Sharp Functional Inequalities and Elliptic Problems on Non-Euclidean Structures

DSc Dissertation

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Dissertation submitted for the degree
Doctor of Sciences (DSc) of the Hungarian Academy of Sciences

Budapest, 2017

Introduction

I) Primary objective. The dissertation treats sharp functional inequalities on curved spaces (non-euclidean structures) with applications in various elliptic partial differential equations (PDEs), combining various elements from geometric analysis, calculus of variations and group theory. The present work resumes my contributions to these fields during the last 8 years, based on 19 papers (and one monograph), all published or accepted for publication in well established mathematical journals; for 8 of them I am the sole author, while the other 11 articles were written with my collaborators.

II) Historical facts. In order to investigate existence, uniqueness/multiplicity of various elliptic PDEs (as Schrödinger, Dirichlet or Neumann problems), fine properties of Sobolev spaces and sharp Sobolev inequalities are needed. For exemplification, let $n \geq 2$, $p \in (1, n)$, and recall the classical Sobolev embedding $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{p^\star}(\mathbb{R}^n)$ with the Sobolev inequality

$$\left(\int_{\mathbb{R}^n} |u|^{p^\star} \mathrm{d}x\right)^{1/p^\star} \leq S_{n,p} \left(\int_{\mathbb{R}^n} |\nabla u|^p \mathrm{d}x\right)^{1/p}, \ \forall u \in W^{1,p}(\mathbb{R}^n), \tag{$\mathbf{S}$}$$

where $S_{n,p} = \pi^{-\frac{1}{2}} n^{-\frac{1}{p}} \left(\frac{p-1}{n-p}\right)^{p'} \left(\frac{\Gamma(1+n/2)\Gamma(n)}{\Gamma(n/p)\Gamma(1+n-n/p)}\right)^{1/n}$ is the sharp constant, $p^\star = \frac{pn}{n-p}$ denotes the critical Sobolev exponent and $p' = \frac{p}{p-1}$ is the conjugate of p, see Talenti [88]. Furthermore, the unique class of extremal functions in ($\mathbf{S}$) is $u_\lambda(x) = \left(\lambda + |x|^{p'}\right)^{1-n/p}$, $\lambda > 0$. Inequality ($\mathbf{S}$) has been established by using a Schwarz symmetrization argument and the Pólya-Szegő inequality, where the sharp isoperimetric inequality in $\mathbb{R}^n$ is deeply explored.

A natural question arose: *what kind of geometric information is encoded into the Sobolev inequality* ($\mathbf{S}$) *whenever the ambient space is curved*? To handle this problem, in the middle of the seventies Aubin [8] initiated the so-called *AB-program* (see Druet and Hebey [34]) whose objective was to establish the optimal values of $A \geq 0$ and $B \geq 0$ in the inequality

$$\left(\int_M |u|^{p^\star} \mathrm{d}V_g\right)^{1/p^\star} \leq A \left(\int_M |\nabla_g u|^p \mathrm{d}V_g\right)^{1/p} + B \left(\int_M |u|^p \mathrm{d}V_g\right)^{1/p}, \ \forall u \in W^{1,p}(M), \tag{$\mathbf{AB}$}$$

where (M, g) is a complete n-dimensional Riemannian manifold, while $\mathrm{d}V_g$ and ∇_g denote the canonical volume form and gradient on (M, g), respectively. It turned out that ($\mathbf{AB}$) deeply depends on the *curvature* of (M, g). For instance, inequality ($\mathbf{AB}$) holds on any n-dimensional Hadamard manifold[1] (M, g) with $A = S_{n,p}$ and $B = 0$ whenever the Cartan-Hadamard conjecture[2] holds on (M, g); such

[1]Simply connected, complete Riemannian manifold with nonpositive sectional curvature.
[2]The validity of the sharp isoperimetric inequality on Hadamard manifolds; see §1.1.2.

cases occur for instance in low-dimensions $n \in \{2, 3, 4\}$, see Druet, Hebey and Vaugon [35]. However, if (M, g) has nonnegative Ricci curvature, inequality (**AB**) holds with $A = S_{n,p}$ and $B = 0$ if and only if (M, g) is isometric to the Euclidean space $\mathbb{R}^n$, see Ledoux [61]. Further contributions in the Riemannian setting can be found in Bakry, Concordet and Ledoux [10], do Carmo and Xia [32], Ni [70], Xia [97], and references therein.

In recent years considerable efforts have been made in order to investigate various nonlinear PDEs involving Laplace-type operators in *curved spaces*. To handle these class of problems, various approaches have been elaborated, like the theory of Ricci flows and optimal mass transportation on Riemannian/Finsler manifolds. One of the main motivations were the study of the famous *Yamabe problem*, see the comprehensive monograph of Hebey [52], and the *Poincaré conjecture* proved by Perelman [76]. In these works sharp geometric and functional inequalities as well as the influence of curvature play crucial roles; see e.g. the Gross-type sharp logarithmic Sobolev inequality in the work of Perelman [76]. Accordingly, this research topic is a very active and flourishing area of the geometric analysis, see e.g. Ambrosio, Gigli and Savaré [5, 6], Lott and Villani [65], Sturm [85, 86], Villani [92].

III) Scientific objectives and brief description of own contributions. In the last few years I was interested *to understand the geometric aspects of certain highly nonlinear phenomena on non-euclidean structures*, by describing *the influence of curvature in some Sobolev-type inequalities and elliptic problems formulated in the language of the calculus of variations*. In the sequel, I intend to briefly describe my results (obtained either as a sole author or in collaboration with my co-authors). In order to avoid technicalities, most of the results are presented in the simplest possible way, although many of them are also valid in more general settings (e.g. on not necessarily reversible Finsler manifolds instead of Riemannian ones). These facts will be highlighted throughout the presentation.

The dissertation contains five chapters.

Chapter 1 is devoted to those fundamental notions and results which are indispensable to have a self-contained character of the present work, recalling elements from the geometry of metric measure spaces (Riemannian/Finsler manifolds and $\mathsf{CD}(K, N)$ spaces) as well as certain variational principles.

Chapter 2 is devoted to interpolation inequalities on curved spaces. We first recall the sharp Gagliardo-Nirenberg interpolation inequality with its limit cases in the flat case $\mathbb{R}^n$, proved by Cordero-Erausquin, Nazaret and Villani [24] and Gentil [45]. In the particular case, this inequality reduces precisely to the sharp Sobolev inequality (**S**). Based on papers [109], [116], [110], [103] and [106], and depending on the *sign of the curvature*, my achievements can be summarized as follows:

- *Positively curved case* (*interpolation inequalities*). It is well-known that any metric measure space verifying the famous curvature-dimension condition $\mathsf{CD}(K, N)$[3] à la Lott-Sturm-Villani supports various geometric inequalities, as Brunn-Minkowski and Bishop-Gromov inequalities. It was a challenging problem, suggested by Villani, whether such non-smooth spaces support functional inequalities. In this context, we prove that the picture is quite rigid: if a $\mathsf{CD}(K, n)$

[3] The curvature-dimension condition $\mathsf{CD}(K, N)$ was introduced by Lott and Villani [65] and Sturm [85, 86] on metric measure spaces. In the case of a Riemannian/Finsler manifold M, the condition $\mathsf{CD}(K, N)$ represents the lower bound $K \in \mathbb{R}$ for the Ricci curvature on M and the upper bound $N \in \mathbb{R}$ for the dimension of M, respectively.

metric measure space (M, d, m) supports the Gagliardo-Nirenberg inequality or any of its limit cases (L^p-logarithmic Sobolev inequality or Faber-Krahn inequality) for some $K \geq 0$ and $n \geq 2$ and an n-density assumption at some point of M, then a *global non-collapsing n-dimensional volume growth* holds, i.e., there exists a universal constant $C_0 > 0$ such that $\mathsf{m}(B(x, \rho)) \geq C_0\rho^n$ for all $x \in M$ and $\rho \geq 0$, where $B(x, \rho) = \{y \in M : d(x, y) < \rho\}$ (see Theorems 2.3, 2.4 and 2.5). Due to the quantitative character of the volume growth estimate, we establish rigidity results on Riemannian/Finsler manifolds with nonnegative Ricci curvature supporting Gagliardo-Nirenberg inequalities via a quantitative Perelman-type homotopy construction. Roughly speaking, once the constant in the Gagliardo-Nirenberg inequality (or in its limit cases) is closer and closer to its optimal Euclidean counterpart, the Riemannian manifold with nonnegative Ricci curvature is topologically closer and closer to the Euclidean space (see Theorem 2.6). In particular, our rigidity result for the L^p-logarithmic Sobolev inequality solves an open problem of Xia [98].

- *Negatively curved case* (*interpolation inequalities*). Inspired by Ni [70] and Perelman [76], we prove that Gagliardo-Nirenberg inequalities hold on n-dimensional Hadamard manifolds with the same sharp constant as in $\mathbb{R}^n$ whenever the Cartan-Hadamard conjecture holds. However, if one expects extremal functions in Gagliardo-Nirenberg inequalities, it turns out that the Hadamard manifold is isometric to the Euclidean space $\mathbb{R}^n$ (see Theorem 2.7).

Chapter 3 deals with famous uncertainty principles on curved spaces. As endpoints of the Caffarelli-Kohn-Nirenberg inequality, we first recall both the *sharp Heisenberg-Pauli-Weyl* and *sharp Hardy-Poincaré uncertainty principles* in the flat case $\mathbb{R}^n$. The own contributions, based on the papers [108], [118], [107] and [104], can be summarized as follows:

- *Positively curved case* (*uncertainty principles*). We prove that on a complete Riemannian manifold (M, g) with nonnegative Ricci curvature the sharp Heisenberg-Pauli-Weyl holds if and only if the manifold is isometric to the Euclidean space of the same dimension (see Theorem 3.4). We note that this result seems to be a strong rigidity in the sense that no quantitative form can be established as in its counterparts from interpolation inequalities.

- *Negatively curved case* (*uncertainty principles*). We first prove that the sharp Heisenberg-Pauli-Weyl uncertainty principle holds on any n-dimensional Hadamard manifold (M, g); however, positive extremals exist if and only if (M, g) is isometric to $\mathbb{R}^n$ (see Theorems 3.5 and 3.6). We emphasize that these sharp results *do not* require the validity of the Cartan-Hadamard conjecture as in the case of interpolation inequalities. Moreover, Theorem 3.6 emends a mistake from Kombe and Özaydin [59] on hyperbolic spaces. We then prove that stronger curvature implies more powerful improvements in the Hardy-Poincaré uncertainty principle on Hadamard manifolds (see Theorems 3.7 and 3.8); the latter result comes from the lack of extremals in the Hardy-Poincaré inequality in $\mathbb{R}^n$. We also prove a sharp Hardy-Poincaré inequality for multiple singularities (see Theorem 3.10) and a sharp Rellich uncertainty principle (see Theorem 3.11).

The next two chapters deal with applications of sharp Sobolev-type inequalities, providing a diversity of existence, uniqueness/multiplicity results for elliptic PDEs both on Finsler and Riemannian manifolds, emphasizing at the same time subtle differences between these two geometric settings.

Chapter 4 treats some elliptic problems on not necessarily Finsler manifolds. The own contributions, based on the papers [120] and [107], can be summarized as follows:

- *Reversibility versus structure of Sobolev spaces.* In order to investigate elliptic problems on Finsler manifolds, basic properties of Sobolev spaces over Finsler manifolds are expected to be valid, like the vector space structure, reflexivity, etc. Surprisingly, the *reversibility constant* $r_F \geq 1$ of a given Finsler manifold (M, F) turns to be decisive at this point. Indeed, we prove that the Sobolev space over (M, F) is a reflexive Banach space whenever $r_F < +\infty$ (see Theorem 4.1), while there are non-compact Finsler manifolds (M, F) with $r_F = +\infty$ for which the 'Sobolev space' over these objects might not be even vector spaces. The latter (counter)example is constructed on the n-dimensional unit ball endowed with a Funk-type Finsler metric (see Theorem 4.2) and highlights the deep difference between Riemannian and Finsler worlds. This set of results refills the missing piece in the theory of Sovolev spaces over non-compact manifolds.

- *Elliptic problems on Finsler-Hadamard manifolds.* We first study an elliptic parameter-depending model problem on a Funk-type manifold which involves the Finsler-Laplacian and a singular nonlinearity. By using variational arguments, we prove that for small parameters, the studied problem has only the trivial solution, while for large parameters, the problem has two distinct non-trivial weak solutions (see Theorem 4.3). We then consider a Poisson problem with a pole/singularity on bounded domains of a Finsler-Hadamard manifold, proving via the Hardy-Poincaré inequality (Chapter 3) uniqueness and further qualitative properties of the solution (see Theorems 4.4 and 4.5). Spectacular results show that the shape of the solution to the Poisson equation fully characterize the curvature of the Finsler manifold (see Theorems 4.6 and 4.7).

Chapter 5 is devoted to various elliptic problems on compact and non-compact Riemannian manifolds. The own contributions, based on the papers [113], [104] and [105] are summarized as follows:

- *Elliptic problems on compact Riemannian manifolds.* By using variational arguments, we prove a sharp bifurcation result (concerning the existence of solutions) for a sublinear eigenvalue problem on compact Riemannian manifolds (see Theorem 5.1), and provide its stability under small perturbations (see Theorem 5.2). These results are applied to establish a sharp Emden-type multiplicity result on an even-dimensional Euclidean space involving a singular term, by reducing the initial problem to a PDE defined on the 1-codimensional unit sphere endowed with its usual Riemannian metric (see Theorem 5.3).

- *Elliptic problems on non-compact Riemannian manifolds.* At first, by applying a sharp multipolar Hardy-Poincaré inequality (Chapter 3), we provide the existence of infinitely many symmetrically distinct solutions for an elliptic problem on the upper hemisphere, involving the natural Laplace-Beltrami operator and two poles/singularities (see Theorem 5.5). This result is obtained by an astounding group-theoretical argument leaning on the solvability of the Rubik cube (see Theorem 5.4). Then, we prove the existence of infinitely many isometry-invariant solutions for a Schrödinger-Maxwell system on Hadamard manifolds which involves an oscillatory nonlinearity near the origin (see Theorem 5.6). Here, the action of the isometry group on the Hadamard manifold plays a crucial role.

IV) Formulation of the Theses (according to the regulation of the Academy). In the sequel we will formulate four Theses which describe the main contributions of the present dissertation:

Thesis 1. (Sharp interpolation inequalities)

- $\mathsf{CD}(K,N)$ metric measure spaces in the sense of Lott-Sturm-Villani (with $K \geq 0$) supporting interpolation inequalities are topologically rigid, having a global non-collapsing volume growth. If the embedding constants in interpolation inequalities are closer and closer to their optimal Euclidean counterparts, the Riemannian manifold with nonnegative Ricci curvature is topologically closer and closer to the Euclidean space.
- Hadamard manifolds support sharp interpolation inequalities whenever the Cartan-Hadamard conjecture holds (e.g. in dimensions 2, 3 and 4). The existence of extremals however imply the flatness of the Hadamard manifolds.

Thesis 2. (Sharp uncertainty principles)

- Hadamard manifolds support the sharp Heisenberg-Pauli-Weyl uncertainty principle (the validity of the Cartan-Hadamard conjecture is not needed). However, a complete Riemannian manifold with nonnegative Ricci curvature supports the sharp Heisenberg-Pauli-Weyl uncertainty principle if and only if the manifold is isometric to the Euclidean space of the same dimension.
- Stronger negative curvature implies more powerful improvements in Hardy-Poincaré and Rellich uncertainty principles on Hadamard manifolds.

Thesis 3. (Elliptic problems on Finsler manifolds)

- Sobolev spaces over arbitrary Finsler manifolds with finite reversibility constants are reflexive Banach spaces. There are however non-compact Finsler manifolds with infinite reversibility constants for which the corresponding Sobolev spaces are not even vector spaces.
- Parameter-depending sublinear elliptic problems on not necessarily reversible Finsler manifolds with finite reversibility constant have two non-zero solutions for enough large parameters. Moreover, the shape of the solution for the unipolar Poisson equation fully characterize the curvature of the Finsler manifold.

Thesis 4. (Elliptic problems on Riemannian manifolds)

- Compactness of Riemannian manifolds in sublinear eigenvalue problems implies sharp bifurcation phenomenon concerning the number of solutions: for small values of the parameter there is only the zero solution, for large parameters there are two non-zero solutions, while the gap interval can be arbitrarily small.
- Non-compactness of Riemannian manifolds can be compensated by certain isometric group actions in order to guarantee multiple, isometry-invariant non-zero solutions for elliptic problems. In particular, the technique of solving the Rubik cube provides symmetrically distinct solutions.

V) Acknowledgments. I am grateful to my friends and collaborators, Francesca Faraci, Csaba Farkas, Shinichi Ohta, Ágoston Róth, Imre Rudas and Csaba Varga for their nice advices and supports over the years.

I thank Zoltán M. Balogh for his friendship and deep discussions about various problems related to Riemannian and sub-Riemannian geometries during my multiple visits at Universität Bern.

I am also indebted to Cédric Villani for stimulating conversations on the L^p-logarithmic Sobolev inequality and to Michel Ledoux for suggesting the study of the whole family of Gagliardo-Nirenberg interpolation inequalities in the non-smooth setting.

I would like to thank my wife Tünde and our children, Marót, Bora, Zonga and Bendegúz, for their patience and constant support.

Most of the results in the present dissertation have been supported by the János Bolyai Research Scholarship of the Hungarian Academy of Sciences (2009-2012 and 2013-2016).

Notations and conventions

Notations:

(M, F)	: Finsler manifold M with the Finsler metric $F : TM \to M$;
(M, g)	: Riemannian manifold M with the inner product g;
d_F	: distance function on the Finsler manifold (M, F);
d_g	: distance function on the Riemannian manifold (M, g);
d_{x_0}	: $d_F(x_0, \cdot)$ or $d_g(x_0, \cdot)$ where $x_0 \in M$;
$\boldsymbol{\nabla}_F$	: gradient on the Finsler manifold (M, F);
∇_g	: gradient on the Riemannian manifold (M, g);
∇	: (usual) Euclidean gradient;
$\boldsymbol{\Delta}_F$	: Finsler-Laplacian operator;
Δ_g	: Laplace-Beltrami operator;
Δ	: (usual) Laplacian operator;
Vol_F	: volume on the Finsler manifold (M, F);
Vol_g	: volume on the Riemannian manifold (M, g);
Vol_e	: Euclidean volume;
$B_F^{\pm}(x_0, r)$	: open forward/backward ball with center x_0 and radius $r > 0$ in the Finsler manifold (M, F);
$B_g(x_0, r)$	: open ball with center x_0 and radius $r > 0$ in a Riemannian manifold (M, g);
$B_e(x_0, r)$	: open ball with center x_0 and radius $r > 0$ in the Euclidean space;
$B(x_0, r)$	: open ball with center x_0 and radius $r > 0$ in a generic metric space (M, d);
B	: Euler beta-function;
Γ	: Euler gamma-function;
ω_n	: volume of the n-dimensional Euclidean unit ball;
L^p	: Lebesgue space over a given set (in a manifold or a metric measure space), $p \geq 1$;
$\Vert \cdot \Vert_{L^p}$	: L^p-Lebesgue norm, $p \geq 1$;
$\mathcal{H}^n$	: n-dimensional Hausdorff measure;
$H_g^1(M)$	: Sobolev space over the Riemannian manifold (M, g);
$I_{\mathbb{R}^n}$	: $n \times n$ identity matrix;
$\cdot^t$	: transpose of a matrix;
$\{a_k\}_k$	: sequence with general term a_k, $k \in \mathbb{N}$.

Conventions:

- When no confusion arises, the norms $\|\cdot\|_{L^p}$ and $\|\cdot\|_{L^p(M)}$ abbreviate:
  (a) $\|\cdot\|_{L^p(M,\mathrm{d}\mathsf{m})}$ on a generic metric measure space (M,d,m);
  (b) $\|\cdot\|_{L^p(M,\mathrm{d}V_g)}$ on the Riemannian manifold (M,g), where $\mathrm{d}V_g$ stands for the canonical Riemannian measure on (M,g);
  (c) $\|\cdot\|_{L^p(M,\mathrm{d}V_F)}$ on the Finsler manifold (M,F), where $\mathrm{d}V_F$ denotes the Busemann-Hausdoff measure on (M,F);
  (d) $\|\cdot\|_{L^p(\mathbb{R}^n,\mathrm{d}x)}$ on the Euclidean/normed space $\mathbb{R}^n$, where $\mathrm{d}x$ is the usual Lebesgue measure.
- When A is not the whole space we are working on, we use the notation $\|u\|_{L^p(A)}$ for the L^p-norm of the function $u : A \to \mathbb{R}$.

Contents

Chapter 1

Fundamental notions and results

This chapter is devoted to those notions and results which will be used throughout the present work.

1.1 Geometry of metric measure spaces

Our results require certain comparison principles and fundamental inequalities within the class of Riemannian, Finsler and $\mathsf{CD}(K,N)$ spaces, respectively.

1.1.1 Non-euclidean structures and comparison principles

1.1.1.1 Smooth setting: Riemannian and Finsler manifolds

Let M be a connected n-dimensional C^∞-manifold and $TM = \bigcup_{x\in M} T_xM$ be its tangent bundle.

Definition 1.1. *The pair (M,F) is called a* Finsler manifold *if the continuous function $F : TM \to [0,\infty)$ satisfies the conditions:*

(a) $F \in C^\infty(TM \setminus \{0\})$;

(b) $F(x,tv) = tF(x,v)$ *for all* $t \geq 0$ *and* $(x,v) \in TM$;

(c) *the* $n \times n$ *matrix*

$$g^v := [g_{ij}(x,v)]_{i,j=1,\dots,n} = \left[\frac{1}{2}\frac{\partial^2}{\partial v^i \partial v^j}F^2(x,v)\right]_{i,j=1,\dots,n}, \quad \text{where } v = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i}, \tag{1.1}$$

is positive definite for all $(x,v) \in TM \setminus \{0\}$. We will denote by g_v the inner product on T_xM induced from (1.1).

If $F(x,tv) = |t|F(x,v)$ for all $t \in \mathbb{R}$ and $(x,v) \in TM$, the Finsler manifold (M,F) is reversible.

If $g_{ij}(x) = g_{ij}(x,v)$ is independent of v then $(M,F) = (M,g)$ is called a *Riemannian manifold.* A *Minkowski space* consists of a finite dimensional vector space V (identified with $\mathbb{R}^n$) and a Minkowski norm which induces a Finsler metric on V by translation, i.e., $F(x,v)$ is independent on the base point x; in such cases we often write $F(v)$ instead of $F(x,v)$. A Finsler manifold (M,F) is called a *locally Minkowski space* if any point in M admits a local coordinate system (x^i) on its neighborhood such that $F(x,v)$ depends only on v and not on x. Another important class of Finsler manifolds is provided by Randers spaces, which will be introduced and widely discussed in Chapter 4.

For every $(x,\alpha) \in T^*M$, the *polar transform* (or, co-metric) of F is given by

$$F^*(x,\alpha) = \sup_{v \in T_xM \setminus \{0\}} \frac{\alpha(v)}{F(x,v)}. \tag{1.2}$$

Note that for every $x \in M$, the function $F^*(x,\cdot)$ is a Minkowski norm on T_x^*M. Since $\alpha \mapsto [F^*(x,\alpha)]^2$ is twice differentiable on $T_x^*M \setminus \{0\}$, we consider the matrix $g_{ij}^*(x,\alpha) := \frac{1}{2}\frac{\partial^2}{\partial\alpha^i \partial\alpha^j}[F^*(x,\alpha)]^2$ for every $\alpha = \sum_{i=1}^n \alpha^i \mathrm{d}x^i \in T_x^*M \setminus \{0\}$ in a local coordinate system (x^i).

Let π^*TM be the pull-back bundle of the tangent bundle TM generated by the natural projection $\pi : TM \setminus \{0\} \to M$, see Bao, Chern and Shen [11]. The vectors of the pull-back bundle π^*TM are denoted by $(v;w)$ with $(x,y) = v \in TM \setminus \{0\}$ and $w \in T_xM$. For simplicity, let $\partial_i|_v = (v;\partial/\partial x^i|_x)$ be the natural local basis for π^*TM, where $v \in T_xM$. One can introduce on π^*TM the *fundamental tensor* g by

$$g_{(x,v)} := g_v = g(\partial_i|_v, \partial_j|_v) = g_{ij}(x,y), \tag{1.3}$$

where $v = y^i(\partial/\partial x^i)|_x$, see (1.1). Unlike the Levi-Civita connection in the Riemannian case, there is no unique natural connection in the Finsler geometry. Among these connections on the pull-back bundle π^*TM, we choose a torsion-free and almost metric-compatible linear connection on π^*TM, the so-called *Chern connection*. The coefficients of the Chern connection are denoted by Γ_{jk}^i, which are instead of the well-known Christoffel symbols from Riemannian geometry. A Finsler manifold is of *Berwald type* if the coefficients $\Gamma_{ij}^k(x,y)$ in natural coordinates are independent of y. It is clear that Riemannian manifolds and (locally) Minkowski spaces are Berwald spaces. The Chern connection induces on π^*TM the *curvature tensor* R. By means of the connection, we also have the *covariant derivative* $D_v u$ of a vector field u in the direction $v \in T_xM$ with reference vector v. A vector field $u = u(t)$ along a curve σ is *parallel* if $D_{\dot\sigma} u = 0$. A C^∞ curve $\sigma : [0,a] \to M$ is a *geodesic* if $D_{\dot\sigma}\dot\sigma = 0$. Geodesics are considered to be parametrized proportionally to arclength. The Finsler manifold is *forward* (resp. *backward*) *complete* if every geodesic segment $\sigma : [0,a] \to M$ can be extended to $[0,\infty)$ (resp. to $(-\infty,a]$). (M,F) is *complete* if it is both forward and backward complete.

Let $u,v \in T_xM$ be two non-collinear vectors and $\mathcal{S} = \mathrm{span}\{u,v\} \subset T_xM$. By means of the curvature tensor R, the *flag curvature* associated with the flag $\{\mathcal{S},v\}$ is

$$\mathbf{K}(\mathcal{S};v) = \frac{g_v(R(U,V)V,U)}{g_v(V,V)g_v(U,U) - g_v^2(U,V)}, \tag{1.4}$$

where $U = (v;u), V = (v;v) \in \pi^*TM$. If (M,F) is Riemannian, the flag curvature reduces to the sectional curvature which depends only on $\mathcal{S}$. If for some $c \in \mathbb{R}$ we have $\mathbf{K}(\mathcal{S};v) \le c$ for every choice of U and V, we say that the flag curvature on (M,F) is bounded above by c and we denote this fact by $\mathbf{K} \le c$. (M,F) is a *Finsler-Hadamard manifold* if it is simply connected, forward complete with $\mathbf{K} \le 0$. A Riemannian Finsler-Hadamard manifold is simply called *Hadamard manifold*.

Take $v \in T_xM$ with $F(x,v) = 1$ and let $\{e_i\}_{i=1}^n$ with $e_n = v$ be an orthonormal basis of (T_xM, g_v)

for g_v from (1.1). Let $\mathcal{S}_i = \mathrm{span}\{e_i, v\}$ for $i = 1, ..., n-1$. Then the *Ricci curvature* of v is defined by $\mathrm{Ric}(v) := \sum_{i=1}^{n-1} \mathbf{K}(\mathcal{S}_i; v)$.

Let $\sigma : [0, r] \to M$ be a piecewise C^∞ curve. The value $L_F(\sigma) = \displaystyle\int_0^r F(\sigma(t), \dot\sigma(t))\,dt$ denotes the *integral length* of σ. For $x_1, x_2 \in M$, denote by $\Lambda(x_1, x_2)$ the set of all piecewise C^∞ curves $\sigma : [0, r] \to M$ such that $\sigma(0) = x_1$ and $\sigma(r) = x_2$. Define the *distance function* $d_F : M \times M \to [0, \infty)$ by

$$d_F(x_1, x_2) = \inf_{\sigma \in \Lambda(x_1, x_2)} L_F(\sigma). \tag{1.5}$$

One clearly has that $d_F(x_1, x_2) = 0$ if and only if $x_1 = x_2$, and d_F verifies the triangle inequality. The open *forward* (resp. *backward*) *metric ball* with center $x_0 \in M$ and radius $\rho > 0$ is defined by $B_F^+(x_0, \rho) = \{x \in M : d_F(x_0, x) < \rho\}$ (resp. $B_F^-(x_0, \rho) = \{x \in M : d_F(x, x_0) < \rho\}$). When $(M, F) = (\mathbb{R}^n, F)$ is a Minkowski space, one has that $d_F(x_1, x_2) = F(x_2 - x_1)$.

Let $\{\partial/\partial x^i\}_{i=1,...,n}$ be a local basis for the tangent bundle TM, and $\{\mathrm{d}x^i\}_{i=1,...,n}$ be its dual basis for T^*M. Consider $B_x(1) = \{y = (y^i) : F(x, y^i \partial/\partial x^i) < 1\} \subset \mathbb{R}^n$. The *Hausdorff volume form* $\mathsf{dm} = \mathrm{d}V_F$ on (M, F) is defined by

$$\mathsf{dm}(x) = \mathrm{d}V_F(x) = \sigma_F(x)\mathrm{d}x^1 \wedge ... \wedge \mathrm{d}x^n, \tag{1.6}$$

where $\sigma_F(x) = \frac{\omega_n}{\mathrm{Vol}_e(B_x(1))}$. Hereafter, $\mathrm{Vol}_e(S)$ and ω_n denote the Euclidean volumes of the set $S \subset \mathbb{R}^n$ and of the n-dimensional unit ball, respectively. The *Finslerian volume* of an open set $S \subset M$ is $\mathrm{Vol}_F(S) = \displaystyle\int_S \mathsf{dm}(x)$. When $(M, F) = (M, g)$ is Riemannian, we simply denote by $\mathrm{d}V_g$ and $\mathrm{Vol}_g(S)$ the *Riemannian measure* and *Riemannian volume* of $S \subset M$, respectively. When $(\mathbb{R}^n, F)$ is a Minkowski space, then on account of (1.6), $\mathrm{Vol}_F(B_F^+(x, \rho)) = \omega_n \rho^n$ for every $\rho > 0$ and $x \in \mathbb{R}^n$.

Let $\{e_i\}_{i=1,...,n}$ be a basis for T_xM and $g_{ij}^v = g_v(e_i, e_j)$. By definition, the *mean distortion* $\mu : TM \setminus \{0\} \to (0, \infty)$ and *mean covariation* $\mathbf{S} : TM \setminus \{0\} \to \mathbb{R}$ are

$$\mu(v) = \frac{\sqrt{\det(g_{ij}^v)}}{\sigma_F} \quad \text{and} \quad \mathbf{S}(x, v) = \frac{\mathrm{d}}{\mathrm{d}t}(\ln \mu(\dot\sigma_v(t)))\big|_{t=0},$$

respectively, where σ_v is the geodesic such that $\sigma_v(0) = x$ and $\dot\sigma_v(0) = v$. We say that (M, F) has *vanishing mean covariation* if $\mathbf{S}(x, v) = 0$ for every $(x, v) \in TM$, and we denote it by $\mathbf{S} = 0$. We note that any Berwald space has vanishing mean covariation, see Shen [82].

For any $c \leq 0$, we introduce the functions

$$\mathbf{s}_c(r) = \begin{cases} r, & \text{if } \ c = 0, \\ \frac{\sinh(r\sqrt{-c})}{\sqrt{-c}}, & \text{if } \ c < 0, \end{cases} \quad \text{and} \quad \mathbf{ct}_c(r) = \begin{cases} \frac{1}{r}, & \text{if } \ c = 0, \\ \sqrt{-c}\coth(r\sqrt{-c}), & \text{if } \ c < 0. \end{cases} \tag{1.7}$$

Consider

$$V_{c,n}(\rho) = n\omega_n \int_0^\rho \mathbf{s}_c^{n-1}(r)\mathrm{d}r.$$

In general, one has for every $x \in M$ that

$$\lim_{\rho \to 0^+} \frac{\mathrm{Vol}_F(B_F^+(x,\rho))}{V_{c,n}(\rho)} = \lim_{\rho \to 0^+} \frac{\mathrm{Vol}_F(B_F^-(x,\rho))}{V_{c,n}(\rho)} = 1. \tag{1.8}$$

We recall the following Bishop-Gromov-type volume comparison result on Finsler manifolds.

Theorem 1.1. (Wu and Xin [96]) *Let (M, F) be an n-dimensional Finsler manifold with $\mathbf{S} = 0$.*

(a) *If $\mathbf{K} \leq c \leq 0$, the function $\rho \mapsto \frac{\mathrm{Vol}_F(B_F^+(x,\rho))}{V_{c,n}(\rho)}$ is non-decreasing for every $x \in M$. In particular, from (1.8) we have*

$$\mathrm{Vol}_F(B_F^+(x,\rho)) \geq V_{c,n}(\rho),\ \forall x \in M, \rho > 0. \tag{1.9}$$

If equality holds in (1.9) for some $x \in M$ and $\rho_0 > 0$, then $\mathbf{K}(\cdot; \dot\gamma_y(t)) = c$ for every $t \in [0, \rho_0)$ and $y \in T_xM$ with $F(x, y) = 1$, where γ_y is the constant speed geodesic with $\gamma_y(0) = x$ and $\dot\gamma_y(0) = y$.

(b) *If (M, F) has nonnegative Ricci curvature, the function $\rho \mapsto \frac{\mathrm{Vol}_F(B_F^+(x,\rho))}{\rho^n}$ is non-increasing for every $x \in M$. In particular, from (1.8) we have*

$$\mathrm{Vol}_F(B_F^+(x,\rho)) \leq \omega_n \rho^n,\ \forall x \in M,\ \rho > 0. \tag{1.10}$$

If equality holds in (1.10), then the flag curvature is identically zero.

The *Legendre transform* $J^* : T^*M \to TM$ associates to each element $\alpha \in T_x^*M$ the unique maximizer on T_xM of the map $y \mapsto \alpha(y) - \frac{1}{2}F^2(x, y)$. This element can also be interpreted as the unique vector $y \in T_xM$ with the properties

$$F(x, y) = F^*(x, \alpha) \text{ and } \alpha(y) = F(x, y)F^*(x, \alpha). \tag{1.11}$$

In particular, if $\alpha = \sum_{i=1}^n \alpha^i \mathrm{d}x^i \in T_x^*M$, one has that

$$J^*(x, \alpha) = \sum_{i=1}^n \frac{\partial}{\partial \alpha_i}\left(\frac{1}{2}\left[F^*(x, \alpha)\right]^2\right)\frac{\partial}{\partial x^i}. \tag{1.12}$$

Let $u : M \to \mathbb{R}$ be a differentiable function in the distributional sense. The *gradient* of u is defined by

$$\boldsymbol{\nabla}_F u(x) = J^*(x, Du(x)), \tag{1.13}$$

where $Du(x) \in T_x^*M$ denotes the (distributional) *derivative* of u at $x \in M$. In local coordinates, we have

$$Du(x) = \sum_{i=1}^n \frac{\partial}{\partial x^i}u(x)\mathrm{d}x^i,\ \boldsymbol{\nabla}_F u(x) = \sum_{i,j=1}^n g_{ij}^*(x, Du(x))\frac{\partial}{\partial x^i}u(x)\frac{\partial}{\partial x^j}. \tag{1.14}$$

In general, $u \mapsto \boldsymbol{\nabla}_F u$ is not linear. If $x_0 \in M$ is fixed, due to Ohta and Sturm [73], one has that

$$F^*(x, Dd_F(x_0, x)) = F(x, \boldsymbol{\nabla}_F d_F(x_0, x)) = Dd_F(x_0, x)(\boldsymbol{\nabla}_F d_F(x_0, x)) = 1 \text{ for a.e. } x \in M. \tag{1.15}$$

Let X be a vector field on M. On account of (1.6), the *divergence* is $\mathrm{div}(X) = \frac{1}{\sigma_F}\frac{\partial}{\partial x^i}(\sigma_F X^i)$ in a local coordinate system (x^i). The *Finsler-Laplace operator*

$$\boldsymbol{\Delta}_F u = \mathrm{div}(\boldsymbol{\nabla}_F u)$$

acts on the space $W^{1,2}_{\mathrm{loc}}(M)$ and for every $v \in C_0^\infty(M)$, one has

$$\int_M v\boldsymbol{\Delta}_F u \mathrm{dm}(x) = -\int_M Dv(\boldsymbol{\nabla}_F u)\mathrm{dm}(x). \tag{1.16}$$

Note that, in general, $\boldsymbol{\Delta}_F(-u) \neq -\boldsymbol{\Delta}_F u$, unless (M,F) is reversible. In particular, for a Riemannian manifold $(M,F) = (M,g)$ the Finsler-Laplace operator is the usual *Laplace-Beltrami operator* $\boldsymbol{\Delta}_F u = \Delta_g u$, while for a Minkowski space $(\mathbb{R}^n, F)$, by using (1.11), we have that $\boldsymbol{\Delta}_F u = \mathrm{div}(F^*(Du)\nabla F^*(Du)) = \mathrm{div}(F(\nabla u)\nabla F(\nabla u))$.

We recall the following Laplacian comparison principle.

Theorem 1.2. (Shen [82], Wu and Xin [96]) *Let (M,F) be an n-dimensional Finsler-Hadamard manifold with $\mathbf{S} = 0$. Let $x_0 \in M$ and $c \leq 0$. Then the following statements hold:*

(a) *if $\mathbf{K} \leq c$ then $\boldsymbol{\Delta}_F d_F(x_0,x) \geq (n-1)\mathbf{ct}_c(d_F(x_0,x))$ for every $x \in M \setminus \{x_0\}$;*

(b) *if $c \leq \mathbf{K}$ then $\boldsymbol{\Delta}_F d_F(x_0,x) \leq (n-1)\mathbf{ct}_c(d_F(x_0,x))$ for every $x \in M \setminus \{x_0\}$.*

1.1.1.2 Non-smooth setting: $\mathsf{CD}(K,N)$ spaces à la Lott-Sturm-Villani

Let (M,d,m) be a metric measure space, i.e., (M,d) is a complete separable metric space and m is a locally finite measure on M endowed with its Borel σ-algebra. In the sequel, we assume that the measure m on M is strictly positive, i.e., $\mathrm{supp}[\mathsf{m}] = M$. As usual, $\mathcal{P}_2(M,d)$ is the L^2-Wasserstein space of probability measures on M, while $\mathcal{P}_2(M,d,\mathsf{m})$ will denote the subspace of m-absolutely continuous measures in $\mathcal{P}_2(M,d)$. (M,d,m) is said to be proper if every bounded and closed subset of M is compact.

For a given number $N \geq 1$, the *Rényi entropy functional* $S_N(\cdot|\mathsf{m}) : \mathcal{P}_2(M,d) \to \mathbb{R}$ with respect to the measure m is defined by

$$S_N(\mu|\mathsf{m}) = -\int_M \rho^{-\frac{1}{N}} \mathrm{d}\mu,$$

ρ being the density of μ^c in $\mu = \mu^c + \mu^s = \rho\mathsf{m} + \mu^s$, where μ^c and μ^s represent the absolutely continuous and singular parts of $\mu \in \mathcal{P}_2(M,d)$, respectively.

Let $K, N \in \mathbb{R}$ be two numbers with $K \geq 0$ and $N \geq 1$. For every $t \in [0,1]$ and $s \geq 0$, consider the function

$$\tau_{K,N}^{(t)}(s) = \begin{cases} +\infty, & \text{if} \quad Ks^2 \geq (N-1)\pi^2; \\ t^{\frac{1}{N}}\left(\sin\left(ts\sqrt{\frac{K}{N-1}}\right) / \sin\left(s\sqrt{\frac{K}{N-1}}\right)\right)^{1-\frac{1}{N}}, & \text{if} \quad 0 < Ks^2 < (N-1)\pi^2; \\ t, & \text{if} \quad Ks^2 = 0. \end{cases}$$

Definition 1.2. (Sturm [85, 86], Lott and Villani [65]) *The space* (M, d, m) *satisfies the* curvature-dimension condition $\mathsf{CD}(K, N)$ *if for every* $\mu_0, \mu_1 \in \mathcal{P}_2(M, d, \mathsf{m})$ *there exists an optimal coupling* γ *of* μ_0, μ_1 *and a geodesic* $\Gamma : [0, 1] \to \mathcal{P}_2(M, d, \mathsf{m})$ *joining* μ_0 *and* μ_1 *such that*

$$S_{N'}(\Gamma(t)|\mathsf{m}) \le -\int_{M\times M} \left[\tau_{K,N'}^{(1-t)}(d(x_0, x_1))\rho_0^{-\frac{1}{N'}}(x_0) + \tau_{K,N'}^{(t)}(d(x_0, x_1))\rho_1^{-\frac{1}{N'}}(x_1)\right] \mathrm{d}\gamma(x_0, x_1)$$

for every $t \in [0, 1]$ *and* $N' \ge N$*, where* ρ_0 *and* ρ_1 *are the densities of* μ_0 *and* μ_1 *with respect to* m.

Clearly, when $K = 0$, the above inequality reduces to the geodesic convexity of $S_{N'}(\cdot|\mathsf{m})$ on the L^2-Wasserstein space $\mathcal{P}_2(M, d, \mathsf{m})$. We note that $\mathsf{CD}(K, N)$ can be defined also for $K < 0$; however, in the present work we do not use it, thus we avoid its definition.

Let $B(x, r) = \{y \in M : d(x, y) < r\}$. The following comparison results hold.

Theorem 1.3. (Sturm [86]) *Let* (M, d, m) *be a metric measure space with strictly positive measure* m *satisfying the curvature-dimension condition* $\mathsf{CD}(K, N)$ *for some* $K \ge 0$ *and* $N > 1$*. Then every bounded set* $S \subset M$ *has finite* m*-measure and the metric spheres* $\partial B(x, r)$ *have zero* m*-measures. Moreover, one has:*

(i) [Generalized Bonnet-Myers theorem] *If* $K > 0$*, then* M =supp[m] *is compact and has diameter less than or equal to* $\sqrt{\frac{N-1}{K}}\pi$.

(ii) [Generalized Bishop-Gromov inequality] *If* $K = 0$*, then for every* $R > r > 0$ *and* $x \in M$*,*

$$\frac{\mathsf{m}(B(x, r))}{r^N} \ge \frac{\mathsf{m}(B(x, R))}{R^N}.$$

Remark 1.1. (Ohta [71]) Let (M, F) be an n-dimensional complete reversible Finsler manifold with $\mathbf{S} = 0$ (endowed with the Busemann-Hausdorff measure $\mathrm{d}\mathsf{m} = \mathrm{d}V_F$). Then the condition $\mathsf{CD}(K, N)$ holds on (M, d_F, m) if and only if $\mathrm{Ric}(v) \ge K$ for every $F(x, v) = 1$ and $\dim(M) \le N$.

The following result will be useful in our proofs.

Lemma 1.1. *Let* (M, d, m) *be a metric measure space which satisfies the curvature-dimension condition* $\mathsf{CD}(0, n)$ *for some* $n \ge 2$*. If*

$$\ell_\infty^{x_0} := \limsup_{\rho\to\infty} \frac{\mathsf{m}(B(x_0, \rho))}{\omega_n \rho^n} \ge a \tag{1.17}$$

for some $x_0 \in M$ *and* $a > 0$*, then*

$$\mathsf{m}(B(x, \rho)) \ge a\omega_n\rho^n, \ \forall x \in M, \ \rho \ge 0.$$

Proof. By fixing $x \in M$ and $\rho > 0$, one obtains successively that

$$\begin{aligned}
\frac{\mathsf{m}(B(x,\rho))}{\omega_n \rho^n} &\geq \limsup_{r\to\infty} \frac{\mathsf{m}(B(x,r))}{\omega_n r^n} && \text{[see Theorem 1.3/(ii)]} \\
&\geq \limsup_{r\to\infty} \frac{\mathsf{m}(B(x_0,(r-d(x_0,x)))}{\omega_n r^n} && [B(x,r)\supset B(x_0,(r-d(x_0,x))] \\
&= \limsup_{r\to\infty} \left(\frac{\mathsf{m}(B(x_0,(r-d(x_0,x)))}{\omega_n (r-d(x_0,x))^n} \cdot \frac{(r-d(x_0,x))^n}{r^n} \right) \\
&= \ell_\infty^{x_0} \geq a. && \text{[cf. (1.17)]}
\end{aligned}$$

□

1.1.2 Cartan-Hadamard conjecture

Let (M,g) be an n-dimensional Hadamard manifold (simply connected complete Riemannian manifold with nonpositive sectional curvature) endowed with its canonical measure $\mathrm{d}V_g$. Although any Hadamard manifold (M,g) is diffeomorphic to $\mathbb{R}^n$, $n = \dim(M)$, see Cartan's theorem (see do Carmo [31]), this is a wide class of non-compact Riemannian manifolds including important geometric objects as Euclidean spaces, hyperbolic spaces, the space of symmetric positive definite matrices endowed with a suitable Killing metric; further examples can be found in Bridson and Haefliger [18], and Jost [55].

Cartan-Hadamard conjecture in n-dimension. (Aubin [8]) *Let (M,g) be an n-dimensional $(n \geq 2)$ Hadamard manifold. Then any compact domain $D \subset M$ with smooth boundary ∂D satisfies the Euclidean-type sharp isoperimetric inequality, i.e.,*

$$\mathrm{Area}_g(\partial D) \geq n\omega_n^{\frac{1}{n}} \mathrm{Vol}_g^{\frac{n-1}{n}}(D). \tag{1.18}$$

Moreover, equality holds in (1.18) if and only if D is isometric to the n-dimensional Euclidean ball with volume $\mathrm{Vol}_g(D)$.

Note that $n\omega_n^{\frac{1}{n}}$ is precisely the isoperimetric ratio in the Euclidean setting. Hereafter, $\mathrm{Area}_g(\partial D)$ stands for the area of ∂D with respect to the metric induced on ∂D by g.

We note that the Cartan-Hadamard conjecture holds on hyperbolic spaces (of any dimension) and on generic Hadamard manifolds in dimension 2 (cf. Beckenbach and Radó [14], and Weil [93]); in dimension 3 (cf. Kleiner [56]); and in dimension 4 (cf. Croke [28]), but it is open for higher dimensions.

For $n \geq 3$, Croke [28] proved a general isoperimetric inequality on n-dimensional Hadamard manifolds: for any bounded domain $D \subset M$ with smooth boundary ∂D, one has that

$$\mathrm{Area}_g(\partial D) \geq C(n)\mathrm{Vol}_g^{\frac{n-1}{n}}(D), \tag{1.19}$$

where

$$C(n) = (n\omega_n)^{1-\frac{1}{n}} \left((n-1)\omega_{n-1} \int_0^{\frac{\pi}{2}} \cos^{\frac{n}{n-2}}(t)\, \sin^{n-2}(t) \mathrm{d}t \right)^{\frac{2}{n}-1}. \tag{1.20}$$

We recall that $C(n) \leq n\omega_n^{\frac{1}{n}}$ for every $n \geq 3$, while equality holds if and only if $n = 4$.

1.2 Variational principles

It is common knowledge that the problem of minimizing a given functional has always been present in the real world in one form or another (minimizing the energy, maximizing the profit). The main objective of the field of calculus of variations is to optimize functionals. In the sequel we recall those abstract methods which are going to be used throughout the dissertation (thus not mentioning such classical tools as Du Bois-Reymond and Weierstrass principles, and Euler-Lagrange equations); for a comprehensive treatment we refer to [117].

1.2.1 Direct variational methods

The following result is a very useful tool in the study of various partial differential equations where no compactness is assumed on the domain of the functional.

Theorem 1.4. (Zeidler [100, p. 154]) *Let X be a reflexive real Banach space, X_0 be a weakly closed, bounded subset of X, and $E : X_0 \to \mathbb{R}$ be a sequentially weak lower semicontinuous function. Then E is bounded from below and its infimum is attained on X_0.*

Remark 1.2. The boundedness of $X_0 \subset X$ in Theorem 1.4 is indispensable, which can be too restrictive in certain applications. In such cases, once the functional $E : X \to \mathbb{R}$ is coercive, i.e., $E(u) \to +\infty$ whenever $\|u\| \to +\infty$, the minimization of E on X can be restricted to a sufficiently large ball of X.

Definition 1.3. Let X be a real Banach space.

(a) A function $E \in C^1(X, \mathbb{R})$ *satisfies the Palais-Smale condition at level* $c \in \mathbb{R}$ (shortly, $(PS)_c$-condition) if every sequence $\{u_k\}_k \subset X$ such that $\lim_{k\to\infty} E(u_k) = c$ and $\lim_{k\to\infty} \|E'(u_k)\| = 0$, possesses a convergent subsequence.

(b) A function $E \in C^1(X, \mathbb{R})$ *satisfies the Palais-Smale condition* (shortly, (PS)-condition) if it satisfies the Palais-Smale condition at every level $c \in \mathbb{R}$.

By a simple application of Ekeland's variational principle, we obtain the next theorem (see [117]):

Theorem 1.5. *Let X be a Banach space and a function $E \in C^1(X, \mathbb{R})$ which is bounded from below. If E satisfies the $(PS)_c$-condition at level $c = \inf_X E$, then c is a critical value of E, i.e., there exists a point $u_0 \in X$ such that $E(u_0) = c$ and u_0 is a critical point of E, i.e., $E'(u_0) = 0$.*

Let X be a real Banach space and X^* its dual, and we denote by $\langle \cdot, \cdot \rangle$ the duality pair between X and X^*. Let $E \in C^1(X, \mathbb{R})$ and $\chi : X \to \mathbb{R} \cup \{+\infty\}$ be a proper (i.e., $\not\equiv +\infty$), convex, lower semi-continuous function. Then, $I = E + \chi$ is a *Szulkin-type functional*, see Szulkin [87]. An element $u \in X$ is called a *critical point of $I = E + \chi$*, if

$$\langle E'(u), v - u \rangle + \chi(v) - \chi(u) \geq 0, \quad \forall v \in X. \tag{1.21}$$

The number $I(u)$ is a *critical value* of I. If $\chi = 0$, the latter critical point notion reduces to the usual notion $E'(u) = 0$. For $u \in D(\chi) = \{u \in X : \chi(u) < \infty\}$ we consider the set

$$\partial\chi(u) = \{x^* \in X^* : \chi(v) - \chi(u) \geq \langle x^*, v - u \rangle, \quad \forall v \in X\}.$$

The set $\partial\chi(u)$ is called the *subdifferential* of χ at u. Note that an equivalent formulation for (1.21) is

$$0 \in E'(u) + \partial\chi(u) \ \text{ in } \ X^*. \tag{1.22}$$

Let G be a group, e its identity element, and let π a representation of G over X, i.e., $\pi(g) \in L(X)$ for each $g \in G$ (where $L(X)$ denotes the set of the linear and bounded operator from X into X), and

a) $\pi(e)u = u$, $\forall u \in X$;

b) $\pi(g_1 g_2)u = \pi(g_1)(\pi(g_2 u))$, $\forall g_1, g_2 \in G$, $u \in X$.

The representation π_* of G over X^* is naturally induced by π by the relation

$$\langle \pi_*(g)v^*, u\rangle = \langle v^*, \pi(g^{-1})u\rangle, \ \forall g \in G, v^* \in X^*, u \in X. \tag{1.23}$$

We often write gu or gv^* instead of $\pi(g)u$ or $\pi_*(g)v^*$, respectively.

A function $h : X \to \mathbb{R}$ is called *G-invariant*, if $h(gu) = h(u)$ for every $u \in X$ and $g \in G$. A subset M of X is called *G-invariant*, if $gM = \{gu : u \in M\} \subseteq M$ for every $g \in G$.

The *fixed point sets* of the group action G on X and X^* are defined as

$$\Sigma = X^G = \{u \in X : gu = u, \ \forall g \in G\} \ \text{ and } \ \Sigma_* = (X^*)^G = \{v^* \in X^* : gv^* = v^*, \ \forall g \in G\}.$$

We conclude this subsection with a non-smooth version of the *principle of symmetric criticality.*

Theorem 1.6. (Kobayashi, Otani [57], Palais [74]) *Let X be a reflexive Banach space and let $I = E + \chi : X \to \mathbb{R} \cup \{+\infty\}$ be a Szulkin-type functional on X. If a compact group G acts linearly and continuously on X, and the functionals E and χ are G-invariant, then the principle of symmetric criticality holds, i.e.,*

$$0 \in (E|_\Sigma)'(u) + \partial(\chi|_\Sigma)(u) \ \text{in} \ \Sigma^* \quad \Longrightarrow \quad 0 \in E'(u) + \partial\chi(u) \ \text{in} \ X^*.$$

1.2.2 Minimax theorems

In the sequel, we recall the simplest version of the Mountain Pass Theorem.

Theorem 1.7. (Ambrosetti and Rabinowitz [3]) *Let X be a Banach space and a functional $E \in C^1(X, \mathbb{R})$ such that*

$$\inf_{\|u-u_0\|=\rho} E(u) \geq \alpha > \max\{E(u_0), E(u_1)\}$$

for some $\alpha \in \mathbb{R}$ and $u_0 \neq u_1 \in X$ with $0 < \rho < \|u_0 - u_1\|$. If E satisfies the $(PS)_c$-condition at level

$$c = \inf_{\gamma\in\Gamma_0} \max_{t\in[0,1]} E(\gamma(t)),$$

where

$$\Gamma_0 = \{\gamma \in C([0,1], X) : \gamma(0) = u_0, \ \gamma(1) = u_1\},$$

then c is a critical value of E with $c \geq \alpha$.

The symmetric version of the Mountain Pass Theorem reads as follows.

Theorem 1.8. (Ambrosetti and Rabinowitz [3]) *Let X be a Banach space and a functional $E \in C^1(X,\mathbb{R})$ satisfying the (PS)-condition such that:*

(i) $E(0) = 0$ *and* $\inf_{\|u\|=\rho} E(u) \geq \alpha$ *for some* $\alpha \in \mathbb{R}$;

(ii) E *is even*;

(iii) *for all finite dimensional subspaces* $\widetilde{X} \subset X$ *there exists* $R := R(\widetilde{X}) > 0$ *such that* $E(u) \leq 0$ *for every* $u \in \widetilde{X}$ *with* $\|u\| \geq R$.

Then E possesses an unbounded sequence of critical values of E characterized by a minimax argument.

If X is a Banach space, we denote by $\mathcal{W}_X$ the class of those functionals $I : X \to \mathbb{R}$ having the property that if $\{u_k\}_k$ is a sequence in X converging weakly to $u \in X$ and $\liminf_{k\to\infty} I(u_k) \leq I(u)$ then $\{u_k\}_k$ has a subsequence strongly converging to u.

According to the well-known three critical points theorem of Pucci and Serrin [77], if a function $E \in C^1(X,\mathbb{R})$ satisfies the (PS)-condition and it has two local minima, then E has at least three distinct critical points. A stability result for the latter statement can be formulated as follows:

Theorem 1.9. (Ricceri [79, Theorem 2]) *Consider the separable and reflexive real Banach space X. Let $I_1 \in C^1(X,\mathbb{R})$ be a coercive, sequentially weakly lower semicontinuous functional belonging to $\mathcal{W}_X$, bounded on each bounded subset of X with a derivative admitting a continuous inverse on X^*; and $I_2 \in C^1(X,\mathbb{R})$ be a functional with compact derivative. Assume that I_1 has a strict local minimum u_0 with $I_1(u_0) = I_2(u_0) = 0$. Setting the numbers*

$$\tau = \max\left\{0, \limsup_{\|u\|\to\infty} \frac{I_2(u)}{I_1(u)}, \limsup_{u\to u_0} \frac{I_2(u)}{I_1(u)}\right\}, \tag{1.24}$$

$$\chi = \sup_{I_1(u)>0} \frac{I_2(u)}{I_1(u)}, \tag{1.25}$$

assume that $\tau < \chi$.

Then, for each compact interval $[a,b] \subset (1/\chi, 1/\tau)$ (with the conventions $1/0 = \infty$ and $1/\infty = 0$) there exists $\kappa > 0$ with the following property: for every $\lambda \in [a,b]$ and every functional $I_3 \in C^1(X,\mathbb{R})$ with compact derivative, there exists $\delta > 0$ such that for each $\mu \in [0,\delta]$, the equation

$$I_1'(u) - \lambda I_2'(u) - \mu I_3'(u) = 0$$

admits at least three solutions in X having norm less than κ.

Chapter 2

Sharp interpolation inequalities

An important role in the theory of geometric functional inequalities is played by Sobolev-type interpolation inequalities. This chapter is devoted to the sharp Gagliardo-Nirenberg interpolation inequality and its limit cases on both positively and negatively curved spaces.

2.1 Interpolation inequalities in the flat case: a short overview

The optimal Gagliardo-Nirenberg inequality in the Euclidean case has been obtained by Del Pino and Dolbeault [30] for a certain range of parameters by using symmetrization arguments. By using optimal mass transportation, Cordero-Erausquin, Nazaret and Villani [24] extended the results from [30] to prove optimal Gagliardo-Nirenberg inequalities on arbitrary normed spaces. In the sequel, we recall the main achievements from [24] and some related results which serve as model cases in the flat case.

Let $\|\cdot\|$ be an arbitrary norm on $\mathbb{R}^n$ (in particular, a reversible Minkowski norm); we may assume that the Lebesgue measure of the unit ball in $(\mathbb{R}^n, \|\cdot\|)$ is the volume of the n-dimensional Euclidean unit ball $\omega_n = \pi^{\frac{n}{2}}/\Gamma\left(\frac{n}{2}+1\right)$. The dual (or polar) norm $\|\cdot\|_*$ of $\|\cdot\|$ is $\|x\|_* = \sup_{\|y\|\leq 1} x\cdot y$, where the dot operator denotes the Euclidean inner product. Let $p\in[1,n)$ and $L^p(\mathbb{R}^n)$ be the Lebesgue space of order p. As usual, we consider the Sobolev spaces

$$\dot{W}^{1,p}(\mathbb{R}^n) = \left\{u\in L^{p^\star}(\mathbb{R}^n) : \nabla u\in L^p(\mathbb{R}^n)\right\} \quad \text{and} \quad W^{1,p}(\mathbb{R}^n) = \left\{u\in L^{p}(\mathbb{R}^n) : \nabla u\in L^p(\mathbb{R}^n)\right\},$$

where $p^\star = \frac{pn}{n-p}$ and ∇ is the gradient operator. If $u\in \dot{W}^{1,p}(\mathbb{R}^n)$, the norm of ∇u is

$$\|\nabla u\|_{L^p} = \left(\int_{\mathbb{R}^n}\|\nabla u(x)\|_*^p \mathrm{d}x\right)^{1/p},$$

where $\mathrm{d}x$ denotes the Lebesgue measure on $\mathbb{R}^n$.

Fix $n\geq 2$, $p\in(1,n)$ and $\alpha\in\left(0,\frac{n}{n-p}\right]\setminus\{1\}$; for every $\lambda>0$, let $h^\lambda_{\alpha,p}(x) = \left(\lambda+(\alpha-1)\|x\|^{p'}\right)_+^{\frac{1}{1-\alpha}}$, $x\in\mathbb{R}^n$, where $p'=\frac{p}{p-1}$ and $r_+=\max\{0,r\}$ for $r\in\mathbb{R}$. The function $h^\lambda_{\alpha,p}$ is positive everywhere for $\alpha>1$, while $h^\lambda_{\alpha,p}$ has always a compact support for $\alpha<1$. The following *optimal Gagliardo-Nirenberg inequalities* are known on normed spaces.

Theorem 2.1. (Cordero-Erausquin, Nazaret and Villani [24, Theorem 4]) *Let $n \geq 2$, $p \in (1,n)$ and $\|\cdot\|$ be an arbitrary norm on $\mathbb{R}^n$.*

- *If $1 < \alpha \leq \frac{n}{n-p}$, then*

$$\|u\|_{L^{\alpha p}} \leq \mathcal{G}_{\alpha,p,n} \|\nabla u\|_{L^p}^{\theta} \|u\|_{L^{\alpha(p-1)+1}}^{1-\theta}, \ \forall u \in \dot{W}^{1,p}(\mathbb{R}^n), \tag{2.1}$$

where

$$\theta := \frac{p^\star(\alpha-1)}{\alpha p(p^\star - \alpha p + \alpha - 1)}, \tag{2.2}$$

and the best constant

$$\mathcal{G}_{\alpha,p,n} := \left(\frac{\alpha-1}{p'}\right)^{\theta} \frac{\left(\frac{p'}{n}\right)^{\frac{\theta}{p}+\frac{\theta}{n}} \left(\frac{\alpha(p-1)+1}{\alpha-1} - \frac{n}{p'}\right)^{\frac{1}{\alpha p}} \left(\frac{\alpha(p-1)+1}{\alpha-1}\right)^{\frac{\theta}{p}-\frac{1}{\alpha p}}}{\left(\omega_n \mathsf{B}\left(\frac{\alpha(p-1)+1}{\alpha-1} - \frac{n}{p'}, \frac{n}{p'}\right)\right)^{\frac{\theta}{n}}}$$

is attained by the family of functions $h_{\alpha,p}^{\lambda}$, $\lambda > 0$;

- *If $0 < \alpha < 1$, then*

$$\|u\|_{L^{\alpha(p-1)+1}} \leq \mathcal{N}_{\alpha,p,n} \|\nabla u\|_{L^p}^{\gamma} \|u\|_{L^{\alpha p}}^{1-\gamma}, \ \forall u \in \dot{W}^{1,p}(\mathbb{R}^n), \tag{2.3}$$

where

$$\gamma := \frac{p^\star(1-\alpha)}{(p^\star - \alpha p)(\alpha p + 1 - \alpha)}, \tag{2.4}$$

and the best constant

$$\mathcal{N}_{\alpha,p,n} := \left(\frac{1-\alpha}{p'}\right)^{\gamma} \frac{\left(\frac{p'}{n}\right)^{\frac{\gamma}{p}+\frac{\gamma}{n}} \left(\frac{\alpha(p-1)+1}{1-\alpha} + \frac{n}{p'}\right)^{\frac{\gamma}{p}-\frac{1}{\alpha(p-1)+1}} \left(\frac{\alpha(p-1)+1}{1-\alpha}\right)^{\frac{1}{\alpha(p-1)+1}}}{\left(\omega_n \mathsf{B}\left(\frac{\alpha(p-1)+1}{1-\alpha}, \frac{n}{p'}\right)\right)^{\frac{\gamma}{n}}}$$

is attained by the family of functions $h_{\alpha,p}^{\lambda}$, $\lambda > 0$.

In one of the borderline cases, i.e., $\alpha = \frac{n}{n-p}$ (thus $\theta = 1$), inequality (2.1) reduces to the optimal Sobolev inequality (**S**), see Talenti [88] in the Euclidean case, and Alvino, Ferone, Lions and Trombetti [2] for normed spaces.

In the other borderline cases, i.e., when $\alpha \to 1$ and $\alpha \to 0$, the inequalities (2.1) and (2.3) degenerate to the *optimal L^p-logarithmic Sobolev inequality* (called also as the entropy-energy inequality involving the Shannon entropy) and *Faber-Krahn-type inequality*, respectively. More precisely, one has

Theorem 2.2. *Let $n \geq 2$, $p \in (1,n)$ and $\|\cdot\|$ be an arbitrary norm on $\mathbb{R}^n$. Then we have:*

- **Limit case I** ($\alpha \to 1$) (Gentil [45, Theorem 1.1]). *One has*

$$\mathbf{Ent}_{dx}(|u|^p) = \int_{\mathbb{R}^n} |u|^p \log |u|^p dx \leq \frac{n}{p} \log\left(\mathcal{L}_{p,n} \|\nabla u\|_{L^p}^p\right), \ \forall u \in W^{1,p}(\mathbb{R}^n), \ \|u\|_{L^p} = 1, \tag{2.5}$$

and the best constant $$\mathcal{L}_{p,n} := \frac{p}{n}\left(\frac{p-1}{e}\right)^{p-1}\left(\omega_n \Gamma\left(\frac{n}{p'}+1\right)\right)^{-\frac{p}{n}}$$

is attained by the family of Gaussian functions

$$l_p^\lambda(x) := \lambda^{\frac{n}{pp'}} \omega_n^{-\frac{1}{p}} \Gamma\left(\frac{n}{p'}+1\right)^{-\frac{1}{p}} e^{-\frac{\lambda}{p}\|x\|^{p'}}, \lambda > 0;$$

- **Limit case II** ($\alpha \to 0$) (Cordero-Erausquin, Nazaret and Villani [24, p. 320]). *One has*

$$\|u\|_{L^1} \leq \mathcal{F}_{p,n}\|\nabla u\|_{L^p}|\mathrm{supp}(u)|^{1-\frac{1}{p^\star}}, \ \forall u \in \dot{W}^{1,p}(\mathbb{R}^n), \tag{2.6}$$

and the best constant

$$\mathcal{F}_{p,n} := \lim_{\alpha \to 0} \mathcal{N}_{\alpha,p,n} = n^{-\frac{1}{p}} \omega_n^{-\frac{1}{n}} (p'+n)^{-\frac{1}{p'}}$$

is attained by the family of functions

$$f_p^\lambda(x) := \lim_{\alpha \to 0} h_{\alpha,p}^\lambda(x) = (\lambda - \|x\|^{p'})_+, \ x \in \mathbb{R}^n,$$

where $\mathrm{supp}(u)$ *stands for the support of* u *and* $|\mathrm{supp}(u)|$ *is its Lebesgue measure.*

Remark 2.1. The families of extremal functions in Theorems 2.1 (with $\alpha \in \left(\frac{1}{p}, \frac{n}{n-p}\right] \setminus \{1\}$) and 2.2 are *uniquely* determined up to translation, constant multiplication and scaling, see [24], [30] and [45]. In the case $0 < \alpha \leq \frac{1}{p}$, the uniqueness of $h_{\alpha,p}^\lambda$ is not known.

2.2 Interpolation inequalities on positively curved spaces: volume non-collapsing

In this section we establish fine topological properties of metric measure spaces à la Lott-Sturm-Villani which support Gagliardo-Nirenberg-type inequalities; the notations are kept from Section 2.1.

Let (M, d, m) be a metric measure space (with a strictly positive Borel measure m) and $\mathrm{Lip}_0(M)$ be the space of Lipschitz functions with compact support on M. For $u \in \mathrm{Lip}_0(M)$, let

$$|\nabla u|_d(x) := \limsup_{y \to x} \frac{|u(y)-u(x)|}{d(x,y)}, \ x \in M. \tag{2.7}$$

Note that $x \mapsto |\nabla u|_d(x)$ is Borel measurable on M for $u \in \mathrm{Lip}_0(M)$.

As before, let $n \geq 2$ be an integer, $p \in (1,n)$ and $\alpha \in \left(0, \frac{n}{n-p}\right] \setminus \{1\}$. Throughout this section we assume that the *lower n-density of the measure* m *at a point* $x_0 \in M$ *is unitary*, i.e.,

$$\liminf_{\rho \to 0} \frac{\mathsf{m}(B(x_0,\rho))}{\omega_n \rho^n} = 1. \tag*{$(\mathbf{D})^n_{x_0}$}$$

Remark 2.2. $(\mathbf{D})^n_{x_0}$ clearly holds for every point x_0 on n-dimensional Riemannian and Finsler manifolds endowed with the canonical Busemann-Hausdorff measure.

2.2.1 Gagliardo-Nirenberg inequalities: cases $\alpha > 1$ and $0 < \alpha < 1$

Theorem 2.3. (Kristály [109]) *Let (M, d, m) be a proper metric measure space which satisfies the curvature-dimension condition $\mathsf{CD}(K, n)$ for some $K \geq 0$ and $n \geq 2$. Let $p \in (1, n)$ and assume that $(\mathbf{D})^n_{x_0}$ holds for some $x_0 \in M$. Then the following statements hold:*

(i) *if $1 < \alpha \leq \frac{n}{n-p}$ and the inequality*

$$\|u\|_{L^{\alpha p}} \leq \mathcal{C}\,\||\nabla u|_d\|^{\theta}_{L^p}\,\|u\|^{1-\theta}_{L^{\alpha(p-1)+1}},\ \forall u \in \mathrm{Lip}_0(M) \qquad (\mathbf{GN1})^{\alpha,p}_{\mathcal{C}}$$

holds for some $\mathcal{C} \geq \mathcal{G}_{\alpha,p,n}$, then $K = 0$ and

$$\mathsf{m}(B(x, \rho)) \geq \left(\frac{\mathcal{G}_{\alpha,p,n}}{\mathcal{C}}\right)^{\frac{n}{\theta}} \omega_n \rho^n, \quad \forall x \in M,\ \rho \geq 0.$$

(ii) *if $0 < \alpha < 1$ and the inequality*

$$\|u\|_{L^{\alpha(p-1)+1}} \leq \mathcal{C}\,\||\nabla u|_d\|^{\gamma}_{L^p}\,\|u\|^{1-\gamma}_{L^{\alpha p}},\ \forall u \in \mathrm{Lip}_0(M) \qquad (\mathbf{GN2})^{\alpha,p}_{\mathcal{C}}$$

holds for some $\mathcal{C} \geq \mathcal{N}_{\alpha,p,n}$, then $K = 0$ and

$$\mathsf{m}(B(x, \rho)) \geq \left(\frac{\mathcal{N}_{\alpha,p,n}}{\mathcal{C}}\right)^{\frac{n}{\gamma}} \omega_n \rho^n, \quad \forall x \in M,\ \rho \geq 0.$$

Proof of (i): $1 < \alpha \leq \frac{n}{n-p}$. The proof is divided into several steps. We clearly may assume that $\mathcal{C} > \mathcal{G}_{\alpha,p,n}$ in $(\mathbf{GN1})^{\alpha,p}_{\mathcal{C}}$; indeed, if $\mathcal{C} = \mathcal{G}_{\alpha,p,n}$ one can consider the subsequent arguments for $\mathcal{C} := \mathcal{G}_{\alpha,p,n} + \varepsilon$ with small $\varepsilon > 0$ and then take $\varepsilon \to 0^+$.

Step 1 ($K = 0$). If we assume that $K > 0$ then the generalized Bonnet-Myers theorem (see Theorem 1.3/(i)) implies that M is compact and $\mathsf{m}(M)$ is finite. Taking the constant map $u(x) = \mathsf{m}(M)$ in $(\mathbf{GN1})^{\alpha,p}_{\mathcal{C}}$ as a test function, we have a contradiction. Therefore, $K = 0$.

Step 2 (*ODE from the optimal Euclidean Gagliardo-Nirenberg inequality*). We consider the optimal Gagliardo-Nirenberg inequality (2.1) in the particular case when the norm is precisely the Euclidean norm $|\cdot|$. After a simple rescaling, one can see that the function $x \mapsto \left(\lambda + |x|^{p'}\right)^{\frac{1}{1-\alpha}}$, $\lambda > 0$, is a family of extremals in (2.1); therefore, we have the following first order ODE

$$\left(\frac{1-\alpha}{\alpha(p-1)+1} h'_G(\lambda)\right)^{\frac{1}{\alpha p}} = \mathcal{G}_{\alpha,p,n}\left(\frac{p'}{\alpha-1}\right)^{\theta}\left(h_G(\lambda) + \frac{\alpha-1}{\alpha(p-1)+1}\lambda h'_G(\lambda)\right)^{\frac{\theta}{p}} h_G^{\frac{1-\theta}{\alpha(p-1)+1}}(\lambda), \quad (2.8)$$

where $h_G : (0, \infty) \to \mathbb{R}$ is given by

$$h_G(\lambda) = \int_{\mathbb{R}^n} \left(\lambda + |x|^{p'}\right)^{\frac{\alpha(p-1)+1}{1-\alpha}} \mathrm{d}x,\ \lambda > 0.$$

For further use, we shall represent the function h_G in two different ways, namely

$$\begin{aligned} h_G(\lambda) &= \omega_n \frac{n}{p'} \mathsf{B}\left(\frac{\alpha(p-1)+1}{\alpha-1} - \frac{n}{p'}, \frac{n}{p'}\right) \lambda^{\frac{\alpha(p-1)+1}{1-\alpha}+\frac{n}{p'}} \\ &= \int_0^\infty \omega_n \rho^n f_G(\lambda,\rho)\mathrm{d}\rho, \end{aligned} \tag{2.9}$$

where

$$f_G(\lambda,\rho) = p' \frac{\alpha(p-1)+1}{\alpha-1} \left(\lambda + \rho^{p'}\right)^{\frac{\alpha p}{1-\alpha}} \rho^{p'-1}. \tag{2.10}$$

STEP 3 (*Differential inequality from* $(\mathbf{GN1})_{\mathcal{C}}^{\alpha,p}$). By the generalized Bishop-Gromov inequality (see Theorem 1.3/(ii)) and hypothesis $(\mathbf{D})_{x_0}^n$, one has that

$$\frac{\mathsf{m}(B(x_0,\rho))}{\omega_n \rho^n} \leq \liminf_{r\to 0} \frac{\mathsf{m}(B(x_0,r))}{\omega_n r^n} = 1,\ \ \rho > 0. \tag{2.11}$$

Inspired by the form of h_G, we consider the function $w_G : (0,\infty) \to \mathbb{R}$ defined by

$$w_G(\lambda) = \int_M \left(\lambda + d^{p'}(x_0,x)\right)^{\frac{\alpha(p-1)+1}{1-\alpha}} \mathrm{d}\mathsf{m}(x),\ \ \lambda > 0.$$

By using the layer cake representation, it follows that w_G is well-defined and of class C^1; indeed,

$$\begin{aligned} w_G(\lambda) &= \int_0^\infty \mathsf{m}\left(\left\{x \in M : \left(\lambda + d^{p'}(x_0,x)\right)^{\frac{\alpha(p-1)+1}{1-\alpha}} > t\right\}\right)\mathrm{d}t \\ &= \int_0^\infty \mathsf{m}(B(x_0,\rho)) f_G(\lambda,\rho)\mathrm{d}\rho \qquad \left[\text{change } t = \left(\lambda+\rho^{p'}\right)^{\frac{\alpha(p-1)+1}{1-\alpha}} \text{ and see (2.10)}\right] \\ &\leq \int_0^\infty \omega_n \rho^n f_G(\lambda,\rho)\mathrm{d}\rho \qquad [\text{see (2.11)}] \\ &= h_G(\lambda), \end{aligned}$$

thus

$$0 < w_G(\lambda) \leq h_G(\lambda) < \infty,\ \ \lambda > 0. \tag{2.12}$$

For every $\lambda > 0$ and $k \in \mathbb{N}$, we consider the function $u_{\lambda,k} : M \to \mathbb{R}$ defined by

$$u_{\lambda,k}(x) = (\min\{0, k - d(x_0,x)\} + 1)_+ \left(\lambda + \max\left\{d(x_0,x), k^{-1}\right\}^{p'}\right)^{\frac{1}{1-\alpha}}.$$

Note that, since (M,d,m) is proper, the set $\mathrm{supp}(u_{\lambda,k}) = \overline{B(x_0,k+1)}$ is compact. Consequently, $u_{\lambda,k} \in \mathrm{Lip}_0(M)$ for every $\lambda > 0$ and $k \in \mathbb{N}$; thus, we can apply these functions in $(\mathbf{GN1})_{\mathcal{C}}^{\alpha,p}$, i.e.,

$$\|u_{\lambda,k}\|_{L^{\alpha p}} \leq \mathcal{C} \, \||\nabla u_{\lambda,k}|_d\|_{L^p}^{\theta} \, \|u_{\lambda,k}\|_{L^{\alpha(p-1)+1}}^{1-\theta}.$$

Moreover,

$$\lim_{k\to\infty} u_{\lambda,k}(x) = \left(\lambda + d^{p'}(x_0,x)\right)^{\frac{1}{1-\alpha}} =: u_\lambda(x).$$

By using the dominated convergence theorem, it turns out from the above inequality that u_λ also verifies $(\mathbf{GN1})_{\mathcal{C}}^{\alpha,p}$, i.e.,

$$\|u_\lambda\|_{L^{\alpha p}} \leq \mathcal{C}\, \||\nabla u_\lambda|_d\|_{L^p}^\theta \, \|u_\lambda\|_{L^{\alpha(p-1)+1}}^{1-\theta}. \tag{2.13}$$

The non-smooth chain rule gives that

$$|\nabla u_\lambda|_d(x) = \frac{p'}{\alpha-1}\left(\lambda + d^{p'}(x_0,x)\right)^{\frac{\alpha}{1-\alpha}} d^{p'-1}(x_0,x)|\nabla d(x_0,\cdot)|_d(x), \ x\in M. \tag{2.14}$$

Since $d(x_0,\cdot)$ is 1-Lipschitz, one has, $|\nabla d(x_0,\cdot)|_d(x) \leq 1$ for all $x \in M\setminus\{x_0\}$. Thus, due to (2.13), (2.14) and the form of the function w_G, we obtain the differential inequality

$$\left(\frac{1-\alpha}{\alpha(p-1)+1}w_G'(\lambda)\right)^{\frac{1}{\alpha p}} \leq \mathcal{C}\left(\frac{p'}{\alpha-1}\right)^\theta \left(w_G(\lambda) + \frac{\alpha-1}{\alpha(p-1)+1}\lambda w_G'(\lambda)\right)^{\frac{\theta}{p}} w_G^{\frac{1-\theta}{\alpha(p-1)+1}}(\lambda). \tag{2.15}$$

Step 4 (*Comparison of w_G and h_G near the origin*). We claim that

$$\lim_{\lambda\to0^+}\frac{w_G(\lambda)}{h_G(\lambda)} = 1. \tag{2.16}$$

By hypothesis $(\mathbf{D})_{x_0}^n$, for every $\varepsilon>0$ there exists $\rho_\varepsilon>0$ such that

$$\mathsf{m}(B(x_0,\rho)) \geq (1-\varepsilon)\omega_n\rho^n, \ \forall\rho\in[0,\rho_\varepsilon]. \tag{2.17}$$

Applying (2.17), one has that

$$\begin{aligned} w_G(\lambda) &= \int_0^\infty \mathsf{m}(B(x_0,\rho))f_G(\lambda,\rho)\mathrm{d}\rho \\ &\geq (1-\varepsilon)\int_0^{\rho_\varepsilon}\omega_n\rho^n f_G(\lambda,\rho)\mathrm{d}\rho = (1-\varepsilon)\lambda^{\frac{\alpha(p-1)+1}{1-\alpha}+\frac{n}{p'}}\int_0^{\rho_\varepsilon\lambda^{-\frac{1}{p'}}}\omega_n\rho^n f_G(1,\rho)\mathrm{d}\rho. \end{aligned}$$

Thus, by the representation (2.9) of h_G and the change of variables it turns out that

$$\liminf_{\lambda\to0^+}\frac{w_G(\lambda)}{h_G(\lambda)} \geq (1-\varepsilon)\liminf_{\lambda\to0^+}\frac{\displaystyle\int_0^{\rho_\varepsilon\lambda^{-\frac{1}{p'}}}\omega_n\rho^n f_G(1,\rho)\mathrm{d}\rho}{\displaystyle\int_0^\infty \omega_n\rho^n f_G(1,\rho)\mathrm{d}\rho} = 1-\varepsilon.$$

The above inequality (with $\varepsilon>0$ arbitrary small) combined with (2.12) proves the claim (2.16).

Step 5 (*Global comparison of w_G and h_G*). We now claim that

$$w_G(\lambda) \geq \left(\frac{\mathcal{G}_{\alpha,p,n}}{\mathcal{C}}\right)^{\frac{n}{\theta}} h_G(\lambda) = \widetilde{h}_G(\lambda), \ \lambda>0. \tag{2.18}$$

Since we assumed that $\mathcal{C} > \mathcal{G}_{\alpha,p,n}$, by (2.16) one has that

$$\lim_{\lambda\to 0^+} \frac{w_G(\lambda)}{\widetilde{h}_G(\lambda)} = \left(\frac{\mathcal{C}}{\mathcal{G}_{\alpha,p,n}}\right)^{\frac{n}{\theta}} > 1.$$

Therefore, there exists $\lambda_0 > 0$ such that $w_G(\lambda) > \widetilde{h}_G(\lambda)$ for every $\lambda \in (0, \lambda_0)$.

By contradiction to (2.18), we assume that there exists $\lambda^\# > 0$ such that $w_G(\lambda^\#) < \widetilde{h}_G(\lambda^\#)$. If

$$\lambda^* = \sup\{0 < \lambda < \lambda^\# : w_G(\lambda) = \widetilde{h}_G(\lambda)\},$$

then $0 < \lambda_0 \le \lambda^* < \lambda^\#$. In particular,

$$w_G(\lambda) \le \widetilde{h}_G(\lambda),\ \forall \lambda \in \left[\lambda^*, \lambda^\#\right].$$

The latter relation and the differential inequality (2.15) imply that for every $\lambda \in [\lambda^*, \lambda^\#]$, we have

$$\left(\frac{1-\alpha}{\alpha(p-1)+1} w_G'(\lambda)\right)^{\frac{1}{\alpha\theta}} \le \mathcal{C}^{\frac{p}{\theta}} \left(\frac{p'}{\alpha-1}\right)^p \left(\widetilde{h}_G(\lambda) + \frac{\alpha-1}{\alpha(p-1)+1}\lambda w_G'(\lambda)\right) \widetilde{h}_G^{\frac{(1-\theta)p}{\theta(\alpha(p-1)+1)}}(\lambda). \tag{2.19}$$

Moreover, since $\widetilde{h}_G(\lambda) = \left(\frac{\mathcal{G}_{\alpha,p,b}}{\mathcal{C}}\right)^{\frac{n}{\theta}} h_G(\lambda)$, the ODE in (2.8) can be equivalently transformed into the equation

$$\left(\frac{1-\alpha}{\alpha(p-1)+1} \widetilde{h}_G'(\lambda)\right)^{\frac{1}{\alpha\theta}} = \mathcal{C}^{\frac{p}{\theta}} \left(\frac{p'}{\alpha-1}\right)^p \left(\widetilde{h}_G(\lambda) + \frac{\alpha-1}{\alpha(p-1)+1}\lambda \widetilde{h}_G'(\lambda)\right) \widetilde{h}_G^{\frac{(1-\theta)p}{\theta(\alpha(p-1)+1)}}(\lambda) \tag{2.20}$$

for every $\lambda > 0$. For $\lambda > 0$ fixed we introduce the increasing function $j_G^\lambda : (0,\infty) \to \mathbb{R}$ defined by

$$j_G^\lambda(t) = \left(\frac{\alpha-1}{\alpha(p-1)+1} t\right)^{\frac{1}{\alpha\theta}} + \mathcal{C}^{\frac{p}{\theta}} \left(\frac{p'}{\alpha-1}\right)^p \frac{\alpha-1}{\alpha(p-1)+1} \lambda \widetilde{h}_G^{\frac{(1-\theta)p}{\theta(\alpha(p-1)+1)}}(\lambda) t.$$

Relations (2.19) and (2.20) can be rewritten into

$$j_G^\lambda(-w_G'(\lambda)) \le \mathcal{C}^{\frac{p}{\theta}} \left(\frac{p'}{\alpha-1}\right)^p \widetilde{h}_G^{1+\frac{(1-\theta)p}{\theta(\alpha(p-1)+1)}}(\lambda) = j_G^\lambda(-\widetilde{h}_G'(\lambda)),\ \forall \lambda \in \left[\lambda^*, \lambda^\#\right],$$

which implies that

$$-w_G'(\lambda) \le -\widetilde{h}_G'(\lambda),\ \forall \lambda \in \left[\lambda^*, \lambda^\#\right],$$

i.e., the function $\widetilde{h}_G - w_G$ is non-increasing in $[\lambda^*, \lambda^\#]$. In particular,

$$0 < (\widetilde{h}_G - w_G)(\lambda^\#) \le (\widetilde{h}_G - w_G)(\lambda^*) = 0,$$

which is a contradiction. This concludes the proof of (2.18).

Step 6 (*Asymptotic volume growth estimate w.r.t. x_0*). We claim that

$$\ell_\infty^{x_0} := \limsup_{\rho\to\infty} \frac{\mathsf{m}(B(x_0,\rho))}{\omega_n\rho^n} \geq \left(\frac{\mathcal{G}_{\alpha,p,n}}{\mathcal{C}}\right)^{\frac{n}{\theta}}. \tag{2.21}$$

By assuming the contrary, there exists $\varepsilon_0 > 0$ such that for some $\rho_0 > 0$,

$$\frac{\mathsf{m}(B(x_0,\rho))}{\omega_n\rho^n} \leq \left(\frac{\mathcal{G}_{\alpha,p,n}}{\mathcal{C}}\right)^{\frac{n}{\theta}} - \varepsilon_0,\ \forall \rho \geq \rho_0.$$

By (2.18) and from the latter relation, we have that

$$\begin{aligned} 0 \ &\leq \ w_G(\lambda) - \left(\frac{\mathcal{G}_{\alpha,p,n}}{\mathcal{C}}\right)^{\frac{n}{\theta}} h_G(\lambda) \\ &= \ \int_0^\infty \left(\frac{\mathsf{m}(B(x_0,\rho))}{\omega_n\rho^n} - \left(\frac{\mathcal{G}_{\alpha,p,n}}{\mathcal{C}}\right)^{\frac{n}{\theta}}\right)\omega_n\rho^n f_G(\lambda,\rho)\mathrm{d}\rho \\ &\leq \ \left(1+\varepsilon_0 - \left(\frac{\mathcal{G}_{\alpha,p,n}}{\mathcal{C}}\right)^{\frac{n}{\theta}}\right)\int_0^{\rho_0}\omega_n\rho^n f_G(\lambda,\rho)\mathrm{d}\rho - \varepsilon_0\int_0^\infty \omega_n\rho^n f_G(\lambda,\rho)\mathrm{d}\rho,\quad \forall\lambda>0. \end{aligned}$$

By (2.9), a suitable rearrangement of the terms in the above relation shows that

$$\varepsilon_0\frac{n}{p'}\mathsf{B}\left(\frac{\alpha(p-1)+1}{\alpha-1} - \frac{n}{p'}, \frac{n}{p'}\right)\lambda^{1+\frac{n}{p'}} \leq \frac{p'}{n+p'}\left(1+\varepsilon_0-\left(\frac{\mathcal{G}_{\alpha,p,n}}{\mathcal{C}}\right)^{\frac{n}{\theta}}\right)\frac{\alpha(p-1)+1}{\alpha-1}\rho_0^{n+p'},\ \forall\lambda>0.$$

If we take the limit $\lambda \to +\infty$ in the last estimate, we obtain a contradiction. Thus, the claim (2.21) is proved and it remains to apply Lemma 1.1, which concludes the proof of Theorem 2.3/(i). □

Proof of (ii): $0 < \alpha < 1$. We shall invoke some of the arguments from the proof of Theorem 2.3/(i), emphasizing that subtle differences arise due to the "dual" nature of the Gagliardo-Nirenberg inequalities $(\mathbf{GN1})_{\mathcal{C}}^{\alpha,p}$ and $(\mathbf{GN2})_{\mathcal{C}}^{\alpha,p}$. As before, we may assume that the inequality $(\mathbf{GN2})_{\mathcal{C}}^{\alpha,p}$ holds with $\mathcal{C} > \mathcal{N}_{\alpha,p,n}$.

Step 1. The fact that $K = 0$ works similarly as in Theorem 2.3/(i).

Step 2. Since $x \mapsto \left(\lambda^{p'} - |x|^{p'}\right)_+^{\frac{1}{1-\alpha}}$ is an extremal function of (2.3) for every $\lambda > 0$, we obtain the ODE

$$\begin{aligned} h_N^{\frac{1}{\alpha(p-1)+1}}(\lambda) \ &= \ \mathcal{N}_{\alpha,p,n}\left(\frac{p'}{1-\alpha}\right)^{\gamma}\left(-h_N(\lambda) + \frac{1-\alpha}{p'(\alpha(p-1)+1)}\lambda h_N'(\lambda)\right)^{\frac{\gamma}{p}} \times \\ &\quad \times \left(\frac{1-\alpha}{p'(\alpha(p-1)+1)}\lambda^{1-p'}h_N'(\lambda)\right)^{\frac{1-\gamma}{\alpha p}}, \end{aligned} \tag{2.22}$$

where the function $h_N : (0,\infty) \to \mathbb{R}$ is defined by

$$h_N(\lambda) = \int_{\mathbb{R}^n}\left(\lambda^{p'} - |x|^{p'}\right)_+^{\frac{\alpha(p-1)+1}{1-\alpha}}\mathrm{d}x,\ \lambda > 0.$$

It is clear that h_N is well-defined and of class C^1 that can be represented as

$$h_N(\lambda) = \omega_n \frac{n}{p'} \mathsf{B}\left(\frac{\alpha(p-1)+1}{1-\alpha}+1, \frac{n}{p'}\right) \lambda^{\frac{\alpha pp'}{1-\alpha}+n+p'} = \int_0^\lambda \omega_n \rho^n f_N(\lambda,\rho)\mathrm{d}\rho,$$

where

$$f_N(\lambda,\rho) = p' \frac{\alpha(p-1)+1}{1-\alpha}\left(\lambda^{p'}-\rho^{p'}\right)^{\frac{\alpha p}{1-\alpha}} \rho^{p'-1},\ \forall \lambda>0,\ \rho\in(0,\lambda). \tag{2.23}$$

Step 3. Consider the function $w_N : (0,\infty) \to \mathbb{R}$ defined by

$$w_N(\lambda) = \int_M \left(\lambda^{p'} - d^{p'}(x_0,x)\right)_+^{\frac{\alpha(p-1)+1}{1-\alpha}} \mathrm{d}\mathsf{m}(x),$$

where $x_0 \in M$ is from $(\mathbf{D})^n_{x_0}$. By the layer cake representation and relations (2.11) and (2.23), w_N is a well-defined positive C^1 function that also fulfills the inequality

$$0 < w_N(\lambda) = \int_0^\lambda \mathsf{m}(B(x_0,\rho)) f_N(\lambda,\rho)\mathrm{d}\rho \le \int_0^\lambda \omega_n \rho^n f_N(\lambda,\rho)\mathrm{d}\rho = h_N(\lambda) < \infty,\ \lambda>0. \tag{2.24}$$

Since $u_\lambda = \left(\lambda^{p'} - d^{p'}(x_0,\cdot)\right)_+^{\frac{1}{1-\alpha}}$ is a Lipschitz function on M with compact support $\overline{B(x_0,\lambda)}$, it belongs to $\mathrm{Lip}_0(M)$. Therefore, we may apply u_λ in $(\mathbf{GN2})^{\alpha,p}_{\mathcal{C}}$ and a similar reasoning as in (2.14) leads to the differential inequality

$$\begin{aligned} w_N^{\frac{1}{\alpha(p-1)+1}}(\lambda) \ &\le\ \mathcal{C}\left(\frac{p'}{1-\alpha}\right)^\gamma \left(-w_N(\lambda) + \frac{1-\alpha}{p'(\alpha(p-1)+1)}\lambda w_N'(\lambda)\right)^{\frac{\gamma}{p}} \times \\ &\times \left(\frac{1-\alpha}{p'(\alpha(p-1)+1)}\lambda^{1-p'} w_N'(\lambda)\right)^{\frac{1-\gamma}{\alpha p}},\ \lambda>0. \end{aligned} \tag{2.25}$$

Step 4. For an arbitrarily fixed $\varepsilon>0$, let $\rho_\varepsilon>0$ from (2.17). If $0<\lambda<\rho_\varepsilon$, one has that

$$w_N(\lambda) = \int_0^\lambda \mathsf{m}(B(x_0,\rho)) f_N(\lambda,\rho)\mathrm{d}\rho \ge (1-\varepsilon)\int_0^\lambda \omega_n \rho^n f_N(\lambda,\rho)\mathrm{d}\rho = (1-\varepsilon)h_N(\lambda).$$

Consequently, the latter relation together with (2.24) implies that

$$\lim_{\lambda\to 0^+} \frac{w_N(\lambda)}{h_N(\lambda)} = 1. \tag{2.26}$$

Step 5. We shall prove that

$$w_N(\lambda) \ge \left(\frac{\mathcal{N}_{\alpha,p,n}}{\mathcal{C}}\right)^{\frac{n}{\gamma}} h_N(\lambda) = \widetilde{h}_N(\lambda),\ \lambda>0. \tag{2.27}$$

By using (2.26), one has that

$$\lim_{\lambda\to 0^+} \frac{w_N(\lambda)}{\widetilde{h}_N(\lambda)} = \left(\frac{\mathcal{C}}{\mathcal{N}_{\alpha,p,n}}\right)^{\frac{n}{\gamma}} > 1,$$

which implies the existence of a number $\lambda_0 > 0$ such that $w_N(\lambda) > \widetilde{h}_N(\lambda)$ for every $\lambda \in (0, \lambda_0)$.

We assume by contradiction that there exists $\lambda^{\#} > 0$ such that $w_N(\lambda^{\#}) < \widetilde{h}_N(\lambda^{\#})$. If $\lambda^* = \sup\{0 < \lambda < \lambda^{\#} : w_N(\lambda) = \widetilde{h}_N(\lambda)\}$, then $0 < \lambda_0 \leq \lambda^* < \lambda^{\#}$ and

$$w_N(\lambda) \leq \widetilde{h}_N(\lambda),\ \forall \lambda \in \left[\lambda^*, \lambda^{\#}\right]. \tag{2.28}$$

For every $\lambda > 0$, let $j_N^\lambda : \left(\frac{p'(\alpha(p-1)+1)}{(1-\alpha)\lambda}, \infty\right) \to \mathbb{R}$ be the function defined by

$$j_N^\lambda(t) = \mathcal{C}\left(\frac{p'}{1-\alpha}\right)^\gamma \left(-1 + \frac{1-\alpha}{p'(\alpha(p-1)+1)}\lambda t\right)^{\frac{\gamma}{p}} \left(\frac{1-\alpha}{p'(\alpha(p-1)+1)}\lambda^{1-p'}t\right)^{\frac{1-\gamma}{\alpha p}}.$$

It is clear that j_N^λ is a well-defined positive increasing function. A direct computation yields that both values

$$(\log w_N)'(\lambda) = \frac{w_N'(\lambda)}{w_N(\lambda)} \quad \text{and} \quad (\log \widetilde{h}_N)'(\lambda) = \frac{\widetilde{h}_N'(\lambda)}{\widetilde{h}_N(\lambda)}$$

are greater than $\frac{p'(\alpha(p-1)+1)}{(1-\alpha)\lambda}$ for every $\lambda > 0$. Taking into account (2.4), we have

$$\frac{1}{\alpha(p-1)+1} - \frac{\gamma}{p} - \frac{1-\gamma}{\alpha p} = -\frac{\gamma}{n};$$

therefore, if we divide the inequality (2.25) by $w_N^{\frac{\gamma}{p}+\frac{1-\gamma}{\alpha p}}(\lambda)$, we obtain that

$$w_N^{-\frac{\gamma}{n}}(\lambda) \leq j_N^\lambda\left((\log w_N)'(\lambda)\right),\ \forall \lambda > 0. \tag{2.29}$$

In a similar manner, by $\widetilde{h}_N(\lambda) = \left(\frac{\mathcal{N}_{\alpha,p,n}}{\mathcal{C}}\right)^{\frac{n}{\gamma}} h_N(\lambda)$ and relation (2.22), we have that

$$\widetilde{h}_N^{-\frac{\gamma}{n}}(\lambda) = j_N^\lambda\left((\log \widetilde{h}_N)'(\lambda)\right),\ \forall \lambda > 0. \tag{2.30}$$

Thus, by (2.28)-(2.30), it turns out that

$$j_N^\lambda\left((\log \widetilde{h}_N)'(\lambda)\right) = \widetilde{h}_N^{-\frac{\gamma}{n}}(\lambda) \leq w_N^{-\frac{\gamma}{n}}(\lambda) \leq j_N^\lambda\left((\log w_N)'(\lambda)\right),\ \forall \lambda \in \left[\lambda^*, \lambda^{\#}\right].$$

Since the inverse of j_N^λ is also increasing, it follows that $(\log \widetilde{h}_N)'(\lambda) \leq (\log w_N)'(\lambda)$ for every $\lambda \in [\lambda^*, \lambda^{\#}]$. Therefore, the function $\lambda \mapsto \log \frac{\widetilde{h}_N(\lambda)}{w_N(\lambda)}$ is non-increasing in the interval $\left[\lambda^*, \lambda^{\#}\right]$. In particular, it follows that

$$0 < \log \frac{\widetilde{h}_N(\lambda^{\#})}{w_N(\lambda^{\#})} \leq \log \frac{\widetilde{h}_N(\lambda^*)}{w_N(\lambda^*)} = 0,$$

a contradiction, which proves the validity of the claim (2.27).

Step 6. We shall prove that

$$\limsup_{\rho\to\infty} \frac{\mathsf{m}(B(x_0,\rho))}{\omega_n \rho^n} \geq \left(\frac{\mathcal{N}_{\alpha,p,n}}{\mathcal{C}}\right)^{\frac{n}{\gamma}}. \tag{2.31}$$

By contradiction, we assume that there exists $\varepsilon_0 > 0$ such that for some $\rho_0 > 0$,

$$\frac{\mathsf{m}(B(x_0,\rho))}{\omega_n\rho^n} \leq \left(\frac{\mathcal{N}_{\alpha,p,n}}{\mathcal{C}}\right)^{\frac{n}{\gamma}} - \varepsilon_0, \ \forall \rho \geq \rho_0.$$

The previous inequality and (2.27) imply that for every $\lambda > \rho_0$,

$$\begin{aligned} 0 &\leq w_N(\lambda) - \left(\frac{\mathcal{N}_{\alpha,p,n}}{\mathcal{C}}\right)^{\frac{n}{\gamma}} h_N(\lambda) = \int_0^\lambda \left(\frac{\mathsf{m}(B(x_0,\rho))}{\omega_n\rho^n} - \left(\frac{\mathcal{N}_{\alpha,p,n}}{\mathcal{C}}\right)^{\frac{n}{\gamma}}\right)\omega_n\rho^n f_N(\lambda,\rho)\mathrm{d}\rho \\ &\leq \left(1+\varepsilon_0 - \left(\frac{\mathcal{N}_{\alpha,p,n}}{\mathcal{C}}\right)^{\frac{n}{\gamma}}\right)\int_0^{\rho_0} \omega_n\rho^n f_N(\lambda,\rho)\mathrm{d}\rho - \varepsilon_0\int_0^\lambda \omega_n\rho^n f_N(\lambda,\rho)\mathrm{d}\rho. \end{aligned}$$

Reorganizing the latter estimate, it follows that for every $\lambda > 0$,

$$\varepsilon_0\frac{n}{p'}\mathsf{B}\left(\frac{\alpha(p-1)+1}{1-\alpha}+1, \frac{n}{p'}\right)\lambda^{n+p'} \leq \frac{p'}{n+p'}\left(1+\varepsilon_0 - \left(\frac{\mathcal{N}_{\alpha,p,n}}{\mathcal{C}}\right)^{\frac{n}{\gamma}}\right)\frac{\alpha(p-1)+1}{1-\alpha}\rho_0^{n+p'}.$$

Once we let $\lambda \to \infty$ in the latter estimate, we get a contradiction. Therefore, (2.31) holds and Lemma 1.1 yields that

$$\frac{\mathsf{m}(B(x,\rho))}{\omega_n\rho^n} \geq \left(\frac{\mathcal{N}_{\alpha,p,n}}{\mathcal{C}}\right)^{\frac{n}{\gamma}}, \ \forall x \in M, \ \rho > 0,$$

which concludes the proof of Theorem 2.3/(ii). □

Remark 2.3. The particular case $p = 2$ and $\alpha = \frac{n}{n-2}$ $(n \geq 3)$ is contained in [116], where a volume doubling property is assumed on metric measure spaces instead of $\mathsf{CD}(K,n)$.

2.2.2 Limit case I ($\alpha \to 1$): L^p-logarithmic Sobolev inequality

Theorem 2.4. (Kristály [109]) *Under the same assumptions as in Theorem 2.3, if*

$$\mathbf{Ent}_{\mathsf{dm}}(|u|^p) = \int_M |u|^p \log|u|^p \mathsf{dm} \leq \frac{n}{p}\log\left(\mathcal{C}\,\||\nabla u|_d\|_{L^p}^p\right), \ \forall u \in \mathrm{Lip}_0(M), \ \|u\|_{L^p} = 1 \qquad (\mathbf{LS})_{\mathcal{C}}^p$$

holds for some $\mathcal{C} \geq \mathcal{L}_{p,n}$, then $K = 0$ and

$$\mathsf{m}(B(x,\rho)) \geq \left(\frac{\mathcal{L}_{p,n}}{\mathcal{C}}\right)^{\frac{n}{p}}\omega_n\rho^n, \ \forall x \in M, \ \rho \geq 0.$$

Proof. We shall assume that $\mathcal{C} > \mathcal{L}_{p,n}$ in $(\mathbf{LS})_{\mathcal{C}}^p$.

Step 1. As in the previous proofs, we obtain that $K = 0$; the only difference is that we shall consider $u(x) = \mathsf{m}(M)^{-1/p}$ as a test function in $(\mathbf{LS})_{\mathcal{C}}^p$, in order to fulfill the normalization $\|u\|_{L^p} = 1$.

Step 2. Since the functions l_p^λ $(\lambda > 0)$ in Theorem 2.2 are extremals in (2.5), once we plug them, we obtain a first order ODE of the form

$$-\log h_L(\lambda) + \lambda\frac{h_L'(\lambda)}{h_L(\lambda)} = \frac{n}{p}\log\left(-\mathcal{L}_{p,n}\left(\frac{p'}{p}\right)^p\lambda^p\frac{h_L'(\lambda)}{h_L(\lambda)}\right), \ \lambda > 0, \qquad (2.32)$$

where $h_L : (0,\infty) \to \mathbb{R}$ is defined by

$$h_L(\lambda) = \int_{\mathbb{R}^n} e^{-\lambda |x|^{p'}} \mathrm{d}x.$$

For later use, we recall that h_L can be represented alternatively by

$$h_L(\lambda) = \frac{2\pi^{\frac{n}{2}}}{p'\lambda^{\frac{n}{p'}}} \cdot \frac{\Gamma\left(\frac{n}{p'}\right)}{\Gamma\left(\frac{n}{2}\right)} = \lambda p' \omega_n \int_0^\infty e^{-\lambda \rho^{p'}} \rho^{n+p'-1} \mathrm{d}\rho = \lambda^{-\frac{n}{p'}} p' \omega_n \int_0^\infty e^{-t^{p'}} t^{n+p'-1} \mathrm{d}t. \tag{2.33}$$

Step 3. Let $w_L : (0,\infty) \to \mathbb{R}$ be defined by

$$w_L(\lambda) = \int_M e^{-\lambda\, d^{p'}(x_0,x)} \mathrm{dm}(x),$$

where $x_0 \in M$ is the element from hypothesis $(\mathbf{D})^n_{x_0}$. Note that w_L is well-defined, positive and differentiable. Indeed, by the layer cake representation, for every $\lambda > 0$ we obtain that

$$\begin{aligned}
w_L(\lambda) &= \int_0^\infty \mathsf{m}\left(\left\{x \in M : e^{-\lambda\, d^{p'}(x_0,x)} > t\right\}\right) \mathrm{d}t \\
&= \int_0^1 \mathsf{m}\left(\left\{x \in M : e^{-\lambda\, d^{p'}(x_0,x)} > t\right\}\right) \mathrm{d}t \\
&= \lambda p' \int_0^\infty \mathsf{m}(B(x_0,\rho)) e^{-\lambda \rho^{p'}} \rho^{p'-1} \mathrm{d}\rho && [\text{change } t = e^{-\lambda \rho^{p'}}] \\
&\leq \lambda p' \omega_n \int_0^\infty e^{-\lambda \rho^{p'}} \rho^{n+p'-1} \mathrm{d}\rho && [\text{see } (2.11)] \\
&= h_L(\lambda) < +\infty.
\end{aligned}$$

Let us consider the family of functions $\widetilde{u}_\lambda : M \to \mathbb{R}$ $(\lambda > 0)$ defined by

$$\widetilde{u}_\lambda(x) = \frac{e^{-\frac{\lambda}{p} d^{p'}(x_0,x)}}{w_L(\lambda)^{\frac{1}{p}}},\ x \in M.$$

It is clear that $\|\widetilde{u}_\lambda\|_{L^p} = 1$ and as in the proof of Theorem 2.3/(i), the function $\widetilde{u}_\lambda$ can be approximated by elements from $\mathrm{Lip}_0(M)$; in fact, $\widetilde{u}_\lambda$ can be used as a test function in $(\mathbf{LS})^p_{\mathcal{C}}$. Thus, plugging $\widetilde{u}_\lambda$ into the inequality $(\mathbf{LS})^p_{\mathcal{C}}$, applying both the non-smooth chain rule and the fact that $|\nabla d(x_0,\cdot)|_d(x) \leq 1$ for every $x \in M \setminus \{x_0\}$, it yields that

$$-\log w_L(\lambda) + \lambda \frac{w_L'(\lambda)}{w_L(\lambda)} \leq \frac{n}{p} \log\left(-\mathcal{C}\left(\frac{p'}{p}\right)^p \lambda^p \frac{w_L'(\lambda)}{w_L(\lambda)}\right),\ \lambda > 0. \tag{2.34}$$

Step 4. We prove that

$$\lim_{\lambda \to +\infty} \frac{w_L(\lambda)}{h_L(\lambda)} = 1. \tag{2.35}$$

For a fixed $\varepsilon > 0$, let $\rho_\varepsilon > 0$ from (2.17). Then one has

$$\begin{aligned} w_L(\lambda) &= \lambda p' \int_0^\infty \mathsf{m}(B(x_0,\rho)) e^{-\lambda \rho^{p'}} \rho^{p'-1} \mathrm{d}\rho \geq \lambda p'(1-\varepsilon)\omega_n \int_0^{\rho_\varepsilon} e^{-\lambda \rho^{p'}} \rho^{n+p'-1} \mathrm{d}\rho \\ &= \lambda^{-\frac{n}{p'}} p'(1-\varepsilon)\omega_n \int_0^{\rho_\varepsilon \lambda^{\frac{1}{p'}}} e^{-t^{p'}} t^{n+p'-1} \mathrm{d}t. \qquad [\text{change } t = \lambda^{\frac{1}{p'}}\rho] \end{aligned}$$

Therefore, by the third representation of h_L (see (2.33)) it turns out that

$$\liminf_{\lambda \to +\infty} \frac{w_L(\lambda)}{h_L(\lambda)} \geq 1 - \varepsilon.$$

The arbitrariness of $\varepsilon > 0$ together with Step 3 implies the validity of (2.35).

Step 5. We claim that

$$w_L(\lambda) \geq \left(\frac{\mathcal{L}_{p,n}}{\mathcal{C}}\right)^{\frac{n}{p}} h_L(\lambda) =: \widetilde{h}_L(\lambda),\ \lambda > 0. \tag{2.36}$$

Since $\mathcal{C} > \mathcal{L}_{p,n}$, by (2.35) it follows that

$$\lim_{\lambda \to +\infty} \frac{w_L(\lambda)}{\widetilde{h}_L(\lambda)} = \left(\frac{\mathcal{C}}{\mathcal{L}_{p,n}}\right)^{\frac{n}{p}} > 1.$$

Consequently, there exists $\widetilde{\lambda} > 0$ such that $w_L(\lambda) > \widetilde{h}_L(\lambda)$ for all $\lambda > \widetilde{\lambda}$. If we introduce the notations

$$W(\lambda) = \log w_L(\lambda) \text{ and } \widetilde{H}(\lambda) = \log \widetilde{h}_L(\lambda),\ \lambda > 0,$$

the latter relation implies that

$$W(\lambda) > \widetilde{H}(\lambda),\ \forall \lambda > \widetilde{\lambda}, \tag{2.37}$$

while relations in (2.34) and (2.32) can be rewritten in terms of W and $\widetilde{H}$ as

$$-W(\lambda) + \lambda W'(\lambda) \leq \frac{n}{p} \log\left(-\mathcal{C}\left(\frac{p'}{p}\right)^p \lambda^p W'(\lambda)\right),\ \lambda > 0, \tag{2.38}$$

and

$$-\widetilde{H}(\lambda) + \lambda \widetilde{H}'(\lambda) = \frac{n}{p} \log\left(-\mathcal{C}\left(\frac{p'}{p}\right)^p \lambda^p \widetilde{H}'(\lambda)\right),\ \lambda > 0. \tag{2.39}$$

Claim (2.36) is proved once we show that $W(\lambda) \geq \widetilde{H}(\lambda)$ for all $\lambda > 0$. By contradiction, we assume there exists $\lambda^\# > 0$ such that $W(\lambda^\#) < \widetilde{H}(\lambda^\#)$. Due to (2.37), $\lambda^\# < \widetilde{\lambda}$. On one hand, let

$$\lambda^* = \inf\{\lambda > \lambda^\# : W(\lambda) = \widetilde{H}(\lambda)\}.$$

In particular,

$$W(\lambda) \leq \widetilde{H}(\lambda),\ \forall \lambda \in \left[\lambda^\#, \lambda^*\right]. \tag{2.40}$$

On the other hand, if we introduce for every $\lambda > 0$ the function $j_L^\lambda : (0,\infty) \to \mathbb{R}$ by

$$j_L^\lambda(t) = \frac{n}{p}\log\left(\mathcal{C}\left(\frac{p'}{p}\right)^p \lambda^p t\right) + \lambda t,\ t > 0,$$

relations (2.38) and (2.39) become

$$-W(\lambda) \le j_L^\lambda(-W'(\lambda)) \text{ and } -\widetilde{H}(\lambda) = j_L^\lambda(-\widetilde{H}'(\lambda)),\ \lambda > 0,$$

respectively. By the previous relations and (2.40) it yields that

$$j_L^\lambda(-\widetilde{H}'(\lambda)) = -\widetilde{H}(\lambda) \le -W(\lambda) \le j_L^\lambda(-W'(\lambda)),\ \forall \lambda \in \left[\lambda^\#, \lambda^*\right].$$

Since j_L^λ is increasing, it follows that $W - \widetilde{H}$ is a non-increasing function on $\left[\lambda^\#, \lambda^*\right]$, which implies

$$0 = (W - \widetilde{H})(\lambda^*) \le (W - \widetilde{H})(\lambda^\#) < 0,$$

a contradiction. This completes the proof of (2.36).

Step 6. We claim that

$$\limsup_{\rho\to\infty} \frac{\mathsf{m}(B(x_0,\rho))}{\omega_n \rho^n} \ge \left(\frac{\mathcal{L}_{p,n}}{\mathcal{C}}\right)^{\frac{n}{p}}. \tag{2.41}$$

By assuming the contrary, there exists $\varepsilon_0 > 0$ such that for some $\rho_0 > 0$,

$$\frac{\mathsf{m}(B(x_0,\rho))}{\omega_n \rho^n} \le \left(\frac{\mathcal{L}_{p,n}}{\mathcal{C}}\right)^{\frac{n}{p}} - \varepsilon_0,\ \forall \rho \ge \rho_0.$$

Combining the latter relation with (2.36) and (2.33), we obtain that

$$\begin{aligned} 0 &\le w_L(\lambda) - \left(\frac{\mathcal{L}_{p,n}}{\mathcal{C}}\right)^{\frac{n}{p}} h_L(\lambda) \\ &\le \lambda p' \int_0^{\rho_0} \mathsf{m}(B(x_0,\rho)) e^{-\lambda\rho^{p'}} \rho^{p'-1} \mathrm{d}\rho + \lambda p' \omega_n \left(\left(\frac{\mathcal{L}_{p,n}}{\mathcal{C}}\right)^{\frac{n}{p}} - \varepsilon_0\right) \int_{\rho_0}^\infty e^{-\lambda\rho^{p'}} \rho^{n+p'-1} \mathrm{d}\rho \\ &\quad -\lambda p' \omega_n \left(\frac{\mathcal{L}_{p,n}}{\mathcal{C}}\right)^{\frac{n}{p}} \int_0^\infty e^{-\lambda\rho^{p'}} \rho^{n+p'-1} \mathrm{d}\rho,\ \forall \lambda > 0. \end{aligned}$$

Rearranging the above inequality, by (2.11) it follows that

$$\varepsilon_0 \int_0^\infty e^{-\lambda\rho^{p'}} \rho^{n+p'-1} \mathrm{d}\rho \le \left(1 - \left(\frac{\mathcal{L}_{p,n}}{\mathcal{C}}\right)^{\frac{n}{p}} + \varepsilon_0\right) \int_0^{\rho_0} e^{-\lambda\rho^{p'}} \rho^{n+p'-1} \mathrm{d}\rho,\ \forall \lambda > 0.$$

Due to (2.33), the latter inequality implies

$$\varepsilon_0 \frac{1}{p' \lambda^{1+\frac{n}{p'}}} \Gamma\left(\frac{n}{p'} + 1\right) \le \left(1 - \left(\frac{\mathcal{L}_{p,n}}{\mathcal{C}}\right)^{\frac{n}{p}} + \varepsilon_0\right) \frac{\rho_0^{n+p'}}{n+p'},\ \forall \lambda > 0.$$

Now, letting $\lambda \to 0^+$ we obtain a contradiction. Therefore, the proof of (2.41) is concluded. Thus, Lemma 1.1 gives that

$$\frac{\mathsf{m}(B(x,\rho))}{\omega_n \rho^n} \geq \left(\frac{\mathcal{L}_{p,n}}{\mathcal{C}}\right)^{\frac{n}{p}}, \ \forall x \in M, \ \rho > 0,$$

which ends the proof of Theorem 2.4. □

2.2.3 Limit case II ($\alpha \to 0$): Faber-Krahn inequality

Theorem 2.5. (Kristály [109]) *Under the same assumptions as in Theorem 2.3, if*

$$\|u\|_{L^1} \leq \mathcal{C}\, \||\nabla u|_d\|_{L^p}\, \mathsf{m}(\mathrm{supp}(u))^{1-\frac{1}{p^\star}}, \ \forall u \in \mathrm{Lip}_0(M) \qquad (\mathbf{FK})^p_{\mathcal{C}}$$

holds for some $\mathcal{C} \geq \mathcal{F}_{p,n}$, then $K = 0$ and

$$\mathsf{m}(B(x,\rho)) \geq \left(\frac{\mathcal{F}_{p,n}}{\mathcal{C}}\right)^n \omega_n \rho^n, \ \forall x \in M, \rho \geq 0.$$

Proof. Similarly as before, we may assume that $\mathcal{C} > \mathcal{F}_{p,n}$.

Step 1. Analogously to Theorem 2.3/(i), it follows that $K = 0$.

Step 2. The function $x \mapsto \left(\lambda^{p'} - |x|^{p'}\right)_+$ being extremal in (2.6) for every $\lambda > 0$, a direct computation shows that

$$h_F(\lambda) \ = \ \mathcal{F}_{p,n} p' \left(-h_F(\lambda) + \frac{1}{p'}\lambda h_F'(\lambda)\right)^{\frac{1}{p}} \left(\frac{1}{p'}\lambda^{1-p'} h_F'(\lambda)\right)^{1-\frac{1}{p^\star}}, \qquad (2.42)$$

where $h_F : (0,\infty) \to \mathbb{R}$ is given by

$$h_F(\lambda) = \int_{\mathbb{R}^n} \left(\lambda^{p'} - |x|^{p'}\right)_+ \mathrm{d}x, \ \lambda > 0.$$

Step 3. Let $x_0 \in M$ from $(\mathbf{D})^n_{x_0}$. Since $u_\lambda = \left(\lambda^{p'} - d^{p'}(x_0,\cdot)\right)_+ \in \mathrm{Lip}_0(M)$, we may insert u_λ into $(\mathbf{FK})^p_{\mathcal{C}}$ obtaining

$$\|u_\lambda\|_{L^1} \leq \mathcal{C}\, \||\nabla u_\lambda|_d\|_{L^p}\, \mathsf{m}(\mathrm{supp}(u_\lambda))^{1-\frac{1}{p^\star}}. \qquad (2.43)$$

At first, we observe that

$$|\nabla u_\lambda|_d(x) = p' d^{p'-1}(x_0,x) |\nabla d(x_0,\cdot)|_d(x) \leq p' d^{p'-1}(x_0,x), \ \forall x \in B(x_0,\lambda),$$

while $|\nabla u_\lambda|_d(x) = 0$ for every $x \notin B(x_0,\lambda)$. Moreover, since the spheres have zero m-measures (see Theorem 1.3), we have that $\mathsf{m}(\mathrm{supp}(u_\lambda)) = \mathsf{m}(\overline{B(x_0,\lambda)}) = \mathsf{m}(B(x_0,\lambda))$. We now introduce the function $w_F : (0,\infty) \to \mathbb{R}$ given by $w_F(\lambda) = \displaystyle\int_M \left(\lambda^{p'} - d^{p'}(x_0,x)\right)_+ \mathrm{d}\mathsf{m}(x)$, $\lambda > 0$. Due to the layer cake representation, one has that

$$\begin{aligned} w_F(\lambda) &= \int_{B(x_0,\lambda)} \left(\lambda^{p'} - d^{p'}(x_0,x)\right) \mathrm{d}m(x) = \lambda^{p'}\mathsf{m}(B(x_0,\lambda)) - \int_{B(x_0,\lambda)} d^{p'}(x_0,x)\mathrm{d}\mathsf{m}(x) \\ &= \lambda^{p'}\mathsf{m}(B(x_0,\lambda)) - p'\int_0^\lambda (\mathsf{m}(B(x_0,\lambda)) - \mathsf{m}(B(x_0,\rho)))\,\rho^{p'-1}\mathrm{d}\rho \qquad [\text{change } t=\rho^{p'}] \\ &= p'\int_0^\lambda \mathsf{m}(B(x_0,\rho))\rho^{p'-1}\mathrm{d}\rho. \end{aligned}$$

Therefore, $\|u_\lambda\|_{L^1} = w_F(\lambda),\ \mathsf{m}(\mathrm{supp}(u_\lambda)) = \mathsf{m}(B(x_0,\lambda)) = \frac{1}{p'}\lambda^{1-p'}w_F'(\lambda)$, and

$$\||\nabla u_\lambda|_d\|_{L^p} \leq p'\left(\int_{B(x_0,\lambda)} d^{p'}(x_0,x)\mathrm{d}\mathsf{m}(x)\right)^{\frac{1}{p}} = p'\left(-w_F(\lambda) + \frac{1}{p'}\lambda w_F'(\lambda)\right)^{\frac{1}{p}}.$$

Consequently, inequality (2.43) takes the form

$$w_F(\lambda) \leq \mathcal{C}p'\left(-w_F(\lambda) + \frac{1}{p'}\lambda w_F'(\lambda)\right)^{\frac{1}{p}} \left(\frac{1}{p'}\lambda^{1-p'}w_F'(\lambda)\right)^{1-\frac{1}{p^\star}},\ \lambda > 0,$$

which is formally (2.25) if $\alpha \to 0$, since $\lim_{\alpha\to 0}\gamma = 1$ and $\lim_{\alpha\to 0}\frac{1-\gamma}{\alpha p} = 1 - \frac{1}{p^\star}$, due to (2.4).

Therefore, we may proceed as in the Steps 4-6 of the proof of Theorem 2.3/(ii), proving that

$$\lim_{\lambda\to 0^+}\frac{w_F(\lambda)}{h_F(\lambda)} = 1,\quad w_F(\lambda) \geq \left(\frac{\mathcal{F}_{p,n}}{\mathcal{C}}\right)^n h_F(\lambda),\ \forall\lambda > 0,$$

and finally

$$\frac{\mathsf{m}(B(x,\rho))}{\omega_n\rho^n} \geq \left(\frac{\mathcal{F}_{p,n}}{\mathcal{C}}\right)^n,\ \forall x\in M,\ \rho > 0,$$

which concludes the proof of Theorem 2.5. □

2.2.4 Rigidities via Munn-Perelman homotopic quantification

We first state an Aubin-Hebey-type result ([8] and [52]) for Gagliardo-Nirenberg inequalities which is valid on generic Riemannian manifolds.

Lemma 2.1. *Let (M,g) be an n-dimensional $(n\geq 2)$ complete Riemannian manifold and $\mathcal{C} > 0$. The following statements hold:*

(i) *if $(\mathbf{GN1})^{\alpha,p}_{\mathcal{C}}$ holds on (M,g) for some $p\in(1,n)$ and $\alpha\in\left(1,\frac{n}{n-p}\right]$, then $\mathcal{C}\geq\mathcal{G}_{\alpha,p,n}$;*

(ii) *if $(\mathbf{GN2})^{\alpha,p}_{\mathcal{C}}$ holds on (M,g) for some $p\in(1,n)$ and $\alpha\in(0,1)$, then $\mathcal{C}\geq\mathcal{N}_{\alpha,p,n}$;*

(iii) *if $(\mathbf{LS})^{p}_{\mathcal{C}}$ holds on (M,g) for some $p\in(1,n)$, then $\mathcal{C}\geq\mathcal{L}_{p,n}$;*

(iv) *if $(\mathbf{FK})^{p}_{\mathcal{C}}$ holds on (M,g) for some $p\in(1,n)$, then $\mathcal{C}\geq\mathcal{F}_{p,n}$.*

Proof. (i) By contradiction, we assume that $(\mathbf{GN1})^{\alpha,p}_{\mathcal{C}}$ holds on (M,g) for some $p\in(1,n)$, $\alpha\in\left(1,\frac{n}{n-p}\right]$, and $\mathcal{C}<\mathcal{G}_{\alpha,p,n}$. Let $x_0\in M$ be fixed arbitrarily. For every $\varepsilon>0$, there exists a local chart

(Ω, φ) of M at the point x_0 and a number $\delta > 0$ such that $\varphi(\Omega) = B_e(0, \delta)$ and the components $g_{ij} = g_{ij}(x)$ of the Riemannian metric g on (Ω, φ) satisfy

$$(1-\varepsilon)\delta_{ij} \leq g_{ij} \leq (1+\varepsilon)\delta_{ij} \tag{2.44}$$

in the sense of bilinear forms; here, δ_{ij} denotes the Kronecker symbol. Since $(\mathbf{GN1})_{\mathcal{C}}^{\alpha,p}$ is valid, relation (2.44) shows that for every $\varepsilon > 0$ small enough, there exists $\delta_\varepsilon > 0$ and $\mathcal{C}_\varepsilon \in (\mathcal{C}, \mathcal{G}_{\alpha,p,n})$ such that

$$\|v\|_{L^{\alpha p}(B_e(0,\delta),\mathrm{d}x)} \leq \mathcal{C}_\varepsilon \|\nabla v\|^{\theta}_{L^p(B_e(0,\delta),\mathrm{d}x)} \|v\|^{1-\theta}_{L^{\alpha(p-1)+1}(B_e(0,\delta),\mathrm{d}x)}, \ \forall \delta \in (0,\delta_\varepsilon), v \in \mathrm{Lip}_0(B_e(0,\delta)). \tag{2.45}$$

Let us fix $u \in \mathrm{Lip}_0(\mathbb{R}^n)$ arbitrarily and set $v_\lambda(x) = \lambda^{\frac{n}{p}} u(\lambda x)$, $\lambda > 0$. For $\lambda > 0$ large enough, one has $v_\lambda \in \mathrm{Lip}_0(B_e(0,\delta))$. If we plug in v_λ into (2.45), by using the scaling properties

$$\|\nabla v_\lambda\|_{L^p(B_e(0,\delta),\mathrm{d}x)} = \lambda \|\nabla u\|_{L^p(\mathbb{R}^n,\mathrm{d}x)} \ \text{ and } \ \|v_\lambda\|_{L^q(B_e(0,\delta),\mathrm{d}x)} = \lambda^{\frac{n}{p}-\frac{n}{q}} \|u\|_{L^q(\mathbb{R}^n,\mathrm{d}x)}, \ \forall q > 0, \tag{2.46}$$

and the form of the number θ (see (2.2)), it follows that

$$\|u\|_{L^{\alpha p}(\mathbb{R}^n,\mathrm{d}x)} \leq \mathcal{C}_\varepsilon \|\nabla u\|^{\theta}_{L^p(\mathbb{R}^n,\mathrm{d}x)} \|u\|^{1-\theta}_{L^{\alpha(p-1)+1}(\mathbb{R}^n,\mathrm{d}x)}.$$

If we insert the extremal function $h^\lambda_{\alpha,p}$ of the optimal Gagliardo-Nirenberg inequality on $\mathbb{R}^n$ $(\alpha > 1)$ into the latter relation, Theorem 2.1 yields that $\mathcal{G}_{\alpha,p,n} \leq \mathcal{C}_\varepsilon$, a contradiction.

The proofs of (ii) (iii) and (iv) are analogous to (i), taking into account in addition to (2.46) that

$$\mathbf{Ent}_{\mathrm{d}x}(|v_\lambda|^p) = \mathbf{Ent}_{\mathrm{d}x}(|u|^p) + n\|u\|^p_{L^p} \log \lambda$$

and

$$\mathcal{H}^n(\mathrm{supp}(v_\lambda)) = \lambda^{-n}\mathcal{H}^n(\mathrm{supp}(u)),$$

respectively. □

Let (M, g) be an n-dimensional $(n \geq 2)$ complete Riemannian manifold with nonnegative Ricci curvature endowed with its canonical volume element $\mathrm{d}V_g$. The *asymptotic volume growth* of (M, g) is defined by

$$\mathrm{AVG}_{(M,g)} = \lim_{r\to\infty} \frac{\mathrm{Vol}_g(B_g(x,r))}{\omega_n r^n}.$$

By the Bishop-Gromov comparison theorem it follows that $\mathrm{AVG}_{(M,g)} \leq 1$ and this number is independent of the point $x \in M$.

Given $k \in \{1, ..., n\}$, let us denote by $\delta_{k,n} > 0$ the smallest positive solution to the equation

$$10^{k+2} C_{k,n}(k) s \left(1 + \frac{s}{2k}\right)^k = 1$$

in the variable s, where

$$C_{k,n}(i) = \begin{cases} 1, & \text{if} \quad i = 0, \\ 3 + 10C_{k,n}(i-1) + (16k)^{n-1}(1 + 10C_{k,n}(i-1))^n, & \text{if} \quad i \in \{1, ..., k\}. \end{cases}$$

We now consider the smooth, bijective and increasing function $h_{k,n} : (0, \delta_{k,n}) \to (1, \infty)$ defined by

$$h_{k,n}(s) = \left[1 - 10^{k+2} C_{k,n}(k) s \left(1 + \frac{s}{2k}\right)^k\right]^{-1}.$$

For every $s > 1$, let

$$\beta(k, s, n) = \begin{cases} 1 - \left[1 + \frac{s^n}{[h_{1,n}^{-1}(s)]^n}\right]^{-1}, & \text{if} \quad k = 1, \\ \max\left\{\beta(1, s, n), \beta(i, 1 + \frac{h_{k,n}^{-1}(s)}{2k}, n) : i = 1, ..., k-1\right\}, & \text{if} \quad k \in \{2, ..., n\}. \end{cases}$$

The constant $\beta(k, s, n)$, which is used to prove the Perelman's maximal volume lemma, denotes the minimum volume growth of (M, g) needed to guarantee that any continuous map $f : \mathbb{S}^k \to B_g(x, \rho)$ has a continuous extension $g : \mathbb{D}^{k+1} \to B_g(x, c\rho)$, where $\mathbb{D}^{k+1} = \{y \in \mathbb{R}^{k+1} : |y| \leq 1\}$ and $\mathbb{S}^k = \partial\mathbb{D}^{k+1}$, see Munn [69]. The non-quantitative form of this construction is due to Perelman [75], who proved that if (M, g) has nonnegative Ricci curvature and the volumes of the balls centered at a fixed point are almost maximal, then M is contractible. We will use the quantitative form of Perelman's construction, introducing

$$\alpha_{MP}(k, n) = \inf_{s \in (1,\infty)} \beta(k, s, n).$$

By construction, $\alpha_{MP}(k, n)$ is non-decreasing in k; for explicit values of $\alpha_{MP}(k, n)$, see Munn [69].

In the sequel we restrict our attention to the L^p-logarithmic Sobolev inequality $(\mathbf{LS})^p_{\mathcal{C}}$ on (M, g) with nonnegative Ricci curvature, by proving that once $\mathcal{C} > 0$ is closer and closer to the optimal Euclidean constant $\mathcal{L}_{p,n}$, the manifold (M, g) approaches topologically more and more to the Euclidean space $\mathbb{R}^n$. To state the result, let $\pi_k(M)$ be the k-th homotopy group of (M, g).

Theorem 2.6. (Kristály [109]) *Let (M, g) be an n-dimensional $(n \geq 2)$ complete Riemannian manifold with nonnegative Ricci curvature, and assume the L^p-logarithmic Sobolev inequality $(\mathbf{LS})^p_{\mathcal{C}}$ holds on (M, g) for some $p \in (1, n)$ and $\mathcal{C} > 0$. Then the following assertions hold:*

(i) $\mathcal{C} \geq \mathcal{L}_{p,n}$;

(ii) *the order of the fundamental group $\pi_1(M)$ is bounded above by* $\left(\frac{\mathcal{C}}{\mathcal{L}_{p,n}}\right)^{\frac{n}{p}}$;

(iii) *if $\mathcal{C} < \alpha_{MP}(k_0, n)^{-\frac{p}{n}} \mathcal{L}_{p,n}$ for some $k_0 \in \{1, ..., n\}$, then $\pi_1(M) = ... = \pi_{k_0}(M) = 0$;*

(iv) *if $\mathcal{C} < \alpha_{MP}(n, n)^{-\frac{p}{n}} \mathcal{L}_{p,n}$, then M is contractible;*

(v) *$\mathcal{C} = \mathcal{L}_{p,n}$ if and only if (M, g) is isometric to the Euclidean space $\mathbb{R}^n$.*

Proof. (i) It follows from Lemma 2.1/(iii), i.e., $\mathcal{C} \geq \mathcal{L}_{p,n}$.

(ii) Anderson [7] and Li [62] stated that if there exists $c_0 > 0$ such that $\text{Vol}_g(B_g(x,\rho)) \geq c_0\omega_n\rho^n$ for every $\rho > 0$, then (M,g) has finite fundamental group $\pi_1(M)$ and its order is bounded above by ${c_0}^{-1}$. Thus it remains to apply Theorem 2.4, due both to Remark 1.1 and to the fact that $|\nabla u|_{d_g} = |\nabla_g u|$, where $|\cdot|$ is the norm coming from the Riemannian metric g.

(iii) Assume that $\mathcal{C} < \alpha_{MP}(k_0,n)^{-\frac{p}{n}}\mathcal{L}_{p,n}$ for some $k_0 \in \{1,...,n\}$. By Theorem 2.4, we have that

$$\text{AVG}_{(M,g)} = \lim_{r\to\infty} \frac{\text{Vol}_g(B_g(x,r))}{\omega_n r^n} \geq \left(\frac{\mathcal{L}_{p,n}}{\mathcal{C}}\right)^{\frac{n}{p}} > \alpha_{MP}(k_0,n) \geq ... \geq \alpha_{MP}(1,n).$$

By Munn [69, Theorem 1.2], it follows that $\pi_1(M) = ... = \pi_{k_0}(M) = 0$.

(iv) If $\mathcal{C} < \alpha_{MP}(n,n)^{-\frac{p}{n}}\mathcal{L}_{p,n}$, then $\pi_1(M) = ... = \pi_n(M) = 0$, which implies the contractibility of M, see e.g. Luft [66].

(v) If $\mathcal{C} = \mathcal{L}_{p,n}$, then by Theorem 2.4 and the Bishop-Gromov volume comparison theorem follows that $\text{Vol}_g(B_g(x,\rho)) = \omega_n\rho^n$ for every $x \in M$ and $\rho > 0$. The equality in Bishop-Gromov theorem implies that (M,g) is isometric to the Euclidean space $\mathbb{R}^n$. The converse trivially holds. $\square$

Remark 2.4. In the study of heat kernel bounds on an n-dimensional complete Riemannian manifold (M,g) with nonnegative Ricci curvature, the logarithmic Sobolev inequality

$$\mathbf{Ent}_{\text{d}V_g}(u^2) \leq \frac{n}{2}\log\left(C\|\nabla_g u\|^2_{L^2(M,\text{d}V_g)}\right), \ \forall u \in C_0^\infty(M), \ \|u\|_{L^2} = 1, \tag{2.47}$$

plays a central role, where $C > 0$. In fact, (2.47) is equivalent to an upper bound of the heat kernel $p_t(x,y)$ on M, i.e.,

$$\sup_{x,y\in M} p_t(x,y) \leq \widetilde{C}t^{-\frac{n}{2}}, \ t > 0, \tag{2.48}$$

for some $\widetilde{C} > 0$. According to Theorem 2.2, the optimal constant in (2.47) for the Euclidean space $\mathbb{R}^n$ is $C = \mathcal{L}_{n,2} = \frac{2}{n\pi e}$; this scale-invariant form on $\mathbb{R}^n$ can be deduced by the famous Gross [50] logarithmic Sobolev inequality

$$\mathbf{Ent}_{\text{d}\gamma_n}(u^2) \leq 2\|\nabla u\|^2_{L^2(\mathbb{R}^n,\text{d}\gamma_n)}, \ \forall u \in C_0^\infty(\mathbb{R}^n), \ \|u\|_{L^2(\mathbb{R}^n,\text{d}\gamma_n)} = 1,$$

where the canonical Gaussian measure γ_n has the density $\delta_n(x) = (2\pi)^{-\frac{n}{2}}e^{-\frac{|x|^2}{2}}$, $x \in \mathbb{R}^n$, see Weissler [94]. Sharp estimates on the heat kernel shows that *on a complete Riemannian manifold (M,g) with nonnegative Ricci curvature the L^2-logarithmic Sobolev inequality* (2.47) *holds with the optimal Euclidean constant $C = \mathcal{L}_{n,2} = \frac{2}{n\pi e}$ if and only if (M,g) is isometric to $\mathbb{R}^n$*, cf. Bakry, Concordet and Ledoux [10], Ni [70], and Li [62]. In this case, $\widetilde{C} = (4\pi)^{-\frac{n}{2}}$ in (2.48).

Remark 2.5. In particular, Theorem 2.6/(v) gives a positive answer to the open problem of Xia [98] concerning the validity of the optimal L^p-logarithmic Sobolev inequality for generic $p \in (1,n)$ in the same geometric context as above. Xia's formulation was deeply motivated by the lack of sharp L^p-estimates ($p \neq 2$) for the heat kernel on Riemannian manifolds with nonnegative Ricci curvature.

Similar results to Theorem 2.6 can be stated also for Gagliardo-Nirenberg inequalities $(\mathbf{GN1})_{\mathcal{C}}^{\alpha,p}$ and $(\mathbf{GN2})_{\mathcal{C}}^{\alpha,p}$, and for Faber-Krahn inequality $(\mathbf{FK})_{\mathcal{C}}^{p}$ with trivial modifications. In particular, we have the next corollary.

Corollary 2.1. *Let (M,g) be an n-dimensional $(n\geq 2)$ complete Riemannian manifold with nonnegative Ricci curvature. The following statements are equivalent:*

(i) $(\mathbf{GN1})_{\mathcal{G}_{\alpha,p,n}}^{\alpha,p}$ *holds on (M,g) for some $p\in(1,n)$ and $\alpha\in\left(1,\frac{n}{n-p}\right]$;*

(ii) $(\mathbf{GN2})_{\mathcal{N}_{\alpha,p,n}}^{\alpha,p}$ *holds on (M,g) for some $p\in(1,n)$ and $\alpha\in(0,1)$;*

(iii) $(\mathbf{LS})_{\mathcal{L}_{p,n}}^{p}$ *holds on (M,g) for some $p\in(1,n)$;*

(iv) $(\mathbf{FK})_{\mathcal{F}_{p,n}}^{p}$ *holds on (M,g) for some $p\in(1,n)$;*

(v) *(M,g) is isometric to the Euclidean space $\mathbb{R}^n$.*

Remark 2.6. (a) The equivalence (i)⇔(v) in Corollary 2.1 is precisely the main result of Xia [97].

(b) A similar rigidity result to Corollary 2.1 can be stated on reversible Finsler manifolds endowed with the natural Busemann-Hausdoff measure $\mathrm{d}V_F$ of (M,F). Indeed, if (M,F) is a reversible Finsler manifold and $u\in\mathrm{Lip}_0(M)$, then relation (2.7) can be interpreted as

$$|\nabla u|_{d_F}(x)=F^*(x,Du(x))\text{ for a.e. }x\in M, \tag{2.49}$$

where $Du(x)\in T_x^*(M)$ is the distributional derivative of u at $x\in M$, see Ohta and Sturm [73]. In fact, by using Remark 1.1, we can replace the notions "Riemannian" and "Euclidean" in Corollary 2.1 by the notions "Berwald" and "Minkowski", respectively.

2.3 Interpolation inequalities on negatively curved spaces: influence of the Cartan-Hadamard conjecture

This section provides negatively curved counterparts for the results obtained in §2.2.4. To do this, let (M,g) be an n-dimensional $(n\geq 2)$ Hadamard manifold endowed with its canonical form $\mathrm{d}V_g$. By using classical Morse theory and density arguments, in order to handle Gagliardo-Nirenberg-type inequalities (and generic Sobolev inequalities) on (M,g), it is enough to consider continuous test functions $u:M\to[0,\infty)$ with compact support $S\subset M$, where S is smooth enough, u being of class C^2 in S and having only non-degenerate critical points in S.

Due to Druet, Hebey and Vaugon [35], we associate with such a function $u:M\to[0,\infty)$ its *Euclidean rearrangement function* $u^*:\mathbb{R}^n\to[0,\infty)$ which is radially symmetric, non-increasing in $|x|$, and for every $t>0$ is defined by

$$\mathrm{Vol}_e\left(\{x\in\mathbb{R}^n:u^*(x)>t\}\right)=\mathrm{Vol}_g\left(\{x\in M:u(x)>t\}\right). \tag{2.50}$$

Here, Vol_e denotes the usual n-dimensional Euclidean volume. By recalling the Croke's constant $C(n)>0$ from (1.20), the following properties are crucial in our further arguments.

Lemma 2.2. *Let (M,g) be an n-dimensional $(n \geq 2)$ Hadamard manifold. Let $u : M \to [0,\infty)$ be a non-zero function with the above properties and $u^* : \mathbb{R}^n \to [0,\infty)$ its Euclidean rearrangement function. Then the following properties hold:*

(i) Volume-preservation: $\mathrm{Vol}_g(\mathrm{supp}(u)) = \mathrm{Vol}_e(\mathrm{supp}(u^*))$;

(ii) Norm-preservation: *for every* $q \in (0,\infty]$, *we have* $\|u\|_{L^q(M)} = \|u^*\|_{L^q(\mathbb{R}^n)}$;

(iii) Pólya-Szegő inequality: *for every* $p \in (1,n)$, *one has*

$$\frac{n\omega_n^{\frac{1}{n}}}{C(n)}\|\nabla_g u\|_{L^p(M)} \geq \|\nabla u^*\|_{L^p(\mathbb{R}^n)}.$$

Moreover, if the Cartan-Hadamard conjecture holds, then

$$\|\nabla_g u\|_{L^p(M)} \geq \|\nabla u^*\|_{L^p(\mathbb{R}^n)}. \tag{2.51}$$

Proof. (i)&(ii) It is clear that u^* is a Lipschitz function with compact support, and by definition, one has $\|u\|_{L^\infty(M)} = \|u^*\|_{L^\infty(\mathbb{R}^n)}$ and $\mathrm{Vol}_g(\mathrm{supp}(u)) = \mathrm{Vol}_e(\mathrm{supp}(u^*))$. If $q \in (0,\infty)$, the layer cake representation immediately implies that $\|u\|_{L^q(M)} = \|u^*\|_{L^q(\mathbb{R}^n)}$.

(iii) We follow the arguments from Hebey [52], Ni [70] and Perelman [76]. For every $0 < t < \|u\|_{L^\infty(M)}$, we consider the level sets $\Lambda_t = u^{-1}(t) \subset S \subset M$ and $\Lambda_t^* = (u^*)^{-1}(t) \subset \mathbb{R}^n$, which are the boundaries of the sets $\{x \in M : u(x) > t\}$ and $\{x \in \mathbb{R}^n : u^*(x) > t\}$, respectively. Since u^* is radially symmetric, the set Λ_t^* is an $(n-1)$-dimensional sphere for every $0 < t < \|u\|_{L^\infty(M)}$. If Area_e denotes the usual $(n-1)$-dimensional Euclidean area, the Euclidean isoperimetric relation gives that

$$\mathrm{Area}_e(\Lambda_t^*) = n\omega_n^{\frac{1}{n}}\mathrm{Vol}_e^{\frac{n-1}{n}}(\{x \in \mathbb{R}^n : u^*(x) > t\}).$$

Due to Croke's estimate (see relation (1.19)) and (2.50), it follows that

$$\begin{aligned}\mathrm{Area}_g(\Lambda_t) &\geq C(n)\mathrm{Vol}_g^{\frac{n-1}{n}}(\{x \in M : u(x) > t\}) = C(n)\mathrm{Vol}_e^{\frac{n-1}{n}}(\{x \in \mathbb{R}^n : u^*(x) > t\}) \\ &= \frac{C(n)}{n\omega_n^{\frac{1}{n}}}\mathrm{Area}_e(\Lambda_t^*). \end{aligned} \tag{2.52}$$

If we introduce the notation

$$V(t) := \mathrm{Vol}_g(\{x \in M : u(x) > t\}) = \mathrm{Vol}_e(\{x \in \mathbb{R}^n : u^*(x) > t\}),$$

the co-area formula gives

$$V'(t) = -\int_{\Lambda_t}\frac{1}{|\nabla_g u|}\mathrm{d}\sigma_g = -\int_{\Lambda_t^*}\frac{1}{|\nabla u^*|}\mathrm{d}\sigma_e, \tag{2.53}$$

where $\mathrm{d}\sigma_g$ (resp. $\mathrm{d}\sigma_e$) denotes the natural $(n-1)$-dimensional Riemannian (resp. Lebesgue) measure induced by $\mathrm{d}V_g$ (resp. $\mathrm{d}x$). Since $|\nabla u^*|$ is constant on the sphere Λ_t^*, by the second relation of (2.53)

it turns out that
$$V'(t) = -\frac{\mathrm{Area}_e(\Lambda_t^*)}{|\nabla u^*(x)|},\ x \in \Lambda_t^*. \tag{2.54}$$
Hölder's inequality and the first relation of (2.53) imply that
$$\mathrm{Area}_g(\Lambda_t) = \int_{\Lambda_t} \mathrm{d}\sigma_g \leq \left(-V'(t)\right)^{\frac{p-1}{p}} \left(\int_{\Lambda_t} |\nabla_g u|^{p-1} \mathrm{d}\sigma_g\right)^{\frac{1}{p}}.$$
Therefore, by (2.52) and (2.54), for every $0 < t < \|u\|_{L^\infty(M)}$ and $x \in \Lambda_t^*$ we have that
$$\begin{aligned}\int_{\Lambda_t} |\nabla_g u|^{p-1} \mathrm{d}\sigma_g &\geq \mathrm{Area}_g^p(\Lambda_t)\left(-V'(t)\right)^{1-p} \geq \left(\frac{C(n)}{n\omega_n^{\frac{1}{n}}}\right)^p \mathrm{Area}_e^p(\Lambda_t^*)\left(\frac{\mathrm{Area}_e(\Lambda_t^*)}{|\nabla u^*(x)|}\right)^{1-p} \\ &= \left(\frac{C(n)}{n\omega_n^{\frac{1}{n}}}\right)^p \int_{\Lambda_t^*} |\nabla u^*|^{p-1} \mathrm{d}\sigma_e.\end{aligned}$$
The latter estimate and the co-area formula give
$$\begin{aligned}\int_M |\nabla_g u|^p \mathrm{d}V_g = \int_0^\infty \int_{\Lambda_t} |\nabla_g u|^{p-1} \mathrm{d}\sigma_g \mathrm{d}t &\geq \left(\frac{C(n)}{n\omega_n^{\frac{1}{n}}}\right)^p \int_0^\infty \int_{\Lambda_t^*} |\nabla u^*|^{p-1} \mathrm{d}\sigma_e \mathrm{d}t \\ &= \left(\frac{C(n)}{n\omega_n^{\frac{1}{n}}}\right)^p \int_{\mathbb{R}^n} |\nabla u^*|^p \mathrm{d}x,\end{aligned} \tag{2.55}$$
which concludes the first part of the proof.

If the Cartan-Hadamard conjecture holds, we can apply (1.18) instead of (1.19), obtaining instead of (2.52) that
$$\mathrm{Area}_g(\Lambda_t) \geq \mathrm{Area}_e(\Lambda_t^*) \ \text{ for every } 0 < t < \|u\|_{L^\infty(M)}, \tag{2.56}$$
and subsequently, $\int_M |\nabla_g u|^p \mathrm{d}V_g \geq \int_{\mathbb{R}^n} |\nabla u^*|^p \mathrm{d}x$, which ends the proof. □

We are in the position to state the main result of this section, where we need the notion introduced in [110]. Given a Riemannian manifold (M,g), a function $u : M \to [0,\infty)$ is *concentrated around* $x_0 \in M$, if for every $0 < t < \|u\|_{L^\infty}$ the level set $\{x \in M : u(x) > t\}$ is a geodesic ball $B_g(x_0, r_t)$ for some $r_t > 0$. Note that in the Euclidean space $\mathbb{R}^n$ the extremal function $h_{\alpha,p}^\lambda$ is concentrated around the origin, cf. Theorems 2.1 and 2.2.

Theorem 2.7. (Farkas, Kristály and Szakál [106]) *Let (M,g) be an n-dimensional $(n \geq 2)$ Hadamard manifold, $p \in (1,n)$ and $\alpha \in \left(1, \frac{n}{n-p}\right]$. Then:*

(i) *the Gagliardo-Nirenberg inequality*
$$\|u\|_{L^{\alpha p}} \leq \mathcal{C}\, \|\nabla_g u\|_{L^p}^\theta\, \|u\|_{L^{\alpha(p-1)+1}}^{1-\theta},\ \forall u \in C_0^\infty(M) \tag{$\mathbf{GN1}$$)_{\mathcal{C}}^{\alpha,p}$}$$
holds for $\mathcal{C} = \left(\frac{n\omega_n^{\frac{1}{n}}}{C(n)}\right)^\theta \mathcal{G}_{\alpha,p,n}$;

(ii) *if the Cartan-Hadamard conjecture holds on (M,g), then the optimal Gagliardo-Nirenberg inequality* $(\mathbf{GN1})^{\alpha,p}_{\mathcal{G}_{\alpha,p,n}}$ *is valid on (M,g), i.e.,*

$$\mathcal{G}^{-1}_{\alpha,p,n} = \inf_{u\in C_0^\infty(M)\setminus\{0\}} \frac{\|\nabla_g u\|^\theta_{L^p}\|u\|^{1-\theta}_{L^{\alpha(p-1)+1}}}{\|u\|_{L^{\alpha p}}}; \tag{2.57}$$

moreover, for a fixed $\alpha \in (1, \frac{n}{n-p}]$, there exists a bounded positive extremal function in $(\mathbf{GN1})^{\alpha,p}_{\mathcal{G}_{\alpha,p,n}}$ *concentrated around x_0 if and only if (M,g) is isometric to the Euclidean space $\mathbb{R}^n$.*

Proof. (i) Let $u : M \to [0,\infty)$ be an arbitrarily fixed test function with the above properties (i.e., it is continuous with a compact support $S \subset M$, S being smooth enough and u of class C^2 in S with only non-degenerate critical points in S). According to Theorem 2.1, the Euclidean rearrangement function $u^* : \mathbb{R}^n \to [0,\infty)$ of u satisfies the optimal Gagliardo-Nirenberg inequality (2.1), thus Lemma 2.2/(ii)-(iii) implies that

$$\begin{aligned}\|u\|_{L^{\alpha p}(M)} &= \|u^*\|_{L^{\alpha p}(\mathbb{R}^n)}\\ &\leq \mathcal{G}_{\alpha,p,n}\|\nabla u^*\|^\theta_{L^p(\mathbb{R}^n)}\|u^*\|^{1-\theta}_{L^{\alpha(p-1)+1}(\mathbb{R}^n)}\\ &\leq \left(\frac{n\omega_n^{\frac{1}{n}}}{C(n)}\right)^\theta \mathcal{G}_{\alpha,p,n}\|\nabla_g u\|^\theta_{L^p(M)}\|u\|^{1-\theta}_{L^{\alpha(p-1)+1}(M)}.\end{aligned}$$

(ii) If the Cartan-Hadamard conjecture holds, then a similar argument as above and (2.51) imply that

$$\begin{aligned}\|u\|_{L^{\alpha p}(M)} &= \|u^*\|_{L^{\alpha p}(\mathbb{R}^n)}\\ &\leq \mathcal{G}_{\alpha,p,n}\|\nabla u^*\|^\theta_{L^p(\mathbb{R}^n)}\|u^*\|^{1-\theta}_{L^{\alpha(p-1)+1}(\mathbb{R}^n)}\\ &\leq \mathcal{G}_{\alpha,p,n}\|\nabla_g u\|^\theta_{L^p(M)}\|u\|^{1-\theta}_{L^{\alpha(p-1)+1}(M)},\end{aligned} \tag{2.58}$$

i.e., $(\mathbf{GN1})^{\alpha,p}_{\mathcal{G}_{\alpha,p,n}}$ holds on (M,g). Moreover, Lemma 2.1 shows that $(\mathbf{GN1})^{\alpha,p}_{\mathcal{C}}$ cannot hold with $\mathcal{C} < G_{\alpha,p,n}$, which ends the proof of the optimality in (2.57).

Let us fix $\alpha \in \left(1, \frac{n}{n-p}\right]$, and assume that there exists a bounded positive extremal function $u : M \to [0,\infty)$ in $(\mathbf{GN1})^{\alpha,p}_{\mathcal{G}_{\alpha,p,n}}$ concentrated around x_0. By rescaling, we may assume that $\|u\|_{L^\infty(M)} = 1$. Since u is an extremal function, we have equalities in relation (2.58) which implies that the Euclidean rearrangement $u^* : \mathbb{R}^n \to [0,\infty)$ of u is an extremal function in the optimal Euclidean Gagliardo-Nirenberg inequality (2.1). Thus, the uniqueness (up to translation, constant multiplication and scaling) of the extremals in (2.1) and $\|u^*\|_{L^\infty(\mathbb{R}^n)} = \|u\|_{L^\infty(M)} = 1$ determine the shape of u^* which is given by $u^*(x) = \left(1 + c_0|x|^{p'}\right)^{\frac{1}{1-\alpha}}$, $x \in \mathbb{R}^n$, for some $c_0 > 0$. By construction, u^* is concentrated around the origin and for every $0 < t < 1$, we have

$$\{x \in \mathbb{R}^n : u^*(x) > t\} = B_e(0, r_t), \tag{2.59}$$

where $r_t = c_0^{-\frac{1}{p'}} \left(t^{1-\alpha}-1\right)^{\frac{1}{p'}}$. We claim that

$$\{x \in M : u(x) > t\} = B_g(x_0, r_t),\ 0 < t < 1. \tag{2.60}$$

By assumption, the function u is concentrated around x_0, thus there exists $r_t' > 0$ such that $\{x \in M : u(x) > t\} = B_g(x_0, r_t')$. We are going to prove that $r_t' = r_t$, which proves the claim.

According to (2.50) and (2.59), one has

$$\begin{aligned} \mathrm{Vol}_g(B_g(x_0, r_t')) &= \mathrm{Vol}_g(\{x \in M : u(x) > t\}) \\ &= \mathrm{Vol}_e(\{x \in \mathbb{R}^n : u^*(x) > t\}) \qquad (2.61) \\ &= \mathrm{Vol}_e(B_e(0, r_t)). \qquad (2.62) \end{aligned}$$

Furthermore, since u is an extremal function in $(\mathbf{GN1})^{\alpha,p}_{\mathcal{G}_{\alpha,p,n}}$, by the equalities in (2.58) and Lemma 2.2/(ii), it turns out that we have actually equality also in the Pólya-Szegő inequality, i.e.,

$$\|\nabla_g u\|_{L^p(M)} = \|\nabla u^*\|_{L^p(\mathbb{R}^n)}.$$

An inspection of the proof of Pólya-Szegő inequality (see Lemma 2.2/(iii)) applied to the functions u and u^* shows that we have also equality in (2.56), i.e., $\mathrm{Area}_g(\Lambda_t) = \mathrm{Area}_e(\Lambda_t^*)$, $0 < t < 1$. In particular, the latter relation, the isoperimetric equality for the pair $(\Lambda_t^*, B_0(r_t))$ and relation (2.50) imply that

$$\begin{aligned} \mathrm{Area}_g(\partial B_g(x_0, r_t')) &= \mathrm{Area}_g(\Lambda_t) = \mathrm{Area}_e(\Lambda_t^*) = n\omega_n^{\frac{1}{n}} \mathrm{Vol}_e^{\frac{n-1}{n}}(\{x \in \mathbb{R}^n : u^*(x) > t\}) \\ &= n\omega_n^{\frac{1}{n}} \mathrm{Vol}_g^{\frac{n-1}{n}}(\{x \in M : u(x) > t\}) \\ &= n\omega_n^{\frac{1}{n}} \mathrm{Vol}_g^{\frac{n-1}{n}}(B_g(x_0, r_t')). \end{aligned}$$

From the validity of the Cartan-Hadamard conjecture (in particular, from the equality case in (1.18)), the above relation implies that the open geodesic ball $\{x \in M : u(x) > t\} = B_g(x_0, r_t')$ is isometric to the n-dimensional Euclidean ball with volume $\mathrm{Vol}_g(B_g(x_0, r_t'))$. On the other hand, by relation (2.61) we actually have that the balls $B_g(x_0, r_t')$ and $B_0(r_t)$ are isometric, thus $r_t' = r_t$, proving the claim (2.60).

On account of (2.60) and (2.50), it follows that $\mathrm{Vol}_g(B_g(x_0, r_t)) = \omega_n r_t^n$, $0 < t < 1$. Since $\lim_{t\to 1} r_t = 0$ and $\lim_{t\to 0} r_t = +\infty$, the continuity of $t \mapsto r_t$ on $(0,1)$ and the latter relation imply that

$$\mathrm{Vol}_g(B_g(x_0, \rho)) = \omega_n \rho^n, \quad \forall \rho > 0. \tag{2.63}$$

By Theorem 1.1 we obtain that the sectional curvature on (M, g) is identically zero, thus (M, g) is isometric to the Euclidean space $\mathbb{R}^n$. $\square$

We state in the sequel (without proof) similar results to Theorem 2.7 concerning $(\mathbf{GN2})^{\alpha,p}_{\mathcal{C}}$, $(\mathbf{LS})^p_{\mathcal{C}}$ and $(\mathbf{FK})^p_{\mathcal{C}}$, respectively. For instance, we have the following result.

Theorem 2.8. *Let (M, g) be an n-dimensional $(n \geq 2)$ Cartan-Hadamard manifold and $p \in (1, n)$. Then:*

(i) *the L^p-logarithmic Sobolev inequality $(\mathbf{LS})^p_{\mathcal{C}}$ holds on (M, g) for $\mathcal{C} = \left(\frac{n\omega_n^{\frac{1}{n}}}{C(n)}\right)^p \mathcal{L}_{p,n}$;*

(ii) *if the Cartan-Hadamard conjecture holds on (M, g), then the optimal L^p-logarithmic Sobolev inequality $(\mathbf{LS})^p_{\mathcal{L}_{p,n}}$ is valid on (M, g), i.e.,*

$$\mathcal{L}_{p,n}^{-1} = \inf_{u \in C_0^\infty(M),\ \|u\|_{L^p}=1} \frac{\|\nabla_g u\|^p_{L^p}}{e^{\frac{p}{n}\mathbf{Ent}_{\mathrm{d}V_g}(|u|^p)}};$$

moreover, there exists a positive extremal function $u \in C_0^\infty(M)$ in $(\mathbf{LS})^p_{\mathcal{L}_{p,n}}$ concentrated around some point $x_0 \in M$ if and only if (M, g) is isometric to $\mathbb{R}^n$.

2.4 Further results and comments

I) Morrey-Sobolev interpolation inequalities on Riemannian manifolds. Let (M, g) be an n-dimensional complete Riemannian manifold and $p > n \geq 2$. For some $C > 0$, we consider on (M, g) the *Morrey-Sobolev interpolation inequality*

$$\|u\|_{L^\infty(M)} \leq C\, \|u\|^{1-\eta}_{L^1(M)}\, \|\nabla_g u\|^\eta_{L^p(M)}, \quad \forall u \in \mathrm{Lip}_0(M), \qquad (\mathbf{MS})_C$$

where

$$\eta = \frac{np}{np + p - n}. \qquad (2.64)$$

By using symmetrization and rearrangement arguments, Talenti [88] proved that if $(M, g) = (\mathbb{R}^n, e)$ is the standard Euclidean space, then $(\mathbf{MS})_{\mathsf{C}(p,n)}$ holds on $\mathbb{R}^n$ with the sharp constant

$$\mathsf{C}(p, n) = (n\omega_n^{\frac{1}{n}})^{-\frac{np'}{n+p'}} \left(\frac{1}{n} + \frac{1}{p'}\right) \left(\frac{1}{n} - \frac{1}{p}\right)^{\frac{(n-1)p'-n}{n+p'}} \left(\mathsf{B}\left(\frac{1-n}{n}p' + 1, p' + 1\right)\right)^{\frac{n}{n+p'}}.$$

By omitting the proofs, similar arguments as in Sections 2.2 and 2.3 lead to the following results.

Theorem 2.9. (Kristály [110]) *Consider the n-dimensional $(n \geq 2)$ complete Riemannian manifold (M, g) with nonnegative Ricci curvature, let $p > n$, and assume that $(\mathbf{MS})_C$ holds on (M, g) for some $C > 0$. Then the following assertions hold:*

(i) *$C \geq \mathsf{C}(p, n)$ and (M, g) has the non-collapsing volume growth property, i.e.,*

$$\mathrm{Vol}_g(B_g(x, \rho)) \geq \left(\frac{\mathsf{C}(p, n)}{C}\right)^{\frac{pn}{p-n}+1} \omega_n \rho^n, \quad \forall x \in M, \rho \geq 0;$$

(ii) *$(\mathbf{MS})_{\mathsf{C}(p,n)}$ holds on (M, g) if and only if (M, g) is isometric to $\mathbb{R}^n$.*

Theorem 2.10. (Kristály [110]) *Consider the n-dimensional $(n \geq 2)$ Hadamard manifold which verifies the Cartan-Hadamard conjecture in the same dimension, and let $p > n$.*

(i) *The Morrey-Sobolev inequality* $(\mathbf{MS})_{\mathsf{C}(p,n)}$ *holds on* (M,g); *moreover,* $\mathsf{C}(p,n)$ *is sharp, i.e.,*

$$\mathsf{C}(p,n)^{-1} = \inf_{u\in \mathrm{Lip}_0(M)\setminus\{0\}} \frac{\|u\|_{L^1(M)}^{1-\eta}\|\nabla_g u\|_{L^p(M)}^{\eta}}{\|u\|_{L^\infty(M)}},$$

where η *is given by* (2.64).

(ii) *Let* $x_0\in M$. *For every* $\kappa>0$ *there exists a nonnegative extremal function* $u\in \mathrm{Lip}_0(M)$ *in* $(\mathbf{MS})_{\mathsf{C}(p,n)}$ *concentrated around* x_0 *and* $\mathcal{H}^n(\mathrm{sprt}\ u)=\kappa$ *if and only if* (M,g) *is isometric to* $\mathbb{R}^n$.

II) Second-order Sobolev inequalities on Riemannian manifolds with nonnegative Ricci curvature. Let (M,g) be an n-dimensional $(n\geq 5)$ complete Riemannian manifold. For some $C>0$, we consider the *second-order Sobolev inequality*

$$\left(\int_M |u|^{2^\sharp}\mathrm{d}V_g\right)^{\frac{2}{2^\sharp}} \leq C\int_M (\Delta_g u)^2\,\mathrm{d}V_g,\quad \forall u\in C_0^\infty(M), \qquad (\mathbf{SS})_C$$

where $2^\sharp=\frac{2n}{n-4}$ is the second-order critical Sobolev exponent. Note that the Euclidean space $\mathbb{R}^n$ supports $(\mathbf{SS})_{K_0}$ for

$$K_0=\left[\pi^2 n(n-4)(n^2-4)\right]^{-1}\left(\frac{\Gamma(n)}{\Gamma\left(\frac{n}{2}\right)}\right)^{4/n}.$$

Moreover, K_0 is optimal, see Edmunds, Fortunato and Janelli [36], and the unique class of extremal functions is $u_{\lambda,x_0}(x)=\left(\lambda+|x-x_0|^2\right)^{\frac{4-n}{2}}$, $x\in\mathbb{R}^n$, where $\lambda>0$ and $x_0\in\mathbb{R}^n$ are arbitrarily fixed.

The only result in the second-order case reads as follows, whose proof follows the line of Theorem 2.3.

Theorem 2.11. (Barbosa and Kristály [103]) *Let* (M,g) *be an* n*-dimensional* $(n\geq 5)$ *complete Riemannian manifold with nonnegative Ricci curvature which satisfies the distance Laplacian growth condition*

$$d_{x_0}\Delta_g d_{x_0}\geq n-5$$

for some $x_0\in M$. *Assume that* (M,g) *supports the second-order Sobolev inequality* $(\mathbf{SS})_C$ *for some* $C>0$. *Then the following properties hold:*

(i) $C\geq K_0$;

(ii) *if in addition* $C\leq \frac{n+2}{n-2}K_0$, *then we have the global volume non-collapsing property*

$$\mathrm{Vol}_g(B_g(x,\rho))\geq (C^{-1}K_0)^{\frac{n}{4}}\omega_n\rho^n,\ \forall x\in M, \rho>0.$$

We conclude the present chapter with some comments and remarks.

III) Non-smooth versus smooth settings. In Section 2.2 we were able to treat interpolation inequalities on any non-smooth metric measure space (M,d,m) verifying the $\mathsf{CD}(K,N)$ condition,

$K \geq 0$. Here, one of the key facts was the *eikonal inequality* $|\nabla d(x_0,\cdot)|_d(x) \leq 1$ for all $x \in M \setminus \{x_0\}$, which is a purely metric relation.

At this point, a natural question arises concerning the validity of (sharp) functional inequalities on generic metric measure spaces which are nonpositively curved (e.g. in the sense of Aleksandrov or Busemann), see Jost [55]. In particular, a purely metric measure approach to this subject – assuming we follow the same line as above – requires a deeper understanding of the following two issues at least. Firstly, we substantially exploited the *co-area formula*; on a generic metric measure space (M, d, m) it is well-known a co-area inequality involving the quantity $|\nabla u|_d$, $u \in \mathrm{Lip}_0(M)$, see Bobkov and Houdré [15, Lemma 3.1], but the equality case requires some regularity of the measure (which are clearly valid on Riemannian and Finsler manifolds with their canonical volume forms). Secondly, the Croke-type isoperimetric inequality (1.19) is indispensable in our arguments; in the generic case, certain restrictions should be made on the *isoperimetric profile* of the metric measure space we are working on; a possible starting point could be the recent works by Martín and Milman [67, 68].

IV) First-order versus higher-order Sobolev inequalities on non-Euclidean structures. With respect to first-order Sobolev inequalities, much less is known about higher-order Sobolev inequalities on curved spaces.

Let (M, g) be an n-dimensional complete Riemannian manifold with nonnegative Ricci curvature and fix $k \in \mathbb{N}$ such that $n > 2k$. Let us consider for some $C > 0$ the k-th order Sobolev inequality

$$\left(\int_M |u|^{\frac{2n}{n-2k}} \mathrm{d}V_g\right)^{\frac{n-2k}{n}} \leq C \int_M (\Delta_g^{k/2} u)^2 \mathrm{d}V_g, \ \forall u \in C_0^\infty(M), \tag{$\mathbf{S}$}_C^k$$

where

$$\Delta_g^{k/2} u = \begin{cases} \Delta_g^{k/2} u, & \text{if } \ k \text{ is even}, \\ |\nabla_g(\Delta_g^{(k-1)/2} u)|, & \text{if } \ k \text{ is odd}. \end{cases}$$

Clearly, we have that $(\mathbf{S})^1_{C^2} = (\mathbf{GN1})_C^{\frac{n}{n-2},2}$ and $(\mathbf{S})^2_C = (\mathbf{SS})_C$. It is far to be clear how is it possible to establish k-th order counterparts of Theorems 2.3 and 2.11 with $k \geq 3$. We note that the optimal Euclidean k-th order Sobolev inequalities are well-known with the optimal constant

$$\Lambda_k = \left[\pi^k n(n-2k)\Pi_{i=1}^{k-1}(n^2 - 4i^2)\right]^{-1} \left(\frac{\Gamma(n)}{\Gamma\left(\frac{n}{2}\right)}\right)^{2k/n},$$

and the unique class of extremals (up to translations and multiplications) is $u_\lambda(x) = \left(\lambda + |x^2|\right)^{\frac{2k-n}{2}}$, $x \in \mathbb{R}^n$, see Cotsiolis and Tavoularis [27], Liu [64]. Once we use $w_\lambda = \left(\lambda + d_{x_0}^2\right)^{\frac{2k-n}{2}}$ for some $x_0 \in M$ as a test-function in $(\mathbf{S})^k_C$, after a multiple application of the chain rule we have to estimate in a sharp way the terms appearing in $\Delta_g^{k/2} w_\lambda$, similar to the eikonal equation $|\nabla_g d_{x_0}| = 1$ and the distance Laplacian comparison $d_{x_0}\Delta_g d_{x_0} \leq n-1$, respectively. To the best of our knowledge, only Theorem 2.11 is available in the literature for a higher-order case on Riemannian manifolds with nonnegative Ricci curvature. In this result the distance Laplacian growth condition $d_{x_0}\Delta_g d_{x_0} \geq n-5$ for some

$x_0 \in M$ is indispensable which shows the genuine second-order character of the studied problem. We note that the first-order counterpart of this condition is the eikonal equation (or eikonal inequality), which trivially holds on any complete Riemannian manifold (or on any metric measure space).

On the other hand, no result is available in the literature for higher-order Sobolev inequalities on Hadamard manifolds, similarly to the results from Section 2.3. The obstacle to extend first-order arguments to higher-order ones is the lack of a suitable Pólya-Szegő inequality for symmetrization. In particular, if (M,g) is an n-dimensional ($n \geq 2$) Hadamard manifold, and $u : M \to [0,\infty)$ is a non-zero function with $u^* : \mathbb{R}^n \to [0,\infty)$ its Euclidean rearrangement function (see Lemma 2.2), we *cannot compare* the terms $\int_M (\Delta_g u)^2 \mathrm{d}V_g$ and $\int_{\mathbb{R}^n} (\Delta u^*)^2 \mathrm{d}x$.

V) Finsler versus Riemannian settings. According to relation (2.49), the global volume non-collapsing properties in Theorems 2.3, 2.4 and 2.5 can be formulated in the Finsler context. However, similar results to those from §2.2.4 concerning rigidities via Munn-Perelman homotopic quantification seems to be valid only in the Riemannian setting. Similar fact is valid concerning sharp interpolation inequalities in the negatively curved setting, see Theorem 2.7, due to the Riemannian character of the Cartan-Hadamard conjecture.

VI) Sharp Sobolev-type inequalities on Riemannian manifolds versus distortion coefficients. Let (M,g) be an n-dimensional ($n \geq 2$) complete non-compact Riemannian manifold. The main challenging question is to find

$$C_{opt}(M,g)^{-1} = \inf_{u \in C_0^\infty(M)\setminus\{0\}} \frac{\|\nabla_g u\|_{L^p}^{\theta} \|u\|_{L^{\alpha(p-1)+1}}^{1-\theta}}{\|u\|_{L^{\alpha p}}},$$

where a, p and θ are from Theorem 2.1. It seems that the constant $C_{opt}(M,g)$ encodes a lot of geometric information about (M,g); indeed, summarizing the results of the present chapter, we know that:

- $C_{opt}(M,g) \geq \mathcal{G}_{\alpha,p,n}$ for any Riemannian manifold (see Lemma 2.1).
- $C_{opt}(M,g) = \mathcal{G}_{\alpha,p,n}$ whenever (M,g) is a Hadamard manifold verifying the Cartan-Hadamard conjecture (see Theorem 2.7);
- $C_{opt}(M,g) > \mathcal{G}_{\alpha,p,n}$ whenever the Ricci curvature is nonnegative and (M,g) is not isometric to $\mathbb{R}^n$ (see Theorem 2.3 and Corollary 2.1).

Although the aforementioned problem seems to be almost impossibly to be resolved in its full generality, some preliminary analysis shows that the *distortion coefficients* of (M,g) should play crucial roles in this study, introduced by Cordero-Erausquin, McCann and Schmuckenschläger [23] and successfully explored in [102] to establish sharp geometric inequalities in the sub-Riemannian setting of the Heisenberg group.

Chapter 3

Sharp uncertainty principles

Uncertainty principles appear in quantum mechanics by simultaneously studying the position and momentum of a given particle. In this chapter we investigate the influence of the curvature on sharp uncertainty principles on Riemannian/Finsler manifolds.

3.1 Uncertainty principles in the flat case: a short overview

Let $p, q \in \mathbb{R}$ and $n \in \mathbb{N}$ be such that

$$0 < q < 2 < p \text{ and } 2 < n < \frac{2(p-q)}{p-2}, \tag{3.1}$$

and denote by $\|\cdot\|$ an arbitrary norm in $\mathbb{R}^n$ and its dual $\|\cdot\|_*$, see Section 2.1. In what follows, we consider the *Caffarelli-Kohn-Nirenberg inequality* (see [19]), i.e.,

$$\left(\int_{\mathbb{R}^n} \|\nabla u(x)\|_*^2 \mathrm{d}x\right) \left(\int_{\mathbb{R}^n} \frac{|u(x)|^{2p-2}}{\|x\|^{2q-2}} \mathrm{d}x\right) \geq \frac{(n-q)^2}{p^2} \left(\int_{\mathbb{R}^n} \frac{|u(x)|^p}{\|x\|^q} \mathrm{d}x\right)^2, \quad \forall u \in C_0^\infty(\mathbb{R}^n). \tag{\textbf{CKN}}$$

One can prove directly the next property.

Theorem 3.1. (Xia [98]) *The constant* $\frac{(n-q)^2}{p^2}$ *is sharp in* (**CKN**) *and the class of extremals* $u_\lambda(x) = \left(\lambda + \|x\|^{2-q}\right)^{\frac{1}{2-p}}$, $\lambda > 0$, *is unique up to scaling factors and translations.*

One of the *endpoints* of (**CKN**) (when $p \to 2$ and $q \to 0$) is the *Heisenberg-Pauli-Weyl principle*

$$\left(\int_{\mathbb{R}^n} \|\nabla u(x)\|_*^2 \mathrm{d}x\right) \left(\int_{\mathbb{R}^n} \|x\|^2 u^2(x) \mathrm{d}x\right) \geq \frac{n^2}{4} \left(\int_{\mathbb{R}^n} u^2(x) \mathrm{d}x\right)^2, \ \forall u \in C_0^\infty(\mathbb{R}^n). \tag{\textbf{HPW}}$$

The Heisenberg-Pauli-Weyl uncertainty principle in quantum mechanics states that the position and momentum of a given particle cannot be accurately determined simultaneously, see Heisenberg [54]. **(HPW)** is the PDE formulation of this principle whose rigorous mathematical formulation is attributed to Pauli and Weyl [95]. It is also known the following result.

Theorem 3.2. *The constant* $\frac{n^2}{4}$ *is sharp in* **(HPW)** *and the class of extremals, provided by the Gaussian functions* $u_\lambda(x) = e^{-\lambda\|x\|^2}$, $\lambda > 0$, *is unique up to scaling factors and translations.*

Another *endpoint* of (**CKN**) (when $p \to 2$ and $q \to 2$) is the famous *Hardy-Poincaré uncertainty principle*

$$\int_{\mathbb{R}^n} \|\nabla u(x)\|_*^2 \mathrm{d}x \geq \frac{(n-2)^2}{4} \int_{\mathbb{R}^n} \frac{u^2(x)}{\|x\|^2} \mathrm{d}x, \quad \forall u \in C_0^\infty(\mathbb{R}^n). \tag{\textbf{HP}}$$

One of the milestones of singular PDEs is the following result (see e.g. Adimurthi, Chaudhuri and Ramaswamy [1], Barbatis, Filippas and Tertikas [12], Brezis and Vázquez [17], Filippas and Tertikas [43], Ghoussoub and Moradifam [48, 49]).

Theorem 3.3. *The constant* $\frac{(n-2)^2}{4}$ *is sharp in* (**HP**)*, but there are no extremal functions.*

3.2 Heisenberg-Pauli-Weyl uncertainty principle on Riemannian manifolds

Since its initial formulation, the Heisenberg-Pauli-Weyl principle is deserving continuously a deep source of inspiration in mathematical physics. Without the sake of completeness, the Heisenberg-Pauli-Weyl principle has been studied in various contexts, among others by Erb [37, 38] and Kombe and Özaydin [58, 59] on compact/complete Riemannian manifolds.

Let (M, g) be an n-dimensional ($n \geq 2$) complete non-compact Riemannian manifold, and $\rho : M \to \mathbb{R}$ be a function such that $|\nabla_g \rho| = 1$ and $\rho \Delta_g \rho \geq C$ for some $C > 0$. In this setting, Kombe and Özaydin [58, 59] proved that

$$\left(\int_M |\nabla_g u|^2 \mathrm{d}V_g\right)\left(\int_M \rho^2 u^2 \mathrm{d}V_g\right) \geq \frac{(C+1)^2}{4}\left(\int_M u^2 \mathrm{d}V_g\right)^2, \ \forall u \in C_0^\infty(M). \tag{3.2}$$

In the Euclidean case, if $\rho = |\cdot|$, then $\Delta|\cdot| = \frac{n-1}{|\cdot|}$ and $C = n-1$, thus (3.2) becomes precisely (**HPW**). When (M, g) is the n-dimensional hyperbolic case, inequality (3.2) also holds for $C = n-1$. In the latter case, Kombe and Özaydin [59] claimed that $\frac{n^2}{4}$ is also sharp and $u = e^{-\lambda d^2}$ is an extremal, where d is the hyperbolic distance. It turns out that this statement is false, as we will explain in §3.2.2.

Accordingly, the purpose of the present section is to describe a complete scenario concerning the sharp Heisenberg-Pauli-Weyl uncertainty principle on complete Riemannian manifolds. To do this, for $x_0 \in M$ fixed, we consider the *Heisenberg-Pauli-Weyl uncertainty principle* on (M, g) as

$$\left(\int_M |\nabla_g u|^2 \mathrm{d}V_g\right)\left(\int_M d_{x_0}^2 u^2 \mathrm{d}V_g\right) \geq \frac{n^2}{4}\left(\int_M u^2 \mathrm{d}V_g\right)^2, \quad \forall u \in C_0^\infty(M). \tag{$\textbf{HPW}_{x_0}$}$$

3.2.1 Positively curved case: strong rigidity

Theorem 3.4. (Kristály [108]) *Let (M, g) be an n-dimensional ($n \geq 2$) complete Riemannian manifold with nonnegative Ricci curvature. The following statements are equivalent:*

(a) $(\textbf{HPW})_{x_0}$ *holds for* some $x_0 \in M$;

(b) $(\textbf{HPW})_{x_0}$ *holds for* every $x_0 \in M$;

(c) (M, g) *is isometric to* $\mathbb{R}^n$.

Proof. Implications (c)⇒(b)⇒(a) trivially hold. The proof of the implication (a)⇒(c) is divided into four steps. Let $x_0 \in M$ be fixed.

STEP 1. If (M,g) is isometric to $\mathbb{R}^n$, then $(\mathbf{HPW})_{x_0}$ can be transformed into the inequality $(\mathbf{HPW})$ for which the standard class of Gaussian functions are extremals, see Theorem 3.2.

For later use, if we consider the function $T:(0,\infty)\to\mathbb{R}$ defined by

$$T(\lambda)=\int_{\mathbb{R}^n} e^{-2\lambda|x|^2}\mathrm{d}x,\ \lambda>0,$$

the equality for the Gaussian extremals in Theorem 3.2 can be rewritten into the form

$$-\lambda T'(\lambda)=\frac{n}{2}T(\lambda),\ \lambda>0. \tag{3.3}$$

Moreover, by the layer cake representation and a change of variables, one has the following representations which are used later:

$$T(\lambda)=4\lambda\omega_n\int_0^\infty \rho^{n+1}e^{-2\lambda\rho^2}\mathrm{d}\rho=\frac{2}{(2\lambda)^{\frac{n}{2}}}\omega_n\int_0^\infty t^{n+1}e^{-t^2}\mathrm{d}t. \tag{3.4}$$

STEP 2. Since $(\mathbf{HPW})_{x_0}$ holds, (M,g) cannot be compact. We consider the class of functions

$$\widetilde{u}_\lambda(x)=e^{-\lambda d_{x_0}^2(x)},\ \lambda>0.$$

Clearly, the function $\widetilde{u}_\lambda$ can be approximated by elements from $C_0^\infty(M)$ for every $\lambda>0$. By inserting $\widetilde{u}_\lambda$ into $(\mathbf{HPW})_{x_0}$, and by using the eikonal equation $|\nabla_g d_{x_0}|=1$ a.e. on M, we obtain the inequality

$$2\lambda\int_M d_{x_0}^2 e^{-2\lambda d_{x_0}^2}\mathrm{d}V_g\geq\frac{n}{2}\int_M e^{-2\lambda d_{x_0}^2}\mathrm{d}V_g,\ \lambda>0. \tag{3.5}$$

We introduce the function $\mathscr{T}:(0,\infty)\to\mathbb{R}$ defined by

$$\mathscr{T}(\lambda)=\int_M e^{-2\lambda d_{x_0}^2}\mathrm{d}V_g,\ \lambda>0.$$

By the layer cake representation, $\mathscr{T}$ can be equivalently rewritten into

$$\begin{aligned}\mathscr{T}(\lambda)&=\int_0^\infty \mathrm{Vol}_g\left(\left\{x\in M: e^{-2\lambda d_{x_0}^2}>t\right\}\right)\mathrm{d}t=\int_0^1 \mathrm{Vol}_g\left(\left\{x\in M: e^{-2\lambda d_{x_0}^2}>t\right\}\right)\mathrm{d}t\\&=4\lambda\int_0^\infty \mathrm{Vol}_g(B_g(x_0,\rho))\rho e^{-2\lambda\rho^2}\mathrm{d}\rho.\end{aligned}$$

Since the Ricci curvature is nonnegative, on account of (1.10), the function $\mathscr{T}$ is well-defined and differentiable. Thus, relation (3.5) is equivalent to

$$-\lambda\mathscr{T}'(\lambda)\geq\frac{n}{2}\mathscr{T}(\lambda),\ \lambda>0. \tag{3.6}$$

STEP 3. We shall prove that
$$\mathscr{T}(\lambda) \geq T(\lambda),\ \forall \lambda > 0. \tag{3.7}$$
By (3.3) and (3.6) it turns out that
$$\frac{\mathscr{T}'(\lambda)}{\mathscr{T}(\lambda)} \leq \frac{T'(\lambda)}{T(\lambda)},\ \forall \lambda > 0.$$
Integrating this inequality, it yields that the function $\lambda \mapsto \frac{\mathscr{T}(\lambda)}{T(\lambda)}$ is non-increasing; in particular,
$$\frac{\mathscr{T}(\lambda)}{T(\lambda)} \geq \liminf_{\lambda\to\infty} \frac{\mathscr{T}(\lambda)}{T(\lambda)},\ \forall \lambda > 0. \tag{3.8}$$
Now, we prove that
$$\liminf_{\lambda\to\infty} \frac{\mathscr{T}(\lambda)}{T(\lambda)} \geq 1. \tag{3.9}$$
Due to relation (1.8), for every $\varepsilon > 0$ one can find $\rho_\varepsilon > 0$ such that
$$\mathrm{Vol}_g(B_g(x_0,\rho)) \geq (1-\varepsilon)\omega_n\rho^n,\ \forall \rho \in [0,\rho_\varepsilon].$$
Consequently, one has
$$\begin{aligned}
\mathscr{T}(\lambda) &= 4\lambda \int_0^\infty \mathrm{Vol}_g(B_g(x_0,\rho))\rho e^{-2\lambda\rho^2}\mathrm{d}\rho \\
&\geq 4\lambda(1-\varepsilon)\omega_n \int_0^{\rho_\varepsilon} \rho^{n+1}e^{-2\lambda\rho^2}\mathrm{d}\rho \\
&= \frac{2}{(2\lambda)^{\frac{n}{2}}}(1-\varepsilon)\omega_n \int_0^{\sqrt{2\lambda}\rho_\varepsilon} t^{n+1}e^{-t^2}\mathrm{d}t. \qquad [\text{change } \sqrt{2\lambda}\rho = t]
\end{aligned}$$
Now, by (3.4), it yields that
$$\liminf_{\lambda\to\infty} \frac{\mathscr{T}(\lambda)}{T(\lambda)} \geq 1-\varepsilon.$$
Since $\varepsilon > 0$ is arbitrary, relation (3.9) holds. This ends the proof of the claim (3.7).

STEP 4. Due to (3.4), relation (3.7) is equivalent to
$$\int_0^\infty \left(\mathrm{Vol}_g(B_g(x_0,\rho)) - \omega_n\rho^n\right)\rho e^{-2\lambda\rho^2}\mathrm{d}\rho \geq 0,\ \forall \lambda > 0.$$
On account of (1.10), we necessarily have that
$$\mathrm{Vol}_g(B_g(x_0,\rho)) = \omega_n\rho^n,\ \forall \rho > 0.$$
Standard arguments show that the latter relation does not depend on $x_0 \in M$, thus by the equality in Theorem 1.1/(b) we have that the sectional curvature is identically zero, which conludes the proof. □

3.2.2 Negatively curved case: curvature versus extremals

Based on (1.7), for every $c \leq 0$, let $\mathbf{D}_c : [0,\infty) \to \mathbb{R}$ be defined by

$$\mathbf{D}_c(\rho) = \begin{cases} 0, & \text{if} \quad \rho = 0, \\ \rho \mathbf{ct}_c(\rho) - 1, & \text{if} \quad \rho > 0. \end{cases}$$

Note that $\mathbf{D}_c \geq 0$. At first, we present a quantitative version of the Heisenberg-Pauli-Weyl principle.

Theorem 3.5. (Kristály [108])] *Let (M,g) be an n-dimensional $(n \geq 2)$ Hadamard manifold such that the sectional curvature is bounded from above by $c \leq 0$. Then*

$$\left(\int_M |\nabla_g u|^2 \mathrm{d}V_g\right)\left(\int_M d_{x_0}^2 u^2 \mathrm{d}V_g\right) \geq \frac{n^2}{4}\left(\int_M \left(1 + \frac{n-1}{n}\mathbf{D}_c(d_{x_0})\right) u^2 \mathrm{d}V_g\right)^2, \ \forall x_0 \in M, u \in C_0^\infty(M).$$

Proof. Let $x_0 \in M$ and $u \in C_0^\infty(M)$ be fixed arbitrarily. According to Theorem 1.2/(a), one has that

$$\begin{aligned} \int_M \Delta_g(d_{x_0}^2) u^2 \mathrm{d}V_g &= 2\int_M (1 + d_{x_0}\Delta_g d_{x_0}) u^2 \mathrm{d}V_g \\ &\geq 2\int_M (1 + (n-1) d_{x_0}\mathbf{ct}_c(d_{x_0})) u^2 \mathrm{d}V_g \\ &= 2n\int_M \left(1 + \frac{n-1}{n}\mathbf{D}_c(d_{x_0})\right) u^2 \mathrm{d}V_g. \end{aligned} \tag{3.10}$$

An integration by parts yields the equality

$$\int_M \Delta_g(d_{x_0}^2) u^2 \mathrm{d}V_g = -\int_M \langle \nabla_g(u^2), \nabla_g(d_{x_0}^2)\rangle \mathrm{d}V_g = -4\int_M u d_{x_0}\langle \nabla_g u, \nabla_g d_{x_0}\rangle \mathrm{d}V_g.$$

By using the eikonal equation $|\nabla_g d_{x_0}| = 1$ a.e. on M, one has that $|\langle \nabla_g u, \nabla_g d_{x_0}\rangle| \leq |\nabla_g u|$. Thus, by Hölder inequality one obtains that

$$\left(\int_M u d_{x_0}\langle \nabla_g u, \nabla_g d_{x_0}\rangle \mathrm{d}V_g\right)^2 \leq \left(\int_M d_{x_0}^2 u^2 \mathrm{d}V_g\right)\left(\int_M |\nabla_g u|^2 \mathrm{d}V_g\right).$$

The latter relation coupled with (3.10) yields the quantitative Heisenberg-Pauli-Weyl principle. □

The main result of this subsection reads as follows.

Theorem 3.6. (Kristály [108]) *Let (M,g) be an n-dimensional $(n \geq 2)$ Hadamard manifold.*

(i) [Sharpness] *The Heisenberg-Pauli-Weyl principle $(\mathbf{HPW})_{x_0}$ holds for every $x_0 \in M$; moreover, $\frac{n^2}{4}$ is sharp, i.e.,*

$$\frac{n^2}{4} = \inf_{u \in C_0^\infty(M)\setminus\{0\}} \frac{\left(\int_M |\nabla_g u|^2 \mathrm{d}V_g\right)\left(\int_M d_{x_0}^2 u^2 \mathrm{d}V_g\right)}{\left(\int_M u^2 \mathrm{d}V_g\right)^2}.$$

(ii) [Extremals] *The following statements are equivalent:*

(a) $\frac{n^2}{4}$ *is attained by a positive extremal in* $(\mathbf{HPW})_{x_0}$ *for* some $x_0 \in M$;

(b) $\frac{n^2}{4}$ *is attained by a positive extremal in* $(\mathbf{HPW})_{x_0}$ *for* every $x_0 \in M$;

(c) (M, g) *is isometric to* $\mathbb{R}^n$.

Proof. (i) Let $x_0 \in M$ be fixed. Since $\mathbf{D}_c \geq 0$, due to Theorem 3.5, the Heisenberg-Pauli-Weyl uncertainty principle $(\mathbf{HPW})_{x_0}$ holds. We prove that the constant $\frac{n^2}{4}$ is optimal in $(\mathbf{HPW})_{x_0}$, by following the Aubin-Hebey argument, see [8], [52], and the arguments from Lemma 2.1. Let

$$\mathsf{C}_{\mathsf{HPW}} = \inf_{u \in C_0^\infty(M) \setminus \{0\}} \frac{\left(\int_M |\nabla_g u|^2 \mathrm{d}V_g\right)\left(\int_M d_{x_0}^2 u^2 \mathrm{d}V_g\right)}{\left(\int_M u^2 \mathrm{d}V_g\right)^2}. \tag{3.11}$$

Since $(\mathbf{HPW})_{x_0}$ holds, then $\mathsf{C}_{\mathsf{HPW}} \geq \frac{n^2}{4}$. Assume that $\mathsf{C}_{\mathsf{HPW}} > \frac{n^2}{4}$. By (3.11), one has

$$\left(\int_M |\nabla_g u|^2 \mathrm{d}V_g\right)\left(\int_M d_{x_0}^2 u^2 \mathrm{d}V_g\right) \geq \mathsf{C}_{\mathsf{HPW}} \left(\int_M u^2 \mathrm{d}V_g\right)^2, \ \forall u \in C_0^\infty(M). \tag{3.12}$$

For every $\varepsilon > 0$, there exists a local chart (Ω, φ) of M at x_0 and a number $\delta > 0$ such that $\varphi(\Omega) = B_e(0, \delta)$, while the components g_{ij} of the metric g satisfy in the sense of bilinear forms the inequalities

$$(1 - \varepsilon)\delta_{ij} \leq g_{ij} \leq (1 + \varepsilon)\delta_{ij}. \tag{3.13}$$

According to (3.12) and (3.13), for $\varepsilon > 0$ small enough, there exists $\widetilde{\delta} > 0$ and $\mathsf{C}'_{\mathsf{HPW}} > \frac{n^2}{4}$ such that for every $\delta \in (0, \widetilde{\delta})$ and $w \in C_0^\infty(B_e(0, \delta))$,

$$\left(\int_{B_e(0,\delta)} |\nabla w|^2 \mathrm{d}x\right)\left(\int_{B_e(0,\delta)} |x|^2 w^2 \mathrm{d}x\right) \geq \mathsf{C}'_{\mathsf{HPW}} \left(\int_{B_e(0,\delta)} w^2 \mathrm{d}x\right)^2. \tag{3.14}$$

Let $u \in C_0^\infty(\mathbb{R}^n)$ be arbitrarily fixed and set $w_\lambda(x) = u(\lambda x)$, $\lambda > 0$. It is clear that $w_\lambda \in C_0^\infty(B_e(0, \delta))$ for large enough $\lambda > 0$. Inserting w_λ into (3.14), and recalling the scaling properties

$$\int_{B_e(0,\delta)} |\nabla w_\lambda|^2 \mathrm{d}x = \lambda^{2-n} \int_{\mathbb{R}^n} |\nabla u|^2 \mathrm{d}x, \ \int_{B_e(0,\delta)} |x|^2 w_\lambda^2 \mathrm{d}x = \lambda^{-2-n} \int_{\mathbb{R}^n} |x|^2 u^2 \mathrm{d}x,$$

and

$$\int_{B_e(0,\delta)} w_\lambda^2 \mathrm{d}x = \lambda^{-n} \int_{\mathbb{R}^n} u^2 \mathrm{d}x,$$

it follows that

$$\left(\int_{\mathbb{R}^n} |\nabla u|^2 \mathrm{d}x\right)\left(\int_{\mathbb{R}^n} |x|^2 u^2 \mathrm{d}x\right) \geq \mathsf{C}'_{\mathsf{HPW}} \left(\int_{\mathbb{R}^n} u^2 \mathrm{d}x\right)^2.$$

In particular, in the latter relation we may substitute the Gaussian function $u(x) = e^{-|x|^2}$, obtaining that $\frac{n^2}{4} \geq \mathsf{C}'_{\mathsf{HPW}}$, a contradiction. Consequently, $\mathsf{C}_{\mathsf{HPW}} = \frac{n^2}{4}$.

(ii) Observe that if (M,g) is isometric to $\mathbb{R}^n$, the sharp Heisenberg-Pauli-Weyl uncertainty principle $(\mathbf{HPW})_{x_0}$ can be equivalently transformed into $(\mathbf{HPW})$ for which the Gaussians $u_\lambda(x) = e^{-\lambda|x|^2}$, $\lambda > 0$, are extremal functions. Thus, the implications (c)$\Rightarrow$(b)$\Rightarrow$(a) hold true.

We now prove (a)$\Rightarrow$(c). Let $u_0 > 0$ be an extremal function in $(\mathbf{HPW})_{x_0}$ for some $x_0 \in M$. In particular, in the estimates in Theorem 3.5 we should have equalities; thus, by (3.10) one has $\mathbf{D}_c \equiv 0$ (i.e., we necessarily have $c = 0$, so the sectional curvature of (M,g) cannot be bounded above by a fixed negative number), and

$$\Delta_g(d_{x_0}^2) = 2n. \tag{3.15}$$

Let us fix $\rho > 0$ arbitrarily. Note that the unit outward pointing normal vector to the sphere $S_g(x_0,\rho) = \partial B_g(x_0,\rho) = \{x \in M : d_g(x_0,x) = \rho\}$ is $\mathbf{n} = \nabla_g d_{x_0}$. Denote by $\mathrm{d}\varsigma_g$ the volume form on $S_g(x_0,\rho)$ induced by $\mathrm{d}V_g$. Applying Stokes' formula and the fact that $\langle \mathbf{n}, \mathbf{n}\rangle = 1$, by (3.15) we have

$$\begin{aligned} 2n\mathrm{Vol}_g(B_g(x_0,\rho)) &= \int_{B_g(x_0,\rho)} \Delta_g(d_{x_0}^2)\mathrm{d}V_g = \int_{B_g(x_0,\rho)} \mathrm{div}(\nabla_g(d_{x_0}^2))\mathrm{d}V_g = \int_{S_g(x_0,\rho)} \langle \mathbf{n}, \nabla_g(d_{x_0}^2)\rangle \mathrm{d}\varsigma_g \\ &= 2\int_{S_g(x_0,\rho)} d_{x_0}\langle \mathbf{n}, \nabla_g d_{x_0}\rangle \mathrm{d}\varsigma_g = 2\rho\int_{S_g(x_0,\rho)} \langle \mathbf{n}, \mathbf{n}\rangle \mathrm{d}\varsigma_g = 2\rho\int_{S_g(x_0,\rho)} \mathrm{d}\varsigma_g \\ &= 2\rho\mathrm{Area}_g(S_g(x_0,\rho)), \end{aligned}$$

where

$$\mathrm{Area}_g(S_g(x_0,\rho)) = \lim_{\varepsilon\to 0^+} \frac{\mathrm{Vol}_g(B_g(x_0,\rho+\varepsilon)) - \mathrm{Vol}_g(B_g(x_0,\rho)}{\varepsilon} := \frac{\mathrm{d}}{\mathrm{d}\rho}\mathrm{Vol}_g(B_g(x_0,\rho))$$

is the surface area of $S_g(x_0,\rho)$. Thus, the above relations imply that

$$\frac{\frac{\mathrm{d}}{\mathrm{d}\rho}\mathrm{Vol}_g(B_g(x_0,\rho)}{\mathrm{Vol}_g(B_g(x_0,\rho))} = \frac{n}{\rho}.$$

By integrating this expression and due to relation (1.8), we conclude that

$$\mathrm{Vol}_g(B_g(x_0,\rho)) = \omega_n\rho^n, \ \forall \rho > 0. \tag{3.16}$$

The equality case of Theorem 1.1/(a) implies that the sectional curvature on (M,g) is identically zero, which concludes the proof. $\square$

Remark 3.1. Implication (a)$\Rightarrow$(c) in Theorem 3.6 has also a geometric proof. Indeed, due to Jost [55, Lemma 2.1.5] and relation (3.15), it follows that we have equality in the CAT(0)-inequality with the reference point $x_0 \in M$, i.e., for *every* geodesic segment $\gamma : [0,1] \to M$ and $s \in [0,1]$, we have that

$$d_g^2(x_0,\gamma(s)) = (1-s)d_g^2(x_0,\gamma(0)) + sd_g^2(x_0,\gamma(1)) - s(1-s)d_g^2(\gamma(0),\gamma(1)).$$

Now, Alexandrov's rigidity result implies that the geodesic triangle formed by the points x_0, $\gamma(0)$ and $\gamma(1)$ is *flat*, see Bridson and Haefliger [18]. Thus, (M,g) is isometric to the Euclidean space $\mathbb{R}^n$.

We conclude this section by discussing the existence of extremals in the Heisenberg-Pauli-Weyl uncertainty principle on hyperbolic spaces. For the hyperbolic space we use the Poincaré ball model $\mathbb{H}^n = B_e(0,1) = \{x \in \mathbb{R}^n : |x| < 1\}$ endowed with the Riemannian metric

$$g_{\rm hyp}(x) = (g_{ij}(x))_{i,j=1,\dots,n} = p^2(x)\delta_{ij},$$

where $p(x) = \frac{2}{1-|x|^2}$. It is well-known that $(\mathbb{H}^n, g_{\rm hyp})$ is a Cartan-Hadamard manifold with constant sectional curvature -1. The volume form is

$$dV_{\mathbb{H}^n}(x) = p^n(x)dx, \tag{3.17}$$

while the hyperbolic gradient and Laplace-Beltrami operator are given by

$$\nabla_{\mathbb{H}^n} u = \frac{\nabla u}{p^2} \text{ and } \Delta_{\mathbb{H}^n} u = p^{-n}{\rm div}(p^{n-2}\nabla u),$$

respectively. The hyperbolic distance between the origin and $x \in \mathbb{H}^n$ is

$$d_{\mathbb{H}^n}(0,x) = \ln\left(\frac{1+|x|}{1-|x|}\right).$$

Recently, Kombe and Özaydin [59] stated a Heisenberg-Pauli-Weyl uncertainty principle on $(\mathbb{H}^n, g_{\rm hyp})$. For completeness, we recall the full statement of Theorem 4.2 from [59]:

"Let $u \in C_0^\infty(\mathbb{H}^n)$, $d = d(x) = d_{\mathbb{H}^n}(0,x)$ and $n > 2$. Then

$$\left(\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^2 dV_{\mathbb{H}^n}\right)\left(\int_{\mathbb{H}^n} d^2u^2 dV_{\mathbb{H}^n}\right) \geq \frac{n^2}{4}\left(\int_{\mathbb{H}^n} u^2 dV_{\mathbb{H}^n}\right)^2. \tag{3.18}$$

Moreover, equality holds in (3.18) if $u(x) = Ae^{-\alpha d^2}$, where $A \in \mathbb{R}$, and

$$\alpha = \frac{n-1}{n-2}\left(n-1+2\pi\frac{C_{n-2}}{C_n}\right) \tag{3.19}$$

with $C_n = \displaystyle\int_{\mathbb{H}^n} e^{-\alpha d^2} dV_{\mathbb{H}^n}$, $\alpha > 0$."

Relation (3.18) holds true, see also Theorem 3.6. However, the statement concerning the *equality* in (3.18) cannot happen, which has the following three independent proofs.

Argument 1 (based on Theorem 3.6). Following Kombe and Özaydin [59], let us assume that the hyperbolic Gaussian $u = e^{-\alpha d^2} > 0$ is an extremal function in (3.18) for some $\alpha > 0$. Due to Theorem 3.6/(ii), it follows that the hyperbolic space $(\mathbb{H}^n, g_{\rm hyp})$ is isometric to the standard Euclidean space $\mathbb{R}^n$, a contradiction.

Argument 2 (based on the non-solvability of (3.19)) . Let $C_n = C_n(\alpha) = \int_{\mathbb{H}^n} e^{-\alpha d^2} \mathrm{d}V_{\mathbb{H}^n}$ be as above. We claim that the nonlinear equation (3.19) *cannot* be solved generically in $\alpha > 0$. For simplicity, we consider only the case $n = 4$; then equation (3.19) reduces to $\alpha = w(\alpha)$, where

$$w(\alpha) := \frac{3}{2}\left(3 + 2\pi \frac{\displaystyle\int_{\mathbb{H}^2} e^{-\alpha d^2} \mathrm{d}V_{\mathbb{H}^2}}{\displaystyle\int_{\mathbb{H}^4} e^{-\alpha d^2} \mathrm{d}V_{\mathbb{H}^4}}\right).$$

Since $w \geq \frac{9}{2}$, the values for α should belong to $[\frac{9}{2}, \infty)$ in order to solve $\alpha = w(\alpha)$. By using the Gauss error function $\mathrm{erf}(s) = \frac{2}{\sqrt{\pi}} \int_0^s e^{-t^2} \mathrm{d}t$, after some elementary computation we obtain that $w(\alpha) \geq 2\alpha + 1$ for every $\alpha \in [4, \infty)$. The latter inequality implies the non-solvability of $\alpha = w(\alpha)$.

Argument 3 (based on Theorem 3.5). Due to Theorem 3.5, for every $u \in C_0^\infty(\mathbb{H}^n)$ one has

$$\left(\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^2 \mathrm{d}V_{\mathbb{H}^n}\right)\left(\int_{\mathbb{H}^n} d^2 u^2 \mathrm{d}V_{\mathbb{H}^n}\right) \geq \frac{n^2}{4}\left(\int_{\mathbb{H}^n} \left(1 + \frac{n-1}{n}\mathbf{D}_{-1}(d)\right) u^2 \mathrm{d}V_{\mathbb{H}^n}\right)^2. \tag{3.20}$$

Since $\mathbf{D}_{-1}(d) \geq 0$, if we have equality in (3.18) for $u = e^{-\alpha d^2}$ for some $\alpha > 0$, we necessarily have in (3.20) the relation $\mathbf{D}_{-1}(\rho) = 0$ for every $\rho \geq 0$ which means that for every $\rho \geq 0$ one has that $0 = \rho \mathbf{ct}_{-1}(\rho) - 1 = \rho \coth(\rho) - 1$, a contradiction. Moreover, in the inequality (3.20) the constant $\frac{n^2}{4}$ is *sharp* and an integration by parts easily shows (by using the exact form of the volume element (3.17)) that the *equality* holds for the hyperbolic Gaussian family of functions $u_\alpha = e^{-\alpha d^2}$, $\alpha > 0$. Therefore, hyperbolic Gaussian functions $u_\lambda = e^{-\lambda d^2}$, $\lambda > 0$, represent the family of extremals for the quantitative Heisenberg-Pauli-Weyl uncertainty principle (3.20), but *not* for the 'pure' Heisenberg-Pauli-Weyl uncertainty (3.18).

3.3 Hardy-Poincaré uncertainty principle on Riemannian manifolds

Depending on the curvature restrictions and number of poles/singularities, in this section we provide sharp Hardy-Poincaré uncertainty principles on Riemannian manifolds.

3.3.1 Unipolar case

At first, we present a quantitative version of the Hardy-Poincaré inequality on Hadamard manifolds.

Theorem 3.7. (Kristály [108]) *Let (M, g) be an n-dimensional ($n \geq 3$) Hadamard manifold with sectional curvature bounded from above by $c \leq 0$. Then for every $x_0 \in M$ and $u \in C_0^\infty(M)$ we have*

$$\int_M |\nabla_g u|^2 \mathrm{d}V_g \geq \frac{(n-2)^2}{4} \int_M \left(1 + \frac{2(n-1)}{n-2}\mathbf{D}_c(d_{x_0})\right) \frac{u^2}{d_{x_0}^2} \mathrm{d}V_g. \tag*{$(\mathbf{HP})_{x_0}$}$$

In addition, the constant $\frac{(n-2)^2}{4}$ is sharp and never attained.

Proof. Let $x_0 \in M$ and $u \in C_0^\infty(M)$ be arbitrarily fixed and $\widetilde{\mu} = \frac{n-2}{2} > 0$. We consider the function $v = d_{x_0}^{\widetilde{\mu}} u$. Thus, for $u = d_{x_0}^{-\widetilde{\mu}} v$ one has that

$$\nabla_g u = -\widetilde{\mu} d_{x_0}^{-\widetilde{\mu}-1} v \nabla_g d_{x_0} + d_{x_0}^{-\widetilde{\mu}} \nabla_g v,$$

which yields the inequality

$$|\nabla_g u|^2 \geq \widetilde{\mu}^2 d_{x_0}^{-2\widetilde{\mu}-2} v^2 |\nabla_g d_{x_0}|^2 - 2\widetilde{\mu} d_{x_0}^{-2\widetilde{\mu}-1} v \langle \nabla_g d_{x_0}, \nabla_g v \rangle.$$

By the eikonal equation $|\nabla_g d_{x_0}| = 1$ a.e. on M, after integrating the latter inequality, we obtain

$$\int_M |\nabla_g u|^2 \mathrm{d}V_g \geq \widetilde{\mu}^2 \int_M d_{x_0}^{-2\widetilde{\mu}-2} v^2 \mathrm{d}V_g + R_0, \tag{3.21}$$

where

$$\begin{aligned} R_0 &:= -2\widetilde{\mu} \int_M d_{x_0}^{-2\widetilde{\mu}-1} v \langle \nabla_g d_{x_0}, \nabla_g v \rangle \mathrm{d}V_g = \frac{1}{2} \int_M \langle \nabla_g (v^2), \nabla_g (d_{x_0}^{-2\widetilde{\mu}}) \rangle \mathrm{d}V_g \\ &= -\frac{1}{2} \int_M v^2 \Delta_g (d_{x_0}^{-2\widetilde{\mu}}) \mathrm{d}V_g = \widetilde{\mu} \int_M v^2 d_{x_0}^{-2\widetilde{\mu}-2} \left(-2\widetilde{\mu} - 1 + d_{x_0} \Delta_g d_{x_0}\right) \mathrm{d}V_g \\ &\geq \frac{(n-1)(n-2)}{2} \int_M \left(d_{x_0} \mathbf{ct}_c(d_{x_0}) - 1\right) \frac{u^2(x)}{d_{x_0}^2} \mathrm{d}V_g, \qquad \text{[see Theorem 1.2]} \\ &= \frac{(n-1)(n-2)}{2} \int_M \mathbf{D}_c(d_{x_0}) \frac{u^2(x)}{d_{x_0}^2} \mathrm{d}V_g, \end{aligned}$$

which completes the first part of the proof.

We shall prove in the sequel that $\widetilde{\mu}^2 = \frac{(n-2)^2}{4}$ is sharp in $(\mathbf{HP})_{x_0}$, i.e.,

$$\frac{(n-2)^2}{4} = \inf_{u \in C_0^\infty(M) \setminus \{0\}} \frac{\displaystyle\int_M |\nabla_g u|^2 \mathrm{d}V_g}{\displaystyle\int_M \left(1 + \frac{2(n-1)}{n-2} \mathbf{D}_c(d_{x_0})\right) \frac{u^2}{d_{x_0}^2} \mathrm{d}V_g}. \tag{3.22}$$

Fix the numbers $R > r > 0$ and a smooth cutoff function $\psi : M \to [0,1]$ with $\mathrm{supp}(\psi) = \overline{B_g(x_0, R)}$ and $\psi(x) = 1$ for $x \in B_g(x_0, r)$; moreover, for every $\varepsilon > 0$, let

$$u_\varepsilon = (\max\{\varepsilon, d_{x_0}\})^{-\widetilde{\mu}}. \tag{3.23}$$

On one hand,

$$\begin{aligned} I_1(\varepsilon) &:= \int_M |\nabla_g (\psi u_\varepsilon)|^2 \mathrm{d}V_g \\ &= \int_{B_g(x_0,r)} |\nabla_g (\psi u_\varepsilon)|^2 \mathrm{d}V_g + \int_{B_g(x_0,R) \setminus B_g(x_0,r)} |\nabla_g (\psi u_\varepsilon)|^2 \mathrm{d}V_g \\ &= \widetilde{\mu}^2 \int_{B_g(x_0,r) \setminus B_g(x_0,\varepsilon)} d_{x_0}^{-2\widetilde{\mu}-2} \mathrm{d}V_g + \widetilde{I}_1(\varepsilon), \end{aligned}$$

where the quantity

$$\widetilde{I}_1(\varepsilon) = \int_{B_g(x_0,R)\setminus B_g(x_0,r)} |\nabla_g(\psi u_\varepsilon)|^2 \mathrm{d}V_g$$

is finite and does *not* depend on $\varepsilon > 0$ whenever $\varepsilon < r$. On the other hand,

$$\begin{aligned} I_2(\varepsilon) &:= \int_M \left(1 + \frac{2(n-1)}{n-2}\mathbf{D}_c(d_{x_0})\right) \frac{(\psi u_\varepsilon)^2}{d_{x_0}^2}\mathrm{d}V_g \geq \int_M \frac{(\psi u_\varepsilon)^2}{d_{x_0}^2}\mathrm{d}V_g \\ &\geq \int_{B_g(x_0,r)\setminus B_g(x_0,\varepsilon)} d_{x_0}^{-2\widetilde{\mu}-2}\mathrm{d}V_g =: \widetilde{I}_2(\varepsilon). \end{aligned}$$

Applying the layer cake representation, we deduce that for $0 < \varepsilon < r$, one has that

$$\begin{aligned} \widetilde{I}_2(\varepsilon) &= \int_{B_g(x_0,r)\setminus B_g(x_0,\varepsilon)} d_{x_0}^{-2\widetilde{\mu}-2}\mathrm{d}V_g = \int_{B_g(x_0,r)\setminus B_g(x_0,\varepsilon)} d_{x_0}^{-n}\mathrm{d}V_g \\ &\geq \int_{r^{-n}}^{\varepsilon^{-n}} \mathrm{Vol}_g(B_g(x_0,\rho^{-\frac{1}{n}}))\mathrm{d}\rho \geq \omega_n \int_{r^{-n}}^{\varepsilon^{-n}} \rho^{-1}\mathrm{d}\rho \qquad \text{[see (1.9)]} \\ &= n\omega_n(\ln r - \ln \varepsilon). \end{aligned}$$

In particular, $\lim_{\varepsilon\to 0^+} \widetilde{I}_2(\varepsilon) = +\infty$. Thus, from the above relations it follows that

$$\begin{aligned} \frac{(n-2)^2}{4} &\leq \inf_{u\in C_0^\infty(M)\setminus\{0\}} \frac{\displaystyle\int_M |\nabla_g u|^2 \mathrm{d}V_g}{\displaystyle\int_M \left(1 + \frac{2(n-1)}{n-2}\mathbf{D}_c(d_{x_0})\right)\frac{u^2}{d_{x_0}^2}\mathrm{d}V_g} \\ &\leq \lim_{\varepsilon\to 0^+} \frac{I_1(\varepsilon)}{I_2(\varepsilon)} \leq \lim_{\varepsilon\to 0^+} \frac{\widetilde{\mu}^2\widetilde{I}_2(\varepsilon) + \widetilde{I}_1(\varepsilon)}{\widetilde{I}_2(\varepsilon)} \\ &= \widetilde{\mu}^2 = \frac{(n-2)^2}{4}, \end{aligned}$$

which concludes the proof of (3.22).

If we assume the function $u_0 \neq 0$ is an extremal in $(\mathbf{HP})_{x_0}$, on one hand, due to (3.21) we have that

$$\int_M d_{x_0}^{-2\widetilde{\mu}} |\nabla_g v_0|^2 \mathrm{d}V_g = 0, \tag{3.24}$$

where $v_0 = d_{x_0}^{\widetilde{\mu}} u_0$. Using (3.24), it follows that v_0 is a constant function, thus $u_0 = c_0 d_{x_0}^{-\widetilde{\mu}}$ for some $c_0 \in \mathbb{R}\setminus\{0\}$. On the other hand, similar estimates as above show (see the function $\widetilde{I}_2$) that

$$\int_M |\nabla_g u_0|^2 \mathrm{d}V_g = \widetilde{\mu}^2 \int_M \frac{u_0^2}{d_{x_0}^2}\mathrm{d}V_g = c_0^2\widetilde{\mu}^2 \int_M d_{x_0}^{-n}\mathrm{d}V_g = +\infty,$$

i.e., $u_0 \notin W^{1,2}(M, \mathrm{d}V_g)$ and $\frac{u_0}{d_{x_0}} \notin L^2(M, \mathrm{d}V_g)$, a contradiction. □

Remark 3.2. Theorem 3.7 provides a quantitative form of the main results from Carron [20] and D'Ambrosio and Dipierro [29].

In the next statement we provide a new type of improved Hardy-Poincaré inequality which shows that *more curvature implies more powerful improvements.*

Theorem 3.8. (Kristály [108]) *Let (M,g) be an n-dimensional $(n \geq 3)$ Hadamard manifold such that its sectional curvature is bounded from above by $c \leq 0$. Then for every $x_0 \in M$ and $u \in C_0^\infty(M)$, we have that*

$$\int_M |\nabla_g u|^2 \mathrm{d}V_g \geq \frac{(n-2)^2}{4} \int_M \frac{u^2}{d_{x_0}^2} \mathrm{d}V_g + \frac{3|c|(n-1)(n-2)}{2} \int_M \frac{u^2}{\pi^2 + |c| d_{x_0}^2} \mathrm{d}V_g.$$

In addition, the constant $\frac{(n-2)^2}{4}$ is sharp (independently by the second term on the right hand side).

Proof. By the continued fraction representation of the function $\rho \mapsto \coth(\rho)$, one has that

$$\rho \coth(\rho) - 1 \geq \frac{3\rho^2}{\pi^2 + \rho^2}, \ \forall \rho > 0.$$

Now, the inequality follows at once from the latter estimate and Theorem 3.7. □

We conclude this subsection by stating a Hardy-Poincaré inequality on Finsler-Hadamard manifolds which will be used in Chapter 5; its proof is similar to Theorem 3.7, thus we omit it.

Theorem 3.9. (Farkas, Kristály és Varga [107]) *Let (M,F) be an n-dimensional $(n \geq 3)$ Finsler-Hadamard manifold with $\mathbf{S} = 0$, and let $x_0 \in M$ be fixed. Then*

$$\int_M [F^*(x, -D(|u|)(x))]^2 \, \mathrm{d}V_F(x) \geq \frac{(n-2)^2}{4} \int_M \frac{u^2(x)}{d_F^2(x_0,x)} \mathrm{d}V_F(x), \ \forall u \in C_0^\infty(M), \tag{3.25}$$

where the constant $\frac{(n-2)^2}{4}$ is optimal and never attained.

3.3.2 Multipolar case

Let $m \geq 2$ and $S = \{x_1, ..., x_m\} \subset M$ be a set of pairwise distinct poles in a Riemannian manifold (M,g). For simplicity of notation, let $d_i = d_g(\cdot, x_i)$ for every $i \in \{1,...,m\}$. A multipolar Hardy-Poincaré theorem on general Riemannian manifolds reads as follows.

Theorem 3.10. (Faraci, Farkas and Kristály [104]) *Let (M,g) be an n-dimensional $(n \geq 3)$ complete Riemannian manifold and $S = \{x_1, ..., x_m\} \subset M$ be the set of pairwise distinct poles, $m \geq 2$. Then*

$$\begin{aligned} \int_M |\nabla_g u|^2 \mathrm{d}V_g \quad &\geq \quad \frac{(n-2)^2}{m^2} \sum_{1\leq i<j\leq m} \int_M \left| \frac{\nabla_g d_i}{d_i} - \frac{\nabla_g d_j}{d_j} \right|^2 u^2 \mathrm{d}V_g \\ &\quad + \frac{n-2}{m} \sum_{i=1}^m \int_M \frac{d_i \Delta_g d_i - (n-1)}{d_i^2} u^2 \mathrm{d}V_g, \quad \forall u \in C_0^\infty(M). \end{aligned} \tag{3.26}$$

Moreover, in the bipolar case (i.e., when $m = 2$), the constant $\frac{(n-2)^2}{m^2} = \frac{(n-2)^2}{4}$ is sharp in (3.26).

Proof. Let $E = \prod_{i=1}^{m} d_i^{2-n}$ and fix $u \in C_0^\infty(M)$ arbitrarily. A direct calculation gives

$$\nabla_g \left(uE^{-\frac{1}{m}}\right) = E^{-\frac{1}{m}}\nabla_g u + \frac{n-2}{m} uE^{-\frac{1}{m}} \sum_{i=1}^{m} \frac{\nabla_g d_i}{d_i} \text{ on } M \setminus \bigcup_{i=1}^{m} (\{x_i\} \cup \mathrm{cut}(x_i)),$$

where $\mathrm{cut}(x_i)$ denotes the cut locus of the point x_i, see do Carmo [31]. Integrating the latter relation, the divergence theorem and eikonal equation give that

$$\begin{aligned}\int_M \left|\nabla_g \left(uE^{-\frac{1}{m}}\right)\right|^2 E^{\frac{2}{m}} \mathrm{d}V_g = & \int_M |\nabla_g u|^2 \mathrm{d}V_g + \frac{(n-2)^2}{m^2} \int_M \left|\sum_{i=1}^{m} \frac{\nabla_g d_i}{d_i}\right|^2 u^2 \mathrm{d}V_g \\ & + \frac{n-2}{m} \sum_{i=1}^{m} \int_M \left\langle \nabla_g u^2, \frac{\nabla_g d_i}{d_i} \right\rangle \mathrm{d}V_g \\ = & \int_M |\nabla_g u|^2 \mathrm{d}V_g + \frac{(n-2)^2}{m^2} \int_M \left|\sum_{i=1}^{m} \frac{\nabla_g d_i}{d_i}\right|^2 u^2 \mathrm{d}V_g \\ & - \frac{n-2}{m} \sum_{i=1}^{m} \int_M \mathrm{div}\left(\frac{\nabla_g d_i}{d_i}\right) u^2 \mathrm{d}V_g.\end{aligned}$$

One clearly has that

$$\mathrm{div}\left(\frac{\nabla_g d_i}{d_i}\right) = \frac{d_i \Delta_g d_i - 1}{d_i^2}, \ i \in \{1, ..., m\}.$$

Thus, an algebraic reorganization of the latter relation provides an Agmon-Allegretto-Piepenbrink-type multipolar representation

$$\begin{aligned} & \int_M |\nabla_g u|^2 \mathrm{d}V_g - \frac{(n-2)^2}{m^2} \sum_{1 \le i < j \le m} \int_M \left|\frac{\nabla_g d_i}{d_i} - \frac{\nabla_g d_j}{d_j}\right|^2 u^2 \mathrm{d}V_g \\ & = \int_M \left|\nabla_g \left(uE^{-1/m}\right)\right|^2 E^{2/m} \mathrm{d}V_g + \frac{n-2}{m} \sum_{i=1}^{m} \mathcal{K}_i(u), \end{aligned} \tag{3.27}$$

where $\mathcal{K}_i(u) = \int_M \frac{d_i \Delta_g d_i - (n-1)}{d_i^2} u^2 \mathrm{d}V_g$. Inequality (3.26) directly follows from (3.27).

In the sequel, we deal with the sharpness of the constant $\mu_H = \frac{(n-2)^2}{m^2}$ in (3.26) when $m = 2$. In this case, the right hand side of (3.26) behaves as $\frac{(n-2)^2}{4} d_g^{-2}(x, x_i)$ whenever $x \to x_i$ and by the local behavior of the geodesic balls we may expect the optimality of $\frac{(n-2)^2}{4}$. In order to be more explicit, let $A_i[r, R] = \{x \in M : r \le d_i(x) \le R\}$ for $r < R$ and $i \in \{1, ..., m\}$. If $0 < r \ll R$, a layer cake representation yields for every $i \in \{1, ..., m\}$ that

$$\begin{aligned}\int_{A_i[r,R]} d_i^{-n} \mathrm{d}V_g & = \frac{\mathrm{Vol}_g(B_g(x_i, R))}{R^n} - \frac{\mathrm{Vol}_g(B_g(x_i, r))}{r^n} + n \int_r^R \mathrm{Vol}_g(B_g(x_i, \rho)))\rho^{-1-n} \mathrm{d}\rho \\ & = o(R) + n\omega_n \log\frac{R}{r}. \end{aligned} \tag{3.28}$$

Let $S=\{x_1,x_2\}$ be the set of distinct poles. Let $\varepsilon\in(0,1)$ be small enough and $B_g(x_1,2\sqrt{\varepsilon})\cap B_g(x_2,2\sqrt{\varepsilon})=\emptyset$. Consider the function

$$u_\varepsilon(x)=\begin{cases}\dfrac{\log\left(\frac{d_i(x)}{\varepsilon^2}\right)}{\log\left(\frac{1}{\varepsilon}\right)}d_i^{\frac{2-n}{2}}(x), & \text{if} \qquad x\in A_i[\varepsilon^2,\varepsilon],\\ \dfrac{2\log\left(\frac{\sqrt{\varepsilon}}{d_i(x)}\right)}{\log\left(\frac{1}{\varepsilon}\right)}d_i^{\frac{2-n}{2}}(x), & \text{if} \qquad x\in A_i[\varepsilon,\sqrt{\varepsilon}],\\ 0, & \text{otherwise,}\end{cases}$$

with $i\in\{1,2\}$. Note that $u_\varepsilon\in C^0(M)$ has the compact support $\bigcup_{i=1}^2 A_i[\varepsilon^2,\sqrt{\varepsilon}]\subset M$ and it can be used as a test function in (3.26). For later use let us introduce the notations $\varepsilon^*:=1/\log^2\left(\frac{1}{\varepsilon}\right)$,

$$\mathcal{I}_\varepsilon:=\int_M|\nabla_g u_\varepsilon|^2\mathrm{d}V_g,\ \mathcal{L}_\varepsilon:=\int_M\frac{\langle\nabla_g d_1,\nabla_g d_2\rangle}{d_1d_2}u_\varepsilon^2\mathrm{d}V_g,\ \mathcal{K}_\varepsilon:=\sum_{i=1}^2\int_M\frac{d_i\Delta_g d_i-(n-1)}{d_i^2}u_\varepsilon^2\mathrm{d}V_g$$

and

$$\mathcal{J}_\varepsilon:=\int_M\left[\frac{1}{d_1^2}+\frac{1}{d_2^2}\right]u_\varepsilon^2\mathrm{d}V_g.$$

By direct computations, one has that

$$\mathcal{I}_\varepsilon-\mu_H\mathcal{J}_\varepsilon=\mathcal{O}(1),\ \mathcal{L}_\varepsilon=\mathcal{O}(\sqrt[4]{\varepsilon}),\ \mathcal{K}_\varepsilon=\mathcal{O}(\sqrt[4]{\varepsilon})\ \text{ as }\ \varepsilon\to 0, \tag{3.29}$$

and

$$\lim_{\varepsilon\to 0}\mathcal{J}_\varepsilon=+\infty. \tag{3.30}$$

Combining relations (3.29) and (3.30) with inequality (3.26), we have that

$$\mu_H\leq\frac{\mathcal{I}_\varepsilon-\frac{n-2}{2}\mathcal{K}_\varepsilon}{\mathcal{J}_\varepsilon-2\mathcal{L}_\varepsilon}\leq\frac{\mathcal{I}_\varepsilon+\frac{n-2}{2}|\mathcal{K}_\varepsilon|}{\mathcal{J}_\varepsilon-2|\mathcal{L}_\varepsilon|}=\frac{\mu_H\mathcal{J}_\varepsilon+\mathcal{O}(1)}{\mathcal{J}_\varepsilon+\mathcal{O}(\sqrt[4]{\varepsilon})}\to\mu_H \text{ as } \varepsilon\to 0,$$

which concludes the proof. □

Remark 3.3. Let us assume in Theorem 3.10 that (M,g) is a Riemannian manifold with sectional curvature verifying $\mathbf{K}\leq c$. By the Laplace comparison theorem we have that

$$\begin{aligned}\int_M|\nabla_g u|^2\mathrm{d}V_g &\geq \frac{(n-2)^2}{m^2}\sum_{1\leq i<j\leq m}\int_M\left|\frac{\nabla_g d_i}{d_i}-\frac{\nabla_g d_j}{d_j}\right|^2u^2\mathrm{d}V_g\\ &+\frac{(n-2)(n-1)}{m}\sum_{i=1}^m\int_M\frac{\mathbf{D}_c(d_i)}{d_i^2}u^2\mathrm{d}V_g,\quad \forall u\in C_0^\infty(M).\end{aligned} \tag{3.31}$$

In addition, if (M,g) is a Hadamard manifold with $\mathbf{K}\leq c\leq 0$, then $\mathbf{D}_c(r)\geq\frac{3|c|r^2}{\pi^2+|c|r^2}$ for all $r\geq 0$. Accordingly, stronger curvature of the Hadamard manifold implies improvement in the multipolar Hardy-Poincaré inequality (3.31), similarly as in the unipolar case, see Theorems 3.7 and 3.8. In particular, if $\mathbf{K}=0$, inequality (3.31) reduces precisely to the main result of Cazacu and Zuazua [21].

A positively curved counterpart of (3.31) can be also stated as a consequence of Theorem 3.10.

Corollary 3.1. (Faraci, Farkas and Kristály [104]) *Let $\mathbb{S}^n_+$ be the open upper hemisphere and $S = \{x_1, ..., x_m\} \subset \mathbb{S}^n_+$ be a set of pairwise distinct poles, where $n \geq 3$ and $m \geq 2$. Let $\beta = \max_{i=\overline{1,m}} d_g(x_0, x_i)$, where $x_0 = (0, ..., 0, 1)$ is the north pole of the sphere $\mathbb{S}^n$ and g is the natural Riemannian metric of $\mathbb{S}^n$ inherited by $\mathbb{R}^{n+1}$. Then, we have the inequality*

$$\|u\|^2_{\mathsf{C}(n,\beta)} \geq \frac{(n-2)^2}{m^2} \sum_{1\leq i<j\leq m} \int_{\mathbb{S}^n_+} \left| \frac{\nabla_g d_i}{d_i} - \frac{\nabla_g d_j}{d_j} \right|^2 u^2 \,\mathrm{d}V_g, \quad \forall u \in H^1_g(\mathbb{S}^n_+), \tag{3.32}$$

where $\|u\|^2_{\mathsf{C}(n,\beta)} = \int_{\mathbb{S}^n_+} |\nabla_g u|^2 \mathrm{d}V_g + \mathsf{C}(n,\beta) \int_{\mathbb{S}^n_+} u^2 \mathrm{d}V_g$ *and* $\mathsf{C}(n,\beta) = (n-1)(n-2)\frac{7\pi^2 - 3\left(\beta+\frac{\pi}{2}\right)^2}{2\pi^2\left(\pi^2 - \left(\beta+\frac{\pi}{2}\right)^2\right)}$.

Proof. Let $M = \mathbb{S}^n$ be the standard unit sphere in $\mathbb{R}^{n+1}$ and the open upper hemisphere $\mathbb{S}^n_+ = \{y = (y_1, ..., y_{n+1}) \in \mathbb{S}^n : y_{n+1} > 0\}$. By Theorem 3.10 we have that

$$\begin{aligned} \int_{\mathbb{S}^n_+} |\nabla_g u|^2 \,\mathrm{d}V_g \quad \geq \quad & \frac{(n-2)^2}{m^2} \sum_{1\leq i<j\leq m} \int_{\mathbb{S}^n_+} \left| \frac{\nabla_g d_i}{d_i} - \frac{\nabla_g d_j}{d_j} \right|^2 u^2 \,\mathrm{d}V_g \\ & + \frac{n-2}{m} \sum_{i=1}^m \int_{\mathbb{S}^n_+} \frac{d_i \Delta_g d_i - (n-1)}{d_i^2} u^2 \,\mathrm{d}V_g, \ \forall u \in C_0^\infty(\mathbb{S}^n_+). \end{aligned}$$

Since $\mathbf{K} \equiv 1$, the two-sided Laplace comparison theorem shows that $\Delta_g d_i = (n-1)\cot(d_i)$.

Fix $u \in C_0^\infty(\mathbb{S}^n_+)$. By using both the Mittag-Leffler expansion

$$\cot(t) = \frac{1}{t} + 2t \sum_{k=1}^\infty \frac{1}{t^2 - \pi^2 k^2}, \ t \in (0, \pi),$$

and the fact that $0 < d_i < \pi$, $i \in \{1, ..., m\}$ (up to null-measured poles), one has that

$$\int_{\mathbb{S}^n_+} \frac{d_i \Delta_g d_i - (n-1)}{d_i^2} u^2 \,\mathrm{d}V_g = -2(n-1) \int_{\mathbb{S}^n_+} \sum_{k=1}^\infty \frac{u^2}{\pi^2 k^2 - d_i^2} \mathrm{d}V_g.$$

Since $d_i < \pi$, we obtain that

$$\int_{\mathbb{S}^n_+} \sum_{k=2}^\infty \frac{u^2}{\pi^2 k^2 - d_i^2} \mathrm{d}V_g \leq \int_{\mathbb{S}^n_+} \sum_{k=2}^\infty \frac{u^2}{\pi^2 k^2 - \pi^2} \mathrm{d}V_g = \frac{3}{4\pi^2} \int_{\mathbb{S}^n_+} u^2 \mathrm{d}V_g.$$

Moreover, since $\beta = \max_{i=\overline{1,m}} d_g(x_0, x_i) < \frac{\pi}{2}$, one can see that for every $x \in \mathbb{S}^n_+$, $d_i(x) = d_g(x, x_i) \leq d_g(x, x_0) + d_g(x_0, x_i) < \frac{\pi}{2} + \beta$. Thus, $\pi^2 - d_i^2 > \pi^2 - \left(\beta + \frac{\pi}{2}\right)^2 > 0$, which implies that

$$\int_{\mathbb{S}^n_+} \frac{u^2}{\pi^2 - d_i^2} \,\mathrm{d}V_g \leq \frac{1}{\pi^2 - \left(\beta + \frac{\pi}{2}\right)^2} \int_{\mathbb{S}^n_+} u^2 \,\mathrm{d}V_g.$$

Combining the above two estimates, we have that

$$\int_{\mathbb{S}^n_+} |\nabla_g u|^2 \, \mathrm{d}V_g + \mathtt{C}(n,\beta) \int_{\mathbb{S}^n_+} u^2 \, \mathrm{d}V_g \geq \frac{(n-2)^2}{m^2} \sum_{1\leq i<j\leq m} \int_{\mathbb{S}^n_+} \left| \frac{\nabla_g d_i}{d_i} - \frac{\nabla_g d_j}{d_j} \right|^2 u^2 \, \mathrm{d}V_g,$$

where

$$\mathtt{C}(n,\beta) = (n-1)(n-2) \frac{7\pi^2 - 3\left(\beta + \frac{\pi}{2}\right)^2}{2\pi^2 \left(\pi^2 - \left(\beta + \frac{\pi}{2}\right)^2\right)}.$$

The latter inequality can be extended to $H^1_g(\mathbb{S}^n_+)$ by standard approximation/density argument. $\square$

3.4 Further results and comments

I) Rellich uncertainty principle on Hadamard manifolds. Second-order Hardy inequalities are referred to *Rellich inequalities* whose most familiar forms can be stated as follows; given $n \geq 5$, one has

$$\int_{\mathbb{R}^n} (\Delta u)^2 \mathrm{d}x \geq \frac{n^2(n-4)^2}{16} \int_{\mathbb{R}^n} \frac{u^2}{|x|^4} \mathrm{d}x, \ \forall u \in C_0^\infty(\mathbb{R}^n), \tag{3.33}$$

$$\int_{\mathbb{R}^n} (\Delta u)^2 \mathrm{d}x \geq \frac{n^2}{4} \int_{\mathbb{R}^n} \frac{|\nabla u|^2}{|x|^2} \mathrm{d}x, \ \forall u \in C_0^\infty(\mathbb{R}^n), \tag{3.34}$$

where both constants $\frac{n^2(n-4)^2}{16}$ and $\frac{n^2}{4}$ are sharp, but are never attained. Their extensions, in the spirit of Theorem 3.8, can be stated as follows.

Theorem 3.11. (Kristály and Repovš [118]) *Let (M,g) be an n-dimensional Hadamard manifold such that its sectional curvature is bounded from above by $c \leq 0$. Let $x_0 \in M$ be fixed arbitrarily.*

(a) *If $n \geq 5$, then for every $u \in C_0^\infty(M)$ one has that*

$$\int_M (\Delta_g u)^2 \mathrm{d}V_g \geq \frac{n^2(n-4)^2}{16} \int_M \frac{u^2}{d_{x_0}^4} \mathrm{d}V_g + \frac{3|c|n(n-1)(n-2)(n-4)}{4} \int_M \frac{u^2}{(\pi^2 + |c|d_{x_0}^2)d_{x_0}^2} \mathrm{d}V_g,$$

where the constant $\frac{n^2(n-4)^2}{16}$ is sharp.

(b) *If $n \geq 9$, then for every $u \in C_0^\infty(M)$ one has that*

$$\int_M (\Delta_g u)^2 \mathrm{d}V_g \geq \frac{n^2}{4} \int_M \frac{|\nabla_g u|^2}{d_{x_0}^2} \mathrm{d}V_g + \frac{3|c|n(n-1)(n-4)^2}{8} \int_M \frac{u^2}{(\pi^2 + |c|d_{x_0}^2)d_{x_0}^2} \mathrm{d}V_g,$$

where the constant $\frac{n^2}{4}$ is sharp.

These results can be obtained also on Finsler manifolds, see Kristály and Repovš [118] (for the reversible case), and Yuan, Zhao and Shen [99] (for the non-reversible case). In these cases, the so-called *Green-deflection* of any $C_0^\infty(M)$ function plays a crucial role, which is automatically verified in the Riemannian setting.

II) Sharpness in the Heisenberg-Pauli-Weyl uncertainty principle. On one hand, if (M, g) is a complete n-dimensional Riemannian manifold with nonnegative Ricci curvature and assume the inequality

$$\left(\int_M |\nabla_g u|^2 \mathrm{d}V_g\right)\left(\int_M d_{x_0}^2 u^2 \mathrm{d}V_g\right) \geq C\left(\int_M u^2 \mathrm{d}V_g\right)^2, \quad \forall u \in C_0^\infty(M), \qquad (\mathbf{HPW})_C$$

holds for some $C \in \left(0, \frac{n^2}{4}\right]$, it is an open problem if one has for some $\alpha > 0$ a global volume non-collapsing property of the type

$$\mathrm{Vol}_g(B_g(x,\rho)) \geq \left(\frac{4C}{n^2}\right)^\alpha \omega_n \rho^n, \ \forall x \in M, \rho > 0.$$

On the other hand, it is remarkable that the sharp Heisenberg-Pauli-Weyl uncertainty principle holds on Hadamard manifolds without requiring the validity of the Cartan-Hadamard conjecture.

III) Hardy-Poincaré uncertainty principle on positively curved spaces. It seems that similar rigidity results for the Hardy-Poincaré inequalities as in Theorem 3.4 cannot be established on nonnegatively curved spaces. The problem comes from the lack of extremal functions in the Euclidean Hardy-Poincaré inequality (see Theorem 3.3) which should serve as a comparison function in the positively curved case.

Chapter 4

Elliptic problems on Finsler manifolds

Various elliptic problems are discussed on Minkowski spaces $(\mathbb{R}^n, F)$, where $F \in C^2(\mathbb{R}^n, [0,\infty))$ is convex and the leading term is given by the nonlinear Finsler-Laplace operator associated with the Minkowski norm F, see Alvino, Ferone, Lions and Trombetti [2], Ferone and Kawohl [42], and references therein. In this class of problems variational arguments are applied, the key roles being played by fine properties of Sobolev spaces, sharpness of Sobolev inequalities, as well as the lower semicontinuity of the energy functionals associated with the studied problems.

In order to have a global approach, the theory of Sobolev spaces has been deeply investigated on metric measure spaces, see Ambrosio, Colombo and Di Marino [4], Cheeger [22], and Hajlasz and Koskela [51]. In [4], the authors proved that if the metric space (X, d) is doubling and separable, and the measure m is finite on bounded sets of X, the Sobolev space $W^{1,2}(X, \mathsf{d}, \mathsf{m})$ is *reflexive*; here, $W^{1,2}(X, \mathsf{d}, \mathsf{m})$ contains functions $u \in L^2(X, \mathsf{m})$ with finite 2-relaxed slope endowed by the norm

$$u \mapsto \left(\int_X |\nabla u|_{*,2}^2 \mathsf{dm} + \int_X u^2 \mathsf{dm} \right)^{1/2},$$

where $|\nabla u|_{*,2}(x)$ denotes the 2-relaxed slope of u at $x \in X$.

This result clearly applies to differential structures as well. Indeed, if (M, F) is a reversible Finsler manifold (in particular, a Riemannian manifold), then for every $x \in M$ and $u \in C_0^\infty(M)$, one has that

$$|\nabla u|_{*,2}(x) = \limsup_{y \to x} \frac{|u(y) - u(x)|}{d_F(y, z)} = |\nabla u|_{d_F}(x) = F^*(x, Du(x)),$$

see also (2.49). Consequently, within the class of reversible Finsler manifolds, the synthetic notion of Sobolev spaces on metric measure spaces (see Ambrosio, Colombo and Di Marino [4], Cheeger [22]) and the analytic notion of Sobolev spaces on Finsler manifolds (see Ge and Shen [44], Ohta and Sturm [73]) coincide. However, Sobolev spaces over non-compact Finsler manifolds may behave pathologically which require a fine analysis based on the so-called *reversibility constant.*

Accordingly, this chapter is devoted to elliptic problems on Finsler manifolds, emphasizing the influence of non-reversibility in some nonlinear phenomena.

4.1 Sobolev spaces on Finsler manifolds: the effect of non-reversibility

Let (M,F) be a Finsler manifold. We consider the number

$$r_F = \sup_{x\in M} r_F(x), \quad \text{where} \quad r_F(x) := \sup_{y\in T_xM\setminus\{0\}} \frac{F(x,y)}{F(x,-y)}, \tag{4.1}$$

called as the *reversibility constant associated with* F, see Rademacher [78]. It is clear that $r_F \geq 1$ (possibly, $r_F = +\infty$), and $r_F = 1$ if and only if (M,F) is reversible. We may define the reversibility constant r_{F^*} associated with the polar transform F^* of F and we observe that $r_{F^*} = r_F$.

The number

$$l_F = \inf_{x\in M} l_F(x), \quad \text{where} \quad l_F(x) := \inf_{y,v,w\in T_xM\setminus\{0\}} \frac{g_{(x,v)}(y,y)}{g_{(x,w)}(y,y)},$$

is the *uniformity constant associated with* F which measures how far (M,F) and (M,F^*) are from Riemannian structures. More precisely, one can see that $l_F \leq 1$, and $l_F = 1$ if and only if (M,F) is a Riemannian manifold. In the same manner, we can define the constant l_{F^*} for F^*, and it follows that $l_{F^*} = l_F$. The definition of l_F in turn shows that

$$\left[F^*(x,t\alpha+(1-t)\beta)\right]^2 \leq t\left[F^*(x,\alpha)\right]^2 + (1-t)\left[F^*(x,\beta)\right]^2 - l_F t(1-t)\left[F^*(x,\beta-\alpha)\right]^2 \tag{4.2}$$

for all $x\in M$, $\alpha,\beta\in T_x^*M$ and $t\in[0,1]$. Furthermore, one can deduce that

$$l_F(x) r_F^2(x) \leq 1, \ \forall x\in M. \tag{4.3}$$

In the sequel, we shall use the canonical Hausdorff measure (1.6) on (M,F), $\mathrm{dm} = \mathrm{d}V_F$. Consider the *Sobolev spaces*

$$W^{1,2}(M,F,\mathsf{m}) := \left\{ u\in W^{1,2}_{\mathrm{loc}}(M) : \int_M \left[F^*(x,Du(x))\right]^2 \mathrm{dm}(x) < +\infty \right\},$$

associated with (M,F) and let $W^{1,2}_0(M,F,\mathsf{m})$ be the closure of $C_0^\infty(M)$ with respect to the norm

$$\|u\|_F := \left(\int_M \left[F^*(x,Du(x))\right]^2 \mathrm{dm}(x) + \int_M u^2(x)\mathrm{dm}(x) \right)^{1/2}. \tag{4.4}$$

Note that $\|\cdot\|_F$ is usually only an asymmetric norm. Denote by

$$F_s(x,y) = \left(\frac{F^2(x,y)+F^2(x,-y)}{2} \right)^{1/2}, \ (x,y)\in TM$$

the *symmetrized* Finsler metric associated with F which induces the reversible Finsler manifold (M,F_s). The symmetrized Finsler metric associated with F^* may be different from F_s^*, i.e., in general $2\left[F_s^*(x,\alpha)\right]^2 \neq \left[F^*(x,\alpha)\right]^2 + \left[F^*(x,-\alpha)\right]^2$. Clearly, if $(M,F)=(M,g)$ is a Riemannian manifold, the Sobolev space $W^{1,2}(M,F,\mathsf{m})$ coincides with the usual Sobolev space $H_g^1(M)$, see Hebey [52].

The first result of this section concerns the case when $r_F < +\infty$.

Theorem 4.1. (Farkas, Kristály and Varga [107]) *Let (M,F) be an n-dimensional $(n \geq 2)$ complete Finsler manifold such that $r_F < +\infty$. Then $(W_0^{1,2}(M,F,\mathsf{m}), \|\cdot\|_{F_s})$ is a reflexive Banach space, while the norm $\|\cdot\|_{F_s}$ and its asymmetric counterpart $\|\cdot\|_F$ are equivalent. In particular,*

$$\left(\frac{1+r_F^2}{2}\right)^{-1/2} \|u\|_F \leq \|u\|_{F_s} \leq \left(\frac{1+r_F^{-2}}{2}\right)^{-1/2} \|u\|_F,\ \forall u \in W_0^{1,2}(M,F,\mathsf{m}). \tag{4.5}$$

Proof. First of all, we note that the norm $\|\cdot\|_{F_s}$ is considered also with respect to the Hausdorff measure $\mathrm{d}\mathsf{m} = \mathrm{d}V_F$ (not with $\mathrm{d}V_{F_s}$), i.e.,

$$\|u\|_{F_s} = \left(\int_M [F_s^*(x,Du(x))]^2\,\mathrm{d}\mathsf{m}(x) + \int_M u^2(x)\mathrm{d}\mathsf{m}(x)\right)^{1/2}. \tag{4.6}$$

On account of the convexity of $\alpha \mapsto [F^*(x,\alpha)]^2$, if $u,v \in W_0^{1,2}(M,F,\mathsf{m})$ then one easily shows that $u+v \in W_0^{1,2}(M,F,\mathsf{m})$. On account of $r_F < \infty$, we have that $cu \in W_0^{1,2}(M,F,\mathsf{m})$ for every $c \in \mathbb{R}$ and $u \in W_0^{1,2}(M,F,\mathsf{m})$. Therefore, $W_0^{1,2}(M,F,\mathsf{m})$ is a vector space over $\mathbb{R}$.

We recall that the norms $\|\cdot\|_{F_s}$ and $\|\cdot\|_F$ are symmetric and not necessarily symmetric, respectively. The definition of the reversibility constant r_F shows that $\|\cdot\|_{F_s}$ and $\|\cdot\|_F$ are equivalent; thus, one has that

$$\left(\frac{1+r_F^2}{2}\right)^{-1/2} F^*(x,\alpha) \leq F_s^*(x,\alpha) \leq \left(\frac{1+r_F^{-2}}{2}\right)^{-1/2} F^*(x,\alpha),\ \forall (x,\alpha) \in T^*M,$$

which implies relation (4.5). Let $L^2(M,\mathsf{m}) := \left\{u : M \to \mathbb{R} : u \text{ is measurable, } \|u\|_{L^2(M,\mathsf{m})} < \infty\right\}$, where

$$\|u\|_{L^2(M,\mathsf{m})} := \left(\int_M u^2(x)\mathrm{d}\mathsf{m}(x)\right)^{1/2}.$$

It is clear that $(L^2(M,\mathsf{m}), \|\cdot\|_{L^2(M,\mathsf{m})})$ is a Hilbert space. Since $\alpha \mapsto [F^*(x,\alpha)]^2$, and consequently $\alpha \mapsto [F_s^*(x,\alpha)]^2$ are (strictly) convex functions, it follows that $(W_0^{1,2}(M,F,\mathsf{m}), \|\cdot\|_{F_s})$ is a closed subspace of the Hilbert space $L^2(M,\mathsf{m})$, thus $(W_0^{1,2}(M,F,\mathsf{m}), \|\cdot\|_{F_s})$ is reflexive. □

In the sequel, we consider some specific examples concerning the applicability of Theorem 4.1.

Example 4.1. (a) *Riemannian manifolds.* If $(M,F) = (M,g)$ is Riemannian, then $r_F = l_F = 1$, thus Theorem 4.1 is well-known by Hebey [52].

(b) *Compact Finsler manifolds.* When the not necessarily reversible Finsler manifold (M,F) is compact, we clearly have that $r_F < +\infty$, thus Theorem 4.1 applies. This particular case is well-known by Ge and Shen [44], and Ohta and Sturm [73].

(c) *Minkowski spaces.* If (M,F) is a Minkowski space, then $r_F < +\infty$, thus Theorem 4.1 applies.

(d) *Randers spaces.* Let M be a manifold and we introduce the Finsler metric $F : TM \to [0,\infty)$ defined by

$$F(x,y) := \sqrt{g_x(y,y)} + \beta_x(y),\ (x,y) \in TM, \tag{4.7}$$

where g is a Riemannian metric on M, β is an 1-form on M, and we assume that

$$\|\beta\|_g(x) = \sqrt{g_x^*(\beta_x, \beta_x)} < 1, \ \forall x \in M.$$

The co-metric g_x^* can be identified as the inverse g_x^{-1} of the symmetric positive definite matrix g_x. Clearly, the Randers space (M, F) in (4.7) is reversible if and only if $\beta = 0$. The canonical measure on (M, F) is

$$\mathrm{d}V_F(x) = \left(1 - \|\beta\|_g^2(x)\right)^{\frac{n+1}{2}} \mathrm{d}V_g(x), \tag{4.8}$$

where $\mathrm{d}V_g(x)$ denotes the canonical Riemannian volume form of g on M.

A direct computation shows (see [120]) that the polar transform of F from (4.7) is

$$F^*(x,\alpha) = \frac{\sqrt{[g_x^*(\alpha,\beta)]^2 + (1 - \|\beta\|_g^2(x))\|\alpha\|_g^2(x)} - g_x^*(\alpha,\beta)}{1 - \|\beta\|_g^2(x)}, \ (x,\alpha) \in T^*M. \tag{4.9}$$

Moreover, the symmetrized Finsler metric and its polar transform are

$$F_s(x,y) = \sqrt{g_x(y,y) + \beta_x^2(y)} \ \text{ and } \ F_s^*(x,\alpha) = \sqrt{\|\alpha\|_g^2(x) - \frac{[g_x^*(\alpha,\beta)]^2}{1 + \|\beta\|_g^2(x)}}. \tag{4.10}$$

Another direct computation gives that

$$r_F(x) = \frac{1 + \|\beta\|_g(x)}{1 - \|\beta\|_g(x)} \ \text{ and } \ l_F(x) = \left(\frac{1 - \|\beta\|_g(x)}{1 + \|\beta\|_g(x)}\right)^2, \ x \in M. \tag{4.11}$$

According to (4.11), we observe that $r_F = \sup_{x \in M} r_F(x)$ can be either finite of infinite, depending on the subtle structure of the Randers space.

For instance, if $\mathbf{S} = 0$, then $r_F < +\infty$ and Theorem 4.1 can be applied. Indeed, if (M, F) is a Randers space with $\mathbf{S} = 0$, Ohta [72] proved that β is a Killing form of constant g-length, i.e., there exists $\beta_0 \in (0,1)$ such that $\|\beta\|_g(x) = \beta_0$ for every $x \in M$. In particular, by (4.11), one has that $r_F = \frac{1+\beta_0}{1-\beta_0} < +\infty$ and $l_F = \left(\frac{1-\beta_0}{1+\beta_0}\right)^2 > 0$.

However, there are cases where Randers spaces provide unbounded reversibility constants; see Theorem 4.2 below which also shows the sharpness of Theorem 4.1.

We consider the usual n-dimensional $(n \geq 2)$ unit ball $B_e(0,1)$ endowed with a Funk-type metric, see Shen [83]. Namely, for every $a \in [0,1]$, we introduce the function $F_a : B_e(0,1) \times \mathbb{R}^n \to \mathbb{R}$,

$$F_a(x,y) = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)}}{1 - |x|^2} + a\frac{\langle x,y\rangle}{1 - |x|^2}, \ x \in B_e(0,1), \ y \in T_x B_e(0,1) = \mathbb{R}^n. \tag{4.12}$$

Hereafter, $|\cdot|$ and $\langle \cdot,\cdot \rangle$ denote the n-dimensional Euclidean norm and inner product, respectively; moreover, let $\mathsf{dm}_a = \mathrm{d}V_{F_a}$ be the volume element. Usual reasonings from Finsler geometry show that $(B_e(0,1), F_a)$ is a Randers space. Moreover, for $a = 0$, the pair $(B_e(0,1), F_0)$ reduces to the well-known Riemannian *Klein model*, while for $a = 1$, the object $(B_e(0,1), F_1)$ is the usual Finslerian *Funk model*.

Theorem 4.2. (Kristály and Rudas [120]) *If $a \in [0,1]$, the following statements are equivalent:*

(i) $W_0^{1,2}(B_e(0,1), F_a, \mathsf{m}_a)$ *is a vector space over* $\mathbb{R}$;

(ii) $r_{F_a} < +\infty$;

(iii) $a \in [0,1)$.

Proof. On $M = B_e(0,1)$ the metric F_a in (4.12) follows from the Klein metric g_K and 1-form β_x, where

$$(g_K)_x(y,y) = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)}}{1-|x|^2}, \quad \beta_x = a\frac{x}{1-|x|^2}.$$

It is clear that

$$(g_K)_{ij} = \frac{\delta_{ij}}{1-|x|^2} + \frac{x_i x_j}{(1-|x|^2)^2},\ i,j \in \{1,...,n\},$$

and $g_K^* = (g_K)^{-1}$ where the elements of the matrix g_K are

$$g_K^{ij} = (1-|x|^2)(\delta_{ij} - x_i x_j),\ i,j \in \{1,...,n\}.$$

Consequently,

$$\|\beta\|_{g_K}(x) = \sqrt{g_K^{ij}(\beta_x^i, \beta_x^j)} = a|x| < 1. \tag{4.13}$$

Therefore, by (4.11) and (4.13), the reversibility constant associated with F_a on $B_e(0,1)$ is

$$r_{F_a} = \sup_{x\in B_e(0,1)} r_{F_a}(x) = \sup_{|x|<1} \frac{1+a|x|}{1-a|x|} = \begin{cases} \dfrac{1+a}{1-a}, & \text{if} \quad a \in [0,1), \\ +\infty, & \text{if} \quad a = 1. \end{cases}$$

Accordingly, (ii) and (iii) are equivalent. The implication (ii)⇒(i) follows by Theorem 3.10.

It remains to prove the implication (i)⇒(iii). Due to (4.8), we have

$$\mathrm{d}V_{F_a}(x) = \left(1 - a^2|x|^2\right)^{\frac{n+1}{2}} \mathrm{d}V_{g_K}(x), \tag{4.14}$$

where the Klein volume form is

$$\mathrm{d}V_{g_K}(x) = \frac{1}{(1-|x|^2)^{\frac{n+1}{2}}}\mathrm{d}x.$$

The polar transform of F_a is

$$F_a^*(x,y) = \frac{\sqrt{(1-|x|^2)(1-a^2|x|^2)|y|^2 - (1-a^2)(1-|x|^2)\langle x,y\rangle^2} - a(1-|x|^2)\langle x,y\rangle}{1-a^2|x|^2}. \tag{4.15}$$

It is clear that $F_a^{**} = F_a$ and $r_{F_a^*} = r_{F_a}$.

By assumption, we know that

$$W_0^{1,2,a}(B_e(0,1)) := W_0^{1,2}(B_e(0,1), F_a, \mathsf{m}_a)$$

is a vector space over $\mathbb{R}$; by contradiction, we also assume that one may have $a = 1$. In this case, F_a

is precisely the Funk metric

$$F_1(x,y) = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)}}{1-|x|^2} + \frac{\langle x,y\rangle}{1-|x|^2},\ x \in B_e(0,1),\ y \in \mathbb{R}^n.$$

Note that the metric F_1 can be obtained by $\left|x + \frac{y}{F_1(x,y)}\right| = 1$, while the distance function associated with F_1 is

$$d_{F_1}(x_1,x_2) = \ln \frac{\sqrt{|x_1-x_2|^2 - (|x_1|^2|x_2|^2 - \langle x_1,x_2\rangle^2)} - \langle x_1, x_2 - x_1\rangle}{\sqrt{|x_1-x_2|^2 - (|x_1|^2|x_2|^2 - \langle x_1,x_2\rangle^2)} - \langle x_2, x_2 - x_1\rangle},\ x_1,x_2 \in B_e(0,1),$$

see Shen [83, p. 141 and p. 4]. In particular, $d_{F_1}(0,x) = -\ln(1-|x|),\ x \in B_e(0,1)$. On one hand, by (1.15) or direct computation via (4.15), we have that

$$F_1^*(x, Dd_{F_1}(0,x)) = 1,\ x \neq 0. \tag{4.16}$$

On the other hand, another direct computation and (4.15) shows that

$$F_1^*(x, -Dd_{F_1}(0,x)) = \frac{1+|x|}{1-|x|},\ x \neq 0. \tag{4.17}$$

Let $u : B_e(0,1) \to \mathbb{R}$ be defined by

$$u(x) = -\sqrt{1-|x|} = -e^{-\frac{d_{F_1}(0,x)}{2}}.$$

It is clear that $u \in W^{1,2}_{\rm loc}(B_e(0,1))$. Since $\mathrm{d}V_{F_1}(x) = \mathrm{d}x$, see (4.14), we have that

$$\int_{B_e(0,1)} u^2(x)\mathrm{d}V_{F_1}(x) = \frac{\omega_n}{n+1}.$$

Therefore, by using the identity $Du(x) = \frac{1}{2}e^{-\frac{d_{F_1}(0,x)}{2}} Dd_{F_1}(0,x)$, equality (4.16) yields that

$$C_1 \ := \int_{B_e(0,1)} [F_1^*(x, Du(x))]^2 \,\mathrm{d}V_{F_1}(x) = \frac{1}{4}\int_{B_e(0,1)} (1-|x|)\mathrm{d}x = \frac{\omega_n}{4(n+1)}.$$

Thus, $\|u\|^2_{F_1} = \frac{5\omega_n}{4(n+1)}$, so $u \in W_0^{1,2,1}(B_e(0,1))$.

However, relation (4.17) implies that

$$C_2 \ := \int_{B_e(0,1)} [F_1^*(x, -Du(x))]^2 \,\mathrm{d}V_{F_1}(x) = \frac{1}{4}\int_{B_e(0,1)} \frac{(1+|x|)^2}{1-|x|}\mathrm{d}x = +\infty,$$

i.e., $-u \notin W_0^{1,2}(B_e(0,1), F_1, \mathsf{m}_1)$, contradicting the vector space structure of the set $W_0^{1,2,1}(B_e(0,1)) = W_0^{1,2}(B_e(0,1), F_1, \mathsf{m}_1)$. $\square$

Remark 4.1. Let $a \in [0,1)$. For every $x \in B_e(0,1)$, one has $0 < 1-a^2 \leq 1-a^2|x|^2 \leq 1$; thus, the volume forms $\mathrm{d}V_{F_a}(x)$ and $\mathrm{d}V_{g_K}(x)$ generate equivalent measures. Moreover, one also has that

$$\frac{1}{(1+a)^2} g_K^*(y,y) \leq [F_a^*(x,y)]^2 \leq \frac{1}{(1-a)^2} g_K^*(y,y),\ x \in B_e(0,1),\ y \in \mathbb{R}^n. \tag{4.18}$$

Consequently,

$$\frac{(1-a^2)^{\frac{n+1}{4}}}{1+a} \|u\|_{H^1_{g_K}} \leq \|u\|_{F_a} \leq \frac{1}{1-a} \|u\|_{H^1_{g_K}},\ u \in C_0^\infty(B_e(0,1)).$$

In particular, the topologies generated by $(W_0^{1,2,a}(B_e(0,1)), \|\cdot\|_{F_a})$ and $(H^1_{g_K}(B_e(0,1)), \|\cdot\|_{H^1_{g_K}})$ are equivalent whenever $a \in [0,1)$. Moreover, a result of Federer and Fleming [40] for the Klein ball model $(B_e(0,1), F_0)$ states that

$$\int_{B_e(0,1)} u^2(x)\mathrm{d}V_{g_K}(x) \leq \frac{4}{(n-1)^2} \int_{B_e(0,1)} g_K^*(Du(x), Du(x))\mathrm{d}V_{g_K}(x),\ \forall C_0^\infty(B_e(0,1)). \tag{4.19}$$

Therefore, the norm $\|\cdot\|_{H^1_{g_K}}$ and the 'gradient' norm over the Klein metric model given by $u \mapsto \|u\|_K = \left(\int_{B_e(0,1)} g_K^*(Du(x), Du(x))\mathrm{d}V_{g_K}(x) \right)^{\frac{1}{2}}$ are also equivalent, i.e.,

$$\|u\|_K \leq \|u\|_{H^1_{g_K}} \leq \left(1 + \frac{4}{(n-1)^2}\right)^{\frac{1}{2}} \|u\|_K. \tag{4.20}$$

Remark 4.2. The space $W_0^{1,2,1}(B_e(0,1))$ is a closed convex cone in $L_1^2(B_e(0,1))$, where $L_a^p(B_e(0,1))$ denotes the usual class of measurable functions $u : B_e(0,1) \to \mathbb{R}$ such that

$$\|u\|_{L_a^p} = \left(\int_{B_e(0,1)} |u(x)|^p \mathrm{d}V_{F_a}(x) \right)^{\frac{1}{p}} < \infty,$$

whenever $1 \leq p < \infty$. L_0^p will be denoted in the usual way by L^p.

4.2 Sublinear problems on the Funk ball

Consider the model problem

$$\begin{cases} -\boldsymbol{\Delta}_{F_a} u(x) = \lambda \kappa(x) h(u(x)) & \text{in} \quad B_e(0,1), \\ u(x) \to 0, & \text{if} \quad |x| \to 1, \end{cases} \tag{$\mathcal{P}_\lambda$}$$

where $a \in [0,1)$, F_a is the Funk-type metric (4.12), $\lambda \geq 0$ is a parameter, $\kappa \in L^1(B_e(0,1)) \cap L^\infty(B_e(0,1))$ and $h : [0,\infty) \to \mathbb{R}$ is a continuous function which fulfills the properties:

(h1) $h(s) = o(s)$ as $s \to 0^+$ and $s \to \infty$;

(h2) $H(s_0) > 0$ for some $s_0 > 0$, where $H(s) = \int_0^s h(t)\mathrm{d}t$.

Note that $h(s) = o(s)$ as $s \to \infty$ implies the sublinearity of h at infinity. Furthermore, due to both (h1) and (h2), the number $c_h := \max_{s>0} \frac{h(s)}{s}$ is well-defined and positive. Since $h(0) = 0$, we can extend h as $h(s) = 0$ for every $s \leq 0$.

Theorem 4.3. (Kristály and Rudas, [120]) *Let $a \in [0,1)$ be fixed, $\kappa \in L^1(B_e(0,1)) \cap L^\infty(B_e(0,1)) \setminus \{0\}$ be a radially symmetric nonnegative function and a continuous function $h : [0,\infty) \to \mathbb{R}$ verifying* (h1) *and* (h2)*. Then, we have:*

(i) *$(\mathcal{P}_\lambda)$ has only the zero solution whenever $0 \leq \lambda < c_h^{-1} \|\kappa\|_{L^\infty}^{-1} \frac{(n-1)^2(1-a^2)^{\frac{n+1}{2}}}{4(1+a)^2}$;*

(ii) *there exists $\widetilde{\lambda} > 0$ such that $(\mathcal{P}_\lambda)$ has at least two distinct non-zero, nonnegative, radially symmetric weak solutions whenever $\lambda > \widetilde{\lambda}$.*

Proof. First of all, by (1.16), an element $u \in W_0^{1,2,a}(B_e(0,1))$ is a weak solution of problem $(\mathcal{P}_\lambda)$ if $u(x) \to 0$ as $|x| \to 1$ and

$$\int_{B_e(0,1)} Dv(\boldsymbol{\nabla}_{F_a} u)\mathrm{d}V_{F_a}(x) = \lambda \int_{B_e(0,1)} \kappa(x)h(u(x))v(x)\mathrm{d}V_{F_a}(x), \ \forall v \in C_0^\infty(B_e(0,1)). \tag{4.21}$$

(i) Let $u \in W_0^{1,2,a}(B_e(0,1))$ be a weak solution to $(\mathcal{P}_\lambda)$. By density reasons, in (4.21) we may use $v = u$ as a test-function, obtaining by means of relations (1.11), (4.14), (4.19) and (4.18) that

$$\begin{aligned}
\int_{B_e(0,1)} \left[F_a^*(x, Du(x))\right]^2 \mathrm{d}V_{F_a}(x) &= \int_{B_e(0,1)} Du(\boldsymbol{\nabla}_{F_a} u)\mathrm{d}V_{F_a}(x) = \lambda \int_{B_e(0,1)} \kappa(x)h(u(x))u(x)\mathrm{d}V_{F_a}(x) \\
&\leq \lambda c_h \|\kappa\|_{L^\infty} \int_{B_e(0,1)} u^2(x)\mathrm{d}V_{g_K}(x) \\
&\leq \frac{4\lambda c_h \|\kappa\|_{L^\infty}}{(n-1)^2} \int_{B_e(0,1)} g_K^*(Du(x), Du(x))\mathrm{d}V_{g_K}(x) \\
&\leq \frac{4\lambda c_h \|\kappa\|_{L^\infty}(1+a)^2}{(n-1)^2(1-a^2)^{\frac{n+1}{2}}} \int_{B_e(0,1)} \left[F_a^*(x, Du(x))\right]^2 \mathrm{d}V_{F_a}(x).
\end{aligned}$$

Consequently, if $0 \leq \lambda < c_h^{-1} \|\kappa\|_{L^\infty}^{-1} \frac{(n-1)^2(1-a^2)^{\frac{n+1}{2}}}{4(1+a)^2}$, u is necessarily 0.

(ii) The proof is divided into several steps.

Step 1. Consider $(H_{g_K}^1(B_e(0,1)), \|\cdot\|_{H_{g_K}^1})$ the usual Riemannian Sobolev space defined on $(B_e(0,1), F_0)$, see Hebey [52]. We observe that the topologies generated by the Sobolev spaces $(W_0^{1,2}(B_e(0,1), F_a, \mathsf{m}_a), \|\cdot\|_{F_a})$ and $(H_{g_K}^1(B_e(0,1)), \|\cdot\|_{H_{g_K}^1})$ are equivalent whenever $a \in [0,1)$. Let $\mathcal{J}_\lambda : H_{g_K}^1(B_e(0,1)) \to \mathbb{R}$ be the energy functional associated with problem $(\mathcal{P}_\lambda)$, i.e.,

$$\mathcal{J}_\lambda(u) = \frac{1}{2}\mathcal{E}(u) - \lambda \mathcal{H}(u),$$

where

$$\mathcal{E}(u) := \int_{B_e(0,1)} [F_a^*(x, Du(x))]^2 \, \mathsf{dm}_a(x) \text{ and } \mathcal{H}(u) := \int_{B_e(0,1)} \kappa(x) H(u(x)) \mathsf{dm}_a(x).$$

Due to (h1), the energy $\mathcal{J}_\lambda$ is well-defined and of class C^1. Furthermore, by relation (1.12), one has that

$$\mathcal{J}'_\lambda(u)(v) = \int_{B_e(0,1)} [Dv(\boldsymbol{\nabla}_{F_a} u)(x) - \lambda\kappa(x) h(u(x)) v(x)] \, \mathsf{dm}_a(x).$$

Accordingly, $\mathcal{J}'_\lambda(u) = 0$ holds if and only if u is a weak solution to problem $(\mathcal{P}_\lambda)$.

Step 2. In spite of the fact that $H^1_{g_K}(B_e(0,1))$ is embedded into $L^p(B_e(0,1))$, $p \in [2, 2^\star)$, see Hebey [52], the embedding is not compact. To recover the compactness, we consider the subspace

$$H_r(B_e(0,1)) = \left\{u \in H^1_{g_K}(B_e(0,1)) : u(x) = u(|x|)\right\}$$

of radially symmetric functions in $H^1_{g_K}(B_e(0,1))$. Using a Strauss-type inequality, the embedding $H_r(B_e(0,1)) \hookrightarrow L^p(B_e(0,1))$ is compact for every $p \in (2, 2^\star)$. Moreover, for every $u \in H_r(B_e(0,1))$, a Strauss-type radial estimate gives that $u(x) \to 0$ as $|x| \to 1$.

Consider the action of the orthogonal group $O(n)$ on the Sobolev space $H^1_{g_K}(B_e(0,1))$ defined by

$$(\tau * u)(x) = u(\tau^{-1}x), \ u \in H^1_{g_K}(B_e(0,1)), \ \tau \in O(n), \ x \in B_e(0,1). \tag{4.22}$$

We observe that the fixed point set of $O(n)$ on $H^1_{g_K}(B_e(0,1))$ is exactly the subspace $H_r(B_e(0,1))$ of the radially symmetric functions in $H^1_{g_K}(B_e(0,1))$, see §1.2.1. Furthermore, F_a^* is $O(n)$-invariant, i.e.,

$$F_a^*(\tau x, \tau y) = F_a^*(x, y), \ \forall \tau \in O(n), \ x \in B_e(0,1), \ y \in \mathbb{R}^n. \tag{4.23}$$

Accordingly, since the chain rule gives $D(\tau * u)(x) = (\tau^{-1})^t Du(\tau^{-1}x) = \tau Du(\tau^{-1}x)$, we have that

$$\begin{aligned}
\mathcal{E}(\tau * u) &= \int_{B_e(0,1)} [F_a^*(x, D(\tau * u)(x))]^2 \, \mathsf{dm}_a(x) \\
&= \int_{B_e(0,1)} \left[F_a^*(x, \tau Du(\tau^{-1}x))\right]^2 \mathsf{dm}_a(x) && [\text{change } \tau^{-1}x = z] \\
&= \int_{B_e(0,1)} [F_a^*(\tau z, \tau Du(z))]^2 \, \mathsf{dm}_a(\tau z) && [\text{see (4.23) and } \mathsf{dm}_a(\tau z) = \mathsf{dm}_a(z)] \\
&= \int_{B_e(0,1)} [F_a^*(z, Du(z))]^2 \, \mathsf{dm}_a(z) \\
&= \mathcal{E}(u),
\end{aligned}$$

for every $\tau \in O(n)$ and $u \in H^1_{g_K}(B_e(0,1))$, which means that $\mathcal{E}$ is $O(n)$-invariant. A similar argument shows that $\mathcal{H}$ is also $O(n)$-invariant and the group $O(n)$ acts isometrically on $H^1_{g_K}(B_e(0,1))$, i.e.,

$$\begin{cases} \mathcal{H}(\tau * u) = \mathcal{H}(u), \\ \|\tau * u\|_{H^1_{g_K}} = \|u\|_{H^1_{g_K}}, \end{cases} \qquad \forall \tau \in O(n),\ u \in H^1_{g_K}(B_e(0,1)).$$

Using these properties, we have that the energy functional $\mathcal{J}_\lambda$ is $O(n)$-invariant. Thus, by the smooth principle of symmetric criticality of Palais (see Theorem 1.6) if follows that the critical points of $\mathcal{R}_\lambda = \mathcal{J}_\lambda|_{H_r(B_e(0,1))}$ are also critical points of $\mathcal{J}_\lambda$. Accordingly, in order to find radially symmetric weak solutions to problem $(\mathcal{P}_\lambda)$, it is sufficient to guarantee critical points for $\mathcal{R}_\lambda$. Let $\mathcal{E}_r$ and $\mathcal{H}_r$ be the restriction of $\mathcal{E}$ and $\mathcal{H}$ to $H_r(B_e(0,1))$, respectively.

Step 3. We assert that

$$\lim_{\substack{u\in H_r(B_e(0,1))\\ \|u\|_{H^1_{g_K}}\to 0}} \frac{\mathcal{H}_r(u)}{\|u\|^2_{H^1_{g_K}}} = \lim_{\substack{u\in H_r(B_e(0,1))\\ \|u\|_{H^1_{g_K}}\to \infty}} \frac{\mathcal{H}_r(u)}{\|u\|^2_{H^1_{g_K}}} = 0. \tag{4.24}$$

Due to (h1), for every $\varepsilon > 0$ there exists $\delta_\varepsilon \in (0,1)$ such that

$$0 \leq |h(s)| \leq \frac{\varepsilon}{\|\kappa\|_{L^\infty}}|s|,\ \forall |s| \leq \delta_\varepsilon,\ |s| \geq \delta_\varepsilon^{-1}. \tag{4.25}$$

Fix $p \in (2, 2^\star)$; clearly, the function $s \mapsto \frac{h(s)}{s^{p-1}}$ is bounded on $[\delta_\varepsilon, \delta_\varepsilon^{-1}]$. Therefore, for some $m_\varepsilon > 0$, one has that

$$0 \leq |h(s)| \leq \frac{\varepsilon}{\|\kappa\|_{L^\infty}}|s| + m_\varepsilon |s|^{p-1}, \quad \forall s \in \mathbb{R}. \tag{4.26}$$

Thus, for every $u \in H_r(B_e(0,1))$, it yields that

$$\begin{aligned} 0 \leq |\mathcal{H}_r(u)| &\leq \int_{B_e(0,1)} \kappa(x)|H(u(x))|\mathrm{d}V_{F_a}(x) \\ &\leq \int_{B_e(0,1)} \kappa(x)\left[\frac{\varepsilon}{2\|\kappa\|_{L^\infty}}u^2(x) + \frac{m_\varepsilon}{p}|u(x)|^p\right]\mathrm{d}V_{g_K}(x) \\ &\leq \int_{B_e(0,1)} \left[\frac{\varepsilon}{2}u^2(x) + \frac{m_\varepsilon}{p}\kappa(x)|u(x)|^p\right]\mathrm{d}V_{g_K}(x) \\ &\leq \frac{\varepsilon}{2}\|u\|^2_{H^1_{g_K}} + \frac{m_\varepsilon}{p}\|\kappa\|_{L^\infty} S_p^p \|u\|^p_{H^1_{g_K}}, \end{aligned}$$

where $S_p > 0$ is the best embedding constant in $H_r(B_e(0,1)) \hookrightarrow L^p(B_e(0,1))$. Accordingly, for every $u \in H_r(B_e(0,1)) \setminus \{0\}$, one obtains that

$$0 \leq \frac{|\mathcal{H}_r(u)|}{\|u\|^2_{H^1_{g_K}}} \leq \frac{\varepsilon}{2} + \frac{m_\varepsilon}{p}\|\kappa\|_{L^\infty} S_p^p \|u\|^{p-2}_{H^1_{g_K}}.$$

Since $p > 2$ and $\varepsilon > 0$ is arbitrarily small, the first limit in (4.24) follows once $\|u\|_{H^1_{g_K}} \to 0$ in $H_r(B_e(0,1))$.

Let $q \in (1,2)$ be fixed. Since $h \in C(\mathbb{R},\mathbb{R})$, there also exists a number $M_\varepsilon > 0$ such that

$$0 \leq \frac{|h(s)|}{s^{q-1}} \leq M_\varepsilon, \quad \forall s \in [\delta_\varepsilon, \delta_\varepsilon^{-1}],$$

where $\delta_\varepsilon \in (0,1)$ is from (4.25). The latter relation together with (4.25) give the inequality

$$0 \leq |h(s)| \leq \frac{\varepsilon}{\|\kappa\|_{L^\infty}}|s| + M_\varepsilon |s|^{q-1}, \ \forall s \in \mathbb{R}.$$

Similarly as above, it yields that

$$0 \leq |\mathcal{H}_r(u)| \quad \leq \quad \frac{\varepsilon}{2}\|u\|^2_{H^1_{g_K}} + \frac{M_\varepsilon}{q}\|\kappa\|_{L^{\frac{2}{2-q}}}\|u\|^q_{H^1_{g_K}}. \tag{4.27}$$

For every $u \in H_r(B_e(0,1)) \setminus \{0\}$, we have that

$$0 \leq \frac{|\mathcal{H}_r(u)|}{\|u\|^2_{H^1_{g_K}}} \quad \leq \quad \frac{\varepsilon}{2} + \frac{M_\varepsilon}{q}\|\kappa\|_{L^{\frac{2}{2-q}}}\|u\|^{q-2}_{H^1_{g_K}}.$$

Since $\varepsilon > 0$ is arbitrary and $q \in (1,2)$, taking the limit $\|u\|_{H^1_{g_K}} \to \infty$ in $H_r(B_e(0,1))$, we obtain the second relation in (4.24).

Step 4. We are going to prove that the functional $\mathcal{R}_\lambda$ is bounded from below, coercive, and satisfies the (PS)-condition on $H_r(B_e(0,1))$ for every $\lambda \geq 0$. At first, by (4.27), it follows that

$$\begin{aligned} \mathcal{R}_\lambda(u) \quad &= \quad \frac{1}{2}\mathcal{E}_r(u) - \lambda\mathcal{H}_r(u) \\ &\geq \quad \frac{(1-a^2)^{\frac{n+1}{2}}}{2(1+a)^2}\|u\|^2_K - \lambda\frac{\varepsilon}{2}\|u\|^2_{H^1_{g_K}} - \lambda\frac{M_\varepsilon}{r}\|\kappa\|_{L^{\frac{2}{2-r}}}\|u\|^r_{H^1_{g_K}}. \end{aligned}$$

Since $\|\cdot\|_{H^1_{g_K}}$ and $\|\cdot\|_K$ are equivalent norms (see (4.20)) and $r < 2$, by choosing $\varepsilon > 0$ sufficiently small, it follows that $\mathcal{R}_\lambda$ is bounded from below and coercive, i.e., $\mathcal{R}_\lambda(u) \to +\infty$ as $\|u\|_{H^1_{g_K}} \to +\infty$.

Now, let $\{u_k\}_k$ be a sequence in $H_r(B_e(0,1))$ such that $\{\mathcal{R}_\lambda(u_k)\}_k$ is bounded and $\|\mathcal{R}'_\lambda(u_k)\|_* \to 0$. Since $\mathcal{R}_\lambda$ is coercive, the sequence $\{u_k\}_k$ is bounded in $H_r(B_e(0,1))$. Therefore, up to a subsequence, we may suppose that $u_k \to u$ weakly in $H_r(B_e(0,1))$ and $u_k \to u$ strongly in $L^p(B_e(0,1))$ for some $u \in H_r(B_e(0,1))$ and $p \in (2, 2^\star)$. In particular, we have that

$$\mathcal{R}'_\lambda(u)(u-u_k) \to 0 \text{ and } \mathcal{R}'_\lambda(u_k)(u-u_k) \to 0 \text{ as } k \to \infty. \tag{4.28}$$

A direct computation gives that

$$\int_{B_e(0,1)} (Du(x) - Du_k(x))(\boldsymbol{\nabla}_{F_a}u(x) - \boldsymbol{\nabla}_{F_a}u_k(x))\mathrm{d}V_{F_a}(x)$$

$$= \mathcal{R}'_\lambda(u)(u-u_k) - R'_\lambda(u_k)(u-u_k) + \lambda\int_{B_e(0,1)} \kappa(x)[h(u_k) - h(u)](u_k - u)\mathrm{d}V_{F_a}(x).$$

By (4.28), the first two terms tend to zero. Moreover, due to (4.26), it follows that

$$\begin{aligned} T &:= \int_{B_e(0,1)} \kappa(x)|h(u_k) - h(u)| \cdot |u_k - u| \mathrm{d}V_{F_a}(x) \\ &\leq \int_{B_e(0,1)} \left(\varepsilon(|u_k| + |u|) + m_\varepsilon \|\kappa\|_{L^\infty}(|u_n|^{p-1} + |u|^{p-1})\right) |u_k - u| \mathrm{d}V_{g_K}(x) \\ &\leq \varepsilon(\|u_k\|_{H^1_{g_K}} + \|u\|_{H^1_{g_K}})\|u_k - u\|_{H^1_{g_K}} + m_\varepsilon \|\kappa\|_{L^\infty}(\|u_k\|^{p-1}_{L^p} + \|u\|^{p-1}_{L^p})\|u_n - u\|_{L^p}. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary small and $u_k \to u$ strongly in $L^p(B_e(0,1))$, the last expression tends to zero.

Moreover, one has the inequality

$$\begin{aligned} \mathcal{E}_r(u - u_k) &= \int_{B_e(0,1)} [F_a^*(Du(x) - Du_k(x))]^2 \, \mathrm{d}V_{F_a}(x) \\ &\leq \left(\frac{1+a}{1-a}\right)^2 \int_{B_e(0,1)} (Du(x) - Du_k(x))(\boldsymbol{\nabla}_{F_a} u(x) - \boldsymbol{\nabla}_{F_a} u_k(x)) \mathrm{d}V_{F_a}(x). \end{aligned}$$

Therefore, $\mathcal{E}_r(u-u_k) \to 0$ as $k \to \infty$, which means in particular (see Remark 4.1) that $\{u_k\}_k$ converges strongly to u in $H_r(B_e(0,1))$.

Step 5. By the assumption on κ and using (h2), one can find a truncation function $u_0 \in H_r(B_e(0,1)) \setminus \{0\}$ such that $\mathcal{H}_r(u_0) > 0$. Accordingly, the number

$$\widetilde{\lambda} := \inf_{\substack{u \in H_r(B_e(0,1)) \\ \mathcal{H}_r(u) > 0}} \frac{\mathcal{E}_r(u)}{2\mathcal{H}_r(u)}$$

is well-defined. By relation (4.24), we obtain that $0 < \widetilde{\lambda} < \infty$. By fixing $\lambda > \widetilde{\lambda}$, there exists $\widetilde{u}_\lambda \in H_r(B_e(0,1))$ with the property $\mathcal{H}_r(\widetilde{u}_\lambda) > 0$ and $\lambda > \dfrac{\mathcal{E}_r(\widetilde{u}_\lambda)}{2\mathcal{H}_r(\widetilde{u}_\lambda)} \geq \widetilde{\lambda}$. Therefore,

$$c_\lambda^1 := \inf_{H_r(B_e(0,1))} \mathcal{R}_\lambda \leq \mathcal{R}_\lambda(\widetilde{u}_\lambda) = \frac{1}{2}\mathcal{E}_r(\widetilde{u}_\lambda) - \lambda \mathcal{H}_r(\widetilde{u}_\lambda) < 0.$$

By Step 4, the functional $\mathcal{R}_\lambda$ is bounded from below and satisfies the (PS)-condition. Thus, c_λ^1 is a critical value of the functional $\mathcal{R}_\lambda$ (see Theorem 1.5), i.e., there exists $u_\lambda^1 \in H_r(B_e(0,1))$ such that $\mathcal{R}_\lambda(u_\lambda^1) = c_\lambda^1 < 0$ and $\mathcal{R}'_\lambda(u_\lambda^1) = 0$. Clearly, one has that $u_\lambda^1 \neq 0$ since $\mathcal{R}_\lambda(0) = 0$.

Let $\lambda > \widetilde{\lambda}$ be fixed. Simple estimates give that

$$\mathcal{R}_\lambda(u) = \frac{1}{2}\mathcal{E}_r(u) - \lambda \mathcal{H}_r(u) \geq \frac{(1-a^2)^{\frac{n+1}{2}}}{4(1+a)^2} \|u\|^2_{H^1_{g_K}} - \lambda \frac{m_\lambda}{p} \|\kappa\|_{L^\infty} S_p^p \|u\|^p_{H^1_{g_K}}, \tag{4.29}$$

where $p \in (2, 2^\star)$ and $m_\lambda := m_\varepsilon > 0$. Consider the number

$$\rho_\lambda := \min\left\{ \frac{\|\widetilde{u}_\lambda\|_{H^1_{g_K}}}{2}, \left(\frac{(1-a^2)^{\frac{n+1}{2}}}{8\lambda \|\kappa\|_{L^\infty} S_p^p m_\lambda (1+a)^2 (1 + \frac{4}{(n-1)^2})} \right)^{\frac{1}{p-2}} \right\}.$$

By (4.29) and Step 5 it follows that

$$\inf_{\|u\|_{H^1_{g_K}}=\rho_\lambda} \mathcal{R}_\lambda(u) = \eta_\lambda > 0 = \mathcal{R}_\lambda(0) > \mathcal{R}_\lambda(\widetilde{u}_\lambda),$$

i.e., $\mathcal{R}_\lambda$ provides the mountain pass geometry (see Theorem 1.7). Due to Step 4, one may apply the Mountain Pass Theorem, resulting the existence of $u^2_\lambda \in H_r(B_e(0,1))$ such that $\mathcal{R}'_\lambda(u^2_\lambda) = 0$ and $\mathcal{R}_\lambda(u^2_\lambda) = c^2_\lambda$, where the number c^2_λ is given by $c^2_\lambda = \inf_{\gamma\in\Gamma_0} \max_{t\in[0,1]} \mathcal{R}_\lambda(\gamma(t))$, with $\Gamma_0 = \{\gamma \in C([0,1]; H_r(B_e(0,1))) : \gamma(0) = 0,\ \gamma(1) = \widetilde{u}_\lambda\}$. Note that

$$c^2_\lambda \geq \inf_{\|u\|_{H^1_{g_K}}=\rho_\lambda} \mathcal{R}_\lambda(u) > 0.$$

Therefore, $0 \neq u^2_\lambda \neq u^1_\lambda$. Since by extension $h(s) = 0$ for every $s \leq 0$, both elements u^1_λ and u^2_λ are nonnegative, which concludes our proof. □

4.3 Unipolar Poisson equations on Finsler-Hadamard manifolds

In this section we establish some qualitative results concerning the model unipolar Poisson problem

$$\begin{cases} \mathcal{L}^\mu_F u = 1 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \qquad (\mathcal{P}^\mu_\Omega)$$

where $\mu \in \mathbb{R}$ is a parameter, Ω is an open bounded domain in an n-dimensional ($n \geq 3$), not necessarily reversible Finsler-Hadamard manifold (M, F) endowed with its usual canonical measure m, $x_0 \in \Omega$ is fixed and

$$\mathcal{L}^\mu_F u = \boldsymbol{\Delta}_F(-u) - \mu \frac{u}{d^2_F(x_0, x)}, \quad u \in W^{1,2}_0(\Omega, F, \mathsf{m}),$$

is the *singular Finsler-Laplace operator.*

At first, we need some preparatory results. We recall the notation $\widetilde{\mu} := \frac{n-2}{2}$ from Section 3.3.

Lemma 4.1. *Let (M, F) be an n-dimensional ($n \geq 3$) Finsler-Hadamard manifold with $\mathbf{S} = 0$ and $l_F > 0$, and let $\Omega \subset M$ be an open domain. If $\mathcal{L}^\mu_F u \leq \mathcal{L}^\mu_F v$ in Ω and $u \leq v$ on $\partial\Omega$, then $u \leq v$ a.e. in Ω whenever $\mu \in [0, l_F r_F^{-2} \widetilde{\mu}^2)$.*

Proof. Assume that $\Omega_+ = \{x \in \Omega : u(x) > v(x)\}$ has a positive m-measure. Multiplying $\mathcal{L}^\mu_F u \leq \mathcal{L}^\mu_F v$ by $(u - v)_+$, relation (1.16) gives that

$$\int_{\Omega_+} (D(-v) - D(-u))(\boldsymbol{\nabla}_F(-v) - \boldsymbol{\nabla}_F(-u))\mathsf{d}\mathsf{m}(x) - \mu \int_{\Omega_+} \frac{(u-v)^2}{d^2_F(x_0, x)} \mathsf{d}\mathsf{m}(x) \leq 0.$$

By (1.13) and the mean value theorem, the definition of l_F yields that

$$\begin{aligned}(D(-v)-D(-u))(\boldsymbol{\nabla}_F(-v)-\boldsymbol{\nabla}_F(-u)) &\geq l_F\left[F^*(x,D(-v)-D(-u))^2=l_F F^*(x,D(u-v))\right]^2\\ &\geq l_F r_F^{-2}\left[F^*(x,-D(u-v))\right]^2,\end{aligned}$$

for every $x\in\Omega_+$. Combining these relations with Theorem 3.9, it follows that

$$\left(l_F r_F^{-2}-\frac{\mu}{\widetilde{\mu}^2}\right)\int_{\Omega_+}\left[F^*(x,-D(u-v)(x))\right]^2\mathsf{dm}(x)\leq 0,$$

which is a contradiction. □

Lemma 4.2. *Let (M,F) be an n-dimensional $(n\geq 3)$ Finsler-Hadamard manifold with $\mathbf{S}=0$ and $l_F>0$, and let $\Omega\subseteq M$ be an open domain and $x_0\in\Omega$ be a fixed point. Then the functional $\mathcal{K}_\mu:W_0^{1,2}(\Omega,F,\mathsf{m})\to\mathbb{R}$ defined by*

$$\mathcal{K}_\mu(u)=\int_\Omega\left[F^*(x,Du(x))\right]^2\mathsf{dm}(x)-\mu\int_\Omega\frac{u^2(x)}{d_F^2(x_0,x)}\mathsf{dm}(x)$$

is positive unless $u=0$ and strictly convex whenever $0\leq\mu<l_F r_F^{-2}\widetilde{\mu}^2$.

Proof. Let $0\leq\mu<l_F r_F^{-2}\widetilde{\mu}^2$ and $x_0\in\Omega$ be arbitrarily fixed. By (4.3), one has that $r_F^2\leq l_F^{-1}<+\infty$. The positivity of $\mathcal{K}_\mu$ follows by Theorem 3.9. Let $0<t<1$ and $u,v\in W_0^{1,2}(\Omega,F,\mathsf{m})$ be fixed where $u\neq v$. Then, by using inequalities (4.2) and

$$F^*(x,D(v-u)(x))\geq r_F^{-1}F^*(x,-D(|v-u|)(x)),\ x\in\Omega,$$

and by applying Theorem 3.9, one has that

$$\begin{aligned}\mathcal{K}_\mu\left(tu+(1-t)v\right) &= \int_\Omega\left[F^*(x,tDu(x)+(1-t)Dv(x))\right]^2\mathsf{dm}(x)-\mu\int_\Omega\frac{(tu+(1-t)v)^2}{d_F^2(x_0,x)}\mathsf{dm}(x)\\ &\leq t\int_\Omega\left[F^*(x,Du(x))\right]^2\mathsf{dm}(x)+(1-t)\int_\Omega\left[F^*(x,Dv(x))\right]^2\mathsf{dm}(x)\\ &\quad -l_F t(1-t)\int_\Omega\left[F^*(x,D(v-u)(x))\right]^2\mathsf{dm}(x)-\mu\int_\Omega\frac{(tu+(1-t)v)^2}{d_F^2(x_0,x)}\mathsf{dm}(x)\\ &= t\mathcal{K}_\mu(u)+(1-t)\mathcal{K}_\mu(v)\\ &\quad -t(1-t)l_F\int_\Omega\left(\left[F^*(x,D(v-u)(x))\right]^2-\mu l_F^{-1}\frac{(v-u)^2}{d_F^2(x_0,x)}\right)\mathsf{dm}(x)\\ &\leq t\mathcal{K}_\mu(u)+(1-t)\mathcal{K}_\mu(v)\\ &\quad -t(1-t)l_F r_F^{-2}\int_\Omega\left(\left[F^*(x,-D|v-u|(x))\right]^2-\mu l_F^{-1}r_F^2\frac{(v-u)^2}{d_F^2(x_0,x)}\right)\mathsf{dm}(x)\\ &< t\mathcal{K}_\mu(u)+(1-t)\mathcal{K}_\mu(v),\end{aligned}$$

which concludes the proof. □

We introduce the *singular energy functional* associated with the operator $\mathcal{L}_F^\mu$ on $W_0^{1,2}(\Omega, F, \mathsf{m})$,

$$\mathcal{E}_\mu(u) = (\mathcal{L}_F^\mu u)(u).$$

According to (1.16), we have that

$$\mathcal{E}_\mu(u) = \int_\Omega [F^*(x, -Du(x))]^2 \,\mathsf{dm}(x) - \mu \int_M \frac{u^2(x)}{d_F^2(x_0, x)} \mathsf{dm}(x) = \mathcal{K}_\mu(-u).$$

Theorem 4.4. (Farkas, Kristály and Varga [107]) *Let (M, F) be an n-dimensional $(n \geq 3)$ Finsler-Hadamard manifold with $\mathbf{S} = 0$ and $l_F > 0$, and let $\Omega \subset M$ be an open bounded domain and $x_0 \in \Omega$ be a fixed point. Then problem $(\mathcal{P}_\Omega^\mu)$ has a unique, nonnegative weak solution for every $\mu \in [0, l_F r_F^{-2} \widetilde{\mu}^2)$.*

Proof. Let $\mu \in [0, l_F r_F^{-2} \widetilde{\mu}^2)$ be fixed and consider the energy functional associated with problem $(\mathcal{P}_\Omega^\mu)$, i.e.,

$$\mathcal{F}_\mu(u) = \frac{1}{2}\mathcal{K}_\mu(-u) - \int_\Omega u(x) \mathsf{dm}(x), \quad u \in W_0^{1,2}(\Omega, F, \mathsf{m}).$$

It is clear that $\mathcal{F}_\mu \in C^1(W_0^{1,2}(\Omega, F, \mathsf{m}), \mathbb{R})$, and its critical points are precisely the weak solutions to problem $(\mathcal{P}_\Omega^\mu)$. Let $R > 0$ be such that $\Omega \subset B_F^+(x_0, R)$. According to Wu and Xin [96, Theorem 7.3], we have that

$$\lambda_1(\Omega) = \inf_{u \in W_0^{1,2}(\Omega, F, \mathsf{m}) \setminus \{0\}} \frac{\displaystyle\int_\Omega [F^*(x, Du(x))]^2 \,\mathsf{dm}(x)}{\displaystyle\int_\Omega u^2(x) \mathsf{dm}(x)} \geq \frac{(n-1)^2}{4R^2 r_F^2}.$$

Consequently, for every $u \in W_0^{1,2}(\Omega, F, \mathsf{m})$, one obtains that

$$\int_\Omega [F^*(x, Du(x))]^2 \,\mathsf{dm}(x) \geq \frac{\lambda_1(\Omega)}{1 + \lambda_1(\Omega)} \|u\|_F^2.$$

Since $\|\cdot\|_F$ and $\|\cdot\|_{F_s}$ are equivalent (see (4.5)), we conclude that $\mathcal{F}_\mu$ is bounded from below and coercive on the reflexive Banach space $(W_0^{1,2}(\Omega, F, \mathsf{m}), \|\cdot\|_{F_s})$, i.e., $\mathcal{F}_\mu(u) \to +\infty$ whenever $\|u\|_{F_s} \to +\infty$. Due to Theorem 1.4 and Remark 1.2, $\mathcal{F}_\mu$ has a global minimum point $u_\mu \in W_0^{1,2}(\Omega, F, \mathsf{m})$. Moreover, due to Lemma 4.2, $\mathcal{F}_\mu$ is strictly convex on $W_0^{1,2}(\Omega, F, \mathsf{m})$, thus the minimum point $u_\mu \in W_0^{1,2}(\Omega, F, \mathsf{m})$ of $\mathcal{F}_\mu$ is unique. Lemma 4.1 implies that $u_\mu \geq 0$. $\square$

In the sequel, we establish some fine properties of the solution to the Poisson problem $(\mathcal{P}_\Omega^\mu)$. We need again further preparatory results.

Lemma 4.3. *Let (M, F) be an n-dimensional $(n \geq 3)$ Finsler-Hadamard manifold, $f \in C^2(0, \infty)$ be a non-increasing function and $x_0 \in M$. Then*

$$\mathcal{L}_F^\mu(f(d_F(x_0, x))) = -f''(d_F(x_0, x)) - f'(d_F(x_0, x)) \cdot \boldsymbol{\Delta}_F d_F(x_0, x) - \mu \frac{f(d_F(x_0, x))}{d_F^2(x_0, x)}, \ x \in M \setminus \{x_0\}.$$

Proof. Since $f' \leq 0$, the claim follows from basic properties of the Legendre transform. Namely,

$$\begin{aligned}
\boldsymbol{\Delta}_F(-f(d_F(x_0,x))) &= \operatorname{div}(\boldsymbol{\nabla}_F(-f(d_F(x_0,x)))) = \operatorname{div}\left(J^*(x, D(-f(d_F(x_0,x))))\right) \\
&= \operatorname{div}(J^*(x, -f'(d_F(x_0,x))Dd_F(x_0,x))) = \operatorname{div}(-f'(d_F(x_0,x))\boldsymbol{\nabla}_F d_F(x_0,x)) \\
&= -f''(d_F(x_0,x)) - f'(d_F(x_0,x)) \cdot \boldsymbol{\Delta}_F d_F(x_0,x),
\end{aligned}$$

which concludes the proof. □

For fixed $\mu \in [0, \widetilde{\mu}^2)$, $c \leq 0$ and $\rho > 0$, we introduce the ordinary differential equation

$$\begin{cases} f''(r) + (n-1)f'(r)\mathbf{ct}_c(r) + \mu\frac{f(r)}{r^2} + 1 = 0, \ \ r \in (0,\rho], \\ f(\rho) = 0, \ \ \displaystyle\int_0^\rho f'(r)^2 r^{n-1} \mathrm{d}r < \infty. \end{cases} \tag{$\mathcal{Q}^\mu_{c,\rho}$}$$

Lemma 4.4. *($\mathcal{Q}^\mu_{c,\rho}$) has a unique nonnegative non-increasing solution of class $C^\infty(0,\rho)$.*

Proof. Although the statement is expected to hold (due to the boundary conditions), we provide its proof which requires elements from Riemannian geometry. We fix $\mu \in [0, \widetilde{\mu}^2)$, $c \leq 0$ and $\rho > 0$. Consider the Riemannian space form (M, g_c) with constant sectional curvature $c \leq 0$, i.e., (M, g_c) is isometric to the Euclidean space when $c = 0$, or (M, g_c) is isometric to the hyperbolic space with sectional curvature $c < 0$. Let $x_0 \in M$ be fixed. Since (M, g_c) verifies the assumptions of Theorem 4.4, the problem

$$\begin{cases} -\Delta_{g_c} u - \mu\frac{u}{d^2_{g_c}(x_0,x)} = 1 & \text{in} \quad B_{g_c}(x_0,\rho), \\ u = 0 & \text{on} \quad \partial B_{g_c}(x_0,\rho), \end{cases} \tag{$\mathcal{R}^\mu_{c,\rho}$}$$

has a unique nonnegative solution u_0 which is nothing but the unique global minimum point of the energy functional $\mathcal{F}_\mu : W_0^{1,2}(B_{g_c}(x_0,\rho), g_c, \mathsf{m}) \to \mathbb{R}$ defined by

$$\mathcal{F}_\mu(u) = \frac{1}{2}\int_{B_{g_c}(x_0,\rho)} |Du(x)|^2_{g_c} \mathsf{dm}(x) - \frac{\mu}{2}\int_{B_{g_c}(x_0,\rho)} \frac{u^2(x)}{d^2_{g_c}(x_0,x)} \mathsf{dm}(x) - \int_{B_{g_c}(x_0,\rho)} u(x)\mathsf{dm}(x).$$

In this particular case, dm denotes the canonical Riemannian volume form on (M, g_c).

Let $u_0^* : B_{g_c}(x_0,\rho) \to [0,\infty)$ be the non-increasing symmetric rearrangement of u_0 in (M, g_c); see Section 2.3 for a similar notion. The Pólya-Szegő and Hardy-Littlewood inequalities (see Baersntein [9] and Lemma 2.2) imply that

$$\int_{B_{g_c}(x_0,\rho)} |Du_0(x)|^2_{g_c} \mathsf{dm}(x) \geq \int_{B_{g_c}(x_0,\rho)} |Du_0^*(x)|^2_{g_c} \mathsf{dm}(x),$$

and

$$\int_{B_{g_c}(x_0,\rho)} \frac{u_0^2(x)}{d^2_{g_c}(x_0,x)} \mathsf{dm}(x) \leq \int_{B_{g_c}(x_0,\rho)} \frac{(u_0^*(x))^2}{d^2_{g_c}(x_0,x)} \mathsf{dm}(x),$$

respectively. Moreover, by the Cavalieri principle, we also have that

$$\int_{B_{g_c}(x_0,\rho)} u_0(x)\mathrm{dm}(x) = \int_{B_{g_c}(x_0,\rho)} u_0^*(x)\mathrm{dm}(x).$$

Therefore, we obtain that $\mathcal{F}_\mu(u_0) \geq \mathcal{F}_\mu(u_0^*)$. Consequently, by the uniqueness of the global minimizer of $\mathcal{F}_\mu$, we have that $u_0 = u_0^*$; thus, its form is $u_0(x) = f(t)$, where $t = d_{g_c}(x_0, x)$ and $f : (0, \rho] \to \mathbb{R}$ is a nonnegative non-increasing function. Clearly, $f(\rho) = 0$ since $u_0(x) = 0$ whenever $d_{g_c}(x_0, x) = \rho$. Moreover, since $u_0 = u_0^* \in W_0^{1,2}(B_{g_c}(x_0, \rho), g_c, \mathrm{m})$, a suitable change of variables gives that $\int_0^\rho \left[f'(r)\right]^2 r^{n-1}\mathrm{d}r < \infty$. By Lemma 4.3 and Theorem 1.2, it follows that the first part of $(\mathcal{R}^\mu_{c,\rho})$ can be transformed into the first part of $(\mathcal{Q}^\mu_{c,\rho})$; in particular, problem $(\mathcal{Q}^\mu_{c,\rho})$ has a nonnegative non-increasing solution. Standard regularity theory implies that $f \in C^\infty(0, \rho)$, see Evans [39, p. 334]. Finally, if we assume that $(\mathcal{Q}^\mu_{c,\rho})$ has two distinct nonnegative non-increasing solutions f_1 and f_2, then both functions $u_i(x) = f_i(d_{g_c}(x_0, x))$ $(i \in \{1, 2\})$ verify $(\mathcal{R}^\mu_{c,\rho})$, which are distinct global minima of the functional $\mathcal{F}_\mu$, a contradiction. $\square$

Remark 4.3. Usually, we are not able to solve explicitly the ODE $(\mathcal{Q}^\mu_{c,\rho})$. However, in some particular cases we have its solution; namely,

$$\sigma_{\mu,\rho,c}(r) = \begin{cases} \frac{1}{\mu+2n}\left(\rho^2\left(\frac{r}{\rho}\right)^{-\tilde{\mu}+\sqrt{\tilde{\mu}^2-\mu}} - r^2\right), & \text{if} \quad c = 0, \\ \displaystyle\int_r^\rho \sinh(s\sqrt{-c})^{-n+1}\int_0^s \sinh(t\sqrt{-c})^{n-1}\mathrm{d}t\mathrm{d}s, & \text{if} \quad c < 0 \text{ and } \mu = 0, \\ W\left(\sqrt{\frac{1}{4}-\mu}, \rho\right)\frac{\sqrt{r}\sinh(\rho)I_{\sqrt{1/4-\mu}}(r)}{\sqrt{\rho}\sinh(r)I_{\sqrt{1/4-\mu}}(\rho)} - W\left(\sqrt{\frac{1}{4}-\mu}, r\right), & \text{if} \quad c = -1,\ n = 3 \text{ and } \mu \in \left[0, \frac{1}{4}\right), \end{cases}$$

where $W : (0, \frac{1}{2}] \times (0, \infty) \to \mathbb{R}$ is given by

$$\begin{aligned} W(\nu, r) \quad = \quad & \frac{2\nu}{(25-4\nu^2)\sin(\nu\pi)\Gamma(\nu)\sinh(r)} \times \\ & \times \left\{(5-2\nu)\,{}_3F_4\left(\left[\frac{3}{4}+\frac{\nu}{2}, \frac{5}{4}+\frac{\nu}{2}, \frac{5}{4}+\frac{\nu}{2}\right]; \left[\frac{3}{2}, 1+\nu, \frac{3}{2}+\nu, \frac{9}{4}+\frac{\nu}{2}\right], r^2\right) \times \right. \\ & \quad \times \left(2^{\nu-2}\sin(\nu\pi)K_\nu(r) + 2^{-\nu-1}\pi I_\nu(r)\right) r^{3+\nu} - \\ & \quad -\nu(5+2\nu)2^{\nu-1}{}_3F_4\left(\left[\frac{3}{4}-\frac{\nu}{2}, \frac{5}{4}-\frac{\nu}{2}, \frac{5}{4}-\frac{\nu}{2}\right]; \left[\frac{3}{2}, 1-\nu, \frac{3}{2}-\nu, \frac{9}{4}-\frac{\nu}{2}\right], r^2\right) \times \\ & \quad \left. \times\, \Gamma(\nu)^2\sin(\nu\pi)I_\nu(r)r^{3-\nu}\right\}. \end{aligned}$$

Here, I_ν and K_ν are the modified Bessel functions of the first and second kinds of order ν, respectively, while ${}_3F_4$ denotes the generalized hypergeometric function.

Theorem 4.5. (Farkas, Kristály and Varga [107]) *Let (M, F) be an n-dimensional $(n \geq 3)$ Finsler-Hadamard manifold with $\mathbf{S} = 0$ and $l_F > 0$, and let $\Omega \subset M$ be an open bounded domain. Let $\mu \in [0, l_F r_F^{-2}\tilde{\mu}^2)$ and $x_0 \in \Omega$ be fixed. If $c_1 \leq \mathbf{K} \leq c_2 \leq 0$, then the unique weak solution u to problem*

$(\mathcal{P}^\mu_\Omega)$ verifies the inequalities

$$\sigma_{\mu,\rho_1,c_1}(d_F(x_0,x)) \leq u(x) \leq \sigma_{\mu,\rho_2,c_2}(d_F(x_0,x)) \text{ for a.e. } x \in B_F^+(x_0,\rho_1),$$

where $\rho_1 = \sup\{\rho > 0 : B_F^+(x_0,\rho) \subset \Omega\}$ and $\rho_2 = \inf\{\rho > 0 : \Omega \subset B_F^+(x_0,\rho)\}$.

In particular, if $\mathbf{K} = c \leq 0$ and $\Omega = B_F^+(x_0,\rho)$ for some $\rho > 0$, then $\sigma_{\mu,\rho,c}(d_F(x_0,\cdot))$ is the unique weak solution to problem $(\mathcal{P}^\mu_{B_F^+(x_0,\rho)})$, being also a pointwise solution in $B_F^+(x_0,\rho) \setminus \{x_0\}$.

Proof. Consider u to be the unique solution to the Poisson problem $(\mathcal{P}^\mu_\Omega)$. We will state that

$$\begin{cases} \mathcal{L}^\mu_F(\sigma_{\mu,\rho_1,c_1}(d_F(x_0,x))) \leq 1 = \mathcal{L}^\mu_F(u) & \text{in} \quad B_F^+(x_0,\rho_1), \\ \sigma_{\mu,\rho_1,c_1}(d_F(x_0,x)) = 0 \leq u(x) & \text{on} \quad \partial B_F^+(x_0,\rho_1), \end{cases}$$

where $\rho_1 := \sup\{\rho > 0 : B_F^+(x_0,\rho) \subset \Omega\}$.

Firstly, since $c_1 \leq \mathbf{K}$, according to Theorem 1.2/(b) and to the fact that σ_{μ,ρ_1,c_1} is non-increasing, by $(\mathcal{Q}^\mu_{c_1,\rho_1})$ it follows that

$$\begin{aligned} 1 &= -\sigma''_{\mu,\rho_1,c_1}(d_F(x_0,x)) - (n-1)\sigma'_{\mu,\rho_1,c_1}(d_F(x_0,x))\mathbf{ct}_{c_1}(d_F(x_0,x)) - \mu\frac{\sigma_{\mu,\rho_1,c_1}(d_F(x_0,x))}{d_F^2(x_0,x)} \\ &\geq -\sigma''_{\mu,\rho_1,c_1}(d_F(x_0,x)) - \sigma'_{\mu,\rho_1,c_1}(d_F(x_0,x))\boldsymbol{\Delta}_F d_F(x_0,x) - \mu\frac{\sigma_{\mu,\rho_1,c_1}(d_F(x_0,x))}{d_F^2(x_0,x)} \\ &= \mathcal{L}^\mu_F(\sigma_{\mu,\rho_1,c_1}(d_F(x_0,x))), \end{aligned}$$

for every $x \in B_F^+(x_0,\rho_1) \setminus \{x_0\}$.

Secondly, since u is nonnegative in Ω, we have that $0 = \sigma_{\mu,\rho_1,c_1}(d_F(x_0,x)) \leq u(x)$ on $\partial B_F^+(x_0,\rho_1)$. We can apply the comparison principle (Lemma 4.1), obtaining the inequality

$$\sigma_{\mu,\rho_1,c_1}(d_F(x_0,x)) \leq u(x) \text{ for a.e. } x \in B_F^+(x_0,\rho_1).$$

In a similar way, by using Theorem 1.2/(a) and $\mathbf{K} \leq c_2$, we prove that

$$\begin{cases} 1 = \mathcal{L}^\mu_F(u) \leq \mathcal{L}^\mu_F(\sigma_{\mu,\rho_2,c_2}(d_F(x_0,x))) & \text{in} \quad \Omega, \\ u(x) = 0 \leq \sigma_{\mu,\rho_2,c_2}(d_F(x_0,x)) & \text{on} \quad \partial\Omega, \end{cases}$$

where $\rho_2 = \inf\{\rho > 0 : \Omega \subset B_F^+(x_0,\rho)\}$. Therefore, by Lemma 4.1 we have that

$$u(x) \leq \sigma_{\mu,\rho_2,c_2}(d_F(x_0,x)) \text{ for a.e. } x \in \Omega.$$

In the particular case when $\mathbf{K} = c \leq 0$ and $\Omega = B_F^+(x_0,\rho)$ for some $\rho > 0$, then $\rho_1 = \rho_2 = \rho$, and the aforementioned arguments imply that $u(x) = \sigma_{\mu,\rho,c}(d_F(x_0,x))$ is the unique (weak) solution to problem $(\mathcal{P}^\mu_{B_F^+(x_0,\rho)})$ which is also a pointwise solution in $B_F^+(x_0,\rho) \setminus \{x_0\}$. $\square$

A simple consequence of Theorem 4.5 is the following corollary.

Corollary 4.1. *Consider the Minkowski space* $(M,F)=(\mathbb{R}^n,\|\cdot\|)$ *and let* $\mu\in[0,l_F r_F^{-2}\widetilde{\mu}^2)$, $x_0\in\mathbb{R}^n$ *and* $\rho>0$ *be fixed. Then* $u=\sigma_{\mu,\rho,0}(\|\cdot-x_0\|)\in C^\infty(B_F^+(x_0,\rho)\setminus\{x_0\})$ *is the unique pointwise solution to problem* $(\mathcal{P}^\mu_{B_F^+(x_0,\rho)})$ *in* $B_F^+(x_0,\rho)\setminus\{x_0\}$.

Proof. $(M,F)=(\mathbb{R}^n,\|\cdot\|)$ being a Minkowski space, it is a Finsler-Hadamard manifold with $\mathbf{S}=0$, $\mathbf{K}=0$ and $l_F>0$. It remains to apply Theorem 4.5. $\square$

Remark 4.4. (i) In addition to the conclusions of Corollary 4.1, one can see that:

(a) $\sigma_{\mu,\rho,0}\in C^1(B_F^+(x_0,\rho))$ if and only if $\mu=0$; and

(b) $\sigma_{\mu,\rho,0}\in C^2(B_F^+(x_0,\rho))$ if and only if $\mu=0$ and $F=\|\cdot\|$ is Euclidean.

(ii) When $(M,F)=(\mathbb{R}^n,\|\cdot\|)$ is a reversible Minkowski space and $\mu=0$, Corollary 4.1 reduces to Theorem 2.1 from Ferone and Kawohl [42].

We establish an estimate for the solution of the singular Poisson equation on *backward* geodesic balls on Minkowski spaces. To do this, we assume that $\sigma_{\mu,r_F^{-1}\rho,0}$ is formally extended beyond $r_F^{-1}\rho$ by the same function, its explicit form being given after the problem $(\mathcal{Q}^\mu_{c,\rho})$. Although problem $(\mathcal{P}^\mu_{B_F^-(x_0,\rho)})$ cannot be solved explicitly in general, the following sharp estimates can be given for its unique solution by means of the reversibility constant r_F.

Proposition 4.1. *Consider the Minkowski space* $(M,F)=(\mathbb{R}^n,\|\cdot\|)$ *and let* $\mu\in[0,l_F r_F^{-2}\widetilde{\mu}^2)$, $x_0\in\mathbb{R}^n$ *and* $\rho>0$ *be fixed. If* $\widetilde{u}_{\mu,\rho}$ *denotes the unique weak solution to problem* $(\mathcal{P}^\mu_{B_F^-(x_0,\rho)})$, *then*

$$(\sigma_{\mu,r_F^{-1}\rho,0}(\|x-x_0\|))_+\leq\widetilde{u}_{\mu,\rho}(x)\leq\sigma_{\mu,r_F\rho,0}(\|x-x_0\|)\ \textit{for a.e. } x\in B_F^-(x_0,\rho).$$

Moreover, the above two bounds coincide if and only if (M,F) *is reversible.*

Proof. The proof immediately follows by the comparison principle (Lemma 4.1), showing that

$$\begin{cases}\mathcal{L}^\mu_F(w^-_{\mu,\rho})=1=\mathcal{L}^\mu_F(w^+_{\mu,\rho}) & \text{in} \quad B_F^-(x_0,\rho),\\[2mm] w^-_{\mu,\rho}\leq 0\leq w^+_{\mu,\rho} & \text{on} \quad \partial B_F^-(x_0,\rho),\end{cases}$$

where $w^-_{\mu,\rho}(x):=\sigma_{\mu,r_F^{-1}\rho,0}(\|x-x_0\|)$ and $w^+_{\mu,\rho}(x):=\sigma_{\mu,r_F\rho,0}(\|x-x_0\|)$, respectively. $\square$

The converse of Theorem 4.5 reads as follows.

Theorem 4.6. (Farkas, Kristály and Varga [107]) *Let* (M,F) *be an* n*-dimensional* $(n\geq 3)$ *Finsler-Hadamard manifold with* $\mathbf{S}=0$, $l_F>0$ *and* $\mathbf{K}\leq c\leq 0$. *Let* $\mu\in[0,l_F r_F^{-2}\widetilde{\mu}^2)$ *and* $x_0\in M$ *be fixed. If the function* $\sigma_{\mu,\rho,c}(d_F(x_0,\cdot))$ *is the unique pointwise solution of* $(\mathcal{P}^\mu_{B_F^+(x_0,\rho)})$ *in* $B_F^+(x_0,\rho)\setminus\{x_0\}$ *for some* $\rho>0$, *then the flag curvature* $\mathbf{K}(\cdot;\dot\gamma_{x_0,y}(t))=c$ *for every* $t\in[0,\rho)$ *and* $y\in T_{x_0}M\setminus\{0\}$, *where* $\gamma_{x_0,y}$ *is the constant speed geodesic with* $\gamma_{x_0,y}(0)=x_0$ *and* $\dot\gamma_{x_0,y}(0)=y$.

Proof. Fix $x_0 \in M$ and assume that the function $u(x) = \sigma_{\mu,\rho,c}(d_F(x_0,x))$ is the unique pointwise solution to problem $(\mathcal{P}^{\mu}_{B^+_F(x_0,\rho)})$ in $B^+_F(x_0,\rho) \setminus \{x_0\}$ for some $\rho > 0$ and $\mu \in [0, l_F r_F^{-2}\widetilde{\mu}^2)$. By Lemma 4.3 and from the fact that $\sigma_{\mu,\rho,c}$ is a solution of $(\mathcal{Q}^{\mu}_{c,\rho})$, one has that pointwisely

$$\boldsymbol{\Delta}_F d_F(x_0,x) = (n-1)\mathbf{ct}_c(d_F(x_0,x)),\ \forall x \in B^+_F(x_0,\rho) \setminus \{x_0\}.$$

It turns out that the latter relation is equivalent to

$$\boldsymbol{\Delta}_F w_c(d_F(x_0,x)) = 1,\ \forall x \in B^+_F(x_0,\rho) \setminus \{x_0\},$$

where

$$w_c(r) = \int_0^r \mathbf{s}_c^{-n+1}(s) \int_0^s \mathbf{s}_c^{n-1}(t)\mathrm{d}t\mathrm{d}s. \tag{4.30}$$

The proof proceeds similarly as in Theorem 3.6/(ii), obtaining that

$$\frac{\mathrm{Vol}_F(B^+_F(x_0,\rho))}{V_{c,n}(\rho)} = \lim_{s\to 0^+} \frac{\mathrm{Vol}_F(B^+_F(x_0,s))}{V_{c,n}(s)} = 1, \quad \rho > 0. \tag{4.31}$$

According to Theorem 1.1/(a), it yields $\mathbf{K}(\cdot;\dot{\gamma}_{x_0,y}(t)) = c$ for every $t \in [0,\rho)$ and $y \in T_{x_0}M$ with $F(x_0,y) = 1$, where $\gamma_{x_0,y}$ is the constant speed geodesic with $\gamma_{x_0,y}(0) = x_0$ and $\dot{\gamma}_{x_0,y}(0) = y$. $\square$

Remark 4.5. In Theorem 4.6 we stated that the flag curvature is *radially* constant with respect to the point $x_0 \in M$. The latter fact means that flag curvature is constant along every geodesics emanating from the point x_0 where the flag-poles are the velocities of the geodesics. However, for general Finsler manifolds, this statement does not imply that the flag curvature $\mathbf{K}$ is fully constant. In the particular case when $(M,F) = (M,g)$ is a Hadamard manifold (i.e., the flag curvature and sectional curvature coincide, thus the flag is not relevant), Theorems 4.5-4.6, and the classification of Riemannian space forms (see do Carmo [31, Theorem 4.1]) give a characterization of the Euclidean and hyperbolic spaces up to isometries by means of the shape of solutions to the Poisson equation $(\mathcal{P}^{\mu}_{\Omega})$ as stated in the next corollary.

Corollary 4.2. *Let (M,g) be an n-dimensional $(n \geq 3)$ Hadamard manifold with sectional curvature bounded above by $c \leq 0$. Then the following statements are equivalent:*

(i) *for some $\mu \in [0,\widetilde{\mu}^2)$ and $x_0 \in M$, the function $\sigma_{\mu,\rho,c}(d_g(x_0,\cdot))$ is the unique pointwise solution to the Poisson equation $(\mathcal{P}^{\mu}_{B_g(x_0,\rho)})$ in $B_g(x_0,\rho) \setminus \{x_0\}$ for every $\rho > 0$;*

(ii) *(M,g) is isometric to the n-dimensional space form with curvature c.*

We note that no full classification is available for Finslerian space forms (i.e., the flag curvature is constant). However, the class of Berwald spaces provide a similar result as Corollary 4.2 in the *flat* case.

Theorem 4.7. (Farkas, Kristály and Varga [107]) *Let (M,F) be an n-dimensional $(n \geq 3)$ Finsler-Hadamard manifold of Berwald type with $l_F > 0$. Then the following statements are equivalent*:

(i) *for some* $\mu \in [0, l_F r_F^{-2} \widetilde{\mu}^2)$ *and* $x_0 \in M$, *the function* $\sigma_{\mu,\rho,0}(d_F(x_0,\cdot))$ *is the unique pointwise solution to the Poisson equation* $(\mathcal{P}^{\mu}_{B_F^+(x_0,\rho)})$ *in* $B_F^+(x_0,\rho) \setminus \{x_0\}$ *for every* $\rho > 0$;

(ii) (M,F) *is isometric to an* n*-dimensional Minkowski space.*

Proof. The implication (ii)$\Rightarrow$(i) is trivial, see Corollary 4.1; it remains to prove that (i)$\Rightarrow$(ii). By the proof of Theorem 4.6, we obtain $\mathrm{Vol}_F(B_F^+(x,\rho)) = V_{0,n}(\rho) = \omega_n \rho^n$ for all $x \in M$ and $\rho > 0$. On account of Theorem 1.1, we conclude that $\mathbf{K} = 0$. Note that every Berwald space with $\mathbf{K} = 0$ is necessarily a locally Minkowski space, see Bao, Chern and Shen [11, Section 10.5]. Therefore, the global volume identity actually implies that (M,F) is isometric to a Minkowski space. $\square$

4.4 Further problems and comments

I) Finiteness of the reversibility constant versus the vector space structure of Sobolev spaces. According to Theorem 4.1, a highly non-trivial problem is to characterize those non-reversible Finsler manifolds for which the Sobolev spaces over them has a vector space structure. In fact, we conjecture that the Sobolev space $W_0^{1,2}(M,F,\mathsf{m})$ on a Finsler manifold (M,F) (with its canonical volume element m) is a vector space if and only if $r_F < +\infty$.

The latter statement is supported by Theorem 4.2, where we proved that $W_0^{1,2}(B_e(0,1),F_a,\mathsf{m}_a)$ has a vector space structure over $\mathbb{R}$ if and only if $r_{F_a} < +\infty$, where F_a is the Funk-type metric. A similar result is also provided on the Finsler-Poincaré disc, see [107]. Note that both examples belong to the class of Randers spaces.

II) Non-smooth critical point theory versus closed convex cones. The case $a = 1$ (Funk model) is not well understood in Section 4.2, since the set $W_0^{1,2,1}(B_e(0,1)) = W_0^{1,2}(B_e(0,1),F_1,\mathsf{m}_1)$ is not a vector space over $\mathbb{R}$. However, we believe that variational problems can also be treated within this context by using elements from the theory of non-smooth Szulkin-type [87] critical points involving the indicator function associated with the closed convex cone $W_0^{1,2,1}(B_e(0,1))$ in $L_1^2(B_e(0,1))$, see Section 1.2 and Szulkin [117, Section 2]. For simplicity of the presentation, we only considered elliptic problems involving sublinear terms at infinity. The above variational arguments seem to work also for elliptic problems involving the Finsler-Laplace operator $\boldsymbol{\Delta}_{F_a}$, $a \in [0,1)$, and for superlinear or oscillatory nonlinear terms as well, see e.g. [114].

Chapter 5

Elliptic problems on Riemannian manifolds

Elliptic problems on Riemannian manifolds have been intensively studied in the last decades. One of the main motivations was the famous Yamabe problem. Indeed, given an n-dimensional ($n \geq 3$) compact/complete Riemannian manifold (M, g), the Yamabe problem concerns the existence of a Riemannian metric g_0 conformal to g for which the scalar curvature is constant. It turns out that this problem can be transformed into an elliptic PDE involving the Laplace-Beltrami operator; namely, the Yamabe problem is equivalent to finding a positive solution $u \in C^\infty(M)$ to

$$-\Delta_g u + \frac{n-2}{4(n-1)} S_g u = \frac{n-2}{4(n-1)} S_{g_0} u^{2^\star - 1}, \tag{$\mathbf{Y}$}$$

where $S := S_g(x)$ is the scalar curvature on (M, g), and $2^\star := 2n/(n-2)$ is the usual critical Sobolev exponent, see e.g. Aubin [8] and Hebey [52]. A similar problem to **(Y)** is the so-called Nirenberg problem on the sphere $\mathbb{S}^n$.

Another class of elliptic problems appears in the case when the right hand side $s \mapsto s^{2^\star - 1}$, $s \geq 0$ of **(Y)** is replaced by some general nonlinear term $s \mapsto f(s)$ satisfying certain growth conditions at the origin and infinity. In particular, such problems arise in mathematical physics, formulated as Klein-Gordon, Schrödinger or Schrödinger-Maxwell equations on Riemannian manifolds, see e.g. Druet and Hebey [33], Hebey and Wei [53], Ghimenti and Micheletti [46], Thizy [89], and references therein.

This chapter is devoted to investigate a diversity of elliptic problems on compact/complete Riemannian manifolds, complementing in some aspects the aforementioned works. In particular, the Riemannian structure, contrary to the Finslerian one, allows us to provide sharp bifurcation phenomena as well as surprising multiplicity results by means of group-theoretical arguments based on Rubik actions and oscillatory behavior of nonlinear functions.

5.1 Sharp sublinear problems on compact Riemannian manifolds

Consider the class of functions

$$\mathcal{F} = \left\{ f \in C(\mathbb{R}_+;\mathbb{R}_+) \setminus \{0\} : \lim_{s\to 0^+} \frac{f(s)}{s} = \lim_{s\to\infty} \frac{f(s)}{s} = 0 \right\},$$

where $\mathbb{R}_+ = [0,\infty)$. For every $f \in \mathcal{F}$, the numbers

$$c_f := \max_{s>0} \frac{f(s)}{s} \text{ and } c_F := \max_{s>0} \frac{2F(s)}{s^2} \tag{5.1}$$

are well-defined and positive, where $F(s) := \int_0^s f(t)\mathrm{d}t$, $s \geq 0$.

5.1.1 Sharp bifurcation on compact Riemannian manifolds

Let (M,g) be an n-dimensional ($n \geq 3$) compact Riemannian manifold and

$$\Lambda_+(M) = \{\alpha \in L^\infty(M) : \mathrm{essinf}_M \alpha > 0\}.$$

For fixed $f \in \mathcal{F}$ and $\alpha, \beta \in \Lambda_+(M)$, we consider the eigenvalue problem

$$-\Delta_g u + \alpha(x)u = \lambda\beta(x)f(u) \text{ on } M, \tag{P_λ}$$

where $\lambda \geq 0$ is a parameter.

Problem (P_λ) has been studied in the pure power case, i.e., when $f(s) = |s|^{p-1}s$, $p > 1$, see Cotsiolis and Iliopoulos [25, 26] for $M = \mathbb{S}^d$, and Vázquez and Véron [91] for general compact Riemannian manifolds. In the aforementioned papers the authors obtained existence and multiplicity of solutions for (P_λ) by means of various variational arguments.

In the sequel, we provide the following sharp bifurcation result.

Theorem 5.1. (Kristály [113]) *Let (M,g) be an n-dimensional ($n \geq 3$) compact Riemannian manifold, $f \in \mathcal{F}$ and $\alpha, \beta \in \Lambda_+(M)$. The following statements hold:*

(i) *for every $0 \leq \lambda < c_f^{-1}\|\beta/\alpha\|_{L^\infty(M)}^{-1}$, problem (P_λ) has only the trivial solution;*

(ii) *for every $\lambda > c_F^{-1}\|\alpha/\beta\|_{L^\infty(M)}$, problem (P_λ) has at least two distinct non-zero nonnegative solutions.*

Proof. Let $f \in \mathcal{F}$ and $\alpha, \beta \in \Lambda_+(M)$. Since $f(0) = 0$, instead of $f : [0,\infty) \to [0,\infty)$, we consider its extension to the whole $\mathbb{R}$, by letting $f(s) = 0$ for every $s \leq 0$. If $u \in H_g^1(M)$ is a solution of (P_λ), it turns to be nonnegative.

First of all, we prove that $c_f > c_F$. Indeed, let $s_0 > 0$ be a maximum point of the function $s \mapsto \frac{2F(s)}{s^2}$, i.e., $c_F = \frac{2F(s_0)}{s_0^2}$. Then, s_0 is a critical point of $s \mapsto \frac{2F(s)}{s^2}$; a simple calculation shows that $f(s_0)s_0 = 2F(s_0)$. Therefore,

$$c_f = \max_{s>0} \frac{f(s)}{s} \geq \frac{f(s_0)}{s_0} = \frac{2F(s_0)}{s_0^2} = c_F.$$

Now, we assume that $c_f = c_F =: C$. Let $\widetilde{s}_0 := \inf\left\{s_0 > 0 : C = \frac{2F(s_0)}{s_0^2}\right\}$. Note that $\widetilde{s}_0 > 0$ thus we may fix $t_0 \in (0, \widetilde{s}_0)$ arbitrarily. In particular, we have that $2F(t_0) < Ct_0^2$. On the other hand, from the definition of c_f, one has $f(s) \leq c_f s = Cs$ for all $s \geq 0$. Combining these facts, we obtain $0 = 2F(\widetilde{s}_0) - C\widetilde{s}_0^2 = (2F(t_0) - Ct_0^2) + 2\int_{t_0}^{\widetilde{s}_0} [f(s) - Cs]\mathrm{d}s < 0$, a contradiction. Therefore, $c_f > c_F$ for every $f \in \mathcal{F}$.

(i) Assume that $u \in H_g^1(M)$ is a solution to (P_λ). Multiplying (P_λ) by the test function $u \in H_g^1(M)$, we obtain

$$\begin{aligned}
\|u\|_\alpha^2 &:= \int_M (\langle \nabla_g u, \nabla_g u\rangle + \alpha(x)u^2)\mathrm{d}V_g = \lambda \int_M \beta(x) f(u)\mathrm{d}V_g \\
&\leq \lambda \|\beta/\alpha\|_{L^\infty(M)} c_f \int_M \alpha(x)u^2 \mathrm{d}V_g \\
&\leq \lambda \|\beta/\alpha\|_{L^\infty(M)} c_f \|u\|_\alpha^2.
\end{aligned}$$

Now, if $0 \leq \lambda < c_f^{-1}\|\beta/\alpha\|_{L^\infty}^{-1}$, the above estimate implies $u = 0$.

(ii) For every $\lambda \geq 0$, let $E_\lambda : H_g^1(M) \to \mathbb{R}$ be the energy functional

$$E_\lambda(u) = I_1(u) - \lambda I_2(u)$$

associated with problem (P_λ), where

$$I_1(u) := \frac{1}{2}\|u\|_\alpha^2 \text{ and } I_2(u) := \int_M \beta(x)F(u(x))\mathrm{d}V_g, \ u \in H_g^1(M). \tag{5.2}$$

It is clear that $I_1, I_2 \in C^1(H_g^1(M), \mathbb{R})$, and every critical point of E_λ is exactly a weak solution to problem (P_λ). Furthermore, a similar reasoning as in Theorem 4.3 (Step 3) shows that $u \mapsto \frac{I_2(u)}{I_1(u)}$ inherits the properties of $f \in \mathcal{F}$; namely,

$$\lim_{u \to 0} \frac{I_2(u)}{I_1(u)} = \lim_{\|u\|_\alpha \to \infty} \frac{I_2(u)}{I_1(u)} = 0. \tag{5.3}$$

Let us fix $s_0 > 0$ such that $F(s_0) > 0$; this choice is possible, due to the fact that $f \in \mathcal{F}$. If $u_{s_0}(x) = s_0$ is the constant function on M, we have that

$$I_2(u_{s_0}) = \|\beta\|_{L^1(M)} F(s_0) > 0 \text{ and } I_1(u_{s_0}) = \|\alpha\|_{L^1} s_0^2 > 0.$$

Thus, we may define the number

$$\lambda^* := \inf_{I_2(u) > 0} \frac{I_1(u)}{I_2(u)}. \tag{5.4}$$

Now, we can apply Theorem 1.9, by choosing $X = H_g^1(M)$, as well as I_1 and I_2 from (5.2). On account of (1.25), it is clear that $\lambda^* = \chi^{-1} > 0$. Standard functional analysis arguments show that the functional I_1 is coercive (see again Theorem 4.3), sequentially weakly lower semicontinuous which belongs to $\mathcal{W}_{H_g^1(M)}$, bounded on each bounded subset of $H_g^1(M)$, and its derivative admits a continuous inverse on $H^1_g(M)^*$. Moreover, I_2 has a compact derivative on $H^1_g(M)$, due to the fact that $H^1_g(M)$ is compactly embedded into $L^q(M)$ for every $q \in [1, 2^\star)$. Moreover, I_1 has a strict global minimum $u_0 = 0$, and $I_1(0) = I_2(0) = 0$. The definition of the number τ (see (1.24)) and (5.3) give that $\tau = 0$. Therefore, on account of Theorem 1.9 (with $I_3 \equiv 0$), we have: for every compact interval $[a, b] \subset (\lambda^*, \infty)$ there exists $\kappa > 0$ such that for each $\lambda \in [a, b]$, the equation $E'_\lambda(u) \equiv I'_1(u) - \lambda I'_2(u) = 0$ admits at least three solutions $u^i_\lambda \in H^1_g(M)$, $i \in \{1, 2, 3\}$, having $H^1_g(M)$-norms less than κ. Moreover, since $H^1_g(M)$ contains the (positive) constant functions on M, we have that

$$\chi = \sup_{I_1(u)>0} \frac{I_2(u)}{I_1(u)} \geq \sup_{s>0} \frac{2\displaystyle\int_M \beta(x)F(s)\mathrm{d}V_g}{\displaystyle\int_M \alpha(x)s^2\mathrm{d}V_g} = c_F \frac{\|\beta\|_{L^1(M)}}{\|\alpha\|_{L^1(M)}} \geq c_F \|\alpha/\beta\|^{-1}_{L^\infty(M)}.$$

Consequently, $\lambda^* = \chi^{-1} \leq c_F^{-1}\|\alpha/\beta\|_{L^\infty(M)}$, thus the above statements are valid for every $\lambda > c_F^{-1}\|\alpha/\beta\|_{L^\infty(M)}$, which ends the proof. □

Remark 5.1. (a) An elementary estimate also shows that $\chi \leq c_F\|\beta/\alpha\|_{L^\infty(M)}$. In conclusion, we have a two-sided estimate for λ^*; namely, we have $c_F^{-1}\|\beta/\alpha\|^{-1}_{L^\infty(M)} \leq \lambda^* \leq c_F^{-1}\|\alpha/\beta\|_{L^\infty(M)}$. In particular, if $\beta/\alpha = c$ for some $c > 0$, then $\lambda^* = c_F^{-1}c^{-1}$.

(b) Note that c_f and c_F may be arbitrary close to each other. Indeed, if $a > 1$ and $f(s) = \min\{\max\{0, s-1\}, a-1\}$, then it is clear that $f \in \mathcal{F}$, and one has $c_f = \frac{a-1}{a}$ and $c_F = \frac{a-1}{a+1}$. Therefore, c_f and c_F become close to each other once a is sufficiently large.

The general form of Theorem 1.9 gives the possibility to show that (P_λ) is stable with respect to small perturbations. Indeed, let us consider the perturbed problem

$$-\Delta_g u + \alpha(x)u = \lambda\beta(x)f(u) + \mu\gamma(x)h(u) \text{ on } M, \qquad (P_{\lambda,\mu})$$

where $\gamma \in L^\infty(M)$, and $h : \mathbb{R} \to \mathbb{R}$ is subcritical, i.e., for some $p \in [1, 2^\star)$ and $c > 0$ we have $|h(s)| \leq c(1 + |s|^p)$ for every $s \in \mathbb{R}$. One can prove that the function $I_3 : H^1_g(M) \to \mathbb{R}$ defined by

$$I_3(u) = \int_M \gamma(x)H(u(x))\mathrm{d}V_g,$$

belongs to $C^1(H^1_g(M), \mathbb{R})$ with compact derivative, where $H(s) := \displaystyle\int_0^s h(t)\mathrm{d}t$. Thus, we can apply Theorem 1.9 in its full generality, which reads as follows.

Theorem 5.2. (Kristály [113]) *Let $f \in \mathcal{F}$ and $\alpha, \beta \in \Lambda_+(M)$ be fixed. Then for every compact interval $[a, b] \subset \left(c_F^{-1}\|\alpha/\beta\|_{L^\infty(M)}, \infty\right)$, there exists $\eta > 0$ with the following property: for every $\lambda \in [a, b]$, for every $\gamma \in L^\infty(M; \mathbb{R})$, and for every subcritical function $h : \mathbb{R} \to \mathbb{R}$, there exists $\delta > 0$ such that for every $\mu \in [0, \delta]$, problem $(P_{\lambda,\mu})$ has at least two distinct non-zero nonnegative weak solutions whose norms in $H^1_g(M)$ are less than η.*

5.1.2 Sharp singular elliptic problem of Emden-type

We present an application of Theorem 5.1 to the singular elliptic problem in the form

$$-\triangle v = \lambda |x|^{-m-2} K(x/|x|) f(|x|^m v), \quad x \in \mathbb{R}^{2m+2} \setminus \{0\}, \tag{S_λ}$$

where $f \in \mathcal{F}$, $K \in L^\infty(\mathbb{S}^{2m+1})$, $m \geq 1$, and $\lambda \geq 0$ is a parameter. Here, $\mathbb{S}^{2m+1}$ denotes the standard $(2m+1)$-dimensional unit sphere. Our result reads as follows.

Theorem 5.3. (Kristály [113]) *Let $f \in \mathcal{F}$ and $K \in \Lambda_+(\mathbb{S}^{2m+1})$, $m \geq 1$ be fixed. Then we have:*

(i) *for every $0 \leq \lambda < c_f^{-1} m^2 \|K\|^{-1}_{L^\infty(\mathbb{S}^{2m+1})}$, problem (S_λ) has only the trivial solution;*

(ii) *for every $\lambda > c_F^{-1} m^2 \|K^{-1}\|_{L^\infty(\mathbb{S}^{2m+1})}$, problem (S_λ) has at least two distinct non-zero nonnegative solutions.*

Proof. At first, we prove (ii). The solutions to (S_λ) are being sought in the particular form

$$v(x) = v(|x|, x/|x|) = u(r, \sigma) = r^{-m} u(\sigma), \tag{5.5}$$

where $(r, \sigma) \in (0, \infty) \times \mathbb{S}^{2m+1}$ are the spherical coordinates in $\mathbb{R}^{2m+2} \setminus \{0\}$. By means of the transformation (5.5), the equation (S_λ) reduces to

$$-\Delta_g u + m^2 u = \lambda K(\sigma) f(u), \ \sigma \in \mathbb{S}^{2m+1},$$

where Δ_g denotes the Laplace-Beltrami operator on $(\mathbb{S}^{2m+1}, g)$ and g is the canonical metric induced by $\mathbb{R}^{2m+2}$. It remains to apply Theorem 5.1/(ii) for $(M, g) = (\mathbb{S}^{2m+1}, g)$, $\alpha = m^2$, and $\beta = K$.

Now, we prove (i). On account of Theorem 5.1/(i) and the aforementioned argument, we expect to have the threshold value $c_f^{-1} m^2 \|K\|^{-1}_{L^\infty(\mathbb{S}^{2m+1})}$ for nonexistence. To see this, let $v \in \mathcal{D}^{1,2}(\mathbb{R}^{2m+2})$ be a solution to (S_λ). We multiply the equation (S_λ) by the test function v and integrate it on $\mathbb{R}^{2m+2}$; by using the sharp Hardy-Poincaré inequality in $\mathbb{R}^{2m+2}$ (see Theorem 3.3), we obtain

$$\begin{aligned}
\int_{\mathbb{R}^{2m+2}} |\nabla v|^2 \mathrm{d}x &= \lambda \int_{\mathbb{R}^{2m+2}} |x|^{-m-2} K(x/|x|) f(|x|^m v) v \mathrm{d}x \\
&\leq \lambda c_f \int_{\mathbb{R}^{2m+2}} |x|^{-2} K(x/|x|) v^2 \mathrm{d}x \\
&\leq \lambda c_f \|K\|_{L^\infty(\mathbb{S}^{2m+1})} \int_{\mathbb{R}^{2m+2}} |x|^{-2} v^2 \mathrm{d}x \\
&\leq \lambda c_f \|K\|_{L^\infty(\mathbb{S}^{2m+1})} \frac{4}{(2m+2-2)^2} \int_{\mathbb{R}^{2m+2}} |\nabla v|^2 \mathrm{d}x \\
&= \lambda c_f \|K\|_{L^\infty(\mathbb{S}^{2m+1})} \frac{1}{m^2} \int_{\mathbb{R}^{2m+2}} |\nabla v|^2 \mathrm{d}x.
\end{aligned}$$

Thus, if $0 \leq \lambda < c_f^{-1} m^2 \|K\|^{-1}_{L^\infty(\mathbb{S}^{2m+1})}$, we have necessarily that $v = 0$, which concludes the proof. $\square$

5.2 Bipolar Schrödinger equations on a hemisphere: multiplicity via Rubik actions

Motivated by molecular physics and quantum chemistry/cosmology, some efforts have been made over the last decades to investigate elliptic phenomena involving multiple singularities. Indeed, such phenomena appear when one tries to describe the behavior of electrons and atomic nuclei in a molecule within the theory of Born-Oppenheimer approximation or Thomas-Fermi theory, where the particles can be modeled as certain pairwise distinct singularities/poles $x_1, ..., x_m \in \mathbb{R}^n$, producing their effect within the form $x \mapsto |x - x_i|^{-1}$, $i \in \{1, ..., m\}$, see e.g. Bosi, Dolbeaut and Esteban [16], Felli, Marchini and Terracini [41], and Lieb [63]. All of the aforementioned works considered the flat/isotropic setting where no external force is present. Once the ambient space structure is perturbed, for instance by a magnetic or gravitational field, the above results do not provide a full description of the physical phenomenon due to the presence of the *curvature*.

We study a simple model on the n-dimensional open upper hemisphere $\mathbb{S}^n_+ = \{x = (x_1, ..., x_{n+1}) \in \mathbb{S}^n : x_{n+1} > 0\}$, by fixing just two different poles, $x_1, x_2 \in \mathbb{S}^n_+$. More precisely, we consider the Dirichlet problem

$$\begin{cases} -\Delta_g u + \mathtt{C}(n,\beta) u = \mu \left| \dfrac{\nabla_g d_1}{d_1} - \dfrac{\nabla_g d_2}{d_2} \right|^2 u + |u|^{p-2} u, & \text{in } \mathbb{S}^n_+ \\ u = 0, & \text{on } \partial \mathbb{S}^n_+, \end{cases} \qquad (P_{\mathbb{S}^n_+})$$

where g is the natural Riemannian structure on the standard unit sphere $\mathbb{S}^n$ inherited by $\mathbb{R}^{n+1}$, $d_i(x) = d_g(x, x_i)$ for $i \in \{1, 2\}$ (as in §3.3.2), $p \in (2, 2^\star)$ and $\mu \in [0, \widetilde{\mu}^2)$ are fixed, and $\mathtt{C}(n, \beta) := (n-1)(n-2)\frac{7\pi^2 - 3\left(\beta + \frac{\pi}{2}\right)^2}{2\pi^2\left(\pi^2 - \left(\beta + \frac{\pi}{2}\right)^2\right)}$. Hereafter, $\widetilde{\mu} = \frac{n-2}{2}$, while $x_0 = (0, ..., 0, 1)$ denotes the north pole of $\mathbb{S}^n$ and $\beta := \max\{d_g(x_0, x_1), d_g(x_0, x_2)\}$. Before stating the main result of this section, we need a specific contruction based on group-theoretical arguments.

5.2.1 Rubik actions: a group-theoretical argument

The goal of this subsection is to provide a generic tool to produce symmetrically different functions belonging to a given Sobolev space by using a suitable splitting of the orthogonal group $O(d+1)$, $d \geq 1$. To handle this problem, we explore the technique of solving the Rubik cube, described in [115] and simplified in [101] for the Heisenberg group. Roughly speaking, $d+1$ corresponds to the number of total 'sides' of the cube, while the sides of the cube are certain blocks in the decomposition subgroup $G = O(d_1) \times ... \times O(d_k)$ with $d_1 + ... + d_k = d + 1$, $d_j \geq 2$ for every $j \in \{1, ..., k\}$.

To be more precise, let $d = 3$ or $d \geq 5$ be fixed, and for every $j \in \{1, ..., t_d\}$, with $t_d = [\frac{d}{2}] + (-1)^{d+1} - 1$, consider the groups

$$G_j^d = \begin{cases} O(j+1) \times O(d - 2j - 1) \times O(j+1), & \text{if } j \neq \frac{d-1}{2}, \\ O\left(\frac{d+1}{2}\right) \times O\left(\frac{d+1}{2}\right), & \text{if } j = \frac{d-1}{2}. \end{cases}$$

Note that $t_4 = 0$. It is clear that a particular G_j^d does not act transitively on the sphere $\mathbb{S}^d$; in terms of the Rubik cube, it is not enough to rotate only one side in order to solve it. However, to recover the transitivity, we shall combine different groups of the type G_j^d; roughly speaking, in the Rubik cube we are rotating (a minimal number of) appropriate sides to solve it. We denote by $\langle G_i^d; G_j^d \rangle$ the group generated by G_i^d and G_j^d.

Theorem 5.4. (Kristály [115]) *Let $d = 3$ or $d \geq 5$ be fixed and $i, j \in \{1, ..., t_d\}$ with $i \neq j$. Then the group $\langle G_i^d; G_j^d \rangle$ acts transitively on $\mathbb{S}^d$.*

Proof. Without loss of generality, we assume that $j > i$. For further use, let $0_k = (0, ..., 0) \in \mathbb{R}^k$, $k \in \{1, ..., d+1\}$. Let us fix $\eta = (\eta_1, \eta_2, \eta_3) \in \mathbb{S}^d$ arbitrarily with $\eta_1, \eta_3 \in \mathbb{R}^{j+1}$ and $\eta_2 \in \mathbb{R}^{d-2j-1}$; clearly, η_2 disappears from η whenever $2j = d - 1$. Taking into account the fact that $O(j+1)$ acts transitively on $\mathbb{S}^j$, there are $g_j^1, g_j^2 \in O(j+1)$ such that if $g_j = g_j^1 \times I_{\mathbb{R}^{d-2j-1}} \times g_j^2 \in G_j^d$, then $g_j\eta = (0_j, |\eta_1|, \eta_2, |\eta_3|, 0_j)$. Since $j - 1 \geq i$, the transitive action of $O(d-2i-1)$ on $\mathbb{S}^{d-2i-2}$ implies the existence of $g_i^1 \in O(d-2i-1)$ such that $g_i^1(0_{j-i-1}, |\eta_1|, \eta_2, |\eta_3|, 0_{j-i-1}) = (1, 0_{d-2i-2})$. Therefore, if $g_i = I_{\mathbb{R}^{i+1}} \times g_i^1 \times I_{\mathbb{R}^{i+1}} \in G_i^d$ then $g_i g_j \eta = (0_{i+1}, 1, 0_{d-i-1}) \in \mathbb{S}^d$.

By repeating the same procedure for another element $\widetilde{\eta} \in \mathbb{S}^d$, there exists $\widetilde{g}_i \in G_i^d$ and $\widetilde{g}_j \in G_j^d$ such that $\widetilde{g}_i\widetilde{g}_j\widetilde{\eta} = (0_{i+1}, 1, 0_{d-i-1}) \in \mathbb{S}^d$. Accordingly, $\eta = g_j^{-1}g_i^{-1}\widetilde{g}_i\widetilde{g}_j\widetilde{\eta} = g_j^{-1}\overline{g}_i\widetilde{g}_j\widetilde{\eta}$, where $\overline{g}_i = g_i^{-1}\widetilde{g}_i \in G_i$, which concludes the proof. □

5.2.2 Multiple solutions for bipolar Schrödinger equations on a hemisphere

The main result of this section reads as follows.

Theorem 5.5. (Faraci, Farkas and Kristály [104]) *Let $\mathbb{S}_+^n$ be the open upper hemisphere ($n \geq 3$), $S = \{x_1, x_2\} \subset \mathbb{S}_+^n$ be the set of poles, $p \in (2, 2^\star)$ and $\mu \in [0, \widetilde{\mu}^2)$ be fixed and $x_0 = (0, ..., 0, 1)$ be the north pole. The following statements hold.*

(i) *Problem $(P_{\mathbb{S}_+^n})$ has infinitely many weak solutions in $H_g^1(\mathbb{S}_+^n)$. In addition, if $x_1 = (a, 0, ..., 0, b)$ and $x_2 = (-a, 0, ..., 0, b)$ for some $a, b \in \mathbb{R}$ with $a^2 + b^2 = 1$ and $b > 0$, then problem $(P_{\mathbb{S}_+^n})$ has a sequence $\{u_k\}_k$ of distinct weak solutions in $H_g^1(\mathbb{S}_+^n)$ of the form*

$$u_k := u_k\left(y_1, \sqrt{y_2^2 + ... + y_n^2}, y_{n+1}\right) = u_k\left(y_1, \sqrt{1 - y_1^2 - y_{n+1}^2}, y_{n+1}\right).$$

(ii) *If $n = 5$ or $n \geq 7$, and $x_1 = (a, 0, ..., 0, b)$, $x_2 = (-a, 0, ..., 0, b)$ for some $a, b \in \mathbb{R}$ with $a^2 + b^2 = 1$ and $b > 0$, then there exists at least $s_n = \left[\frac{n}{2}\right] + (-1)^{n-1} - 2$ sequences of sign-changing weak solutions to $(P_{\mathbb{S}_+^n})$ in $H_g^1(\mathbb{S}_+^n)$ whose elements mutually differ by their symmetries.*

Proof. Fix $\mu \in [0, \widetilde{\mu}^2)$ arbitrarily. The energy functional $\mathcal{E} : H_g^1(\mathbb{S}_+^n) \to \mathbb{R}$ associated with problem $(P_{\mathbb{S}_+^n})$ is

$$\mathcal{E}(u) = \frac{1}{2}\|u\|^2_{\mathsf{C}(n,\beta)} - \frac{\mu}{2}\int_{\mathbb{S}_+^n}\left|\frac{\nabla_g d_1}{d_1} - \frac{\nabla_g d_2}{d_2}\right|^2 u^2 \,\mathrm{d}V_g - \frac{1}{p}\int_{\mathbb{S}_+^n}|u|^p \,\mathrm{d}V_g.$$

It is clear that $\mathcal{E} \in C^1(H_g^1(\mathbb{S}_+^n), \mathbb{R})$ and its critical points are precisely the weak solutions to $(P_{\mathbb{S}_+^n})$.

(i) We note that the embedding $H^1_g(\mathbb{S}^n_+) \hookrightarrow L^p(\mathbb{S}^n_+)$ is compact for every $p \in (2, 2^\star)$, see e.g. Hebey [52]. By means of Corollary 3.1, one can easily prove that the functional $\mathcal{E}$ has the mountain pass geometry and is even, i.e., $\mathcal{E}(-u) = \mathcal{E}(u)$ for every $u \in H^1_g(\mathbb{S}^n_+)$. Accordinly, a simple reasoning shows that $\mathcal{E}$ satisfies the assumptions of the symmetric version of the Mountain Pass Theorem (see Theorem 1.8), thus there exists a sequence of distinct critical points of $\mathcal{E}$ which are weak solutions to problem $(P_{\mathbb{S}^n_+})$ in $H^1_g(\mathbb{S}^n_+)$.

Let $x_1 = (a, 0, ..., 0, b)$ and $x_2 = (-a, 0, ..., 0, b)$ fixed for some $a, b \in \mathbb{R}$ with $a^2 + b^2 = 1$ and $b > 0$. We note that in this case $\beta = d_g(x_0, x_1) = d_g(x_0, x_2) = \arccos(b)$. We shall prove that the energy functional $\mathcal{E}$ is G_0-invariant, where $G_0 := \mathrm{id}_\mathbb{R} \times O(n-1) \times \mathrm{id}_\mathbb{R}$. Hereafter, the action of G_0 on $H^1_g(\mathbb{S}^n_+)$ is given by

$$(\zeta \circ u)(x) = u(\zeta^{-1}x), \quad \forall u \in H^1_g(\mathbb{S}^n_+), \zeta \in G_0, x \in \mathbb{S}^n_+.$$

Since $\zeta \in G_0$ is an isometry on $\mathbb{R}^{n+1}$, a change of variables easily implies that

$$u \mapsto \frac{1}{2}\|u\|^2_{\mathsf{C}(n,\beta)} - \frac{1}{p}\int_{\mathbb{S}^n_+} |u|^p \, \mathrm{d}V_g$$

is G_0-invariant. Thus, it remains to deal with the G_0-invariance of the functional

$$u \mapsto \int_{\mathbb{S}^n_+} \left| \frac{\nabla_g d_1}{d_1} - \frac{\nabla_g d_2}{d_2} \right|^2 u^2 \, \mathrm{d}V_g.$$

We recall that

$$\left| \frac{\nabla_g d_1}{d_1} - \frac{\nabla_g d_2}{d_2} \right|^2 = \frac{1}{d_1^2} + \frac{1}{d_2^2} - 2\frac{\langle \nabla_g d_1, \nabla_g d_2 \rangle}{d_1 d_2}$$

and $\nabla_g d_g(\cdot, y)(x) = -\frac{\exp_x^{-1}(y)}{d_g(x,y)}$ for every distinct points $x, y \in \mathbb{S}^n_+$. Spherical calculus shows that

$$\exp_x^{-1}(x_i) = \frac{d_i(x_i - x\cos(d_i))}{\sin(d_i)}, \qquad i \in \{1,2\},\ x \in \mathbb{S}^n_+ \setminus \{x_i\}.$$

Therefore,

$$\nabla_g d_i(x) = \nabla_g d_g(x, x_i) = -\frac{\exp_x^{-1}(x_i)}{d_i} = \frac{x\cos(d_i) - x_i}{\sin(d_i)}, \qquad i \in \{1,2\},\ x \in \mathbb{S}^n_+ \setminus \{x_i\}. \tag{5.6}$$

Let $\zeta \in G_0$, $i \in \{1, 2\}$ and $x \in \mathbb{S}^n_+ \setminus \{x_i\}$ be fixed. Since $\zeta x_i = x_i$ and ζ is an isometry for the metric d_g, it follows that

$$d_i(\zeta x) = d_g(\zeta x, x_i) = d_g(\zeta x, \zeta x_i) = d_g(x, x_i) = d_i(x),$$

and by applying (5.6), one has $\langle \nabla_g d_g(\zeta x, x_1), \nabla_g d_g(\zeta x, x_2)\rangle = \langle \nabla_g d_g(x, x_1), \nabla_g d_g(x, x_2)\rangle$. Therefore, the above properties (combined with a trivial change of variables) imply that the energy functional $\mathcal{E}$ is G_0-invariant, i.e., $\mathcal{E}(\zeta \circ u) = \mathcal{E}(u)$ for every $u \in H^1_g(\mathbb{S}^n_+)$ and $\zeta \in G_0$.

We can apply the same variational argument as above (see Theorem 1.8) to the functional $\mathcal{E}_0 = \mathcal{E}|_{H_{G_0}(\mathbb{S}^n_+)}$, where

$$H_{G_0}(\mathbb{S}^n_+) = \left\{ u \in H^1_g(\mathbb{S}^n_+) : \zeta \circ u = u,\ \forall \zeta \in G_0 \right\}.$$

Accordingly, one can find a sequence $\{u_k\}_k \subset H_{G_0}(\mathbb{S}^n_+)$ of pairwise distinct critical points of $\mathcal{E}_0$. Moreover, due to the smooth principle of symmetric criticality of Palais (see Theorem 1.6), the critical points of $\mathcal{E}_0$ are also critical points of the original energy functional $\mathcal{E}$, thus weak solutions to problem $(P_{\mathbb{S}^n_+})$. Since u_k are G_0-invariant functions, they have the form

$$u_k := u_k\left(y_1, \sqrt{y_2^2+...+y_n^2}, y_{n+1}\right) = u_k\left(y_1, \sqrt{1-y_1^2-y_{n+1}^2}, y_{n+1}\right), \quad k \in \mathbb{N}.$$

(ii) Let $n = 5$ or $n \geq 7$ be fixed, and denote by $s_n = \left[\frac{n}{2}\right] + (-1)^{n-1} - 2$. (Note that $s_6 = 0$.) For every $j \in \{1, ..., s_n\}$ we consider

$$G_j^n = \begin{cases} O(j+1) \times O(n-2j-3) \times O(j+1), & \text{if} \quad j \neq \frac{n-3}{2}, \\ O\left(\frac{n-1}{2}\right) \times O\left(\frac{n-1}{2}\right), & \text{if} \quad j = \frac{n-3}{2}, \end{cases}$$

and

$$\tau_j = \begin{cases} \begin{bmatrix} 0 & 0 & I_{\mathbb{R}^{j+1}} \\ 0 & I_{\mathbb{R}^{n-2j-3}} & 0 \\ I_{\mathbb{R}^{j+1}} & 0 & 0 \end{bmatrix}, & \text{if} \quad j \neq \frac{n-3}{2}, \\ \begin{bmatrix} 0 & I_{\mathbb{R}^{\frac{n-1}{2}}} \\ I_{\mathbb{R}^{\frac{n-1}{2}}} & 0 \end{bmatrix}, & \text{if} \quad j = \frac{n-3}{2}. \end{cases}$$

Note that $\tau_j \notin G_j^n$, $\tau_j G_j^n \tau_j^{-1} = G_j^n$ and $\tau_j^2 = \mathrm{id}_{\mathbb{R}^{n-1}}$. Let us introduce the group

$$G_{j,\tau_j}^n = \mathrm{id}_{\mathbb{R}} \times \langle G_j^n, \tau_j \rangle \times \mathrm{id}_{\mathbb{R}} \subset O(n+1).$$

The latter properties show that we have two types of elements in G_{j,τ_j}^n: either of the type $\widetilde{G}_j^n = \mathrm{id}_{\mathbb{R}} \times G_j^n \times \mathrm{id}_{\mathbb{R}}$ or $\mathrm{id}_{\mathbb{R}} \times \tau_j G_j^n \times \mathrm{id}_{\mathbb{R}}$. Following the idea of Bartsch and Willem [13], we introduce the action of the group G_{j,τ_j}^n on the space $H_g^1(\mathbb{S}^n_+)$ by

$$(\widetilde{\zeta} \circledast u)(x) = \begin{cases} u(\zeta^{-1}x), & \text{if} \quad \widetilde{\zeta} = \zeta \in \widetilde{G}_j^n, \\ -u(\zeta^{-1}\widetilde{\tau}_j^{-1}x), & \text{if} \quad \widetilde{\zeta} = \widetilde{\tau}_j\zeta \in G_{j,\tau_j}^n \setminus \widetilde{G}_j^n, \end{cases} \tag{5.7}$$

for every $\zeta \in \widetilde{G}_j^n = \mathrm{id}_{\mathbb{R}} \times G_j^n \times \mathrm{id}_{\mathbb{R}}$, $\widetilde{\tau}_j = \mathrm{id}_{\mathbb{R}} \times \tau_j \times \mathrm{id}_{\mathbb{R}}$, $u \in H_g^1(\mathbb{S}^n_+)$ and $x \in \mathbb{S}^n_+$. We define the subspace

$$H_{G_{j,\tau_j}^n}(\mathbb{S}^n_+) := \left\{u \in H_g^1(\mathbb{S}^n_+) : \widetilde{\zeta} \circledast u = u, \ \forall \widetilde{\zeta} \in G_{j,\tau_j}^n\right\}$$

of $H_g^1(\mathbb{S}^n_+)$ that consists of all symmetric points with respect to the compact group G_{j,τ_j}^n. By (5.7) and Theorem 5.4 (applied for $d = n-2$) we obtain that for every $j \neq k \in \{1, 2, ..., s_n\}$ one has that

$$H_{G^n_{j,\tau_j}}(\mathbb{S}^n_+) \cap H_{G^n_{k,\tau_k}}(\mathbb{S}^n_+) = \{0\}. \tag{5.8}$$

In a similar way as above, we can prove that the energy functional $\mathcal{E}$ is G^n_{j,τ_j}-invariant for every $j \in \{1, ..., s_n\}$ (note that $\mathcal{E}$ is an even functional), where the group action on $H^1_g(\mathbb{S}^n_+)$ is given by (5.7). Therefore, for every $j \in \{1, ..., s_n\}$ there exists a sequence $\{u^j_k\}_k \subset H_{G^n_{j,\tau_j}}(\mathbb{S}^n_+)$ of distinct critical points of $\mathcal{E}_j = \mathcal{E}|_{H_{G^n_{j,\tau_j}}(\mathbb{S}^n_+)}$. Again by the smooth principle of symmetric criticality (see Theorem 1.6), $\{u^j_k\}_k \subset H_{G^n_{j,\tau_j}}(\mathbb{S}^n_+)$ are distinct critical points also for $\mathcal{E}$, thus weak solutions to problem $(P_{\mathbb{S}^n_+})$. It is clear that every u^j_k is sign-changing (see (5.7)) and according to (5.8), elements in different sequences have mutually different symmetry properties. This concludes the proof. □

5.3 Schrödinger-Maxwell equations on Hadamard manifolds: multiplicity via oscillation

The Schrödinger-Maxwell system

$$\begin{cases} -\frac{\hbar^2}{2m}\Delta u + \omega u + eu\varphi = f(x,u) & \text{in} \quad \mathbb{R}^3, \\ -\Delta\varphi = 4\pi e u^2 & \text{in} \quad \mathbb{R}^3, \end{cases} \tag{5.9}$$

describes the statical behavior of a charged non-relativistic quantum mechanical particle interacting with the electromagnetic field. More precisely, the unknown terms $u : \mathbb{R}^3 \to \mathbb{R}$ and $\varphi : \mathbb{R}^3 \to \mathbb{R}$ are the fields associated with the particle and the electric potential, respectively. Hereafter, the quantities m, e, ω and $\hbar$ are the mass, charge, phase, and Planck's constant, respectively, while $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function verifying some growth conditions. In fact, system (5.9) comes from the evolutionary nonlinear Schrödinger equation by using a Lyapunov-Schmidt reduction.

Motivated by certain physical phenomena, Schrödinger-Maxwell systems has been studied in the last few years on n-dimensional *compact Riemannian manifolds*, where $2 \leq n \leq 5$, see Druet and Hebey [33], Hebey and Wei [53], Ghimenti and Micheletti [46, 47], and Thizy [89, 90]. More precisely, in the aforementioned papers various forms of the system

$$\begin{cases} -\frac{\hbar^2}{2m}\Delta_g u + \omega u + eu\varphi = f(u) & \text{in} \quad M, \\ -\Delta_g\varphi + \varphi = 4\pi e u^2 & \text{in} \quad M, \end{cases} \tag{5.10}$$

has been considered, where (M, g) is a compact Riemannian manifold and f has a certain nonlinear growth. As expected, the compactness of (M, g) played a crucial role in these investigations.

The purpose of the present section is to provide a multiplicity result for the Maxwell-Schrödinger system

$$\begin{cases} -\Delta_g u + u + eu\varphi = \alpha(x) f(u) & \text{in} \quad M, \\ -\Delta_g \varphi + \varphi = qu^2 & \text{in} \quad M, \end{cases} \tag{$\mathcal{SM}$}$$

whenever (M, g) is a *Hadamard manifold* and $f : [0, \infty) \to \mathbb{R}$ is a continuous function that verifies the assumptions:

(f_0^1) $-\infty < \liminf_{s\to 0} \frac{F(s)}{s^2} \leq \limsup_{s\to 0} \frac{F(s)}{s^2} = +\infty$, where $F(s) = \int_0^s f(t)\mathrm{d}t$;

(f_0^2) there exists a sequence $\{s_k\}_k \subset (0, 1)$ converging to 0 such that $f(s_k) < 0$, $k \in \mathbb{N}$.

Note that (f_0^1) and (f_0^2) imply an oscillatory behavior of the function f near the origin. Since (M, g) is not compact, we shall explore certain actions of its group of isometries in order to regain some compactness. To do this, we denote by $\mathrm{Isom}_g(M)$ the group of isometries of (M, g) and let G be a subgroup of $\mathrm{Isom}_g(M)$. The function $u : M \to \mathbb{R}$ is said to be *radially symmetric with respect to* $x_0 \in M$ if u depends on $d_g(x_0, \cdot)$. The *fixed point set of* G on M is given by $\mathrm{Fix}_M(G) = \{x \in M : \sigma(x) = x \text{ for all } \sigma \in G\}$. For a given $x_0 \in M$, we formulate the following hypotheses.

($\boldsymbol{H_G^{x_0}}$) *The group G is a compact connected subgroup of $\mathrm{Isom}_g(M)$ such that $\mathrm{Fix}_M(G) = \{x_0\}$.*

The main result of the present section reads as follows.

Theorem 5.6. (Farkas and Kristály [105]) *Let (M, g) be an n-dimensional $(3 \leq n \leq 5)$ homogeneous Hadamard manifold, $x_0 \in M$ be fixed, $\alpha \in L^1(M) \cap L^\infty(M)$ be a non-zero nonnegative radially symmetric function with respect to x_0 and $G \subset \mathrm{Isom}_g(M)$ be a group that satisfies the hypothesis* **($\boldsymbol{H_G^{x_0}}$)***. If $f : [0, \infty) \to \mathbb{R}$ is a continuous function satisfying (f_0^1) and (f_0^2), then there exists a sequence $\{(u_k^0, \varphi_{u_k^0})\}_k \subset H_g^1(M) \times H_g^1(M)$ of distinct nonnegative G-invariant weak solutions to $(\mathcal{SM})$ such that*

$$\lim_{k\to\infty} \|u_k^0\|_{H_g^1(M)} = \lim_{k\to\infty} \|\varphi_{u_k^0}\|_{H_g^1(M)} = 0.$$

The proof of Theorem 5.6 is a logical puzzle which is assembled by several pieces. At first, we prove that the system $(\mathcal{SM})$ can be discussed by variational arguments, reducing it to the detection of critical points of a specific energy functional (see §5.3.1). Then we consider an auxiliary, closely related problem to $(\mathcal{SM})$ by locating in a precise way its solutions (see §5.3.2). Finally, we put all these results together to produce multiple solutions to $(\mathcal{SM})$.

In the sequel, we assume the hypotheses of Theorem 5.6 are verified.

5.3.1 Variational formulation of the Maxwell-Schrödinger system

We define the energy functional $\mathcal{J} : H_g^1(M) \times H_g^1(M) \to \mathbb{R}$ associated with system $(\mathcal{SM})$, namely,

$$\mathcal{J}(u, \varphi) = \frac{1}{2}\|u\|_{H_g^1(M)}^2 + \frac{e}{2}\int_M \varphi u^2 \mathrm{d}V_g - \frac{e}{4q}\int_M |\nabla_g \varphi|^2 \mathrm{d}V_g - \frac{e}{4q}\int_M \varphi^2 \mathrm{d}V_g - \int_M \alpha(x) F(u) \mathrm{d}V_g.$$

The functional $\mathcal{J}$ is well-defined and of class C^1 on $H_g^1(M) \times H_g^1(M)$. To see this, we have to consider the second and fifth terms from $\mathcal{J}$; the other terms trivially verify the required properties. First, a comparison principle and suitable Sobolev embeddings give that there exists $C > 0$ such that for every $(u, \varphi) \in H_g^1(M) \times H_g^1(M)$,

$$0 \leq \int_M \varphi u^2 \mathrm{d}V_g \leq \left(\int_M \varphi^{2^\star} \mathrm{d}V_g \right)^{\frac{1}{2^\star}} \left(\int_M |u|^{\frac{4n}{n+2}} \mathrm{d}V_g \right)^{1-\frac{1}{2^\star}} \leq C \|\varphi\|_{H_g^1(M)} \|u\|^2_{H_g^1(M)} < \infty,$$

where we used $3 \leq n \leq 5$. Since f is subcritical, we have that the functional $\mathcal{F} : H_g^1(M) \to \mathbb{R}$ defined by

$$\mathcal{F}(u) := \int_M \alpha(x) F(u) \mathrm{d}V_g,$$

is well-defined and $\mathcal{F} \in C^1(H_g^1(M), \mathbb{R})$. The following observation is trivial.

Step 1. *The pair $(u, \varphi) \in H_g^1(M) \times H_g^1(M)$ is a weak solution to $(\mathcal{SM})$ if and only if (u, φ) is a critical point of $\mathcal{J}$.*

Due to Lax-Milgram theorem, we introduce the map $\varphi_u : H_g^1(M) \to H_g^1(M)$ by associating with every $u \in H_g^1(M)$ the unique solution $\varphi = \varphi_u$ to the Maxwell equation

$$-\Delta_g \varphi + \varphi = q u^2.$$

We recall some important properties of the function $u \mapsto \varphi_u$ which are straightforward adaptations of the Euclidean case (see [119]) to the Riemannian setting:

$$\|\varphi_u\|^2_{H_g^1(M)} = q \int_M \varphi_u u^2 \mathrm{d}V_g, \quad \varphi_u \geq 0; \tag{5.11}$$

$$u \mapsto \int_M \varphi_u u^2 \mathrm{d}V_g \text{ is convex;} \tag{5.12}$$

$$\int_M (u\varphi_u - v\varphi_v)(u - v) \mathrm{d}V_g \geq 0, \ \forall u, v \in H_g^1(M). \tag{5.13}$$

The "one-variable" energy functional $\mathcal{E}_\lambda : H_g^1(M) \to \mathbb{R}$ associated with system $(\mathcal{SM})$ is defined by

$$\mathcal{E}(u) := \frac{1}{2} \|u\|^2_{H_g^1(M)} + \frac{e}{4} \int_M \varphi_u u^2 \mathrm{d}V_g - \mathcal{F}(u). \tag{5.14}$$

By using standard variational arguments, one can perform the next step.

Step 2. *The pair $(u, \varphi) \in H_g^1(M) \times H_g^1(M)$ is a critical point of $\mathcal{J}$ if and only if u is a critical point of $\mathcal{E}$ and $\varphi = \varphi_u$.* Moreover, we have that

$$\mathcal{E}'(u)(v) = \int_M (\langle \nabla_g u, \nabla_g v \rangle + uv + e\varphi_u uv) \mathrm{d}V_g - \int_M \alpha(x) f(u) v \mathrm{d}V_g, \ \forall v \in H_g^1(M). \tag{5.15}$$

Let $x_0 \in M$, $G \subset \mathrm{Isom}_g(M)$ and $\alpha \in L^1(M) \cap L^\infty(M)$ as in the hypotheses. The action of G on

$H_g^1(M)$ is defined by

$$(\sigma * u)(x) = u(\sigma^{-1}(x)),\ \forall \sigma \in G,\ u \in H_g^1(M),\ x \in M, \tag{5.16}$$

where $\sigma^{-1} : M \to M$ is the inverse of the isometry σ. Let

$$H_{g,G}^1(M) = \{u \in H_g^1(M) : \sigma * u = u,\ \forall \sigma \in G\}$$

be the subspace of G-invariant functions of $H_g^1(M)$ and $\mathcal{E}_G : H_{g,G}^1(M) \to \mathbb{R}$ be the restriction of the energy functional $\mathcal{E}$ to $H_{g,G}^1(M)$. The following statement is crucial in our investigation.

Step 3. *If $u_G \in H_{g,G}^1(M)$ is a critical point of $\mathcal{E}_G$, then it is a critical point also for $\mathcal{E}$ and φ_{u_G} is G-invariant.*

Proof. Due to relation (5.16), the group G acts continuously on $H_g^1(M)$. We claim that $\mathcal{E}$ is G-invariant. To prove this, let $u \in H_g^1(M)$ and $\sigma \in G$ be fixed. Since $\sigma : M \to M$ is an isometry on M, we have by (5.16) and the chain rule that $\nabla_g(\sigma * u)(x) = D\sigma_{\sigma^{-1}(x)}\nabla_g u(\sigma^{-1}(x))$ for every $x \in M$, where $D\sigma_{\sigma^{-1}(x)} : T_{\sigma^{-1}(x)}M \to T_xM$ denotes the differential of σ at the point $\sigma^{-1}(x)$. The (signed) Jacobian determinant of σ is 1 and $D\sigma_{\sigma^{-1}(x)}$ preserves inner products; thus, by relation (5.16) and by applying the change of variables $y = \sigma^{-1}(x)$, it turns out that

$$\begin{aligned}
\|\sigma * u\|^2_{H_g^1(M)} &= \int_M \left(|\nabla_g(\sigma * u)(x)|_x^2 + (\sigma * u)^2(x)\right) \mathrm{d}V_g(x) \\
&= \int_M \left(|\nabla_g u(\sigma^{-1}(x))|^2_{\sigma^{-1}(x)} + u^2(\sigma^{-1}(x))\right) \mathrm{d}V_g(x) = \int_M \left(|\nabla_g u(y)|_y^2 + u^2(y)\right) \mathrm{d}V_g(y) \\
&= \|u\|^2_{H_g^1(M)}.
\end{aligned}$$

A change of variable and the properties of the function α give that

$$\begin{aligned}
\mathcal{F}(\sigma * u) &= \int_M \alpha(x)F((\sigma * u)(x))\mathrm{d}V_g(x) = \int_M \alpha(x)F\left(u(\sigma^{-1}(x))\right) \mathrm{d}V_g(x) = \int_M \alpha(y)F(u(y))\mathrm{d}V_g(y) \\
&= \mathcal{F}(u).
\end{aligned}$$

We now consider the Maxwell equation $-\Delta_g\varphi_{\sigma*u} + \varphi_{\sigma*u} = q(\sigma u)^2$ which reads pointwisely $-\Delta_g\varphi_{\sigma*u}(y) + \varphi_{\sigma*u}(y) = qu(\sigma^{-1}(y))^2$, $y \in M$. After a change of variables one has that $-\Delta_g\varphi_{\sigma*u}(\sigma(x)) + \varphi_{\sigma*u}(\sigma(x)) = qu^2(x)$, $x \in M$, which means by the uniqueness that $\varphi_{\sigma*u}(\sigma(x)) = \varphi_u(x)$. Therefore,

$$\int_M \varphi_{\sigma*u}(x)(\sigma * u)^2(x)\mathrm{d}V_g(x) = \int_M \varphi_u(\sigma^{-1}(x))u^2(\sigma^{-1}(x))\mathrm{d}V_g(x) = \int_M \varphi_u(y)u^2(y)\mathrm{d}V_g(y),$$

which proves the G-invariance of $u \mapsto \displaystyle\int_M \varphi_u u^2 \mathrm{d}V_g$. Since the fixed point set of $H_g^1(M)$ for G is $H_{g,G}^1(M)$, the principle of symmetric criticality (see Theorem 1.6) shows that every critical point $u_G \in H_{g,G}^1(M)$ of $\mathcal{E}_G$ is also a critical point of $\mathcal{E}$. Moreover, from the above uniqueness argument, for every $\sigma \in G$ and $x \in M$ we have $\varphi_{u_G}(\sigma * x) = \varphi_{\sigma * u_G}(\sigma x) = \varphi_{u_G}(x)$, i.e., φ_{u_G} is G-invariant. $\square$

By Steps 1–3, we have the following implications: for an element $u \in H^1_{g,G}(M)$,

$$\mathcal{E}'_G(u) = 0 \ \Rightarrow\ \mathcal{E}'(u) = 0 \ \Leftrightarrow\ \mathcal{J}'(u, \varphi_u) = 0 \ \Leftrightarrow (u, \varphi_u) \text{ is a weak solution of } (\mathcal{SM}). \tag{5.17}$$

In order to guarantee G-invariant weak solutions to $(\mathcal{SM})$, it is enough to produce critical points for the energy functional $\mathcal{E}_G : H^1_{g,G}(M) \to \mathbb{R}$. Since the embedding $H^1_g(M) \hookrightarrow L^p(M)$ is only continuous for every $p \in [2, 2^\star]$, we adapt the next Lions-type result in order to regain some compactness by exploring the presence of group symmetries.

Proposition 5.1. (Skrzypczak and Tintarev [84, Theorem 1.3 and Proposition 3.1]) *Let (M, g) be an n-dimensional ($n \geq 3$) homogeneous Hadamard manifold and G be a compact connected subgroup of* $\mathrm{Isom}_g(M)$ *such that* $\mathrm{Fix}_M(G)$ *is a singleton. Then $H^1_{g,G}(M)$ is compactly embedded into $L^p(M)$ for every $p \in (2, 2^\star)$.*

5.3.2 Truncation technique

This subsection treats an auxiliary Schrödinger-Maxwell system

$$\begin{cases} -\Delta_g u + u + eu\varphi = \alpha(x)\widetilde{f}(u) & \text{in} \quad M, \\ -\Delta_g \varphi + \varphi = qu^2 & \text{in} \quad M, \end{cases} \tag*{$(\widetilde{\mathcal{SM}})$,}$$

where the following assumptions hold:

$(\widetilde{f}_1)$ $\widetilde{f} : [0, \infty) \to \mathbb{R}$ is a bounded function such that $\widetilde{f}(0) = 0$;

$(\widetilde{f}_2)$ there are $0 < a \leq b$ such that $\widetilde{f}(s) \leq 0$ for all $s \in [a, b]$.

Let $\widetilde{\mathcal{E}}$ be the "one-variable" energy functional associated with system $(\widetilde{\mathcal{SM}})$, and $\widetilde{\mathcal{E}}_G$ be the restriction of $\widetilde{\mathcal{E}}$ to the set $H^1_{g,G}(M)$. It is clear that $\widetilde{\mathcal{E}}$ is well-defined. Consider the number $b \in \mathbb{R}$ from $(\widetilde{f}_2)$; for further use, we introduce the sets

$$W^b = \left\{u \in H^1_g(M) : \|u\|_{L^\infty(M)} \leq b\right\} \quad \text{and} \quad W^b_G = W^b \cap H^1_{g,G}(M).$$

Lemma 5.1. *Let (M, g) be an n-dimensional ($3 \leq n \leq 5$) homogeneous Hadamard manifold, $x_0 \in M$ be fixed, $\alpha \in L^1(M) \cap L^\infty(M)$ be a non-zero nonnegative radially symmetric function with respect to x_0 and $G \subset \mathrm{Isom}_g(M)$ be a group that satisfies the hypothesis $(\boldsymbol{H^{x_0}_G})$. If $\widetilde{f} : [0, \infty) \to \mathbb{R}$ is a continuous function satisfying $(\widetilde{f}_1)$ and $(\widetilde{f}_2)$, then:*

(i) *the infimum of $\widetilde{\mathcal{E}}_G$ on W^b_G is attained at an element $u_G \in W^b_G$;*

(ii) *$u_G(x) \in [0, a]$ a.e. $x \in M$;*

(iii) *(u_G, φ_{u_G}) is a weak solution to system $(\widetilde{\mathcal{SM}})$.*

Proof. (i) By using the same method as in the proof of Theorem 4.3 and Proposition 5.1, the functional $\widetilde{\mathcal{E}}_G$ is sequentially weakly lower semicontinuous and bounded from below on $H^1_{g,G}(M)$. The set W^b_G is convex and closed in $H^1_{g,G}(M)$, thus weakly closed. Therefore, the claim directly follows (see Theorem 1.4 and Remark 1.2); let $u_G \in W^b_G$ be the infimum of $\widetilde{\mathcal{E}}_G$ on W^b_G.

(ii) We consider the function $\gamma(s) = \min(s_+, a)$ and set $w = \gamma \circ u_G$. Since γ is Lipschitz continuous, then $w \in H^1_g(M)$ (see Hebey, [52, Proposition 2.5, page 24]). We claim that $w \in H^1_{g,G}(M)$. Indeed,

$$(\sigma * w)(x) = w\left(\sigma^{-1}(x)\right) = (\gamma \circ u_G)\left(\sigma^{-1}(x)\right) = \gamma\left(u_G\left(\sigma^{-1}(x)\right)\right) = \gamma(u_G(x)) = w(x),$$

for every $x \in M$ and $\sigma \in G$. By construction, we clearly have that $w \in W^b_G$.

Consider $A = \{x \in M : u_G(x) \notin [0, a]\}$ and suppose that the Riemannian measure of A is positive. If

$$A_1 = \{x \in A : u_G(x) < 0\} \text{ and } A_2 = \{x \in A : u_G(x) > a\},$$

one has that $A = A_1 \cup A_2$, and from the construction we have $w(x) = u_G(x)$ for all $x \in M \setminus A$, $w(x) = 0$ for all $x \in A_1$, and $w(x) = a$ for all $x \in A_2$. The latter facts show that

$$\begin{aligned}\widetilde{\mathcal{E}}_G(w) - \widetilde{\mathcal{E}}_G(u_G) = &-\frac{1}{2}\int_A |\nabla_g u_G|^2 \mathrm{d}V_g + \frac{1}{2}\int_A (w^2 - u_G^2)\mathrm{d}V_g + \frac{e}{4}\int_A (\varphi_w w^2 - \varphi_{u_G} u_G^2)\,\mathrm{d}V_g \\ &- \int_A \alpha(x)\left(\widetilde{F}(w) - \widetilde{F}(u_G)\right)\,\mathrm{d}V_g.\end{aligned}$$

Here, $\widetilde{F}(s) = \int_0^s \widetilde{f}(t)\mathrm{d}t$. We observe that

$$\int_A \left(w^2 - u_G^2\right)\mathrm{d}V_g = -\int_{A_1} u_G^2 \mathrm{d}V_g + \int_{A_2} \left(a^2 - u_G^2\right)\mathrm{d}V_g \le 0.$$

It is also clear that $\int_{A_1} \alpha(x)(\widetilde{F}(w) - \widetilde{F}(u_G))\mathrm{d}V_g = 0$, and due to the mean value theorem and $(\widetilde{f}_2)$ we have that $\int_{A_2} \alpha(x)(\widetilde{F}(w) - \widetilde{F}(u_G))\mathrm{d}V_g \ge 0$. Furthermore,

$$\int_A (\varphi_w w^2 - \varphi_{u_G} u_G^2)\mathrm{d}V_g = -\int_{A_1} \varphi_{u_G} u_G^2 \mathrm{d}V_g + \int_{A_2} (\varphi_w w^2 - \varphi_{u_G} u_G^2)\mathrm{d}V_g,$$

and since $0 \le w \le u_G$, we have that

$$\int_{A_2} (\varphi_w w^2 - \varphi_{u_G} u_G^2)\mathrm{d}V_g \le 0.$$

Combining the above estimates, we have $\widetilde{\mathcal{E}}_G(w) - \widetilde{\mathcal{E}}_G(u_G) \le 0$.

On the other hand, since $w \in W^b_G$ then $\widetilde{\mathcal{E}}_G(w) \ge \widetilde{\mathcal{E}}_G(u_G) = \inf_{W^b_G} \widetilde{\mathcal{E}}_G$, thus we necessarily have that

$$\int_{A_1} u_G^2 \mathrm{d}V_g = \int_{A_2} (a^2 - u_G^2)\mathrm{d}V_g = 0,$$

which implies that the Riemannian measure of A should be zero, a contradiction.

(iii) The proof is divided into two steps.

Step 1. $\widetilde{\mathcal{E}}'(u_G)(w-u_G)\geq 0$ *for all* $w\in W^b$. It is clear that the set W^b is closed and convex in $H^1_g(M)$. Let χ_{W^b} be the indicator function of the set W^b (i.e., $\chi_{W^b}(u)=0$ if $u\in W^b$, and $\chi_{W^b}(u)=+\infty$ otherwise) and consider the Szulkin-type functional $\mathcal{I}:H^1_g(M)\to\mathbb{R}\cup\{+\infty\}$ given by $\mathcal{I}=\widetilde{\mathcal{E}}+\chi_{W^b}$. On account of the definition of the set W^b_G, the restriction of χ_{W^b} to $H^1_{g,G}(M)$ is precisely the indicator function $\chi_{W^b_G}$ of the set W^b_G. By (i), since u_G is a local minimum point of $\widetilde{\mathcal{E}}_G$ relative to the set W^b_G, it is also a local minimum point of the Szulkin-type functional $\mathcal{I}_G=\widetilde{\mathcal{E}}_G+\chi_{W^b_G}$ on $H^1_{g,G}(M)$. In particular, u_G is a critical point of $\mathcal{I}_G$ in the sense of Szulkin (see Section 1.2), i.e.,

$$0\in\widetilde{\mathcal{E}}'_G(u_G)+\partial\chi_{W^b_G}(u_G)\text{ in }\left(H^1_{g,G}(M)\right)^*.$$

By exploring the compactness of the group G, we may apply the non-smooth principle of symmetric criticality for Szulkin-type functionals (see Theorem 1.6), obtaining that

$$0\in\widetilde{\mathcal{E}}'(u_G)+\partial\chi_{W^b}(u_G)\text{ in }\left(H^1_g(M)\right)^*.$$

Consequently, the claim follows since for every $w\in W^b$ we have that

$$0\leq\widetilde{\mathcal{E}}'(u_G)(w-u_G)+\chi_{W^b}(w)-\chi_{W^b}(u_G).$$

Step 2. (u_G,φ_{u_G}) *is a weak solution to the system* $(\widetilde{\mathcal{SM}})$. By assumption $(\widetilde{f}_1)$ it is clear that $C_{\mathtt{m}}=\sup_{s\in\mathbb{R}}|\widetilde{f}(s)|<\infty$. The previous step and (5.15) imply that

$$\begin{aligned}0\ &\leq\ \int_M\langle\nabla_g u_G,\nabla_g(w-u_G)\rangle\mathrm{d}V_g+\int_M u_G(w-u_G)\mathrm{d}V_g\\ &\quad+e\int_M u_G\varphi_{u_G}(w-u_G)\mathrm{d}V_g-\int_M\alpha(x)\widetilde{f}(u_G)(w-u_G)\mathrm{d}V_g,\ \forall w\in W^b.\end{aligned}$$

Let us define the truncation function $\zeta(s)=\operatorname{sgn}(s)\min\{|s|,b\}$. Since ζ is Lipschitz continuous and $\zeta(0)=0$, then for fixed $\varepsilon>0$ and $v\in H^1_g(M)$ the function $w_\zeta=\zeta\circ(u_G+\varepsilon v)$ belongs to $H^1_g(M)$, see Hebey [52, Proposition 2.5, page 24]. By construction, $w_\zeta\in W^b$. Now, standard estimates for the test-function $w=w_\zeta$ yield

$$0\leq\int_M\langle\nabla_g u_G,\nabla_g v\rangle\mathrm{d}V_g+\int_M u_G v\mathrm{d}V_g+e\int_M u_G\varphi_{u_G}v\mathrm{d}V_g-\int_M\alpha(x)\widetilde{f}(u_G)v\mathrm{d}V_g.$$

Replacing v by $(-v)$, it follows that

$$0=\int_M\langle\nabla_g u_G,\nabla_g v\rangle\mathrm{d}V_g+\int_M u_G v\mathrm{d}V_g+e\int_M u_G\varphi_{u_G}v\mathrm{d}V_g-\int_M\alpha(x)\widetilde{f}(u_G)v\mathrm{d}V_g,$$

i.e., $\widetilde{\mathcal{E}}'(u_G)=0$. Thus (u_G,φ_{u_G}) is a G-invariant weak solution to $(\widetilde{\mathcal{SM}})$. $\square$

Now, we are ready to conclude the proof of Theorem 5.6. Let $s > 0$, $0 < r < \rho$ and $A_{x_0}[r,\rho] = B_g(x_0, \rho + r) \setminus B_g(x_0, \rho - r)$ be an annulus-type domain. We also define the function $w_s : M \to \mathbb{R}$ by

$$w_s(x) = \begin{cases} 0, & x \in M \setminus A_{x_0}[r,\rho], \\ s, & x \in A_{x_0}[r/2,\rho], \\ \dfrac{2s}{r}\left(r - |d_g(x_0,x) - \rho|\right), & x \in A_{x_0}[r,\rho] \setminus A_{x_0}[r/2,\rho]. \end{cases}$$

Note that $(\boldsymbol{H_G^{x_0}})$ implies $w_s \in H^1_{g,G}(M)$.

Due to (f_0^2) and the continuity of f one can fix two sequences $\{\theta_k\}_k, \{\eta_k\}_k$ such that $\lim_{k\to+\infty} \theta_k = \lim_{k\to+\infty} \eta_k = 0$, and for every $k \in \mathbb{N}$,

$$0 < \theta_{k+1} < \eta_k < s_k < \theta_k < 1, \tag{5.18}$$

$$f(s) \leq 0 \text{ for every } s \in [\eta_k, \theta_k], \tag{5.19}$$

Consider the truncation function $f_k(s) = f(\min(s,\theta_k))$. Since $f(0) = 0$ (by (f_0^1) and (f_0^2)), then $f_k(0) = 0$ and we may extend continuously the function f_k to the whole real line by $f_k(s) = 0$ if $s \leq 0$. For every $s \in \mathbb{R}$ and $k \in \mathbb{N}$, we define $F_k(s) = \int_0^s f_k(t)\mathrm{d}t$. It is clear that f_k satisfies the assumptions $(\widetilde{f}_1)$ and $(\widetilde{f}_2)$. Thus, applying Lemma 5.1 to the function f_k, $k \in \mathbb{N}$, the system

$$\begin{cases} -\Delta_g u + u + eu\varphi = \alpha(x) f_k(u) & \text{in} \quad M, \\ -\Delta_g \varphi + \varphi = qu^2 & \text{in} \quad M, \end{cases} \tag{5.20}$$

has a G-invariant weak solution $(u_k^0, \varphi_{u_k^0}) \in H^1_{g,G}(M) \times H^1_{g,G}(M)$ such that

$$u_k^0 \in [0, \eta_k] \text{ a.e. } x \in M, \tag{5.21}$$

$$u_k^0 \text{ is the infimum of the functional } \mathcal{E}_k \text{ on the set } W_G^{\theta_k}, \tag{5.22}$$

where

$$\mathcal{E}_k(u) = \frac{1}{2}\|u\|^2_{H_g^1(M)} + \frac{e}{4}\int_M \varphi_u u^2 \mathrm{d}V_g - \int_M \alpha(x) F_k(u)\mathrm{d}V_g.$$

By (5.21), $(u_k^0, \varphi_{u_k^0}) \in H^1_{g,G}(M) \times H^1_{g,G}(M)$ is also a weak solution to the initial system $(\mathcal{SM})$.

It remains to prove the existence of infinitely many distinct elements in the sequence $\{(u_k^0, \varphi_{u_k^0})\}_k$. Note that there exist $0 < r < \rho$ such that $\operatorname{essinf}_{A_{x_0}[r,\rho]}\alpha > 0$. For simplicity, let $D = A_{x_0}[r,\rho]$ and $K = A_{x_0}[r/2,\rho]$. By (f_0^1) there exist $l_0 > 0$ and $\delta \in (0,\theta_1)$ such that

$$F(s) \geq -l_0 s^2 \text{ for every } s \in (0,\delta). \tag{5.23}$$

Assumption (f_0^1) implies the existence of a non-increasing sequence $\{\widetilde{s}_k\}_k \subset (0,\delta)$ such that $\widetilde{s}_k \leq \eta_k$

and

$$F(\widetilde{s}_k) > L_0 \widetilde{s}_k^2, \ \forall k \in \mathbb{N}, \tag{5.24}$$

where $L_0 > 0$ is large enough, e.g.,

$$L_0 \mathrm{essinf}_K \alpha > \frac{1}{2}\left(1 + \frac{4}{r^2}\right) \mathrm{Vol}_g(D) + \frac{e}{4}\|\varphi_\delta\|_{L^1(D)} + l_0 \|\alpha\|_{L^1(M)}. \tag{5.25}$$

Note that

$$\mathcal{E}_k(w_{\widetilde{s}_k}) = \frac{1}{2}\|w_{\widetilde{s}_k}\|^2_{H^1_g(M)} + \frac{e}{4} I_k - J_k,$$

where

$$I_k = \int_D \varphi_{w_{\widetilde{s}_k}} w^2_{\widetilde{s}_k} \mathrm{d}V_g \text{ and } J_k = \int_D \alpha(x) F_k(w_{\widetilde{s}_k}) \mathrm{d}V_g.$$

Observe that $I_k \leq \widetilde{s}_k^2 \|\varphi_\delta\|_{L^1(D)}, \ k \in \mathbb{N}$. Moreover, by (5.23) and (5.24) we have that

$$J_k \geq L_0 \widetilde{s}_k^2 \mathrm{essinf}_K \alpha - l_0 \widetilde{s}_k^2 \|\alpha\|_{L^1(M)}, \ k \in \mathbb{N}.$$

Therefore,

$$\mathcal{E}_k(w_{\widetilde{s}_k}) \leq \widetilde{s}_k^2 \left(\frac{1}{2}\left(1 + \frac{4}{r^2}\right) \mathrm{Vol}_g(D) + \frac{e}{4}\|\varphi_\delta\|_{L^1(D)} + l_0 \|\alpha\|_{L^1(M)} - L_0 \mathrm{essinf}_K \alpha \right).$$

Thus, on one hand, by (5.25) we have that

$$\mathcal{E}_k(u_k^0) = \inf_{W_G^{\theta_k}} \mathcal{E}_k \leq \mathcal{E}_k(w_{\widetilde{s}_k}) < 0. \tag{5.26}$$

On the other hand, by (5.18) and (5.21) we obtain that

$$\mathcal{E}_k(u_k^0) \geq -\int_M \alpha(x) F_k(u_k^0) \mathrm{d}V_g = -\int_M \alpha(x) F(u_k^0) \mathrm{d}V_g \geq -\|\alpha\|_{L^1(M)} \max_{s \in [0,1]} |f(s)| \eta_k, \ k \in \mathbb{N}.$$

Combining the latter relations, it yields that

$$\lim_{k \to +\infty} \mathcal{E}_k(u_k^0) = 0.$$

Since $\mathcal{E}_k(u_k^0) = \mathcal{E}_1(u_k^0)$ for all $k \in \mathbb{N}$, we obtain that the sequence $\{u_k^0\}_k$ contains infinitely many distinct elements. In particular, by (5.26) we have that

$$\frac{1}{2}\|u_k^0\|^2_{H^1_g(M)} \leq \|\alpha\|_{L^1(M)} \max_{s \in [0,1]} |f(s)| \eta_k,$$

which implies that $\lim_{k \to \infty} \|u_k^0\|_{H^1_g(M)} = 0$. Recalling (5.11), we also have $\lim_{k \to \infty} \|\varphi_{u_k^0}\|_{H^1_g(M)} = 0$, which concludes the proof. $\square$

We close this section with some examples for which Theorem 5.6 can be applied.

- *Euclidean spaces.* If $(M,g) = (\mathbb{R}^n, g_e)$ is the usual Euclidean space, then $x_0 = 0$ and $G = SO(n_1) \times ... \times SO(n_l)$ with $n_k \geq 2$, $k \in \{1,...,l\}$ and $n_1 + ... + n_l = n$, satisfy $(\boldsymbol{H_G^{x_0}})$. Here, $SO(k)$ denotes the special orthogonal group in dimension k. Indeed, we have $\mathrm{Fix}_{\mathbb{R}^n}(G) = \{0\}$.

- *Hyperbolic spaces.* Let us consider the Poincaré ball model $\mathbb{H}^n = B_e(0,1) = \{x \in \mathbb{R}^n : |x| < 1\}$ from §3.2.2. Hypothesis $(\boldsymbol{H_G^{x_0}})$ is verified with the same choices as above.

- *Symmetric positive definite matrices.* Let $\mathrm{Sym}(n,\mathbb{R})$ be the set of real-valued symmetric $n \times n$ matrices, $\mathrm{P}(n,\mathbb{R}) \subset \mathrm{Sym}(n,\mathbb{R})$ be the $\frac{n(n+1)}{2}$-dimensional cone of symmetric positive definite matrices, and $\mathrm{P}(n,\mathbb{R})_1$ be the subspace of matrices in $\mathrm{P}(n,\mathbb{R})$ with determinant one. The set $\mathrm{P}(n,\mathbb{R})$ is endowed with the scalar product

$$\langle\langle U,V\rangle\rangle_X = \mathrm{Tr}(X^{-1}VX^{-1}U), \quad \forall X \in \mathrm{P}(n,\mathbb{R}),\ U,V \in T_X(\mathrm{P}(n,\mathbb{R})) \simeq \mathrm{Sym}(n,\mathbb{R}),$$

where $\mathrm{Tr}(Y)$ denotes the trace of $Y \in \mathrm{Sym}(n,\mathbb{R})$, and let us denote by $d_H : \mathrm{P}(n,\mathbb{R}) \times \mathrm{P}(n,\mathbb{R}) \to \mathbb{R}$ the induced metric function. The pair $(\mathrm{P}(n,\mathbb{R}), \langle\langle\cdot,\cdot\rangle\rangle)$ is a Hadamard manifold, see [112] and Lang [60, Chapter XII]. Note that $\mathrm{P}(n,\mathbb{R})_1$ is a convex totally geodesic submanifold of $\mathrm{P}(n,\mathbb{R})$ and the special linear group $SL(n,\mathbb{R})$ leaves $\mathrm{P}(n,\mathbb{R})_1$ invariant and acts transitively on it; thus $(\mathrm{P}(n,\mathbb{R})_1, \langle\langle\cdot,\cdot\rangle\rangle)$ is itself a homogeneous Hadamard manifold, see Bridson and Haefliger [18, Chapter II.10]. Moreover, for every $\sigma \in SL(n,\mathbb{R})$, the map $[\sigma] : \mathrm{P}(n,\mathbb{R})_1 \to \mathrm{P}(n,\mathbb{R})_1$ defined by $[\sigma](X) = \sigma X \sigma^t$, is an isometry.

  Let $G = SO(n)$. One can prove that

$$\mathrm{Fix}_{\mathrm{P}(n,\mathbb{R})_1}(G) = \{I_{\mathbb{R}^n}\}.$$

  On one hand, it is clear that $I_{\mathbb{R}^n} \in \mathrm{Fix}_{\mathrm{P}(n,\mathbb{R})_1}(G)$; indeed, for every $\sigma \in G$ we have that

$$[\sigma](I_{\mathbb{R}^n}) = \sigma I_{\mathbb{R}^n} \sigma^t = \sigma\sigma^t = I_{\mathbb{R}^n}.$$

  On the other hand, if $X_0 \in \mathrm{Fix}_{\mathrm{P}(n,\mathbb{R})_1}(G)$, then it turns out that $\sigma X_0 = X_0 \sigma$ for every $\sigma \in G$. By using elementary matrices from G, the latter relation implies that $X_0 = cI_{\mathbb{R}^n}$ for some $c \in \mathbb{R}$. Since $X_0 \in \mathrm{P}(n,\mathbb{R})_1$, we necessarily have $c = 1$.

5.4 Further results and comments

I) Sublinear problems on compact Riemannian manifolds: the gap interval. In Theorem 5.1, we proved that c_f and c_F may be arbitrary close to each other, thus the gap interval $\left[c_f^{-1}\|\beta/\alpha\|_{L^\infty}^{-1}, c_F^{-1}\|\alpha/\beta\|_{L^\infty}\right]$ can be arbitrarily small, but never degenerated. It is not clear what can be said about the number of solutions to the problem (P_λ) when λ belongs to the latter interval.

II) Infinitely many solutions: Finsler versus Riemannian settings. We note that the arguments in the proof of Theorem 5.5 cannot be applied in generic Finsler manifolds to produce infinitely many solutions to elliptic problems. Indeed, we recall that we used the symmetric version of the Mountain Pass Theorem, which required the *evenness* of the energy functional associated to the studied problem. Now, if we consider the Randers space $(B_e(0,1); F_a)$ from Section 4.2 with the metric F_a defined by (4.12), we observe that the energy functional

$$u \mapsto \int_{B_e(0,1)} [F_a^*(x, Du(x))]^2 \, \mathrm{d}V_{F_a}(x)$$

is not even, thus it is *not* G^n_{j,τ_k}-invariant (see the action (5.7)); the latter follows by the fact that F_a is not reversible unless $a = 0$, which corresponds to the Riemannian (non-Finslerian) Klein model.

III) Schrödinger-Maxwell systems of Poisson type. Beside the Schrödinger-Maxwell system $(\mathcal{SM})$ involving oscillatory nonlinear terms, is it possible to treat other systems. For instance, if we consider the Schrödinger-Maxwell system with a Poisson-type term, we can prove

Theorem 5.7. (Farkas and Kristály [105]) *Let (M, g) be an n-dimensional $(3 \leq n \leq 6)$ homogeneous Hadamard manifold, and $\alpha \in L^2(M)$ be a nonnegative function. Then there exists a unique, nonnegative weak solution $(u_0, \varphi_0) \in H^1_g(M) \times H^1_g(M)$ to the system*

$$\begin{cases} -\Delta_g u + u + eu\varphi = \alpha(x) & \text{in} \quad M, \\ -\Delta_g \varphi + \varphi = qu^2 & \text{in} \quad M. \end{cases} \tag{$\mathcal{SM}$}$$

Moreover, if $x_0 \in M$ is fixed and α is radially symmetric with respect to x_0, then (u_0, φ_0) is G-invariant with respect to any group $G \subset \mathrm{Isom}_g(M)$ which satisfies $(\boldsymbol{H_G^{x_0}})$.

IV) Schrödinger-Poisson systems with arbitrary growth nonlinearity. For simplicity, let $\Omega \subset \mathbb{R}^n$ be an open bounded domain, $n \geq 2$ (Ω can also be a subset of a complete Riemannian manifold). We consider the model Schrödinger-Poisson system

$$\begin{cases} -\Delta u = \varphi^p & \text{in} \quad \Omega, \\ -\Delta \varphi = f(u) & \text{in} \quad \Omega, \\ u = \varphi = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{$\mathcal{SP}$}$$

where

$$\begin{cases} 0 < p, & \text{if} \quad n = 2, \\ 0 < p < \dfrac{2}{n-2}, & \text{if} \quad n \geq 3, \end{cases} \tag{5.27}$$

and the continuous function $f : \mathbb{R} \to \mathbb{R}$ fulfills the hypotheses:

(H_0^1) $-\infty < \liminf_{s\to 0} \frac{F(s)}{|s|^{\frac{p+1}{p}}} \leq \limsup_{s\to 0} \frac{F(s)}{|s|^{\frac{p+1}{p}}} = +\infty$, where $F(s) = \displaystyle\int_0^s f(t)\mathrm{d}t$, $s \in \mathbb{R}$,

(H_0^2) there exist two sequences $\{a_k\}_k$ and $\{b_k\}_k$ in $(0,\infty)$ with $b_{k+1} < a_k < b_k$, $\lim_{k\to\infty} b_k = 0$ such that

$$\text{sgn}(s)f(s) \leq 0, \;\forall |s| \in [a_k, b_k],$$

(H_0^3) $\lim_{k\to\infty} \frac{a_k}{b_k} = 0$ and $\lim_{k\to\infty} \frac{\max_{[-a_k,a_k]} F}{b_k^{\frac{p+1}{p}}} = 0$.

Hypotheses (H_0^1) $-$ (H_0^3) imply an oscillatory behaviour of f near the origin. By using the general variational principle of Ricceri [80, 81], one can prove the following result, which constitutes a kind of counterpart for Theorem 5.6:

Theorem 5.8. (Kristály [114]) *Assume that* (5.27) *holds and* $f \in C(\mathbb{R},\mathbb{R})$ *fulfills* (H_0^1) $-$ (H_0^3). *Then, system* ($\mathcal{SP}$) *possesses a sequence* $\{(u_k,\varphi_k)\}_k \subset X \times X$ *of distinct (strong) solutions which satisfy*

$$\lim_{k\to\infty} \|u_k\|_X = \lim_{k\to\infty} \|\varphi_k\|_X = \lim_{k\to\infty} \|u_k\|_{L^\infty(\Omega)} = \lim_{k\to\infty} \|\varphi_k\|_{L^\infty(\Omega)} = 0,$$

where $X = W^{2,\frac{p+1}{p}}(\Omega) \cap W_0^{1,\frac{p+1}{p}}(\Omega)$.

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