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Tightness results for several variants of the First Fit bin packing algorithm (with help of weighting functions)

DSc Dissertation

György Dósa

University of Pannonia

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Contents

Chapter 1

Preface

Bin Packing is a classical area of Combinatorial Optimization, and First Fit (FF for short) is among the most famous algorithms. The bulk of the dissertation contains several tightness results of the author which are connected with different versions of FF. These tight results are found after forty years of the definition of the algorithm.

These results are as follows. First we consider the FFD algorithm, which is the ordered version of the FF algorithm. For any OPT value (i.e. the number of the used bins in an optimal solution) we determine at most how many bins are used by FFD. These tight values were known only for few OPT values before. By this investigation a long-standing open question is answered. Then we deal with FF, and we again give a complete characterization of the worst-case behavior of the algorithm: given the value of the optimum, we determine how large the number of the bins given by FF can be. This result includes the determination of the absolute approximation ratio of FF, a question that was open also for forty years. Then we consider the parameterized version of FF, where the size of each item is bounded above by $1/d$, for a given integer d. We make the same complete characterization of FF, answering an open question after forty years. Then we deal with FF for cardinality-constrained bin packing, where each bin can contain only a bounded number of items. For this problem we determine the tight value of the asymptotic approximation ratio of the FF algorithm, answering again a question that was open for about forty years. In the last chapter we consider two special models: Batched bin packing and Graph-bin packing. We show that by application of the FFD algorithm improved results can be obtained.

In almost all cases, we use several kinds of weighting functions. We will come across with such weighting function that was used previously, but we use it in a new, more effective form. Further ones have some new feature, composed from several parts, such kinds of weighting functions were never applied before. By the deeper understanding and more sophisticated usage of these weighting functions we became able to determine the above mentioned tight results.

Since the total of the complete proofs would be long and technical we rather give only the key ideas of the proofs. The omitted details are put into the Appendix.

In any of the listed results the author's contribution was significant; however, some of the results have been achieved in part with coauthors. At the beginning of each chapter it is made clear what is the contribution of the present author. The author is grateful for the help obtained from his coauthors and other colleagues. Special thanks are due to the "fathers", who discovered, introduced and established this nice Combinatorial Optimization area: Bin Packing!

Chapter 2

Introduction

Given a set of items with sizes $p_1, p_2, \ldots, p_n$ (positive rational numbers between 0 and 1), the Bin Packing problem (BP for short) asks for the minimum number of unit-capacity bins, into which all items can be packed in such a way that the sum of the sizes of items being packed into each bin (called level of the bin) is at most 1. The problem belongs to the area of combinatorial optimization and it is well-known to be NP-hard (see e.g. [14] or [43]). BP was "born" in the early seventies, Johnson's thesis [53] on bin packing together with Graham's work on scheduling [46, 47] belong to the early influential works that started and formed the whole area of *approximation algorithms*. These kinds of algorithms run in polynomial time, and they find optimal or near-optimal solutions, in some sense.

Similarly to other problems, generally two main versions of BP are considered, the *offline* and the *online* case. In the offline case all information about the data are known before running an algorithm, while in the online case (applying the so-called List Model) the items are revealed one by one, and each item must be packed without any information on the later items. In this dissertation we consider only the offline case, although algorithm FF (which will be defined soon) can be seen also as an online algorithm. Thus, we suppose that all items are known before the algorithm starts to make the packing.

Johnson considered several "Fit-type" algorithms in his thesis, like First Fit, Best Fit, and others. We define these algorithms below. (Algorithm First Fit probably appears first in the work of Ullman in 1971 [74].)

The *First Fit* (FF for short) algorithm packs the items in a given order. The next item is always packed into the first bin where it fits, possibly opening a new bin if the item does not fit into any currently open bin. We call the algorithm *First Fit Decreasing* (FFD for short) if the order of the items is decreasing by sizes. In the *parameterized* version of bin packing (and the parameterized version of FF), it holds that $p_i \leq 1/d$ for every $i \in \{1, ..., n\}$ with some integer $d \geq 1$. Moreover, in case of the *Cardinality Constrained Bin Packing* problem ($CCBP$ for short), an integer parameter k is also given and the further condition is that each bin can contain at most k items.

Algorithm *Best Fit* (BF for short) packs the next item into the most loaded bin (i.e. the bin with the highest level) where it fits. If no bin can accommodate the new item, it is packed into a new bin. There are also many other algorithms (see [14] for a review), but we will deal only with some versions of FF and BF in this dissertation.

The efficiency of a bin packing algorithm usually is measured by approximation ratio. Its two main versions are the asymptotic and the absolute approximation ratios. We define them below. Let L denote the set of the items to be packed. (If the set is ordered, it is often called a *list*.) Let OPT denote an optimal offline algorithm, and A an arbitrary packing algorithm. Let $OPT(L)$ and $A(L)$ denote the number of bins created by them to pack the items of L, respectively. The absolute and asymptotic approximation ratio of A are defined as

$$R_{abs}(A) = \sup_L \{A(L)/OPT(L)\},$$

and

$$R_{as}(A) = \lim_{n\to\infty} \sup_L \{A(L)/OPT(L) \mid OPT(L) \geq n\}.$$

Later on, if it does not make any confusion, we simply omit the letter L. It turns out that the asymptotic ratio of many algorithms can be determined quite easily, but determining the tight value of the absolute ratio is much harder.

The proof that the asymptotic approximation ratio of FF bin packing is at most 1.7 was given by Ullman [74] and the matching lower bound by Garey et al. and Johnson et al. [45, 55]. These results were among the first results on approximation algorithms. An estimate about the absolute ratio appeared only in 1994, when Simchi-Levy [73] proved that it is at most 1.75 for both FF and BF. A detailed description about how these bounds have been improved in time will be given in Chapter 4. We note here only the final results about these, as we found it in our present papers [23, 24]: The tight absolute bound for FF is given as $FF \leq 1.7 \cdot OPT$. In fact, the truth is the following: For any input L it holds that

$$FF \leq \lfloor 1.7 \cdot OPT \rfloor,$$

moreover for any integer m, there is an input L such that $OPT(L) = m$ and $BF(L) = FF(L) = \lfloor 1.7 \cdot OPT(L) \rfloor$. (Algorithm BF can be seen as a generalization of FF, as it is explained in [24] in detail. It follows that any lower bound for FF applies immediately to BF, and any upper bound for BF applies also for FF.) We emphasize that in this way we give a *complete characterization* for the worst-case performance of FF. Such information was previously known only for few OPT values. In this way a question is completely answered, which question was open for forty years.

Let us consider FFD now. Johnson in his doctoral thesis [53] proved in 1973 that $FFD(L) \leq 11/9 \cdot OPT(L) + 4$ holds for every list L of items. He also showed that the asymptotic coefficient $11/9$ is tight. There were some attempts to decrease the additive term, we will see the complete history about this in the next chapter (Chapter 3). The final value is found as $FFD(L) \leq 11/9 \cdot OPT(L) + 6/9$ in our papers [20, 21]. Considering the absolute bound, it is trivial that if $OPT = 1$, then FFD creates also only one bin, while if $OPT = 2$, in the worst case FFD creates 3 bins, so the absolute bound is at least $3/2$, and it is easy to see that it is not bigger. Nevertheless, there are some questions which seem to be quite trivial, but they aren't. For example, does there exist an instance L of bin packing for which $OPT(L) = 5$ and $FFD(L) = 7$ hold? Not having the tight upper bound at hand, even this innocent-looking problem was unsolved until our recent work [79]; actually the answer is negative. So, the complete answer regarding the worst-case performance of FFD also needs to determine the next: for any given m, if $OPT = m$, what is the biggest possible value of FFD? The answer is the next. On one hand,

$$FFD(L) \leq \lfloor 11/9 \cdot OPT(L) + 6/9 \rfloor$$

holds, and on the other hand, for any integer m there is an L for which $OPT(L) = m$ and the above inequality holds with equality. It means that we could make a complete characterization regarding FFD. This question was open also for forty years. As the complete proof would need 70 pages, some parts of the proof will be put into Appendix A and Appendix B.

In Chapter 5 we deal with the parameterized model where each item has a size bounded from above $p_i \leq 1/d$ $(i = 1, ..., n)$, for a given integer d. Here we give the complete characterization of the FF algorithm in the above sense: how large can FF be in the worst case if OPT is given. This question is answered after about forty years.

In Chapter 6 we deal with bin packing with cardinality constraints ($BPCC$), there we give a short introduction to the $BPCC$ model about the existing results. In this model there is an upper bound $k \geq 2$ on the number of items that can be packed into each bin, additionally to the standard constraint on the total size of items packed into a bin. We study the algorithm FF, that acts on a list of items, packing each item into the first bin that contains at most $k-1$ items and has sufficient space for the item. We present a complete analysis of its asymptotic approximation ratio for all values of $k \geq 3$. Prior to this work, only the tight bound for $k = 2$ was known. After forty years that the problem $BPCC$ and the natural algorithm FF for it were introduced, its tight asymptotic approximation ratio for all values of k is finally found.

Finally in the last chapter (Chapter 7) we consider two related, but more difficult models, where applying FFD (as a servant algorithm) we can make efficient approximation. Actually, with my co-authors we have defined and investigated several new models in the recent years.

a, Bin packing with rejection [49, 30], here any item can be rejected (i.e. it is not packed) by some cost incurred. The total cost is the number of used bins plus the total penalty, given for the unpacked items, which is to be minimized.

b, A general model of scheduling with machine cost [32].

c, A bin covering model with a general cost function [7].

d, Batch scheduling with nonidentical job sizes [34]. In this model there are jobs with different sizes (processing times), these jobs can be scheduled in batches on a single machine or on parallel machines. The makespan is to be minimized. (This model is a common generalization of bin packing and parallel machine scheduling.) The parallel machine case was never considered before, and for the single machine case we gave an improved algorithm.

e, Three new versions of the selfish bin packing game [64, 81, 29].

f, Several new reassignment models on two related machines, for example in [80].

g, Black and White bin packing [3, 4, 5]. Here the items have size and color (black or white), and in any bin, two consecutive items cannot have the same color.

h, Colorful bin packing [28]. This is a generalization of Black and White bin packing for more colors, with a similar constraint.

i, The Graph-Bin Packing Problem [11], here a graph should be packed into another graph under several natural constraints. This is a very general model, a generalization of many other known models.

j, Multiprofessor Scheduling [35]. This is also a general model, a common generalization of several scheduling models.

In many cases when a new model is defined and investigated for the first time, it is a natural idea to apply the adaptation of "classical" algorithms, trying whether they can perform well also for the new problem. And it often turns out that they are really efficient, we made such experiments in case of many of the models listed above. There is not enough space here to describe all the above models and results, thus we consider only one of them (model i), where we deal with Batched bin packing and Graph-bin packing. These models will be defined and treated in the last chapter in details. We will see that an appropriate application of FFD performs quite well; we determine the tight asymptotic approximation ratio of the adopted algorithm.

To shed light on the "usefulness" of algorithm FF a bit more, we mention three further results.

(i) In our paper [33] we consider the "LIB" constrained online bin packing model (this model is not listed above). The "LIB" constraint means that in any bin, no item can be packed on the top of some smaller item. We concentrated only on algorithm FF (since this algorithm was proven to have the best performance ratio) and we gave a more careful analysis of FF under the LIB constraint. It is shown that the (absolute) approximation ratio of FF is not worse than $2 + 1/6 \approx 2.1666$ for the problem, which improves the previous best upper bound 2.5 of Epstein [37].

(ii) Another interesting case is model d, listed above. Here we revisit the bounded batch scheduling problem with nonidentical job sizes on single or parallel identical machines, with the objective of minimizing the makespan. For the single machine case, we present an algorithm which calls an online algorithm P (chosen arbitrarily) for the one-dimensional bin packing problem as a sub-procedure, and prove that its worst-case ratio is the same as the absolute worst-case performance of P. Our (natural) choice is to apply FF as the servant algorithm. Hence, we gain an algorithm with worst-case ratio 1.7, which is better than any known upper bound on this problem. Since this is a combined model and we would not like to expand the length of this dissertation too much (with defining the basic definitions and results also for scheduling), the whole introduction of this combined model and the improved result for the single machine case is put into Appendix E, about Batch Scheduling.

(iii) We mention also our paper [6] in which we determine the tight absolute approximation ratio for online bin packing. We already know that the tight absolute ratio of FF is 1.7, but one can detect that FF perform much better in average. So, it is a good idea to change FF, but only very slightly: When FF would create a new 2-bin with certain properties (not detailed here), sometimes we instead open a new bin for the next item (that would be packed into a 2-bin by FF, i.e. it would be packed into a bin as a second item), and keep this bin only for a later large item to be packed here. This algorithm is the first algorithm which has absolute approximation ratio exactly $5/3$, which is the value of the tight lower bound.

Above, we mentioned only directly relevant previous work for bin packing. Of course, there is much more work on this topic; in particular, for the offline case there exist asymptotic approximation schemes for this problem [76, 56] as well as many other (classical or new) algorithms.

In the online case the known best lower bound is 1.5403 by [2], and the actual best upper bound is 1.58889 due to Seiden [69], see also [86, 68, 77]. We refer to the classical survey [14], another survey concentrating on online algorithms [16], the recent surveys on several variants of bin packing [17, 18, 19], or to the recent book [78]. The interested reader can find more details about FF in [36].

Finally some words about the proofs: We apply case analysis; within this, in many cases we need the help of weighting functions. Such weighting functions were already used in the proof of the asymptotic approximation ratio of FF, see [14]. This approach is frequently used for many different problems and algorithms. We note that even in this dissertation, we need to find and apply many different kinds of weighting functions; later we will see them in details.

Chapter 3

The tight bound of algorithm FFD

First Fit Decreasing is a classical bin packing algorithm: the items are ordered by non-increasing size, and then in this order the next item is always packed into the first bin where it fits. For an instance L, $FFD(L)$ and $OPT(L)$ denote the number of bins used by algorithm FFD and by an optimal algorithm, respectively. We are going to show that

$$FFD(L) \leq 11/9 \cdot OPT(L) + 6/9.$$

This result is best possible, as we described it in the Introduction. The asymptotic coefficient $11/9$ was proved already in 1973 by Johnson, but the tight bound of the additive constant was an open question for four decades.

We apply in this section the results of [20] and [21]. The former paper of Dosa gives the tight bound as a conjecture, and divides the proof into two main cases. The proof of the first main case is completely given in [20], and the first steps are also made in the other main case (by making a possible classification of the items, and a corresponding weighting function). (Paper [20] is an extended version of a conference talk, and could not contain a long proof.) The complete proof is then appeared in [21]. This proof is 70 pages long (it is only about 50 pages in the journal, by decreasing the letter size). Regarding paper [21], the contribution of the author of this dissertation is approximately 75%. For the first main case [21] gives a new classification (5 classes instead of 6 classes as in [20]), and so the proof of the first main case is shorter. And [21] provides also a new classification for the second main case, and gives a complete proof also for this second main case.

History. Now, let us see the history of the upper bound of FFD. Johnson in his doctoral thesis [53] proved in 1973 that

$$FFD(L) \leq 11/9 \cdot OPT(L) + C$$

holds with $C = 4$ for every list L of items. He also showed that the asymptotic coefficient $11/9$ is tight; but the value of the additive constant C has quite a different story, the problem of determining the best possible C remained open for many decades. (Note that Johnson's thesis did not focus on the tight value of the additive constant at all, but contained many great results, making the basis of later investigations.)

It took more than ten years until Baker [9] published a slightly simpler proof of the upper bound $11/9 \cdot OPT(L) + C$ and showed that it is valid with $C \leq 3$. Then in 1991, Yue [87] proved it with $C = 1$; and in 2000 Li and Yue proved in [62] how the bound could be tightened further down to $C = 7/9$. In that paper the authors also conjectured that the tight additive constant is $5/9$. In [20], however, Dosa showed that $6/9$ is a lower bound for C in general (and the paper also gives a sketch of a possible proof that the bound is tight). After all those efforts the long-standing open problem concerning the least value of the additive constant is answered in [21], giving the first complete proof of the statement

$$FFD(L) \leq 11/9 \cdot OPT(L) + 6/9, \tag{3.1}$$

i.e. this is *the* tight bound. In fact, the paper proves much more.

It is trivial that if $OPT = 1$, then FFD creates also only one bin. If $OPT = 2$, in the worst case FFD creates 3 bins, but never opens more. But what happens if OPT is bigger? For instance, does there exist an instance L for which $OPT(L) = 5$ and $FFD(L) = 7$ hold? Not having the tight upper bound at hand, even this innocent-looking problem was unsolved until the recent work [79]; actually the answer is negative.

We note that from our statement (3.1), for each integer m one can determine the largest k such that $OPT(L) = m$ and $FFD(L) = k$ hold for some problem instance L. The complete table of these tight upper bounds is in [20]. In order to obtain a handy formula, let us write $OPT(L)$ in the form $OPT(L) = 9n + i$ where n is an integer and, in a somewhat unusual way, i is taken from the range $2 \leq i \leq 10$. Then

$$FFD(L) \leq \begin{cases} 11n + i + 1, & 2 \leq i \leq 5; \\ 11n + i + 2, & 6 \leq i \leq 10; \end{cases} \tag{3.2}$$

or, in equivalent form,

$$FFD(L) \leq \begin{cases} 11/9 \cdot OPT(L) + 5/9, & i = 2; \\ 11/9 \cdot OPT(L) + 3/9, & i = 3; \\ 11/9 \cdot OPT(L) + 1/9, & i = 4; \\ 11/9 \cdot OPT(L) - 1/9, & i = 5; \\ 11/9 \cdot OPT(L) + 6/9, & i = 6; \\ 11/9 \cdot OPT(L) + 4/9, & i = 7; \\ 11/9 \cdot OPT(L) + 2/9, & i = 8; \\ 11/9 \cdot OPT(L) + 0/9, & i = 9; \\ 11/9 \cdot OPT(L) - 2/9, & i = 10; \end{cases} \tag{3.3}$$

and the bounds are tight for all values of n and i. Since $FFD(L)$ always is an integer, (3.1) can be written in the form $FFD(L) \leq \lfloor 11/9 \cdot OPT(L) + 6/9 \rfloor$, and hence one can easily see that (3.1) implies (3.2) and (3.3). We emphasize that *infinitely many elements* of this table were not known previously. (If one took the usual residue classes from 0 to 8, then the previous table would be a little more complex, since there is a "jump" regarding the additive constant where i steps from 5 to 6, and also from 1 to 2, but no jump occurs when i changes from 8 to 0.) Only the trivial case of $n = 0$ and $i = 1$ is not covered by the table; then $OPT(L) = FFD(L) = 1$.

Tightness. For completeness, here we describe instances which prove that $\lfloor 11/9 \cdot OPT(L) + 6/9 \rfloor$ is really a lower bound for all values of OPT. The basic case is $OPT \equiv 6$ (mod 9), i.e. the residue class for which the upper bound in (3.1) is an integer. (Our example below is a modification of the one in [14], Chapter 2, page 16.) Afterwards, we shall describe tight instances for the other residue classes, too. The case of $OPT \equiv 2$ (mod 9), i.e. the other place where the additive constant jumps, requires special care; the remaining seven cases turn out to be trivial consequences.

Let ε be chosen to be a sufficiently small positive real number, i.e. let $0 < \varepsilon < 1/8$.

For the case of $OPT \equiv 6 \pmod 9$, let L be the instance composed from the following $9n + 6$ fully packed bins (and the items therein):

- $6n + 4$ bins packed with $\{1/2 + \varepsilon, 1/4 + \varepsilon, 1/4 - 2\varepsilon\}$;
- $3n + 2$ bins packed with $\{1/4 + 2\varepsilon, 1/4 + 2\varepsilon, 1/4 - 2\varepsilon, 1/4 - 2\varepsilon\}$.

The table below displays how FFD distributes these items into bins.

# of bins	$6n+4$	$2n+1$	1	$3n+1$	1
items	A, B	C, C, C	C, D, D, D	D, D, D, D	D
bin level	$3/4+3\varepsilon$	$3/4+3\varepsilon$	$1-5\varepsilon$	$1-8\varepsilon$	$1/4-2\varepsilon$

ITEM CODE: A: $1/2+\varepsilon$; B: $1/4+2\varepsilon$; C: $1/4+\varepsilon$; D: $1/4-2\varepsilon$

Consequently, $FFD(L) = 11n + 8$, whereas $OPT(L) = 9n + 6$.

Let us see now the class with $i = 2$. Here the example given for $i = 6$ can be modified as follows. Let L be the instance composed from the items being packed into the following optimal bins:

- $6n + 1$ bins packed with $\{1/2 + \varepsilon, 1/4 + \varepsilon, 1/4 - 2\varepsilon\}$;
- $3n + 1$ bins packed with $\{1/4 + 2\varepsilon, 1/4 + 2\varepsilon, 1/4 - 2\varepsilon, 1/4 - 2\varepsilon\}$.

On the other hand, the following table displays how FFD distributes these items into bins.

# of bins	$6n+1$	1	$2n-1$	1	$3n$	1
items	A, B	B,C,C	C, C, C	C, C, D, D	D, D, D, D	D
bin level	$3/4+3\varepsilon$	$3/4+4\varepsilon$	$3/4+3\varepsilon$	$1-2\varepsilon$	$1-8\varepsilon$	$1/4-2\varepsilon$

ITEM CODE: A: $1/2+\varepsilon$; B: $1/4+2\varepsilon$; C: $1/4+\varepsilon$; D: $1/4-2\varepsilon$

Consequently, $FFD(L) = 11n + 3 = 11/9 \cdot OPT(L) + 5/9$, whereas $OPT(L) = 9n + 2$.

The tight examples for $3 \leq i \leq 5$ are obtained by adding $i - 2$ items of size 1 to the example for $i = 2$, and the examples for $7 \leq i \leq 10$ are obtained by adding $i - 6$ items of size 1 to the example for $i = 6$.

The rest of this chapter is devoted to the proof of the upper bound. This proof is neither easy nor short. We do believe that a really short proof does not exist (although it is possible that the current proof could be shortened a little bit). The problem seems like looking for a needle in a haystack when the needle is not even there. As an even closer analogue, the presented proof is similar to the situation where something important but little thing (the suspected counterexample) is searched for in a big house. Since we do not know where it should be, we must look into every room, in every room we must look into every cupboard, and in every cupboard into every case (in fact, there will be many cases). We shall find bigger rooms with many places to consider, and also smaller rooms, i.e. simpler cases. We can be sure that the searched thing is not in the house only if we look into every place and find it nowhere.

3.1 Preliminaries

The goal of this chapter is to prove the following theorem.

Theorem 3.1.1 $9\,FFD(L) \le 11\,OPT(L) + 6$.

The proof is split into four sections. In the current one we present some general observations, whereas the arguments in the other sections will consider classifications of items according to their sizes.

It is trivial that the assertion of the theorem is equivalent to (3.1). Since $FFD(L)$ and $OPT(L)$ are integers, it will suffice to show that there does not exist any instance for which

$$9\,FFD(L) \ge 11\,OPT(L) + 7 \tag{3.4}$$

holds.

For every set of sizes $a_1 > a_2 > \cdots > a_l$, any problem instance can be represented with a "configuration vector" $(n_1, n_2, \ldots, n_l)$ of length l, where $n_i \ge 0$ is the number of items of size a_i in the instance. Suppose on the contrary that the theorem is false, and let L be a minimal counterexample. L is called *minimal*, if taking all item-sizes $a_1 > a_2 > \cdots > a_l$ in L, the configuration vector of L is lexicographically minimal among all counterexamples that only contain items of sizes in $\{a_1, a_2, \ldots, a_l\}$.

Definition 3.1.1 *Let us say that an item is larger than another item if the former appears before the latter in the sorted order. Similarly, an item is called smaller than another item if the former appears after the latter in the sorted order. (Hence these relations are defined also between items of the same size.)*

Observation 3.1.1 $OPT(L) \ge 2$ *and* $FFD(L) \ge 4$.

Proof. It is trivial that $OPT(L) \ge 2$ must hold. Then (3.4) means $9\,FFD(L) \ge 11\,OPT(L) + 7 \ge 29$, thus $FFD(L) \ge 4$ also holds. □

Let us choose an arbitrary (but fixed) optimal solution, and then denote the **optimal bins** as B_i^* for $i = 1, \ldots, OPT(L)$, and the **FFD bins** as B_j for $j = 1, \ldots, FFD(L)$. The sum of the sizes of items being packed into a bin will be referred to as the **level** of the bin in question, and will be denoted as $l\left(B_i^*\right)$ and $l\left(B_j\right)$, respectively. For an item U, we shall often denote by $bin(U)$ the bin into which U is packed by algorithm FFD.

From the minimality of the counterexample it follows that the last FFD bin contains only one item, and no item arrives after this item, thus the only one item in the last FFD bin is the last (and smallest) item. This specific item, and also its size, will be denoted as X. Let the items (and their sizes) be p_k for $k = 1, \ldots, n$, where n stands for the number of items. We suppose without loss of generality that the sizes of the items are non-increasing, i.e. $p_1 \ge p_2 \ge \cdots \ge p_n = X$.

We also introduce the following notation. Let the k-th item of the i-th optimal bin be denoted as $A_{i,k}^*$ for every $i = 1, \ldots, OPT(L)$, and analogously let the k-th item of the j-th FFD bin be denoted as $A_{j,k}$ for every $j = 1, \ldots, FFD(L)$. (Depending on context, we will sometimes use more than one notation for the same item, and in all cases the notation is both for the item and its size.) We assume without loss of generality that for every i and every $k_1 < k_2$ the inequality $A^*_{i,k_1} \geq A^*_{i,k_2}$ holds, and A^*_{i,k_1} arrives before A^*_{i,k_2} in the order of the items. Similarly, $A_{j,k_1} \geq A_{j,k_2}$ follows from the FFD rule for every j and every $k_1 < k_2$, moreover A_{j,k_1} arrives before A_{j,k_2} in the order of the items. A bin is called a k-bin if it contains exactly k items.

Clearly, $\sum_{k=1}^{n} p_k \leq OPT(L)$ holds because all the items fit in the optimal packing into $OPT(L)$ optimal bins. Note that item X does not fit into any previous FFD bin, thus we get

$$l\,(B_i) > 1 - X, \quad i = 1, \ldots, FFD(L) - 1. \tag{3.5}$$

Lemma 3.1.1 $X > \frac{FFD(L)-OPT(L)-1}{FFD(L)-2} \geq 2/11.$

Proof. We apply (3.5) to get

$$\begin{aligned} OPT(L) \geq \sum_{k=1}^{n} p_k = (X + l\,(B_1)) + \sum_{i=2}^{FFD(L)-1} l\,(B_i) \\ > 1 + (1 - X)\,(FFD(L) - 2)\,, \end{aligned}$$

from which the first inequality follows, while the second inequality is equivalent to (3.4). □

Corollary 3.1.1 $FFD(L) > OPT(L) + 1.$

Proof. This fact follows from (3.4) and Observation 3.1.1, as

$$\begin{aligned} FFD(L) \geq \frac{11}{9} OPT(L) + 7/9 = \frac{2}{9} OPT(L) + OPT(L) + 7/9 \\ \geq OPT(L) + 4/9 + 7/9 > OPT(L) + 1. \end{aligned}$$

□

From Lemma 3.1.1 we also see that no bin can contain more than five items, since the smallest item is $X > 2/11 > 1/6$.

Corollary 3.1.2 $X > \frac{\lceil 11/9 \cdot OPT(L) + 7/9 \rceil - OPT(L) - 1}{\lceil 11/9 \cdot OPT(L) + 7/9 \rceil - 2}.$

Proof. We apply (3.4), Lemma 3.1.1, and the facts that $FFD(L)$ is an integer and the ratio $\frac{FFD(L)-OPT(L)-1}{FFD(L)-2}$ is an increasing function with respect to $FFD(L)$. □

Definition 3.1.1 *We say that an FFD bin B_j dominates an optimal bin B^*_i if there exists an injective (but not necessarily surjective) mapping $f : B^*_i \to B_j$ such that $p_k \leq f(p_k)$ holds for every element $p_k \in B^*_i$.*

Lemma 3.1.2 *(Domination Lemma) There are no bins B_j and B^*_i such that B_j dominates B^*_i.*

Proof. First we note that the FFD packing always has the following special property: Omitting the items being packed into a specific FFD bin, and running again FFD for the remaining items, we get the same packing for them. Suppose that the FFD bin B_j dominates the optimal bin B_i^*. Let every item $x \in B_i^*$ satisfying $x \neq f(x)$ be swapped in the optimal packing with its image $f(x)$. Having done this swapping procedure, B_j still dominates the optimal bin B_i^*, and all items of B_i^* are packed into B_j in the FFD packing. If there are further items in B_j, let these items be also (re)packed into the optimal bin B_i^*. Then the other optimal bins (the original place of these items) will have fewer items. Thus, finally the FFD bin B_j and the optimal bin B_i^* will have the same contents. Then omitting just their items, we get a smaller counterexample, what is a contradiction to our minimality assumption. □

Lemma 3.1.3 *Each optimal bin contains at least three items.*

Proof. By Lemma 3.1.2 an optimal bin with a single item cannot occur, since it will be dominated by the FFD bin which has this item.

Suppose now that the optimal bin B_i^* contains only two items, Y and Z, where $Y \geq Z$. If Y and Z are packed into the same bin by FFD, we get a contradiction to the Domination Lemma. Assume that Y and Z are not packed together by FFD, and Y is packed before Z. If Y is not a first item of an FFD bin, then its bin contains an item which is not smaller than Z, and we are done. Otherwise, if Z is packed into a bin of a larger index than the bin of Y, then since Z does not fit into the bin of Y, it is known that an earlier item joined Y in that bin. Finally, if Z is packed into a bin of a smaller index than the bin of Y, then the first item of this bin is not smaller than Y. In any case we got a contradiction to the Domination Lemma. Hence each optimal bin contains at least three items. □

Lemma 3.1.4 *Each FFD bin but the last one contains at least two items.*

Proof. Suppose that an FFD bin B_j contains just one element for some $1 \leq j \leq FFD(L) - 1$. Let this item be Y, and suppose that Y is packed into the optimal bin B_i^*. There exists another item, say Z, in this optimal bin. Then $Y + Z \leq 1$, implying that $Y + X \leq Y + Z \leq 1$ (since X is the smallest item), thus the last item X fits into this FFD bin, a contradiction. □

Lemma 3.1.5 $X < 1/4$.

Proof. Suppose that $X \geq 1/4$. Then each (optimal or FFD) bin contains either at most three items, or four items if each of these four items has size exactly $1/4$. Let K be the number of optimal bins containing four items; let us call these bins as special (optimal) bins, and the other optimal bins as ordinary optimal bins. The items being in some special or ordinary bin will be called as special and ordinary items, respectively. It holds that the size of any special item is exactly $1/4$ (but there can be ordinary items with size $1/4$, as well). Note that in case $X > 1/4$ there is no special item. We assume without loss of generality that all special items arrive after all ordinary items (including ordinary items of size $1/4$). This can be assumed by possibly swapping the location of items of size $1/4$ in the optimal solution.

From Lemma 3.1.3 we get that every ordinary optimal bin contains exactly three items, thus the number of items is $3(OPT(L)-K)+4K = 3\,OPT(L)+K$. Moreover, the size of each item is at most $1/2$, and the items of the special optimal bins are (among) the smallest items.

Suppose that there are two consecutive FFD bins B_α and B_β ($\beta = \alpha+1$) where B_α and B_β contain three and two items, respectively. If $A_{\alpha,1}+A_{\alpha,2} \geq A_{\beta,1}+A_{\beta,2}$, then because $A_{\alpha,3}$ fits into the α-th bin and $A_{\alpha,3} \geq X$, it follows that X fits into the β-th bin, a contradiction. Thus $A_{\alpha,1}+A_{\alpha,2} < A_{\beta,1}+A_{\beta,2}$ must hold. Since $A_{\alpha,1} \geq A_{\beta,1}$, it follows that $A_{\alpha,2} < A_{\beta,2}$. Thus $A_{\beta,2}$ is packed before $A_{\alpha,2}$, and it did not fit into the α-th bin, therefore $A_{\alpha,1}+A_{\beta,2} > 1$ and the bigger of them exceeds $1/2$, a contradiction.

Hence, the FFD bins at the beginning contain two items, the FFD bins after them contain three items, and then there can be some FFD bins with four items (each item in these latter bins has size exactly $1/4$); and finally the last FFD bin contains only one item. Let $n_i \geq 0$ be the number of the FFD i-bins, for $i = 2,3,4$. Then $n_2+n_3+n_4+1 = FFD(L)$, and the number of the items is $3\,FFD(L)-n_2+n_4-2$.

Suppose that $n_2 > 0$. Then, since the sum of the sizes of any two items is at most 1, the first $2n_2$ items (in the non-increasing order) are packed pairwise into the first n_2 FFD bins, and therefore

$$p_{2n_2-1}+p_{2n_2}+X > 1 \tag{3.6}$$

holds (recall that p_k denotes the size of the k-th item). Also recall that an item is said to be *larger* than another item if the former appears before the latter in the sorted order.

Now let us consider the largest item, say item Y, which is a second item in some optimal bin, i.e. consider the largest $A^*_{i,2}$ item for the optimal bins.

It follows that item Y cannot occur later than the $(OPT(L)-K+1)$-th item (since an item can precede it only if it is a first item in some optimal 3-bin), thus $A^*_{i,2} = p_{k_2}$ for some $k_2 \leq OPT(L)-K+1$. Let $A^*_{i,1} = p_{k_1}$ and $A^*_{i,3} = p_{k_3}$. Then $k_1 < k_2 < k_3$ must hold for these indices, since we assume that the items in an optimal bin are sorted. The inequality

$$p_{OPT(L)-K}+p_{OPT(L)-K+1}+X \leq p_{k_2-1}+p_{k_2}+X \leq p_{k_1}+p_{k_2}+p_{k_3} \leq 1 \tag{3.7}$$

holds because of the non-increasing order of the items, and since X is the smallest item. Comparing (3.6) and (3.7) we see that in case of $n_2 > 0$ we gain the next upper bound on n_2: $OPT(L) \geq OPT(L)-K \geq 2n_2$. Also, the inequalities trivially hold if $n_2 = 0$, thus

$$OPT(L) \geq OPT(L)-K \geq 2n_2$$

hold in both cases. We state that $\min\{K, n_4\} = 0$, i.e. K and n_4 cannot be positive at the same time. Indeed, otherwise there exist both FFD and optimal bins with four items in each, and then all these items have size exactly $1/4$, contradicting the Domination Lemma.

First suppose that $K = 0$. Then the number of items is

$$\begin{aligned} 3OPT(L) &= 3FFD(L)-n_2+n_4-2 \geq 3FFD(L)-OPT(L)/2-2 \\ &\geq 3\,(11/9\cdot OPT(L)+7/9)-OPT(L)/2-2 \\ &= 19/6\cdot OPT(L)+1/3 > 3OPT(L), \end{aligned}$$

a contradiction.

Now suppose that $K > 0$; then $n_4 = 0$ follows. Recall that there are $4K > 0$ special items. We will need an upper bound on K. For this purpose consider the moment in the FFD packing when all ordinary items are just packed. Let j be the number of the opened bins at this moment, and let Z be the first item in the last (i.e. j-th) opened bin. If $Z = X$ then there is no special item, a contradiction. Thus Z precedes X. If $Z = 1/4$, then no special item can be packed into the first $j-1$ bins (since Z did not fit there either) and there are at least four special items, thus $n_4 > 0$, a contradiction. Thus $Z > 1/4$ follows. Since $n_4 = 0$, only one bin, namely the last FFD bin will be opened in the future, thus $j = FFD(L) - 1$. There cannot be now two FFD bins each having level at most a half. It follows that at most one special item fits into any opened bin at this time, with at most one exception (and at most two special items fit into any opened bin). Thus at most $j + 1$ special items will be packed in the future into the already opened bins, moreover one special item (i.e. X) will be packed into a new (i.e. the last FFD) bin, therefore $FFD(L) + 1 \geq 4K$ follows. Then we get

$$\begin{aligned}
3OPT(L) + K &= 3FFD(L) - n_2 - 2 \\
&= \frac{FFD(L)+1}{8} + \frac{23}{8}FFD(L) - n_2 - \frac{17}{8} \\
&\geq K/2 + \frac{23}{8}(11/9 \cdot OPT(L) + 7/9) - (OPT(L) - K)/2 - \frac{17}{8} \\
&= (\frac{23}{8} \cdot \frac{11}{9} - \frac{1}{2})OPT(L) + \frac{23}{8} \cdot \frac{7}{9} - \frac{17}{8} + K \\
&= \frac{217}{72}OPT(L) + \frac{1}{9} + K > 3OPT(L) + K,
\end{aligned}$$

a contradiction completing the proof of the lemma. □

Remark 3.1.2 *This proof would be a little bit easier if we prove only $X \leq 1/4$. Moreover it seems hard to decrease the upper estimate on X further to $1/4 - c$ for some constant $c > 0$.*

At this point we already know that X must lie in the interval $(2/11, 1/4)$. The following lemma will be very useful.

Lemma 3.1.6 (i) *It holds that $OPT(L) \geq 8$.*
(ii) *If $X \leq 1/5$, then $OPT(L) = 10$, $OPT(L) = 14$, or $OPT(L) \geq 18$.*

Proof. Applying Corollary 3.1.2, for $2 \leq OPT(L) \leq 17$ we get the following tables:

$OPT(L) =$	2	3	4	5	6	7	8	9	10
$11\,OPT(L) + 7$	29	40	51	62	73	84	95	106	117
$\lceil(11\,OPT(L) + 7)/9\rceil$	4	5	6	7	9	10	11	12	13
$X >$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{2}{7}$	$\frac{2}{8}$	$\frac{2}{9}$	$\frac{2}{10}$	$\frac{2}{11}$

$OPT(L) =$	11	12	13	14	15	16	17
$11\,OPT(L) + 7$	128	139	150	161	172	183	194
$\lceil(11\,OPT(L) + 7)/9\rceil$	15	16	17	18	20	21	22
$X >$	$\frac{3}{13}$	$\frac{3}{14}$	$\frac{3}{15}$	$\frac{3}{16}$	$\frac{4}{18}$	$\frac{4}{19}$	$\frac{4}{20}$

In case of $OPT(L) \in \{2,3,4,6,7\}$, $X > 1/4$ would follow, contradicting the previous lemma. Reference [79] contains the proof that in case of $OPT(L) = 5$ the value of $FFD(L)$ is at most 6, which contradicts (3.4), but we give a simplified proof for completeness in Appendix A. Thus $OPT(L) \geq 8$ follows. Now assume that $X \leq 1/5$ holds. Then, if $OPT(L) \leq 18$, according to the previous table, only the following cases are possible: $OPT(L) = 10$, $OPT(L) = 14$, or $OPT(L) \geq 18$. □

Remark 3.1.3 *Let us consider cases $OPT(L) = 10 + 4k$ where $k \in \{0,1,2\}$. From (3.4) it follows that in these cases $FFD(L) \geq 13+5k$ holds. If even $FFD(L) \geq 14+5k$ held, then from Lemma 3.1.1 we would get $X > \frac{(14+5k)-(10+4k)-1}{(14+5k)-2} = \frac{k+3}{5k+12} > 1/5$. Thus supposing $X \leq 1/5$ and $OPT(L) = 10 + 4k$ for some $k \in \{0,1,2\}$, only $FFD(L) = 13 + 5k$ remains to be considered. We will show later that this case is impossible.*

During the packing process, we say that a bin is an **open bin** if there is at least one item already packed into it. (All through the packing process, algorithm FFD keeps open each bin which is already opened, and all bins will be closed only after packing the last item.) An item which is packed into the currently last open bin is called a **regular item**, otherwise the item is called a **fallback item**. A bin is denoted as (A,B,C) bin if A, B, and C are items, and exactly these items are packed into that bin. Analogous notation will be used for bins containing fewer or more than three items, too.

Lemma 3.1.7 *Let B and C be two consecutive items in the (ordered) list L of the items, B preceding C. If C is packed into a (G,C) FFD-bin, where $G > 1/2$, then the following two properties hold.*
(i) If B and C have the same size, then B is packed into a (H,B) FFD-bin where H and G are two consecutive items in this order, and H and G have equal size.
(ii) If B is bigger than C, then $B > 1 - G$.

Proof. Suppose that $B \leq 1 - G$. Since B arrives before C, this means that B fits into $bin(G)$. It follows that B is packed into an earlier bin than $bin(G)$. Since $G > 1/2$, it means that there are $k \geq 1$ consecutive items $H_1 \geq H_2 \geq \cdots \geq H_k \geq G > 1/2$, packed one by one into different, consecutive bins, such that B is packed into $bin(H_1)$ (and there may be further bins before $bin(H_1)$). Let us denote B as B_1 in the following. Since $H_1 + B_1 + X \geq G + C + X > 1$, hence $bin(B_1)$ contains exactly two items, i.e. it is a (H_1, B_1) bin. Consider the moment when B_1 is packed. Since the next item C is packed into $bin(G)$, and $H_i + C \leq H_1 + B_1 \leq 1$ for any $2 \leq i \leq k$, there must be a second item, say B_i, in any $bin(H_i)$ for $2 \leq i \leq k$ at this time, otherwise C would be packed there. Then $B_i \geq C$ trivially follows. Later no further item can be packed into any $bin(H_i)$ (as $H_i + B_i + X \geq G + C + X > 1$), thus any $bin(H_i)$ has exactly two items, i.e. any $bin(H_i)$ is a (H_i, B_i) bin, for $1 \leq i \leq k$.

We shall use the following terms: the items of the set $\{B_1, B_2, \ldots, B_k, C\}$ are the *considered items*, and their bins are the *considered bins*; the items being packed into non-considered bins are *ordinary* items, the bins with smaller index than (H_1, B_1) are *earlier* bins, and the bins with bigger index than (G, C) are *later* bins.

Suppose that $H_1 > G$. We claim that decreasing the sizes of items $H_1, H_2, \ldots, H_k$ to the size of G (referred to as *change*), exactly the considered items will be packed as second items into the considered bins (possibly in another order), no more items will be packed there, and each ordinary item coming before C will be packed into the same bin as before. Suppose that this claim is not true, and let A be the first item for which the claim fails.

Case 1, Item A is an ordinary item, coming before C. If before the change A was packed into an earlier bin, then A is packed into the same bin, as the change of the sizes of $H_1, H_2, \ldots, H_k$ and the packing of the considered items (before A) cannot violate the packing of A. Otherwise before the change A is packed into a later bin. Since before the change at the time of packing A there is only one item, namely G in $bin(G)$, and A was not packed there, $G + A > 1$ holds. Thus after the change A is packed again into the same later bin.

Case 2, Item A is a considered item. It cannot be packed into an earlier bin, since the ordinary items that are before A in the list are already packed there in the same bins as before the change. On the other hand, since $1 \geq H_i + B_i \geq G + B_i$, and at this time no ordinary item is packed into a considered bin, A fits into a considered bin (as exactly $k + 1$ considered items are to be packed into the same number of considered bins). As $B_i \geq C$ and $G + C + X > 1$, it is clear that any considered bin remains a 2-bin.

We got a contradiction, thus the claim follows. It means that there exists a smaller counterexample, which is a contradiction. It follows that $H_1 = H_2 = \cdots = H_k = G$. Then $k > 1$ is impossible, since then the largest B_i item (which is the first in the sorted order) must be packed into the bin of H_1 instead of $bin(H_i)$.

Summarizing the results so far, we proved that in case $B \leq 1 - G$ it follows that B is packed into an (H, B) FFD bin where H and G are two consecutive items in this order, H and G have equal size, and (H, B) and (G, C) are two consecutive bins.

Then $C < B \leq 1 - G$ is impossible, because then decreasing the size of B to the size of C, B would be packed into the same bin (as C does not fit into any earlier bin), and we would get a smaller counterexample. It follows that in case $C \leq B \leq 1 - G$ either $B = C$ or $B > 1 - G$, thus we proved both (i) and (ii). □

Corollary 3.1.4 *Given a (G, C) FFD bin, where $G > 1/2$, there are no items of size in $(C, 2X]$.*

Proof. From Lemma 3.1.7 (ii) it follows that there is no item in $(C, 1 - G]$. Since in any optimal bin there are at least three items, it follows that $G \leq 1 - 2X$, thus $2X \leq 1 - G$, and we get $(C, 2X] \subseteq (C, 1 - G]$. □

Corollary 3.1.5 *Let (G, C) denote an FFD bin, and (G, A, B) denote an optimal bin, where $G > 1/2$, G denotes the same item in the two bins, but A, B and C are different items. Then any item bigger than C is also bigger than $A + B$.*

Proof. The claim follows directly from Lemma 3.1.7 (ii). □

The last corollary will often be used in the later sections, since with its help many cases can be omitted from consideration. We also state here a trivial observation, which will be used several times.

Observation 3.1.2 *Let (B, A_1, A_2, A_3) and (G, A_4, A_5) denote two optimal bins, where $B > A_4 + A_5$. Then the five items A_1, A_2, A_3, A_4, A_5 fit into a common bin.*

This part of the work could be done generally to prepare the proof of the main result. Now we need to introduce classifications on the items.

3.2 Case $1/5 < X < 1/4$.

We put the items into some classes according to their sizes. Let Z be the smallest regular item in interval $(\frac{1-X}{3}, \frac{1}{3}]$, if there exists at least one such item; otherwise let $Z = 1/3$. (The definition of Z is a core matter here, by use of Z we can reduce the number of necessary classes from 6 to 5). The classes are defined only on the basis of the values of X and Z. The classes are called **giant, big, medium, small,** and **tiny**, and their item sets are denoted as G, B, M, S, T, respectively. We also assign weights to the items being in the classes, as follows.

Name	Class	Weight
Giant	$\frac{1}{2} < G$	23
Big	$\frac{1-X}{2} < B \leq \frac{1}{2}$	18
Medium	$\frac{1-Z}{2} < M \leq \frac{1-X}{2}$	15
Small	$Z \leq S \leq \frac{1-Z}{2}$	12
Tiny	$X \leq T < Z$	9

The classification of the items
in case $1/5 < X < 1/4$.

Observation 3.2.1 *The classes are well defined. Furthermore, if set S is not empty, then there exists at least one item in class S with size exactly Z. Moreover, the size of each regular T item is at most $\frac{1-X}{3}$.*

Proof. Class T is well defined, since $X < \frac{1-X}{3} < Z$. Class S is well defined, since $Z \leq \frac{1-Z}{2}$. The further classes are also well defined, by $0 < X < Z$. Now suppose that there exists at least one item in class S. If $Z = 1/3$, then $Z = \frac{1-Z}{2}$, thus any S item has size $Z = 1/3$. Otherwise there exists at least one regular item in the interval $(\frac{1-X}{3}, \frac{1}{3}]$, the smallest one has size Z, and it belongs to class S. Now let A denote a regular T item. Since A is a T item, it follows that $A < Z$, but Z is the smallest regular item in $(\frac{1-X}{3}, \frac{1}{3}]$, thus A cannot be in this interval and therefore $A \leq \frac{1-X}{3}$ follows. □

Observation 3.2.2 *For the ranges of X, Z, $\frac{1-Z}{2}$ and $\frac{1-X}{2}$ (the boundaries of the classes above) the following inequalities are valid:*

$$1/5 < X < 1/4 < Z \leq 1/3 \leq \frac{1-Z}{2} < 3/8 < \frac{1-X}{2} < 2/5 < 2X.$$

Especially, $\frac{1-X}{2} < 2X$ holds.

Proof. Each inequality can be checked easily. □

Let $\delta = (1 - X)/36$. In the next lemma we find a relation between the sizes and the weights of the items.

Observation 3.2.3 *The size of any item is strictly larger than δ times the weight of the item, except possibly the giant items.*

Proof. The size of a tiny item is at least X, while δ times its weight is $9\delta = \frac{1-X}{4} < X$ by Observation 3.2.2. The size of a small item is at least $Z > \frac{1-X}{3}$, while δ times its weight is $12\delta = \frac{1-X}{3}$. The size of a medium item is bigger than $\frac{1-Z}{2} \geq 1/3$, while $15\delta = \frac{15}{36}(1-X) < \frac{15}{36} \cdot \frac{4}{5} = \frac{1}{3}$. The size of a big item exceeds $\frac{1-X}{2}$, while δ times its weight is just $18\delta = \frac{1-X}{2}$. □

For nonnegative integers $c_1, \ldots, c_5$ and real c_6 we write $c_1 G + c_2 B + c_3 M + c_4 S + c_5 T > c_6$ to express that the inequality holds substituting (the size of) **any** c_1 large, c_2 big, c_3 medium, c_4 small, and c_5 tiny items. We only require that the items must be distinct, some (or all) of them may have the same size. Other kinds of inequalities involving any specific types of items are meant analogously. For example, $3S > 1 - X$ holds since no S item is smaller than $Z > \frac{1-X}{3}$, thus substituting any three S items, the sum of their sizes is bigger than $1 - X$.

Observation 3.2.4 *The inequalities $G \leq 1 - 2X$ and $G + M \leq 1$ hold for any G and M items. Moreover no item has size in $(1 - 2X, 1]$.*

Proof. Since each optimal bin contains at least three items, $G \leq 1 - 2X$ is valid. The second statement follows from the facts that there is at least $2X$ room in a bin which contains only a G item at that time, and $M \leq \frac{1-X}{2} < 2X$ holds for any M item, by Observation 3.2.2. Since $G \leq 1 - 2X$ holds for any giant item, and G is the biggest class, the size of any item is at most $1 - 2X$. □

Let us introduce some further notation. A (C_1, C_2) bin denotes an FFD bin-type, where C_1 and C_2 are two different specified classes. For example (G, S) means an FFD bin which contains exactly two items, a G item and an S item. 3-bins and 4-bins (i.e. bins containing 3 or 4 items) will be denoted similarly.

We also use the term A-bin, where A is some class, to mean that some of the items in the bin belongs to A. For example, G-bin (B-bin, M-bin, ...) denotes a bin which contains a G item (at least one B item, M item, ..., respectively), among other items.

We will often use properties of the FFD packing, similar to the one highlighted in the previous observation, in order to reduce the number of possible bin-types.

We denote the weight of an item A as $w(A)$, the sum of the weights of all items as $w(L)$, and the weight of an optimal or FFD bin as $w(B^*)$ or $w(B)$, respectively. We define the **reserve** of an optimal bin as $r(B^*) = 44 - w(B^*)$. When we define the weights of the classes, we do it in such a way that no optimal bin will have weight more than 44, i.e. the reserves of all optimal bins are nonnegative, and almost all of the optimal bins have positive reserve. Define the **surplus** of an FFD bin as $sur(B) = w(B) - 36$, if this value is nonnegative. Otherwise, let $short(B) = 36 - w(B)$ be called its **shortage**. In case if the weight of every FFD bin is at least 36 (i.e. there is no shortage) and the reserve of all optimal bins is nonnegative, we easily get that

$$36\, FFD(L) \leq \sum_{k=1}^{FFD(L)} w\left(B_k\right) = w(L) = \sum_{k=1}^{OPT(L)} w\left(B_k^*\right) \leq \frac{44}{36} \cdot 36\, OPT(L),$$

and the proof is done for such instances. Unfortunately, FFD bins with less weight (i.e. with some shortage) may exist. But we prove that the total shortage can be **covered** by the reserves of the optimal bins plus the surplus of the other FFD bins, plus a required additive constant 27 (see below). In this section the weight of the smallest class is $w(T) = w(X) = 9$. Thus, the shortage of the last FFD bin, which contains only the last item X, is $36 - w(X) = 27$, therefore the additive constant just covers the shortage of the last FFD bin.

Let $sur(L)$ and $res(L)$ be the total value of the surplus and reserve of all FFD and optimal bins, respectively, let the required additive constant 27 be denoted as $rex(L)$, and finally let $short(L)$ be the total value of the shortage given by all FFD bins. Then we have

$$w(L) = \sum_{k=1}^{FFD(L)} w(B_k) = 36 \cdot FFD(L) + sur(L) - short(L), \tag{3.8}$$

$$w(L) = \sum_{k=1}^{OPT(L)} w(B_k^*) = 44 \cdot OPT(L) - res(L). \tag{3.9}$$

Suppose that

$$res(L) + sur(L) + rex(L) \geq short(L) \tag{3.10}$$

holds. Then applying (3.8) and (3.9), we have

$$\begin{aligned} 36 \cdot FFD(L) &= w(L) - sur(L) + short(L) \\ &\leq w(L) + res(L) + rex(L) \\ &= 44 \cdot OPT(L) + 27. \end{aligned}$$

Dividing by 36 and considering that $27/36 < 7/9$ we get our main result. Thus in the remaining part of this section our goal is to prove (3.10).

Let us see what kind of bins are possible, in terms of the distribution of the classes. First we list all possible optimal bins, and then all possible FFD bins. In the last rows r means the reserve of the optimal bins, while s denotes the value of the surplus or shortage of the FFD bins. If s is positive, then it means surplus, and if it is negative, then it means shortage. The value of reserve, surplus or shortage of the bins can easily be computed from the weights of the classes. The weights are as given above, but we also include the weights here in the left column for the sake of easier checking of reserve, surplus or shortage of a bin. The very last column for FFD corresponds to the last bin, and there can be only one such bin. We use the required additive constant to cover the shortage of the last FFD bin, thus we write 0 there for the remaining shortage.

Lemma 3.2.1 *Only the following bin-types are possible.*

OPT

23	G	1	1																
18	B			1	1	1	1												
15	M			1				2	1	1	1	1							
12	S	1			2	1			2	1			3	2	2	1	1		
9	T	1	2	1		1	2	1		1	3	2		2	1	3	2	4	3
	r	0	3	2	2	5	8	5	5	8	2	11	8	2	11	5	14	8	17

FFD

23	G	1	1	1	1				1	1									
18	B	1				2	1	1			1	1	1	1					
15	M		1				1				1				2	1	1	1	1
12	S			1				1	1			2	1			2	1		
9	T				1				1	2	1		1	2	1		1	3	2
	s	5	2	-1	-4	0	-3	-6	8	5	6	6	3	0	3	3	0	6	-3

12	S	3	2	2	1	1		
9	T		2	1	3	2	4	1
	s	0	6	-3	3	-6	0	0

Proof. First let us check what kind of optimal bins are possible. Each optimal bin contains three or four items. If the bin contains a giant or big item, then three further items do not fit into it, since $G + 3X > B + 3X > \frac{1-X}{2} + 3X = \frac{1}{2} + \frac{5}{2}X > 1$. Thus, any optimal G-bin or B-bin contains exactly two further items.

In a G-bin there cannot occur a B or M item, since $G+B+X > G+M+X > \frac{1}{2}+\frac{1}{3}+X > 1$. Two S items cannot be in a G-bin, since $S \geq Z > 1/4$. Then only (G, S, T) and $(G, 2T)$ optimal G-bins are possible. Two B items cannot be in an optimal bin, because $2B + X > 1$. Since $B+M+S > 2M+S > 1$, if an optimal B-bin contains also an M item, then its third item cannot be an S item, only a T item.

Let us consider the optimal bins whose largest item is some M item. Using $2M + 2X > 2M + S > 1$, if the bin contains two M items, then only one T item can fit into it. Suppose the bin contains one M item. Using $M + S + 2T > \frac{1-Z}{2} + Z + 2X > \frac{1}{2} + \frac{5}{2}X > 1$, if the bin contains one M item and also at least one S item, then two further items cannot fit into the bin, thus only an $(M, 3T)$ bin is possible if the M-bin is a 4-bin.

If the bin contains items only from the classes S and T, then $3S + X > 1$ implies that there cannot be a 4-bin that contains three S items.

Now let us consider the possible FFD bins. Each FFD bin contains at least two, and at most four items. Considering the 2-bins, any G-bin is possible, except $(2G)$ bins. If there is a (B, T) bin, the T item must be regular, and therefore its size is at most $\frac{1-X}{3}$ by Observation 3.2.1. Then X would fit into such a bin, since $B + \frac{1-X}{3} + X < 1$ holds. Thus a (B, T) FFD bin is impossible, but any other B-bin (without a G item) is possible among the 2-bins. There cannot be further 2-bins, since $2M + X \leq 1$. Considering the 3-bins and 4-bins, almost all bin-types are possible which we have got among the optimal bin-types. We can exclude the $(3T)$ bin-type from the FFD bins, by the following reason: If there is such a bin, then each T item in this bin must be regular. Thus all of them are no larger than $\frac{1-X}{3}$ by Observation 3.2.1, and then X would fit into this bin. □

Observation 3.2.5 *If there exists at least one (G, T) bin, then there are no $S \cup M$ items. If there exists at least one (G, S) bin, then there are no M items. In any case, if there exists at least one (G, T) bin or (G, S) bin, then all (G, T) bins or all (G, S) bins are identical in terms of the item sizes in them.*

Proof. Suppose that there exists a (G, T) bin in the FFD packing. It follows from Corollary 3.1.4 that there is no item with size in $[Z, 2X]$. Since $Z \leq S < M \leq \frac{1-X}{2} < 2X$, there are no $S \cup M$ items. If there exists a (G, S) bin, similarly it follows that there are no M items. The third claim follows from Lemma 3.1.7. Suppose for example that there exists a (G, T) bin; let T_0 denote the smallest T item for which there exists a (G, T) bin. Then as we have seen, there is no item with size in $(T_0, 2X]$, thus if (G, T_1) is another (G, T) bin, then T_0 and T_1 have the same size, and then Lemma 3.1.7 (i) can be applied. □

Some situations can be eliminated by considering just the class of the largest items in the bins. For this, we simplify the notation to $(A, .)$ bin, where A is any class, meaning that the bin contains at least one item from class A, but contains no item from any higher classes. For example, $(M, .)$ denotes a bin which contains at least one M item, but contains neither a G item nor a B item.

Observation 3.2.6 *If there exists at least one $(S, .)$ FFD bin, then the last $(S, .)$ FFD bin contains an S item with size exactly Z.*

Proof. If $Z = 1/3$, then any S item has size exactly Z, thus the claim follows. Suppose that $Z < 1/3$. Observation 3.2.1 implies that there is an S item with size exactly Z. Suppose that all S items in the last $(S, .)$ bin are greater than Z. Then each item with size Z must be in earlier bins, thus all of them are fallback items. But this is a contradiction, since Z is defined as the size of a regular item (if $Z < 1/3$). □

Using also the previous observation, we can exclude several combinations of bin-types.

Observation 3.2.7 *The following bin-types cannot occur at the same time: Two $(B, .)$ bins, both having only one B item; or two $(M, .)$ bins, both having only one M item; or two $(S, .)$ bins, both having only at most two S items. Consequently two $(B, .)$ bins or two $(M, .)$ bins or two $(S, .)$ bins, both having shortage, cannot occur simultaneously.*

Proof. Two B-bins with shortage cannot occur, since these bins can be only of type (B, M) or (B, S), but two B items fit into a common bin. By the same reason, two $(M, .)$ bins both with only one M item cannot occur. Suppose that there are two $(S, .)$ bins, both having at most two S items. Then by Observation 3.2.6, there is an S item with size exactly Z in the last S-bin, and this item would fit into the first $(S, .)$ bin under consideration, since two S items and a third S item with size Z fit into a common bin. □

Now we introduce another notion: We say that $\{(G, A), (G, B, C)\}$ is a **cobin** (couple of bins) if (G, A) is an FFD bin, (G, B, C) is an optimal bin, and G denotes the same item in these two bins (i.e. (G, A) is the FFD bin of item G, and (G, B, C) is the optimal bin of item G).

Lemma 3.2.2 *Suppose that there is no (G, T) FFD bin, and there is no $\{(G, S), (G, S, T)\}$ cobin. Then the inequality (3.10) holds.*

Proof. We have seen in Lemma 3.2.1 that only (G, S, T) or $(G, 2T)$ optimal G-bins are possible. Let the G items of the (G, S, T) and $(G, 2T)$ optimal bins be denoted as G_1 and G_2, respectively. Now we decrease the weights of the G_1 items by 2, and increase the weights of the G_2 items by 1.

As a result, any optimal bin has at least 2 reserve. Since there are at least eight optimal bins by Lemma 3.1.6, we have at least 16 reserve in total.

Let us consider now the FFD bins. There is no (G, T) FFD bin. Moreover since there is no $\{(G, S), (G, S, T)\}$ cobin, the G item of any (G, S) FFD bin is of type G_2, thus the bin has no shortage. Any other G-bin had at least 2 surplus before the change of the weights, thus it follows that no FFD G-bin has shortage after the change of the weights.

Now let us count the total shortage caused by the other FFD bins.

We get at most 6 shortage by a $(B,.)$ FFD bin, at most 3 shortage by an $(M,.)$ bin, and at most 6 shortage caused by an $(S,.)$ FFD bin, applying Observation 3.2.7. Thus we have at most 15 shortage altogether, which is covered by the 16 reserve. □

Lemma 3.2.3 *If there exists a (G,T) FFD bin, then the inequality (3.10) holds.*

Proof. Suppose that there exists a (G,T) FFD bin. Then by Observation 3.2.5 there are neither S nor M items. Moreover by Observation 3.2.4, no item has size in $(1-2X, 1]$. Thus only the bins shown in the table below are possible in the optimal and FFD packing, respectively. (We increased the weight of the G items.)

OPT

26	G	1			
18	B		1		
9	T	2	2	4	3
	r	0	8	8	17

FFD

G	1	1		1			
B	1		2		1		
T		1		2	2	4	1
s	8	-1	0	8	0	0	0

Then only the (G,T) FFD bins cause shortage. From Observation 3.2.5 it follows that the T items of the (G,T) FFD bins have equal size, and also the G items of the (G,T) FFD bins have equal size. Let us denote the G and T items of the (G,T) FFD bins as G' and T', respectively.

Now let us increase the weight of the T' items by 1, and decrease the weight of the $G \setminus G'$ items by 8. As a result, any FFD bin has exactly zero shortage. In order to prove the claim, it is important that if a $(G,2T)$ bin in the optimal packing has a T' item (even if just one), then the G item must belong to $G \setminus G'$ (as $G'+T'+X > 1$). Thus no optimal bin has shortage. □

In the following we suppose that a (G,T) FFD bin does not exist.

Lemma 3.2.4 *If there exists a $\{(G,S),(G,S,T)\}$ cobin, then inequality (3.10) holds.*

Proof. Suppose that there is a $\{(G,S),(G,S,T)\}$ cobin. From Observation 3.2.5 it follows that there is no M item, and from Lemma 3.1.7(i) it follows, that the G items of the (G,S) FFD bins have the same size and also the S items of these bins have the same size. Let us denote these items as G' and S', respectively. Since there exists a $\{(G',S'),(G',S,T)\}$ cobin, it follows from Corollary 3.1.5 that each item bigger than S' has size bigger than the sum of sizes of the S and T items in the cobin. Note that this property holds for any cobin. It follows that each item which is bigger than S', must be also bigger than $Z+X$. This means that $G > B > Z+X > T+X$ holds for any G, B and T items. Moreover, there is no $(B,2S)$ bin, since $B+2S > (Z+X)+2Z = 3Z+X > 3\cdot\frac{1-X}{3}+X = 1$. Thus only the following bins are possible. We increased the weight of class B by 4.

OPT

23	G	1	1										
22	B			1	1								
12	S	1		1		3	2	2	1	1			
9	T	1	2	1	2		2	1	3	2	4	3	
	r	0	3	1	4	8	2	11	5	14	8	17	

FFD

23	G	1	1			1	1									
22	B	1		2	1			1	1							
12	S		1		1	1		1		3	2	2	1	1		
9	T					1	2	1	2		2	1	3	2	4	1
	s	9	-1	8	-2	8	5	7	4	0	6	-3	3	-6	0	0

Let us decrease now the weight of the $G \setminus G'$ items by 1 and increase the weight of the S' items by 1. We find the cases for which the resulting weights prove the required inequality.

Since the weight of the G' items is not decreased, the (G', S') FFD bins have no shortage, and any other FFD G-bin has positive surplus. Let us see what happened regarding the optimal bins.

If S' is packed into some optimal bin which is not a G-bin, the reserve of the bin did not become negative. If S' is packed into some (G, S', T) optimal bin, then the G item of this optimal bin cannot be a G' item, since $G' + S' + X > 1$. Thus the zero reserve of the bin is decreased by 1 and at the same time also increased by 1, thus it remains 0.

Then there can remain at most 8 shortage: a (B, S) FFD bin can cause 2 shortage, and by a $(2S, T)$ or $(S, 2T)$ FFD bin we can have at most 6 further shortage (since these bin-types cannot occur at the same time by Observation 3.2.7). Therefore, we need not consider bins whose reserve or surplus is at least 8. Moreover the following FFD bins are also impossible:

$(G, 2T)$ or $(B, 2T)$ bin. — Suppose there exists a $(G, 2T)$ FFD bin. Then there cannot be a (B, S) FFD bin, since the S item of this bin would be packed into the G-bin before the two T items because $S \leq 2T$. By the same reason neither $(2S, T)$ nor $(S, 2T)$ FFD bins can occur, thus there is no shortage. The same explanation is valid to show that there cannot be a $(B, 2T)$ bin.

(G, S, T) bin. — Suppose there exists a (G, S, T) bin. Then we claim that there is no (B, S) FFD bin. If there was a (B, S) bin, the S item of this bin would arrive only after the S item of the (G, S, T) FFD bin, since any S item fits into a bin containing only a G item. Then the last item X would fit into the (B, S) bin. Therefore (as there is no (B, S) bin) at most 6 shortage can occur, and it is covered by the 7 reserve of the (G, S, T) bin.

(B, S, T) bin. — If there exists a (B, S, T) bin, then there is no (B, S) FFD bin by Observation 3.2.7, and the (B, S, T) bin has 7 surplus, which covers the possible 6 shortage.

$(2S, 2T)$ or $(S, 3T)$ bin. — If there exists a $(2S, 2T)$ bin, it has 6 surplus, and then there are neither $(2S, T)$ nor $(S, 2T)$ FFD bins by Observation 3.2.7, thus the possible 2 shortage of the (B, S) bin is covered. An $(S, 3T)$ bin is impossible by the same reason.

Thus only the following bin-types remain. We further increase the weight of B items by 1.

OPT

23	G	1	1					
23	B			1	1			
12	S	1		1		3	2	1
9	T	1	2	1	2		2	3
	r	0	3	0	3	8	2	5

FFD

23	G	1						
23	B		1					
12	S	1	1	3	2	1		
9	T				1	2	4	1
	s	-1	-1	0	-3	-6	0	0

Considering these possible FFD bins, we see that all G items are packed into (G', S') FFD bins, i.e. set $G \setminus G'$ is empty. (Recall that the weight of the G' items remained 23.) At this point we revert to the old weight of the S' items to be 12 (as it is written in the table above).

We realize that there is at most one B item, packed into a (B, S) FFD bin. If this situation occurs, let the size of this S item be denoted by S''. Then the S'' item comes after the S' items of the (G', S') FFD bins, by Observation 3.2.4 ($G+M \leq 1$). Then all S' and S'' items have the same size, otherwise the sizes of the S' items can be decreased to be S'', and we get a contradiction to our minimality assumption regarding the input.

The total shortage is 3 or 6 (if there exists a $(2S, T)$ or an $(S, 2T)$ FFD bin), plus the number of (G', S') and (B, S') FFD bins. As we *cover* the shortage of the FFD bins by the reserves of the optimal bins, we define the *free reserve* of an optimal bin as the reserve of the bin minus the number of the S' items in the bin.

We state that there are at least three $\{(G', S'), (G', S, T)\}$ cobins. Indeed, suppose that there are $1 \leq k_1 \leq 2$ cobins of type $\{(G', S'), (G', S, T)\}$ and $k_2 \geq 0$ cobins of type $\{(G', S'), (G', 2T)\}$. Then the total shortage is at most 6 (caused by a $(2S, T)$ or an $(S, 2T)$ FFD bin) plus $k_1 + k_2$ by the cobins plus 1 if there exists a (B, S) FFD bin. Thus the total shortage is at most $7 + k_1 + k_2$.

On the other hand, there are at least 8 optimal bins. Among them, there are k_1 G-bins with zero reserve, and k_2 G-bins with 3 reserve each. There can be at most one B-bin with possibly zero reserve, and every other optimal bin has at least 2 reserve. If $k_1 + k_2 \leq 7$, then we have at least $3k_2 + 2(7 - k_1 - k_2) = k_1 + k_2 - 3k_1 + 14 \geq k_1 + k_2 + 8$ reserve, otherwise if $k_1 + k_2 \geq 8$, then $k_2 \geq 6$, and hence we have at least $3k_2 = k_1 + k_2 - k_1 + 2k_2 \geq k_1 + k_2 - 2 + 12$ reserve. Thus all shortage is covered, a contradiction. It follows that there are at least three $\{(G', S'), (G', S, T)\}$ cobins.

Recall that any item bigger than S' has size bigger than the sum of sizes of the S and T items in *any* $\{(G', S'), (G', S, T)\}$ cobin (see the beginning of the proof of this lemma). Thus denoting by S_1 and S_2 the two smallest S items, it follows that $G > B > S_i + X$ holds for $i = 1, 2$.

If there were neither $(2S, T)$ nor $(S, 2T)$ FFD bins, then the shortage of the (G', S') and (B, S') FFD bins would be covered by the optimal bins of the S' items, thus one of these FFD bins must exist (the two cannot exist at the same time).

Case 1 There exists an $(S, 2T)$ FFD bin. Then let the T items being in this bin be denoted as T_i, $i = 1, 2$. The S item in this bin is the last regular S item, thus it is the smallest S item, and it has size Z (this claim follows from Observation 3.2.6). Then $T_1 + T_2 + Z + X > 1$ holds. We claim that no T_i item can be packed into a (B, S, T) optimal bin. Indeed, suppose that for example T_1 is packed into a (B, S, T) optimal bin. Recall that $B > S_i + X > T_2 + X$ holds. Thus we get $1 \geq B + Z + T_1 > (T_2 + X) + Z + T_1 > 1$, a contradiction. By the same reason, a T_i item cannot be packed into a (G, S, T) optimal bin. Suppose that a T_i item, say T_1 is packed into a $(2S, 2T)$ optimal bin. Since the other T item in the bin is not smaller than the size of X, moreover the total size of the two S items in the bin is bigger than $Z + T_2$, we got contradiction to $T_1 + T_2 + Z + X > 1$. Thus no T_i item is packed into a $(2S, 2T)$ optimal bin.

Since $G > B > S_i + X \geq Z + X$ holds ($i = 1, 2$), the two T_i items cannot be packed into the same $(G, 2T)$ or $(B, 2T)$ bin (otherwise we would get a contradiction to $T_1 + T_2 + Z + X > 1$.) Finally if both T_i items are packed into an $(S, 3T)$ optimal bin, then the sum of sizes of the third T item and the S item is at least $Z + X$ and we got a contradiction again.

After excluding the listed cases (since any T_i item is *really* packed into some optimal bin), only the next cases are possible: Each T_i item is packed into some $(G, 2T)$, $(B, 2T)$ or $(S, 3T)$ optimal bin, these two bins are different, and each of these two bins has at least 3 free reserve, thus they cover the total shortage caused by the $(S, 2T)$ FFD bin. (The further shortage caused by the (G', S') and (B, S') FFD bins is covered by the optimal bins of the S' items.)

Case 2 There exists a $(2S, T)$ FFD bin. Then the non-covered shortage is only 3. Thus no optimal bin can have at least 3 free reserve, and only the following possible bins remain:

OPT

23	G	1		
23	B		1	
12	S	1	1	2
9	T	1	1	2
	r	0	0	2

FFD

23	G	1					
23	B		1				
12	S	1	1	3	2		
9	T				1	4	1
	s	-1	-1	0	-3	0	0

Since $S' \leq 2X$, and $G \leq 1 - 2X$ (by Observations 3.2.2 and 3.2.4), the first S item is packed into the first (G', S') FFD bin; in other words, the S' items are the largest S items, and the S' items are packed one by one into a (G', S') or (B, S') FFD bin, and only then come the remaining (smaller) S items. Consider the $(2S, T)$ FFD bin, and suppose that the S items in the bin are S_1 and S_2, the two smallest S items (and recall that $G > B > S_i + X$, $i = 1, 2$ holds). Then similarly as in Case 1, T_1 can be packed into no optimal bin, a contradiction.

Thus it follows that one of S_1 and S_2 is a fallback S item. Then it is packed surely into some $(3S)$ FFD bin, hence there exists a $(3S)$ FFD bin with a fallback S item. We obtain that the first S item being in the first $(3S)$ bin is bigger than $1/3$, therefore $S' > 1/3$. Then two S' items cannot be packed into the same $(2S, 2T)$ bin, but each S' is packed into some $(2S, 2T)$ optimal bin. Since there are at least three G items, thus there are at least three S' items, and so we have the sufficient 3 free reserve in three $(2S, 2T)$ optimal bins. □

Thus we have proved that in the case $1/5 < X < 1/4$ the theorem is valid.

3.3 Case $2/11 < X \leq 1/5$, and $OPT \leq 18$.

If $X \leq 1/5$, the general investigations (apart from one very hard branch) will be significantly easier under the assumption $OPT \geq 19$, by making use of the fact that in most cases a large number of optimal bins provides us with large total reserve. In this sense, less flexibility makes the cases with small OPT values harder, despite that they look like “just something finite”. Therefore, in this section our goal is to prove that if $X \leq 1/5$, then $OPT \leq 18$ is impossible. We briefly mention that in this case we again use some classifications and weighting functions. All remained details are put into Appendix B, to simplify the treatment in this basic part of the dissertation.

3.4 Case $2/11 < X \leq 1/5$, and $OPT \geq 19$

In this case we redefine the classes of the items and their weights as follows: Let Z be the smallest regular item in $(\frac{1-X}{4}, \frac{1}{4}]$ if there exists such an item, otherwise let Z be $1/4$. Then we introduce a refinement of the classification, with two kinds of big items (denoted by B and C) and two kinds of medium items (denoted by M and N, hence using the letters next to B and M to the second types of 'big' and 'medium', respectively). We give weights to the items as below, somewhat differently from those in the previous sections.

	Name			Weight
$\frac{1}{2} <$	G			23
$\frac{1-X}{2} <$	B	$\leq$	$\frac{1}{2}$	18
$\frac{1-Z}{2} <$	C	$\leq$	$\frac{1-X}{2}$	16
$\frac{1}{3} <$	M	$\leq$	$\frac{1-Z}{2}$	14
$\frac{1-X}{3} <$	N	$\leq$	$\frac{1}{3}$	12
$\frac{1-Z}{3} <$	S	$\leq$	$\frac{1-X}{3}$	10
$Z \leq$	U	$\leq$	$\frac{1-Z}{3}$	9
$X \leq$	V	$<$	Z	8

The classification of the items in case $2/11 < X \leq 1/5$

Observation 3.4.1 *The classes are well-defined. Moreover, if there exists a U item, then the smallest U has size exactly Z. Furthermore the size of each regular V item is at most $\frac{1-X}{4}$.*

Proof. The classes are well-defined, by the following inequalities: $X \leq \frac{1-X}{4} < Z \leq \frac{1-Z}{3} < \frac{1}{3} < \frac{1-Z}{2}$. Now suppose that there exists at least one item in class U. If $Z = 1/4$, then $Z = \frac{1-Z}{3}$, thus any U item has size $Z = 1/4$. Otherwise there exists a regular item in the interval $(\frac{1-X}{4}, \frac{1}{4}]$, the smallest one has size Z, and it belongs to class U. Now let A denote a regular V item. Since A is a V item, it follows that $A < Z$, but Z is the smallest regular item in interval $(\frac{1-X}{4}, \frac{1}{4}]$, thus A cannot be in this interval and hence $A \leq \frac{1-X}{4}$ follows. □

Observation 3.4.2 *For the ranges of X, Z, $\frac{1-Z}{3}$, $\frac{1-X}{3}$, $\frac{1-Z}{2}$, and $\frac{1-X}{2}$ (i.e., for the boundaries of the classes) the following inequalities hold:*

$$2/11 < X \leq 1/5 < Z \leq 1/4 \leq \frac{1-Z}{3} < 4/15 \leq \frac{1-X}{3} < \frac{1}{3} < \frac{3}{8} \leq \frac{1-Z}{2} < \frac{2}{5} \leq \frac{1-X}{2} < \frac{9}{22}.$$

Proof. Each inequality can be checked easily. □

Observation 3.4.3 *The inequalities $G \leq 1 - 2X$ and $G + N \leq 1$ hold for any G and N items.*

Proof. Since each optimal bin contains at least three items, $G \leq 1 - 2X$ is valid also in this section. The second inequality follows from the facts that there is at least $2X$ room in a bin which contains only a G item at that time, and $N \leq 1/3 < 2X$ holds for any N item. □

We define the reserve, surplus or shortage of a bin as before. According to the definition of the classes, we obtain:

Lemma 3.4.1 *Only the next bins are possible.*

OPT

G	1	1	1	1	1	1	1
N	1	1					
S			1	1			
U	1		1		2	1	
V		1		1		1	2
r	0	1	2	3	3	4	5

B	1	1	1	1	1	1	1	1	1	1	1	1	1	1
C	1													
M		1	1	1										
N					2	1	1	1						
S		1				1			2	1	1			
U			1				1			1		2	1	
V	1			1				1			1		1	2
r	2	2	3	4	2	4	5	6	6	7	8	8	9	10

1	1
1	
2	3
1	2

C	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1
M		1	1	1	1										
N		1				2	1	1	1						
S			1				1			2	1	1			
U				1				1			1		2	1	
V	1				1				1			1		1	2
r	4	2	4	5	6	4	6	7	8	8	9	10	10	11	12

1	1	1	1
1			
	2	1	
2	1	2	3
2	2	3	4

M	2	2	2	2	1	1	1	1	1	1	1	1	1	1
N	1				2	1	1	1						
S		1				1			2	1	1			
U			1				1			1		2	1	
V				1				1			1		1	2
r	4	6	7	8	6	8	9	10	10	11	12	12	13	14

1	1	1	1	1	1	1	1
1	1						
		1	1				
1		1		3	2	1	
1	2	1	2		1	2	3
1	2	3	4	3	4	5	6

N	3	2	2	2	1	1	1	1	1	1
S		1			2	1	1			
U			1			1		2	1	
V				1			1		1	2
r	8	10	11	12	12	13	14	14	15	16

2	2	2	2	1	1	1	1	1	1	1	1
1				2	1	1	1				
	2	1			2	1		3	2	1	
1		1	2	1		1	2		1	2	3
2	2	3	4	4	4	5	6	5	6	7	8

S	3	2	2	1	1	1
U		1		2	1	
V			1		1	2
r	14	15	16	16	17	18

3	2	2	2	1	1	1	1
	2	1		3	2	1	
1		1	2		1	2	3
6	6	7	8	7	8	9	10

1
4
2

U	3	2	1	
V		1	2	3
r	17	18	19	20

4	3	2	1	
	1	2	3	4
8	9	10	11	12

3	2	1	
2	3	4	5
1	2	3	4

FFD

G	1	1	1	1	1	1	1				
B	1							2	1	1	1
C		1							1		
M			1							1	
N				1							1
S					1						
U						1					
V							1				
s	5	3	1	-1	-3	-4	-5	0	-2	-4	-6

G	1	1	1	1	1	1	1
B							
C							
M							
N	1	1					
S			1	1			
U	1		1		2	1	
V		1		1		1	2
s	8	7	6	5	5	4	3

B	1	1	1	1	1	1	1	1	1	1	1	1	1	1
C	1													
M		1	1	1										
N					2	1	1	1						
S		1				1			2	1	1			
U			1				1			1		2	1	
V	1			1				1			1		1	2
s	6	6	5	4	6	4	3	2	2	1	0	0	-1	-2

B	1	1
C		
M		
N		
S		
U	1	
V	2	3
s	7	6

C	2	1	1	1	1	1	1	1	1	1	1	1	1	1
M		1	1	1	1									
N		1				2	1	1	1					
S			1				1			2	1	1		
U				1				1			1		2	1
V	1				1				1			1		1
s	4	6	4	3	2	4	2	1	0	0	-1	-2	-2	-3

C	1	1	1	1
M				
N				
S	1			
U		2	1	
V	2	1	2	3
s	6	6	5	4

M	2	2	2	2	1	1	1	1	1	1	1	1	1
N	1				2	1	1	1					
S		1				1			2	1	1		
U			1				1			1		2	1
V				1				1			1		1
s	4	2	1	0	2	0	-1	-2	-2	-3	-4	-4	-5

M	1	1	1	1	1	1	1	1
N	1	1						
S			1	1				
U	1		1		3	2	1	
V	1	2	1	2		1	2	3
s	7	6	5	4	5	4	3	2

N	3	2	2	2	1	1	1
S		1			2	1	
U			1			1	2
V				1			
s	0	-2	-3	-4	-4	-5	-6

N	2	2	2	2	1	1	1	1	1	1	1	1
S	1				2	1	1	1				
U		2	1			2	1		3	2	1	
V	1		1	2	1		1	2		1	2	3
s	6	6	5	4	4	4	3	2	3	2	1	0

S	3	2	2	2	1	1	1	1
U		2	1		3	2	1	
V	1		1	2		1	2	3
s	2	2	1	0	1	0	-1	-2

S	1
U	
V	4
s	6

S				
U	4	3	2	1
V		1	2	3
s	0	-1	-2	-3

S				
U	3	2	1	
V	2	3	4	5
s	7	6	5	4

S	
U	
V	1
s	-1

Proof. The statement can be proven according to the definition of the classes. We will investigate each bin-type, and during this investigation we will apply Observation 3.4.2 several times. First let us consider the possible optimal bins. Any optimal bin contains at least three and at most five items. First we find all possible 3-bins, then the possible 4-bins and finally the possible 5-bins. The bins are listed in the table in lexicographic order. Let us consider now the possible 3-bins.

Since $G + M + X > 1$, a G-bin cannot contain a B, C or M item. Since $G + N + S > G + 2S > \frac{1}{2} + 2 \cdot \frac{1}{4} = 1$ holds by Observation 3.4.2, if there is an N or an S item in the G-bin, then the third item is surely an U or a V item.

Let us see the $(B, .)$-bins. Such a bin can contain only one B item. If the second item is a C item, then the third item is a V item, as $2C+U > 1$. If there is an M item in the bin, then the third item is smaller than an N item, since $B + M + N > \frac{1-X}{2} + \frac{1}{3} + \frac{1-X}{3} = \frac{7}{6} - \frac{5}{6}X \geq 1$.

Let us consider the possible $(C, .)$ bins. If two C items are in the bin then the third item is a V item. Suppose that the bin contains one C item. If there is also an M item in the bin, then applying $C + 2M > 3M > 1$, the third item is smaller than an M item.

If the 3-bin does not contain a C or bigger item, a $(3M)$ bin is impossible.

Now let us find all possible 4-bins. A G-bin cannot be a 4-bin. If the biggest item is a B item, then applying $S > \frac{1-Z}{3} \geq 1/4$ we get $B + S + 2X > \frac{1-X}{2} + \frac{1}{4} + 2X = \frac{3}{2}X + \frac{3}{4} > \frac{3}{11} + \frac{3}{4} > 1$, and it follows that the second biggest item cannot be an S or bigger item. A $(B, 2U, V)$ bin is not possible either.

Suppose that the bin is a $(C, .)$-bin. Then applying $C + N + 2X > \frac{1-Z}{2} + \frac{1-X}{3} + 2X > \frac{3}{8} + \frac{1}{3} + \frac{5}{3}X > \frac{3}{8} + \frac{1}{3} + \frac{10}{33} = \frac{89}{88}$, the second biggest item is smaller than an N item. Moreover, $C + S + U + V > \frac{1-Z}{2} + \frac{1-Z}{3} + Z + X = X + \frac{1}{6}Z + \frac{5}{6} > 1$, thus if there is an S item in the bin then the other two items are V items. Also, $C + 3U > \frac{1-Z}{2} + 3Z = \frac{5}{2}Z + \frac{1}{2} > 1$ is valid, thus a $(C, 3U)$ bin is not possible either.

Let us consider the $(M, .)$-bins among the 4-bins. We have $2M + 2X > 1$, thus the bin can contain only one M item. Suppose that the second biggest item is an N item. Then the third biggest item cannot be an S item, since $M+N+S+V > M+2S+V > \frac{1}{3}+2\cdot\frac{1}{4}+X > \frac{5}{6}+X > 1$, moreover two U items cannot be in the bin, since $M+N+2U > M+S+2U > \frac{1}{3}+\frac{1-Z}{3}+2Z = \frac{5}{3}Z+\frac{2}{3} > 1$.

If the second biggest item in the bin is not bigger than an S item, then we already have seen that $(M, 2S, V)$ or $(M, S, 2U)$ bins are impossible.

Let us consider the remaining 4-bins. Since $3N + X > 1$ holds, three N items cannot be in the bin. Since $3S + U > 1$ holds, if there are three N or S items in the bin, then the fourth item is a V item.

Now let us consider the possible 5-bins. Applying $N + 4X > \frac{1-X}{3} + 4X = \frac{11}{3}X + \frac{1}{3} > 1$, it follows that in a 5-bin there is no N or bigger item. Since $S + U + 3X > \frac{1-Z}{3} + Z + 3X = 3X + \frac{2}{3}Z + \frac{1}{3} > \frac{11}{3}X + \frac{1}{3} > 1$, if there is an S item in the bin, then the other four items are V items. Finally a $(4U, V)$ bin is not possible.

Now let us find the possible FFD bins. The differences between the possible optimal and FFD bins are as follows: On one hand, 2-bins may also occur, but in this case the level of the bin must be bigger than $1 - X$. On the other hand, any optimal bin-type is possible, except the bins that have level surely at most $1 - X$, since such a bin would have enough room to accommodate the last X item.

Among the FFD 2-bins, any bin is possible which contains one G item. If the 2-bin contains one B item, the second item cannot be an S or smaller item, as $B + S + X \leq \frac{1}{2} + \frac{1-X}{3} + X =$

$\frac{2}{3}X + \frac{5}{6} \leq \frac{2}{15} + \frac{5}{6} < 1$. Moreover, both items cannot be smaller than a B item, as $2C + X \leq 1$.

Any 5-bin is possible. From the set of 4-bins we can exclude all $(4V)$ bins. The reason is that here the V items would be regular as $V < Z \leq 1/4$, thus all have sizes at most $\frac{1-X}{4}$, according to Observation 3.4.1, and then X would fit into the bin.

We can also exclude the next 3-bins:

– $(C, 2V)$ FFD bin. In such a bin, the V items would be regular, thus both would have sizes at most $\frac{1-X}{4}$, according to Observation 3.4.1. Since $C \leq \frac{1-X}{2}$ also holds, there would remain at least X empty room in the bin, and this is a contradiction. By the same reason no $(M, 2V)$, $(N, 2V)$, $(S, 2V)$, $(U, 2V)$ bin is possible.

– (N, S, V) or (N, U, V) FFD bin. In such a bin, the V item would be regular, thus would have size at most $\frac{1-X}{4}$. Then $N+U+V+X \leq N+S+V+X \leq \frac{1}{3}+\frac{1-X}{3}+\frac{1-X}{4}+X = \frac{11}{12}+\frac{5}{12}X \leq 1$ would hold, a contradiction.

– Moreover, if all three items are at most S items, then item X naturally fits into the bin, thus such bins are not possible either. □

Observation 3.4.4 *No optimal bin has shortage, and all optimal non-G bins have at least one reserve.*

Note that now the weight of the smallest item is only 8, thus the additive constant 27 does not cover the 28 shortage of the last FFD bin, i.e. there remains 1 shortage in the last FFD bin. Thus in this section we must prove the next inequality:

$$res(I) + sur(I) + rex(I) \geq short(I) + 1. \tag{3.11}$$

Observation 3.4.5 *If there exists at least one (G, V) FFD bin, then there are no $U \cup S \cup N$ items. If there exists at least one (G, U) FFD bin, then there are no $S \cup N$ items. If there exists at least one (G, S) FFD bin, then there are no N items. In any case, if there exists at least one FFD bin with two items, one of which is a G item and the other one a $V \cup U \cup S \cup N$ item, then all these bins are identical in terms of the item sizes in them.*

Proof. Suppose that there exists a (G, V) bin in the FFD packing. It follows from Corollary 3.1.4 that there is no item with size in $(T, 2X]$. Since $U < S < N \leq 1/3 < 2X$, there are no $U \cup S \cup N$ items. The next two statements follow similarly. The last claim follows from Lemma 3.1.7. Suppose for example that there exists a (G, V) bin; let V_0 denote the smallest V item for which there exists a (G, V) bin. Then there is no item with size in $(V_0, 2X]$, thus if (G, V_1) is another (G, V) bin, then V_0 and V_1 have the same size, and then Lemma 3.1.7 (i) can be applied. □

Observation 3.4.6 *If at least one U item occurs, then there is a U item with size exactly Z in the last $(U, .)$ FFD bin (if there exists at least one $(U, .)$ FFD bin).*

Proof. The proof is almost the same as the proof of Observation 3.2.6. If $Z = 1/4$, then any U item has size exactly Z, thus the claim follows. Suppose that $Z < 1/4$. Observation 3.4.1 implies that there is a U item with size exactly Z. Suppose that all U items in the last $(U, .)$ bin are greater than Z. Then each item with size Z must be in earlier bins, thus all of them are fallback items. But this is a contradiction, since Z is defined as the size of a regular item (if $Z < 1/4$). □

In the next observation we exclude the possibility of many non-G FFD bins with shortage at the same time.

Observation 3.4.7 *The following pairs of bins cannot occur: Two $(B,.)$ bins, both having only one B item; or two $(C,.)$ bins, both having only one C item; or two $(M,.)$ bins, both having only one M item; or two $(N,.)$ bins, both having only at most two N items; or two $(S,.)$ bins, both having only at most two S items; or two $(U,.)$ bins, both having only at most three U items. Consequently two $(B,.)$ bins or two $(C,.)$ bins or two $(M,.)$ bins or two $(N,.)$ bins or two $(S,.)$ bins or two $(U,.)$ bins, both having shortage, cannot occur at the same time.*

Proof. The proof is the same as for the corresponding Observation 3.2.7. □

Observation 3.4.8 *If there exists an $(N,3V)$ FFD bin, then neither an $(S,.)$ nor a $(U,.)$ FFD bin can occur.*

Proof. Suppose that there exists an $(N,3V)$ FFD bin. Then no S or U item is packed into any later bin than the $(N,3V)$ FFD bin, since the S or U item would be packed into the $(N,3V)$ bin before the V items. □

Theorem 3.4.1 *Suppose that there is no (G,V) FFD bin, and there are no $\{(G,U),(G,2U)\}$, $\{(G,U),(G,U,V)\}$, $\{(G,S),(G,S,U)\}$, $\{(G,S),(G,S,V)\}$, $\{(G,S),(G,2U)\}$, $\{(G,N),(G,N,V)\}$ cobins, and there is at most one $\{(G,N),(G,N,U)\}$ cobin. Then (3.11) holds.*

Proof. Let us consider an arbitrary FFD G-bin. If the bin has shortage s, we increase the weight of the G item of the bin by s; and if the bin has surplus s, we decrease the weight of the G item of the bin by s.

If there is no $\{(G,N),(G,N,U)\}$ cobin, then the assumption of the lemma means (before the modification) that if an FFD G-bin had shortage k, then the optimal bin of the G item had reserve at least $k+1$, and any G item of some (G,N,U) optimal bin is packed into such FFD bin that has positive surplus. Thus after the modification no FFD G-bin has shortage and each optimal bin has positive reserve.

If there is one $\{(G,N),(G,N,U)\}$ cobin, the only exception is that there is exactly one optimal bin with shortage 1, and any other optimal bin has positive reserve.

Since we have at least 19 optimal bins, we have at least $18-1=17$ reserve, and the shortage caused by all FFD G-bins is already covered. Now let us see how much shortage can be caused by the FFD non-G-bins.

a, a $(B,.)$ 2-bin can cause at most 6,
b, a $(B,.)$ 3-bin can cause at most 2,
c, a $(C,.)$ bin can cause at most 3,
d, an $(M,.)$ bin can cause at most 5,
e, an $(N,.)$ bin can cause at most 6,

f, an $(S,.)$ bin can cause at most 2,
g, a $(U,.)$ bin can cause at most 3,
h, and finally there is 1 shortage in the last FFD bin.

Case 1. Suppose that there is a (B,U,V) or $(B,2V)$ FFD bin. Then there cannot be a $(B,.)$ 2-bin by Observation 3.4.7. FFD bins of type $(C,.)$ cannot occur either, since a C item would fit into the (B,U,V) or $(B,2V)$ bin before the smaller items. By the same reason an $(M,.)$ bin is impossible.

Thus the total uncovered shortage (as they were listed in bin-types a,–h,) is at most $0+2+0+0+6+2+3+1=14$, thus it is covered by the 17 reserve of the optimal bins. Thus we conclude that a $(B,.)$ 3-bin with shortage cannot exist.

Case 2. Suppose that there is a $(C,.)$ bin with shortage, i.e. there is a (C,S,U) or (C,S,V) or $(C,2U)$ or (C,U,V) FFD bin. Then there cannot be FFD bins of type $(M,.)$ or $(N,.)$, since an M or N item would fit into any of the previous $(C,.)$ bins in consideration, before the smaller items. Thus the total uncovered shortage is at most $6+0+3+0+0+2+3+1=15$, and it is covered by the 17 reserve of the optimal bins. Hence we can suppose in the following that a $(C,.)$ bin with shortage does not exist.

Case 3. Suppose that there is an $(M,.)$ bin with shortage. Then there cannot be a (B,N) FFD bin, since an M item would fit there before the N item. Thus the shortage caused by a $(B,.)$ 2-bin is at most 4.

If the $(M,.)$ bin causing the shortage is of type (M,N,U) or (M,N,V), then the total shortage is at most $4+0+0+2+6+2+3+1=18$, and it can be 18 only if there exists an $(N,2U)$ bin with shortage 6. But in this case an $(S,.)$-bin (with shortage) cannot exist, as the S item of such a bin would fit into the $(N,2U)$ bin before the U items. Thus the total shortage is in fact smaller, it is at most 17, thus it is covered.

Otherwise the $(M,.)$ bin that causes the shortage is of type $(M,2S)$, (M,S,U), (M,S,V), $(M,2U)$ or (M,U,V). In this case there is no $(N,.)$-bin (with shortage), since the N item would fit into the $(M,.)$ bin before the smaller items. Thus the total shortage is at most $4+0+0+5+0+2+3+1=15$, hence it is covered. Thus we suppose in the following that an $(M,.)$ bin with shortage does not exist.

Case 4. After excluding bins of Cases 1–3, the total uncovered shortage is at most $6+0+0+0+6+2+3+1=18$, and it can be really 18 only if there exist both an $(N,2U)$ bin with shortage 6 and an $(S,3V)$ bin with shortage 2. These bins cannot exist at the same time, however, thus the total shortage is at most 17, and it is again covered by the 17 reserve of the optimal bins.

Our proof is done. □

Lemma 3.4.2 *If (G,V) FFD bins exist, then (3.11) holds.*

Proof. Suppose that there is at least one (G,V) FFD bin. Then from Observation 3.4.5 it follows that there is no N, S, U item. The possible optimal and FFD bins that remain are listed in the following tables; we redefine also the weights of the items:

OPT (The possible optimal bins are as follows:)

28	G	1														
18	B		1	1	1	1										
16	C		1				2	1	1	1						
16	M			1				1			2	1	1			
8	V	2	1	1	3	2	1	1	3	2	1	3	2	5	4	3
	r	0	2	2	2	10	4	4	4	12	4	4	12	4	12	20

FFD (The possible FFD bins are as follows:)

28	G	1	1	1	1				1											
18	B	1				2	1	1		1	1	1	1							
16	C		1				1			1				2	1	1				
16	M			1				1			1				1		2	1		
8	V				1				2	1	1	2	3	1	1	3	1	3	5	1
	s	10	8	8	0	0	−2	−2	8	6	6	−2	6	4	4	4	4	4	4	−1

Since there can be at most one FFD B-bin which contains only one B item, we have only at most 3 shortage caused by a $(B,.)$ bin and the last FFD bin. Thus there cannot be such a bin with at least 3 reserve or surplus. Deleting them only the following possible bins remain:

OPT

28	G	1			
18	B		1	1	1
16	C		1		
16	M			1	
8	V	2	1	1	3
	r	0	2	2	2

FFD

28	G	1					
18	B		2	1	1	1	
16	C			1			
16	M				1		
8	V	1				2	1
	s	0	0	−2	−2	−2	−1

There can be at most one optimal bin with positive reserve, otherwise the total shortage is covered. Thus there is at most one optimal B-bin, i.e. there is at most one B item. Let the number of G items be k; since there are at least 19 optimal bins, it follows that $k \geq 18$. Then there are at least $2k$ V items in the optimal bins, while there are at most $k+3$ V items in the FFD bins, this is naturally a contradiction as $k+3 < 2k$. □

From this point in the proof we suppose that there is no (G, V) FFD bin.

Lemma 3.4.3 *If a $\{(G, U), (G, 2U)\}$ or $\{(G, U), (G, U, V)\}$ cobin exists, then (3.11) holds.*

Proof. From Observation 3.4.5 it follows that all G and U items in the (G, U) FFD bins have the same sizes; let us denote them as G' and U', respectively. From Observation 3.4.5 it follows that there are neither N nor S items. From the existence of the cobin and Corollary 3.1.5 it follows that each M item is bigger than $Z + X > \frac{1-X}{4} + X = \frac{3}{4}X + \frac{1}{4}$. Then there cannot be (B, M, U) or $(M, 3U)$ bins, since $B+M+U > \frac{1-X}{2} + \frac{3}{4}X + \frac{1}{4} + \frac{1-X}{4} = 1$ and $M+3U > \frac{3}{4}X + \frac{1}{4} + \frac{3(1-X)}{4} = 1$. Then we increase the weight of some types and we get the following possible bins:

OPT

26	G	1	1	1																
18	B				1	1	1	1	1	1	1									
17	C				1							2	1	1	1	1	1	1	1	1
17	M					1							1	1						
9	U	2	1				2	1		1			1		2	1		2	1	
8	V		1	2	1	1		1	2	2	3	1		1		1	2	1	2	3
r	r	0	1	2	1	1	8	9	10	1	2	2	1	2	9	10	11	1	2	3

17	M	2	2	1	1	1	1	1	1													
9	U	1		2	1		2	1		3	2	1		4	3	2	1		3	2	1	
8	V		1		1	2	1	2	3		1	2	3		1	2	3	4	2	3	4	5
r	r	1	2	9	10	11	1	2	3	17	18	19	20	8	9	10	11	12	1	2	3	4

FFD

26	G	1	1	1	1				1	1	1							
18	B	1				2	1	1				1	1	1	1	1	1	1
17	C		1				1					1						
17	M			1				1					1					
9	U				1				2	1				2	1		1	
8	V									1	2	1	1		1	2	2	3
	s	8	7	7	-1	0	-1	-1	8	7	6	7	7	0	-1	-2	7	6

17	C	2	1	1	1	1	1	1	1							
17	M		1	1						2	2	1	1	1	1	1
9	U		1		2	1	2	1		1		2	1	2	1	
8	V	1		1		1	1	2	3		1		1	1	2	3
	s	6	7	6	-1	-2	7	6	5	7	6	-1	-2	7	6	5

9	U	4	3	2	1	3	2	1		
8	V		1	2	3	2	3	4	5	1
	s	0	-1	-2	-3	7	6	5	4	-1

Now we distinguish between two cases.

Case 1. A $\{(G,U),(G,2U)\}$ cobin does not exist, but there exists a $\{(G,U),(G,U,V)\}$ cobin.

Let us increase the weight of the G' items by 1. Then there is no shortage by the (G',U') FFD bins. Since there is no $\{(G,U),(G,2U)\}$ cobin, no optimal bin has shortage.

We claim that the total shortage (caused by the FFD bins) is at most 6. We can see it as follows.

There cannot be two FFD bins with only one B item in each by Observation 3.4.7.

If there exists a $(B,2V)$ FFD bin, then there cannot be another B-bin with shortage, and there cannot be a regular C or M or U item after this bin, thus the shortage is at most $2+1$ (the $+1$ comes from the last FFD bin). Thus we can have at most 1 shortage caused by some B-bin.

If there is a (C,U,V) FFD bin, then there cannot be a U item in some later bin, since $C+2U \le \frac{1-X}{2} + \frac{2(1-X)}{3} = \frac{7}{6} - \frac{7}{6}X < 1$. Then the total uncovered shortage is at most $1+2+1$, caused by a B-bin, the (C,U,V) bin and the last FFD bin. The situation is the same if there exists an (M,U,V) FFD bin. Otherwise we can have at most 1 shortage because of a $(C,.)$ or $(M,.)$ FFD bin, since a $(C,2U)$ bin and an $(M,2U)$ bin cannot exist at the same time. Then there can be at most $1+1+3+1=6$ shortage by a $(B,.)$ bin, by a $(C,.)$ or $(M,.)$ bin, by a $(U,.)$ bin and by the last FFD bin.

Deleting each bin that has at least 6 reserve or surplus, only the following bins remain.

OPT

26	G	1	1	1																			
18	B				1	1	1	1															
17	C				1				2	1	1	1	1	1									
17	M					1				1	1				2	2	1	1	1				
9	U	2	1				1			1		2	1		1		2	1		3	2	1	
8	V		1	2	1	1	2	3	1		1	1	2	3		1	1	2	3	2	3	4	5
r	r	0	1	2	1	1	1	2	2	1	2	1	2	3	1	2	1	2	3	1	2	3	4

FFD

26	G	1												
18	B		2	1	1	1	1	1						
17	C			1					1	1	1			
17	M				1							1	1	1
9	U	1				2	1		2	1		2	1	
8	V						1	2		1	3		1	3
	s	-1	0	-1	-1	0	-1	-2	-1	-2	5	-1	-2	5

9	U	4	3	2	1	1		
8	V		1	2	3	4	5	1
	s	0	-1	-2	-3	5	4	-1

Then all G items are G' items, and it follows that there is no $(G,2U)$ optimal bin. Now we estimate the number of V items from above in the FFD bins. Hence, let us consider the FFD V-bins. Among them there cannot be two $(B,.)$ or $(C,.)$ or $(M,.)$ FFD bins at the same time, and it follows that there can be at most three V items in the mentioned bins. There can be at most four V items in some $(U,.)$ FFD bin, since there can be at most one $(U,.)$ bin with at most three U items by Observation 3.4.7. There can be at most one $(5V)$ FFD bin, since otherwise the total shortage is covered. Thus the number of V items is at most $3+4+5+1=13$, where the last "+1" stands for the last item. It follows that there are at least $19-13=6$ optimal bins containing no V item. Considering them in the table above we can realize that any such bin has at least one reserve, and the total shortage is covered.

Case 2. There exists a $\{(G,U),(G,2U)\}$ cobin. Then we can apply Corollary 3.1.5 in a stronger way, i.e. since $M < C \leq \frac{1-X}{2} \leq 2U$ holds, neither M nor C items exist, thus any bin containing an M or C item can be excluded from the possible bins. The following bins remain:

OPT

26	G	1	1	1																		
18	B				1	1	1	1	1													
9	U	2	1		2	1		1		3	2	1		4	3	2	1		3	2	1	
8	V		1	2		1	2	2	3		1	2	3		1	2	3	4	2	3	4	5
r	r	0	1	2	8	9	10	1	2	17	18	19	20	8	9	10	11	12	1	2	3	4

FFD

26	G	1	1		1	1	1					
18	B	1		2				1	1	1	1	1
9	U		1		2	1		2	1		1	
8	V					1	2		1	2	2	3
	s	8	-1	0	8	7	6	0	-1	-2	7	6

9	U	4	3	2	1	3	2	1		
8	V		1	2	3	2	3	4	5	1
	s	0	-1	-2	-3	7	6	5	4	-1

From the existence of the $\{(G, U), (G, 2U)\}$ cobin, $U' > 2Z - X$ follows.

Now we increase the weight of any U' item by 1, and decrease the weight of the $G \setminus G'$ items by 2. After this the (G', U') FFD bins have no shortage. Let us consider the optimal bins. Since $G' + U' + X > 1$, an item U' is not packed into an optimal G'-bin. Thus if U' is packed into some optimal G-bin, then the bin has no shortage. Suppose that U' is packed into an optimal non-G-bin. Since $U' + 2U + 2V > (2Z - X) + 2Z + 2X > 1$ holds, U' cannot be packed into a $(3U, 2V)$ optimal bin. Any other optimal non-G-bin has at least as large reserve as the number of U items packed into it. Thus no optimal bin has shortage.

We claim that the total shortage caused by the FFD bins is at most 4. Indeed, if there exists a (B, U, V) or a $(B, 2V)$ FFD bin, then there cannot be a $(U, .)$ bin, since $B+U+Z \le \frac{1}{2}+\frac{1-Z}{3}+Z = \frac{2}{3}Z + \frac{5}{6} \le 1$, applying also Observation 3.4.6. Thus in this case the total shortage is at most 3. On the other hand, if a $(B, .)$ bin with shortage does not exist, then the total shortage with modified weights is at most 4.

Let us delete each bin which has at least 4 reserve or surplus (with the modified weights); then only the following bins remain:

OPT

26	G	1	1	1					
18	B				1	1			
9	U	2	1		1		3	2	1
8	V		1	2	2	3	2	3	4
r	r	0	1	2	1	2	1	2	3

FFD

G	1									
B		2	1	1	1					
U	1		2	1		4	3	2	1	
V				1	2		1	2	3	1
s	-1	0	0	-1	-2	0	-1	-2	-3	-1

It turned out that any FFD G-bin is a (G', U') bin, hence each G item is a G' item. Thus no U' item is packed into an optimal G-bin. Since $B + U' + 2V > \frac{1-X}{2} + (2Z - X) + 2X > 1$, thus a U' item cannot be packed into a $(B, U, 2V)$ optimal bin, and we have already seen that a U' item cannot be packed into a $(3U, 2V)$ optimal bin. Moreover $2U' + 3V > 2\,(2Z - X) + 3X = 4Z + X > 1$, i.e. two U' items cannot be packed into the same $(2U, 3V)$ optimal bin. It follows that there is at most one U' item in any optimal bin and each optimal bin except the $(G, 2U)$ bins has positive reserve.

There can be at most three optimal bins with positive reserve, otherwise the total shortage is covered. Since there are at least 19 optimal bins, it follows that there are at least $19 - 3 = 16$ optimal G-bins. Thus there are at least 16 FFD (G', U') bins, and hence the number of U' items is at least 16. These U' items are packed one by one into different optimal non-G bins, serving at least 16 reserve, a contradiction. $\square$

Lemma 3.4.4 *If there exists a* $\{(G,S),(G,S,U)\}$ *cobin, then (3.11) holds.*

Proof. Suppose that there exists a $\{(G,S),(G,S,U)\}$ cobin. Note that $S+U > \frac{1-Z}{3}+Z = \frac{1}{3}+\frac{2}{3}Z > \frac{1}{3}+\frac{2}{3}\cdot\frac{1-X}{4} = \frac{1}{2}-\frac{1}{6}X$. Since $N < M < C \leq \frac{1-X}{2} < S+U$ holds, from Corollary 3.1.5 it follows that there are no N, M, C items, and each B item is bigger than $\frac{1}{3}+\frac{2}{3}Z > \frac{1}{2}-\frac{X}{6}$. Then there cannot be $(B,U,2V)$ and $(B,3V)$ bins, since $B+3V > \frac{1}{2}-\frac{X}{4}+3X = \frac{1}{2}+\frac{11}{4}X > 1$. A $(B,2S)$ bin is not possible either, since $B+2S > \frac{1}{3}+\frac{2}{3}Z+2\cdot\frac{1-Z}{3} = 1$. Note further that a (G,U) FFD bin is not possible, since then by Observation 3.4.5 no S item would exist. Only the next possible bins remain (with redefined weights):

OPT

25	G	1	1	1	1	1					
24	B						1	1	1	1	1
10	S	1	1				1	1			
9	U	1		2	1		1		2	1	
8	V		1		1	2		1		1	2
	r	0	1	1	2	3	1	2	2	3	4

10	S	3	2	2	1	1	1	3	2	2	2	1	1	1	1	1
9	U		1		2	1			2	1		3	2	1		
8	V			1		1	2	1		1	2		1	2	3	4
	r	14	15	16	16	17	18	6	6	7	8	7	8	9	10	2

9	U	3	2	1		4	3	2	1		3	2	1	
8	V		1	2	3		1	2	3	4	2	3	4	5
	r	17	18	19	20	8	9	10	11	12	1	2	3	4

FFD

25	G	1	1		1	1	1	1	1					
24	B	1		2						1	1	1	1	1
10	S		1		1	1				1	1			
9	U				1		2	1		1		2	1	
8	V					1		1	2		1		1	2
	s	13	−1	12	8	7	7	6	5	7	6	6	5	4

10	S	3	2	2	2	1	1	1	1	1									
9	U		2	1		3	2	1			4	3	2	1	3	2	1		
8	V	1		1	2		1	2	3	4		1	2	3	2	3	4	5	1
	s	2	2	1	0	1	0	−1	−2	6	0	−1	−2	−3	7	6	5	4	−1

Then again, we denote the G and S items of the (G,S) FFD-bins as G' and S', respectively; they have equal sizes by Observation 3.4.5. From the existence of a $\{(G',S'),(G',S,U)\}$ cobin, it follows that $S' > \frac{1-Z}{3}+Z-X$ holds.

Now we increase the weight of the S' items by 1, and decrease the weight of the $G \setminus G'$ items by 1.

As a result, no (G',S') FFD-bin has shortage. Let us see the optimal bins. If S' is packed into some optimal G-bin, then this G item is not a G' item, thus the optimal bin has no shortage. If S' is packed into some other optimal bin, the reserve of the bin before the modification of the weight was at least the number of S items in it. Thus no optimal bin has shorage.

The total shortage caused by the FFD bins is at most 4. (This statement can be proved similarly as it was proved in the previous lemma.) Let us delete all bins which have at least 4 reserve or surplus. Only the following bins remain:

OPT

25	G	1	1	1	1	1									
24	B						1	1	1	1					
10	S	1	1				1	1			3	1			
9	U	1		2	1		1		2	1			3	2	1
8	V		1		1	2		1		1	1	4	2	3	4
	r	0	1	1	2	3	1	2	2	3	6	2	1	2	3

FFD

25	G	1													
24	B														
10	S	1	3	2	2	2	1	1	1	1					
9	U			2	1		3	2	1		4	3	2	1	
8	V		1		1	2		1	2	3		1	2	3	1
	s	-1	2	2	1	0	1	0	-1	-2	0	-1	-2	-3	-1

Since no B item remained in the FFD bins, it follows that there is no B item at all. The following bins remain:

Since each remaining FFD G-bin is a (G', S') bin, it follows that each G item is a G' item. Thus each S' item is packed into a non-G optimal bin. Since $S' + 4V > (\frac{1-Z}{3} + Z - X) + 4X = \frac{1}{3} + \frac{2}{3}Z + 3X > \frac{1}{3} + \frac{11}{3}X > 1$, an S' item cannot be packed into an $(S, 4V)$ optimal bin. Moreover $S' + 2S + V > (\frac{1-Z}{3} + Z - X) + 2 \cdot \frac{1-Z}{3} + X = 1$ also holds, thus an S' item cannot be packed into a $(3S, V)$ optimal bin. Thus there is no S' item at all (as there is no S' item in the optimal bins), contradicting our assumption that there exists a $\{(G', S'), (G', S, U)\}$ cobin. $\square$

From now on we suppose that a $\{(G, S), (G, S, U)\}$ cobin does not exist.

Lemma 3.4.5 *If there exists a $\{(G, S), (G, S, V)\}$ or $\{(G, S), (G, 2U)\}$ cobin, then (3.11) holds.*

Proof. Suppose that there is at least one $\{(G, S), (G, S, V)\}$ or $\{(G, S), (G, 2U)\}$ cobin. Then $S + V \geq \frac{1}{4} + X$, $2U > \frac{1-X}{2}$, while $\frac{1}{2} + 2X > 1 - X$, thus $\frac{1}{4} + X > \frac{1-X}{2}$ holds. By the existence of the cobin, applying Corollary 3.1.5 it follows that there are no N, M, C items, since $N < M < C \leq \frac{1-X}{2} < \min\{S + V, 2U\}$. There cannot be a (G, U) FFD bin since then there would be no S item by Observation 3.4.5. Thus the next possible bins remain (we increased the weight of the G items by 1):

OPT

24	G	1	1	1	1	1								
18	B						1	1	1	1	1	1	1	1
10	S	1	1				2	1	1					
9	U	1		2	1			1		2	1		1	
8	V		1		1	2			1		1	2	2	3
	r	1	2	2	3	4	6	7	8	8	9	10	1	2

10	S	3	2	2	1	1	1	3	2	2	2	1	1	1	1	1
9	U		1		2	1			2	1		3	2	1		
8	V			1		1	2	1		1	2		1	2	3	4
	r	14	15	16	16	17	18	6	6	7	8	7	8	9	10	2

9	U	3	2	1		4	3	2	1		3	2	1	
8	V		1	2	3		1	2	3	4	2	3	4	5
	r	17	18	19	20	8	9	10	11	12	1	2	3	4

FFD

24	G	1	1		1	1	1	1	1								
18	B	1		2						1	1	1	1	1	1	1	1
10	S		1		1	1				2	1	1					
9	U				1		2	1			1		2	1		1	
8	V					1		1	2			1		1	2	2	3
	s	6	-2	0	7	6	6	5	4	2	1	0	0	-1	-2	7	6

10	S	3	2	2	2	1	1	1	1	1									
9	U		2	1		3	2	1			4	3	2	1	3	2	1		
8	V	1		1	2		1	2	3	4		1	2	3	2	3	4	5	1
	s	2	2	1	0	1	0	-1	-2	6	0	-1	-2	-3	7	6	5	4	-1

Now we denote again by G' and S', respectively, the G and S items of the (G,S) FFD bins; these items have equal sizes. Now we increase the weights of both the G' and S' items by 1, and decrease the weights of the $G \setminus G'$ items by 4. After this modification no (G',S') FFD bin has shortage. We claim that any optimal bin still has at least 1 reserve. This claim trivially holds for any non-G optimal bin. The statement also holds if the G-bin does not contain an S item. In case of a (G,S,V) optimal bin, both G and S items cannot be of types G' and S', respectively, since $G'+S'+X>1$. Finally in case of the (G,S,U) optimal bins the G item cannot be a G' item, since we already know that there is no $\{(G,S),(G,S,U)\}$ cobin. Thus we have at least 19 reserve in the optimal bins. On the other hand, the total shortage in the FFD bins is at most $2+2+3+1=8$ by a $(B,.)$ bin, by an $(S,.)$ bin, by a $(U,.)$ bin, and by the last FFD bin, thus it is covered. $\square$

Lemma 3.4.6 *If there exists at most one $\{(G,N),(G,N,U)\}$ cobin, moreover there exists a cobin of type $\{(G,N),(G,N,V)\}$, then (3.11) holds.*

Proof. Suppose that there exist a $\{(G,N),(G,N,V)\}$ cobin, and at the same time there exists at most one $\{(G,N),(G,N,U)\}$ cobin. By Observation 3.4.5 it follows that there cannot be a (G,U) or a (G,S) FFD bin (since then there would be no N item). The G and N items of the (G,N) FFD bins have the same sizes, and we denote them as G' and N'. Since $N+V > \frac{1-X}{3}+X = \frac{1}{3}+\frac{2}{3}X$, furthermore $M < C \le \frac{1-X}{2} < \frac{1}{3}+\frac{2}{3}X$ holds, it follows from Corollary 3.1.5 that there is no M or C item, and each B (if any) is bigger than $\frac{1-X}{3}+X$. Then there cannot be a $(B,2N)$ bin, since $B+2N > \frac{1-X}{3}+X+\frac{2(1-X)}{3} = 1$. The $(B,U,2V)$ or $(B,3V)$ bins are impossible, too, since $B+3V > \frac{1-X}{3}+X+3X = \frac{1}{3}+\frac{11}{3}X > 1$. The possible bins (with new weight $w(B)=20$) are as follows:

OPT

23	G	1	1	1	1	1	1	1									
20	B								1	1	1	1	1	1	1	1	1
12	N	1	1						1	1	1						
10	S			1	1				1			2	1	1			
9	U	1		1		2	1			1			1		2	1	
8	V		1		1		1	2			1			1		1	2
r	r	0	1	2	3	3	4	5	2	3	4	4	5	6	6	7	8

12	N	3	2	2	2	1	1	1	1	1	1	2	2	2	2	1	1	1	1	1	1	1	1
10	S		1			2	1	1				1				2	1	1	1				
9	U			1			1		2	1			2	1			2	1		3	2	1	
8	V				1			1		1	2	1		1	2	1		1	2		1	2	3
r	r	8	10	11	12	12	13	14	14	15	16	2	2	3	4	4	4	5	6	5	6	7	8

10	S	3	2	2	1	1	1	3	2	2	2	1	1	1	1	1
9	U		1		2	1			2	1		3	2	1		
8	V			1		1	2	1		1	2		1	2	3	4
r	r	14	15	16	16	17	18	6	6	7	8	7	8	9	10	2

9	U	3	2	1		4	3	2	1		3	2	1	
8	V		1	2	3		1	2	3	4	2	3	4	5
r	r	17	18	19	20	8	9	10	11	12	1	2	3	4

FFD

23	G	1	1			1	1	1	1	1	1	1									
20	B	1		2	1								1	1	1	1	1	1	1	1	1
12	N		1		1	1	1						1	1	1						
10	S							1	1				1			2	1	1			
9	U					1		1		2	1			1			1		2	1	
8	V						1		1		1	2			1			1		1	2
r	s	7	-1	4	-4	8	7	6	5	5	4	3	6	5	4	4	3	2	2	1	0

12	N	3	2	2	2	1	1	1	2	2	2	2	1	1	1	1	1	1	1	1
10	S		1			2	1		1				2	1	1	1				
9	U			1			1	2		2	1			2	1		3	2	1	
8	V				1				1		1	2	1		1	2		1	2	3
r	s	0	-2	-3	-4	-4	-5	-6	6	6	5	4	4	4	3	2	3	2	1	0

10	S	3	2	2	2	1	1	1	1	1									
9	U		2	1		3	2	1			4	3	2	1	3	2	1		
8	V	1		1	2		1	2	3	4		1	2	3	2	3	4	5	1
r	s	2	2	1	0	1	0	-1	-2	6	0	-1	-2	-3	7	6	5	4	-1

Now we decrease the weight of the $G \setminus G'$ items by 3, we increase the weight of the N' items by $2/3$, and also increase the weight of the G' items by $1/3$. Then the shortage of the (G', N') FFD bins is covered.

Let us see the optimal bins. In any optimal bin there remains at least $2/3$ reserve, except that there can be at most one (G', N, U) bin, it has $1/3$ shortage. As there are at least 19 optimal bins, we have at least $-1/3+18\cdot 2/3=35/3$ reserve in total.

If there is no (B,N) FFD bin, then the total shortage of the FFD bins is at most $6+3+1=10$ (since an $(S,.)$ and a $(U,.)$ FFD bin, both having shortage cannot exist at the same time), and this shortage is covered since in the optimal bins we found more reserve in total.

Otherwise there exists a (B,N) FFD bin, let the items of this bin be denoted as (B'',N'').

If there is also a (G,B) FFD bin, it covers the shortage of the (B,N) FFD bin and we are done again.

Otherwise there is no (G,B) FFD bin. Then the B'' item is the last B item, and the size of the N'' item is just the same as the size of the N' items. Thus it follows that an N' item (or the N'' item) cannot be packed into an optimal B-bin.

First suppose that there is no $\{(G,N),(G,N,U)\}$ cobin. Then in the optimal bin of the B'' item we have at least 2 reserve, in any other optimal bin we have at least $2/3$ reserve, thus we have at least $2+18\cdot 2/3=14$ reserve in total. This amount of reserve is just enough to cover the possible shortage of the FFD bins, which is at most $4+6+3+1=14$; we are done again.

Finally suppose that there is exactly one $\{(G,N),(G,N,U)\}$ cobin. Then it follows from Corollary 3.1.5 that the size of each B is bigger than $\frac{1-Z}{3}+Z$, as $N>\frac{1-Z}{3}$ and $U\geq Z$. Thus there is no (B,N,S) bin, since $B+N+S>B+2S>(\frac{1-Z}{3}+Z)+2\cdot\frac{1-Z}{3}=1$. In this case in the optimal bin of the B'' item we have at least 3 reserve, there is one optimal bin with $1/3$ shortage, and in any other optimal bin we have at least $2/3$ reserve, hence we have again at least $3-1/3+17\cdot 2/3=14$ reserve in total, which is enough to cover the shortage of the FFD bins. □

Now we are almost done, only one more lemma remained, but the proof of this last lemma will be the hardest one. We also need a new definition.

Definition 3.4.1 *Let H be an arbitrary nonempty set of items, and $A\in H$ an item. We say that item A is packed into an FFD bin for the i-th* ***attempt*** *(regarding H), if there are exactly $i-1\geq 0$ H items preceding A in the order of the items that are packed into some later bins than $bin(A)$ (i.e. bins with bigger index). It means that $i-1$ H items are tried to be packed into $bin(A)$ before A, none of them fits there, but then A fits into the bin. If $i=1$, we say that A is packed for the first attempt.*

Lemma 3.4.7 *If at least two $\{(G,N),(G,N,U)\}$ cobins exist, then (3.11) holds.*

Proof. Suppose that there exist two $\{(G,N),(G,N,U)\}$ cobins. Note that from this condition it follows that there are at least two U items. The G and N items of the (G,N) FFD bins have equal sizes, they are denoted as G' and N', respectively. Let us denote the items in the optimal bins of the two $\{(G,N),(G,N,U)\}$ cobins as N_{01} and U_{01}, and N_{02} and U_{02}, respectively. It follows that $N'>N_{0i}+U_{0i}-X$ holds for $i=1,2$. Since there are (G,N) FFD bins, by Observation 3.4.5 it follows that neither (G,S) nor (G,U) FFD bins are possible. Since $M<C\leq\frac{1-X}{2}<\frac{1-X}{3}+\frac{1-X}{4}<N_{0i}+U_{0i}$ (for $i=1,2$), it follows from Corollary 3.1.5 that there is no M or C item, and each B (if any) is bigger than $N_{0i}+U_{0i}>\frac{1-X}{3}+Z>\frac{7}{12}(1-X)$. There are no $(B,2N)$, (B,N,S), $(B,2S)$ bins, since $B+2S>(\frac{1-Z}{3}+Z)+2\cdot\frac{1-Z}{3}=1$. Further, there can be neither $(B,U,2V)$ nor $(B,3V)$ bins, since $B+3V>(\frac{1-X}{3}+X)+3X=\frac{1}{3}+\frac{11}{3}X>1$.

An N' item cannot be packed into a $(2N,S,V)$ or $(N,2S,V)$ optimal bin, since $N'+N+S+V>N'+2S+V>(N_{01}+U_{01}-X)+2S+X>3\cdot\frac{1-Z}{3}+Z=1$.

Thus the only possible bins are as follows, where we increased the weights of the B items by 5 and the weights of the S items by 1.

OPT

23	G	1	1	1	1	1	1	1
23	B							
12	N	1	1					
11	S			1	1			
9	U	1		1		2	1	
8	V		1		1		1	2
	r	0	1	1	2	3	4	5

23	G							
23	B	1	1	1	1	1	1	1
12	N	1	1					
11	S			1	1			
9	U	1		1		2	1	
8	V		1		1		1	2
	r	0	1	1	2	3	4	5

12	N	3	2	2	2	1	1	1	1	1	1
11	S		1			2	1	1			
9	U			1			1		2	1	
8	V				1			1		1	2
	r	8	9	11	12	10	12	13	14	15	16

12	N	2	2	2	2	1	1	1	1	1	1	1	1
11	S	1				2	1	1	1				
9	U		2	1			2	1		3	2	1	
8	V	1		1	2	1		1	2		1	2	3
	r	1	2	3	4	2	3	4	5	5	6	7	8

11	S	3	2	2	1	1	1
9	U		1		2	1	
8	V			1		1	2
	r	11	13	14	15	16	17

11	S	3	2	2	2	1	1	1	1
9	U		2	1		3	2	1	
8	V	1		1	2		1	2	3
	r	3	4	5	6	6	7	8	9

11	S	1
9	U	
8	V	4
	r	1

9	U	3	2	1	
8	V		1	2	3
	r	17	18	19	20

9	U	4	3	2	1	
8	V		1	2	3	4
	r	8	9	10	11	12

9	U	3	2	1	
8	V	2	3	4	5
	r	1	2	3	4

FFD

23	G	1	1		
23	B	1		2	1
12	N		1		1
11	S				
9	U				
8	V				
	s	10	-1	10	-1

23	G	1	1	1	1	1	1	1
23	B							
12	N	1	1					
11	S			1	1			
9	U	1		1		2	1	
8	V		1		1		1	2
	s	8	7	7	6	5	4	3

23	G							
23	B	1	1	1	1	1	1	1
12	N	1	1					
11	S			1	1			
9	U	1		1		2	1	
8	V		1		1		1	2
	s	8	7	7	6	5	4	3

12	N	3	2	2	2	1	1	1
11	S		1			2	1	
9	U			1			1	2
8	V				1			
	s	0	-1	-3	-4	-2	-4	-6

12	N	2	2	2	2	1	1	1	1	1	1	1	1
11	S	1				2	1	1	1				
9	U		2	1			2	1		3	2	1	
8	V	1		1	2	1		1	2		1	2	3
	s	7	6	5	4	6	5	4	3	3	2	1	0

11	S	3	2	2	2	1	1	1	1
9	U		2	1		3	2	1	
8	V	1		1	2		1	2	3
	s	5	4	3	2	2	1	0	-1

11	S	1
9	U	
8	V	4
	s	7

11	S				
9	U	4	3	2	1
8	V		1	2	3
	s	0	-1	-2	-3

11	S				
9	U	3	2	1	
8	V	2	3	4	5
	s	7	6	5	4

11	S	
9	U	
8	V	1
	s	-1

Let $H = N \cup S$. Recalling Definition 3.4.1, and checking the possible FFD bins, also taking into account that $S < N \leq 1/3 < 2X$ hold and that the size of any G item is at most $1 - 2X$, it follows that each H item is packed for the first attempt.

Observation 3.4.9 *Consider a certain FFD* 3*-bin, where there is one N item, or there are two N items, and the remained item(s) are from some smaller class(es). Then this N item is the smallest N item, or these two N items are the two smallest N items.*

Proof. Any N item in some earlier bin is packed for the first attempt, thus it precedes any N item in the considered bin. There is no other $(N, .)$ bin with at most two N items, by Observation 3.4.7. $\square$

Case 1. There are neither $(2N', 2U)$ nor $(2N', U, V)$ optimal bins. Suppose there exists a (B, N) FFD bin (there can be at most one such bin); let the items of this bin be denoted as (B', N''). Then since any N' item is packed for the first attempt, it follows that item N'' comes after the N' items. Thus by the minimality assumption on the items it follows that item N'' has the same size as the N' items (otherwise the size of the N' items could be decreased to the size of N''). Thus if there exists a (B, N) FFD bin, then also N'' will be denoted as N'.

Now we increase the weight of the N' items by 1, decrease the weights of the $G \setminus G'$ items by 3, and also decrease the weights of the $B \setminus B'$ items by 3. As a result, no FFD G-bin or B-bin has shortage. Considering the optimal bins, we conclude that any optimal G-bin has positive reserve, except the (G', N, U) optimal bins (where the N item cannot be an N' item), this bin has zero reserve. Also, any optimal B-bin has positive reserve, except the (B', N, U) optimal bin if any (then the N item of this bin cannot be an N' item), and this bin has zero reserve. Since no N' item is packed into a $(2N, S, V)$ or $(N, 2S, V)$ optimal bin, moreover there are no $(2N', 2U)$ or $(2N', U, V)$ optimal bins, in any optimal bin there remains positive reserve. Moreover if the optimal bin contains $k \geq 1$ N' items, there remains in the bin at least k reserve.

On the other hand, it is easy to check that the total shortage is at most 10 in the FFD bins. To finish the proof, we distinguish between two subcases, as follows. If there exist at most nine N' items, then the total number of G' and B' items is also at most nine, therefore the reserve is zero only in at most nine optimal bins, thus (since $OPT \geq 19$) we have totally at least 10 reserve in the optimal bins, which covers the shortage. Otherwise there are at least ten N' items, and then we again have at least 10 reserve in the optimal bins of the N' items, which covers the shortage.

Case 2. There exists a $(2N', 2U)$ or a $(2N', U, V)$ optimal bin. First we increase the weight of the S items to 12, and decrease the weights of the B items to 22. Then the tables of the possible bins look as follows:

OPT

23	G	1	1	1	1	1	1	1							
22	B								1	1	1	1	1	1	1
12	N	1	1						1	1					
12	S			1	1						1	1			
9	U	1		1		2	1		1		1		2	1	
8	V		1		1		1	2		1		1		1	2
	r	0	1	0	1	3	4	5	1	2	1	2	4	5	6

12	N	3	2	2	2	1	1	1	1	1	1	2	2	2	2	1	1	1	1	1	1	1	1
12	S		1			2	1	1				1				2	1	1	1				
9	U			1			1		2	1			2	1			2	1		3	2	1	
8	V				1			1		1	2	1		1	2	1		1	2		1	2	3
	r	8	8	11	12	8	11	12	14	15	16	0	2	3	4	0	2	3	4	5	6	7	8

12	S	3	2	2	1	1	1	3	2	2	2	1	1	1	1	1
9	U		1		2	1			2	1		3	2	1		
8	V			1		1	2	1		1	2		1	2	3	4
	r	8	11	12	14	15	16	0	2	3	4	5	6	7	8	0

9	U	3	2	1		4	3	2	1		3	2	1	
8	V		1	2	3		1	2	3	4	2	3	4	5
	r	17	18	19	20	8	9	10	11	12	1	2	3	4

FFD

23	G	1	1			1	1	1	1	1	1	1							
22	B	1		2	1								1	1	1	1	1	1	1
12	N		1		1	1	1						1	1					
12	S							1	1						1	1			
9	U					1		1		2	1		1		1		2	1	
8	V						1		1		1	2		1		1		1	2
	s	9	−1	8	−2	8	7	8	7	5	4	3	7	6	7	6	4	3	2

12	N	3	2	2	2	1	1	1	2	2	2	2	1	1	1	1	1	1	1	1
12	S		1			2	1		1				2	1	1	1				
9	U			1			1	2		2	1			2	1		3	2	1	
8	V				1				1		1	2	1		1	2		1	2	3
	s	0	0	−3	−4	0	−3	−6	8	6	5	4	8	6	5	4	3	2	1	0

12	S	3	2	2	2	1	1	1	1	1									
9	U		2	1		3	2	1			4	3	2	1	3	2	1		
8	V	1		1	2		1	2	3	4		1	2	3	2	3	4	5	1
	s	8	6	5	4	3	2	1	0	8	0	−1	−2	−3	7	6	5	4	−1

Suppose that there exists a (B, N) FFD bin. Since any N is packed for the first attempt, the N item of this bin comes after the N' items, thus $N' > 1/2 - X$ holds, contradicting the condition of Case 2. Thus a (B, N) FFD bin is impossible.

Suppose that there exists a $(2N, V)$ FFD bin. Then these N items are the two smallest N items, by Observation 3.4.9. It follows that two N items, a U item and a further U or V item cannot be packed into one bin, contradicting the assumption of Case 2. Thus there is no $(2N, V)$ FFD bin.

Now we decrease the weight of the $G \setminus G'$ items by 1, and increase the weight of the N' items by 1. As a result, no (G', N') FFD bin has shortage, and no optimal bin has shortage (as no N' item is packed into a $(2N, S, V)$ or $(N, 2S, V)$ bin. Now we cover the shortage caused by some $(N, 2U)$, (N, S, U) or $(2N, U)$ FFD bin. Clearly there can be at most one of them by Observation 3.4.7. Let the U item (or items) of this bin be called as **bad U** item(s). Now we increase the weight of the bad U items by 3, and we show below that the reserve of the optimal bins of the bad U items do not become negative.

Case of (N,2U) FFD bin. Let the items of the bin be denoted as (N_1, U_1, U_2), where $U_1 \geq U_2$. Then N_1 is the smallest N item by Observation 3.4.9. Thus $N_1 \leq 1/2 - Z$ holds because of the existence of the $\{(G, N), (G, N, U)\}$ cobin. We know from Observation 3.4.7 that there is no other $(N, .)$ bin, except the $(3N)$ bins. If there is an S item in some bin earlier than the (N_1, U_1, U_2) bin (for example if there exists a (G, S, U) FFD bin), N_1 would fit into any such bin before the S item, a contradiction. It follows that there is no S item in the earlier bins. There is no $(S, .)$ bin either, since one S item from such a bin would fit into the (N_1, U_1, U_2) bin before the U items. Thus in this case there is no S item at all.

Suppose that there exists a (B, N, U_1) optimal bin (the case of (B, N, U_2) bin is similar). Since $B > N_{01} + U_{01} \geq N_1 + X$ and $N > U_2$ hold, the level of the (B, N, U_1) optimal bin would be bigger than $(N_1 + X) + U_2 + U_1 > 1$, a contradiction.

Let U' denote any of U_1 and U_2. It follows that U' cannot be packed into a (B, N, U) optimal bin, and similar arguments show that U' cannot occur in any (G, N, U), $(2N, 2U)$ or $(2N, U, V)$ optimal bin. Moreover, since $1 < N_1+U_1+U_2+X \leq (1/2-Z)+\frac{1-Z}{3}+U_2+X = \frac{5}{6}-\frac{4}{3}Z+U_2+X$, thus $U_1 \geq U_2 > \frac{4}{3}Z+\frac{1}{6}-X$ holds. Then $U'+U+3V > (\frac{4}{3}Z+\frac{1}{6}-X)+Z+3X = 2X+\frac{7}{3}Z+\frac{1}{6} > 2X + \frac{7}{3}\frac{1-X}{4} + \frac{1}{6} = \frac{17}{12}X + \frac{3}{4} > \frac{17}{12}\frac{2}{11} + \frac{3}{4} = \frac{133}{132} > 1$, thus no U' item can be packed into a $(3U, 2V)$ or $(2U, 3V)$ optimal bin. Since $G > B > N_1 + X$, neither U_1 nor U_2 can occur in optimal bins $(G, 2U)$ and $(B, 2U)$, and they cannot be simultaneously in $(N, 3U)$ or $(N, 2U, V)$ optimal bins. Thus, checking one by one the possible optimal bins of the bad U items, we find 3 reserve in the optimal bin of each bad U, or 6 reserve if they are in a common bin, to cover the increased weights of the bad items.

Case of (N,S,U) FFD bin. Let the items be denoted as (N_1, S_1, U_1). Similarly as in the case of the $(N, 2U)$ bin, it follows that N_1 is the smallest N item, thus $N_1 \leq 1/2 - Z$ holds as above. It follows also that S_1 is the only one S item. (If there was an S item in some earlier bin, N_1 would be packed there instead of the S item; and if there was an S item in some later bin, this S item would fit into the (N_1, S_1, U_1) bin before the U item.) Now $1 < N_1 + S_1 + U_1 + X \leq (1/2 - Z) + \frac{1-X}{3} + U_1 + X = \frac{2}{3}X - Z + U_1 + \frac{5}{6}$ holds, thus $U_1 > Z + \frac{1}{6} - \frac{2}{3}X$ is valid. Then $U_1 + U + 3V > (Z + \frac{1}{6} - \frac{2}{3}X) + Z + 3X = \frac{7}{3}X + 2Z + \frac{1}{6} > \frac{7}{3}X + \frac{1-X}{2} + \frac{1}{6} = \frac{11}{6}X + \frac{2}{3} > 1$, thus U_1 cannot be in a $(3U, 2V)$ or $(2U, 3V)$ optimal bin.

The inequalities $G > B > N_1 + X > S_1 + X$ hold similarly as in the previous case, thus U_1 cannot be packed into a (G, N, U), (G, S, U), (B, N, U), (B, S, U), $(2N, 2U)$, $(2N, U, V)$, $(N, S, 2U)$ or (N, S, U, V) optimal bin (and there is only one S item, i.e. there is no S-bin with at least two S items). Thus we find 3 reserve in the optimal bin of U_1.

Case of (2N,U) FFD bin. Let the items be denoted as (N_1, N_2, U_1), where $N_1 \geq N_2$. Again it holds that N_1 and N_2 are the two smallest N items by Observation 3.4.9. Also, similarly as in the case of the $(N, 2U)$ FFD bin, there is no S item. Suppose that U_1 is packed into a $(3U, 2V)$ or $(2U, 3V)$ optimal bin. Then $N_1+N_2+U_1+X > 1 \geq U_1+Z+3X$, i.e. $N_1+N_2 > Z+2X$. Since $N' > N_{0i}+U_{0i}-X$ holds for $i = 1, 2$, and N_1 and N_2 are the two smallest N items, it follows that $N' > N_1+Z-X \geq N_2+Z-X$. Thus $2N'+U+V > (N_1+Z-X)+(N_2+Z-X)+Z+X = (N_1 + N_2) + 3Z - X > 4Z + X > 1$, a contradiction. We have obtained that U_1 cannot be packed into a $(3U, 2V)$ or $(2U, 3V)$ optimal bin.

Applying again that there exist at least two cobins, $G > B > N_1+X \geq N_2+X$ hold. Thus U_1 cannot be packed into a (G, N, U), (B, N, U), $(2N, 2U)$ or $(2N, U, V)$ optimal bin. In any other optimal bin where U_1 can occur we find 3 reserve.

Summarizing the previous results, the shortage caused by the (G', N') FFD bins is covered, and also the shortage of the FFD bin of the bad U item(s) is covered. Since a (B, N) bin or a $(2N, V)$ FFD bin is impossible in Case 2, at most 4 shortage remains, namely at most 3 by the $(U, .)$ FFD bins and 1 in the last FFD bin.

Let us delete each bin which has at least 4 surplus or reserve. (We must be careful during the calculation of the reserve of the optimal bins, as we must take into account the increased weights of the N' and bad U items). The next bins remain:

OPT

23	G	1	1	1	1	1	1						
22	B							1	1	1	1	1	1
12	N	1	1					1	1				
12	S			1	1					1	1		
9	U	1		1		2	1	1		1		2	1
8	V		1		1		1		1		1		1
	r	0	1	0	1	3	4	1	2	1	2	4	5

12	N	2	2	2	2	1	1	1	1	1	1	1											
12	S	1				2	1	1	1				3	2	2	1	1	1					
9	U		2	1			2	1		3	2	1		2	1	3	2		4	3	3	2	1
8	V	1		1	2	1		1	2		1	2	1		1		1	4		1	2	3	4
	r	0	2	3	4	0	2	3	4	5	6	7	0	2	3	5	6	0	8	9	1	2	3

FFD

23	G	1	1	1		
22	B				1	1
12	N	1				
12	S					
9	U		1		1	
8	V		1	2	1	2
	s	-1	4	3	3	2

12	N	3	2	2	1	1	1	1	1	1	1									
12	S		1		2	1						1	1	1	1					
9	U			1		1	2	3	2	1		3	2	1		4	3	2	1	
8	V								1	2	3		1	2	3		1	2	3	1
	s	0	0	-3	0	-3	-6	3	2	1	0	3	2	1	0	0	-1	-2	-3	-1

Now we can exclude some more FFD bins as follows:

- $(N, U, 2V)$ or $(S, U, 2V)$ FFD bins are impossible, since then there is no U item in some later bin (as it would fit into this bin before the two V items), thus there is no $(U, .)$ bin with shortage, and the 1 surplus of the $(N, U, 2V)$ or $(S, U, 2V)$ bin covers the remaining 1 shortage of the last FFD bin.

- (G, U, V), $(G, 2V)$, (B, U, V), $(B, 2V)$ bins are impossible, since an N item always fits into a bin which contains only a G item; thus if such a bin exists, then there would not be any N item different from the N' items (but such N must occur in the optimal bin of the cobin). Then no B-bin remains in the FFD bins, thus there is no B item at all. The following bins remain:

OPT

23	G	1	1	1	1	1	1
12	N	1	1				
12	S			1	1		
9	U	1		1		2	1
8	V		1		1		1
	r	0	1	0	1	3	4

12	N	2	2	2	2	1	1	1	1	1	1	1											
12	S	1				2	1	1	1				3	2	2	1	1	1					
9	U		2	1			2	1		3	2	1		2	1	3	2		4	3	3	2	1
8	V	1		1	2	1		1	2		1	2	1		1		1	4		1	2	3	4
	r	0	2	3	4	0	2	3	4	5	6	7	0	2	3	5	6	0	8	9	1	2	3

FFD

23	G	1																	
12	N	1	3	2	2	1	1	1	1	1	1								
12	S			1		2	1					1	1	1					
9	U				1		1	2	3	2		3	2		4	3	2	1	
8	V									1	3		1	3		1	2	3	1
	s	-1	0	0	-3	0	-3	-6	3	2	0	3	2	0	0	-1	-2	-3	-1

Now we exclude some further bins as follows.

- There cannot be an $(N, 3V)$ or $(S, 3V)$ FFD bin. If such a bin exits, then there would be no U item at all, since there is no U item in the earlier bins, and cannot be in the later bins either (but there must be at least one U item by the condition of this Case 2).

- Suppose there exists an $(N, 2U, V)$ FFD bin with surplus 2; let the items be denoted as (N_1, U_1, U_2, V_1). Then N_1 is the smallest N item, there is no S item, and there is no bad U item. If there is no $(U, .)$ FFD bin with shortage, then no shortage remains. Otherwise there exists a $(U, .)$ bin with shortage. Then the last (and thus smallest) U item does not fit into the $(N, 2U, V)$ bin before the V item. It follows that $N_1 + U_1 + U_2 + Z > 1$ holds.

We state that neither U_1 nor U_2 can be packed into a (G, N, U) optimal bin. Suppose for example that U_1 is packed into a (G, N, U) optimal bin. Since $G > N_1 + Z$ and $N > U_2$, we get that the level of the (G, N, U) optimal bin is bigger than $(N_1 + Z) + U_2 + U_1 > 1$, a contradiction, thus the claim follows.

Similarly we get that neither U_1 nor U_2 can be packed into a $(2N, 2U)$ optimal bin, and they cannot be packed into any optimal S-bin (since there is no S item). Neither U_1 nor U_2 can be packed into a $(3U, 2V)$ optimal bin (indeed, then a U and two V items would also fit here, but N_1 and Z do not fit into a bin with U_1 and U_2, thus then $N_1 + Z > Z + 2X$, implying $1/3 \geq N > 2X$, which is a contradiction). Thus in the optimal bins of U_1 and U_2 we find at least 2 further total reserve, therefore the total shortage is covered.

- Similarly it follows that there is no $(S, 2U, V)$ FFD bin. If such a bin exists, then it contains the smallest S item, and there is no $(N, .)$ FFD bin with shortage. (If an (N, S, U) FFD bin exists, then there is no S in the later bins.) Thus there is no bad U item. There must be a $(U, .)$ bin with shortage, otherwise the total shortage is covered. Then $S_1 + U_1 + U_2 + Z > 1$ holds with the items, and we again find at least 2 more reserve in the optimal bins of the U_1 and U_2 items.

- Thus no V item remained before the $(U, .)$ FFD bins. We conclude that there are at most four V items, since from Observation 3.4.7 it follows that there can be at most one $(U, .)$ FFD bin with at most three U items.

- There can be neither an $(S, 4V)$ nor a $(U, 4V)$ optimal bin. Indeed, if such a bin exists, then there are four V items, these items can be only in a $(U, 3V)$ FFD bin (here $U = Z$) and in the last FFD bin, and then the smallest U and the four V items do not fit into a bin, a contradiction. Thus only the following bins remain:

OPT

23	G	1	1	1	1	1	1
12	N	1	1				
12	S			1	1		
9	U	1		1		2	1
8	V		1		1		1
	r	0	1	0	1	3	4

12	N	2	2	2	2	1	1	1	1	1	1	1									
12	S	1				2	1	1	1				3	2	2	1	1				
9	U		2	1			2	1		3	2	1		2	1	3	2	4	3	3	2
8	V	1		1	2	1		1	2		1	2	1		1		1		1	2	3
	r	0	2	3	4	0	2	3	4	5	6	7	0	2	3	5	6	8	9	1	2

FFD

23	G	1												
12	N	1	3	2	2	1	1	1	1						
12	S			1		2	1			1					
9	U				1		1	2	3	3	4	3	2	1	
8	V											1	2	3	1
	s	-1	0	0	-3	0	-3	-6	3	3	0	-1	-2	-3	-1

Suppose there exists an $(S, 3U)$ FFD bin, with 3 surplus. Then there must be also a $(U, 3V)$ FFD bin, otherwise the total shortage is covered. Because of the existence of the $(S, 3U)$ FFD bin, there is no bad U item. There cannot be an optimal bin with positive reserve, since then the total shortage would be covered (recall that any G item is a G' item, thus there is no N' item in an optimal G-bin) therefore only the following optimal bins are possible: (G, N, U), (G, S, U), $(2N, S, V)$, $(2N, 2U)$, $(N, 2S, V)$, and $(3S, V)$. Since there are four V items, and any remaining possible optimal V-bin contains exactly one V item, and also contains at least one S item, it follows that there are at least four S items. On the other hand, since there can be at most one $(N, .)$ FFD bin with at most two N items, and there is only one $(S, 3U)$ FFD bin, it follows that there can be at most three S items. This is a contradiction, thus there is no $(S, 3U)$ FFD bin. (The existence of an $(N, 3U)$ FFD bin could be excluded similarly but we do not need this in the following.)

After excluding the $(S, 3U)$ FFD bin we make the following consideration: If there exists an S item, then exactly one FFD bin must occur from the bin-types listed below:

a, a $(2N, S)$ bin with the two smallest N items, and the only one S item,

b, an $(N, 2S)$ bin with the smallest N item, and the only two S items,

c, an (N, S, U) bin with the smallest N item, the only one S item and the biggest U item.

It follows that $(2N, S, V)$, $(N, 2S, V)$, $(3S, V)$ optimal bins are impossible as the items do not fit into one bin, and there exist at most two S items. By excluding these bins we have the possibility to increase the weight of the V items to be $w(V) = 9$. Then the following possible bins remain:

OPT

23	G	1	1	1	1	1	1
12	N	1	1				
12	S			1	1		
9	U	1		1		2	1
9	V		1		1		1
	r	0	0	0	0	3	3

12	N	2	2	2	1	1	1	1	1	1								
12	S				1	1	1				2	2	1	1				
9	U	2	1		2	1		3	2	1	2	1	3	2	4	3	3	2
9	V		1	2		1	2		1	2		1		1		1	2	3
	r	2	2	2	2	2	2	5	5	5	2	2	5	5	8	8	-1	-1

FFD

23	G	1												
12	N	1	3	2	2	1	1	1	1					
12	S			1		2	1							
9	U				1		1	2	3	4	3	2	1	
9	V										1	2	3	1
	s	-1	0	0	-3	0	-3	-6	3	0	0	0	0	0

Now we have no shortage in the $(U, .)$ FFD bins, there is no shortage in the last FFD bin, all shortage caused by (G, N) FFD bins is covered by the optimal bins of the N' items, and also the shortage caused by the bad U items is covered. Thus the total shortage is covered, except for the shortage created now in the possible $(3U, 2V)$ or $(2U, 3V)$ optimal bins.

If there is a $(U, 3V)$ FFD bin, then the smallest U item and the four V items do not fit into one bin, thus there is no optimal 5-bin, and no shortage remains, therefore a $(U, 3V)$ FFD bin is impossible.

Observation 3.4.10 *There are at most three V items. Moreover there is exactly one optimal bin with shortage, a $(3U, 2V)$ optimal bin or a $(2U, 3V)$ optimal bin.*

Proof. There can be at most three V items, since V items are only in one $(U, .)$ FFD bin, and item X in the last bin. Since there can be at most three V items, $(3U, 2V)$ and $(2U, 3V)$ optimal bins cannot occur at the same time, and there can be at most one such bin. If there is no such bin, then the total shortage would be covered. □

Thus there is only one optimal bin with 1 shortage, i.e. the uncovered shortage is only one. There cannot be an optimal bin different from $(3U, 2V)$ and $(2U, 3V)$ and containing at least two V items, since then there would be four V items. Thus $(2N, 2V)$, $(N, S, 2V)$, $(N, U, 2V)$ optimal bins are impossible. Let us delete these bins, furthermore let us delete any bin which has positive reserve or surplus, since they would cover the possible shortage. The following possible bins remain:

OPT

23	G	1	1	1	1	1	1				
12	N	1	1					2	2		
12	S			1	1						
9	U	1		1		2	1	2	1	3	2
9	V		1		1		1		1	2	3
	r	0	0	0	0	3	3	2	2	-1	-1

FFD

23	G	1										
12	N	1	3	2	2	1	1	1				
12	S			1		2	1					
9	U				1		1	2	4	3	2	
9	V									1	2	1
	s	-1	0	0	-3	0	-3	-6	0	0	0	0

We conclude that

a, Each N' item is packed into a $(2N', 2U)$ or $(2N', U, V)$ optimal bin, and both N items are N' items in these bins (otherwise positive reserve would remain in the bin).

b, In any $(G, 2U)$ optimal bin or (G, U, V) optimal bin there is exactly one bad U item (otherwise there would be positive reserve) and a bad U item can occur only in such an optimal bin.

Upon the considerations above, we introduce a new notation H for all N items, S items and bad U items. By the simplified notation we get fewer bin-types, more exactly, several different bin-types can be handled in a simplified way.

OPT

23	G	1	1				
13	N'			2	2		
12	H	1	1				
9	U	1		2	1	3	2
9	V		1		1	2	3
	r	0	0	0	0	-1	-1

FFD

G	1					
N'	1					
H		3				
U			4	3	2	
V				1	2	1
s	0	0	0	0	0	0

Let U_0 denote the first U item in the bin before the last FFD bin (i.e. the first U item in the $(3U, V)$ FFD bin or in the $(2U, 2V)$ FFD bin).

Claim 3.4.1 *There exists a fallback U item after U_0.*

Proof. Otherwise, since the five smallest items do not fit into one bin, there is no optimal 5-bin, thus there is no uncovered shortage, a contradiction. □

Thus there must be fallback U item after U_0. (A V item cannot be fallback, since three U items plus an item with size at most Z always fit into a bin). Such a fallback U item can be packed only into some $(4U)$ FFD bin.

Now we introduce one more definition. We say that a U item is **big** if it is bigger than $1/4$ and it is not a bad U item. (Bad U items (if there are any) precede the big U items, thus the bad U items are also bigger than $1/4$.) The U items which are not bigger than $1/4$ are called **small** U items. Note that a big U item cannot be fallback, thus any fallback U item is surely a small U item, but small regular U items can exist. Then it follows that there exists at least one $(4U)$ FFD bin, and the first item in the first $(4U)$ FFD bin is a big U item (otherwise there would be no fallback item).

It is easy to check that a big U item cannot be packed into a (G, H, U), $(3U, 2V)$ or $(2U, 3V)$ optimal bin; the reason for the last two types is $1/4 + Z + 3X > 1/4 + \frac{1-X}{4} + 3X = \frac{11}{4}X + \frac{1}{2} > 1$. Thus a big U item can be only in a $(2N', 2U)$ or $(2N', U, V)$ optimal bin, and only one big U can be packed into such a bin.

Claim 3.4.2 *At the actual state of the FFD packing just after U_0 has been packed, each $(4U)$ FFD bin except the last one contains three or four items. Furthermore at this time all $(4U)$ bins containing three items are located before the $(4U)$ FFD bins containing four items, and the last $(4U)$ bin contains U_0 only.*

Proof. Since the sizes of the items are decreasing, and any three U items fit into a bin, it follows that the sums of the first three items in the $(4U)$ bins are decreasing. Thus if there are already four items in some $(4U)$ bin just when U_0 comes, this U_0 item would fit into any later bin, if there are only three items there. □

Let i denote the number of 3-bins among the final $(4U)$ FFD bins just after packing U_0. Moreover let the number of big U items be $3k + l$, where $k \geq 0$ and $l \in \{0, 1, 2\}$.

Claim 3.4.3 *The number of fallback U items is $k - 1$ or k if $l = 0$, and it is k or $k + 1$ if $l = 1$ or $l = 2$.*

Proof. Let us consider the FFD packing, how the $(4U)$ bins are packed. First three big U items are packed into one bin, then the next three big U are packed into the next bin, and so on, and finally in the $(k + 1)$-st $(4U)$ bin there are l big U items. Then come the small U items, and four small U items always fit into a common bin. If $l = 0$, then there will be $k - 1$ fallback items in the first $k - 1$ $(4U)$ bins, and there can be also one more fallback item in the k-th $(4U)$ bin. If $l > 0$, then there will be k fallback items in the first k $(4U)$ bins, and there can be also one more fallback item in next $(4U)$ bin. □

Corollary 3.4.2 *For the number i of 3-bins defined above in terms of U_0 we have $1 \leq i \leq k + 1$.*

To finish the proof of Case 2 of this lemma, we consider two subcases as follows.

Subcase 2.1. There exists a $(2U, 3V)$ optimal bin. Let the items be denoted as $(U_1, U_2, V_1, V_2, V_3)$. Since there are three V items in the bin, there is no other V-bin in the optimal packing. Moreover there is a $(2U, 2V)$ FFD bin, since otherwise the number of V items would be only two. The possible bins in this subcase are:

OPT

23	G	1		
13	N’		2	
12	H	1		
9	U	1	2	2
9	V			3
	r	0	0	−1

FFD

G	1				
N’	1				
H		3			
U			4	2	
V				2	1
r	0	0	0	0	0

Let us consider an arbitrary big U item, denoted as U_b.

Claim 3.4.4 *For any big U item U_b, there are at least three different small U items $U_{b,1}, U_{b,2}, U_{b,3}$, such that U_b together with any two further big or small U items and any item from $U_{b,1}, U_{b,2}, U_{b,3}$ fit into a common bin; furthermore for different choices of U_b the sets of the corresponding three small U items are mutually disjoint.*

Proof. Recall that U_b is surely packed into an optimal bin $(2N', 2U)$. Let the items be denoted in this bin as $(N', N', U_b, U_{b,1})$; here $U_{b,1}$ is a small U item. The two N' items of this optimal bin are packed into two (G', N') FFD bins. On the other hand, any G' item is packed into a (G', H, U) optimal bin, where the U item is a small U. Let us denote the optimal bins of the two G' items as $(G', A_1, U_{b,2})$ and $(G', A_2, U_{b,3})$. (We do not use different letters for the two different G' items, since they have the same size.) Then A_1 and A_2 are H items, and both $U_{b,2}$ and $U_{b,3}$ are small U items. Thus, all the items considered fit into four optimal bins: $(G', A_1, U_{b,2})$, $(G', A_2, U_{b,3})$, $(N', N', U_b, U_{b,1})$ and $(U_1, U_2, V_1, V_2, V_3)$. On the other hand, since $G' + N' + V_1 > 1$ and $G' + N' + V_2 > 1$ hold, furthermore $U_1 + U_2 + U_{b,s} + U_{b,t} + V_3 > 1$ holds with any two distinct $s, t \in \{1, 2, 3\}$, it follows that the remaining items fit into one common bin, i.e. $A_1 + A_2 + U_b + U_{b,s} \leq 1$ holds with any $s \in \{1, 2, 3\}$. Then the statement follows from the fact that both items A_1 and A_2 are at least as large as a big U item. □

Let us now consider first the simpler case when $i = 1$. Let U_b denote the first U item in the first $(4U)$ FFD bin, which is naturally a big U item. Then $U_{b,1}$, $U_{b,2}$, or $U_{b,3}$ cannot be the second or third item in this bin, since then the next U item fits into this bin, thus there would be no fallback item in this bin. After packing three (big or small) U items into the first $(4U)$ bin, no further item fits here until the arrival of U_0, moreover U_0 does not fit here either, and only the first $(4U)$ bin remains a 3-bin after the packing of U_0. Recall that U_0 is the first item in the bin before the last FFD bin, and this bin is of type $(2U, 2V)$ in this subcase. Thus exactly two U items remain after the packing of U_0 (the fallback U item that is packed into the first $(4U)$ bin and the other U item in the bin of U_0). On the other hand all of $U_{b,1}$, $U_{b,2}$, and $U_{b,3}$ remain after U_0 (since any of them fits into the first $(4U)$ bin as the fourth item), a contradiction.

Now suppose that $i > 1$. Let U_b denote the first U item in the i-th $(4U)$ FFD bin; this item is surely a big U item. Then, similarly as above, it follows that $U_{b,1}$, $U_{b,2}$, or $U_{b,3}$ cannot be the second or third item in this bin. Moreover, if U_c denotes any big U item before U_b, we similarly obtain that $U_{c,1}$, $U_{c,2}$, or $U_{c,3}$ cannot be the second or third item in the i-th $(4U)$ FFD bin (because $U_c \geq U_b$). The $U_{b,s}$ and $U_{c,s}$ items ($s = 1, 2, 3$) cannot be in the first $i-1$ $(4U)$ FFD bins among the first three items, because all these items in the first $i - 1$ $(4U)$ FFD bins are big U items. Moreover any such item ($U_{b,s}$ or $U_{c,s}$, $s = 1, 2, 3$) fits into the i-th $(4U)$ FFD bin as the fourth item (using again that $U_c \geq U_b$). Thus it follows that all these items remain after U_0. Thus at least $3(3(i-1)+1) = 9i-6$ small U items remain after U_0; but on the other hand the number of such items is exactly $i+1$ (the i fallback U items after U_0, and the other U item in the bin of U_0), thus $i + 1 \geq 9i - 6$ holds, from which we get $i = 0$, a contradiction.

Subcase 2.2. There exists a $(3U, 2V)$ optimal bin. In this case a similar proof works, just we have to do it in a little bit more careful way. Suppose that a big U item is packed into a $(2N', 2U)$ optimal bin. Consider the two G' items which are placed in two FFD bins together with the two N' items. If both of them are packed in (G', H, U) optimal bins, we call the big U item **nice** (thus in Subcase 2.1 each big U item was nice). Otherwise (if a big U item is packed into a $(2N', U, V)$ optimal bin, or a big U item is packed into a $(2N', 2U)$ optimal bin but a G' item from the FFD bin of one N' item of this optimal bin is packed into a (G', H, V) optimal bin), we call the big U item **freaky**.

There is at most one freaky big U item, since there is at most one optimal V-bin different from the $(3U, 2V)$ optimal bin as the number of V items is at most three. Let the items in the $(3U, 2V)$ optimal bin be denoted as $(U_1, U_2, U_3, V_1, V_2)$; here the items U_j $(j = 1, 2, 3)$ are small U items. Now the possible bins are as follows:

	OPT						FFD						
23	G	1	1				G	1					
13	N'			2	2		N'	1					
12	H	1	1				H		3				
9	U	1		2	1	3	U			4	3	2	
9	V		1		1	2	V				1	2	1
	r	0	0	0	0	-1	s	0	0	0	0	0	0

Claim 3.4.5 *(a) If a big U item, say U_b is nice, then there exist (at least) six different small U items $U_{b,1}$,$U_{b,2}$,$U_{b,3}$,U_1,U_2,U_3, such that U_b together with any one of them and with any two further big or small U items fit into a common bin, moreover the first three small U items $U_{b,1}$, $U_{b,2}$, and $U_{b,3}$ are different for different nice big U items.*
(b) If a big U item, say U_c is freaky, then there exist (at least) five different small U items $U_{c,1}, U_{c,2}$, U_1, U_2, U_3, such that U_c together with any one of them and with any two further big or small U items fit into a common bin, moreover the first two small U items $U_{c,1}$ and $U_{c,2}$ are different from any small U items $U_{b,j}$ $(j = 1, 2, 3)$ associated with a nice big U item U_b.

Proof. Let a (nice or freaky) big U item be denoted again as U_b. The optimal bin of U_b can be a $(2N', 2U)$ or a $(2N', U, V)$ optimal bin, thus let the items being in the bin be denoted as $(N', N', U_b, U_{b,1})$, where $U_{b,1}$ is a small U or V item. Consider the optimal bins of the G' items that are packed together in the FFD packing with the previous two N' items. Let these optimal bins be denoted as $(G', A_1, U_{b,2})$ and $(G', A_2, U_{b,3})$ optimal bins, where A_1 and A_2 are H items, and either both $U_{b,2}$ and $U_{b,3}$ are small U items or one of them is a small U item and the other is a V item. Since there are at most three V items, there is at most one V item among $U_{b,1}$,$U_{b,2}$ and $U_{b,3}$, and there is no V item among them if U_b is nice. Since $G' + N' + V_1 > 1$ and $G' + N' + V_2 > 1$ hold, furthermore the sum of any five items among U_1,U_2,U_3, and $U_{b,1}$,$U_{b,2}$,$U_{b,3}$ is bigger than 1 (because at most one V item can occur among the six items, and the other items are U items, and $4U + X > 1$), it follows that the remaining items fit into one common bin, i.e. both inequalities $A_1 + A_2 + U_b + U_{b,j} \leq 1$ and $A_1 + A_2 + U_b + U_j \leq 1$ hold for all $j \in \{1, 2, 3\}$. □

Now we are ready to finish the proof of this Subcase 2.2, which also completes the proof of the theorem.

Suppose first that $i > 1$. Then the first U item is big in the i-th $(4U)$ FFD bin, and there are three big U items in each previous $(4U)$ FFD bin. Among these $3(i-1)+1$ big U items there are at least $3(i-1)$ nice big U items. Let U_b denote any nice big U item among them. Then, since the sizes of the big U items are decreasing, $U_{b,1}$, $U_{b,2}$, or $U_{b,3}$ cannot be the second or third item in the i-th $(4U)$ FFD bin, they are not among the first three items in the first $i-1$ $(4U)$ FFD bins, and any of them ($U_{b,j}$ for any $j = 1, 2, 3$) fits into the i-th $(4U)$ FFD bin as the fourth item. It follows that all these items remain after U_0. Thus at least $9(i-1)$ U items remain after U_0, but on the other hand there are at most $i + 2$ U items after U_0, namely the i fallback U items after U_0 plus at most two further U items in the bin of U_0.

It follows that $i + 2 \geq 9i - 9$ holds, thus we get $i \leq 1$, a contradiction. Thus only the case $i = 1$ remains to be considered. In this case there remain at most 3 U items after U_0. On the other hand, the first item in the first $(4U)$ FFD bin is a big U item (nice or freaky), thus even if this big U item is freaky, there are at least five small U items (in this case we must take into account also the small U items of the $(3U, 2V)$ optimal bin), such that any of them must be after U_0. This is a contradiction, as well. □

Chapter 4

The tight bound of First Fit Algorithm

The results of this chapter are published in papers [23, 24]. Both papers are written together with Jiri Sgall, and the contributions of the authors are indivisible (approximately 50%-50%).

In this chapter we give the tight bound of the First Fit algorithm (FF for short). This algorithm packs each item into the first bin where it fits, possibly opening a new bin if the item does not fit into any currently open bin. The proof that the asymptotic approximation ratio of FF is 1.7 given by Ullman [74] and subsequent works by Garey et al. and Johnson et al. [45, 55] were among the first results on approximation algorithms.

Here we prove that also the *absolute* approximation ratio for FF is exactly 1.7. This means that if the optimum needs OPT bins, First Fit always uses at most $\lfloor 1.7 \cdot OPT \rfloor$ bins. Thus we settle this open problem after about 40 years. Furthermore we show matching lower bounds for *all* values of OPT, i.e., we give instances on which FF uses exactly $\lfloor 1.7 \cdot OPT \rfloor$ bins. Such matching upper and lower bounds were previously known only for finitely many and small values of OPT. Thus our results not only give the exact worst case for all values of OPT, but actually even give the first infinite sequence of values of OPT for which the exact worst-case performance of FF is known.

We note here, that the same (tight) bounds hold also for algorithm Best Fit (BF for short), as we proved this in our recent work [24], but the proof for BF is much more complicated than that is for FF, thus we restrict our attention here only for the tight result of FF.

In the previous chapter we have analyzed the FFD algorithm, which behaves like FF but receives the items on the input sorted from the largest one to the smallest, and proved that while the asymptotic approximation ratio is equal to $11/9$, regarding the absolute bound we need an additive constant, its tight value is $2/3$. That is, $\frac{11}{9}OPT + \frac{2}{3}$ bins are sufficient for FFD, but this number of bins is actually also necessary for some instances for infinitely many values of OPT. Thus for FFD, the asymptotic and absolute approximation ratios are not equal. In light of this result, it is rather surprising that for FF the asymptotic and absolute approximation ratios *are* equal and no additive term is needed.

History and related work. The upper bound on FF was first shown by Ullman in 1971 [74]; he proved that for any instance, $FF \leq 1.7 \cdot OPT + 3$, where FF and OPT denote the number of bins used by FF and the optimum, respectively. Still in seventies, the additive term was improved first in [45] to 2 and then in [44] to $FF \leq \lceil 1.7 \cdot OPT \rceil$; due to integrality of FF and OPT this is equivalent to $FF \leq 1.7 \cdot OPT + 0.9$. Recently the additive term of the asymptotic bound was improved to $FF \leq 1.7 \cdot OPT + 0.7$ in [85].

The absolute approximation ratio of FF got some attention recently. A significant step towards settling the question of the absolute approximation ratio was the upper bound of 1.75 by Simchi-Levy [73]. This was improved independently by Xia and Tan [85] and Boyar, Dosa and Epstein [10] to $12/7 \approx 1.7143$ and recently by Németh to $101/59 \approx 1.7119$ [65].

For the lower bound, the early works give examples both for the asymptotic and absolute ratios. The example for the asymptotic bound gives $FF = 1.7 \cdot OPT$ whenever $OPT = 10k + 1$, thus it shows that the asymptotic upper bound of 1.7 is tight, see [74, 45, 55]. For the absolute ratio, an example is given with $FF = 17$ and $OPT = 10$, which shows that the absolute approximation ratio cannot be better than 1.7 [45, 55]. (Also an example with $FF = 34$ and $OPT = 20$ is claimed, but it seems that this example has never been published.)

From these (recent) improvements, one can distinguish *two different possible ways* to reach the tight result. One option is to decrease the absolute ratio, where we do not use any additive constant. The upper bound of Simchy-Levi is $7/4$, this bound is decreased then to $12/7 \approx 1.7143$ by [85] and [10], and later to $101/59 \approx 1.7119$ by [65]. However, it turns out, that as we go down to approach to 1.7, more and more counterexamples appear which all must be excluded one by one; naturally, on this way we cannot arrive at the tight bound.

Another way is to decrease the additive constant. After the seminal work of $FF \leq 1.7 \cdot OPT + 0.9$ given in [44], the tight value of the additive constant (taking into account also the lower bound) could be only 0.9 or 0.8 or 0.7, ..., or finally 0, due to integrality of FF and OPT. The authors in [85] could go down to 0.7. What is the smallest possible value of the additive? We show below how we can go down to 0.1 quite easily. Our key tool is a weighting function. Surprisingly, it is the same weighting function that was applied in the early seventies, but we use it in a new, more efficient form.

To obtain the tight value, i.e. excluding the last 0.1, needs a bit more and tricky work, considering also some combinatorial properties of the packing, if the packing is very close to the worst scenario.

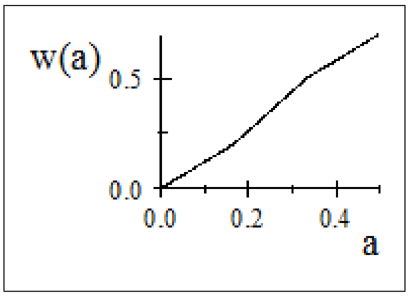
4.1 Main ideas

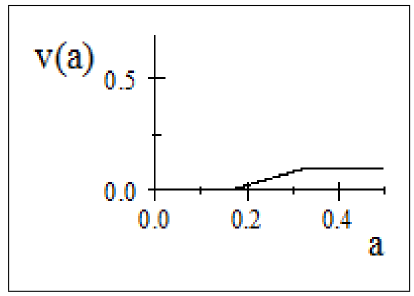
Once the asymptotic bound with a small additive constant is shown, a natural approach to improve absolute upper bounds is to study fixed small values of OPT and to exclude the possibility of a higher absolute ratio for them. Indeed, solving a few such cases necessarily improves upper bounds on the absolute ratio—but cannot give a tight result. Of course, this is still far from trivial: Even for a fixed OPT, each such problem seems to lead to a new and more extensive case analysis. Instead of joining this race of incremental results, we choose a different approach to attack arbitrarily large values of OPT directly.

The first important step is a combination of amortization and weight function analysis. To illustrate our technique, we now present a new short proof of the asymptotic ratio 1.7 for FF. It uses the same weight function as the traditional analysis of FF. To use amortization, we split the weight of each item into two parts. (This partition was first applied in [70], for giving a short proof for the asymptotic performance of BF.) Identifying an item a with its size, the weight of a is its scaled size $\frac{6}{5}a$ plus the bonus $v(a)$ defined as

$$v(a) = \begin{cases} 0 & \text{if } a \le \frac{1}{6}\,, \\ \frac{3}{5}(a - \frac{1}{6}) & \text{if } a \in \left(\frac{1}{6}, \frac{1}{3}\right)\,, \\ 0.1 & \text{if } a \in \left[\frac{1}{3}, \frac{1}{2}\right]\,, \\ 0.4 & \text{if } a > \frac{1}{2}\,. \end{cases}$$

Below we give a picture about the weight function as well as about the bonus, focusing on the range between 0 and $1/2$ only.





Note that there is a discontinuity only at $a = 1/2$. For a set of items B, $v(B) = \sum_{a \in B} v(a)$ denotes the total bonus and $s(B) = \sum_{a \in B} a$ the total size.

It is very easy to observe that the weight of any bin B, i.e., of any set with $s(B) \le 1$, is at most 1.7: The scaled size of B is at most 1.2, so we only need to check that $v(B) \le 0.5$. If B contains no item larger than $1/2$, there are at most 5 items with non-zero $v(a)$ and $v(a) \le 0.1$ for each of them. Otherwise the large item has bonus 0.4; there are at most two other items with non-zero bonus and it is easy to check that their total bonus is at most 0.1.

Consider an instance I. The previous bound implies that the weight of the whole instance $\frac{6}{5}s(I) + v(I)$ is at most $1.7 \cdot OPT$. The key part is to show that, *on average*, the weight of each FF bin is at least 1 (with a few exception). For this, the key observation is in the next lemma.

Lemma 4.1.1 *Let B, C be two bins in the FF packing such that $s(B) \ge 2/3$, C contains at least two items, and B is opened before C. Then $\frac{6}{5}s(B) + v(C) \ge 1$.*

Proof. Since C is after B in the FF packing, C contains two items c and c' that do not fit in B, i.e., $c, c' > 1 - s(B)$. If $s(B) \ge 5/6$ then the lemma follows trivially without considering $v(C)$. In the remaining case, let $x \in (0, \frac{1}{6}]$ be such that $s(B) = \frac{5}{6} - x$. Thus $c, c' > \frac{1}{6} + x$ and $v(c), v(c') > \frac{3}{5}x$. We get $\frac{6}{5}s(B) + v(c) + v(c') > \frac{6}{5}(\frac{5}{6} - x) + \frac{3}{5}x + \frac{3}{5}x = 1$. □

Consider any FF-bin B with a single item. If $s(B) > 1/2$, then $b(B) = 0.4$ and $\frac{6}{5}s(B) + v(B) > 1$. Furthermore, at most one FF-bin has $s(B) \le 1/2$, by the definition of FF.

Now consider FF-bins with two or more items. Similarly, at most one of them has size less than $2/3$: If we have one such bin, any item in any later bin is larger than $1/3$ and thus any later bin with two items is larger than $2/3$. Now we use Lemma 4.1.1 for every FF bin B with two or more items and $s(B) \geq 2/3$ (except for the last such bin); the bin C is chosen as the next bin with the same properties.

Summing the bounds for bins with a single item plus the bounds from Lemma 4.1.1 for bins with two or more items (note that each bin is used at most once as B and at most once as C), we obtain that $\frac{6}{5}s(I) + v(I) \geq FF - 3$. The additive constant 3 comes from the fact that we did not bound the weight of at most three FF-bins: (i) one bin with a single item and $s(B) \leq 1/2$, (ii) one bin with two or more items and $s(B) < 2/3$, and (iii) the last bin with two or more items. Combining this with the previous bound on the total weight, we obtain $FF - 3 \leq \frac{6}{5}s(I) + v(I) \leq 1.7 \cdot OPT$ and the asymptotic bound follows.

By a bit more careful analysis, we can toward decrease the additive constant (after examining the remaining three bins in the FF packing) but cannot remove it completely. To obtain the tight bound, we need to analyze different types of bins in the FF packing quite carefully. In the typical worst case, FF packing starts by bins with five or more items of size around $1/6$ (bigger or smaller), followed by $OPT/2$ bins with two items slightly larger or smaller than $1/3$, and ends by OPT bins with a single item slightly larger than $1/2$. We analyze these three types of bins separately. To handle various possible situations we slightly modify the weight function (see Definition 4.2.2) and the amortization lemma (see Lemma 4.2.5).

The most delicate part of the proof analyzes the FF bins containing three or four items—or rather shows that they cannot play an important role in the worst case; here it is important that the amortization uses the bonus of only two items and thus the bins with three or four items are "wasteful". In the final steps of the proof, the parity of the items of size around $1/3$ comes into play: Typically they come in pairs, as described above, but for odd values of OPT one of them is missing (or is in a FF bin of 3 or more items), and this allows us to remove the last 0.1 of the additive term. Our analysis sketched above still leaves a few values of OPT that need to be analyzed separately. However, with our framework of the general proof, even this is relatively simple compared to the previous proofs in this area. The upper bound proof is presented in the next subsection.

Notations. Let us fix an instance I with items $a_1, \ldots a_n$ and denote the number of bins in the FF and optimal solutions by FF and OPT, respectively. We will often identify an item and its size. For a set of items A, let $s(A) = \sum_{a \in A} a$, i.e., the total size of items in A and also for a set of bins $\mathcal{A}$, let $s(\mathcal{A}) = \sum_{A \in \mathcal{A}} s(A)$. Furthermore, let $S = s(I)$ be the total size of all items of I. Obviously $S \leq OPT$.

The bins in the FF packing are ordered by the time they are opened (i.e., the first item is packed into them). We refer to this order when we say that one bin is before or after another one, or when we speak about the first or last bin.

A bin is called a *k-bin* or *k^+-bin*, if it contains exactly k items or at least k items, respectively, for an integer k. An item is called k-item if FF packs it into a k-bin.

We classify the the FF bins into three groups. If a 2^+-bin B satisfies $s(B) \geq 5/6$, it is a *big* bin; $\mathcal{B}$ denotes the set of all big bins and β their number. Any other 2^+-bin C is a *common* bin; $\mathcal{C}$ denotes the set of all common bins and γ their number. Finally, any 1-bin D is a *dedicated* bin; $\mathcal{D}$ denotes the set of all dedicated bins and δ their number. The items in big, common, and dedicated bins are called B-items, C-items, and D-items, respectively. Finally, let $C2$-items be the items in common 2-bins. The common and dedicated bins are typically denoted by C and D, and C-items and D-items by c and d (with indices and other decorations). We use B for generic bins (typically big or common) and b for items that may be in big or common bins. If there exists a D-item with size at most $1/2$, denote it d_0; otherwise d_0 is undefined. We shall see in Lemma 4.2.1(i) that there is at most one such item.

4.2 The upper bound proof for FF

First we state a few basic properties of FF packings. Assumption 4.2.1 as well as all parts of Lemma 4.2.1 are known and easy facts used explicitly or implicitly in previous works on FF including [73, 85, 10].

Assumption 4.2.1 *We assume, without loss of generality, that no two items a_i and a_j are packed into the same bin both in FF and OPT solutions.*

This is w.l.o.g., since any two such items may be replaced by a single item of size $a_i + a_j$ that arrives at the time of arrival of the first of the original items. It is easy to see that both FF and OPT solutions are unchanged (except for this replacement).

Lemma 4.2.1 *In the FF packing the following holds:*
(i) The sum of sizes of any two FF-bins is greater than 1. The total size of any $k \geq 2$ FF-bins is greater than $k/2$.
(ii) The D-items are packed into different optimal bins. Thus $\delta \leq OPT$.
(iii) There is at most one common bin C_0 with $s(C_0) \leq 2/3$. Furthermore, if $s(C_0) = 2/3 - 2x$ for $x \geq 0$ then for any other 2^+-bin (i.e., any other common or big bin) B we have $s(B) > 2/3 + x$; in addition, if B is opened after C_0, then $s(B) > 2/3 + 4x$.
(iv) If $k \geq 3$, then the total size of k arbitrary 2^+-bins is greater than $\frac{2}{3}k$.
(v) Suppose that $k \geq 1$, we have $k+1$ FF-bins $B_1, B_2, \dots, B_k, B$, in this order, and such that B is a k^+-bin. Then the sum of the sizes of these $k+1$ bins is greater than k.

Proof. (i): The first item in any FF-bin does not fit in any previous bin, thus the sum of their sizes is greater than 1 already at the time when the second bin is opened. For k bins, order the bins cyclically and sum the inequalities $s(B_i) + s(B_j) > 1$ for pairs of adjacent bins.

(ii): Follows from (i), as the size of each D-item equals the size of its dedicated FF-bin.

(iii): If B is after C_0, then it contains only items of size larger than $1 - s(C_0) = 1/3 + 2x$; since it contains two items, $s(B) > 2/3 + 4x$ follows. If B is before C_0, then notice that C_0 contains an item of size at most $s(C_0)/2 = 1/3 - x$; This item was not packed into B, thus it follows that $s(B) > 2/3 + x$.

(iv): Follows immediately from (iii).

(v): Let x be the minimum of $s(B_i)$, $i = 1, \dots, k$. Then by the FF-rule, any item in bin B is larger than $1 - x$. Since there are at least k items in bin B, we have $s(B) + \sum_{i=1}^{k} s(B_i) > k(1-x) + kx = k$. □

Now we assume that the instance violates the absolute ratio 1.7 and derive some easy consequences that exclude some degenerate cases. The first claim, $OPT \geq 7$, follows from [10, 85]; we include its proof for completeness. Note that the values of $1.7 \cdot OPT$ are multiples of 0.1 and FF is an integer, thus $FF > 1.7 \cdot OPT$ implies $FF \geq 1.7 \cdot OPT + 0.1$. Typically we derive a contradiction with the fact $S \leq OPT$ stated above.

Lemma 4.2.2 *If $FF > 1.7 \cdot OPT$ then the following holds:*
(i) $OPT \geq 7$.
(ii) No common bin C has size $s(C) \leq 1/2$.
(iii) The number of dedicated bins is bounded by $\delta \geq 3$.
(iv) The number of common bins is bounded by $\gamma \geq OPT/2 + 1 > 4$. If $FF \geq 1.7 \cdot OPT + \tau/10$ for some integer $\tau \geq 1$ then $\gamma > (OPT + \tau)/2$.

Proof. (i): If $OPT \in \{3, 4, 5, 6\}$ and $FF > 1.7 \cdot OPT$ then we can verify that both $FF \geq 2 \cdot OPT - 1$ and $FF \geq OPT + 3$. Using Lemma 4.2.1(ii), the number of 2^+-bins is $\beta + \gamma = FF - \delta \geq FF - OPT \geq 3$. Thus we can use Lemma 4.2.1(iv) and obtain a contradiction:

$$S > \frac{2}{3}(\beta + \gamma) + \frac{1}{2}\delta = \frac{1}{6}(\beta + \gamma) + \frac{1}{2}FF \geq \frac{1}{6} \cdot 3 + \frac{1}{2}(2 \cdot OPT - 1) = OPT\,.$$

Above we supposed that $\delta \neq 1$. If eventually $\delta = 1$, then $\beta + \gamma \geq 4$ trivially holds. In this case we count a 2^+-bin into the set of dedicated bins; thus δ grows by one, and $\beta + \gamma$ decreases by one. After the modification $\delta = 2$ and $\beta + \gamma \geq 3$, thus the above calculation works. We possibly apply this calculation also in later parts (ii) and (iii) if needed.

If $OPT = 2$ and $FF > 1.7 \cdot OPT$ then $FF \geq 4$, and by Lemma 4.2.1(i) we have $S > 4 \cdot \frac{1}{2} = OPT$, a contradiction. For $OPT = 1$, FF is trivially optimal.

(ii): Suppose that $s(C_0) \leq 1/2$ for a contradiction. Lemma 4.2.1(iii) implies that any big or common bin C before C_0 has $s(C) \geq 3/4$. Furthermore, any bin after C_0 is a D-bin (as it can contain only items larger than $1/2$) and by Lemma 4.2.1(i), the total size of C_0 and all D-bins is at least $(\delta + 1)/2$. Thus we can obtain a contradiction by using $OPT \geq 7$ from (i) and $\delta \leq OPT$ from Lemma 4.2.1(ii) as follows:

$$\begin{aligned} S &> \frac{3}{4}(\beta + \gamma - 1) + \frac{1}{2}(\delta + 1) = \frac{3}{4}FF - \frac{1}{4}(\delta + 1) \\ &\geq \frac{3}{4}\left(\frac{17}{10}OPT + \frac{1}{10}\right) - \frac{1}{4}(OPT + 1) = \frac{41}{40}OPT - \frac{7}{40} \geq OPT\,. \end{aligned}$$

(iii): Suppose for a contradiction that $\delta \leq 2$. Then each FF-bin contains at least two items, except for at most two dedicated FF-bins. Since $OPT \geq 7$ from (i), we can apply Lemma 4.2.1(iv) for the $FF - 2 \geq 3$ of 2^+-bins and Lemma 4.2.1(i) for the remaining two bins, and thus we obtain a contradiction as follows:

$$S > \frac{2}{3}(FF - 2) + 1 \geq \frac{2}{3}\left(\frac{17}{10}OPT + \frac{1}{10} - 2\right) + 1 = \frac{17}{15}OPT - \frac{4}{15} > OPT\,.$$

(iv): To obtain the first bound from the second one, use $\tau = 1$ and the integrality of OPT. Now suppose for a contradiction that $\gamma \leq (OPT + \tau)/2$. If $\gamma \geq 3$, then we use Lemma 4.2.1(iv) for the common bins, Lemma 4.2.1(i) for the dedicated bins, and the fact that the remaining bins are big, and we obtain

$$\begin{aligned} S &> \frac{5}{6}(FF - \gamma - \delta) + \frac{2}{3}\gamma + \frac{1}{2}\delta = \frac{5}{6}FF - \frac{1}{6}\gamma - \frac{1}{3}\delta \\ &\geq \frac{5}{6}\left(\frac{17}{10}OPT + \frac{\tau}{10}\right) - \frac{OPT + \tau}{12} - \frac{1}{3}OPT = OPT\,, \end{aligned}$$

a contradiction. If $\gamma \leq 2$ then

$$\begin{aligned} S &> \frac{5}{6}(FF - \delta - 2) + \frac{1}{2}(\delta + 2) = \frac{5}{6}FF - \frac{1}{3}(\delta + 2) \\ &\geq \frac{5}{6}\left(\frac{17}{10}OPT + \frac{\tau}{10}\right) - \frac{1}{3}OPT - \frac{2}{3} \geq \frac{13}{12}OPT - \frac{7}{12} \geq OPT\,, \end{aligned}$$

using $\tau \geq 1$ and (i) in the last step, and we obtain a contradiction as well. □

The weight function and the main lemma. Now we introduce the main ingredients of our analysis: the modified weight function and the main lemma that is used for the amortized analysis of the weight of FF bins. As in the simple proof in the introduction and previous bin packing literature, our ultimate goal is to prove that each OPT-bin has weight at most 1.7 and each FF-bin has an amortized (average) weight at least 1.

It is convenient to describe the weight of each item a in two parts. The first part, $r(a)$, is called the regular (part of the) weight, and it is proportional to the size of a; it is the same as in the simple proof. The other part, $v(a)$ is called the bonus and it is modified so that it depends both on the size of a and the type of FF-bin where a is packed. B-items have no bonus. C-items have bonus equal to 0 for items of size at most $1/6$, equal to 0.1 for items of size at least $1/3$, and linearly interpolated between these values. D-items have bonus 0.4 if they have size at least $1/2$ and slightly smaller if they have smaller size (this concerns only the single item d_0).

Compared to the simple proof and the previous literature, we make several modifications to the weight function. These are mostly a matter of convenience and simplification of the case analysis in the proof. First, we move the bonus from the items larger than $1/2$ to the D-items. Mostly these are actually the same items, except for d_0. As we shall see later, in the tight cases, each OPT-bin contains a D-item and this change allows a more uniform analysis. Second, we decrease some of the weights that we do not use in the proof, namely we do not put any bonus on B-items and decrease the bonus on d_0 (this is necessary to guarantee that its OPT-bin has weight at most 1.7; however, in tight cases d_0 is very close to $1/2$).

The third change is essential in our last step of the proof where we remove the remaining additive constant of 0.1. We define a set of at most two exceptional C-items whose bonus is decreased to 0. Since they are in different 3^+-bins in the FF packing, this does not change the analysis of the FF packing significantly. On the other hand, the exceptional items are chosen so that one OPT-bin is guaranteed to have weight at most 1.6, which is exactly the necessary improvement. Formally we define the exceptional items as follows:

Definition 4.2.1 *If $OPT \equiv 7 (\bmod 10)$ and there exists an OPT-bin E that contains no C2-item, then fix any such bin E for the rest of the proof. Otherwise E is undefined. If E contains at most two C-items with size larger than $1/6$, denote the set of these items E'. Otherwise (if there are three or more C-items in E or no E exists) put $E' = \emptyset$. Let us call E the* exceptional *bin and the items in E' the* exceptional *items.*

Note that there is at most one exceptional item in each FF-bin by Assumption 4.2.1. Later we shall show that in a potential counterexample with $FF = 1.7 \cdot OPT + 0.1$ the bin E exists.

Definition 4.2.2 *The weight function is defined as follows:*

For a B-item b we define $v(b) = 0.$

For a C-item c we define $$v(c) = \begin{cases} 0 & \text{if } c \le \frac{1}{6} \text{ or } c \in E', \\ \frac{3}{5}(c - \frac{1}{6}) & \text{if } c \in \left[\frac{1}{6}, \frac{1}{3}\right] \text{ and } c \notin E', \\ 0.1 & \text{if } c \ge \frac{1}{3} \text{ and } c \notin E'. \end{cases}$$

For a D-item d we define $$v(d) = \begin{cases} 0.4 & \text{if } d \ge \frac{1}{2}, \\ 0.4 - \frac{3}{5}(\frac{1}{2} - d) & \text{if } d < \frac{1}{2}. \end{cases}$$

For every item a we define $r(a) = \frac{6}{5}a$ *and* $w(a) = r(a) + v(a)$.

For a set of items A and a set of bins $\mathcal{A}$, let $w(A)$ and $w(\mathcal{A})$ denote the total weight of all items in A or $\mathcal{A}$; similarly for r and v. Furthermore, let $W = w(I)$ be the total weight of I.

In Definition 4.2.2 the function v is continuous on the boundary of the cases. Furthermore, if we have a set A of k C-items with size in $[\frac{1}{6}, \frac{1}{3}]$ (and no of them is exceptional item), then the definition implies that its bonus is exactly $v(A) = \frac{3}{5}\left(s(A) - \frac{k}{6}\right)$. More generally, if A contains k items, each of size at least $1/6$ and no D-item, then we get an upper bound $v(A) \le \frac{3}{5}\left(s(A) - \frac{k}{6}\right)$.

First we analyze the weight of the OPT-bins.

Lemma 4.2.3 *For every optimal bin A its weight $w(A)$ can be bounded as follows:*
(i) $w(A) \le 1.7$.
(ii) If E is the exceptional OPT-bin then $w(E) \le 1.6$.
(iii) If A contains no D-item, then $w(A) \le 1.5$.

Proof. In all cases $r(A) \le 1.2$, thus it remains to bound $v(A)$. We distinguish three cases:

Case 1: A contains no D-item. Either it contains at least 4 items with non-zero bonus, in which case their total bonus is at most $v(A) \le \frac{3}{5}(s(A) - \frac{4}{6}) \le \frac{3}{5} \cdot \frac{2}{6} = 0.2$. Or else it contains at most 3 items with non-zero bonus and $v(A) \le 0.3$. In both subcases (iii) follows and thus (ii) also holds if $E = A$.

Case 2: A contains a D-item larger than $1/2$. The bonus of the D-item is 0.4. In addition, A contains at most 2 items larger than $1/6$ of total size y at most $y < 1/2$; they may be B-items or C-items. If $E = A$ then they have no bonus and both (i) and (ii) hold. Otherwise we have $v(A) \le 0.4 + \frac{3}{5}(y - \frac{2}{6}) < 0.4 + \frac{3}{5} \cdot \frac{1}{6} = 0.5$ and (i) follows.

Case 3: A contains d_0. Let the size of d_0 be $\frac{1}{2} - x$ for $x \ge 0$. We have $v(d_0) = 0.4 - \frac{3}{5}x$. We distinguish two subcases.

Case 3.1: A contains at most two items other than d_0 and larger than $1/6$. Then their total size is at most $\frac{1}{2} + x$. If $E = A$ then they have no bonus and both (i) and (ii) hold. Otherwise their bonus is at most $0.1 + \frac{3}{5}x$ and (i) holds.

Case 3.2: If A contains at least three items other than d_0 and larger than $1/6$. Then their total bonus is at most $\frac{3}{5}x$, thus $v(A) \le 0.4$ and both (i) and (ii) hold. (This subcase may also happen if $E = A$, but there is no need to distinguish this in the proof.) □

Next we analyze the weight of FF-bins. The case of big and dedicated bins is easy:

Lemma 4.2.4 *(i) The total weight of the big bins is $w(\mathcal{B}) \geq \beta$.*
(ii) The total weight of the dedicated bins is $w(\mathcal{D}) > \delta$.

Proof. (i): For every big bin B, $w(B) = r(B) = \frac{6}{5}s(B) \geq \frac{6}{5} \cdot \frac{5}{6} = 1$.

(ii): If d_0 is undefined then for every dedicated bin D, $w(D) = \frac{6}{5}s(D) + 0.4 > \frac{6}{5} \cdot \frac{1}{2} + 0.4 = 1$ and the claim follows. If d_0 exists and has size $\frac{1}{2} - x$ for $x \geq 0$, then every other D-item has size strictly larger than $\frac{1}{2} + x$. We also have $\delta \geq 3$ by Lemma 4.2.2(iii). Thus

$$w(\mathcal{D}) > (\delta - 1)\left(\frac{6}{5}\left(\frac{1}{2} + x\right) + 0.4\right) + \frac{6}{5}\left(\frac{1}{2} - x\right) + 0.4 - \frac{3}{5}x = \delta + \left((\delta - 1)\frac{6}{5} - \frac{6}{5} - \frac{3}{5}\right)x \geq \delta.$$

□

Now we focus on the common FF-bins. The next lemma gives the key insight for the amortized analysis. It shows that for most common bins, the regular weight of the bin plus the bonus of the *next* common bin is at least 1. A similar method was used for the analysis of BF in [70]. For the rest of the upper bound proof, number the common bins as $C_1, \ldots, C_\gamma$, in the order of their opening. The bins $C_2, \ldots, C_{\gamma-1}$ are called *inner* common bins. Note that there are some inner common bins, as $\gamma \geq 5$ by Lemma 4.2.2(iv).

Lemma 4.2.5 *Let $i = 2, \ldots, \gamma$ be such that $s(C_{i-1}) \geq 2/3$. Then there exist two items $c, c' \in C_i \setminus E'$ and for any such items*

$$r(C_{i-1}) + v(c) + v(c') \geq 1\,.$$

Thus we have $r(C_{i-1}) + v(C_i) \geq 1$.

Proof. If C_i is a 2-bin, then it contains no exceptional item. If C_i is a 3^+-bin, then it contains at most one exceptional item by Assumption 4.2.1. In both cases c and c' exist. Since C_{i-1} is common, the size of this bin is smaller than $5/6$ and it is at least $2/3$ by the assumption of the lemma. Let $x \in (0, \frac{1}{6}]$ be such that $s(C_{i-1}) = \frac{5}{6} - x$. Thus $c, c' > \frac{1}{6} + x$ and $v(c), v(c') > \frac{3}{5}x$. We get $r(C_{i-1}) + v(c) + v(c') > \frac{6}{5}(\frac{5}{6} - x) + \frac{3}{5}x + \frac{3}{5}x = 1$. □

The outline of the rest of the proof is this: We will prove that the common FF-bins have total weight at least $\gamma - 0.2$ (note that this follows almost immediately from Lemma 4.2.5 if there is no common bin smaller than $2/3$). Since the total weight of the dedicated bins is strictly greater than δ, this implies $W > FF - 0.2$. Together with $W \leq 1.7 \cdot OPT$ now $FF \leq 1.7 \cdot OPT + 0.1$ follows. However, $FF = 1.7 \cdot OPT + 0.1$ can hold only if $OPT \equiv 7 (\bmod 10)$. Then we show that the exceptional bin is defined, thus $W \leq 1.7 \cdot OPT - 0.1$ and we save the last 0.1.

The analysis is considerably harder in case when the last common bin is smaller than $2/3$, thus we will distinguish between these two main cases later: the last common bin is big or small.

Lemma 4.2.6 *Suppose $w(\mathcal{C}) \geq \gamma - 0.2$. Then*
(i) $FF \leq 1.7 \cdot OPT + 0.1$, and
(ii) if the exceptional bin E is defined, then $FF \leq 1.7 \cdot OPT$.

Proof. By Lemma 4.2.4 and the assumption we have $W > \beta + (\gamma - 0.2) + \delta = FF - 0.2$. By Lemma 4.2.3(i) we have $W \le 1.7 \cdot OPT$. Thus $FF - 0.2 < W \le 1.7 \cdot OPT$. Since FF and OPT are integers, (i) follows. If E is defined then by Lemma 4.2.3(i) and (ii) we have $W \le 1.7 \cdot OPT - 0.1$. Thus $FF - 0.2 < W \le 1.7 \cdot OPT - 0.1$ and (ii) follows. □

To decrease the bound by the last one tenth, we only need to show that the exceptional OPT-bin is defined. First yet another auxiliary lemma:

Lemma 4.2.7 *Suppose that every OPT-bin contains a D-item. Then no OPT bin contains two 2-items c_1 and c_2.*

Proof. For contradiction, assume we have such c_1 and c_2 and number them so that the FF-bin of c_1 is before the FF-bin of c_2. (Note that by Assumption 4.2.1, c_1 and c_2 are not in the same FF-bin.) Let c_3 be the other item in the FF-bin of c_1. Since c_2 was not packed into this bin, which contains only c_1 and c_3, we have $c_1 + c_2 + c_3 > 1$. This implies that c_3 cannot be in the OPT-bin of c_1 and c_2. Every OPT-bin contains a D-item by the assumption; let d_1 be the D-item in the OPT-bin of c_1 and c_2 and d_3 the D-item in the OPT-bin of c_3. By Lemma 4.2.1(i), $d_1 + d_3 > 1$ and thus $c_1 + c_2 + c_3 + d_1 + d_3 > 2$. As all these items are in two OPT-bins, this is a contradiction. □

Proposition 4.2.1 *Suppose that $w(\mathcal{C}) \ge \gamma - 0.2$. Then $FF \le 1.7 \cdot OPT$.*

Proof. By Lemma 4.2.6 (i) we have $FF \le 1.7 \cdot OPT + 0.1$. If $OPT \not\equiv 7 \bmod(10)$ then by checking all the other residue classes we can verify that $1.7 \cdot OPT + 0.1$ is non-integral. Thus $FF \le 1.7 \cdot OPT + 0.1$ implies $FF \le 1.7 \cdot OPT$ and we are done. It remains to handle the case when $OPT \equiv 7 \bmod(10)$ and $FF = 1.7 \cdot OPT + 0.1$. First we claim that every OPT-bin contains a D-item and thus $\delta = OPT$. If some OPT-bin does not contain a D-item, its weight is at most 1.5 by Lemma 4.2.3(iii). Thus $W \le 1.7 \cdot OPT - 0.2$. Since $FF > W - 0.2$, we obtain $FF \le 1.7 \cdot OPT$, a contradiction. Lemma 4.2.7 now implies that no OPT-bin contains two $C2$-items. Note that OPT is odd, as $OPT \equiv 7 \bmod(10)$. On the other hand, the number of $C2$-items is even (in any FF-bin there are either zero or two $C2$-items). Thus some OPT-bin contains no $C2$-item. This bin satisfies all the conditions of Definition 4.2.1 of the exceptional bin. Thus E is defined and by Lemma 4.2.6(ii) the proposition follows. □

4.2.1 The last common bin is large

Now we prove that if the last common bin is large, then the total weight of the common bins is $w(\mathcal{C}) \ge \gamma - 0.2$. By this, our proof is complete in this case.

Lemma 4.2.8 *If $s(C_\gamma) \ge 2/3$, then the total weight of the common bins is $w(\mathcal{C}) \ge \gamma - 0.2$.*

Proof. First consider the case when every common bin has size at least $2/3$. We apply Lemma 4.2.5 for every $i = 2, \dots, \gamma$. The regular weight of the last bin is at least $r(C_\gamma) \ge \frac{6}{5} \cdot \frac{2}{3} = 0.8$. Summing all of these inequalities we obtain

$$w(\mathcal{C}) = \sum_{i=1}^{\gamma} w(C_i) \ge r(C_\gamma) + \sum_{i=2}^{\gamma} (r(C_{i-1}) + v(C_i)) \ge 0.8 + (\gamma - 1) = \gamma - 0.2.$$

Now suppose that $s(C_k) = 2/3 - x$ for $x > 0$ and $1 \leq k < \gamma$. Using Lemma 4.2.1(iii), each C_j, $j > k$, contains (exactly) two items larger than $1/3 + x$. Thus $v(C_j) = 0.2$ and also $s(C_j) > 2/3 + 2x$ which implies $\sum_{i=k}^{\gamma} s(C_i) > (\gamma + 1 - k)\frac{2}{3}$. Combining these we have $r(C_k) + \sum_{j=k+1}^{\gamma} w(C_j) \geq (\gamma + 1 - k) - 0.2$. Adding the last inequality and the inequalities $r(C_{i-1}) + v(C_i) \geq 1$ from Lemma 4.2.5 for $i = 2, \ldots, k$, the lemma follows also in this case. □

We gained the next Theorem:

Theorem 4.2.1 *For any instance of bin packing, in the last common bin is large, then* $FF \leq 1.7 \cdot OPT$.

4.2.2 The last common bin is small

Suppose the last common bin is small, i.e. $s(C_\gamma) < 2/3$. The analysis of this case goes along the line of the case when the last common bin is large. This investigation is a bit more technical, so it is put into Appendix C.

4.3 Tight lower bound

What do we know at this point? From one hand, it turned out that $FF \leq \lfloor 1.7 \cdot OPT \rfloor$, by the integrality of FF, see Theorem 4.2.1. On the other hand, from works [74, 45, 55] we know such inputs for which $FF = 1.7 \cdot OPT$ whenever $OPT = 10k + 1$. In fact, there is some gap between the bounds. What can we do to close the gap? We made two different efforts in papers [23, 24], as we describe it below.

The first effort. In [23] we realized that by a small modification of the instance from [45, 55], we almost can close the gap. The case is as follows: The original construction is quite intricate. Fortunately—and perhaps also surprisingly—it is sufficient to carefully analyze the high-level structure of the instance, add to it a few new jobs, and carefully position them in the input instance. In the original construction we have an input instance L with three regions of items. In the first region there are items of size close to $1/6$, in the second region come items close to $1/3$, and in the third region there are items with the equal size $1/2 + \delta$, for a small $\delta > 0$. We did not modify the items in this list, only add some new items before or after L, and also in between the three regions of L. In this way we needed to review the properties of L with the focus on the resulting FF packing in each region; the details within each region are somewhat delicate but fortunately we can use that part as a black box. After adding the new items to the original list L, we gain the instance I with the next properties:

Theorem 4.3.1 *For all integers* $k \geq 1$ *and* $0 \leq i \leq 9$*, there exists an instance* I *such that* $OPT = 10k + i$ *and the lower bound in the top row of the next table holds. The bottom row of the table gives the upper bounds for a comparison.*

$i =$	0	1	2	3	4	5	6	7	8	9
$FF \geq 17k+$	-1	1	3	4	6	8	10	11	13	15
$FF \leq \lfloor 17k + 1.7i \rfloor = 17k+$	0	1	3	5	6	8	10	11	13	15

Furthermore, for $i = 1, \ldots, 9$ *there exist instances with* $OPT = i$ *and* $FF = \lfloor 1.7 \cdot i \rfloor$.

The proof shows how the new items should be chosen and added to the original list L. The proof of the theorem is omitted here, as below we show a better way, to close the gap completely. But note, that we already got the tight bounds for 8 residue classes out of 10. The exceptions are where $i = 0$ and $i = 3$. Fortunately we can get the tight result also in these cases, see below.

The second, and last effort. In [24] we found a brand new construction. This construction is much simpler than the original one, and makes possible to get the tight result for each residue class. (Note that the purpose of the original construction was not to get the tight absolute bound, it wanted to only consider the asymptotic performance.)

The high level scheme of the lower bound for $OPT = 10k$ is this: For a tiny $\varepsilon > 0$, the instance consists of OPT items of size approximately $1/6$, followed by OPT items of size approximately $1/3$, followed by OPT items of size $1/2 + \varepsilon$. The optimum packs in each bin one item from each group. FF packs the items of size about $1/6$ into $2k$ bins with 5 items, with the exception of the first and last of these bins that will have 6 and 4 items, respectively. The items of size about $1/3$ are packed in pairs. To guarantee this packing, the sizes of items differ from $1/6$ and $1/3$ in both directions by a small amount δ_i which is exponentially decreasing, but greater than ε for all i. This guarantees that only the item with the largest δ_i in a bin is relevant for its final size and this in turn enables us to order the items so that no additional later item fits into these bins.

Theorem 4.3.2 *For all values of OPT, there exists an instance I with $FF = BF = \lfloor 1.7 \cdot OPT \rfloor$.*

Proof. We prove the theorem for $OPT = 10k$ and $OPT = 10k + 3$, $k \geq 1$. For the other values, the theorem follows from the results of [23] where we used another construction.

Let $\delta > 0$ be sufficiently small ($\delta = 1/50$ will be sufficient). Let $\delta_j = \delta/4^j$ and $\varepsilon < \delta_{10k+4}$.

The instance I contains the following items, reordered as described later: Items $b_j^+ = 1/6 + \delta_j$ and $c_j^- = 1/3 - \delta_j - \varepsilon$ for $j = 1, \ldots, \lfloor OPT/2 \rfloor$, items $b_j^- = 1/6 - \delta_j$ and $c_j^+ = 1/3 + \delta_j - \varepsilon$ for $j = 0, \ldots, \lceil OPT/2 \rceil - 1$, and OPT items of size $1/2 + \varepsilon$. Note the shifted indices in the two subsets of items; this is important for the construction.

The optimal packing uses bins $\{b_j^+, c_j^-, 1/2 + \varepsilon\}$, $j = 1, \ldots, \lfloor OPT/2 \rfloor$, and $\{b_j^-, c_j^+, 1/2 + \varepsilon\}$, $j = 0, \ldots, \lceil OPT/2 \rceil - 1$. All these bins have size exactly 1 and their number is OPT.

Now we describe the sequence of FF-bins for the case of $OPT = 10k$. The items are then issued in the order of FF-bins they are packed into, thus the BF and FF packings coincide. The first $2k$ bins $B_1, \ldots, B_{2k}$ contain all the b_j^+ and b_j^- items. Bin B_1 contains the 6 smallest items $b_0^-, b_1^-, \ldots, b_5^-$, bin B_{2k} contains the 4 largest items $b_1^+, b_2^+, b_3^+, b_4^+$. Each remaining bin B_i, $i = 2, \ldots, 2k-1$ contains items b_{i+3}^+ and b_{i+4}^- (i.e., the largest and the smallest among the remaining ones of size about $1/6$) and some other three items chosen arbitrarily from the remaining items b_j^+ and b_j^-. We need to verify that the first fit packing indeed behaves this way. Since δ is sufficiently small, the size of B_1 is close to 1 and no other item fits there. For $i = 2, \ldots, 2k - 1$, it is crucial that all items of size at most $1/6 - \delta_{i+3}$ are packed into previous bins $B_{i'}$, $i' < i$. First, this implies that $s(B_i) \geq b_{i+3}^+ + b_{i+4}^- + 3b_{i+5}^- > 5/6 + \delta_{i+3}/2$. Second, this in turn implies that all items packed in later bins $B_{i'}$, $i' > i$ have size at least $1/6 - \delta_{i+5} > 1 - (5/6 + \delta_{i+3}/2) > 1 - s(B_i)$ and indeed they cannot be packed in B_i. The following part of FF packing contains $5k$ bins $C_1, \ldots, C_{5k}$ and C_i contains items c_{i-1}^+ and c_i^-. First note that no c_j^+ or c_j^- item fits into B_i, $i < 2k$, as $s(B_i) > 5/6$. Also, $s(B_{2k}) > 2/3 + \delta_1 + \delta_2$, thus no c_j^+ or c_j^- item fits into B_{2k}. Similarly as in the previous segment the fast decreasing δ_i and small ε yield $s(C_i) > 2/3 + 2\delta_i$, which guarantees that no later item c^+_{j-1} or c^-_j, $j > i$, fits there. Finally, the last segment of FF packing contains $10k$ bins with a single item $1/2 + \varepsilon$; all the bins have size more than $1/2$ so these items are packed separately. Altogether the FF packing contains $2k + 5k + 10k = 17k$ bins as needed.

It remains to describe the modification of the construction for $OPT = 10k + 3$. The items and OPT packing are already described; they are the same as in the instance for $OPT = 10k$ plus the new items $b^+_{5k+1}, c^-_{5k+1}, b^-_{5k}, b^-_{5k+1}, c^+_{5k}, c^+_{5k+1}$ and three items $1/2 + \varepsilon$. We pack items from the instance for $OPT = 10k$ as above, with the exception of b^+_{2k+3} which we replace by new b^-_{5k}. This creates no problems, as b^+_{2k+3} is among the arbitrarily assigned items for $OPT = 10k$, i.e., it is not the largest item in any B_i and its size is not relevant in any calculations. We pack the remaining items as follows: We create a bin $\widehat{B} = \{b^+_{2k+3}, b^+_{5k+1}, b^-_{5k+1}, c^+_{5k+1}\}$ and insert it between B_{2k-1} and B_{2k}. None of these items fit in the previous bins, as the smallest one of them has size $1/6 - \delta_{5k+1}$ and $s(B_{2k-1}) > 5/6 + \delta_{2k+2}/2$. Furthermore, $s(\widehat{B}) > 5/6$, thus no item from B_{2k} and later bins fits into $\widehat{B}$. Next we add $C_{5k+1} = \{c^+_{5k}, c^-_{5k+1}\}$ after C_{5k}, following the pattern of bins C_i, and three bins with single items of size $1/2 + \varepsilon$. Thus $FF = 17k + 5$ and $\lfloor 1.7 \cdot OPT \rfloor = \lfloor 1.7(10k + 3) \rfloor = 17k + 5$ and we are done. $\square$

Remark 4.3.1 *While we have only shown how to obtain an instance for $OPT = 10k + 3$ from the one for $OPT = 10k$, analogous construction can be used to modify the instance for any OPT to one for $OPT + 3$ with additional 5 FF-bins. Thus we can get instances for any OPT in a uniform way as follows: For any $k \geq 1$ and $i = 0, \ldots 9$ use the instance for $OPT = 10k$ and repeat the transformation to construct an instance for $OPT = 10k + 3i$ with $FF = 17k + 15i = \lfloor 1.7 \cdot OPT \rfloor$. (The equality is easy to check for all residues.) Since the instances for $k \leq 2$ can be constructed trivially, we can get the lower bound instances also for the remaining small values of OPT.*

Chapter 5

The tight absolute bound of First Fit in the parameterized case

The results of this chapter are from [25]. The contribution of this chapter is reached by the author of the dissertation.

In this chapter we consider the *parameterized* case, i.e. we suppose that the size of any item is at most $p_i \leq 1/d$, where $d \geq 1$ is some fixed integer. (In case $d = 1$ we get back the original problem.)

The tight asymptotic approximation ratio of First Fit was known from the early seventies, also in the parameterized case. Then the tight absolute bound of FF was found recently for $d = 1$ (as it is described in the previous chapter). In this chapter we give the tight absolute approximation ratio of First Fit for any $d \geq 2$. In fact, we again do more. For *any* value of OPT (the number of bins in an optimum solution) we determine that exactly how large FF (the number of bins created by First Fit) can be in the worst case.

For the asymptotic bound of FF, it was already known in [53] and [55] (see also [14] about this), that

$$R_{as}(FF) = \frac{d+1}{d}, \text{ if } d > 1.$$

We do not know about further results, regarding the absolute bound of FF in the parameterized case. Thus the tight absolute bound is found about after forty years.

We note that the proof for the tight bound of FF in the general case ($d = 1$, previous chapter and [23, 24]) is a bit difficult, it needs a tricky handle of the old weighing function, which is used previously to prove the asymptotic bound. Now, the proof for $d > 1$ is simple: We apply a lower bound construction in the parameterized case, which is very similar to that which was used in the general case (i.e. for $d = 1$, the construction is in the proof of Theorem 4.3.2). On the other hand, the upper bound is straightforward and we do not need any tricky technique for it.

Below we give the tight absolute bound in a more detailed form. Let $d > 1$ and $OPT \geq 1$ be arbitrary integers. Then let $OPT = k \cdot d + r$, where both k and r are integers, and $1 \leq r \leq d$. This condition that r is between 1 and d seems a bit unnatural, but this treatment will be convenient for us to give the tight bound, as follows.

Theorem 5.0.3 *(i), If $OPT = k \cdot d + 1$, then the maximum number of possible FF bins is exactly*

$$FF_d \leq \left\lfloor \frac{d+1}{d} OPT \right\rfloor = OPT + k, \tag{5.1}$$

(ii), Otherwise, if $OPT = k \cdot d + r$, where $2 \leq r \leq d$, the maximum number of possible FF bins is exactly

$$FF_d \leq \left\lceil \frac{d+1}{d} OPT \right\rceil = OPT + k + 1. \tag{5.2}$$

Note that the same *lower bounds* hold for algorithm BF, as well. But the *tightness* of these bounds for BF remain open now.

Notation. Let us fix a list L with items $p_1, \ldots, p_n$ and denote the number of bins in the FF and optimal solutions by FF and OPT, respectively. We will identify the items with their sizes. Let P be the total size of all items of L. Obviously $P \leq OPT$.

A bin is called a k^+-bin, if it contains at least k items (after packing all items of the list). The bins in the FF packing are ordered by the time they are opened (i.e., when the first item is packed into them). Expressions like "earlier", "later", refer to this ordering.

First we prove several properties of the FF packing which will be used in later calculations.

Claim 5.0.1 *In the FF packing, the total size of any $d+1$ d^+-bins is strictly bigger than d.*

Proof. Let the minimum level among the first d bins in consideration be x, for some $0 < x \leq 1$. If $x = 1$, then the statement holds (since the last bin also has at least one item with positive size). Otherwise $x < 1$. Let bin B denote the last d^+-bin. Then by the FF rule, any item in B, is bigger than $1 - x$. Since there are at least d items in B, the sum of levels of the considered $d+1$ bins is bigger than $d \cdot x + d(1-x) = d$. □

Corollary 5.0.1 *In the FF packing, the total size of any j d^+-bins is strictly bigger than $j \cdot \frac{d}{d+1}$, if $j \geq d+1$.*

Proof. The statement is valid for any chosen $d+1$ bins among the j bins. Thus the claim follows by average argument. □

Claim 5.0.2 *Any bin except the last FF bin is a d^+-bin.*

Proof. Consider one such bin, say B. Any item has size at most $1/d$, thus at least d items fit into B, no matter how big or small these items are. Since there is also (at least one) item in the last bin which does not fit into B, it follows that there are at least d items in B. □

Claim 5.0.3 *The total level of any two FF bins is bigger than* 1.

Proof. Immediately follows from the FF rule. □

Claim 5.0.4 *Let us consider a packing for which it holds that considering any two bins B_i and B_j, $i < j$, no item being in B_j fits into B_i. Then this packing is the outcome of an FF packing.*

Proof. FF creates this packing, if the items are taken in the next list: First we put into the list the items of the first bin, in any order. Then we continue the list with the items of the second bin, in any order, and so on. □

5.1 The proof of the bound

First we make some pretreatment to clear such cases when OPT is small, i.e. when $1 \leq OPT \leq 3$. The first three paragraphs give upper bounds, then we show the tightness, and finally we show that the bounds we get for small OPT values are the same that are in Theorem 5.0.3.

5.1.1 Small cases

If $OPT = 1$, we get $FF = 1$, for any $d \geq 2$.

If $OPT = 2$, $FF \geq 4$ contradicts to Claim 5.0.3, applying it for any two pairs of bins. Thus $FF \leq 3$.

If $OPT = 3$, we state $FF \leq 4$. For the sake of contradiction suppose $FF \geq 5$. If $d = 2$, applying Claim 5.0.1 and Claim 5.0.2, the total level of the first three bins is bigger than 2, and applying Claim 5.0.3 for the last two bins, their total level is bigger than 1, thus the total level of all bins is bigger than 3, a contradiction. Similarly, if $d = 3$, the total level of the first four bins bins is bigger than 3, we got contradiction again. Finally if $d \geq 4$, there is an item in the last bin with size at most $1/4$, thus the level of any previous bin is bigger than $3/4$, and thus the total level is again bigger than $4 \cdot 3/4 = 3$, a contradiction. Thus we conclude that $FF \leq 4$.

Now we show that the above upper bounds for the number of FF bins are tight. This claim is trivially true if $OPT = 1$. If $2 \leq OPT \leq 3$, we construct the next input. Any optimal bin will contain $d + 1$ items, with almost the same sizes, and any optimal bin is completely full. In any optimal bin the first $d - 1$ items (called big items) will have slightly bigger sizes, and the two last items (called small items) will have slightly smaller sizes. Moreover the total size of two small items is bigger than the size of a big item. Then in the list the items are ordered by non-increasing sizes. Then d big items are packed into the first bin by FF, and there remains a small room, which is not enough to accommodate a small item, so this small room remains unused. From this fact it follows that FF cannot create an optimal packing, and it uses an extra bin (but no more).

The exact sizes are the next: Each small item has size $\frac{1}{d+3/2}$ (which is slightly smaller than $\frac{1}{d+1}$). Thus the big items share $1 - \frac{2}{d+3/2} = \frac{2d-1}{2d+3}$ total size, i.e. the size of any big item is $\frac{2d-1}{(2d+3)(d-1)} < \frac{1}{d}$. The last inequality holds since $2d^2 - d < (2d+3)(d-1) = 2d^2 + d - 3$. Also holds that the size of a big item is slightly bigger than $\frac{1}{d+1}$. The total size of two small items is strictly bigger than one big item, since $\frac{2}{d+3/2} - \frac{2d-1}{(2d+3)(d-1)} = \frac{2d-3}{(2d+3)(d-1)} > 0$. From the above calculations we conclude the next:

a, d big items will be packed into the first FF bin, and there remains a small unused room in this bin, where no more item fits.

b, If $OPT = 2$, then $d-2$ big items will be packed into the second FF bin, and so all big items are packed. There remained four small items, three of them fit into the second FF bin (since $d-1$ big items and 2 small items fit together into a bin, it follows that $d-2$ big items and 3 small items fit together into a bin), and the last small item opens a third bin.

c, Suppose $OPT = 3$. If $d \geq 3$, then d big items will be packed also into the second FF bin, and no more item fits here. There remained $3(d-1) - 2d = d-3$ big items, these are packed into the third FF bin. At least four (but not all) small items fit here, and the further small items fit into the fourth bin. Finally if $d = 2$, then there are altogether three big items, and one big item is packed into the second FF bin. Then two small items are packed into this bin, and they fill the second FF bin. There remain four small items, they do not fit into the third FF bin (since all items do not fit into three bins as there remained a small unused room in the first FF bin), but three small items fit into the third FF bin, and the last small item opens a new bin.

Finally, we show that these upper bounds are the same as given in Theorem 5.0.3. Thus let $OPT = k \cdot d + r$, where $1 \leq r \leq d$. Note that $d \geq 2$. First let $OPT = 2$. No matter how large is d, only the next case is possible: $d \geq 2$, $k = 0$ and $r = 2$. Thus we need to substitute into (5.2), and we get $FF_d \leq OPT + k + 1 = 2 + 0 + 1 = 3$. Now let $OPT = 3$. If $d = 2$, we get $k = 1$ and $r = 1$. Substituting into (5.1), we get $FF_d \leq OPT + k = 3 + 1 = 4$. Otherwise $d \geq 3$, then $k = 0$ and $r = 3$. Substituting into (5.2) we get $FF_d \leq OPT + k + 1 = 3 + 0 + 1 = 4$.

In all cases we got the same bounds as stated in Theorem 5.0.3.

After considering the small cases, in the subsequent part of the paper it suffices only to consider when $OPT \geq 4$.

5.1.2 The lower bound construction

We assume that $OPT \geq 4$, as the small cases are already treated.

In the lower bound construction we have three different item types. In each of the i-th OPT-bin $(1 \leq i \leq OPT)$, there are $d+1$ items, as

- $d-1$ copies from $A = 1/(d+1) + \delta$,
- $B_i = 1/(d+1) - (d-1)\delta - \varepsilon_i$, and
- $C_i = 1/(d+1) + \varepsilon_i$,

where $\varepsilon_i = (\frac{1}{d})^i \cdot \varepsilon$, $\varepsilon = \frac{1}{5(d+1)}$, and $\delta = \varepsilon_{OPT}$.

We will see that all items in the input are smaller than $1/d$. Let us realize that the sequence of ε_i (quickly) decreases, the biggest one is $\varepsilon_1 = \varepsilon/d$, and the smallest one is the last one among them, i.e. $\varepsilon_{OPT} = \delta$. Furthermore any A and C items are bigger than $1/(d+1)$, and any B items are smaller than $1/(d+1)$, moreover the sizes of the B_i items are increasing, and the sizes of the C_i items are decreasing, as i grows. Note also that the smallest C_i item is $C_{OPT} = 1/(d+1) + \varepsilon_{OPT} = 1/(d+1) + \delta = A$, i.e. the A items and C_{OPT} are the smallest items except the B_i items.

Claim 5.1.1 *The biggest item is still smaller than $1/d$.*

Proof. Recall that $\varepsilon < 1/(d+1)$. The biggest item is $C_1 = 1/(d+1) + \varepsilon_1 = 1/(d+1) + \varepsilon/d < \frac{1}{d+1} + \frac{1}{d(d+1)} = 1/d$. $\square$

Claim 5.1.2 *The next inequalities are valid.*
(a), $(d-1)A + B_1 + B_2 \leq 1$,
(b), $(d-1)A + B_1 + B_2 + B_3 > 1$,
(c), $(d-2)A + B_{k+1} + C_{k-1} + B_{k+2} > 1$, *for any* $2 \leq k \leq OPT - 2$,
(d), $(d-2)A + B_{OPT} + C_{OPT-2} + A > 1$.

Proof. In all cases we use the definition of ε, and $\delta = \varepsilon_{OPT} \leq \varepsilon_i$ $(1 \leq i \leq OPT)$.

a, We get

$$\begin{aligned}
&(d-1)A + B_1 + B_2 \\
&= (d-1)(\frac{1}{d+1} + \delta) + (\frac{1}{d+1} - (d-1)\delta - \varepsilon_1) + (\frac{1}{d+1} - (d-1)\delta - \varepsilon_2) \\
&= 1 - (d-1)\delta - (\varepsilon_1 + \varepsilon_2) < 1.
\end{aligned}$$

b, Using $d\varepsilon_3 = \varepsilon_2$ we get

$$\begin{aligned}
&(d-1)A + B_1 + B_2 + B_3 \\
&= (d-1)(\frac{1}{d+1} + \delta) + \sum_{i=1}^{3}(\frac{1}{d+1} - (d-1)\delta - \varepsilon_i) \\
&= \frac{d+2}{d+1} - 2(d-1)\delta - (\varepsilon_1 + \varepsilon_2 + \varepsilon_3) \\
&> \frac{d+2}{d+1} - 2d\varepsilon_3 - (\varepsilon_1 + \varepsilon_2 + \varepsilon_3) = \frac{d+2}{d+1} - (\varepsilon_1 + 3\varepsilon_2 + \varepsilon_3) \\
&> \frac{d+2}{d+1} - 5\varepsilon_1 = 1 + \frac{1}{d+1} - \frac{5}{d}\varepsilon = 1 + \frac{1}{d+1} - \frac{1}{d(d+1)} > 1.
\end{aligned}$$

c, Using $d\delta \leq \varepsilon_{k+1}$ we get

$$\begin{aligned}
&(d-2)A + B_{k+1} + C_{k-1} + B_{k+2} \\
&= (d-2)(\frac{1}{d+1} + \delta) + \sum_{i=1}^{2}(\frac{1}{d+1} - (d-1)\delta - \varepsilon_{k+i}) + (\frac{1}{d+1} + \varepsilon_{k-1}) \\
&= 1 + \varepsilon_{k-1} - \varepsilon_{k+1} - \varepsilon_{k+2} - d\delta > 1 + \varepsilon_{k-1} - 3\varepsilon_{k+1} = 1 + \varepsilon_{k-1}(1 - \frac{3}{d^2}) > 1.
\end{aligned}$$

d, We simply get

$$\begin{aligned}
&(d-2)A + B_{OPT} + C_{OPT-2} + A \\
&= (d-1)(\frac{1}{d+1} + \delta) + (\frac{1}{d+1} - (d-1)\delta - \varepsilon_{OPT}) + (\frac{1}{d+1} + \varepsilon_{OPT-2}) \\
&= 1 + \varepsilon_{OPT-2} - \varepsilon_{OPT} > 1.
\end{aligned}$$

□

After this pretreatment, we are ready to get the lower bound for the FF packing.

On one hand, it is trivial, that the items of the construction, completely fill OPT bins. Now we create the FF bins one by one (according to Claim 5.0.4).

In the first FF bin there are $(d-1)$ copies from the A items, moreover B_1 and B_2 (i.e. the two smallest B items). These items fit into one bin according to Claim 5.1.2(a). On the other hand, no more item fits into the bin by Claim 5.1.2(b), since at this point B_3 is the smallest unpacked item, and this item does not fit into the bin.

In the second FF bin there are $(d-2)$ copies from the A items, moreover B_3 and C_1. In the third FF bin there are $(d-2)$ copies from the A items, moreover B_4 and C_2. Generally, in the k-th FF bin, there are $(d-2)$ copies from the A items, moreover B_{k+1} and C_{k-1}, for $2 \leq k \leq OPT-1$.

Consider any such bin (the k-th bin in the FF packing, with $2 \leq k \leq OPT-1$). We prove that no more item (which is not packed yet) fits into the bin. First let us consider the case where $2 \leq k \leq OPT-2$. Then B_{k+2} is the smallest unpacked item, and this item does not fit into the bin, by Claim 5.1.2(c). Now let $k = OPT-1$. Then all B_i items are already packed, thus the smallest unpacked item is $C_{OPT} = A$, and this item does not fit into the bin, by Claim 5.1.2(d).

Up to this point, we have created $OPT-1$ bins, and all B items are packed. Moreover, all C items are already packed, except the last two C items. The number of already packed A items is $(d-1)+(d-2)(OPT-2) = (d-2)OPT-(d-3)$. Hence, the number of not packed A items is exactly $(d-1)OPT-(d-2)OPT+(d-3) = OPT+(d-3)$. Together with the not packed two C items, we have altogether $OPT-1+d$ items, all are slightly bigger than $1/(d+1)$, and all are smaller than $1/d$ by Claim 5.1.1, and none of them fits into any previous bin.

With these items we fill $\lceil \frac{OPT-1+d}{d} \rceil$ bins, more exactly the last bin can contain less than d items (but at least one item), and all previous bins will contain exactly d items.

Thus the number of FF bins is exactly

$$(OPT-1)+\left\lceil \frac{OPT-1+d}{d} \right\rceil = OPT+\left\lceil \frac{OPT-1}{d} \right\rceil.$$

Now, distinguishing the residual classes, we get the following.

If $OPT = k \cdot d + 1$, then the number of FF bins is exactly

$$FF_d = OPT + k,$$

otherwise, if $OPT = k \cdot d + r$, where $2 \leq r \leq d$, the number of FF bins is exactly

$$FF_d = OPT + \left\lceil \frac{k \cdot d + r - 1}{d} \right\rceil = OPT + k + \left\lceil \frac{r-1}{d} \right\rceil = OPT + k + 1.$$

5.1.3 The tight upper bound

Now we determine the tight upper bound. Recall that $OPT \geq 4$, as the smaller cases are already treated.

Case 1, $OPT = k \cdot d + 1$. If $k = 0$, then $OPT = 1$, a contradiction. Thus $k \geq 1$. We claim $FF_d \leq OPT + k = k(d+1) + 1$. Suppose for the sake of contradiction, that there exists a list L, for which $FF_d \geq k(d+1) + 2$. We apply Corollary 5.0.1 and Claim 5.0.2 for the first $k(d+1) \geq d+1$ bins. The total level of these bins is bigger than $k(d+1) \cdot \frac{d}{d+1} = kd$. Moreover the total level of the last two bins is bigger than 1, by Claim 5.0.3. Thus the total size of the items $P > kd + 1 = OPT$, a contradiction.

Case 2, $OPT = k \cdot d + r$, where $2 \leq r \leq d$, and $k \geq 1$. We claim that $FF_d \leq OPT + k + 1 = (k \cdot d + r) + k + 1 = k(d+1) + r + 1$. Suppose for contradiction, that there exists a list for which $FF_d \geq k(d+1) + r + 2$. We apply Corollary 5.0.1 for the first $k(d+1) + r \geq d+1$ bins, all these bins are d^+-bins by Claim 5.0.2. Thus their total level is bigger than $(k(d+1)+r) \cdot \frac{d}{d+1} = kd + \frac{dr}{d+1}$. Moreover the total level of the last two bins is bigger than 1, by Claim 5.0.3. Thus the total size of the items is

$$P > kd + \frac{dr}{d+1} + 1 = kd + \frac{dr + d + 1}{d+1} > kd + \frac{dr + r}{d+1} = kd + r = OPT,$$

a contradiction.

Case 3, $OPT = k \cdot d + r$, where $2 \leq r \leq d$, and $k = 0$. In other words, $2 \leq r = OPT \leq d$. We claim that $FF_d \leq r + 1 = OPT + 1$. Suppose to the contrary, that there exists a list for which $FF_d \geq OPT + 2$. We complete the input with $d \cdot (d - OPT + 1)$ items with equally $1/d$ sizes, and these items will be put to the end of the list L. The new list is denoted by L'. We denote the new values of the optimum and the FF packing by OPT' and FF'_d, respectively. Naturally, $OPT' = OPT + (d - OPT + 1) = d + 1$.

Now let us consider the FF'_d packing. Let the last bin in the FF_d packing be denoted by B. Since there is an item in B (with size at most $1/d$), it follows that the level of any previous bin is bigger than $1 - 1/d$. Now, when the new items come, none of them fits into any previously opened bin, except B. Since the new items completely fill $(d - OPT + 1)$ bins, and B is not empty, they do not fit into B, and at most $(d - OPT)$ newly opened bins.

Thus exactly $(d - OPT + 1)$ new bins will be opened, and thus $FF'_d = FF_d + (d - OPT + 1) \geq OPT + 2 + (d - OPT + 1) = d + 3$. This contradicts to what we have proved in Case 1.

Chapter 6

The tight asymptotic bound of First Fit with cardinality constraints

The results of this chapter are from [26]. All results of [26] are contained (together with further results) in [27]. The contribution of the author of this dissertation regarding the results of this chapter is approximately 75%.

In bin packing with cardinality constraints (BPCC), there is a global parameter $k \geq 2$, which is an upper bound (called the cardinality constraint) on the number of items that can be packed into each bin, additionally to the standard constraint on the total size of items packed into a bin. In this chapter the items are denoted by $1, 2, \ldots, n$, where item i has a size $s_i > 0$ associated with it. In many applications of bin packing, the assumption that a bin can contain any number of items is not realistic, and bounding the number of items as well as their total size provides a more accurate modeling of the problem. BPCC was studied both in the offline and online environments [58, 59, 57, 13, 1, 38, 40, 42].

Here we study the algorithm First Fit (FF). This algorithm processes the input items one by one. Each item is packed into the a bin of the smallest index where it can be packed. An item i can be packed into bin B if the packing is possible both with respect to the total size of items already packed into that bin and with respect to the number of packed items, i.e., the bin contains items of total size at most $1 - s_i$ and it contains at most $k - 1$ items. We present a complete analysis of its asymptotic approximation ratio for all values of $k \geq 3$. Prior to this work, only the tight bound for $k = 2$ was known. After almost forty years after the problem BPCC and the natural algorithm First Fit for it were introduced, its tight asymptotic competitive ratio is for all values of k is finally found.

Approximation algorithms were designed for the offline version of BPCC (which is strongly NP-hard for $k \geq 3$) [58, 57, 13, 40], and the problem has an asymptotic fully polynomial approximation scheme (AFPTAS) [13, 40]. Using elementary bounds, it was shown by Krause, Shen, and Schwetman [58] that FF has an asymptotic approximation ratio of at most $2.7 - \frac{2.4}{k}$. For $k \to \infty$, it can be deduced that the asymptotic approximation ratio is 2.7 also since this is a special case of vector bin packing (with two dimensions) [44]. The case $k = 2$ is solvable using matching techniques in the offline scenario, but it is not completely resolved in the online scenario, and the best possible asymptotic approximation ratio is in $[1.42764, 1.44721]$ [61, 1, 42]. For larger k, there is an approximation algorithm of approximation ratio at most 2 [1], and improved algorithms (that have smaller asymptotic approximation ratios than $\min\{2, 2.7 - \frac{2.4}{k}\}$) are known for $k = 3, 4, 5, 6$ [38].

For comparison we note that tight asymptotic approximation ratio of the cardinality constrained variant of the Harmonic algorithm [60] (that partitions items into k classes and packs each class independently, such that the classes are $I_\ell = (\frac{1}{\ell+1}, \frac{1}{\ell}]$ for $1 \le \ell \le k-1$ and $I_k = (0, \frac{1}{k}]$, and for any $1 \le \ell \le k$, each bin of I_ℓ, possibly except for the last such bin, receives exactly ℓ items) is the same as for FF for $2 \le k \le 4$, and slightly smaller for any $k \ge 5$, see [38]. Known lower bounds on the competitive ratio do not exceed those known for standard bin packing [86, 77, 2, 42]. A related problem is called *class constrained bin packing* [39, 71, 72, 84]. In that problem each item has a color, and a bin cannot contain items of more than k colors (for a fixed parameter k). BPCC is the special case of that problem where all items have distinct colors.

Value of k	FF	prev. UB for FF	best known UB
2	1.5 [58]	1.5 [58]	1.44721 [1]
3	1.8333	1.9 [58]	1.75 [38]
4	2	2.1 [58]	1.86842 [38]
5	2.1333	2.22 [58]	1.93719 [38]
6	2.2222	2.3 [58]	1.99306 [38]
7	2.2857	2.35714 [58]	2 [1]
8	2.3333	2.4 [58]	2 [1]
9	2.3704	2.43333 [58]	2 [1]
10	2.4	2.46 [58]	2 [1]
11	2.4273	2.481818 [58]	2 [1]
12	2.45	2.5 [58]	2 [1]

Table 6.1: Bounds for $2 \le k \le 12$. The second column contains the tight asymptotic approximation ratio of FF, the third column contains the previous upper bound on FF's asymptotic approximation ratio, and the last column contains the asymptotic approximation ratio of the current best algorithm. Entries without a citation are those proved here.

Below we provide a complete analysis of the famous and natural algorithm FF with respect to the asymptotic approximation ratio. We find that the asymptotic approximation ratio of FF is $2.5 - \frac{2}{k}$ for $k = 3, 4$, $\frac{8(k-1)}{3k} = \frac{8}{3} - \frac{8}{3k}$ for $4 \le k \le 10$, and $2.7 - \frac{3}{k}$ for $k \ge 10$ (recall that the values $k = 4$ and $k = 10$ are included in two cases each). Interestingly, introducing cardinality constraints (with sufficiently large values of k) results in an increase of many approximation ratios by 1 [58, 55, 60, 38]. In particular, the asymptotic approximation ratio of the cardinality constrained FF has an approximation ratio that is larger by 1 than its approximation ratios for standard bin packing. (Harmonic has a slightly smaller approximation ratio of 2.69103.) Moreover, it can be verified that the worst-case examples of Harmonic are valid (but not tight) for FF.

While FF is a frequently studied natural algorithm, its exact asymptotic approximation ratio as a function of k was unknown. While it is not difficult to show an upper bound of 2.7 for all values of k [58, 44], providing such a tight analysis as a function of k turns out to be quite difficult. Intuitively, it initially seems that the asymptotic approximation ratio should simply increase by $\frac{k-3}{k}$

compared to the approximation ratio of FF for standard bin packing. The reason for this is that in the well-known worst-case example of [55], it is possible to define an optimal packing that packs three items into each bin, leaving space for $k-3$ very small items that can still be packed into each bin of the optimal solution, while these items can arrive first, in which case FF will pack them into their own bins (such that k items will be packed into each bin). As it turns out, this input can be used for $k \geq 10$, but for $k \leq 9$ there are worse inputs. Our first attempt was to adapt the weight function that was used to prove an upper bound on the asymptotic approximation ratio of FF for standard bin packing [55]. Such an adaptation is quite tricky for the cases where $10 \leq k \leq 19$, and in particular, items of sizes in $[0.2, 0.3)$ require a special treatment. Additionally, bins of an optimal solution that contain two items that FF does not pack into bins containing k items also require a special treatment, which is very different from the known analysis. While the cases where $k \in \{2,3,4,5\}$ are sufficiently straightforward to deal with, in the cases $k \in \{6,7,8,9\}$, a completely new weight function was needed. Intuitively, some of the difficulty is caused by the fact that in these cases the worst-case examples contain two very different types of bins packed by optimal solutions. In particular, in the case $k = 9$, it turned out that items whose sizes are approximately 0.2 or 0.3 are most difficult to treat, and therefore one of the weight functions is partitioned into seven cases. Similar cases are also used here in a weight function defined for the cases $10 \leq k \leq 19$. In summary, while the approach seems similar to that of other work, it is in fact quite different and challenging.

Notation. Below we see a bin as a set of items, and for a bin B, we let $s(B) = \sum_{i \in B} s_i$ be its level.

6.1 Upper bounds

In this section we prove upper bounds on the the asymptotic approximation ratio. In the analysis, a bin of FF that has j items for $j \leq k$ is called a j-bin, and a bin whose number of items is in $[j, k-1]$ for some $1 \leq j < k$ is called a j^+-bin. For a bin packed by FF, a later bin is a bin that was opened after the current bin was opened (it appears later in the ordering of FF) and an earlier bin is a bin that was opened before the current one (it appears earlier in the ordering of FF). When we discuss an item packed into some bin and the "further items" of a bin, we mean all the other items packed into the same the bin, where the list of items of a bin is not ordered according to the times that they were considered by FF.

Given a function f defined on items or on item sizes and a subset of items X, $f(X)$ is defined as the sum of images of the items in X under f. We start with several lemmas that will assist in the upper bounds proofs.

Lemma 6.1.1 *Let $1 \leq j \leq k-1$. Every j^+-bin B except for at most one bin has level above $\frac{j}{j+1}$.*

Proof. Assume that there exists a j^+-bin B whose level is at most $\frac{j}{j+1}$. All later j^+-bins only have items of sizes above $\frac{1}{j+1}$ (as they could not be packed into B), and each such bin has at least j items, so their levels are above $\frac{j}{j+1}$. □

In what follows, we often use the following partition of item sizes into classes. Items of sizes at most $\frac{1}{6}$ are called **tiny**. Items of sizes in $(\frac{1}{6}, \frac{1}{4}]$ are called **small**, items of sizes in $(\frac{1}{4}, \frac{1}{3}]$ are called

medium, items of sizes in $(\frac{1}{3}, \frac{1}{2}]$ are called **big**, and other items (of sizes above $\frac{1}{2}$) are called **huge**. We will use weights for the analysis of FF, and huge items will always have weight 1, and in most cases this will not be stated in the definitions of weights (i.e., we will define weights only for items that are not huge).

Lemma 6.1.2 *Let $k \geq 3$. Except for at most one bin, any 1-bin has a huge item. Consider 2-bins that do not have huge items. Except for at most two bins, any such 2-bin has a big item, and its other item is big or medium.*

Proof. By Lemma 6.1.1, any bin except for at most one bin has a level above $\frac{1}{2}$, and all such 1-bins have huge items. Moreover, by the same lemma, all 2-bins except for at most one bin have levels above $\frac{2}{3}$. Consider 2-bins with levels above $\frac{2}{3}$ and no huge items. Any such bin must have an item of size above $\frac{1}{3}$. Moreover, since it has no huge item, it must have a big item. Assume that there is such a bin whose smaller item is no larger than $\frac{1}{4}$ and consider the first such bin B. Its level does not exceed $\frac{3}{4}$, and thus any later bin cannot have an item of size at most $\frac{1}{4}$, as such an item could be packed into B. Thus, later 2-bins without huge items only have medium and big items. We find that all 2-bins without huge items, except possibly for a bin with load no larger than $\frac{2}{3}$, and one additional bin, have the described contents. □

Lemma 6.1.3 *Let $k \geq 4$. Except for at most one bin, any 3^+-bin that has a level of at most $\frac{5}{6}$ has no tiny items.*

Proof. If all 3^+-bins have levels above $\frac{5}{6}$, we are done. Otherwise, consider the first 3^+-bin B of level at most $\frac{5}{6}$. Any later bin cannot have an item of size at most $\frac{1}{6}$, as such an item could be packed into B. □

Lemma 6.1.4 *Let $k \geq 4$. Except for at most one bin, any 3-bin has at least one item of size above $\frac{1}{4}$. Except for at most two bins, any 3-bin without a huge item has one of the following structures.*

- *The bin has no tiny items, and at least one of its items is medium or big.*
- *The bin has level above $\frac{5}{6}$, it has exactly one tiny item, and at least one big item.*

Proof. By Lemma 6.1.1, all 3-bins except for at most one bin have levels above $\frac{3}{4}$, so at least one item has size above $\frac{1}{4}$. By Lemma 6.1.3, for all 3-bins except for at most one bin, if the bin has at least one tiny item, its level is above $\frac{5}{6}$. Thus it can contain at most one tiny item, and if it has a tiny item, then it must contain at least one big item, given its level. □

Lemma 6.1.5 *Let $k \geq 5$. Except for at most one bin, any 4-bin without huge items has one of the following structures.*

- *The bin has no tiny items.*
- *The bin has level above $\frac{5}{6}$, and it has at least one big item.*

- *The bin has level above $\frac{5}{6}$, and it has exactly one tiny item.*
- *The bin has level above $\frac{5}{6}$, and it has two tiny items, a medium item, and another item that is small or medium.*

Proof. By Lemma 6.1.3, it is sufficient to consider a bin of level above $\frac{5}{6}$. If the bin has no big items, it can have at most two tiny items, and if it has exactly two tiny items, the remaining items cannot be both small. □

6.1.1 The cases $k = 3, 4, 5$

We use the following weight function for $k = 3, 4$. The variable a denotes the size of an item.

$$w(a) = \begin{cases} \frac{1}{k} & \text{if} \quad 0 < a \le \frac{1}{4}, \quad \text{(tiny or small)} \\ \frac{1}{2} & \text{if} \quad \frac{1}{4} < a \le \frac{1}{2}, \quad \text{(medium or big)} \end{cases}$$

We analyze the total weight of a bin of an optimal solution OPT, and of a bin packed by FF.

Lemma 6.1.6 *For $k = 3, 4$, any bin of OPT has weight of at most $\frac{3}{2} + \frac{k-2}{k} = \frac{5}{2} - \frac{2}{k}$.*

Proof. If the bin has no huge item, then the total weight is at most $\frac{k}{2} \le \frac{5}{2} - \frac{2}{k}$ for $k = 3, 4$. If the bin has a huge item, then it can have at most one item of weight $\frac{1}{2}$, and the other items have weights of $\frac{1}{k}$. □

Lemma 6.1.7 *For $k = 3, 4$, except for at most k bins, any bin of FF has weight of at least* 1.

Proof. As the weight of any item is at least $\frac{1}{k}$, any k-bin has a weight of at least 1. Any bin containing a huge item also has a weight of at least 1. Thus, by Lemma 6.1.2, at most one 1-bin has weight below 1. Consider the 2^+-bins that do not contain huge items. By Lemma 6.1.2, except for at most two bins, any 2-bin without a huge item has two medium or big items, and the total weight of such a bin is 1. In the case $k = 4$, by Lemma 6.1.4, except for at most one 3-bin, any 3-bin has at least one item of size above $\frac{1}{4}$. Thus, such a bin has at least one item of weight at least $\frac{1}{2}$, and two other items whose total weight is at least $\frac{1}{2}$. □

We showed that for any input I, $W(I) \le (\frac{5}{2} - \frac{2}{k})OPT(I)$ and $W(I) \ge FF(I) - k$, thus $FF(I) \le (\frac{5}{2} - \frac{2}{k}) - k$ for $k = 3, 4$.

Next, we define a weight function for $k = 5$, as a function of the item sizes.

$$w(a) = \begin{cases} 3/15 & \text{if} \quad a \le 1/6, \quad \text{(tiny)} \\ 4/15 & \text{if} \quad 1/6 < a \le 1/4, \quad \text{(small)} \\ 7/15 & \text{if} \quad 1/4 < a \le 1/3, \quad \text{(medium)} \\ 8/15 & \text{if} \quad 1/3 < a \le 1/2, \quad \text{(big)} \end{cases}$$

We will show that the weight of any bin of OPT is at most $32/15$, while the weight of any bin of FF is at least 1, except for a constant number of special bins.

Lemma 6.1.8 *For every bin B of OPT, $w(B) \leq \frac{32}{15}$ holds.*

Proof. Bin B can contain at most one huge item. First, assume first that B contains a huge item. If it also contains a big item, then every remaining item is tiny and has weight $\frac{3}{15}$. The total weight is therefore at most $1 + \frac{8}{15} + 3 \cdot \frac{3}{15} = \frac{32}{15}$. If B does not contain a big item (in addition to the huge item), then it can have at most two items that are medium or small, out of which at most one can be medium, and the remaining items have weights of $\frac{3}{15}$. In this case the total weight is at most $1 + \frac{7}{15} + \frac{4}{15} + 2 \cdot \frac{3}{15} = \frac{32}{15}$.

If B does not contain a huge item, then it can contain at most three items of sizes above $\frac{1}{4}$, out of which at most two can have sizes above $\frac{1}{3}$, and the remaining items have weights of at most $\frac{4}{15}$. The total weight is at most $2 \cdot \frac{8}{15} + \frac{7}{15} + 2 \cdot \frac{4}{15} = \frac{31}{15}$. □

Now we consider the bins created by FF.

Lemma 6.1.9 *For $k = 5$, except for at most 6 bins, any bin of FF has weight of at least 1.*

Proof. As the weight of any item is at least $\frac{1}{5}$, any 5-bin has weight of at least 1. Any bin containing a huge item also has a weight of at least 1. Thus, by Lemma 6.1.2, at most one 1-bin has weight below 1. Consider the 2^+-bins that do not contain huge items. By Lemma 6.1.2, except for at most two bins, any 2-bin without a huge item has a big item and another item that is medium or big, and the total weight of such a bin is 1 or $\frac{16}{15}$. By Lemma 6.1.4, except for at most two bins, any 3-bin has one of two forms. In the first form, all three items have weights of at least $\frac{4}{15}$, and at least one item has a weight of at least $\frac{7}{15}$, giving a total of at least 1. In the second form, one item has weight $\frac{3}{15}$, one item has weight $\frac{8}{15}$, and the third item has a weight of at least $\frac{4}{15}$, giving a total of at least 1 again. By Lemma 6.1.5, except for at most one bin, any 4-bin has one of four forms. In the first form, all four items have weights of at least $\frac{4}{15}$, giving a total above 1. In the second form, one item has weight $\frac{8}{15}$, and each remaining item has weight of at least $\frac{3}{15}$, giving a total above 1 again. In the third form, one item has weight $\frac{3}{15}$, and each remaining item has weight of at least $\frac{4}{15}$, giving a total of at least 1. In the fourth form, two items have weights of $\frac{3}{15}$, one item has weight $\frac{7}{15}$, and one item of weight at least $\frac{4}{15}$, giving a total above 1 again. □

We showed that for any input I, $W(I) \leq \frac{32}{15} OPT(I)$ and $W(I) \geq FF(I) - 7$, thus $FF(I) \leq \frac{32}{15} OPT(I) - 6$, for $k = 5$.

Theorem 6.1.1 *The asymptotic approximation ratios of FF for $k = 3, 4, 5$ are at most $\frac{11}{6}$, 2, and $\frac{32}{15}$, respectively.*

6.1.2 The cases $k = 6, 7, 8$

In this case the classification into item types items remain the same, but the weights of such items are defined differently. We define the weights of items with sizes at most $1/2$. The weight $w(a)$ of any item of size $a \leq \frac{1}{2}$ consists of three parts. The first part is the ground weight, the second part is the scaled size, and the third part is the bonus. Each part is non-negative. The ground weight of any item is $\frac{1}{k}$, and we let $g(a) = 1/k$. This ensures that the weight of any item (no matter how small it is) is at least $1/k$. The scaled size of an item of size $a \leq \frac{1}{2}$, is defined by $s(a) = \frac{2(2k-11)}{3k} a$.

The bonus of an item of size a, denoted by $b(a)$ is defined as follows.

$$b(a) = \begin{cases} 0 & \text{if} \quad a \le 1/6 \\ \frac{2(2k-11)}{3k}(a - \frac{1}{6}) + \frac{10-k}{9k} = \frac{2(2k-11)}{3k}a + \frac{7-k}{3k} & \text{if} \quad 1/6 < a \le 1/4 \\ \frac{2(2k-11)}{3k}(a - \frac{1}{4}) + \frac{3}{2k} = \frac{2(2k-11)}{3k}a + \frac{10-k}{3k} & \text{if} \quad 1/4 < a \le 1/3 \\ \frac{2}{k} & \text{if} \quad 1/3 < a \le 1/2 \end{cases}$$

where we call the items in the classes after each other as tiny, small, medium and big, respectively.

As $k \ge 6$, we have $\frac{2(2k-11)}{3k} > 0$, and therefore the function $s(a)$ is monotonically increasing. The bonus of a medium item is at least $\frac{3}{2k}$ and at most $\frac{\frac{2}{3}(2k-11)}{3k} + \frac{10-k}{3k} = \frac{k+8}{9k} < \frac{2}{k}$, and the bonus of a small item is at least $\frac{10-k}{9k}$ and at most $\frac{\frac{2}{4}(2k-11)}{3k} + \frac{7-k}{3k} = \frac{1}{2k} < \frac{3}{2k}$. The value of the bonus is zero if $a \le 1/6$, and the bonus is a constant value ($\frac{2}{k}$) for $a \in (1/3, 1/2]$. We get that $b(a)$ (and therefore also $w(a)$) is a piecewise linear function that is monotonically non-decreasing for $a \in (0, 1/2]$. The weight of an item of size $a \le 1/2$, is $w(a) = g(a) + s(a) + b(a)$. The weight function has the discontinuity points, $1/6$, $1/4$, $1/3$, and $1/2$ (this is not the same set of discontinuity points as in the weight function of FF for standard bin packing [55]).

Now we first find some properties of the weighting, and then we are able to establish the asymptotic bound.

Lemma 6.1.10 *For every bin B of OPT, $w(B) \le \frac{8(k-1)}{3k}$ holds.*

Proof. **Case 1:** B contains no huge item. The bin can contain at most k items, thus the total ground weight is at most 1. Similarly, the total scaled size is at most $\frac{2(2k-11)}{3k}$. Thus, it remains to bound $b(B)$, it suffices to show that the total bonus of the items in the bin is at most $b(A) \le \frac{8k-8}{3k} - 1 - \frac{2(2k-11)}{3k} = \frac{k+14}{3k}$.

If B has two big items in the bin, then there can be at most one further item with a positive bonus, and $b(A) \le 3 \cdot \frac{2}{k} \le \frac{k+14}{3k}$, for $k \ge 4$. If there is only one big item in the bin, there can be at most three further items having positive bonuses. Then $b(A) \le \frac{2}{k} + 3 \cdot \frac{k+8}{9k} = \frac{k+14}{3k}$. Now suppose that any item of B has size at most $1/3$. There can be at most five items in the bin having positive bonuses, and there can be at most three medium items among them. Thus the total bonus is at most $b(A) \le 3 \cdot \frac{k+8}{9k} + 2 \cdot \frac{1}{2k} = \frac{k+11}{3k}$.

Case 2: B contains a huge item. Recall that the weight of the huge item is 1, and its size is bigger than $\frac{1}{2}$. There can be at most $k-1$ further items in the bin, their total ground weight is at most $\frac{k-1}{k}$, and their total scaled size is at most $\frac{2(2k-11)}{3k} \cdot \frac{1}{2}$. Thus, it suffices to show that the total bonus of the further items in the bin is at most $\frac{8k-8}{3k} - 1 - \frac{k-1}{k} - \frac{2k-11}{3k} = \frac{2}{k}$. The total size of remaining items is below $\frac{1}{2}$, thus the bin can contain at most two items with positive bonuses. Moreover, if B contains only one item with a positive bonus, then this bonus is at most $\frac{2}{k}$, and we are done. Otherwise, if it contains two items of positive bonuses, none of them can be big, and at least one of them is small. If both are small, then their total bonus is at most $\frac{1}{k}$. We are left with the case that B contains items of sizes a_1 and a_2 where $\frac{1}{6} < a_1 \le \frac{1}{4} < a_2 \le \frac{1}{3}$. Then applying $a_1 + a_2 < \frac{1}{2}$, we get that the total bonus is

$$\frac{2\,(2k-11)}{3k}a_1 + \frac{7-k}{3k} + \frac{2\,(2k-11)}{3k}a_2 + \frac{10-k}{3k} \le \frac{2\,(2k-11)}{3k} \cdot \frac{1}{2} + \frac{17-2k}{3k} = \frac{2}{k}.$$

□

Now, we find a lower bound on the total weight of the bins created by FF for an input L and a given $k \in \{6, 7, 8\}$. The total weight of 1-bins is at least their number minus 1, as all 1-bins except for possibly one bin have huge items. The total weight of k-bins is at least their number. We will show that for each one of the four sets: 2-bins, 3-bins, 4-bins, and 5^+-bins, the total weight of items packed into bins of this set is at least the number of such bins minus 2 (for 5^+-bins it is at least their number minus 1). This will show that $W \geq FF(L) - 8$. Since the weight of every bin that contains a huge item is at least 1, we can restrict the analysis to bins that do not contain such items, and for $2 \leq j \leq k-1$ we will only consider j-bins that have no huge items.

Claim 6.1.1 *Every 5^+-bin of level above $\frac{5}{6}$ has weight of at least 1, and the total weight of 5^+-bins is at least their number minus 1.*

Proof. Consider a j-bin A where $5 \leq j \leq k-1$. The ground weight of its items is $\frac{j}{k}$, and their scaled size is at least $\frac{5}{6} \cdot \frac{2(2k-11)}{3k}$. If $j = 5$, then at least one item has a positive bonus (otherwise the total size is at most $\frac{5}{6}$), and the weight of the bin is $w(A) = g(A) + s(A) + b(A) \geq \frac{5}{k} + \frac{5}{6} \cdot \frac{2(2k-11)}{3k} + \frac{10-k}{9k} = 1$, since the value of any positive bonus is at least $\frac{10-k}{9k}$. Otherwise, $k \geq 6$, so the ground weight is at least $\frac{6}{k}$, and we are done since $\frac{1}{k} \geq \frac{10-k}{9k}$.

By Lemma 6.1.1, all 5^+-bins, except for at most one bin, have levels above $\frac{5}{6}$. Since there is at most one 5^+-bin whose level is at most $\frac{5}{6}$, and all 5^+-bins with level above $\frac{5}{6}$ have weights of at least 1, we find that the total weight of 5^+-bins is at least their number minus 1. □

It remain to consider only the 2-bins, 3-bins, and 4-bins. For all of these cases we consider two subcases. We will show that if the level of a bin is sufficiently large (above $\frac{3}{4}$ for 2-bins, and above $\frac{5}{6}$ otherwise), then the total weight of the bin is at least 1. Then, we will consider j-bins of smaller levels for for $j = 2, 3, 4$.

Lemma 6.1.11 *The weight of any 2-bin B of level above $\frac{3}{4}$ is at least 1.*

Proof. As B has no huge item, it must have a big item and another item that is either medium or big. The ground weight is $\frac{2}{k}$, and the scaled size is at least $\frac{2(2k-11)}{3k} \cdot \frac{3}{4}$. The total bonus is at least $\frac{2}{k} + \frac{3}{2k} = \frac{7}{2k}$. The total weight is therefore at least $\frac{2}{k} + \frac{2k-11}{2k} + \frac{7}{2k} = \frac{4+2k-11+7}{2k} = 1$. □

Lemma 6.1.12 *Let $j \in \{3, 4\}$. The weight of any j-bin B of level above $\frac{5}{6}$ is at least 1.*

Proof. The scaled size of the bin is at least $\frac{2(2k-11)}{3k} \cdot \frac{5}{6} = \frac{10k-55}{9k} = 1 + \frac{k-55}{9k}$. For $j = 3$, the bin has ground weight $\frac{3}{k}$, and for $j = 4$, the bin has ground weight $\frac{4}{k}$. If the bin has at least two items of sizes above $\frac{1}{4}$, then their combined bonuses are at least $\frac{3}{k}$, and the total weight is at least $\frac{3}{k} + 1 + \frac{k-55}{9k} + \frac{3}{k} = 1 + \frac{k-1}{9k} > 1$. Similarly, if a bin has a big item and at least one other item with a positive bonus, their combined bonuses are at least $\frac{2}{k} + \frac{10-k}{9k} = \frac{28-k}{9k}$, and the total weight is at least $\frac{3}{k} + 1 + \frac{k-55}{9k} + \frac{28-k}{9k} = 1$. The remaining cases are considered separately for $j = 3$ and $j = 4$.

Let $j = 3$. A bin that has a big item and two tiny items has level of at most $\frac{5}{6}$. A bin that has a medium item and two items that are no larger than $\frac{1}{4}$ also has level of at most $\frac{5}{6}$. Thus for $j = 3$ there are no additional cases and we are done. Let $j = 4$. A bin that has a big item has weight of at least 1, as $\frac{4}{k}+1+\frac{k-55}{9k}+\frac{2}{k}>1$. We are left with the case where $j=4$, the bin has no big items, and it has at most one medium item. If the bin has one medium item, then (since the size of a medium item and three tiny items is at most $\frac{5}{6}$), it must have also one small item and their total bonus is at least $\frac{3}{2k}+\frac{10-k}{9k}=\frac{47-2k}{18k}$, and the bin has weight of at least $\frac{4}{k}+1+\frac{k-55}{9k}+\frac{47-2k}{18k}=1+\frac{1}{2k}>1$. Finally, if it has no medium items, then it must have at least three small items. If there are four small items, then their total size is above $\frac{5}{6}$. If there is a tiny item, then the total size of the remaining items is at least $\frac{2}{3}$. Thus, the total bonus of the small items is at least $\frac{2(2k-11)}{3k}\cdot\frac{2}{3}+\frac{3(7-k)}{3k}=\frac{19-k}{9k}$. In this case the weight of the bin is at least $\frac{4}{k}+1+\frac{k-55}{9k}+\frac{19-k}{9k}=1$. □

Lemma 6.1.13 *The total weight of the* 2*-bins of levels in* $(\frac{2}{3},\frac{3}{4}]$ *is at least their number minus* 1.

Proof. Consider two consecutive 2-bins of levels in $(\frac{2}{3},\frac{3}{4}]$, B_i and B_j (these bins become consecutive if we remove all other kinds of bins from the list of bins). We prove that $g(B_i)+s(B_i)+b(B_j)\geq 1$. Let the level of B_i be $2/3+x$ with some $0<x\leq 1/12$. There are two items packed into B_j, their sizes are above $1/3-x\geq\frac{1}{4}$ (since they were not packed into B_i), and moreover one of them must be big (and the other one is either medium or big). As the bonus function is monotonically non-decreasing, the bonus of the smaller item is at least the bonus of an item of size $1/3-x$, and their total bonus is at least $\frac{2}{k}+\frac{2(2k-11)}{3k}\cdot(1/3-x)+\frac{10-k}{3k}=\frac{26+k-6x(2k-11)}{9k}$. We get

$$g(B_i)+s(B_i)+b(B_j)=\frac{2}{k}+\frac{2\,(2k-11)}{3k}\cdot(2/3+x)+\frac{26+k}{9k}-\frac{2x(2k-11)}{3k}=1.$$

The number of pairs i,j that are considered is the number of considered bins minus 1 and the claim follows. □

Since there is at most one 2-bin whose level is at most $\frac{2}{3}$, and all 2-bins with level above $\frac{3}{4}$ have weights of at least 1, we find that the total weight of 2-bins is at least their number minus 2.

Lemma 6.1.14 *The total weight of the* 3*-bins of levels in* $(\frac{3}{4},\frac{5}{6}]$ *is at least their number minus* 1.

Proof. Suppose that B_i and B_j are two consecutive 3-bins. We prove that $g(B_i)+s(B_i)+b(B_j)\geq 1$. Let the level of B_i be $3/4+x$ with some $0<x\leq 1/12$. Then there are three items in B_j, of sizes $a_1\geq a_2\geq a_3$, such that all are bigger than $1/4-x$, and in particular, all are bigger than $1/6$. At least one of them must be also bigger than $1/4$, otherwise the level of the bin is at most $3/4$. We have $g(B_i)+s(B_i)=\frac{3}{k}+\frac{2(2k-11)}{3k}\cdot(\frac{3}{4}+x)=\frac{2k-5}{2k}+\frac{2x(2k-11)}{3k}$. Thus, it is sufficient to show $b(B_j)\geq\frac{5}{2k}-\frac{2x(2k-11)}{3k}$. This holds if $a_2>\frac{1}{4}$, as the bonus of a medium or big item is at least $\frac{3}{2k}$. If none of the items is big, using $a_1+a_2+a_3\geq\frac{3}{4}$ we have $b(B_j)\geq\frac{2(2k-11)}{3k}(a_1+a_2+a_3)+2\cdot\frac{7-k}{3k}+\frac{10-k}{3k}\geq\frac{2(2k-11)}{3k}\cdot\frac{3}{4}+\frac{24-3k}{3k}=\frac{5}{2k}$.

We are left with the case that $a_1>\frac{1}{3}$, and $\frac{1}{4}-x<a_3\leq a_2\leq\frac{1}{4}$. The bonus of each small item is at least $\frac{2(1/4-x)(2k-11)}{3k}+\frac{7-k}{3k}$, and $b(B_j)\geq\frac{2}{k}+2(\frac{2(2k-11)}{3k}(\frac{1}{4}-x)+\frac{7-k}{3k})=\frac{3}{k}-2\frac{2x(2k-11)}{3k}=\frac{5}{2k}-\frac{2x(2k-11)}{3k}+\frac{1}{2k}-\frac{2x(2k-11)}{3k}$, and by using $x\leq\frac{1}{12}$, we get $\frac{1}{2k}-\frac{2x(2k-11)}{3k}\geq\frac{1}{2k}-\frac{2k-11}{18k}=\frac{20-2k}{18k}\geq 0$.
□

Since there is at most one 3-bin whose level is at most $\frac{3}{4}$, and all 3-bins with level above $\frac{5}{6}$ have weights of at least 1, we find that the total weight of 3-bins is at least their number minus 2.

Lemma 6.1.15 *The total weight of the* 4*-bins of levels in* $(\frac{4}{5}, \frac{5}{6}]$ *is at least their number minus* 1.

Proof. Suppose that B_i and B_j are two consecutive such 4-bins. We prove that $g(B_i) + s(B_i) + b(B_j) \geq 1$. Let the size of B_i be $5/6 - x$ with some $0 \leq x < 1/30$. Then there are four items in B_j, of sizes $a_1 \geq a_2 \geq a_3 \geq a_4$, all are bigger than $1/6 + x$. If any of them is also bigger than $1/4$, we have

$$g(B_i) + s(B_i) + b(B_j) \geq \frac{4}{k} + \frac{2\,(2k-11)}{3k} \cdot \frac{4}{5} + \frac{3}{2k} = \frac{32k-11}{30k} \geq 1,$$

since $k \geq 6$ and we are done. Otherwise all four items are small, and we get

$$g(B_i) + s(B_i) + b(B_j) \geq \frac{4}{k} + \frac{4\,(2k-11)}{3k} \cdot \frac{4}{5} + 4 \cdot \frac{(7-k)}{3k} = \frac{60 + 32k - 176 + 140 - 20k}{15k}$$
$$= \frac{12k+24}{15k} = \frac{4}{5} + \frac{8}{5k} \geq 1\,.$$

using the property that the levels of B_i and B_j are both above $\frac{4}{5}$ (which gives a lower bound on the total size of these items) and $k \leq 8$.

□

Since there is at most one 4-bin whose level is at most $\frac{4}{5}$, and all 4-bins with level above $\frac{5}{6}$ have weights of at least 1, we find that the total weight of 4-bins is at least their number minus 2.

We proved $FF(L) - 8 \leq W \leq (8/3 - 8/(3k))OPT(L)$.

Theorem 6.1.2 *The asymptotic approximation ratio of* FF *for any* $6 \leq k \leq 8$ *is at most* $8/3 - 8/(3k)$.

6.1.3 The case $k = 9$

We analyzed the cases $3 \leq k \leq 8$, and it remains to analyze the asymptotic approximation ratio for the cases $k \geq 9$, which are more complicated. We consider the case $k = 9$ separately, as the proofs for other values of k fail in this case. We combine methods from all other proofs here. The weighting function is similar to that is used in the previous section, in the sense that it has discontinuity points at $1/6$ and $1/3$. It is also similar to the weighting function used in the next section as the intervals for small and medium sizes are divided to two parts in the same way. Items that are packed in k-bins in the packing of FF will be treated separately. These items are called α-items, and the weight of every such item will be equal to $\frac{1}{k}$ in all remaining cases. Items that are not α-items will be called *additional* items. We will distinguish the bins of OPT according to the number of additional items packed into them, and define weights based on this. Since α-items always have weights of $\frac{1}{k}$, bins (of OPT and of FF) that contain k such items will have weights of exactly 1. Thus, in the analysis we focus on bins of OPT having at least one additional item. We have different weights for the cases of a small number of additional items (one or two), and a larger number of additional items (at least three). As we will see later, the worst-case examples consist of optimal solutions where every bin has two or three additional items. This separation in the definition of weights is needed due to the case that a bin of OPT has two additional items, where one of them is huge, and it also has $k - 2$ α-items. In this case, the smaller additional item has a special role. This item cannot have a very large weight even if its size is almost $\frac{1}{2}$. Instead of splitting the item classes further (which would result in many additional cases in the proofs), we distinguish such items from other items of similar sizes. We now define the weights.

Case a. Consider bins of OPT containing one or two additional items (and the remaining items are α-items). Such bins are called γ-bins, and the additional items packed into such bins (in OPT) are called γ-items. Huge γ items (called γ_1-items) have weights of 1, and other γ items (called γ_2-items) have weights of $\frac{16}{27}$. Obviously, a bin that has two γ-items cannot have two γ_1-items, but it may have two γ_2-items.

Case b. Consider the other bins of OPT (each containing *at least* three additional items). Each such bin has at most six α-items, we call it a ϕ-bin, and its additional items are called ϕ-items. The weighting function of the ϕ-items is more complicated. The weight of any huge ϕ-item is 1 as usual. The weight of a ϕ-item of size $a \le 1/2$ is $w(a) = s(a) + b(a)$, where $s(a) = \frac{32}{27}a$ is called the scaled size, and $b(a)$ is the bonus of the item. Note that there is no ground weight in this case. Below we give the bonus function of the ϕ-items of sizes no larger than $1/2$. The functions $b(a)$ and $w(a)$ are piecewise linear, and the breakpoints where it is continuous are $1/5$ and $3/10$. The small items and medium items are split into two classes.

$$b(a) = \begin{cases} 0 & \text{if} \quad a \le 1/6 \\ \frac{32}{27}(a - \frac{1}{6}) + \frac{1}{81} = \frac{32}{27}a - \frac{5}{27} & \text{if} \quad 1/6 < a \le 1/5 \\ -\frac{28}{27}(a - \frac{1}{5}) + \frac{7}{135} = -\frac{28}{27}a + \frac{7}{27} & \text{if} \quad 1/5 < a \le 1/4 \\ -\frac{28}{27}(a - \frac{1}{4}) + \frac{1}{9} = -\frac{28}{27}a + \frac{10}{27} & \text{if} \quad 1/4 < a \le 3/10 \\ \frac{32}{27}(a - \frac{3}{10}) + \frac{8}{135} = \frac{32}{27}a - \frac{8}{27} & \text{if} \quad 3/10 < a \le 1/3 \\ \frac{1}{9} & \text{if} \quad 1/3 < a \le 1/2 \end{cases}$$

where we call the items in the classes after each other as tiny, very small, larger small, smaller medium, larger medium and big, respectively.

The value of the bonus is zero if $a \le 1/6$ and it is constant ($\frac{1}{9}$) between $1/3$ and $1/2$. The bonus function is not continuous at the points $1/6$, $1/4$, and $1/3$, it is monotonically increasing in $(1/6, 1/5)$ and in $(3/10, 1/3)$, and it is monotonically decreasing in $(1/5, 3/10)$ (which is less typical for weight functions). Nevertheless, the weight function remains monotonically increasing for the whole interval $0 < a \le 1/2$, and the value of the bonus is nonnegative for the whole interval.

We state several additional properties of the bonus function. For small items (very small and larger small items, i.e., items of sizes in $(1/6, 1/4]$), the maximum value of the bonus is given for $a = 1/5$, and the bonus at this point is $7/135$. The bonus of very small items is at least $\frac{1}{81}$, and the smallest bonus of larger small items is zero. For smaller medium items, the bonus decreases from $\frac{1}{9}$ to $\frac{8}{135}$, and for larger medium items, the bonus increases from $\frac{8}{135}$ to $\frac{8}{81}$. The weight of a big ϕ-item is at least $41/81$, the weight of a ϕ-item with size more than $1/4$ is at least $11/27$, and for any ϕ-item with size $0 < a \le 1$, and for any γ_2-item, the next inequality holds: $w(a) \ge \frac{32}{27}a > \frac{7}{6}a$. This is true since bonuses of ϕ-items are non-negative, and since the size of any γ_2-item is at most $\frac{1}{2}$, while its weight is $\frac{16}{27}$.

Properties of the weighting and the asymptotic bound

Lemma 6.1.16 *For every bin B of OPT, $w(B) \le 8/3 - 8/27 = 2 + \frac{10}{27} = \frac{64}{27}$ holds.*

Proof. Consider the case that B is a γ-bin. In this case B has one item that is a γ_1-item or a γ_2-item, that is, its weight is 1 or $\frac{16}{27}$, possibly also a γ_2-item of weight $\frac{16}{27}$, and each remaining item is an α-item and has weight $\frac{1}{9}$. Thus, the total weight is at most $1 + \frac{16}{27} + \frac{7}{9} = \frac{64}{27}$. It remains to consider the case that B is a ϕ-bin. It contains at most six α-items, of total weight at most $6/9$. Thus it suffices to show that the total weight of the ϕ-items is at most $\frac{64}{27} - \frac{6}{9} = \frac{46}{27}$ (no matter how many ϕ-items there are), or that it is at most $\frac{64}{27}$ minus $\frac{1}{9}$ times the number of α-items.

First, assume that B contains no huge item. The total scaled size of the ϕ-items is at most $32/27$. It suffices to show that the total bonus of ϕ-items the bin is at most $46/27-32/27 = 14/27$. Since the bonus is zero if the size of the item is at most $1/6$, it follows that at most five items can have positive bonuses. Moreover at most three items can have sizes above $1/4$, and the bonus of each such item is at most $\frac{1}{9}$, while the bonus of any other item is at most $7/135$. Thus the total bonus of the bin is at most $3 \cdot \frac{1}{9} + 2 \cdot \frac{7}{135} = 59/135 < 14/27$.

Next, assume that B contains a huge item. The weight of a huge item is exactly 1. We will show that if there are six α-items, then the total weight of the further additional items of B is at most $19/27$, and consider also the case that the number of α items is smaller. Since the total size of remaining additional items is below $\frac{1}{2}$, their scaled size is at most $\frac{16}{27}$, and it suffices to show that their total bonus is at most $19/27 - 16/27 = 1/9$. Since only items of size above $\frac{1}{6}$ have positive bonuses, there can be at most two further items in the bin having positive bonuses. If there is only one further item having positive bonus, we are done, since no bonus is above $\frac{1}{9}$. If there are two items with bonuses, but there are at most five α-items, then the total weight of α-items is at most $\frac{5}{9}$, and we are done as well. Thus, it is left to consider the case where there are two further ϕ-items in the bin both having positive bonuses, and there are no other ϕ-items packed into B except for the huge item and these two items. Let their sizes be denoted as a_1 and a_2, where $1/6 < a_1 \leq a_2$, and thus $a_2 < 1/3$ as $a_1 + a_2 < 1/2$. The claim holds if $a_2 \leq 1/4$, since then the total bonus is at most $2 \cdot 7/135 = \frac{14}{135} < \frac{1}{9}$. Thus the only remaining case is where the item of size a_1 is small and the item of size a_2 is a medium item. We will show $w(a_1) + w(a_2) \leq \frac{19}{27}$. There are three cases, as $a_1 > \frac{1}{5}$ and $a_2 > 0.3$ cannot hold simultaneously. In all cases $s(a_1) + s(a_2) = \frac{32}{27}(a_1 + a_2)$ and $a_1 + a_2 < \frac{1}{2}$.

If $a_1 \leq \frac{1}{5}$ and $a_2 \leq \frac{3}{10}$, $w(a_1)+w(a_2) = \frac{32}{27}(a_1+a_2)+\frac{32}{27}a_1 - \frac{28}{27}a_2 + \frac{5}{27} = \frac{64}{27}a_1 + \frac{4}{27}a_2 + \frac{5}{27} \leq \frac{64}{27}a_1 + \frac{4}{27}(\frac{1}{2} - a_1) + \frac{5}{27} = \frac{60}{27}a_1 + \frac{7}{27} \leq \frac{19}{27}$. If $a_1 > \frac{1}{5}$ and $a_2 \leq \frac{3}{10}$, $w(a_1) + w(a_2) = \frac{32}{27}(a_1 + a_2) - \frac{28}{27}(a_1 + a_2) + \frac{17}{27} = \frac{4}{27}(a_1 + a_2) + \frac{17}{27} \leq \frac{19}{27}$. If $a_1 \leq \frac{1}{5}$ and $a_2 > \frac{3}{10}$, $w(a_1) + w(a_2) = \frac{32}{27}(a_1 + a_2) + \frac{32}{27}(a_1 + a_2) - \frac{13}{27} = \frac{64}{27}(a_1 + a_2) - \frac{13}{27} \leq \frac{19}{27}$. $\square$

Now, we will analyze the total weight of the bins of FF. Once again, we split the analysis according to the number of items in these bins. The 9-bins have weight of 1, and any bin with a huge item (a ϕ-item or a γ_1-item) has a weights of at least 1. Thus, we neglect all bins containing items of size above $\frac{1}{2}$ from the analysis. At most one 1-bin is left, and we neglect that bin (if it exists) as well. In what follows we analyze 2^+-bins of FF that contain items of sizes in $(0, \frac{1}{2}]$. These bins only contain ϕ-items and γ_2-items.

Lemma 6.1.17 *The weight of any bin with level above $\frac{6}{7}$ is at least* 1. *There is at most one 6^+-bin whose weight is below* 1.

Proof. For any ϕ-item and for any γ_2-item, the weight of the item is at least $\frac{32}{27}$ times the size of the item. Since except for at most one bin, the level of 6^+-bin is above $\frac{6}{7}$ (by Lemma 6.1.1), the weights of these bins, except for at most one bin, are no smaller than 1 as $\frac{32}{27} \cdot \frac{6}{7} > 1$. $\square$

In the following we concentrate on the 2-bins, 3-bins, 4-bins, and 5-bins. We start with analyzing bins containing a γ_2-item.

Lemma 6.1.18 *Except for at most four bins, the weight of any bin whose number of items is in* $[2,5]$*, and that has a* γ_2*-item, is at least* 1*.*

Proof. A bin that has two γ_2-items has weight at least $2 \cdot \frac{16}{27} > 1$, and thus we consider bins that contain one γ_2-item and the remaining (at least one and at most four) items are ϕ-items. Assume by contradiction that there are at least five bins with γ_2-items having weights strictly below 1.

Any bin that has a γ_2-item and a ϕ-item of size above $\frac{1}{4}$ has total weight of at least 1, since any ϕ-item of size above $\frac{1}{4}$ has weight at least $\frac{11}{27}$, and each γ_2-item has weight $\frac{16}{27}$. Moreover, since the weight of any item is at least $\frac{32}{27}$ times its size, if the level of a bin is at least $\frac{27}{32}$, then its total weight is at least 1. By Lemma 6.1.2, except for at most two 2-bins, every 2-bin has only items of sizes above $\frac{1}{4}$, and therefore they have weights of at least 1. We find that there exist three bins with 3, 4, or 5 items each, levels below $\frac{27}{32}$, and weights below 1. Let such three bins be denoted by B_i, B_j, and B_r (where the bins appear in this order in the sequence of bins of FF). Any item of B_j and B_r has size above $\frac{5}{32}$, as these item could not be packed into B_i. Such an item has weight of at least $\frac{5}{27}$, so a bin with a γ_2-item and weight below 1 can have at most two such items. Thus, B_j and B_r are 3-bins.

We split the analysis to the cases where B_j has a level above $\frac{5}{6}$, and the case that it does not. If it has a level above $\frac{5}{6}$, then the total size of its ϕ-items is above $\frac{1}{3}$ (as the γ_2-item has size of at most $\frac{1}{2}$), and at least one of them has size above $\frac{1}{6}$. If at least one of the two ϕ-items has size above $\frac{1}{5}$, then their total size is above $\frac{5}{32} + \frac{1}{5} = \frac{57}{160}$, and the weight of all items is at least $\frac{16}{27} + \frac{32}{27} \cdot \frac{57}{160} > 1$. Otherwise, there is an item that has a bonus of at least $\frac{1}{81}$, and the total weight is at least $\frac{16}{27} + \frac{32}{27} \cdot \frac{1}{3} + \frac{1}{81} = 1$. Therefore, we find that B_j has a level of at most $\frac{5}{6}$. However, in this case the two ϕ-items of B_r have sizes above $\frac{1}{6}$. The weight of such an item is at least $\frac{32}{27} \cdot \frac{1}{6} + \frac{1}{81} = \frac{17}{81}$, and the total weight of B_r is at least $\frac{16}{27} + 2 \cdot \frac{17}{81} > 1$, a contradiction again. $\square$

It is left to analyze with bins containing only ϕ-items of sizes at most $\frac{1}{2}$, that are 2-bins, 3-bins, 4-bins, and 5-bins. Before analyzing their total weights we discuss some properties of ϕ-items.

Lemma 6.1.19 *Consider two* ϕ*-items of sizes* $a_1 \le a_2 \le 1/2$*. If* $1 \ge a_1 + a_2 > 1 - a_1$ *holds, then the total weight of the two items is at least* 1*.*

Proof. We have $a_1 > \frac{1-a_2}{2} \ge \frac{1}{4}$. If $a_1 > \frac{1}{3}$, then both items are big, and since the weight of any big item is at least $\frac{41}{81} > 1/2$, the claim holds in this case. Next, we consider the case $1/4 < a_1 \le 1/3$, where $a_2 > 1 - 2a_1 \ge 1/3$, thus the item of size a_2 is big.

If a_1 is smaller medium, then let $a_1 = 1/4 + x$ for some $0 < x \le 1/20$, and $a_1 + a_2 > 1 - a_1 = \frac{3}{4} - x$. The total weight of the items is

$$\frac{32}{27}(a_1 + a_2) + b(a_1) + b(a_2) \ge \frac{32}{27}(3/4 - x) - \frac{28}{27}x + \frac{1}{9} + \frac{1}{9} = \frac{10}{9} - \frac{20}{9}x \ge 1.$$

If a_1 is larger medium, then let $a_1 = 3/10 + x$ for some $0 < x \le 1/30$, and $a_1 + a_2 > 1 - a_1 = \frac{7}{10} - x$. We get

$$\frac{32}{27}(a_1 + a_2) + b(a_1) + b(a_2) \ge \frac{32}{27}(7/10 - x) + \frac{32}{27}x + \frac{8}{135} + \frac{1}{9} = 1.$$

□

Lemma 6.1.20 *Consider three ϕ-items of sizes $a_1 \le a_2 \le a_3 \le 1/2$. If $1 \ge a_1 + a_2 + a_3 > 1 - a_1$ holds, then the total weight of the three items is at least* 1.

Proof. We have $\frac{3}{2}(a_1 + a_2) \ge 2a_1 + a_2 > 1 - a_3 \ge \frac{1}{2}$, thus $a_1 + a_2 > \frac{1}{3}$.

If $a_1 > \frac{1}{4}$, then the claim holds since the weight of an item with size above $1/4$ is at least $11/27$ (so the total weight is at least $\frac{11}{9}$). In what follows we assume that $a_1 \le 1/4$, and thus $a_1 + a_2 + a_3 > 1 - a_1 \ge \frac{3}{4}$. If the largest item is big, then its bonus is $\frac{1}{9}$, and the total weight of the three items is at least $\frac{32}{27}(a_1 + a_2 + a_3) + \frac{1}{9} \ge \frac{32}{27} \cdot \frac{3}{4} + \frac{1}{9} = 1$. If the largest item is not big, i.e., $a_3 \le \frac{1}{3}$, then using $\frac{2}{3} \ge 2a_3 \ge a_2 + a_3 > 1 - 2a_1 \ge \frac{1}{2}$ we get $a_1 > \frac{1}{6}$ and $a_3 > \frac{1}{4}$, thus the smallest item is small, and the largest item is medium. If there are two medium items, then the bonus of each one of them is at least $\frac{8}{135}$, and the total weight is at least $\frac{32}{27} \cdot \frac{3}{4} + \frac{16}{135} = \frac{136}{135} > 1$. We are left with the case where the smallest two items are small and the largest item is medium. We find that $2a_1 > 1 - a_2 - a_3 \ge 1 - \frac{1}{4} - \frac{1}{3} = \frac{5}{12}$, thus the smallest item is larger small, and so is the item of size a_2.

If the largest item is smaller medium, we have a total weight of $\frac{32}{27}(a_1 + a_2 + a_3) - \frac{28}{27}(a_1 + a_2 + a_3) + \frac{24}{27} \ge \frac{4}{27} \cdot \frac{3}{4} + \frac{24}{27} = 1$. If the largest item is larger medium, we will use $a_1 + a_2 > \frac{2}{3}(1 - a_3)$. The total weight is at least $\frac{32}{27}(a_1 + a_2 + a_3) - \frac{28}{27}(a_1 + a_2) + \frac{14}{27} + \frac{32}{27}a_3 - \frac{8}{27} = \frac{4}{27}(a_1 + a_2) + \frac{64}{27}a_3 + \frac{2}{9} \ge \frac{4}{27} \cdot \frac{2}{3}(1 - a_3) + \frac{64}{27}a_3 + \frac{2}{9} = \frac{184}{81}a_3 + \frac{26}{81} \ge \frac{184}{81} \cdot \frac{3}{10} + \frac{26}{81} = \frac{406}{405} > 1$. □

Lemma 6.1.21 *Consider four ϕ-items of sizes $a_1 \le a_2 \le a_3 \le a_4 \le 1/2$. If $1 \ge a_1 + a_2 + a_3 + a_4 > 1 - a_1$ holds, then the total weight of the four items is at least* 1.

Proof. First, consider the case $a_1 \le \frac{1}{6}$. In this case $a_1 + a_2 + a_3 + a_4 > 1 - a_1 \ge \frac{5}{6}$, and if at least one item is very small, medium, or big, then there is at least one item with a bonus of at least $\frac{1}{81}$, and the total weight is at least $\frac{32}{27} \cdot \frac{5}{6} + \frac{1}{81} = 1$. Otherwise, all items are larger small and tiny. At least three items must be larger small as $2a_1 + a_2 + a_3 + a_4 > 1$, which is impossible in the case where $a_2 \le \frac{1}{6}$ and $a_4 \le \frac{1}{4}$. The total weight if all four items are larger small is at least $\frac{4}{27} \cdot (a_1 + a_2 + a_3 + a_4) + \frac{28}{27} > 1$. Otherwise, the total weight is at least $\frac{32}{27}a_1 + \frac{4}{27} \cdot (a_2 + a_3 + a_4) + \frac{21}{27} > \frac{32}{27}a_1 + \frac{4}{27} \cdot (1 - 2a_1) + \frac{21}{27} = \frac{8}{9}a_1 + \frac{25}{27}$. Since the three largest items are larger small, $a_2 + a_3 + a_4 \le \frac{3}{4}$, and $2a_1 > 1 - (a_2 + a_3 + a_4) \ge \frac{1}{4}$, so $a_1 > \frac{1}{8}$. We find that the total weight is at least $\frac{28}{27} > 1$.

Next, consider the case $a_1 > 1/6$. It follows that the total weight of the three smallest items is at least $\frac{32}{27} \cdot \frac{3}{6} = \frac{16}{27}$. If the biggest item is bigger than $1/4$, the total weight is at least $\frac{16}{27} + \frac{11}{27} = 1$. It is left to consider only the case where $1/6 < a_1 \le a_2 \le a_3 \le a_4 \le 1/4$, i.e., all items are small.

Note that the weight of a larger small item is at least $\frac{32}{27} \cdot \frac{1}{5} + \frac{7}{135} = \frac{13}{45}$. If all four items are larger small, then their total weight is above 1. Otherwise, the smallest item is very small. The total size of the items is above $1 - a_1$, and the bonus of the smallest item is $\frac{32}{27}a_1 - \frac{5}{27}$. The total weight of the four items is at least $\frac{32}{27}(1 - a_1) + \frac{32}{27}a_1 - \frac{5}{27} = 1$. □

Lemma 6.1.22 *Consider five ϕ-items of sizes $a_1 \le a_2 \le a_3 \le a_4 \le a_5 \le 1/2$. If $1 \ge a_1 + a_2 + a_3 + a_4 + a_5 > 1 - a_1$ holds, then the total weight of the five items is at least* 1.

Proof. If $a_1 \leq \frac{1}{6}$, then $a_1+a_2+a_3+a_4+a_5 > 1-a_1 \geq \frac{5}{6}$ holds. Otherwise, $a_1+a_2+a_3+a_4+a_5 \geq 5a_1 > \frac{5}{6}$ holds too. If at least one item is not larger small, then its bonus is at least $\frac{1}{81}$ and the total weight is at least $\frac{32}{27} \cdot \frac{5}{6} + \frac{1}{81} = 1$. If all items are larger small, then their total size is above 1, contradicting the assumption. □

Lemma 6.1.23 *The total weight of the* 2*-bins,* 3*-bins,* 4*-bins, and* 5*-bins, containing* ϕ*-bins is at least their number minus* 1*.*

Proof. Consider the bins of FF whose numbers of items is in $[2, 5]$, that contain only ϕ-items, and their weights are below 1. Obviously these bins have no huge items. If the level of a given bin is bigger than 1 minus the size of the smallest item in the bin, then the weight of the bin is at least 1 by the previous lemmas. Thus, we only consider bins that do not satisfy this property. If there is at most one bin to consider, then we are done. Otherwise, in the list of remaining bins, consider two consecutive bins B_i and B_j (such that B_j appears after B_i in the ordering of FF). Let i_1 denote the smallest item of B_i and j_1 the smallest item of B_j (breaking ties in favor of items of smaller indices). Let $S = s(B_i)$. Consider the set X consisting of j_1 and the items of B_i excluding i_1. We will show $w(X) \geq 1$. Applying this property to every such consecutive pair of bins will show that the total weight is at least the number of bins in the list of remaining bins minus 1. If $S - s_{i_1} + s_{j_1} > 1$, then their total weight is above $\frac{32}{27} > 1$. Otherwise, we have the following properties. First, $s_{j_1} > 1 - S$ since j_1 was not packed into B_i. Additionally, by assumption, $S \leq 1 - s_{i_1}$. Therefore $s_{j_1} > s_{i_1}$. Let s' be the size of the smallest item in X. We have $s' \geq s_{i_1}$ as no item of B_i is smaller than s_{i_1} and $s_{j_1} > s_{i_1}$. We find, $s(X) = (S - s_{i_1} + s_{j_1}) + s' \geq S + s_{j_1} > 1$. Thus, the set X satisfies the condition of one of Lemmas 6.1.19, 6.1.20, 6.1.21, 6.1.22 (the lemma where the considered number of items is equal to that of this set - which is equal to the number of items of B_i and therefore it is in $\{2, 3, 4, 5\}$), and the total weight of this set is at least 1. □

We proved $FF(L) - 7 \leq W \leq (64/27)OPT(L)$.

Theorem 6.1.3 *The asymptotic approximation ratio of* FF *for* $k = 9$ *is at most* $64/27$*.*

6.1.4 The cases $k \geq 10$

The cases where $k \geq 10$ are studied similarly to previous cases, thus, to shorten the length of the dissertation, this last part of the investigation is put into Appendix D. The main result of this omitted part is summarized in the next theorem:

Theorem 6.1.4 *The asymptotic approximation ratio of* FF *for any* $k \geq 10$ *is at most* $2.7 - 3/k$*.*

6.2 The tight lower bounds for FF

Here we complete the analysis and provide examples showing that the asymptotic approximation ratios cannot be smaller that the bounds proved in the previous section. For completeness we include a lower bound for $k = 2$ as well. In the statement of the theorem the cases $k = 4, 10$ are included in two cases.

Theorem 6.2.1 *The asymptotic approximation ratio of FF satisfies the following properties.*

- *It is at least* $2.5 - \frac{2}{k}$ *for* $k = 2, 3, 4$.
- *It is at least* $\frac{8(k-1)}{3k} = \frac{8}{3} - \frac{8}{3k}$ *for* $4 \leq k \leq 10$.
- *It is at least* $2.7 - \frac{3}{k}$ *for* $k \geq 10$.

Proof. **The cases** $2 \leq k \leq 4$**.**

Let $\ell \geq 0$ be a large integer, and let $0 < \varepsilon < \frac{1}{9k}$. Consider an input consisting of $2k(k-2)\ell$ items of size ε each (smallest items), $2k\ell$ items of size $\frac{1}{2} - k\varepsilon > \frac{1}{3}$ each (medium size items), and $2k\ell$ items of size $\frac{1}{2} + \varepsilon$ each (largest items). The items are presented in this order. FF creates $2(k-2)\ell$ bins containing k smallest items each. Then, as further items are larger than $\frac{1}{3}$, FF creates $k\ell$ bins containing pairs of medium size items, and as the remaining items are larger than $\frac{1}{2}$, the largest items are packed into $2k\ell$ dedicated bins. For this input L_ℓ, $OPT(L_\ell) = 2k\ell$, since it is possible to pack a largest item, a medium size item, and $k-2$ smallest items into a bin as $\frac{1}{2} + \varepsilon + \frac{1}{2} - k\varepsilon + (k-2)\varepsilon < 1$, while $FF(L_\ell) = 2(k-2)\ell + k\ell + 2k\ell = 5k\ell - 4\ell$. This shows that the asymptotic approximation ratio of FF is at least $2.5 - \frac{2}{k}$, that is, at least $\frac{11}{6}$ for $k = 3$ and at least 2 for $k = 4$. The example is valid for $k = 2$ too, giving the value 1.5 (in this case there are no smallest items).

The cases $5 \leq k \leq 10$**.**

Let ℓ be a positive integer divisible by k, let $0 < \varepsilon < \frac{1}{120}$ and $\delta < \frac{\varepsilon}{3^{\ell+4}}$ be small positive values, and consider the following input. There are 3ℓ items of size $\frac{1}{2} + \delta$, ℓ items of size $\frac{1}{2} - 10\delta$, ℓ items of size $\frac{1}{4} + 20\delta$, ℓ items of size $\frac{1}{4} - 30\delta$, $(3k-8)\ell$ items of size δ, and for $1 \leq p \leq \ell$ there is a pair of items of sizes $\frac{1}{4} + \frac{\varepsilon}{3^p}$ and $\frac{1}{4} - \frac{\varepsilon}{3^p} - 10\delta$. Since $\delta < \varepsilon < \frac{1}{120}$, all sizes are strictly positive. An optimal solution has three types of bins. There are ℓ bins with an item of size $\frac{1}{2} + \delta$, an item of size $\frac{1}{2} - 10\delta$, and $k - 2 \leq 8$ items of size δ each. There are ℓ bins with an item of size $\frac{1}{2} + \delta$, an item of size $\frac{1}{4} + 20\delta$, an item of size $\frac{1}{4} - 30\delta$ and $k - 3 \leq 7$ items of size δ each. Finally, there are ℓ bins, where the pth bin has an item of size $\frac{1}{2} + \delta$, the pair of items of sizes $\frac{1}{4} + \frac{\varepsilon}{3^p}$ and $\frac{1}{4} - \frac{\varepsilon}{3^p} - 10\delta$, and $k - 3 \leq 7$ items of size δ each. Remove the items of sizes $\frac{1}{4} + \frac{\varepsilon}{3^\ell}$ and $\frac{1}{4} - 10\delta - \frac{\varepsilon}{3}$, and one item of size $\frac{1}{4} - 30\delta$ from the input. Obviously, an optimal solution still requires at most 3ℓ bins. For $1 \leq p \leq \ell - 1$, the items of sizes $\frac{1}{4} + \frac{\varepsilon}{3^p}$ and $\frac{1}{4} - 10\delta - \frac{\varepsilon}{3^{p+1}}$ are called a modified pair of index p.

The items are presented to FF in the following order. First, all items of size δ are presented and packed into $(3k-8)\frac{\ell}{k}$ bins that cannot be used again. Next, for $1 \leq p \leq \ell - 1$, the modified pair of items of index p is presented, followed by an item of size $\frac{1}{4} - 30\delta$. The total size of these three items is $\frac{1}{4} + \frac{\varepsilon}{3^p} + \frac{1}{4} - 10\delta - \frac{\varepsilon}{3^{p+1}} + \frac{1}{4} - 30\delta = \frac{3}{4} - 40\delta + \frac{2\varepsilon}{3^{p+1}} > \frac{3}{4} - \frac{40\varepsilon}{3^{\ell+4}} + \frac{2\varepsilon}{3^{p+1}} \geq \frac{3}{4} - \frac{40\varepsilon}{3^{p+5}} + \frac{2\varepsilon}{3^{p+1}} > \frac{3}{4} + \frac{\varepsilon}{2 \cdot 3^p}$, while further items have sizes of $\frac{1}{2} + \delta > \frac{1}{2} - 10\delta > \frac{1}{4} + 20\delta > \frac{1}{4}$, $\frac{1}{4} + \frac{\varepsilon}{3^{p'}} > \frac{1}{4}$, $\frac{1}{4} - 30\delta > \frac{1}{4} - 10\delta - \frac{20\varepsilon}{3^{\ell+4}} \geq \frac{1}{4} - 10\delta - \frac{\varepsilon}{3^{\ell+1}} > \frac{1}{4} - 10\delta - \frac{\varepsilon}{3^{p'+1}}$, and $\frac{1}{4} - 10\delta - \frac{\varepsilon}{3^{p'+1}}$, where $p' \geq p+1$ (there are additional modified pairs arriving later only if $p < \ell - 1$). We have $\frac{3}{4} + \frac{\varepsilon}{2 \cdot 3^p} + \frac{1}{4} - \frac{\varepsilon}{3^{p'+1}} - 10\delta \geq 1 + \frac{\varepsilon}{2 \cdot 3^p} - \frac{\varepsilon}{3^{p+2}} - \frac{10\varepsilon}{3^{p+5}} = 1 + \frac{\varepsilon(3^5/2 - 3^3 - 10)}{3^{p+5}} > 1$. This proves that after a bin of a modified pair and an item of size $\frac{1}{4} - 30\delta$ is created, no other items can be packed into that bin. When no modified pairs remain, pairs of items of sizes $\frac{1}{2} - 10\delta$ and $\frac{1}{4} + 20\delta$ are presented (there are ℓ such pairs). Each bin receives such a pair, whose total size is $\frac{3}{4} + 10\delta$. Since all remaining items have sizes above $\frac{1}{4}$, each created bin will not be used for other items. Finally, all remaining items (of sizes $\frac{1}{2} + \delta$) are packed into dedicated bins. The total number of bins is $(3k-8)\frac{\ell}{k} + \ell - 1 + \ell + 3\ell = \frac{8k-8}{k}\ell - 1$. Since an optimal solution has at most 3ℓ bins, we find that the asymptotic approximation ratio is at least $\frac{8(k-1)}{3k}$.

The cases $k \geq 10$.

After having got the tight lower bound construction for standard bin packing, given in Theorem 4.3.2 (in the chapter that deals with algorithm FF for standard bin packing), we can now apply this construction for the cardinality constrained model. In that place of the dissertation there is an input for any $OPT = 10m$, for which $FF = 17m$. (We changed the letter k to m in the claim, since in that chapter k is "only" an integer, but in this chapter k has a special meaning.)

We adapt this lower bound example of FF by adding a large number of tiny items. The original construction is such that every bin of OPT has a big item whose size is $\frac{1}{2} + \varepsilon$, and it holds that every optimal bin contains exactly 3 items. We replace the big items with slightly smaller big items of sizes $\frac{1}{2} + \varepsilon/2$. Then, any optimal bin receives also $k-3$ tiny items of sizes $\frac{\varepsilon}{2k}$. Naturally, this modification is possible both with respect to size and to the number of packed items. The tiny items are presented to FF before other items, so they are packed into bins containing k items each, that cannot be used for other items.

The items of sizes $\frac{1}{2} + \varepsilon/2$ are presented last and must be packed into dedicated bins, as any previous bin either has k items or total size above $\frac{1}{2}$. Thus, the modified construction gives a lower bound of $1.7 + \frac{k-3}{k} = 2.7 - \frac{3}{k}$ on the asymptotic approximation ratio of FF. □

Chapter 7

Batched Bin Packing and Graph-Bin Packing

The results of this chapter are from [22], thus the contribution of this chapter is reached by the author of the dissertation. In this last chapter we revisit the Batched Bin Packing problem (abbreviated as BBP). In this model items come in K consecutive batches, and the items of the earlier batches must be packed without any knowledge of later batches. The model is introduced in [48]. Let L be the set of items where $L = B_1 \cup ... \cup B_K$, where $B_i \cap B_j = \varnothing$ if $i \neq j$. Note that for any i, B_i may be empty. We say that B_i is the i-th *batch* of the input. It is assumed that for any $1 \leq i < j \leq K$, the i-th batch is revealed before the j-th batch. As soon as a batch is revealed, the items in the batch must be irrevocably packed, this part of the packing procedure being called the i-th phase. If $K = 1$, we get back to the offline packing problem. If every batch contains only one item, we get the online packing problem (with the only difference being that the number of items is known). Thus the *batched bin packing* problem is in some sense a common generalization of the offline and online bin packing problem. It seems that no other paper has considered this model so far, except [22], the work of the present author.

We give the first approximation algorithm for the case $K = 2$, with tight asymptotic approximation ratio 1.5833, while the known lower bound of the model is 1.378.

Let us consider some possible applications: An office moves from one building to another one. There are two rooms in the office, an inner and an outer room. It is possible to carry out the furniture from the inner room only through the outer room. It is very important that the documents of the staff of one room should not be mixed up with the documents of the other room, thus the staff make the decision that first all the furniture (and documents therein) from the outer room are carried out and packed into several trucks, and only after this can the remaining furniture from the inner room be handled in the same way. Here the furniture of the two rooms form the batches, and the trucks play the role of the bins.

Another situation that may occur is: A factory moves from one country to another country. First the machines (i.e. items) are transported by train to a transfer point, then the machines are unpacked from the wagons, and they are packed into trucks, because the target point is among the hills and there is no railway to the destination. Since the factory is large, the items of the factory are transported by several trains, hence it is possible that a part of the input arrives at the transfer point on Monday, the next batch arrives on Tuesday, and so on. Then it is natural to start to pack the first batch on Monday. Then on Tuesday, the next batch is packed. At any time when a truck is full, the items are transported.

Note that in the above-mentioned applications, two versions of the batched bin packing can be distinguished. In the first application (moving) the bins of the first batch *cannot* be used to pack the items of the next batch, but in the next application (transporting items of a factory) they are *allowed* to be used. We call these two versions the *augmenting* model (the bins that are used in the i-th phase can also be used in the j-th phase, for any $i < j$, to pack other items if they fit), while in the *disjunctive* model the bins that are used in some phase cannot be used in any later phase. (At this point the latter model does not seem to be very interesting or attractive, but we will need this model later on.) Thus in the augmenting model the algorithms can choose to combine items from different batches, while in the disjunctive model they cannot.

The algorithms which are applied in the two different models are called augmenting and disjunctive algorithms, respectively. Note that a disjunctive algorithm simply packs the batches independently. The asymptotic approximation ratio of an (augmenting or disjunctive) algorithm A is defined in an appropriate way. The number of the bins used by algorithm A is compared with the solution of an offline optimal algorithm OPT. Note that OPT is allowed to pack together the items from different batches.

For the BBP problem, in [48] the authors just investigate the augmenting model in the special case when $K = 2$. The authors prove that $R_{as}(A) \geq r \approx 1.3871$ is a lower bound for the problem, where r is a solution of equation $r/(r-1) - 3 = \ln r/(2r-2)$, thus the asymptotic approximation ratio of any algorithm A is at least this value. We approach the problem from the opposite side: We give the first algorithm for the same special case, when $K = 2$. The tight asymptotic approximation ratio of the algorithm is $19/12 \approx 1.5833$, thus it remains below the asymptotic approximation ratio of the best-known BP algorithm (i.e. the algorithm SH of Seiden which has asymptotic approximation ratio of 1.58889, see [69]). Comparing to SH, our algorithm is very simple. We define it for the disjunctive model, but it can be also applied for the augmenting model in a natural way.

Another model and connection: an improved result. The *bin packing with conflicts* (BPC for short) is another generalization of the BP problem: several pairs of items are in conflict, which means that the two items are not allowed to be packed into the same bin; see e.g. [52] or [41]. A more general version called the *graph-bin packing* problem (abbreviated by GBP) is defined in [11], and (the simplified version of this problem) is as follows. Given a graph, with lower and upper bounds on the edges and weights on the points, the weight of a point is called the size. The points of the graph, also called items, are to be packed into unit capacity bins. The total size of the items in any bin can be at most 1, as usual. But some additional constraints must also be satisfied. Namely, given any two points, say a and b, if they are connected in the graph by an edge, and the lower and upper bounds of this edge are l and u, respectively, and a and b are packed into some bins B_i and B_j, respectively, then the indices of the bins must satisfy $l \leq |i - j| \leq u$. Note that for the BPC problem, $u = \infty$ and $l \in \{0, 1\}$ for any edge.

In the case of the GBP problem [11], among several results, an approximation algorithm is given with absolute approximation ratio 3, for the special case where there are only lower bounds on the edges (i.e. $u = \infty$ for any edge), and the graph has chromatic number 2, i.e. it is bipartite. Surprisingly, with the application of our algorithm (which we defined for the BBP problem), we are able to get an improved algorithm for the graph-bin packing problem for this special case. We improve the previous 3 upper bound to 2.5833, not just in the asymptotic, but also in the absolute sense.

Notation Recall that a bin is called a k-bin if it contains exactly k items. The level of a bin, denoted by $l(B)$, is the sum of the sizes of the items in the bin. In this chapter the total size of all the items will be denoted by P.

7.1 An upper bound for Batched Bin Packing for $K = 2$

Now we will introduce an approximation algorithm for the batched bin packing problem, for the special case where $K = 2$. In fact, we apply algorithm FFD for the two batches separately; i.e. we pack the items in batch B_1 by FFD, and we also apply FFD for the second batch B_2, independently of the first batch (thus we consider the disjunctive model; i.e. we do not use the bins of the first batch to pack the items of the second batch).

This natural adaptation of FFD will be denoted by $FFD(B_1, B_2)$. The two independent packings will be denoted by $FFD(B_1)$ and $FFD(B_2)$, respectively. Then $FFD(B_1, B_2) = FFD(B_1) + FFD(B_2)$. Let the optimum value of the relaxed offline problem (where the items of set $B_1 \cup B_2$ can be packed freely together) be denoted by OPT. We will call the solution of this relaxed problem the optimal solution. Then when we measure the quality of our packing, we will compare $FFD(B_1, B_2)$ with OPT. Note that $OPT \geq P$ trivially holds. Next, we will show that

$$FFD(B_1, B_2) \leq \frac{19}{12} OPT + 2, \tag{7.1}$$

and the approximation ratio $19/12$ is tight for the algorithm. We group the items into classes. Now let a denote an arbitrary item. Then a is called tiny, small, medium or grand, if $a \leq 1/4$, $1/4 < a \leq 1/3$, $1/3 < a \leq 1/2$, or $1/2 < a$, respectively. The classes, and the items therein are denoted by T, S, M and N, respectively (letter G will be used for another meaning, which is why we use the letter N for graNd items). If a medium item and a grand item share a bin (possibly with some further items), we call them a *pair*.

The next lemma and the corollary provide some insight into the way the FFD algorithm works. Both are folklore results, but we present them and their proofs for completeness.

Lemma 7.1.1 *Consider an arbitrary list of items L in the BP problem, and let N and M denote the set of the grand and medium-sized items, respectively. Let $p = p(L)$ be the maximum number of (n, m) pairs, where $n \in N$ and $m \in M$ and no item appears in more than one pair. Then algorithm FFD creates exactly p pairs.*

Proof. Suppose that the statement is not true, and let L be a minimal counterexample, i.e. a list when FFD makes r pairs, where $r < p$. It follows that L does not have any item smaller than a medium item, since after deleting these items FFD makes the same number of pairs, and $p(L)$ does not change. Now consider all feasible packings of L, where there are exactly p pairs. Let us suppose that there are t such feasible packings. For the k-th packing among these, denote the pairs by $(n_{1k}, m_{1k}), ..., (n_{pk}, m_{pk})$, where the pairs are listed in lexicographically decreasing order, i.e. if $n_{ik} > n_{jk}$ for some $1 \leq i, j \leq p$, then (n_{ik}, m_{ik}) precedes (n_{jk}, m_{jk}), and also, if $n_{ik} = n_{jk}$

and $m_{ik} > m_{jk}$. Note that there can be some other items in the list. There may be several grand items that are single items in their bins, and there may be also some other bins containing one or two medium-sized items. Now we choose all the packings among the t packings, for which n_{1k} has the largest possible size, say, and we have $t_1 \le t$ such packings. Then among the t_1 packings, we choose all the packings for which m_{1k} has the largest possible size, and let us fix one such packing, and let k simply denote again the index of the chosen packing, and we will refer to this packing below as the chosen packing. We claim that there are no $n_0 \in N$ and $m_0 \in M$, a grand and a medium-sized item, respectively, such that $n_0 > n_{1k}$, and n_0 and m_0 fit into a common bin. Suppose for the sake of contradiction that such an (n_0, m_0) pair exists. It follows that n_0 is a single item in its bin. If m_0 is different from any m_{ik}, $1 \le i \le p$, this would contradict the fact that p is the maximum number of pairs, as we found $p+1$ pairs. Otherwise $m_0 = m_{ik}$, for some $1 \le i \le p$. Then, in the chosen packing we take off m_{ik} from its actual bin, and pack it into the bin of n_0 (thus replace the pair (n_{ik}, m_{ik}) by a new pair (n_0, m_{ik})), and leave unchanged the bin of any other item. This packing has p pairs but differs from any packings in the t possible packings, which is a contradiction. Thus our claim is proved. Now let us see how FFD works. First FFD packs the grand items in order of decreasing size. Consider the first bin with a grand item of size n_{1k} (k is the index of the chosen packing). Because of the above claim, FFD cannot pack a medium item into some earlier bin, but it will find that a medium item of size m_{1k} fits into this bin. (It also holds that no medium-sized item with size bigger than m_{1k} will fit into this bin.) Thus FFD will pack n_{1k} and m_{1k} together. (More exactly, FFD packs a certain grand item n, and a medium item m into a common bin, where the sizes of n and n_{1k}, and the sizes of m and m_{1k} are equal. For the sake of simplicity we will assume that $n = n_{1k}$ and $m = m_{1k}$; if this is not the case, we can swap n and n_{1k}, and also m and m_{1k} in the packing.) We claim that by deleting n_{1k} and m_{1k} from the input, we get a smaller counterexample. Indeed, FFD will create the same packing for the remaining items in the sense that any non-deleted item will be packed together with the same item as before; or alone, if the item was alone in the previous packing. It follows that in the smaller list FFD creates exactly $r - 1$ pairs. However, in the modified list at least $p - 1$ pairs can be created. Thus we conclude that our list was not minimal, which is a contradiction. □

Corollary 7.1.1 *For any list of items L in the BP problem, if the smallest item in L is still larger than $1/3$, then algorithm FFD makes an optimal packing.*

Proof. Note that no bin can contain more than two items, no two grand items fit into a common bin, and any two medium items fit into a common bin. Now let us consider a list L, and all possible packings of L, where $p = p(L)$ means the maximum number of pairs in the packings. We can easily see that if we make an arbitrary packing with the only restrictions that the number of pairs is chosen to be p, and there are no two bins with some single medium items, then the packing is optimal. Since FFD creates p pairs by Lemma 7.1.1, and FFD never creates two bins with single medium items, it follows that FFD makes an optimal packing. □

Now we prove our main result.

Theorem 7.1.1 $FFD(B_1, B_2) \le \frac{19}{12} \cdot OPT + 2$*, and the asymptotic bound is tight for the algorithm.*

Proof. First we note that the additive term is necessary here, as the statement is not valid without any additive term. Its smallest value may be smaller than 2, but if we use 2, this is just enough to apply this theorem in the next section. In the calculation of the proof of Theorem 7.2.1, the 2 additive term will just disappear, and this makes it possible to get an absolute bound there. Thus using a bigger additive term in the present theorem would not be sufficient later, but using a smaller additive term would make the proof harder. Suppose that the statement is not true; then let us choose a minimal counterexample (in the sense of the number of items in $B_1 \cup B_2$). It then follows that in the last $FFD(B_1)$ bin there is only one item, denoted by X, and also in the last $FFD(B_2)$ bin there is only one item, denoted by Y. We also know that each of them (i.e. X and Y) is the last item in the sorted list of FFD in its batch. The reason is that if there were some items after X in batch B_1, or there were some items after Y in batch B_2, these final items could be removed from the input; the numbers of FFD bins for the batches remain the same, while OPT does not increase. Let B_1^i and B_2^j denote the bins in the two independent FFD packings of batches B_1 and B_2, respectively. Then $1 \leq i \leq FFD(B_1)$ and $1 \leq j \leq FFD(B_2)$. Moreover, let B_k^* denote the optimal bins where $1 \leq k \leq OPT$. We assign weights for the items. The weight of any bin, denoted by $w(B_1^i)$ or $w(B_2^j)$ or $w(B_k^*)$, is the sum of weights of the items being packed into the bin. The total weight of the items will be denoted by W. The following analysis is similar to that in [21]. If the weight of any optimal bin is larger than $\frac{19}{12}$, we call the difference between the weight of the bin and $\frac{19}{12}$ a **shortfall**. If the weight of an FFD bin is smaller than 1, we say that there is a **shortage** in the bin. The value of the shortage is defined as 1 minus the weight of the bin. Otherwise, if the weight of an FFD bin is larger than 1, we say that there is **surplus** in this bin, it being defined as the weight of the bin minus 1. Next, let f, h and u denote the total shortfall, total shortage and total surplus, respectively. Note that all these values are nonnegative. Then we get

$$\begin{aligned}
&FFD(B_1) + FFD(B_2) \\
&= \sum_i w(B_1^i) + \sum_j w(B_2^j) + h - u \\
&= W + h - u = \sum_k w(B_k^*) + h - u \\
&\leq \frac{19}{12} \cdot OPT + h - u + f.
\end{aligned}$$

Hence, if

$$h - u + f \leq 2 \tag{7.2}$$

holds, then the statement of the lemma follows. In most cases it turns out that $f = 0$, so we just need to verify that $h \leq 2$. Before considering the cases one by one, we note that the level of each bin of the first batch is above $1 - X$, except possibly for the last such bin. Similarly, all bins of the second batch, except possibly for the last such bin, have levels above $1 - Y$. Moreover, all item sizes for the first batch are at least X, and for the second batch they are at least Y (given the properties stated in the second paragraph of the proof of this theorem). Now let us examine the different cases.

Case 1. $X \leq 1/3$ and $Y \leq 1/3$. In any FFD bin the level is larger than $2/3$, except of course the last bins. Let B_1^α and B_2^β denote any FFD bin in the two packings, where $1 \leq \alpha \leq FFD(B_1) - 1$ and $1 \leq \beta \leq FFD(B_2) - 1$, respectively. Then the following statement (without using the weights) is true:

$$
\begin{aligned}
&FFD(B_1) + FFD(B_2) \\
&\leq \frac{3}{2}\left(\sum_{\alpha} l(B_1^{\alpha}) + \sum_{\beta} l(B_2^{\beta})\right) + 2 \\
&\leq \frac{3}{2}P + 2 \leq \frac{3}{2}OPT + 2.
\end{aligned}
$$

Case 2. $X > 1/3$ and $Y > 1/3$. Here we use the following simple weights: Let the weights of any grand item and any medium item be 1 and $1/2$, respectively. Then in the optimum packing of the relaxed problem each bin contains at most two items, and two grand items do not fit into the same bin, so the weight of any optimal bin is at most $3/2$, it follows that $f = 0$. However, in the $FFD(B_1)$ and $FFD(B_2)$ packings, if a bin does not contain a grand item, then the bin must contain two medium items, except the two last bins, hence the weight of any FFD bin is at least 1, except for the two last FFD bins. Thus there are two FFD bins (the last FFD bins of the two batches) with a shortage, $h \leq 2$ holds, and we are done.

Case 3. $X > 1/3$ and $Y \leq 1/3$. Then we use different weights for sets B_1 and B_2. For set B_1, we again use $w(N) = 1$ and $w(M) = 1/2$, for any grand or medium item. Note that set B_1 does not have any smaller item. Then in the $FFD(B_1)$ packing the weight of any bin is at least 1, except for the last bin, and in the last bin a shortage of at most $1/2$ is created. The weights of the B_2 items will be defined differently in the next subcases.

Subcase 3.1. $1/4 < Y \leq 1/3$. In this case in set B_2 all items are larger than $1/4$, thus any item in set $B_1 \cup B_2$ is also larger than $1/4$. It follows that if a grand item is packed into a bin, then no two other items can be packed into that bin. Now we define the weights of the B_2 items as follows: Let any B_2 item, say v, be denoted as a Z item, if in the optimal packing, v is packed together with a grand B_1 item. The weight of a Z item is $7/12$. Note that a Z item is surely smaller than $1/2$. Let any other $v \in B_2 \setminus Z$ item be called a regular item, and let $w(v) = \frac{19}{12}v$. First we show that the weight of any optimal bin is at most $19/12$. If the bin contains a grand B_1 item, then the bin can contain at most one other item. It is a medium B_1 item with weight $1/2$, or it is a Z-item, with weight $7/12$, and the weight of the bin is at most $19/12$. Now consider an optimal bin which does not contain a grand B_1 item. If the bin contains one or two medium B_1 items, then the remaining room in the bin is not larger than $2/3$ or $1/3$, respectively, so the total weight of the bin is at most $1/2 + 19/12 \cdot 2/3 < 19/12$ or $1 + 19/12 \cdot 1/3 < 19/12$, respectively. Lastly, if the optimal bin does not contain any B_1 item, then the weight of the bin is at most $19/12$, since the total weight is at most $19/12$-times the total size of the items in the bin. It follows that $f = 0$. Now we will show that the weight of any $FFD(B_2)$ bin is at least 1, with at most two exceptions. One exceptional bin is the last $FFD(B_2)$ bin, where the shortage is smaller than 1. The level of any other bin is larger than $1 - Y \geq 2/3$. If there is no Z-item in the bin, then the weight of the bin is at least $2/3 \cdot 19/12 > 1$. If the bin contains two Z items, the weight of the bin is at least $7/6$. Hence suppose that the bin contains exactly one Z-item, this item being smaller than $1/2$. Since the level of this bin is larger than $2/3$, the bin must contain at least one other item v, which has a size at least $Y > 1/4$. Note that the weight of any regular item is at least $19/12 \cdot 1/4$. If there are two regular items in the bin, the weight is greater than 1. If there is one regular item in the bin, the weight is at least $7/12 + 19/12 \cdot 1/4 = 47/48$, i.e. the shortage is at most $1/48$. We claim that two bins, denoted by (z_1, v_1), (z_2, v_2), cannot occur, where both have a shortage, and z_1, z_2 are Z items and v_1, v_2 are regular items. Suppose there are two such bins. If any of v_1, v_2 is larger than $1/3$, then the weight of its bin is larger than $7/12 + 19/12 \cdot 1/3 > 1$, which is a contradiction. Thus both v_1, v_2 have sizes of at most $1/3$. If any of z_1, z_2 had a size of at most $1/3$, then its bin would have a level of at most $2/3$, which is a contradiction. Thus both z_1, z_2 have sizes above $1/3$ (and at most $1/2$). It follows that z_1 would end up together with an item of at least the size of z_2, which is a contradiction. Thus the claim follows. We have proved that the total shortage is at most $1 + 1/48$ in the $FFD(B_2)$ bins. Then from $f = 0$ and $h \leq 1/2 + 1 + 1/48$, (7.2) follows.

Subcase 3.2. $1/6 < Y \leq 1/4$. Then any items in $B_1 \cup B_2$ are larger than $1/6$, thus no three items can share a bin with a grand item. Moreover a bin containing a grand item and a medium item does not contain any other item. We will define the weights of the B_2 items as follows. Let a B_2 item, say v, be called irregular if v is packed into some optimal bin together with a grand B_1 item. We further divide the set of irregular items. If v is the only one B_2 item in the optimal bin (besides the grand B_1 item), v is denoted as a Z-item. Otherwise, if there are two irregular items in the optimal bin of the grand B_1 item, these two items will be denoted as a U item and a V item, respectively, where the U item is not smaller than the V item. Note that any Z item is smaller than $1/2$, any U item is smaller than $1/3$, and any V item is smaller than $1/4$. Their weights are $7/12$, $1/3$, and $1/4$, respectively. Each remaining item $v \in B_2 \setminus (Z \cup U \cup V)$ is called a regular B_2 item, and is denoted as an R item. For any R item, $w(v) = \frac{19}{12}v$. Since the size of any item is larger than $1/6$, the weight of any R item is larger than $19/72 > 1/4$. It follows that the weight of any (regular or irregular) B_2 item is at least $1/4$. We will introduce a notation for the $FFD(B_2)$ bins. For example, if the bin contains one Z-item and one U-item, we will denote this bin as a (Z, U)-bin. If the bin contains one U-item, one V-item and an R-item, the bin is denoted as (U, V, R). A similar notation is used for any kind of the $FFD(B_2)$ bins, according to the classes of the items in the bin. First, we will show that the weight of any optimal bin is at most $19/12$. Suppose the bin contains a grand B_1 item. If the bin just contains this item, the claim holds. If the bin contains a grand B_1 item and a medium-sized B_1 item, then no further item can fit into the bin, and the weight is only 1.5. Otherwise the optimal bin contains a grand B_1 item, and one or two B_2 items. If it contains only one B_2 item, it is a Z item with weight $7/12$. Otherwise it contains a U and a V item with total weight $1/3 + 1/4 = 7/12$. In each case the claim holds. Now suppose that the bin does not contain a grand B_1 item. It means that the bin contains two, one or zero medium-sized B_1 items, and several regular items. In this case we find in the same way as in Subcase 3.1 that the weight of the bin is at most $19/12$. Since no optimal bin has a shortage, we have shown that $f = 0$. Now we will show that the weight of any $FFD(B_2)$ bin is at least 1, except for a finite number of exceptional bins, and in the exceptional bins the total shortage is at most 1.5. Recall that for the $FFD(B_2)$ packing, except the last bin (which is one of the exceptional bins), the level of any bin is strictly larger than $1 - Y \geq 3/4$. If there are only regular B_2-items in the bin, then the weight of the bin is at least $3/4 \cdot 19/12 = \frac{19}{16} > 1$. If the $FFD(B_2)$ bin contains at least four items, then the weight of the bin is again at least 1. If the bin contains only one item (and this is not the last bin), this item is surely a grand item; then it follows that this item is a regular B_2 item, and this case has already been considered. So it just remains to consider the cases where an $FFD(B_2)$ bin contains two or three items, and not just regular items. First, consider the 2-bins among the $FFD(B_2)$ bins. Since there is at least one irregular item in the bin, only the following bin types can occur, which are listed in the table below.

Z	2	1	1	1					
U		1			2	1	1		
V			1			1		2	1
R				1			1		1

The (Z,V), $(2U)$, (U,V), $(2V)$ bin types are impossible, since the level of the bin would be at most $3/4$. With a $(2Z)$ bin, the weight of the bin is $7/6$. For the (V,R) bins, the size of the R item is at least $1/2$; otherwise the level of the bin remains below $3/4$, and the weight of the bin is at least $\frac{1}{4}+1/2\cdot\frac{19}{12}=\frac{25}{24}>1$. There remains the following: (Z,U), (Z,R) and (U,R). We will show that there can be at most one bin from each type with a shortage. - (Z,U) type. The weight of such a bin is $\frac{7}{12}+\frac{1}{3}=\frac{11}{12}$; i.e. each such bin has a shortage of $\frac{1}{12}$. Since $U<1/3$, the Z item must be larger than $1/3$; otherwise the level of the bin remains below $3/4$. Recall that a Z item has a size of at most $1/2$. Suppose there are two such bins, denoted by (z_1,u_1) and (z_2,u_2). Then FFD would not open a new bin for z_2, which is a contradiction. - (Z,R) type. If the size of the R item is at least $1/3$, then the weight of the bin is is at least $7/12+1/3\cdot 19/12=\frac{10}{9}>1$. Otherwise R is smaller. The Z item must be larger than $1/3$ (but at most $1/2$); otherwise the level of the bin cannot be above $3/4$. It follows that there cannot be two (Z,R) bins with $R<1/3$ (since FFD would not open a new bin for the second Z item). Thus there can be at most one (Z,R) bin with a shortage. The size of the R item is bigger than $1/4$; otherwise the level of the bin cannot be above $3/4$. So the weight of such a bin is at least $7/12+1/4\cdot 19/12=47/48$. Moreover, from such a bin type we can have a shortage of at most $1/48$. - (U,R) type. Since the size of the U item is at most $1/3$, and the level of the bin is above $3/4$, the size of the R item is above $3/4-1/3=5/12$. Thus the weight of the bin is at least $1/3+5/12\cdot 19/12=143/144$. We have seen that the R item is the larger item in the bin. If the R item has a size of at least $1/2$, the weight of the bin is at least $\frac{1}{3}+\frac{1}{2}\cdot 19/12=\frac{9}{8}>1$. It follows that two (U,R) bins, both having a shortage, cannot occur since FFD would not open a new bin for the later R item. It follows that the shortage is at most $1/144$ from this type. Summarizing the results, there can be a shortage of at most $1/12+1/48+1/144=1/9$ created by the 2-bins. Now let us consider the 3-bins, containing at least one irregular item. If the bin contains a Z-item and two other items, then the weight is at least $7/12+2\cdot 1/4>1$. In the remaining cases the (one or two) irregular items in the bin are at most small items. If there is a regular item in the bin with size at least $1/3$, the weight of the bin is at least $2\cdot 1/4+1/3\cdot 19/12=37/36>1$. It follows that all three items in the bin are at most small items. If there are three small items in the bin, it means that the bin is of type $(2U,R)$, or $(U,2R)$ (since V is tiny, and there is at least one irregular item in the bin). In these cases the weight is at least $2/3+1/4\cdot 19/12=17/16>1$, or $1/3+2/4\cdot 19/12=9/8>1$. The case that all three items are tiny is impossible, since then the level of the bin is at most $3/4$. We conclude that any remaining bin contains at least one small item, but at most two small items, and the other items are tiny. But there can be at most one such bin, since three small items always fit into one bin, hence the third small item among them cannot open a new bin. The weight of such a bin is still at least $3/4$, since the smallest weight is at least $1/4$. Thus with some 3-bin we can have a shortage of at most $1/4$. Thus the total shortage is at most the following: $1/2$ caused by the last $FFD(B_1)$ bin; moreover at most $3/4$ caused by the last $FFD(B_2)$ bin (since this bin contains a tiny item); and a further shortage of at most $1/9+1/4$ is caused by some other $FFD(B_2)$ bins (some 2-bins and a 3-bin). This is altogether a total shortage of at most $h\leq 1/2+3/4+1/9+1/4<2$, so (7.2) follows.

Subcase 3.3. $1/7 < Y \leq 1/6$. For the B_2 items we introduce the following weighting. (We call the items of the appropriate classes as $C, D, ..., J$ items). As regards the B_2 items, we say that an item a is in some higher class than item b, if they are in different classes and $a > b$. We will use the notation $H + C > 5/14 + 1/7$, which means that the total size of any item from class H and any item from class C is larger than the right hand side. The statement trivially follows from the definition of the classes. Similar notations are also used for the other classes.

Class	Weight	ratio of $w(a)/a$
$2/3 < J \leq 1$	1	$w(a)/a < 3/2$
$1/2 < I \leq 2/3$	10/12	$w(a)/a < 5/3$
$5/14 < H \leq 1/2$	7/12	$w(a)/a < 5/3$
$1/3 < G \leq 5/14$	5/12	$w(a)/a < 5/4$
$1/4 < F \leq 1/3$	4/12	$w(a)/a < 4/3$
$4/21 < E \leq 1/4$	3/12	$w(a)/a < 4/3$
$1/6 < D \leq 4/21$	2.5/12	$w(a)/a < 5/4$
$1/7 < C \leq 1/6$	2/12	$w(a)/a < 7/6$

First, we will show that the weight of any optimal bin is at most $19/12$, except for the (N, M, C) optimal bins. We will analyze them according to the largest item of B_2.

case a, Suppose the bin contains a grand B_1 item and no other B_1 item. If the bin contains a H item, then no more items fit into the bin since $H + C > 5/14 + 1/7 = 1/2$. If the bin contains a G item, then only one C item fits into the bin as another item. In both these cases the total weight is at most $19/12$. If the bin contains an F item, then only one other item can be in the bin; and as this item is not larger than an E item, the total weight is at most $19/12$. If the bin contains an E item (and no item from some higher class), then at most two other items can be in the bin. If there are two such items, both must be C items, since $C + D + E > 1/7 + 1/6 + 4/21 = 1/2$, and the total weight is again at most $19/12$. Lastly, there are at most three other items in the bin, and not all three items can be D items. Thus the total weight is at most $1 + 2 \cdot 5/24 + 2/12 = 19/12$.

case b, The bin contains two medium-sized B_1 items. If the bin does not contain more items, its weight is just 1. Otherwise it can contain at most two other items which are B_2 items. If the bin contains an F item, or an E item, no other item fits into the bin, since $2/3 + 4/21 + 1/7 = 1$. In these cases we are done. Otherwise the bin has at most two other items, and the total weight is smaller than $19/12$.

case c, The bin contains one medium-sized B_1 item, and no more B_1 items. If the bin contains an I item, then only one C item fits as another item in the bin, and the total weight is at most $1/2 + 10/12 + 2/12 = 18/12$. If the bin contains a H item, and one other item, this one is at most an F item, and the total weight is at most $1/2 + 7/12 + 4/12 = 17/12$. Suppose the bin contains a H item and two other items. Then both other items must be C items, since otherwise the total size is more than $1/3 + 5/14 + 1/6 + 1/7 = 1$. Thus the total weight is $1/2 + 7/12 + 4/12 = 17/12$. Otherwise the bin contains a medium-sized B_1 item, and all other items are B_2 items, and none of them comes from classes H or above. For these classes $w(a)/a < 4/3$ holds, and the total size of the B_2 items in the bin is at most $2/3$, so we find that the total weight of the bin is at most $1/2 + 2/3 \cdot 4/3 = 25/18 < 19/12$.

case d, The optimal bin contains only several B_2 items. If the bin contains a J or a I item, its size is more than $1/2$, and its weight is at most 1, and the calculation is similar to case a, replacing the J or I item by a grand B_1 item. If the bin contains two H items, then at most one other item fits into the bin which is at most an F item, and the total weight is at most $2 \cdot 7/12 + 4/12 < 19/12$. Suppose the bin contains one H item. The total weight is at most $7/12 + 9/14 \cdot 4/3 = 121/84 < 19/12$, since the total size of the other B_2 items in the bin is at most $9/14$, and the $w(a)/a$ ratio for these items is at most $4/3$. The calculation is similar if all the items are from smaller classes.

case e, The weight of an (N, M) bin is (only) 1.5.

case f, The optimal bin contains a grand B_1 item and also a medium-sized B_1 item, and some other item. The latter can be only a C item, since no other item fits into the bin. Now let us consider an (N, M, C) optimal bin. The weight of the bin is $3/2 + 2/12 = 20/12$, which means that the bin has a shortfall of $1/12$.

We have investigated all the cases and it turned out that any (N, M, C) optimal bin has a shortfall of exactly $1/12$, and no other bin has a shortfall. Next, we consider the FFD bins. The $FFD(B_1)$ bins have a shortage of at most $\frac{1}{2}$ in total. Now will we show that any $FFD(B_2)$ bin has a weight of at least 1, apart from at most five bins. Note that the level of any $FFD(B_2)$ bin (except for the last one) is larger than $1 - Y \geq 5/6$. In this last $FFD(B_2)$ bin we have a shortage of $5/6$, since the Y item is a C item. Now we will consider the other $FFD(B_2)$ bins. If there is a J item in the bin, the weight is at least 1. Otherwise, if there is an I item in the bin, it cannot be the only one item in the bin since the level of the bin is above $5/6$. Since the smallest weight is $2/12$, the weight of the bin is at least 1. In the following, we will suppose that any item in the bin is at most a H item. For this we will need the following claim:

Claim. If (i) there are two H or G items in the bin, or (ii) there are three F items in the bin, or (iii) there are four E items in the bin, or (iv) there are five D items in the bin, or (v) there are six C items in the bin; the weight of the bin is at least 1.

Proof of the claim: The claim trivially holds in all cases except (i). Hence, let us consider case (i). If at least one of the two items in question is a H item, the total weight is at least 1. Otherwise both items are G items. Their total size is at most $5/7 < 5/6$, so there must be another item in the bin, with weight at least $2/12$, and the claim also holds in this case. □

Now we will return to the remaining cases. Since any item in the bin is from classes C to H, it only remains to consider that (a) there is exactly one H or G item in the bin (and other items from smaller classes), or (b) there are at most two F items in the bin (and some other items from smaller classes), or (c) there are at most three E items in the bin (and some other items from smaller classes), or (d) there are at most four D items in the bin (and some other items from smaller classes), or (e) there are at most five C items in the bin. Case (e) is not possible, since the level of the bin would be too small. Our main consideration here is that there can be at most one bin from any other type (a)-(d). This is true, since two H or G items always fit into a common bin, three F items always fit into a common bin, and so on. It follows that there is at most one bin with a shortage from any type (a)-(d). Note that $w(a) \geq a$ holds for each class. Since the levels of these bins are larger than $5/6$, the shortage in any such bin is at most $1/6$. Thus the total shortage of the $FFD(B_2)$ packing is at most $4/6$ from the types (a)-(d), and there is an additional shortage of $5/6$ in the last $FFD(B_2)$ bin, which means altogether a shortage of at most $9/6 = 3/2$. We claim that $f \leq u$ holds. We have a shortfall only in the (N, M, C) optimal bins. But for any such optimal bin there exists an (N, M) bin with weight 1.5 in the $FFD(B_1)$ packing, according to Lemma 7.1.1. We can show this as follows. Let the number of the (N, M, C) optimal bins be q. Then the maximum number of (N, M) pairs in the list is $p \geq q$. Then in the $FFD(B_1)$ packing exactly

p pairs, i.e. (N, M) bins will be created. It follows that if there are $l \geq 0$ optimal bins of type (N, M, C), having a shortage of $l/12$ in total, then there exists l bins in the $FFD(B_1)$ packing, providing a surplus of $l/2$ in total. For the value of h we get $h \leq 1/2 + 3/2 = 2$, where $1/2$ stands for the last $FFD(B_1)$ bin, and $3/2$ stands for the $FFD(B_2)$ bins. Thus we conclude that $h - u + f \leq 2$, i.e. (7.2) holds.

Subcase 3.4. $Y \leq 1/7$. Let each item $v \in B_2$ has a weight $w(v) = \frac{7}{6} \cdot v$. Now consider the optimal bins. If the bin does not contain any B_1 item, the weight of the bin is at most $7/6 < 19/12$. If the bin does not contain a grand B_1 item, but contains one or two medium-sized items, the weight is at most $1/2 + 2/3 \cdot 7/6 = 23/18$ or $1 + 1/3 \cdot 7/6 = 25/18 < 3/2 < 19/12$, respectively. If the bin contains a grand B_1 item and does not contain any medium-sized B_1 item, its weight is at most $1 + 1/2 \cdot 7/6 = 19/12$. Lastly suppose that the bin contains a grand B_1 item, a medium-sized B_1 item, and possibly some other B_2 items. The weight of an optimal bin like this is at most $1 + 1/2 + \frac{7}{6} \cdot \frac{1}{6} = 61/36$, which means that any such bin can have a shortfall of at most $1/9$. But for any such optimal bin there exists a (N, M) bin with weight 1.5 in the $FFD(B_1)$ packing, according to Lemma 7.1.1. In a similar way as at the end of Subcase 3.3., we find that the total shortfall of the optimal bins is covered by the total surplus of the $FFD(B_1)$ bins. The weight of any other optimal bin is at most $19/12$; and, moreover, any other $FFD(B_1)$ bin has weight at least 1, except the last bin, which can have a shortage of $\frac{1}{2}$. The weight of any $FFD(B_2)$ bin, except the last one, is at least 1. As we saw earlier, $f \leq u$ holds. For the value of h we get $h \leq \frac{1}{2} + 1$, where $1/2$ and 1 stand for the last $FFD(B_1)$ bin and last $FFD(B_2)$ bin, respectively. Thus we conclude that (7.2) holds.

Case 4. $X \leq 1/3$ and $Y > 1/3$. This case can be treated in exactly the same way as Case 3.

Finally we will prove that the (asymptotic) bound is tight for the algorithm. Let B_1 consist of $12n$ items of size $1/4 + \varepsilon$, and $12n$ items of size $1/4 - 2\varepsilon$ (with an appropriate choice of a small ε), and let B_2 consist of $12n$ grand items of size $1/2 + \varepsilon$. Then $OPT(B_1, B_2) = 12n$, while $FFD(B_1) = 4n + 3n$, $FFD(B_2) = 12n$; hence $FFD(B_1, B_2) = 19n$, so the statement follows. Moreover, it is easy to see that applying the same construction, but with $OPT = 12n + 9$, we get the following. $FFD(B_1) = (4n + 3) + (3n + 3)$, $FFD(B_2) = 12n + 9$, and so $FFD(B_1, B_2) = 19n + 15 = \frac{19}{12}(12n + 9) + 3/4$, hence the additive term in the theorem cannot be smaller than $3/4$. □

Now let us discuss the connection between the augmenting and the disjunctive models of BBP. We restrict our investigations here to just considering the behavior of the FFD algorithm in the two models. If we apply the FFD algorithm in the augmenting model in the way that we *are allowed, but we do not use* the bins of the first batch, this algorithm is an $19/12$-approximation algorithm. Do we get a better algorithm in the sense of the asymptotic bound, if applying the FFD algorithm, we also use the bins of the first batch, since it is allowed? The answer is no, since this lower bound construction we just have seen in the end of proof of Theorem 7.1.1 also works for the augmenting model (i.e. FFD is allowed to use any bins of the first batch, but it cannot). Let $R_a(FFD)$ and $R_d(FFD)$ denote the (asymptotic) approximation ratios of the appropriate versions of FFD in the two models in question, respectively (considering only $K = 2$). We proved in Theorem 7.1.1 that $R_d(FFD) = 19/12$, and from the previous note we have $R_a(FFD) \geq R_d(FFD)$. Now we show that the opposite inequality also holds.

Lemma 7.1.2 *$R_a(FFD) \leq R_d(FFD)$, if $K = 2$.*

Proof. Let FFD' denote FFD for the augmenting model (for the disjoint model, call it simply FFD). Now consider an output of FFD' for a list L'. Let L be the list of items we get from L', deleting the items of the second batch that FFD' placed into bins of the first batch. We have $FFD'(L') = FFD(L)$, while $OPT(L) \leq OPT(L')$, since $L \subseteq L'$. We know that $FFD(L) \leq R_d(FFD) \cdot OPT(L) + C$ (for an additive constant $C \geq 0$), and therefore $FFD'(L') \leq R_d(FFD) \cdot OPT(L') + C$. This proves that $R_a(FFD) \leq R_d(FFD)$. $\square$

Corollary 7.1.2 *$R_a(FFD) = R_d(FFD)$, if $K = 2$.*

The statement of the corollary, is the reason why we proved the asymptotic approximation bound $19/12$ in the disjunctive model. As we cannot get any gain to construct the proof in the augmenting model, only the proof would require more effort. Proposing and investigating some other algorithm than FFD for the BBP problem is also an interesting direction for further study.

Remark 7.1.1 *(i) The precise connection between the BP problem and the BBP problem is not completely understood. For the BBP problem with $K = 2$, we presented the first approximation algorithm, with asymptotic approximation ratio $19/12 \approx 1.5833$, which is strictly smaller than the best known asymptotic approximation ratio for online BP; i.e. 1.5888 ([77]). For $K = 2$ there still remains a gap between the lower and upper bounds. It would be nice to tighten this gap.*
(ii) It does not help in this sense, if we apply algorithm FFD for $K = 3$ in the disjunctive model. It is obvious that its asymptotic approximation ratio is at least $5/3$, so this algorithm is not too efficient. Let n be the number of optimal bins, and let the following items be in any optimal bin: $1/2+\varepsilon$, $1/3+\varepsilon$, $1/6-2\varepsilon$. Then let the batches contain the tiny items, the medium items and grand items, in this order. It is easy to see that $FFD(B_1, B_2, B_3)/OPT = 1/6+1/2+1 = 5/3$. Or, if the first half of the first batch contains n items of sizes $1/12+\varepsilon$, while the second half of the first batch contains n items with sizes $1/12 - 3\varepsilon$, we get the slightly better ratio $FFD(B_1, B_2, B_3)/OPT = 1/11 + 1/12 + 1/2 + 1 = \frac{221}{132} \approx 1.6742$. And what happens if $K \to \infty$, and we apply the FFD algorithm for each batch in the disjunctive model? As each batch can consist of a single item, we can apply the lower bound construction of the FF algorithm, and we get a tight (asymptotic and also absolute) approximation ratio 1.7; for details, see [23, 24].
(iii) Further, applying any algorithm in the disjunctive model, as the number of batches grows, it can be shown that the approximation ratio must be at least that of the harmonic algorithm, namely 1.69103. We can see this as follows. In the tight worst case example of the $Harmonic(K)$ algorithm, each optimal bin contains an item of size $1/2 + \varepsilon$, an item of size $1/3 + \varepsilon$, an item of size $1/7 + \varepsilon$, an item of size $1/43 + \varepsilon$, and so on. (After defining the initial couple of items, the remaining room in the bin, not considering the small epsilons, is just $1/k$ for some k. Then the next item type is $1/(k + 1) + \varepsilon$). This sequence of items contains K item types. Then in the BBP problem, choosing the same K, we let the i-th smallest items be in the i-th batch. Since these items have the same size, no algorithm can pack them better than the $Harmonic(K)$ algorithm, and the statement follows.

7.2 An improved result for the GBP problem

In [11], a general packing algorithm is given for the GBP problem, with absolute approximation ratio 3, in the case where the graph which is to be packed is bipartite. Now we will present an improved algorithm with absolute approximation ratio $19/12 + 1 = 31/12 \approx 2.5833$, for this restricted case.

Let $G(A, B, E)$ be the bipartite graph in question, where $V = A \cup B$ is the set of points (also called items) of the bipartite graph, and E is the set of edges. Any point $v \in V$ has a size, denoted by $0 < s(v) \leq 1$. Moreover for any $e(a, b)$ edge ($a \in A, b \in B$) an integer lower bound will be denoted by $d(a, b) \geq 0$. In this special case of the GBP problem, the goal is as follows: Let us pack the items into as small a number of bins as possible, in such a way that for any two items $a \in A, b \in B$, if they are packed into bins B_i and B_j, respectively, the indices of their bins must satisfy $|i - j| \geq d(a, b)$, and the total size of items in any bin cannot exceed 1. Let d denote the maximum of the prescribed lower bounds, i.e. let $d = \max\{d(a, b) \mid e(a, b) \in E\}$. (We suppose that $d \geq 1$, otherwise there is no lower bound restriction.) In this section, let OPT be the number of bins in an optimum packing. Then it naturally follows that $OPT \geq LB_1 = d + 1$.

Let OPT_R denote the optimum value of the relaxed problem, where the lower bounds on the edges are neglected; i.e. we simply consider the packing problem of items $A \cup B$. It trivially follows that $OPT \geq OPT_R$. Moreover if we apply our BPP algorithm for the two sets of nodes after each other, Theorem 7.1.1 holds, hence $FFD(A, B) \leq \frac{19}{12} \cdot OPT_R + 2$. Now we will introduce a very simple algorithm called $Master$ with the absolute approximation ratio of $31/12$. The number of bins created will also be denoted by $Master$.

Algorithm Master

1. Pack items of A by bin packing algorithm FFD. The packing (and also the number of the bins used) will be be denoted by $FFD(A)$. Now we pack items of B into new bins (independently of the packing of set A); let the packing of set B; and let the number of bins used be denoted by $FFD(B)$. In this step, the lower bound constraints are totally neglected.

2. We leave $d - 1$ empty bins between the two packings; end.

Note that we are quite liberal when performing this Step 2. However, by applying Step 2, the packing of Master naturally will be feasible, so we do not need to deal with the lower bound restrictions. Roughly saying, Step 2 increases the absolute approximation ratio by 1, as is shown in the following theorem.

Theorem 7.2.1 *The absolute approximation ratio of the Master algorithm is at most* $31/12$*.*

Proof. For any input, it follows that

$$\begin{aligned} Master &= FFD(A) + FFD(B) + d - 1 \\ &= FFD(A, B) + d - 1 \\ &\leq 19/12 \cdot OPT_R + 2 + (d - 1) \\ &= 19/12 \cdot OPT_R + LB_1 \leq 31/12 \cdot OPT. \end{aligned}$$

□

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Chapter 8

Appendix

8.1 Appendix A

Here we give a simplified, short proof of the following result proved originally in [79].

Theorem 8.1.1 *If $OPT = 5$ then $FFD \leq 6$.*

Proof. Suppose for a contradiction that L is a minimal counterexample such that $OPT = 5$, and $FFD = 7$. From Lemmas 3.1.1 and 3.1.5 it follows that the last item, X is between the next bounds: $1/5 < X < 1/4$. Since all items fit in the optimal packing into five optimal bins, it follows that $\sum_{k=1}^{n} p_k \leq 5$. Moreover X does not fit into any previous FFD bin, thus we get

$$l\left(B_i\right) > 1 - X, \quad i = 1, \ldots, 6. \tag{A.1}$$

□

Recall that each optimal bin contains at least three items, thus the total number of items is at least 15. Moreover each FFD bin but the last one contains at least two items. Since $X > 1/5$ holds for the smallest item, no (optimal or FFD) bin contains more than four items.

Lemma 8.1.1 *The level of each FFD bin is strictly less than $4X$, thus each FFD bin contains at most three items.*

Proof. Suppose that the level of B_j is at least $4X$. Applying (A.1), it follows that

$$5 \geq \sum_{k=1}^{n} p_k = X + l\left(B_j\right) + \sum_{i \neq j,\, i \neq 7} l\left(B_i\right) > X + 4X + 5\left(1 - X\right) = 5,$$

which is a contradiction. □

In the following we use A_{ik} to represent the k-th item in the i-th FFD bin and A_{ik}^{*} to represent the k-th item in the i-th optimal bin.

Lemma 8.1.2 *Suppose that B_i and B_j contains two and three elements, respectively. Then $i < j$.*

Proof. Suppose to the contrary that B_i and B_j are two consecutive bins (i.e. $j = i + 1$), and B_i contains three elements, while B_j contains only two. Suppose that $A_{i1}+A_{i2} \geq A_{j1}+A_{j2}$. Because A_{i3} fits into the i-th bin, and $A_{i3} \geq X$, it follows that X fits into the j-th bin, a contradiction. Thus $A_{i1} + A_{i2} < A_{j1} + A_{j2}$. Moreover, from $A_{i1} \geq A_{j1}$ it follows that $A_{i2} < A_{j2}$. Thus A_{j2} is packed before A_{i2}, and it did not fit into the i-th bin, thus $A_{i1} + A_{j2} > 1$ while $A_{i1} + A_{i2} + A_{i3} \leq 1$. Hence $A_{j1} \geq A_{j2} > A_{i2} + A_{i3} \geq 2X$, and then $A_{j1} + A_{j2} \geq 4X$, contradicting Lemma 8.1.1. □

By Lemma 8.1.2, the FFD bins at the beginning contain two items, the following ones contain three items, and the last one contains only one, namely the last item. Then the 6-th FFD bin must contain three items (otherwise the number of items is at most $6 \cdot 2 + 1 = 13$).

Lemma 8.1.3 *Each item coming after $A_{6,1}$ is regular.*

Proof. For the sake of simplicity, let $A_{6,1}$ be denoted as U. Suppose that Z comes later than U (but not necessarily right after U), and is packed into some previous bin. Since B_6 contains three items, it follows that $l\,(B_6) \geq U + 2X$. On the other hand the level of the bin where Z is packed, is more than $1 - U + Z \geq 1 - U + X$. Using (A.1), we get

$$5 \geq \sum_{k=1}^{n} p_k > X + (1 - U + X) + (U + 2X) + 4\,(1 - X) = 5,$$

a contradiction. □

It follows that the three items of the 6-th FFD bin are three consecutive items, and the next item is the last item X, which is packed into the last FFD bin. We get the next

Corollary 8.1.1 *The four smallest items cannot fit into one bin, and thus each optimal bin contains exactly three items.*

Now we are ready to prove Theorem 8.1.1. Since each optimal bin contains exactly three items, and each FFD bin contains two or three items except the last FFD bin, the number of items is exactly $n = 15$, and the FFD bins contain 2-2-2-2-3-3-1 items.

Consider the fourth FFD bin (it is a 2-bin). We know that $A_{41}+A_{42}+X > 1$. Item A_{41} cannot be earlier than p_4, thus $A_{41} \leq p_4$, and A_{42} comes after A_{41}, thus $A_{42} \leq p_5$, therefore $p_4+p_5+X > 1$. Suppose that items A^*_{i1}, $i = 1, \ldots, 5$ are not the five biggest items. Then there exists some j, for which A^*_{j2} comes not later than the fifth item, i.e. $A^*_{j2} \geq p_5$. It follows that $A^*_{j1} \geq p_4$, but then $1 < p_4+p_5+X \leq A^*_{j1}+A^*_{j2}+A^*_{j3} \leq 1$, a contradiction. Thus items A^*_{i1}, $i = 1, \ldots, 5$ are the first five items; we can suppose without loss of generality that $A^*_{i1} = p_i$ for $i = 1, \ldots, 5$. Then $A^*_{j2} = p_6$ for some j, and it follows that $p_5+p_6+X \leq A^*_{j1}+A^*_{j2}+A^*_{j3} \leq 1$. Thus the first item in the fourth FFD bin cannot be p_5 or some later item (because then $A_{41} + A_{42} + X \leq p_5 + p_6 + X \leq 1$). We have seen that A_{41} cannot be earlier than p_4, hence $A_{41} = p_4$, i.e. the first four items are packed into different FFD bins. Thus from the FFD rule it follows that $p_1 \geq p_2 \geq p_3 > 1/2$. Then the sum of the sizes of the other four items being in the optimal bins of p_1 and p_2 is less than 1, thus the four smallest items fit together in one bin, contradicting Corollary 8.1.1 and completing the proof.

8.2 Appendix B

Here we give the omitted case (Case $2/11 < X \leq 1/5$, and $OPT \leq 18$) from the proof of the FFD algorithm.

If $X \leq 1/5$, the general investigations (apart from one very hard branch) will be significantly easier under the assumption $OPT \geq 19$, by making use of the fact that in most cases a large number of optimal bins provides us with large total reserve. In this sense, less flexibility makes the cases with small OPT values harder, despite that they look like "just something finite".

Therefore, in this section our goal is to prove that if $X \leq 1/5$, then $OPT \leq 18$ is impossible. Due to our previous calculations (given in Remark 3.1.3) only the following cases remain to consider: $OPT = 10 + 4k$, while $FFD = 13 + 5k$, for some $k \in \{0, 1, 2\}$, thus these equalities are assumed in this section. First we find some properties that must hold in this case. Since all items fit in the optimal packing into $10+4k$ optimal bins, it follows that $\sum_{k=1}^{n} p_k \leq 10+4k$. Recall that item X does not fit into any previous FFD bin, thus we get

$$l(B_i) > 1 - X, \quad i = 1, \ldots, 12 + 5k. \tag{8.1}$$

Lemma 8.2.1 *For $1 \leq j \leq 3$, the sum of the levels of any j bins among the first $12+5k$ FFD bins is strictly less than $(j-1)(1-X)+5X$.*

Proof. Suppose that there are j bins among the first $12 + 5k$ FFD bins with total level at least $(j-1)(1-X)+5X$. The level of any other FFD bin, except the last one, is bigger than $1 - X$, thus we get

$$\begin{aligned} 10 + 4k \geq \sum_{k=1}^{n} p_k &> (j-1)(1-X) + 5X + (12 + 5k - j)(1-X) + X \\ &= 10 + 4k + (k+1)(1-5X) \geq 10 + 4k, \end{aligned}$$

which is a contradiction. □

Corollary 8.2.1 (a) *Let B_i be an arbitrary FFD bin. Then $l(B_i) < 5X$, thus each FFD bin contains at most four items.*
(b) *Let B_i and B_j be two arbitrary FFD bins. Then $l(B_i) + l(B_j) < 1 + 4X \leq 2 - X$.*
(c) *Let B_i, B_j and B_k be three arbitrary FFD bins. Then $l(B_i) + l(B_j) + l(B_k) < 2 + 3X$.*

Proof. We apply the previous lemma with $j = 1$, $j = 2$, or $j = 3$, and the fact $X \leq 1/5$ (and that the level of the last FFD bin is only X). □

Lemma 8.2.2 *Let B_i and B_j with $1 \leq i \leq 12+5k$ and $j = i+1$ be two consecutive FFD bins at any time of executing the algorithm before X has arrived. Let the next item be U. If B_i contains more items than B_j, then U fits into B_j.*

Proof. If B_j contains zero or one item, the statement follows. From Corollary 8.2.1 (a) we already know that no FFD bin contains five or more items. Thus, it follows that B_j contains two or three items.

Case 1. B_i contains four items. Because $A_{i,2} + A_{i,3} + A_{i,4} \geq 3X > 1/2 \geq A_{j,2}$, it follows that $A_{j,2}$ would fit into the previous bin if B_i contained only $A_{i,1}$ at this moment. Thus $A_{i,2}$ comes before $A_{j,2}$ and therefore $A_{i,2} \geq A_{j,2}$. Suppose first that B_j contains two items. Then U fits into the j-th bin since $U \leq A_{i,3}$. Now suppose that B_j contains three items right before the arrival of U. If $A_{j,3} > A_{i,3}$ then $A_{j,3}$ comes before $A_{i,3}$ and does not fit into the i-th bin, thus $A_{j,3} > A_{i,3} + A_{i,4} \geq 2X$. Then the level of the j-th bin is at least $A_{j,1} + A_{j,2} + A_{j,3} \geq 3A_{j,3} \geq 6X$, a contradiction. Thus $A_{j,3} \leq A_{i,3}$. Because $U \leq A_{i,4}$, the item U fits into the j-th bin.

Case 2. B_i contains three items; then B_j contains two items. If $A_{i,1} + A_{i,2} \geq A_{j,1} + A_{j,2}$, then because $A_{i,3}$ fits into the i-th bin, and $A_{i,3} \geq U$, it follows that U fits into the j-th bin. Now assume that $A_{i,1} + A_{i,2} < A_{j,1} + A_{j,2}$. Because $A_{i,1} \geq A_{j,1}$, also $A_{i,2} < A_{j,2}$ follows. Thus $A_{j,2}$ is packed before $A_{i,2}$, and it did not fit into the i-th bin, thus $A_{i,1} + A_{j,2} > 1$ and $A_{j,1} \geq A_{j,2} > 1 - A_{i,1}$. If $A_{i,1} \geq 3X$, then the level of the i-th bin is at least $A_{i,1} + 2X \geq 5X$, this contradicts Corollary 8.2.1 (a). Thus let $A_{i,1} = 3X - \alpha$ with some $\alpha > 0$. Then

$$l(B_i) + l(B_j) > 3X - \alpha + 2X + 2(1 - 3X + \alpha) = 2 - X + \alpha > 2 - X$$

contradicting Corollary 8.2.1 (b). □

Corollary 8.2.2 *Let B_i and B_j be two consecutive FFD bins with $j = i + 1 < 13 + 5k$. Then having executed the algorithm, B_i cannot contain more items than B_j.*

Lemma 8.2.3 *Suppose that some bin, say B_i, contains four items after the execution of algorithm FFD. Let $U = A_{i,1}$ be the first item in this bin. Then there is no fallback item after U.*

Proof. Suppose that Z comes later than U (not necessarily right after U), and is packed into some previous bin. Because B_i contains four items, it follows that $l(B_i) \geq U + 3X$. On the other hand it holds that the level of $bin(Z)$ exceeds $1 - U + Z \geq 1 - U + X$. We get

$$l(B_i) + l(bin(Z)) > (1 - U + X) + (U + 3X) = 1 + 4X,$$

contradicting Corollary 8.2.1 (b). □

It follows that, in a bin B_i with four items, the items are four consecutive ones, i.e. they are in the order right after each other, and are packed into this bin. Since no five items can be packed into the same FFD bin, the next item, say U_1, is packed into the next bin. If this next FFD bin is not the last FFD bin, then this bin also will contain four items according to Corollary 8.2.2, thus U_1 and the next three items are packed into one bin, and so on.

Definition 8.2.1 *A fallback item Z is called Y**-fallback** if Z comes later than Y (not necessarily right after Y), Y is a first item in some FFD bin, and Z is packed into an earlier bin than $bin(Y)$.*

Lemma 8.2.4 *If $Y \leq 2X$, then there can be at most one Y-fallback item.*

Proof. Suppose Z is a Y-fallback item. By definition, Y is the first item in its bin. Then, since $Y \leq 2X$, and since no item of size bigger than $1 - 2X$ exists, it follows that the bin where Z will be packed contains at this moment at least two items. It cannot contain at this moment three items, since then we get a contradiction to Lemma 8.2.3. Thus $bin(Z)$ already contains exactly two items just when Z is packed there, let these items be A and B; and Z is the last item in this bin.

Suppose that there is another Y-fallback item, say U. Then $bin(Z)$ and $bin(U)$ are different bins, since U and Z are both third items in their bins, moreover both $bin(Z)$ and $bin(U)$ are earlier bins than $bin(Y)$. At the moment when U is packed, $bin(U)$ already contains exactly two items, let they be C and D; and U is the last item in this bin.

From Corollary 8.2.2 and Lemma 8.2.3 it follows that after the entire running also $bin(Y)$ contains exactly three items; let the other two items in this bin be S and T. We assume, without loss of generality, that S precedes T and that $bin(Z)$ is a bin earlier than $bin(U)$.

Case 1. U comes before Z. Then since U is not packed into $bin(Z)$, it follows that U did not fit into $bin(Z)$, thus $l\,(bin(Z)) > 1 - U + Z \geq 1 - U + X$. Also it follows that $l\,(bin(U)) > 1 - Y + U \geq 1 - 2X + U$, thus we get

$$l\,(bin(Z)) + l\,(bin(U)) > (1 - U + X) + (1 - 2X + U) = 2 - X,$$

which contradicts Corollary 8.2.1 (b).

Thus in the following we suppose that Z comes before U.

Case 2. Z comes after S (where S is the second item in $bin(Y)$). Then U comes also after S, since Z precedes U. Then $A + B + Y > 1$, and $C + D + S > 1$, since Y did not fit into $bin(Z)$, and S did not fit into $bin(U)$. Thus we get

$$\begin{aligned}
&l\,(bin(Z)) + l\,(bin(U)) + l\,(bin(Y)) \\
&= (A + B + Z) + (C + D + U) + (Y + S + T) \\
&= (A + B + Y) + (C + D + S) + (Z + U + T) \\
&> 1 + 1 + 3X,
\end{aligned}$$

contradicting Corollary 8.2.1 (c).

Case 3. Z comes before S and T. Then it follows that U must precede both S and T (since when the first item among U, S and T comes, $bin(Z)$ contains three items, and $bin(U)$ contains only two items, thus the just arriving item fits into $bin(U)$ by Lemma 8.2.2). Then since D is not packed into $bin(Z)$ (no matter whether B is already packed here or not), it follows that $A + B + D > 1$, furthermore $C \geq Y$, and $\min\{Z, U\} \geq \max\{S, T\}$, moreover $Y + S + T > 1 - X$, hence we get

$$\begin{aligned}
l\,(bin(Z)) + l\,(bin(U)) &= (A + B + Z) + (C + D + U) \\
&= (A + B + D) + (C + Z + U) \\
&> 1 + (Y + S + T) \\
&> 1 + 1 - X = 2 - X,
\end{aligned}$$

contradicting Corollary 8.2.1 (b). □

Corollary 8.2.3 *Suppose that $Y \leq 2X$ and Z is a Y-fallback item. Then both $bin(Z)$ and $bin(Y)$ are 3-bins.*

After this treatment, let Y denote in this section the first item in the bin before the last FFD bin, i.e. $Y = A_{12+5k,1}$. We already know that $Y > 1/5$, otherwise there would be five items in $bin(Y)$, contradicting Corollary 8.2.1 (a).

Lemma 8.2.5 *The size of Y is at most $1/3$.*

Proof. Suppose to the contrary that $Y > 1/3$. Let g denote the number of items bigger than a half (which are called *giant* items). Let l be the number of *large* items, where an A is defined to be large if $1/3 < A \leq 1/2$. Since the g giant items are packed into different FFD bins, it follows that there are at least $2\,(11 + 5k - g) + 1$ large items in the next FFD bins (and some more can also occur in the bins of the giant items). In an optimal packing a giant and a large item cannot be packed into a common bin (since there are at least three items in each optimal bin, and too small room remains for the third item), the giant items are packed into different bins, and at most two large items can be packed into an optimal bin. It follows that $OPT \geq 12 + 5k$, i.e. even the giant and large items cannot fit into $10 + 4k$ optimal bins, a contradiction. □

Remark 8.2.2 *The stronger claim $A_{11+4k,1} \leq 1/3$ similarly holds but we will not use this property.*

Lemma 8.2.6 *The five smallest items do not fit into one bin, thus each (FFD or optimal) bin contains at most four items.*

Proof. Consider $bin(Y)$, i.e. the $(12 + 5k)$-th FFD bin.

By $Y \leq 1/3$ and Corollary 8.2.1 (a), $bin(Y)$ contains three or four items.

If $bin(Y)$ is a 4-bin, then these four items plus item X in the last FFD bin are the last five items, according to Lemma 8.2.3. It follows that the five smallest items do not fit into one bin.

Suppose that $bin(Y)$ is a 3-bin. We have seen in Lemma 8.2.5 that $Y \leq 1/3 \leq 2X$. Then by Lemma 8.2.4 there can be at most one fallback item after Y, thus the three items in $bin(Y)$, plus the possible fallback item after Y, plus the last item X are the four or five smallest items, and they cannot fit into one bin, and the statement follows. □

Now let Q be the set of items which come after Y. There are at most four items in Q. Thus essentially it remains to show in the next two subsections that the following situation is impossible: $OPT = 10 + 4k$, $FFD = 13 + 5k$, and there are at most four items with sizes at most $1/5$. We mainly will follow the lines of the third section (where all items were bigger than $1/5$).

Subcase $Y > 1/4$.

Recall that Y is the first item in the bin before the last FFD bin. We have seen in Lemma 8.2.5 that $Y \leq 1/3$. The bin of Y cannot contain five items, by Corollary 8.2.1 (a). Then it follows that $bin(Y)$ contains three or four items. So the following cases are possible:

Case a, $bin(Y)$ is a 4-bin. Then there is no fallback item after Y. Since $Y > 1/4$, it follows that $bin(Y)$ is the only one 4-bin in the FFD packing. (The previous bin contains fewer than four items.)

Case b, $bin(Y)$ is a 3-bin, and there is no fallback item after Y. In this case the four smallest items do not fit into a common bin, thus there is no 4-bin at all.

Case c, $bin(Y)$ is a 3-bin, and there exists exactly one fallback item after Y.

Note that in cases b, and c, $Y > \frac{1-X}{3} > 1/4$ holds, otherwise after three items X still fits into $bin(Y)$. We use the next classification on the items of $L \backslash Q$. (Note that Y is the smallest item in the set $L \backslash Q$.) The Q items will get weight 9.

Name	Class	Weight
Big	$\frac{1-X}{2} < B$	18
Medium	$\frac{1-Y}{2} < M \leq \frac{1-X}{2}$	15
Small	$Y \leq S \leq \frac{1-Y}{2}$	12

Classification on items $L \backslash Q$.

Lemma 8.2.7 *Only the optimal bin-types listed below are possible. Moreover, no* 4*-bin exists in Case b, and neither an* $(M, S, 2Q)$ *nor a* $(3S, Q)$ *optimal bin is possible in Case c. Furthermore there is at least* 8 *reserve in the optimal bins.*

OPT

18	B	1			1	1	1							1						
15	M		1		1			2	1	1					1	1				
12	S	2	2	3		1			1		2	1			1		3	2	1	
9	Q				1	1	2	1	1	2	1	2	3	3	2	3	1	2	3	4
	r	2	5	8	2	5	8	5	8	11	11	14	17	-1	-1	2	-1	2	5	8

Proof. Each optimal bin contains three or four items, and it can contain four items only if it contains also (one or more) Q items. If the bin does not contain Q items, then it cannot contain two or more M or B items, thus in this case only $(B, 2S)$, $(M, 2S)$ or $(3S)$ bins are possible. Now let us consider the Q-bins. Among the 3-bins a $(2B, Q)$ bin is impossible. Now let us consider the 4-bins. (Every 4-bin must contain a Q-item since $Y > 1/4$.)

From $B + S + 2Q > \frac{1-X}{2} + Y + 2X > \frac{3}{4} + \frac{3}{2}X > 1$ it follows that if the 4-bin contains a B item, then only a $(B, 3Q)$ bin is possible. A 4-bin cannot contain two or more M items since the remaining room in the bin is smaller than $Y < 2X$. If the bin contains one M item, then an $(M, 2S, Q)$ bin is impossible since $M + 2S + Q > \frac{1-Y}{2} + 2Y + X = \frac{1}{2} + \frac{3}{2}Y + X > 1$.

Regarding Case c, we know that $Y > \frac{1-X}{3}$ holds, hence a $(3S, Q)$ bin is impossible; and an $(M, S, 2Q)$ bin is also impossible, since $M + S + 2Q > \frac{1-Y}{2} + Y + 2X = \frac{1}{2} + \frac{1}{2}Y + 2X > \frac{1}{2} + \frac{1-X}{6} + 2X = \frac{2}{3} + \frac{11}{6}X > 1$.

Considering the possible optimal bins, only some 4-bins can have shortage, and only if there is at least one Q item in the bin. Every other bin has at least 2 reserve. Since there are at least ten optimal bins, the total reserve is at least $6 \cdot 2 - 4 = 8$. □

Let b be the number of (B, M) and (B, S) FFD bins. We say that a B item is a *big* B if it is bigger than a half.

Observation 8.2.1 *We have* $b \leq 5$*, and there is at least* 5 *reserve in the optimal bin of each big* B *item.*

Proof. In the first $b-1$ bins among the (B, M) and (B, S) FFD bins the B items are big B items, otherwise the two last B items in consideration fit into a common bin. Each big B item is packed into a 3-bin in the optimal packing, and there must be at least one Q item in the bin as $Y > 1/4$. Since there are only at most four Q items, $b \leq 5$ follows. Regarding the second statement, a big B item cannot be packed into a (B, M, Q) bin either, thus a big B can be packed only into a (B, S, Q) or $(B, 2Q)$ optimal bin, thus the assertion follows. □

Now we consider the three possible cases after each other:
Case a. $bin(Y)$ is a 4-bin. We list the possible FFD bins below. The Q items are only in the last two FFD bins, and $bin(Y)$ is a $(S, 3Q)$ bin. An FFD 2-bin must contain some B item, since $2M + X \leq 1$. Thus only the following FFD bins are possible:

18	B	2	1	1	1				
15	M		1			1			
12	S			1	2	2	3	1	
9	Q							3	1
	s	0	-3	-6	6	3	0	3	0

Here $bin(Y)$ has 3 surplus. Only (B, M) and (B, S) FFD bins have shortage.

If $b \leq 1$, then the total shortage is at most $6 - 3 = 3$ because of the FFD bins, and we have at least 8 reserve in the optimal bins, thus the shortage is covered.

Otherwise $1 < b \leq 5$. The total shortage is at most $6b - 3$ in the FFD bins, while in the optimum packing there are at least $b - 1$ bins containing at least one Q item together with a big B item, and having at least 5 reserve in each, there are at most $5 - b$ other Q-bins with shortage 1 in each, and there are at least six further optimal bins with at least 2 reserve in each. Thus the total reserve is at least $5(b - 1) - (5 - b) + 6 \cdot 2 = 6b + 2$, hence we are done.

Case b. $bin(Y)$ is a 3-bin, and there is no fallback item after Y. Then there is no (optimal or FFD) 4-bin at all, since no four items fit into one bin. The possible FFD bins are the same as in case a, except $bin(Y)$.

18	B	2	1	1	1				
15	M		1			1			
12	S			1	2	2	3	1	
9	Q							2	1
	s	0	-3	-6	6	3	0	-6	0

Now the three Q items are in the last FFD bin and in the previous bin. Since there is no 4-bin in the optimal packing each optimal bin has at least 2 reserve, thus we have in total at least 20 reserve from the optimal bins. Only (B, M) and (B, S) FFD bins, and the $(S, 2Q)$ FFD bin have shortage, thus if $b \leq 2$ then the total shortage is at most 18, and thus it is covered. Hence suppose that $b > 2$. Then there are at least two FFD bins of type (B, M) or (B, S); let these bins and the items in them be denoted as (B_1, A_1) and (B_2, A_2), where B_1 and B_2 are B items, while A_1 and A_2 are M or S items. Then $B_1 + B_2 > 1$, otherwise they would be packed into the same bin. Let us consider the optimal bins of B_1 and B_2; these are some 3-bins, let they be denoted as (B_1, C_1, C_2) and (B_2, C_3, C_4), where C_1, C_2, C_3 and C_4 are some items. Then from the inequalities $B_1 + C_1 + C_2 \leq 1$, $B_2 + C_3 + C_4 \leq 1$ and $B_1 + B_2 > 1$, it follows that $C_1 + C_2 + C_3 + C_4 \leq 1$, which is a contradiction since no four items fit into a common bin.

Case c. $bin(Y)$ is a 3-bin, and there exists one fallback item after Y. Let this fallback item be denoted as Z. There are four Q items, namely the Y-fallback item Z, the item X, and two more

Q-items in $bin(Y)$, let these latter two be denoted as Q_1 and Q_2. From Corollary 8.2.3 it follows that $bin(Z)$ is a 3-bin. This bin can be only a (B, M, Q), (B, S, Q), $(2M, Q)$ or (M, S, Q) bin, since $2B + X > 1$ and $2S + Y \leq 1$. Since $Y + Q_1 + Q_2 + X > 1$ and these are the four smallest items in $L \setminus \{Z\}$, no four items of $L \setminus \{Z\}$ fit into one bin. Since $X \leq Z$, it also holds that no four items of $L \setminus \{X\}$ fit into one bin. Thus there can be at most one optimal 4-bin, and then both Z and X are packed into this bin. The possible FFD bins are as follows:

18	B	2	1	1	1	1			1				
15	M		1		1		2	1		1			
12	S			1		1		1	2	2	3	1	
9	Q				1	1	1	1				2	1
	s	0	-3	-6	6	3	3	0	6	3	0	-6	0

Recall that in this case neither $(M, S, 2Q)$ nor $(3S, Q)$ optimal bins exist, only a $(B, 3Q)$ optimal bin can have shortage, and naturally there can be at most one such bin. In any other optimal bin there is at least 2 reserve, thus in the optimal bins we have in total at least $9 \cdot 2 - 1 = 17$ reserve. If $b \leq 1$, then the total shortage is covered.

Suppose $b \geq 2$. Then we apply similar calculation as in case b. There are at least two FFD bins of type (B, M) or (B, S); let these bins and the items in them be denoted as (B_1, A_1) and (B_2, A_2), where B_1 and B_2 are B items, while A_1 and A_2 are M or S items. Then $B_1 + B_2 > 1$ holds. Let the optimal bins of B_1 and B_2 be denoted as (B_1, C_1, C_2) and (B_2, C_3, C_4), where C_1, C_2, C_3 and C_4 are some items. Then it follows that $C_1 + C_2 + C_3 + C_4 \leq 1$. (If $bin(B_1)$ or $bin(B_2)$ in the optimal packing would be a 4-bin, then similarly we get at least five items that fit into one bin, a contradiction.) Thus Z and X are among C_1, C_2, C_3 and C_4, and these Q items are packed into the optimal 3-bins of B_1 and B_2. Since there are only four Q items, it follows that there is no $(B, 3Q)$ optimal bin. Then there is at least $10 \cdot 2 = 20$ reserve in the optimal bins, thus all shortage is covered if $b = 2$.

Thus $3 \leq b \leq 5$ follows. Then similarly to the previous calculations we get that there are at least three FFD bins of type (B, M) or (B, S), let the B items be denoted as B_i for $i = 1, 2, 3$. The sum of any two among them is bigger than 1. Each B_i is packed into a 3-bin in the optimal packing, let these optimal bins be denoted as (B_1, C_1, C_2), (B_2, C_3, C_4), and (B_3, C_5, C_6). Then the next inequalities hold: $C_1 + C_2 + C_3 + C_4 \leq 1$, $C_1 + C_2 + C_5 + C_6 \leq 1$ and $C_3 + C_4 + C_5 + C_6 \leq 1$. Then Z (and X) must be in each of the sets $\{C_1, C_2, C_3, C_4\}$, $\{C_1, C_2, C_5, C_6\}$, and $\{C_3, C_4, C_5, C_6\}$. This is a contradiction.

Subcase $1/5 < Y \leq 1/4$

In this subcase we use the same classification for items $L \backslash Q$ which was already used for the whole item-set in Section 3 (where X was supposed to be between $1/5$ and $1/4$), but now Y has the role that X had in Section 3. The items which come after Y are denoted by Q. Note that $Y > 1/5$, and the last item X (which is a Q item) is surely smaller than $1/5$, but some Q items coming after Y and before X can be smaller or bigger than $1/5$ (but we do not make further investigations or assumptions regarding the sizes of these items).

Since $Y \leq 1/4$, $bin(Y)$ is definitely a 4-bin, and there is no fallback item after Y. There are exactly four Q items (taking also X into account).

Let now Z be the smallest regular item in the interval $(\frac{1-Y}{3}, \frac{1}{3}]$, if there exists at least one such item; otherwise let $Z = 1/3$.

The same optimal bins as in Section 3 are again possible, and in any optimal bin Q can stand in the place of a T item. Some further optimal bin-types are also possible if a bin contains one or more Q items, and there can be at most 4 new bins at the same time, since there are only 4 new items.

By our minimality assumption on the items, it holds that in each optimal bin there is at least one item from $L \backslash Q$ (otherwise by deleting the Q items we could get a smaller counterexample with $OPT' \le 9 + 4k$ and $FFD' = 12 + 5k$).

We apply the proof of the previous section, by some modifications where needed.

Name	Class	Weight
Giant	$\frac{1}{2} < G$	23
Big	$\frac{1-Y}{2} < B \le \frac{1}{2}$	18
Medium	$\frac{1-Z}{2} < M \le \frac{1-Y}{2}$	15
Small	$Z \le S \le \frac{1-Z}{2}$	12
Tiny	$Y \le T < Z$	9

Lemma 8.2.8 *Only the following bin-types are possible.*

OPT1, optimal bins without a Q item

23	G	1	1				
18	B			1	1	1	1
15	M			1			
12	S	1			2	1	
9	T	1	2	1		1	2
	r	0	3	2	2	5	8

23	G					
18	B					
15	M	2	1	1	1	1
12	S		2	1		
9	T	1		1	3	2
	r	5	5	8	2	11

23	G							
18	B							
15	M							
12	S	3	2	2	1	1		
9	T		2	1	3	2	4	3
	r	8	2	11	5	14	8	17

OPT2, optimal 3-bins containing at least one Q item

23	G	1	1	1														
18	B				2	1	1	1	1									
15	M					1				2	1	1	1					
12	S	1					1				1			2	1	1		
9	T		1					1				1			1		2	1
9	Q	1	1	2	1	1	1	1	2	1	1	1	2	1	1	2	1	2
	r	0	3	3	−1	2	5	8	8	5	8	11	11	11	14	14	17	17

OPT3, optimal 4-bins containing at least one Q item

18	B	1				1	1						1					
15	M		1					1	1					1				
12	S			1		1		1		2	1				3	2	1	
9	T				1		1		1		1	2	2	2		1	2	3
9	Q	3	3	3	3	2	2	2	2	2	2	2	1	1	1	1	1	1
	r	−1	2	5	8	−4	−1	−1	2	2	5	8	−1	2	−1	2	5	8

FFD

23	G	1	1	1	1				1	1										
18	B	1				2	1	1			1	1	1	1						
15	M		1				1				1				2	1	1	1	1	
12	S			1				1	1			2	1			2	1			
9	T				1				1	2	1		1	2	1		1	3	2	
	s	5	2	-1	-4	0	-3	-6	8	5	6	6	3	0	3	3	0	6	-3	

12	S	3	2	2	1	1			
9	T		2	1	3	2	4	1	
9	Q							3	1
	s	0	6	-3	3	-6	0	0	0

Proof. In the first table OPT1, we list all (and the same) bin-types as in the previous section. In the second table OPT2, we list the new possible optimal 3-bins, if the bin contains one or more Q items. Since $M > \frac{1-Z}{2} \geq 1/3$, the (G, B, Q) and (G, M, Q) bins are impossible. If the 3-bin does not contain a G item and contains at least one Q item, excluding $(3Q)$ any bin-type is possible which is later in the lexicographical order than $(2B, Q)$, and this bin is also possible, since $2B + Q$ can be smaller than 1.

The possible optimal 4-bins are listed in Table OPT3. A 4-bin cannot contain a G item. A 4-bin cannot contain only (the four) Q items. If a 4-bin contains three Q items, the fourth item cannot be a G item. Suppose that the bin contains two Q items. Since $M > 1/3$, the further items cannot be B or M items. Finally suppose that the 4-bin contains only one Q item. Then (B, S, T, Q) or (M, S, T, Q) bins are impossible, since $M + S + T + Q > \frac{1-Z}{2} + Z + Y + X = \frac{1}{2} + \frac{1}{2}Z + Y + X > \frac{1}{2} + \frac{1-Y}{6} + Y + X = X + \frac{5}{6}Y + \frac{2}{3} > X + 5/6 > 1$. Thus if the bin contains a B or an M item, the remaining two items must be T items. If the bin does not contain a B or M item, then any remaining case is possible.

Regarding the possible FFD bins, the same bin-types are possible as in the previous section, except that there is a bin just before the last FFD bin with one Y and three Q items. □

We emphasize again that we do not state that all such bin-types occur, we simply listed all bin-types which we could not exclude from the consideration. Now, since the possible bin-types are almost the same as in the previous section (where $X > 1/5$ was assumed), we can almost repeat the proof that we have done there. Since now some further bin-types are also possible, we must make a more careful analysis, but on the other hand we have the advantage that now more types of optimal bins occur, and this makes the investigation easier. We need the next observation which is very similar to what we have stated in the previous section.

Observation 8.2.2 *If at least one S item occurs, then there is an S item with size exactly Z in the last $(S, .)$ FFD bin (if there exists an $(S, .)$ FFD bin). Moreover two $(B, .)$ bins or two $(M, .)$ bins or two $(S, .)$ bins, both having shortage, cannot occur at the same time.*

Proof. The proof is the same as the proof of Observations 3.2.6 and 3.2.7. □

Lemma 8.2.9 *Suppose that there is no (G, T) FFD bin, there is no $\{(G, S), (G, S, T)\}$ cobin, and there is no $\{(G, S), (G, S, Q)\}$ cobin. Then all shortage is covered.*

Proof. Let us denote the G and S items of the (G, S) FFD bins as G' and S', respectively. Let us increase the weight of the G' items by 1 and decrease the weight of the $G \setminus G'$ items by 2.

By the assumption of the lemma any G' item is packed into a $(G, 2T)$ or (G, T, Q) or $(G, 2Q)$ optimal bin. Each such bin still has (after the modification) at least 2 reserve.

Now consider a (G, S, T) or (G, S, Q) optimal bin. Since the G item of this bin cannot be a G' item (again by the assumption of this lemma), thus (after the modification of the weights of the G and $G \setminus G'$ items) the bin has 2 reserve.

It follows that the (G, S) FFD bins do not have shortage, and each optimal non-Q bin has at least 2 reserve. Let the optimal bins having at least 2 reserve be called **good** bins, the other optimal bins be called **bad** bins. Then there are at most four bad bins, since there are four Q items.

We state that we have at least 8 reserve in total. Indeed, using that $OPT \geq 10$, if there is no bad bin, then we have at least $10 \cdot 2 = 20$ reserve. If there is only one bad bin (in the worst case it has 4 shortage) we have at least $9 \cdot 2 - 4 = 14$ reserve. If there are two bad bins, we have at least $8 \cdot 2 - 2 \cdot 4 = 8$ reserve. If there are three bad bins, then only one of them can be a $(B, S, 2Q)$ bin with 4 shortage, since there are only four Q items, and the shortage of any other bad bin is only 1, thus we have at least $7 \cdot 2 - (4 + 1 + 1) = 8$ reserve. Finally, if there are four bad bins, then the shortage of any bad bin is 1, and we have at least $6 \cdot 2 - 4 = 8$ reserve.

Now we are going to calculate how much shortage can be caused by the FFD bins. A (B, M) and a (B, S) FFD bin cannot occur at the same time, furthermore if there exists an $(M, 2T)$ FFD bin (there can be only one) then there cannot be any $(S, .)$ FFD bin (with shortage). Thus, applying Observation 8.2.2, if there exists a (B, S) and also a $(2S, T)$ FFD bin then there can be at most 12 shortage in the worst case.

Case a. Suppose there exists an $(M, 2T)$ FFD bin. Then there cannot be a (B, S) FFD bin (since an M item fits into the bin of the B item, before the S item), and there can be neither a $(2S, T)$ nor an $(S, 2T)$ bin, since the first S item of such a bin would be packed into the $(M, 2T)$ bin after the M item before the two T items. Thus if there exists an $(M, 2T)$ FFD bin, the total shortage is at most 6, thus it is covered by the total reserve.

We suppose in the following that an $(M, 2T)$ FFD bin does not exist.

If there are neither (B, M) nor (B, S) FFD bins, then the uncovered shortage is at most 6, hence it would be covered by the reserve. Thus a (B, M) or a (B, S) FFD bin must exist. Let these items be denoted as B' and S', or B' and M', respectively. Then B' is the last regular B item (since any two B items fit into a common bin), any $(2B)$ FFD bin may appear before it, and any B item from such a bin precedes B'. We state that B' is either the smallest or the second smallest B item. Indeed, any B item smaller than B' can be packed only into some (G, B) FFD bin, and there can be at most one such bin, otherwise the shortage is covered. Thus $2B > 1 - X$, and consequently a $(2B, Q)$ optimal bin is impossible; it follows that there is not bad 3-bin. A $(B, 3Q)$ optimal bin (with 1 shortage) is not possible either, since then there can be at most one other bad bin with one Q item (and 1 shortage), thus we have at least $8 \cdot 2 - 2 = 14$ reserve, and it would cover the total shortage.

Case b. Suppose that a $(3S, Q)$ bad bin does not exist. We state that there can be at most one bad 4-bin. Indeed, each remaining possible bad bin can be only of type $(B, S, 2Q)$, $(B, T, 2Q)$, $(M, S, 2Q)$ or $(B, 2T, Q)$. Suppose there are two such bins, both containing B items, thus the two bins are, say, (B_1, A_1, A_2, A_3) and (B_2, A_4, A_5, A_6). Then using that no five items fit into a common bin, it follows that any three of the eight items fit into a bin, thus for example $1 \geq B_1 + B_2 + A_1 > 1$, a contradiction. If (at least) one of the two bins is $(M, S, 2Q)$, then the biggest item among the eight items is a B or M item, the second biggest is an M item, and there is also an S item among the eight items, hence using $2M + S > 1$ we get again a contradiction. Thus there can be at most one bad bin among the optimal bins, and so we get at least $9 \cdot 2 - 4 = 14$ reserve in the optimal bins (as we have seen before), thus the total shortage is covered.

We suppose in the following that there exists a $(3S, Q)$ optimal bad bin.

Case c. Suppose there exists an $(S, 2T)$ FFD bin. By the definition of Z, the S item in this bin has size Z, thus it is an S item with smallest size; and any T item is smaller than any S item; hence we get $3S + X > 1$. It follows that a $(3S, Q)$ optimal bin is impossible, a contradiction.

Thus there is no $(S, 2T)$ FFD bin.

If there is no $(2S, T)$ FFD bin either, then the total shortage is at most 6 and it is covered, thus there must be a $(2S, T)$ FFD bin; it follows that this is the last $(S, .)$ bin, by Observation 8.2.2. Let the items of this bin be denoted as (S_1, S_2, T_1), where S_1 precedes S_2. There also exists a (B, S) FFD bin, otherwise the total shortage is covered.

Suppose that $S_1 \leq 2X$. Then according to Lemma 8.2.4 (and also taking into account that any bin later than the (S_1, S_2, T_1) bin is a 4-bin or the last FFD bin) there is at most one fallback item after S_1. If there is a fallback S item after S_1, say S_3, then S_1, S_2 and S_3 are the three smallest S items, and they together are not smaller than $S_1 + S_2 + T_1 > 1 - X$, thus a $(3S, Q)$ optimal bin is impossible, a contradiction. If there is a fallback T item after S_1, or no fallback item occurs after S_1, then S_1 and S_2 are the two smallest S items, and again it follows that there is no $(3S, Q)$ optimal bin, which is a contradiction. Thus $S_1 > 2X$ follows.

The uncovered shortage is only 9, on the other hand the total reserve is at least 8, hence there cannot occur any FFD bin with positive surplus. Because a (B, S) bin exists, a $(B, 2T)$ FFD bin does not exist. Furthermore there is no M item in bins later than the (B, S) bin. Only the next FFD bins remain possible:

FFD

23	G	1	1							
18	B			2	1					
15	M	1								
12	S		1		1	3	2			
9	T						1	4	1	
9	Q								3	1
	s	2	-1	0	-6	0	-3	0	0	0

Consider a $(3S)$ FFD bin. Since any two S items fit into the same bin, it follows that the first two items in the $(3S)$ bin are not smaller than S_1, which is bigger than $2X$. Thus the sum of the three S items in this bin is bigger than $2X + 2X + X = 5X$, contradicting Corollary 8.2.1 (a). It follows that a $(3S)$ FFD bin does not exist. We state that an S item in some (G, S) or (B, S) FFD bin cannot be smaller than S_1. This can be seen as follows. If a G item is packed with a Q item, the reserve increases (as there are fewer bad bins) and we are done. Otherwise, $G \leq 1 - 2Y$ and $S \leq 2Y$, so already the first S item can be packed with a G item. It follows that the two S items in the $(2S, T)$ FFD bin are the two smallest S items, and then it follows again that a $(3S, Q)$ optimal bin cannot occur, a contradiction. $\square$

Lemma 8.2.10 *If there exists a (G,T) FFD bin, then all shortage is covered.*

Proof. Suppose that there exists a (G,T) FFD bin. Then by Lemma 3.1.7, the G items from the (G,T) bins have the same size, and also the T items; let us denote them again by G' and T', respectively. From Corollary 3.1.4 it follows that the T' items are the biggest T items, since any item bigger than T' is also bigger than $2X$, but $T < 2X$; furthermore if there exists an S item, then $S > 2X$ must hold. From this fact it follows that an S item cannot be in a G-bin (the total size of the items in the bin would be more than $1/2 + 2X + X > 1$) thus the optimal bin of a G' item can be only a $(G, 2T)$, a (G,T,Q) or a $(G,2Q)$ bin.

Case a. There exists a $\{(G,T),(G,2T)\}$ cobin. Let the two T items of the optimal bin of one such cobin be denoted as T_1 and T_2. Then by Corollary 3.1.5, since $\frac{1-Y}{2} < 2Y \leq 2T$, there are neither S nor M items, and each B item is bigger than $T_1 + T_2 \geq 2Y$. Suppose there exists an optimal 4-bin which contains a B item. Note that items T_1 and T_2 are packed into *another* optimal bin, since they are packed together into a 3-bin, namely together with a G' item. Then from Observation 3.1.2 it follows that five items fit into a common bin, a contradiction. Thus there is no optimal 4-bin which contains a B item. We state that a $(2B,Q)$ bin is not possible either. Indeed, if there exists a $(2B,Q)$ optimal bin, then using $B > 2Y$, the inequality $4Y + X \leq 1$ follows. But then the five smallest items, Y and the four Q items would fit into one bin. Thus only the next bins are possible (we changed the original weights of the classes):

OPT

26	G	1				1	1							
18	B		1					1	1					
9	T	2	2	4	3	1		1		2	1	1	2	3
9	Q					1	2	1	2	1	2	3	2	1
	r	0	8	8	17	0	0	8	8	17	17	8	8	8

FFD

26	G	1	1		1				
18	B	1		2		1			
9	T		1		2	2	4	1	
9	Q							3	1
	s	8	−1	0	8	0	0	0	0

Now we increase the weight of the T' items by 1 and reduce the weight of the $G \setminus G'$ items by 2. Then no FFD bin has shortage. Let us consider the optimal bins. Before the modification of the weights no optimal bin had shortage. Since there are at most four T items in any optimal bin, no shortage is caused if the bin had at least 4 reserve or if the bin does not contain a T item. Thus shortage could be created only in $(G, 2T)$ or (G,T,Q) optimal bins if at least one T item in the bin is a T' item. But then the G item of this bin cannot be a G' item (since $G' + T' + X > 1$), thus the reserve of the optimal bin is decreased by at most 2, and at the same time it is also increased by 2. Thus there is no shortage.

Case b. There is no $\{(G,T),(G,2T)\}$ cobin, but there exists a $\{(G,T),(G,T,Q)\}$ cobin. Let the items of the optimal bin of this cobin be denoted as G_1, T_1 and Q_1. According to Corollary 3.1.5, every item of size above T' is also above $X + Y$. Then, since $2X + 2Y + Z > 2X + 2Y + \frac{1-Y}{3} = 2X + \frac{5}{3}Y + \frac{1}{3} > 2X + \frac{2}{3} > 1$ holds, it follows that $S \leq \frac{1-Z}{2} < X + Y$, thus there is no S item, and each M or B item is bigger than $T_1 + Q_1$. Then from Observation 3.1.2 it follows that there is no optimal 4-bin which contains a B or M item, since substituting this B or M item with T_1 and Q_1 (which are in another optimal bin) five items would be packed into one bin. Only the next bins are possible (we changed the original weights of the G items):

OPT

24	G	1							1	1												
18	B		1	1							2	1	1	1								
15	M		1		2	1						1			2	1	1					
9	T	2	1	2	1	2	4	3	1				1			1		2	1	1	2	3
9	Q								1	2	1	1	1	2	1	1	2	1	2	3	2	1
	r	2	2	8	5	11	8	17	2	2	-1	2	8	8	5	11	11	17	17	8	8	8

FFD

24	G	1	1	1			1								
18	B	1			2	1		1	1						
15	M		1			1		1		2	1	1			
9	T			1			2	1	2	1	3	2	4	1	
9	Q													3	1
	s	6	3	-3	0	-3	6	6	0	3	6	-3	0	0	0

Suppose there is only one (G, T) FFD bin. If there is no (B, M) FFD bin, then the total shortage is at most 6 caused by the (G, T) bin and an $(M, 2T)$ bin. On the other hand there are at least ten optimal bins, and there are among them at most four $(2B, Q)$ bins (since there are only four Q items), thus there is at least $6 \cdot 2 - 4$ reserve, hence the shortage is covered. If there exist both (B, M) and (G, B) FFD bins, then there is 3 further shortage, and also 6 surplus; we are done again. Finally if there is a (B, M) FFD bin but there is no (G, B) FFD bin, the B item of the (B, M) FFD bin is the smallest B item. Then a $(2B, Q)$ optimal bin is impossible. In this case the total shortage is at most 9, and on the other hand we have at least $10 \cdot 2$ reserve, the shortage is again covered.

Now suppose that there are at least two (G, T) FFD bins. Let the optimal bin of the G item of the second (G, T) FFD bin denoted as a (G_2, A, C) bin, then from Corollary 3.1.5 it follows that any B item is bigger than $A + C$.

Now suppose that there exists a $(2B, Q)$ optimal bin; let the items of this bin be denoted as B_1, B_2 and Q_2. Then the (G_1, T_1, Q_1), (G_2, A, C) and (B_1, B_2, Q_2) bins are three different optimal bins, where $B_1 > T_1 + Q_1$ and $B_2 > A + C$. Thus we obtain the contradiction that the five items A, C, T_1, Q_1, Q_2 fit into one bin. It follows that there is no $(2B, Q)$ optimal bin.

Since there is no $\{(G, T), (G, 2T)\}$ cobin, the optimal bin of any G' item contains a Q item, thus there are at most four (G, T) FFD bins. Then the total shortage is at most $4 \cdot 3 + 3 + 3 = 18$ (by the (G, T) bins, and possibly by a (B, M) and by an $(M, 2T)$ bin), and on the other hand we have at least 20 reserve by the optimal bins; we are done.

Case c. There is neither a $\{(G, T), (G, 2T)\}$ nor a $\{(G, T), (G, T, Q)\}$ cobin. Since we have seen in the beginning of the proof of this lemma that (G, S, T) and (G, S, Q) bins are impossible, the condition of this Case c means that there is one or there are two $\{(G, T), (G, 2Q)\}$ cobins, and there is no further (G, T) bin.

Each S, M, or B is bigger than $2X$ (since any S, M or B is bigger than the sum of the sizes of the two Q items being in one $(G, 2Q)$ bin, by Corollary 3.1.5).

Thus there cannot be a G-bin containing also an S item. Since $6X > 1$, the following (optimal or FFD) bins are impossible: $(B, 2S)$, $(M, 2S)$, $(3S)$, $(2S, 2T)$, $(B, S, 2Q)$, $(M, S, 2Q)$, $(2S, 2Q)$, $(3S, Q)$, $(2S, T, Q)$. There is no optimal bin containing three Q items, since there are four Q items and at least two of them are in a $(G, 2Q)$ optimal bin.

Now we increase the weight of the S items to be 15. The following bins remain possible:

OPT1

23	G	1												
18	B		1	1	1									
15	M		1			2	1	1	1					
15	S			1			1			2	1	1		
9	T	2	1	1	2	1	1	3	2	1	3	2	4	3
	r	3	2	2	8	5	5	2	11	5	2	11	8	17

OPT2

23	G	1	1													
18	B			2	1	1	1	1								
15	M				1				2	1	1	1				
15	S					1				1			2	1	1	
9	T	1					1				1			1		2	1
9	Q	1	2	1	1	1	1	2	1	1	1	2	1	1	2	1	2
	r	3	3	-1	2	2	8	8	5	5	11	11	5	11	11	17	17

OPT3

18	B	1				1			
15	M		1				1		
15	S			1				1	
9	T	1	1	1	2	2	2	2	3
9	Q	2	2	2	2	1	1	1	1
	r	-1	2	2	8	-1	2	2	8

FFD

23	G	1	1	1	1				1													
18	B	1				2	1	1		1	1	1										
15	M		1				1			1			2	1	1	1						
15	S			1				1			1			1			2	1	1			
9	T				1				2	1	1	2	1	1	3	2	1	3	2	4	1	
9	Q																				3	1
	s	5	2	2	-4	0	-3	-3	5	6	6	0	3	3	6	-3	3	6	-3	0	0	0

Then the shortage is at most 14 in the FFD bins. Indeed, there can be at most two (G, T) FFD bins, since the optimal bin of such a G item is surely a $(G, 2Q)$ bin. Moreover (B, M) and (B, S) FFD bins cannot occur at the same time, and there can be only one such bin, furthermore $(M, 2T)$ and $(S, 2T)$ bins cannot occur at the same time, and there can be only one of them. Thus we have at most $2 \cdot 4 + 3 + 3 = 14$ shortage in total in the FFD bins.

In each optimal bin there is at least 2 reserve, except in the $(2B, Q)$, $(B, T, 2Q)$ and $(B, 2T, Q)$

optimal bins. If such a bin occurs, there can be at most two of them (since at least two Q items are in a $(G, 2Q)$ optimal bin, and only two further Q items remain). Thus we have at least $8 \cdot 2 - 2 = 14$ reserve, and it is enough to cover the total shortage.

In the following we suppose that a (G, T) FFD bin does not exist. $\square$

Lemma 8.2.11 *If there exists a* $\{(G, S), (G, S, T)\}$ *cobin, then all shortage is covered.*

Proof. Suppose that there exists a $\{(G, S), (G, S, T)\}$ cobin; let the S and T item of the optimal bin of this cobin be denoted as S_1 and T_1, respectively. Consider the (G, S) FFD bins. From Lemma 3.1.7(i) we know that all of their G items have equal size, and all of their S items also have equal size. Let us denote these items as G' and S', respectively. Since there exists a $\{(G', S'), (G', S_1, T_1)\}$ cobin, it follows (Corollary 3.1.5) that each item bigger than S' has size bigger than $S_1 + T_1 \geq Z + Y$. Then there cannot be an M item, since $M \leq \frac{1-Y}{2} < 2Y < Z + Y$. Also it follows by the same reason that $B > S_1 + T_1 \geq Z + Y$ holds for each B item. Hence there is no $(B, 2S)$ bin, since a B item is bigger than $Z + Y$, and each S item is at least Z, thus a B and two S items have total size bigger than $3Z + Y > 3 \cdot \frac{1-Y}{3} + Y = 1$.

There cannot be a 4-bin which contains a B item (otherwise replacing this B with S_1 and T_1 five items would fit into a bin, contradicting Observation 3.1.2).

Bin-type $(2B, Q)$ is impossible since $2B + Q \geq 2(Z + Y) + X \geq 4Y + X > 1$.

Thus only the next bins are possible (we increased the weight of class B to 23):

OPT1

23	G	1	1									
23	B			1	1							
12	S	1		1		3	2	2	1	1		
9	T	1	2	1	2		2	1	3	2	4	3
	r	0	3	0	3	8	2	11	5	14	8	17

OPT2

23	G	1	1	1								
23	B				1	1	1					
12	S	1			1			2	1	1		
9	T		1			1			1		2	1
9	Q	1	1	2	1	1	2	1	1	2	1	2
	r	0	3	3	0	3	3	11	14	14	17	17

OPT3

12	S	1		2	1		3	2	1	
9	T		1		1	2		1	2	3
9	Q	3	3	2	2	2	1	1	1	1
	r	5	8	2	5	8	-1	2	5	8

FFD

23	G	1	1			1	1										
23	B	1		2	1			1	1								
12	S		1		1	1		1		3	2	2	1	1			
9	T					1	2	1	2		2	1	3	2	4	1	
9	Q															3	1
	s	10	-1	10	-1	8	5	8	5	0	6	-3	3	-6	0	0	0

Since there exists a $\{(G', S'), (G', S, T)\}$ cobin, $S' > Z + Y - X$ follows.

If there exists a (B, S) FFD bin, let for a moment the size of this S item be denoted by S''. Then the S'' item comes after the S' items. (Indeed, an S item always fits into a bin which contains only a G' item at that moment, since $G' \leq 1 - 2Y$ because of the existence of the $\{(G', S'), (G', S, T)\}$ cobin, and $S \leq 2Y$ holds.) Then it can be supposed that all these S items (the S' items and the S'' item) have the same size, namely S', otherwise the size of S' items can be decreased to be S'', contradicting our minimality assumption. Thus, if there exists a (B, S) FFD bin (there can be only one), let the items of this bin be denoted as B' and S', respectively.

No optimal bin has shortage, except the $(3S, Q)$ bins.

Now we increase the weight of the S' items by one, and decrease the weight of the $G \setminus \{G'\}$ items by 1, and also decrease the weight of the $B \setminus \{B'\}$ items by 1.

After this modification the (G', S') FFD bins have no shortage, and if there exists a (B', S') FFD bin, it has no shortage either. Let us see the possible optimal bins of the S' items.

If an S' is packed into some (G, S, T) or (G, S, Q) optimal bin, then the G item of this optimal bin cannot be a G' item since $G' + S' + X > 1$, thus the bin does not have shortage. If an S' is packed into a (B, S, T) or (B, S, Q) optimal bin, then the B item of this optimal bin cannot be a B' item, thus the bin does not have shortage. Suppose that an S' is packed into some other optimal bin, say B_i^*, where B_i^* contains k of the S or S' items in total. Each such optimal bin has at least k surplus in the table (i.e. before the modification of the weights), thus the bin does not have shortage.

It follows that the total shortage caused by the S' items is *covered* in this way. There could be problem only with the $(3S, Q)$ bins, since they do not have reserve to cover the shortage of the S' item if an S' is packed into such a bin. But an S' cannot be in a $(3S, Q)$ bin, since $S' + 2S + Q > (Z + Y - X) + 2Z + X = Y + 3Z > 1$.

Then the total uncovered shortage can be at most 7. Indeed, if there is no $(3S, Q)$ optimal bin, then there can be 3 or 6 shortage by a $(2S, V)$ or $(S, 2V)$ FFD bin (since these bin-types cannot occur at the same time). If there exists a $(3S, Q)$ optimal bin, then a $(S, 2T)$ FFD bin cannot occur, so there can be at most 3 shortage by a $(2S, T)$ FFD bin, and at most 4 by the $(3S, Q)$ optimal bins, since there are only four Q items.

Then we can delete from consideration each bin which has at least 7 reserve or surplus. The following FFD bins are also impossible:

A $(G, 2T)$ or $(B, 2T)$ bin, because then there cannot be an S item in some later bin (since $S \leq 2T$) thus $(2S, V)$ and $(S, 2V)$ FFD bins do not exist, and the possible 4 shortage caused by the $(3S, Q)$ bins is covered by the surplus of the $(G, 2T)$ or $(B, 2T)$ FFD bin.

There cannot be a $(2S, 2T)$ FFD bin, it has only 6 surplus, but if there is such a bin, then there is no $(S, .)$ FFD bin with shortage by Observation 8.2.2, and the 6 surplus covers the at most 4 shortage caused by the $(3S, Q)$ optimal bins. Thus only the following possible bin-types remain:

OPT

23	G	1	1						1	1	1									
23	B			1	1							1	1	1						
12	S	1		1		3	2	1	1			1			1	2	1	3	2	1
9	T	1	2	1	2		2	3		1			1				1		1	2
9	Q								1	1	2	1	1	2	3	2	2	1	1	1
	r	0	3	0	3	8	2	5	0	3	3	0	3	3	5	2	5	-1	2	5

FFD

23	G	1								
23	B		1							
12	S	1	1	3	2	1	1			
9	T				1	3	2	4	1	
9	Q								3	1
	s	-1	-1	0	-3	3	-6	0	0	0

Let us have a look at the remaining possible FFD bins. Each G item is packed into (G', S') FFD bins, thus every G item is a G' item, with equal size. Since there exists a $\{(G', S'), (G', S, T)\}$ cobin, it follows that $G' + Z + Y \leq 1$.

Since every G item is a G' item, all shortage caused by the (G', S') FFD bins are covered by the optimal bins of the S' items. If there exists a (B', S') bin, then the shortage of this bin is also covered by the optimal bin of the S' item of this bin. Moreover if there exists a B item, then there is only one B item, packed into a (B', S') FFD bin.

Suppose there exists an $(S, 2T)$ FFD bin. Let the items being in this bin be denoted as (S_2, T_2, T_3), where T_2 precedes T_3. Then S_2 is the last regular S item, hence it is the smallest S item, thus it has size Z. Then $3S + X > T_2 + T_3 + Z + X > 1$ holds. Thus there is no $(3S, Q)$ optimal bin. Further, a $(2S, T)$ FFD bin is not possible either, thus the uncovered shortage is 6.

We state that neither T_2 nor T_3 can be packed into a (G, S, T) optimal bin. Indeed, suppose that for example T_2 is packed into a (G, S, T) optimal bin. We have seen in the beginning of the proof that $G > B > Z + Y$ holds for any G or B items. Thus the sum of the sizes of the items in the (G, S, T) bin is bigger than $(Z + Y) + Z + T_2 > (T_3 + X) + Z + T_2 > 1$, a contradiction.

By the same reason, neither T_2 nor T_3 can be packed into a (B, S, T), $(2S, 2T)$ or $(2S, T, Q)$ optimal bin, moreover they cannot be packed into the same $(G, 2T)$, $(B, 2T)$, $(S, 3T)$ or $(S, 2T, Q)$ optimal bin.

Thus T_2 and T_3 are packed into two different $(G, 2T)$, $(B, 2T)$, $(S, 3T)$, (G, T, Q), (B, T, Q), $(S, T, 2Q)$, or $(S, 2T, Q)$ optimal bins. Each of these bins has at least 3 reserve, therefore they cover the total shortage caused by the $(S, 2T)$ FFD bin. Since we have got a contradiction, we suppose in the following that an $(S, 2T)$ FFD bin does not exists.

Now we show that a $(2S, T)$ FFD bin and a $(3S, Q)$ optimal bin cannot exist at the same time. Thus suppose that there exists a $(2S, T)$ FFD bin; let its items be denoted as (S_2, S_3,T_2). If S_2 and S_3 are the two smallest S items, then there cannot be a $(3S, Q)$ optimal bin. If $S_2 \leq 2X$, then there can be at most one fallback item after S_2. If there is no fallback item after S_2, or the fallback item is a T item, then again S_2 and S_3 are the two smallest S items. If there is a fallback S item after S_2, then this fallback S item with S_2 and S_3 are the three smallest S items, thus a $(3S, Q)$ bin is impossible. Finally, suppose that $S_2 > 2X$. Any $(3S)$ FFD bin precedes the $(2S, T)$ bin by

Observation 8.2.2, and the first two S items in a $(3S)$ FFD bin precede S_2, thus the level of any $(3S)$ bin would be bigger than $5X$, a contradiction to Corollary 8.2.1 (a). Thus there is no $(3S)$ FFD bin. We have already seen in the beginning of the proof of this lemma that the S' items are the largest S items, thus none of them can be smaller than S_2. Thus it follows again that (if an $(S, 2T)$ FFD bin exists, then) S_2 and S_3 are the two smallest S items, consequently there is no $(3S, Q)$ optimal bin.

It follows that the uncovered shortage is at most 4 (and it can be 4 only if there are four $(3S, Q)$ optimal bins). Thus there cannot be any optimal bin which has at least 4 reserve. (While calculating the reserve, we must take into account that the shortage of the S' items are covered by the optimal bins of the S' items.) The optimal bins having 3 reserve and containing at least one Q item are also impossible, because then there can be at most three $(3S, Q)$ optimal bins, or a $(2S, T)$ FFD bin. Thus only the next possible bins remain:

OPT

23	G	1	1				1				
23	B			1	1			1			
12	S	1		1		2	1	1	2	3	2
9	T	1	2	1	2	2					1
9	Q						1	1	2	1	1
	r	0	3	0	3	2	0	0	2	-1	2

FFD

23	G	1							
23	B		1						
12	S	1	1	3	2	1			
9	T				1	3	4	1	
9	Q							3	1
	s	-1	-1	0	-3	3	0	0	0

Since we have seen that an S' cannot be packed into a $(3S, Q)$ optimal bin, nor can it be packed into a G-bin or B-bin, it follows that an S' item can occur only in a $(2S, 2T)$, $(2S, 2Q)$, or $(2S, T, Q)$ optimal bin.

Subcase $S' > 1/3$. Then using $S' + S + 2T > 1/3 + Z + 2Y > 1/3 + \frac{1-Y}{3} + 2Y = \frac{5}{3}Y + \frac{2}{3} > 1$, we obtain that S' cannot be in a $(2S, 2T)$ bin. Thus S' can occur only in a $(2S, 2Q)$ or $(2S, T, Q)$ bin, and only one S' can be in such a bin. If there are at least three S' items, then their shortage is covered by the optimal bins of these S' items; further we have at least 3 more reserve, and there are at least three Q items in these optimal bins. Thus there can be at most one $(3S, Q)$ optimal bin or a $(2S, T)$ FFD bin, the total shortage is covered, and we are done. Suppose next that the number of S' items is $1 \leq k \leq 2$. (There must be at least one S' item because of the existence of the cobin.) Then the S' items are packed into $k \leq 2$ different optimal bins. Moreover there are altogether $k \leq 2$ further optimal bins containing a G or a B item. Also, there are at most $4 - k \leq 3$ $(3S, Q)$ optimal bins. Thus at least $10 - 7 = 3$ further optimal bins remain, each having 2 reserve, thus the total shortage is covered and we are done.

Subcase $S' \leq 1/3$. Then there is no fallback item in the $(3S)$ FFD bins, i.e. these S items have non-increasing sizes. Suppose that there are neither $(2S, T)$ nor $(S, 3T)$ FFD bins. Then there is no shortage in the FFD bins. On the other hand there cannot be any $(3S, Q)$ optimal bin either, since an S' item cannot be in a $(3S,Q)$ optimal bin, moreover all $S \setminus S'$ items are in $(3S)$ FFD bins, and the three smallest S items are in the last $(3S)$ FFD bin where X does not fit.

It follows that there exists a $(2S,T)$ or an $(S,3T)$ FFD bin. Suppose there exists an $(S,3T)$ FFD bin. Then there is no $(2S,T)$ FFD bin, and there can be at most one $(3S,Q)$ optimal bin. (Indeed, if there are two $(3S,Q)$ optimal bins, then the sum of the sizes of the S items in one $(3S,Q)$ bin is not smaller than the sum of the sizes of the S items of the last $(3S)$ FFD bin.)

Thus it follows that there is a $(2S,T)$ FFD bin; let its items be denoted as (S_2,S_3,T_2). Then S_2, and S_3 are the two smallest S items.

We have seen that there is no $(3S,Q)$ optimal bin, since a $(2S,T)$ FFD bin and a $(3S,Q)$ optimal bin cannot occur at the same time.

It means that the uncovered shortage is only 3. Thus no optimal bin can have reserve at least 3. (Let us recall that the shortage of the S' items are covered by the optimal bins of the S' items.) Thus only the next possible bins remain:

OPT

23	G	1			1				
23	B		1			1			
12	S	1	1	2	1	1	2	2	
9	T	1	1	2				1	
9	Q				1	1	2	1	
	r	0	0	2	0	0	2	2	

FFD

G	1						
B		1					
S	1	1	3	2			
T				1	4	1	
Q						3	1
s	-1	-1	0	-3	0	0	0

Suppose there is only one (G',S') FFD bin. Then there are at most two S' items. Calculating the total shortage using the original weights of the S' items, we have at most $2+3=5$ shortage in the FFD bins. On the other hand there are at least ten optimal bins, there are only at most two G-, or B-bins among them, and any other optimal bin has 2 reserve which covers the shortage. Thus there are at least two (G',S') FFD bins. Since any G' is packed into a (G,S,T) or a (G,S,Q) optimal bin, there are at least two cobins where the optimal bin of the cobin is a (G,S,T) or a (G,S,Q) bin.

It follows from Corollary 3.1.5, that $G > B > S_2 + X$, and $G > B > S_3 + X$, as S_2 and S_3 are the two smallest S items. Then similarly as before, T_2 can be packed into no optimal bin, a contradiction. $\square$

Lemma 8.2.12 *If there is no $\{(G,S),(G,S,T)\}$ cobin but there exists a $\{(G,S),(G,S,Q)\}$ cobin, then all shortage is covered.*

Proof. Suppose that there is no $\{(G,S),(G,S,T)\}$ cobin, but there is a $\{(G,S),(G,S,Q)\}$ cobin. Let the S and Q item of the optimal bin of this cobin be denoted as S_1 and Q_1, respectively. In the (G,S) FFD bins all of the G items have equal size and all of the S items also have equal size, by Lemma 3.1.7(i). Denote these items again as G' and S', respectively. Since there exists a $\{(G',S'),(G',S,Q)\}$ cobin (applying Corollary 3.1.5) it follows that each item bigger than S' has size bigger than $S_1+Q_1 \geq Z+X$. This means that there cannot be an M item, since $M \leq \frac{1-Y}{2} < Z+X$. (The inequality holds because $2Z+2X+Y > 2\frac{1-Y}{3}+2X+Y = 2X+\frac{1}{3}Y+\frac{2}{3} > 1$.) Also it follows by the same reason that $G > B > S_1+Q_1 \geq Z+X$ holds for each G and B item. Since no five items fit into a bin, applying Observation 3.1.2, there cannot be a 4-bin which contains a B item. Thus only the next bins are possible:

OPT1

23	G	1	1										
18	B			1	1	1							
12	S	1		2	1		3	2	2	1	1		
9	T	1	2		1	2		2	1	3	2	4	3
	r	0	3	2	5	8	8	2	11	5	14	8	17

OPT2

23	G	1	1	1									
18	B				2	1	1	1					
12	S	1				1			2	1	1		
9	T		1				1			1		2	1
9	Q	1	1	2	1	1	1	2	1	1	2	1	2
	r	0	3	3	-1	5	8	8	11	14	14	17	17

OPT3

12	S	1		2	1		3	2	1	
9	T		1		1	2		1	2	3
9	Q	3	3	2	2	2	1	1	1	1
	r	5	8	2	5	8	-1	2	5	8

FFD

23	G	1	1			1	1											
18	B	1		2	1			1	1	1								
12	S		1		1	1		2	1		3	2	2	1	1			
9	T					1	2		1	2		2	1	3	2	4	1	
9	Q																3	1
	s	5	-1	0	-6	8	5	6	3	0	0	6	-3	3	-6	0	0	0

Now we increase the weight of the G' items by 1, and decrease the weight of the $G \setminus G'$ items by 2. As a result, the FFD G-bins do not have shortage. In any optimal G-bin there remains 2 reserve, with just one exception: If a G' item is packed into some (G, S, Q) optimal bin, then 1 shortage is created in this optimal bin, but this bin uses one Q item. (If there exists a (G, S, T) optimal bin, the G item in this bin cannot be a G' item since there is no $\{(G, S), (G, S, T)\}$ cobin.) Thus if there is shortage in some optimal bin, then it is only 1, and each such bin contains a Q item; if there is no shortage in the optimal bin, then it has at least 2 reserve, thus there are at least 6 optimal bins with at least 2 reserve in each, and so we have at least $6 \cdot 2 - 4 = 8$ reserve.

If there is no (B, S) FFD bin, then there is at most 6 shortage in the FFD bins, and we are done. Thus there exists a (B, S) FFD bin; naturally there can be only one such bin. We state that the size of the S item of the (B, S) FFD bin is S'. Indeed, because of the existence of a $\{(G', S'), (G', S, Q)\}$ cobin, $G' + Z + X \leq 1$ holds for the size of the G' items, and on the other hand any S item is smaller than $Z + X$ since $S \leq \frac{1-Z}{2} < Z + X$ follows from $3Z + 2X > 3 \cdot \frac{1-Y}{3} + 2X = 1 - Y + 2X > 1$. Thus the S item of the (B, S) FFD bin comes after the S item of the (G', S') FFD bins, as any S item would fit into the bin where there is only a G' item. Consequently the size of the S item of the (B, S) FFD bin must be exactly S' because of our minimality assumption on the items.

Since there exists a (B, S') FFD bin, $S' > 1/2 - X$ holds.

The FFD bins can cause at most 12 shortage in total. There cannot be any FFD bin with surplus at least 4 because we already covered the shortage of the (G', S') FFD bins and the optimal bins have 8 further reserve, thus 4 further surplus would be enough to cover the shortage.

Furthermore a $(B, 2S)$, (B, S, T), $(B, 2T)$ FFD bin cannot occur, because of the existence of the (B, S) FFD bin.

There is no $(S, 3T)$ FFD bin, because then there would not be any $(S, .)$ FFD bin with shortage. By the same reason a $(G, 2T)$ FFD bin is not possible either, since $S \leq 2Y$ holds, thus if there is such bin then there cannot be any $(S, .)$ FFD bin (with shortage.) Only the following FFD bins remain:

FFD

23	G	1	1									
18	B	1		2	1							
12	S		1		1	3	2	1				
9	T						1	2	4	1		
9	Q									3	1
	s	5	-1	0	-6	0	-3	-6	0	0	0

Then there must be a $(2S, T)$ or $(S, 2T)$ FFD bin, otherwise the total shortage is covered. We claim that $3S + X > 1$ holds. Indeed, if there exists an $(S, 2T)$ FFD bin, then its S item is the smallest S, and thus $3S + X > 1$. Suppose that there exists a $(2S, T)$ FFD bin; let its items be denoted as (S_2, S_3, T_1). If these are the two smallest S items, the statement follows similarly. Otherwise there is a fallback S item, smaller than S_2 in another bin. Such an S item can occur only in a $(3S)$ FFD bin, thus such a bin exists. If $S_2 > 2X$, then the level of the $(3S)$ bin is at least $2S_2 + X > 5X$, since the first two S items in the $(3S)$ bin precede S_2. This would be a contradiction, thus $S_2 \leq 2X$ holds. Then by Lemma 8.2.4 there is at most one (i.e. now there is exactly one) fallback item after S_2, thus the fallback S item and the two S items in the $(2S, T)$ bin are the three smallest S items; and then $3S + X > S_2 + S_3 + T_1 > 1$ follows.

Then it follows that a $(3S, Q)$ optimal bin is impossible. A $(B, 2S)$ optimal bin cannot occur either, since a B item is bigger than $S_1 + Q_1$ (where this S_1 item is packed into some other optimal bin), but three S items and a Q item cannot fit into a bin.

There can be at most one (G, B) FFD bin, since one such bin has 3 surplus, and two such bins would cover the 4 shortage that remained. Since any B item in some $(2B)$ FFD bin precedes the B item of the (B, S) FFD bin, it follows that there can be at most one B item smaller than the B item of the (B, S) FFD bin, namely the B item of the (G, B) FFD bin. Thus a $(2B, Q)$ bin is impossible. It follows that in each remaining optimal B-bin there is at least 5 reserve.

Regarding the optimal bins it still holds that there can be at most four bins with 1 shortage in each, and there is at least 2 reserve in any other optimal bin. We have seen before that there exists a (B, S) FFD bin, thus there is at least one B item. In the optimal bin of this B item there is at least 5 reserve, thus we have at least 11 reserve in total in the optimal bins (and we have exactly 12 shortage in the FFD bins).

We draw the following conclusions: There must be an $(S, 2T)$ FFD bin with 6 shortage. There is only one B item, namely the B item of the (B, S) FFD bin. There are exactly four optimal bins with shortage, it can be only of type (G, S, Q), it means that there are exactly four (G, S, Q) optimal bins. Thus all Q items are in (G, S, Q) optimal bins, i.e. there is no other optimal Q-bin.

There is one optimal B-bin, and any optimal bin has exactly 2 reserve except the B-bin and the Q-bins.

Since each G item is packed into (G', S') FFD bins (all other FFD G-bin types are excluded), any G item is a G' item; recall that $w(G') = 24$. Then a (G, S, T) optimal bin is impossible since there is no $\{(G, S), (G, S, T)\}$ cobin. Hence only the following bin-types remain:

OPT

24	G'	1			1
18	B		1		
12	S		1	2	1
9	T	2	1	2	
9	Q				1
	r	2	5	2	-1

FFD

G'	1						
B		1					
S	1	1	3	1			
T				2	4	1	
Q						3	1
r	0	-6	0	-6	0	0	0

Recall that any G item is a G' item. By the four (G, S, Q) optimal bins there are at least four (G, S) FFD bins. There is also a (B, S) FFD bin, thus there are at least five S' items.

An S' item cannot be in a (B, S, T) optimal bin (since $B + S' + X > 1$), thus an S' item can be only in a $(2S, 2T)$ optimal bin. We have seen that $S' > 1/2 - X$ holds, thus there can be only one S' item in some $(2S, 2T)$ optimal bin. Thus there are at least five $(2S, 2T)$ optimal bins, and there is exactly one (B, S, T) optimal bin. There is no further optimal bin, for otherwise the total shortage would be covered. It means that there are exactly four (G, S, Q), one (B, S, T) and five $(2S, 2T)$ optimal bins.

Then considering any T item of the $(S, 2T)$ FFD bin, this item cannot be packed into the (B, S, T) optimal bin (since $B > S_1 + Q_1$, and S_1 is bigger than the other T item in the $(S, 2T)$ FFD bin), and it cannot be packed into a $(2S, 2T)$ optimal bin either, thus it can be packed into no optimal bin, a final contradiction completing the proof for $2/11 < X \leq 1/5$ and $OPT \leq 18$. $\square$

8.3 Appendix C, the omitted part of proof of FF's tight bound

Here we provide the remained part of the complete proof for the tight ratio of FF. We will use the results of Chapter 4, so we continue here with the analysis of the case when the last common bin is small.

Suppose that the size of the last common bin is smaller than $2/3$. For the rest of the upper bound proof, fix $x > 0$ so that $s(C_\gamma) = \frac{2}{3} - 2x$. Lemma 4.2.2(ii) implies $s(C_\gamma) > 1/2$ and thus $x < 1/12$.

Since now the regular weight of the last bin is smaller than 0.8, we need to compensate for this. This is indeed possible due to the fact that now Lemma 4.2.1(iii) implies that the inner common bins are larger than $2/3 + x$ and this allows us to improve the bounds of Lemma 4.2.5 by an amount proportional to x.

Note that C_i, $i > 1$, cannot be a 5^+-bin: Since $s(C_1) < 5/6$, all items in C_i have size larger than $1/6$ and five of them would add up to more than $5/6$, contradicting the assumption that C_i is a common bin.

Lemma 8.3.1 *For $i = 2, \ldots, \gamma - 1$ we have the following bounds: If C_i is a 2-bin or a 3-bin, then $r(C_{i-1}) + v(C_i) \geq 1 + \frac{3}{5}x$. If C_i is a 4-bin, then $r(C_{i-1}) + v(C_i) \geq 1 + \frac{3}{10}x$.*

Proof. Let y be such that $s(C_{i-1}) = \frac{5}{6} - y$. Since C_{i-1} is a common bin, $y > 0$. On the other hand, by Lemma 4.2.1(iii) the size of C_{i-1} is greater than $\frac{2}{3} + x$ and thus also $y < \frac{1}{6} - x$. Note that $r(C_{i-1}) = \frac{6}{5}(\frac{5}{6} - y) = 1 - \frac{6}{5}y$ and that every item $c \in C_i$ satisfies $c > \frac{1}{6} + y$.

Case 1: C_i is a 2-bin. Then C_i contains at least one item c of size larger than $1/3$ as otherwise $s(C_{i-1}) \leq 2/3$ contradicting Lemma 4.2.1(iii) together with the assumption that $S(C_\gamma) < 2/3$. The other item c' in C_i satisfies $c' > \frac{1}{6} + y$. Thus

$$r(C_{i-1}) + v(C_i) \geq 1 - \frac{6}{5}y + \frac{3}{5}y + 0.1 = 1.1 - \frac{3}{5}y \geq 1.1 - \frac{3}{5}\left(\frac{1}{6} - x\right) = 1 + \frac{3}{5}x\,.$$

Case 2: C_i is a 3-bin. Suppose that C_i contains an item $c > 1/3$. Then the remaining two items in C_i have size at least $\frac{1}{6} + y$ and we obtain

$$r(C_{i-1}) + v(C_i) \geq 1 - \frac{6}{5}y + \frac{3}{5}(y + y) + 0.1 = 1.1 \geq 1 + \frac{3}{5}x$$

since $x < 1/12$. Otherwise all three items in C_i have size at most $1/3$. We claim that one of them has size at least $\frac{1}{6} + x$, as otherwise, using $x < 1/12$, we have $s(C_i) < 3(\frac{1}{6} + x) = \frac{1}{2} + 3x < \frac{2}{3} + x$, contradicting Lemma 4.2.1(iii). Now we get

$$r(C_{i-1}) + v(C_i) \geq 1 - \frac{6}{5}y + \frac{3}{5}(y + y + x) = 1 + \frac{3}{5}x\,.$$

Case 3: Suppose C_i is a 4-bin. All items in C_i are small (i.e. have sizes between $1/6$ and $1/3$), as otherwise $s(C_i) \geq \frac{1}{3} + 3 \cdot \frac{1}{6} = \frac{5}{6}$, contradicting the assumption that C_i is a common bin. As $s(C_i) > \frac{2}{3} + x$ by Lemma 4.2.1(iii), there must be two items with total size at least $\frac{1}{3} + \frac{x}{2}$ and their total bonus is at least $\frac{3}{5} \cdot \frac{x}{2}$. Thus

$$r(C_{i-1}) + v(C_i) \geq 1 - \frac{6}{5}y + \frac{3}{5}\left(y + y + \frac{x}{2}\right) = 1 + \frac{3}{10}x\,.$$

□

Let γ_k denote the number of k-bins that do not contain an exceptional item among the inner common bins, i.e., among $C_2, \dots, C_{\gamma-1}$. Let $\alpha = 2(\gamma_2 + \gamma_3) + \gamma_4$.

Lemma 8.3.2 *Suppose that $s(C_\gamma) < 2/3$. The following holds:*
(i) If $\alpha \geq 8$ then the total weight of the common bins is at least $w(\mathcal{C}) \geq \gamma - 0.2$.
(ii) If $\alpha \geq 4$ then the total weight of the common bins is at least $w(\mathcal{C}) \geq \gamma - 0.3$.

Proof. We apply Lemma 8.3.1 for any $i = 1, \dots, \gamma - 2$ such that C_{i+1} does not contain an exceptional item. Otherwise, i.e., if C_{i+1} contains an exceptional item and also for $i = \gamma - 1$ we apply Lemma 4.2.5. Summing all the resulting bounds on $r(C_i) + v(C_{i+1})$ and $r(C_\gamma) = 0.8 - \frac{12}{5}x$ we obtain that the total weight of the common bins is

$$s(\mathcal{C}) \geq \gamma - 1 + (\gamma_2 + \gamma_3)\frac{3}{5}x + \gamma_4\frac{3}{10}x + (0.8 - \frac{12}{5}x) = \gamma - 0.2 + \frac{(3\alpha - 24)x}{10}.$$

For $\alpha \geq 8$ we have $3\alpha \geq 24$ and (i) follows. For $\alpha \geq 4$ we use $x < 1/12$, which gives $(3\alpha - 24)x \geq -12x \geq -1$ and (ii) follows. □

Theorem 8.3.1 *For any instance of bin packing, $FF \leq 1.7 \cdot OPT$.*

Proof. If $s(C_\gamma) \geq 2/3$ then the theorem follows by Theorem 4.2.1. Thus assume $s(C_\gamma) < \frac{2}{3}$ and $FF \geq 1.7 \cdot OPT + 0.1$. We distinguish several cases and in each we derive a contradiction or prove the theorem statement $FF \leq 1.7 \cdot OPT$, leading to an indirect proof as well.

Case 1: $OPT \geq 21$. By Lemma 4.2.2(iv) we have $\gamma \geq 12$. Thus there are at least 10 inner common bins and at most 2 of them have an exceptional item. Thus $\alpha \geq \gamma_2 + \gamma_3 + \gamma_4 \geq 8$ and $w(C) \geq \gamma - 0.2$ by Lemma 8.3.2(i). Now Lemma 4.2.6 and Proposition 4.2.1 imply the theorem.

Case 2: $OPT \geq 8$, $OPT \not\equiv 4 (\text{mod } 10)$, and $OPT \not\equiv 7 (\text{mod } 10)$. Then $FF \geq 1.7 \cdot OPT + 0.3$, thus we can use Lemma 4.2.2(iv) with $\tau = 3$ and we obtain $\gamma \geq 6$. There are no exceptional items, since $OPT \not\equiv 7 (\text{mod } 10)$, and thus $\alpha \geq 4$. Lemma 8.3.2(ii) implies $W > \beta + (\gamma - 0.3) + \delta = FF - 0.3 \geq 1.7 \cdot OPT$, a contradiction.

Case 3: $OPT = 14$. Then $FF = 24$. Lemma 4.2.2(iv) with $\tau = 2$ implies $\gamma \geq 9$. There are no exceptional items, thus $\gamma_2 + \gamma_3 + \gamma_4 \geq 7$. If $\gamma_2 + \gamma_3 \geq 1$ then $\alpha \geq 8$ and the theorem follows by Lemma 8.3.2(i). In the remaining case $\gamma_4 \geq 7$, thus there are seven 4-bins among the common bins. Using Lemma 4.2.1(v) for five of these common 4-bins, Lemma 4.2.1(iv) for some four of the remaining $\gamma - 5 \geq 4$ common bins and Lemma 4.2.1(i) for the remaining $24 - 9 = 15$ bins we get

$$S > 4 + 4 \cdot \frac{2}{3} + 15 \cdot \frac{1}{2} > 14 = OPT\,,$$

a contradiction.

Case 4: $OPT = 17$. Then $FF = 29$. Lemma 4.2.2(iv) gives $\gamma \geq 10$. Thus $\gamma_2 + \gamma_3 + \gamma_4 \geq 6$. If $\gamma_4 \leq 4$ then $\alpha \geq 2(6 - \gamma_4) + \gamma_4 \geq 8$ and the theorem follows by Lemma 8.3.2(i). Otherwise there are five 4-bins among the common bins. Using Lemma 4.2.1(v) for these five 4-bins, Lemma 4.2.1(iv) for some five of the remaining $\gamma - 5 \geq 5$ common bins and Lemma 4.2.1(i) for the remaining $29 - 10 = 19$ bins we get

$$S > 4 + 5 \cdot \frac{2}{3} + 19 \cdot \frac{1}{2} > 17 = OPT\,,$$

a contradiction.

Case 5: $OPT = 7$. Then $FF = 12$. First we claim that $\delta = 7$. Otherwise $S > 6 \cdot \frac{2}{3} + 6 \cdot \frac{1}{2} = 7$, a contradiction. It follows that there are at most three 2-bins in the FF packing, since by Lemma 4.2.7 no OPT-bin can contain two 2-items. Next we claim that no two FF-bins have total size greater than or equal to $3/2$. Otherwise there remain at least three 2^+-bins in the FF packing and $S > \frac{3}{2} + 3 \cdot \frac{2}{3} + 7 \cdot \frac{1}{2} = 7$, a contradiction. Since there are five 2^+-bins and at most three 2-bins, there have to be at least two 3^+-bins. Let C be the last 3^+-bin and B some bin before it. Then C contains three items of size larger than $1 - s(B)$ and $s(B) + s(C) \geq s(B) + 3(1 - s(B)) = 3 - 2 \cdot s(B)$. Since no two bins have total size $3/2$ or more, this implies $s(B) \geq 3/4$. Furthermore, this implies that there is a single bin before C, as otherwise there would again be two bins with total size at least $3/2$. I.e., B is the first bin, C is the second bin and there are exactly three 2-bins. C has at least three items and they are packed into different OPT-bins by Assumption 4.2.1. We claim that one of these three bins contains both a 2-item c and a D-item d with size $d > 1/2$: Each OPT-bin contains a D-item and there is at most one D item of size at most $1/2$; furthermore, there is at most one OPT-bin not containing a 2-item, as there are six 2-items in the three 2-bins. Thus the condition excludes at most two OPT-bins. Fix c' to be an item from C packed with such a c and d in the same OPT-bin. Note that c and d are in later FF-bins than C, as B and C are the first bins and they are 3^+-bins. We have $c' + c < 1/2$ as they are packed with $d > 1/2$ in an OPT-bin. On the other hand we claim that $s(C) - c' < 1/2$: otherwise we note that $c' > 1 - s(B)$, as c' was not packed in B and thus $s(B) + s(C) > s(B) + c' + 1/2 > s(B) + (1 - s(B)) + 1/2 = 3/2$, contradicting the first claim in the proof. Thus $s(C) + c = (s(C) - c') + (c' + c) < 1/2 + 1/2 = 1$ and FF should have packed c into C, which is the final contradiction. □

We note that the last case of $OPT = 7$ is also covered in the manuscript [65], we have included it for completeness.

8.4 Appendix D, the omitted part of FF's tightness proof in the CCBP model

Here we provide the omitted part of the complete investigation of the tight bound of FF for the cardinality constrained model. We have divided the investigation to parts regarding the global constant k. For $k = 2$ the tight bound of the asymptotic ratio was known in advance, so this investigation is not given in this dissertation. For the cases of $3 \leq k \leq 5$, we used simple weight function, it is a piece-wise constant function. For $6 \leq k \leq 8$, a more difficult weight function was useful. The hardest part of the investigation is for case $k = 9$. And we provide here the omitted cases, that is, the cases for $k \geq 10$.

The cases where $k \geq 10$ are studied similarly to previous cases. In this case we also distinguish the definitions of weights based on the bins of OPT according to the number of additional items packed into these bins. The weight of an α-item remains $\frac{1}{k}$.

Case a. Consider bins of OPT containing one or two additional items (and the remaining items are α-items). Such bins are called γ-bins again, and the additional items in the bin are called γ-items. Huge γ-items are called γ_1-items and they have weights of 1. Other γ-items are called γ_2-items again. If $10 \leq k \leq 19$, then the weight of the γ_2-item is $\frac{7}{10} - \frac{1}{k}$, and otherwise (if $k \geq 20$), then its weight is $\frac{13}{20} = 0.65$. The smallest weight of a γ_2-item is 0.6, and its size is at most $\frac{1}{2}$, therefore the ratio between the weight of such an item and its size is at least 1.2.

Case b. Consider the remaining bins of OPT (containing at least three additional items). Each such bin has at most $k - 3$ α-items, and we call it a ϕ-bin. The additional items packed into ϕ-bins are called ϕ-items. The weighting function of the ϕ-items is again more complicated. The weight of any huge ϕ-item is 1 again. The weight of a ϕ-item of size $a \leq 1/2$ is $w(a) = s(a) + b(a)$, where $s(a) = \frac{6}{5}a$ is the scaled size, and $b(a)$ is the bonus of the item. Below we give the bonus function of the ϕ-items of sizes no larger than $1/2$.

For $k \geq 20$, the classical weighting function of FF [55] is appropriate, in this case the bonus function is defined as follows.

$$b(a) = \begin{cases} 0 & \text{if} \quad a \leq 1/6 \\ \frac{3}{5}(a - \frac{1}{6}) = 0.6a - 0.1 & \text{if } 1/6 < a \leq 1/3 \\ 0.1 & \text{if } 1/3 < a \leq 1/2 \end{cases}$$

where we items are called in the classes as tiny, small or medium, and big, respectively.

The weight function in this case is continuous in the interval $(0, \frac{1}{2})$. The bonus is piecewise linear (and so is the weight function). In the interval $(\frac{1}{6}, \frac{1}{3})$, the bonus increases from 0 to 0.1.

For $10 \leq k \leq 19$, we use additional modifications to the classic weight function. In some of the cases the bonus function is still equal to the one in the classic analysis. More specifically, these are the cases where the size is in $(1/6, 1/5]$ and $(3/10, 1/3]$. In these intervals the slope of the weight function is 1.8, i.e, the slope of the bonus function is 0.6. The bonus function and the weight function are piecewise linear, and continuous in $(0, \frac{1}{4})$ and $(\frac{1}{4}, \frac{1}{2})$. The partition into types is as in the case $k = 9$.

$$b(a) = \begin{cases} 0 & \text{if} \quad a \le 1/6 \\ \frac{3}{5}(a - \frac{1}{6}) = 0.6a - 0.1 & \text{if} \quad 1/6 < a \le 1/5 \\ (1.6 - \frac{20}{k})(a - \frac{1}{5}) + 0.02 = (1.6 - \frac{20}{k})a - 0.3 + \frac{4}{k} & \text{if} \quad 1/5 < a \le 1/4 \\ (1.6 - \frac{20}{k})(a - \frac{1}{4}) + \frac{1}{k} = (1.6 - \frac{20}{k})a - 0.4 + \frac{6}{k} & \text{if} \quad 1/4 < a \le 3/10 \\ \frac{3}{5}(a - \frac{3}{10}) + 0.08 = 0.6a - 0.1 & \text{if} \quad 3/10 < a \le 1/3 \\ 0.1 & \text{if} \quad 1/3 < a \le 1/2 \end{cases}$$

where we call the items in the classes after each other as tiny, very small, larger small, smaller medium, larger medium and big, respectively.

This bonus function is monotonically non-decreasing for $k \ge 13$, but not in the cases $k = 10, 11, 12$; whereas the resulting weight function is monotonically increasing for $10 \le k \le 19$. The value of the bonus is zero for $a \le 1/6$ and it is 0.1 between $1/3$ and $1/2$. We have $b(\frac{1}{5}) = 0.02$ (and $w(\frac{1}{5}) = 0.26$), $b(\frac{1}{4}) = 0.1 - \frac{1}{k}$, thus for $a \in (\frac{1}{5}, \frac{1}{4}]$ the bonus is in $[0.1 - \frac{1}{k}, 0.02)$ for $k = 10, 11, 12$ and in $(0.02, 0.1 - \frac{1}{k}]$ for $13 \le k \le 19$. For $a \in (\frac{1}{4}, 0.3]$ the bonus is in $[0.08, \frac{1}{k})$ for $k = 10, 11, 12$ and in $(\frac{1}{k}, 0.08]$ for $13 \le k \le 19$ (we have $w(0.3) = 0.44$). The bonus of an item of size above $\frac{1}{4}$ is always above 0.05.

For $k \ge 10$, since the bonus function is non-negative, for any ϕ-item of size $0 < a \le \frac{1}{2}$, $w(a) \ge \frac{6}{5}a$ holds. The bonus of every item of size in $(0, \frac{1}{2}]$ is in $[0, 0.1]$. The weight of a big item is at least 0.5. The weight of a medium item is at least $0.3 + \frac{1}{k} > 0.35$ for $k \le 19$ and at least 0.35 for $k \ge 20$.

Now we find properties of the weighting and then we give the asymptotic bound.

Lemma 8.4.1 *For every bin B of OPT, $w(B) \le 2.7 - 3/k$ holds.*

Proof. The claim holds for bins having only α-items. For a γ-bin, if there is just one γ-item, then the total weight is at most $\frac{k-1}{k} + 1 < 2$. Otherwise, if $k \le 19$, then the total weight is at most $\frac{k-2}{k} + 1 + 0.7 - \frac{1}{k} = 2.7 - \frac{3}{k}$, and if $k \ge 20$, then the total weight is at most $\frac{k-2}{k} + 1 + 0.65 = 2.65 - \frac{2}{k} \le 2.7 - \frac{3}{k}$.

Next, consider ϕ-bins. For $k \ge 20$, the proof follows from the standard analysis [55], and we include it for completeness. There are at most $k - 3$ α-items, and their total weight never exceeds $\frac{k-3}{k}$. If a bin does not contain a huge ϕ-item, then it has at most five ϕ-items of positive bonuses (each bonus is at most 0.1), and their scaled size is at most 1.2. This gives a total weight of at most $1 - \frac{3}{k} + 1.2 + 0.5 = 2.7 - \frac{3}{k}$. Note that this total weight cannot be achieved as both situations where there are $k - 3$ α-items and five ϕ-items cannot occur simultaneously. If a bin contains a huge item, then there are at most two (other) ϕ-items with positive bonuses. The scaled size of all ϕ-items except for the huge item is at most 0.6, and the total weight excluding the bonuses of ϕ-items is at most $1 + \frac{k-3}{k} + 0.6 = 2.6 - \frac{3}{k}$. If there is only one ϕ-item with a positive bonus, then the total weight is at most $2.7 - \frac{3}{k}$ again. Assume that there are two items with positive bonuses. None of these items can be larger than $\frac{1}{3}$, as their total size is below $\frac{1}{2}$. If both items have bonuses of 0.6 times their sizes minus 0.1, then their total bonus is at most $0.6 \cdot \frac{1}{2} - 0.2 = 0.1$. In the cases where $k \ge 20$, this is the only remaining option (as each of these items is small or medium), and we are left with the cases where $k \le 19$, and moreover, in the remaining case there are two items with positive bonuses, and these bonuses are not both equal to the sizes times 0.6 minus 0.1. Let $a_1 \leq a_2$ be the sizes of the items. We have $a_2 \in (0.2, 0.3]$ (otherwise either both items are very small, or the larger item is larger medium and the smaller one is very small, and both items have bonuses of the form 0.6 times the size minus 0.1, a case that was analyzed earlier). Thus, the larger item of the two is either larger small or smaller medium. We will bound the total weight of the two items and show that it does not exceed 0.7. Since the weight function is monotonically non-decreasing, we analyze $w(a_2) + w(\frac{1}{2} - a_2)$. If $\frac{1}{5} < a_2 \leq \frac{1}{4}$, then $w(a_2) + w(\frac{1}{2} - a_2) = (2.8 - 20/k)\frac{1}{2} - 0.7 + \frac{10}{k} = 1.4 - 10/k - 0.7 + 10/k = 0.7$. The case $\frac{1}{4} < a_2 \leq 0.3$ is symmetric. □

Now, we bound the total weight of the bins of FF. Once again we split the analysis into several cases according to the number of items packed into the bins. In this case we can also neglect k-bins and 1-bins, as the total weight of a k bin is 1, and all items of size above $\frac{1}{2}$ have weights of 1. Moreover, any bin that contains a huge item is removed from the analysis. Thus, we are left with 2^+-bins that do not contain such items. Additionally, the weight of any bin with level at least $5/6$ is at least 1, as the weight of any ϕ-item and of a γ_2-item is at least $6/5$-times the size of the item. Since there can be at most one 5^+-bin whose level is below $5/6$, the weight of any 5^+-bin (except for at most one bin) is at least 1. In the following we concentrate on the 2-bins, 3-bins and 4-bins.

Lemma 8.4.2 *The weight of any* 2*-bin containing a* γ_2*-item is at least* 1*, except for at most one bin. The weight of any* 3*-bin or* 4*-bin, containing a* γ_2*-item, is at least* 1*, except for at most one bin.*

Proof. Assume that at least two bins have γ_2-items, and each one has weight below 1. Denote them by B_i and B_j such that B_j appears after B_i in the ordering of FF. Each of them can have at most one γ_2-item, as the total weight of two γ_2 items is above 1. None of them has a level of at least $\frac{5}{6}$, as in such a case the weight is at least 1.

Assume that both these bins are 2-bins. The total weight of a γ_2-item and a ϕ-item of size above $\frac{1}{4}$ is at least $0.35 + 0.65 = 1$ for $k \geq 20$, and at least $0.3 + \frac{1}{k} + 0.7 - \frac{1}{k} = 1$ for $k \leq 19$. Thus, each of these 2-bins has a ϕ-item of size at most $\frac{1}{4}$ (as there is only one γ_2-item packed into each of the two bins). We find $s(B_i) \leq \frac{3}{4}$, as the size of the γ_2-item is at most $\frac{1}{2}$, and therefore B_j cannot have any item of size at most $\frac{1}{4}$, a contradiction.

Next, assume that B_i and B_j contain 3 or 4 items each and have weights below 1, such that each of them contains one γ_2-item, and the other items are ϕ-items. Similarly to the proof for 2-bins, none of them has a ϕ-item of size above $\frac{1}{4}$. If all the ϕ-items of B_j have sizes of at least $1/6$, then their total size is at least $\frac{1}{3}$, and their total weight is at least $\frac{6}{5} \cdot \frac{1}{3} = 0.4$, and we reach a contradiction, since the γ_2-item of that bin has weight of at least 0.6. Otherwise, since B_j has an item of size below $\frac{1}{6}$, the level of B_i is above $5/6$, a contradiction. □

We are left with bins containing only ϕ-items that are not huge.

Lemma 8.4.3 *Consider two* ϕ*-items of sizes* $a_1 \leq a_2 \leq 1/2$*. If* $1 \geq a_1 + a_2 > 1 - a_1$ *holds, then the total weight of the two items is at least* 1*.*

Proof. We have $a_1 > (1 - a_2)/2 \geq \frac{1}{4}$. If both items have sizes at least $1/3$, since $w(1/3) = 1/2$, the claim holds, since w is monotonically non-decreasing. Thus it suffices to consider the case

$1/4 < a_1 \leq 1/3$. In this case $a_2 > 1-2a_1 \geq \frac{1}{3}$. If $k \geq 20$, then the total weight of the two items is $1.2(a_1+a_2)+(0.6a_1-0.1)+0.1 = 1.8a_1+1.2a_2 = 0.9(2a_1+a_2)+0.3a_2 > 0.9\cdot 1+0.3\cdot\frac{1}{3} = 1$. If $k \leq 19$, we consider two cases. If $a_1 > 0.3$, then the calculation is the same as for $k \geq 20$. Otherwise, the total weight of the two items is $1.2(a_1+a_2)+((1.6-20/k)a_1-0.4+\frac{6}{k})+0.1 = (2.8-20/k)a_1+1.2a_2-0.3+\frac{6}{k} > (2.8-20/k)a_1+1.2(1-2a_1)-0.3+\frac{6}{k} = (0.4-20/k)a_1+0.9+6/k \geq (0.4-20/k)\cdot 0.3+0.9+\frac{6}{k} = 1.02 > 1$, since $0.4-20/k < 0$ and $a_1 \leq 0.3$. □

Lemma 8.4.4 *Consider three ϕ-items of sizes $a_1 \leq a_2 \leq a_3 \leq 1/2$. If $1 \geq a_1+a_2+a_3 > 1-a_1$ holds, then the total weight of the three items is at least* 1.

Proof. We have $4a_3 \geq 2a_1+a_2+a_3 > 1$, so $a_3 > \frac{1}{4}$. If $a_1 > \frac{1}{4}$, then the claim holds since the weight of an item with size above $1/4$ is at least 0.35. If $a_1 \leq \frac{1}{6}$, then $a_1+a_2+a_3 > \frac{5}{6}$, and the total weight is at least 1. Thus, we are left with the case $\frac{1}{6} < a_1 \leq \frac{1}{4}$, and thus $a_1+a_2+a_3 > \frac{3}{4}$. If $a_3 > \frac{1}{3}$, then its bonus is 0.1, and the total weight of the three items is at least $\frac{6}{5}\cdot\frac{3}{4}+0.1 = 1$. We are left with the case $\frac{1}{6} < a_1 \leq a_2 \leq a_3 \leq \frac{1}{3}$. We find that in the case $k \geq 20$, as all three items have sizes in $(\frac{1}{6}, \frac{1}{3}]$, the total weight of the items is $1.8(a_1+a_2+a_3)-0.3 > 1.8\cdot\frac{3}{4}-0.3 = 1.05 > 1$. We are left with the case $k \leq 19$. If $a_2 > \frac{1}{4}$, then since the bonus of any item of size above $\frac{1}{4}$ is above $\frac{1}{20}$, the total weight of the items is at least $1.2\cdot\frac{3}{4}+2\cdot 0.05 = 1$. If $a_1 \leq \frac{1}{5}$, then $a_1+a_2+a_3 > \frac{4}{5}$, and since the bonus of the largest item is above $\frac{1}{20}$, we get a total weight of at least $1.2\cdot\frac{4}{5}+0.05 = 1.01 > 1$. We are therefore left with the case $\frac{1}{5} \leq a_1 \leq a_2 \leq \frac{1}{4}$. If $a_3 \leq 0.3$, then the total weight is at least $(2.8-20/k)\frac{3}{4}+2(-0.3+4/k)+(-0.4+6/k) = 1.1-1/k \geq 1$. If $a_3 > 0.3$, then the total weight is $(2.8-20/k)(a_1+a_2)+1.8a_3+2(-0.3+4/k)-0.1 > (2.8-20/k)a_1+1.8a_3-0.7+8/k+(2.8-20/k)(1-2a_1-a_3) = (20/k-2.8)a_1+(20/k-1)a_3+2.1-12/k \geq (20/k-2.8)/4+(20/k-1)\cdot 0.3+2.1-12/k = 1.1-1/k \geq 1$ since $k \geq 10$, $a_1 \leq \frac{1}{4}$, and $a_3 \geq 0.3$. □

Lemma 8.4.5 *Consider four ϕ-items of sizes $a_1 \leq a_2 \leq a_3 \leq a_4 \leq 1/2$. If $1 \geq a_1+a_2+a_3+a_4 > 1-a_1$ holds, then the total weight of the four items is at least* 1.

Proof. We have $5a_4 \geq 2a_1+a_2+a_3+a_4 > 1$, so $a_4 > \frac{1}{5}$. If $a_1 \leq \frac{1}{6}$, then $a_1+a_2+a_3+a_4 > \frac{5}{6}$, and the total weight is at least 1. Otherwise we find $a_1+a_2+a_3+a_4 \geq \max\{1-a_1, 4a_1\} \geq \frac{4}{5}$. If $a_4 > \frac{1}{4}$, then its bonus is above $\frac{1}{20}$, and the total weight is at least $1.2\cdot\frac{4}{5}+0.05 > 1$. Thus, $\frac{1}{5} < a_4 \leq \frac{1}{4}$. If $k \geq 20$, as the sizes of all items are in $(\frac{1}{6}, \frac{1}{4}]$, the total weight of all four items is $1.8(a_1+a_2+a_3+a_4)-0.4 \geq 1.04 > 1$. We are left with the case $k \leq 19$. If $a_1 > \frac{1}{5}$, then the total weight of all four items is $(2.8-20/k)(a_1+a_2+a_3+a_4)-1.2+16/k \geq (2.8-20/k)\cdot 0.8-1.2+16/k = 1.04 > 1$. Otherwise, $\frac{1}{6} < a_1 \leq \frac{1}{5} < a_4 \leq \frac{1}{4}$. If $a_2 > \frac{1}{5}$, then the total weight is $1.8a_1-0.1+(2.8-20/k)(a_2+a_3+a_4)-0.9+12/k > 1.8a_1+(2.8-20/k)(1-2a_1)-1+12/k = a_1(40/k-3.8)+1.8-8/k$. If $40/k-3.8$ is non-negative, then using $8/k \leq 0.8$ we find a total weight of at least 1. Otherwise, using $a_1 \leq \frac{1}{5}$, we find a total weight of at least $(40/k-3.8)\cdot\frac{1}{5}+1.8-8/k = 1.04 > 1$. If $a_2 \leq \frac{1}{5}$, then the scaled size is $1.2(a_1+a_2+a_3+a_4) > 1.2(1-a_1)$. Thus the bonus of the two smallest items is $0.6(a_1+a_2)-0.2 \geq 1.2a_1-0.2$. Thus, the total weight is at least 1. □

Lemma 8.4.6 *The total weight of the* 2*-bins,* 3*-bins and* 4*-bins of FF that contain only ϕ-items is at least their number minus* 1.

Proof. The proof is the same as for Lemma 6.1.23 (the only difference is that 5-bins are not considered). □

We proved $FF(L) - 5 \leq W \leq (2.7 - 3/k)OPT(L)$, thus we proved the next theorem.

Theorem 8.4.1 *The asymptotic approximation ratio of FF for any $k \geq 10$ is at most $2.7 - 3/k$.*

8.5 Appendix E, about Batch Scheduling

Here we consider the bounded batch scheduling problem with nonidentical job sizes on a single machine, with the objective of minimizing the makespan. We present an algorithm which calls an online algorithm $\mathcal{P}$ (chosen arbitrarily) for the one-dimensional bin-packing problem as a sub-procedure, and prove that its worst-case ratio is the same as the absolute worst-case performance of $\mathcal{P}$. Hence, there exists an algorithm with worst-case ratio $\frac{17}{10}$, which is better than any known upper bound on this problem. The content of this Appendix E is from [34].

The problem is defined in [75]. We are given a non-empty set of jobs $\mathcal{J} = \{J_1, J_2, \ldots, J_n\}$. For $j = 1, \ldots, n$ the processing time and size of J_j is p_j and s_j, respectively. There are $m \geq 1$ machines $M_1, M_2, \ldots, M_m$ with the same capacity B. Each machine can simultaneously process a number of jobs as a batch as long as the total size of jobs in the batch is no greater than B. The processing time of a batch, which is also called the *length*, is the maximum of the processing times of jobs contained in the batch.

Without loss of the generality, we will assume that $B = 1$ and $s_j \leq 1$ for all j. All jobs are available at time 0 and no preemption is allowed. Here, the objective is to minimize the makespan, i.e. the maximum completion time of all jobs. Now denote by $C^{\mathcal{A}}(\mathcal{J})$ and $C^*(\mathcal{J})$ the makespan of the schedule produced by algorithm $\mathcal{A}$ and of the optimal schedule, respectively. For measuring the performance of an algorithm, traditional terminology in scheduling is "worst-case ratio" while that in bin packing is (asymptotic or absolute) "approximation ratio". Now, as we consider a combined model, we will use the term (asymptotic or absolute) performance ratio, and this term will be used also for the packing or scheduling models. Thus, the *absolute performance ratio* of $\mathcal{A}$ is then defined as

$$\inf\{r \mid \frac{C^{\mathcal{A}}(\mathcal{J})}{C^*(\mathcal{J})} \leq r \text{ for all } \mathcal{J}\},$$

while the *asymptotic performance ratio* for algorithm $\mathcal{P}$ is defined as

$$\lim_{N\to\infty} \inf\{r \mid \frac{C^{\mathcal{P}}(\mathcal{I})}{C^*(\mathcal{I})} \leq r \text{ for all } \mathcal{I} \text{ with } C^*(\mathcal{I}) \geq N\}.$$

The definitions are standard, and can be found for instance in [14].

8.5.1 Related work

The class of problems considered here contains many fundamental combinatorial optimization problems as special cases. If the size of each job is 1, then the batch scheduling problem on parallel identical machines simplifies to the classical scheduling problem on parallel identical machines with the objective to minimize the makespan. The latter problem is NP-hard even if the number m of machines is just 2 [43]. Graham proposed algorithms called *List Scheduling* (LS) and *Longest Processing Time first* (LPT) in his seminal works [46, 47]. Assuming that an order of the jobs has been chosen, algorithm LS assigns the first unprocessed job in the sequence to the machine which can complete it as early as possible, where ties are broken arbitrarily. Algorithm LPT first sorts all jobs in descending order of their processing times, and then assigns the jobs by the LS algorithm. The absolute performance ratios of LS and LPT are $2 - \frac{1}{m}$ and $\frac{4}{3} - \frac{1}{3m}$, respectively. The problem also admits a *Fully Polynomial Time Approximation Scheme* (FPTAS) when m is a fixed number and a *Polynomial Time Approximation Scheme* (PTAS) when m is part of the input [67, 50]. If all jobs have the same processing time p, then the length of each batch in any schedule is p. Thus the batch scheduling problem on a single machine reduces to the one-dimensional bin-packing problem. Now let $\mathcal{P}$ denote a bin-packing algorithm. We will use $C^{\mathcal{P}}(\mathcal{I})$ to denote the number of bins employed when $\mathcal{P}$ is applied to item set $\mathcal{I}$, and $C^*(\mathcal{I})$ to denote the number of bins employed for an optimal packing of $\mathcal{I}$.

An interesting feature of the bin-packing problem is that the asymptotic performance ratio and the absolute performance ratio of an algorithm may be different. Concerning the variants of batch scheduling studied here, currently we cannot prove or disprove that the best possible asymptotic and absolute performance ratios are equal.

In the *online* version of the bin-packing problem [14], items arrive according to some unknown list (even the length of the list is unknown until the very end), and the next item must be packed into a bin promptly when it arrives, without any information about the remaining items. Algorithms which can solve the online version are called online algorithms. Clearly, FF can be interpreted as an online algorithm (it applies no look-ahead), but FFD does not have this feature. There exist online algorithms with *asymptotic* performance ratios better than that of FF, such as *Refined FF* [86] and *Harmonic* [60].

The online algorithm proposed by Seiden [69] with an asymptotic ratio of at most 1.58889 is the best one known so far. An online problem has an *asymptotic (absolute) lower bound* ρ if no online algorithm has an asymptotic (absolute) performance ratio smaller than ρ. The first asymptotic lower bound $\frac{3}{2}$ for the one-dimensional bin-packing problem was given by Yao [86], and the current best asymptotic lower bound is 1.54037 [2]. To the best of our knowledge, no result on an absolute lower bound has been reported before the publication of our article [34] which contains the results of this section, i.e. before 2014. (Then, in 2015 we published our paper [6] which deals with this question, and gives an algorithm with the tight $5/3$ bound. In fact, this algorithm is a modification of algorithm FF.)

Research on batch scheduling problems is motivated by burn-in operations in semi-conductor manufacturing, and dates back to the 1980's [51]. Several variants of batch scheduling have been proposed, and most of them can be classified into three types: *unbounded batch model*, where the capacity of a batch is infinity; *bounded batch model with identical job sizes*; and *bounded batch model with nonidentical job sizes*. Clearly, problems in the last category are the most difficult ones. Below, we will briefly survey some results on the classical batch scheduling problems with the objective of minimizing the makespan. More results on other objectives and more complex paradigms such as nonidentical release times and online batching models can be found in [12, 66], and references therein.

If the batch has an unbounded capacity, combining all the jobs into a single batch is an optimal choice for both the single and parallel machine cases. For a bounded batch with identical job sizes, the problem is still polynomially solvable for the single machine case. If there is more than one machine, the problem becomes NP-hard, but it still admits a PTAS, see Li et al. [63].

The bounded batch problem with nonidentical job sizes on a single machine contains the one-dimensional bin-packing problem as a special case. Hence, there is no polynomial-time (offline) algorithm with absolute performance ratio smaller than $\frac{3}{2}$ unless $P = NP$. Uzsoy [75] proposed four algorithms for this problem. One of them is called LPT–FF, as this algorithm can be viewed as a combination of LPT and FF (performing them one after the other, see details later in a more general setting).

Zhang et al. [83] proved that among the four algorithms, $LPT\text{–}FF$ is the only one that has a finite absolute performance ratio, and that its absolute performance ratio is at most 2. In the same paper, Zhang et al. proposed another algorithm called $MLPT\text{–}FF$, which has an absolute performance ratio of $\frac{7}{4}$. As far as we know, no algorithm was proved to have an absolute performance ratio better than $\frac{7}{4}$ before the present study (and there are no results at all on approximation algorithms for the parallel machine case).

Below, we revisit the bounded batch scheduling problem with nonidentical job sizes. For the single machine case, we use a novel technique to learn the relationship between the performance of batch scheduling algorithms and bin-packing algorithms; in this way we can improve the absolute performance ratio of $LPT\text{–}FF$ to $\frac{17}{10}$ with a very concise proof (Corollary 8.5.1). It has long been believed that the asymptotic performance ratio is better for measuring the performance of a bin-packing algorithm [17]. Our approach shows that the absolute performance ratio can be of help as well. For completeness, we will also give a short proof that $\frac{5}{3}$ is an absolute lower bound for one-dimensional online bin-packing (Theorem 8.5.2). In this Appendix E we deal only with the single machine case, the investigation of the parallel machines case can be found in [34].

8.5.2 The single machine case

Next, we will generalize the idea of $LPT\text{–}FF$ to get better upper bounds for batch scheduling on a single machine. For an instance of batch scheduling with job set $\mathcal{J} = \{J_1, J_2, \ldots, J_n\}$, we introduce the *induced instance* of the bin-packing problem with the item set $\mathcal{I}_\mathcal{J} = \{I_1, I_2, \ldots, I_n\}$ and bin capacity 1, where item I_j corresponds to job J_j. For any j, $1 \leq j \leq n$, the size of I_j is just s_j, i.e. the size of J_j. Clearly, items whose corresponding jobs are assigned to a batch in any schedule of $\mathcal{J}$ can be packed into a bin; and also conversely, any subset of items fitting in a bin may form a batch.

Now we introduce the algorithm $LPT\text{–}\mathcal{P}$ which calls an online bin-packing algorithm $\mathcal{P}$ as a sub-procedure. This point may actually come as a surprise: we have complete information about the (offline) problem instance, but after some preprocessing we intentionally restrict ourselves and solve a seemingly harder problem, as if it was online. Eventually the upper bound derived from this approach turns out to be better than the one given by any previously known purely offline method.

Algorithm $LPT\text{–}\mathcal{P}$

- Step 1. Sort the jobs in the non-increasing order of their processing times, i.e. $p_1 \geq p_2 \geq \ldots \geq p_n$.
- Step 2. Given bin size $B = 1$, apply the bin-packing algorithm $\mathcal{P}$ to pack the induced item list $\mathcal{I}_\mathcal{J}$ (where the j-th item has size s_j). Jobs whose corresponding items are packed into the same bin form a batch.
- Step 3. Schedule the batches without idle time on the machine in any order.

Theorem 8.5.1 *If $\mathcal{P}$ is an online algorithm for the bin-packing problem, then the absolute performance ratio of the algorithm $LPT\text{–}\mathcal{P}$ for batch scheduling with nonidentical job sizes on a single machine is the same as the absolute performance ratio of $\mathcal{P}$.*

Proof. Let $\mathcal{J} = \{J_1, J_2, \ldots, J_n\}$ be any instance of single machine batch scheduling, and let $\mathcal{I}_{\mathcal{J}} = \{I_1, I_2, \ldots, I_n\}$ be the corresponding induced instance of bin-packing. We denote by $N_{\mathcal{J}}$ the number of different processing times of jobs in $\mathcal{J}$. If $N_{\mathcal{J}} = 1$, i.e. all jobs have the same processing time, say p, then all batches have the same length p; and those restricted instances of batch scheduling are in one-to-one correspondence with the instances of bin-packing. Hence both the optimum for $\mathcal{J}$ and the value of $C^{LPT-\mathcal{P}}(\mathcal{J})$ are exactly p times the optimum for $\mathcal{I}_{\mathcal{J}}$ and the value of $C^{\mathcal{P}}(\mathcal{I}_{\mathcal{J}})$, respectively. This implies that the absolute performance ratio of LPT–$\mathcal{P}$ cannot be better than the absolute performance ratio of $\mathcal{P}$. We still have to prove that it is not larger. The above observations verify this, for all instances $\mathcal{J}$ with $N_{\mathcal{J}} = 1$.

Let the absolute performance ratio of $\mathcal{P}$ be r. Suppose, for a contradiction, that there exists some instance $\mathcal{J}$ of single machine batch scheduling with

$$C^{LPT-\mathcal{P}}(\mathcal{J}) > rC^*(\mathcal{J}). \tag{8.2}$$

We certainly must have $N_{\mathcal{J}} \geq 2$. From all possible choices of $\mathcal{J}$ we select one $\{J_1, J_2, \ldots, J_n\}$ with the smallest $N_{\mathcal{J}}$. We will modify $\mathcal{J}$ to an instance $\mathcal{J}'$ which has $N_{\mathcal{J}'} = N_{\mathcal{J}} - 1$ and still satisfies the inequality $C^{LPT-\mathcal{P}}(\mathcal{J}') > rC^*(\mathcal{J}')$. This contradiction in the choice of $\mathcal{J}$ completes the proof of the theorem.

Let the largest and second largest processing times of jobs be x and y, respectively, and let $\mathcal{J}_x$ and n_x denote the set and the number of jobs having processing time x. Since $N_{\mathcal{J}} > 1$, both x and y are well-defined and we have $n_x < n$. To create $\mathcal{J}'$, our strategy is to decrease the processing time of the jobs in $\mathcal{J}_x$ from x to y, while keeping their sizes unchanged. In this way the induced instance of bin-packing remains exactly the same, and consequently LPT–$\mathcal{P}$ creates the same batch partition for $\mathcal{J}$ and $\mathcal{J}'$ because $\mathcal{P}$ only takes the item sizes into account.

Denote by b_x the number of batches of length x in the schedule generated by LPT–$\mathcal{P}$ for $\mathcal{J}$, and let b_x^* be the number of batches of length x in a schedule that has been chosen arbitrarily from the optimal ones. Let us run LPT–$\mathcal{P}$ on the restricted instance $\mathcal{J}_x$, which is a starting segment of $\mathcal{J}$ in the list of items for $\mathcal{P}$ after Step 1. Then we obtain

$$b_x = C^{\mathcal{P}}(\mathcal{I}_{\mathcal{J}_x}) \leq rC^*(\mathcal{I}_{\mathcal{J}_x}) \leq rb_x^* \tag{8.3}$$

because $\mathcal{P}$ is an online algorithm, $N_{\mathcal{J}_x} = 1 < N_{\mathcal{J}}$, and there is enough room for the n_x items inside b_x^* bins.

Since the modification from $\mathcal{J}$ to $\mathcal{J}'$ keeps the batch partition the same, the makespan obtained from LPT–$\mathcal{P}$ is decreased by exactly $(x - y)b_x$. This means

$$C^{LPT-\mathcal{P}}(\mathcal{J}') = C^{LPT-\mathcal{P}}(\mathcal{J}) - (x - y)b_x. \tag{8.4}$$

Indeed, the length of batches containing at least one longest item decreases from x to y, while the lengths of the other batches remain unchanged.

However, we get a feasible schedule of $\mathcal{J}'$ by keeping the assignment of jobs to batches in the same way as that in the chosen optimal schedule of $\mathcal{J}$. Then the length of b_x^* batches which contain jobs of $\mathcal{J}_x$ decreases from x to y, while the length of all the other batches remains unchanged. Hence

$$C^*(\mathcal{J}') \leq C^*(\mathcal{J}) - (x - y)b_x^*. \tag{8.5}$$

Combining the above points, we find that

$$\frac{C^{LPT-\mathcal{P}}(\mathcal{J}')}{C^*(\mathcal{J}')} \geq \frac{C^{LPT-\mathcal{P}}(\mathcal{J}) - (x-y)b_x}{C^*(\mathcal{J}) - (x-y)b_x^*} > \frac{rC^*(\mathcal{J}) - (x-y)rb_x^*}{C^*(\mathcal{J}) - (x-y)b_x^*} = r,$$

where (8.4) and (8.5) are applied in the first inequality, and (8.2) and (8.3) are applied in the second inequality. With the choice of $\mathcal{J}$, however, the leftmost side should be at most r because $N_{\mathcal{J}'} < N_{\mathcal{J}}$. This contradiction completes the proof of the theorem. □

Since the tight absolute performance ratio of FF is $\frac{17}{10}$ [23], from Theorem 8.5.1 we get the following.

Corollary 8.5.1 *The tight absolute performance ratio of LPT–FF is $\frac{17}{10}$. As a consequence, batch scheduling with nonidentical job sizes on a single machine admits a polynomial-time $\frac{17}{10}$-approximation.*

This result improves the estimate in [83]. Moreover, this upper bound was the current champion before the publication of our newer work [6]. To improve LPT–FF further, the simplest idea seems to be to use an online bin-packing algorithm with better absolute performance ratio. Interestingly enough, however, although there were several known online algorithms whose asymptotic performance ratio is smaller than that of FF, none of them has a smaller absolute performance ratio. In fact, we have the following general lower bound, namely $5/3$, on the absolute performance ratio of any online algorithm. After the submission of this paper, Zhang [82] kindly informed us that he proved the same result about two decades ago but he never published it. The instances used in the proof below bear some similarity with the instance used in [55] to show that the absolute performance ratio of FF is no smaller than $5/3$.

Theorem 8.5.2 *There is no online bin-packing algorithm with absolute performance ratio smaller than $\frac{5}{3}$.*

Proof. To prove the statement we construct a list $S \cup M \cup L$ of small, medium-sized, and large items. It will be decided online whether the input is just S, or $S \cup M$, or the entire $S \cup M \cup L$. To prove the statement we construct the following adversary list. Let the list begin with 6 *small* items of size $\frac{1}{6} - 2\varepsilon$ each (where ε is a very small positive value). If an online algorithm $\mathcal{P}$ uses at least two bins, then the list is terminated, and $\frac{C^{\mathcal{P}}(\mathcal{I})}{C^*(\mathcal{I})} \geq 2 > \frac{5}{3}$. Otherwise, the list is continued with 6 *medium-sized* items of size $\frac{1}{3} + \varepsilon$. In the optimal packing, two medium-sized items and two small items can be packed into the same bin, thus $C^*(\mathcal{I}) = 3$. (But in the current situation, the empty space 12ε is too small for a medium-sized item.) Therefore, to avoid the absolute performance ratio of $\mathcal{P}$ becoming no smaller than $\frac{5}{3}$, $\mathcal{P}$ must pack all medium-sized items into 3 bins. Then the list is completed with 6 *big* items of size $\frac{1}{2} + \varepsilon$. Algorithm $\mathcal{P}$ must pack each of them into a new bin. Hence, $C^{\mathcal{P}}(\mathcal{I}) = 10$. In the optimal packing, however, each bin contains a small item, a medium-sized item and a big item. Thus, $C^*(\mathcal{I}) = 6$ and $\frac{C^{\mathcal{P}}(\mathcal{I})}{C^*(\mathcal{I})} = \frac{5}{3}$. □

Although the lower bound given in Theorem 8.5.2 is smaller than the absolute performance ratio of FF, we conjectured that this lower bound of $5/3$ for online bin-packing cannot be improved further. And soon it turned out that the conjecture is true, as in our paper [6] we gave an online bin packing algorithm with absolute approximation ratio $5/3$. The name of the algorithm is FT (as ”five third”). Thus the next corollary also holds:

Corollary 8.5.2 *The tight absolute performance ratio of $LPT–FT$ is $\frac{5}{3}$. As a consequence, batch scheduling with nonidentical job sizes on a single machine admits a polynomial-time $\frac{5}{3}$-approximation.*

But it is still not the limit of approximability for batch scheduling with nonidentical job sizes on a single machine when some polynomial-time algorithms other than $LPT–\mathcal{P}$ are considered. In such cases we know only the smaller lower bound $\frac{3}{2}$ unless $P = NP$ (see e.g. [14]), since the problem is a generalization of the one-dimensional bin-packing problem.

Some remarks. Above we reviewed the bounded batch scheduling problem with nonidentical job sizes on a single machine. We applied a one-dimensional bin-packing algorithm $\mathcal{P}$ as a subprocedure. We proved that if $\mathcal{P}$ is an online bin-packing algorithm, then the absolute performance ratio of $LPT–\mathcal{P}$ is equal to the absolute performance ratio of $\mathcal{P}$. Based on the fact that FF has a performance ratio of $\frac{17}{10}$, (or based on the fact taht FT has a performance ratio of $\frac{5}{3}$), we derived the same improved absolute performance ratio for the batch scheduling problem on a single machine. A remarkable aspect of this approach is that it creates an *offline* batch scheduling algorithm from an *online* bin-packing algorithm, by combining it with some preprocessing.

There are many interesting problems which deserve further study. The main one is to improve the performance of algorithms for the single machine case or to prove that the constant $\frac{5}{3}$ is tight. From the conclusions above a more effective algorithm would require that some new ideas concerning the assignment of jobs to batches be introduced. Next, is it possible to generalize the approaches presented here to other variants of batch scheduling? Is some analogue of the above-mentioned relation between the performance of $\mathcal{P}$ and $LPT–\mathcal{P}$ valid for other variants of the problem?

Our paper [34] also proposes some new and interesting tasks on bin-packing problems. Although studies on the absolute performance ratio of bin-packing algorithms were almost neglected earlier, it can be seen from the results drawn here that various results on it are indeed useful. For instance, knowing that the FF algorithm has a smaller absolute performance ratio when dealing with items of size smaller than $\frac{1}{2}$, we can also state for batch scheduling on a single machine that the algorithm performs better in the setting where all the jobs are of size smaller than $\frac{1}{2}$.