Answers to the referee report of László Györfi

I would like to thank László Györfi for his positive report, for a careful reading of my dissertation and for the pertinent questions he posed. I will answer each of them in detail below.

Question 1: *Knowing the distributions of the price process, is there any result how to construct ϕ^* or its approximation?*

Standard numerical maximization procedures are applicable in the one-step case. They may have multiple local optima though due to the lack of (strict) concavity.

In the multistep-case, however, a dynamic programming procedure needs to be performed which is rather costly. I see hope only for Ω finite (say, with a tree structure such as binomial or trinomial trees): in this case one-step maximization can be combined with dynamic programming easily and the optimal strategy can be found.

For general Ω, it seems feasible to approximate the probability space with finite ones Ω_n. However, the respective optimal strategies do not necessarily converge (due to the lack of uniqueness). I expect that such a sequence of strategies will have a condensation point that is optimal on Ω but I know of no such result in the literature. It seems to require rather tedious estimates, in the spirit of Theorem 2.49.

Question 2: *Not knowing the distributions of the price process, the problem is more difficult, because the components of the price process have positive growth rate, therefore the components of the price-difference process are not stationary. Is there any result how to estimate ϕ^* or its approximation, if the relative price processes S^j_{t+1}/S^j_t are stationary and ergodic, $j = 1, \ldots, d$?*

This question leads quite far into uncharted waters. Let us define the simplex $\Sigma := \{x \in \mathbb{R}^d_+ : \sum_{j=1}^d x^j = 1\}$. We will use multiplicative parametrization where strategies π lie in Σ and π^j represents the proportion of wealth allocated to asset j. For simplicity, let $d := 2$ and consider only constant proportion strategies. That is, the strategy is described by $\pi \in [0, 1]$ representing the constant proportion of wealth allocated to the stock (the rest is allocated to the bond). Assuming interest rate $r \geq 0$, a standard problem would be to consider maximizing

$$\liminf_{T\to\infty} \frac{1}{T} \ln Eu(V_T(\pi))$$

with $u(x) := x^p/p$, $p < 1$, $p \neq 0$ and $V_T(\pi) = V_0 \prod_{t=1}^T (\pi(S_t/S_{t-1}) + (1-\pi)(1+r))$, the wealth corresponding to strategy π. This can be rewritten as a risk-sensitive control problem. These are well-studied for Markovian S, however, the usual Bellmann equation approach requires the knowledge of the distribution. If S is a Markov chain then a stochastic approximation scheme has been proposed in [*] which could perhaps be adapted to determine the optimal π^*, without knowing the distributions.

The setting of [*] looks, however, very restrictive and it is unclear how to develop implementable stochastic approximation schemes for risk-sensitive cost functions in general. To highlight the degree of difficulty, a similitude can be formulated as follows: if u is logarithmic then we stay in the realm of ergodic control (laws of large numbers) while for u a power function, we enter the arena of risk-sensitive control (large deviations). It would be important to cover the case of non-logarithmic u since these correspond better to the observed behaviour of market paticipants.

Here I reflected only on the case of concave u. The non-concave case looks completely out of reach.

Question 3: *Is it possible to show that*

$$Eu(z + X_T^{z,\phi^*} - B) > Eu\left(z + \sum_{j=1}^{d} X_T^{z/d,\phi_j^*} - B\right) \tag{1}$$

if the components of the price process are independent ?

The answer is no, in general. Let $z = B = 0$, $T = 1$ and $d = 2$, that is, we consider a one-step model with two assets, 0 initial capital and 0 reference random variable. Furthermore let ΔS_1^i, $i = 1, 2$ be independent. If we choose the popular exponential utility $u(x) = -e^{-x}$, $x \in \mathbb{R}$ it is clear that, for any strategies ϕ_1, ϕ_2 (representing the holdings in assets 1 and 2) $Eu(\phi_1 \Delta S_1^1 + \phi_2 \Delta S_1^2) = -Eu(\phi_1 \Delta S_1^1) Eu(\phi_2 \Delta S_1^2)$. This means that utility maximization can be performed separately in the two assets. In particular, the pair of minimzers for trading in the respective single assets, ϕ_1^*, ϕ_2^*, provide the global minimizer in the two-asset problem as well. That is, equality holds in (1).

Clearly, if we relax the independence hypothesis on ΔS_1^i then strict inequality may arise in (1). For instance, if $\Delta S_1^1 = \Delta S_1^2$ is Gaussian with unit mean and unit variance then the maximizer for the single assets is $\phi_1^* = \phi_2^* = 1$, by direct calculation. However, the optimizer for the market with both assets is any ψ_1, ψ_2 with $\psi_1 + \psi_2 = 1$. Thus, (1) holds in this case since

$$-Ee^{-\psi_1 \Delta S_1^1 - \psi_2 \Delta S_1^2} = -Ee^{-\Delta S_1^1} > -Ee^{-2\Delta S_1^1} = -Ee^{-\phi_1^* \Delta S_1^1 - \phi_2^* \Delta S_1^2},$$

again by direct calculation.

Question 4: *Is there any result on a trading strategy ϕ, which has been derived from a non-concave utility such that the wealth process $X_t^{z,\phi}$, $t = 1, \ldots, T$ has good growth rate and risk properties?*

I am unaware of any such result about the growth rate. As far as the risk properties are concerned: the utility function u can itself provide a measure of risk (corresponding to the preferences of the given agent) and ϕ with maximal $Eu(X_T^{z,\phi})$ is a strategy that has the best risk profile at time T in this sense. By the dynamic programming principle, this property is also time consistent, that is, at any time t, the best portfolio choice for $t+1, t+2, \ldots, T$ is $\phi_{t+1}, \ldots, \phi_T$, starting from the present wealth is $X_t^{z,\phi}$. However, all this is closely linked with u. I am unaware of a result stating a "good risk property" for a u-independent criteria (e.g. for variance or other central moments).

Question 5: *In a real trading situation the transactions are executed with a positive delay δ. What happens if ϕ_t is measurable with respect to $\mathcal{F}_{t-\delta}$?*

When we assume this delayed setting the arguments go through without modification up to Theorem 4.18. However, when the existence of an optimizing strategy needs to be established we use that, roughly speaking, "anything" can be replicated by a stochastic integral. When we are allowed to use only strategies with a delay, this replication property becomes extremely delicate. We would need, e.g., that, for any function f, the functional $f(dQ/dP)$ is replicable. In the almost sure sense such a result certainly fails in general.

One may, however, try to replicate in the sense of probability laws, i.e. find a (delayed) ϕ such that the law of $f(dQ/dP)$ is the same as that of the stochastic integral with respect to ϕ. There is little chance for proving this, even in specific models.

In the setting of Section 5 (illiquid markets), one may prove Theorem 5.2 with the delayed filtration $\mathcal{F}_{t-\delta}$ replacing the original one $\mathcal{F}_t$ in the definition of admissible strategies, without any changes in the proof.

Question 6: *Do we lose generality, if the strategies are piecewise constant?*

One may formulate the problem of expected utility over piecewise constant strategies (i.e. where the strategies ϕ are step functions). There are two natural questions: Does the optimization problem have the same value ? It is attained by a piecewise constant strategy ?

For the second question there is no hope for a positive answer in general, since the portfolio values corresponding to piecewise constant strategies do not form a closed set in any reasonable topology hence one cannot expect to find an optimizer in it (only in its appropriate closure which is the whole set of strategies anyway).

The first question can be answered in the affirmative under appropriate technical assumptions. Let $\mathcal{A}_p$ denote the class of piecewise constant strategies with $\int_0^T \phi_t \, dt = 0$ and assume $B = 0$ and $G_t(x) = x^2$ for simplicity. Without going into details, if the price process S is uniformly bounded and the concave utility u is bounded above and satisfies $u(x) \geq -c|x|^\kappa$ for some $c > 0$ and $\kappa > 1$ then

$$\sup_{\phi \in \mathcal{A}_p} Eu(V_T(\phi)) = \sup_{\phi \in \mathcal{A}'} Eu(V_T(\phi))$$

holds.

[*] A. Basu, T. Bhattacharyya, V. Borkar. A learning algorithm for risk-sensitive cost. *Math. Oper. Res.*, 33:880–898, 2008.

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