tools for systematically evaluating, prioritizing, and integrating desired transit functionalities and the so-called Advanced Public Transportation System (APTS) capabilities.

In the *PROMETHEE* (Preference Ranking Organization METHod for Enrichment Evaluations) *method* (Brans and Vincke [6]; Brans et al. [7]), a valued preference relationship based on a generalization of the notion of criteria is constructed and a preference index is defined, then a valued outranking graph is obtained. According to the preference index, *PROMETHEE I* provides a partial pre-order, however, *PROMETHEE II* offers a complete preorder on all alternatives. As concerns criteria, six types of generalized criteria (U-shaped, V-shaped, Gaussian and user-defined ones) and corresponding preference functions are considered with *preference* and *indifference thresholds*. In the paper of Gupta et al. [36], the *PROMETHEE* technique has been used for a cement company to select the logistic service providers (LSPs) and to demonstrate its easy and effective use. The paper of Godvin and Ezhilvannan [33] deals with a warehouse layout optimization problem concerning distance reduction and travel time minimization. They also searched for a flexible tool in order to optimize a layout functionally taking into account the fluctuations in demand and the levels of inventory. An optimization of arrangements of the departments in the warehouse is also presented. The locations are selected using *PROMETHEE II*.

The *simple additive weighting* (SAW), or *weighted sum model* (WSM) is originated with Fishburn [29]. In this well-known method, the weighting factors $[s_1, s_2, \dots, s_m]^{\mathrm{T}}$, are directly assigned to the criteria by the DM. It is important to note that SAW can only be used if all data are expressed in exactly the same unit. The single performance scores with their weighting factors are aggregated into a single performance metric. SAW selects the most preferred alternative, denoted by A^*, which has the maximum weighted outcome, as shown by (A.23). For an alternative A_i, the "total" importance is calculated, i.e., when all criteria are considered simultaneously [44] (hereafter, for convenience, we index the values in a decision table as a_{ij}):

$$A^* = \left\{ A_i \middle| \max_i \sum_{j=1}^{m} s_j a_{ij} \right\}, \quad i = 1, 2, \dots, n. \tag{A.23}$$

SAW is one of the most widely used MCDA methods because of its simplicity. However, it also has some limitations. SAW requires all criteria values to be both numerical and comparable, which will trigger the quantification problem for the qualitative criteria and a normalization problem for all elements included in the decision matrix. Quantification and the normalization methods used put a serious impact on the final results. Moreover, SAW is quite sensitive to the weighting factors. Using SAW, Bureika [10] analyzed a traction rolling-stock employed for freight transportation. Technical data of the rolling-stock were given and operation costs were estimated. The efficiency of the selected locomotives operated on a given railway line was determined by the SAW technique. Locomotives were evaluated from points of view of technical, economic and ecological parameters.

The *multiplicative weighting method* (MWM), or *weighted product method* (WPM) is originated with Miller and Starr [67]. In this method, the weighting factors $[s_1, s_2, \dots, s_m]^{\mathrm{T}}$ are assigned to the criteria by the DM in a direct way, like in the SAW method. Here, the performance score of each alternative is raised to the power determined by the corresponding weighting factor. This method selects the most preferred alternative, which has the largest value, when preference is put on larger criteria values:

$$A^* = \left\{ A_i \middle| \max \prod_{j=1}^{m} a_{ij}^{w_j} \right\}, \quad i = 1, 2, \dots, n, \tag{A.24}$$

Due to the exponential nature of (A.24), all criteria values should be greater than 1, in order to ensure monotonicity [44]. Zavadskas et al. [109] reported that alternative design solutions of buildings were successfully evaluated by using the WPM method. A comprehensive case study about how to rank the facades for public and commercial buildings was also presented.

The *TOPSIS* (Technique for Order Preference by Similarity to an Ideal Solution) *method* is presented first in [44]. The basic principle is that the chosen alternative should have the shortest distance from the ideal solution and the farthest distance from the negative ideal solution. The *TOPSIS* procedure consists of the following steps; see e.g. in Tzeng et al. [95]:

(1) Construct the normalized evaluation matrix with the elements r_{ij}:

$$r_{ij} = a_{ij} / \sqrt{\sum_{i=1}^{J} a_{ij}^2}, \qquad j = 1, 2, \ldots, J, \quad i = 1, 2, \ldots, n. \tag{A.25}$$

(2) Construct the weighted normalized evaluation matrix with the elements v_{ij}:

$$v_{ij} = s_i r_{ij}, \qquad j = 1, 2, \ldots, J, \quad i = 1, 2, \ldots, n, \tag{A.26}$$

where s_i is the weighting factor of the ith attribute and $\sum_{i=1}^{n} s_i = 1$.

(3) Determine the positive and the negative ideal solutions:

$$A^* = \{v_1^*, \ldots, v_n^*\} = \{(j \max v_{ij} | i \in I'), (j \min v_{ij} | i \in I^n)\}, \tag{A.27}$$
$$A^- = \{v_1^-, \ldots, v_n^-\} = \{(j \min v_{ij} | i \in I'), (j \max v_{ij} | i \in I^n)\}, \tag{A.28}$$

where I' is associated with benefit criteria and I^n is associated with cost criteria.

(4) Obtain the separation measures using the n-dimensional Euclidean distance. The separation of each alternative from the ideal and from the negative-ideal solution, respectively, is given as:

$$D_j^* = \sqrt{\sum_{i=1}^{n} (v_{ij} - v_i^*)^2} \quad \text{and} \quad D_j^- = \sqrt{\sum_{i=1}^{n} (v_{ij} - v_i^-)^2} \quad j = 1, 2, \ldots, J. \tag{A.29}$$

(5) Calculate the relative closeness (similarity) S_j^* of alternative A_j with respect to the ideal solution A^* as:

$$S_j^* = D_j^- / (D_j^* + D_j^-), \quad j = 1, \ldots, J. \tag{A.30}$$

(6) Rank the preference order of the alternatives based on the relative closeness to the ideal alternative, i.e. according to the ascending order of S_j^*. There have been a great number of applications of *TOPSIS* for a variety of transportation problems, e.g. for evaluating an urban rail transit network (Turnisa et al. [94]); for selecting an appropriate logistics center location (Erhayman et al. [27]); to measuring the so-called overall construction project success and taking into consideration pre-production plan parameters as well (Pinter and Psunder [74]); and for a GIS supported planning of urban infrastructure development (Coutinho-Rodrigues et al. [20]).

The *VIKOR* (VIse Kriterijumska Optimizacija i kompromisno Resenje) *method* determines the compromise solution, and is able to establish the stability of decision performance by replacing the compromise solution obtained with initial weights. It was originally developed by Opricovic [70] to solve decision problems with conflicting and noncommensurable (different units) criteria, assuming that compromise is acceptable for conflict resolution, the decision maker wants a solution that is the closest to the ideal, and the alternatives are evaluated according to all established criteria. *VIKOR* ranks alternatives and determines the solution termed *compromise solution* that is the closest to the ideal one. Find the description of the algorithm e.g., in (Opricovic and Tzeng, [71]), or in (Tzeng et al. [95]). Dag and Önder [22] detected 10 criteria for a facility location problem of a production company. These were “raw material supply", "proximity to customer", "proximity to airport", "proximity to harbor", "transportation cost", "availability of skilled labor", "labor cost", "proximity to industrial zone", "government facilities" and "construction cost". By applying *VIKOR method* they determined the appropriate location providing the most company's satisfaction for the criteria identified. Kuo et al. [53] proposed a hybrid MCDM model to analyze the transport policy decision-making process and *VIKOR* was used to select the best transport mode. The railway mode was proven to be the best choice among the four transport modes from both the stakeholders and the scholars perspectives.

A.2.4 Prioritization methods (Scaling methods) with applications

A specific sub-class of MCDA methods is called *prioritization methods* or *scaling methods*, since, fundamentally, they are used to prioritize the decision alternatives. The majority of these methods is focusing on the *pairwise comparison matrix* (PCM) $\mathbf{A} = [a_{ij}]$, which is central to the *analytic hierarchy process* (AHP) *method* founded by Saaty [81]; see a detailed discussion about the AHP methodology in Chapter 2. There are two different approaches depending upon how to derive implicit *weights* (priority scores), $w_1, w_2, \ldots, w_n$, from a matrix $\mathbf{A}$, entries of which are positive numbers. A *vector* of the *weights*, $\mathbf{w} = [w_i]$, $w_i > 0$, $i = 1, \ldots, n$, may be extracted using an *eigenvalue-eigenvector formulation* or *extremal methods*. A comparison of these approaches can be found in (Golany and Kress [34]).

Saaty [81] developed a procedure what he called the *eigenvector method* (EV). He proposed the principal right eigenvector of $\mathbf{A}$ (Perron-eigenvector) to be the weight (priority) vector $\mathbf{w}$, whose components are appropriate to prioritize the decision alternatives. To find this vector, the eigenvalue-eigenvector problem:

$$\mathbf{A}\mathbf{w} = \lambda_{\max}\mathbf{w}, \quad \mathbf{e}^{\mathrm{T}}\mathbf{w} = 1, \tag{A.31}$$

is to be solved, where $\lambda_{\max}$ is the principal (Perron) eigenvalue of matrix $\mathbf{A}$ and $\mathbf{e}^{\mathrm{T}} = [1, \ldots, 1]$. If the DM provided consistent judgments, then $\lambda_{\max} = n$, otherwise $\lambda_{\max} > n$, where n is the order of matrix $\mathbf{A}$. It was shown by various researchers that for small deviations around the consistent ratio estimates for the entries of $\mathbf{A}$, the EV method produces a reasonably good approximation of the priority vector. If, however, the estimates encompass large inconsistencies, then the solutions to problem (A.31) are deemed not to be satisfactory and the judgment process should be revised.

The EM has been criticized both from prioritization and consistency points of view and some new techniques have been developed. Jensen [46] proposed the *direct least-squares method* (DLS) that minimizes the familiar Euclidean distance:

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} \left(a_{ij} - w_j/w_i\right)^2, \tag{A.32}$$

$$\text{subject to} \sum_{i=1}^{n} w_i = 1.$$

to obtain the weights, w_i. Due to the numerical nature of the solution procedure, however, the initial values of the iteration should be in a close neighborhood of a stationary point in order to find a possible feasible solution (at least a local minima). Another problem, inherent in the DLS method, is that it usually leads to a non-convex optimization procedure. In their efforts to modeling and optimizing transportation demand, Cascetta [13] and Gupta and Shah [37] applied such generalized least-squares algorithms to optimize so called origin-destination (OD) matrices.

Chu et al. [19] proposed the *weighted least-squares method* (WLS) as a modification of the *direct least-squares method* (DLS). The WLS method minimizes a distance function in a L_2-norm sense defined for the elements of the unknown weight (priority) vector $\mathbf{w}$ with known judgment ratios a_{ij}. The WLS method thus considers the following constrained non-linear optimization problem:

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} (a_{ij} w_j - w_i)^2 \tag{A.33}$$

$$\text{subject to } \sum_{i=1}^{n} w_i = 1.$$

Above minimization problem with equality constraint can be transformed into a system of linear equations by differentiating the Lagrangian of (A.33) and then equating it to zero. It was shown by Blankmeyer [5], that following this common way, the WLS provides a unique, strictly positive solution for the weights w_i.

The *logarithmic least-squares method* (LLS) also utilizes an L_2 metric in defining the objective function of the following optimization problem:

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} [\ln a_{ij} - (\ln w_i - \ln w_j)]^2 \tag{A.34}$$

$$\text{subject to } \prod_{i=1}^{n} w_i = 1, \quad w_i > 0, \quad i = 1, 2, \ldots, n.$$

Crawford and Williams [21] have shown that the solution to problem (A.34) is unique and it can be found simply by taking the geometric means of the rows of matrix $\mathbf{A}$:

$$w_i = \prod_{j=1}^{n} a_{ij}^{1/n}, \quad i = 1, 2, \ldots, n. \tag{A.35}$$

The *chi-square method*, or minimum χ^2 method for the matrix $\mathbf{A}$ developed by Jensen [45] is to minimize the sum of linear ratios over the positive orthant:

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{a_{ij} - w_j / w_i}{w_j / w_i} \right)^2 . \tag{A.36}$$

It is worth to mention that strictly speaking (A.36) is not an index of Pearson's goodness-of-fit because a_{ij} is not an observed frequency and w_i/w_j is not an expected (theoretical) frequency value. An application of this approach directed to find a solution to a strategic decision problem in supply chain management (Nakagawa and Sekitani [68]).

The *logarithmic goal programming* (LGP) *method* was developed by Bryson [9]. This approach requires the weights (priorities) to assume values so that the following equalities are satisfied:

$$a_{ij} - (w_i/w_j) \left(\delta_{ij}^{+} / \delta_{ij}^{-} \right) = 0, \quad i, j = 1, 2, \ldots, n, \quad i > j, \tag{A.37}$$

where $\delta_{ij}^{+} \geq 1$ and $\delta_{ij}^{-} \geq 1$ are the *deviation variables* which cannot be greater than 1 at one at a time. The weights w_i, are obtained as the solutions of the following linear goal programming problem with logarithmic constraints:

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\ln \delta_{ij}^{+} + \ln \delta_{ij}^{-} \right) \tag{A.38}$$

$$\text{subject to} \quad \ln w_i - \ln w_j + \ln \delta_{ij}^{+} - \ln \delta_{ij}^{-} = \ln a_{ij} \quad i, j = 1, 2, \ldots, n, \quad j > i,$$

where each δ_{ij}^{-} and δ_{ij}^{+} is nonnegative. As an application of this method, Dutta et al. [26] employed LGP to show how would railway passengers choose the 'best' option of a set of origin-destination pairs of trains from among several alternatives available for their journeys.

Gass and Rapcsák [31] developed the s*ingular value decomposition method* (SVD) and they claimed that the appropriate weights be obtained from the SVD decomposition of matrix $\mathbf{A}$ as:

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} \left(a_{ij} - w_j^{\mathrm{SVD}} / w_i^{\mathrm{SVD}} \right)^2 . \tag{A.39}$$

For this purpose, they provided a low-rank approximation $\mathbf{A}_{[1]} = \alpha_1 \mathbf{u} \mathbf{v}^{\mathrm{T}}$ of $\mathbf{A}$, where the $\mathbf{A}_{[1]}$ matrix is the 'best' rank-one approximation of matrix $\mathbf{A}$ in the Frobenius-norm sense, by first solving the distance minimization problem (A.39), then generating the (rank-one) left and right singular vectors. According to this concept, the vectors associated with the largest singular value of $\mathbf{A}_{[1]}$ yield the "theoretically justified" weights:

$$w_i^{\mathrm{SVD}} = \frac{u_i + 1/v_i}{\sum\limits_{j=1}^{n} (u_j + 1/v_j)}, \quad i, j = 1, 2, \ldots, n. \tag{A.40}$$

A successful real-world application of this method was related to a personnel selection problem. Mamat and Daniel [56] have shown how to select the most favorable person from ten candidates who will be acting as a member of a transportation engineering faculty.

The *fuzzy preference programming* (FPP) *method* was developed by Mikhailov [66]. As its first step, it investigates whether or not matrix $\mathbf{A}$ is consistent. Then FPP describes the equations: $a_{ij} w_j - w_i = 0$ for all $i, j = 1, 2, \ldots, n$, $j > i$, which can be represented as a system of $m = n(n-1)/2$ linear equations. Then, the following system can easily be solved for the w_i's:

$$\mathbf{R}\mathbf{w} = 0. \tag{A.41}$$

If $\mathbf{A}$ is inconsistent, it is necessary to find values of w_i, such that (A.41) is approximately satisfied, i.e., $\mathbf{R}\mathbf{w} \approx 0$. The FPP method geometrically represents (A.41) as an intersection of fuzzy hyper-lines and transforms the prioritization problem to an optimization one. It determines the values of the weights so that they correspond to the point with the highest value of the *measure of intersection*, a concept that was introduced by Mikhailov [66]. This way, the prioritization problem could be reduced to such a fuzzy programming problem that can easily be solved as a standard linear program:

$$\max \mu. \tag{A.42}$$

$$\text{subject to} \quad \mu d_j^{+} + R_j w_i \leq d_j^{+},$$

$$\mu d_j^- - R_j w_i \leq d_j^-, \quad j = 1, 2, \ldots, m, \quad 0 \leq \mu \leq 1.$$
$$\sum_{i=1}^{n} w_i = 1, \quad w_i > 0, \quad i = 1, 2, \ldots, n,$$

where the left and right *tolerance values*, d_j^- and d_j^+, represent an admissible interval for the approximate satisfaction of the *crisp equality*: $R_j w_i = 0$. The parameter μ stands for a specific consistency index of the FPP. The paper of Vermote et al. [98] used FPP and addressed the problem of "lack of customized routing networks to convey freight over the road" by proposing general principles to elaborate a regional freight route network. The paper recommends the multi-actor multi-criteria assessment (MAMCA) tool to incorporate stakeholder objectives in the evaluation of possible freight network scenarios. Such a combined use of FPP with MAMCA would assist policymakers in building consensus among multiple actors when implementing transportation projects.

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Appendix B

Derivation of the characteristic polynomial, $p_{\rm n}^{\rm P}(\lambda)$, of the perturbed PCM, $\mathbf{A}_{\rm p}$

In order to obtain the characteristic polynomial, $p_{\rm n}^{\rm P}(\lambda)$, of the perturbed PCM, $\mathbf{A}_{\rm p}$; see expression (3.8), write the determinant of the modified matrix $\mathbf{K}_{\rm p}(\lambda)$ given by (3.9), in the following form

$$\begin{aligned}\det \mathbf{K}_{\rm p}(\lambda) &= \det\left[\left(\lambda \mathbf{I}_n + \mathbf{U}_{\rm p}\mathbf{V}_{\rm p}^{\rm T}\right) - \mathbf{e}\mathbf{e}^{\rm T}\right] = \\ &= \det\left(\lambda \mathbf{I}_n + \mathbf{U}_{\rm p}\mathbf{V}_{\rm p}^{\rm T}\right)\det\left[\mathbf{I}_n - \left(\lambda \mathbf{I}_n + \mathbf{U}_{\rm p}\mathbf{V}_{\rm p}^{\rm T}\right)^{-1}\mathbf{e}\mathbf{e}^{\rm T}\right]. \qquad \text{(B1)}\end{aligned}$$

It is easy to show that

$$\det\left[\mathbf{I}_n + \mathbf{W}\mathbf{Z}^{\rm T}\right] = \det\left[\mathbf{I}_m + \mathbf{Z}^{\rm T}\mathbf{W}\right],$$

where $\mathbf{W}$ is an $n \times m$ matrix and $\mathbf{Z}^{\rm T}$ is also an $m \times n$ matrix. Rewriting (B1), then using (3.10) and (3.11) we get

$$\begin{aligned}\det \mathbf{K}_{\rm p}(\lambda) &= \det\left(\lambda \mathbf{I}_n + \mathbf{U}_{\rm p}\mathbf{V}_{\rm p}^{\rm T}\right)\left[1 - \mathbf{e}^{\rm T}\left(\lambda \mathbf{I}_n + \mathbf{U}_{\rm p}\mathbf{V}_{\rm p}^{\rm T}\right)^{-1}\mathbf{e}\right] = \\ &= \det \mathbf{T}_{\rm p}(\lambda)\left[1 - \mathbf{e}^{\rm T}\mathbf{T}_{\rm p}^{-1}(\lambda)\mathbf{e}\right]. \qquad \text{(B2)}\end{aligned}$$

The inverse of a matrix modified by a low-rank matrix may be written in the following form (see in the paper of Woodbury [B2]):

$$\mathbf{T}_{\rm p}^{-1}(\lambda) = \left(\lambda \mathbf{I}_n + \mathbf{U}_{\rm p}\mathbf{V}_{\rm p}^{\rm T}\right)^{-1} = \frac{1}{\lambda}\mathbf{I}_n - \frac{1}{\lambda}\mathbf{U}_{\rm p}\left(\lambda \mathbf{I}_2 + \mathbf{V}_{\rm p}^{\rm T}\mathbf{U}_{\rm p}\right)^{-1}\mathbf{V}_{\rm p}^{\rm T}. \qquad \text{(B3)}$$

Using (B2) and performing the necessary operations in (B3) (see in Farkas, Rózsa and Stubnya [B1, p. 426]), the characteristic polynomial of the perturbed PCM is obtained in the form

$$p_n^{\rm P}(\lambda) = \lambda^{n-3}\left\{\lambda^3 - n\lambda^2 + (n-1)\sum_{i=1}^{n-1}(1-\delta_i)\left(1 - \frac{1}{\delta_i}\right) - \sum_{i=1}^{n-1}\left(1 - \frac{1}{\delta_i}\right)\sum_{i=1}^{n-1}(1-\delta_i)\right\}. \qquad \text{(B4)}$$

Remark. It can be shown, that if the number of the rows (and their corresponding columns) which contain at least one perturbed pair of elements in the specific PCM, $\mathbf{A}$, [see matrix (3.7)] is $m \le (n-1)/2$, then the rank of matrix $\mathbf{A}$ increases by $2m$, i.e., the multiplicity of the zero eigenvalues becomes $n - 2m - 1$, and we obtain an equation of degree $2m + 1$ for the nonzero eigenvalues.

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Appendix C

Development of the principal eigenvector of the simple perturbed PCM, $\mathbf{A}_{\mathrm{S}}$

To develop the principal eigenvector of the simple perturbed PCM, $\mathbf{A}_{\mathrm{S}}$, calculate the rank-one matrix

$$\text{adj }(r\mathbf{I}_n - \mathbf{A}_{\mathrm{S}}) = \mathbf{a}\mathbf{b}^{\mathrm{T}}, \tag{C1}$$

any column of which produces the principal (right) eigenvector. First, we give the proof of the following lemma that refers to the calculation of the adjoint of a modified matrix.

Lemma C.1 *If $\mathbf{T}_{\mathrm{P}}$ is a nonsingular matrix of order n, furthermore, $\mathbf{a}$ and $\mathbf{b}$ are column vectors of order n, then the adjoint of the modified matrix $\mathbf{T}_{\mathrm{P}} - \mathbf{a}\mathbf{b}^{\mathrm{T}}$ can be obtained in the form* (Elsner and Rózsa [C1]):

$$\text{adj}\left[\mathbf{T}_{\mathrm{P}} - \mathbf{a}\mathbf{b}^{\mathrm{T}}\right] = \text{adj}\,\mathbf{T}_{\mathrm{P}}\left\{\left(1 - \mathbf{b}^{\mathrm{T}}\mathbf{T}_{\mathrm{P}}^{-1}\mathbf{a}\right)\mathbf{I}_n + \mathbf{a}\mathbf{b}^{\mathrm{T}}\mathbf{T}_{\mathrm{P}}^{-1}\right\}. \tag{C2}$$

Proof. By the Sherman-Morrison formula [see the paper C2, p. 126], the inverse of the modified nonsingular matrix $\mathbf{T}_{\mathrm{P}} - \mathbf{a}\mathbf{b}^{\mathrm{T}}$ exists if

$$1 - \mathbf{b}^{\mathrm{T}}\mathbf{T}_{\mathrm{P}}^{-1}\mathbf{a} \neq 0,$$

and it can be written as

$$(\mathbf{T}_{\mathrm{P}} - \mathbf{a}\mathbf{b}^{\mathrm{T}})^{-1} = \mathbf{T}_{\mathrm{P}}^{-1} + \frac{\mathbf{T}_{\mathrm{P}}^{-1}\mathbf{a}\mathbf{b}^{\mathrm{T}}\mathbf{T}_{\mathrm{P}}^{-1}}{1 - \mathbf{b}^{\mathrm{T}}\mathbf{T}_{\mathrm{P}}^{-1}\mathbf{a}}. \tag{C3}$$

By (B2), the determinant of a nonsingular matrix $\mathbf{T}_{\mathrm{P}}$ modified by a rank-one matrix $\mathbf{a}\mathbf{b}^{\mathrm{T}}$ is given as

$$\det\left[\mathbf{T}_{\mathrm{P}} - \mathbf{a}\mathbf{b}^{\mathrm{T}}\right] = \left(1 - \mathbf{b}^{\mathrm{T}}\mathbf{T}_{\mathrm{P}}^{-1}\mathbf{a}\right)\det\mathbf{T}_{\mathrm{P}}. \tag{C4}$$

Multiplying (C4) by the inverse (C3), the formula (C2) for the adjoint follows. □

Corollary C.1 *Since the determinant is a continuous function of its elements,* (C2) *is valid also in the case, if* $1 - \mathbf{b}^{\mathrm{T}}\mathbf{T}_{\mathrm{P}}^{-1}\mathbf{a} = 0$, i.e.,

$$\text{adj}\left[\mathbf{T}_{\mathrm{P}} - \mathbf{a}\mathbf{b}^{\mathrm{T}}\right] = (\text{adj}\,\mathbf{T}_{\mathrm{P}})\,\mathbf{a}\mathbf{b}^{\mathrm{T}}\mathbf{T}_{\mathrm{P}}^{-1}, \quad \text{if} \quad 1 - \mathbf{b}^{\mathrm{T}}\mathbf{T}_{\mathrm{P}}^{-1}\mathbf{a} = 0. \tag{C5}$$

Now applying these results for the *simple perturbed* PCM, $\mathbf{A}_{\mathrm{S}}$, and making use of (3.16), it is easy to show that

$$\text{adj}\,[\lambda\mathbf{I}_n - \mathbf{A}_{\mathrm{S}}] = \mathbf{D}\left\{\text{adj}\left[\lambda\mathbf{I}_n - \mathbf{D}^{-1}\mathbf{A}_{\mathrm{S}}\mathbf{D}\right]\right\}\mathbf{D}^{-1} = \mathbf{D}\left\{\text{adj}\,[\mathbf{K}_{\mathrm{S}}(\lambda)]\right\}\mathbf{D}^{-1}. \tag{C6}$$

Introduce the notation $\mathbf{P}_{\mathrm{S}}(\lambda) = \text{adj}[\mathbf{K}_{\mathrm{S}}(\lambda)]$. According to (C2), by letting $\mathbf{a} = \mathbf{b} = \mathbf{e}$ and using (3.19) we can write that

$$\mathbf{P}_{\mathrm{S}}(\lambda) = \text{adj}\,[\mathbf{K}_{\mathrm{S}}(\lambda)] = \text{adj}\left[\mathbf{T}_{\mathrm{S}}(\lambda) - \mathbf{e}\mathbf{e}^{\mathrm{T}}\right]. \tag{C7}$$

Substituting r for λ, by (3.22) and (3.21), it is obvious that $1-\mathbf{e}^{\mathrm{T}}\mathbf{T}_{\mathrm{S}}^{-1}(r)\mathbf{e}=0$. Thus, (C5) can be applied, and for the adjoint $\mathbf{P}_{\mathrm{S}}(r)$ we have

$$\mathbf{P}_{\mathrm{S}}(r)=\left[p_{ij}^{\mathrm{S}}(r)\right]=\text{adj }[\mathbf{K}_{\mathrm{S}}(r)]=\{\text{adj }\mathbf{T}_{\mathrm{S}}(r)\}\,\mathbf{e}\mathbf{e}^{\mathrm{T}}\mathbf{T}_{\mathrm{S}}^{-1}(r). \tag{C8}$$

Consequently, $\mathbf{P}_{\mathrm{S}}(r)$ is a rank-one matrix, therefore, any (column) vector of adj$[\mathbf{T}_{\mathrm{S}}(r)]$ is the principal eigenvector corresponding to the maximal eigenvalue r of the simple perturbed PCM, $\mathbf{A}_{\mathrm{S}}$. Hence, using (3.18), (C6), and (C8), the eigenvectors $u_{ij}^{\mathrm{S}}(r)$ given by formulas (3.24), (3.25), and (3.26) can be obtained from

$$\text{adj}\,(r\mathbf{I}_n-\mathbf{A}_{\mathrm{S}})=$$

$$=\mathbf{D}\left\{r^{n-3}\begin{bmatrix} r^2+(\delta-1)r \\ r^2-\left(1-\dfrac{1}{\delta}\right)r \\ r^2+Q \\ \vdots \\ r^2+Q\end{bmatrix}\begin{bmatrix}\dfrac{r-\left(1-\frac{1}{\delta}\right)}{r^2+Q}, & \dfrac{r+\delta-1}{r^2+Q}, & \dfrac{1}{r}, & \ldots, & \dfrac{1}{r}\end{bmatrix}\right\}\mathbf{D}^{-1} \tag{C9}$$

as the kth column of $\mathbf{P}_{\mathrm{S}}(r)$ is premultiplied by $\mathbf{D}$ and is multiplied by x_{k-1}, $k=1,2,\ldots,n$. In (C9), Q is given by (3.21).

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Appendix D

Deriving the Spectrum of Augmented Pairwise Comparison Matrices

D.1 Simple Perturbed Pairwise Comparison Matrices of Augmented Form

An augmented PCM occurs in the AHP when any of the alternatives (say the *k*th alternative) is repeated in the course of the decision process. The repeated alternative is called a *replica* or a *copy*.

Definition D.1 If a simple perturbed PCM is bordered by one of its columns and by its corresponding row, it is called a *bordered* PCM, denoted by $\mathbf{A}_{\mathrm{B}}$.

In the following we determine the components of the principal eigenvector of such a bordered PCM. Assume that a simple perturbed PCM is bordered by its kth column and by its corresponding row $(k = 1, 2, \ldots, n)$. The bordered PCM, $\mathbf{A}_{\mathrm{B}}$, may be written in the following partitioned form

$$\mathbf{A}_{\mathrm{B}} = \begin{bmatrix} \mathbf{A}_{\mathrm{S}} & \mathbf{A}_{\mathrm{S}}\mathbf{e}_k^{(n)} \\ \mathbf{e}_k^{(n)\mathrm{T}}\mathbf{A}_{\mathrm{S}} & 1 \end{bmatrix}, \tag{D.1}$$

where $\mathbf{e}_{\mathrm{k}}^{(\mathrm{n})}$ is the kth unit vector of order n:

$$\mathbf{e}_k^{(n)\mathrm{T}} = \begin{bmatrix} 1 & k & n \\ 0, \ldots, 0, & 1, & 0, \ldots, 0 \end{bmatrix},$$

and $\mathbf{A}_{\mathrm{S}}$ is given by (3.14). By factoring $\mathbf{A}_{\mathrm{B}}$ as

$$\mathbf{A}_{\mathrm{B}} = \begin{bmatrix} \mathbf{I}_n & 0 \\ \mathbf{e}_k^{(n)\mathrm{T}}\mathbf{A}_{\mathrm{S}} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{A}_{\mathrm{S}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{I}_n & \mathbf{e}_k^{(n)} \\ 0 & 1 \end{bmatrix}, \tag{D.2}$$

the eigenvalue problem of $\mathbf{A}_{\mathrm{B}}$ leads to the equation

$$\left[\lambda \mathbf{I}_{n+1} - \mathbf{A}_{\mathrm{B}}\right] \mathbf{u}^{\mathrm{B}} = 0, \tag{D.3}$$

where $\mathbf{u}^{\mathrm{B}}$ is a vector of order $n+1$. It is easy to show that (D.3) can be written in the following form

$$\begin{bmatrix} \lambda \mathbf{I}_n - \mathbf{A}_{\mathrm{S}} & -\lambda \mathbf{e}_k^{(n)} \\ -\lambda \mathbf{e}_k^{(n)\mathrm{T}} & 2\lambda \end{bmatrix} \left(\mathbf{I}_{n+1} + \mathbf{e}_k^{(n+1)}\mathbf{e}_{n+1}^{(n+1)\mathrm{T}}\right) \mathbf{u}^{\mathrm{B}} = 0. \tag{D.4}$$

Introducing now the characteristic polynomial of $\mathbf{A}_{\mathrm{B}}$, $p_{n+1}^{\mathrm{B}}(\lambda)$, we get

$$p_{n+1}^{\mathrm{B}}(\lambda, k) \equiv \det \begin{bmatrix} \lambda \mathbf{I}_n - \mathbf{A}_{\mathrm{S}} & -\lambda \mathbf{e}_k^{(n)} \\ -\lambda \mathbf{e}_k^{(n)\mathrm{T}} & 2\lambda \end{bmatrix}, \tag{D.5}$$

and the characteristic equation for the bordered perturbed matrix, $\mathbf{A}_{\mathrm{B}}$, can be written as

$$p_{n+1}^{\mathrm{B}}(\lambda, k) = 0, \quad k = 1, 2, \ldots, n. \tag{D.6}$$

Let $h(k)$, $k = 1, 2, \ldots, n$, denote the maximal eigenvalue of $\mathbf{A}_{\mathrm{B}}$. The characteristic polynomial of $\mathbf{A}_{\mathrm{B}}$ is expressed as a function of k, because it depends on the column of $\mathbf{A}_{\mathrm{S}}$ that is being repeated. In section D.4, it is shown that (D.6) leads to the equations:

$$h^3(j) - (n+1)h^2(j) - 2(n-2)Q = 0, \quad \text{if} \quad j = 1, 2, \tag{D.7}$$

and

$$h^3(q) - (n+1)h^2(q) - (n-1)Q = 0, \quad \text{if} \quad q = 3, 4, \ldots, n, \tag{D.8}$$

where Q is given by equation (3.21). In section D.4 it is shown that the elements $u_{ik}^{\mathrm{B}}(h(k))$, of the principal eigenvector for the augmented perturbed case are as follows

$$\begin{bmatrix} u_{11}^{\mathrm{B}}[h(1)] \\ u_{21}^{\mathrm{B}}[h(1)] \\ \vdots \\ u_{i1}^{\mathrm{B}}[h(1)] \\ \vdots \\ u_{n+1,1}^{\mathrm{B}}[h(1)] \end{bmatrix} = \begin{bmatrix} h^{n-2}(1)[[h(1) - (n-1)] \\ \dfrac{1}{x_1} 2h^{n-3}(1) \left\{ h(1) - \left(1 - \dfrac{1}{\delta}\right)[h(1) - (n-2)] \right\} \\ \vdots \\ \dfrac{1}{x_{i-1}} 2h^{n-3}(1) \left\{ h(1) - \left(1 - \dfrac{1}{\delta}\right) \right\} \\ \vdots \\ h^{n-2}(1)[h(1) - (n-1)] \end{bmatrix}; \; i = 3, 4, \ldots, n, \tag{D.9}$$

$$\begin{bmatrix} u_{12}^{\mathrm{B}}[h(2)] \\ u_{22}^{\mathrm{B}}[h(2)] \\ \vdots \\ u_{i2}^{\mathrm{B}}[h(2)] \\ \vdots \\ u_{n+1,2}^{\mathrm{B}}[h(2)] \end{bmatrix} = \begin{bmatrix} 2h^{n-3}(2)\left\{ h(2) + (\delta - 1)[h(2) - (n-2)] \right\} \\ \dfrac{1}{x_1} h^{n-2}(2)[h(2) - (n-1)] \\ \vdots \\ \dfrac{1}{x_{i-1}} 2h^{n-3}(2)[h(2) + (\delta + 1)] \\ \vdots \\ \dfrac{1}{x_1} h^{n-2}(2)[h(2) - (n-1)] \end{bmatrix} x_1; \; i = 3, 4, \ldots, n, \tag{D.10}$$

and

$$\begin{bmatrix} u_{1q}^{\mathrm{B}}[h(q)] \\ u_{2q}^{\mathrm{B}}[h(q)] \\ \vdots \\ u_{iq}^{\mathrm{B}}[h(q)] \\ \vdots \\ u_{n+1,q}^{\mathrm{B}}[h(q)] \end{bmatrix} = \begin{bmatrix} 2h^{n-3}(q)[h(q) + (\delta - 1)] \\ \dfrac{1}{x_1} 2h^{n-3}(q) \left\{ h(q) - \left(1 - \dfrac{1}{\delta}\right) \right\} \\ \vdots \\ \dfrac{1}{x_{i-1}} 2h^{n-2}(q) \left\{ \dfrac{h(q) - 2}{n-1} \right\} \\ \vdots \\ \dfrac{1}{x_{q-1}} 2h^{n-2}(q) \left\{ \dfrac{h(q) - 2}{n-1} \right\} \end{bmatrix} x_{q-1}; \; i, q = 3, 4, \ldots, n. \tag{D.11}$$

D.2 The Issue of Rank Reversal

The concept of rank reversal is introduced in this section. Consider the simple perturbed matrix $\mathbf{A}_{\mathrm{S}}$ defined by (3.14) and the augmented perturbed matrix $\mathbf{A}_{\mathrm{B}}$ defined by (D.1). In the specific versus the augmented perturbed case the maximal eigenvalue $h(k)$, $k = 1, 2, \ldots, n$, of the bordered matrix $\mathbf{A}_{\mathrm{B}}$ is dependent upon which column k (and its corresponding row) of the simple perturbed matrix $\mathbf{A}_{\mathrm{S}}$ is repeated. When $k = 1, 2$, then the maximal eigenvalue $h(j)$, $j = 1, 2$, of $\mathbf{A}_{\mathrm{B}}$ can be obtained from (D.7), whereas for $k = 3, 4, \ldots, n$, the maximal eigenvalue $h(q)$, $q = 3, 4, \ldots, n$, can be obtained from (D.8). In section D.4 it is verified that $h(k) > r$ and $h(k) > n + 1$, $(n \geq 3)$, $k = 1, 2, \ldots, n$, hold. Consider the case when any of the columns, $q = 3, 4, \ldots, n$, – say the nth – of the simple perturbed PCM is repeated. Hence, in this section, the components of the eigenvector (D.11) are applied.

Suppose that for two consecutive elements, u_i and u_{i+1} of the principal eigenvector of a *specific* PCM, i.e. for matrix $\mathbf{A}$ the relation

$$u_i < u_{i+1} \tag{D.12}$$

holds. In addition, assume that for the corresponding two elements, $u^{\mathrm{B}}_{\mathrm{in}}(h(n))$ and $u^{\mathrm{B}}_{i+1,n}(h(n))$, of the *bordered* PCM an opposite relation, i.e.

$$u^{\mathrm{B}}_{in}(h(n)) > u^{\mathrm{B}}_{i+1,n}(h(n)) \tag{D.13}$$

holds. If this case occurs, then, the rank order of the alternatives A_i and A_{i+1} has been reversed. This phenomenon is commonly called a *rank reversal* of the alternatives.

As is well known in the cardinal theory of preferences an opposite order of the corresponding components of the principal eigenvector *cannot* be yielded. Yet, in contrast to this, we will demonstrate the occurrence of such rank reversals in the AHP between the alternatives A_1 and A_2. For this purpose, it is sufficient to compare the order of the first two components of the principal right eigenvector.

For the specific case, the maximal eigenvalue of $\mathbf{A}$ equals n. The first two components of the principal eigenvector of $\mathbf{A}$ are as follows [cf. (3.5)]

$$1; \qquad \frac{1}{x_1} \tag{D.14}$$

i.e., the components of the principal eigenvector are monotonously increasing for $x_1 < 1$, whereas they are monotonously decreasing for $x_1 > 1$. By (D.11), it can be seen that the first two components of the principal eigenvector of the bordered perturbed PCM, $\mathbf{A}_{\mathrm{B}}$, have the same forms as those of yielded in (3.26), if the maximal eigenvalue of the simple perturbed PCM, r, is replaced by the maximal eigenvalue of the bordered perturbed PCM, $h(k)$. In Theorem D.1, necessary and sufficient condition is given for the occurrence of a rank reversal in the specific versus the augmented perturbed case.

Theorem D.1 *Let $\mathbf{A} = [a_{ij}]$ be a transitive (consistent) paired comparison matrix of order n, $n \geq 3$. Between the alternatives A_1 and A_2, when the elements a_{12} and a_{21} of $\mathbf{A}$ are perturbed and column k (and its corresponding row), $k = 3, 4, \ldots, n$, is repeated, a rank reversal occurs if and only if*

$$1 > x_1 > \frac{h(k) - 1 + \dfrac{1}{\delta}}{h(k) - 1 + \delta} = 1 - \frac{\delta - \dfrac{1}{\delta}}{\delta + [h(k) - 1]}, \quad \text{for} \quad \delta > 1, \quad k = 3, 4, \ldots, n, \tag{D.15}$$

or

$$1 < x_1 < \frac{h(k) - 1 + \dfrac{1}{\delta}}{h(k) - 1 + \delta} = 1 + \frac{\dfrac{1}{\delta} - \delta}{\delta + [h(k) - 1]}, \quad \text{for} \quad 0 < \delta < 1, \quad k = 3, 4, \ldots, n. \tag{D.16}$$

Proof. Using the expression for the adjoint of a modified matrix (see Elsner and Rózsa [D1]) and performing some algebraic manipulations with the first two elements of the kth column of adj $[h(k)\mathbf{I}_{n+1} - \mathbf{D}^{-1}\mathbf{A}_{\mathrm{B}}\mathbf{D}]$, $k = 3, 4, \ldots, n$, and letting $k = n$, i.e., the cofactors corresponding to the first two elements of the nth row $[h(k)\mathbf{I}_{n+1} - \mathbf{D}^{-1}\mathbf{A}_{\mathrm{B}}\mathbf{D}]$ are obtained as [cf. (D.11)]

$$\begin{aligned} \left\{\text{adj } [h(n)\mathbf{I}_{n+1} - \mathbf{D}^{-1}\mathbf{A}_{\mathrm{B}}\mathbf{D}]\right\}_{1n} &= 2h^{n-3}(n)[h(n) - (1 - \delta)]; \\ \left\{\text{adj } [h(n)\mathbf{I}_{n+1} - \mathbf{D}^{-1}\mathbf{A}_{\mathrm{B}}\mathbf{D}]\right\}_{2n} &= 2h^{n-3}(n)\left[h(n) - \left(1 - \frac{1}{\delta}\right)\right]. \end{aligned} \tag{D.17}$$

Premultiplying and multiplying (D.17) by $\mathbf{D}$ and $\mathbf{D}^{-1}$, respectively, the first two components of the principal right eigenvector of the bordered PCM, $\mathbf{A}_{\mathrm{B}}$, are proportional to

$$h(n) - 1 + \delta; \quad \frac{1}{x_1}\left[h(n) - 1 + \frac{1}{\delta}\right]. \tag{D.18}$$

A rank reversal occurs if the elements in (D.18) are monotonously decreasing for $x_1 < 1$, or they are monotonously increasing for $x_1 > 1$ [cf. (D.14)]. Depending on whether δ is greater than unity, or δ is less than unity, two cases are distinguished:

(i) if $\delta > 1$ and $x_1 < 1$, then the elements in (D.18) are monotonously *decreasing* if x_1 resides in the interval given by (D.15), and

(ii) if $0 < \delta < 1$ and $x_1 > 1$, then the elements in (D.18) are monotonously *increasing* if x_1 resides in the interval given by (D.16).

This means that the condition is *necessary*. Furthermore, since all operations in the proof can be performed in the opposite direction, the condition is *sufficient* as well.

We note that according to (3.21), (D.7) and (D.8), $h(k)$ is dependent on the value of δ. This fact, however, has no impact on the *existence* of the intervals (D.15) and (D.16), over which a rank reversal occurs. □

The analysis of a possible rank reversal for the case when any of the columns $k = 1, 2$, and its corresponding row is being repeated, can be made similarly.

D.3 An Illustrative Sample Example

As an illustration, consider the following 3×3 transitive PCM with its maximal λ and associated $\mathbf{u}$:

$$\mathbf{A} = \begin{bmatrix} 1 & 3/2 & 3 \\ 2/3 & 1 & 2 \\ 1/3 & 1/2 & 1 \end{bmatrix}, \quad \lambda = 3.000 = n, \quad \mathbf{u} = \begin{bmatrix} 0.500 \\ 0.333 \\ 0.167 \end{bmatrix}.$$

By repeating its *n*th column and row (i.e the third column and its associated row) we get the bordered matrix:

$$\mathbf{A}_{\mathrm{B}} = \begin{bmatrix} 1 & 3/2 & 3 & 3 \\ 2/3 & 1 & 2 & 2 \\ 1/3 & 1/2 & 1 & 1 \\ 1/3 & 1/2 & 1 & 1 \end{bmatrix}, \quad \lambda = 4.000 = n, \quad \mathbf{u}^{\mathrm{B}} = \begin{bmatrix} 0.428 \\ 0.286 \\ 0.143 \\ 0.143 \end{bmatrix}.$$

As it can be seen the bordered PCM has remained transitive and the alternatives A_3 and A_4 are in a tie. No rank reversal has occurred between the alternatives A_1 and A_2.

Let $\delta = 1/6$. Introducing this SR perturbation at the pair of elements a_{12} and a_{21} of the transitive matrix $\mathbf{A}$ we get the simple perturbed matrix $\mathbf{A}_\mathrm{S}$:

$$\mathbf{A}_\mathrm{S} = \begin{bmatrix} 1 & 1/4 & 3 \\ 4 & 1 & 2 \\ 1/3 & 1/2 & 1 \end{bmatrix}, \quad r = 3.367, \quad \mathbf{u}^\mathrm{S} = \begin{bmatrix} 0.263 \\ 0.578 \\ 0.159 \end{bmatrix}.$$

By repeating the nth column and row (i.e the third column and its associated row) of the simple perturbed matrix $\mathbf{A}_\mathrm{S}$ we get the bordered PCM, $\mathbf{A}_\mathrm{B}$:

$$\mathbf{A}_\mathrm{B} = \begin{bmatrix} 1 & 1/4 & 3 & 3 \\ 4 & 1 & 2 & 2 \\ 1/3 & 1/2 & 1 & 1 \\ 1/3 & 1/2 & 1 & 1 \end{bmatrix}, \quad h(n) = 4.424, \quad \mathbf{u}^\mathrm{B} = \begin{bmatrix} 0.267 \\ 0.467 \\ 0.133 \\ 0.133 \end{bmatrix}.$$

From $\mathbf{u}^\mathrm{B}$ it is apparent, that a rank reversal has occurred between the alternatives A_1 and A_2 in the specific versus the augmented perturbed case. This result conforms to Theorem D.1, as $a_{12} = 1.5$ and $\delta = 1/6 < 1$. With these values, by (D.16), the interval for x_1 over which a reversal certainly occurs between A_1 and A_2 in the specific versus the augmented perturbed case is: $1 < x_1 < 1.6237$. We remark that the occurrence of such a reversal, so that there is *no* reversal between the specific versus the simple perturbed case may also occur but if and only if either of the columns $k = 1, 2$ is repeated.

D.4 Theoretical Derivations for Finding the Principal Right Eigenvector of $\mathbf{A}_\mathrm{B}$

The characteristic polynomial of the bordered PCM, $\mathbf{A}_\mathrm{B}$, is given by the determinant (D.5). The eigenvectors of $\mathbf{A}_\mathrm{B}$ can be obtained from (D.4). The coefficient matrix can be considered as the characteristic matrix $\lambda\mathbf{I}_n - \mathbf{A}_\mathrm{S}$ bordered by one column and row as that of the matrix in (D.4). First, some properties of bordered matrices are given. Consider the bordered matrix

$$\mathbf{M} = \begin{bmatrix} \mathbf{G} & \mathbf{h} \\ \mathbf{c}^\mathrm{T} & d \end{bmatrix}. \tag{D.19}$$

If $\mathbf{G}$ is nonsingular, it is easy to show that

$$\det \mathbf{M} = (\det \mathbf{G}) \left(d - \mathbf{c}^\mathrm{T}\mathbf{G}^{-1}\mathbf{h}\right), \tag{D.20}$$

or

$$\det \mathbf{M} = d(\det \mathbf{G}) - \mathbf{c}^\mathrm{T}(\mathrm{adj}\, \mathbf{G})\mathbf{h}. \tag{D.21}$$

Furthermore, the inverse $\mathbf{M}^{-1}$ can be written as

$$\mathbf{M}^{-1} = \begin{bmatrix} \mathbf{G}^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -\mathbf{G}^{-1} & \mathbf{h} \\ & 1 \end{bmatrix} \left(d - \mathbf{c}^\mathrm{T}\mathbf{G}^{-1}\mathbf{h}\right)^{-1} \left[-\mathbf{c}^\mathrm{T}\mathbf{G}^{-1} \quad 1\right]. \tag{D.22}$$

Multiplying by the determinant (D.20), the adjoint is obtained in the following form

$$\mathrm{adj}\, \mathbf{M} = \left(d - \mathbf{c}^\mathrm{T}\mathbf{G}^{-1}\mathbf{h}\right) \begin{bmatrix} \mathrm{adj}\, \mathbf{G} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -(\mathrm{adj}\, \mathbf{G})\mathbf{h} \\ \det \mathbf{G} \end{bmatrix} \left[-\mathbf{c}^\mathrm{T}\mathbf{G}^{-1} \quad 1\right]. \tag{D.23}$$

Let us now apply these results for the *bordered* PCM, $\mathbf{A}_\mathrm{B}$. Using (D.21) to calculate the polynomial $p^\mathrm{B}_{n+1}(\lambda, k)$ described in (D.5), we obtain

$$p^\mathrm{B}_{n+1}(\lambda, k) \equiv 2\lambda \det [\lambda \mathbf{I}_n - \mathbf{A}_\mathrm{S}] - \lambda^2 \mathbf{e}_k^\mathrm{T} \mathrm{adj}\, [\lambda \mathbf{I}_n - \mathbf{A}_\mathrm{S}] \mathbf{e}_k. \tag{D.24}$$

From (D.24), by using (3.15) and (3.25), with the adjoint, adj $[\mathbf{K}_\mathrm{S}(\lambda)]$, we get

$$p^\mathrm{B}_{n+1}(\lambda, k) \equiv 2\lambda \left(2p^\mathrm{S}_n(\lambda) - \lambda p^\mathrm{S}_{kk}(\lambda)\right) = 0. \tag{D.25}$$

where the polynomial $p^\mathrm{S}_n(\lambda)$ is given by (3.13). For the polynomials $p^\mathrm{S}_{kk}(\lambda)$, by using $\mathbf{P}_\mathrm{S}(\lambda) = $ adj $[\mathbf{K}_\mathrm{S}(\lambda)]$ and $\mathbf{P}_\mathrm{S}(r) = $ adj $[\mathbf{K}_\mathrm{S}(r)]$, a short calculation produces the following expressions:

$$p_{11}(\lambda) = p_{22}(\lambda) = \lambda^{n-1} - (n-1)\lambda^{n-2}, \tag{D.26}$$

and

$$p_{qq}(\lambda) = \lambda^{n-1} - (n-1)\lambda^{n-2} - (n-3)Q\lambda^{n-4}, \quad q = 3, 4, \ldots, n, \tag{D.27}$$

where Q is given by (3.21). Finally, if the first column $(k = 1)$, or the second column $(k = 2)$ of the simple perturbed PCM is repeated, the substitutions of (3.20), (D.26) and (D.27) into (D.25) yield the equation

$$\lambda^{n+1} - (n+1)\lambda^n - 2(n-2)Q\lambda^{n-2} = 0, \tag{D.28}$$

and similarly, if any of the columns $3 \le k \le n$ of the simple perturbed PCM is repeated we obtain

$$\lambda^{n+1} - (n+1)\lambda^n - (n-1)Q\lambda^{n-2} = 0. \tag{D.29}$$

From (D.26) and (D.27), it is obvious that the maximal eigenvalue of $\mathbf{A}_\mathrm{B}$ is dependent upon which of the columns of $\mathbf{A}_\mathrm{S}$ is being repeated. Let the maximal eigenvalue of $\mathbf{A}_\mathrm{B}$ be denoted by $h(j)$, $j = 1, 2$, and by $h(q)$, $q = 3, 4, \ldots, n$, respectively. The maximal eigenvalue of $\mathbf{A}_\mathrm{B}$ can now easily be calculated, as equations (D.28) and (D.29) produce equations (D.7) and (D.8).

Let us now show that $h(k) > r$ holds. Substituting r for λ in (D.25), we get that

$$p^\mathrm{B}_{n+1}(\lambda, k) < -r^\mathrm{S} p^\mathrm{S}_{kk}(r) < 0,$$

since all elements of adj $(r\mathbf{I}_n - \mathbf{A}_\mathrm{S}) = \left[p^\mathrm{S}_{ij}(r)\right]$ are positive. Furthermore, since the leading terms of both of the polynomials, $p^\mathrm{S}_n(\lambda)$ and $p^\mathrm{S}_{kk}(\lambda)$ are equal to 1, from (D.25), for a sufficiently large λ, obviously, $p^\mathrm{S}_{n+1}(\lambda, k) > 0$. Therefore, indeed, for the maximal zero of $p^\mathrm{B}_{n+1}(\lambda, k)$, $h(k) > r$, $k = 1, 2, \ldots, n$, holds. From (D.28) and (D.29), it is easy to see that $h(k) > n + 1$, $k = 1, 2, \ldots, n$, holds, as well, since $Q > 0$.

The principal eigenvectors, $\mathbf{u}^\mathrm{B}(h(k))$ of the bordered matrix $\mathbf{A}_\mathrm{B}$ can be determined from (D.4) when substituting $h(k)$ into it. Let us first introduce the vectors $\mathbf{v}(h(k))$ as

$$\mathbf{v}(h(k)) = \left(\mathbf{I}_{n+1} + \mathbf{e}_k^{(n+1)} \mathbf{e}_{n+1}^{(n+1)\mathrm{T}}\right) \mathbf{u}^\mathrm{B}(h(k)). \tag{D.30}$$

Then, by expressing (D.30) for $\mathbf{u}^\mathrm{B}(h(k))$ we obtain

$$\mathbf{u}^\mathrm{B}(h(k)) = \left(\mathbf{I}_{n+1} - \mathbf{e}_k^{(n+1)} \mathbf{e}_{n+1}^{(n+1)\mathrm{T}}\right) \mathbf{v}^\mathrm{B}(h(k)). \tag{D.31}$$

In order to solve (D.4), consider the properties of its coefficient matrix. Since $h(k) > r$, it follows that $h(k)\mathbf{I}_n - \mathbf{A}_\mathrm{S}$ is nonsingular, hence the rank of the bordered coefficient matrix is

n, and therefore, the solution is equal to any column of its adjoint. By choosing the $(n+1)$st column of the adjoint and applying (D.23) we get

$$\mathbf{v}^{\mathrm{B}}(h(k)) = \begin{bmatrix} -\mathrm{adj}\left(\lambda \mathbf{I}_n - \mathbf{A}_{\mathrm{S}}\right)\left(-\lambda \mathbf{e}_k^{(n)}\right) \\ \det\left(\lambda \mathbf{I}_n - \mathbf{A}_{\mathrm{S}}\right) \end{bmatrix}, \quad \text{with } \lambda = h(k). \tag{D.32}$$

For the adjoint, we may write

$$\mathrm{adj}\left[\lambda \mathbf{I}_n - \mathbf{A}_{\mathrm{S}}\right] = \mathbf{D}\left\{\mathrm{adj}\left[\mathbf{T}_{\mathrm{S}}(\lambda) - \mathbf{e}\mathbf{e}^{\mathrm{T}}\right]\right\}\mathbf{D}^{-1} = \mathbf{D}\mathbf{P}_{\mathrm{S}}(\lambda)\mathbf{D}^{-1}. \tag{D.33}$$

Substituting into (D.32), then into (D.31), $\mathbf{u}^{\mathrm{B}}(h(k))$ yields

$$\mathbf{u}^{\mathrm{B}}(h(k)) = \left(\mathbf{I}_{n+1} - \mathbf{e}_k^{(n+1)}\mathbf{e}_{n+1}^{(n+1)\mathrm{T}}\right)\begin{bmatrix} \lambda \mathbf{D}\mathbf{P}_{\mathrm{S}}(\lambda)\mathbf{D}^{-1}\mathbf{e}_k^{(n)} \\ p_n^{\mathrm{S}}(\lambda) \end{bmatrix}, \quad \text{with } \lambda = h(k). \tag{D.34}$$

From (D.25), the relation for $h(k)$ is

$$h(k) = \frac{2p_n^{\mathrm{S}}(h(k))}{p_{kk}^{\mathrm{S}}(h(k))}. \tag{D.35}$$

Substituting into (D.34) and factoring out

$$\frac{p_n^{\mathrm{S}}(h(k))}{p_{kk}^{\mathrm{S}}(h(k))},$$

we obtain

$$\mathbf{u}^{\mathrm{B}}(h(k)) = \frac{p_n^{\mathrm{S}}(h(k))}{p_{kk}^{\mathrm{S}}(h(k))}\left\{\begin{bmatrix} 2\mathbf{D}\left[p_{ik}^{\mathrm{S}}(h(k))\right]\mathbf{D}^{-1}\mathbf{e}_k^{(n)} \\ p_{kk}^{\mathrm{S}}(h(k)) \end{bmatrix} - \begin{bmatrix} \mathbf{e}_k^{(n)}p_{kk}^{\mathrm{S}}(h(k)) \\ 0 \end{bmatrix}\right\}, \tag{D.36}$$

and hence

$$\mathbf{u}^{\mathrm{B}}(h(k)) = \frac{p_n^{\mathrm{S}}(h(k))}{p_{kk}^{\mathrm{S}}(h(k))}\begin{bmatrix} 2p_{1k}^{\mathrm{S}}(h(k)) \\ \frac{1}{x_1}2p_{2k}^{\mathrm{S}}(h(k)) \\ \vdots \\ \frac{1}{x_{k-2}}p_{k-1,k}^{\mathrm{S}}(h(k)) \\ \frac{1}{x_{k-1}}p_{kk}^{\mathrm{S}}(h(k)) \\ \frac{1}{x_k}2p_{k+1,k}^{\mathrm{S}}(h(k)) \\ \vdots \\ \frac{1}{x_{n-1}}2p_{nk}^{\mathrm{S}}(h(k)) \\ \frac{1}{x_{k-1}}p_{kk}^{\mathrm{S}}(h(k)) \end{bmatrix} x_{k-1}. \tag{D.37}$$

By omitting the factors

$$\frac{p_n^{\mathrm{S}}(h(k))}{p_{kk}^{\mathrm{S}}(h(k))},$$

and then substituting the corresponding expressions for $p_{ik}^{\mathrm{S}}(h(k)),\ i,k = 1,2,\ldots,n$, the eigenvectors (D.9), (D.10) and (D.11) are obtained.

References

[D1] Elsner, L. and Rózsa, P.: "On eigenvectors and adjoints of modified matrices." *Linear and Multilinear Algebra*. **10.** (1981), 235–247.

Appendix E

The recursive LS algorithm (triple R-I iteration) is implemented in Mathematica code and presented below. Notations in the program are in complete accordance to those used in the dissertation.

Recursive Least-Squares Algorithm for SR matrices

```
(*Enter the SR matrix A, then calculate its Perron-eigenvalue
λmax[A] and its normalized right and left Perron-eigenvectors,
umax[A] and vmax[A]*)

MatrixForm[AA_]
λmax[AA_]:=First@Eigenvalues[AA]
umax[AA_]:=(u=First@Eigenvectors[AA])/Total[u]
vmax[AA_]:=(u=First@Eigenvectors@Transpose[AA])/Total[u]
  Print[Row[{A, λmax[A], umax[A], vmax[A]}]];

(*Using the N-K method, find the stationary vector w^(0) at the initial
step k=0 of the LS algorithm, then construct the transitive matrix B_0
and compute the approximation error S[w^(0)]*)

StepZero[A_,B_,φ0_]:=Module[{n=Length[A], k=0},
    EE=ConstantArray[1,{n,n}]; ee=ConstantArray[1,{n}];
    H0=A;
    w^(0)[[1]];
    S[w_]:=Norm[A-B];
  fr=FindRoot[(W0^-2(H0-W0^-1.EE.W0)-
        Transpose[H0-W0^-1.EE.W0]W0^-2).W0.ee,{w, w[φ0]}];
W0=DiagonalMatrix[w^(0)=w/.fr];
B0=Inverse[W0].EE.W0;
  Print[Row[{w^(0), Inverse[w^(0)], W0, Inverse[W0], B0, λmax[B0], umax[B0],
        vmax[B0], S[w^(0)]}]
    If["w^(0) is infeasible"; Return["New initial guess"]]];

(*Perform the LS Recursion, by updating matrix H_k in each step of the
recursion and find the iterate w^(k), k=1,..., then test for feasibility
of these steps and compute w^(k), Inverse[w^(k)], W_k, Inverse[W_k], H_k, λmax[H_k],
umax[H_k], vmax[H_k], B_k, λmax[B_k], umax[B_k], vmax[B_k], S[w̃^(k)]*)

k=1;
NKIteration[H_,w_,W_,B_,φ0_]:=
      Module[{}, H_k:=W_{k-1}.H_{k-1}.Inverse[W_{k-1}];
    w^(k)[1]];
```

```
frr=FindRoot[(W_k^-2(W_{k-1}.H_{k-1}.W_{k-1}^-1-W_k^-1.EE.W_k)-
      Transpose[W_{k-1}.H_{k-1}.W_{k-1}^-1-W_k^-1.EE.W_k]W_k^-2).W_k.ee, {w, w[φ0]}];
  W_k=DiagonalMatrix[w^(k)=w/.frr];
  B_k=Inverse[W_k].B_{k-1}.W_k;
  S[w̃^(k)_]=Norm[A-B_k];
  If[(S[w̃^(k)]-S[w^(0)]) < 0, [δ[k_]:=0;Return["step is unsuccessful"]]];
    While[
      Test, Which[
        k > 2, If[Abs[S[w̃^(k-1)]-S[w̃^(k)]] < Abs[S[w̃^(k-1)]-S[w̃^(k-1-1)]], δ[1_,k_]:=1,
          δ[1_,k_]:=0];
        k > 3, If[Abs[S[w̃^(k-1)]-S[w̃^(k)]] < Abs[S[w̃^(k-1+1)]-S[w̃^(k-1-1)]],
          δ[1_,k_]:=1, δ[1_,k_]:=0];
        k > 5, If[Abs[S[w̃^(k-1)]-S[w̃^(k)]] < Abs[S[w̃^(k-1+2)]-S[w̃^(k-1-1)]],
          δ[1_,k_]:=1, δ[1_,k_]:=0];
        k > 7, If[Abs[S[w̃^(k-1)]-S[w̃^(k)]] < Abs[S[w̃^(k-1+3)]-S[w̃^(k-1-1)]],
          δ[1_,k_]:=1, δ[1_,k_]:=0]]];
            Print[Row[{δ[1_,k_], w^(k), Inverse[w^(k)], W_k, Inverse[W_k], H_k,
            λmax[H_k], umax[H_k], vmax[H_k], B_k, λmax[B_k], umax[B_k], vmax[B_k],
            S[w̃^(k)]}];
              If["w^(k) is infeasible"; Return["New initial
              guess"]]]; k++];

(*Write the termination steps of the algorithm: q+(r-1); l=1, r=1;
or l=2, r=1,2; or l=3, r=1,2,3; or l=4, r=1,2,3,4, and identify case
(i), or (ii), or (iii), or (iv), then compute the outputs: H_{q+(r-1)},
H GeomMean, w^(q+(r-1)), Inverse[w^(q+(r-1))], W_{q+(r-1)}, Inverse[W_{q+(r-1)}], w̃^(q+(r-1)),
Inverse[w̃^(q+(r-1))], W̃_{q+(r-1)}, Inverse[W̃_{q+(r-1)}], W̃ GeomMean, B̃_{q+(r-1)}, B̃ GeomMean,
λmax[H GeomMean], umax[H GeomMean], vmax[H GeomMean], λmax[W̃ GeomMean],
umax[W̃ GeomMean], vmax[W̃ GeomMean], λmax[B̃ GeomMean], umax[B̃ GeomMean],
vmax[B̃ GeomMean] and the GeomMean matrix error ν_g^2*)

  Test=Which[ε=10^-6; Which[
    k >=1, If[(Tr[W_k-IdentityMatrix[n]] < ε),
      (l=1;
      Print[Row[{"Case (i), q="<>ToString[k], q=k,
        H_q=H_k,
        H GeomMean=MatrixPower[Product[H_{q+r-1}, {r, 1, 1}], 1],
        w^(q)=w^(k), w^-1(q)=Inverse[w^(k)], W_q=W_k, W_q^-1=Inverse[W_k],
        w̃^(k)=Product[w^(i), {i, 0, k-1}], W̃_k=Product[W_i, {i, 0, k-1}],
        W̃_q=W̃_k,
        W̃ GeomMean=MatrixPower[Product[W̃_{q+r-1}, {r, 1, 1}], 1],
        B̃_q=Inverse[W̃_q].Inverse[W_k].EE.W_k.W̃_q,
```

```
      B̃ GeomMean=MatrixPower[Product[B̃_{q+r-1}, {r, 1, 1}], 1],
      λmax[H GeomMean], umax[HGeomMean], vmax[H GeomMean],
      λmax[W̃ GeomMean], umax[W̃ GeomMean], vmax[W̃ GeomMean],
      λmax[B̃ GeomMean], umax[B̃ GeomMean], vmax[B̃ GeomMean],
      ν_g^2=(Transpose[ee].(H GeomMean−
           Transpose[H GeomMean]).ee)^2}]]))],
  k > 3, If[(Norm[W_k−W_{k-2}] < ε) & (Norm[W_{k-1}−W_{k-3}] < ε),
    (l=2;
    Print[Row[{"Case (ii), q="<>ToString[k], q=k,
      H_q=H_k, H_{q+1}=H_{k-1},
      H GeomMean=MatrixPower[Product[H_{q+r-1}, {r, 1, 2}], 1/2],
      w^(q)=w^(k), w^{-1(q)}=Inverse[w^(k)], w^(q+1)=w^(k-1), w^{-1(q+1)}=Inverse[w^(k-1)],
      W_q=W_k, W_q^{-1}=Inverse[W_k], W_{q+1}=W_{k-1}, W_{q+1}^{-1}=Inverse[W_{k-1}],
      w̃^(k)=Product[w^(i), {i, 0, k−1}], W̃_k=Product[W_i, {i, 0, k−1}],
      w̃^(k-1)=Product[w^(i), {i, 0, k}], W̃_{k-1}=Product[W_i, {i, 0, k}],
      W̃_q=W̃_k, W̃_{q+1}=W̃_{k-1},
      W̃ GeomMean=MatrixPower[Product[W̃_{q+r-1}, {r, 1, 2}], 1/2],
      B̃_q=Inverse[W̃_q].Inverse[W_{k-1}].EE.W_{k-1}.W̃_q,
      B̃_{q+1}=Inverse[W̃_{q+1}].Inverse[W_k].EE.W_k.W̃_{q+1},
      B̃ GeomMean=MatrixPower[Product[B̃_{q+r-1}, {r, 1, 2}], 1/2],
      λmax[H GeomMean], umax[H GeomMean], vmax[H GeomMean],
      λmax[W̃ GeomMean], umax[W̃ GeomMean], vmax[W̃ GeomMean],
      λmax[B̃ GeomMean], umax[B̃ GeomMean], vmax[B̃ GeomMean],
      ν_g^2=(Transpose[ee].(H GeomMean−
           Transpose[H GeomMean]).ee)^2}]]))],
  k > 5, If[(Norm[W_k−W_{k-3}] < ε) & (Norm[W_{k-1}−W_{k-4}] < ε) & (Norm[W_{k-2}−W_{k-5}] < ε),
    (l=3;
    Print[Row[{"Case (iii), q="<>ToString[k], q=k,
      H_q=H_k, H_{q+1}=H_{k-2}, H_{q+2}=H_{k-1},
      H GeomMean=MatrixPower[Product[H_{q+r-1}, {r, 1, 3}], 1/3],
      w^(q)=w^(k), w^{-1(q)}=Inverse[w^(k)], w^(q+1)=w^(k-2), w^{-1(q+1)}=Inverse[w^(k-2)],
      w^(q+2)=w^(k-1), w^{-1(q+2)}=Inverse[w^(k-1)],
      W_q=W_k, W_q^{-1}=Inverse[W_k], W_{q+1}=W_{k-2}, W_{q+1}^{-1}=Inverse[W_{k-2}], W_{q+2}=W_{k-1},
      W_{q+2}^{-1}=Inverse[W_{k-1}],
      w̃^(k)=Product[w^(i), {i, 0, k−1}], W̃_k=Product[W_i, {i, 0, k−1}],
      w̃^(k-1)=Product[w^(i), {i, 0, k−2}], W̃_{k-1}=Product[W_i, {i, 0, k−2}],
      w̃^(k-2)=Product[w^(i), {i, 0, k}], W̃_{k-2}=Product[W_i, {i, 0, k}],
      W̃_q=W̃_k, W̃_{q+1}=W̃_{k-2}, W̃_{q+2}=W̃_{k-1},
```

```
W̃ GeomMean=MatrixPower[Product[W̃_{q+r−1},{r,1,3}],1/3],
B̃_q=Inverse[W̃_q].Inverse[W_{k−2}].EE.W_{k−2}.W̃_q,
B̃_{q+1}=Inverse[W̃_{q+1}].Inverse[W_{k−1}].EE.W_{k−1}.W̃_{q+1},
B̃_{q+2}=Inverse[W̃_{q+2}].Inverse[W_k].EE.W_k.W̃_{q+2},
B̃ GeomMean=MatrixPower[Product[B̃_{q+r−1},{r,1,3}],1/3],
λmax[H GeomMean], umax[H GeomMean], vmax[H GeomMean],
λmax[W̃ GeomMean], umax[W̃ GeomMean], vmax[W̃ GeomMean],
λmax[B̃ GeomMean], umax[B̃ GeomMean], vmax[B̃ GeomMean],
ν_g^2=(Transpose[ee].(H GeomMean−
    Transpose[H GeomMean]).ee)^2}]])],
k > 7, If[(Norm[W_k−W_{k−4}] < ε) & (Norm[W_{k−1}−W_{k−5}] < ε) & (Norm[W_{k−2}−W_{k−6}] < ε)
  & (Norm[W_{k−3}−W_{k−7}] < ε),
  (l=4;
  Print[Row[{"Case (iv), q="<>ToString[k], q=k,
   H_q=H_k, H_{q+1}=H_{k−3}, H_{q+2}=H_{k−2}, H_{q+3}=H_{k−1},
   H GeomMean=MatrixPower[Product[H_{q+r−1},{r,1,4}],1/4],
   w^(q)=w^(k), w^{−1(q)}=Inverse[w^(k)], w^(q+1)=w^(k−3), w^{−1(q+1)}=Inverse[w^(k−3)],
   w^(q+2)=w^(k−2), w^{−1(q+2)}=Inverse[w^(k−2)], w^(q+3)=w^(k−1),
   w^{−1(q+3)}=Inverse[w^(k−1)],
   W_q=W_k, W_q^{−1}=Inverse[W_k], W_{q+1}=W_{k−3}, W_{q+1}^{−1}=Inverse[W_{k−3}], W_{q+2}=W_{k−2},
   W_{q+2}^{−1}=Inverse[W_{k−2}], W_{q+3}=W_{k−1}, W_{q+3}^{−1}=Inverse[W_{k−1}],
   w̃^(k)=Product[w^(i),{i,0,k−1}], W̃_k=Product[W_i,{i,0,k−1}],
   w̃^(k−1)=Product[w^(i),{i,0,k−2}], W̃_{k−1}=Product[W_i,{i,0,k−2}],
   w̃^(k−2)=Product[w^(i),{i,0,k−3}], W̃_{k−2}=Product[W_i,{i,0,k−3}],
   w̃^(k−3)=Product[w^(i),{i,0,k}], W̃_{k−3}=Product[W_i,{i,0,k}],
   W̃_q=W̃_k, W̃_{q+1}=W̃_{k−3}, W̃_{q+2}=W̃_{k−2}, W̃_{q+3}=W̃_{k−1},
   W̃ GeomMean=MatrixPower[Product[W̃_{q+r−1},{r,1,4}],1/4],
   B̃_q=Inverse[W̃_q].Inverse[W_{k−3}].EE.W_{k−3}.W̃_q,
   B̃_{q+1}=Inverse[W̃_{q+1}].Inverse[W_{k−2}].EE.W_{k−2}.W̃_{q+1},
   B̃_{q+2}=Inverse[W̃_{q+2}].Inverse[W_{k−1}].EE.W_{k−1}.W̃_{q+2},
   B̃_{q+3}=Inverse[W̃_{q+3}].Inverse[W_k].EE.W_k.W̃_{q+3},
   B̃ GeomMean=MatrixPower[Product[B̃_{q+r−1},{r,1,4}],1/4],
   λmax[H GeomMean], umax[H GeomMean], vmax[H GeomMean],
   λmax[W̃ GeomMean], umax[W̃ GeomMean], vmax[W̃ GeomMean],
   λmax[B̃ GeomMean], umax[B̃ GeomMean], vmax[B̃ GeomMean],
   ν_g^2=(Transpose[ee].(H GeomMean−
       Transpose[H GeomMean]).ee)^2}]])],
True, k++])];
```

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