Gráfok metszési számai és a k-halmaz probléma
Doktori disszertáció

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Tartalomjegyzék

1. fejezet

Summary in Hungarian

1.1. Bevezetés

Az értekezés nagy részében gráfok metszési számainak a tulajdonságait vizsgáljuk, és kapcsolatot keresünk más gráf paraméterekkel és tulajdonságokkal. Az utolsó fejezetben a metszési számokkal szorosan összefüggő k-halmaz problémával kapcsolatos eredményeket ismertetjük.

Egy gráf lerajzolása a síkra egy olyan reprezentáció, ahol a csúcsoknak pontok, az éleknek pedig a megfelelő pontokat összekötő görbék felelnek meg. Ha nem okoz félreértést, a szövegben nem teszünk különbséget a gráf csúcsa és az őt reprezentáló pont, illetve az él és az őt reprezentáló görbe között. A lerajzolásoknál feltesszük, hogy (i) semelyik él sem tartalmaz a belsejében csúcsot, (ii) bármely két élnek véges sok közös belső pontja van és ezek mindegyikében metszik egymást, (iii) három vagy több él nem metszi egymást egy pontban.

Egy G gráf *metszési száma* $\textsc{cr}(G)$ az él-metszések minimális száma G összes lerajzolására. A metszési szám vizsgálatát 1944-ben Turán Pál kezdeményezte egy gyakorlati probléma kapcsán. Munkaszolgálatosként téglával megrakott vasúti kocsikat kellett tologatniuk a kemencéktől a raktárépületekig. Az igazán komoly nehézséget a kereszteződések okozták. Ha n kemence és m raktárépület van és minden kemence és raktár között van sín, akkor a legjobb esetben $\textsc{cr}(K_{n,m})$ kereszteződés van.

Gráfok metszési számának meghatározása nagyon nehéz, részben a különböző lerajzolások óriási nagy száma és áttekinthetetlensége miatt. Garey és Johnson be is látták, hogy a metszési szám meghatározása NP-teljes feladat

[GJ83]. Csak nagyon kicsi vagy nagyon speciális gráfok metszési számát tudták eddig pontosan meghatározni. Általában viszonylag könnyű „kitalálni” egy gráf legjobb lerajzolását, és az alsó korlát bizonyítása okoz gondot. Legtöbbször sok lépésben, egyre kifinomultabb leszámlálásokon keresztül közeledünk a célhoz [LVWW04], [AF05], [BS06], [AGOR06].

Például az említett $\mathrm{CR}(K_{n,m})$ metszési szám értéke Zarankiewicz [Z54] sejtése szerint $\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor$. A sejtett legjobb lerajzolás a következő. A egyik osztály n csúcsa közül tegyünk $\lfloor n/2 \rfloor$ darabot a pozitív x-tengelyre, a többit a negatív x-tengelyre, és hasonlóan, a másik osztály m csúcsa közül tegyünk $\lfloor m/2 \rfloor$ darabot a pozitív y-tengelyre, a többit a negatív y-tengelyre. A éleket egyenes szakaszként húzzuk be a megfelelő pontok között. Zarankiewicz [Z54] sejtését Kleitman [K70] igazolta abban az esetben, ha m legfeljebb 6, és Woodall [W93] $m = 7$, $n \leq 10$ esetén. Általában a legjobb alsó korlát $m^2n^2/20$ körül van.

A metszési számok vizsgálata újabb lendületet kapott, amikor Leighton munkássága [L83] nyomán kiderült, hogy a metszési számoknak nagy jelentősége van a nyomtatott áramkörök tervezésénél. Leighton, és tőle függetlenül Ajtai, Chvátal, Newborn és Szemerédi [ACNS82] igazolták a következő, Metszési Lemmának nevezett egyenlőtlenséget. Ha a G gráfnak n csúcsa és $e \geq 4n$ éle van, akkor $\mathrm{CR}(G) \geq \frac{1}{64}\frac{e^3}{n^2}$. Ez a korlát a konstanstól eltekintve nem javítható, a jelenleg ismert legjobb konstansok megtalálhatóak a dolgozat első fejezetében.

Megjegyezzük, hogy $e = 3n-6$ esetén még elképzelhető, hogy $\mathrm{CR}(G) = 0$, ezért szükséges az élek számát alulról korlátozni. De az $e \geq 4n$ feltétel helyett bármilyen $e > cn$, $c > 3$ feltétellel is kimondhattuk volna a Metszési Lemmát, viszont minél közelebb van c a 3-hoz, annál kisebb számot kell írnunk az $\frac{1}{64}$ együttható helyére.

A Metszési Lemma akkor került különösen az érdeklődés középpontjába, amikor 1995-ben Székely László egy új módszer segítségével egy új alkalmazási területet talált. Számos nehéz vagy nehéznek tartott korlát a geometriai illeszkedések témakörében egyszerűen következik a Metszési Lemmából. Ennek illusztrálására tekintsük a Szemerédi-Trotter tételt [ST83], amely szerint n pont és m egyenes között legfeljebb $O(n^{2/3}m^{2/3} + n + m)$ illeszkedés lehet. Ez a korlát is a konstanstól eltekintve pontos. Székely módszere a következő: Tekintsünk n pontot és m egyenest, és tegyük föl hogy I illeszkedés van köztük. Definiáljunk egy lerajzolt gráfot, amelynek csúcsai a pontok, és mindegyik egyenesen kössük össze a szomszédos pontokat. Így egy n csúcsú és $I - m$ élű gráfot kapunk. A Metszési Lemma alapján ha

$I-m \geq 4n$ akkor a metszési szám legalább $\frac{1}{64}\frac{(I-m)^3}{n^2}$. Ugyanakkor a metszések mindegyike az m egyenes egyik metszéspontja, tehát legfeljebb $\binom{m}{2}$ metszés lehet. Innen azonnal adódik a tétel. A dolgozat második fejezetében megtalálhatóak a részletek.

Székely módszerét azóta messzemenően általánosították, és nagyon sok helyen alkalmazták, [SST99], [PS98], [E02], [D98].

Az egyik fontos alkalmazási terület a k-halmaz probléma. Adott n pont a síkon, k-halmaznak nevezünk egy olyan részhalmazt, amely szeparálható a maradék $n-k$ ponttól egy egyenessel. A k-halmaz problémakör legfontosabb kérdése az, hogy egy n pontú halmaznak legfeljebb hány k-halmaza lehet. A problémát először Erdős, Lovász, Simmons és Straus [L71], [ELSS73] vetették fel, majd Edelsbrunner és Welzl [EW85], [EW86] vették észre a feladat, különösen a duális verzió fontosságát a számítógépes grafikában és geometriai algoritmusok analízisében és tervezésében. A probléma duális verziója a következő. Adott n nem függőleges egyenes a síkon, legfeljebb hány olyan metszéspont van, amely alatt $k-1$ egyenes van. Ettől kezdve a problémával nagyon sokan foglalkoztak, mind az elméleti jelentősége, mind a gyakorlati alkalmazások miatt. A problémát magasabb dimenzióban is vizsgálták, de a pontos korlát megállapítása szinte reménytelennek tűnik, még a síkon is.

A továbbiakban részletezzük a disszertáció főbb eredményeit.

1.2. A Metszési Lemma és élesítései

Metszési Lemma ([ACNS82], [L83]) *Minden n csúcsú és $e \geq 4n$ élű G gráfra $\text{CR}(G) \geq \frac{1}{64}\frac{e^3}{n^2}$ és ez a korlát konstanstól eltekintve nem javítható.*

Talán a legegyszerűbb bizonyítás a következő ötleten alapul: Tudjuk, hogy egy síkgráfnak legfeljebb $3n-6$ éle lehet. Ebből könnyen következik, hogy minden gráfra $\text{CR}(G) \geq e-(3n-6)$, hiszen egy maximális síkbarajzolt részgráf valamelyik élét minden további él metszi. Ezután tekintsünk egy tetszőleges G gráfot, és vegyük egy véletlen, ritka részgráfját, vagyis akkora valószínűséggel válasszuk ki a csúcsait, hogy az általuk feszített részgráfnak várhatóan alig több éle legyen mint egy síkgráfnak. Erre a részgráfra pedig alkalmazzuk az előbbi egyszerű egyenlőtlenséget. Innen az eredmény egy kis számolással következik; a részletek megtalálhatóak a második fejezetben.

Azt, hogy a korlát nagyságrendileg nem javítható, legegyszerűbben egy olyan gráf mutatja, amely egyforma, körülbelül $\frac{e}{n}$ méretű teljes gráfok diszjunkt uniója. Kicsit pontosabban: osszuk az n csúcsot $\frac{2e}{n}$ méretű blokkokba, minden blokkon belül húzzuk be az összes élt, a blokkok között pedig ne legyen él. Ennek a gráfnak n csúcsa és körülbelül e éle van, mindegyik blokk metszési száma $c\frac{e^4}{n^4}$, $\frac{n^2}{2e}$ darab blokk van és lerajzolhatóak a blokkok úgy, hogy a különböző blokkok élei nem metszik egymást. Így azt kapjuk, hogy a metszési szám $c\frac{e^3}{n^2}$ valamilyen c konstansra.

Nézzük meg egy kicsit közelebbről a $\mathrm{CR}(G) \geq e-(3n-6)$ egyenlőtlenséget! Egy másik lehetséges bizonyítása az élek számára vonatkozó indukció. Ha $e \leq 3n-6$, akkor az állítás nyilvánvaló, ha pedig $e > 3n-6$, akkor a gráf nem síkgráf. Hagyjunk el egy élet, amin van metszés, és használjuk az indukciós feltevést. A korlát nem javítható, ha e $3n-6$ közelében van, de nagyobb e esetén vélhetően nem pontos, hiszen ha a gráfnak sok éle van, akkor kell lenni olyan élnek is, amin nem egy, hanem több metszés van. Ennek elhagyásával erősebb korlátot bizonyíthatunk. Ezen az úton először Pach Jánossal indultunk el 1995-ben. Legyen $e_k(n)$ egy n csúcsú gráf éleinek maximális száma, amely úgy lerajzolható, hogy minden élen legfeljebb k metszés van. Világos, hogy $e_0(n) = 3n-6$. Beláttuk, hogy ha $0 \leq k \leq 4$, akkor $e_k(n) \leq (k+3)(n-2)$. Ennek segítségével a $\mathrm{CR}(G) \geq e-(3n-6)$ egyenlőtlenségnél jóval erősebb $\mathrm{CR}(G) \geq 5e-25n$ egyenlőtlenséget, és ezt felhasználva az $\frac{1}{64}$ konstans helyett $\frac{1}{33.75}$-et kaphatunk. Beláttuk azt is, hogy az $e_1(n) \leq 4n-8$ és $e_2(n) \leq 5n-10$ egyenlőtlenségek pontosak, viszont $e_3(n)$ esetében már nem találkozott az alsó és felső korlátunk. A másik gyenge pontja az említett korlátoknak az, hogy csak olyan lerajzolásokat engedhettünk meg, amelyekben bármely két élnek csak egy közös pontja van, amely vagy közös végpont vagy metszéspont. Pach Jánossal, Radoš Radoičić-csel és Tardos Gáborral pontosítottuk a korlátot, és általánosítottuk az eredményeket olyan lerajzolásokra is, ahol az élek akármilyen sokszor metszhetik egymást.

1. Tétel. *Ha egy n csúcsú gráf lerajzolható úgy hogy bármelyik élén legfeljebb 3 metszés van, akkor az élek száma legfeljebb $5.5(n-2)$. Ez a korlát egy additív konstanstól eltekintve pontos.*

Az 1. Tétel és további észrevételek felhasználásával azt kaphatjuk, hogy minden n csúcsú és e élű G gráfra $\mathrm{CR}(G) \geq 4e - \frac{103}{6}n$, és végül, ezt az egyenlőtlenséget használva, a Metszési Lemmát a következő formában kapjuk, az eddig ismert legjobb konstanssal.

2. Tétel. *Minden n csúcsú és $e \geq 18n$ élű G gráfra $\mathrm{CR}(G) \geq 0.032\frac{e^3}{n^2}$.*

Azt is beláttuk, hogy a fenti állítás már nem teljesül, ha a konstans helyére 0.09-et írunk.

De vajon egyáltalán beszélhetünk „legjobb” konstansról? Pach Jánossal és Joel Spencerrel, Erdős és Guy régi sejtését igazolva bebizonyítottuk, hogy igen, a következő értelemben: Legyen $\kappa(n,e)$ az n csúcsú és e élű gráfok metszési számának a minimuma, azaz

$$\kappa(n,e) = \min_{\substack{n(G)=n \\ e(G)=e}} \operatorname{CR}(G).$$

3. Tétel. *Ha $n \ll e \ll n^2$, akkor a*

$$\lim_{n\to\infty} \kappa(n,e)\frac{n^2}{e^3} = C > 0$$

határérték létezik.

Az $a \ll b$ jelölés azt jelenti, hogy $a = o(b)$. A 2. Tétel és az utána levő megjegyzés alapján tehát $0.032 < C < 0.09$. Azt nem tudtuk eldönteni, hogy valóban szükség van-e az $n \ll e \ll n^2$ feltételre. Elképzelhető, hogy a jóval gyengébb $C_1 n < e < C_2 n^2$ feltétel is elegendő. Ha igen, akkor $C_1 > 3$, hiszen $\kappa(n, 3n) = 6$. Ugyanakkor a teljes gráf metszési számára ismert alsó korlát [G72] alapján láthatjuk, hogy $e = \binom{n}{2}$ helyettesítéssel a tételben szereplő C konstansnál nagyobb számot kapunk, tehát a feltételben $C_2 < \frac{1}{2}$.

Egy gráf *vastagsága* (bisection width, $b(G)$) azon élek minimális száma, amelyek elhagyásával a gráf két, közel egyforma (legalább $n/3$ csúcsú) részre bomlik föl. Ez a paraméter rendkívül hasznos a rekurzív algoritmusok tervezésében és elemzésében, és rekurzív bizonyításokban. Pach, Shahrokhi és Szegedy bizonyították a következő összefüggést a vastagság és a metszési szám között. Tetszőleges G n csúcsú gráfra amelyben a csúcsok fokai $d_1, d_2, \ldots, d_n$,

$$b(G) \leq 10\sqrt{\operatorname{CR}(G)} + 2\sqrt{\sum_{i=1}^{n} d_i^2}. \tag{1.1}$$

Ez durván szólva azt jelenti, hogy egy kis metszési számú gráfnak a vastagsága is kicsi. Ebben az értelemben ez a síkgráfokra vonatkozó Lipton-Tarjan szeparátor tétel [LT79] általánosítása. Ezt felhasználva belátható, hogy azoknak a gráfoknak, amelyeknek a metszési száma e^3/n^2 közelében van, nagyon speciális struktúrájuk van, nagyon hasonlóak a már említett példához. A csúcsok beoszthatóak nagyságrendileg egyforma nagy, ce/n méretű blokkokra úgy, hogy a blokkok pozitív sűrűségű részgráfokat feszítenek, míg a blokkok közötti élek halmaza összesen is csak egy elhanyagolható része az összes élnek. Tehát ha a gráfnak valamilyen olyan tulajdonsága van amely lehetetlenné teszi a pozitív sűrűségű részgráfokat, akkor a metszési számra adott ce^3/n^2 alsó korlát javítható.

Ez az észrevétel vezetett az extremális gráfelmélet és a metszési számok közötti összefüggés felismeréséhez. Nézzünk erre egy példát! Ismert [R58], hogy egy n csúcsú, 4 hosszú kört nem tartalmazó gráfnak legfeljebb $cn^{3/2}$ éle van.

4. Tétel. *Ha a G gráfnak n csúcsa és e éle van, és G nem tartalmaz 4 hosszú kört, akkor*

$$\mathrm{CR}(G) \geq c\frac{e^4}{n^3},$$

és ez a korlát nagyságrendileg nem javítható.

A példa, ami azt mutatja, hogy a korlát nem javítható, nagyon hasonló az általános példához, amely megfelelő méretű teljes gráfok uniója. Itt megfelelő méretű *extremális gráfok* unióját kell tekinteni.

Hasonló eredményeket kaptunk más tiltott részgráfok esetén, valamint olyan gráfokra, amelyekben nincs rövid kör, illetve általában minden olyan öröklődő gráftulajdonságra, amelynél a maximális élszám $o(n^2)$. Azokban az esetekben, amikor a megfelelő extremális gráfelméleti feladatban az élek maximális száma nagyságrendileg ismert, ott a metszési számokra is nagyságrendileg nem javítható korlátot kapunk.

1.3. Egyéb metszési számok

Egy gráf metszési számán általában a „metszések minimális számát” [BL84] értjük a gráf összes lerajzolására. Azonban ez a definíció pontosításra szorul. Bizonyos szerzők felteszik, hogy egy lerajzolásban az élek csak egyenes szakaszok lehetnek [J71]. Más esetekben görbék is megengedettek. Ha az élek görbék, feltehetjük, hogy bármely két élnek csak egy közös pontja van [WB78], [B91] [PT97], közös végpont vagy metszéspont, illetve megengedhetjük, hogy akárhány metszéspontjuk legyen [T70], [GJ83], [SSSV97]. Ez utóbbi esetben ráadásul számolhatjuk a metszéspontokat, vagy a metsző élpárokat függetlenül attól, hogy az élpár hányszor metszi egymást. Sőt számolhatjuk csak azokat az élpárokat, amelyek *páratlan sokszor* metszik egymást. A tisztázatlan definíciók miatt hibás, hiányos, vagy részben egymásnak ellentmondó állítások jelentek meg, és a kutatók többsége nem volt tudatában annak, hogy ezek az árnyalatnyi definíciós különbségek lényeges eltérést eredményezhetnek. Pach Jánossal rendet kívántunk teremteni ezen a téren. Tekintsük a következő négy metszési számot.

$\textsc{lin-cr}(G)$, az *egyenes vonalú metszési szám*: a metszéspontok minimális száma a G gráf olyan lerajzolásaira, ahol az élek *egyenes szakaszok*

$\textsc{cr}(G)$ a *közönséges metszési szám*: a metszéspontok minimális száma a G gráf olyan lerajzolásaira, ahol az élek *tetszőleges görbék*.

$\textsc{pair-cr}(G)$ a *pár-metszési szám*: a metsző élpárok minimális száma a G gráf olyan lerajzolásaira, ahol az élek *tetszőleges görbék*.

$\textsc{odd-cr}(G)$ a *páratlan-metszési szám*: az egymást *páratlan sokszor* metsző élpárok minimális száma a G gráf olyan lerajzolásaira, ahol az élek *tetszőleges görbék*.

Mindig feltesszük, hogy egy él nem megy át egy csúcson, és két él közös pontja vagy közös végpont vagy metszés (tehát nem érinthetik egymást).

Nyilvánvalóan $\textsc{cr}(G)$ a „szokásos" metszési szám, ezen dolgozat második fejezete is erről a metszési számról szól. A definíciókból az is nyilvánvaló, hogy

$$\textsc{odd-cr}(G) \leq \textsc{pair-cr}(G) \leq \textsc{cr}(G) \leq \textsc{lin-cr}(G)$$

minden G gráfra. Nem nehéz belátni, hogy ha $\textsc{odd-cr}(G) = 0$, akkor G nem tartalmazhat topologikus K_5 és $K_{3,3}$ részgráfokat, tehát Kuratowski tétele alapján síkgráf. Viszont ekkor $\textsc{cr}(G) = \textsc{pair-cr}(G) = 0$, sőt Fáry tétele miatt $\textsc{lin-cr}(G) = 0$.

Bienstock és Dean [BD93] találtak olyan G_i gráfokat, amelyekre $\textsc{cr}(G_i) = 4$, de $\textsc{lin-cr}(G_i)$ tetszőlegesen nagy. Tehát az egy nagyon lényeges különbség, hogy az élek egyenes szakaszok vagy görbék, $\textsc{cr}(G)$ értékének semmilyen függvényével sem lehet $\textsc{lin-cr}(G)$ értékét felülről becsülni.

Pach Jánossal bebizonyítottuk, hogy a másik három metszési számra nem ez a helyzet, $\textsc{odd-cr}(G)$ illetve $\textsc{pair-cr}(G)$ segítségével felülről becsülhető $\textsc{cr}(G)$.

5. Tétel. *Minden G gráfra*

$$\textsc{cr}(G) < 2 \cdot \textsc{odd-cr}(G)^2.$$

A bizonyítás lényege a következő: Induljunk ki egy lerajzolásból, amely $\textsc{odd-cr}(G)$ szempontjából optimális, vagyis ahol a páratlan sok pontban metsző élpárok száma éppen $\textsc{odd-cr}(G)$. Nevezzük *páros élnek* azokat az éleket, amelyeket minden más él páros sokszor metsz, a többi élet meg nevezzük *páratlan élnek*. Megmutatjuk, hogy a gráf úgy is lerajzolható hogy a *páros éleket* egyáltalán ne metssze semmi. Ezután már könnyű elérni hogy a *páratlan élek* közül bármely kettő legfeljebb egyszer metssze egymást. Mivel a páratlan élek száma legfeljebb $2 \cdot \textsc{odd-cr}(G)$, az így kapott lerajzolásban valóban kevesebb mint $2 \cdot \textsc{odd-cr}(G)^2$ metszéspont van.

Mivel $\textsc{pair-cr}(G) \geq \textsc{odd-cr}(G)$, nyilvánvalóan $\textsc{cr}(G) < 2 \cdot \textsc{pair-cr}(G)^2$ is teljesül. Ezt a korlátot Pavel Valtr megjavította, bebizonyította, hogy minden gráfra $\textsc{cr}(G) \leq c \cdot \textsc{pair-cr}(G)^2 / \log \textsc{pair-cr}(G)$. Ezt sikerült tovább javítanom.

6. Tétel. *Minden G gráfra*

$$\textsc{cr}(G) \leq c \cdot \textsc{pair-cr}(G)^2 / \log^2 \textsc{pair-cr}(G).$$

Ez a korlát valószínűleg még mindig nagyon gyenge, különös tekintettel arra, hogy lehetséges hogy minden G gráfra $\textsc{cr}(G) = \textsc{pair-cr}(G)$.

Schaefer, Štefankovič és Pelsmajer [PSS06] egy gyönyörű konstrukció segítségével megmutatták, hogy $\textsc{odd-cr}(G)$ viszont már nem mindig egyenlő a másik két metszési számmal, mutattak olyan G gráfot, amelyre $\textsc{odd-cr}(G) < \textsc{pair-cr}(G)$, pontosabban tetszőleges $\varepsilon > 0$ konstanshoz konstruáltak olyan $G = G_\varepsilon$ gráfot amelyre $\textsc{odd-cr}(G) < (\sqrt{3}/2 + \varepsilon) \cdot \textsc{pair-cr}(G)$. Ezen a korláton sikerült minimálisan javítanom egy egészen más konstrukcióval.

7. Tétel. *Minden $\varepsilon > 0$ konstanshoz létezik olyan $G = G_\varepsilon$ gráf amelyre*

$$\textsc{odd-cr}(G) < \left(\frac{3\sqrt{5}}{2} - \frac{5}{2} + \varepsilon \right) \textsc{pair-cr}(G).$$

.

1.4. Véletlen gráfok metszési számai

Már tudjuk, hogy egy n csúcsú és $e \geq 4n$ élű gráf metszési száma nagyságrendileg e^3/n^2 és e^2 között van. De vajon mennyi egy *tipikus* gráf metszési száma? Legyen $G(n,p)$ egy n csúcsú véletlen gráf, amelynek bármely két csúcsa között egymástól függetlenül p valószínűséggel húzunk be élt. Az élek számának várható értéke $e = p\binom{n}{2}$. Pach Jánossal bebizonyítottuk, hogy $\mathrm{CR}(G)$ várható értéke nagyságrendileg a maximális e^2 közelében van, sőt értéke majdnem biztosan a várható érték közelében van.

8. Tétel. *Legyen $G(n,p)$ egy n csúcsú véletlen gráf p él-valószínűséggel, és legyen $e = p\binom{n}{2}$, az élek számának várható értéke. Ha $e \geq 10n$, akkor majdnem biztosan*

$$\mathrm{CR}(G) \geq \frac{e^2}{4000}.$$

9. Tétel. *Ugyanezekkel a jelölésekkel*

$$\Pr\left[|\mathrm{CR}(G) - \mathrm{E}[\mathrm{CR}(G)]| > 3\alpha e^{3/2}\right] < 3\exp(-\alpha^2/4)$$

teljesül minden olyan α számra, amelyre $(e/4)^3\exp(-e/4) \leq \alpha \leq \sqrt{e}$.

Mivel $\mathrm{LIN\text{-}CR}(G) \geq \mathrm{CR}(G)$, a 8. Tétel állítása $\mathrm{LIN\text{-}CR}(G)$-re is teljesül. A bizonyítás ismét a már említett, a metszési szám és vastagság közötti összefüggésen alapul. Ennek alapjan egy gráf metszési száma alulról becsülhető a vastagság segítségével. A vastagság várható értéke egy véletlen gráfnál pedig könnyen becsülhető.

Joel Spencerrel tovább vizsgáltuk véletlen gráfok metszési számait, és több irányba is általánosítottuk és pontosítottuk a 8. Tételt. A 9. Tétel állítása teljesül a többi metszési számra is, ugyanazzal a bizonyítással.

Legyen

$$\kappa_{\mathrm{LIN\text{-}CR}}(n,p) = \frac{\mathrm{E}\left[\mathrm{LIN\text{-}CR}(G)\right]}{e^2}, \quad \kappa_{\mathrm{CR}}(n,p) = \frac{\mathrm{E}\left[\mathrm{CR}(G)\right]}{e^2},$$

$$\kappa_{\mathrm{PAIR\text{-}CR}}(n,p) = \frac{\mathrm{E}\left[\mathrm{PAIR\text{-}CR}(G)\right]}{e^2}, \quad \kappa_{\mathrm{ODD\text{-}CR}}(n,p) = \frac{\mathrm{E}\left[\mathrm{ODD\text{-}CR}(G)\right]}{e^2},$$

ahol $G = G(n,p)$. A definíció alapján nyilvánvaló hogy $\kappa_{\mathrm{ODD\text{-}CR}}(n,p) \leq \kappa_{\mathrm{PAIR\text{-}CR}}(n,p) \leq \kappa_{\mathrm{CR}}(n,p) \leq \kappa_{\mathrm{LIN\text{-}CR}}(n,p)$ minden n-re és p-re.

10. Tétel. *Tetszőleges $n > 0$-ra $\kappa_{\mathrm{LIN\text{-}CR}}(n,p)$, $\kappa_{\mathrm{CR}}(n,p)$, $\kappa_{\mathrm{PAIR\text{-}CR}}(n,p)$ és $\kappa_{\mathrm{ODD\text{-}CR}}(n,p)$ p növekvő, folytonos függvényei.*

11. Tétel. *Tetszőleges $\varepsilon > 0$ esetén legyen $p = p(n) = n^{\varepsilon-1}$, ekkor*

$$\liminf_{n\to\infty} \kappa_{\mathrm{PAIR\text{-}CR}}(n,p) > 0, \quad \liminf_{n\to\infty} \kappa_{\mathrm{ODD\text{-}CR}}(n,p) > 0.$$

Itt az a fő különbség a 8. Tételhez képest hogy a már említett, a gráf vastagságán alapuló bizonyítási módszert nem alkalmazhattuk, mivel a gráf vastagsága és $\text{CR}(G)$ közötti Pach-Shahrokhi-Szegedy egyenlőtlenség (1.1) megfelelője nem ismert $\text{PAIR-CR}(G)$-vel, illetve $\text{ODD-CR}(G)$-vel. Csak jóval gyengébb, számunkra használhatatlan egyenlőtlenség ismert. Ezért más módszert kellett alkalmazni. Bebizonyítottuk, hogy nagy valószínűséggel nagyon sok topologikus K_5 található $G(n,p)$-ben, mindegyik egy-egy metszést jelent, és ebből becsüljük $\text{PAIR-CR}(G)$ és $\text{ODD-CR}(G)$ értékét. Ezt viszont csak $e > n^{\varepsilon+1}$ esetén tudtuk alkalmazni, míg a 8. Tételben elég volt feltenni, hogy $e > 10n$. A következő tétel azt mutatja, hogy ezt a feltételt lényegesen gyengíthetjük.

12. Tétel. *Tetszőleges $c > 1$ esetén legyen $p = p(n) = c/n$, ekkor*

$$\liminf_{n\to\infty} \kappa_{\text{CR}}(n,p) > 0$$

A 12. Tétel természetesen $\kappa_{\text{LIN-CR}}(n,p)$-re is teljesül. Itt viszont sokkal erősebb állítást is be tudtunk látni: rögzített n-re $\kappa_{\text{LIN-CR}}(n,p)$ mint p függvénye nagyon gyorsan eléri a maximumat.

13. Tétel. *Ha $p = p(n) \gg \frac{\ln n}{n}$ akkor*

$$\lim_{n\to\infty} \kappa_{\text{LIN-CR}}(n,p) = \lim_{n\to\infty} \kappa_{\text{LIN-CR}}(n,1) = \lim_{n\to\infty} \frac{\text{LIN-CR}(K_n)}{\binom{n}{2}^2}$$

1.5. A k-halmaz probléma

Ebben a fejezetben minden ponthalmazról feltesszük, hogy általános helyzetben van, vagyis nincs három pont egy egyenesen. Legyen P egy n pontú halmaz a síkon. Egy k elemű részhalmazt *k-halmaznak* nevezünk, ha elválaszthatók a többi $n-k$ ponttól egy egyenessel. A kérdés az, hogy egy n elemű ponthalmaznak legfeljebb hány k-halmaza lehet. Ez a kombinatorikus geometria talán egyik legizgalmasabb, máig megoldatlan kérdése. A problémát átfogalmazhatjuk a következő módon. A P halmaz egy pontpárját *k-élnek* nevezzük, ha az általuk meghatározott egyenes egyik oldalán $k-1$, másik oldalán $n-k-1$ pont van. Nem nehéz belátni hogy a k-élek és a k-halmazok száma megegyezik, így vizsgálhatjuk a k-élek számát is.

Egy alkalmas duális transzformációt alkalmazva a pontokból egyenesek, a k-élekből pedig olyan metszéspontok lesznek, amelyek alatt pontosan $k-1$

egyenes van, fölötte pedig $n-k-1$. Így kapjuk a k-szint problémát, amely lényegében ekvivalens a k-halmaz problémával. Adott n általános helyzetű egyenes (semelyik három nem metszi egymás ugyanabban a pontban), egyik sem függőleges. Tekintsük az egyenesek azon pontjainak a halmazát, amelyek alatt pontosan k másik egyenes van. Ez a halmaz az egyeneseken levő nyílt intervallumokból áll, amelyeknek a végpontjai a metszéspontok. A *k-adik szint* ennek a halmaznak a relatív lezártja, vagyis hozzávesszük a metszéspontokat is. Így egy x-monoton töröttvonalat kapunk, amely minden metszéspontban kanyarodik. A k-adik szint *bonyolultsága* vagy *hossza* az őt alkotó intervallumok száma, vagyis a kanyarok száma plusz egy. A k-szint probléma az, hogy k általános, nem függőleges egyenes halmazában legfeljebb mekkora lehet a k-adik szint bonyolultsága.

A k-halmaz problémát először Erdős, Lovász, Simmons és Straus [L71], [ELSS73] vetették fel, és bebizonyították az $O(n\sqrt{k})$ felső korlátot. Ezenkívül konstruáltak olyan ponthalmazt, amelynek $\Omega\,(n\log k)$ k-halmaza van. Edelsbrunner és Welzl [EW85], [EW86] fogalmazták meg először a probléma duális verzióját, és ők vették észre a probléma fontosságát geometriai algoritmusok elemzésében. Az alsó és felső korlátokon érdemben nem tudtak javítani. Annak ellenére, hogy a problémát intenzíven vizsgálták, a felső korlátot csak 20 évvel később Pach, Steiger és Szemerédi [PSS92] tudta megjavítani, egy $\log^* k$ faktorral. Bebizonyították, hogy egy n pontú halmaznak legfeljebb $O(n\sqrt{k}/\log^* k)$ k-halmaza van. Végül 1998-ban Tamal Dey [D98] ért el áttörést, Székely már említett módszerét és a Metszési Lemmát zseniálisan alkalmazva. Az ő felső korlátja $O(n\sqrt[3]{k})$.

Az $\Omega\,(n\log k)$ alsó korlátot viszont nem sikerült megjavítani, kivéve a kisebb javításokat a konstans szorzón [EW85], [E92], [E98], és sokan azt sejtették, hogy ez a korlát az igazság közelében van. Ezt sikerült 2000-ben megcáfolni.

14. Tétel. *Tetszőleges n, k, $n \geq 2k > 0$ számokhoz létezik olyan n pontú halmaz a síkon, amely k-halmazainak a száma*

$$ne^{\Omega\left(\sqrt{\log k}\right)}.$$

A probléma természetesen általánosítható magasabb dimenzióra is, és ott még sokkal kevesebbet tudunk. A leginkább vizsgált eset az, amikor $k=n/2$, azaz n páros. A kérdés ebben az esetben úgy is fogalmazható, hogy

n általános helyzetű pont halmazát a d dimenziós térben hány különböző módon lehet félbevágni egy hipersíkkal. Jelöljük ezt a számot $f_d(n)$-nel. Ezzel a jelöléssel az előbb említett legjobb korlátok a síkban $f_2(n) = O(n^{4/3})$, illetve $f_2(n) = n \exp(\Omega(\sqrt{\log n}))$. Az nyilvánvaló, hogy $f_d(n) = O(n^d)$. Három dimenzióban az első javítást Bárány, Füredi és Lovász [BFL90] érték el, bebizonyították, hogy $f_3(n) = O(n^{3-1/343})$. Ezt javította Aronov, Chazelle, Edelsbrunner, Guibas, Sharir és Wenger [ACE91], Eppstein [E93], majd Dey és Edelsbrunner [DE94]. A jelenleg ismert legjobb felső korlát, $f_3(n) = O(n^{5/2})$, Sharir, Smorodinsky és Tardos [SST99] eredménye. Nemrég Matoušek, Sharir, Smorodinsky és Wagner [MSSW06] általánosította a három dimenziós bizonyítási módszereket négy dimenzióra, az ő eredményük $f_4(n) = O(n^{4-2/45})$. Ennél magasabb dimenzióban a legjobb felső korlátot Živaljević és Vrećica [ZV92] algebrai topológiai eredményéből (sokszínű Tverberg tétel) Alon, Bárány, Füredi és Kleitman [ABFK92] vezette le, ennek értelmében $f_d(n) = O(n^{d-c_d})$, ahol $c_d = (4d-3)^{-d}$.

A legjobb alsó korlát minden dimenzióban a 14. Tétel egyszerű következménye.

15. Tétel. *Tetszőleges páros n-re és $d \geq 2$-re*

$$f_d(n) = n^{d-1} e^{\Omega\left(\sqrt{\log n}\right)}.$$

Térjünk vissza a síkra. Radoš Radoičić-csel a k-szint problémának egy általánosítását vizsgáltuk. Tekintsünk n általános helyzetű egyenest. Egy x-monoton töröttvonal, amely az egyenesek szakaszaiból áll, *hossza* az őt alkotó intervallumok száma, vagyis a rajta levő kanyarok száma plusz 1. Sharir vetette föl a kérdést, hogy mekkora $h(n)$, egy ilyen töröttvonal maximális hossza. Ez a kérdés tehát annyiban általánosabb a k-szint problémánál, hogy az itt vizsgált töröttvonalaknak nem feltétlenül kell minden metszéspontban kanyarodni.

Sharir és Meggido [E87] mutatták meg, hogy $h(n) = \Omega(n^{3/2})$, Matoušek [M91] $\Omega(n^{5/3})$-re javította a korlátot. Ezt javítottuk tovább.

16. Tétel. $h(n) = \Omega(n^{7/4})$.

Azóta Balogh, Regev, Smyth, Steiger és Szegedy [BRSSS04] ezt az eredményt jelentősen tovább javította, az ő alsó korlátjuk $h(n) = \Omega(n^{2-(d/\sqrt{\log n})})$ valamilyen d konstansra.

A feladatra $\binom{n}{2}$ triviális felső korlát és a ma ismert legjobb, majdnem triviális korlát ennek lényegében a fele.

Köszönetnyilvánítás

Nagyon hálás vagyok Pach Jánosnak, aki már 18 éve végtelen türelemmel és fáradhatatlanul segíti munkámat.

Köszönöm tanáraim, Elekes György, Richard Pollack és Thiry Imréné, társszerzőim, Bóna Miklós, ifj. Böröczky Károly, Csizmadia György, Adrian Dumitrescu, Fejes Tóth Gábor, Gyárfás András, Károlyi Gyula, Keszegh Balazs, Jan Kynčl, Daniel Kleitman, Pálvölgyi Dömötör, Rom Pinchasi, Radoš Radoičić, Micha Sharir, Solymosi József, Joel Spencer, Tardos Gábor, Torsten Thiele és Pavel Valtr, és még sok más munkatársam segítségét, bíztatását, támogatását és a közös munkát.

Chapter 2

Improving the Crossing Lemma

This chapter is based on the manuscript [PRTT06]. Twenty years ago, Ajtai, Chvátal, Newborn, Szemerédi, and, independently, Leighton discovered that the crossing number of any graph with v vertices and $e > 4v$ edges is at least ce^3/v^2, where $c > 0$ is an absolute constant. This result, known as the 'Crossing Lemma,' has found many important applications in discrete and computational geometry. It is tight up to a multiplicative constant. Here we improve the best known value of the constant by showing that the result holds with $c > 1024/31827 > 0.032$. The proof has two new ingredients, interesting on their own right. We show that (1) if a graph can be drawn in the plane so that every edge crosses at most 3 others, then its number of edges cannot exceed $5.5(v-2)$; and (2) the crossing number of any graph is at least $\frac{7}{3}e - \frac{25}{3}(v-2)$. Both bounds are tight up to an additive constant (the latter one in the range $4v \leq e \leq 5v$).

2.1 Introduction

Unless stated otherwise, the graphs considered in this paper have no loops or parallel edges. The number of vertices and number of edges of a graph G are denoted by $v(G)$ and $e(G)$, respectively. We say that G is *drawn* in the plane if its vertices are represented by distinct points and its edges by (possibly intersecting) Jordan arcs connecting the corresponding point pairs. If it leads to no confusion, in terminology and notation we make no distinction between the vertices of G and the corresponding points, or between the edges and the corresponding Jordan arcs. We always assume that in a drawing (a) no edge passes through a vertex different from its endpoints, (b) no three edges cross at the same point, (c) any two edges have only a finite number of interior points in common, and at these points they properly cross, i.e., one of the edges passes from one side of the other edge to the other side (see [P99], [P04]). A *crossing* between two edges is their common interior point (if it exists). The *crossing number* of G, denoted by $\mathrm{CR}(G)$, is the minimum number of crossings in a drawing of G satisfying the above conditions.

Ajtai, Chvátal, Newborn, and Szemerédi [ACNS82] and, independently, Leighton [L83] have proved the following result, which is usually referred to as the 'Crossing Lemma.' The crossing number of any graph with v vertices and $e > 4v$ edges satisfies

$$\mathrm{CR}(G) \geq \frac{1}{64}\frac{e^3}{v^2}.$$

This result, which is tight apart from the value of the constant, has found many applications in combinatorial geometry, convexity, number theory, and VLSI design (see [L83], [S98], [PS98], [ENR00], [STT02], [PT02]). In particular, it has played a pivotal role in obtaining the best known upper bound on the number of k-sets [D98] and lower bound on the number of distinct distances determined by n points in the plane [ST01], [KT04]. According to a conjecture of Erdős and Guy [EG73], which was verified in [PST00], as long as $e/v \to \infty$ and $e/v^2 \to 0$, the limit

$$\lim_{v\to\infty} \min_{\substack{v(G)=v\\ e(G)=e}} \frac{\mathrm{CR}(G)}{e^3/v^2}$$

exists. The best known upper and lower bounds for this constant (roughly 0.09 and $1/33.75 \approx 0.029$, resp.) were obtained in [PT97].

All known proofs of the Crossing Lemma are based on the trivial inequality $\mathrm{CR}(H) \geq e(H) - (3v(H) - 6)$, which is an immediate corollary of Euler's Polyhedral Formula ($v(H) > 2$). Applying this statement inductively to all small (and, mostly sparse) subgraphs $H \subseteq G$ or to a randomly selected one, the lemma follows. The main idea in [PT97] was to obtain stronger inequalities for the *sparse* subgraphs H, which have led to better lower bounds on the crossing numbers of *all* graphs G. In the present paper we follow the same approach.

For $k \geq 0$, let $e_k(v)$ denote the maximum number of edges in a graph of $v \geq 2$ vertices that can be drawn in the plane so that every edge is involved in at most k crossings. By Euler's Formula, we have $e_0(v) = 3(v-2)$. Pach and

Tóth [PT97] proved that $e_k(v) \le (k+3)(v-2)$, for $0 \le k \le 3$. Moreover, for $0 \le k \le 2$, these bounds are tight for infinitely many values of v. However, for $k = 3$, there was a gap between the lower and upper estimates. Our first theorem, whose proof is presented in Section 2.2, fills this gap.

Theorem 2.1.1. *Let G be a graph on $v \ge 3$ vertices that can be drawn in the plane so that each of its edges crosses at most three others. Then we have*

$$e(G) \le 5.5(v-2).$$

Consequently, the maximum number of edges over all such graphs satisfies $e_3(v) \le 5.5(v-2)$, and this bound is tight up to an additive constant.

As we have pointed out before, the inequality $e_0(v) \le 3(v-2)$ immediately implies that if a graph G of v vertices has more than $3(v-2)$ edges, then every edge beyond this threshold contributes at least one to $\textsc{cr}(G)$. Similarly, it follows from inequality $e_1(v) \le 4(v-2)$ that, if $e(G) \ge 4(v-2)$, then every edge beyond $4(v-2)$ must contribute an additional crossing to $\textsc{cr}(G)$ (i.e., altogether at least two crossings). Summarizing, we obtain that

$$\begin{aligned}\textsc{cr}(G) &\ge (e(G) - 3\,(v(G)-2)) + (e(G) - 4\,(v(G)-2)) \\ &\ge 2e(G) - 7\,(v(G)-2)\end{aligned}$$

holds for every graph G. Both components of this inequality are tight, so one might expect that their combination cannot be improved either, at least in the range when $e(G)$ is not much larger that $4(v-2)$. However, this is not the case, as is shown by our next result, proved in Section 2.3.

Theorem 2.1.2. *The crossing number of any graph G with $v(G) \ge 3$ vertices and $e(G)$ edges satisfies*

$$\textsc{cr}(G) \ge \frac{7}{3}e(G) - \frac{25}{3}(v(G)-2).$$

This bound is tight up to an additive constant whenever $4\,(v(G)-2) \le e(G) \le 5\,(v(G)-2)$.

As an application of the above two theorems, in Section 2.4 we establish the following improved version of the Crossing Lemma.

Theorem 2.1.3. *The crossing number of any graph G satisfies*

$$\text{CR}(G) \geq \frac{1}{31.1} \frac{e^3(G)}{v^2(G)} - 1.06 v(G).$$

If $e(G) \geq \frac{103}{16} v(G)$, we also have

$$\text{CR}(G) \geq \frac{1024}{31827} \frac{e^3(G)}{v^2(G)}.$$

Note for comparison that $1024/31827 \approx 1/31.08 \approx 0.032$.

In the last section, we adapt the ideas of Székely [S98] to deduce some consequences of Theorem 2.1.3, including an improved version of the Szemerédi-Trotter theorem [ST83] on the maximum number of incidences between n points and m lines. We also discuss some open problems and make a few conjectures and concluding remarks.

All drawings considered in this paper satisfy the condition that *any pair of edges have at most one point in common.* This may be either an endpoint or a proper crossing. It is well known and easy to see that every drawing of a graph G that minimizes the number of crossings meets this requirement. Thus, in the proofs of Theorems 2.1.2 and 2.1.3, we can make this assumption without loss of generality. However, it is not so obvious whether the same restriction can be justified in the case of Theorem 2.1.1. Indeed, in [PT97], the bound $e(G) \leq (k+3)(v(G)-2)$ was proved only for graphs that can be drawn with at most $k \leq 4$ crossings per edge and which satisfy this extra condition. To prove Theorem 2.1.1 in its full generality, we have to establish the following simple statement.

Lemma 2.1.4. *Let $k \leq 3$, and let G be a graph of v vertices that can be drawn in the plane so that each of its edges participates in at most k crossings.*

In any drawing with this property that minimizes the total number of crossings, every pair of edges have at most one point in common.

Proof: Suppose for contradiction that some pair of edges, e and f, have at least two points in common, A and B. At least one of these points, say B, must be a proper crossing. First, try to swap the portions of e and f between A and B, and modify the new drawing in small neighborhoods of A and B so as to reduce the number of crossings between the two edges. Clearly, during this process the number of crossings along any other edge distinct from e and f remains unchanged. The only possible problem that may arise is that after the operation either e or f (say e) will participate in more than k crossings. In this case, before the operation there were at least two more crossings inside the portion of f between A and B, than inside the portion of e between A and B. Since f participated in at most three crossings (at most two, not counting B), we conclude that in the original drawing the portion of e between A and B contained no crossing. If this is the case, instead of swapping the two portions, replace the portion of f between A and B by an arc that runs very close to the portion of e between A and B, without intersecting it. □

It is interesting to note that the above argument fails for $k \geq 4$, as shown in Figure 2.1.

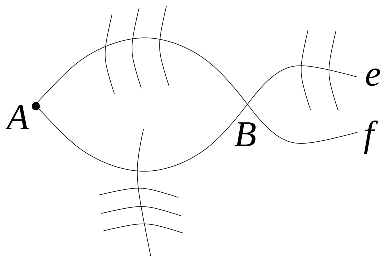

Figure 2.1: Two adjacent edges e and f cross, each participating in exactly 4 crossings.

2.2 Proof of Theorem 2.1.1

We use induction on v. For $v \leq 4$, the statement is trivial. Let $v > 4$, and suppose that the theorem has already been proved for graphs having fewer than v vertices.

Let $\mathcal{G}$ denote the set of all triples $(G, G', \mathcal{D})$ where G is a graph of v vertices, $\mathcal{D}$ is a drawing of G in the plane such that every edge of G crosses at most *three* others (and every pair of edges have at most one point in common), and G' is a *planar subgraph* of G with $V(G') = V(G)$ that satisfies the condition that no two arcs in $\mathcal{D}$ representing edges of G' cross each other. Let $\mathcal{G}' \subset \mathcal{G}$ consist of all elements $(G, G', \mathcal{D}) \in \mathcal{G}$ for which the number of edges of G is maximum. Finally, let $\mathcal{G}'' \subset \mathcal{G}'$ consist of all elements of $\mathcal{G}'$ for which the number of edges of G' is maximum. Fix a triple $(G, G', \mathcal{D}) \in \mathcal{G}''$ such that the total number of crossings in $\mathcal{D}$ along all edges of G' is as small as possible. This triple remains fixed throughout the whole argument. The term *face*, unless explicitly stated otherwise, refers to a face of the planar drawing of G' induced by $\mathcal{D}$. For any face Φ (of G'), let $|\Phi|$ denote its number of sides, i.e., the number of edges of G' along the boundary of Φ, where every edge whose both sides belong to the interior of Φ is counted twice. Notice that $|\Phi| \geq 3$ for every face Φ, unless G' consists of a single edge, in which case $v(G) \leq 4$, a contradiction.

It follows from the maximality of G' that every edge e of G that does not belong to G' (in short, $e \in G - G'$) crosses at least one edge of G'. The closed portion between an endpoint of e and the nearest crossing of e with an edge of G' is called a *half-edge*. We orient every half-edge from its endpoint which is a vertex of G (and G') towards its other end sitting in the interior of an edge of G'. Clearly, every edge $e \in G - G'$ has two oriented half-edges. Every half-edge lies in a face Φ and contains at most *two* crossings with edges of G in its interior. The *extension* of a half-edge is the edge of $G - G'$ it belongs to. The set of half-edges belonging to a face Φ is denoted by $H(\Phi)$.

Lemma 2.2.1. *Let Φ be a face of G', and let g be one of its sides. Then $H(\Phi)$ cannot contain two non-crossing half-edges, both of which end on g and cross two other edges of G (that are not necessarily the same).*

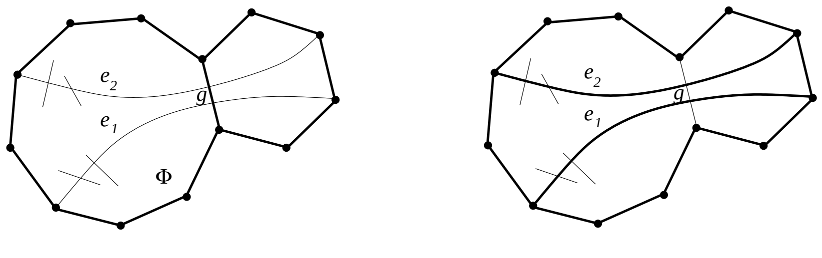

Figure 2.2: Lemma 2.2.1; the edges of G' are drawn in bold.

Proof: Let e_1 and e_2 denote the extensions of two non-crossing half-edges in Φ that end on g. Both half-edges cross two edges of G, so their extensions cannot cross any other edge apart from g. Removing g from G' and adding e_1 and e_2, we would obtain a larger plane subgraph of G, contradicting the maximality of G'. $\square$

A face Φ of G' is called *simple* if its boundary is connected and it does not contain any isolated vertex of G' in its interior.

Lemma 2.2.2. *The number of half-edges in any simple face Φ satisfies*

$$|H(\Phi)| \leq 3|\Phi| - 6.$$

Proof: For an induction argument to go through, it will be more convenient to prove the lemma for more general configurations. Slightly abusing the terminology and the notation, we prove the inequality $|H(\Phi)| \leq 3|\Phi| - 6$, for any simple 'face' Φ with $|\Phi| \geq 3$ (Φ may have nothing to do with G or G') and for any set of oriented 'half-edges' $H(\Phi)$ contained in Φ that satisfy the following conditions:

(i) Every half-edge in $H(\Phi)$ emanates from a vertex of Φ and ends at an edge of Φ not incident to that vertex.

(ii) The number of half-edges ending at any edge of Φ is at most *three*.

(iii) Every half-edge belonging to $H(\Phi)$ crosses at most *two* others.

(iv) If there are two non-crossing half-edges in $H(\Phi)$, each crossing *two* other elements of $H(\Phi)$, then they cannot end at the same edge of Φ.

By definition, conditions (i)–(iii) are satisfied for 'real' faces and half-edges associated with the triple $(G, G', \mathcal{D})$, while (iv) follows from Lemma 2.2.1.

Assume without loss of generality that the boundary of Φ is a simple cycle. If this is not the case, replace each vertex of Φ encountered more than once during a full counter-clockwise tour around the boundary of Φ by as many copies as many times it is visited, and replace each edge of Φ whose both sides belong to Φ by two edges running very close to it. Obviously, the number of sides of the resulting 'face' will be the same as that of the original.

We proceed by induction on $s = |\Phi|$. We start with the case $s = 3$. Denote the vertices of Φ by A, B, and C. Let a, b, and c denote the number of half-edges in Φ, emanating from A, B, and C, respectively. Without loss of generality, we can assume that $a \geq b \geq c$. By (i), every half-edge must end in the interior of the edge opposite to its starting point. Thus, by (ii), we have $a \leq 3$. Every half-edge emanating from C must cross all half-edges emanating from A and B. Hence, by (iii), if $a + b > 2$, we must have $c = 0$. Similarly, if $a = 3$, then $b = 0$ must hold. The only set of values satisfying the above constraints, for which we have $a+b+c > 3s-6 = 3$, is $a = b = 2$ and $c = 0$. In this case, both half-edges emanating from A end in the interior of the edge BC and both cross the two half-edges emanating from B, which contradicts condition (iv).

Now let $s > 3$, and suppose that the statement has already been proved for faces with fewer than s sides.

Given a half-edge $h \in H(\Phi)$, its endpoints divide the boundary of Φ into two pieces. Consider all of these pieces over all elements of $H(\Phi)$, and let $\mathcal{R}$ be the set of those pieces that have the smallest number of vertices in their interiors. Pick R, a minimal element of $\mathcal{R}$ by containment. R is defined by a half-edge $e = AE$, where A is a vertex of Φ and E is an interior point of an edge g of Φ (see Figure 2.3). Let P denote the set of all half-edges in Φ that start at A and end on g. Clearly, we have $e \in P$ and, by (ii), $1 \le |P| \le 3$. By the minimality of R, every element of P other than e ends outside R. Let Q denote the set of half-edges in Φ that cross e. We claim that every element $h \in Q$ crosses all half-edges in P. Indeed, otherwise h would start at an interior vertex of R and end at a point of g outside R. However, in this case the piece of the boundary of Φ defined by h, which contains E, would have fewer interior vertices than R, contradicting the choice of R.

Thus, if $|P| = 3$ then, by (iii), Q must be empty. If $|P| = 2$ then, by (iv), $|Q| \le 1$, and if $|P| = 1$ then, by (iii), $|Q| \le 2$. Therefore, we always have $|P \cup Q| \le 3$.

Let $\overline{\Phi}$ denote the 'face' obtained from Φ as follows. Replace the arc R by the half-edge e. Remove all vertices and edges in R, and regard the union of e and the part of g not belonging to R as a single new edge (see Figure 2.3). By the definition of $\mathcal{R}$, the resulting face has $s' \ge 3$ sides. By (i), we have $s' < s$. Consider the set of half-edges $H(\overline{\Phi}) = H(\Phi) \setminus (P \cup Q)$. None of the elements of this set crosses e, so, by the minimality of R, all of them lie in $\overline{\Phi}$. They meet the conditions (i)–(iv), so one can apply the induction hypothesis to conclude that

$$|H(\Phi)| \le |H(\overline{\Phi})| + 3 \le (3s'-6) + 3 \le 3s - 6,$$

as claimed. □

Return to the proof of Theorem 2.1.1. A simple face Φ of G' is said to be *triangular* if $|\Phi| = 3$, otherwise it is a *big face*.

By Lemma 2.2.2, we have $|H(\Phi)| \le 3$, for any triangular face Φ. A triangular face Φ is called an *i-triangle* if $|H(\Phi)| = i$ $(0 \le i \le 3)$. A 3-

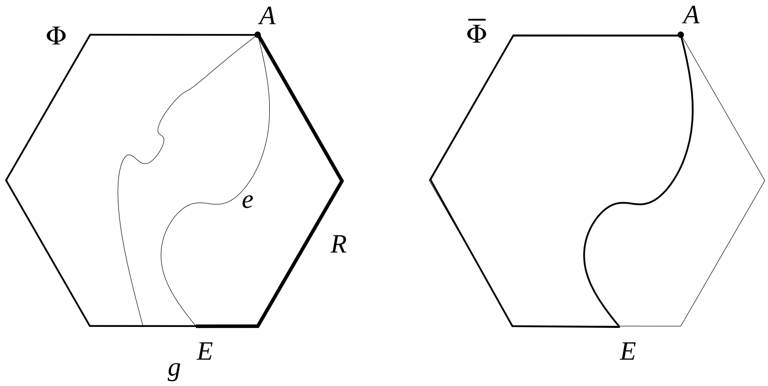

Figure 2.3: Induction step in the proof of Lemma 2.2.2.

triangle is a $3X$*-triangle* if one half-edge emanates from each of its vertices. Otherwise, it is a $3Y$*-triangle*.

If Φ is a $3Y$-triangle, then at least two of its half-edges must end at the same side. The face adjacent to Φ along this side is called the *neighbor* of Φ.

An edge of $G - G'$ is said to be *perfect* if it starts and ends in 3-triangles and all the faces it passes through are triangular. The neighbor Ψ of a $3Y$-triangle Φ is called a *strong neighbor* if either it is a 0-triangle or it is a 1-triangle and the extension of one of the half-edges in $H(\Phi)$ ends in Ψ.

Lemma 2.2.3. *Let Φ be a 3-triangle. If the extensions of at least two half-edges in $H(\Phi)$ are perfect, then Φ is a $3Y$-triangle with a strong neighbor.*

Proof: If Φ is a $3X$-triangle, then the extension of none of its half-edges is perfect (see Figure 2.4a). Indeed, observe that if Φ is a $3X$-triangle, then it has three mutually crossing half-edges, so that their extensions do not have any additional crossing and they must end in a face adjacent to Φ. Moreover, no other edges of G can enter a $3X$-triangle.

Therefore, Φ is a $3Y$-triangle. It has a unique neighbor Ψ, which, by the assumptions in the lemma, must be a triangle. We use a tedious case analysis, illustrated by Figure 2.4, to prove that Ψ is a strong neighbor. We only sketch the argument. The set of extensions of the half-edges in $H(\Phi)$ is denoted by H.

Case 1. One half-edge $f \in H(\Phi)$ emanates from a different vertex than the other two. Then the extension $\overline{f} \in H$ of f is not perfect (see Figure

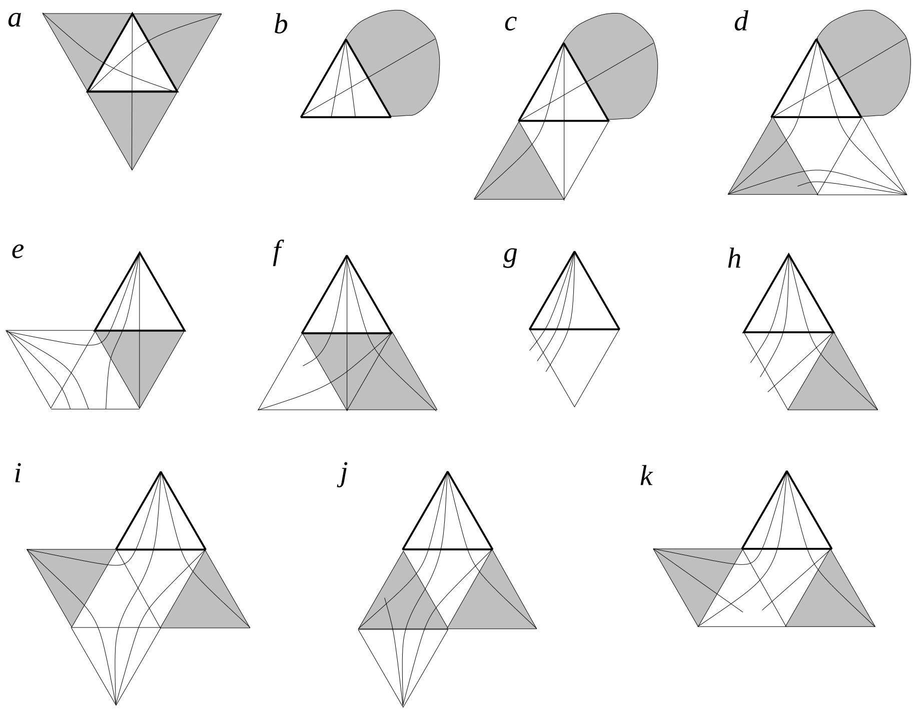

Figure 2.4: Proof of Lemma 2.2.3; triangles that are shaded are not 3-triangles.

2.4b). We have to distinguish further cases, depending on where the other two edges end, to conclude that at least one of them cannot be perfect either (see Figure 2.4cd). An interested reader can find a thorough outline of this case in Appendix 1.

Case 2. All half-edges of $H(\Phi)$ emanate from the same vertex.
Subcase 2.1. Some edge $e \in H$ ends in Ψ. Then Ψ is not a 3-triangle, so e is not perfect. If the other two edges are perfect, then Ψ is a 1-triangle (see Figure 2.4ef).

Subcase 2.2. None of the edges in H end in Ψ. Suppose Ψ is not a 0-triangle. Then some edge $e \in H$ must leave Ψ through a different side than the other two edges $f, g \in H$ do (see Figure 2.4g). Then e cannot be perfect (see Figure 2.4h). We have to distinguish three cases, depending on whether f, g, or neither of them end in the triangle next to Ψ. In each of these cases, one can show that f and g cannot be perfect simultaneously (see Figure 2.4ijk).
$\square$

Claim A. Suppose that Ψ is a simple face of G' with $|\Psi| = 4$ and $|H(\Psi)| = 6$. Then there are seven combinatorially different possibilities for the arrangement of Ψ and the half-edges, as shown in Figure 2.5.

The proof of Claim A is a straightforward case analysis, carried out in Appendix 2.

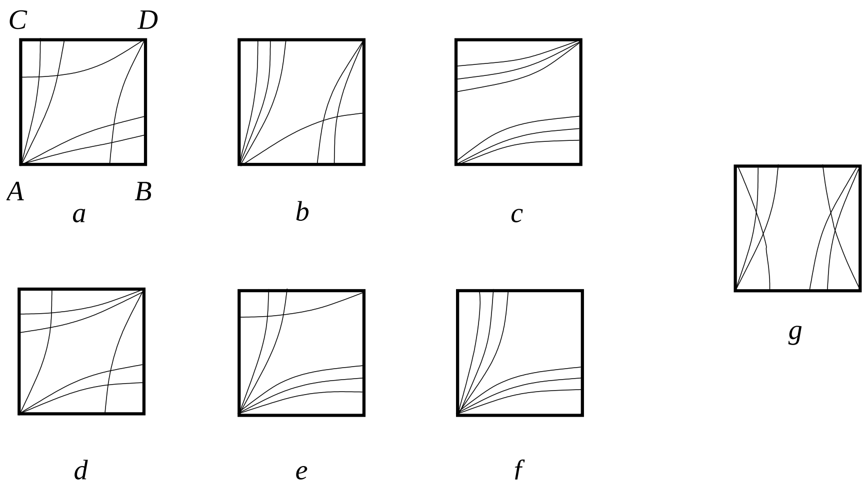

Figure 2.5: Seven different types of quadrilateral faces.

Lemma 2.2.4. *Let Ψ be a simple face of G' with $|\Psi| = 4$ and $|H(\Psi)| = 6$, and suppose that the arrangement of half-edges in Ψ is* not *homeomorphic with configuration (g) on Figure 2.5. Then we have*

$$E(G) < 5.5\,(v(G) - 2)\,.$$

Proof: Notice that one of the diagonals of Φ, denoted by $e = AB$, can be added in the interior of Φ without creating any crossing with the half-edges in Ψ or with other potentially existing edges of $G - G'$ that may enter Φ. Thus, by the maximality of G (more precisely, by the fact that $(G, G', \mathcal{D}) \in \mathcal{G}'$), we may assume that that A and B are connected by an edge e' of G. Obviously, e' must lie entirely outside of Ψ. (See Figure 2.6, for an illustration.) We may also assume that $e' \in G'$ and that it does not cross any edge of G, otherwise replacing e' by e in G, we would obtain a contradiction with the maximality of G' (more precisely, with the fact that $(G, G', \mathcal{D}) \in \mathcal{G}''$ and the total number of crossings along all edges of G' is as small as possible).

Let G_1 (resp. G_2) denote the subgraph of G induced by A, B, and all vertices in the interior (resp. exterior) of the 'lens' enclosed by e and e' (see Figure 2.6). Clearly, we have $v(G) = v(G_1) + v(G_2) - 2$ and $e(G) = e(G_1) + e(G_2) - 1$. As e' and e run in the exterior and in the interior of Ψ, resp., both $v(G_1)$ and $v(G_2)$ are strictly smaller than $v(G)$. Therefore, we can apply the induction hypothesis to G_1 and G_2 to obtain that

$$\begin{aligned} e(G) = e(G_1) + e(G_2) - 1 &\le 5.5\,(v(G_1) - 2) + 5.5\,(v(G_2) - 2) - 1 \\ &< 5.5\,(v(G) - 2)\,, \end{aligned}$$

as required. $\square$

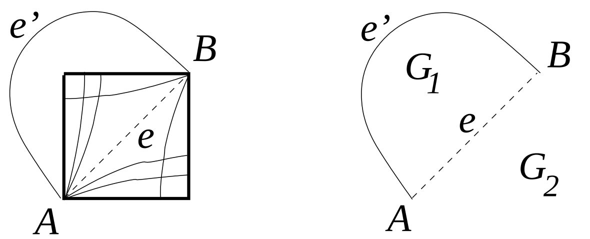

Figure 2.6: Proof of Lemma 2.2.4.

In view of the last lemma, from now on we may and will assume that in every simple quadrilateral face that contains 6 half-edges, these half-edges form an arrangement homeomorphic to configuration (g) on Figure 2.5.

We define a bipartite multigraph $M = (V_1 \cup V_2, E)$ with vertex classes V_1 and V_2, where V_1 is the set of 3-triangles and V_2 is the set of all other faces of G'. For each vertex (3-triangle) $\Phi \in V_1$, separately, we add to the edge set E of M some edges incident to Φ, according to the following rules.

- *Rule 0*: Connect Φ to an adjacent triangular face Ψ by two parallel edges if Ψ is a 0-triangle.
- *Rule 1*: Connect Φ to any (not necessarily adjacent) 1-triangle Ψ by two parallel edges if there is an edge of $G - G'$ that starts in Φ and ends in Ψ.
- *Rule 2*: Connect Φ to any (not necessarily adjacent) 2-triangle Ψ by a single edge if there is an edge of $G - G'$ that starts in Φ and ends in Ψ.
- *Rule 3*: If the extension e of a half-edge in $H(\Phi)$ passes through or ends in a big face, we may connect Φ by a single edge to the first such big face along e. However, we use this last rule only to bring the degree of Φ in M up to 2. In particular, if we have applied Rules 0 or 1, for some Φ, we do not apply Rule 3. Similarly, in no case do we apply Rule 3 for all three half-edges in $H(\Phi)$.

Notice that, besides Rules 0 and 1, the application of Rule 3 can also yield parallel edges if two half-edges in $H(\Phi)$ reach the same big face. However, we never create three parallel edges in M.

Let $d(\Phi)$ denote the degree of vertex Φ in M.

Lemma 2.2.5. *For any $\Phi \in V_1$, we have $d(\Phi) \geq 2$.*

Proof: We can disregard the restriction on the use of Rule 3, since it only applies if $d(\Phi)$ has already reached 2. If the extension e of a half-edge in $H(\Phi)$ is not perfect, then e yields a (possible) edge of M incident to Φ according to one of the Rules 1, 2, or 3. We get two edges this way, unless the extensions of at least two of the half-edges in $H(\Phi)$ are perfect. In this latter case, Lemma 2.2.3 applies and either Rule 0 or Rule 1 provides two parallel edges of M connecting Φ to its strong neighbor. $\square$

To complete the proof of Theorem 2.1.1, we have to estimate from *above* the degrees of the vertices belonging to V_2 in M. If $\Psi \in V_2$ is a 1-triangle or a 2-triangle, we have $d(\Psi) \leq 2$. Every 0-triangle Ψ is adjacent to at most three 3-triangles, so its degree satisfies $d(\Psi) \leq 6$. The following lemma establishes a bound for big faces.

Lemma 2.2.6. *For any big face $\Psi \in V_2$, we have $d(\Psi) \leq 2|\Psi|$. Moreover, if Ψ is a simple quadrilateral face with six half-edges forming an arrangement homeomorphic to the one depicted in Figure 2.5g, we have $d(\Psi) \leq 4$.*

Proof: Every edge of M incident to Ψ corresponds to an edge of $G - G'$ that starts in some 3-triangle and enters Ψ. Different edges of M correspond to different edges of $G - G'$ (or opposite orientations of the same edge). Since any side of Ψ crosses at most 3 edges of $G - G'$, we obtain the weaker bound $d(\Psi) \leq 3|\Psi|$. If Ψ is a simple quadrilateral face satisfying the conditions in the second part of the lemma, then two of its sides do not cross any edge of $G - G'$, hence we have $d(\Psi) \leq 6$. The stronger bounds stated in the lemma immediately follow from the fact that, even if some side of a big face Ψ is crossed by three edges of $G - G'$, they can contribute only at most 2 to the degree of Ψ.

To verify this fact, consider a fixed side g of Ψ, and suppose that it crosses three edges of $G - G'$. These crossings do not contribute to the degree of Ψ if both sides of g belong to the interior of Ψ; so we assume that this is not the case. Every edge e that crosses g is divided by g into two pieces. If the piece incident to the exterior side of g passes through a big face or does not end in a 3-triangle, then e does not contribute to $d(\Psi)$. Therefore, we may assume that all three such edge pieces pass through only triangular faces and end in 3-triangles (hence, excluding all but the cases a, g, j and k in Figure 2.7). A case analysis shows that either at least one of these edge pieces ends in a 3-triangle which has a strong neighbor (see Figure 2.7gjk), or all of them end in the same 3-triangle (see Figure 2.7a). In either case, the corresponding three edges contribute at most two to the degree of Ψ.

The details of the case analysis are omitted, but they can be reconstructed from Figure 2.7, where the circular arc, together with the horizontal segment, represents the boundary of Ψ. Dark-shaded triangles are not 3-triangles, while light-shaded triangles are $3Y$-triangles with a strong neighbor. We omitted the cases where the three edges crossing g leave the triangular face adjacent to g through the same other edge g'. These cases can be handled by removing the edge g and considering the resulting big face and the three edges crossing the side g' of this face. Applying this reduction twice if necessary we reduce this case to one of the other cases. $\square$

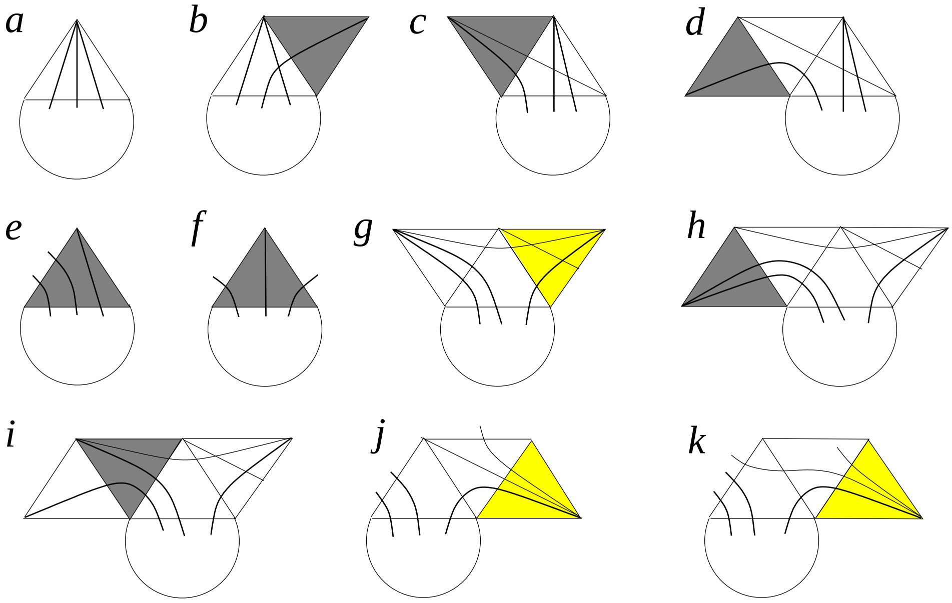

Figure 2.7: Proof of Lemma 2.2.6; dark-shaded triangles (bcdefhi) and light-shaded triangles (gjk).

For any face Φ, let $t(\Phi)$ and $\bar{t}(\Phi)$ denote the number of triangles and diagonals, resp., in a triangulation of Φ. Thus, if the sum of the number of isolated vertices of G' that lie in the interior of Φ and the number of connected components of the boundary of Φ is k, we have $t(\Phi) = |\Phi| + 2k - 4$ and $\bar{t}(\Phi) = |\Phi| + 3k - 6$.

We introduce the notation $\overline{d}(\Phi) := -d(\Phi)$ for $\Phi \in V_1$, and $\overline{d}(\Psi) := d(\Psi)$ for $\Psi \in V_2$. Let $V := V_1 \cup V_2$ denote the set of all faces of G'. Then the fact that the sum of degrees of the vertices must be the same on both sides of M, can be expressed by the equation

$$\sum_{\Phi \in V} \overline{d}(\Phi) = 0.$$

Lemma 2.2.7. *For every face $\Phi \in V$, we have*

$$|H(\Phi)| + \frac{1}{4}\overline{d}(\Phi) \leq \frac{5}{2}t(\Phi) + 2\bar{t}(\Phi).$$

Proof: The proof is by straightforward case analysis, based on the previous lemmas.

If Φ is triangular, we have $\bar{t}(\Phi) = 0$, $t(\Phi) = 1$, so that $\frac{5}{2}t(\Phi) + 2\bar{t}(\Phi) = \frac{5}{2}$. For a 3-triangle Φ, by Lemma 2.2.5, we have $|H(\Phi)| + \frac{1}{4}\bar{d}(\Phi) \leq 3 + \frac{1}{4}(-2) = \frac{5}{2}$. For a 2-triangle Φ, we have $|H(\Phi)| + \frac{1}{4}\bar{d}(\Phi) \leq 2 + \frac{1}{4}(2) = \frac{5}{2}$. For a 1-triangle Φ, we have $|H(\Phi)| + \frac{1}{4}\bar{d}(\Phi) \leq 1 + \frac{1}{4}(2) = \frac{3}{2}$, and for a 0-triangle Φ, we have $|H(\Phi)| + \frac{1}{4}\bar{d}(\Phi) \leq 0 + \frac{1}{4}(6) = \frac{3}{2}$.

If Φ is a simple face with $|\Phi| \geq 5$ sides, we have $t(\Phi) = |\Phi| - 2$ and $\bar{t}(\Phi) = |\Phi| - 3$, so that $\frac{5}{2}t(\Phi) + 2\bar{t}(\Phi) = \frac{9}{2}|\Phi| - 11$. It follows from Lemmas 2.2.2 and 2.2.6 that $|H(\Phi)| \leq 3|\Phi| - 6$ and $\bar{d}(\Phi) = d(\Phi) \leq 2|\Phi|$. Thus, we have

$$|H(\Phi)| + \frac{1}{4}\bar{d}(\Phi) \leq \frac{7}{2}|\Phi| - 6 \leq \frac{9}{2}|\Phi| - 11.$$

If Φ is a simple face with $|\Phi| = 4$, we have $t(\Phi) = 2$, $\bar{t}(\Phi) = 1$, so that $\frac{5}{2}t(\Phi) + 2\bar{t}(\Phi) = 7$. By Lemmas 2.2.2 and 2.2.6, we obtain $|H(\Phi)| \leq 6$ and $\bar{d}(\Phi) = d(\Phi) \leq 8$. If $|H(\Phi)| \leq 5$, then $|H(\Phi)| + \frac{1}{4}\bar{d}(\Phi) \leq 5 + \frac{1}{4}(8) = 7$. If $|H(\Phi)| = 6$, then by Lemma 2.2.6 $\bar{d}(\Phi) = d(\Phi) \leq 4$ and $|H(\Phi)| + \frac{1}{4}\bar{d}(\Phi) \leq 6 + \frac{1}{4}(4) = 7$.

Finally, assume that Φ is not a simple face, i.e., its boundary is not connected or it contains at least one isolated vertex of G' in its interior. In this case, we have $t(\Phi) \geq |\Phi|$, $\bar{t}(\Phi) \geq |\Phi|$, so that $\frac{5}{2}t(\Phi) + 2\bar{t}(\Phi) \geq \frac{9}{2}|\Phi|$. By Lemma 2.2.6, we now obtain $\bar{d}(\Phi) = d(\Phi) \leq 2\Phi$. Lemma 2.2.2 does not apply here, but we have $|H(\Phi)| \leq 3|\Phi|$, because every half-edge in $H(\Phi)$ ends at an edge of Φ. Hence, we have $|H(\Phi)| + \frac{1}{4}\bar{d}(\Phi) \leq 3|\Phi| + \frac{1}{4}(2|\Phi|) = \frac{7}{2}|\Phi|$. $\square$

Now we can easily complete the proof of Theorem 2.1.1. Since every edge of $G - G'$ gives rise to two half-edges, we have

$$e(G) - e(G') = \frac{1}{2}\sum_{\Phi \in V} |H(\Phi)| = \frac{1}{2}\sum_{\Phi \in V}\left(|H(\Phi)| + \frac{1}{4}\bar{d}(\Phi)\right)$$

$$\leq \frac{5}{4}\sum_{\Phi \in V} t(\Phi) + \sum_{\Phi \in V} \bar{t}(\Phi),$$

where the inequality holds by Lemma 2.2.7. We obviously have $\sum_{\Phi \in V} t(\Phi) = 2\,(v(G) - 2)$, which is equal to the total number of faces in any triangulation of G'. In order to obtain such a triangulation from G', one needs to add $\sum_{\Phi \in V} \bar{t}(\Phi)$ edges. Hence, we have $\sum_{\Phi \in V} \bar{t}(\Phi) = 3(v(G) - 2) - e(G')$. Notice that triangulating each face separately may create a triangulation of the plane containing some parallel edges, but this has no effect on the number of triangles or the number of edges. Now the theorem follows by simple calculation:

$$e(G) = e(G') + (e(G) - e(G'))$$
$$\leq e(G') + \frac{5}{4}2\,(v(G) - 2) + (3\,(v(G) - 2) - e(G')) = 5.5\,(v(G) - 2)\,.$$

This completes the proof of the inequality in Theorem 2.1.1.

We close this section by presenting a construction which shows that the result is not far from being tight.

Proposition 1. *For every $v \equiv 0$ $(mod\ 6)$, $v \geq 12$, there exists a graph G with v vertices and $5.5(v-2) - 4$ edges that can be drawn in the plane so that each of its edges crosses at most three others. That is, for these values we have $e_3(v) \geq 5.5v - 15$.*

Proof: Let T_q denote a hexagonal tiling of a vertical cylindrical surface with $q \geq 1$ horizontal *layers*, each consisting of 3 hexagonal faces wrapped around the cylinder (see Figure 2.8). Notice that the top and the bottom face of the cylinder are also hexagonal. Let V_q be the set of all the vertices of the tiles. To each face except the top and the bottom one, add 8 diagonals (all but one main diagonal). Finally, add all diagonals to the top and the bottom face that do not yield parallel edges. This means adding 6 edges on both the top and the bottom face, as depicted in Figure 2.8. The resulting graph G_q is drawn on the surface of the cylinder with each edge crossing at most 3 other edges. We have $v(G_q) = 6q + 6$ and $e(G_q) = 33q + 18 = 5.5v(G_q) - 15$. □

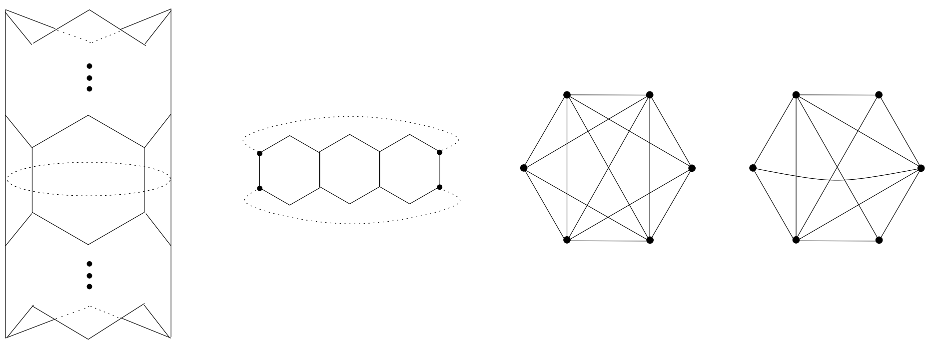

Figure 2.8: The vertical cylindrical surface, a layer, side-face, top-face, and bottom-face.

2.3 Proof of Theorem 2.1.2

For any graph G drawn in the plane, let G^{free} denote the subgraph of G on the same vertex set, consisting of all crossing-free edges. Let $\triangle(G^{\text{free}})$ denote the number of triangular faces of G^{free}, containing no vertex of G in their interiors.

Lemma 2.3.1. *Let G be a graph on $v(G) \geq 3$ vertices, which is drawn in the plane so that none of its edges crosses two others. Then the number of edges of G satisfies*

$$e(G) \leq 4(v(G) - 2) - \frac{1}{2}\triangle(G^{\text{free}}).$$

Proof: We can assume without loss of generality that G^{free} is *maximal* in the following sense: if two vertices, u and v, can be connected by a Jordan arc that does not cross any edge of G, then G^{free} contains an edge uv between these vertices. We can also assume that G is 3-*connected*. Otherwise, we can conclude by induction on $v(G)$, as follows. Let $G = G_1 \cup G_2$ be a decomposition of G into subgraphs on fewer than $v(G)$ vertices, where G_1 and G_2 share at most 2 vertices. Clearly, we have $(v(G_1) - 2) + (v(G_2) - 2) \leq v(G) - 2$, $e(G_1) + e(G_2) \geq e(G)$, and $\triangle(G_1^{\text{free}}) + \triangle(G_2^{\text{free}}) \geq \triangle(G^{\text{free}})$. Therefore, applying the induction hypothesis to G_1 and G_2 separately, we obtain that the statement of the lemma holds for G.

Observe that if two edges uv and zw cross each other, then u and z, say, can be connected by a Jordan arc running very close to the union of the edges uv and zv, without crossing any edge of G. Thus, it follows from the maximality of G^{free} that uz, and similarly zv, vw, and wu, are edges of G^{free}. Moreover, the quadrilateral $uzvw$ containing the crossing pair of edges uv, zw must be a *face* of G^{free}. To see this, it is enough to observe that the 3-connectivity of G implies that this quadrilateral cannot contain any vertex of G in its interior. Thus, all edges in $G - G^{\text{free}}$ are diagonals of quadrilateral faces of G^{free}. Letting $q(G^{\text{free}})$ denote the number of quadrilateral faces of G^{free}, we obtain

$$e(G^{\text{free}}) + 2q(G^{\text{free}}) - e(G) \geq 0.$$

Let $f(G^{\text{free}})$ denote the total number of faces of G^{free}. Then we have

$$f(G^{\text{free}}) - q(G^{\text{free}}) - \triangle(G^{\text{free}}) \geq 0$$

and, by Euler's Formula,

$$v(G) + f(G^{\text{free}}) - e(G^{\text{free}}) - 2 \geq 0.$$

Double counting the pairs (σ, a), where σ is a face of G^{free} and a is an edge of σ, we obtain

$$2e(G^{\text{free}}) - 4f(G^{\text{free}}) + \triangle(G^{\text{free}}) \geq 0.$$

Multiplying the above four inequalities by the coefficients 1, 2, 4 and 3/2, respectively, and adding them up, the lemma follows. □

Instead of Theorem 2.1.2, we establish a slightly stronger claim.

Lemma 2.3.2. *Let G be a graph on $v(G) \geq 3$ vertices, which is drawn in the plane with $x(G)$ crossings. Then we have*

$$x(G) \geq \frac{7}{3}e(G) - \frac{25}{3}(v(G) - 2) + \frac{2}{3}\triangle(G^{\text{free}}).$$

Proof: We use induction on $x(G) + v(G)$. As in the proof of Lemma 2.3.1, we can assume that G is 3-*connected* and that G^{free} is *maximal* in the sense that whenever the points u and v can be connected by a Jordan arc without crossing any edge of G, the edge uv belongs to G^{free}. We distinguish four cases.

Case 1. G contains an edge that crosses at least 3 other edges.

Let a be such an edge, and let G_0 be the subgraph of G obtained by removing a. Now we have, $e(G_0) = e(G) - 1$, $x(G_0) \leq x(G) - 3$, and $\triangle(G_0^{\text{free}}) \geq \triangle(G^{\text{free}})$. Applying the induction hypothesis to G_0, we get

$$x(G) - 3 \geq \frac{7}{3}\left(e(G) - 1\right) - \frac{25}{3}\left(v(G) - 2\right) + \frac{2}{3}\triangle(G^{\text{free}}),$$

which implies the statement of the lemma.

Case 2. Every edge in G crosses at most one other edge.

Lemma 2.3.1 yields

$$4\left(v(G) - 2\right) - \frac{1}{2}\triangle(G^{\text{free}}) \geq e(G).$$

The statement is obtained by multiplying this inequality by 4/3 and adding to it the simple inequality $x(G) \geq e(G) - 3\left(v(G) - 2\right)$ mentioned in the Introduction.

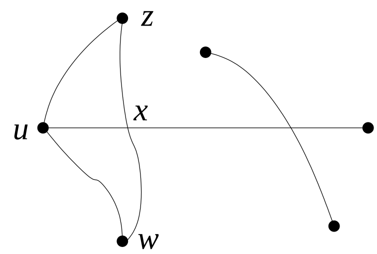

Figure 2.9: Proof of Lemma 2.3.2, Case 3.

Case 3. There exists an edge e of G that crosses two other edges, one of which does not cross any other edge of G.

Let zw be an edge that crosses e at point x and does not participate in any other crossing. Let u denote the endpoint of e for which the piece of e between x and u is crossing-free. Notice that u can be connected to both z and w by noncrossing Jordan arcs, without crossing any edge of G. Therefore, by the maximality of G^{free}, the edges uz and uw must belong to G^{free}. Let G_0 be the subgraph of G obtained by removing the edge e. We have $e(G_0) = e(G) - 1$ and $x(G_0) = x(G) - 2$. Clearly, G_0^{free} contains zw and all edges in G^{free}. By the 3-connectivity of G, the triangle uzw must be a triangular face of G_0^{free}, so that we have $\triangle(G_0^{\text{free}}) \geq \triangle(G^{\text{free}}) + 1$. Applying the induction hypothesis to G_0, we obtain

$$x(G) \geq \frac{7}{3}e(G) - \frac{25}{3}\left(v(G) - 2\right) + \frac{2}{3}\triangle(G^{\text{free}}) + \frac{1}{3},$$

which is better than what we need.

Case 4. There exists an edge a of G that crosses precisely two other edges, b and c, and each of these edges also participates in precisely two crossings.

Subcase 4.1. b and c do not cross each other.

Let G_0 be the subgraph of G obtained by removing b. Clearly, we have $e(G_0) = e(G) - 1$, $x(G_0) = x(G) - 2$, and $\triangle(G_0^{\text{free}}) \geq \triangle(G^{\text{free}})$. Notice that c is an edge of G_0 that crosses two other edges; one of them is a, which is crossed by no other edge of G_0. Thus, we can apply to G_0 the last inequality in the analysis of Case 3 to conclude that

$$x(G) - 2 \geq \frac{7}{3}\left(e(G) - 1\right) - \frac{25}{3}\left(v(G) - 2\right) + \frac{2}{3}\triangle(G^{\text{free}}) + \frac{1}{3},$$

which is precisely what we need.

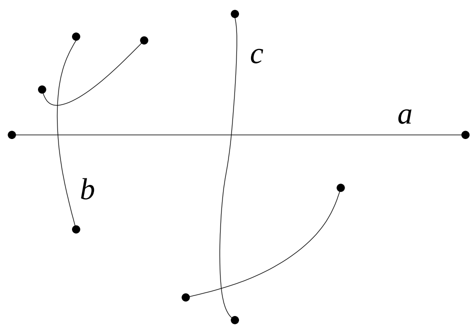

Figure 2.10: Proof of Lemma 2.3.2, Subcase 4.1.

Subcase 4.2. b and c cross each other.

The three crossing edges, a, b, and c can be drawn on the sphere in two topologically different ways. If the closed curve formed by segments of the three edges separates two of the endpoints of the three edges from the other four, then the graph is not 3-connected as the vertices on the two sides of this closed curve are only connected by two edges (see the configuration on the left-hand side of Figure 2.11). So it is enough to consider the configuration depicted on the right-hand side of Figure 2.11. By the maximality condition, G^{free} must contain the six dashed edges in the figure. Note that a, b, and c are not crossed by any additional edges, so all other edges of G contained in the hexagon Φ, formed by the dashed edges, must be contained in one of the triangular or quadrilateral faces of the arrangement, and the existence of such edges contradicts the 3-connectedness of G. Thus, Φ is a face of G^{free}, and the only edges of G inside this face are a, b, and c. Let G_0 be the graph obtained from G by removing the edges a, b, c, and inserting a new vertex in the interior of Φ, which is connected to every vertex of Φ by crossing-free edges. We have $v(G_0) = v(G) + 1$ and $x(G_0) = x(G) - 3$, so that we can apply the induction hypothesis to G_0. Obviously, we have $e(G_0) = e(G) + 3$ and $\triangle(G_0^{\text{free}}) = \triangle(G^{\text{free}}) + 6$. Thus, we obtain

$$x(G) - 3 \geq \frac{7}{3}\,(e(G) + 3) - \frac{25}{3}\,(v(G) - 1) + \frac{2}{3}\left(\triangle(G^{\text{free}}) + 6\right),$$

which is much stronger than the inequality in the lemma. □

The tightness of Theorem 2.1.2 is discussed at the end of the last section.

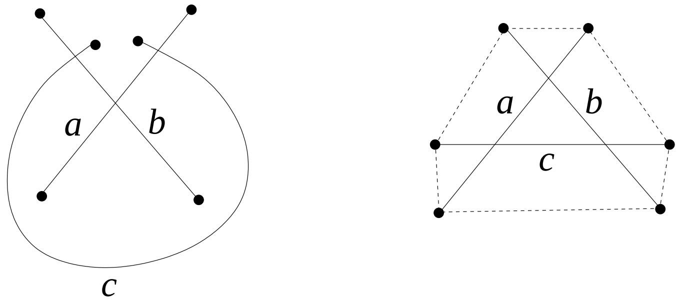

Figure 2.11: Proof of Lemma 2.3.2, Subcase 4.2.

2.4 Proof of Theorem 2.1.3

Our proof is based on the following consequence of Theorems 2.1.1 and 2.1.2.

Corollary 2.4.1. *The crossing number of any graph G of at least 3 vertices satisfies*

$$\mathrm{CR}(G) \geq 4e(G) - \frac{103}{6}(v(G) - 2).$$

Proof: If G has at most $5(v(G)-2)$ edges, then the statement directly follows from Theorem 2. If G has more than $5(v(G)-2)$ edges, fix one of its drawings in which the number of crossings is minimum. Delete the edges of G one by one until we obtain a graph G_0 with $5(v(G)-2)$ edges. At each stage, delete one of the edges that participates in the largest number of crossings in the current drawing. Using the inequality $e_2(v) \leq 5(v-2)$ proved in [PT97] and quoted in Section 2.1, at the time of its removal every edge has at least *three* crossings. Moreover, by Theorem 1, with the possible exception of the at most $\frac{1}{2}(v(G)-2)$ edges deleted last, every edge has at least *four* crossings. Thus, the total number of deleted crossings is at least

$$4(e(G) - 5(v(G) - 2)) - \frac{1}{2}(v(G) - 2) = 4e(G) - \frac{41}{2}(v(G) - 2).$$

On the other hand, applying Theorem 2.1.2 to G_0, we obtain that the number of crossings *not* removed during the algorithm is at least

$$\mathrm{CR}(G_0) \geq \frac{10}{3}(v(G) - 2).$$

Summing up these two estimates, the result follows. □

Now we can easily complete the proof of Theorem 2.1.3. Let G be a graph drawn in the plane with $\text{CR}(G)$ crossings, and suppose that $e(G) \geq \frac{103}{16}v(G)$.

Construct a *random* subgraph $G' \subseteq G$ by selecting each vertex of G independently with probability

$$p = \frac{103}{16}\frac{v(G)}{e(G)} \leq 1,$$

and letting G' be the subgraph of G induced by the selected vertices. The expected number of vertices of G' is $E[v(G')] = pv(G)$. Similarly, $E[e(G')] = p^2 e(G)$. The expected number of crossings in the drawing of G' inherited from G is $p^4\text{CR}(G)$, and the expected value of the crossing number of G' is even smaller.

By Corollary 2.4.1, $\text{CR}(G') \geq 4e(G') - \frac{103}{6}v(G')$ holds for every G'. (Note that after getting rid of the constant term in Corollary 2.4.1, we do not have to assume any more that $v(G') \geq 3$; the above inequality is true for every G'.) Taking expectations, we obtain

$$p^4\text{CR}(G) \geq E[\text{CR}(G')] \geq 4E[e(G')] - \frac{103}{6}E[v(G')] = 4p^2e(G) - \frac{103}{6}pv(G).$$

This implies that

$$\text{CR}(G) \geq \frac{1024}{31827}\frac{e^3(G)}{v^2(G)} \geq \frac{1}{31.1}\frac{e^3(G)}{v^2(G)},$$

provided that $e(G) \geq \frac{103}{16}v(G)$.

To obtain an unconditional lower bound on the crossing number of any graph G, we need different estimates when $e(G) < \frac{103}{16}v(G)$. Comparing the bounds in Theorem 2.1.2 and in Corollary 2.4.1 with the trivial estimates $\text{CR}(G) \geq 0$ and $\text{CR}(G) \geq e - 3(v(G) - 2)$, a case analysis shows that

$$\frac{1024}{31827}\frac{e^3(G)}{v^2(G)} - \text{CR}(G) \leq 1.06v(G).$$

The maximum is attained for a graph G with $e(G) = 4(v(G) - 2)$ and $\text{CR}(G) = v(G) - 2$. In conclusion,

$$\text{CR}(G) \geq \frac{1024}{31827}\frac{e^3(G)}{v^2(G)} - 1.06v(G) \geq \frac{1}{31.1}e^3(G)v^2(G) - 1.06v(G)$$

holds for every graph G. This completes the proof of Theorem 2.1.3.

Remark. In chapter 4 we introduce two variants of the crossing number. The *pair-crossing number* (resp. the *odd crossing number*) of G is defined as the minimum number of pairs of non-adjacent edges that cross (resp. cross an odd number of times) over all drawings of G. These parameters are at most as large as $\text{CR}(G)$.

The original proofs of the Crossing Lemma readily generalize to the new crossing numbers, see Theorem 4.1.5, and it follows that both of them are at least $\frac{1}{64}\frac{e^3(G)}{v^2(G)}$, provided that $e(G) \geq 4v(G)$. We have been unable to extend our proof of Theorem 2.1.3 to these parameters.

2.5 Applications, problems, remarks

Every improvement of the Crossing Lemma automatically leads to improved bounds in all of its applications. For completeness and future reference, we include some immediate corollaries of Theorem 2.1.3 with a sketch of computations.

First, we plug Theorem 2.1.3 into Székely's method [S98] to improve the coefficient of the main term in the Szemerédi-Trotter theorem [ST83], [CE90], [PT97].

Corollary 2.5.1. *Given m points and n lines in the Euclidean plane, the number of incidences between them is at most $2.5m^{2/3}n^{2/3} + m + n$.*

Proof: We can assume that every line and every point is involved in at least one incidence, and that $n \geq m$, by duality. Since the statement is true for $m = 1$, we have to check it only for $m \geq 2$.

Define a graph G drawn in the plane such that the vertex set of G is the given set of m points, and join two points with an edge drawn as a straight-line segment if the two points are consecutive along one of the lines. Let I denote the total number of incidences between the given m points and n lines. Then $v(G) = m$ and $e(G) = I - n$. Since every edge belongs to one of the n lines, $\text{CR}(G) \leq \binom{n}{2}$. Applying Theorem 2.1.2 to G, we obtain that $\frac{1}{31.1}\frac{(I-n)^3}{m^2} - 1.06m \leq \text{CR}(G) \leq \binom{n}{2}$. Using that $n \geq m \geq 2$, easy calculation shows that

$$I - n \leq \sqrt[3]{15.55m^2n^2 + 33m^3} \leq \sqrt[3]{15.55}n^{2/3}m^{2/3} + m,$$

which implies the statement. □

It was shown in [PT97] that Corollary 2.5.1 does not remain true if we replace the constant 2.5 by 0.42 .

Theorem 2.1.3 readily generalizes to multigraphs with bounded edge multiplicity, improving the constant in Székely's result [S98].

Corollary 2.5.2. *Let G be a multigraph with maximum edge multiplicity m. Then*

$$\mathrm{CR}(G) \geq \frac{1}{31.1} \frac{e^3(G)}{mv^2(G)} - 1.06m^2 v(G).$$

Proof: Define a random simple subgraph G' of G as follows. For each pair of vertices v_1, v_2 of G, let $e_1, e_2, \ldots e_k$ be the edges connecting them. With probability $1 - k/m$, G' will not contain any edge between v_1 and v_2. With probability k/m, G' contains precisely one such edge, and the probability that this edge is e_i is $1/m$ $(1 \leq i \leq k)$. Applying Theorem 3 to G' and taking expectations, the result follows. □

Next, we state here the improvement of another result in [PT97].

Corollary 2.5.3. *Let G be a graph drawn in the plane so that every edge is crossed by at most k others, for some $k \geq 1$, and every pair of edges have at most one point in common. Then*

$$e(G) \leq 3.95\sqrt{k}v(G).$$

Proof: For $k \leq 2$, the result is weaker than the bounds given in [PT97]. Assume that $k \geq 3$, and consider a drawing of G such that every edge crosses at most k others. Let x denote the number of crossings in this drawing. If $e(G) < \frac{103}{6}v(G)$, then there is nothing to prove. If $e(G) \geq \frac{103}{6}v(G)$, then using Theorem 2.1.3, we obtain

$$\frac{1024}{31827} \frac{e^3(G)}{v^2(G)} \leq \mathrm{CR}(G) \leq x \leq \frac{e(G)k}{2},$$

and the result follows. □

Recall that $e_k(v)$ was defined as the maximum number of edges that a graph of v vertices can have if it can be drawn in the plane with at most k crossings per edge. We define some other closely related functions. Let $e^*_k(v)$

denote the maximum number of edges of a graph of v vertices which has a drawing that satisfies the above requirement and, in addition, every pair of its edges meet at most once (either at an endpoint or at a proper crossing). We define $\overline{e}_k(v)$ and $\overline{e}^*_k(v)$ analogously, with the only difference that now the maximums are taken over all *triangle-free* graphs with v vertices.

It was mentioned in the Introduction (see Lemma 2.1.4) that $e_k(v) = e^*_k(v)$ for $0 \leq k \leq 3$, and that $e^*_k(v) \leq (k+3)(v-2)$ for $0 \leq k \leq 4$ [PT97]. For $0 \leq k \leq 2$, the last inequality is tight for infinitely many values of v. Our Theorem 2.1.1 shows that this is not the case for $k = 3$.

Conjecture 1. *We have $e_k(v) = e^*_k(v)$ for every k and v.*

Using the proof technique of Theorem 2.1.1, it is not hard to improve the bound $e^*_4(v) \leq 7(v-2)$. In particular, in this case Lemma 2.2.2 holds with $3(|\Phi|-2)$ replaced by $4(|\Phi|-2)$. Moreover, an easy case analysis shows that every triangular face Φ with four half-edges satisfies at least one of the following two conditions:

1. The extension of at least one of the half-edges in Φ either ends in a triangular face with fewer than four half-edges, or enters a big face.

2. Φ is adjacent to an empty triangle.

Based on this observation, one can modify the arguments in Section 2.2 to obtain the upper bound $e^*_4(v) \leq (7 - \frac{1}{9})v - O(1)$.

Conjecture 2. $e^*_4(v) \leq 6v - O(1)$.

As for the other two functions, we have $\overline{e}_k(v) = \overline{e}^*_k(v)$ for $0 \leq k \leq 3$, and $\overline{e}^*_k(v) \leq (k+2)(v-2)$ for $0 \leq k \leq 2$. If $0 \leq k \leq 1$, these bounds are attained for infinitely many values of v. These estimates were applied by Czabarka et al. [CSSV06] to obtain some lower bounds on the so-called *biplanar crossing number* of complete graphs.

Given a triangle-free graph drawn in the plane so that every edge crosses at most 2 others, an easy case analysis shows that each quadrilateral face that contains four half-edges is adjacent to a face which is either non-quadrilateral or does not have four half-edges[1]. As in the proof of Theorem 2.1.1 (before

[1]This statement actually holds under the assumption that G and G' are maximal, in the sense described at the beginning of Section 2.2.

Lemma 2.2.5), we can use a properly defined bipartite multigraph M to establish the bound

$$\overline{e}_2(v) \leq \left(4 - \frac{1}{10}\right) v - O(1).$$

Conjecture 3. $\overline{e}_2(v) \leq 3.5v - O(1)$.

The coefficient 3.5 in the above conjecture cannot be improved as shown by the triangle-free (actually bipartite!) graph in Figure 2.12, whose vertex set is the set of vertices of a $4 \times v/4$ grid.

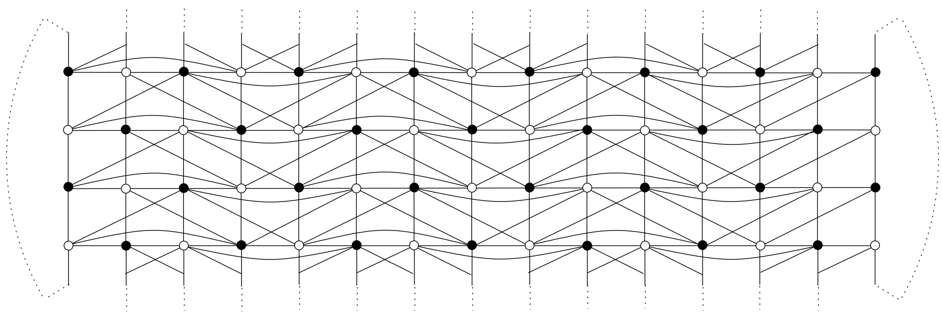

Figure 2.12: $\overline{e}_2(v) \geq 3.5v - 16$.

Let $\mathrm{cr}(v, e)$ denote the minimal crossing number of a graph with $v \geq 3$ vertices and e edges. Clearly, we have $\textsc{cr}(v, e) = 0$, whenever $e \leq 3(v-2)$, and $\textsc{cr}(v, e) = e - 3(v-2)$ for $3(v-2) \leq e \leq 4(v-2)$. To see that these values are indeed attained by the function, consider the graph constructed in [PT97], which (if v is a multiple of 4) can be obtained from a planar graph with v vertices, $2(v-2)$ edges, and $v-2$ quadrilateral faces, by adding the diagonals of the faces. If $e < 4(v-2)$, delete as many edges participating in a crossing, as necessary.

In the next interval, i.e., when $4(v-2) \leq e \leq 5(v-2)$, Theorem 2.1.2 gives tight bound on $\textsc{cr}(v, e)$ up to an additive constant. To see this, consider a planar graph with only pentagonal and quadrilateral faces and add all diagonals in every face. If no two faces of the original planar graph shared more than a vertex or an edge, for the resulting graph the inequality of Theorem 2.1.2 holds with equality. For certain values of v and e, no such construction exists, but we only lose a constant.

If $5(v-2) \leq e \leq 5.5(v-2)$, the best known bound, $\text{CR}(v,e) \geq 3e - \frac{35}{3}(v-2)$, follows from Theorem 2.1.2, while for $e \geq 5.5(v-2)$ the best known bound is either the one in Corollary 2.4.1 or the one in Theorem 2.1.3. We do not believe that any of these bounds are optimal.

Conjecture 4. $\text{CR}(v,e) \geq \frac{25}{6}e - \frac{35}{2}(v-2)$.

Note that, if true, this bound is tight up to an additive constant for $5(v-2) \leq e \leq 6(v-2)$. To see this, consider a planar graph with only pentagonal and hexagonal faces and add all diagonals of all faces. If no two faces of the planar graph shared more than a vertex or an edge, the resulting graph shows that Conjecture 4 cannot be improved. As a first step toward settling this conjecture, we can show the following statement, similar to Lemma 2.3.1.

Lemma 2.5.4. *Let G be a graph on $v(G) \geq 3$ vertices drawn in the plane so that every edge is involved in at most two crossings. Then*

$$e(G) \leq 5(v(G)-2) - \triangle(G^{\text{free}}).$$

2.6 Appendix 1: Case 1 in the proof of Lemma 2.2.3

Our proof will be a straightforward case analysis. Recall that Φ is a $3Y$-triangle with the unique neighbor Ψ, which is also a triangle. Let A, B, and C be the vertices of Φ, and let f, g, and h denote the half-edges in $H(\Phi)$. Here, g and h emanate from vertex A, while f starts at vertex B. Next, we introduce a new notation: given a vertex V in a face Υ of G', let $d_\Upsilon(V)$ denote the number of half edges in $H(\Upsilon)$ that emanate from V.

Let Ξ denote the face of G' that is adjacent to Φ along side AC. First, we claim that the extension $\overline{f} \in H$ of f is not perfect. Indeed, otherwise Ξ is a triangle and edge $\overline{f}$, having crossed half-edges g and h, as well as side AC, must end in D, the vertex of Ξ opposite to AC. Since $\overline{f} = BD$ cannot be crossed by any other edge, we have $d_\Xi(A) = d_\Xi(C) = 0$. Aside from $\overline{f}$, the extension of any half-edge in $H(\Xi)$, that emanates from vertex D, has to exit Ξ through side AC and enter Φ. It cannot exit Φ through side AB (it would cross four edges AC, h, g, and AB, in this order), nor it can end at vertex B

(there are no parallel edges). Hence, it must exit Φ through side BC, which is already crossed by the extensions of g and h (see Figure 2.13i). Therefore, $d_\Xi(D) \leq 2$, and Ξ is not a 3-triangle, contradicting the assumption that $\overline{f}$ is perfect.

Next, suppose that the extensions $\overline{g}$ and $\overline{h}$ of g and h, respectively, are perfect. We distinguish two cases, based on where these two edges end.

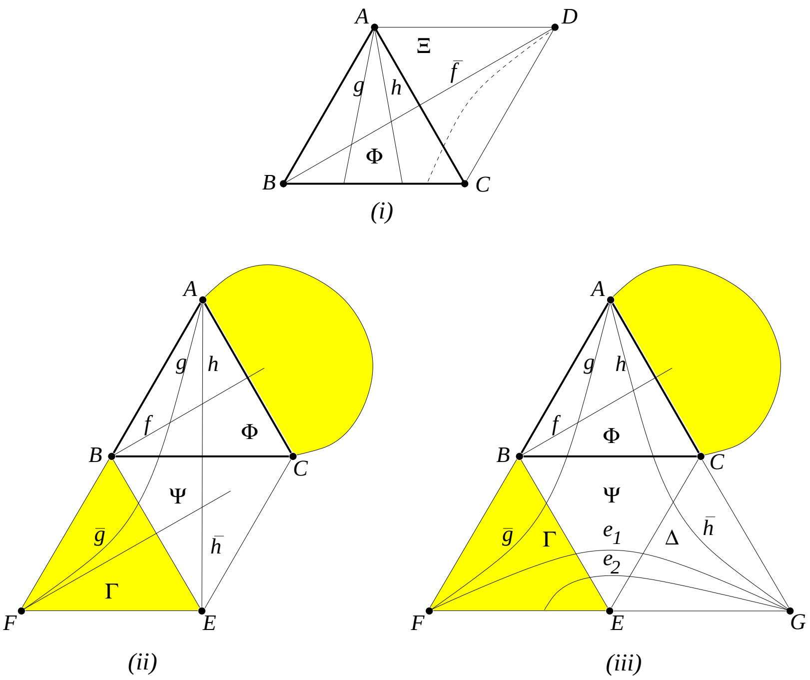

Figure 2.13: Proof of Lemma 2.2.3, Case 1. (A more detailed version of Figure 2.4 bcd.)

Subcase 1.1. $\overline{h}$ ends in E, the vertex of Ψ opposite to BC (see Figure 2.13ii). Since there are no parallel edges, $\overline{g}$ has to exit Φ across the side BE. Having already crossed three other edges, $\overline{g}$ must end in Γ, the face of G' adjacent to Ψ along BE. By the assumption that $\overline{g}$ is perfect, we conclude that Γ is a triangle, and we let F be the vertex of Γ, where $\overline{g}$ ends. Since $\overline{g}$ cannot be crossed by any other edge, then $d_\Gamma(B) = d_\Gamma(E) = 0$.

Aside from $\overline{g}$, the extension of any half-edge in $H(\Gamma)$, that emanates from vertex F, exits Γ through side BE and enters Ψ. It cannot exit Ψ through side BC, since $\overline{f}$ already crosses three other edges. Thus, it has to cross $\overline{h}$, that is already crossed by f and BC (see Figure 2.13ii). Therefore, $d_\Gamma(F) \leq 2$, and Γ is not a 3-triangle, contradicting the assumption that $\overline{g}$ is perfect.

The symmetric case, when $\overline{g}$ ends in E (and $\overline{h}$ exits Ψ through side CE), can be handled similarly.

Subcase 1.2. $\overline{g}$ exits Ψ through BE and $\overline{h}$ exits Ψ through CE (see Figure 2.13iii). Both of these edges already cross three other edges, so $\overline{g}$ ends in Γ, the face of G' adjacent to Ψ along BE, and $\overline{h}$ ends in Δ, the face of G' adjacent to Ψ along CE. Both Γ and Δ are triangles by assumption. Let F and G denote the vertices of Γ and Δ, where g and h end respectively. As before, we easily conclude that $d_\Gamma(B) = d_\Gamma(E) = 0$ and $d_\Delta(C) = d_\Delta(E) = 0$. Since $\overline{h}$ is perfect, then Δ is a 3-triangle and $d_\Delta(G) = 3$. Let e_1 and e_2 denote the extensions of the half-edges contributing to $d_\Delta(G)$ (other than $\overline{h}$). These edges exit Δ through CE and enter Ψ. Neither of them can exit Ψ through BC or end in B, since $\overline{f}$, $\overline{g}$, and $\overline{h}$ already cross three other edges. Hence, e_1 and e_2 exit Ψ through BE and enter Γ. Now, there are only two possibilities: either e_1 and e_2 both exit Γ through FE; or one of them ends in vertex F, while the other exits Γ through FE (see Figure 2.13iii). In both cases, BE is crossed by three edges (e_1, e_2, $\overline{g}$), and $d_\Gamma(F) \leq 2$. Therefore, Γ is not a 3-triangle, contradicting the assumption that $\overline{g}$ is perfect. $\square$

2.7 Appendix 2: Proof of Claim A

Recall that Ψ is a simple face of G' with $|\Psi| = 4$ and $|H(\Psi)| = 6$. Next, we introduce some notation. Let A, B, C, and D denote the vertices of Ψ, and let d_V be the *degree* of $V \in \{A, B, C, D\}$ in Ψ, that is, the number of half edges in $H(\Psi)$ incident to vertex V. Encode each half-edge by its *type*, consisting of the initial vertex and the side of Ψ where it ends. So, for example, a half-edge of type $A(BC)$ connects vertex A with the side BC. Finally, let Δ denote the maximum degree of all the vertices of Ψ.

Case 1. $\Delta = 6$.

Suppose that $d_A = \Delta$. Since at most three half-edges can exit Ψ through the same side, there is only one possibility, depicted in Figure 2.5a.

Case 2. $\Delta = 5$.

Let A be the vertex of degree 5. Three of the half-edges incident to A exit through the same side, say BC, and two through the side CD. The remaining half-edge of $H(\Psi)$ cannot have its endpoint on AB or on BC, and it cannot emanate from B. Therefore, it has to be of type $C(AD)$ (see Figure 2.5b).

Case 3. $\Delta = 4$.

Let $d_A = \Delta$. There are two possibilities:

Case 3.1. Two of the half-edges incident to A exit Ψ through side BC, while the other two exit through side CD. If there is a half-edge incident to B, it should exit through CD. However, then the remaining half-edge cannot be drawn: clearly, it cannot start at C or D, and if it starts at B, then the two half-edges incident to B have to be of type $B(CD)$, forcing at least four crossings on CD. Similarly, no half-edge can be incident to D. Therefore, the remaining two half-edges both emanate from C. By Lemma 2.2.1, they should exit Ψ through different sides, giving Figure 2.5c.

Case 3.2. There are three half-edges in $H(\Psi)$ of type $A(CD)$ and one of type $A(BC)$. Then the remaining two half-edges cannot have their endpoints on AD, CD, or in D. So, they are both of type $C(AB)$ (see Figure 2.5d).

Case 4. $\Delta = 3$.

Let A be a vertex of degree 3. Again, there are two possibilities (up to symmetry).

Case 4.1. All three half-edges incident to A are of the same type, say $A(BC)$. The remaining three half-edges of $H(\Psi)$ cannot have their endpoints on AB, on BC, or in B. Therefore, all of them are of type $C(AD)$, as shown in Figure 2.5e.

Case 4.2. Two half-edges incident to A are of type $A(BC)$, while the remaining one is of type $A(CD)$.

If there is a half-edge incident to B, it can only be of type $B(CD)$, Then, by Lemma 2.2.1, there are no more half-edge emanating from B. Moreover, no half-edge is incident to C; otherwise, any half-edge from C would cross the existing half-edge of type $B(CD)$, whose extension already crosses three other edges. Similarly, at most one half-edge emanates from D (extensions of the half-edges of type $A(BC)$ already cross two other edges). This contradicts $|H(\Psi)| = 6$.

If there is a half-edge incident to D, it can only be of type $D(BC)$, and it has to be the unique half-edge of this type. The remaining two half-edges of $H(\Psi)$ must be incident to C. None of them can exit Ψ through AB, so they are both of type $C(AD)$. However, then the extension of the existing half-edge of type A(CD) crosses four other edges.

Therefore, we can assume that there are two half-edges of type $A(BC)$, one of type $A(CD)$, and the other three half-edges are incident to C. It is impossible that all three are of type $C(AD)$, since they would all cross the half-edge of type $A(CD)$. Moreover, by Lemma 2.2.1, at most one can be of type $C(AB)$. Therefore, one is of type $C(AB)$ and two are of type $C(AD)$, see Figure 2.5f.

Case 5. $\Delta = 2$.

First, suppose that for every vertex of degree two the two half-edges incident to it exit Ψ through different sides. Also, assume that $d_A = 2$, i.e., there is a half-edge of type $A(BC)$ and a half-edge of type $A(CD)$. If B is of degree two, then there is a half-edge of type $B(CD)$ and a half-edge of type $B(AD)$. Now, it is easy to see that at most one further half-edge can be added, either of type $C(AD)$ or of type $D(BC)$, contradicting $|H(\Psi)| = 6$. If C is of degree two, for each of the four types: $A(BC)$, $A(CD)$, $C(AB)$, $C(AD)$, there is a unique half-edge of this type, whose extension is already crossed by two edges. Any additional half-edge emanating from either B or D would have to cross three of the above mentioned half-edges before reaching a side of Ψ. Hence, if $d_C = 2$, then $d_B = d_D = 0$, contradicting $|H(\Psi)| = 6$.

Now, we can assume that there is a vertex (say, A) of degree two, such that both half-edges incident to it have the same type, say $A(CD)$. It follows from Lemma 2.2.1 that $d_D \leq 1$. If $d_D = 0$, then $|H(\Psi)| = 6$ implies $d_B = d_C = 2$. Let us consider the two half-edges emanating from B. At most one of them is of type $B(CD)$. Furthermore, by Lemma 2.2.1, at most one of them is of type $B(AD)$. So, we have exactly one half-edge of type $B(CD)$ and one half-edge of type $B(AD)$. Any half-edge incident to C would have to either cross three half-edges before reaching AD, or cross the existing half-edge of type $B(AD)$, whose extension already crosses three other edges. Therefore, we obtain $d_C = 0$, a contradiction.

We are left with the case when there are two half-edges of type $A(CD)$, and $d_D = 1$. If the half-edge incident to D is of type $D(BC)$, then $d_C = 0$, which, together with $d_B \leq 2$, gives $|H(\Psi)| \leq 5$, a contradiction. Therefore, the half-edge incident to D has type $D(AB)$. In this case, the half-edges incident to B or C cannot end on AD, so the possible types are $B(CD)$ and $C(AB)$. Since CD is already crossed by two edges, there is at most one half-edge of type $B(CD)$. So, there are two half-edges of type $C(AB)$, see Figure 2.5g. This concludes the proof of Claim A. □

Chapter 3

New bounds for crossing numbers

This chapter is based on the manuscript [PST00]. The *crossing number*, $\text{CR}(G)$, of a graph G is the least number of crossing points in any drawing of G in the plane. Denote by $\kappa(n, e)$ the minimum of $\text{CR}(G)$ taken over all graphs with n vertices and at least e edges. We prove a conjecture of P. Erdős and R. Guy by showing that $\kappa(n, e)n^2/e^3$ tends to a positive constant as $n \to \infty$ and $n \ll e \ll n^2$. Similar results hold for graph drawings on any other surface of fixed genus.

We prove better bounds for graphs satisfying some monotone properties. In particular, we show that if G is a graph with n vertices and $e \geq 4n$ edges, which does not contain a cycle of length *four* (resp. *six*), then its crossing number is at least ce^4/n^3 (resp. ce^5/n^4), where $c > 0$ is a suitable constant. These results cannot be improved, apart from the value of the constant. This settles a question of M. Simonovits.

3.1 Introduction

Let G be a simple undirected graph with $n(G)$ nodes (vertices) and $e(G)$ edges. A *drawing* of G in the *plane* is a mapping f that assigns to each vertex of G a distinct point in the plane and to each edge uv a continuous arc connecting $f(u)$ and $f(v)$, not passing through the image of any other vertex. For simplicity, the arc assigned to uv is also called an *edge*, and if this leads to no confusion, it is also denoted by uv. We assume that no three edges have an interior point in common. The *crossing number*, $\textsc{cr}(G)$, of G is the minimum number of crossing points in any drawing of G.

The determination of $\textsc{cr}(G)$ is an NP-complete problem [GJ83]. It was discovered by Leighton [L84] that the crossing number can be used to estimate the chip area required for the VLSI circuit layout of a graph. He proved the following general lower bound for $\textsc{cr}(G)$, which was discovered independently by Ajtai, Chvátal, Newborn, and Szemerédi. The best known constant, $1/33.75$, in the theorem is due to Pach and Tóth.

Theorem A. [ACNS82], [L84], [PT97] *Let G be a graph with $n(G) = n$ nodes and $e(G) = e$ edges, $e \geq 7.5n$. Then we have*

$$\textsc{cr}(G) \geq \frac{1}{33.75} \frac{e^3}{n^2}.$$

Theorem A can be used to deduce the best known upper bounds for the number of unit distances determined by n points in the plane [S98], for the number of different ways how a line can split a set of n points into two equal parts [D98], and it has some other interesting corollaries [PS98].

It is easy to see that the bound in Theorem A is tight, apart from the value of the constant. However, as it was suggested by Miklós Simonovits [S97], it may be possible to strengthen the theorem for some special classes of graphs, e.g., for graphs not containing some fixed, so-called *forbidden* subgraph. In Sections 3.2 and 3.3 of the present paper we verify this conjecture.

A graph property $\mathcal{P}$ is said to be *monotone* if

- whenever a graph G satisfies $\mathcal{P}$, then every subgraph of G also satisfies $\mathcal{P}$;
- whenever G_1 and G_2 satisfy $\mathcal{P}$, then their disjoint union also satisfies $\mathcal{P}$.

For any monotone property $\mathcal{P}$, let $\mathrm{ex}(n, \mathcal{P})$ denote the maximum number of edges that a graph of n vertices can have if it satisfies $\mathcal{P}$. In the special case when $\mathcal{P}$ is the property that the graph does not contain a subgraph isomorphic to a fixed forbidden subgraph H, we write $\mathrm{ex}(n, H)$ for $\mathrm{ex}(n, \mathcal{P})$.

Theorem 3.1.1. *Let $\mathcal{P}$ be a monotone graph property with* $\mathrm{ex}(n, \mathcal{P}) = O(n^{1+\alpha})$ *for some $\alpha > 0$.*

Then there exist two constants $c, c' > 0$ such that the crossing number of any graph G with property $\mathcal{P}$, which has n vertices and $e \geq cn \log^2 n$ edges, satisfies

$$\text{CR}(G) \geq c' \frac{e^{2+1/\alpha}}{n^{1+1/\alpha}}.$$

If $\text{ex}(n, \mathcal{P}) = \Theta(n^{1+\alpha})$, then this bound is asymptotically tight, up to a constant factor.

In some interesting special cases when we know the precise order of magnitude of the function $\text{ex}(n, \mathcal{P})$, we obtain some slightly stronger results. The *girth* of a graph is the length of its shortest cycle.

Theorem 3.1.2. *Let G be a graph with n vertices and $e \geq 4n$ edges, whose girth is larger than $2r$, for some $r > 0$ integer. Then the crossing number of G satisfies*

$$\text{CR}(G) \geq c_r \frac{e^{r+2}}{n^{r+1}},$$

where $c_r > 0$ is a suitable constant. For $r = 2, 3,$ and 5, these bounds are asymptotically tight, up to a constant factor.

What happens if the girth of G is larger than $2r + 1$? Since one can destroy every odd cycle of a graph by deleting at most half of its edges, even in this case we cannot expect an asymptotically better lower bound for the crossing number of G than the bound given in Theorem 3.1.2.

Theorem 3.1.3. *Let G be a graph with n vertices and $e \geq 4n$ edges, which does not contain a complete bipartite subgraph $K_{r,s}$ with r and s vertices in its classes, $s \geq r$.*

Then the crossing number of G satisfies

$$\text{CR}(G) \geq c_{r,s} \frac{e^{3+1/(r-1)}}{n^{2+1/(r-1)}},$$

where $c_{r,s} > 0$ is a suitable constant. These bounds are tight up to a constant factor if $r = 2, 3,$ or if r is arbitrary and $s > (r-1)!$.

The *bisection width*, $b(G)$, of a graph G is defined as the minimum number of edges whose removal splits the graph into two roughly equal subgraphs. More precisely, $b(G)$ is the minimum number of edges running between V_1

and V_2, over all partitions of the vertex set of G into two parts $V_1 \cup V_2$ such that $|V_1|, |V_2| \geq n(G)/3$.

Leighton [L83] observed that there is an intimate relationship between the bisection width and the crossing number of a graph, which is based on the Lipton–Tarjan separator theorem for planar graphs [LT79]. The proofs of Theorems 3.1.1-3.1.3 are based on repeated application of the following version of this relationship.

Theorem B. [PSS96] *Let G be a graph of n vertices, whose degrees are $d_1, d_2, \ldots, d_n$. Then*

$$b(G) \leq 10\sqrt{\text{CR}(G)} + 2\sqrt{\sum_{i=1}^{n} d_i^2}.$$

Let $\kappa(n, e)$ denote the minimum crossing number of a graph G with n vertices and at least e edges. That is,

$$\kappa(n, e) = \min_{\substack{n(G) = n \\ e(G) \geq e}} \text{CR}(G).$$

It follows from Theorem A that, for $e \geq 4n$, $\kappa(n, e)n^2/e^3$ is bounded from below and from above by two positive constants. Paul Erdős and Richard K. Guy [EG73] conjectured that if $e \gg n$ then $\lim \kappa(n, e)n^2/e^3$ exists. (We use the notation $f(n) \gg g(n)$ to express that $\lim_{n\to\infty} f(n)/g(n) = \infty$.) In Section 3.4, we settle this problem.

Theorem 3.1.4. *If $n \ll e \ll n^2$, then*

$$\lim_{n\to\infty} \kappa(n, e)\frac{n^2}{e^3} = C > 0$$

exists.

We call the constant $C > 0$ in Theorem 3.1.4 the *midrange crossing constant*. It is necessary to limit the range of e from below and from above. (See the Remark at the end of Section 3.4.)

All of the above problems can be reformulated for graph drawings on other surfaces. Let S_g denote a torus with g holes, i.e., a compact oriented surface of *genus* g with no boundary. Define $\text{cr}_g(G)$, the crossing number of

G on S_g, as the minimum number of crossing points in any drawing of G on S_g. Let

$$\kappa_g(n,e) = \min_{\substack{n(G)=n \\ e(G)\geq e}} \text{CR}_g(G).$$

With this notation, $\text{cr}_0(G)$ is the planar crossing number and $\kappa_0(n,e) = \kappa(n,e)$.

In Section 3.5, we prove that there is a midrange crossing constant for graph drawings on any surface S_g of fixed genus $g \geq 0$.

Theorem 3.1.5. *For every $g \geq 0$, if $n \ll e \ll n^2$ then the limit*

$$\lim_{n\to\infty} \kappa_g(n,e)\frac{n^2}{e^3}$$

exists and is equal to the constant $C > 0$ in Theorem 3.1.4.

To prove this result, we have to generalize Theorem B.

Theorem 3.1.6. *Let G be a graph of n vertices, whose degrees are $d_1, d_2, \ldots,$ d_n. Then*

$$b(G) \leq 300(1+g^{3/4})\sqrt{\text{CR}_g(G) + \sum_{i=1}^{n} d_i^2}.$$

3.2 Crossing numbers and monotone properties – Proof of Theorem 3.1.1

Let $\mathcal{P}$ be a monotone graph property with $\text{ex}(n,\mathcal{P}) \leq An^{1+\alpha}$, for some $A, \alpha > 0$. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, where $|V(G)| = n(G) = n$ and $|E(G)| = e(G) = e$. Suppose that G satisfies property $\mathcal{P}$ and $e \geq cn\log^2 n$. To prove Theorem 3.1.1, we assume that

$$\text{cr}(G) < c'\frac{e^{2+1/\alpha}}{n^{1+1/\alpha}},$$

and, if c and c' are suitable constant, we will obtain a contradiction.

We break G into smaller components, according to the following procedure.

Decomposition Algorithm

Step 0. **Let** $G^0 = G, G_1^0 = G, M_0 = 1, m_0 = 1$.

Suppose that we have already executed Step i, and that the resulting graph, G^i, consists of M_i components, $G_1^i, G_2^i, \ldots, G_{M_i}^i$, each of at most $(2/3)^i n$ vertices. Assume, without loss of generality, that the first m_i components of G^i have at least $(2/3)^{i+1}n$ vertices and the remaining $M_i - m_i$ have fewer. Then

$$(2/3)^{i+1}n(G) \leq n(G_j^i) \leq (2/3)^i n(G) \quad (j = 1, 2, \ldots, m_i).$$

Thus, we have that $m_i \leq (3/2)^{i+1}$.

Step $i+1$. **If**

$$(2/3)^i < \frac{1}{(2A)^{1/\alpha}} \cdot \frac{e^{1/\alpha}}{n^{1+1/\alpha}}, \tag{3.1}$$

then stop. (1) is called the *stopping rule.*

Else, for $j = 1, 2, \ldots, m_i$, delete $b(G_j^i)$ edges from G_j^i such that G_j^i falls into two components, each of at most $(2/3)n(G_j^i)$ vertices. Let G^{i+1} denote the resulting graph on the original set of n vertices. Clearly, each component of G^{i+1} has at most $(2/3)^{i+1}n$ vertices.

Suppose that the Decomposition Algorithm terminates in Step $k+1$. If $k > 0$, then

$$(2/3)^k < \frac{1}{(2A)^{1/\alpha}} \cdot \frac{e^{1/\alpha}}{n^{1+1/\alpha}} \leq (2/3)^{k-1}.$$

First, we give an upper bound on the total number of edges deleted from G.

Using that, for any non-negative reals $a_1, a_2, \ldots, a_m$,

$$\sum_{j=1}^{m} \sqrt{a_j} \leq \sqrt{m \sum_{j=1}^{m} a_j}, \tag{3.2}$$

we obtain that, for any $0 \leq i < k$,

$$\sum_{j=1}^{m_i} \sqrt{\mathrm{CR}(G_j^i)} \leq \sqrt{m_i \sum_{j=1}^{m_i} \mathrm{CR}(G_j^i)} \leq \sqrt{(3/2)^{i+1}} \sqrt{\mathrm{CR}(G)}$$

$$< \sqrt{(3/2)^{i+1}}\sqrt{\frac{c'e^{2+1/\alpha}}{n^{1+1/\alpha}}}.$$

Denoting by $d(v, G^i_j)$ the degree of vertex v in G^i_j, we have

$$\sum_{j=1}^{m_i}\sqrt{\sum_{v\in V(G^i_j)} d^2(v, G^i_j)} \le \sqrt{(3/2)^{i+1}}\sqrt{\sum_{v\in V(G^i)} d^2(v, G^i)}$$

$$\le \sqrt{(3/2)^{i+1}}\sqrt{\max_{v\in V(G^i)} d(v, G^i)\sum_{v\in V(G^i)} d(v, G^i)}$$

$$\le \sqrt{(3/2)^{i+1}}\sqrt{(2/3)^i n(2e)} = \sqrt{3en}.$$

In view of Theorem B in the Introduction, the total number of edges deleted during the procedure is

$$\sum_{i=0}^{k-1}\sum_{j=1}^{m_i} b(G^i_j) \le 10\sum_{i=0}^{k-1}\sum_{j=1}^{m_i}\sqrt{\mathrm{CR}(G^i_j)} + 2\sum_{i=0}^{k-1}\sum_{j=1}^{m_i}\sqrt{\sum_{v\in V(G^i_j)} d^2(v, G^i_j)}$$

$$< 10\sqrt{c'}\sqrt{\frac{e^{2+1/\alpha}}{n^{1+1/\alpha}}}\sum_{i=0}^{k-1}\sqrt{(3/2)^i} + 2k\sqrt{3en}$$

$$\le 250\sqrt{c'}\sqrt{\frac{e^{2+1/\alpha}}{n^{1+1/\alpha}}}\sqrt{(2A)^{1/\alpha}\frac{n^{1+1/\alpha}}{e^{1/\alpha}}} + 2k\sqrt{3en} \le \frac{e}{2},$$

provided that c' is sufficiently small and c is sufficiently large.

Therefore, the number of edges of the graph G^k obtained in the final Step of the algorithm satisfies

$$e(G^k) \ge \frac{e}{2}.$$

(Note that this inequality trivially holds if the algorithm terminates in the very first Step, i.e., when $k = 0$.)

Next we give a lower bound on $e(G^k)$. The number of vertices of each connected component of G^k satisfies

$$n(G^k_j) \le (2/3)^k n < \frac{1}{(2A)^{1/\alpha}} \cdot \frac{e^{1/\alpha}}{n^{1+1/\alpha}} n = \left(\frac{e}{2An}\right)^{1/\alpha} \quad (j = 1, 2, \ldots, M_k).$$

Since each G_j^k has property $\mathcal{P}$, it follows that

$$e(G_j^k) \leq An^{1+\alpha}(G_j^k) < An(G_j^k) \cdot \frac{e}{2An}.$$

Therefore, for the total number of edges of G_k, we have

$$e(G^k) = \sum_{j=1}^{M_k} e(G_j^k) < A\frac{e}{2An}\sum_{j=1}^{M_k} n(G_j^k) = \frac{e}{2},$$

the desired contradiction. This proves the bound of Theorem 3.1.1.

It remains to show that the bound is tight up to a constant factor. Suppose that $\mathrm{ex}(n,\mathcal{P}) \geq A'n^{1+\alpha}$ For every e ($cn < e \leq An^{1+\alpha}$), we construct a graph G of at most n vertices and at least e edges, which has property $\mathcal{P}$ and crossing number

$$\mathrm{cr}(G) \leq c''\frac{e^{2+1/\alpha}}{n^{1+1/\alpha}},$$

for a suitable constant $c'' = c''(A', \alpha)$.

Let

$$k = \left\lceil \frac{2e}{A'n} \right\rceil^{\frac{1}{\alpha}},$$

and let G_k denote a graph of k vertices and at least $A'k^{1+\alpha}$ edges, which has property $\mathcal{P}$. Clearly,

$$\mathrm{CR}(G_k) \leq e^2(G_k) \leq (Ak^{1+\alpha})^2 = A^2k^{2+2\alpha}.$$

Let G be the union of $\lfloor n/k \rfloor$ disjoint copies of G_k. Then $n(G) = \lfloor n/k \rfloor k \leq n$,

$$e(G) = \left\lfloor \frac{n}{k} \right\rfloor e(G_k) \geq \frac{n}{2k}A'kk^{\alpha} \geq e,$$

$$\mathrm{CR}(G) = \left\lfloor \frac{n}{k} \right\rfloor \mathrm{CR}(G_k) \leq \frac{n}{k}A^2k^{2+2\alpha} \leq A^2 n\left(2\left(\frac{2e}{A'n}\right)^{\frac{1}{\alpha}}\right)^{1+2\alpha}$$

$$= \frac{2^{3+2\alpha+1/\alpha}A^2}{(A')^{2+1/\alpha}} \cdot \frac{e^{2+1/\alpha}}{n^{1+1/\alpha}},$$

as required. □

3.3 Forbidden subgraphs – Proofs of Theorems 3.1.2 and 3.1.3

In Section 3.2, we established Theorem 3.1.1 under the assumption $e \geq cn \log^2 n$, where c is a suitable constant depending on property $\mathcal{P}$. It seems very likely that the same result is true for every $e \geq cn$. The appearance of the $\log^2 n$ factor was due to the fact that to estimate the total number of edges deleted during the DECOMPOSITION ALGORITHM, we applied Theorem B. We used a poor upper bound on the term $\sum d_i^2$, because some of the degrees d_i may be very large. However, in some interesting special cases, this difficulty can be avoided by a simple trick. We can split each vertex of high degree into vertices of 'average degree,' unless the new graph ceases to have property $\mathcal{P}$.

We illustrate this technique by proving the following result, which is the $r = s = 2$ special case of Theorem 3.1.3 and a slight modification of Theorem 3.1.2 for $r = 2$.

Theorem 3.3.1. *Let G be a $K_{2,2}$-free (C_4-free) graph with $n(G) = n$ vertices and $e(G) = e$ edges, $e \geq 1000n$. Then*

$$\mathrm{CR}(G) \geq \frac{1}{10^8} \frac{e^4}{n^3}.$$

This bound is tight up to a constant factor.

Proof. Let G be a graph with n vertices and $e \geq 1000n$ edges, which does not contain $K_{2,2}$ as a subgraph. Suppose, in order to obtain a contradiction, that

$$\mathrm{CR}(G) < \frac{1}{10^8} \frac{e^4}{n^3},$$

and G is drawn in the plane with $\mathrm{CR}(G)$ crossings.

First, we split every vertex of G whose degree exceeds $\overline{d} := 2e/n$ into vertices of degree at most $\overline{d}$, as follows. Let v be a vertex of G with degree $d(v, G) = d(v) = d > \overline{d}$, and let $vw_1, vw_2, \ldots, vw_d$ be the edges incident to v, listed in clockwise order. Replace v by $\lceil d/\overline{d} \rceil$ new vertices, $v_1, v_2, \ldots, v_{\lceil d/\overline{d} \rceil}$, placed in clockwise order on a very small circle around v. Without introducing any new crossings, connect w_j to v_i if and only if $\overline{d}(i-1) < j \leq \overline{d}i$ ($1 \leq j \leq d, 1 \leq i \leq \lceil d/\overline{d} \rceil$). Repeat this procedure for every vertex whose degree exceeds $\overline{d}$, and denote the resulting graph by G'.

Obviously, G' is also $K_{2,2}$-free, $e(G') = e(G) = e$, and

$$\text{CR}(G') \le \text{CR}(G) < \frac{1}{10^8}\frac{e^4(G)}{n^3(G)}.$$

Since all but at most n vertices of G' have degree $\overline{d}$, we have $n(G') < 2n(G) = 2n$.

Apply the Decomposition Algorithm described in the previous section to the graph G' with the difference that, instead of (1), use the following stopping rule: STOP in Step $i+1$ if

$$\left(\frac{2}{3}\right)^i < \frac{e^2(G')}{16n^3(G')}.$$

Suppose that the algorithm terminates in Step $k+1$. If $k > 0$, then

$$(2/3)^k < \frac{e^2(G')}{16n^3(G')} \le (2/3)^{k-1}.$$

Just like in the proof of Theorem 3.1.1, for every $i < k$, we have that

$$\sum_{j=1}^{m_i} \sqrt{\text{CR}(G_j^i)} \le \sqrt{(3/2)^{i+1}}\sqrt{\text{CR}(G)} < \frac{1}{10^4}\sqrt{(3/2)^{i+1}}\frac{e^2}{n^{3/2}}$$

and, using the fact that the maximum degree in G' is at most $\overline{d}$,

$$\sum_{j=1}^{m_i} \sqrt{\sum_{v\in V(G_j^i)} d^2(v, G_j^i)} \le \sqrt{(3/2)^{i+1}}\sqrt{\sum_{v\in V(G')} d^2(v, G')}$$

$$\le \sqrt{(3/2)^{i+1}}\sqrt{\overline{d}2e(G')} \le 2\sqrt{(3/2)^{i+1}}\frac{e}{\sqrt{n}}.$$

Hence, by Theorem B, the total number of edges deleted during the algorithm is

$$\sum_{i=0}^{k-1}\sum_{j=1}^{m_i} b(G_j^i) \le 10\sum_{i=0}^{k-1}\sum_{j=1}^{m_i}\sqrt{\text{CR}(G_j^i)} + 2\sum_{i=0}^{k-1}\sum_{j=1}^{m_i}\sqrt{\sum_{v\in V(G_j^i)} d^2(v, G_j^i)}$$

$$< \frac{1}{1000}\frac{e^2}{n^{3/2}}\sum_{i=0}^{k-1}\sqrt{(3/2)^{i+1}} + 4\frac{e}{\sqrt{n}}\sum_{i=0}^{k-1}\sqrt{(3/2)^{i+1}}$$

$$= \sqrt{3/2}\frac{\sqrt{(3/2)^k}-1}{\sqrt{3/2}-1}\left(\frac{e^2}{1000n^{3/2}}+\frac{4e}{\sqrt{n}}\right)$$

$$< 100\frac{n^{3/2}}{e}\left(\frac{e^2}{1000n^{3/2}}+\frac{4e}{\sqrt{n}}\right) < \frac{e}{10}+400n < \frac{e}{2}.$$

Therefore, for the resulting graph,

$$e(G^k) \geq \frac{e}{2}.$$

On the other hand, each component of G^k has relatively few vertices:

$$n(G_j^k) < (2/3)^k n(G') < \frac{e^2}{16n^2(G')} = \frac{e^2}{16n^2(G^k)} \quad (j = 1, 2, \ldots, M_k).$$

Claim C. [R58] *Let* $\mathrm{ex}(n, K_{2,2})$ *denote the maximum number of edges that a* $K_{2,2}$*-free graph with* n *vertices can have. Then*

$$\mathrm{ex}(n, K_{2,2}) \leq \frac{n\left(1+\sqrt{4n-3}\right)}{4} \leq n^{3/2}.$$

Applying the Claim to each G_k^j, we obtain

$$e(G_j^k) \leq n^{3/2}(G_j^k) < n(G_j^k)\cdot\sqrt{\frac{e^2}{16n^2(G^k)}},$$

therefore,

$$e(G^k) = \sum_{j=1}^{M_k} e(G_j^k) < \frac{e}{4n(G^k)}\sum_{j=1}^{M_k} n(G_j^k) = \frac{e}{4},$$

the desired contradiction. The tightness of Theorem 3.3.1 immediately follows from the fact that Theorem 3.1.1 was tight. □

Theorems 3.1.2 and 3.1.3 can be proved similarly. It is enough to notice that splitting a vertex of high degree does not decrease the girth of a graph G and does not create a subgraph isomorphic to $K_{r,s}$. Instead of Claim C, now we need

Claim C′. [BS74], [Br66], [Be66], [S66], [W91] *For a fixed positive integer* r*, let* $\mathcal{G}_{2r}$ *denote the property that the girth of a graph is larger than* $2r$.

Then the maximum number of edges of a graph with n vertices, which has property $\mathcal{G}_{2r}$, satisfies

$$\mathrm{ex}(n, \mathcal{G}_{2r}) = O(n^{1+1/r}).$$

For $r = 2, 3$ and 5, this bound is tight.

Claim C″. [KST54], [F96], [ER62], [Br66], [ARS99] *For any integers $s \geq r \geq 2$, the maximum number of edges of a $K_{r,s}$-free graph of n vertices, satisfies*

$$\mathrm{ex}(n, K_{r,s}) = O(n^{2-1/r}).$$

This bound is tight for $s > (r-1)!$.

In case $r = 3$, we obtain the following slight generalization of Theorem 3.1.2.

Theorem 3.3.2. *Let G be a graph of n vertices and $e \geq 4n$ edges, which contains no cycle C_6 of length 6.*

Then, for a suitable constant $c_6' > 0$, we have

$$\textsc{cr}(G) \geq c_6' \frac{e^5}{n^4}.$$

To establish Theorem 3.3.2, it is enough to modify the proof of Theorem 3.1.2 at one point. Before splitting the high-degree vertices of G and running the Decomposition Algorithm, we have to turn G into a bipartite graph, by deleting at most half of its edges. After that, splitting a vertex cannot create a C_6, and the rest of the above argument shows that the crossing number of the remaining graph still exceeds $c_6' \frac{e^5}{n^4}$.

We do not see, however, how to obtain the analogous generalization of Theorem 3.1.2 for $r > 3$.

3.4 Midrange crossing constant in the plane – Proof of Theorem 3.1.4

Lemma 3.4.1. (i) *For any $a > 0$, the limit*

$$\gamma[a] = \lim_{n \to \infty} \frac{\kappa(n, na)}{n}$$

exists and is finite.

(ii) $\gamma[a]$ *is a convex continuous function.*
(iii) *For any* $a \geq 4$, $1 > \delta > 0$,

$$\gamma[a] - \gamma[a(1-\delta)] \leq \gamma[a(1+\delta)] - \gamma[a] \leq 10^3 \delta \gamma[a].$$

by taking the limit as $n \to \infty$

Proof. Clearly, any two graphs, G_1 and G_2, can be drawn in the plane so that the edges of G_1 do not intersect the edges of G_2. Therefore,

$$\kappa(n_1 + n_2, e_1 + e_2) \leq \kappa(n_1, e_1) + \kappa(n_2, e_2). \tag{3.3}$$

In particular, the function $f_a(n) = \kappa(n, na)$ is subadditive and hence the limit

$$\gamma[a] = \lim_{n \to \infty} \frac{\kappa(n, na)}{n}$$

exists and is finite for every fixed $a > 0$. It also follows from (3.3) that for any $a, b > 0$ and $1 > \alpha > 0$, if n and αn are both integers,

$$\kappa(n, (\alpha a + (1-\alpha)b)n) \leq \kappa(\alpha n, \alpha a n) + \kappa((1-\alpha)n, (1-\alpha)bn),$$

so for any $1 > \alpha > 0$ rational,

$$\gamma[\alpha a + (1-\alpha)b] \leq \alpha\gamma[a] + (1-\alpha)\gamma[b].$$

But since the function $\gamma[a]$ is monotone increasing, it follows that for *any* $1 > \alpha > 0$,

$$\gamma[\alpha a + (1-\alpha)b] \leq \alpha\gamma[a] + (1-\alpha)\gamma[b]. \tag{3.4}$$

That is, the function $\gamma[a]$ is *convex*. In particular, for every $1 > \delta > 0$, we have

$$\gamma[a] - \gamma[a(1-\delta)] \leq \gamma[a(1+\delta)] - \gamma[a].$$

It is known that for any $a \geq 4$,

$$\frac{a^3 n}{100} \leq \kappa(n, an) \leq a^3 n \;\; \Rightarrow \;\; \frac{a^3}{100} \leq \gamma[a] \leq a^3 \tag{3.5}$$

(see e.g. [PT97]). Let $a \geq 4$, $1 > \delta > 0$. By (3.4),

$$\gamma[a(1+\delta)] \leq (1-\delta)\gamma[a] + \delta\gamma[2a].$$

Therefore, using (3.5),

$$\gamma[a(1+\delta)] - \gamma[a] \leq \delta\gamma[2a] \leq \delta 8a^3 < 10^3\delta\gamma[a]. \ \Box$$

Set

$$C := \limsup_{a\to\infty} \frac{\gamma[a]}{a^3}.$$

By (3.5), we have that $C < 1$.

Lemma 3.4.2. *For any* $0 < \epsilon < 1$, *there exists* $N = N(\epsilon)$ *such that* $\kappa(n,e) > C\frac{e^3}{n^2}(1-\epsilon)$, *whenever* $\min\{n, e/n, n^2/e\} > N$.

Proof. Let $A > \frac{10^9}{\epsilon^3}$ be a rational number satisfying

$$\frac{\gamma[A]}{A^3} > C(1 - \frac{\epsilon}{10}). \tag{3.6}$$

Let $N = N(\epsilon) \geq A$ such that, if $n > N$, $e = nA'$, and $|A - A'| \leq A\epsilon$, then

$$\kappa(n,e) > \gamma[A'](1 - \frac{\epsilon}{10})n. \tag{3.7}$$

Let n and e be fixed, $\min\{n, e/n, n^2/e\} > N$ and let $G = (V, E)$ be a graph with $|V| = n$ vertices and $|E| = e$ edges, drawn in the plane with $\kappa(n,e)$ crossings. Set $p = An/e$. Let U be a randomly chosen subset of V with $\Pr[v \in U] = p$, independently for all $v \in V$. Let $\nu = |U|$, and let η (resp. ξ) be the number of edges (resp. crossings) in the (drawing of the) subgraph of G induced by the elements of U.

ν has mean pn and variance $p(1-p)n \leq pn$, so, by the Chebyshev Inequality,

$$\Pr\left[|\nu - pn| > \frac{\epsilon}{10^4}pn\right] < \frac{\epsilon}{10}.$$

Write $\eta = \sum I_{uv}$, where the sum is taken over all edges $uv = vu \in E$, and I_{uv} denotes the indicator for the event $u, v \in U$. Obviously, $E[\eta] = \sum_{uv\in E} E[I_{uv}] = ep^2$. We decompose

$$\text{Var}\,[\eta] = \sum_{uv\in E} \text{Var}\,[I_{uv}] + \sum_{uv,uw\in E} Cov\,[I_{uv}, I_{uw}]\,,$$

as $\text{Cov}[I_{uv}, I_{wz}] = 0$ when all four indices are distinct. As always with indicators, we have

$$\sum_{uv\in E} \text{Var}\,[I_{uv}] \leq \sum_{uv\in E} E\,[I_{uv}] = E\,[\eta] = ep^2.$$

Using the bound $\mathrm{Cov}[I_{uv}, I_{uw}] \le E[I_{uv}I_{uw}] = p^3$, we obtain

$$\mathrm{Var}\,[\eta] \le p^2 e + p^3 \sum_{v\in V} \binom{d(v)}{2},$$

where $d(v)$ is the degree of vertex v in G. But $\sum_{v\in V} d(v) = 2e$ and all $d(v) < n$, so

$$\sum_{v\in V} \binom{d(v)}{2} \le \frac{1}{2} \sum_{v\in V} d^2(v) \le en.$$

Thus, we have

$$\mathrm{Var}\,[\eta] \le p^2 e + p^3 en \le 2p^3 en,$$

as $pn = An^2/e \ge 1$. Again, by the Chebyshev Inequality,

$$\Pr\left[|\eta - p^2 e| > \frac{\epsilon}{10^4} p^2 e\right] < \frac{\epsilon}{10}.$$

With probability at least $1 - \frac{\epsilon}{5}$,

$$pn(1 - \frac{\epsilon}{10^4}) < \nu < pn(1 + \frac{\epsilon}{10^4})$$

and

$$p^2 e(1 - \frac{\epsilon}{10^4}) < \eta < p^2 e(1 + \frac{\epsilon}{10^4}),$$

so with probability at least $1 - \frac{\epsilon}{5}$,

$$A(1 - \frac{3\epsilon}{10^4}) < \frac{\eta}{\nu} = A' < A(1 + \frac{3\epsilon}{10^4}).$$

Therefore, in view of (3.7), with probability at least $1 - \frac{\epsilon}{5}$, the subgraph of G induced by U has at least $pn(1 - \frac{\epsilon}{10})\gamma[A'](1 - \frac{\epsilon}{10})$ crossings. But then, we have

$$\begin{aligned} E\,[\xi] &\ge (1 - \frac{\epsilon}{5}) pn(1 - \frac{\epsilon}{10})\gamma[A'](1 - \frac{\epsilon}{10}) \\ &\ge (1 - \frac{\epsilon}{5}) pn(1 - \frac{\epsilon}{10})\gamma[A](1 - \frac{3\epsilon}{10})(1 - \frac{\epsilon}{10}) \\ &\ge (1 - \frac{\epsilon}{5}) pn(1 - \frac{\epsilon}{10}) CA^3 (1 - \frac{\epsilon}{10})(1 - \frac{3\epsilon}{10})(1 - \frac{\epsilon}{10}) \ge (1 - \epsilon) CA^3 pn, \end{aligned}$$

where the second and third inequalities follow from Lemma 3.4.1(iii) and from the choice of A, respectively.

On the other hand,

$$E[\xi] = p^4 \kappa(n, e),$$

as every crossing lies in U with probability p^4. Thus

$$\kappa(n, e) \geq (1 - \epsilon) \frac{pnCA^3}{p^4} = C \frac{e^3}{n^2}(1 - \epsilon)$$

as desired. □

To complete the proof of Theorem 3.1.4, we have to establish the "counterpart" of Lemma 3.4.2.

Lemma 3.4.3. *For any* $1 > \epsilon > 0$, *there exists* $M = M(\epsilon)$ *such that* $\kappa(n, e) < C\frac{e^3}{n^2}(1 + \epsilon)$, *whenever* $\min\{n, e/n, n^2/e\} > M$.

Proof. Let $A > \frac{10^4}{\epsilon^2}$ be a rational number satisfying

$$C(1 - \frac{\epsilon}{10}) < \frac{\gamma[A]}{A^3} < C(1 + \frac{\epsilon}{10}).$$

Let $M_1 = M_1(\epsilon) \geq A$ such that, if $n > M_1$ and $e = nA$, then

$$CA^3 n(1 - \frac{\epsilon}{5}) < \kappa(n, e) < CA^3 n(1 + \frac{\epsilon}{5}).$$

Let $G_1 = G_1(n_1, e_1)$ be a graph with $n_1 > M_1$ vertices, $e_1 = An_1$ edges, and suppose that G_1 is drawn in the plane with $\kappa(n_1, e_1)$ crossings, where $CA^3 n_1(1-\frac{\epsilon}{5}) < \kappa(n_1, e_1) < CA^3 n_1(1+\frac{\epsilon}{5})$. For each vertex v of G_1 with degree $d(v) > A^{3/2}$, we do the following. Let $d(v) = rA^{3/2} + s$, where $0 \leq s < A^{3/2}$. Substitute v with $r + 1$ vertices, each of degree $A^{3/2}$, except one which has degree s, each drawn very close to the original position of v. Clearly, this can be done without creating any additional crossing. We obtain a graph $G_2(n_2, e_2)$ such that

$$n_1 \leq n_2 \leq n_1(1 + \frac{2}{\sqrt{A}}) \leq n_1(1 + \frac{\epsilon}{10}),$$

$e_2 = e_1$, and G_2 is drawn in the plane with $\kappa(n_1, e_1)$ crossings.

Suppose that n and e are fixed, $\min\{n, e/n, n^2/e\} > M(\epsilon) = \frac{10M_1}{\epsilon}$. Let

$$L = \frac{e/n}{e_2/n_2} \quad \text{and} \quad K = \frac{n^2/e}{n_2^2/e_2},$$

so that

$$n = KLn_2 \text{ and } e = KL^2e_2.$$

Let

$$\tilde{L} = \left\lfloor L(1+\frac{\epsilon}{10})\right\rfloor \text{ and } \tilde{K} = \left\lfloor K(1-\frac{\epsilon}{10})\right\rfloor$$

and let

$$\tilde{n} = \tilde{K}\tilde{L}n_2 \text{ and } \tilde{e} = \tilde{K}\tilde{L}^2e_2.$$

Then $n(1-\frac{\epsilon}{5}) < \tilde{n} < n$ and $e_2 < \tilde{e} \leq e_2(1+\frac{\epsilon}{4})$, so we have $\kappa(n,e) < \kappa(\tilde{n},\tilde{e})$.

Substitute each vertex of G_2 with $\tilde{L}$ very close vertices, and substitute each edge of G_2 with the corresponding $\tilde{L}^2$ edges, all running very close to the original edge. Make $\tilde{K}$ copies of this drawing, each separated from the others. This way we got a graph $\tilde{G}(\tilde{n},\tilde{e})$ drawn in the plane. We estimate the number of crossings X in this drawing.

A crossing in the original drawing of G_2 corresponds to $\tilde{K}\tilde{L}^4$ crossings in the present drawing of $\tilde{G}$. For any two edges of G_2 with common endpoint, uv and uw, the edges arise from them have at most $\tilde{K}\tilde{L}^4$ crossings with each other. So

$$X \leq \tilde{K}\tilde{L}^4\left(\kappa(n_1,e_1) + \sum_{v\in V(G_2)}\binom{d(v)}{2}\right)$$

But $\sum_{v\in V(G_2)} d(v) = 2e_2$ and $d(v) \leq A^{3/2}$, so

$$\sum_{v\in V(G_2)}\binom{d(v)}{2} < 3A^{5/2}n_2.$$

Therefore,

$$\kappa(n,e) < \kappa(\tilde{n},\tilde{e}) \leq c < \tilde{K}\tilde{L}^4\kappa(n_1,e_1) + \tilde{K}\tilde{L}^4 3A^{5/2}n_2 < \tilde{K}\tilde{L}^4\kappa(n_1,e_1)(1+\frac{\epsilon}{10})$$

$$< \tilde{K}\tilde{L}^4CA^3n_1(1+\frac{\epsilon}{5})(1+\frac{\epsilon}{10}) = \tilde{K}\tilde{L}^4C\frac{e_1^3}{n_1^2}(1+\frac{\epsilon}{5})(1+\frac{\epsilon}{10})$$

$$< KL^4C\frac{e_2^3}{n_2^2}(1+\frac{\epsilon}{10})^6(1+\frac{\epsilon}{5})(1+\frac{\epsilon}{10}) < C(1+\epsilon)\frac{e^3}{n^2}. \quad \square$$

Remark.

We cannot decide whether Theorem 3.1.4 remains true under the weaker condition that $C_1n \leq e \leq C_2n^2$ for suitable positive constants C_1 and C_2. If the answer were in the affirmative, then, clearly, $C_1 > 3$. We would also have that $C_2 < 1/2$, because, by [G72], for $e = \binom{n}{2}$, $\mathrm{CR}(K_n) > (\frac{1}{10}-\epsilon)\frac{e^3}{n^2}$ for any $\epsilon > 0$ if n is large enough.

3.5 Midrange crossing constants on other surfaces – Proof of Theorem 3.1.5

Lemma 3.5.1. *For any integer $g \geq 0$ and for any $1 > \epsilon > 0$, there exists $N = N(g, \epsilon)$ such that $\kappa_g(n, e) > C\frac{e^3}{n^2}(1-\epsilon)$, whenever $\min\{n, e/n, n^{3/2}/e\} > N$.*

Proof. For $g = 0$, the assertion follows from Lemma 3.4.2. Suppose that $g > 0$ is fixed and we have already proved the lemma for $g-1$. For any $\epsilon > 0$, let $N(g, \epsilon) = \frac{10^5}{\epsilon^2} g N(g-1, \epsilon/10)$. Suppose, in order to get a contradiction, that $\min\{n, e/n, n^{3/2}/e\} > N$, and let $G(n, e)$ be a graph drawn on S_g with $\mathrm{cr}_g(G) = \kappa_g(n, e) < C\frac{e^3}{n^2}(1-\epsilon)$ crossings.

As long as there is an edge with at least $4C\frac{e^2}{n^2}$ crossings, delete it. Let the resulting graph be $G_1(n_1, e_1)$. Suppose that we deleted e' edges. Then G_1 has $n_1 = n$ vertices, $e_1 = e - e'$ edges, and the number of crossings in the resulting drawing of G_1 is at most $\mathrm{cr}_g(G) - 4C\frac{e^2}{n^2}e'$. Therefore, $e' < e/4$, so $e \geq e_1 \geq 3e/4$. It is not hard to check that $\mathrm{cr}_g(G_1) < C\frac{e_1^3}{n_1^2}(1-\epsilon)$ and G_1 contains no edge with more than $4C\frac{e^2}{n^2} < 8C\frac{e_1^2}{n_1^2}$ crossings.

Consider all cycles of G_1, as they are drawn on S_g. If each cycle is *trivial*, i.e., each cycle is contractible to a point of S_g, then every connected component of G is contractible to a point. That is, in this case, our drawing of G on S_g is equivalent to a drawing of G_1 on the plane. Consequently, $\mathrm{CR}_{g-1}(G_1) \leq \mathrm{cr}_0(G_1) < C\frac{e^3}{n^2}(1-\epsilon)$ contradicting the induction hypothesis.

Suppose that there is a non-trivial (i.e., non-contractible) cycle $\mathcal{C}$ of G_1 with at most $\frac{\epsilon}{80C}\frac{n_1^2}{e_1}$ edges. Clearly, $\mathcal{C}$ contains a non-trivial closed curve, $\mathcal{C}'$, which does not intersect itself. The total number of crossings along $\mathcal{C}'$ is at most

$$\frac{\epsilon}{80C}\frac{n_1^2}{e_1}8C\frac{e_1^2}{n_1^2} = \frac{\epsilon}{10}e_1.$$

Delete all edges that cross $\mathcal{C}'$. Cut S_g along $\mathcal{C}'$. Replace every vertex (resp. edge) $\mathcal{C}'$ by two vertices, one on each side of the cut. Every edge of G arriving at a vertex v of $\mathcal{C}'$ from a given side of the cut will be connected to the copy of v lying on the same side. Thus, we obtain a graph $G_2(n_2, e_2)$, drawn with fewer than $\mathrm{cr}_g(G_1)$ crossings. Attaching a half-sphere to each side of the cut, we obtain either a surface of genus $g-1$ or two surfaces whose genuses are smaller than g. We discuss only the former case (the calculation in the latter one is very similar). Since we doubled at most $\frac{\epsilon}{80C}\frac{n_1^2}{e_1} = \epsilon n_1 \frac{n_1}{e_1}\frac{1}{80C} < \epsilon n_1 \frac{1}{N} < n_1 \frac{\epsilon}{10}$ vertices and deleted at most $\frac{\epsilon}{10}e$ edges, we have $n_2 \leq n_1(1+\frac{\epsilon}{10})$ and $e_2 \geq e_1(1-\frac{\epsilon}{10})$. In the resulting drawing there are fewer than $\mathrm{cr}_g(G_1)$ crossings, therefore

$$\mathrm{cr}_{g-1}(G_2) < \mathrm{cr}_g(G_1) < C\frac{e_1^3}{n_1^2}(1-\epsilon) \leq C\frac{e_2^3}{n_2^2}(1-\epsilon)(1-\frac{\epsilon}{10})^{-3}(1+\frac{\epsilon}{10})^2$$
$$\leq C\frac{e_2^3}{n_2^2}(1-\frac{\epsilon}{10}),$$

contradicting the induction hypothesis.

Thus, we can assume that every non-trivial cycle of G_1 contains at least $\frac{\epsilon}{80C}\frac{n_1^2}{e_1}$ edges. For each vertex v of G_1 with degree $d(v) > \frac{10e_1}{\epsilon n_1}$, we do the following. Let $d(v) = r\frac{10e_1}{\epsilon n_1} + s$, where $0 \leq s < \frac{10e_1}{\epsilon n_1}$. Without creating any new crossing, replace v by $r+1$ nearby vertices, each of degree $\frac{10e_1}{\epsilon n}$, except one, whose degree is s. We obtain a graph $G_3(n_3, e_3)$ drawn on S_g with $n_1 \leq n_3 \leq n_1(1+\frac{\epsilon}{5})$, $e_3 = e_1$, and with the same number of crossings as G_1. Hence,

$$\mathrm{CR}_g(G_3) \leq \mathrm{CR}_g(G_1) \leq C\frac{e_1^3}{n_1^2}(1-\epsilon) \leq C\frac{e_3^3}{n_3^2}(1-\epsilon)(1+\frac{\epsilon}{5})^2 \leq C\frac{e_3^3}{n_3^2}(1-\frac{\epsilon}{2}).$$

The maximum degree D in G_3 cannot exceed $\frac{10e_1}{\epsilon n_1} < \frac{18e_3}{\epsilon n_3}$, and the length of each non-trivial cycle is at least $\frac{\epsilon}{80C}\frac{n_1^2}{e_1} \geq \frac{\epsilon}{100C}\frac{n_3^2}{e_3}$. Apply to G_3 the Decomposition Algorithm described in Section 3.2 with the difference that, instead of (1), use the following stopping rule: stop in Step $i+1$ if

$$(2/3)^i < \frac{\epsilon}{100C}\frac{n_3}{e_3}.$$

Suppose that the algorithm terminates in Step $k+1$. Then

$$(2/3)^k < \frac{\epsilon}{100C}\frac{n_3}{e_3} \leq (2/3)^{k-1}.$$

First, we give an upper bound on the total number of edges deleted from G_3. Let $G^0 = G_1^0 = G_3$ and $m_0 = 1$. Using (2), we obtain that, for every $0 \leq i < k$,

$$\sum_{j=1}^{m_i}\sqrt{\mathrm{CR}_g(G_j^i)} \leq \sqrt{m_i\sum_{j=1}^{m_i}\mathrm{CR}_g(G_j^i)} \leq \sqrt{(3/2)^{i+1}}\sqrt{\mathrm{CR}_g(G_3)}$$

$$\leq \sqrt{(3/2)^{i+1}}\sqrt{C\frac{e_3^3}{n_3^2}(1-\frac{\epsilon}{2})}.$$

Denoting by $d(v, G_j^i)$ the degree of vertex v in G_j^i, we have

$$\sum_{j=1}^{m_i}\sqrt{\sum_{v\in V(G_j^i)} d^2(v, G_j^i)} \leq \sqrt{(3/2)^{i+1}}\sqrt{\sum_{v\in V(G^i)} d^2(v, G^i)}$$

$$\leq \sqrt{(3/2)^{i+1}}\sqrt{\max_{v\in V(G^i)} d(v, G^i) \sum_{v\in V(G^i)} d(v, G^i)}$$

$$\leq \sqrt{(3/2)^{i+1}}\sqrt{\frac{18e_3^3}{\epsilon n_3^2}(2e_3)} = 12\sqrt{(3/2)^{i+1}}\frac{e_3}{\sqrt{\epsilon n_3}}.$$

By Theorem 3.1.6 (proved in the last section), the total number of edges deleted during the algorithm is

$$\sum_{i=0}^{k-1}\sum_{j=1}^{m_i} b(G_j^i) \leq 300(1+g^{3/4})\sum_{i=0}^{k-1}\sum_{j=1}^{m_i}\sqrt{\mathrm{CR}_g(G_j^i) + \sum_{v\in V(G_j^i)} d^2(v, G_j^i)}$$

$$\leq 300(1+g^{3/4})\sum_{i=0}^{k-1}\sum_{j=1}^{m_i}\sqrt{\mathrm{CR}_g(G_j^i)} + 300(1+g^{3/4})\sum_{i=0}^{k-1}\sum_{j=1}^{m_i}\sqrt{\sum_{v\in V(G_j^i)} d^2(v, G_j^i)}$$

$$\leq 300(1+g^{3/4})\sum_{i=0}^{k-1}\sqrt{(3/2)^{i+1}}\left(\sqrt{C\frac{e_3^3}{n_3^2}(1-\frac{\epsilon}{2})} + 6\frac{e_3}{\sqrt{\epsilon n_3}}\right)$$

$$\leq 300(1+g^{3/4})\sqrt{3/2}\frac{\sqrt{(3/2)^k}-1}{\sqrt{3/2}-1}\left(\sqrt{C\frac{e_3^3}{n_3^2}(1-\frac{\epsilon}{2})} + 6\frac{e_3}{\sqrt{\epsilon n_3}}\right)$$

$$\leq 2000(1+g^{3/4})\sqrt{\frac{C}{\epsilon}}\sqrt{\frac{e}{n}}\left(\sqrt{C\frac{e_3^3}{n_3^2}(1-\frac{\epsilon}{2})} + 6\frac{e_3}{\sqrt{\epsilon n_3}}\right) \leq e_3\frac{\epsilon}{10}.$$

Therefore, the number of edges $e(G^k)$ of the graph G^k obtained in the final STEP of the algorithm satisfies $e(G^k) \geq e_3(1-\frac{\epsilon}{10})$. Consider the drawing of G^k on S_g inherited from the drawing of G_3. Each connected component of G^k has fewer than $\frac{\epsilon}{100C}\frac{n_3^2}{e_3}$ vertices, therefore, each cycle of G^k, as drawn on S_g, is contractible to a point. Consequently, this drawing is equivalent to a planar drawing of G^k. Hence,

$$\text{CR}_{g-1}(G^k) \leq \text{CR}_0(G^k) \leq \text{CR}_g(G_3) \leq C\frac{e_3^3}{n_3^2}(1-\frac{\epsilon}{2})$$

$$\leq C\frac{e^3(G^k)}{n^2(G^k)}(1-\frac{\epsilon}{2})(1-\frac{\epsilon}{10})^{-3} < C\frac{e^3(G^k)}{n^2(G^k)}(1-\frac{\epsilon}{10}),$$

a contradiction. This concludes the proof of Lemma 3.5.1. □

Lemma 3.5.2. *For any integer $g \geq 0$ and for any $\epsilon > 0$, there exists $N' = N'(g, \epsilon)$ such that $\kappa_g(n, e) > C\frac{e^3}{n^2}(1-\epsilon)$, whenever $\min\{n, e/n, n^2/e\} > N'$.*

Proof. The proof is analogous to that of Lemma 3.4.2. □

Lemma 3.5.3. *For any integer $g \geq 0$ and for any $\epsilon > 0$, there exists $M = M(g, \epsilon)$ such that $\kappa_g(n, e) < C\frac{e^3}{n^2}(1+\epsilon)$, whenever $\min\{n, e/n, n^2/e\} > M$.*

Proof. Clearly, for any graph G and for any $g \geq 0$, we have $\text{CR}_0(G) \geq \text{CR}_g(G)$. Therefore, Lemma 3.5.3 is a direct consequence of Lemma 3.4.3. □

Theorem 3.1.5 now readily follows from Lemmas 3.5.2 and 3.5.3.

3.6 A separator theorem – Proof of Theorem 3.1.6

For the proof of Theorem 3.1.6, we need a slight variation of the notion of bisection width. The *weak bisection width*, $\overline{b}(G)$, of a graph G is defined as the minimum number of edges whose removal splits the graph into two components, each of size at least $|V(G)|/5$. That is,

$$\overline{b}(G) = \min_{|V_A|,|V_B| \geq n/5} |E(V_A, V_B)|,$$

where $E(V_A, V_B)$ denotes the number of edges between V_A and V_B, and the minimum is taken over all partitions $V(G) = V_A \cup V_B$ with $|V_A|, |V_B| \geq |V(G)|/5$.

Lemma 3.6.1. *For any graph G, we have*

$$\overline{b}(G) \leq b(G) \leq 2\max_{H \subset G} \overline{b}(H).$$

Proof. The first inequality is obviously true. To prove the second one, let $|V(G)| = n$ and consider a partition $V(G) = V_A \cup V_B$ such that $n/5 \leq |V_A|, |V_B| \leq 4n/5$ and $|E(V_A, V_B)| = \overline{b}(G)$. Suppose that $|V_A| \leq |V_B|$. If $n/3 \leq |V_A|$, then $b(G) = \overline{b}(G)$ and we are done. So we can assume that $n/5 \leq |V_A| \leq n/3$ and $2n/3 \leq |V_B| \leq 4n/5$.

Let H be the subgraph of G induced by V_B. By definition, there is a partition $V_B = V_B' \cup V_B''$ such that $|V_B|/5 \leq |V_B'|, |V_B''| \leq 4|V_B|/5$ and $|E(V_B', V_B'')| = \overline{b}(H)$. We can assume that $|V_B'| \leq |V_B''|$. Then

$$\frac{n}{3} \leq \frac{|V_B|}{2} \leq |V_B''| \leq \frac{4|V_B|}{5} \leq \frac{16n}{25} < \frac{2n}{3}.$$

Letting $V_1 = V_A \cup V_B'$ and $V_2 = V_B''$, we have $V(G) = V_1 \cup V_2$, $n/3 \leq |V_1|, |V_2| \leq 2n/3$,

$$|E(V_1, V_2)| \leq |E(V_A, V_B)| + |E(V_B', V_B'')| \leq \overline{b}(G) + \overline{b}(H),$$

and the result follows. □

Theorem 3.1.6 is an immediate consequence of Lemma 3.6.1 and the following statement.

Theorem 3.6.2. *Let G be a graph with n vertices of degrees $d_1, d_2, \ldots, d_n$. Then*

$$\overline{b}(G) \leq 150(1 + g^{3/4})\sqrt{cr_g(G) + \sum_{i=1}^{n} d_i^2}.$$

Proof. Clearly, we can assume that G contains no isolated vertices, that is, $d_i > 0$ for all $1 \leq i \leq n$. Consider a drawing of G on S_g with exactly $\text{CR}_g(G)$ crossings. Let $v_1, v_2, \ldots, v_n$ be the vertices of G with degrees $d_1, d_2, \ldots, d_n$, respectively. Introduce a new vertex at each crossing. Denote the set of these vertices by V_0. Replace each $v_i \in V(G)$ $(i = 1, 2 \ldots, n)$ by a set V_i of vertices forming a $d_i \times d_i$ piece of a square grid, in which each vertex is connected to its horizontal and vertical neighbors. Let each edge incident to v_i be hooked up to distinct vertices along one side of the boundary of V_i without creating any crossing. These d_i vertices will be called the *special boundary vertices* of V_i.

Thus, we obtain a graph H of $\sum_{i=0}^{n} |V_i| = \text{CR}_g(G) + \sum_{i=1}^{n} d_i^2$ vertices and no crossing (see Fig. 3.1.). For each $1 \leq i \leq n$, assign weight $1/d_i$ to each special boundary vertex of V_i. Assign weight 0 to all other vertices of H. For any subset ν of the vertex set of H, let $w(\nu)$ denote the total weight of the

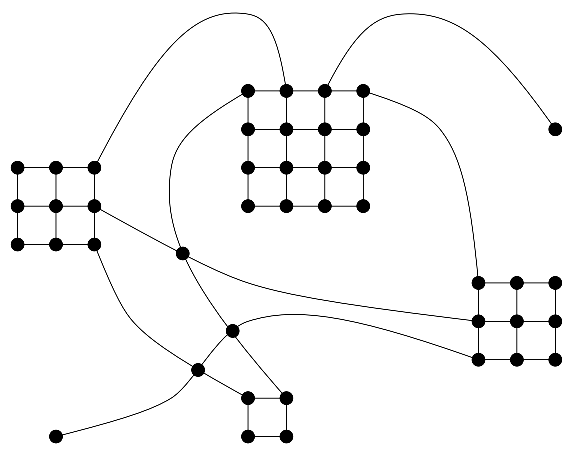

Figure 3.1: Replace the vertices by square grids

vertices belonging to ν. With this notation, $w(V_i) = 1$ for each $1 \leq i \leq n$. Consequently, $w(V(H)) = n$.

Since H is drawn on S_g without crossing, H does not contain K_α as a minor, where $\alpha = \lfloor 4+4\sqrt{g} \rfloor$ [RY68]. Then, by a result of Alon, Seymour, and Thomas [AST90], the vertices of H can be partitioned into three sets, A, B and C, such that $w(A), w(B) \geq n/3$, $|C| \leq 25(1+g^{3/4})\sqrt{\mathrm{CR}_g(G) + \sum_{i=1}^{n} d_i^2}$, and there is no edge from A to B. Let $A_i = A \cap V_i$, $B_i = B \cap V_i$, $C_i = C \cap V_i$ $(i = 0, 1, \ldots, n)$.

For any $1 \leq i \leq n$, we say that V_i is of *type A* (resp. *type B*) if $w(A_i) \geq 5/6$ (resp. $w(B_i) \geq 5/6$), and it is of *type C,* otherwise.

Define a partition $V(G) = V_A \cup V_B$ of the vertex set of G, as follows. For any $1 \leq i \leq n$, let $v_i \in V_A$ (resp. $v_i \in V_B$) if V_i is of type A (resp. type B). The remaining vertices, $\{v_i \mid V_i \text{ is of type } C\}$ are assigned either to V_A or to V_B so as to minimize $||V_A| - |V_B||$.

Claim 3.6.3. $n/5 \leq |V_A|, |V_B| \leq 4n/5$

To prove the claim, define another partition $V(H) = \overline{A} \cup \overline{B} \cup \overline{C}$ such that $\overline{A} \cap V_i = A \cap V_i$ and $\overline{B} \cap V_i = B \cap V_i$, for $i = 0$ and for every V_i of type C. If V_i is of type A (resp. type B), then let $V_i = \overline{A}_i \subset \overline{A}$ (resp. $V_i = \overline{B}_i \subset \overline{B}$), finally, let $\overline{C} = V(H) - \overline{A} - \overline{B}$.

For any V_i of type A, $w(\overline{A}_i) - w(A_i) \leq w(A_i)/5$. Similarly, for any V_i of type B, $w(\overline{B}_i) - w(B_i) \leq w(B_i)/5$. Therefore,

$$|w(\overline{A}) - w(A)| \leq \max\{w(A), w(B)\}/5 \leq 2n/15.$$

Hence, $n/5 \leq w(\overline{A}) \leq 4n/5$ and, analogously, $n/5 \leq w(\overline{B}) \leq 4n/5$. In particular, $|w(\overline{A}) - w(\overline{B})| \leq 3n/5$. Using the minimality of $||V_A| - |V_B||$, we obtain that $||V_A| - |V_B|| \leq 3n/5$, which implies Claim 3.6.3.

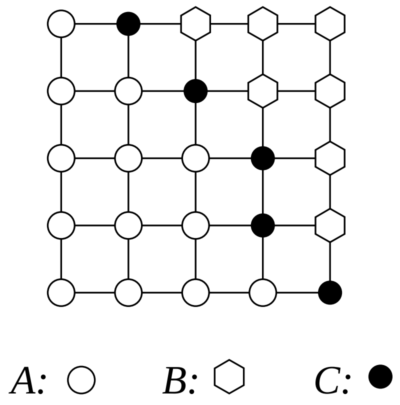

Figure 3.2: Switch the uv segment of e and f.

Claim 3.6.4. *For any $1 \leq i \leq n$,*
(i) *if V_i is of type A (resp. of type B), then $w(B_i)d_i \leq |C_i|$ (resp. $w(A_i)d_i \leq |C_i|$);*
(ii) *if V_i is of type C, then $d_i/6 \leq |C_i|$.*

In V_i, every connected component belonging to A_i is separated from every connected component belonging to B_i by vertices in C_i. There are $w(A_i)d_i$ (resp. $w(B_i)d_i$) special boundary vertices in V_i, which belong to A_i (resp. B_i). It can be shown by an easy case analysis that the number of separating points $|C_i| \geq \min\{w(A_i), w(B_i)\}d_i$, and Claim 3.6.4 follows (see Fig. 3.2.).

In order to establish Theorem 3.6.2 (and hence Theorem 3.1.6), it remains to prove the following statement.

Claim 3.6.5. *The total number of edges between V_A to V_B satisfies*

$$|E(V_A, V_B)| \leq 150(1 + g^{3/4})\sqrt{\mathrm{CR}_g(G) + \sum_{i=1}^{n} d_i^2}.$$

To see this, denote by E_0 the set of all edges of H adjacent to at least one element of C_0. For any $1 \le i \le n$, define $E_i \subset E(H)$ as follows. If V_i is of type A (resp. type B), let E_i consist of all edges leaving V_i and adjacent to a special boundary vertex belonging to B_i (resp. A_i). If V_i is of type C, let all edges leaving V_i belong to E_i.

For any $1 \le i \le n$, let E_i' denote the set of edges of G corresponding to the elements of E_i $(0 \le i \le n)$. Clearly, we have $|E_i'| \le |E_i|$, because distinct edges of G give rise to distinct edges of H. It is easy to see that every edge between V_A and V_B belongs to $\cup_{i=0}^n E_i'$.

Obviously, $|E_0'| \le |E_0| \le 4|C_0|$. By Claim 3.6.4, if V_i is of type A or of type B, then $|E_i'| \le |E_i| \le |C_i|$. If V_i is of type C, then $|E_i'| \le |E_i| = d_i \le 6|C_i|$. Therefore,

$$|E(V_A, V_B)| \le |\cup_{i=0}^n E_i'| \le \sum_{i=0}^n |E_i| \le 6|C| \le 150(1 + g^{3/4})\sqrt{\mathrm{CR}_g(G) + \sum_{i=1}^n d_i^2}.$$

This concludes the proof of Claim 3.6.5 and hence Theorem 3.6.2 and Theorem 3.1.6. □

Acknowledgement. We would like to express our gratitude to Zoltán Szabó for his help in writing Section 3.5, and to László Székely for many very useful remarks.

Chapter 4

Which crossing number is it, anyway?

This chapter is based on the papers [PT98], [T06] and part of [PT00a].

A *drawing* of a graph G is a mapping which assigns to each vertex a point of the plane and to each edge a simple continuous arc connecting the corresponding two points. The crossing number $\text{CR}(G)$ of a graph G is the minimum possible number of edge-crossings in a drawing of G, the pair-crossing number $\text{PAIR-CR}(G)$ is the minimum possible number of crossing pairs of edges in a drawing of G, and the odd-crossing number $\text{ODD-CR}(G)$ is the minimum number of pairs of edges that cross an odd number of times. Clearly, $\text{ODD-CR}(G) \leq \text{PAIR-CR}(G) \leq \text{CR}(G)$.

We prove that the largest of these numbers (the crossing number) cannot exceed twice the square of the smallest (the odd-crossing number). Our proof is based on the following generalization of an old result of Hanani, which is of independent interest. Let G be a graph and let E_0 be a subset of its edges such that there is a drawing of G, in which every edge belonging to E_0 crosses any other edge an even number of times. Then G can be redrawn so that the elements of E_0 are not involved in any crossing.

We prove a better inequality for the crossing number in terms of the pair-crossing number; slightly improving the bound of Valtr, we show that if the pair-crossing number of G is k, then its crossing number is at most $O(k^2/\log^2 k)$.

We construct graphs with $0.855\text{PAIR-CR}(G) \geq \text{ODD-CR}(G)$. This improves the bound of Schaefer and Štefankovič.

We show that the determination of each of these parameters is an NP-hard problem and it is NP-complete in the case of the crossing number and the odd-crossing number.

Finally, we introduce even more variants of the crossing number prove some inequalities and pose some open questions.

4.1 Introduction

The crossing number of a graph G is usually defined as "the minimum number of edge crossings in any drawing of G in the plane" [BL84]. However, one has to be careful with this definition, because it can be interpreted in several ways. Sometimes it is assumed that in a proper drawing no two edges cross more than once, and if two edges share an endpoint, they cannot have another point in common ([WB78], [B91]). Many authors do not make this assumption ([T70], [GJ83], [SSSV97]). If two edges are allowed to cross several times, we may count their intersections with multiplicity or without. We may also wish to impose some further restrictions on the drawings (e.g., the edges must be straight-line segments [J71], or polygonal paths of length at most k [BD93]). No matter what definition we use, the determination of the crossing number of a graph appears to be an extremely difficult task ([GJ83], [B91]). In fact, we do not even know the asymptotic value of *any* of the above quantities for the complete graph K_n with n vertices and for the complete bipartite graph $K_{n,n}$ with $2n$ vertices, as n tends to infinity [RT97]. The latter question, raised more than fifty years ago, is often referred to as Turán's Brick Factory Problem [T77] or as Zarankiewicz's problem [G69].

In the present paper, we investigate the relationship between various crossing numbers. First we agree on the terminology.

A *drawing* of a simple undirected graph is a mapping f that assigns to each vertex a distinct point in the plane and to each edge uv a continuous arc (i.e., a homeomorphic image of a closed interval) connecting $f(u)$ and $f(v)$, not passing through the image of any other vertex. For simplicity, the arc assigned to uv is called an *edge* of the drawing, and if this leads to no confusion, it is also denoted by uv. We assume that no three edges have an interior point in common, and if two edges share an interior point p, then they cross at p. We also assume that any two edges of a drawing have a only a finite number of *crossings* (common interior points). A common endpoint of two edges does not count as a crossing.

Definition. Let G be a simple undirected graph.

(i) The *rectilinear crossing number* of G, LIN-CR(G), is the minimum number of crossings in any drawing of G, in which every edge is represented by a straight-line segment.

(ii) The *crossing number* of G, CR(G), is the minimum number of edge crossings in any drawing of G.

(iii) The *pairwise crossing number* of G, PAIR-CR(G), is the minimum number of pairs of edges (e, e') such that e and e' determine at least one crossing, over all drawings of G. (That is, now crossings are counted *without* multiplicities.)

(iv) The *odd-crossing number* of G, ODD-CR(G), is the minimum number of pairs of edges (e, e') such that e and e' cross an odd number of times.

Clearly, we have

$$\text{ODD-CR}(G) \leq \text{PAIR-CR}(G) \leq \text{CR}(G) \leq \text{LIN-CR}(G),$$

It was shown by Bienstock and Dean [BD93] that there are graphs with crossing number 4, whose rectilinear crossing numbers are arbitrarily large. On the other hand, we cannot rule out the possibility that

$$\text{ODD-CR}(G) = \text{PAIR-CR}(G) = \text{CR}(G)$$

for every graph G. The only result in this direction is the following remarkable theorem of Hanani and Tutte (see also [LPS97]).

Theorem A. [Ch34], [T70] *If a graph G can be drawn in the plane so that any two edges which do not share an endpoint cross an even number of times, then G is planar.*

For a generalization of this result to other surfaces, see [CN99].

In a fixed drawing of a graph G, an edge is called *even* if it crosses every other edge an *even* number of times. It follows from Theorem A that if all edges of G are even, i.e., if ODD-CR(G) = 0, then CR(G) = 0. (In this case, by Fáry's theorem [F48], we also have LIN-CR(G) = 0.) In the next section, we establish the following generalization of this statement.

Theorem 4.1.1. *For a fixed drawing of a graph G, let $G_0 \subseteq G$ denote the subgraph formed by all even edges.*

Then G can be drawn in such a way that the edges belonging to G_0 are not involved in any crossing.

At the end of the next section, we show how Theorem 4.1.1 implies that if the odd-crossing number of a graph is bounded, then its crossing number cannot be arbitrarily large. More precisely, we prove

Theorem 4.1.2. *The crossing number of any graph G satisfies*

$$\textsc{cr}(G) \leq 2\left(\textsc{odd-cr}(G)\right)^2.$$

Since $\textsc{pair-cr}(G) \geq \textsc{odd-cr}(G)$ for every graph G, it follows from Theorem 4.1.2 that for any G, if $\textsc{pair-cr}(G) = k$, then $\textsc{cr}(G) \leq 2k^2$. Valtr [V05] managed to improve this bound to $\textsc{cr}(G) \leq 2k^2/\log k$. Based on the ideas of Valtr, we give a further little improvement.

Theorem 4.1.3. *For any graph G, if $\textsc{pair-cr}(G) = k$, then*

$$\textsc{cr}(G) \leq 9k^2/\log^2 k.$$

Theorem 4.1.2 states that if $\textsc{odd-cr}(G) = k$, then $\textsc{cr}(G) \leq 2k^2$ and this is the best known bound. (Obviously it follows that $\textsc{pair-cr}(G) \leq 2k^2$ and this is also the best known bound.) On the other hand, Pelsmajer, Schaefer and Štefankovič [PSS06] proved that $\textsc{odd-cr}(G)$ and $\textsc{pair-cr}(G)$ are not necessarity equal, they constructed a series of graphs with $\textsc{odd-cr}(G) \leq (\frac{\sqrt{3}}{2} + o(1)) \cdot \textsc{pair-cr}(G)$. We slightly improve their bound with a completely different construction.

Theorem 4.1.4. *There is a series of graphs G with*

$$\textsc{odd-cr}(G) < \left(\frac{3\sqrt{5}}{2} - \frac{5}{2} + o(1)\right) \cdot \textsc{pair-cr}(G).$$

.

Since $\textsc{pair-cr}(G) \leq \textsc{cr}(G)$, Theorem 4.1.4 holds also for $\textsc{cr}(G)$ instead of $\textsc{pair-cr}(G)$. Moreover, the whole argument works, without any change.

It was discovered by Leighton [L84] that the crossing number can be used to obtain a lower bound on the chip area required for the VLSI circuit layout of a graph. For this purpose, he proved the following general lower bound for $\textsc{cr}(G)$, which was discovered independently by Ajtai, Chvátal, Newborn, and Szemerédi. The best known constant, 1/33.75, in the theorem is due to Pach and Tóth.

Theorem B. [ACNS82], [L84], [PT97] *Let G be a graph with vertex set $V(G)$ and edge set $E(G)$ such that $|E(G)| \geq 7.5|V(G)|$. Then we have*

$$\textsc{cr}(G) \geq \frac{1}{33.75} \frac{|E(G)|^3}{|V(G)|^2}.$$

In Section 4.5, we prove that a similar inequality holds for the odd-crossing number.

Theorem 4.1.5. *Let G be a graph with vertex set $V(G)$ and edge set $E(G)$ such that $|E(G)| \geq 4|V(G)|$. Then we have*

$$\textsc{odd-cr}(G) \geq \frac{1}{64} \frac{|E(G)|^3}{|V(G)|^2}.$$

It was shown by Garey and Johnson [GJ83] that, given a graph G and an integer K, it is an NP-complete problem to decide whether $\textsc{cr}(G) \leq K$. In the last section we show that the same is true for the odd-crossing number.

Theorem 4.1.6. *Given a graph G and an integer K, it is an NP-complete problem to decide whether $\textsc{odd-cr}(G) \leq K$.*

We can not prove the same for the pair-crossing number. (See Remark at the end of Section 4.1.6.)

4.2 Proofs of Theorems 4.1.1 and 4.1.2

First we establish Theorem 4.1.1. The proof somewhat resembles a proof of Kuratowski's theorem (see [BM76]).

Suppose that Theorem 4.1.1 is false. Then there exists a graph G with vertex set $V(G) = V$ and edge set $E(G) = E$, and there is a subset $E_0 \subseteq E$ such that G has a drawing, in which every edge in E_0 is even, but there is no drawing, in which none of these edges is involved in any crossing. Let us fix a *minimal counterexample* to Theorem 4.1.1, i.e., a pair (G, E_0) such that there exists no other pair $(\overline{G}, \overline{E}_0)$, $\overline{E}_0 \subseteq \overline{E}$, with the above property, for which the triple $(|\overline{E}|, |\overline{E}_0|, |\overline{V}|)$ would precede $(|E|, |E_0|, |V|)$ in the lexicographic ordering. In particular, it follows from the minimality of (G, E_0) that G is connected.

If it leads to no confusion, throughout this section G will stand both for the graph and for a particular drawing, in which all edges of E_0 are even. Let $G_0 = (V, E_0)$. A path (resp. cycle) in G is said to be an E_0-*path* (resp. E_0-*cycle*), if all of its edges belong to E_0. Two edges are called *independent*, if they do not share an endpoint.

Claim 4.2.1. *G and $G_0 = (V, E_0)$ satisfy the following properties.*
(i) *There is no vertex of degree 1 in G_0.*
(ii) *There are no two adjacent vertices of degree* 2 *in G_0.*
(iii) *In any subdivision of K_5 or $K_{3,3}$ contained in G, there are two paths representing independent edges, such that neither of them is an E_0-path.*

Proof. If v has degree 1 in $G_0 = (V, E_0)$, and $uv \in E_0$, then $(G, E_0 \setminus \{uv\})$ is another counterexample, (lexicographically) smaller than (G, E_0). If u, v both have degree 2 in G_0 and $uv \in E_0$, then contract the edge uv and remove all multiple edges (that is, keep only one copy of each edge), to obtain a smaller counterexample. Finally, part (iii) is an immediate corollary to Theorem A. □

Let C be any E_0-cycle of G. A connected subgraph $B \subset G$ is a *bridge* of C (in G) if it consists of either a single edge whose endpoints belong to $V(C)$, or of a connected component of $G - V(C)$ together with all edges connecting it to C. The endpoints of these edges in C are called the *endpoints of* bridge B. (See also [BM76].) In the following, $P(x, y)$ will always denote a path in G between two vertices, x and y.

Claim 4.2.2. *G contains an E_0-cycle which has at least two bridges.*

Proof. First we show that there is an E_0-cycle with a chord which is either a single E_0-edge or an E_0-path of length two.

Delete all isolated vertices of G_0. For every vertex v, which is adjacent to exactly two vertices, u and w, in G_0, replace uv, vw, and v with the single edge uw. Call the resulting multigraph $\widehat{G}_0$. By Claim 4.2.1, the degree of every vertex of $\widehat{G}_0$ is at least 3.

Let $P = x_0 x_1 \ldots x_m$ be a longest path in $\widehat{G}_0$. Vertex x_0 has at least 3 neighbors, and, by the maximality of the path, all of them are on P. Hence, for some $1 < i < j$, $x_0 x_i$ and $x_0 x_j$ are edges of $\widehat{G}_0$. Then $x_0 x_1 \ldots x_j$ is a cycle with chord $x_0 x_i$ in $\widehat{G}_0$. Since every edge of $\widehat{G}_0$ arose from either an edge or a path of length two in G_0, the corresponding edges of G_0 form a cycle C with a chord c which is either a single edge or an E_0-path of length 2.

If C has at least two bridges, then we are done. Assume it has only one bridge, B. Now c is not a single edge, otherwise B would be identical with c, and $G = G_0 = C \cup c$ is *not* a counterexample. Therefore, we can assume that c is an E_0-path xvy of length 2.

The points x and y divide C into two complementary paths (arcs). If two vertices of C, a and b (different from x and y) do not belong to the same arc, we say that the pair $\{x, y\}$ *separates* a from b on C. Equivalently, the pair $\{a, b\}$ separates x from y.

We distinguish three cases.

CASE 1. *B has no two endpoints separated by the pair $\{x, y\}$.*

Let $P(x, y)$ denote the arc of C containing no endpoint of B in its interior. Let G' be the graph obtained from G by replacing $P(x, y)$ with a single edge xy, and let $E_0' = E_0 \cup \{xy\}$. It is easy to see that (G', E_0') is also a counterexample. By the minimality of (G, E_0), we have that $G = G'$, i.e., $P(x, y)$ is a single edge $xy \in E_0$.

Swapping xy with the chord xvy, we obtain an E_0-cycle C' with a chord xy. Therefore, C' has at least two bridges, and Claim 4.2.2 is true.

CASE 2. *There is a path $P(a, b) \subset B$, not passing through v, which connects two points, a and $b \in V(C)$, separated by the pair $\{x, y\}$.*

Since v and $P(a, b)$ belong to the same bridge, there is a path $P(v, q) \subset B$ connecting v to an interior point q of $P(a, b)$. Then G contains a subdivision of $K_{3,3}$ with vertex classes $\{x, y, q\}$ and $\{a, b, v\}$. Moreover, all paths representing the edges of $K_{3,3}$ belong to E_0, with the possible exceptions of those adjacent to q. This contradicts Claim 4.2.1 (iii), which shows that this case cannot occur.

CASE 3. *Every path in B, whose endpoints are separated on C by the pair $\{x, y\}$, passes through v.*

Let $P_1(x, y)$ and $P_2(x, y)$ denote the two complementary arcs of C, and let B_i be the union of all paths in B, which connect an internal point of $P_i(x, y)$ to x, v, or y.

Suppose first that $B = B_1 \cup B_2$. Then, by the minimality (G, E_0), $G - B_i$, for $i = 1, 2$, has a drawing where no edge belonging to E_0 is involved in any crossing. In particular, in this drawing, xvy and the edges of C are not crossed by any edge, so we can assume that all curves representing the edges of B_i lie in the region bounded by $P_i(x, y)$ and xvy $(i = 1, 2)$. Redrawing $G - B_2$, if necessary, so that C and xvy are mapped to exactly the same curves as in the drawing of $G - B_1$, the two drawings can be combined to give a drawing of G, contradicting our assumption that (G, E_0) is a counterexample.

We are left with the case when $B \neq B_1 \cup B_2$. Then there is a vertex s of B which can not be reached from any internal point of $P_i(x, y)$ without passing through x, v, or y $(i = 1, 2)$. Swapping $P_1(x, y)$ with xvy, we obtain an E_0-cycle C' with a chord $P_1(x, y)$, which can be arbitrarily long. C' has at least two bridges, because $P_1(x, y)$ and s do not be in the same bridge. $\square$

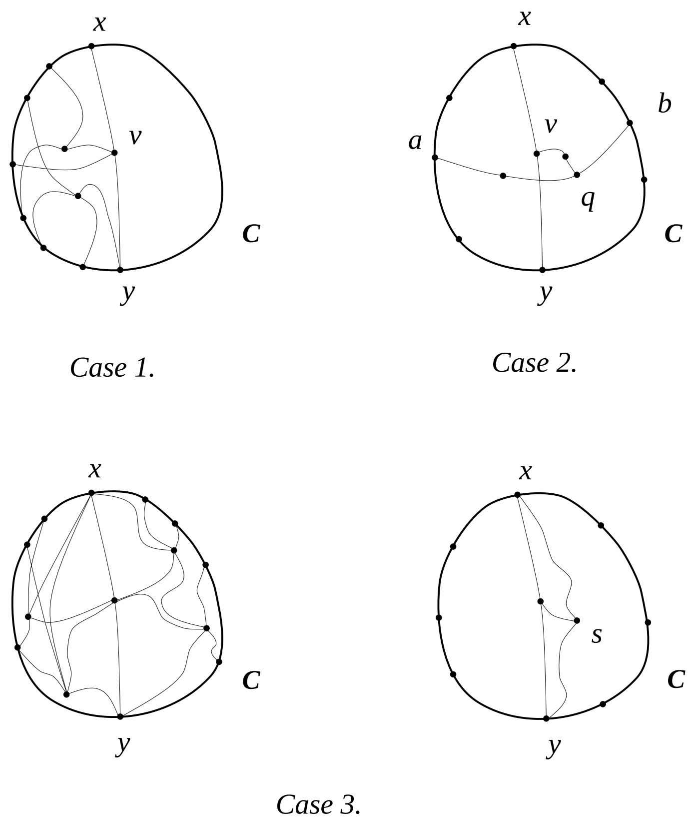

Figure 4.1: Proof of Claim 4.2.2

In the sequel, let C denote a fixed E_0-cycle of G which has at least two bridges.

Claim 4.2.3. *C has at least three bridges.*

Proof. Suppose there are only two bridges of C, B_1 and B_2. By the minimality of G, $G - B_1$ (resp. $G - B_2$) can be drawn in the plane so that none of its edges belonging to E_0 is involved in any crossing. In particular, in this drawing none of the edges of C is involved in any crossing, therefore B_2 (resp. B_1) lies entirely on one side of C, say, in its interior (resp. exterior). But then we can combine the two drawings and get a drawing of G. It is a contradiction since G is assumed to be a counterexample. $\square$

Let B_1 and B_2 be two bridges of C. By the minimality of (G, E_0), the graph $C \cup B_1 \cup B_2$ can be drawn in the plane so that none of its edges belonging to E_0 participates in any crossing. If in all such drawings B_1 and B_2 are on different sides of C, then B_1 and B_2 are said to be *conflicting*.

Claim 4.2.4. *C has exactly three bridges, at least one of which is a single edge.*

Proof. Construct a graph Γ whose vertices correspond to the bridges of C, and two vertices are connected by an edge if and only if the corresponding bridges are conflicting. By the minimality of (G, E_0), after the removal of any bridge the remaining graph can be drawn in the plane so that none of its edges belonging to E_0 is involved in any crossing. In other words, if we delete any vertex of Γ , it becomes two-colorable (the two colors correspond to the bridges inside and outside C). Therefore, any odd cycle of Γ passes through every vertex of Γ, hence Γ itself is an odd cycle.

Fix now any drawing of G, in which all edges belonging to E_0 are even. The closed curve representing C divides the plane into connected cells. Color them with black and white so that no two cells that share a boundary arc receive the same color.

Let B_i be a bridge of C. We need the following observation, which is an immediate consequence of the fact that every edge of B_i crosses all edges of C an even number of times. Assume that in a small neighborhood of one of its endpoints some edge of B_i runs in the black (white) region. Then *every* edge of B_i is black (resp. white) in a sufficiently small neighborhood of *both* of its endpoints. In this case, B_i is said to be a *black* (resp. *white*) *bridge*. Every non-endpoint of a black (white) bridge must lie in the black (resp. white) region.

Since Γ is an odd cycle, it has two consecutive vertices such that the corresponding bridges, say, B_1 and B_2, are conflicting and they are of the same color, say, black. We will specify two edges, $b_1 \in E(B_1)$ and $b_2 \in E(B_2)$. We distinguish two cases.

Suppose first that B_1 and B_2 have a common endpoint v. In a small neighborhood of v, all edges of B_1 and B_2 emanating from v are disjoint and run in the black region. Therefore, we can find two consecutive edges, b_1 and b_2, in the cyclic order around v such that $b_i \in B_i,\ i = 1, 2$. In this case, set $w_1 = w_2 = v$.

Suppose next that B_1 and B_2 do not have a common endpoint. Let $v_i v_{i+1} \dots v_j$ be a piece of C such that v_i is an endpoint of B_1, v_j is an endpoint of B_2, and no v_k $(i < k < j)$ is an endpoint of either B_1 or B_2. There may be several edges of B_1 adjacent to v_i, which lie in the black region in a small neighborhood of v_i; let b_1 denote the *last* one in the cyclic order from the initial piece of $v_i v_{i-1}$ to that of $v_i v_{i+1}$. Similarly, let b_2 denote the *first* edge of B_2 emanating from v_j in the cyclic order from the initial piece of $v_j v_{j-1}$ to that of $v_j v_{j+1}$. Now set $w_1 = v_i$ and $w_2 = v_j$.

Consider the drawing of $C \cup B_1 \cup B_2$ inherited from the original drawing of G. In this drawing, all edges belonging to $E_0 \cap (E(C) \cup E(B_1) \cup E(B_2))$ are even. We distinguish three cases depending on whether B_1 and B_2 are single edges, and in each case we slightly modify the graph $C \cup B_1 \cup B_2$ and its drawing. The modified graph and its drawing will be denoted by $\overline{G} = (\overline{V}, \overline{E})$, and we will also specify a set of edges $\overline{E}_0 \subseteq \overline{E}$.

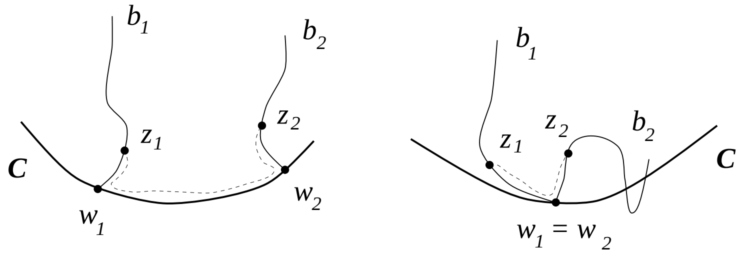

Figure 4.2: Proof of Claim 4.2.4 Case 1

Case 1. *Both B_1 and B_2 are single edges.*

Then $E(B_i) = \{b_i\} = \{w_i u_i\},\ i = 1, 2$. Split b_i into two edges by adding an extra vertex z_i very close to w_i, $i = 1, 2$. Connect z_1 and z_2 by an edge running very close to the path $z_1 w_1 ... w_2 z_2$, but not intersecting it (see Fig. 4.2), and denote the resulting graph drawing by $\overline{G}$. Since b_1 and b_2 are conflicting, at least one of them (say, b_1) belongs to E_0. Then set $\overline{E}_0 = E(C) \cup \{w_1 z_1, z_1 u_1\}$.

CASE 2. *B_1 is a single edge, B_2 is not.*

Then $E(B_1) = \{b_1\} = \{w_1u_1\}$, $E(B_2) \supset \{b_2\} = \{w_2z_2\}$, where $u_1 \in V(C)$ and $z_2 \notin V(C)$. Split b_1 into two edges by adding a vertex z_1 very close to w_1. As before, connect z_1 and z_2 by an edge running very close to the path $z_1w_1...w_2z_2$, and denote the resulting graph drawing by $\overline{G}$. If $b_1 \in E_0$ then set $\overline{E}_0 = E(C) \cup \{w_1z_1, z_1u_1\}$. Otherwise, let $\overline{E}_0 = E_0 \cap (E(C) \cup E(B_2))$, i.e., we leave the set of specified edges unchanged.

CASE 3. *Neither B_1 nor B_2 is a single edge.*

Then $E(B_i) \supset \{b_i\} = \{w_iz_i\}$, where $z_i \notin V(C)$, for $i = 1, 2$. Connect z_1 and z_2 by an edge running very close to the path $z_1w_1...w_2z_2$, and denote the resulting graph drawing by $\overline{G}$. As in the previous case, let us leave the set of specified edges unchanged, i.e., set $\overline{E}_0 = E_0 \cap (E(C) \cup E(B_1) \cup E(B_2))$.

It follows from the construction that in the above drawing of $\overline{G}$, every edge belonging to $\overline{E}_0$ is even. Recall that B_1 and B_2 were conflicting (see the last paragraph before Claim 4.2.4), which implies that in every drawing of $\overline{G}$ with the property that no edge in $\overline{E}_0$ is involved in any crossing, z_1 and z_2 lie on different sides of C. However, $z_1z_2 \in E(\overline{G}) = \overline{E}$, proving that $(\overline{G}, \overline{E}_0)$ is also a counterexample to Theorem 4.1.1.

Suppose, to obtain a contradiction, that C has more than three bridges in G. Since Γ is an odd cycle, the number of bridges is odd, i.e., C has at least five bridges. In the construction of $\overline{G}$, we kept only two of these bridges, so we deleted at least three bridges, hence at least three edges. In Cases 1 and 2, we added at most two new edges. Thus, in these cases, $|E(\overline{G})| = |\overline{E}| < |E|$, contradicting our assumption that (G, E_0) is a minimal counterexample.

The only remaining possibility is that C has exactly five bridges, all of which are single edges. It follows from the structure of Γ that at least three of these bridges (edges) belong to E_0. On the other hand, $\overline{G}$ has only two edges not in C that belong to $\overline{E}_0$. Thus, in this case, $|\overline{E}| = |E|$, but $|\overline{E}_0| < |E_0|$. This again contradicts the minimality of our counterexample.

Therefore, we can assume that C has exactly three bridges in G, B_1, B_2, and B_3. If none of them is a single edge, then we can add one edge (as in Case 3) and delete a bridge, which contains more than one edge, to obtain a counterexample smaller than (G, E_0). $\square$

Claim 4.2.5. *C has at least two bridges which are single edges.*

Proof. Assume, to obtain a contradiction, that C has only one bridge which consists of a single edge. Take a closer look at the transformation in the proof of Claim 4.2.4. By deleting B_3 and adding one, two, or three edges, we obtained another counterexample $(\overline{G}, \overline{E_0})$.

If B_1 or B_2 was the bridge consisting of a single edge, then we added two edges (cf. Case 2 in the proof of Claim 4.2.4) and deleted B_3, which had at least three edges. This contradicts the assumption that (G, E_0) was a minimal counterexample.

Therefore, we can assume that B_3 consists of a single edge xy. Then, during the above transformation we deleted B_3 and added an edge that does not belong to $\overline{E_0}$ (cf. Case 3). Therefore, using the minimality of (G, E_0) again, we obtain that $xy \notin E_0$.

Since B_1 and B_3 are conflicting, it follows that there is an E_0-path $P(a, b) \subset B_1$ whose endpoints, a and b, separate x and y on C. Let $P_x(a, b)$ and $P_y(a, b)$ denote the two complementary arcs of C between a and b, containing x and y, respectively.

We distinguish two cases.

CASE 1. *All endpoints of B_2 belong to the same arc, $P_x(a, b)$ or $P_y(a, b)$.*

By symmetry, we can assume that all endpoints of B_2 are on $P_x(a, b)$. Then all endpoints of B_1 must also belong to $P_x(a, b)$. Indeed, if an endpoint of B_1 did not lie on this arc, then we could delete all edges of B_1 adjacent to it and obtain a smaller counterexample.

Consider the graph $\overline{G}$ constructed in the proof of Claim 4.2.4. In this graph, y is adjacent to only two vertices, y' and y'', both of which belong to C. Let G' denote the graph obtained from $\overline{G}$ by deleting y and replacing the E_0-path $y'yy''$ by a single edge $y'y''$. Set $E'_0 = E_0 \setminus \{yy', yy''\} \cup \{y'y''\}$. Clearly, (G', E'_0) is a counterexample to Theorem 4.1.1, which precedes (G, E_0), contradicting the minimality of (G, E_0).

CASE 2. *There exists a path $P(p, q) \subseteq B_2$ such that p and q are interior points of $P_x(a, b)$ and $P_y(a, b)$, respectively.*

Consider again the graph $\overline{G}$. Clearly, B_1 contains a path connecting b_1 to some internal point r of $P(a, b)$. (Note that r may be an endpoint of b_1. Moreover, b_1 may belong to $P(a, b)$.) Similarly, B_2 contains a path connecting b_2 to some internal point s of $P(p, q)$. However, in this case, $\overline{G}$ contains a subdivision of $K_{3,3}$ with vertex classes $\{a, b, s\}$ and $\{p, q, r\}$. Furthermore, with the exception of the paths incident to s, all paths representing the edges of $K_{3,3}$ belong to E_0. However, this contradicts Claim 4.2.1 (iii). □

Now we can complete the proof of Theorem 4.1.1. By Claims 4.2.4 and 4.2.5, C has precisely three pairwise conflicting bridges B_i, $(i = 1, 2, 3)$ in G.

Two of them, say, B_1 and B_2, are single edges, xy and ab, respectively. Since B_1 and B_2 are conflicting, at least one of them, say xy, is in E_0.

Using the fact that B_3 is in conflict with $xy \in E_0$, we obtain that it contains a path connecting a pair of points $\{p, q\} \subset V(C)$ which separates x from y. Similarly, since B_3 is in conflict with ab, it also contains a path connecting a pair of points $\{p', q'\} \subset V(C)$ which separates a from b, and this path belongs to E_0 unless $ab \in E_0$. According to the position of these paths, we can distinguish four different cases up to symmetry (see Fig. 4.2). $P(p, q)$ always stands for a path connecting p and q, whose internal vertices do not belong to C.

CASE 1. *B_3 contains a path $P(p, q)$; $p, q \in V(C)$, such that the pair $\{p, q\}$ separates a from b and x from y, and ab or $P(p, q)$ belongs to E_0.*

Then G has a subdivision of $K_{3,3}$ with vertex classes $\{a, p, y\}$ and $\{b, q, x\}$. Moreover, with the exception of ab or $P(p, q)$, all paths representing the edges of $K_{3,3}$ belong to E_0. This contradicts Claim 4.2.1 (iii).

CASE 2. *B_3 contains three internally disjoint paths, $P(a, r)$, $P(p, r)$ and $P(q, r)$, such that r does not belong to C; the pair $\{p, q\}$ separates b from the set $\{a, x, y\}$; and ab or $P(p, r) \cup P(q, r)$ belongs to E_0.*

Then G *properly* contains a subdivision of $K_{3,3}$ with vertex classes $\{x, r, b\}$, and $\{a, p, q\}$. It is easy to see that deleting from G the arc of C between a and y which does not contain $\{x, p, b, q\}$, we obtain a smaller counterexample. Thus, this case cannot occur.

CASE 3. *B_3 contains three internally disjoint paths, $P(p, r)$, $P(q, r)$, and $P(y, r)$, such that r does not belong to C; the pair $\{p, q\}$ separates x from the set $\{a, b, y\}$; and at least one of ab, $P(p, r) \cup P(y, r)$ and $P(q, r) \cup P(y, r)$ belongs to E_0.*

Then G *properly* contains a subdivision of $K_{3,3}$ with vertex classes $\{x, r, b\}$, and $\{y, p, q\}$. If ab belongs to E_0, then deleting from G the arc of C between a and y which does not contain $\{p, x, q, b\}$, we obtain a smaller counterexample. If ab does not belong to E_0, but, say, $P(p, r) \cup P(y, r)$ does, then, by the minimality of (G, E_0), all paths depicted in Fig. 4.2 (3) are single edges, and G has no further edges. However, this case cannot occur, because here b and q are two adjacent vertices of degree 2 in G_0, contradicting Claim 4.2.1 (ii).

CASE 4. *The endpoints of B_3 are a, b, x, y.*

Since B_2, and B_3 are conficting, B_3 contains two intersecting paths,

$P(a,b)$ and $P(x,y)$, such that either ab or $P(x,y)$ belongs to E_0. It follows from the minimality of our counterexample that $P(a,b)$ and $P(x,y)$ have only one vertex in common. Denoting it with r, we can write $P(a,b) = P(a,r) \cup P(b,r)$ and $P(x,y) = P(x,r) \cup P(y,r)$. Then G contains a subdivision of K_5 induced by a,b,x,y,r. Moreover, with the exception of ab, $P(a,r)$, and $P(b,r)$, all paths representing the edges of $K_{3,3}$ belong to E_0. This contradicts Claim 4.2.1 (iii).

In each case, we arrived at a contradiction. Thus, there exists no (minimal) counterexample (G, E_0) to Theorem 4.1.1. The proof of Theorem 4.1.1 is complete. □

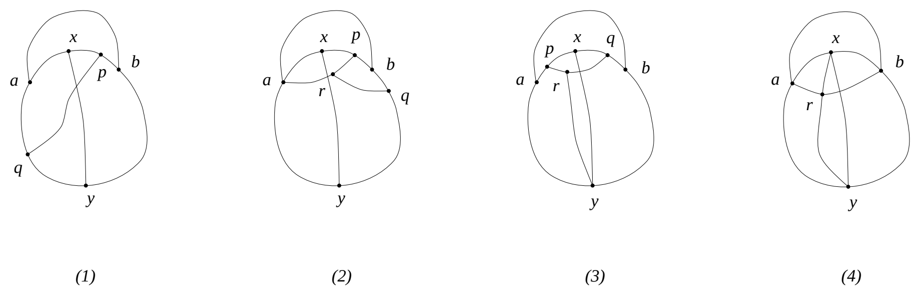

Figure 4.3: Cases 1–4 in the proof of Theorem 4.1.1

Theorem 4.1.2 is an easy corollary to Theorem 4.1.1. Let $G = (V,E)$ be a simple graph drawn in the plane with $\lambda = \textsc{odd-cr}(G)$ pairs of edges that cross an odd number of times. Let $E_0 \subset E$ denote the set of *even* edges in this drawing. Since every edge not in E_0 crosses at least one other edge an odd number of times, we obtain that

$$|E \setminus E_0| \le 2\lambda.$$

By Theorem 4.1.1, there exists a drawing of G, in which no edge of E_0 is involved in any crossing. Pick a drawing with this property such that the total number of crossing points between all pairs of edges not in E_0 is minimal. Notice that in this drawing, any two edges cross at most once. Therefore, the number of crossings is at most

$$\binom{|E \setminus E_0|}{2} \le \binom{2\lambda}{2} \le 2\lambda^2,$$

and Theorem 4.1.2 follows.

4.3 Proof of the Theorem 4.1.3

Let G be a graph, PAIR-CR$(G) = k$ and take a drawing of G which has exactly k crossing pairs of edges. Let t be a parameter, to be defined later. We distinguish three types of edges. An edge e is

good if it is not crossed by any other edge;

light if it is crossed by at least one and at most t other edges;

heavy if it is crossed by more than t other edges.

We will apply the following result of Schaefer and Štefankovič [SS04].

Lemma A. (Schaefer and Štefankovič, 2004) *Suppose that a graph is drawn in the plane, and edge e is crossed by m other edges. If there are at least 2^m crossings on e, then the drawing can be modified such that (i) the number of crossings between any two edges does not increase, and (ii) the number of crossings on e decreases.*

Return to the proof of Theorem 4.1.3. Suppose that there is a light edge that has at least 2^t crossings. Then we can modify the drawing according to the Lemma A. This modification does not increase the number of crossings on any edge and does not introduce new pairs of crossing edges. On the other hand, it decreases the total number of crossings, so after finitely many applications, all light edges have less than 2^t crossings.

Now we apply two other types of redrawing steps.

Suppose that in our drawing two heavy edges e and f cross at least twice and let u and v be two crossings. Then switch the uv segment of e and f. This way (i) we reduced the number of crossings between e and f and (ii) the total number of crossings on any other edge remains the same.

Observe that this way we could have introduced self-crossings, in this case remove the loop formed by the self-crossing edge. This way (i) the number of crossings on any edge does not increase, and (ii) the total number of crossings decreases.

Apply the above redrawing steps as long as there are two heavy edges that cross more than once or there is a self-crossing edge. Since the total number of crossings decreases in each step, after finitely many applications any two heavy edges will cross at most once and no edge crosses itself.

Now count the number of crossings for the drawing obtained. Originaly there were k pairs of crossing edges. A heavy edge crosses more than t other edges, so there are less than $2k/t$ heavy edges. The total number of

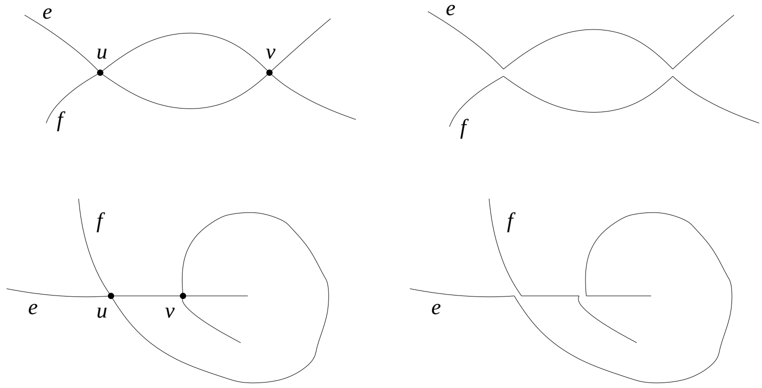

Figure 4.4: Switch the uv segment of e and f.

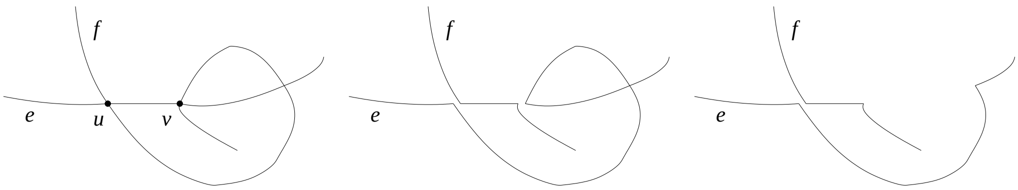

Figure 4.5: Switch the uv segment of e and f and remove the self-crossing.

light edges is at most $2k$. Each light edge has less than 2^t crossings, so the total number of crossings on the light edges is less than $2k2^t$. On the other hand, since any two heavy edges cross at most once, we have less than $\binom{2k/t}{2}$ heavy-heavy crossings. So, for the the total number of crossings C we have

$$\text{CR}(G) \leq C < k2^{t+1} + \binom{2k/t}{2} < k2^{t+1} + 2k^2/t^2.$$

Set $t = \log k/2$, we obtain $\text{CR}(G) < 9k^2/\log^2 k$. $\square$

4.4 Proof of Theorem 4.1.4

The idea and sketch of the construction.

In the descriprion we use weights on the edges of the graph. If we substitute each weighted edge by an appropriate number of parallel paths, say, each of length two, we can obtain an unweighted simple graph whose ratio of the pair-crossing and odd-crossing numbers is arbitrarily close to that of the weighted construction.

First of all, take a "frame" F, which is a cycle K with very heavy edges, together with a vertex V connected to all vertices of the cycle, also with very heavy edges. In the optimal drawings the edges of F do not participate in any crossing, and we can assume that V is drawn *outside* the cycle K. Therefore, all additional edges and vertices of the graph will be *inside* K.

We have four further vertices, each connected to three different vertices of the frame-cycle K. These three edges have weights 1, 1, w respectively, with some $1 < w < 2$. Each one of these four vertices, together with the adjacent three edges, and the frame F, is called a *component* of the construction.

If we take any *two* of the components, it is easy to see how to draw them optimally, both in the odd-crossing and pair-crossing sense. See Figure 4.6. The point is that if we take all four components, we can still draw them such that each of the six pairs are drawn optimally, in the odd-crossing sense. See Figure 4.7. On the other hand, it is easy to see that it is impossible to draw all six pairs optimally in the pair-crossing sense, some pairs will not have their best drawing. See Figure 4.8. Note that we did not indicate vertex V of the frame.

We get the best result with $w = \frac{\sqrt{5}+1}{2}$. Actually, we will see that among any *three* components there is a pair which is not drawn optimally in the pair-crossing sense. So, we could take the union of just three components, but that gives a weaker bound.

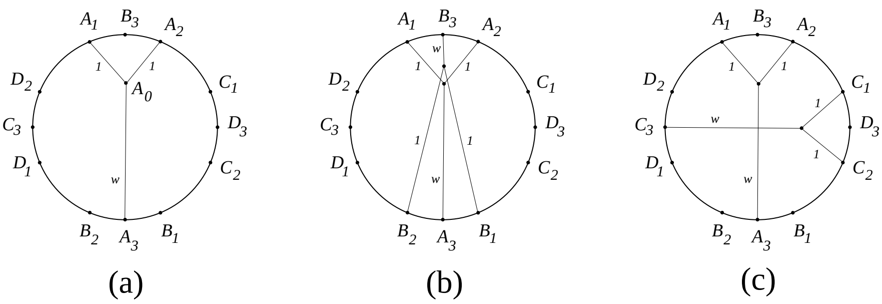

Figure 4.6: (a) Component A (b), (c) Optimal drawings of the pairs (A, B) and (A, C), resp.

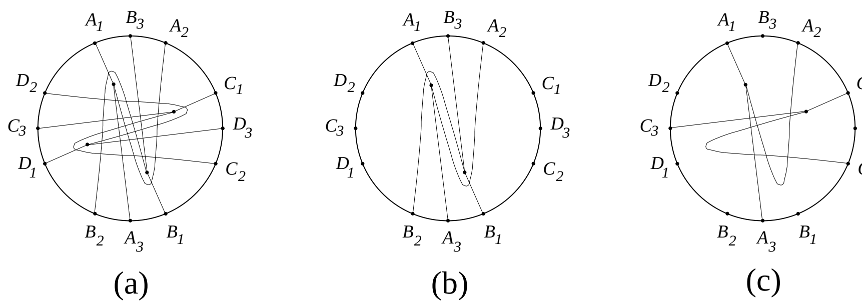

Figure 4.7: (a) Optimal drawing of G in the odd-crossing sense (b), (c) The pairs (A, B) and (A, C) resp. from the same drawing.

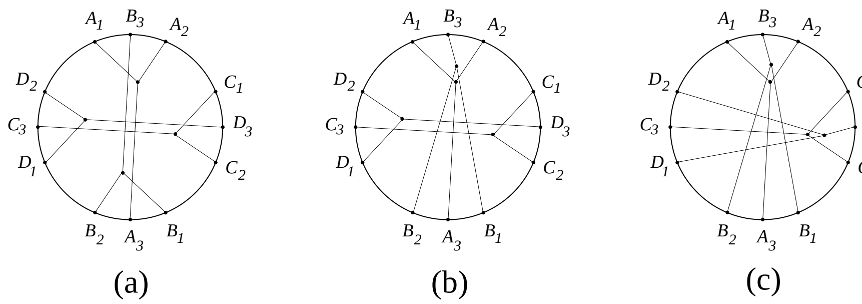

Figure 4.8: (a), (b) Cases 1 and 2 of Lemma 2, resp., optimal drawings of G in the pair-crossing sense (c) Case 3, not optimal drawing.

Proof of Theorem 4.1.4.

A *weighted graph* G is a graph with positive weights on its edges. For any edge e let $w(e)$ denote its weight. For any fixed drawing $\mathcal{G}$ of G, the *pair-crossing value* $\text{PAIR-CR}(\mathcal{G}) = \sum w(e)w(e')$ where the sum goes over all crossing pairs of edges e, e'. The *odd-crossing value* $\text{ODD-CR}(\mathcal{G}) = \sum w(e)w(e')$ where the sum goes over all pairs of edges e, e' that cross an odd number of times.

The pair-crossing number (resp. odd-crossing number) is the minimum of the pair-crossing value (resp. odd-crossing value) over all drawings. That is,

$$\text{PAIR-CR}(G) = \min_{\text{over all drawings}} \sum_{\substack{\text{for all crossing pairs}\\ \text{of edges } e, e'}} w(e)w(e'),$$

$$\text{ODD-CR}(G) = \min_{\text{over all drawings}} \sum_{\substack{\text{for all pairs of edges } e, e'\\ \text{that cross an odd number of times}}} w(e)w(e').$$

Theorem 4.4.1. *There exists a weighted graph G with* $\text{PAIR-CR}(G) = (\frac{3\sqrt{5}}{2} - \frac{5}{2}) \cdot \text{ODD-CR}(G)$.

Proof of Theorem 4.4.1. First we define the weighted graph G. Take nine vertices, A_1, B_3, A_2, C_1, D_3, C_2, B_1, A_3, B_2, D_1, C_3, D_2 which form cycle K in this order. Vertex V is connected to all of the nine vertices of K. These vertices and edges form the "frame" F. All edges of F have extremely large weights, therefore, they do not participate in any crossing in an optimal drawing. We can assume without loss of generality that V is drawn *outside* the cycle K, so all further edges and vertices of G will be *inside* K.

There are four more vertices, A_0, B_0, C_0, D_0, and for $X = A, B, C, D$, X_0 is connected to X_1, X_2, and X_3. The weight $w(X_0X_1) = w(X_0X_2) = 1$ and $w(X_0X_3) = w = \frac{\sqrt{5}+1}{2}$. Graph X is a subgraph of G, induced by the frame and X_0. See Figure 4.4.1. Finally, for any $X, Y = A, B, C, D$, $X \neq Y$, let $\text{PAIR-CR}(X, Y) = \text{PAIR-CR}(X \cup Y)$, and $\text{ODD-CR}(X, Y) = \text{ODD-CR}(X \cup Y)$.

First we find all these crossing numbers. Moreover, we also find out the second smallest pair-crossing values.

Start with $A \cup C$. Since the path $A_1B_3A_2$ is not intersected by any edge in an optimal drawing, we can contract it to one vertex, without changing the pair-crossing number, so now $A_1 = A_2$. Consider the edges $e_1 = A_1A_0$ and $e_2 = A_2A_0$. Now they connect the same vertices. Suppose that they do not go parallel in an optimal drawing. Let $w^*(e_1)$ (resp. $v^*(e_2)$) be the sum of the weights of the edges crossing e_1 (resp. e_2) and assume without loss of generality that $w^*(e_1) \leq w^*(e_2)$. Then draw e_2 parallel with e_1, the drawing obtained is at least as good as the original drawing was, so it is optimal as well. Therefore, we can assume without loss of generality that e_1 and e_2 go parallel in an optimal drawing, so we can substitute them by one edge of weight 2. Similarly, we can contract the path $C_1D_3C_2$ and substitute the edges C_1C_0 and C_2C_0 by one edge of weight 2. Now we have a very simple graph, whose pair-crossing number is immediate, we have two paths $C_1C_0C_3$ and $A_1A_0A_3$, which have to cross each other, and on both paths one edge has weight w the other one has weight 2. Since $w < 2$, in the optimal drawing the edges A_0A_3 and C_0C_3 will cross each other and no other edges cross so we have $\textsc{pair-cr}(A, C) = w^2$. Moreover, it is also clear that the second smallest pair-crossing value is $2w$.

The same argument holds for $\textsc{odd-cr}(A, C)$, moreover, by symmetry, we can argue exactly the same way for the pairs (A, D), (B, C), and (B, D).

Now we determine $\textsc{pair-cr}(A, B)$ and the second smallest pair-crossing value. The edges $a_1 = A_0A_1$, $a_2 = A_0A_2$, $a_3 = A_0A_3$ divide the interior of F into three regions R_1, R_2 and R_3. Number them in such a way that for $i = 1, 2, 3$, a_i is outside R_i. Once we place B_0 into one of these regions, it is clear how to draw the edges $b_1 = B_0BB_1$, $b_2 = B_0BB_2$, $b_3 = B_0BB_3$ to get the best of the possible drawings. If B_0 is in R_1 or in R_2, we get the pair-crossing value $2w$, but if we place B_0 in R_3, then we get 2. Again, the same argument holds for $\textsc{odd-cr}(A, B)$, and by symmetry, the situation is the same with the pair (C, D).

Lemma 4.4.2.

$$\textsc{odd-cr}(G) = 4w^2 + 4.$$

Proof of Lemma 4.4.2. We have $\textsc{odd-cr}(G) \geq \textsc{odd-cr}(A, B) + \textsc{odd-cr}$ (A, C) $+\textsc{odd-cr}(A, D) + \textsc{odd-cr}(B, C) + \textsc{odd-cr}(B, D) + \textsc{odd-cr}(C, D)$ $= 4w^2+4$, and there is a drawing (see Fig. 4.7) with exactly this odd-crossing value. □

Lemma 4.4.3.

$$\textsc{pair-cr}(G) = 4w^2 + 4w.$$

Proof of Lemma 4.4.3. The argument, except for the exact calculation, should be clear from the figures. While we have a drawing which is optimal for all six pairs in the odd-crossing sense (see Fig. 4.7), in the pair-crossing sense some of the pairs will not be optimal, they have to take at least the second smallest pair-crossing value. We start with an observation that in any *triple* at least one pair is not optimal. Then we will distinguish three cases.

Take a drawing $\mathcal{G}$ of G. Suppose that we have a drawing $\mathcal{G}$ of G where the pairs (A, C) and (A, D) are drawn optimally, that is, $\textsc{pair-cr}(\mathcal{A}, \mathcal{C}) = \textsc{pair-cr}(\mathcal{A}, \mathcal{D}) = w^2$. Recall that the edges $a_1 = A_0A_1$, $a_2 = A_0A_2$, $a_3 = A_0A_3$ divide the interior of F into three regions R_1, R_2 and R_3. It follows from the above argument that $C_0 \in R_1$, $D_0 \in R_2$. But then the pair (C, D) is not drawn optimally, that is, PAIR-CR$(\mathcal{C}, \mathcal{D}) > 2$, so we have PAIR-CR$(\mathcal{C}, \mathcal{D}) \geq 2w$. In other words, it is impossible that all three pairs (A, C), (A, D), (C, D) are drawn optimally at the same time. By symmetry, this observation holds for any triple of A, B, C, D.

We have to distinguish three cases.

Case 1. Neither (A, B), nor (C, D) are drawn optimally. In this case, PAIR-CR$(\mathcal{A}, \mathcal{B}) > 2$ so by the above argument we have PAIR-CR$(\mathcal{A}, \mathcal{B}) \geq 2w$, and similarly PAIR-CR$(\mathcal{C}, \mathcal{D}) \geq 2w$. For all other pairs we have pair-crossing value at least w^2, therefore, PAIR-CR$(\mathcal{G})$ = PAIR-CR$(\mathcal{A}, \mathcal{B})$ + PAIR-CR$(\mathcal{A}, \mathcal{C})$ + PAIR-CR$(\mathcal{A}, \mathcal{D})$+PAIR-CR$(\mathcal{B}, \mathcal{C})$+PAIR-CR$(\mathcal{B}, \mathcal{D})$+PAIR-CR$(\mathcal{C}, \mathcal{D}) \geq 4w^2+4w$.

Case 2. (A, B) is drawn optimally, (C, D) is not. Since (A, B) is drawn optimally, one of the pairs (A, C) and (B, C) and one of the pairs (A, D) and (B, D) is not drawn optimally so we have PAIR-CR$(\mathcal{A}, \mathcal{C})$ + PAIR-CR$(\mathcal{B}, \mathcal{C}) \geq w^2+2w$ and analogously PAIR-CR$(\mathcal{A}, \mathcal{D})$+PAIR-CR$(\mathcal{B}, \mathcal{D}) \geq w^2+2w$ therefore, PAIR-CR$(\mathcal{G})$ = PAIR-CR$(\mathcal{A}, \mathcal{B})$ +PAIR-CR$(\mathcal{A}, \mathcal{C})$ +PAIR-CR$(\mathcal{A}, \mathcal{D})$ +PAIR-CR $(\mathcal{B}, \mathcal{C})$ +PAIR-CR$(\mathcal{B}, \mathcal{D})$ +PAIR-CR$(\mathcal{C}, \mathcal{D}) \geq 2w^2 + 6w + 2 = 4w^2 + 4w$. The last equality can be verified by solving the quadratic equation.

Case 3. Both (A, B) and (C, D) are drawn optimally. If none of the other four pairs is optimal, then we have PAIR-CR$(\mathcal{G})$ = PAIR-CR$(\mathcal{A}, \mathcal{B})$ +PAIR-CR$(\mathcal{A}, \mathcal{C})$ +PAIR-CR$(\mathcal{A}, \mathcal{D})$ +PAIR-CR$(\mathcal{B}, \mathcal{C})$ +PAIR-CR$(\mathcal{B}, \mathcal{D})$ +PAIR-CR$(\mathcal{C}, \mathcal{D}) \geq 8w + 4 = 4w^2 + 4w$. So we can assume that one of them, say (A, C) is drawn optimally, that is, PAIR-CR$(\mathcal{A}, \mathcal{C}) = w^2$. Since in any triple we have at least one non-optimal pair, we have PAIR-CR$(\mathcal{B}, \mathcal{C}) \geq 2w$ and PAIR-CR$(\mathcal{A}, \mathcal{D}) \geq 2w$. We estimate PAIR-CR$(\mathcal{B}, \mathcal{D})$ now.

Again, the edges $a_1 = A_0A_1$, $a_2 = A_0A_2$, $a_3 = A_0A_3$ of A divide the interior of F into three regions R_1, R_2 and R_3 with R_i is the one to the opposite of a_i. Similarly define the regions Q_1, Q_2, Q_3 for C. Since (A, C) is drawn optimally, R_3 and Q_3 are disjoint. Since (A, B) is drawn optimally, $B_0 \in R_3$, and since (C, D) is also drawn optimally, $D_0 \in Q_3$. See Figure 4.8. Now it is not hard to see that the edge D_0D_1 either crosses A_0A_1, A_0A_2, and B_0B_3, or B_0B_1, B_0B_2, and A_0A_3. The same holds for the edge D_0D_1, so PAIR-CR$(\mathcal{A}, \mathcal{D})$ + PAIR-CR$(\mathcal{B}, \mathcal{D}) \geq 2w + 4$. So we have PAIR-CR$(\mathcal{G})$ = PAIR-CR$(\mathcal{A}, \mathcal{B})$ +PAIR-CR$(\mathcal{A}, \mathcal{C})$ +PAIR-CR$(\mathcal{A}, \mathcal{D})$ +PAIR-CR$(\mathcal{B}, \mathcal{C})$ +PAIR-CR $(\mathcal{B}, \mathcal{D})$ +PAIR-CR$(\mathcal{C}, \mathcal{D}) \geq w^2 + 4w + 8 > 4w^2 + 4w$. This concludes the proof of Lemma 4.4.3. $\square$.

Now we have

$$\frac{\text{ODD-CR}(G)}{\text{PAIR-CR}(G)} = \frac{4w^2+4}{4w^2+4w} = \frac{-5}{2} + \frac{3\sqrt{5}}{2},$$

and Theorem 4.4.1 follows immediately. □

Return to the proof of Theorem 4.1.4. Let $\varepsilon > 0$ an arbitrary small number. Let p and q be positive integers with the property that $w(1+\frac{\varepsilon}{10}) > \frac{p}{q} > w(1-\frac{\varepsilon}{10})$. Let G_ε be the following graph. In the weighted graph G of Lemma 4.4.3, (i) substitute each edge $e = XY$ of weight 1 with q paths between X and Y, each of length 2, (ii) substitute each edge $e = XY$ of weight w with p paths between X and Y, each of length 2, and (iii) substitute each edge $e = XY$ of the frame F with a huge number of paths between X and Y, each of length 2. Then

$$\frac{\text{ODD-CR}(G_\varepsilon)}{\text{PAIR-CR}(G_\varepsilon)} < \frac{\text{ODD-CR}(G)}{\text{PAIR-CR}(G)}(1+\varepsilon) < \frac{-5}{2} + \frac{3\sqrt{5}}{2} + \varepsilon.$$

4.5 Proof of Theorem 4.1.5

The proofs of Theorem B readily generalize to this case. We include a short argument, for completeness.

First, we show that for any graph G,

$$\text{ODD-CR}(G) \geq |E(G)| - 3|V(G)|. \tag{4.1}$$

If $|E(G)| \leq 3|V(G)|$, then (1) is trivially true. Let $|E(G)| > 3|V(G)|$ and suppose that (1) holds for any graph with $|V(G)|$ vertices and less than $|E(G)|$ edges. Consider a drawing of G with exactly $\text{ODD-CR}(G)$ pairs of edges crossing an odd number of times. Since $|E(G)| > 3|V(G)|$, G is not planar, so by Theorem A, $\text{ODD-CR}(G) \geq 1$. Let $\overline{G}$ denote the the graph obtained from G by deleting one edge that crosses at least one other edge an odd number of times. Applying the induction hypothesis to $\overline{G}$, we get

$$\text{ODD-CR}(G) \geq \text{ODD-CR}(\overline{G}) + 1 \geq |E(\overline{G})| - 3|V(\overline{G})| + 1 = |E(G)| - 3|V(G)|,$$

as required.

To prove Theorem 4.1.5, fix a drawing of G with exactly $\text{ODD-CR}(G)$ pairs of edges crossing an odd number of times, and suppose that $|E(G)| \geq$

$4|V(G)|$. Construct a *random* subgraph $G' \subseteq G$ by selecting each vertex of G independently with probability p, and letting G' be the subgraph induced by the selected vertices. The expected number of vertices of G', $\mathrm{Exp}[|V(G')|] = p|V(G)|$. Similarly, $\mathrm{Exp}[|E(G')|] = p^2|E(G)|$. The expected number of pairs of edges that cross an odd number of times in the drawing of G' inherited from G is $p^4\textsc{odd-cr}(G)$, hence the expected value of the odd-crossing number of G' cannot be larger than this.

By (1), $\textsc{odd-cr}(G') \geq |E(G')| - 3|V(G')|$ for every particular G'. Taking expectations,

$$p^4\textsc{odd-cr}(G) \geq \mathrm{Exp}[\textsc{odd-cr}(G')] \geq \mathrm{Exp}[|E(G')|] - 3\mathrm{Exp}[|V(G')|]$$

$$= p^2|E(G)| - 3p|V(G)|.$$

Setting $p = 4|V(G)|/|E(G)|$ we obtain

$$\textsc{odd-cr}(G) \geq \frac{1}{64}\frac{|E(G)|^3}{|V(G)|^2}, \tag{4.2}$$

whenever $|E(G)| \geq 4|V(G)|$. $\square$

Remarks. 1. In case $|E(G)| \geq 6|V(G)|$, Theorem 4.1.2 trivially follows from Theorem 4.1.5. Indeed, for any graph G,

$$\textsc{cr}(G) \leq \binom{|E(G)|}{2} < |E(G)|^2/2.$$

If $|E(G)| \geq 6|V(G)|$ then Theorem 4.1.5 implies

$$2(\textsc{odd-cr}(G))^2 \geq 2 \cdot \left(\frac{1}{64}\frac{|E(G)|^3}{|V(G)|^2}\right)^2 \geq \frac{|E(G)|^2}{2} > \textsc{cr}(G).$$

2. Using the fact that Theorem A guarantees, in any non-planar graph, the existence of two *independent* edges that cross an odd number of times, the above proof gives the same lower bound, $(1/64)|E(G)|^3/|V(G)|^2$, for the minimum number of pairs of *independent* edges that cross an odd number of times. This result is somewhat stronger than Theorem 4.1.5, because here we do not count any odd crossing between two edges that share an endpoint.

4.6 Proof of Theorem 4.1.6

First, we prove that the ODD CROSSING NUMBER PROBLEM, ODD-CR$(G) \leq K$, is in NP, and then we show that there is an NP-complete problem that can be reduced to it in polynomial time.

Fix a graph G with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set E. Every drawing $\mathcal{D}$ of G can be represented by an $\binom{|E|}{2}$-dimensional $(0,1)$-vector $\bar{X}_{\mathcal{D}}(G)$, in which each coordinate is assigned to an unordered pair of edges $\{e, f\} \subseteq E$, and is equal to 1 if and only if e and f cross an odd number of times. That is,

$$\bar{X}_{\mathcal{D}}(G) = (x_{\mathcal{D}}\{e, f\})_{e \neq f; e, f \in E},$$

where, for every $e, f \in E$,

$$x_{\mathcal{D}}\{e, f\} = \begin{cases} 0 \text{ if } e \text{ and } f \text{ cross an even number of times,} \\ 1 \text{ if } e \text{ and } f \text{ cross an odd number of times.} \end{cases}$$

We say that two drawings of G, $\mathcal{D}$ and $\mathcal{D}'$ are *equivalent* if they are represented by the same vector, i.e., if $\bar{X}_{\mathcal{D}}(G) = \bar{X}_{\mathcal{D}'}(G)$. An $\binom{|E|}{2}$-dimensional $(0,1)$-vector $\bar{X}$ is said to be *realizable* if there exists a drawing $\mathcal{D}$ of G such that $\bar{X}_{\mathcal{D}}(G) = \bar{X}$

Using an idea of Tutte [T70], it is not hard to describe the set of all realizable vectors of G. We need some further notation. For any $v \in V$, $g \in E$, let

$$\bar{Y}_{v,g} = (y\{e, f\})_{e \neq f; e, f \in E},$$

where

$$y\{e, f\} = \begin{cases} 1 \text{ if } e = g \text{ and } f \text{ is adjacent to } v, \\ \quad \text{or } f = g \text{ and } e \text{ is adjacent to } v, \\ 0 \text{ otherwise.} \end{cases}$$

Let Φ denote the vector space over GF(2) generated by the vectors $\bar{Y}_{v,g}$, i.e.,

$$\Phi = \left\langle \bar{Y}_{v,g} \mid v \in V,\ g \in E \right\rangle_{\text{gen}} \subset \{0,1\}^{\binom{|E|}{2}}.$$

Place the vertices $v_1, v_2, \ldots, v_n$ on a circle in this clockwise order so that they form a regular n-gon, and connect v_i and v_j $(i \neq j)$ by a straight-line segment if and only if $v_i v_j \in E$. This drawing is said to be the *convex* drawing of G, and is denoted by $\mathcal{C}$.

For any $1 \le i \le n$ let d_i be the degree of v_i and let $e^i_1, e^i_2, \ldots, e^i_{d_i}$ be the list of edges adjacent to v_i, in clockwise in the convex drawing of G. Let $\sigma_i : \{1, 2, \ldots, d_i\} \to \{1, 2, \ldots, d_i\}$ be any permutation. Define

$$\bar{Z}_{v_i,\sigma_i} = (z\{e, f\})_{e \neq f; e,f \in E}\,,$$

where

$$z\{e, f\} = \begin{cases} 1 \text{ if } e = e^i_\alpha,\ f = e^i_\beta \text{ and } (\alpha - \beta)(\sigma_i(\alpha) - \sigma_i(\beta)) < 0, \\ 0 \text{ otherwise.} \end{cases}$$

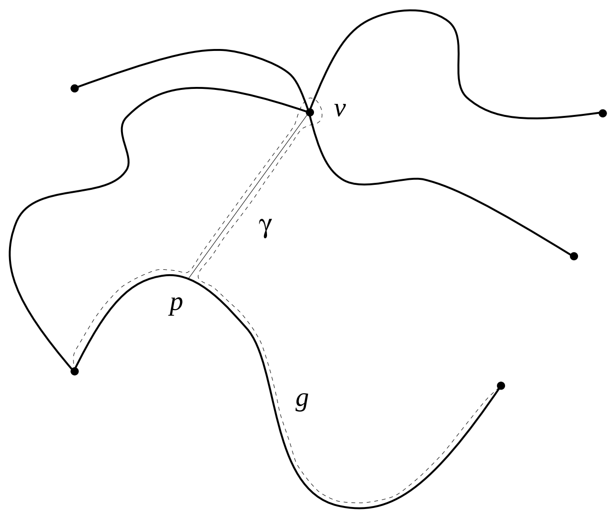

Figure 4.9: The first redrawing operation.

Lemma 4.6.1. *Let* Φ *denote the vector space over* GF(2) *generated by the vectors* $\bar{Y}_{v,g},\ v \in V,\ g \in E$*, let* $\bar{X}_{\mathcal{C}}(G)$ *be the* $(0,1)$*-vector representing the convex drawing of* G*, and let*

$$\Gamma = \left\{ \sum_{i=1}^{n} \bar{Z}_{v_i,\sigma_i} \mid \sigma_i \textit{ is any permutation } \{1, 2, \ldots, d_i\} \to \{1, 2, \ldots, d_i\} \right\}.$$

Then the set of all realizable vectors of G is

$$\Psi = \bar{X}_{\mathcal{C}}(G) + \Gamma + \Phi,$$

where the sum is taken mod 2*.*

Proof. Let $\mathcal{D}$ be any drawing of G, let $v \in V, g \in E$. Consider the following two operations:

(i) Choose a simple smooth arc γ connecting any internal point p of g to v such that it does not pass through any vertex, is not tangent to any edge, and crosses every edge a finite number of times. Replace a small piece of g containing p by a path going around v and running extremely close to γ (see Fig. 4.9). The $(0,1)$-vector representing this new drawing is

$$\bar{X}_{\mathcal{E}}(G) = \bar{X}_{\mathcal{D}}(G) + \bar{Y}_{v,g} \qquad (\text{mod } 2).$$

(ii) Let σ_i be the clockwise order of $e_1^i, e_2^i, \ldots, e_{d_i}^i$ as they emanate from v_i in drawing $\mathcal{D}$. Change the clockwise order of edges as they emanate from v_i to $e_1^i, e_2^i, \ldots, e_{d_i}^i$ in a small neighborhood of v_i. (See Fig. 4.10.) The $(0,1)$-vector representing this new drawing is

$$\bar{X}_{\mathcal{F}}(G) = \bar{X}_{\mathcal{D}}(G) + \bar{Z}_{v_i,\sigma_i} \qquad (\text{mod } 2).$$

This shows that any vector in Ψ is realizable.

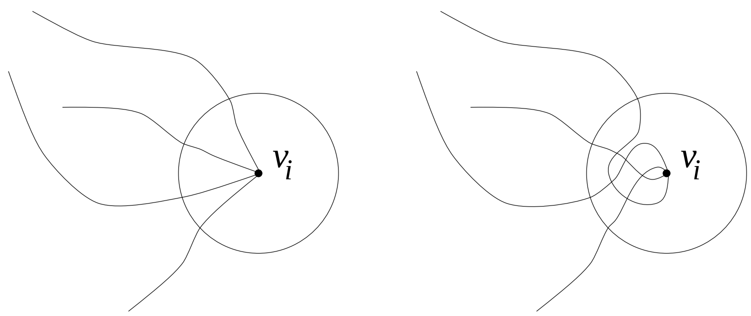

Figure 4.10: The second redrawing operation.

Next we prove that $\bar{X}_{\mathcal{D}}(G) \in \Psi$, for any drawing $\mathcal{D}$ of G. Using a topological transformation of the plane, if necessary, we can assume without loss of generality that the vertices of G, $v_1, v_2, \ldots, v_n$, form a regular n-gon, in this clockwise order. First, for every $1 \le i \le n$, in a small neighborhood of v_i, change the clockwise order of edges as they emanate from v_i to $e_1^i, e_2^i, \ldots, e_{d_i}^i$ such that in a very small neighborhood of v_i, each egde $v_i v_j$ is represented by the corresponding part of the segment $v_i v_j$.

Then, pick an edge $g = v_i v_j$, and transform it into the straight-line segment between v_i and v_j, by continuous deformation. Performing this operation for all edges, one by one, we obtain $\mathcal{C}$ the convex drawing of G.

Let $\mathcal{D}'$ denote the drawing after the first step. Then,

$$\bar{X}_{\mathcal{D}'}(G) = \bar{X}_{\mathcal{D}}(G) + \sum_{i=1}^{n} \bar{Z}_{v_i,\sigma_i} \qquad (\text{mod } 2)$$

for some permutations $\sigma^1, \sigma^2, \ldots, \sigma^n$.

During the second step, the representation vector of the drawing changes whenever a deforming edge g hits a vertex v. Let $\mathcal{E}$ and $\mathcal{F}$ denote the drawing immediately before and after this event. Clearly,

$$\bar{X}_{\mathcal{F}}(G) = \bar{X}_{\mathcal{E}}(G) + \bar{Y}_{v,g} \qquad (\text{mod } 2).$$

Finally, we obtain

$$\bar{X}_{\mathcal{C}}(G) = \bar{X}_{\mathcal{D}}(G) + \bar{Y} \qquad (\text{mod } 2),$$

for some $\bar{Y} \in \Phi$, hence

$$\bar{X}_{\mathcal{D}}(G) \in \bar{X}_{\mathcal{C}}(G) + \bar{Y} = \Psi. \qquad \square$$

Now we are in a position to prove that the Odd Crossing Number Problem is in NP. Suppose that odd-cr$(G) \le K$. Then, by Lemma 4.6.1, there is a realizable vector $\bar{Y} \in \Psi$ such that all but at most K coordinates of $\bar{Y}$ are 0. We can give the vector $\bar{Y}$ in the form

$$\bar{Y} = \bar{X}_{\mathcal{C}}(G) + \sum_{i=1}^{n} \bar{Z}_{v_i,\sigma_i} + \sum_{v \in V, g \in E} \alpha_{(v,g)} \bar{Y}_{v,g} \qquad (\text{mod } 2),$$

where $\alpha_{(v,g)} \in \{0,1\}$ and $\sigma_i : \{1, 2, \ldots, d_i\} \to \{1, 2, \ldots, d_i\}$ are permutations. Clearly, the correctness of this equation can be checked in polynomial time. Thus, the Odd Crossing Number Problem is in NP.

The Optimal Linear Arrangement Problem is the following. We have a graph $G = (V, E)$ and an integer K, is there a one to one function $\sigma : V \to \{1, 2, \ldots, |V|\}$ such that $\sum_{uv\in E} |\sigma(u) - \sigma(v)| \le K$?

Notice that the Odd Crossing Number Problem for simple graphs is equivalent to the same problem for *multigraphs*, i.e., when the graph G may have multiple (parallel) edges. Indeed, we can remove all multiplicities by introducing new vertices along the edges of G. For any graph $\overline{G}$ obtained from G by subdividing one (or more) of its edges, we have

$$\text{ODD-CR}(\overline{G}) = \text{ODD-CR}(G).$$

Lemma 4.6.2. *The* Optimal Linear Arrangement Problem *can be reduced to the* Odd Crossing Number Problem *in polynomial time.*

Proof. Suppose we are given an instance $G = (V, E)$, K, and we want to decide if there exists a one-to one function $\sigma : V \to \{1, 2, \ldots, |V|\}$ such that $\sum_{uv\in E} |\sigma(u) - \sigma(v)| \le K$. Let $V = \{v_1, v_2, \ldots, v_n\}$ and assume without loss of generality that G is connected. We construct a multigraph G'_K and a number K' such that the answer to our Optimal Linear Arrangement Problem is affirmative if and only if $\text{ODD-CR}(G'_K) \le K'$.

Let $G'_K = (V', E')$, where $V' = V_1 \cup V_2 \cup \{u, w\}$, $E = E_1 \cup E_2 \cup E_3$,

$$V_1 = \{u_i \mid 1 \le i \le n\}, \quad V_2 = \{w_i \mid 1 \le i \le n\},$$

$$E_1 = \{|E|^2 \text{ copies of } u_i w_i \mid 1 \le i \le n\},$$

$$E_2 = \{u_i w_j \mid i < j \text{ and } v_i w_j \in E\},$$

$$E_3 = \{K^2|E|^2 \text{ copies of } uw,\ uu_i,\ ww_i,\ 1 \le i \le n\},$$

and let

$$K' = |E|^2(K - |E|) + |E|^2 - 1.$$

Suppose first that there exists a bijection $\sigma : V \to \{1, 2, \ldots, |V|\}$ such that $\sum_{uv\in E} |\sigma(u) - \sigma(v)| \le K$. We construct a drawing of G' with at most K' pairs of crossing edges. Place u_i at $(1, \sigma(v_i))$, w_i at $(0, \sigma(v_i))$, u at $(2, 0)$, and w at $(-1, 0)$. Represent all single edges by straight-line segments and all multiple edges by pairwise disjoint curves running very close to the corresponding straight line segment. It is easy to see that the total number of crossing pairs of edges is at most

$$\sum_{uv\in E} (|\sigma(u) - \sigma(v)| - 1)|E|^2 + |E|^2 - 1 \le |E|^2(K - |E|) + |E|^2 - 1 = K'.$$

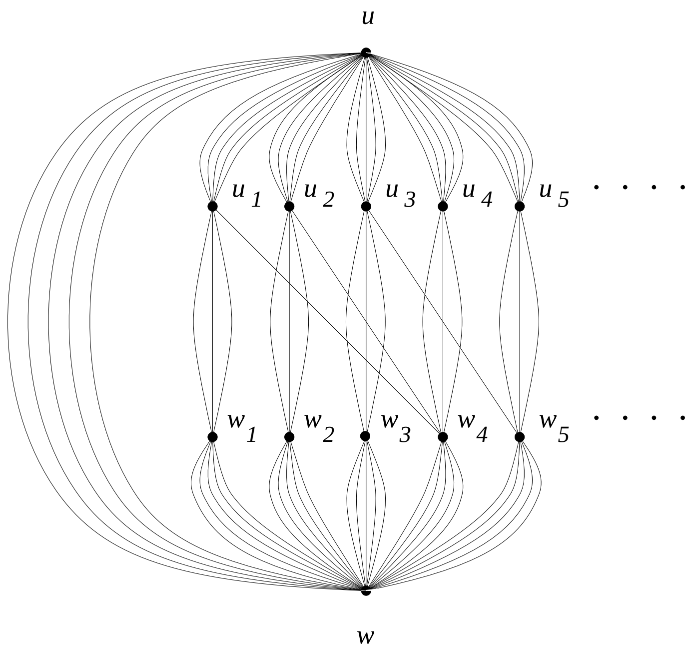

Figure 4.11: The multigraph G'_K.

Next, suppose that ODD-CR$(G'_K) \leq K'$. We show, using some simple transformations, that there is another drawing of G' generated by a function σ in the way described above, which has at most K' pairs of edges that cross an odd number of times. Consider a drawing of G'_K with at most K' pairs of edges that cross an odd number of times.

(a) We can assume that any two parallel edges, e and f, are drawn very close to each other, so that they are openly disjoint, and any other edge crosses both of them the same number of times. Indeed, if e and f are drawn differently, then replacing either e by an arc running very close to f, or f by an arc running very close to e, we obtain a new drawing of G which has at most as many pairs of edges that cross an odd number of times as the original drawing.

(b) Any two edges $e, f \in E_1 \cup E_3$ must cross an even number of times. Indeed, otherwise, by (a), we can assume that each of the at least $|E|^2$ edges parallel (or identical) to e crosses each of the at least $|E|^2$ edges parallel (or identical) to f an odd number of times. This implies that the number of edge pairs that cross an odd number of times is at least $|E|^4 > K'$, a contradiction.

(c) No edge of G'_K can cross any edge between u and w an odd number of times. Otherwise, by (a), the number of pairs of edges that cross an odd number of times would be at least $K^2|E|^2 > K'$, which is impossible.

(d) Let e be any edge between u and w, and let f_i (resp. g_i) be any edge whose endpoints are u and u_i (resp. w and w_i), $1 \le i \le n$. If for some $i \ne j$, the edges (e, f_i, f_j) emanate from u in clockwise order, then (e, g_i, g_j) must emanate from v in counter-clockwise order.

To see this, consider a cycle C formed by f_i, e, g_i, and any edge connecting u_i and w_i. The closed curve representing this cycle divides the plane into connected cells. As in the proof of Theorem 4.1.1, color these cells with black and white so that no two cells that share a boundary arc receive the same color. Let P be a path formed by f_j, g_j, and any edge between u_j and w_j. Suppose that in a small neighborhood of u, f_j is in, say, the black region. Then, in a small neighborhood of w, g_j must also lie in the black region, because, by (b), every edge of P crosses (every edge of) C an even number of times.

(e) Suppose that $e, f_1, f_2, \ldots, f_n$ emanate from u in the clockwise order $e, f_{\alpha(1)}, f_{\alpha(2)}, \ldots, f_{\alpha(n)}$. Then, by (d), $e, g_1, g_2, \ldots, g_n$ must emanate from w in the reverse order $e, g_{\alpha(n)}, g_{\alpha(n-1)}, \ldots, g_{\alpha(1)}$. Let $\sigma(v_i) = \alpha^{-1}(i)$, $1 \le i \le n$.

We *claim* that for every $u_i w_j \in E_2$, there are at least $(|\sigma(v_i) - \sigma(v_j)| - 1)|E|^2$ edges in G'_K that cross $u_i w_j$ an odd number of times. To see this, it is enough to show that for every $r < s < t$, if $v_{\alpha(r)} v_{\alpha(t)} \in E$, then the edge $e_{rt} := u_{\alpha(r)} w_{\alpha(t)}$ must cross the path $P_s := f_{\alpha(s)} \cup e_{\alpha(s)} \cup g_{\alpha(s)}$ an odd number of times, where $e_{\alpha(s)}$ denotes any edge between $u_{\alpha(s)}$ and $w_{\alpha(s)}$. As before, color the cells determined by the closed curve $P_s \cup e$ with black and white. It follows from (d) that if in a small neighborhood of u, $f_{\alpha(r)} \cup e_{rt} \cup g_{\alpha(t)}$ is in the black region, then in a small neighborhood of w it is in the white region. In view of (b) and (c), this implies that e_{rt} crosses at least one of the edges $f_{\alpha(s)}$, $e_{\alpha(s)}$, and $g_{\alpha(s)}$ an odd number of times. In each case, we are done, and our claim is true.

Therefore, we have

$$\sum_{uv \in E} (|\sigma(u) - \sigma(v)| - 1)|E|^2 \le \text{ODD-CR}(G'_K) \le K' = |E|^2(K - |E|) + |E|^2 - 1,$$

which implies that

$$\sum_{uv \in E} (|\sigma(u) - \sigma(v)| \le K,$$

as desired. □

With Lemma 4.6.2, the proof of Theorem 4.1.6 (ii) is complete, because the Optimal Linear Arrangement Problem is known to be NP-complete [GJS76].

4.7 Even More Crossing Numbers

We can further modify each of the above crossing numbers, by applying one of the following rules:

Rule $+$ **:** Consider only those drawings where two edges with a common endpoint do not cross each other.

Rule 0 **:** Two edges with a common endpoint are allowed to cross and their crossing counts.

Rule $-$: Two edges with a common endpoint are allowed to cross, but their crossing does not count.

In the previous definitions we have always used Rule 0. If we apply Rule + (Rule $-$) in the definition of the crossing numbers, then we indicate it by using the corresponding subscript, as shown in the table below. This gives us an array of nine different crossing numbers. It is easy to see that in a drawing of a graph, which minimizes the number of crossing points, any two edges have at most one point in common (see e.g. [RT97]). Therefore, $\text{CR}_+(G) = \text{CR}(G)$, which slightly simplifies the picture.

Rule +	$\text{ODD-CR}_+(G)$	$\text{PAIR-CR}_+(G)$	$\text{CR}(G)$
Rule 0	$\text{ODD-CR}(G)$	$\text{PAIR-CR}(G)$	
Rule $-$	$\text{ODD-CR}_-(G)$	$\text{PAIR-CR}_-(G)$	$\text{CR}_-(G)$

Figure 4.12: Modifications of the crossing number

Moving from left to right or from bottom to top in this array, the numbers do not decrease. It is not hard to generalize (1) to each of these crossing numbers. We obtain (as in in [PT97]) that

$$\textsc{odd-cr}_-(G) \geq \frac{1}{64}\frac{e^3}{n^2},$$

for any graph G with n vertices and with $e \geq 4n$ edges. We cannot prove anything else about $\textsc{odd-cr}_-(G)$, $\textsc{pair-cr}_-(G)$, and $\textsc{cr}_-(G)$. We conjecture that these values are very close to $\textsc{cr}(G)$, if not the same. That is, we believe that by letting pairs of *incident* edges cross an arbitrary number of times, we cannot effectively reduce the total number of crossings between *independent* pairs of edges. The weakest open questions are the following.

Problem. *Do there exist suitable functions f_1, f_2, f_3 such that every graph G satisfies*

(i) $$\textsc{odd-cr}(G) \leq f_1(\textsc{odd-cr}_-(G)),$$

(ii) $$\textsc{pair-cr}(G) \leq f_2(\textsc{pair-cr}_-(G)),$$

(iii) $$\textsc{cr}(G) \leq f_3(\textsc{cr}_-(G)) \ ?$$

Remark. We can prove that the Pair Crossing Number Problem, $\textsc{pair-cr}(G) \leq K$, is NP-hard. The proof is analogous to the proofs of the corresponding results for the crossing number (see [GJ83]) and for the odd-crossing number (see Lemma 4.6.2).

On the other hand, we could not generalize Lemma 4.6.1 for $\textsc{pair-cr}(G)$. With a completely different approach, Schaefer, Sedgwick and Štefankovič [SSS03] managed to prove that the Pair Crossing Number Problem is also in NP.

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Chapter 5

Crossing numbers of random graphs

The *crossing number* of G is the minimum number of crossing points in any drawing of G. We consider the following two other parameters. The *rectilinear crossing number* is the minimum number of crossing points in any drawing of G, with straight line segments as edges. The *pair-crossing number* of G is the minimum number of pairs of crossing edges over all drawings of G. The *odd-crossing number* of G is the minimum number of pairs of edges that cross an odd number of times. We prove several results on the expected values of these parameters of a random graph.

5.1 Introduction

A *drawing* of a graph G is a mapping which assigns to each vertex a point of the plane and to each edge a simple continuous arc connecting the corresponding two points. We assume that in a drawing no three edges (arcs) cross at the same point, and the edges do not pass through any vertex. The *crossing number* $\text{CR}(G)$ of G is the minimum number of crossing points in any drawing of G. We consider the following two variants of the crossing number. The *rectilinear crossing number* $\text{LIN-CR}(G)$ is the minimum number of crossing points in any drawing of G, with straight line segments as edges. The *pair-crossing number* or *pair-crossing number*, $\text{PAIR-CR}(G)$, of G is the minimum number of crossing pairs of edges over all drawings of G. The *odd-crossing number* of G is the minimum number of pairs of edges that cross an odd number of times. Clearly, $\text{ODD-CR}(G) \leq \text{PAIR-CR}(G) \leq \text{CR}(G) \leq \text{LIN-CR}(G)$.

The determination of the crossing numbers is extremely difficult. Even the crossing numbers of the complete graphs are not known. Let

$$\gamma_{\text{ODD-CR}} = \lim_{n\to\infty} \frac{\text{ODD-CR}(K_n)}{\binom{n}{2}^2}, \quad \gamma_{\text{PAIR-CR}} = \lim_{n\to\infty} \frac{\text{PAIR-CR}(K_n)}{\binom{n}{2}^2},$$
$$\gamma_{\text{CR}} = \lim_{n\to\infty} \frac{\text{CR}(K_n)}{\binom{n}{2}^2}, \quad \gamma_{\text{LIN-CR}} = \lim_{n\to\infty} \frac{\text{LIN-CR}(K_n)}{\binom{n}{2}^2}.$$

These limits are known to exist [RT97] and the best known bounds are $1/30 \leq \gamma_{\text{ODD-CR}} \leq 1/16$, $1/30 \leq \gamma_{\text{PAIR-CR}} \leq 1/16$, $1/20 \leq \gamma_{\text{CR}} \leq 1/16$, $0.06327 \leq \gamma_{\text{LIN-CR}} \leq 0.0639$ [AGOR06], [BDG00] (see also [G72, RT97]).

In this paper we investigate the crossing numbers of *random graphs.* Let $G = G(n,p)$ be a *random* graph with n vertices, whose edges are chosen independently with probability p. Let e denote the *expected number* of edges of G, i.e., $e = p\binom{n}{2}$. We shall always have $e \to \infty$ (indeed, $p = \Omega(n^{-1})$) so that G almost surely has $e(1+o(1))$ edges.

In [PT00a] it was shown that if $e > 10n$, then almost surely we have $\text{CR}(G) \geq \frac{e^2}{4000}$. Consequently, almost surely we also have $\text{LIN-CR}(G) \geq \frac{e^2}{4000}$. As we always *can* draw a graph with straight lines the crossing number (in any form) is never larger than the number of pairs of edges and the expected number of pairs of edges is $\sim \frac{e^2}{2}$ Our interest will be in those regions of p for which the various crossing numbers are, asymptotically, a positive proportion of the number of pairs of edges.

Let

$$\kappa_{\text{LIN-CR}}(n,p) = \frac{\mathrm{E}\left[\text{LIN-CR}(G)\right]}{e^2}, \quad \kappa_{\text{CR}}(n,p) = \frac{\mathrm{E}\left[\text{CR}(G)\right]}{e^2},$$
$$\kappa_{\text{PAIR-CR}}(n,p) = \frac{\mathrm{E}\left[\text{PAIR-CR}(G)\right]}{e^2}, \quad \kappa_{\text{ODD-CR}}(n,p) = \frac{\mathrm{E}\left[\text{ODD-CR}(G)\right]}{e^2}.$$

We have $\kappa_{\text{PAIR-CR}}(n,p) \leq \kappa_{\text{CR}}(n,p) \leq \kappa_{\text{LIN-CR}}(n,p)$ for any n, p.

Theorem 5.1.1. *For any fixed n, $\kappa_{\text{LIN-CR}}(n,p)$, $\kappa_{\text{CR}}(n,p)$, $\kappa_{\text{PAIR-CR}}(n,p)$ are increasing, continous functions of p.*

With Theorem 5.1.1 we may express (roughly) our two central concerns. At which $p = p(n)$ are $\kappa_{\text{LIN-CR}}(n,p)$, $\kappa_{\text{CR}}(n,p)$, $\kappa_{\text{PAIR-CR}}(n,p)$ bounded away from zero? At which $p = p(n)$ are $\kappa_{\text{LIN-CR}}(n,p)$, $\kappa_{\text{CR}}(n,p)$, $\kappa_{\text{PAIR-CR}}(n,p)$, $\kappa_{\text{ODD-CR}}(n,p)$ close to the values $\gamma_{\text{LIN-CR}}, \gamma_{\text{CR}}, \gamma_{\text{PAIR-CR}}, \gamma_{\text{ODD-CR}}$, respectively? Our results for these three crossing numbers shall be quite different. We are uncertain whether or not that represents the reality of the situation. The following relatively simple result shows basically that for $p = \frac{1}{n}$ all three crossing numbers are asymptotically negligible and that for $p = \frac{c}{n}$ with $c > 1$ fixed the three crossing numbers have not reached their limiting values.

Theorem 5.1.2. *1.* $\limsup_{n\to\infty} \kappa_{\text{LIN-CR}}(n, c/n) = 0$ *for* $c \le 1$

2. $\limsup_{n\to\infty} \kappa_{\text{CR}}(n, c/n) = 0$ *for* $c \le 1$

3. $\limsup_{n\to\infty} \kappa_{\text{PAIR-CR}}(n, c/n) = 0$ *for* $c \le 1$

4. $\limsup_{n\to\infty} \kappa_{\text{ODD-CR}}(n, c/n) = 0$ *for* $c \le 1$

5. $\lim_{c\to 1} \limsup_{n\to\infty} \kappa_{\text{LIN-CR}}(n, c/n) = 0$

6. $\lim_{c\to 1} \limsup_{n\to\infty} \kappa_{\text{CR}}(n, c/n) = 0$

7. $\lim_{c\to 1} \limsup_{n\to\infty} \kappa_{\text{PAIR-CR}}(n, c/n) = 0$

8. $\lim_{c\to 1} \limsup_{n\to\infty} \kappa_{\text{ODD-CR}}(n, c/n) = 0$

9. $\limsup_{n\to\infty} \kappa_{\text{LIN-CR}}(n, c/n) < \gamma_{\text{LIN-CR}}$ *for all* c

10. $\limsup_{n\to\infty} \kappa_{\text{CR}}(n, c/n) < \gamma_{\text{CR}}$ *for all* c

11. $\limsup_{n\to\infty} \kappa_{\text{PAIR-CR}}(n, c/n) < \gamma_{\text{PAIR-CR}}$ *for all* c

12. $\limsup_{n\to\infty} \kappa_{\text{PAIR-CR}}(n, c/n) < \gamma_{\text{ODD-CR}}$ *for all* c

Theorem 5.1.2 gives only upper bounds for the various crossing numbers. The main results of this paper, given in Theorems 5.1.3, 5.1.4, 5.1.5, deal with lower bounds for the three crossing numbers. Our weakest result is for the pair-crossing number and the odd-crossing number.

Theorem 5.1.3. *For any* $\varepsilon > 0$, $p = p(n) = n^{\varepsilon - 1}$,

$$\liminf_{n\to\infty} \kappa_{\text{PAIR-CR}}(n,p) > 0, \quad \liminf_{n\to\infty} \kappa_{\text{ODD-CR}}(n,p) > 0.$$

For the crossing number we have a much stronger result.

Theorem 5.1.4. *For any $c > 1$ with $p = p(n) = c/n$*

$$\liminf_{n\to\infty} \kappa_{\text{CR}}(n, p) > 0$$

As LIN-CR$(G) \geq$ CR(G) the lower bound of Theorem 5.1.4 applies also to the rectilinear crossing number. Our most surprising result is that with the rectilinear crossing number one reaches an asymptotically best limit in relatively short time.

Theorem 5.1.5. *If $p = p(n) \gg \frac{\ln n}{n}$ then*

$$\lim_{n\to\infty} \kappa_{\text{LIN-CR}}(n, p) = \gamma_{\text{LIN-CR}}(n, p)$$

5.2 Upper Bounds

First we prove Theorem 5.1.1. Let f be any real valued function on graphs. Then with $G \sim G(n, p)$

$$E[f(G)] = \sum_H f(H) p^{e(H)} (1-p)^{\binom{n}{2} - e(H)}$$

where H runs over the labelled graphs on n vertices and $e(H)$ is the number of edges of H. This is a polynomial and hence a continous function of p, giving the second part of Theorem 5.1.1. We argue that $\kappa_{\text{CR}}(n, p)$ is an increasing function of p, the arguments in the other cases are identical. For $0 \leq p, q \leq 1$ we may view $G(n, pq)$ as a two step process, first creating $G(n, p)$ and then taking each edge from $G(n, p)$ with probability q. After the first stage consider a drawing with the minimal number of crossings X, so that $E[X] = \kappa_{\text{CR}}(n, p)$. Now keep that drawing but take each edge with probability q. Each crossing is still in the new picture with probability q^2. This gives a drawing of $G(n, pq)$ with expected number of crossings $q^2 \kappa_{\text{CR}}(n, p)$. We do not claim this drawing is optimal, but it does give the desired upper bound as $E[\text{CR}(G(n, pq))] \leq q^2 E[\text{CR}(G(n, p))]$, completing Theorem 5.1.1.

The first eight parts of Theorem 5.1.2 will come as no surprise to those familiar with random graphs as in the classic papers of Erdős and Rényi it was shown that with $p = \frac{c}{n}$ the random graph $G(n, p)$ is almost surely planar when $c < 1$. Our argument is a bit technical, however, as we must bound the expected crossing number.

We prove only part 1, for $c < 1$. Parts 2, 3, and 4 for $c < 1$ follow immediately, since they involve smaller crossing numbers. The statements for $c=1$ follow from part 5. Fix $c<1$, set $p=\frac{c}{n}$ and $X=\text{LIN-CR}(G)$ with $G\sim G(n,p)$. Let Y be the number of cycles of G and Z the number of edges of G. Then we claim $X\leq YZ$. Remove from G one edge from each cycle. This leaves a forest which can be drawn with straight lines and no crossings. Now add back in those Y edges as straight lines. At worst they could hit every edge, giving $\leq YZ$ crossings. With $c<1$ $E[Y]=\sum_{i=3}^{n}\frac{(n)_i}{2i}p^i<\sum_{i=3}^{\infty}c^i$ is bounded by a constant, say A. As Z has Binomial Distribution standard bounds give, say, $\Pr[Z>10n]<\alpha^{-n}$ for some explicit $\alpha>1$. As $X\leq n^4$ always, $X\leq 10nY+n^4\chi(Z>10n)$ where χ is the indicator random variable. Thus $E[X]\leq 10An+n^4\alpha^{-n}=o(n^2)$. This completes the proof of parts 1, 2, and 3 for the case $c<1$.

Now fix $c=1+\varepsilon$ with ε positive and small. Set $p=\frac{1+\varepsilon}{n}$, $p'=\frac{1-\varepsilon}{n}$ and let p^* satisfy $p'+p^*-p'p^*=p$ so that $p^*\sim\frac{2\varepsilon}{n}$. We may consider $G(n,p)$ as the union of independently chosen $G(n,p')$ and $G(n,p^*)$. Say the first has rectilinear crossing number X and Y edges and the second has Z edges. Then their union has rectilinear crossing number at most $X+Z(Y+Z)$ as we can draw $G(n,p')$ optimally and assume all other pairs of edges do intersect. But $E[X]=o(n^2)$ and it is easy to show that $E(Z(Y+Z))\sim E(Z)(E(Y+Z))\sim\frac{1}{2}n^2\varepsilon(1+\varepsilon)$. Thus

$$E[\text{LIN-CR}(G)]\leq(1+o(1))\frac{1}{2}\varepsilon(1+\varepsilon)n^2$$

from which part 5 of Theorem 5.1.2 follows. Parts 6, 7, and 8 then also follow as they involve smaller crossing numbers.

The final four parts of Theorem 5.1.2 are also natural to those familiar with random graphs. For $c>1$ fixed $G(n,\frac{c}{n})$ has a "giant component" with $\Omega(n)$ vertices. Outside the giant component there are $\Omega(n)$ edges all lying in trees or unicylic components. These edges may be drawn with no crossings and that will involve a positive proportion of the potential edge crossings. Again, our argument will be a bit technical as we must deal with expectations. We state the argument only for rectilinear crossing number but it is the same in all four cases.

We first note a deterministic result: Let G be *any* graph on n vertices with e edges. Then

$$\frac{\text{LIN-CR}(G)}{e^2}\leq\frac{4\cdot\text{LIN-CR}(K_n)}{(n)_4}$$

Fix a drawing of K_n with $\text{LIN-CR}(K_n)$ crossings. Define a random drawing of G by randomly mapping its n vertices bijectively to the n vertices of the drawing. Let e_1, e_2 be two edges of G with no common vertex, there being at most $e^2/2$ such unordered pairs. They may be mapped to a particular crossing of the drawing of K_n in eight ways, so they have probability $8 \cdot \text{LIN-CR}(K_n)/(n)_4$ of being mapped to a crossing. Now the expected number of crossings of G in this random drawing is at most, by Linearity of Expectation, $\frac{e^2}{2}\frac{8 \cdot \text{LIN-CR}(K_n)}{(n)_4}$ and thus there exists a drawing of G with at most that many crossings.

As the right hand side approaches $\gamma_{\text{LIN-CR}}$ we have

$$\frac{\text{LIN-CR}(G)}{e^2} \leq \gamma_{\text{LIN-CR}} + o(1)$$

where the $o(1)$ term approaches zero in n, uniformly over all graphs G.

With $c > 0$ fixed (this argument is only needed for $c > 1$ but works for all positive c), $p = \frac{c}{n}$ and $G \sim G(n,p)$ let X denote the number of edges and Y denote the number of isolated edges. The savings comes from noting that isolated edges can always be added to a graph with no additional crossings. Thus

$$E[\text{LIN-CR}(G)] \leq E[(X-Y)^2](\gamma_{\text{LIN-CR}} + o(1))$$

Here $E[X] \sim \frac{c}{2}n$ and $E[Y] = \binom{n}{2}p(1-p)^{2n-4} \sim \frac{c}{2}e^{-2c}n$ and elementary calculations give

$$E[(X-Y)^2] \sim E[X-Y]^2 \sim [\frac{c}{2}(1-e^{-2c})n]^2$$

With $e := p\binom{n}{2} \sim \frac{c}{2}n$ we have

$$\frac{E[\text{LIN-CR}(G)]}{e^2} \leq \gamma_{\text{LIN-CR}}(1-e^{-2c})^2(1+o(1))$$

`Comments and Open Questions.` We note that as c approaches infinity the $(1-e^{2c})^2$ term above approaches one. The above bound may be improved somewhat by letting Y denote the edges in isolated trees and unicyclic components and there are even further improvements possible. Still, all these improvements seem to approach one as c approaches infinity. This leads to an intriguing conjecture: If $p(n) \gg \frac{1}{n}$ then $\kappa_{\text{LIN-CR}}(n,p) \to \gamma_{\text{LIN-CR}}$. One may make the same conjecture for all three variants of the crossing number. Indeed, this entire paper may be viewed as an attempt (thus far unsuccessful) of the authors to resolve these conjectures.

We conjecture that for any $c \geq 0$, the limits $\lim_{n\to\infty} \kappa_{\text{LIN-CR}}(n, c/n)$, $\lim_{n\to\infty} \kappa_{\text{CR}}(n, c/n)$, and $\lim_{n\to\infty} \kappa_{\text{PAIR-CR}}(n, c/n)$ exist. This follows from Theorem 5.1.2, for $c \leq 1$. If this conjecture is true, it is not hard to see that the functions $f_{\text{LIN-CR}}(c) = \lim_{n\to\infty} \kappa_{\text{LIN-CR}}(n, c/n)$, $f_{\text{CR}}(c) = \lim_{n\to\infty} \kappa_{\text{CR}}(n, c/n)$, and $f_{\text{PAIR-CR}}(c) = \lim_{n\to\infty} \kappa_{\text{PAIR-CR}}(n, c/n)$ are continous and increasing for all $c \geq 0$.

5.3 The pair-crossing and odd-crossing number

Here we prove Theorem 5.1.3 for the odd-crossing number, the statement for the pair-crossing number follows immediately. Fix $\varepsilon > 0$ and set $p = p(n) = n^{\varepsilon-1}$. Our object is to show

$$\liminf_{n\to\infty} \kappa_{\text{ODD-CR}}(n, p) > 0$$

This is equivalent to showing that for n sufficiently large

$$E[\text{ODD-CR}(G(n, p))] > \delta n^4 p^2$$

for some δ dependent only on ε. For $L \geq 1$ we let $K_5(L)$ denote the following graph:

- There are five vertices $x_1, \ldots, x_5$
- For each distinct pair x_i, x_j there is a path between them of length L.

There are no other vertices nor edges so $K_5(L)$ has $5 + 10(L-1)$ vertices and $10L$ edges. Note that $K_5(L)$ is a topological K_5. Hence, by Hanani's theorem [Ch34], in any drawing of $K_5(L)$ there must be two edges that cross an odd number of times. We shall fix L such that $L\varepsilon > 1$. We shall show that G contains many $K_5(L)$. Each $K_5(L)$ will force at least one odd-crossing pair of edges. With L fixed this is a positive (albeit only $0.01L^{-2}$) proportion of the square of the number of edges involved. When this is carefully counted over all $K_5(L)$ we shall see that the total number of odd-crossing pairs is at least this constant times the square of the total number of edges.

We use three results about the almost sure behavior of $G(n, p)$. In the third K is any fixed constant.

1. Every vertex has degree $\sim np$.

2. Between every pair of distinct vertices there are $\sim n^{L-1}p^L$ paths of length L.

3. For any distinct $x, y, z_1, \ldots, z_K$ there are $\sim n^{L-1}p^L$ paths of length L between x and y that do not use any of the z_j.

The first result holds whenever $np \gg \ln n$ and follows from basic Large Deviation bounds on the degree of a vertex. Both the first and the second result are examples of a more general result [S90] on counting extensions. For the third we note from [S90] that the probability that the number of paths of length L between fixed x and y is not in $[(1-\epsilon)n^{L-1}p^L, (1+\epsilon)n^{L-1}p^L]$ is exponentially small. Fix $x, y, z_1, \ldots, z_K$. Consider L-paths from x to y on G with $z_1, \ldots, z_K$ deleted, which has distribution $G(n-K, p)$. The K has negligible effect and so with exponentially small failure this number is as desired – hence almost surely it is as desired for all $O(n^{K+2})$ choices of $x, y, z_1, \ldots, z_K$.

Now we count the $K_5(L)$. There are $\binom{n}{5} \sim \frac{1}{5!}n^5$ choices for $x_1, \ldots, x_5$. We want to know how many ways we can add the ten paths, say $P_1, \ldots, P_{10}$, one between each pair of vertices. Suppose we have already selected $P_1, \ldots, P_i$. Then a constant number of vertices have been taken. Thus the number of choices for P_{i+1} is $\sim n^{L-1}p^L$. So we are making ten choices and each time we have $\sim n^{L-1}p^L$ choices so the total number of choices is $\sim [n^{L-1}p^L]^{10}$. This gives a total of $\sim \frac{1}{5!}n^{10L-5}p^{10L}$ copies of $K_5(L)$. For each one we count one pair of odd-crossing edges. Now consider an odd-crossing pair, say, edges uv and wz. How many $K_5(L)$ do they lie on? Renumbering for convenience say the path from x_1 to x_2 has u as its i-th and v as its $i+1$-st point and the path from x_3 to x_4 has w as its j-th and z as its $j+1$-st point. There are L^2 choices for i, j. Now fix u, v, w, z and i, j. From the first property there are $\sim (np)^i$ paths of length i starting at u, $\sim (np)^{L-i-1}$ paths of length $L-i$ starting at v and similarly for w, z. Further these numbers are not asymptotically effected when we require that they miss a fixed number of points. So we extend u, v, w, z to some x_1, x_2, x_3, x_4 in $\sim (np)^{2(L-1)}$ ways. We have n choices for x_5 and then $\sim (n^{L-1}p^L)^8$ ways to complete the remaining eight paths forming $K_5(L)$. Thus edges uv,wz lie on $\sim L^2 n^{10L-9}p^{10L-2}$ different $K_5(L)$. So each odd-crossing pair has been counted at most that many times and hence the number of odd-crossing pairs is at least asymptotically

$$\frac{\frac{1}{5!}n^{10L-5}p^{10L}}{L^2 n^{10L-9}p^{10L-2}} = \frac{1}{120L^2}n^4p^2$$

as desired.

`Comments and Open Questions.` As we must take $L > \varepsilon^{-1}$ the constant $\frac{1}{120}L^{-2}$ in this result goes to zero as $\varepsilon \to 0$. This is in surprising contrast to the crossing number $\text{CR}(G)$ discussed in the next section. That crossing number becomes a positive proportion of the square of the number of edges already at $p = \frac{c}{n}$ when $c > 1$. Can the odd-crossing number (pair-crossing number) and the crossing number have such different behavior? We doubt it. As mentioned in Chapter 4, we cannot rule out the possibility that the pair-crossing number and the crossing number are always exactly the same. This is not the case with the odd-crossing number. We can certainly make the weaker conjecture that the expectation of the pair-crossing number of $G(n,p)$ becomes $\Omega(n^4p^2)$ already at $p = \frac{1+\epsilon}{n}$. We further note that we have no idea at which p $\kappa_{\text{PAIR-CR}}(n,p)$ gets within $o(1)$ of its limit $\gamma_{\text{PAIR-CR}}$.

5.4 The crossing number

Here we prove Theorem 5.1.4. Fix $c > 1$ and set $p = \frac{c}{n}$. Let $G \sim G(n,p)$. Our object is to show

$$\liminf_{n\to\infty} \frac{E[\text{CR}(G)]}{\left[\binom{n}{2}p\right]^2} > 0$$

As c is constant this is equivalent to showing that for n sufficiently large

$$E[\text{CR}(G)] > \delta n^2$$

for some δ dependent only on c.

We begin by reviewing in outline form the argument of Pach and Tóth [PT00a] which requires that c be a sufficiently large constant. We will see why their argument does not work for $c = 1 + \epsilon$ with $\epsilon > 0$ small and then how a modification of their argument, combined with results on $G(n,p)$, does work.

Define the *bisection width* of G, denoted by $b(G)$, as the minimal number of edges running between T (top) and B (bottom) over all partitions of the vertex set into two disjoint parts $V = T \cup B$ such that $\frac{2}{3}|V| \geq |T|, |B| \geq \frac{1}{3}|V|$. (The specific constant $\frac{2}{3}$ is not essential here, we need only to assure that the sizes of T and B are within a constant factor.) Leighton observed that there is an intimate relationship between the bisection width and the crossing number of a graph [L84], which is based on the Lipton-Tarjan separator theorem for planar graphs [LT79]. The following version of this relationship was obtained by Pach, Shahrokhi, and Szegedy [PSS96]. Let G be a graph on vertex set V with d_v denoting the degree of vertex v. Then

$$b(G) \leq 10\sqrt{\mathrm{CR}(G)} + 2\sqrt{\sum_{v \in V(G)} d_v^2}$$

With $G \sim G(n, \frac{c}{n})$, $E[d_v^2] \sim c^2 = O(1)$ and almost surely $2\sqrt{\sum_{v\in V} d_v^2} = O(\sqrt{n})$ which proves to be negligible. For c a *large* constant basic probabilistic methods give that almost surely *every* partition $V = T \cup B$ with $\frac{2}{3}|V| \geq |T|, |B| \geq \frac{1}{3}|V|$ has a constant proportion of the edges running between them. That is, almost surely $b(G) = \Omega(n)$. Hence almost surely $\mathrm{CR}(G) = \Omega(n^2)$.

Now suppose $c = 1 + \epsilon$ with $\epsilon > 0$ small. The difficulty is: almost surely $b(G)$ is zero! Why? From classic Erdős-Rényi results G will have a "giant component" of size $\sim kn$ with $k = k(c)$ and all other components will have size $O(\ln n)$. The function $k = k(c)$ was given explicitly by Erdős and Rényi but we need here only to note that $\lim_{c\to 1^+} k(c) = 0$. For ϵ a small (actually, not so small) but fixed constant and $c = 1 + \epsilon$ the giant component has size kn with $k < \frac{2}{3}$. Place the giant component in the top T. Now take all other components sequentially. Add them to the top T if $|T|$ remains below $\frac{2}{3}n$, otherwise place them in the bottom B. This gives a partition with $\frac{2}{3}|V| \geq |T|, |B| \geq \frac{1}{3}|V|$ and no edges running between T and B.

Our approach shall be to show, effectively, that the giant component of $G(n, \frac{c}{n})$ has high bisection width for any $c > 1$. To do this we employ an "enhancement" approach which we take, with only slight modification, from the work of M. Łuczak and C. McDiarmid [LM01].

Theorem 5.4.1. *Let V be a set of m vertices. Let T be a tree on V. Let G be the random graph on V with edge probability $p = \frac{a}{m}$. For $a > 0$ fixed almost surely*

$$b(T \cup G) = \Omega(m)$$

That is, there exists $\eta > 0$ dependent only on a such that $\Pr[b(T \cup G) \leq m\eta]$ *approaches zero as m approaches infinity.*

Consider partitions $V = V_1 \cup V_2$ such that T has at most $m\eta$ cut edges. For $i \leq m\eta$ we can choose i cut edges in at most $\binom{m-1}{i} \leq \binom{m}{i}$ ways and orient them (selecting one endpoint for V_1 and the other for V_2) in at most 2^i ways. As T is connected, these choices determine the partition. Hence the number of such partitions is at most

$$\sum_{i \leq m\eta} \binom{m}{i} 2^i \leq 2^{m\eta} \sum_{i \leq m\eta} \binom{m}{i} \leq 2^{m(\eta + H(\eta))}$$

where $H(\eta) := -\eta \log_2 \eta - (1-\eta)\log_2(1-\eta)$ is the standard Entropy function.

Now fix a partition $V = V_1 \cup V_2$ with $\frac{m}{3} \leq |V_i| \leq \frac{2m}{3}$ for $i = 1, 2$. Let $b((V_1, V_2); G)$ denote the number of edges $\{x, y\}$ of G with x, y in different V_i. Then

$$b((V_1, V_2); G) \sim B\left[|V_1| \cdot |V_2|, \frac{a}{m}\right]$$

where B is the Binomial Distribution. As $|V_1| \cdot |V_2| \geq \frac{2}{9}m^2$

$$\Pr[b((V_1, V_2); G) \leq m\eta] \leq \Pr\left[B(\frac{2}{9}m^2, \frac{a}{m}) \leq m\eta\right]$$

This last large deviation probability can be bounded in a number of ways. For our purposes let us assume $\eta < \frac{a}{9}$ and use that (see, e.g., the appendix of [AS92]) $\Pr[B(n,p) \leq \frac{1}{2}np] < e^{-np/8}$.

We select $\eta > 0$ such that $\eta < \frac{a}{9}$ and

$$(\ln 2)(\eta + H(\eta)) < \frac{a}{36}$$

Each of the at most $2^{m(\eta + H(\eta))}$ partitions of V which has fewer than $m\eta$ cut edges with respect to T, has probability less than $\exp[-\frac{1}{8}\frac{2a}{9}m]$ of having fewer than $m\eta$ cut edges with respect to G, and so the expected number of partitions with fewer than $m\eta$ cut edges with respect to both T and G is at most the product. Our selection of η insures that the product approaches zero, completing the proof of Theorem 5.4.1.

Now we prove Theorem 5.1.4. Let $c > 1$ be fixed. Fix c_1, a with $1 < c_1 < c$ and $c_1 + a = c$, for definiteness we may take $c_1 = \frac{1+c}{2}$ and $a = \frac{c-1}{2}$. Set $p = \frac{c}{n}$, $p_1 = \frac{c_1}{n}$ and p_2 such that $p = p_1 + p_2 - p_1 p_2$. Note $p_2 \sim \frac{a}{n}$. We may regard $G \sim G(n,p)$ as the union of independently chosen $G_1 \sim G(n, p_1)$ and $G_2 \sim G(n, p_2)$. On G_1 almost surely there exists a "giant component" X with $|X| \sim d_1 n$ where d_1 is an explicit function of c_1 given in the classic Erdős-Rényi papers. Set $m = |X|$. Then $p_2 m \sim a d_1$. Then $G_1|_X$ contains some tree T. From Theorem 5.4.1 there is a constant η such that almost surely $(T \cup G_2)|_X$ has bisection width at least $m\eta \sim n(d_1\eta)$. As adding edges can only increase the bisection width $b(G|_X) > n(d_1\eta)(1 - o(1))$. Applying the basic relationship between bisection width and crossing numbers gives $\mathrm{CR}(G|_X) = \Omega(n^2)$. Hence $\mathrm{CR}(G) \geq \mathrm{CR}(G|_X) = \Omega(n^2)$.

`Comments and Open Questions:` From Theorem 5.1.1 we know that $\kappa_{\mathrm{CR}}(n, p) \to \gamma_{\mathrm{CR}}$ as $p \to 1$ and we have just shown that $\kappa_{\mathrm{CR}}(n, \frac{c}{n})$ is bounded from below. How large does $p = p(n)$ need to be so that $\kappa_{\mathrm{CR}}(n, p(n)) \sim \gamma_{\mathrm{CR}}$? We have already conjectured that for any $p = p(n)$ with $np \to \infty$ we have $\kappa_{\mathrm{CR}}(n, p) \to \gamma_{\mathrm{CR}}$. But we cannot even show that $\kappa_{\mathrm{CR}}(n, p) \to \gamma_{\mathrm{CR}}$ when $p < 1$ is a constant. Suppose (which is surely true though we are unable to show it) that $\lim_n \kappa_{\mathrm{CR}}(n, \frac{c}{n})$ exists and call it $f_{\mathrm{CR}}(c)$. Then $f_{\mathrm{CR}}(c)$ would be increasing so $\lim_{c\to\infty} f_{\mathrm{CR}}(c)$ would exist but might be a value strictly less than γ_{CR}. Would there be a second (or even a third or more) region (something like $p = \Theta(n^{-1/2})$ or, more likely, $p = \Theta(1)$) where $\kappa_{\mathrm{CR}}(n, p)$ increases (in some asymptotic sense) until it finally reaches γ_{CR}?

5.5 The rectilinear crossing number

Here we show Theorem 5.1.5. An *order type* of the points $x_1, x_2, \ldots, x_n$ in the plane (with no three colinear) is a list of orientations of all triples $x_i x_j x_k$, $i < j < k$ [GP86]. Elementary geometry gives that the order type of the four triples $x_i x_j x_k, x_i x_j x_l, x_i x_k x_l, x_j x_k x_l$ determines whether or not the straight line segments $x_i x_j$ and $x_k x_l$ intersect. Let X be the set of all order types of the points $x_1, x_2, \ldots, x_n$ in the plane. We shall make critical use of a result of Goodman and Pollack [GP86, GP89] that $|X| < n^{6n}$. We note that the Goodman-Pollack result is derived from the Milnor-Thom theorem, a now classical and very deep result concerning algebraic varieties.

First, however, we examine a fixed order type $\xi \in X$. For any graph G with vertices $v_1, \ldots, v_n$ let $\mathrm{LIN\text{-}CR}_\xi(G)$ denote the number of crossings in the straight line drawing of G where v_i is placed at x_i in the plane and $x_1, \ldots, x_n$ have order type ξ.

Theorem 5.5.1. *Let $G(n, p)$ be a random graph with vertices $v_1, v_2, \ldots v_n$, with edge probability $0 < p = p(n) < 1$, and let $e = p\binom{n}{2}$. Then*

$$\Pr\left[|\mathrm{LIN\text{-}CR}_\xi(G) - \mathrm{E}[\mathrm{LIN\text{-}CR}_\xi(G)]| > 3\alpha e^{3/2}\right] < 3\exp(-\alpha^2/4)$$

holds for every α satisfying $(e/4)^3 \exp(-e/4) \leq \alpha \leq \sqrt{e}$.

Proof: We follow the approach of Pach and Tóth [PT00a]. (We note that general polynomial concentration results of Kim and Vu [KV00] could also be used.) Let $e_1, e_2, \ldots, e_{\binom{n}{2}}$ be the edges of the complete graph on $V(G)$. Define another random graph G^* on the same vertex set, as follows. If G has at most $2e$ edges, let $G^* = G$. Otherwise, there is an $i < \binom{n}{2}$ so that $|\{e_1, e_2, \ldots, e_i\} \cap E(G)| = 2e$, and set $E(G^*) = \{e_1, e_2, \ldots, e_i\} \cap E(G)$. Finally, let $f(G) = \textsc{lin-cr}_\xi(G^*)$.

The addition of any edge to G can modify the value of f by at most $2e$. Following the terminology of Alon–Kim–Spencer [AKS97], we say that the *effect* of every edge is at most $2e$. The *variance* of any edge is defined as $p(1-p)$ times the square of its effect. Therefore, the *total variance* cannot exceed

$$\sigma^2 = \binom{n}{2} p(2e)^2 = 4e^3.$$

Applying the Martingale Inequality of [AKS97], which is a variant of Azuma's Inequality [A67] (see also [AS92]), we obtain that for any positive $\alpha \leq \sigma/e = 2\sqrt{e}$,

$$\Pr\left[|f(G) - \mathrm{E}[f(G)]| > \alpha\sigma = 2\alpha e^{3/2}\right] < 2\exp(-\alpha^2/4).$$

Our goal is to establish a similar bound for $\textsc{lin-cr}_\xi(G)$ in place of $f(G)$. Obviously,

$$\Pr\left[f(G) \neq \textsc{lin-cr}_\xi(G)\right] \leq \Pr\left[G \neq G^*\right] < \exp(-e/4).$$

Thus, we have

$$|\mathrm{E}\left[f(G)\right] - \mathrm{E}\left[\textsc{lin-cr}_\xi(G)\right]| \leq \Pr\left[f(G) \neq \textsc{lin-cr}_\xi(G)\right] \max \textsc{lin-cr}_\xi(G) \leq$$

$$\exp\left(-e/4\right)\frac{n^4}{8} \leq \alpha e^{3/2},$$

whenever $\alpha \geq (e/4)^3 \exp(-e/4)$ (say). Therefore,

$$\Pr\left[|\textsc{lin-cr}_\xi(G) - \mathrm{E}[\textsc{lin-cr}_\xi(G)]| > 3\alpha e^{3/2}\right] \leq$$

$$\Pr\left[\textsc{lin-cr}_\xi(G) \neq f(G)\right] + \Pr\left[|f(G) - \mathrm{E}[f(G)]| > 2\alpha e^{3/2}\right] \leq$$

$$\exp(-e/4) + 2\exp(-\alpha^2/4).$$

If $\alpha \leq \sqrt{e}$, the last sum is at most $3\exp(-\alpha^2/4)$, as required. This concludes the proof of Theorem 5.5.1.

Now we can prove Theorem 5.1.5. Fix $p = p(n)$ with $p(n) \gg \frac{\ln n}{n}$ and $G \sim G(n,p)$. Set $e = p\binom{n}{2}$. Let $C_n = \kappa_{\text{LIN-CR}}(n, 1)$. Since $\xi \in X$, $\mathrm{E}\left[\text{LIN-CR}_\xi(G)\right] = p^2\text{LIN-CR}_\xi(K_n) \geq C_n e^2$. Let $\varepsilon > 0$ be arbitrarily small, but fixed. Then

$$\Pr\left[\text{LIN-CR}(G) < (C_n - \varepsilon)e^2\right]$$

$$\leq \sum_{\xi \in X} \Pr\left[\text{LIN-CR}_\xi(G) < \mathrm{E}[\text{LIN-CR}_\xi(G)] - \varepsilon e^2\right].$$

We apply Theorem 5.5.1 with $3\alpha e^{3/2} = \varepsilon e^2$ so that $\alpha^2/4 = \frac{1}{36}\varepsilon^2 e$. The growth rate of $p(n)$ insures that this is $o(n^{-6n})$ for any fixed positive ε. The Goodman-Pollack result critically bounds $|X| \leq n^{6n}$. Hence the sum goes to zero, as desired.

`Comments and Open Questions:` We have not been able to determine if the condition $p \gg \frac{\ln n}{n}$ in Theorem 5.1.5 is necessary. We have conjectured that for any $p = p(n)$ with $np \to \infty$ we already have $\lim_{n\to\infty} \kappa_{\text{LIN-CR}}(n,p) = \gamma_{\text{LIN-CR}}$. While the theorem of Goodman-Pollack itself cannot be improved asymptotically [A86], it might be the case that there are few (in some sense) near optimal drawings so that the $n^{-\Theta(n)}$ error probability used in the proof of Theorem 5.1.5 may not be fully necessary. This, however, remains highly speculative.

The concentration result Theorem 5.5.1 holds for the other crossing numbers as well, with essentially the same proof.

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Chapter 6

The k-set problem

6.1 Point sets with many k-sets

For any n, k, $n \geq 2k > 0$, we construct a set of n points in the plane with $ne^{\Omega\left(\sqrt{\log k}\right)}$ k-sets. This improves the bounds of Erdős, Lovász, et al. As a consequence, we also improve the lower bound for the number of halving hyperplanes in higher dimensions.

6.1.1 Introduction

For a set P of n points in the d-dimensional space R^d, a *k-set* is subset $P' \subset P$ such that $P' = P \cap H$ for some open half-space H, and $|P'| = k$. The problem is to determine the maximum number of k-sets of an n-point set in R^d. Even in the most studied two dimensional case, we are very far from the solution, and in higher dimensions even less is known.

The first results in the two dimensional case are due to Lovász, and Erdős, Lovász, Simmons and Straus [L71], [ELSS73]. They established an upper bound $O(n\sqrt{k})$, and a lower bound $\Omega\,(n \log k)$. Despite great interest in this problem [GP84], [W86], [E87], [S91], [EVW97], [AACS98], partly due to its importance in the analysis of geometric algorithms [EW86], [CP86], [CSY87], [E87], there was no progress until the very small improvement due to Pach, Steiger and Szemerédi [PSS92]. They improved the upper bound to $O(n\sqrt{k}/\log^* k)$. Recently, Dey [D98] obtained an essential improvement of the upper bound; his bound is $O(n\sqrt[3]{k})$. There was no improvement on the lower bound of Erdős et al., besides little improvements on the constant

[EW85], [E92], [E98].

Theorem 6.1.1. *For any n, k, $n \geq 2k > 0$, there exists a set of n points in the plane with $ne^{\Omega\left(\sqrt{\log k}\right)}$ k-sets.*

In the dual setting, Theorem 6.1.1 gives an arrangement of n lines such that the complexity of the k-th level (the number of intersection points having exactly k lines above them) is $ne^{\Omega\left(\sqrt{\log k}\right)}$. A similar bound was obtained by Klawe, Paterson and Pippenger [KPP82] for the complexity of the median level ($k = n/2$) in *pseudoline arrangements* (see also [GP93]). However, our construction seems to be essentially different.

Definition 1. Let $n > d \geq 2$, $n - d$ even, and let P be a set of n points in R^d in general position (no $d+1$ of them lie in the same hyperplane). A hyperplane determined by d points of P is called a *halving hyperplane* (resp. *halving line* for $d = 2$ and *halving plane* for $d = 3$) if it has exactly $(n-d)/2$ points of P on both sides.

In the plane, there is a one-to-one correspondence between complementary pairs of $n/2$-sets and halving lines [AG86] and for any fixed d, the number of halving hyperplanes is proportional to the number of $\lfloor n/2 \rfloor$-sets [E87], [DE94]. Theorem 6.1.1 is based on the following result.

Theorem 6.1.2. *For any $n > 0$ even, there exists a set of n points in the plane with $ne^{\Omega\left(\sqrt{\log n}\right)}$ halving lines.*

The k-set problem in space seems even harder than in the plane. The most interesting and studied case is $k = n/2$, i. e. finding the maximum number of *halving planes*. The first nontrivial upper bound was given by Bárány, Füredi and Lovász [BFL90]. It was improved by Aronov et al. [ACE91], Eppstein [E93] and then by Dey and Edelsbrunner [DE94] (see also [AACS98]). The best known bound, $O(n^{5/2})$, was found very recently by Sharir, Smorodinsky and Tardos [SST99]. In $d > 3$ dimensions, the trivial upper bound, $O(n^d)$ was only very slightly improved, to $O(n^{d-\varepsilon_d})$ by Živaljević and Vrećica [ZV92] (see also [ABFK92]). The best known lower bound in $d \geq 3$ dimensions, $\Omega\left(n^{d-1}\log n\right)$ follows directly from the lower bound in the plane, as described in [E87]. Using Theorem 6.1.1 and the method shown in [E87], we obtain an immediate improvement.

Theorem 6.1.3. *For any $n > 0$, $d \geq 2$, there exists a set of n points in R^d with $n^{d-1}e^{\Omega\left(\sqrt{\log n}\right)}$ halving hyperplanes.*

6.1.2 Idea of the construction

It is not hard to see and and shown in the next subsection that it is enough to consider the case $k = n/2$, i. e. the case of *halving lines*. Then the construction for other values of k can be obtained easily.

We construct a sequence of point sets, $V_0, V_1, V_2, \ldots$, recursively. For $i = 0, 1, 2, \ldots$ V_i has n_i points and at least m_i halving lines. Suppose that we already have V_{i-1} with parameters n_{i-1} and m_{i-1}. We can assume that none of the lines determined by the points is horizontal. Replace each of the points $v \in V_{i-1}$ by $a = a_i$ points, $v_1, v_2, \ldots, v_a$, lying from left to right on a short horizontal segment very close to v. Let the resulting point set be V'_{i-1}. Now we have an_{i-1} points. If the line uw is a halving line of V_{i-1} then $u_1w_a, u_2w_{a-1}, \ldots u_aw_1$ are all halving lines of V'_{i-1} (Fig. 6.1). Therefore, we get am_{i-1} halving lines. Clearly, this recursive construction would give only $m_i = O(n_i)$.

Now suppose that for each $v \in V_{i-1}$, the points $v_1, v_2, \ldots, v_a$ replacing v are placed *equidistantly* on the corresponding very short horizontal segment. Let uw be a fixed halving line of V_{i-1}. Suppose also that u lies higher than w. Then the corresponding a halving lines of V'_{i-1}, $u_1w_a, u_2w_{a-1}, \ldots u_aw_1$ pass through the same point q (Fig. 6.1). Add two more points, x and y to V'_{i-1}. Let x be a point on the horizontal line through q, very close to q and to the left of it, and let y be anywhere on the left side of the oriented line $\overline{xu_1}$ and on the right side of $\overline{xw_1}$. Then, $u_1w_a, u_2w_{a-1}, \ldots u_aw_1$ are not halving lines any more, since they have two more points on one of their sides than on the other. Observe, however, that the lines $xu_1, xu_2, \ldots, xu_a$ and $xw_1, xw_2, \ldots xw_a$ are all halving lines now. Consequently, by adding two extra points, we obtain $2a$ halving lines corresponding to the original halving line uw, instead of a, as in V'_{i-1}. We would like to add those extra points similarly for each pair $u, w \in V_{i-1}$, whenever uw is a halving line of V_{i-1}. The problem is that these extra points x and y work very well *locally* for uw, but they might ruin the other halving lines as they might be on their same side.

Once u and w are replaced by the a equidistant points, q is given, and we have very little freedom in choosing the location of x. On the other hand, we have much more freedom with y. The only way we can essentially relocate

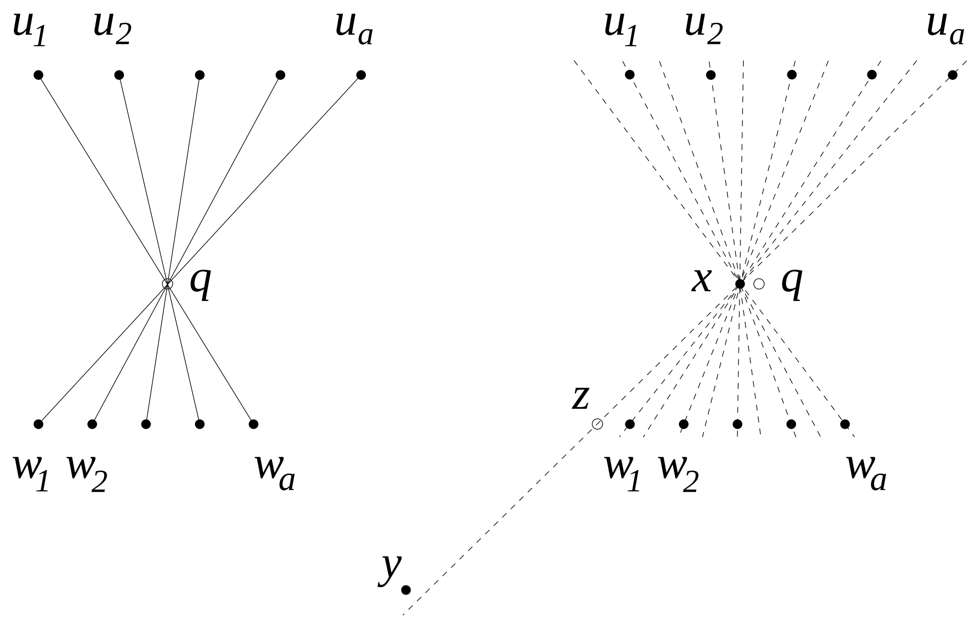

Figure 6.1: Every original halving line is replaced by $2a$ halving lines

q and hence x, is to change the distance between the consecutive points replacing u and v. In our construction, we place the extra points x and y for each halving-pair $u, w \in V_{i-1}$ and introduce some further extra points, in such a way that none of the halving lines is ruined. So, finally every original halving line is replaced by $2a$ halving lines, and the number of points is just slightly more than a times the original number of points. More precisely, $m_i = 2am_{i-1}$ and $n_i \approx an_{i-1}$. With a proper choice of $a = a_i$, this will give the desired bound.

6.1.3 Proofs of Theorems 6.1.1 and 6.1.2

First we show how Theorem 6.1.1 follows from Theorem 6.1.2, and then we prove Theorem 6.1.2.

Proof of Theorem 6.1.1. Let n, k, be fixed, $n \geq 2k > 0$, let $m = \lfloor n/2k \rfloor$, and let $m' = n - 2km$. Let $X_1, X_2, \ldots, X_m$ be the vertices of a regular m-gon, inscribed in a unit circle with center C. Let $\varepsilon > 0$ be very small and let $X_i(\varepsilon)$ be the ε-neighborhood of X_i ($i = 1, 2, \ldots, m$), and $C(\varepsilon)$ be the ε-neighborhood of C.

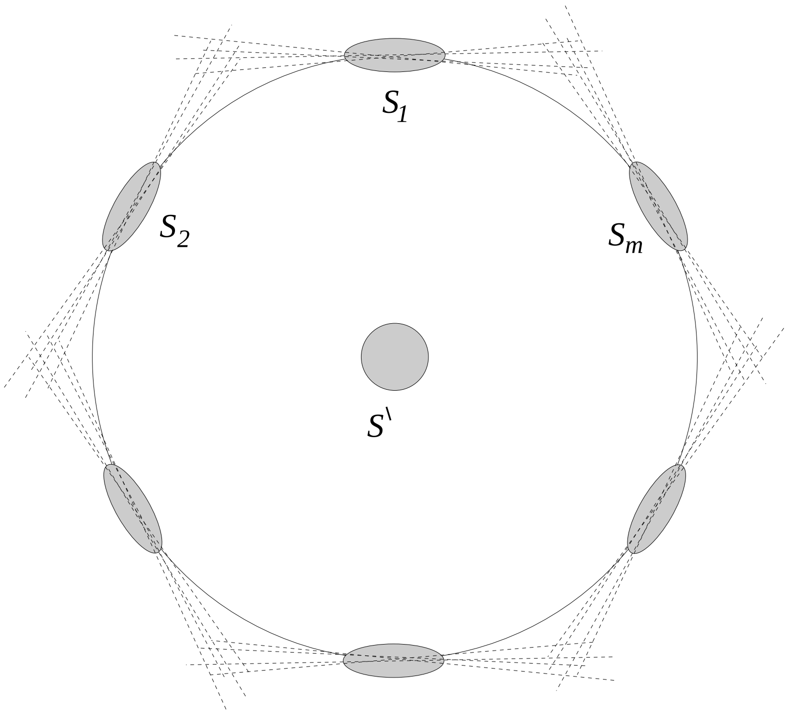

Figure 6.2: Construction for general k

By Theorem 6.1.2, there exists a $2k$-element point set S, with $2ke^{\Omega\left(\sqrt{\log k}\right)}$ halving lines. For any $1 \leq i \leq m$ apply a suitable affine transformation A_i to S such that $A_i(S) = S_i \subset X_i(\varepsilon)$ and for any halving line ℓ of S_i, all $X_j(\varepsilon)$, $1 \leq j \leq m$, $j \neq i$, are on the same side of ℓ. Finally, let S' be a set of m' points in $C(\varepsilon)$. Then the set $T = S' \cup_{i=1}^{m} S_i$ has $m2k + m' = n$ points and $m2ke^{\Omega\left(\sqrt{\log k}\right)} = ne^{\Omega\left(\sqrt{\log k}\right)}$ k-sets (Fig. 6.1.3). $\square$

Definition 2. For a positive integer a and $\varepsilon > 0$, let $P(a, \varepsilon)$ be a set of a equidistant points lying on a horizontal line such that the distance between the first and last points is ε. Then $P(a, \varepsilon)$ is called an (a, ε)-*progression*. We say that a point p is *replaced* by an (a, ε)-progression, if p is identical to one of the points in the progression.

Definition 3. A *geometric graph* G is a graph drawn in the plane by (possibly crossing) straight line segments, i.e., it is defined as a pair $G =$

(V, E), where V is a set of points in general position (no three on a line) in the plane and E is a set of closed segments whose endpoints belong to V (see also [PA95]).

Proof of Theorem 6.1.2. We construct a sequence of geometric graphs $G_0(V_0, E_0)$, $G_1(V_1, E_1)$, $G_2(V_2, E_2)$, ..., recursively with the property, that for any i, every edge $e \in E_i$ is a halving line of V_i. For $i = 0, 1, 2, \ldots$, G_i has $|V_i| = n_i$ vertices and $|E_i| = m_i$ edges. Denote the *maximum degree* of a vertex in G_i by d_i.

Let G_0 have two vertices (points) and an edge connecting them. Suppose that we have already constructed G_{i-1}. Assume without loss of generality that no edge of G_{i-1} is horizontal. Let $\varepsilon = \varepsilon_i > 0$ be very small, and let $v_1, v_2, \ldots, v_{n_{i-1}}$ be the vertices of G_{i-1}. The graph $G_i(V_i, E_i)$ is constructed in three steps.

STEP 1. For $j = 1, 2, \ldots n_{i-1}$, replace v_j by an (a_i, ε^j)-progression. The exact value of $a = a_i$ will be specified later. The resulting point set is V'_{i-1}.

STEP 2. Let e be an element of E_{i-1} with endpoints u and w. Then, for some $1 \leq \alpha, \beta \leq n_{i-1}$, we have $u = v_\alpha$, $w = v_\beta$. Suppose without loss of generality that $\alpha < \beta$. Denote the points of the arithmetic progression replacing u (resp. w) by $u_1, u_2, \ldots, u_a$ (resp. $w_1, w_2, \ldots, w_a$). Let q be the intersection of the lines $u_1 w_a, u_2 w_{a-1}, \ldots u_a w_1$ (Fig. 6.1). Add two more points, x and y to the point set as follows.

Place x so that xq is horizontal, x is to the left of q and the distance $\overline{xq}$ is so small that for $1 \leq j < a$, the line xu_j separates $w_1, w_2, \ldots w_{a-j}$ from $w_{a-j+1}, \ldots w_a$, and similarly, the line xw_j separates $u_1, u_2, \ldots u_{a-j}$ from $u_{a-j+1}, \ldots u_a$.

Finally, let z be the intersection point of the line xu_a with the line passing through $w_1, w_2, \ldots, w_a$, and place y so that the vectors $\overrightarrow{qz}$ and $\overrightarrow{zy}$ are equal. (see Fig. 6.1).

Add the edges $\{xu_1, xu_2, \ldots, xu_a, xw_1, xw_2, \ldots, xw_a\}$ to E_i.

Since ε is very small and $\alpha < \beta$, we obtain that x and y are in a small neighborhood of w. Moreover, $w_1, w_2, \ldots, w_a$ must be very close to the midpoint of the segment xy. Therefore, any line vw, with $w \in \{w_1, w_2, \ldots, w_a\}$,

$v \in V'_{i-1}$, and $v \notin \{u_1, u_2, \ldots, u_a\}$, intersects the segment xy very close to its midpoint, in particular, it separates x and y.

Execute STEP 2 for every edge $e \in E_{i-1}$.

STEP 3. Let u be an element of V_{i-1}. In STEP 1, we replaced u by an (a, ε^j)-progression, say $\{u_1, u_2, \ldots, u_a\}$, from left to right. In STEP 2, we possibly placed some pairs of points in a small neighborhood of u. Denote the number of those points by $2D$. For each edge of G_{i-1} adjacent to u, we placed zero or two points in the neighborhood of u, and the number of those edges is at most d_{i-1}. Therefore, we have $D \leq d_{i-1}$.

Place $d_{i-1} - D$ points on the line of $\{u_1, u_2, \ldots, u_a\}$, to the left of u_1, such that their distance from u_1 is between ε and 2ε. Analogously, place $d_{i-1} - D$ points on the line of $\{u_1, u_2, \ldots, u_a\}$, to the right of u_a, such that their distance from u_a is between ε and 2ε (see Fig. 6.1.3).

Execute STEP 3 for every vertex $u \in V_{i-1}$, and finally, perturb the points very slightly so that they are in general position. Let $G_i(V_i, E_i)$ be the resulting geometric graph.

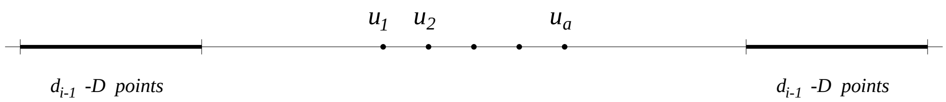

Figure 6.3: Each vertex is replaced by $a + 2d_{i-1}$ points

Claim 6.1.4. *All edges in E_i, introduced in* STEP 2*, are halving lines of V_i.*

Proof of Claim 6.1.4. Let $e \in E_{i-1}$ be any edge of G_{i-1} with endpoints $u, w \in V_{i-1}$. Use the notations introduced in STEP 2. Let $1 \leq j \leq a$. We know that the line xu_j separates $w_1, w_2, \ldots w_{a-j}$ from $w_{a-j+1}, \ldots w_a$. Therefore, it is a halving line of the point set $\{x, y, u_1, u_2, \ldots, u_a, w_1, w_2, \ldots, w_a\}$. All the other points in the neighborhoods of u and w are introduced in pairs, one on each side of the line xu_j. Since uw is a halving line of V_{i-1}, there are exactly $(n_{i-1} - 2)/2$ points of V_{i-1} on both sides of uw, and each of them are replaced by exactly $a + 2d_{i-1}$ points in their small neighborhoods. Therefore, we can conclude that the number of points of V_i, lying on different sides of uw are the same. $\Box$

Each vertex of G_{i-1} is replaced by $a + 2d_{i-1}$ points. Therefore, $|V_i| = n_i = (a + 2d_{i-1})n_{i-1}$. For each edge $e \in E_{i-1}$, we introduced $2a$ edges in E_i. Consequently, $|E_i| = m_i = 2am_{i-1}$. Let $a = 4d_{i-1}$. Then we have

$$n_i = 6d_{i-1}n_{i-1}, \tag{6.1}$$

$$m_i = 8d_{i-1}m_{i-1}. \tag{6.2}$$

Now we calculate d_i. There are three types of points in V_i.

1. Those points which are introduced in Step 1. They have the same degree in G_i as the original point in G_{i-1}. Hence, the maximum degree of those points is d_{i-1}.

2. Those points which are introduced in Step 2. Half of them have degree zero, the other half has degree $2a = 8d_{i-1}$.

3. Those points which are introduced in Step 3. They all have degree zero.

Therefore, for $i > 0$, the maximum degree is $d_i = 8d_{i-1}$. Since $d_0 = 1$, we have $d_i = 8^i$. Using (6.1) and $n_0 = 2$,

$$n_i = 2 \cdot 6^i \cdot 8^{1+2+\cdots+(i-1)} = 8^{\frac{i^2}{2}+(\log_8 6-\frac{1}{2})i+\frac{1}{3}}.$$

Analogously, using (6.2) and $m_0 = 1$,

$$m_i = 8^i \cdot 8^{1+2+\cdots+(i-1)} = 8^{\frac{i^2}{2}+\frac{i}{2}}.$$

Therefore,

$$m_i = n_i 8^{(1-\log_8 6)i-\frac{1}{3}} = n_i e^{\Omega\left(\sqrt{\log n_i}\right)}.$$

This proves Theorem 6.1.2 if n is of the form $2 \cdot 6^i \cdot 8^{1+2+\cdots+(i-1)}$ for some $i \geq 0$. It is not hard to extend the result for every n, using the following easy and well known results [L71], [ELSS73], [E87]. Let $f(n)$ be the maximum number of halving lines of a set of n points in the plane.

Claim 6.1.5. *For $a, n > 0$,* (i) $f(an) \geq af(n)$*, and* (ii) $f(n+2) \geq f(n)$.

Proof of Claim 6.1.5. Let P be a set of n points with $f(n)$ halving lines and suppose that no line determined by the points of P is horizontal. For (i), replace each point of P by an (a, ε^j)-progression. (See also the previous section and Fig. 6.1.)

For (ii), add two points to P, one very far from P to the left and one very far to the right. Then all halving lines of P are halving lines of the new point set. $\square$

This concludes the proof of Theorem 6.1.2.

6.1.4 Proof of Theorem 6.1.3

Let $f_d(n)$ be the maximum number of halving hyperplanes of a set of n points in R^d.

Claim 6.1.6. *For $n > 0$, $f_d(n+2) \geq f_d(n)$.*

Proof of Claim 6.1.6. The proof is analogous to the proof of Claim 6.1.5 (ii). $\square$

Suppose for simplicity that d is even. For d odd, the proof is analogous. By Claim 6.1.6, we can assume without loss of generality that n is divisible by 6. Let P_1 be a set of $n/3$ points in the intersection of the hyperplanes $x_1 = 0$ and $x_2 = 1$ such that no $d-1$ of them lie in a common $d-3$ dimensional affine subspace. Let $P_2 = -P_1$ that is, P_2 is the reflection of P_1 about the origin. Any hyperplane that contains the x_1-axis and avoids P_1, also avoids P_2 and cuts the set $P_1 \cup P_2$ into two equal subsets. Let P_3 be a set of $n/3$ points in the plane spanned by the x_1 and x_d axes, with $ne^{\Omega\left(\sqrt{\log n}\right)}$ halving lines, such that the points of P_3 are very close to the origin, and all halving lines have very little angles with the x_1-axis. Now any hyperplane which contains a halving line of P_3 and avoids $P_1 \cup P_2$, is a halving hyperplane of the set $P_1 \cup P_2 \cup P_3$. Since for any halving line of P_3, there are $\Omega\left(n^{d-2}\right)$ combinatorially different such hyperplanes, Theorem 6.1.3 follows.

Remarks. 1. The proof of Theorems 6.1.1 and 6.1.2 imply the lower bound $ne^{0.282\sqrt{\ln k}-2.1}$ for the number of k-sets. If we use a better choice for the value of a_i, a proper ordering of the vertices of G_{i-1} before Step 1, and place the additional points in Step 3 more carefully, we can obtain the lower bound $ne^{0.744\sqrt{\ln k}-2.7} > \frac{n}{20}2^{\sqrt{\ln k}}$.

2. Based on Theorem 6.1.3 and the proof of Theorem 6.1.1, it is not hard to construct an n-element point set in R^d with $nk^{d-2}e^{\Omega\left(\sqrt{\log k}\right)}$ k-sets.

6.2 Monotone paths in line arrangements

We show that for any n there is an arrangement of n lines which contain an x-monotone path of length $\Omega(n^{7/4})$.

6.2.1 Introduction

Properties of line arrangements in the plane (see [PA95]) have been intensively studied, partly because of their importance in the construction and analysis of geometric algorithms (see [E87]). One of the most important and studied such problems is the k-level problem. The k-level of an arrangement of n lines is the closure of the the set of points of the lines with the property that there are exactly k lines pass below them. The k-level of a line arrangement is an x-monotone polygon (path) which has a turn in each of the line intersections on it. Its length is the number of turns plus one, which is called the complexity of the k-level. The k-level problem asks for the maximum complexity of the k-level in an arrangement of n lines. The best known upper bound is $O(n\sqrt[3]{k})$ [D98], and the best known lower bound is $ne^{\Omega\left(\sqrt{\log k}\right)}$ (see Theorem 6.1.1) for any $n \geq 2k$.

In this note we consider a generalization of this problem, when the polygon does not necessarily have a turn in each of the intersections on it. In other words, we want to find the maximum length of an x-monotone path in an arrangement of n lines in the plane. The length of the path is the number of turns plus one. Sharir (see [EG89], [E87]) established an $\Omega(n^{3/2})$ lower bound. Matoušek [M91] improved it to $\Omega(n^{5/3})$. Yamamoto et. al. [YKII88] found an interesting application of this problem.

Theorem 6.2.1. *For any $n > 0$ there exists an arrangement of n lines which contain a monotone path of length $\Omega(n^{7/4})$.*

Obviously, there are at most $\binom{n}{2}$ intersection points in any arrangement of n lines, so a monotone path has length at most $\binom{n}{2} + 1$. We very slightly improve this trivial upper bound (see Remarks).

6.2.2 Proof of Theorem 6.2.1

We construct an arrangement of at most n lines which contain a monotone path of length $\Omega(n^{7/4})$. We define it in three steps. For any arrangement $\mathcal{A}$ of lines, $|\mathcal{A}|$ denotes the number of lines in $\mathcal{A}$.

Step 1. For any $m > 0$, let $\mathcal{A}^1_m$ be an arrangement of $2m$ lines, arranged into two bundles of m parallel lines, called the row bundle $\mathcal{R}^1_m$ and the column bundle $\mathcal{C}^1_m$. More precisely, let

$$\mathcal{R}^1_m = \{(y = i) \mid i = 1, 2, \ldots m\},$$

$$\mathcal{C}^1_m = \{(x - y = i) \mid i = 1, 2, \ldots m\},$$

and let $\mathcal{A}^1_m = \mathcal{R}^1_m \cup \mathcal{C}^1_m$. Clearly, there is a monotone path of length $2m$ in this arrangement (see Fig. 6.2.2).

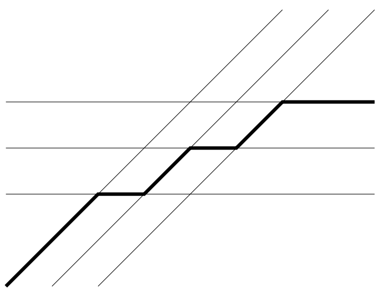

Figure 6.4: A monotone path of length $2m$

Step 2. Suppose for simplicity that $\sqrt{m}$ is an integer. Define $\mathcal{A}^2_m$, an arrangement of $3m-1$ lines, arranged into four bundles of parallel lines. Let $\varepsilon > 0$ very small, say, $\varepsilon < \frac{1}{10\sqrt{m}}$. $\mathcal{A}^2_m = \mathcal{R}^2_m \cup \mathcal{C}^2_m \cup \mathcal{U}^2_m \cup \mathcal{V}^2_m$. $\mathcal{R}^2_m$ and $\mathcal{C}^2_m$ are further subdivided into sub-bundles. $\mathcal{R}^2_m = \bigcup_{j=1}^{\sqrt{m}} \mathcal{R}^2_m(j)$ where

$$\mathcal{R}^2_m(j) = \left\{(y = \varepsilon j + \varepsilon^2 j') \mid j' = 1, 2, \ldots \sqrt{m}\right\}.$$

$\mathcal{R}^2_m(j)$ is called the *j-th row*.

Similarly, $\mathcal{C}^2_m = \bigcup_{i=1}^{\sqrt{m}} \mathcal{C}^2_m(i)$ where

$$\mathcal{C}^2_m(i) = \left\{(x - y = i + \varepsilon^2 i') \mid i' = 1, 2, \ldots \sqrt{m}\right\}.$$

$\mathcal{C}^2_m(i)$ is called the *i-th column*.

Clearly, any row $\mathcal{R}^2_m(j)$ and column $\mathcal{C}^2_m(i)$ form an arrangement isomorphic to $\mathcal{A}^1_{\sqrt{m}}$, so in the intersection of any row and column we have a monotone path of length $2\sqrt{m}$. The lines in $\mathcal{U}^2_m$ and $\mathcal{V}^2_m$ allow us to link all these monotone paths.

$$\mathcal{U}^2_m = \left\{\ell_{i,j} \mid i = 1, 2, \ldots \sqrt{m}, j = 1, 2, \ldots \sqrt{m} - 1\right\}$$

where

$$\ell_{i,j} = (2x - y = 2(i + (j + \frac{1}{2})\varepsilon) - (j + \frac{1}{2})\varepsilon = 2i + (j + \frac{1}{2})\varepsilon).$$

$$\mathcal{V}^2_m = \left\{\ell'_i \mid i = 1, 2, \ldots \sqrt{m} - 1\right\}$$

where

$$\ell'_i = (2x + y = 2i + 1).$$

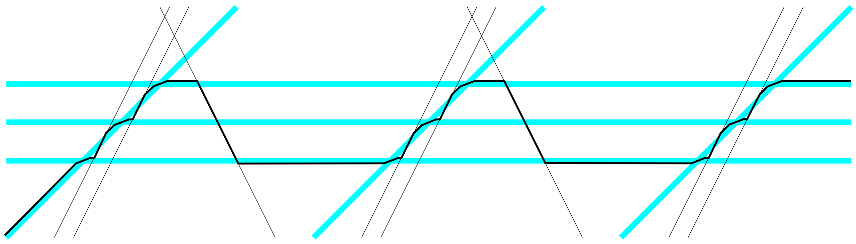

Figure 6.5: A monotone path of length at least $m^{3/2}$

Now we have the following monotone path. Start with a monotone path of length $2\sqrt{m}$ in the intersection of the first row and first column. We leave the intersection on the highest line in the first row. Then we use $\ell_{1,1}$ to go up to the highest line in the first column, below its intersection with the second row, and then we go along the monotone path of length $2\sqrt{m}$ in the intersection of the second row and first column. After leaving the intersection, we use $\ell_{1,2}$ to reach again the highest line in the first column, and we continue analogously, until leaving the intersection of the last row and first column. Then we go down on ℓ'_1 to the lowest line of the first row, and proceed similarly along the second column, then the third column, until the last column. This path includes a monotone path of length $2\sqrt{m}$ in the intersection of each row and column. Therefore, the length is at least $2m\sqrt{m} > m^{3/2}$ (see Fig. 6.5).

Step 3. First we define $\mathcal{A}^3_m = \mathcal{R}^3_m \cup \mathcal{C}^3_m \cup \mathcal{U}^3_m \cup \mathcal{V}^3_m$, $|\mathcal{A}^3_m| < 6m$. Assume that $\sqrt[4]{m}$ is an integer. $\mathcal{R}^3_m$ is divided into $\sqrt{m}$ bundles of $\sqrt{m}$ parallel lines, called the *rows*, and each row is further subdivided into $\sqrt[4]{m}$ sub-bundles of $\sqrt[4]{m}$ parallel lines. More precisely, $\mathcal{R}^3_m = \bigcup_{i=1}^{\sqrt{m}} \mathcal{R}^3_m(i)$ and $\mathcal{R}^3_m(i)$ is called the *i-th row*, $\mathcal{R}^3_m(i) = \bigcup_{j=1}^{\sqrt[4]{m}} \mathcal{R}^3_m(i,j)$ where

$$\mathcal{R}^3_m(i,j) = \left\{(y = i + \varepsilon^2 j + \varepsilon^3 k) \mid k = 1, 2, \ldots \sqrt[4]{m}\right\},$$

so

$$\mathcal{R}^3_m(i) = \left\{(y = i + \varepsilon^2 j + \varepsilon^3 k) \mid j = 1, 2, \ldots, \sqrt[4]{m},\ k = 1, 2, \ldots \sqrt[4]{m}\right\}.$$

Similarly, $\mathcal{C}^3_m = \bigcup_{i=1}^{\sqrt{m}} \mathcal{C}^3_m(i)$ and $\mathcal{C}^3_m(i)$ is called the *i-th column*, $\mathcal{C}^3_m(i) = \bigcup_{j=1}^{\sqrt[4]{m}} \mathcal{C}^3_m(i,j)$ where

$$\mathcal{C}^3_m(i,j) = \left\{(x - y = i + \varepsilon j + \varepsilon^3 k) \mid k = 1, 2, \ldots \sqrt[4]{m}\right\},$$

so

$$\mathcal{C}^3_m(i) = \left\{(x - y = i + \varepsilon j + \varepsilon^3 k) \mid j = 1, 2, \ldots, \sqrt[4]{m},\ k = 1, 2, \ldots \sqrt[4]{m}\right\}.$$

Consider any row $\mathcal{R}^3_m(i)$ and column $\mathcal{C}^3_m(i')$. The arrangement $\mathcal{R}^3_m(i) \cup \mathcal{C}^3_m(i')$ is isomorphic to $\mathcal{R}^2_{\sqrt{m}} \cup \mathcal{C}^2_{\sqrt{m}}$ from the arrangement $\mathcal{A}^2_{\sqrt{m}}$. Let $\mathcal{U}^3_m(i,i')$ (resp. $\mathcal{V}^3_m(i,i')$) be the copy of $\mathcal{U}^2_{\sqrt{m}}$ (resp. $\mathcal{V}^2_{\sqrt{m}}$) under the same isomorphism. Let

$$\mathcal{U}^3_m = \bigcup_{i=1}^{\sqrt{m}} \bigcup_{i'=1}^{\sqrt{m}} \mathcal{U}^3_m(i,i'),$$

and

$$\mathcal{V}^3_m = \bigcup_{i=1}^{\sqrt{m}} \bigcup_{i'=1}^{\sqrt{m}} \mathcal{V}^3_m(i,i').$$

In other words, for any row $\mathcal{R}^3_m(i)$ and column $\mathcal{C}^3_m(i')$, add the lines corresponding to $\mathcal{U}^2_{\sqrt{m}}$ and $\mathcal{V}^2_{\sqrt{m}}$ so that we get an arrangement isomorphic to $\mathcal{A}^2_{\sqrt{m}}$.

Because of the slopes of the lines in $\mathcal{U}_m^3(i,i')$, $\mathcal{U}_m^3(i,i') = \mathcal{U}_m^3(i+2,i'-1)$ and $|\mathcal{U}_m^3(i,i')| < \sqrt{m}$, therefore, $|\mathcal{U}_m^3| < 3\sqrt{m}\sqrt{m} = 3m$. Similarly, $\mathcal{V}_m^3(i,i') = \mathcal{V}_m^3(i+2,i'-3)$ and $|\mathcal{V}_m^3(i,i')| < \sqrt[4]{m}$, therefore, $|\mathcal{V}_m^3| < 5\sqrt{m}\sqrt[4]{m} < m$. Cearly, $|\mathcal{R}_m^3| = |\mathcal{C}_m^3| = m$, so $|\mathcal{A}_m^3| < 6m$ (see Fig. 6.6).

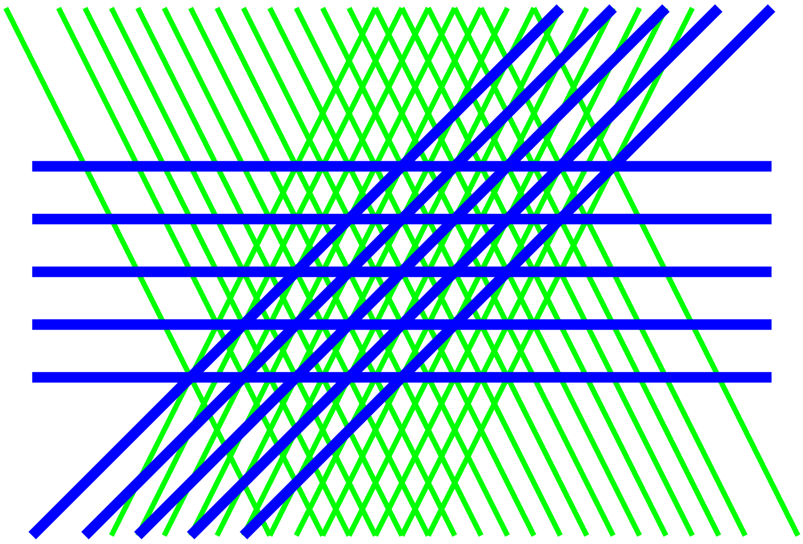

Figure 6.6: Our construction.

In $\mathcal{A}_m^3$, in the crossing of any row and column we have an arrangement isomorphic to $\mathcal{A}_{\sqrt{m}}^2$, so there is a monotone path of length at least $(\sqrt{m})^{3/2} = m^{3/4}$. We want to link all of them with some additional lines, just like in the construction of $\mathcal{A}_m^2$. The problem is that the crossing of row $\mathcal{R}_m^3(i)$ and column $\mathcal{C}_m^3(i')$ is exactly below the crossing of $\mathcal{R}_m^3(i+1)$ and $\mathcal{C}_m^3(i'-1)$. Let T be an affine transformation, $T(x,y) = (x+\sqrt{\varepsilon}y, y)$ and let $\mathcal{B}_m^3 = T(\mathcal{A}_m^3)$. It is not hard to see that all lines with positive (resp. negative) slopes will still have positive (resp. negative) slopes. So, in the crossing of any row and column of $\mathcal{B}_m^3$ we still have a monotone path of length $m^{3/4}$. But now, for ε small enough, say, $\varepsilon < \frac{1}{10m}$, all crossings of the rows and columns can be separated from each other by vertical lines. These lines can be perturbed to lines of very large positive or negative slopes, such that they can be used to link the monotone paths in consecutive crossings. Let $\mathcal{L}$ be the set of these lines. Then $|\mathcal{L}| = m-1$, so $|\mathcal{B}_m^3 \cup \mathcal{L}| < 7m$.

There are m disjoint row-column crossings, and in each of them we have a monotone path of length at least $m^{3/4}$ so the monotone path containing all of them has length at least $m^{3/4}m = m^{7/4}$.

Remarks. 1. As mentioned in the introduction, a monotone path has length at most $\binom{n}{2} + 1$ in any arrangement of n lines. This can be improved by the following observation. Take a monotone path of length $5m$ and divide it into m intervals, each of length 5. Notice that above or below each of these intervals there is a crossing of the lines which is not on the path. Therefore, if there are n lines and a monotone path of length k, then $\binom{n}{2} \geq k - 1 + \lfloor k/5 \rfloor$ so $5n^2/12 > k$. With a slightly more careful analysis one can show that $n^2/4 > k$, but we were unable to give a $o(n^2)$ upper bound.

2. If instead of the number of turns, we define the length of the path as the number of intersection points on it, it is easy to construct an arrangement of n lines with a monotone path of length $\Omega(n^2)$.

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