Referee report on the dissertation
Optimal investment: expected utility and beyond
by Miklós Rásonyi

The dissertation considers the central topic of mathematical finance, which is the characterization of optimal investment strategies such that the optimality is formulated by expected utility. The basic model in the study is that the investor has access for d assets, and at any time he can rebalance his wealth between the assets in a self-financing way. It means that multi-asset and multi-period strategies are taken into account. Notice that most of the related literature deals with single period (step) strategies, only.

Denote by $S_t = (S_t^1, \dots, S_t^d)$, $t = 0, 1, \dots$, the $\mathbb{R}^d$ valued price process of the d assets. A strategy $\phi = \{\phi_t, t = 0, 1, \dots\}$ is an $\mathbb{R}^d$ valued process such that the j-th component ϕ_t^j of ϕ_t tells how much stocks the investor buys or sells at time t from asset j, $j = 1, \dots, d$. Chapter 2 is on the Expected Utility Theory (EUT) for discrete time model, namely it is on the existence of optimal strategy for not necessarily concave utility function u. For a fixed time horizon T and for an unknown reference random variable B, let $X_T^{z,\phi}$ be the wealth at time T, having initial capital z and using the strategy ϕ. For bounded above u, Theorem 2.1 contains conditions, under which there exists a strategy ϕ^* such that

$$\sup_{\phi} \mathbb{E}\left\{u(z + X_T^{z,\phi} - B)\right\} = \mathbb{E}\left\{u(z + X_T^{z,\phi^*} - B)\right\}.$$

It is shown by counter examples that the conditions cannot be weakened. Thus, Theorem 2.1 is sharp. For unbounded u, one has to include more conditions. Theorem 2.18 shows the interesting feature of the optimal strategy ϕ^* such that it is "one-step optimal", i.e., the optimization can be done step-wise. Notice that the results are completely distribution-free, the only condition on the price process is that it satisfies the so called no-arbitrage condition, otherwise the conditions are only about the utility function, the choice of which is in the hand of the investor.
Q1: Knowing the distributions of the price process, is there any result how to construct ϕ^* or its approximation?
Q2: Not knowing the distributions of the price process, the problem is more difficult, because the components of the price process have positive growth rate, therefore the components of the price-difference process are not stationary. Is there any result how to estimate ϕ^* or its approximation, if the relative price processes S_{t+1}^j / S_t^j are stationary and ergodic (or memoryless like in the Black-Scholes model and in the Cox-Ross-Rubinstein model), $j = 1, \dots, d$?

Chapter 3 is on the Cumulative Prospect Theory (CPT). The CPT is an extension of EUT such that for CPT, the investor takes into account separately the positive and negative values of the utilities $u(z+X_T^{z,\phi}-B)$. Here we can see some additional arguments, why in the related literature the utility function is concave for gains and it is convex for losses. Theorems 3.4 and 3.16 formulate conditions, under which optimal strategy exists for bounded above and unbounded utility, respectively. These theorems have interesting consequences (see Examples 3.20-3.23).

Q3: This question is on the possible performance gain of multi-asset strategy with respect to the single asset ones. For initial capital z/d and reference random variable B/d, let $X_T^{z/d,\phi_j^*}$ be the optimal single asset strategy driven by asset j, $j = 1, \dots, d$. Obviously,

$$\mathbb{E}\left\{u(z + X_T^{z,\phi^*} - B)\right\} \geq \mathbb{E}\left\{u\left(z + \sum_{j=1}^{d} X_T^{z/d,\phi_j^*} - B\right)\right\}.$$

Is it possible to show that

$$\mathbb{E}\left\{u(z + X_T^{z,\phi^*} - B)\right\} > \mathbb{E}\left\{u\left(z + \sum_{j=1}^{d} X_T^{z/d,\phi_j^*} - B\right)\right\}$$

if the components of the price process are independent?

Q4: Is there any result on a trading strategy ϕ, which has been derived from a non-concave utility such that the wealth process $X_t^{z,\phi}, t = 1, \dots$ has good growth rate and risk properties?

Chapter 4 is on the continuous time setup such that the evolution of the wealth is defined by a stochastic integral. Because of the existence of stochastic integral, the analysis becomes technically more involved, for example, one has to assume that the price process is semi-martingale. After several negative findings, Theorems 4.16 and 4.18 present general assumptions, under which optimal strategy exists.

Q5: In a real trading situation the transactions are executed with a positive delay δ. What happens if ϕ_t is measurable with respect to $\mathcal{F}_{t-\delta}$?

In Chapters 2-4, the transactions (buying or selling assets) don't influence the prices. In practice, this condition is approximately satisfied if the capitalizations of the corresponding assets are large and the trader is small. Chapter 5 considers the case of illiquid markets, when buying increases the execution price, while selling decreases it. The results are formulated in the continuous time model. For the readers, maybe they would be more transparent in the discrete time model. Most of all, I liked this chapter, because it is related to the problem of transaction cost. If $\alpha = 1$ and H_t is almost constant, then it is the model of the fixed transaction cost, where the transaction cost is approximately proportional to the traded volume (amount of stocks). Notice, that Assumption 5.3 excludes $\alpha = 1$. For the case of proportional transaction cost, the cost is constant times the traded wealth. Interestingly, for proportional transaction cost and for log-utility, the optimal strategy is not "one-step optimal" even in the case of memoryless relative price processes. For concave utility, Theorem 5.12 is on the existence optimal strategy for illiquid market.

Q6: Do we loose generality, if the strategies are piecewise constant?

The dissertation is based on eight articles. In most part of the work the notations were unified, however, some of them remained different, for example, strategies, set of strategies, inner product, etc. The preliminary results are summarized in Chapter 1 and in the

Appendix, causing many back and forth jumps for the reader. I would prefer to have the Appendix at the end of Chapter 1.

Summary: The dissertation contains solutions of several challenging and important problems of mathematical finance. I acknowledge that the results are novel and in some cases pioneering. I learned a lot from the smart proofs and from the tricky counter examples, which use a huge arsenal of techniques from stochastic analysis. Miklós Rásonyi is one of the internationally recognized leading experts of the field. Based on the results in the dissertation, I recommend that Miklós Rásonyi be awarded by the title Doctor of the Hungarian Academy of Sciences.

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