Convex bodies and their approximations

D.Sc. dissertation

Ferenc Fodor

Bolyai Institute
University of Szeged
Hungary

2019

Ajánlás

Ezt a disszertációt szüleimnek: Fodor Ferencné Zahoran Olgának és idősebb Fodor Ferencnek ajánlom nagyon sok szeretettel és hálával. Az ő áldozatvállalásuk nélkül sohasem lehettem volna matematikus.

Acknowledgements

First of all, I would like to thank my co-authors: Károly J. Böröczky, Daniel Hug, Péter Kevei, and Viktor Vígh for their valuable contributions. I wish to express my deep gratitude to my doctoral supervisor András Bezdek under whose guidance I have become a research mathematician. I am also grateful to Tamás Szőnyi who helped me a lot in the early stages of my studies. The friendship and support of Ted Bisztriczky has always been an invaluable asset and inspiration for me. I also owe thanks to the late Paul Erdős for his support. I am thankful for the valuable collaboration with my former students Gergely Ambrus, Viktor Vígh and Tamás Zarnócz; I consider it a privilege to have been their advisor.

My family has always supported me throughout my career and, in particular, during the composition of this dissertation. Heartfelt thanks are due to my wife Mária Bakti who endured with great patience the long hours I spent on this work, to my mother Olga Fodor Ferencné Zahoran, my father Ferenc Fodor and my brother László Fodor for always being there for me when I needed them most.

Contents

Chapter 1

Summary

This dissertation is part of my application for the *Doctor of the Hungarian Academy of Sciences* (D.Sc., in Hungarian: MTA doktora) title. The topic of this work belongs to the theory of convex bodies, and to the theory of approximations (best and random) of convex bodies by polytopes and similar objects. The dissertation is based on six of my papers, each written with co-authors, that appeared in high-quality refereed international mathematical journals. These papers are the following (in alphabetical order): Böröczky and Fodor [BF19], Böröczky, Fodor and Hug [BFH10], Böröczky, Fodor and Hug [BFH13], Fodor, Kevei and Vígh [FKV14], Fodor and Vígh [FV12], and Fodor and Vígh [FV18] (citations refer to the Bibliography at the end of this dissertation). The paper [BF19] investigates a generalization of the classical Minkowski problem which is one of the fundamental questions in the theory of convex bodies. The papers [BFH10, BFH13] are about approximations of convex bodies by random polytopes and polyhedral sets in three different settings. In particular, [BFH10] studies weighted random approximations of convex bodies by random polytopes contained in the body, and by applying certain polarity arguments, also approximations by random polyhedral sets containing the body. The article [BFH13] is about approximations of convex bodies by random polytopes whose vertices are chosen from the boundary of the body. The three papers [FKV14, FV12, FV18] concern the approximation properties convex of bodies that are intersections of congruent closed balls (so-called spindle convex or hyperconvex bodies) both in the random setting [FKV14, FV18] and for best approximations [FV12]. The three papers [BFH10, BFH13, FKV14] are about asymptotic results on expectations of various geometric quantities of random polytopes, polyhedra and disc-polygons, while [FV18] contains asymptotic bounds on the variance of some of these quantities. Some of my other papers ([BFRV09, BFV10, FHZ16]) on random approximations about asymptotic bounds on variance and laws of large numbers are not used in this dissertation, but they are briefly mentioned in the historical overview. The problems discussed in this work belong to the rapidly developing fields of Convexity and Stochastic Geometry that are intricately interlaced. In our arguments we use a combination of methods from Geometry, Analysis and Probability.

The dissertation is organized as follows. Chapter 2 is an introduction: Section 2.1 contains a summary of our results along with a brief overview of the history of the relevant parts of the theory. In Section 2.2 we introduce some of the most important terms and notations used throughout this work.

Chapter 3 is based on the paper [BF19]. We solve the existence part of the L_p dual Minkowski problem for $p > 1$ and $q > 0$, which, in the absolutely continuous case, constitutes solving the associated Monge-Ampère equation. We also examine the regularity properties of the solutions for certain measures.

Chapter 4 is based on parts of the paper [BFH10] in which we investigate weighted volume approximations of general convex bodies by inscribed random polytopes.

Chapter 5 is based on parts of the paper [BFH10] where we deal with mean width approximations of convex bodies by circumscribed random polytopes.

Chapter 6 is based on the paper [BFH13]. In this chapter we study the properties of the intrinsic volumes of random polytopes whose vertices are selected from the boundary of a convex body.

In Chapters 7 and 8 we investigate approximations of sufficiently round convex bodies in the plane by convex disc-polygons, which are objects that arise as intersections of congruent circular discs. In particular, Chapter 7 is based on the papers [FKV14] and [FV18], where we consider random approximations by inscribed random disc-polygons in the plane. Chapter 8 is based on the paper [FV12] in which we investigate the properties of best approximations of planar convex bodies by disc-polygons.

Below we state the main results of this dissertation in the form of six *Theses*. Since the following Theses are intended for a wider readership, they are phrased with the minimal use of mathematical symbols. The mathematically precise statements of our results are formulated in the individual chapters of this work.

Thesis 1. *We have solved the existence part of the L_p dual Minkowski problem for $p > 1$ and $q > 0$, which, in essence, constitutes solving the associated Monge-Ampère equation if the considered measure is absolutely continuous with respect to the Hausdorff measure on the sphere. We also examine the smoothness of the solutions using the regularity properties of the Monge-Ampère equation.*

Thesis 1 is supported by Theorems 3.1.1, 3.1.2, and Theorems 3.1.3, 3.1.4 and 3.1.5. In particular, Theorem 3.1.1 establishes the existence of the solution of the L_p dual Minkowski problem for $p > 1$ and $q > 0$ for discrete measures, and Theorem 3.1.2 deals with the case of general measures. Theorems 3.1.3, 3.1.4 and 3.1.5 establish smoothness properties of the solution in the case when the measure is absolutely continuous with respect to the $(n-1)$-dimensional Hausdorff measure on the unit sphere. (The detailed proofs of the smoothness results are not included in this dissertation, for the arguments see [BF19].)

Thesis 2. *We have established an asymptotic formula in d-dimensional Euclidean space for the expectation of the difference of weighted volume of a general convex body and a random polytope which is the convex hull of n identically distributed independent random points chosen from the convex body according to a given probability density function, as n tends to infinity. It is assumed that both the weight function and the probability density function are continuous and the probability density function is positive in a neighbourhood of the boundary of the convex body.*

Thesis 2 is supported by Theorem 4.1.1 in Chapter 4. We note that Theorem 4.1.1 implies Corollary 4.1.2, which provides an asymptotic formula for the expectation of the number of vertices of the random polytope. Theorem 4.1.1 and Corollary 4.1.2 are later used to prove Theorem 5.1.1 and Theorem 5.1.2 in Chapter 5 but they are important in themselves being the most general version of a sequence of earlier results. Their significance is partly due to the fact that there is no regularity or smoothness condition on the boundary of the convex body and both the weight function and the probability density function are very general.

Thesis 3. *We have established an asymptotic formula in d-dimensional Euclidean space for the expectation of the mean width difference of a general convex body and a random polyhedral set containing the convex body where the random polyhedral set is the intersection of n identically distributed independent random closed half-spaces, each containing the convex body and selected according to a prescribed probability density, as n tends to infinity.*

Thesis 3 is supported by Theorem 5.1.1. As a corollary of Theorem 5.1.1, we also obtain an asymptotic formula for the expected number of facets of the random polyhedral set as n tends to infinity, cf. Theorem 5.1.2. We note that, in fact, we have proved the much more general statements in Theorem 5.2.2 and Theorem 5.2.3. The significance of the result of Thesis 3 is due to the fact that previously there has been very little information about circumscribed random polytopes compared to the vast literature of the inscribed case, and that there are no requirements for the regularity or smoothness of the boundary of the convex body.

Thesis 4. *We have established an asymptotic formula in d-dimensional Euclidean space for the expectation of the difference of the intrinsic volumes of a convex body that has a rolling ball and a random polytope which is the convex hull of n identically distributed independent random points chosen from the boundary of the convex body according to a given continuous and positive probability density.*

Thesis 4 is supported by Theorem 6.1.2. We note that examples show that the condition that the convex body has a rolling ball cannot be dropped without losing the validity of the asymptotic formula. The result of Thesis 4 is an extension of earlier theorems of Reitzner [Rei02], Schütt and Werner [SW03], however, the methods used in the proof are quite different.

Thesis 5. *We have proved asymptotic formulae in the Euclidean plane for sufficiently round and smooth convex discs for the expectation of the number of vertices, area difference and perimeter difference of the convex disc and a random disc-polygon generated by n independent uniform random points selected from the convex disc, as n tends to infinity. We have also proved asymptotic estimates on the variance of the missed area and the number of vertices. Furthermore, we give analogous results for a circumscribed model.*

Thesis 5 is supported by Theorems 7.1.1, 7.1.2, 7.1.3, and Theorems 7.2.1, 7.2.2, and Theorem 7.2.6, Corollary 7.2.7. The term *sufficiently round* means that there is a positive radius R such that the convex disc can be represented as the intersection of a family of radius R closed circular discs. The random disc-polygons arise as the intersections of all radius R closed circular discs containing n independent uniform random points chosen from the convex disc. Theorems 7.1.1, 7.1.2, and 7.1.3 are the disc-polygonal analogues of the celebrated results of Rényi and Sulanke [RS63, RS64] for the random approximations of smooth convex discs by uniform random polygons. Moreover, they are also generalizations of the results of Rényi and Sulanke in the sense that for a sufficiently smooth convex disc they converge to them as the radius R tends to infinity. Theorems 7.2.1 and 7.2.2 provide asymptotic bounds on the variance of the number of vertices and missed area for smooth convex discs and circles, respectively. Theorem 7.2.6 and Corollary 7.2.7 present a circumscribed model and certain analogues of the inscribed results.

Thesis 6. *We have established asymptotic formulae for the approximation orders of sufficiently round and smooth convex discs in the Euclidean plane by inscribed and circumscribed disc-polygons with n vertices in the sense of area, perimeter and Hausdorff distance, as n tends to infinity.*

Thesis 6 is supported by Theorem 8.1.1. This result is a disc-polygonal analogue and generalization of the classical theorem proved by McClure and Vitale [MV75], originally stated by L. Fejes Tóth [FT53], for the approximation orders of convex discs by inscribed and circumscribed polygons with n vertices in the sense of area, perimeter and Hausdorff distance, as n tends to infinity.

Chapter 2

Introduction

2.1 History and overview of results

The classical Minkowski problem in the Brunn-Minkowski theory of convex bodies is concerned with the characterization of the so-called surface area measure. The surface area measure of a convex body K is a Borel measure on the unit sphere S^{n-1} such that for any Borel set η, the measure of η is defined as the $n-1$ dimensional Hausdorff measure of its inverse image under the spherical image map. The (classical) Minkowski problem asks for necessary and sufficient conditions for a Borel measure on S^{n-1} to be the surface area measure of a convex body. A particularly important case of the Minkowski problem is for discrete measures. Let $P \subset \mathbb{R}^n$ be an n-dimensional polytope, which is defined as the convex hull of a finite number of points in $\mathbb{R}^n$ provided $\mathrm{int} P \neq \emptyset$. Those faces whose dimension is $n-1$ are called facets. A polytope P has a finite number of facets and the union of facets covers the boundary of P. The surface area measure of P is a discrete measure on the sphere that is concentrated on the outer unit normals of the facets. The measure of a Borel set η on S^{n-1} is equal to the sum of the surface areas of the facets of P whose outer unit normals are contained in η.

The (discrete) Minkowski problem asks the following: let μ be a discrete positive Borel measure on S^{n-1}. Under what conditions does there exist a polytope P such that its surface area measure is μ? Furthermore, if such a P exists, is it unique? This polytopal version, along with the case when the surface area measure of K is absolutely continuous with respect to the spherical Lebesgue measure, was solved by Minkowski [Min97, Min03]. He also proved the uniqueness of the solution. For general measures the problem was solved by Alexandrov [Ale38, Ale39] and independently by Fenchel and Jensen. The argument for existence uses the Alexandrov variational formula of the surface area measure, and the uniqueness employs the Minkowski inequality for mixed volumes. In summary, the necessary and sufficient conditions for the existence of the solution of the Minkowski problem for μ are that for any linear subspace $L \leq \mathbb{R}^n$ with $\dim L \leq n-1$, $\mu(L \cap S^{n-1}) < \mu(S^{n-1})$, and that the centre of mass of μ is at the origin.

Similar questions have been posed, and at least partially solved, for other measures associated with convex bodies in the Brunn-Minkowski theory, for example, the integral curvature measure of Alexandrov, or the L_p surface area measure introduced by Lutwak [Lut93b], where the case $p=1$ is the classical surface area measure, and the $p=0$ case is the cone volume measure (logarithmic Minkowski problem). For a detailed description of these measures and their associated Minkowski problems, and further references, see, for example, the book [Sch14] by Schneider, and the paper [HLYZ16] by Huang, Lutwak, Yang and Zhang.

Lutwak built the dual Brunn-Minkowski theory in the 1970s as a "dual" counterpart of the classical theory. Although there is no formal duality between the classical and dual theories, one can say roughly that in the dual theory the radial function plays a similar role as the support function in the classical theory. The dual Brunn-Minkowski theory concerns the class of compact star shaped sets of $\mathbb{R}^n$. Convex bodies are naturally star shaped with respect to any of their points.

The qth dual intrinsic volumes for convex bodies containing the origin in their interior were defined by Lutwak [Lut75], whose definition works for all real q. His definition is via an integral formula involving the qth power of the radial function (for the precise definition see (3.1.2)). We note that dual intrinsic volumes for integers $q = 0, \dots, n$ are the coefficients of the dual Steiner polynomial for star shaped compact sets, where the radial sum replaces the Minkowski sum. The qth dual intrinsic volumes, which arise as coefficients naturally satisfy (3.1.2), and this provides the possibility to extend their definition for arbitrary real q in the case when the origin is in the interior of the body.

Huang, Lutwak, Yang, Zhang [HLYZ16] and Lutwak, Yang, Zhang [LYZ18] defined, with the help of the reverse radial Gauss map, the qth dual curvature measures by means of an integral formula involving the qth power of the radial function; for the precise definition we refer to 3.1.3. We note that the so-called cone volume measure and Alexandrov's integral curvature measure can both be represented as dual curvature measures. Furthermore, the qth dual curvature measure is a natural extension of the cone volume measure also in the variational sense, see Corollary 4.8 of Huang, Lutwak, Yang, Zhang [HLYZ16].

For integers $q = 0, \dots, n$, dual curvature measures arise in a similar way as in the Brunn-Minkowski theory by means of localized dual Steiner polynomials. These measures satisfy (3.1.3), and hence their definition can be extended for $q \in \mathbb{R}$. Huang, Lutwak, Yang and Zhang [HLYZ16] proved that the qth dual curvature measure of a convex body containing the origin in its interior can also be obtained from the qth dual intrinsic volume by means of an Alexandrov-type variational formula.

Lutwak, Yang, Zhang [LYZ18] introduced a more general version of dual curvature measures where a star shaped set Q (called the parameter body) containing the origin in its interior is also involved; for a precise definition see (3.2.9). The parameter body Q acts as a gauge, and its advantage is, for example, in the equiaffine invariant formula (3.1.10).

The L_p dual curvature measures emerged recently [LYZ18] as a family of geometric measures which unify several important families of measures in the Brunn-Minkowski theory and its dual theory of convex bodies. They were also introduced by Lutwak, Yang and Zhang [LYZ18] using the $-p$th power of the support function and qth dual curvature measure (see (3.1.11)). They provide a common framework for several other geometric measures of the (L_p) Brunn-Minkowski theory and the dual theory: L_p surface area measures, L_p integral curvature measures, and dual curvature measures, cf. [LYZ18]. L_p dual curvature measures also arise from Alexandrov-type variational formulas for the dual intrinsic volumes as proved by Lutwak, Yang and Zhang, see Theorem 6.5 in [LYZ18].

In [LYZ18] Lutwak, Yang and Zhang introduced the L_p dual Minkowski existence problem: Find necessary and sufficient conditions that for fixed $p, q \in \mathbb{R}$ and star body Q containing the origin in its interior and a given Borel measure μ on S^{n-1} there exists a convex body K such that μ is the L_p dual qth curvature measure of K. As they note in [LYZ18], this version of the Minkowski problem includes earlier considered other variants (L_p Minkowski problem, dual Minkowski problem, L_p Aleksandrov problem) for special choices of the parameters p and q. When Q is the unit ball and μ is absolutely continuous with density function f, then the L_p dual Minkowski problem constitutes solving the associated Monge-Ampère equation (3.1.12), and in the case of general Q, the somewhat more complicated Monge-Ampère equation (3.1.13).

The case of the L_p dual Minkowski problem for even measures (that are symmetric with respect to the origin) has received much attention, but since this topic diverges from our direction we do not discuss it here in detail. Instead, we refer to Böröczky, Lutwak, Yang, Zhang [BLYZ13] concerning the L_p surface area measure, Böröczky, Lutwak, Yang, Zhang, Zhao [BLY$^+$], Jiang Wu [JW17] and Henk, Pollehn [HP18], Zhao [Zha18] concerning the qth dual curvature measure, and Huang, Zhao [HZ18] for the L_p dual curvature measure for detailed discussion of history and recent results.

We briefly discuss the known results about the L_p dual Minkowski problem in Section 3.1, but for that we need some more formal definitions and notations.

Our main results about the existence part of the L_p dual Minkowski problem are contained in Theorems 3.1.1 and 3.1.2. In particular, Theorem 3.1.2 states that if the measure μ is not concentrated on any closed hemisphere of S^{n-1}, then there exists a convex body K containing the origin such that its L_p dual curvature measure is μ.

We prove Theorem 3.1.2 in several stages. In this dissertation, we only present the proof in the simpler case when the parameter body Q is the unit ball. The general case for an arbitrary parameter body containing the origin in its interior and having a sufficiently smooth boundary is treated in Section 6 of [BF19] on pages 8008–8015.

One of the important ingredients of the proof is the extension for $q > 0$ of the qth dual intrinsic volumes, qth dual curvature measures and L_p dual qth dual curvature measures for convex bodies that may contain the origin on their boundary. We spend Section 3.2 with investigating the properties of these extended notions.

In Section 3.3 we prove Theorem 3.1.1 for the simpler case when the parameter body Q is the unit ball. Theorem 3.1.1 is the discrete version of the main Theorem 3.1.2. The proof of Theorem 3.1.1 follows a variational argument. Before embarking on the actual proof of Theorem 3.1.2 (for $Q = B^n$), we investigate the properties of L_p dual curvature measures in Section 3.4. The proof of Theorem 3.1.2 is contained in Section 3.5 and it by means of weak approximation by discrete measures.

Theorems 3.1.3, 3.1.4 and 3.1.5 establish smoothness properties of the solution of the L_p dual Minkowski problem for measures that are absolutely continuous with respect to the surface area measure. In this case, the solution of the problem constitutes solving a Monge-Ampère type partial differential equation. In this dissertation we do not give the proofs of the statements on the smoothness of the solution but the detailed arguments can be found in Section 7 of [BF19]. The proof uses Caffarelli's results [Caf90a, Caf90b] on the regularity of the solutions of the Monge-Ampère equation.

We continue this section with a brief overview of the relevant parts of the history of random and best approximations of convex bodies by polytopes in various models, and we describe the main results of this type contained in this dissertation without the use of complicated notations. The precise (and formal) statements of results can be found in the first sections of the subsequent chapters.

Approximation of complicated mathematical objects by simpler ones is an age-old method that has been used extensively in many mathematical disciplines. In this dissertation, we investigate approximations of convex bodies in Euclidean d-space $\mathbb{R}^d$. We note that the use of d for dimension instead of n is natural in the context of approximations when n is reserved for the number of points or hyperplanes. We use different classes of geometric objects (convex bodies themselves) for the approximations such as polytopes, polyhedral sets, and intersections of congruent closed balls. In the larger part of this work we consider random approximations, that is, the approximating objects are produced by some random process. However, in the last chapter we describe best approximations of certain convex discs in the plane by convex disc-polygons.

There is a vast literature about both random and best approximations of convex bodies. In this short overview we concentrate only on those specific topics that are directly related to our own work presented in this dissertation. For a more comprehensive treatment of the subject we refer the reader to the works listed at the end of this section.

Approximations of convex bodies by random polytopes, polyhedral sets, etc. is at the intersection of Convexity and Stochastic Geometry. The beginnings of Stochastic Geometry are frequently attributed to two classical problems: the Buffon needle problem, and Sylvester's four point problem; a historical overview can be found, for example, in the book by Schneider and Weil [SW08, Section 8.1], and in the survey paper by Weil and Wieacker [WW93].

One of the most common models of random polytopes is the following. Let K be a convex body in $\mathbb{R}^d$. The convex hull $K_{(n)}$ of n independent, identically distributed random points in K chosen according to the uniform distribution is a (uniform) random polytope contained in K. This is usually called the uniform model. Sometimes it is said that the random polytope is inscribed in K although its vertices are not assumed to be on the boundary of K in general.

The famous four-point problem of Sylvester [Syl64] is considered a starting point of an extensive investigation of random polytopes of this type. Beside specific probabilities as in Sylvester's problem, important objects of study are expectations, variances and distributions of various geometric functionals associated with the random polytope. Typical examples of such functionals are volume, other intrinsic volumes, and the number of i-dimensional faces.

In their ground-breaking papers [RS63] and [RS64], Rényi and Sulanke investigated random polytopes in the Euclidean plane and proved asymptotic results for the expectations of basic functionals of random polytopes in a convex domain K in the cases where K is either sufficiently smooth or a convex polygon; for some specific statements of Rényi and Sulanke see Section 7.1. Since then a significant part of results have been in the form of asymptotic formulae as the number n of random points tends to infinity. We also follow this path in this dissertation.

In the last few decades, much effort has been devoted to exploring the properties of the uniform model of a random polytope contained in a d-dimensional convex body K. From the extensive literature of this subject we select two specific topics that are directly related to our results presented in this dissertation.

To give a concrete example of such an asymptotic formula, we quote here the result concerning the expectation of the volume difference $V(K) - V(K_{(n)})$ of K and $K_{(n)}$. The following formula holds for all convex bodies $K \subset \mathbb{R}^d$ of unit volume

$$\lim_{n\to\infty}(V(K) - \mathbb{E}V(K_{(n)})) \cdot n^{\frac{2}{d+1}} = c_d \cdot \int_{\partial K} \kappa(x)^{\frac{1}{d+1}} \mathcal{H}^{d-1}(dx), \tag{2.1.1}$$

where c_d is an absolute constant depending only on d (defined in (4.1.1)), and $\kappa(x)$ is the generalized Gaussian curvature (see Section 2.2.2 for precise definition) at the boundary point $x \in \partial K$, and $\mathcal{H}^{d-1}$ denotes the $(d-1)$-dimensional Hausdorff measure. We note that the integral on the right-hand side of (2.1.1) is called the affine surface area of K. The affine surface area turns out to be a fundamental quantity which plays an important role in the theory of convex bodies, for more information see [Sch14, Section 10.5].

Rényi and Sulanke [RS63] proved (2.1.1) in the planar case when the boundary of the convex body is three times continuously differentiable and has strictly positive curvature everywhere, for the specific formula in the plane see also (7.1.2). Wieacker [Wie78] extended this result for the case when K is the d-dimensional unit ball, and Affentranger investigated even non-uniform distributions in the ball. Bárány [Bár92] established (2.1.1) for d-dimensional convex bodies with three times continuously differentiable boundary and strictly positive Gaussian curvature. Finally, Schütt [Sch94] removed the smoothness condition on the boundary of K. In Chapter 4 we further extend (2.1.1) in the following way. We consider a generalized version of the uniform model of a random polytope in a d-dimensional convex body K, where the random points are chosen from K not necessarily uniformly but according to a given probability density function. Furthermore, instead of the volume difference of the convex body and the random polytope we consider the weighted volume difference where we use a quite general weight function. The main result of Chapter 4, which is from the paper by Böröczky, Fodor and Hug [BFH10], is the asymptotic formula for the expectation of the weighted volume difference of K and the (non-uniform) random polytope, stated in Theorem 4.1.1. Moreover, this result implies, through a well-known argument of Efron, an asymptotic formula for the expected number of vertices of the random polytope, formulated in Corollary 4.1.2. We also note that our proof of Theorem 4.1.1 makes Schütt's proof complete, see the more detailed explanation in Section 4.1.

An asymptotic formula for the expectation of the mean width difference of K and a uniform random polytope was proved by Schneider and Wieacker [SW80] when the boundary of K is three times continuously differentiable and has positive Gaussian curvature everywhere. The assumption of smoothness was relaxed by Böröczky, Fodor, Reitzner and Vígh [BFRV09]. Although it is not included in this dissertation, we note that in our recent paper [FHZ16] by Fodor, Hug and Ziebarth, we generalized this asymptotic formula for the case of non-uniform probability distributions and weighted mean width difference for convex bodies that have a rolling ball using the methods of the papers by Böröczky, Fodor, Reitzner and Vígh [BFRV09] and Böröczky, Fodor and Hug [BFH10].

Although in this dissertation we only consider first order type results, we note that recently even variance estimates, laws of large numbers, and central limit theorems have been proved in different models in a number of papers, for instance by Bárány, Böröczky, Fodor, Hug, Reitzner, Schreiber, Vígh, Vu, Yukich and Ziebarth; see [BR10b], [BV07], [Rei03], [Rei05], [SY08] [Vu05], [Vu06], [BFV10], [BFRV09], [FHZ16].

We do not intend to give a thorough overview of second order type results here, but we mention three papers, of which I am a co-author of, in which we have recently established asymptotic results on the variance of various quantities of random polytopes and also laws of large numbers. In particular, in the paper Böröczky, Fodor, Reitzner and Vígh [BFRV09] we proved matching lower and upper bounds for the order of magnitude of the variance, and also the law of large numbers, of the mean width of uniform random polytopes in a convex body that has a rolling ball. This is analogous to the results of Reitzner [Rei03] and Bárány and Reitzner [BR10a]: Reitzner [Rei03] proved the law of large numbers for the volume of random polytopes in convex bodies with twice continuously differentiable boundary and everywhere positive Gaussian curvature with the help of an optimal upper bound on the variance of the volume, also shown in [Rei03]. Bárány and Reitzner proved a matching lower bound for the variance of volume for arbitrary convex bodies. Further, we mention that in the paper Bárány, Fodor and Vígh [BFV10] we established matching asymptotic lower and upper bounds on the order of magnitude of the variance of all intrinsic volumes of uniform random polytopes contained in a convex body whose boundary is twice continuously differentiable and has positive Gaussian curvature everywhere. The proof of the lower bound in [BFV10] is based on an idea, originally from Reitzner [Rei05] and also used in Böröczy, Fodor, Reitzner and Vígh [BFRV09], that we can define small independent caps and show that the variance is already quite large in these caps. The proof of the upper bound is based on the Economic Cap Covering Theorem of Bárány and Larman [BL88] and Bárány [Bár89], and the Efron–Stein jackknife inequality [ES81]. Both arguments are very different from the ones presented in this dissertation. Finally, we add that in our recent paper [FHZ16] by Fodor, Hug and Ziebarth, we proved an upper bound of optimal order for the variance of the weighted mean width of a non-uniform random polytope in a convex body that has a rolling ball using a similar argument as in Böröczky, Fodor, Reitzner and Vígh [BFRV09].

In Chapter 5 we consider random polyhedral sets containing a general d-dimensional convex body K. It is well-known that a polytope can be represented as the intersection of closed half-spaces. The intersection of a finite number of closed half-spaces is called a polyhedral set, or polyhedron for short. Thus, it is a natural way to generate random polytopes (more precisely, random polyhedral sets) as the intersection of a finite number of random closed half-spaces selected according to some given probability distribution. If we select closed half-spaces which all contain a convex body K, then their intersection will also contain K, and thus we obtain a random polyhedral set circumscribed about K.

One such model of random polyhedral sets (in the plane) was suggested and investigated in the paper of Rényi and Sulanke [RS68]. Subsequently, this circumscribed model has not received as much attention as the inscribed case so there is considerably less information about it in the literature.

Since polar duality turns the convex hull of a finite number of points contained in a convex body K into the intersection of a finite number of closed half-spaces containing K, one can regard a circumscribed random polyhedral set, in an intuitive sense, as a "dual" of an inscribed random polytope. This duality relation can be made precise, but we will see in Chapter 5.2 that the exact connection between the two models is more complicated than it seems at first sight.

In Chapter 5 we consider the following probability model (and also some more general versions of it). Let μ be the unique rigid motion invariant Borel measure on the space of hyperplanes of $\mathbb{R}^d$ which is normalized in a way that the measure of the set of hyperplanes meeting a convex body M is always equal to the mean width of M. For a convex body K, let $\mathcal{H}_K$ be the set of hyperplanes whose distance from K is at most 1 but they are disjoint from the interior of K. Then the restriction of $\frac{1}{2}\mu$ to $\mathcal{H}_K$ is a probability measure. Take n independent random hyperplanes chosen according to this probability measure from $\mathcal{H}_K$ and consider the closed half-spaces bounded by them that contain K. The intersection $K^{(n)}$ of these half-spaces provides a model of a random polyhedral set containing K. As $K^{(n)}$ can be unbounded with positive probability, we investigate its intersection with a suitable convex body which contains K in its interior. This only affects some constants in our results but not the asymptotic behaviour.

The main results of Chapter 5, which are from the paper by Böröczky, Fodor and Hug [BFH10], are the asymptotic formula of Theorem 5.1.1 for the expectation of the mean width difference of K and $K^{(n)} \cap K_1$, where K_1 is the set whose points are at most distance 1 from K (radius 1 parallel domain of K, see Section 2.2), and the asymptotic formula of Theorem 5.1.2 for the expectation of the number of facets of $K^{(n)} \cap P$, where P is a polytope containing K in its interior. These (and some more general, see Theorems 5.2.2 and 5.2.3) results will be achieved with the help of Theorem 4.1.1 and Corollary 4.1.2 from Chapter 4 on weighted volume approximation of a given convex body by inscribed random polytopes using polarity. In all these results, no regularity or curvature assumptions on K are required. We remark that the use of polarity to connect certain quantities of inscribed polytopes to those of circumscribed ones goes back to Ziezold [Zie70]. Glasauer and Gruber [GG97] used polarity to connect the mean width and volume, and they used this relation for proving asymptotic formulae for best approximations of convex bodies.

Earlier results on this model include the paper [Zie70] by Ziezold who investigated circumscribed polygons in the plane, and the doctoral dissertation [Kal90] of Kaltenbach who proved asymptotic formulae for the expectation of the volume difference and for the expectation of the number of vertices of circumscribed random polytopes around a convex body K, under the assumption that the boundary of K is three times continuously differentiable and has positive Gaussian curvature everywhere. Böröczky and Schneider [BS10] established upper and lower bounds for the expectation of the mean width difference for a general convex body K. Furthermore, they also proved asymptotic formulae for the expected number of vertices and facets of the circumscribed random polytope, and an asymptotic formula for the expectation of the mean width difference, under the assumption that the reference body K is a simplicial polytope with a given number of facets.

We remark that in the paper [FHZ16] by Fodor, Hug and Ziebarth, we have proved an asymptotic formula for the expectation of the volume difference of a circumscribed random polytope and the parent convex body K under a very weak smoothness condition that requires that K slides freely in a ball. This result, which is an extension of the corresponding theorem of Kaltenbach [Kal90], was achieved using a similar argument as in [BFH10] (presented in Chapter 5). Furthermore, we have also proved an asymptotic upper bound for the variance of the volume of the circumscribed random polytope, and the strong law of large numbers in [FHZ16].

In Chapter 6 we investigate yet another model of random polytopes. Instead of choosing the random points from all of K, we sample random points only from the boundary of K according to a given probability density. The convex hull of these points provides a probability model of a random polytope that we consider in Chapter 6. This (inscribed) model has not been explored to the same extent as the previously discussed uniform model. Our main focus is on the convergence of the expectation of the intrinsic volumes of such a random polytope. The main result of Chapter 6, which is from the paper [BFH13] by Böröczky, Fodor and Hug, stated in Theorem 6.1.2, extends previous works of Reitzner [Rei02] and Schütt and Werner [SW03] by relaxing the regularity assumptions on K. In fact, for $j = 1, \dots, d$, Reitzner [Rei02] established an asymptotic formula for the expectation of the difference of the jth intrinsic volumes of the random polytope and the parent convex body for the case when the parent body has twice continuously differentiable boundary and everywhere positive Gaussian curvature, cf Theorem 6.1.1. In the case of volume, Schütt and Werner [SW03] extended the asymptotic formula (6.1.1) of Reitzner to convex bodies that have a rolling ball and, at the same time, slide freely in a ball, for precise definitions see Section 2.2. In Chapter 6 we extend this asymptotic formula for convex bodies that have a rolling ball in the case of all intrinsic volumes. This is not an easy task as the speed of convergence depends in an essential way on the boundary structure of K. Our approach, which is different from those of Reitzner [Rei02] and Schütt and Werner [SW03], refines arguments that have been developed in [BFH10] by Böröczky, Fodor and Hug (and presented in Chapter 4 of this dissertation) to establish first order results for the aforementioned model of a random polytope in a convex body K, and it combines geometric and probabilistic ideas. Examples show that the existence of a rolling ball cannot be deleted from Theorem 6.1.2 while maintaining the validity of the asymptotic formula. We further note, that we also prove general lower and upper bounds in the case of mean width in Theorem 6.1.3.

In Chapters 7 and 8 we consider approximations of sufficiently round convex bodies in the Euclidean plane by intersections of congruent closed circular discs, in direct analogy to how polytopes are produced as intersections of closed half-spaces. Of course, not all convex discs can be approximated by intersections of equal size balls; such bodies must satisfy special conditions. The natural objects that can be approximated by radius $R > 0$ closed balls in $\mathbb{R}^d$ are the so-called *R-spindle convex* or *R-hyperconvex bodies*, which are convex bodies that can be represented as the intersection of a family of radius R closed balls, for more precise definition, basic properties and references see Sections 7.1 and 7.1.1. A convex body that is the intersection of a finite number of radius R closed balls is called a *ball-polyhedron*, and in the planar case, a *disc-polygon*. We remark that the property that a convex body K in $\mathbb{R}^d$ is R-spindle convex is equivalent to the fact that it is a Minkowski (vector) summand of the d-dimensional ball of radius R, and that it slides freely in a ball of radius R (see Section 2.2 for the definition). Requiring that a convex body slides freely in a ball is a common enough regularity condition on its boundary so R-spindle convex bodies are fairly common in approximation problems. For the literature on properties of Minkowski summands of balls, we refer to Schneider [Sch14, Sections 3.1 and 3.2].

In Chapter 7 we consider the following probability model. We take n independent random points from an R-spindle convex disc S according to the uniform probability distribution. Then the intersection S_n of all closed radius R circular discs containing these random points yields a model of a random disc-polygon in S. The main results on expectations are are Theorems 7.1.1, 7.1.2 and 7.1.3. In particular, Theorem 7.1.1 provides an asymptotic formula for the expectation of the area difference of S and S_n and another formula for the expected number of vertices of S_n under the condition that the boundary of S is twice continuously differentiable and its curvature is strictly larger than $1/R$ everywhere. Theorem 7.1.2 is an asymptotic formula for the expectation of the perimeter difference of S and S_n under stronger differentiability conditions of the boundary of S. Finally, Theorem 7.1.3 gives similar asymptotic formulae to the ones in Theorems 7.1.1 and 7.1.2 in the special case when $R = 1$ and S is the unit circle. The ideas of the proofs of Theorems 7.1.1, 7.1.2 and 7.1.3 go back to Rényi and Sulanke [RS63, RS64], however, the details are much more difficult as we have to use integral geometric ideas for circles instead of lines. We also show that our results for R-spindle convex discs reproduce, in the limit as $R \to \infty$, the corresponding results of Rényi and Sulanke in the case when the boundary of K is sufficiently smooth, cf. Section 7.1.2. Thus, our results can be rightfully considered as generalizations of those.

In Section 7.2 we study the variance of the number of vertices and the missed area in the cases when either the spindle convex disc has a twice continuously differentiable boundary and the radius of the approximating circles is strictly larger than the maximum of the curvature radius, see Theorem 7.2.1, or the spindle convex disc is a circle of fixed radius that is equal to the radius of the approximating circles, see Theorem 7.2.2. The proofs depend on the Efron-Stein inequality and the general idea of the argument is based on works of Reitzner.

Finally, in Chapter 8 we investigate best approximations of spindle convex discs in the plane by disc-polygons in various settings. We consider both inscribed and circumscribed disc-polygons. Here, inscribed means that we select the vertices of the disc-polygon from the boundary of S, while circumscribed means that the sides of the disc-polygon are tangent to the boundary of S. We measure the efficiency of the approximation by three measures of distance: area deviation, perimeter deviation and Hausdorff distance. We seek to find the minimum of the distance between S and the inscribed or circumscribed disc-polygons with n vertices according to the selected measure. Since finding the actual minimum for general S and n is prohibitively difficult, as it is common in these approximations problems, we establish asymptotic formulae for the order of approximation as n tends to infinity. The main result of Chapter 8 is a set of such asymptotic formulae stated in Theorem 8.1.1, which are from the paper [FV12]. In the cases where one approximates a (linearly) convex disc in the plane by inscribed and circumscribed convex polygons of a given number of vertices with respect to area deviation, perimeter deviation and Hausdorff distance, asymptotic formulae for the order of approximation were given by L. Fejes Tóth in [FT53]. These asymptotic formulae were later proved by McClure and Vitale in [MV75]. Our results in Theorem 8.1.1 are the spindle convex analogues of the corresponding results of L. Fejes Tóth and McClure and Vitale. Furthermore, in the case when the boundary of S is twice continuously differentiable and has strictly positive curvature everywhere, then the asymptotic formulae in Theorem 8.1.1 reproduce those of L. Fejes Tóth and McClure and Vitale as the radius R of the disc-polygons tends to infinity, so they can be considered as the generalizations of the classical results from [FT53] and [MV75]. The proof of Theorem 8.1.1 uses an analytic framework developed by McClure and Vitale combined with geometric arguments.

Finally, we list some of the literature where one can find more details of the topics discussed above. Due to the large number of contributions, any such list can only be incomplete, so our suggestions should be considered only as starting points if one wishes to learn more about a particular problem.

We must begin with the classical book by Santaló [San46] which is a standard reference in geometric problems of probabilistic nature. The recent monograph of Schneider and Weil [SW08] provides an excellent introduction to Stochastic Geometry and the integral geometric methods used in problems of geometric probability along with a large number of references for further study. As surveys on random polytopes, we suggest the following papers by Bárány [Bár08], Hug [Hug13], Reitzner [Rei10], Schneider [Sch88, Sch18], Weil and Wieacker [WW93].

For an early reference on asymptotic aspects of best approximation of convex bodies by polytopes, see the book of L. Fejes Tóth [FT53]. For a more recent introduction into this topic and for references, we suggest the book by Gruber [Gru07, Chapter 11]. The following survey papers contain a detailed list of contributions: Bronshteĭn [Bro07], Gruber [Gru83] and [Gru93]. The paper by Gruber [Gru97] provides a comparison of best and random approximations of convex bodies.

2.2 Notations and basic definitions

In this section we set some general notations and conventions used throughout the dissertation. Due to the slightly different settings in the individual chapters, there are some variations in certain notations in order to avoid collisions; these variations are kept to the necessary minimum, and they are introduced only at the beginning of the chapter they pertain to. In each topic we use the notation prevalent in the particular subject.

For a comprehensive treatment of the theory of convex bodies, we refer the reader to the books by Schneider [Sch14] and Gruber [Gru07].

2.2.1 General notations

In this dissertation we work in d-dimensional Euclidean space which we denote by $\mathbb{R}^d$, or when n denotes the dimension, then by $\mathbb{R}^n$. As common in the literature, we do not distinguish between points of the Euclidean space and vectors of the underlying vector space if this does not lead to confusion. Generally, we use small-case (Latin) letters to denote points (or vectors) and capitals to denote sets of points. Greek letters are usually constants unless otherwise noted. For a point set $X \subseteq \mathbb{R}^d$, we write $\operatorname{cl} X$ for the closure of X, $\operatorname{int} x$ for the interior of X, X^C for the complement set of X, and ∂X for the boundary of X.

We use $\langle \cdot, \cdot \rangle$ for the Euclidean scalar product, and the induced norm is written as $\| \cdot \|$. The d-dimensional unit ball centred at the origin o is denoted by B^d and its boundary is S^{d-1}.

A convex body $K \subset \mathbb{R}^d$ is a compact, convex set with interior points. In the special case that $d = 2$, a convex body is also called a convex disc.

Let V denote volume and $\mathcal{H}^j$ denote the j-dimensional Hausdorff measure. We write $\alpha_d := V(B^d)$ and $\omega_d := \mathcal{H}^{d-1}(S^{d-1}) = d\alpha_d$. In particular, if $d = 2$, then for the area we also use the notation $A(\cdot)$, and for the perimeter (the $\mathcal{H}^1$ measure of the boundary or arclength) $\operatorname{Per}(\cdot)$.

For two sets $X, Y \subseteq \mathbb{R}^d$, the Minkowski sum $X + Y$ of X and Y is defined as

$$X + Y := \{x + y : x \in X \text{ and } y \in Y\}.$$

It is known that if both X and Y are convex sets then $X + Y$ is also a convex set. For a convex body K and a real number $\lambda \geq 0$, the Minkowski sum $K + \lambda B^d$ is called the radius λ parallel domain of K, and it is denoted by K_λ. One can think of K_λ as the set of points in $\mathbb{R}^d$ whose Euclidean distance from K does not exceed λ. We note that parallel domains of convex bodies play a very important role in the theory of convex bodies.

There are various ways to define a measure of distance between convex bodies. We frequently use the so-called Hausdorff distance of compact sets which is defined the following way. For two compact sets $A, B \subset \mathbb{R}^d$, the Hausdorff distance is

$$d_H(A, B) := \min\{\lambda \geq 0 : A \subseteq B_\lambda \text{ and } B \subseteq A_\lambda\}.$$

It is known that the set of compact sets of $\mathbb{R}^d$ with the Hausdorff distance form a locally compact and complete metric space of which the set of compact convex sets is a closed subspace in the induced topology by d_H. For more information on the Hausdorff metric, see, for example, [Sch14, Section 1.8].

For a convex body $K \subset \mathbb{R}^d$ and a unit vector $u \in S^{d-1}$, the width $w_K(u)$ of K in the direction of u is the defined as the distance of the two unique supporting hyperplanes of K orthogonal to u. The mean width of K is the average of $w_K(u)$ over S^{d-1}, that is, $W(K) = \omega_d^{-1} \int_{S^{d-1}} w_K(u)\mathcal{H}^{d-1}(du)$.

We frequently compare the order of magnitude of functions and use the following common notations. For two functions $f(n)$ and $g(n)$ defined on the set of positive integers, we write $f(n) \sim g(n)$ if $\lim_{n\to\infty} f(n)/g(n) = 1$. For two real functions h_1 and h_2 defined on the same space, we write $h_1 \ll h_2$ or $h_1 = O(h_2)$ if there exists a positive constant γ with the property that $|h_1| \leq \gamma \cdot h_2$. We also use the common Landau symbol $o(\cdot)$ in the dissertation.

2.2.2 Differentiability and regularity conditions

A hyperplane H supports the convex body K at the boundary point $x \in \partial K$ if $x \in H \cap K$ and K is contained in one of the closed half-spaces determined by H. It is well-known that a convex body K has a supporting hyperplane at each boundary point. The supporting hyperplane may not be unique though. We say that a boundary point $x \in \partial K$ is smooth if there is a unique supporting hyperplane of K at x. (Non-smooth boundary points are called singular.)

Let H be a supporting hyperplane of K at $x \in \partial K$. A unit vector $u \in S^{d-1}$ is an outer unit normal of K if it is normal to H and points to the open half-space of H that does not contain K. The outer unit normal may not be unique. If $x \in \partial K$ is a smooth boundary point, then there exists a unique outer unit normal vector of K at x, which we denote by $u(x)$ or $\nu(x)$, in some cases when it fits other notations better, by u_x.

A significant percentage of results on polytopal approximation of convex bodies involve some kind of regularity or differentiability conditions on the boundary of the convex body. In some cases, differentiability assumptions are not only technical conditions which are required by the techniques used in the proof but they are essential to the behaviour of the random polytope. It is always an important question to determine whether a particular smoothness condition is essential or not, and if not, then try to weaken it as much as possible.

The most common differentiability condition used in results about approximation by polytopes is that we require ∂K to be C^k smooth for some $k \geq 1$. More precisely, we say that ∂K is C^k smooth for some $k \geq 1$ if ∂K is a C^k submanifold of $\mathbb{R}^d$ (k times continuously differentiable everywhere). Moreover, ∂K is C^k_+ if it is C^k and, in addition, its Gauss-Kronecker curvature is strictly positive everywhere. We remark that if ∂K is C^2 smooth, then that makes it possible to use tools from differential geometry. We will use such differentiability conditions, for example, in Chapters 7 and 8.

The following are also common smoothness conditions on the boundary of a convex body, and we use them, for example, in Chapter 6. Let $K, L \subset \mathbb{R}^d$ be convex bodies. We say that L slides freely in K, if for any $x \in \partial K$, there exists a $p \in \mathbb{R}^d$ such that $x \in L + p$ and $L + p \subseteq K$, see [Sch14, Section 3.2]. In the special case when L is a ball B, then B rolls (or slides) freely in K, and if K is a ball B, then L slides freely in B. Note that if K has a rolling ball, then each one of its boundary points is smooth. Moreover, it is known that the existence of a rolling ball in K is equivalent to the Lipschitz continuity of the outer unit normal function $u(x) : \partial K \to S^{d-1}$ (see D. Hug [Hug00]). On the other hand, it was proved by Blaschke that if the boundary of K is twice continuously differentiable everywhere, then K has a rolling ball (see D. Hug [Hug00] or K. Leichtweiss [Lei98]).

In some cases strict differentiability conditions are not essential in the sense that a particular asymptotic formula remains valid under slightly weaker conditions. In Chapters 4, 5 and 6 we use the following notions of generalized second order differentiability of ∂K.

Let $x \in \partial K$ be a smooth boundary point. Assume that K is oriented in $\mathbb{R}^d$ (using a suitable rigid motion) such that $x = o$ and $x_d = 0$ is a supporting hyperplane of K at x. Under these conditions, $u = (0, \ldots, 0, -1)$ is an outer unit normal of K at x. Then for a suitably small $\varepsilon > 0$ and a neighbourhood U of x, the boundary of K can be represented as follows

$$\partial K \cap U = \{-u(x) f(z) : z \in (x_d = 0) \cap \varepsilon B^d\},$$

where f is a non-negative real valued function which is convex on the set $(x_d = 0) \cap \varepsilon B^d$ and $f(o) = 0$.

We introduce a notion of generalized second order differentiability of ∂K at x where we will call ∂K differentiable twice in the generalized sense if f has a second order Taylor expansion at x. More precisely, if there exists a positive semi-definite quadratic form $Q(z)$ with the property that

$$f(z) = \frac{1}{2} Q(z) + o(\|z\|^2), \tag{2.2.1}$$

as $z \to o$, then we say that ∂K is twice differentiable in the generalized sense at x. In this case x is called a normal boundary point of K.

The eigenvalues of Q are called the generalized principal curvatures of ∂K at x, and they are denoted by $k_1(x), \dots, k_{d-1}(x)$. Furthermore, we will need the normalized jth elementary symmetric functions of the generalized curvatures which are defined as follows:

$$H_j(x) = \binom{d-1}{j}^{-1} \sum_{1\le i_1<\cdots<i_j\le d-1} k_{i_1}(x)\cdots k_{i_j}(x)$$

for $j \in \{1, \dots, d-1\}$, and let $H_0(x) := 1$. In particular, $H_{d-1}(x)$ is the generalized Gauss-Kronecker curvature and $H_1(x)$ is the (generalized) mean curvature of ∂K at x. For brevity of notation we sometimes write $\kappa(x)$ for the Gauss-Kronecker curvature, that is, $\kappa(x) = H_{d-1}(x)$. When we use any of the $H_i(x)$, then tacitly assume that x is a normal boundary point.

One reason why this notion of generalized second order differentiability is important is that most boundary points of a convex body do possess this property as it was shown by Alexandrov. More precisely, the boundary of a convex body is differentiable in this generalized sense in almost all points with respect to $\mathcal{H}^{d-1}$. For more information on this topic we refer to Note 3 of Section 1.5, and Section 2.6 of [Sch14], and also to Section 2.2 of [Gru07].

2.2.3 Intrinsic volumes

It is well-known that the volume of the radius $\lambda \ge 0$ parallel domain of a convex body K is a polynomial of degree d of λ; this polynomial is frequently referred to as the Steiner's polynomial of K. The intrinsic volumes arise as suitably normalized coefficients of this polynomial in the following way:

$$V(K + \lambda B^d) = \sum_{j=0}^{d} \lambda^{d-j} \alpha_{d-j} V_j(K).$$

The intrinsic volumes carry important geometric information about K, and some of them are actually equal to (constant times) some familiar quantities. In particular, $V_d(K)$ is the volume of K, $V_{d-1}(K)$ is one half times the surface area of K, $V_1(K)$ is a constant times the mean width of K, and $V_0(K) = 1$. This particular normalization of the coefficients of the Steiner formula was introduced by McMullen in [McM75]. It has the advantage that the intrinsic volumes are independent of the dimension of the ambient space. Another version of the Steiner formula is also frequently used

$$V(K + \lambda B^d) = \sum_{j=0}^{d} \lambda^j \binom{d}{j} W_j(K),$$

where the $W_j(K)$, $j = 0, \dots, d$ are called the *Quermassintegrals* of K.

Due to the works of Cauchy and Kubota, it is known that the intrinsic volumes can also be written as mean projection volumes as follows. Let $\mathcal{L}_j^d$ denote the Grassmanian of all j-dimensional linear subspaces of $\mathbb{R}^d$. Let ν_j be the unique Haar probability measure on $\mathcal{L}_j^d$. For $L \in \mathcal{L}_j^d$, denote by $K|L$ the orthogonal projection of K into L. Since L is j-dimensional, the j-th intrinsic volume $V_j(K|L)$ of $K|L$ is simply the j-dimensional volume (Lebesgue measure) of $K|L$. Kubota's formulae state the following:

$$V_j(K) = \frac{\binom{d}{j}\alpha_d}{\alpha_j \alpha_{d-j}} \int_{\mathcal{L}_j^d} V_j(K|L)\nu_j(dL)$$

for $j \in \{1, \ldots, d-1\}$.

Finally, we note that general curvature and surface area measures arise by the so-called localizations of the Quermassintegrals. Since these measures will only be used in Chapter 3 in the context of the Minkokwski problem, we give a short historical introduction to them only there. The same applies to the dual Brunn-Minkowski theory and the related dual intrinsic volumes and associated dual curvature measures.

Chapter 3

The L_p dual Minkowski problem

The contents of this chapter is based on parts of the paper [BF19] by K.J. Böröczky and F. Fodor, *The L_p dual Minkowski problem for $p > 1$ and $q > 0$*, J. Differential Equations **266** (2019), no. 12, 7980–8033. (DOI 10.1016/j.jde.2018.12.020)

3.1 Introduction

In this chapter our setting is the Euclidean n-space $\mathbb{R}^n$ with $n \geq 2$. We use the notation $\kappa_n = V(B^n)$ for the volume of the unit ball. Recall that for a convex compact set $K \subset \mathbb{R}^n$, the support function $h_K(u) : S^{n-1} \to \mathbb{R}$ is defined as $h_K(u) = \max\{\langle x, u\rangle : x \in K\}$. For $u \in S^{n-1}$, the face of K with exterior unit normal u is $F(K, u) = \{x \in K : \langle x, u\rangle = h_K(u)\}$. For $x \in \partial K$, let the spherical image of x be defined as $\boldsymbol{\nu}_K(\{x\}) = \{u \in S^{n-1} : h_K(u) = \langle x, u\rangle\}$. For a Borel set $\eta \subset S^{n-1}$, the reverse spherical image is

$$\boldsymbol{\nu}_K^{-1}(\eta) = \{x \in \partial K : \boldsymbol{\nu}_K(x) \cap \eta \neq \emptyset\} = \cup_{u\in\eta} F(K, u).$$

If K has a unique supporting hyperplane at x, then we say that K is smooth at x, and in this case $\boldsymbol{\nu}_K(\{x\})$ contains exactly one element that we denote by $\nu_K(x)$ and call it the exterior unit normal of K at x.

The classical Minkowski problem seeks to characterize surface area measures. The surface area measure of a convex body can be defined in a direct way as follows. Let $\partial' K$ denote the subset of the boundary of K where there is a unique outer unit normal vector. It is well-known that $\partial K \backslash \partial' K$ is the countable union of compact sets of finite $\mathcal{H}^{n-2}$-measure (see Schneider [Sch14, Theorem 2.2.5]), and hence $\partial' K$ is Borel and $\mathcal{H}^{n-1}(\partial K \setminus \partial' K) = 0$. Then $\nu_K : \partial' K \to S^{n-1}$ is a function that is usually called the spherical Gauss map, and ν_K is continuous on $\partial' K$. The surface area measure of K, denoted by $S(K, \cdot)$, is a Borel measure on S^{n-1} such that for any Borel set $\eta \subset S^{n-1}$, we have $S(K, \eta) = \mathcal{H}^{n-1}(\boldsymbol{\nu}_K^{-1}(\eta))$. It is an important property of the surface area measure that it satisfies Minkowski's variational formula

$$\lim_{\varepsilon\to 0^+} \frac{V(K + \varepsilon L) - V(K)}{\varepsilon} = \int_{S^{n-1}} h_L \, dS(K, \cdot) \tag{3.1.1}$$

for any convex body $L \subset \mathbb{R}^n$.

The classical Minkowski problem asks for necessary and sufficient conditions for a Borel measure on S^{n-1} to be the surface area measure of a convex body. A particularly important case of the Minkowski problem is for discrete measures. Let $P \subset \mathbb{R}^n$ be a polytope, which is defined as the convex hull of a finite number of points in $\mathbb{R}^n$ provided $\text{int}P \neq \emptyset$. Those faces whose dimension is $n-1$ are called facets. A polytope P has a finite number of facets and the union of facets covers the boundary of P. Let $u_1, \dots, u_k \in S^{n-1}$ be the exterior unit normal vectors of the facets of P. Then $S(P, \cdot)$ is a discrete measure on S^{n-1} concentrated on the set $\{u_1, \dots, u_k\}$, and $S(P, \{u_i\}) = \mathcal{H}^{n-1}(F(P, u_i))$, $i = 1, \dots, k$. The Minkowski problem asks the following: let μ be a discrete positive Borel measure on S^{n-1}. Under what conditions does there exist a polytope P such that $\mu = S(P, \cdot)$? Furthermore, if such a P exists, is it unique? This polytopal version, along with the case when the surface area measure of K is absolutely continuous with respect to the spherical Lebesgue measure, was solved by Minkowski [Min97, Min03]. He also proved the uniqueness of the solution. For general measures the problem was solved by Alexandrov [Ale38, Ale39] and independently by Fenchel and Jensen. The argument for existence uses the Alexandrov variational formula of the surface area measure, and the uniqueness employs the Minkowski inequality for mixed volumes. In summary, the necessary and sufficient conditions for the existence of the solution of the Minkowski problem for μ are that for any linear subspace $L \leq \mathbb{R}^n$ with $\dim L \leq n-1$, $\mu(L \cap S^{n-1}) < \mu(S^{n-1})$, and that the centre of mass of μ is at the origin, that is, $\int_{S^{n-1}} u \, \mu(du) = 0$.

Similar questions have been posed for $K \in \mathcal{K}_o^n$, and at least partially solved, for other measures associated with convex bodies in the Brunn-Minkowski theory, for example, the integral curvature measure $J(K, \cdot)$ of Alexandrov (see (3.1.5) below), or the L_p surface area measure $dS_p(K, \cdot) = h_K^{1-p} dS(K, \cdot)$ for $p \in \mathbb{R}$ introduced by Lutwak [Lut93b], where $S_1(K, \cdot) = S(K, \cdot)$ $(p = 1)$ is the classical surface area measure, and $S_0(K, \cdot)$ $(p = 0)$ is the cone volume measure (logarithmic Minkowski problem). Here some care is needed if $p > 1$, when we only consider the case $o \in \partial K$ if the resulting L_p surface area measure $S_p(K, \cdot)$ is finite. For a detailed overview of these measures and their associated Minkowski problems and further references see, for example, Schneider [Sch14], and Huang, Lutwak, Yang and Zhang [HLYZ16].

Lutwak built the dual Brunn-Minkowski theory in the 1970s as a "dual" counterpart of the classical theory. Although there is no formal duality between the classical and dual theories, one can say roughly that in the dual theory the radial function plays a similar role as the support function in the classical theory. The dual Brunn-Minkowski theory concerns the class of compact star shaped sets of $\mathbb{R}^n$. A compact set $S \subset \mathbb{R}^n$ is star shaped with respect to a point $p \in S$ if for all $s \in S$, the segment $[p, s]$ is contained in S. We denote the class of compact sets in $\mathbb{R}^n$ that are star shaped with respect to o by $\mathcal{S}_o^n$, and the set of those elements of $\mathcal{S}_o^n$ that contain o in their interiors are denoted by $\mathcal{S}_{(o)}^n$. Clearly, $\mathcal{K}_o^n \subset \mathcal{S}_o^n$ and $\mathcal{K}_{(o)}^n \subset \mathcal{S}_{(o)}^n$. For a star shaped set $S \in \mathcal{S}_o^n$, we define the radial function of S as $\varrho_S(u) = \max\{t \geq 0 : tu \in S\}$ for $u \in S^{n-1}$.

Dual intrinsic volumes for convex bodies $K \in \mathcal{K}_{(o)}^n$ were defined by Lutwak [Lut75] whose definition works for all $q \in \mathbb{R}$. For $q > 0$, we extend Lutwak's definition of the qth dual intrinsic volume $\widetilde{V}_q(\cdot)$ to a compact convex set $K \in \mathcal{K}_o^n$ as

$$\widetilde{V}_q(K) = \frac{1}{n} \int_{S^{n-1}} \varrho_K^q(u) \, \mathcal{H}^{n-1}(du), \tag{3.1.2}$$

which is normalized in such a way that $\widetilde{V}_n(K) = V(K)$. We note that ϱ_K is continuous at all $u \in S^{n-1}$ but a compact set of $\mathcal{H}^{n-1}$-measure zero (see Lemma 3.2.1). We observe that $\widetilde{V}_q(K) = 0$ if $\dim K \leq n-1$, and $\widetilde{V}_q(K) > 0$ if K is full dimensional. We note that dual intrinsic volumes for $q = 0, \ldots, d$ are the coefficients of the dual Steiner polynomial for star shaped compact sets, where the radial sum replaces the Minkowski sum. The qth dual intrinsic volumes, which arise as coefficients naturally satisfy (3.1.2), and this provides the possibility to extend their definition for arbitrary $q \in \mathbb{R}$ in the case when $o \in \mathrm{int}K$ and for $q > 0$ when $o \in K$.

Extending the definition of Huang, Lutwak, Yang, Zhang [HLYZ16] and Lutwak, Yang, Zhang [LYZ18] for $K \in \mathcal{K}^n_{(o)}$, if $K \in \mathcal{K}^n_o$ and $\eta \subset S^{n-1}$ is a Borel set, then the reverse radial Gauss image of η is

$$\boldsymbol{\alpha}^*_K(\eta) = \{u \in S^{n-1} : \varrho_K(u)u \in F(K,v) \text{ for some } v \in \eta\} = \{u \in S^{n-1} : \varrho_K(u)u \in \boldsymbol{\nu}_K^{-1}(\eta)\},$$

which is Lebesgue measurable according to Lemma 3.2.3. For the measurability of $\boldsymbol{\alpha}^*_K(\eta)$ in the case $K \in \mathcal{K}^n_{(0)}$, see [Sch14, Lemma 2.2.4]. For a convex body $K \in \mathcal{K}^n_o$ and $q \in \mathbb{R}$, the qth dual curvature measure $\widetilde{C}_q(K,\cdot)$ is a Borel measure on S^{n-1} defined in [HLYZ16] as

$$\widetilde{C}_q(K,\eta) = \frac{1}{n}\int_{\boldsymbol{\alpha}^*_K(\eta)} \varrho_K^q(u)\mathcal{H}^{n-1}(du). \tag{3.1.3}$$

Similar to the case of qth dual intrinsic volumes, the notion of qth dual curvature measures can be extended to compact convex sets $K \in \mathcal{K}^n_o$ when $q > 0$ using (3.1.3). Here if $\dim K \leq n-1$, then $\widetilde{C}_q(K,\cdot)$ is the trivial measure. We note that the so-called cone volume measure $V(K,\cdot) = \frac{1}{n}\, S_0(K,\cdot) = \frac{1}{n}\, h_K S(K,\cdot)$, and Alexandrov's integral curvature measure $J(K,\cdot)$ can both be represented as dual curvature measures as

$$\begin{aligned} V(K,\cdot) = \tfrac{1}{n}\, S_0(K,\cdot) &= \widetilde{C}_n(K,\cdot) && (3.1.4)\\ J(K^*,\cdot) &= \widetilde{C}_0(K,\cdot) \text{ provided } o \in \mathrm{int}K. && (3.1.5)\end{aligned}$$

Based on Alexandrov's integral curvature measure, the L_p Alexandrov integral curvature measure

$$dJ_p(K,\cdot) = \varrho_K^p\, dJ(K,\cdot)$$

was introduced by Huang, Lutwak, Yang, Zhang [HLYZ18] for $p \in \mathbb{R}$ and $K \in \mathcal{K}^n_{(o)}$.

We note that the qth dual curvature measure is a natural extension of the cone volume measure $V(K,\cdot) = \frac{1}{n}\, h_K S(K,\cdot)$ also in the variational sense, Corollary 4.8 of Huang, Lutwak, Yang, Zhang [HLYZ16] states the following generalization of Minkowski's formula (3.1.1). For arbitrary convex bodies $K, L \in \mathcal{K}^n_{(o)}$, we have

$$\lim_{\varepsilon\to 0^+} \frac{\widetilde{V}_q(K+\varepsilon L) - \widetilde{V}_q(K)}{\varepsilon} = \int_{S^{n-1}} \frac{h_L}{h_K}\, d\widetilde{C}_q(K,\cdot). \tag{3.1.6}$$

In this paper, we actually do not use (3.1.6), but use Lemma 3.3.3, which is a variational formula in the sense of Alexandrov for dual curvature measures of polytopes.

For integers $q = 0, \ldots, n$, dual curvature measures arise in a similar way as in the Brunn-Minkowski theory by means of localized dual Steiner polynomials. These measures satisfy (3.1.3), and hence their definition can be extended for $q \in \mathbb{R}$. Huang, Lutwak, Yang and Zhang [HLYZ16] proved that the qth dual curvature measure of a convex body $K \in \mathcal{K}^n_{(o)}$ can also be obtained from the qth dual intrinsic volume by means of an Alexandrov-type variational formula.

Lutwak, Yang, Zhang [LYZ18] introduced a more general version of the dual curvature measure where a star shaped set $Q \in \mathcal{S}^n_{(o)}$ is also involved; namely, for a Borel set $\eta \subset S^{n-1}$, $q \in \mathbb{R}$ and $K \in \mathcal{K}^n_{(o)}$, we have

$$\widetilde{C}_q(K, Q, \eta) = \frac{1}{n} \int_{\boldsymbol{\alpha}^*_K(\eta)} \varrho^q_K(u) \varrho^{n-q}_Q(u) \mathcal{H}^{n-1}(du) \tag{3.1.7}$$

and the associated qth dual intrinsic volume with parameter body Q is

$$\widetilde{V}_q(K, Q) = \widetilde{C}_q(K, Q, S^{n-1}) = \frac{1}{n} \int_{S^{n-1}} \varrho^q_K(u) \varrho^{n-q}_Q(u) \mathcal{H}^{n-1}(du). \tag{3.1.8}$$

According to Lemma 5.1 in [LYZ18], if $q \neq 0$ and the Borel function $g : S^{n-1} \to \mathbb{R}$ is bounded, then

$$\int_{S^{n-1}} g(u)\, d\widetilde{C}_q(K, Q, u) = \frac{1}{n} \int_{\partial' K} g(\nu_K(x)) \langle \nu_K(x), x\rangle \|x\|^{q-n}_Q \, d\mathcal{H}^{n-1}(x) \tag{3.1.9}$$

where $\|x\|_Q = \min\{\lambda \geq 0 : \lambda x \in Q\}$ is a continuous, even and 1-homogeneous function satisfying $\|x\|_Q > 0$ for $x \neq o$. The advantage of introducing the star body Q is apparent in the equiaffine invariant formula (see Theorem 6.8 in [LYZ18]) stating that if $\varphi \in \mathrm{SL}(n, \mathbb{R})$, then

$$\int_{S^{n-1}} g(u)\, d\widetilde{C}_q(\varphi K, \varphi Q, u) = \int_{S^{n-1}} g\left(\frac{\varphi^{-t} u}{\|\varphi^{-t} u\|}\right) d\widetilde{C}_q(K, Q, u), \tag{3.1.10}$$

where φ^{-t} denotes the transpose of the inverse of φ.

For $q > 0$, we extend these notions and fundamental observations to any convex body containing the origin on its boundary. In particular, for $q > 0$, $K \in \mathcal{K}^n_o$ and $Q \in \mathcal{S}^n_{(o)}$, we can define the associated curvature measure by (3.1.7) and the associated dual intrinsic volume by (3.1.8), where $\widetilde{C}_q(K, Q, \cdot)$ is a finite Borel measure on S^{n-1}, and $\widetilde{V}_q(K, Q, \cdot)$ is finite. In addition, for $q > 0$, we extend (3.1.9) in Lemma 6.1 on page 8008 in [BF19] and (3.1.10) in Lemma 6.5 on page 8009 in [BF19] to any $K \in \mathcal{K}^n_o$.

L_p dual curvature measures were also introduced by Lutwak, Yang and Zhang [LYZ18]. They provide a common framework that unifies several other geometric measures of the (L_p) Brunn-Minkowski theory and the dual theory: L_p surface area measures, L_p integral curvature measures, and dual curvature measures, cf. [LYZ18]. For $q \in \mathbb{R}$, $Q \in \mathcal{S}^n_{(o)}$, $p \in \mathbb{R}$ and $K \in \mathcal{K}^n_{(o)}$, we define the L_p qth dual curvature measure $\widetilde{C}_{p,q}(K, Q, \cdot)$ of K with respect to Q by the formula

$$d\widetilde{C}_{p,q}(K, Q, \cdot) = h^{-p}_K d\widetilde{C}_q(K, Q, \cdot). \tag{3.1.11}$$

While we also discuss the measures $\widetilde{C}_{p,q}(K, Q, \cdot)$ involving a $Q \in \mathcal{S}^n_{(o)}$, we concentrate on $\widetilde{C}_{p,q}(K, \cdot)$ in this paper, which represents many fundamental measures associated to a $K \in \mathcal{K}^n_{(o)}$. Basic examples are

$$\widetilde{C}_{p,n}(K, \cdot) \;=\; S_p(K, \cdot)$$

$$\begin{aligned}\widetilde{C}_{0,q}(K,\cdot) &= \widetilde{C}_q(K,\cdot)\\ \widetilde{C}_{p,0}(K,\cdot) &= J_p(K^*,\cdot).\end{aligned}$$

Alexandrov-type variational formulas for the dual intrinsic volumes were proved by Lutwak, Yang and Zhang, cf. Theorem 6.5 in [LYZ18], which produce the L_p dual curvature measures $\widetilde{C}_{p,q}(K,Q,\cdot)$. In this paper we will use a simpler variational formula, cf. Lemma 3.3.3 for the qth dual intrinsic volumes that we specialize for our particular setting.

In Problem 8.1 in [LYZ18] the authors introduced the L_p dual Minkowski existence problem: Find necessary and sufficient conditions that for fixed $p,q\in\mathbb{R}$ and $Q\in\mathcal{S}^n_{(o)}$ and a given Borel measure μ on S^{n-1} there exists a convex body $K\in\mathcal{K}^n_{(o)}$ such that $\mu=\widetilde{C}_{p,q}(K,Q,\cdot)$. As they note in [LYZ18], this version of the Minkowski problem includes earlier considered other variants (L_p Minkowski problem, dual Minkowski problem, L_p Aleksandrov problem) for special choices of the parameters. For $Q=B^n$ and an absolutely continuous measure μ with density function f, the L_p dual Minkowski problem constitutes solving the Monge-Ampère equation

$$\det(\nabla^2 h + h\,\mathrm{Id}) = \tfrac{1}{n}\, h^{p-1}\cdot(\|\nabla h\|^2+h^2)^{\frac{n-q}{2}}\cdot f \tag{3.1.12}$$

for the non-negative L_1 Borel function f with $\int_{S^{n-1}} f\,d\mathcal{H}^{n-1}>0$ (see (93) on page 8016 in Section 7 of [BF19]). Actually, if $Q\in\mathcal{S}^n_{(o)}$, then the related Monge-Ampère equation is (see (94) on page 8016 in Section 7 of [BF19])

$$\det(\nabla^2 h(u) + h(u)\,\mathrm{Id}) = \tfrac{1}{n}\, h(u)^{p-1}\|\nabla h(u)+h(u)\,u\|_Q^{n-q}\cdot f(u). \tag{3.1.13}$$

The case of the L_p dual Minkowski problem for even measures has received much attention but is not discussed here, see Böröczky, Lutwak, Yang, Zhang [BLYZ13] concerning the L_p surface area $S_p(K,\cdot)$, Böröczky, Lutwak, Yang, Zhang, Zhao [BLY$^+$], Jiang Wu [JW17] and Henk, Pollehn [HP18], Zhao [Zha18] concerning the qth dual curvature measure $\widetilde{C}_q(K,\cdot)$, and Huang, Zhao [HZ18] concerning the L_p dual curvature measure for detailed discussion of history and recent results.

Let us indicate the known solutions of the L_p dual Minkowski problem when only mild conditions are imposed on the given measure μ or on the function f in (3.1.12). We do not state the exact conditions, rather aim at a general overview. For any finite Borel measure μ on S^{n-1} such that the measure of any open hemi-sphere is positive, we have that

- if $p>0$ and $p\neq 1,n$, then $\mu=S_p(K,\cdot)=n\widetilde{C}_{p,n}(K,\cdot)$ for some $K\in\mathcal{K}^n_o$, where the case $p>1$ and $p\neq n$ is due to Chou, Wang [CW06] and Hug, Lutwak, Yang, Zhang [HLYZ05], while the case $0<p<1$ is due to Chen, Li, Zhu [CLZ17];

- if $p\geq 0$ and $q<0$, then $\mu=\widetilde{C}_{p,q}(K,\cdot)$ for some $K\in\mathcal{K}^n_o$ where the case $p=0$ ($\mu=\widetilde{C}_q(K,\cdot)$) is due to Zhao [Zha17] (see also Li, Sheng, Wang [LSW]), and the case $p>0$ is due to Huang, Zhao [HZ18] and Gardner, Hug, Xing, Ye, Weil [GHW$^+$19].

In addition, if $p>q$ and f is C^α for $\alpha\in(0,1]$, then (3.1.12) has a unique positive $C^{2,\alpha}$ solution according to Huang, Zhao [HZ18].

Naturally, the L_p dual Monge-Ampère equation (3.1.12) has a solution in the above cases for any non-negative L_1 function f whose integral on any open hemi-sphere is positive. In addition, if $-n < p \le 0$ and f is any non-negative $L_{\frac{n}{n+p}}$ function on S^{n-1} such that $\int_{S^{n-1}} f\,d\mu > 0$, then (3.1.12) has a solution, where the case $p = 0$ is due to Chen, Li, Zhu [CLZ19], and the case $p \in (-n, 0)$ is due to Bianchi, Böröczky, Colesanti, Yang[BBCY19].

We also note that if $p \le 0$ and μ is discrete such that any n elements of suppμ are independent vectors, then $\mu = S_p(K, \cdot) = n \cdot \widetilde{C}_{p,n}(P, \cdot)$ for some polytope $P \in \mathcal{K}^n_{(o)}$ according to Zhu [Zhu15, Zhu17].

In this chapter, we first solve the discrete L_p dual Minkowski problem if $p > 1$ and $q > 0$.

Theorem 3.1.1 (Böröczky, Fodor [BF19], Theorem 1.1 on page 7986)**.** *Let $Q \in \mathcal{S}^n_{(o)}$, $p > 1$ and $q > 0$ with $p \neq q$, and let μ be a discrete measure on S^{n-1} that is not concentrated on any closed hemisphere. Then there exists a polytope $P \in \mathcal{K}^n_{(o)}$ such that $\widetilde{C}_{p,q}(P, Q, \cdot) = \mu$.*

Remark If $p > q$, then the solution is unique according to Theorem 8.3 in Lutwak, Yang and Zhang [LYZ18].

We note that, in fact, we prove the existence of a polytope $P_0 \in \mathcal{K}^n_{(o)}$ satisfying

$$\widetilde{V}_q(P_0, Q)^{-1}\widetilde{C}_{p,q}(P_0, Q, \cdot) = \mu,$$

which P_0 exists even if $p = q$ (see Theorem 3.3.1).

Let us turn to a general, possibly non-discrete Borel measure μ on S^{n-1}. As the example at the end of the paper by Hug, Lutwak, Yang, Zhang[HLYZ05] shows, even if μ has a positive continuous density function with respect to the Hausdorff measure on S^{n-1}, for $q = n$ and $1 < p < n$, it may happen that the only solution K has the origin on its boundary. In this case, h_K has some zero on S^{n-1} even if it occurs with negative exponent in $\widetilde{C}_{p,q}(K, \cdot)$. Therefore if $Q \in \mathcal{S}^n_{(o)}$, $p > 1$ and $q > 0$, the natural form the L_p dual Minkowski problem is the following (see Chou, Wang [CW06] and Hug, Lutwak, Yang, Zhang [HLYZ05] if $q = n$). For a given non-trivial finite Borel measure μ, find a convex body $K \in \mathcal{K}^n_o$ such that

$$d\widetilde{C}_q(K, Q, \cdot) = h_K^p d\mu. \tag{3.1.14}$$

It is natural to assume that $\mathcal{H}^{n-1}(\Xi_K) = 0$ in (3.1.14) for

$$\Xi_K = \{x \in \partial K : \text{there exists exterior normal } u \in S^{n-1} \text{ at } x \text{ with } h_K(u) = 0\}, \tag{3.1.15}$$

which property ensures that the surface area measure $S(K, \cdot)$ is absolutely continuous with respect to $\widetilde{C}_q(K, Q, \cdot)$ (see Corollary 6.2 on page 8009 in [BF19]). Actually, if $q = n$ and $Q = B^n$, then $d\widetilde{C}_n(K, \cdot) = \frac{1}{n}\, h_K\, dS(K, \cdot)$, and [CW06] and [HLYZ05] consider the problem

$$dS(K, \cdot) = nh_K^{p-1} d\mu, \tag{3.1.16}$$

where the results of [HLYZ05] about (3.1.16) yield the uniqueness of the solution in (3.1.16) for $q = n$, $p > 1$ and $Q = B^n$ only under the condition $\mathcal{H}^{n-1}(\Xi_K) = 0$ (see Section 3.4 for more detailed discussion).

Theorem 3.1.2 (Böröczky, Fodor [BF19], Theorem 1.2 on page 7987)**.** *Let $Q \in \mathcal{S}^n_{(o)}$, $p > 1$ and $q > 0$ with $p \neq q$, and let μ be a finite Borel measure on S^{n-1} that is not concentrated on any closed hemisphere. Then there exists a $K \in \mathcal{K}^n_o$ with $\mathcal{H}^{n-1}(\Xi_K) = 0$ and* $\mathrm{int}K \neq \emptyset$ *such that $d\widetilde{C}_q(K, Q, \cdot) = h^p_K d\mu$, where $K \in \mathcal{K}^n_{(o)}$ provided $p > q$.*

The solution in Theorem 3.1.2 is known to be unique in some cases:

- if $p > q$ and μ is discrete (K is a polytope) according to Lutwak, Yang and Zhang [LYZ18],
- if $p > q$, Q is a ball and μ has a C^α density function f for $\alpha \in (0, 1]$ according to Huang, Zhao [HZ18],
- if $p > 1$, Q is a ball and $q = n$ according to Hug, Lutwak, Yang, Zhang[HLYZ05].

For Theorem 3.1.2, in fact, we prove the existence of a convex body $K_0 \in \mathcal{K}^n_o$ such that

$$\widetilde{V}_q(K_0, Q)^{-1}\, d\widetilde{C}_q(K_0, Q, \cdot) = h^p_K\, d\mu,$$

which K_0 exists even if $p = q$ (see Theorem 3.5.2).

The other main results of the paper [BF19] concern the smoothness properties of the solutions of the L_p dual Minkowski problem in the case when $\widetilde{C}_q(K, Q, \cdot)$ is absolutely continuous with respect to the Hausdorff measure on S^{n-1}. We only state these theorems here and we refer to Section 7 of [BF19] for the detailed arguments.

Concerning regularity, we prove the following statements based on Caffarelli [Caf90a, Caf90b] (see Section 7 of [BF19]). We note that if ∂Q is C^2_+ for $Q \in \mathcal{S}^n_{(o)}$, then Q is convex.

Theorem 3.1.3 (Böröczky, Fodor [BF19], Theorem 1.3 on page 7987)**.** *Let $p > 1$, $q > 0$, $Q \in \mathcal{S}^n_{(o)}$, $0 < c_1 < c_2$ and let $K \in \mathcal{K}^n_o$ with $\mathcal{H}^{n-1}(\Xi_K) = 0$ and* $\mathrm{int}K \neq \emptyset$ *be such that*

$$d\widetilde{C}_q(K, Q, \cdot) = h^p_K f\, d\mathcal{H}^{n-1}$$

for some Borel function f on S^{n-1} satisfying $c_1 \leq f \leq c_2$.

(i) *$\partial K \backslash \Xi_K = \{z \in \partial K : h_K(u) > 0$ for all $u \in N(K, z)\}$ and $\partial K \backslash \Xi_K$ is C^1 and contains no segment, moreover h_K is C^1 on $\mathbb{R}^n \backslash N(K, o)$.*

(ii) *If f is continuous, then each $u \in S^{n-1} \backslash N(K, o)$ has a neighbourhood U on S^{n-1} such that the restriction of h_K to U is $C^{1,\alpha}$ for any $\alpha \in (0, 1)$.*

(iii) *If f is in $C^\alpha(S^{n-1})$ for some $\alpha \in (0, 1)$, and ∂Q is C^2_+, then $\partial K \backslash \Xi_K$ is C^2_+, and each $u \in S^{n-1} \backslash N(K, o)$ has a neighbourhood where the restriction of h_K is $C^{2,\alpha}$.*

We note that in Theorem 3.1.3 (ii), the same neighbourhood U of u works for every $\alpha \in (0, 1)$. In addition, Theorem 3.1.3 (i) yields that for any convex $W \subset \mathbb{R}^n \backslash N(K, o)$, $h_K(u + v) < h_K(u) + h_K(v)$ for independent $u, v \in W$. For the case $o \in \mathrm{int}\, K$ in Theorem 3.1.3, see the more appealing statements in Theorem 3.1.5.

We recall that according to Theorem 3.1.2, if $p > q > 0$ and $p > 1$, then $K \in \mathcal{K}^n_{(o)}$ holds for the solution K of the L_p dual Minkowski problem. On the other hand, Example 7.1 on page 8015 in [BF19] shows that if $1 < p < q$, then the solution K of the L_p dual Minkowski problem provided by Theorem 3.1.2 may satisfy that $o \in \partial K$ and o is not a smooth point. Next we show that K is still strictly convex in this case, at least if $q \leq n$.

Theorem 3.1.4 (Böröczky, Fodor [BF19], Theorem 1.4 on page 7987)**.** *If* $1 < p < q \leq n$, $Q \in \mathcal{S}^n_{(o)}$, $0 < c_1 < c_2$ *and* $K \in \mathcal{K}^n_o$ *with* $\mathcal{H}^{n-1}(\Xi_K) = 0$ *and* $\mathrm{int}K \neq \emptyset$ *be such that*

$$d\widetilde{C}_q(K, Q, \cdot) = h^p_K f \, d\mathcal{H}^{n-1}$$

for some Borel function f *on* S^{n-1} *satisfying* $c_1 \leq f \leq c_2$*, then* K *is strictly convex; or equivalently,* h_K *is* C^1 *on* $\mathbb{R}^n \backslash o$*.*

If $q = n$, then Theorems 3.1.3 and 3.1.4 are due to Chou, Wang [CW06]. We do not know whether Theorem 3.1.4 holds if $q > n$ (see the comments at the end of Section 7 of [BF19]).

We note that if $o \in \mathrm{int}K$, then the ideas leading to Theorem 3.1.3 work for any $p, q \in \mathbb{R}$.

Theorem 3.1.5 (Böröczky, Fodor [BF19], Theorem 1.5 on page 7988)**.** *Let* $p, q \in \mathbb{R}$, $Q \in \mathcal{S}^n_{(o)}$, $0 < c_1 < c_2$ *and let* $K \in \mathcal{K}^n_{(o)}$ *be such that*

$$d\widetilde{C}_{p,q}(K, Q, \cdot) = f \, d\mathcal{H}^{n-1}$$

for some Borel function f *on* S^{n-1} *satisfying* $c_1 \leq f \leq c_2$*. We have that*

(i) K *is smooth and strictly convex, and* h_K *is* C^1 *on* $\mathbb{R}^n \backslash \{o\}$*;*

(ii) *if* f *is continuous, then the restriction of* h_K *to* S^{n-1} *is in* $C^{1,\alpha}$ *for any* $\alpha \in (0, 1)$*;*

(iii) *if* $f \in C^\alpha(S^{n-1})$ *for* $\alpha \in (0, 1)$*, and* ∂Q *is* C^2_+*, then* ∂K *is* C^2_+*, and* h_K *is* $C^{2,\alpha}$ *on* S^{n-1}*.*

The rest of this chapter is organized as follows. We discuss properties of dual curvature measures in Section 3.2 extending some statements for the case when $o \in \partial K$ and $q > 0$. We prove Theorem 3.1.1 in Section 3.3 only for $Q = B^n$ in order to simplify and shorten the presentation. Fundamental properties of L_p dual curvature measures are considered in Section 3.4, and we use all these results to prove Theorem 3.1.2 for $Q = B^n$ in Section 3.5. Finally, we note that in the case of general Q, Theorem 3.1.1 and Theorem 3.1.2 are proved in Section 6 of [BF19].

3.2 On the dual curvature measure

The goal of this section is for $q > 0$, to extend the results of Huang, Lutwak, Yang and Zhang [HLYZ16] about the dual curvature measure $\widetilde{C}_q(K, \cdot)$ when $K \in \mathcal{K}^n_{(o)}$ to the case when $K \in \mathcal{K}^n_o$. For any measure, we take the measure of the empty set to be zero.

For any compact convex set K in $\mathbb{R}^n$ and $z \in \partial K$, we write $N(K, z)$ to denote the normal cone at z; namely,

$$N(K, z) = \{y \in \mathbb{R}^n : \langle y, x - z\rangle \leq 0 \text{ for } x \in K\}.$$

If $z \in \mathrm{int}K$, then simply $N(K, z) = \{o\}$. For compact, convex sets $K, L \subset \mathbb{R}^n$, we define their Hausdorff distance as

$$\delta_H(K, L) := \sup_{u \in S^{n-1}} |h_K(u) - h_L(u)|.$$

It is a metric on the space of compact convex sets, and the induced metric space is locally compact according to the Blaschke selection theorem. For basic properties of Hausdorff distance we refer to Schneider [Sch14], and also to Gruber [Gru07].

First we extend Lemma 3.3 in [HLYZ16]. Let $K \in \mathcal{K}_o^n$ with $\mathrm{int}K \neq \emptyset$. We recall that the so-called singular points $z \in \partial K$ where $\dim N(K,z) \geq 2$ form a Borel set of zero $\mathcal{H}^{n-1}$ measure, and hence its complement $\partial' K$ of smooth points is also a Borel set. For $z \in \partial' K$, we write $\nu_K(z)$ to denote the unique exterior normal at z. In addition, for any $z \in \partial K$, we define $\boldsymbol{\nu}_K(z) = N(K,z) \cap S^{n-1}$, and hence $\boldsymbol{\nu}_K^{-1}(\eta) = \cup_{u\in\eta} F(K,u)$ is the total inverse Gauss image of a Borel set $\eta \subset S^{n-1}$; namely, the set of all $z \in \partial K$ with $N(K,z) \cap \eta \neq \emptyset$. In particular, if $o \in \partial K$, then we have

$$\Xi_K = \boldsymbol{\nu}_K^{-1}\left(N(K,o) \cap S^{n-1}\right). \tag{3.2.1}$$

If $o \in \mathrm{int}K$, then $\Xi_K = \emptyset$. We also observe that the dual of $N(K,o)$ is

$$N(K,o)^* = \{y \in \mathbb{R}^n : \langle y, x\rangle \leq 0 \text{ for } x \in N(K,o)\} = \mathrm{cl}\{\lambda x : \lambda \geq 0 \text{ and } x \in K\},$$

and hence

$$\Xi_K = K \cap \partial N(K,o)^*. \tag{3.2.2}$$

If $o \in \mathrm{int}K$, then simply $N(K,o)^* = \mathbb{R}^n$. The following properties of ϱ_K readily follow from the definition.

Lemma 3.2.1. *If $K \in \mathcal{K}_o^n$, then ϱ_K is upper semicontinuous. In addition, if $\dim K \leq n-1$, then $\varrho_K(u) = 0$ for $u \in S^{n-1} \backslash \mathrm{lin}\, K$, and if $\mathrm{int}\, K \neq \emptyset$, then ϱ_K is continuous on $S^{n-1} \backslash \partial N(K,o)^*$ and $\varrho_K(u) = 0$ for $u \in S^{n-1} \backslash N(K,o)^*$.*

For $q > 0$, we extend Lutwak's definition of the qth dual intrinsic volume $\widetilde{V}_q(\cdot)$ to a compact convex set $K \in \mathcal{K}_o^n$ as

$$\widetilde{V}_q(K) = \frac{1}{n}\int_{S^{n-1}} \varrho_K^q(u)\, \mathcal{H}^{n-1}(du), \tag{3.2.3}$$

and hence $\widetilde{V}_n(K) = V(K)$. It follows from Lemma 3.2.1 that $\widetilde{V}_q(K)$ is well-defined and $\widetilde{V}_q(K) = 0$ if $\dim K \leq n-1$.

For a $K \in \mathcal{K}_o^n$ with $\mathrm{int}K \neq \emptyset$ and a Borel set $\eta \subset S^{n-1}$, let

$$\boldsymbol{\alpha}_K^*(\eta) = \{u \in S^{n-1} : \varrho_K(u)u \in F(K,v) \text{ for some } v \in \eta\} = \{u \in S^{n-1} : \varrho_K(u)u \in \boldsymbol{\nu}_K^{-1}(\eta)\}.$$

Following Huang, Lutwak, Yang, Zhang [HLYZ16] and Lutwak, Yang, Zhang [LYZ18], the set $\boldsymbol{\alpha}_K^*(\eta)$ is called the reverse radial Gauss image of η.

Lemma 3.2.2. *If $K \in \mathcal{K}_o^n$ with $\mathrm{int}K \neq \emptyset$, then*

$$S^{n-1} \cap (\mathrm{int}N(K,o)^*) \subset \boldsymbol{\alpha}_K^*\left(S^{n-1} \backslash N(K,o)\right) \subset S^{n-1} \cap N(K,o)^*, \tag{3.2.4}$$

$$\boldsymbol{\alpha}_K^*\left(S^{n-1} \cap N(K,o)\right) = S^{n-1} \backslash \left(\mathrm{int}N(K,o)^*\right). \tag{3.2.5}$$

Proof. If $o \in \operatorname{int} K$, then $N(K,o) = \{o\}$, and hence the statements are trivial. Therefore we assume that $o \in \partial K$.

It follows from (3.2.2) that

$$(\operatorname{int} N(K,o)^*) \cap \partial K = \{x \in \partial K : h_K(v) > 0 \text{ for all } v \in \boldsymbol{\nu}_K(x)\}. \tag{3.2.6}$$

Now (3.2.6) yields directly the first containment relation of (3.2.4), and $K \subset N(K,o)^*$ implies the second containment relation.

To prove (3.2.5), let $u, v \in S^{n-1}$ be such that $\varrho_K(u)u \in F(K,v)$. If $v \in N(K,o) \cap S^{n-1}$, then $F(K,v) \subset \Xi_K$, thus (3.2.6) yields that $u \notin \operatorname{int} N(K,o)^*$. On the other hand, if $u \notin \operatorname{int} N(K,o)^*$, then either $u \notin N(K,o)^*$, and hence $\varrho_K(u) = 0$, or $u \in \partial N(K,o)^*$, therefore $\varrho_K(u)u \in \Xi_K$ in both cases. We conclude $v \in N(K,o)$, and in turn (3.2.5). $\square$

We note that the radial map $\tilde{\pi} : \mathbb{R}^n \backslash \{o\} \to S^{n-1}$, $\tilde{\pi}(x) = x/\|x\|$ is locally Lipschitz. We write $\tilde{\pi}_K$ to denote the restriction of $\tilde{\pi}$ onto the $(n-1)$-dimensional Lipschitz manifold $(\partial K)\backslash \Xi_K = (\partial K) \cap \operatorname{int} N(K,o)^*$. For any $z \in (\partial' K)\backslash \Xi_K$, the Jacobian of $\tilde{\pi}_K$ at z is

$$\langle \nu_K(z), \tilde{\pi}_K(z)\rangle \|z\|^{-(n-1)} = \langle \nu_K(z), z\rangle \|z\|^{-n}. \tag{3.2.7}$$

Lemma 3.2.3. *If $K \in \mathcal{K}_o^n$ with $\operatorname{int} K \neq \emptyset$ and $\eta \subset S^{n-1}$ is a Borel set, then $\boldsymbol{\alpha}_K^*(\eta) \subset S^{n-1}$ is Lebesgue measurable.*

Proof. Since $\boldsymbol{\alpha}_K^*(\eta \cap N(K,o)) \cap \boldsymbol{\alpha}_K^*(\eta \backslash N(K,o)) \subset \partial N(K,o)^* \cap S^{n-1}$ by Lemma 3.2.2, and $\mathcal{H}^{n-1}(\partial N(K,o)^* \cap S^{n-1}) = 0$, it is equivalent to prove that both $\boldsymbol{\alpha}_K^*(\eta \cap N(K,o))$ and $\boldsymbol{\alpha}_K^*(\eta \backslash N(K,o))$ are Lebesgue measurable.

If $\eta \cap N(K,o) \neq \emptyset$, then we claim that

$$S^{n-1} \backslash N(K,o)^* \subset \boldsymbol{\alpha}_K^*(\eta \cap N(K,o)) \subset S^{n-1} \backslash \operatorname{int} N(K,o)^*. \tag{3.2.8}$$

The second containment relation follows from Lemma 3.2.2. For the first containment relation in (3.2.8), let $v \in \eta \cap N(K,o)$. Since $o \in F(K,v)$ and $\varrho_K(u) = 0$ for $u \in S^{n-1} \backslash N(K,o)^*$, it follows that $S^{n-1} \backslash N(K,o)^* \subset \boldsymbol{\alpha}_K^*(\{v\})$. Thus we have (3.2.8), and in turn $\eta \cap N(K,o)$ is Lebesgue measurable.

Next we consider $\eta \backslash N(K,o)$. Since $\partial' K$ is Borel, we have that $\sigma_K = \partial' K \cap \operatorname{int} N(K,o)^*$ is Borel, as well. We write $\tilde{\nu}_K : \sigma_K \to S^{n-1} \backslash N(K,o)$ to denote the restriction of ν_K to σ_K. As $\tilde{\nu}_K$ is continuous on σ_K, we deduce that $\tilde{\nu}_K^{-1}(\eta \backslash N(K,o))$ is Borel. In addition, $\tilde{\pi}_K$ is also continuous on $\partial K \cap \operatorname{int} N(K,o)^*$, thus $\tilde{\pi}_K \circ \tilde{\nu}_K^{-1}(\eta \backslash N(K,o))$ is also Borel. Since

$$\begin{aligned} \tilde{\pi}_K \circ \tilde{\nu}_K^{-1}(\eta \backslash N(K,o)) &\subset \boldsymbol{\alpha}_K^*(\eta \backslash N(K,o)) \\ &\subset \tilde{\pi}_K \circ \tilde{\nu}_K^{-1}(\eta \backslash N(K,o)) \cup \tilde{\pi}_K \Big((\partial K \cap \operatorname{int} N(K,o)^*) \backslash \partial' K \Big). \end{aligned}$$

Here $\mathcal{H}^{n-1}\Big((\partial K \cap \operatorname{int} N(K,o)^*) \backslash \partial' K \Big) = 0$ and $\tilde{\pi}_K$ is locally Lipschitz, therefore $\boldsymbol{\alpha}_K^*(\eta \backslash N(K,o))$ is Lebesgue measurable, as well. $\square$

Extending the definition in Huang, Lutwak, Yang, Zhang [HLYZ16], for a convex compact set $K \in \mathcal{K}_o^n$ and $q > 0$, the qth dual curvature measure $\widetilde{C}_q(K, \cdot)$ is a Borel measure on S^{n-1} defined in a way such that if $\eta \subset S^{n-1}$ is Borel, then

$$\widetilde{C}_q(K, \eta) = 0 \quad \text{if } \dim K \leq n-1, \text{ and} \tag{3.2.9}$$

$$\widetilde{C}_q(K,\eta) \;=\; \frac{1}{n}\int_{\boldsymbol{\alpha}_K^*(\eta)} \varrho_K^q(u)\,d\mathcal{H}^{n-1}(u) \;\text{ if } \operatorname{int} K \neq \emptyset. \tag{3.2.10}$$

Here, if $\operatorname{int} K \neq \emptyset$, then ϱ_K is continuous on $S^{n-1}\backslash \partial N(K,o)^*$, therefore $\widetilde{C}_q(K,\cdot)$ is well-defined by Lemma 3.2.3.

Since $\varrho_K(u) = 0$ for $u \in S^{n-1}\backslash N(K,o)^*$, and $\mathcal{H}^{n-1}(S^{n-1} \cap \partial N(K,o)^*) = 0$, we deduce from (3.2.5) that if $q > 0$, then

$$\widetilde{C}_q\left(K, N(K,o) \cap S^{n-1}\right) = \widetilde{C}_q\left(K, \{u \in S^{n-1} : h_K(u) = 0\}\right) = 0. \tag{3.2.11}$$

For $u \in S^{n-1}$, we write $r_K(u) = \varrho_K(u)u \in \partial K$. Since $\tilde{\pi}_K$ is locally Lipschitz, $\mathcal{H}^{n-1}$ almost all points of $S^{n-1} \cap (\operatorname{int} N(K,o)^*)$ are in the image of $(\partial' K) \cap (\operatorname{int} N(K,o)^*)$ by $\tilde{\pi}_K$. Therefore for $\mathcal{H}^{n-1}$ almost all points $u \in S^{n-1} \cap (\operatorname{int} N(K,o)^*)$, there is a unique exterior unit normal $\alpha_K(u)$ at $r_K(u) \in \partial K$. Here α_K is the so-called reverse radial Gauss map. For the other points $u \in S^{n-1} \cap (\operatorname{int} N(K,o)^*)$, we just choose an exterior unit normal $\alpha_K(u)$ at $r_K(u) \in \partial K$. The extensions of Lemma 3.3 and Lemma 3.4 in [HLYZ16] to the case when the origin may lie on the boundary of convex bodies are the following.

Lemma 3.2.4. *If $q > 0$, $K \in \mathcal{K}_o^n$ with $\operatorname{int} K \neq \emptyset$, and $g : S^{n-1} \to [0,\infty)$ is Borel measurable, then*

$$\begin{aligned}
\int_{S^{n-1}} g(u)\,d\widetilde{C}_q(K,u) &= \frac{1}{n}\int_{S^{n-1}\cap(\operatorname{int} N(K,o)^*)} g(\alpha_K(u))\varrho_K(u)^q\,d\mathcal{H}^{n-1}(u) && (3.2.12)\\
&= \frac{1}{n}\int_{\partial' K\backslash \Xi_K} g(\nu_K(x))\langle \nu_K(x), x\rangle \|x\|^{q-n}\,d\mathcal{H}^{n-1}(x), && (3.2.13)\\
&= \frac{1}{n}\int_{\partial' K} g(\nu_K(x))\langle \nu_K(x), x\rangle \|x\|^{q-n}\,d\mathcal{H}^{n-1}(x) && (3.2.14)
\end{aligned}$$

Proof. To prove (3.2.12), the integral of g can be approximated by integrals of finite linear combinations of characteristic functions of Borel sets of S^{n-1}, and hence we may assume that $g = \mathbf{1}_\eta$ for a Borel set $\eta \subset S^{n-1}$. In this case,

$$\int_{S^{n-1}\cap N(K,o)} \mathbf{1}_\eta\,d\widetilde{C}_q(K,\cdot) = 0$$

by (3.2.11), and

$$\int_{S^{n-1}\backslash N(K,o)} \mathbf{1}_\eta\,d\widetilde{C}_q(K,\cdot) = \widetilde{C}_q(K, \eta\backslash N(K,o)) = \int_{S^{n-1}\cap(\operatorname{int} N(K,o)^*)} \mathbf{1}_\eta(\alpha_K(u))\varrho_K(u)^q\,d\mathcal{H}^{n-1}(u)$$

by (3.2.4) and the definition of $\widetilde{C}_q(K,\cdot)$, verifying (3.2.12).

In turn, (3.2.12) yields (3.2.13) by (3.2.7). For (3.2.14), we observe that if $x \in \Xi_K \cap \partial' K$, then $\langle \nu_K(x), x\rangle = 0$. $\square$

Now we prove that the qth dual curvature measure is continuous on $\mathcal{K}_o^n$ for $q > 0$.

Lemma 3.2.5. *For $q > 0$, $\widetilde{V}_q(K)$ is a continuous function of $K \in \mathcal{K}_o^n$ with respect to the Hausdorff distance.*

Proof. Let $R > 0$ be such that $K \subset \text{int} R B^n$. Let $K_m \in \mathcal{K}_o^n$ be a sequence of compact convex sets tending to K with respect to Hausdorff distance. In particular, we may assume that $K_m \subset RB^n$ for all K_m.

If $\dim K \leq n-1$, then there exists $v \in S^{n-1}$ such that $K \subset v^\perp$, where $v^\perp$ denotes the orthogonal (linear) complement of v. For $t \in [0,1)$, we write

$$\Psi(v,t) = \{x \in \mathbb{R}^n : |\langle v, x\rangle| \leq t\}$$

to denote the closed region of width $2t$ between two hyperplanes orthogonal to v and symmetric to 0.

There exists a $t_0 \in (0,1)$ such that for any $t \in (0,t_0)$ and $v \in S^{n-1}$ it holds that $\mathcal{H}^{n-1}(S^{n-1} \cap \Psi(v,t)) < 3t(n-1)\kappa_{n-1}$.

Let $\varepsilon \in (0,t_0)$. We claim that there exists an m_ε such that for all $m > m_\varepsilon$ and for any $u \in S^{n-1} \backslash \Psi(v,\varepsilon)$, we have

$$\varrho_{K_m}(u) \leq \varepsilon. \tag{3.2.15}$$

Since $K_m \to K$ in the Hausdorff metric, there exists an index m_ε such that for all $m > m_\varepsilon$ it holds that $K_m \subset K + \varepsilon^2 B^n \subset \Psi(v,\varepsilon^2)$. Then if $u \in S^{n-1} \backslash \Psi(v,\varepsilon)$, then

$$\varepsilon^2 \geq \langle v, \varrho_{K_m}(u)u\rangle = \varrho_{K_m}(u)\langle v,u\rangle \geq \varrho_{K_m}(u)\cdot\varepsilon,$$

yielding (3.2.15). We deduce from (3.2.15) and $K_m \subset RB^n$ that for any $\varepsilon \in (0,t_0)$, if $m > m_\varepsilon$, then

$$\begin{aligned}\widetilde{V}_q(K_m) &\leq \int_{S^{n-1}\backslash\Psi(v,\varepsilon)} \varepsilon^q \, d\mathcal{H}^{n-1}(u) + \int_{S^{n-1}\cap\Psi(v,\varepsilon)} R^q \, d\mathcal{H}^{n-1}(u) \\ &\leq n\kappa_n\varepsilon^q + 3\varepsilon(n-1)\kappa_{n-1}R^q,\end{aligned}$$

therefore $\lim_{m\to\infty} \widetilde{V}_q(K_m) = 0 = \widetilde{V}_q(K)$.

Next, let $\text{int}\, K \neq \emptyset$ such that $o \in \partial K$. Since the functions $\varrho_{K_m}(u)$, $m = 1, \ldots$ are uniformly bounded, by Lebesgue's dominated convergence theorem it is sufficient to prove that

$$\lim_{m\to\infty} \varrho_{K_m}(u) = \varrho_K(u) \quad \text{for } u \in S^{n-1}\backslash\partial N(K,o)^*, \tag{3.2.16}$$

as $\mathcal{H}^{n-1}(S^{n-1} \cap \partial N(K,o)^*) = 0$. Now, let $\varepsilon \in [0,1)$.

Case 1. Let $u \in S^{n-1} \cap \text{int}\, N(K,o)^*$.

Then $\varrho_K(u) > 0$, and $(1-\varepsilon)\varrho_K(u)u \in \text{int}\, K$ and $(1+\varepsilon)\varrho_K(u)u \notin K$. Thus, there exists an index $m(u,\varepsilon) > 0$ such that for all $m > m(u,\varepsilon)$ it holds that $(1-\varepsilon)\varrho_K(u)u \in K_m$ and $(1+\varepsilon)\varrho_K(u)u \notin K_m$, or in other words,

$$(1-\varepsilon)\varrho_K(u) \leq \varrho_{K_m}(u) \leq (1+\varepsilon)\varrho_K(u),$$

which in turn yields (3.2.16) in this case.

Case 2. Let $u \in S^{n-1}\backslash N(K,o)^*$.

Then $\varrho_K(u) = 0$, and there exists $v \in S^{n-1} \cap \text{int}\, N(K,o)$ such that $\langle u,v\rangle > 0$. As $K_m \to K$, there exists an index $m(u,v,\varepsilon) > 0$ such that for all $m > m(u,v,\varepsilon)$ it holds that $K_m \subset K + \varepsilon\langle u,v\rangle B^n$, and thus $h_{K_m}(v) < \varepsilon\langle u,v\rangle$. Therefore, for all $m > m(u,v,\varepsilon)$,

$$\varepsilon\langle u,v\rangle > h_{K_m}(v) \geq \langle \varrho_{K_m}(u)u, v\rangle = \varrho_{K_m}(u)\langle u,v\rangle.$$

This yields that $\varrho_{K_m}(u) < \varepsilon$ for all $m > m(u, v, \varepsilon)$, and thus (3.2.16) holds by $\varrho_K(u) = 0$.

Finally, let $\operatorname{int} K \neq \emptyset$ and $o \in \operatorname{int} K$. The argument for this case is analogous to the one used above in Case 1. □

The following Proposition 3.2.6 extends Lemma 3.6 from Huang, Lutwak, Yang, Zhang [HLYZ16] about $K \in \mathcal{K}^n_{(o)}$ to the case when $K \in \mathcal{K}^n_o$.

Proposition 3.2.6. *If $q > 0$, and $\{K_m\}$, $m \in \mathbb{N}$, tends to K for $K_m, K \in \mathcal{K}^n_o$, then $\widetilde{C}_q(K_m, \cdot)$ tends weakly to $\widetilde{C}_q(K, \cdot)$.*

Proof. Since any element of $\mathcal{K}^n_o$ can be approximated by elements of $\mathcal{K}^n_{(o)}$, we may assume that each $K_m \in \mathcal{K}^n_{(o)}$. We fix $R > 0$ such that $K \subset \operatorname{int} R\, B^n$, and hence we may also assume that $K_m \subset RB^n$ for all K_m. We need to prove that if $g : S^{n-1} \to \mathbb{R}$ is continuous, then

$$\lim_{m\to\infty} \int_{S^{n-1}} g(u)\, d\widetilde{C}_q(K_m, u) = \int_{S^{n-1}} g(u)\, d\widetilde{C}_q(K, u) \tag{3.2.17}$$

First we assume that $o \in \partial K$. If $\dim K \leq n-1$, then $\widetilde{C}_q(K, \cdot)$ is the constant zero measure by (3.2.9). Since $\widetilde{C}_q(K_m, S^{n-1}) = \widetilde{V}_q(K_m)$ tends to zero according to Lemma 3.2.5, we conclude (3.2.17) in this case.

Therefore we may assume that $\operatorname{int} K \neq \emptyset$ and $o \in \partial K$. To simplify notation, we set

$$\sigma = N(K, o)^*.$$

According to Lemma 3.2.4, (3.2.17) is equivalent to

$$\lim_{m\to\infty} \int_{S^{n-1}} g(\alpha_{K_m}(u))\varrho_{K_m}(u)^q\, d\mathcal{H}^{n-1}(u) = \int_{S^{n-1}\cap(\operatorname{int}\sigma)} g(\alpha_K(u))\varrho_K(u)^q\, d\mathcal{H}^{n-1}(u). \tag{3.2.18}$$

Since $\tilde{\pi}_K$ is Lipschitz and $\mathcal{H}^{n-1}(S^{n-1} \cap (\partial\sigma)) = 0$, to verify (3.2.18), and in turn (3.2.17), it is sufficient to prove

$$\lim_{m\to\infty} \int_{\tilde{\pi}_K((\operatorname{int}\sigma)\cap\partial' K)} g(\alpha_{K_m}(u))\varrho_{K_m}(u)^q\, d\mathcal{H}^{n-1}(u) = \int_{\tilde{\pi}_K((\operatorname{int}\sigma)\cap\partial' K)} g(\alpha_K(u))\varrho_K(u)^q\, d\mathcal{H}^{n-1}(u) \tag{3.2.19}$$

$$\lim_{m\to\infty} \int_{S^{n-1}\setminus\sigma} g(\alpha_{K_m}(u))\varrho_{K_m}(u)^q\, d\mathcal{H}^{n-1}(u) = 0. \tag{3.2.20}$$

To prove (3.2.19) and (3.2.20), it follows from $K_m \subset RB^n$, the continuity of g and Lemma 3.2.5 that there exists $M > 0$ such that

$$\begin{array}{rcll} |\varrho_{K_m}(u)| & \leq & R & \text{for } u \in S^{n-1}, \\ |g(u)| & \leq & M & \text{for } u \in S^{n-1}, \\ \widetilde{C}_q(K_m, S^{n-1}) & \leq & M & \text{for } m \in \mathbb{N} \end{array} \tag{3.2.21}$$

We deduce from (3.2.21) that Lebesgue's Dominated Convergence Theorem applies both in (3.2.19) and (3.2.20). For (3.2.19), let $u \in \tilde{\pi}_K((\operatorname{int}\sigma)\cap\partial' K)$. Readily, $\lim_{m\to\infty} \varrho_{K_m}(u)^q = \varrho_K(u)^q$. Since $\alpha_K(u)$ is the unique normal at $\varrho_K(u)u \in \partial' K$, we have $\lim_{m\to\infty} \alpha_{K_m}(u) =$

$\alpha_K(u)$, and hence $\lim_{m\to\infty} g(\alpha_{K_m}(u)) = g(\alpha_K(u))$ by the continuity of g. In turn, we conclude (3.2.19) by Lebesgue's Dominated Convergence Theorem.

Turning to (3.2.20), it follows from Lebesgue's Dominated Convergence Theorem, $q>0$ and (3.2.21) that it is sufficient to prove that if $\varepsilon>0$ and $u\in S^{n-1}\backslash\sigma$, then

$$\varrho_{K_m}(u)\leq\varepsilon \tag{3.2.22}$$

for $m\geq m_0$ where m_0 depends on $u,\{K_m\},\varepsilon$. Since $u\not\in\sigma=N(K,o)^*$, there exists $v\in N(K,o)$ such that $\langle v,u\rangle=\delta>0$. As $h_K(v)=0$ and K_m tends to K, there exists m_0 such that $h_{K_m}(v)\leq\delta\varepsilon$ if $m\geq m_0$. In particular, if $m\geq m_0$, then

$$\varepsilon\delta\geq h_{K_m}(v)\geq\langle v,\varrho_K(u)u\rangle=\varrho_K(u)\delta,$$

yielding (3.2.22), and in turn (3.2.20).

Finally, the argument leading to (3.2.19) implies (3.2.17) also in the case when $o\in \mathrm{int}K$, completing the proof of Proposition 3.2.6. $\square$

3.3 Proof of Theorem 3.1.1 for $Q=B^n$

To verify Theorem 3.1.1, we prove the following statement, which also holds if $p=q$.

Theorem 3.3.1. *Let $p>1$ and $q>0$, and let μ be a discrete measure on S^{n-1} that is not concentrated on any closed hemisphere. Then there exists a polytope $P\in\mathcal{K}^n_{(o)}$ such that $\widetilde{V}_q(P)^{-1}\widetilde{C}_{p,q}(P,\cdot)=\mu$.*

We recall that $\tilde{\pi}:\mathbb{R}^n\backslash\{o\}\to S^{n-1}$ is the radial projection, and for a convex body K in $\mathbb{R}^n$ and $u\in S^{n-1}$, the face of K with exterior unit normal u is the set

$$F(K,u)=\{x\in K:\,\langle x,u\rangle=h_K(u)\}.$$

We observe that if $P\in\mathcal{K}^n_o$ is a polytope with $\mathrm{int}\,P\neq\emptyset$, and $v_1,\ldots,v_l\in S^{n-1}$ are the exterior normals of the facets of P not containing the origin, then

$$\begin{aligned}\mathrm{supp}\,\widetilde{C}_q(P,\cdot) &= \{v_1,\ldots,v_l\},\ \text{and}\\ \widetilde{C}_q(P,\{v_i\}) &= \frac{1}{n}\int_{\tilde{\pi}(F(P,v_i))}\varrho_P^q(u)\,d\mathcal{H}^{n-1}(u)\qquad\text{for } i=1,\ldots,l.\end{aligned}\tag{3.3.1}$$

Let $p>1$, $q>0$ and μ be a discrete measure on S^{n-1} that is not concentrated on any closed hemi-sphere. Let $\mathrm{supp}\,\mu=\{u_1,\ldots,u_k\}$, and let $\mu(\{u_i\})=\alpha_i>0$, $i=1,\ldots,k$. For any $z=(t_1,\ldots,t_k)\in(\mathbb{R}_{\geq0})^k$, we define

$$\begin{aligned}\Phi(z) &= \sum_{i=1}^k\alpha_i t_i^p,\\ P(z) &= \{x\in\mathbb{R}^n:\,\langle x,u_i\rangle\leq t_i\,\forall i=1,\ldots,k\},\\ \Psi(z) &= \widetilde{V}_q(P(z)).\end{aligned}\tag{3.3.2}$$

Since $\alpha_i > 0$ for $i = 1, \ldots, k$, the set $Z = \{z \in (\mathbb{R}_{\geq 0})^k : \Phi(z) = 1\}$ is compact, and hence Lemma 3.2.5 yields the existence of $z_0 \in Z$ such that

$$\Psi(z_0) = \max\{\Psi(z) : z \in Z\}.$$

We prove that $o \in \operatorname{int} P(z_0)$ and there exists $\lambda_0 > 0$ such that

$$\widetilde{V}_q(\lambda_0 P(z_0))^{-1} \widetilde{C}_{p,q}(\lambda_0 P(z_0), \cdot) = \mu.$$

Lemma 3.3.2. *If $p > 1$ and $q > 0$, then $o \in \operatorname{int} P(z_0)$.*

Proof. It is clear from the construction that $o \in P(z_0)$. We assume that $o \in \partial P(z_0)$, and seek a contradiction. Without loss of generality, we may assume that $z_0 = (t_1, \ldots, t_k) \in (\mathbb{R}_{\geq 0})^k$, where there exists $1 \leq m < k$ such that $t_1 = \ldots = t_m = 0$ and $t_{m+1}, \ldots, t_k > 0$. For sufficiently small $t > 0$, we define

$$\begin{aligned} \tilde{z}_t &= \left(\overbrace{0, \ldots, 0}^{m}, (t_{m+1}^p - \alpha t^p)^{\frac{1}{p}}, \ldots, (t_k^p - \alpha t^p)^{\frac{1}{p}}\right) \quad \text{for } \alpha = \tfrac{\alpha_1 + \ldots + \alpha_m}{\alpha_{m+1} + \ldots + \alpha_k}, \text{ and} \\ z_t &= \left(\overbrace{t, \ldots, t}^{m}, (t_{m+1}^p - \alpha t^p)^{\frac{1}{p}}, \ldots, (t_k^p - \alpha t^p)^{\frac{1}{p}}\right). \end{aligned}$$

Simple substitution shows that $\Phi(z_t) = 1$, so $z_t \in Z$.

We prove that there exist $\tilde{t}_0, \tilde{c}_1, \tilde{c}_2 > 0$ depending on p, q, μ and z_0 such that if $t \in (0, \tilde{t}_0]$, then

$$\Psi(\tilde{z}_t) \geq \Psi(z_0) - \tilde{c}_1 t^p, \tag{3.3.3}$$

$$\Psi(z_t) \geq \Psi(\tilde{z}_t) + \tilde{c}_2 t, \tag{3.3.4}$$

therefore

$$\Psi(z_t) \geq \Psi(z_0) - \tilde{c}_1 t^p + \tilde{c}_2 t. \tag{3.3.5}$$

We choose $R > 0$ such that $P(z_0) \subset \operatorname{int} RB^n$ and $R \geq \max\{t_{m+1}, \ldots, t_k\}$.

We start with proving (3.3.3), and set $\varrho_0 = \min\{t_{m+1}, \ldots, t_k\}$. We frequently use the following form of Bernoulli's inequality that says that if $\tau \in (0, 1)$ and $\eta > 0$, then

$$(1 - \tau)^\eta \geq 1 - \max\{1, \eta\} \cdot \tau. \tag{3.3.6}$$

It follows from (3.3.6) and $\varrho_0 \leq t_i \leq R$, $i = m+1, \ldots, k$, that there exist $s_0, c_0 > 0$, depending on z_0, μ and p such that if $t \in (0, s_0)$, then

$$(t_i^p - \alpha t^p)^{\frac{1}{p}} > t_i - c_0 t^p > \varrho_0 / 2 \quad \text{for } i = m+1, \ldots, k. \tag{3.3.7}$$

We consider the cone $N(P(z_0), o)^* = \{x \in \mathbb{R}^n : \langle x, u_i \rangle \leq 0 \ \forall i = 1, \ldots, m\}$ that satisfies that $\varrho_{P(z_0)}(u) > 0$ for $u \in S^{n-1}$ if and only if $u \in N(P(z_0), o)^*$. It follows from (3.3.7) that $\varrho_{P(\tilde{z}_t)}(u) > 0$ for $u \in S^{n-1}$ also if and only if $u \in N(P(z_0), o)^*$, and even $\varrho_{P(\tilde{z}_t)}(u) > \varrho_0/2$ in this case.

Let $u \in N(P(z_0), o)^* \cap S^{n-1}$, and hence $\varrho_{P(\tilde{z}_t)}(u) \cdot u$ lies in a facet $F(P(\tilde{z}_t), u_i)$ for an $i \in \{m+1, \ldots, k\}$, thus

$$\langle \varrho_{P(\tilde{z}_t)}(u)u, u_i \rangle = (t_i^p - \alpha t^p)^{\frac{1}{p}} > t_i - c_0 t^p > \varrho_0/2.$$

Combining the last estimate with $\varrho_{P(\tilde{z}_t)}(u) \leq R$, we deduce that $\langle u, u_i \rangle \geq \frac{\varrho_0}{2R}$. Let $s > 0$ be defined by $\langle su, u_i \rangle = t_i$. Then $s \geq \varrho_{P(z_0)(u)}$, and hence

$$s - \varrho_{P(\tilde{z}_t)}(u) = \frac{\langle su, u_i \rangle - \langle \varrho_{P(\tilde{z}_t)}(u)u, u_i \rangle}{\langle u, u_i \rangle} \leq \frac{t_i - (t_i - c_0 t^p)}{\langle u, u_i \rangle} \leq \frac{2Rc_0}{\varrho_0} \cdot t^p,$$

thus $\varrho_{P(\tilde{z}_t)}(u) \geq \varrho_{P(z_0)}(u) - \frac{2Rc_0}{\varrho_0} \cdot t^p$. We choose $t_0 > 0$ with $t_0 \leq s_0$ depending on z_0 and p such that $\frac{2Rc_0}{\varrho_0} \cdot t_0^p < \varrho_0/2$. Since $\varrho_0 \leq \varrho_{P(z_0)}(u) \leq R$, we deduce from (3.3.6) that there exists $c_1 > 0$ depending on μ, z_0, q and p that if $t \in (0, t_0)$ and $u \in C \cap S^{n-1}$, then

$$\varrho_{P(\tilde{z}_t)}(u)^q \geq \left(\varrho_{P(z_0)}(u) - \frac{2Rc_0}{\varrho_0} \cdot t^p \right)^q \geq \varrho_{P(z_0)}(u)^q - c_1 \cdot t^p,$$

which yields (3.3.3) by (3.2.3) and by taking into account that $N(P(\tilde{z}_t), o)^* = N(P(z_0), o)^*$.

The main idea of the proof of (3.3.4) is that we construct a set $\widetilde{G}_t \subset S^{n-1}$ for sufficiently small $t > 0$ whose $\mathcal{H}^{n-1}$ measure is of order t, and if $u \in \widetilde{G}_t$, then $\varrho_{P(z_t)}(u) \geq r$ for a suitable constant $r > 0$ while $\varrho_{P(\tilde{z}_t)}(u) = 0$. In order to show that the constants involved really depend only on p, q μ and $P(z_0)$, we start to set them with respect to $P(z_0)$.

We may assume, possibly after reindexing $u_1, \ldots, u_m$, that $\dim F(P(z_0), u_1) = n-1$. In particular, there exist $r > 0$ and $y_0 \in F(P(z_0), u_1) \backslash \{o\}$ such that

$$\langle y_0, u_i \rangle \leq h_{P(z_0)}(u_i) - 8r \quad \text{for } i = 2, \ldots, k.$$

For $v = y_0/\|y_0\| \in S^{n-1} \cap u_1^\perp$, we consider $y = y_0 + 4rv$, and hence $4r \leq \|y\| \leq R$, and

$$\langle y, u_i \rangle \leq h_{P(z_0)}(u_i) - 4r \quad \text{for } i = 2, \ldots, k.$$

Note that $P(\tilde{z}_t) \to P(z_0)$ as $t \to 0^+$ and also $P(\tilde{z}_t) \subset P(z_0)$ for $t > 0$. Therefore there exists a positive $t_1 \leq \min\{r, t_0\}$, depending only on p, q, μ and z_0 such that if $t \in (0, t_1]$, then

$$\langle y, u_i \rangle \leq h_{P(\tilde{z}_t)}(u_i) - 2r \quad \text{for } i = 2, \ldots, k \quad \text{and} \quad P(z_t) \subset RB^n. \tag{3.3.8}$$

For two vectors $a, b \in \mathbb{R}^n$, we denote by $[a, b]$ ((a, b)) the closed (open) segment with endpoints a and b. Let the $(n-2)$-dimensional unit ball G be defined as

$$G = u_1^\perp \cap v^\perp \cap B^n.$$

Then we have that $y + rG \subset F(P(z_0), u_1)$ and $(y + rG) + r[o, u_1] \subset y + 2rB^n$. Let $G_t = (y + rG) + t(o, u_1]$ be the $(n-1)$-dimensional right spherical cylinder of height $t < \min\{t_1, r\}$, whose base $y + rG$ does not belong to G_t. We deduce from (3.3.8) and $h_{P(z_t)}(u_1) = t$ that $G_t \subset P(z_t) \backslash N(P(z_0), o)^* \subset P(z_t) \backslash P(\tilde{z}_t)$.

Let $\widetilde{G}_t$ be the the radial projection of G_t to S^{n-1}. For $x \in G_t$, we have $\langle x, v\rangle = \|y\| \geq 4r$ and $\|x\| \leq R$, therefore

$$\mathcal{H}^{n-1}(\widetilde{G}_t) = \int_{G_t} \left\langle \frac{x}{\|x\|}, v \right\rangle \|x\|^{-(n-1)}\, d\mathcal{H}^{n-1}(x)$$
$$\geq \frac{4r\mathcal{H}^{n-1}(G_t)}{R^n} = \frac{4r \cdot r^{n-2}\kappa_{n-2}}{R^n} \cdot t = \frac{4r^{n-1}\kappa_{n-2}}{R^n} \cdot t.$$

Since $\varrho_{P(\tilde{z}_t)}(u) \leq \varrho_{P(z_t)}(u)$ for all $u \in S^{n-1}$, and if $u \in \widetilde{G}_t$, then $\varrho_{P(z_t)}(u) \geq \|y\| \geq 4r$ and $\varrho_{P(\tilde{z}_t)}(u) = 0$, we deduce that

$$\begin{aligned}
\Psi(z_t) &= \frac{1}{n}\int_{S^{n-1}} \varrho^q_{P(z_t)}(u) d\mathcal{H}^{n-1}(u) \\
&= \frac{1}{n}\int_{S^{n-1}\setminus\widetilde{G}_t} \varrho^q_{P(z_t)}(u) d\mathcal{H}^{n-1}(u) + \frac{1}{n}\int_{\widetilde{G}_t} \varrho^q_{P(z_t)}(u) d\mathcal{H}^{n-1}(u) \\
&\geq \frac{1}{n}\int_{S^{n-1}} \varrho^q_{P(\tilde{z}_t)}(u) d\mathcal{H}^{n-1}(u) + \frac{1}{n}\int_{\widetilde{G}_t} \varrho^q_{P(z_t)}(u) d\mathcal{H}^{n-1}(u) \\
&\geq \Psi(\tilde{z}_t) + \frac{(4r)^q \cdot 4r^{n-1}\kappa_{n-2}}{nR^n} \cdot t,
\end{aligned}$$

which proves (3.3.4). Combining (3.3.3) and (3.3.4), we obtain (3.3.5).

Finally, we deduce from $p > 1$ and (3.3.5) that if $t > 0$ is sufficiently small, then $\Psi(P(z_t)) > \Psi(P(z_0))$, which contradicts the optimality of z_0, and yields Lemma 3.3.2. $\square$

As we already know that $o \in \operatorname{int} P(z_0)$ by Lemma 3.3.2, we can freely decrease $h_{P(z_0)}(u_i)$ for $i = 1, \ldots, k$, and increase it if $\dim F(P(z_0), u_i) = n - 1$. To control what happens to $\Psi(z)$ when we perturb $P(z_0)$, we use Lemma 3.3.3, which is a consequence of Theorem 4.4 in [HLYZ16]. Let $\mathbb{R}_+$ denote set of the positive real numbers.

Lemma 3.3.3 (Huang, Lutwak, Yang, Zhang, [HLYZ16])**.** *If $q \neq 0$, $\eta \in (0,1)$ and $z_t = (z_1(t), \ldots, z_k(t)) \in \mathbb{R}^k_+$ for $t \in (-\eta, \eta)$ are such that $\lim_{t\to 0^+} \frac{z_i(t)-z_i(0)}{t} = z_i'(0) \in \mathbb{R}$ for $i = 1, \ldots, k$ exists, then the $P(z_t)$ defined in* (3.3.2) *satisfies that*

$$\lim_{t\to 0^+} \frac{\widetilde{V}_q(P(z_t)) - \widetilde{V}_q(P(z_0))}{t} = q\sum_{i=1}^{k} \frac{z_i'(0)}{h_{P(z_0)}(u_i)} \cdot \widetilde{C}_q(P(z_0), \{u_i\}).$$

For the sake of completeness, in Section 6 we prove a general version of Lemma 3.3.3 about the variation of $\widetilde{V}_q(P(z(t)), Q)$ in the case when Q is an arbitrary star body, cf. Lemma 6.7 on page 8011 in [BF19].

We note that $\operatorname{supp} C_q(P(z_0), \cdot) \subset \{u_1, \ldots, u_k\}$, where $\widetilde{C}_q(P(z_0), \{u_i\}) > 0$ if and only if
$\dim F(P(z_0), u_i) = n - 1$.

Lemma 3.3.4. *If $p > 1$ and $q > 0$, then* $\dim F(P(z_0), u_i) = n - 1$ *for $i = 1, \ldots, k$.*

Proof. We suppose that $\dim F(P(z_0), u_1) < n-1$, and seek a contradiction. We may assume that $\dim F(P(z_0), u_k) = n-1$. For small $t \geq 0$, we consider

$$\tilde{z}(t) = (t_1 - t, t_2, \dots, t_k),$$

and $\theta(t) = \Phi(P(\tilde{z}(t))$. In particular, $\theta(0) = 1$ and $\theta'(0) = -p\alpha_1 t_1^{p-1}$, and hence

$$z(t) = \theta(t)^{-1/p}\tilde{z}(t) = (z_1(t), \dots, z_k(t)) \in Z$$

satisfies $\frac{d}{dt}\theta(t)^{-1/p}|_{t=0^+} = \alpha_1 t_1^{p-1}$ and $z_i'(0) = \alpha_1 t_1^{p-1} t_i > 0$ for $i = 2, \dots, k$. We deduce from Lemma 3.3.3 and $\widetilde{C}_q(P(z_0), \{u_1\}) = 0$ that

$$\begin{aligned}\lim_{t\to 0^+} \frac{\widetilde{V}_q(P(z(t))) - \widetilde{V}_q(P(z_0))}{t} &= q\sum_{i=2}^{k} \frac{z_i'(0)}{h_{P(z_0)}(u_i)} \cdot \widetilde{C}_q(P(z_0), \{u_i\}) \\ &\geq \frac{q\, z_k'(0)}{h_{P(z_0)}(u_k)} \cdot \widetilde{C}_q(P(z_0), \{u_k\}) > 0,\end{aligned}$$

therefore $\widetilde{V}_q(P(z(t))) > \widetilde{V}_q(P(z_0))$ for small $t > 0$. This contradicts the optimality of z_0, and proves Lemma 3.3.4. $\square$

Proof of Theorem 3.3.1 According to Lemmas 3.3.2 and 3.3.4,
we have $\dim F(P(z_0), u_i) = n-1$ for $i = 1, \dots, k$, $o \in \operatorname{int} P(z_0)$ and $h_{P(z_0)}(u_i) = t_i$ for $i = 1, \dots, k$. Let $(g_1, \dots, g_k) \in \mathbb{R}^k$ satisfying $\sum_{i=1}^k g_i \alpha_i t_i^{p-1} = 0$ such that not all g_i are zero. If $t \in (-\varepsilon, \varepsilon)$ for small $\varepsilon > 0$, then consider

$$\tilde{z}(t) = (t_1 + g_1 t, \dots, t_k + g_k t),$$

and $\theta(t) = \Phi(P(\tilde{z}(t))$. In particular, $\theta(0) = 1$ and

$$\theta'(0) = p\sum_{i=1}^{k} g_i \alpha_i t_i^{p-1} = 0.$$

Therefore

$$z(t) = \theta(t)^{-1/p}\tilde{z}(t) = (z_1(t), \dots, z_k(t)) \in Z$$

satisfies $\frac{d}{dt}\theta(t)^{-1/p}|_{t=0} = 0$ and $z_i'(0) = g_i$ for $i = 1, \dots, k$. We deduce from Lemma 3.3.3 and $h_{P(z_0)}(u_i) = t_i$ for $i = 1, \dots, k$ that

$$\lim_{t\to 0} \frac{\widetilde{V}_q(P(z(t))) - \widetilde{V}_q(P(z_0))}{t} = q\sum_{i=1}^{k} \frac{g_i}{t_i} \cdot \widetilde{C}_q(P(z_0), \{u_i\}).$$

Since $\widetilde{V}_q(P(z(t)))$ attains its maximum at $t = 0$ by the optimality of z_0, we have

$$\sum_{i=1}^{k} \frac{g_i}{t_i} \cdot \widetilde{C}_q(P(z_0), \{u_i\}) = 0. \tag{3.3.9}$$

In particular, (3.3.9) holds whenever $(g_1,\ldots,g_k)\in\mathbb{R}^k\backslash\{o\}$ satisfies $\sum_{i=1}^k g_i\alpha_i t_i^{p-1}=0$, or in other words, there exists a $\lambda\in\mathbb{R}$ such that

$$\lambda\cdot\frac{\widetilde{C}_q(P(z_0),\{u_i\})}{t_i}=\alpha_i t_i^{p-1}\quad\text{for } i=1,\ldots,k.$$

Since $\lambda>0$ and $p>1$, there exists a $\lambda_0>0$ such that $\lambda=\lambda_0^{-p}\widetilde{V}_q(P(z_0))$, and hence

$$\alpha_i=\widetilde{V}_q(\lambda_0P(z_0))^{-1}h_{\lambda_0P(z_0)}(u_i)^{-p}\widetilde{C}_q(\lambda_0P(z_0),\{u_i\})\quad\text{for } i=1,\ldots,k.$$

In other words,

$$\mu=\widetilde{V}_q(\lambda_0P(z_0))^{-1}h_{\lambda_0P(z_0)}(u_i)^{-p}\widetilde{C}_q(\lambda_0P(z_0),\cdot).$$

This finishes the proof of Theorem 3.3.1. □

Proof of Theorem 3.1.1 in the case of $Q=B^n$ We have $p\neq q$. According to Theorem 3.3.1, there exists a polytope $P_0\in\mathcal{K}^n_{(o)}$ such that $\widetilde{V}_q(P_0)^{-1}\widetilde{C}_{p,q}(P_0,\cdot)=\mu$. For $\lambda=\widetilde{V}_q(P_0)^{\frac{-1}{q-p}}$ and $P=\lambda P_0$, we have

$$\widetilde{C}_{p,q}(P,\cdot)=\lambda^{q-p}\widetilde{C}_{p,q}(P_0,\cdot)=\widetilde{V}_q(P_0)^{-1}\widetilde{C}_{p,q}(P_0,\cdot)=\mu.$$

□

3.4 On the L_p dual curvature measures

According to Lemma 5.1 in Lutwak, Yang, Zhang [LYZ18], if $K\in\mathcal{K}^n_{(o)}$, $p\in\mathbb{R}$ and $q>0$, then for any Borel function $g:\,S^{n-1}\to\mathbb{R}$, we have that

$$\int_{S^{n-1}}g(u)\,d\widetilde{C}_{p,q}(K,u)=\frac{1}{n}\int_{\partial' K}g(\nu_K(x))\langle\nu_K(x),x\rangle^{1-p}\|x\|^{q-n}\,d\mathcal{H}^{n-1}(x).\tag{3.4.1}$$

As a simple consequence of Lemma 3.2.4, we can partially extend (3.4.1) to allow $o\in\partial K$.

Corollary 3.4.1. *If $p>1$, $q>0$, $K\in\mathcal{K}^n_o$ with $\mathrm{int}K\neq\emptyset$, $\widetilde{C}_{p,q}(K,S^{n-1})<\infty$ and $\mathcal{H}^{n-1}(\Xi_K)=0$, and the Borel function $g:\,S^{n-1}\to\mathbb{R}$ is bounded, then*

$$\int_{S^{n-1}}g(u)\,d\widetilde{C}_{p,q}(K,u)=\frac{1}{n}\int_{\partial' K}g(\nu_K(x))\langle\nu_K(x),x\rangle^{1-p}\|x\|^{q-n}\,d\mathcal{H}^{n-1}(x).$$

Proof. Knowing that $\widetilde{C}_{p,q}(K,S^{n-1})<\infty$, it follows from Lemma 3.2.4 and $\mathcal{H}^{n-1}(\Xi_K)=0$ that

$$\begin{aligned}\int_{S^{n-1}}g(u)\,d\widetilde{C}_{p,q}(K,u)&=\int_{S^{n-1}}g(u)h_K(u)^{-p}\,d\widetilde{C}_q(K,u)\\&=\frac{1}{n}\int_{\partial' K\backslash\Xi_K}g(\nu_K(x))h_K(\nu_K(x))^{-p}\langle\nu_K(x),x\rangle\|x\|^{q-n}\,d\mathcal{H}^{n-1}(x)\\&=\frac{1}{n}\int_{\partial' K}g(\nu_K(x))\langle\nu_K(x),x\rangle^{1-p}\|x\|^{q-n}\,d\mathcal{H}^{n-1}(x).\end{aligned}$$

□

Next, we prove a basic estimate on the inradius of K in terms of its total L_p dual curvature measure. For a convex body $K \in \mathcal{K}^n_{(o)}$, we write $r(K)$ to denote the maximal radius of balls contained in K. Since $o \in K$, Steinhagen's theorem yields the existence of $w \in S^{n-1}$ such that

$$|\langle x, w\rangle| \le 2nr(K) \quad \text{for } x \in K. \tag{3.4.2}$$

Lemma 3.4.2. *For $n \ge 2$, $p > 1$ and $q > 0$, there exists a constant $c > 0$ depending only on p, q, n such that if $K \in \mathcal{K}^n_{(o)}$, then*

$$\widetilde{C}_{p,q}(K, S^{n-1}) \ge c \cdot r(K)^{-p} \cdot \widetilde{V}_q(K).$$

Proof. We may assume that $r(K) = 1$, and hence (3.4.2) yields the existence of $w \in S^{n-1}$ such that

$$|\langle x, w\rangle| \le 2n \quad \text{for } x \in K. \tag{3.4.3}$$

Let $\widetilde{K} = K|w^\perp$ be the orthogonal projection of K to the hyperplane $w^\perp$, and hence the radial function $\varrho_{\widetilde{K}}$ is positive and continuous on $w^\perp \cap S^{n-1}$. We consider the concave function f and the convex function g on $\widetilde{K} = K|w^\perp$ such that

$$K = \left\{ y + tw : \, y \in \widetilde{K} \text{ and } g(y) \le t \le f(y) \right\}.$$

We divide $w^\perp \cap S^{n-1}$ into pairwise disjoint Borel sets $\widetilde{\Omega}_1, \ldots, \widetilde{\Omega}_m$ of positive $\mathcal{H}^{n-2}$ measure such that for each $\widetilde{\Omega}_i$, there exists a $\varrho_i > 0$ satisfying

$$\varrho_i/2 \le \varrho_{\widetilde{K}}(u) \le \varrho_i \quad \text{for } u \in \widetilde{\Omega}_i. \tag{3.4.4}$$

For any $i = 1, \ldots, m$, we consider

$$\begin{aligned} \Omega_i &= \left\{ u\cos\alpha + w\sin\alpha : \, u \in \widetilde{\Omega}_i \text{ and } \alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \right\} \subset S^{n-1}, \\ \Psi_i &= \{\varrho_K(u)u : \, u \in \Omega_i\} \subset \partial K. \end{aligned}$$

It follows that $S^{n-1}\backslash\{w, -w\}$ is divided into the pairwise disjoint Borel sets $\Omega_1, \ldots, \Omega_m$, and $\partial K \backslash \{f(o)w, g(o)w\}$ is divided into the pairwise disjoint Borel sets $\Psi_1, \ldots, \Psi_m$.

According to (3.4.1) and Lemma 3.2.4, to verify Lemma 3.4.2, it is sufficient to prove that there exists a constant $c > 0$ depending only on n, p, q such that if $i = 1, \ldots, m$, then

$$\int_{\partial' K \cap \Psi_i} \langle \nu_K(x), x\rangle^{1-p} \|x\|^{q-n} \, d\mathcal{H}^{n-1}(x) \ge c \int_{\partial' K \cap \Psi_i} \langle \nu_K(x), x\rangle \|x\|^{q-n} \, d\mathcal{H}^{n-1}(x). \tag{3.4.5}$$

We define

$$R = 4(2n)^2. \tag{3.4.6}$$

Case 1. $\varrho_i \le R$

If $\varrho_i \le R$ and $x \in \partial' K \cap \Psi_i$, then (3.4.3) yields that

$$\langle \nu_K(x), x\rangle \le \|x\| \le R + 2n \quad \text{for } x \in \Psi_i,$$

and hence $\langle \nu_K(x), x\rangle^{1-p} \ge \langle \nu_K(x), x\rangle (R+2n)^{-p}$. Therefore we may choose $c = (R+2n)^{-p}$ in (3.4.5).

Case 2. $\varrho_i > R$

If $\varrho_i > R$, then consider the set

$$\Phi_i = \left\{tu : u \in \widetilde{\Omega}_i \text{ and } 0 < t \le \varrho_i/4\right\} \subset \Psi_i|w^\perp,$$

and subdivide Ψ_i into

$$\begin{aligned} \Psi_i^0 &= \{y + f(y)w : y \in \Phi_i\} \cup \{y + g(y)w : y \in \Phi_i\} \subset \Psi_i \cap \left(\tfrac{\varrho_i}{4} + 2n\right) B^n, \text{ and} \\ \Psi_i^1 &= \Psi_i \backslash \Psi_i^0 \subset \Psi_i \backslash \left(\tfrac{\varrho_i}{4} B^n\right). \end{aligned}$$

We claim that

$$\langle \nu_K(x), x\rangle \le 6n \quad \text{for } x \in \partial' K \cap \Psi_i^0. \tag{3.4.7}$$

We observe that $x = y + tw$ for some $y \in \Phi_i$ and $t \in [-2n, 2n]$, and $s = f(2y)$ satisfies $s \in [-2n, 2n]$ and $2y + sw \in \Psi_i$. It follows that

$$\langle \nu_K(x), 2y + sw\rangle \le \langle \nu_K(x), x\rangle = \langle \nu_K(x), y + tw\rangle,$$

and hence

$$\langle \nu_K(x), y\rangle \le \langle \nu_K(x), tw\rangle - \langle \nu_K(x), sw\rangle \le 4n.$$

We conclude that $\langle \nu_K(x), y + tw\rangle = \langle \nu_K(x), y\rangle + \langle \nu_K(x), tw\rangle \le 6n$, in accordance with (3.4.7).

In turn, (3.4.7) yields that $\langle \nu_K(x), x\rangle^{1-p} \ge \langle \nu_K(x), x\rangle(6n)^{-p}$ for $x \in \partial' K \cap \Psi_i^0$, and hence

$$\int_{\partial' K \cap \Psi_i^0} \langle \nu_K(x), x\rangle^{1-p} \|x\|^{q-n}\, d\mathcal{H}^{n-1}(x) \ge (6n)^{-p} \int_{\partial' K \cap \Psi_i^0} \langle \nu_K(x), x\rangle \|x\|^{q-n}\, d\mathcal{H}^{n-1}(x). \tag{3.4.8}$$

Next, we prove the existence of $\gamma_1 > 0$ depending on n, p, q such that

$$\int_{\partial' K \cap \Psi_i^0} \langle \nu_K(x), x\rangle^{1-p} \|x\|^{q-n}\, d\mathcal{H}^{n-1}(x) \ge \begin{cases} \gamma_1 \mathcal{H}^{n-2}(\widetilde{\Omega}_i)\varrho_i^{q-1} & \text{if } \quad q > 1 \\ \gamma_1 \mathcal{H}^{n-2}(\widetilde{\Omega}_i) & \text{if } \quad q \in (0, 1] \end{cases}. \tag{3.4.9}$$

Let us consider $x = y + f(y)w \in \Psi_i^0 \cap \partial' K$ for some $y \in \Phi_i \backslash (2nB^n)$. Since $\|y\| \le \|x\| \le 2\|y\|$ by (3.4.3), it follows from (3.4.7) that

$$\langle \nu_K(x), x\rangle^{1-p} \|x\|^{q-n} \ge (6n)^{1-p} \min\{1, 2^{q-n}\}\, \|y\|^{q-n}.$$

Therefore there exists $\gamma_2 > 0$ depending on n, p, q such that

$$\begin{aligned} \int_{\partial' K \cap \Psi_i^0} \langle \nu_K(x), x\rangle^{1-p} \|x\|^{q-n}\, d\mathcal{H}^{n-1}(x) &\ge \gamma_2 \int_{\Phi_i \backslash (2nB^n)} \|y\|^{q-n}\, d\mathcal{H}^{n-1}(x) \\ &= \gamma_2 \mathcal{H}^{n-2}(\widetilde{\Omega}_i) \int_{2n}^{\varrho_i/4} t^{q-n} t^{n-2}\, dt \\ &= \gamma_2 \mathcal{H}^{n-2}(\widetilde{\Omega}_i) \int_{2n}^{\varrho_i/4} t^{q-2}\, dt, \end{aligned}$$

and in turn we conclude (3.4.9).

The final part of the argument is the estimate

$$\int_{\partial' K\cap\Psi_i^1}\langle\nu_K(x),x\rangle\|x\|^{q-n}\,d\mathcal{H}^{n-1}(x)\leq 2^q16n\cdot\mathcal{H}^{n-2}(\widetilde{\Omega}_i)\cdot\varrho_i^{q-1}. \tag{3.4.10}$$

Let $\Omega_i^1=\pi_K(\Psi_i^1)$. If $x=y+sw\in\Psi_i^1$ for $y\in(\Psi_i|w^\perp)\setminus\Phi_i$, then $y\in(\Psi_i|w^\perp)\setminus(\frac{\varrho_i}{4}B^n)$ and $|s|\leq 2n$. It follows that $|\tan\alpha|\leq\frac{2n}{\varrho_i/4}=\frac{8n}{\varrho_i}$ for the angle α of x and y. In particular,

$$\Omega_i^1\subset\pi_K\left(\widetilde{\Omega}_i+\left[\frac{-8n}{\varrho_i},\frac{8n}{\varrho_i}\right]\cdot w\right)$$

which, in turn, yields that

$$\mathcal{H}^{n-1}(\Omega_i^1)\leq\frac{16n}{\varrho_i}\,\mathcal{H}^{n-2}(\widetilde{\Omega}_i).$$

We deduce from (3.2.7) and from the fact that $\|x\|\leq\varrho_i+2n\leq 2\varrho_i$ for $x\in\Psi_i^1$ that

$$\int_{\partial' K\cap\Psi_i^1}\langle\nu_K(x),x\rangle\|x\|^{q-n}\,d\mathcal{H}^{n-1}(x)=\int_{\Omega_i^1}\varrho_K(u)^q\,d\mathcal{H}^{n-1}(u)\leq\frac{16n}{\varrho_i}\,\mathcal{H}^{n-2}(\widetilde{\Omega}_i)\cdot(2\varrho_i)^q,$$

yielding (3.4.10).

We deduce from (3.4.9) and (3.4.10) the existence of $\gamma_3>0$ depending on n,p,q such that

$$\int_{\partial' K\cap\Psi_i^0}\langle\nu_K(x),x\rangle^{1-p}\|x\|^{q-n}\,d\mathcal{H}^{n-1}(x)\geq\gamma_3\int_{\partial' K\cap\Psi_i^1}\langle\nu_K(x),x\rangle\|x\|^{q-n}\,d\mathcal{H}^{n-1}(x). \tag{3.4.11}$$

Combining (3.4.8) and (3.4.11) implies (3.4.5) if $\varrho_i>R$, as well, completing the proof of Lemma 3.4.2. □

Next we investigate the limit of convex bodies with bounded L_p dual curvature measure in Lemmas 3.4.3 and 3.4.4.

Lemma 3.4.3. *If $p>1$, $0<q\leq p$ and $K_m\in\mathcal{K}^n_{(o)}$ for $m\in\mathbb{N}$ tend to $K\in\mathcal{K}^n_o$ with $\mathrm{int}K\neq\emptyset$ such that $\widetilde{C}_{p,q}(K_m,S^{n-1})$ stays bounded, then $K\in\mathcal{K}^n_{(o)}$.*

Proof. Let us suppose that $o\in\partial K$, and seek a contradiction. We claim that there exists a vector $w\in\mathrm{int}N(K,o)^*$ such that $-w\in N(K,o)\cap S^{n-1}$. If this property fails, then $(-N(K,o))\cap\mathrm{int}N(K,o)^*=\emptyset$, and hence the Hahn-Banach theorem yields the existence of a vector $v\in S^{n-1}$ such that $\langle v,u\rangle\leq 0$ if $u\in N(K,o)^*$, and $\langle v,u\rangle\geq 0$ if $u\in -N(K,o)$, and hence $v\in N(K,o)^*$. Since $\langle v,v\rangle=1>0$ contradicts $\langle v,u\rangle\leq 0$ if $u\in N(K,o)^*$, we conclude the existence of the required w.

To simplify notation, we set $B(r)=w^\perp\cap(rB^n)$ for $r>0$. The conditions in Lemma 3.4.3 and (3.4.1) yield the existence of some $M>0$ such that for each K_m, we have that

$$M\ >\ \widetilde{C}_{p,q}(K_m,S^{n-1})=\frac{1}{n}\int_{\partial' K_m}\langle\nu(K_m,x),x\rangle^{1-p}\|x\|^{q-n}\,d\mathcal{H}^{n-1}(x) \tag{3.4.12}$$

$$\geq \frac{1}{n}\int_{\partial' K_m\cap B^n} \|x\|^{1-n+q-p}\, d\mathcal{H}^{n-1}(x) \geq \frac{1}{n}\int_{\partial' K_m\cap B^n} \|x\|^{1-n}\, d\mathcal{H}^{n-1}(x).$$

We note that since $K_m \to K$ and $o \in \partial K$, for sufficiently large m, $\partial' K_m \cap B^n \neq \emptyset$ and the right-hand side of (3.4.12) is greater than zero. As $w \in \mathrm{int}N(K,o)^*$ and $w \in N(K,o)$, there exist a $\varrho \in (0,1)$ and a non-negative convex function f on $B(2\varrho)$ with $f(o) = 0$ such that

$$U = \{z + f(z)w : z \in B(2\varrho)\} \subset \partial K.$$

In particular, there exist an $\eta > 0$ such that

$$\|x|w^\perp\| \geq 2\eta\|x\| \text{ for } x \in U. \tag{3.4.13}$$

We may assume that $\varrho \in (0,1)$ is small enough to ensure that $U \subset \mathrm{int}B^n$.

Since $\int_{B(\varrho)} \|z\|^{1-n} d\mathcal{H}^{n-1}(z) = \infty$, there exists some $\delta \in (0,\varrho)$ such that

$$\frac{1}{n}\int_{B(\varrho)\backslash B(\delta)} \left(\frac{\|z\|}{\eta}\right)^{1-n} d\mathcal{H}^{n-1}(z) > M. \tag{3.4.14}$$

There exist and an m_0 such that if $m > m_0$, then for some convex function f_m on $B(\varrho)$, we have

$$U_m = \{z + f_m(z)w : z \in B(\varrho)\backslash B(\delta)\} \subset (\partial K_m) \cap (\mathrm{int}B^n), \tag{3.4.15}$$

and (compare (3.4.13))

$$\|z\| \geq \eta\|z + f_m(z)w\| \text{ for } z \in B(\varrho)\backslash B(\delta). \tag{3.4.16}$$

We deduce from (3.4.12), (3.4.15) and (3.4.16), and finally from (3.4.14) that

$$\begin{aligned} M &> \frac{1}{n}\int_{U_m} \|x\|^{1-n}\, d\mathcal{H}^{n-1}(x) \geq \frac{1}{n}\int_{B(\varrho)\backslash B(\delta)} \|z + f_m(z)w\|^{1-n} d\mathcal{H}^{n-1}(z) \\ &\geq \frac{1}{n}\int_{B(\varrho)\backslash B(\delta)} \left(\frac{\|z\|}{\eta}\right)^{1-n} d\mathcal{H}^{n-1}(z) > M. \end{aligned}$$

This is a contradiction, and in turn we conclude Lemma 3.4.3. □

Lemma 3.4.4. *If $p > 1$, $q > 0$ and $K_m \in \mathcal{K}^n_{(o)}$ for $m \in \mathbb{N}$ tend to $K \in \mathcal{K}^n_o$ with $\mathrm{int}K \neq \emptyset$ such that $\widetilde{C}_{p,q}(K_m, S^{n-1})$ stays bounded, then $\mathcal{H}^{n-1}(\Xi_K) = 0$.*

Proof. We fix a point $z \in \mathrm{int}K$, and for any bounded $X \subset \mathbb{R}^n\backslash\{z\}$, we define the set

$$\sigma(X) = \{z + \lambda(x - z) : x \in X \text{ and } \lambda > 0\}.$$

We observe that $\sigma(X)$ is open if $X \subset \partial K$ is relatively open, and $\sigma(X) \cup \{o\}$ is closed if X is compact.

We will use the weak continuity of the $(n-1)$th curvature measure. In particular, according to Theorem 4.2.1 and Theorem 4.2.3 in Schneider [Sch14], if $\beta \subset \mathbb{R}^n$ is open, then

$$\liminf_{m\to\infty} \mathcal{H}^{n-1}(\beta \cap \partial K_m) \geq \mathcal{H}^{n-1}(\beta \cap \partial K). \tag{3.4.17}$$

Let us suppose, on the contrary, that $\mathcal{H}^{n-1}(\Xi_K) > 0$, and hence $o \in \partial K$, and seek a contradiction. Choose some large $M, R > 0$, and a compact set $\widetilde{\Xi} \subset \Xi_K \backslash \{o\}$ such that

$$\begin{aligned} K_m &\subset RB^n, \\ \widetilde{C}_{p,q}(K_m, S^{n-1}) &\leq M \quad \text{for } m \in \mathbb{N}, \\ \mathcal{H}^{n-1}(\widetilde{\Xi}) &= \omega > 0. \end{aligned}$$

Now there exists some $\eta > 0$ such that

(i) $(\eta B^n) \cap \sigma(\widetilde{\Xi} + \eta B^n) = \emptyset$.

Since $p > 1$, we may choose $\varepsilon > 0$ such that

$$\frac{(2\varepsilon)^{1-p}}{n} \cdot \min\{\eta^{q-n}, R^{q-n}\} \cdot (\omega/2) > M. \tag{3.4.18}$$

We have $\mathcal{H}^{n-1}(\widetilde{\Xi} \cap \partial' K) = \omega$. For any $x \in \widetilde{\Xi} \cap \partial' K$, there exists $r_x \in (0, \eta)$ such that

$$h_K(u) \leq \varepsilon \;\text{ if } u \in S^{n-1} \text{ is exterior normal at } y \in \partial K \cap (x + r_x B^n), \tag{3.4.19}$$

and we define $B_x = \mathrm{int}(x + r_x B^n)$. Let

$$\mathcal{U} = \bigcup_{x \in \widetilde{\Xi} \cap \partial' K} (B_x \cap \partial K),$$

which is a relatively open subset of ∂K satisfying

(a) $(\eta B^n) \cap \sigma(\mathcal{U}) = \emptyset$,

(b) $\mathcal{H}^{n-1}(\mathcal{U}) \geq \omega$,

(c) $h_K(u) \leq \varepsilon$ if $u \in S^{n-1}$ is exterior normal at $x \in \mathrm{cl}\,\mathcal{U}$.

It follows that (applying (3.4.17) in the case (b')) that there exists m_0 such that if $m \geq m_0$, then

(a') $\|x\| \geq \eta$ if $x \in \sigma(\mathcal{U}) \cap \partial K_m$,

(b') $\mathcal{H}^{n-1}(\sigma(\mathcal{U}) \cap \partial K_m) \geq \omega/2$,

(c') $h_K(u) \leq 2\varepsilon$ if $u \in S^{n-1}$ is exterior normal at $x \in \sigma(\mathcal{U}) \cap \partial K_m$.

For any $x \in \sigma(\mathcal{U}) \cap \partial K_m$, (a') and $K_m \subset RB^n$ yield that

$$\|x\|^{q-n} \geq \min\{\eta^{q-n}, R^{q-n}\}.$$

It follows first by (3.4.1), then by (b'), (c') and (3.4.18), that

$$M \geq \widetilde{C}_{p,q}(K_m, S^{n-1}) \geq \frac{1}{n} \int_{\sigma(\mathcal{U}) \cap \partial' K_m} \langle \nu_K(x), x \rangle^{1-p} \|x\|^{q-n} \, d\mathcal{H}^{n-1}(x) > M.$$

This contradiction proves Lemma 3.4.4. □

3.5 Theorem 3.1.2 for general convex bodies if $Q = B^n$

For $w \in S^{n-1}$ and $\alpha \in (-1,1)$, we write

$$\Omega(w,\alpha) = \{u \in S^{n-1} : \langle u, w\rangle > \alpha\}.$$

The following is a simple but useful observation.

Lemma 3.5.1. *For a finite Borel measure μ on S^{n-1} not concentrated on a closed hemi-sphere, there exists $t \in (0,1)$ such that for any $w \in S^{n-1}$, we have $\mu(\Omega(w,t)) > t$.*

First we prove the following variant of Theorem 3.1.2 involving the dual intrinsic volume.

Theorem 3.5.2. *For $p > 1$ and $q > 0$, and finite Borel measure μ on S^{n-1} not concentrated on a closed hemi-sphere, there exists a convex body $K \in \mathcal{K}_o^n$ with $\mathrm{int}K \neq \emptyset$ and $\mathcal{H}^{n-1}(\Xi_K) = 0$ such that*

$$\widetilde{V}_q(K) h_K^p d\mu = d\widetilde{C}_q(K,\cdot),$$

and in addition, $K \in \mathcal{K}_{(o)}^n$ if $p \geq q$.

Proof. We choose a sequence of discrete measures μ_m tending to μ that are not concentrated on any closed hemispheres. It follows from Theorem 3.3.1, that there exists polytope $P_m \in \mathcal{K}_{(o)}^n$ such that

$$d\mu_m = \frac{1}{\widetilde{V}_q(P_m)}\, d\widetilde{C}_{p,q}(P_m,\cdot) = \frac{h_{P_m}^{-p}}{\widetilde{V}_q(P_m)}\, d\widetilde{C}_q(P_m,\cdot) \tag{3.5.1}$$

for each m, and hence we may assume that

$$\frac{\widetilde{C}_{p,q}(P_m, S^{n-1})}{\widetilde{V}_q(P_m)} < 2\mu(S^{n-1}). \tag{3.5.2}$$

We claim that there exists $R > 0$ such that

$$P_m \subset RB^n. \tag{3.5.3}$$

We prove (3.5.3) by contradiction, thus we suppose that $R_m = \max_{x\in P_m} \|x\|$ tends to infinity. We choose $v_m \in S^{n-1}$ such that $R_m v_m \in P_m$, and we may assume by possibly taking a subsequence that v_m tends to $v \in S^{n-1}$. We deduce from Lemma 3.5.1 that there exist $s, t > 0$ such that $\mu(\Omega(v,2t)) > 2s$. As v_m tends to $v \in S^{n-1}$ and μ_m tends weakly to μ, we may also assume that $\Omega(v,2t) \subset \Omega(v_m,t)$ and $\mu_m(\Omega(v,2t)) > s$, therefore $\mu_m(\Omega(v_m,t)) > s$ for each m. Since $h_{P_m}(u) \geq \langle R_m v_m, u\rangle \geq R_m t$ for $u \in \Omega(v_m,t)$, we deduce from (3.5.1) that

$$s < \mu_m(\Omega(v_m,t)) = \int_{\Omega(v_m,t)} \frac{h_{P_m}^{-p}(u)}{\widetilde{V}_q(P_m)}\, d\widetilde{C}_q(P_m,u) \leq R_m^{-p} t^{-p} \frac{\widetilde{C}_q(P_m,S^{n-1})}{\widetilde{V}_q(P_m)} \leq R_m^{-p} t^{-p}.$$

In particular, $R_m^p \leq s^{-1}t^{-p}$, contradicting the fact that R_m tends to infinity, and in turn proving (3.5.3).

It follows from (3.5.3) that P_m tends to a compact convex set $K \in \mathcal{K}_o^n$ with $K \subset RB^n$. We deduce from (3.5.2) and Lemma 3.4.2 that $r(K) > 0$.

We observe that $h_{P_m}^p$ tends uniformly to h_K^p, and hence also $\widetilde{V}_q(P_m)h_{P_m}^p$ tends uniformly to $\widetilde{V}_q(K)h_K^p$ by Lemma 3.2.5. Therefore given any continuous function f, we have

$$\lim_{m\to\infty} \int_{S^{n-1}} f(u)\widetilde{V}_q(P_m)h_{P_m}^p(u)\, d\mu_m = \int_{S^{n-1}} f(u)\widetilde{V}_q(K)h_K^p(u)\, d\mu.$$

It follows from Proposition 3.2.6 that the dual curvature measure $\widetilde{C}_q(P_m,\cdot)$ tends weakly to $\widetilde{C}_q(K,\cdot)$, thus (3.5.1) yields

$$\int_{S^{n-1}} f(u)\widetilde{V}_q(K)h_K^p(u)\, d\mu = \int_{S^{n-1}} f(u)\, d\widetilde{C}_q(K,u).$$

Since the last property holds for all continuous function f, we conclude that

$$\widetilde{V}_q(K)h_K^p\, d\mu = d\widetilde{C}_q(K,\cdot),$$

as it is required.

Having (3.5.2) at hand, Lemma 3.4.4 yields that $\mathcal{H}^{n-1}(\Xi_K) = 0$, and Lemma 3.4.3 implies that if $p \geq q$, then $K \in \mathcal{K}_{(o)}^n$. $\square$

Proof of Theorem 3.1.2 in the case of $Q = B^n$ Let $p > 1$, $q > 0$ and $p \neq q$. According to Theorem 3.5.2, there exists a $K_0 \in \mathcal{K}_{(o)}^n$ with $\mathrm{int}K_0 \neq \emptyset$ and $\mathcal{H}^{n-1}(\Xi_{K_0}) = 0$ such that $\widetilde{V}_q(K_0)^{-1}\widetilde{C}_{p,q}(K_0,\cdot) = \mu$. For $\lambda = \widetilde{V}_q(K_0)^{\frac{-1}{q-p}}$ and $K = \lambda K_0$, we have

$$\widetilde{C}_{p,q}(K,\cdot) = \lambda^{q-p}\widetilde{C}_{p,q}(K_0,\cdot) = \widetilde{V}_q(K_0)^{-1}\widetilde{C}_{p,q}(K_0,\cdot) = \mu.$$

It follows from Theorem 3.5.2 that $o \in \mathrm{int}K$ if $p > q$. $\square$

Chapter 4

Weighted volume approximation by inscribed polytopes

This chapter of the dissertation is based on parts of the paper [BFH10] by K.J. Böröczky, F. Fodor, and D. Hug, *The mean width of random polytopes circumscribed around a convex body*, J. Lond. Math. Soc. (2) **81** (2010), no. 2, 499–523. (DOI 10.1112/jlms/jdp077)

4.1 Introduction and results

For a given convex body, we introduce a class of inscribed random polytopes. Let C be a convex body in $\mathbb{R}^d$, let ϱ be a bounded, nonnegative, measurable function on C, and let $\mathcal{H}^d \llcorner C$ denote the restriction of $\mathcal{H}^d$ to C. Assuming that $\int_C \varrho(x)\, \mathcal{H}^d(dx) > 0$, we choose random points from C according to the probability measure

$$\mathbb{P}_{\varrho,C} := \left(\int_C \varrho(x)\, dx \right)^{-1} \varrho\, \mathcal{H}^d \llcorner C.$$

Expectation with respect to $\mathbb{P}_{\varrho,C}$ is denoted by $\mathbb{E}_{\varrho,C}$. The convex hull of n independent and identically distributed random points with distribution $\mathbb{P}_{\varrho,C}$ is denoted by $K_{(n)}$ if ϱ is clear from the context. This yields a general model of an inscribed random polytope.

In order to state our results, we define the constant

$$c_d = \frac{(d^2+d+2)(d^2+1)}{2(d+3)\cdot(d+1)!} \Gamma\left(\frac{d^2+1}{d+1}\right) \left(\frac{d+1}{\alpha_{d-1}}\right)^{2/(d+1)} \tag{4.1.1}$$

(cf. J.A. Wieacker) [Wie78]. In the following, we simply write dx instead of $\mathcal{H}^d(dx)$.

Generalizing a result by C. Schütt [Sch94], we prove the following theorem.

Theorem 4.1.1 (Böröczky, Fodor, Hug [BFH10, Theorem 3.1 on page 502])**.** *For a convex body K in $\mathbb{R}^d$, a probability density function ϱ on K, and an integrable function $\lambda : K \to \mathbb{R}$ such that, on a neighbourhood of ∂K relative to K, λ and ϱ are continuous and ϱ is positive, we have*

$$\lim_{n\to\infty} n^{\frac{2}{d+1}}\, \mathbb{E}_{\varrho,K} \int_{K\setminus K_{(n)}} \lambda(x)\, dx = c_d \int_{\partial K} \varrho(x)^{\frac{-2}{d+1}} \lambda(x) \kappa(x)^{\frac{1}{d+1}}\, \mathcal{H}^{d-1}(dx), \tag{4.1.2}$$

where c_d is defined in (4.1.1).

The limit on the right-hand side of (4.1.2) depends only on the values of ϱ and λ on the boundary of K. In particular, we may prescribe any continuous, positive function ϱ on ∂K. Then any continuous extension of ϱ to a probability density on K (there always exists such an extension) will satisfy Theorem 4.1.1 with the prescribed values of ϱ on the right-hand side.

Our proof of Theorem 4.1.1 is inspired by the approach in [Sch94], where the special case $\varrho \equiv \lambda \equiv 1$ is considered. We note that for [Sch94, Lemma 2], which is crucial for the proof in [Sch94], no explicit proof is provided, but reference is given to an analogous result in an unpublished note by M. Schmuckenschläger. Beside missing a factor $\frac{1}{2}$, Lemma 2 does not hold in the generality stated in [Sch94]. For instance, it is not true for simplices. Most probably, this gap can be overcome, but still our approach to prove Theorem 4.1.1, where [Sch94, Lemma 2] is replaced by the present more elementary Lemma 4.2.2, might be of some interest.

The present partially new approach to Theorem 4.1.1 involves also some other interesting new features. In particular, we do not need the concept of a Macbeath region. An outline of the proof is given below. It should also be emphasized that the generality of Theorem 4.1.1 is needed for our study of circumscribed random polyhedral sets via duality in Chapter 5.

Let $f_i(P)$, $i \in \{0, \ldots, d-1\}$, denote the number of i-dimensional faces of a polyhedral set P. A classical argument going back to Efron [Efr65] shows that

$$\mathbb{E}_{\varrho,K}\left(f_0(K_{(n)})\right) = n \cdot \mathbb{E}_{\varrho,K} \int_{K \setminus K_{(n-1)}} \varrho(x)\, dx,$$

which yields the following consequence of Theorem 4.1.1.

Corollary 4.1.2 (Böröczky, Fodor, Hug [BFH10, Corollary 3.2 on page 503]). *For a convex body K in $\mathbb{R}^d$, and for a probability density function ϱ on K which is continuous and positive on a neighbourhood of ∂K relative to K, we have*

$$\lim_{n\to\infty} n^{-\frac{d-1}{d+1}}\, \mathbb{E}_{\varrho,K}(f_0(K_{(n)})) = c_d \int_{\partial K} \varrho(x)^{\frac{d-1}{d+1}} \kappa(x)^{\frac{1}{d+1}}\, \mathcal{H}^{d-1}(dx),$$

where c_d is defined in (4.1.1).

The proof of Theorem 4.1.1 is obtained through the following intermediate steps. Details are provided in Section 4.2. Since the convex body K is fixed, we write $\mathbb{E}_\varrho$ and $\mathbb{P}_\varrho$ instead of $\mathbb{E}_{\varrho,K}$ and $\mathbb{P}_{\varrho,K}$, respectively. The basic observation to prove Theorem 4.1.1 is that

$$\mathbb{E}_\varrho \int_{K \setminus K_{(n)}} \lambda(x)\, dx = \int_K \mathbb{P}_\varrho\left(x \notin K_{(n)}\right) \lambda(x)\, dx, \tag{4.1.3}$$

which is an immediate consequence of Fubini's theorem. Throughout the proof, we may assume that $o \in \operatorname{int}(K)$. The asymptotic behaviour, as $n \to \infty$, of the right-hand side of (4.1.3) is determined by points $x \in K$ which are sufficiently close to the boundary of K. In order to give this statement a precise meaning, scaled copies of K are introduced as follows. For $t \in (0,1)$, we define $K_t := (1-t)K$ and $y_t := (1-t)y$ for $y \in \partial K$. In Lemma 4.2.3, we show that

$$\lim_{n\to\infty} n^{\frac{2}{d+1}} \int_{K_{n^{-1/(d+1)}}} \mathbb{P}_\varrho\left(x \notin K_{(n)}\right) \lambda(x)\, dx = 0.$$

This limit relation is based on a geometric estimate of $\mathbb{P}_\varrho\left(x \notin K_{(n)}\right)$, provided in Lemma 4.2.1, and on a disintegration result stated as Lemma 4.2.2.

For $y \in \partial K$, we write $u(y)$ for some exterior unit normal of K at y. This exterior unit normal is uniquely determined for $\mathcal{H}^{d-1}$-almost all boundary points of K. Applying the disintegration result again and using Lebesgue's dominated convergence result, we finally get

$$\lim_{n\to\infty} n^{\frac{2}{d+1}} \mathbb{E}_\varrho \int_{K\setminus K_{(n)}} \lambda(x)\, dx = \int_{\partial K} \lambda(y) J_\varrho(y)\, \mathcal{H}^{d-1}(dy),$$

where

$$J_\varrho(y) = \lim_{n\to\infty} \int_0^{n^{-1/(d+1)}} n^{\frac{2}{d+1}} \langle y, u(y)\rangle \mathbb{P}_\varrho\left(y_t \notin K_{(n)}\right)\, dt$$

for $\mathcal{H}^{d-1}$-almost all $y \in \partial K$. For the subsequent analysis, it is sufficient to consider a small cap of K at a normal boundary point $y \in \partial K$. The case $\kappa(y) = 0$ is treated in Lemma 4.2.4. The main case is $\kappa(y) > 0$. Here we reparametrize y_t as $\tilde{y}_s$, in terms of the probability content of a small cap of K whose bounding hyperplane passes through y_t. This implies that

$$J_\varrho(y) = (d+1)^{-\frac{d-1}{d+1}} \alpha_{d-1}^{-\frac{2}{d+1}} \varrho(y)^{\frac{-2}{d+1}} \kappa(y)^{\frac{1}{d+1}} \lim_{n\to\infty} \int_0^{n^{-1/2}} n^{\frac{2}{d+1}} \mathbb{P}_\varrho\left(\tilde{y}_s \notin K_{(n)}\right) s^{-\frac{d-1}{d+1}}\, ds,$$

cf. (4.2.26). It is then a crucial step in the proof to show that the remaining integral asymptotically is independent of the particular convex body K, and thus the limit of the integral is the same as for a Euclidean ball (see Lemma 4.2.6). To achieve this, the integral is first approximated, up to a prescribed error of order $\varepsilon > 0$, by replacing $\mathbb{P}_\varrho\left(\tilde{y}_s \notin K_{(n)}\right)$ by the probability of an event that depends only on a small cap of K at y and on a small number of random points. This important step is accomplished in Lemma 4.2.5. For the proofs of Lemmas 4.2.5 and 4.2.6 it is essential that the boundary of K near the normal boundary point y can be suitably approximated by the osculating paraboloid of K at y.

4.2 Proof of Theorem 4.1.1

To start with the actual proof, we fix some further notation which will be used in this chapter. For $y \in \partial K$ and $t \in (0,1)$, we define the cap $C(y,t) := \{x \in K : \langle u(y), x\rangle \geq \langle u(y), y_t\rangle\}$ whose bounding hyperplane passes through y_t and has normal $u(y)$. For $u \in \mathbb{R}^d \setminus \{o\}$ and $t \in \mathbb{R}$, we define the hyperplane $H(u,t) := \{x \in \mathbb{R}^d : \langle x, u\rangle = t\}$, and the closed half-spaces $H^+(u,t) := \{x \in \mathbb{R}^d : \langle x, u\rangle \geq t\}$ and $H^-(u,t) := \{x \in \mathbb{R}^d : \langle x, u\rangle \leq t\}$ bounded by $H(u,t)$. We further put $\mathbb{R}^+ := [0,\infty)$.

For $y \in \partial K$, we denote by $r(y)$ the maximal number $r \geq 0$ such that $y - ru(y) + rB^d \subset K$. This number is called the interior reach of the boundary point y. It is well known that

$r(y) > 0$ for $\mathcal{H}^{d-1}$-almost all $y \in \partial K$. If $r(y) > 0$, then there is a unique tangent plane of K at y. In particular, $r(y) \leq r(K)$ where $r(K)$ is the inradius of K.

Finally, we observe that there exists a constant $\gamma_0 \in (0,1)$ such that for $y \in \partial K$, we have

$$|\langle y, u(y)\rangle| \geq \gamma_0 \|y\|, \text{ and hence } \|y|u(y)^\perp\| \leq \sqrt{1-\gamma_0^2} \cdot \|y\|, \tag{4.2.1}$$

where $y|u^\perp$ denotes the orthogonal projection of y onto the orthogonal complement of the vector $u \in \mathbb{R}^d \setminus \{o\}$. Subsequently, we always assume that $n \in \mathbb{N}$.

Lemma 4.2.1. *There exists a constant $\delta > 0$, depending on K and ϱ, such that, if $y \in \partial K$ and $t \in (0, \delta)$, then*

$$\mathbb{P}_\varrho\left(y_t \notin K_{(n)}\right) \ll \left(1 - \gamma_1 r(y)^{\frac{d-1}{2}} t^{\frac{d+1}{2}}\right)^n.$$

Remarks. 1. *In addition, we may assume that on $K \setminus \operatorname{int}(K_\delta)$, both functions ϱ, λ are continuous, ϱ is positive and $\gamma_1 r(K)^{\frac{d-1}{2}} \delta^{\frac{d+1}{2}} < 1$. Further, we always choose $\delta < 1$.*

2. *In the following, we will use the notion of a "coordinate corner". Given an orthonormal basis in a linear i-dimensional subspace L, the corresponding $(i-1)$-dimensional coordinate planes cut L into 2^i convex cones, which we call coordinate corners (with respect to L and the given basis).*

Proof of Lemma 4.2.1. If $r(y) = 0$, then there is nothing to prove. So let $r(y) > 0$, thence $u(y)$ is uniquely determined. Choose an orthonormal basis in $u(y)^\perp$, and let $\Theta'_1, \ldots, \Theta'_{2^{d-1}}$ be the corresponding coordinate corners in $u(y)^\perp$. For $i = 1, \ldots, 2^{d-1}$ and $t \in [0,1]$, we define

$$\Theta_{i,t} := C(y,t) \cap \left(y_t + \left[\Theta'_i, \mathbb{R}^+ y\right]\right).$$

If $\delta > 0$ is small enough to ensure that $\varrho > 0$ is positive and continuous in a neighbourhood (relative to K) of ∂K, then

$$\int_{\Theta_{i,t}} \varrho(x)\, dx \geq \gamma_2\, V(\Theta_{i,t}).$$

If $y_t \notin K_{(n)}$ and $o \in K_{(n)}$, then there exists a hyperplane H through y_t, bounding the half-spaces H^- and H^+, for which $K_{(n)} \subset H^-$. Moreover, there is some $i \in \{1, \ldots, 2^{d-1}\}$ such that $\Theta_{i,t} \subset H^+$. Therefore

$$\mathbb{P}_\varrho\left(y_t \notin K_{(n)}, o \in K_{(n)}\right) \ll \sum_{i=1}^{2^{d-1}} \left(1 - \gamma_2 V(\Theta_{i,t})\right)^n. \tag{4.2.2}$$

Finally, we prove

$$V(\Theta_{i,t}) \gg r(y)^{\frac{d-1}{2}} t^{\frac{d+1}{2}}, \tag{4.2.3}$$

for $i = 1, \ldots, 2^{d-1}$. According to (4.2.1), there exist positive constants γ_3, γ_4 with $\gamma_3 \leq 1$ such that if $t \leq \gamma_3 r(y)$, then $(y_t + \Theta'_i) \cap K$ contains a $(d-1)$-ball of radius at least

$$\gamma_4 \sqrt{r(y)^2 - (r(y)-t)^2} \geq \gamma_4 \sqrt{r(y) t},$$

and we are done. On the other hand, if $t \geq \gamma_3 r(y)$, then

$$V(\Theta_{i,t}) \gg t^d \gg r(y)^{\frac{d-1}{2}} t^{\frac{d+1}{2}}.$$

To deal with the case $o \notin K_{(n)}$, we observe that there exists a positive constant $\gamma_5 \in (0,1)$ such that the probability measure of each of the 2^d coordinate corners of $\mathbb{R}^d$ is at least γ_5. If $o \notin K_{(n)}$, then $\{x_1, \dots, x_n\}$ is disjoint from one of these coordinate corners, and hence

$$\mathbb{P}_\varrho(o \notin K_{(n)}) \leq 2^d(1-\gamma_5)^n. \tag{4.2.4}$$

Now the assertion follows from (4.2.2), (4.2.3) and (4.2.4). □

Subsequently, the estimate of Lemma 4.2.1 will be used, for instance, to restrict the domain of integration on the right-hand side of (4.1.3) (cf. Lemma 4.2.3) and to justify an application of Lebesgue's dominated convergence theorem (see (4.2.9)). For these applications, we also need that if $c > 0$ is such that $\omega := c\,\delta^{(d+1)/2} < 1$, then

$$\int_0^\delta \left(1 - c\,t^{\frac{d+1}{2}}\right)^n dt = \frac{2}{d+1}\,c^{\frac{-2}{d+1}} \int_0^\omega s^{\frac{2}{d+1}-1}(1-s)^n\,ds \ll c^{\frac{-2}{d+1}} \cdot n^{\frac{-2}{d+1}}, \tag{4.2.5}$$

where we use that $(1-s)^n \leq e^{-ns}$ for $s \in [0,1]$ and $n \in \mathbb{N}$.

The next lemma will allow us to decompose integrals in a suitable way.

Lemma 4.2.2. *If* $0 \leq t_0 \leq t_1 < \delta$ *and* $h : K \to [0,\infty]$ *is a measurable function, then*

$$\int_{K_{t_0} \setminus K_{t_1}} h(x)\,dx = \int_{\partial K} \int_{t_0}^{t_1} (1-t)^{d-1} \langle y, u(y)\rangle h(y_t)\,dt\,\mathcal{H}^{d-1}(dy).$$

Proof. The map $T : \partial K \times [t_0, t_1] \to K_{t_0} \setminus K_{t_1}$, $(y,t) \mapsto (1-t)y$, provides a bilipschitz parametrization of $K_{t_0} \setminus K_{t_1}$ with $(1-t)y = y_t \in \partial K_t$. The Jacobian of T, for $\mathcal{H}^{d-1}$-almost all $y \in \partial K$ and $t \in [t_0, t_1]$, is given by $JT(y,t) = (1-t)^{d-1}\langle y, u(y)\rangle$, where $u(y)$ is the ($\mathcal{H}^{d-1}$-almost everywhere) unique exterior unit normal of ∂K at y. The assertion now follows from Federer's area/coarea theorem (see [Fed69]). □

In the following, we will use the important fact that, for $\alpha > -1$,

$$\int_{\partial K} r(y)^\alpha\,\mathcal{H}^{d-1}(dy) < \infty, \tag{4.2.6}$$

which is a result due to C. Schütt and E. Werner [SW90].

By decomposing λ into its positive and its negative parts, we can henceforth assume that λ is a nonnegative, integrable function.

Lemma 4.2.3. *As* n *tends to infinity, it holds that*

$$\int_{K_{n^{-1/(d+1)}}} \mathbb{P}_\varrho\left(x \notin K_{(n)}\right)\lambda(x)\,dx = o\left(n^{\frac{-2}{d+1}}\right).$$

Proof. Let $\delta > 0$ be chosen as in Lemma 4.2.1 and the subsequent remark. First, we consider a point x in K_δ. Let ω be the minimal distance between the points of ∂K and K_δ, and let $z_1, \dots, z_k$ be a maximal family of points in $K \setminus \mathrm{int}(K_\delta)$ such that $\|z_i - z_j\| \geq \frac{\omega}{4}$ for $i \neq j$. We define $p_0 > 0$ by

$$p_0 := \min\left\{\mathbb{P}_\varrho\left(z_i + \tfrac{\omega}{4}\, B^d\right) : i = 1, \dots, k\right\}.$$

Let $x \in K_\delta$. If $x \notin K_{(n)}$, then there is some $u \in S^{d-1}$ such that $x \in H^+(u,t)$ and $K_{(n)} \subset \mathrm{int}(H^-(u,t))$. Since $x \in K_\delta$, we obtain that $K_{(n)} \subset \mathrm{int}(H^-(u, h(K_\delta, u)))$. If $z \in H(u, h(K_\delta, u)) \cap \partial K_\delta$, then

$$z + \frac{\omega}{2} u + \frac{\omega}{2} B^d \subset K \cap H^+(u, h(K_\delta, u)).$$

By the maximality of the set $\{z_1, \dots, z_k\}$, we have

$$\{z_1, \dots, z_k\} \cap \left(z + \frac{\omega}{2} u + \frac{\omega}{4} B^d\right) \neq \emptyset.$$

Let z_j lie in the intersection. Then $z_j + \frac{\omega}{4} B^d \subset H^+(u, h(K_\delta, u))$, and hence $x_i \notin z_j + \frac{\omega}{4} B^d$ for $i = 1, \dots, n$. This implies that, for $x \in K_\delta$,

$$\mathbb{P}_\varrho\left(x \notin K_{(n)}\right) \leq k(1 - p_0)^n. \tag{4.2.7}$$

Put $\varepsilon := (2(d^2-1))^{-1}$ and let $n \geq \delta^{-(d+1)}$. For $y \in \partial K$ we show that

$$\int_{n^{-1/(d+1)}}^{\delta} \mathbb{P}_\varrho\left(y_t \notin K_{(n)}\right)\, dt \ll r(y)^{-\frac{d}{d+1}} n^{\frac{-2}{d+1} - \varepsilon}. \tag{4.2.8}$$

In fact, if $r(y) \leq n^{-(d+1)\varepsilon}$, then Lemma 4.2.1 and (4.2.5) yield

$$\begin{aligned}
\int_{n^{-1/(d+1)}}^{\delta} \mathbb{P}_\varrho\left(y_t \notin K_{(n)}\right)\, dt &\leq \int_0^\delta \left(1 - \gamma_1 r(y)^{\frac{d-1}{2}} t^{\frac{d+1}{2}}\right)^n dt \\
&\ll r(y)^{-\frac{d-1}{d+1}} n^{-\frac{2}{d+1}} \\
&\leq r(y)^{-\frac{d}{d+1}} n^{-\frac{2}{d+1} - \varepsilon},
\end{aligned}$$

where the assumption on $r(y)$ is used for the last estimate.

If $r(y) \geq n^{-(d+1)\varepsilon}$ and $n \geq n_0$, where n_0 depends on K, ϱ and λ, then Lemma 4.2.1 implies for all $t \in (n^{-1/(d+1)}, \delta)$ that

$$\begin{aligned}
\mathbb{P}_\varrho\left(y_t \notin K_{(n)}\right) &\ll \left(1 - \gamma_1 n^{-\frac{d^2-1}{2}\varepsilon - \frac{1}{2}}\right)^n \\
&= (1 - \gamma_1 n^{-3/4})^n \\
&\leq e^{-\gamma_1 n^{1/4}} \\
&\leq r(K)^{-\frac{d}{d+1}} n^{\frac{-2}{d+1} - \varepsilon},
\end{aligned}$$

which again yields (4.2.8). In particular, writing I to denote the integral in Lemma 4.2.3, we obtain from Lemma 4.2.2, (4.2.7), (4.2.8) and (4.2.6) that

$$\begin{aligned} I &\ll \int_{K_\delta} \mathbb{P}_\varrho \left(x \notin K_{(n)} \right) \lambda(x)\, dx + \int_{\partial K} \int_{n^{-1/(d+1)}}^{\delta} \mathbb{P}_\varrho \left(y_t \notin K_{(n)} \right) \, dt \, \mathcal{H}^{d-1}(dy) \\ &\ll k(1-p_0)^n + \int_{\partial K} r(y)^{-\frac{d}{d+1}} n^{\frac{-2}{d+1}-\varepsilon} \, \mathcal{H}^{d-1}(dy) \ll n^{\frac{-2}{d+1}-\varepsilon}, \end{aligned}$$

where we also used the fact that λ is integrable on K and bounded on $K \setminus K_\delta$. This is the required estimate. $\square$

It follows from (4.1.3), Lemma 4.2.3 and Lemma 4.2.2 that

$$\begin{aligned} &\lim_{n\to\infty} n^{\frac{2}{d+1}} \, \mathbb{E}_\varrho \int_{K \setminus K_{(n)}} \lambda(x)\, dx \\ &\quad = \lim_{n\to\infty} n^{\frac{2}{d+1}} \int_K \mathbb{P}_\varrho \left(x \notin K_{(n)} \right) \lambda(x)\, dx \\ &\quad = \lim_{n\to\infty} \int_{\partial K} \int_0^{n^{-1/(d+1)}} n^{\frac{2}{d+1}} (1-t)^{d-1} \langle y, u(y) \rangle \mathbb{P}_\varrho \left(y_t \notin K_{(n)} \right) \lambda(y_t)\, dt \, \mathcal{H}^{d-1}(dy). \end{aligned}$$

Lemma 4.2.1 and (4.2.5) imply that, if $y \in \partial K$ and $r(y) > 0$, then

$$\int_0^{n^{-1/(d+1)}} n^{\frac{2}{d+1}} \mathbb{P}_\varrho \left(y_t \notin K_{(n)} \right) \langle y, u(y) \rangle \lambda(y_t)\, dt \ll r(y)^{-\frac{d-1}{d+1}}.$$

Therefore, by (4.2.6) and since λ is bounded and continuous in a neighbourhood of ∂K, we may apply Lebesgue's dominated convergence theorem, and thus

$$\lim_{n\to\infty} n^{\frac{2}{d+1}} \mathbb{E}_\varrho \int_{K \setminus K_{(n)}} \lambda(x)\, dx = \int_{\partial K} \lambda(y) J_\varrho(y) \, \mathcal{H}^{d-1}(dy), \tag{4.2.9}$$

where

$$J_\varrho(y) := \lim_{n\to\infty} \int_0^{n^{-1/(d+1)}} n^{\frac{2}{d+1}} \langle y, u(y) \rangle \mathbb{P}_\varrho \left(y_t \notin K_{(n)} \right) \, dt,$$

for $\mathcal{H}^{d-1}$-almost all $y \in \partial K$.

Lemma 4.2.4. *If $y \in \partial K$ is a normal boundary point of K with $\kappa(y) = 0$, then $J_\varrho(y) = 0$.*

Proof. In view of the estimate (4.2.4), it is sufficient to prove that for any given $\varepsilon > 0$,

$$\int_0^{n^{-1/(d+1)}} n^{\frac{2}{d+1}} \mathbb{P}_\varrho \left(y_t \notin K_{(n)}, o \in K_{(n)} \right) \, dt \ll \varepsilon, \tag{4.2.10}$$

if n is sufficiently large. We choose the coordinate axes in $u(y)^\perp$ parallel to the principal curvature directions of K at y, and denote by $\Theta'_1, \ldots, \Theta'_{2^{d-1}}$ the corresponding coordinate corners. For $i = 1, \ldots, 2^{d-1}$ and $t \in (0, n^{-1/(d+1)})$, let

$$\Theta_{i,t} := C(y,t) \cap \left(y_t + \left[\Theta'_i, \mathbb{R}^+ y \right] \right),$$

and hence, if n is large enough, then

$$\int_{\Theta_{i,t}} \varrho(x)\,dx \gg V(\Theta_{i,t}),$$

since ϱ is continuous and positive near ∂K. If $y_t \notin K_n$ and $o \in K_{(n)}$, then there exists a half-space H^- which contains $K_{(n)}$ and for which $y_t \in \partial H^-$. Moreover, for some $i \in \{1, \ldots, 2^{d-1}\}$ the interior of H^- is disjoint from $\Theta_{i,t}$. Hence, as in the proof of Lemma 4.2.1, it holds that

$$\mathbb{P}_\varrho\left(y_t \notin K_{(n)}, o \in K_{(n)}\right) \ll \sum_{i=1}^{2^{d-1}} \left(1 - \gamma_6 V(\Theta_{i,t})\right)^n. \tag{4.2.11}$$

Since ∂K is twice differentiable in the generalized sense at y, we have $r(y) > 0$. By assumption, $\kappa(y) = 0$, therefore one principal curvature at y is zero, and hence less than $\varepsilon^{d+1} r(y)^{d-2}$. In particular, there exists $\delta' \in (0, \delta)$, which by (4.2.1) depends only on y and ε, such that, if $i \in \{1, \ldots, 2^{d-1}\}$ and $t \in (0, \delta')$, then

$$\mathcal{H}^{d-1}\left((y_t + \Theta_i') \cap K\right) \gg \sqrt{t\varepsilon^{-(d+1)} r(y)^{-(d-2)}} \cdot \sqrt{tr(y)}^{\,d-2},$$

and thus $V(\Theta_{i,t}) \gg \varepsilon^{-(d+1)/2} t^{(d+1)/2}$. Therefore (4.2.10) follows from (4.2.5) and (4.2.11). □

Next we consider the case of a normal boundary point $y \in \partial K$ with $\kappa(y) > 0$. First, we prove that $J_\varrho(y)$ depends only on the random points near y (see Lemma 4.2.5). In a second step, we compare the simplified expression obtained for $J_\varrho(y)$ with the corresponding expression which is obtained if K is a ball.

We start by reparametrizing y_t in terms of the probability measure of the corresponding cap. For $t \in (0, n^{-1/(d+1)})$, where $n \geq n_0$ is sufficiently large so that ϱ is positive and continuous on $C(y,t)$, for all $y \in \partial K$, we put

$$\tilde{y}_s := y_t,$$

where, for given $s > 0$ (sufficiently small), the corresponding $t = t(s)$ is determined by the relation

$$s = \int_{C(y,t)} \varrho(x)\,dx. \tag{4.2.12}$$

It is easy to see that the right-hand side of (4.2.12) is a continuous and strictly increasing function $s = s(t)$ of t, if $t > 0$ is sufficiently small. This implies that, for a given $s > 0$ (sufficiently small), there is a unique $t(s)$ such that (4.2.12) is satisfied.

Moreover, observe that

$$\frac{ds}{dt} = \langle u(y), y\rangle \int_{H(y,t)\cap K} \varrho(x)\,\mathcal{H}^{d-1}(dx) \tag{4.2.13}$$

for $t \in (0, n^{-1/(d+1)})$. We further define

$$\widetilde{C}(y,s) := C(y,t) \qquad \text{and} \qquad \widetilde{H}(y,s) := \{x \in \mathbb{R}^d : \langle u(y), x\rangle = \langle u(y), \tilde{y}_s\rangle\},$$

where $t = t(s)$.

Let Q denote the second fundamental form of ∂K at y (cf. (2.2.1)), considered as a function on $u(y)^\perp$. We define

$$E := \{z \in u(y)^\perp : Q(z) \le 1\}$$

and put $u := u(y)$. Choosing a suitable orthonormal basis $v_1, \dots, v_{d-1}$ of $u(y)^\perp$, we have

$$Q(z) = \sum_{i=1}^{d-1} k_i(y) z_i^2,$$

where $k_i(y)$, $i = 1, \dots, d-1$, are the generalized principal curvatures of K at y and where $z = z_1 v_1 + \ldots + z_{d-1} v_{d-1}$. Since y is a normal boundary point of K, there is a nondecreasing function $\mu : (0, \infty) \to \mathbb{R}$ with $\lim_{r \to 0^+} \mu(r) = 1$ such that

$$\frac{\mu(r)^{-1}}{\sqrt{2r}}(K(u,r) + ru - y) \subset E \subset \frac{\mu(r)}{\sqrt{2r}}(K(u,r) + ru - y), \tag{4.2.14}$$

where $K(u,r) := K \cap H(u, h(K,u) - r)$. In the following, $\mu_i : (0, \infty) \to \mathbb{R}$, $i = 1, 2, \ldots$, always denote nondecreasing functions with $\lim_{r \to 0^+} \mu(r) = 1$. Applying (4.2.14) and Fubini's theorem, we get

$$V(K \cap H^+(u, h(K,u) - r)) = \mu_1(r) \frac{(2r)^{\frac{d+1}{2}}}{d+1} \alpha_{d-1} \kappa(y)^{-\frac{1}{2}},$$

which yields

$$s(t) = \mu_2(t) \frac{(2t\langle y, u\rangle)^{\frac{d+1}{2}}}{d+1} \alpha_{d-1} \kappa(y)^{-\frac{1}{2}} \varrho(y), \tag{4.2.15}$$

since ϱ is continuous at y. Moreover, defining

$$\eta := (d+1)^{\frac{1}{d+1}} \alpha_{d-1}^{-\frac{1}{d+1}} \varrho(y)^{\frac{-1}{d+1}} \kappa(y)^{\frac{1}{2(d+1)}},$$

we obtain

$$\lim_{s \to 0^+} s^{\frac{-1}{d+1}} [(\widetilde{H}(y,s) \cap K) - \tilde{y}_s] = \eta \cdot E \tag{4.2.16}$$

in the sense of the Hausdorff metric on compact convex sets (see Schneider [Sch14] or Gruber [Gru07]). Here we also use the fact that

$$\lim_{s \to 0^+} s^{-\frac{1}{d+1}} (\tilde{y}_s - \langle \tilde{y}_s, u\rangle u) = o. \tag{4.2.17}$$

Now it follows from (4.2.13) and (4.2.16) that (4.2.9) turns into

$$J_\varrho(y) = (d+1)^{-\frac{d-1}{d+1}} \alpha_{d-1}^{-\frac{2}{d+1}} \varrho(y)^{\frac{-2}{d+1}} \kappa(y)^{\frac{1}{d+1}} \lim_{n \to \infty} \int_0^{g(n,y)} n^{\frac{2}{d+1}} \mathbb{P}_\varrho \left(\tilde{y}_s \notin K_{(n)}\right) s^{-\frac{d-1}{d+1}} \, ds,$$

where

$$\lim_{n \to \infty} n^{\frac{1}{2}} g(n,y) = (d+1)^{-1} \alpha_{d-1} \varrho(y) (2\langle u(y), y\rangle)^{\frac{d+1}{2}} \kappa(y)^{-\frac{1}{2}}.$$

The rest of the proof is devoted to identifying the asymptotic behaviour of the integral. First, we adjust the domain of integration and the integrand in a suitable way. In a second step, the resulting expression is compared to the case where K is the unit ball. We recall that $x_1,\dots,x_n$ are random points in K, and we put $\Xi_n := \{x_1,\dots,x_n\}$, and hence $K_{(n)} = [\Xi_n]$. Let $\#X$ denote the cardinality of a finite set $X \subset \mathbb{R}^d$.

Lemma 4.2.5. *For $\varepsilon \in (0,1)$, there exist $\alpha,\beta > 1$ and an integer $k > 1$, depending only on ε and d, with the following property. If $y \in \partial K$ is a normal boundary point of K with $\kappa(y) > 0$ and if $n > n_0$, where n_0 depends on $\varepsilon, y, K, \varrho$, then*

$$\int_0^{g(n,y)} \mathbb{P}_\varrho\left(\tilde{y}_s \notin K_{(n)}\right) s^{-\frac{d-1}{d+1}}\, ds = \int_{\varepsilon^{(d+1)/2}/n}^{\alpha/n} \varphi(K,y,\varrho,\varepsilon,s) s^{-\frac{d-1}{d+1}}\, ds + O\left(\frac{\varepsilon}{n^{\frac{2}{d+1}}}\right),$$

where

$$\varphi(K,y,\varrho,\varepsilon,s) = \mathbb{P}_\varrho\left(\left(\tilde{y}_s \notin [\widetilde{C}(y,\beta s)\cap \Xi_n]\right) \text{and } \left(\#(\widetilde{C}(y,\beta s)\cap \Xi_n) \le k\right)\right).$$

Proof. Let Q be the second fundamental form of ∂K at the normal boundary point y, and let $v_1,\dots,v_{d-1}$ be an orthonormal basis of $u(y)^\perp$ with respect to Q, as described above. Let $\Theta'_1,\dots,\Theta'_{2^{d-1}}$ be the corresponding coordinate corners, and, for $i = 1,\dots,2^{d-1}$ and for $s \in (0, n^{-1/2})$, put

$$\widetilde{\Theta}_{i,s} := \widetilde{C}(y,s) \cap \left(\tilde{y}_s + \left[\Theta'_i, \mathbb{R}^+ y\right]\right).$$

Let A_s, $s > 0$, be the affine map of $\mathbb{R}^d$ with $A_s(y) = y$ for which the associated linear map $\widetilde{A}_s$ is determined by $\widetilde{A}_s(v) = s^{\frac{1}{d+1}} v$, for $v \in u^\perp$, and $\widetilde{A}_s(u) = s^{\frac{2}{d+1}} u$. Then $\det(\widetilde{A}_s) = s$ and $A_{s^{-1}}(\widetilde{C}(y,s))$ converges in the Hausdorff metric, as $s \to 0^+$, to the cap $\widetilde{C}(y)$ of the osculating paraboloid of K at y having volume $\varrho(y)^{-1}$. Here we use the assumptions that ϱ is continuous at y, $\varrho(y) > 0$ and relation (4.2.12). Let $\lambda > 0$ be such that $\tilde{y} := y - \lambda u \in \partial\widetilde{C}(y)$. Then $A_{s^{-1}}(\widetilde{\Theta}_{i,s})$ converges in the Hausdorff metric, as $s \to 0^+$, to $\widetilde{C}(y) \cap (\tilde{y} + [\Theta'_i, \mathbb{R}^+ u])$, since (4.2.17) is satisfied. Since ϱ is continuous and positive at y, we thus get

$$\begin{aligned}
\lim_{s\to 0^+} s^{-1} \int_{\widetilde{\Theta}_{i,s}} \varrho(x)\, dx &= \lim_{s\to 0^+} s^{-1} V(\widetilde{\Theta}_{i,s})\varrho(y) \\
&= \lim_{s\to 0^+} V(A_{s^{-1}}(\widetilde{\Theta}_{i,s}))\varrho(y) \\
&= V(\widetilde{C}(y) \cap (\tilde{y} + [\Theta'_i, \mathbb{R}^+ u]))\varrho(y) \\
&= 2^{-(d-1)} V(\widetilde{C}(y))\varrho(y) \\
&= 2^{-(d-1)} \lim_{s\to 0^+} V(A_{s^{-1}}(\widetilde{C}(y,s))\varrho(y) \\
&= 2^{-(d-1)} \lim_{s\to 0^+} s^{-1} V(\widetilde{C}(y,s))\varrho(y) \\
&= 2^{-(d-1)} \lim_{s\to 0^+} s^{-1} \int_{\widetilde{C}(y,s)} \varrho(x)\, dx \\
&= 2^{-(d-1)},
\end{aligned}$$

that is,

$$\lim_{s\to 0^+} s^{-1}\int_{\widetilde{\Theta}_{i,s}} \varrho(x)\,dx = 2^{-(d-1)}. \tag{4.2.18}$$

Let $\alpha > 1$ be chosen such that

$$2^{d-1+2d/(d+1)}\int_{2^{-d}\alpha}^{\infty} e^{-x}x^{\frac{2}{d+1}-1}\,dx \le \varepsilon.$$

Then we first choose $\beta \ge (16(d-1))^{d+1}$ such that

$$2^{d-1}e^{-d^{-1}2^{-(d+3)}\beta^{\frac{1}{d+1}}\varepsilon^{\frac{d+1}{2}}} \le \frac{\varepsilon}{\alpha^{\frac{2}{d+1}}},$$

and then we fix an integer $k > 1$ such that

$$\frac{(\alpha\beta)^k}{k!} \le \frac{\varepsilon}{\alpha^{\frac{2}{d+1}}}.$$

Lemma 4.2.5 follows from the following three statements, which we will prove assuming that n is sufficiently large.

(i)

$$\int_0^{g(n,y)} \mathbb{P}_\varrho\left(\tilde{y}_s \notin K_{(n)}\right) s^{-\frac{d-1}{d+1}}\,ds = \int_{\varepsilon^{(d+1)/2}/n}^{\alpha/n} \mathbb{P}_\varrho\left(\tilde{y}_s \notin K_{(n)}\right) s^{-\frac{d-1}{d+1}}\,ds + O\left(\frac{\varepsilon}{n^{\frac{2}{d+1}}}\right).$$

(ii) If $\varepsilon^{(d+1)/2}/n < s < \alpha/n$, then

$$\mathbb{P}_\varrho\left(\#\left(\widetilde{C}(y,\beta s)\cap \Xi_n\right) \ge k\right) = O\left(\frac{\varepsilon}{\alpha^{\frac{2}{d+1}}}\right).$$

(iii) If $\varepsilon^{(d+1)/2}/n < s < \alpha/n$, then

$$\mathbb{P}_\varrho\left(\tilde{y}_s \notin K_{(n)}\right) = \mathbb{P}_\varrho\left(\tilde{y}_s \notin \left[\widetilde{C}(y,\beta s)\cap \Xi_n\right]\right) + O\left(\frac{\varepsilon}{\alpha^{\frac{2}{d+1}}}\right).$$

To prove (i), we first observe that

$$\int_0^{\varepsilon^{(d+1)/2}/n} \mathbb{P}_\varrho\left(\tilde{y}_s \notin K_{(n)}\right) s^{-\frac{d-1}{d+1}}\,ds \le \int_0^{\varepsilon^{(d+1)/2}/n} s^{-\frac{d-1}{d+1}}\,ds \ll \frac{\varepsilon}{n^{\frac{2}{d+1}}}.$$

If $\alpha/n < s < g(n,y)$, $o \in K_{(n)}$, $\tilde{y}_s \notin K_{(n)}$ and if n is sufficiently large, then there is some $i \in \{1,\dots,2^{d-1}\}$ such that $\widetilde{\Theta}_{i,s}\cap K_{(n)} = \emptyset$, and hence (4.2.4) and (4.2.18) yield

$$\mathbb{P}_\varrho\left(\tilde{y}_s \notin K_{(n)}\right) \ll 2^{d-1}(1-2^{-d}s)^n \le 2^{d-1}e^{-2^{-d}ns}. \tag{4.2.19}$$

Therefore, by the definition of α, we get

$$\begin{aligned}\int_{\alpha/n}^{g(n,y)} \mathbb{P}_\varrho\left(\tilde{y}_s \notin K_{(n)}\right) s^{-\frac{d-1}{d+1}}\, ds &\ll 2^{d-1}\int_{\alpha/n}^{\infty} e^{-2^{-d}ns} s^{\frac{2}{d+1}-1}\, ds\\ &= 2^{d-1}2^{2d/(d+1)} n^{-\frac{2}{d+1}} \int_{2^{-d}\alpha}^{\infty} e^{-x} x^{\frac{2}{d+1}-1}\, dx\\ &\le \varepsilon\, n^{-\frac{2}{d+1}},\end{aligned}$$

which verifies (i).

Next (ii) simply follows from (4.2.12) as, if $s < \alpha/n$, then

$$\mathbb{P}_\varrho\left(\#\left(\widetilde{C}(y,\beta s)\cap \Xi_n\right) \ge k\right) = \binom{n}{k}(\beta s)^k \le \binom{n}{k}\left(\frac{\alpha\beta}{n}\right)^k < \frac{(\alpha\beta)^k}{k!} \le \frac{\varepsilon}{\alpha^{\frac{2}{d+1}}}.$$

Now we prove (iii). To this end, for s in the given range, our plan is to construct sets $\widetilde{\Omega}_{1,s},\dots,\widetilde{\Omega}_{2^{d-1},s} \subset K$ such that

$$\int_{\widetilde{\Omega}_{i,s}} \varrho(x)\, dx \ge d^{-1}2^{-(d+3)}\beta^{\frac{1}{d+1}} s, \qquad \text{for } \ i = 1,\dots,2^{d-1}, \tag{4.2.20}$$

and, if $\tilde{y}_s \in K_{(n)}$ but $\tilde{y}_s \notin \left[\widetilde{C}(y,\beta s)\cap\Xi_n\right]$, then $\Xi_n \cap \widetilde{\Omega}_{i,s} = \emptyset$ for some $i \in \{1,\dots,2^{d-1}\}$.

For $i = 1,\dots,2^{d-1}$, let $w_i \in \Theta_i'$ be the vector whose coordinates (up to sign) in the basis $v_1,\dots,v_{d-1}$ are

$$w_i := \left(\sqrt{\beta}s\right)^{\frac{1}{d+1}} \frac{\eta}{2\sqrt{d-1}}\left(\pm\frac{1}{\sqrt{k_1(y)}},\dots,\pm\frac{1}{\sqrt{k_{d-1}(y)}}\right).$$

Further, for $i = 1,\dots,2^{d-1}$ we define

$$\widetilde{\Omega}_{i,s} = [\tilde{y}_{\sqrt{\beta}\, s} + w_i, K\cap(\tilde{y}_s + \Theta_i')].$$

Then, if $s > 0$ is small enough, we have $\tilde{y}_{\sqrt{\beta}\, s} + w_i \in K$, and hence $\widetilde{\Omega}_{i,s} \subset K$. Here we use the fact that

$$w_i \in (\sqrt{\beta}s)^{\frac{1}{d+1}}\frac{1}{2}\eta E$$

and therefore, by (4.2.16),

$$\tilde{y}_{\sqrt{\beta}s} + w_i \in \widetilde{H}(y,\sqrt{\beta}s)\cap K \subset K.$$

Recall that $\tilde{y}_s = (1-t)y$, where s and t are related by (4.2.15). Hence, if $s,t > 0$ are sufficiently small,

$$\langle u(y), \tilde{y}_s - \tilde{y}_{\sqrt{\beta}\, s}\rangle > \frac{\beta^{\frac{1}{d+1}}-1}{2}\langle u(y), y-\tilde{y}_s\rangle > \frac{\beta^{\frac{1}{d+1}}}{4}\langle u(y), y-\tilde{y}_s\rangle, \tag{4.2.21}$$

since $\beta \ge 2^{d+1}$; moreover,

$$\langle u(y), y-\tilde{y}_s\rangle\cdot\mathcal{H}^{d-1}\left(K\cap(\tilde{y}_s+\Theta_i')\right) \ge V(\widetilde{\Theta}_{i,s}). \tag{4.2.22}$$

Combining (4.2.21), (4.2.22), and (4.2.18) together with the continuity of ϱ at y with $\varrho(y) > 0$, we get

$$\begin{aligned}\int_{\widetilde{\Omega}_{i,s}} \varrho(x)\,dx &\geq \frac{1}{\sqrt{2}}\frac{1}{d}\varrho(y)\langle u(y), \tilde{y}_s - \tilde{y}_{\sqrt{\beta}\,s}\rangle \mathcal{H}^{d-1}\left(K \cap (\tilde{y}_s + \Theta'_i)\right) \\ &\geq \frac{\beta^{\frac{1}{d+1}}}{4}\frac{1}{\sqrt{2}d}V(\widetilde{\Theta}_{i,s}) \\ &\geq \frac{\beta^{\frac{1}{d+1}}}{4}\frac{1}{2d}\int_{\widetilde{\Theta}_{i,s}} \varrho(x)\,dx \\ &\geq \frac{\beta^{\frac{1}{d+1}}s}{8d\,2^d},\end{aligned}$$

which proves (4.2.20).

It is still left to prove that, if $\tilde{y}_s \in K_{(n)}$ but $\tilde{y}_s \notin \left[\widetilde{C}(y,\beta s) \cap \Xi_n\right]$, then $\Xi_n \cap \widetilde{\Omega}_{i,s} = \emptyset$ for some $i \in \{1, \ldots, 2^{d-1}\}$. So we assume that $\tilde{y}_s \in K_{(n)}$ but $\tilde{y}_s \notin \left[\widetilde{C}(y,\beta s) \cap \Xi_n\right]$. Then there exist $a \in \left[\widetilde{C}(y,\beta s) \cap \Xi_n\right]$ and $b \in K_{(n)} \setminus \widetilde{C}(y,\beta s)$ such that $\tilde{y}_s \in [a,b]$, and hence there exists a hyperplane H containing $\tilde{y}_s$ and bounding the half-spaces H^+ and H^- such that $\widetilde{C}(y,\beta s) \cap \Xi_n \subset \mathrm{int}(H^+)$ and $b \in \mathrm{int}(H^-)$.

Next we show that there exists $q \in [\tilde{y}_s, b]$ such that

$$q \in H^- \cap \left(\tilde{y}_{\sqrt{\beta}\,s} + \frac{\eta}{2\sqrt{d-1}}(\sqrt{\beta}s)^{\frac{1}{d+1}}E\right). \tag{4.2.23}$$

In fact, define $q := [\tilde{y}_s, b] \cap \widetilde{H}(y, \sqrt{\beta}s)$ and $q' := [\tilde{y}_s, b] \cap \widetilde{H}(y, \beta s)$. Since $a \in H^+$ and $\tilde{y}_s \in H$, it follows that $q \in H^-$. From (4.2.16) we get

$$\widetilde{H}(y,\beta s) \cap K \subset \tilde{y}_{\beta s} + 2\beta^{\frac{1}{d+1}}s^{\frac{1}{d+1}}\eta E. \tag{4.2.24}$$

By (4.2.15),

$$\begin{aligned}\langle u(y), \tilde{y}_s - \tilde{y}_{\beta\,s}\rangle &< \frac{\beta^{\frac{2}{d+1}}}{\beta^{\frac{2}{d+1}}-1}\cdot\frac{\beta^{\frac{2}{d+1}}-1}{\beta^{\frac{2}{d+1}}-\beta^{\frac{1}{d+1}}}\langle u(y), \tilde{y}_{\sqrt{\beta}\,s} - \tilde{y}_{\beta s}\rangle \\ &< \frac{\beta^{\frac{1}{d+1}}}{\beta^{\frac{1}{d+1}}-1}\langle u(y), \tilde{y}_{\sqrt{\beta}\,s} - \tilde{y}_{\beta s}\rangle.\end{aligned} \tag{4.2.25}$$

Furthermore, by elementary geometry

$$\frac{\|q - \tilde{y}_{\sqrt{\beta}s}\|}{\|q' - \tilde{y}_{\beta s}\|} = \frac{\langle u, \tilde{y}_s - \tilde{y}_{\sqrt{\beta}s}\rangle}{\langle u, \tilde{y}_s - \tilde{y}_{\beta s}\rangle}.$$

Then, by (4.2.24) and (4.2.25),

$$q \in \tilde{y}_{\sqrt{\beta}s} + \frac{\langle u, \tilde{y}_s - \tilde{y}_{\sqrt{\beta}s}\rangle}{\langle u, \tilde{y}_s - \tilde{y}_{\beta s}\rangle}\cdot 2(\beta s)^{\frac{1}{d+1}}\eta E$$

$$
\begin{aligned}
&\subset \tilde{y}_{\sqrt{\beta}s} + \left(1 - \frac{\langle u, \tilde{y}_{\sqrt{\beta}s} - \tilde{y}_{\beta s}\rangle}{\langle u, \tilde{y}_s - \tilde{y}_{\beta s}\rangle}\right) \cdot 2\beta^{\frac{1}{d+1}} s^{\frac{1}{d+1}} \eta E \\
&\subset \tilde{y}_{\sqrt{\beta}s} + 2s^{\frac{1}{d+1}} \eta E \\
&\subset \tilde{y}_{\sqrt{\beta}s} + \frac{1}{2\sqrt{d-1}} (\sqrt{\beta}s)^{\frac{1}{d+1}} \eta E,
\end{aligned}
$$

where $\beta \geq (16(d-1))^{d+1}$ is used for the last inclusion. Now there exists some $i \in \{1, \ldots, 2^{d-1}\}$ such that $\tilde{y}_s + \Theta'_i \subset H^-$, and hence $q + \Theta'_i \subset H^-$. By (4.2.23) this finally yields

$$\tilde{y}_{\sqrt{\beta}s} + w_i \subset q + \Theta'_i \subset H^-.$$

Therefore, we obtain that $\widetilde{\Omega}_{i,s} \cap \Xi_n = \emptyset$.

Finally, (iii) follows as, if $\varepsilon^{(d+1)/2}/n < s < \alpha/n$, then

$$
\begin{aligned}
0 \leq \mathbb{P}_\varrho &\left(\tilde{y}_s \notin \left[\widetilde{C}(y, \beta s) \cap \Xi_n\right]\right) - \mathbb{P}_\varrho \left(\tilde{y}_s \notin K_{(n)}\right) \\
&\leq \sum_{i=1}^{2^{d-1}} (1 - \int_{\widetilde{\Omega}_{i,s}} \varrho(x)\, dx)^n \\
&\leq \sum_{i=1}^{2^{d-1}} e^{-n \int_{\widetilde{\Omega}_{i,s}} \varrho(x)\, dx} \\
&\leq 2^{d-1} e^{-d^{-1} 2^{-(d+3)} \beta^{\frac{1}{d+1}} \varepsilon^{\frac{d+1}{2}}} \\
&\leq \varepsilon\, \alpha^{-\frac{2}{d+1}},
\end{aligned}
$$

by the choice of β. □

Remark. *As a consequence of the proof of Lemma 4.2.5, it follows that*

$$J_\varrho(y) = (d+1)^{-\frac{d-1}{d+1}} \alpha_{d-1}^{-\frac{2}{d+1}} \varrho(y)^{\frac{-2}{d+1}} \kappa(y)^{\frac{1}{d+1}} \lim_{n\to\infty} \int_0^{n^{-1/2}} n^{\frac{2}{d+1}} \mathbb{P}_\varrho \left(\tilde{y}_s \notin K_{(n)}\right) s^{-\frac{d-1}{d+1}}\, ds. \tag{4.2.26}$$

In fact, since $g(n, y) \ll n^{-1/2}$, it is sufficient to show that

$$\lim_{n\to\infty} n^{\frac{2}{d+1}} \int_{c_1 n^{-1/2}}^{c_2 n^{-1/2}} \mathbb{P}_\varrho \left(\tilde{y}_s \notin K_{(n)}\right) s^{-\frac{d-1}{d+1}}\, ds = 0$$

for any two constants $0 < c_1 \leq c_2 < \infty$. Since the estimate (4.2.19) can be applied, we get

$$
\begin{aligned}
n^{\frac{2}{d+1}} \int_{c_1 n^{-1/2}}^{c_2 n^{-1/2}} \mathbb{P}_\varrho \left(\tilde{y}_s \notin K_{(n)}\right) s^{-\frac{d-1}{d+1}}\, ds &\ll n^{\frac{2}{d+1}} \int_{c_1 n^{-1/2}}^{c_2 n^{-1/2}} e^{-2^{-d} n s} s^{\frac{2}{d+1}-1}\, ds \\
&\ll \int_{2^{-d} c_1 n^{1/2}}^{2^{-d} c_2 n^{1/2}} e^{-r} r^{\frac{2}{d+1}-1}\, dr,
\end{aligned}
$$

from which the conclusion follows.

Subsequently, we write $\mathbf{1}$ to denote the constant one function on $\mathbb{R}^d$. For the unit ball B^d, we recall that $B^d_{(n)}$ denotes the convex hull of n random points distributed uniformly and independently in B^d. We fix a point $w \in \partial B^d$, and for $s \in (0, \frac{1}{2})$, define $\tilde{w}_s := t \cdot w$, where $t \in (0,1)$ is chosen such that

$$s = \alpha_d^{-1} \cdot V(\{x \in B^d : \langle x, w\rangle \geq \langle \tilde{w}_s, w\rangle\}).$$

By a classical result due to J.A. Wieacker [Wie78],

$$\lim_{n\to\infty} n^{\frac{2}{d+1}} \mathbb{E}_{\mathbf{1},B^d} V(B^d \setminus B^d_{(n)}) = c_d\, \omega_d\, \alpha_d^{\frac{2}{d+1}},$$

where the constant c_d is given in (4.1.1). Hence, it follows from (4.2.9), (4.2.26) and the preceding remark that

$$\lim_{n\to\infty} \int_0^{n^{-1/2}} n^{\frac{2}{d+1}} \mathbb{P}_{\mathbf{1},B^d} \left(\tilde{w}_s \notin B^d_{(n)}\right) s^{-\frac{d-1}{d+1}}\, ds = c_d\, (d+1)^{\frac{d-1}{d+1}} \alpha_{d-1}^{\frac{2}{d+1}}. \tag{4.2.27}$$

We are now going to show that the same limit is obtained if B^d is replaced by the convex body K and if a normal boundary point y of K with positive Gauss curvature is considered instead of $w \in \partial B^d$.

Lemma 4.2.6. *If $y \in \partial K$ is a normal boundary point of K satisfying $\kappa(y) > 0$, then*

$$\lim_{n\to\infty} \int_0^{n^{-1/2}} n^{\frac{2}{d+1}} \mathbb{P}_\varrho \left(\tilde{y}_s \notin K_{(n)}\right) s^{-\frac{d-1}{d+1}}\, ds = c_d (d+1)^{\frac{d-1}{d+1}} \alpha_{d-1}^{\frac{2}{d+1}}.$$

Proof. Let $\varepsilon \in (0,1)$ be arbitrarily chosen. According to Lemma 4.2.5 and its notation and by the preceding remark, if n is sufficiently large, we have

$$\begin{aligned}\int_0^{n^{-1/2}} \mathbb{P}_\varrho \left(\tilde{y}_s \notin K_{(n)}\right) s^{-\frac{d-1}{d+1}}\, ds &= O\left(\frac{\varepsilon}{n^{\frac{2}{d+1}}}\right) + \sum_{i=0}^{k} \binom{n}{i} \int_{\varepsilon^{(d+1)/2}/n}^{\alpha/n} (\beta s)^i (1-\beta s)^{n-i} \\ &\quad \times \mathbb{P}_{\varrho, \widetilde{C}(y,\beta s)} \left(\tilde{y}_s \notin \widetilde{C}(y,\beta s)_{(i)}\right) s^{-\frac{d-1}{d+1}}\, ds. \end{aligned} \tag{4.2.28}$$

We fix a unit vector p, and consider the reference paraboloid Ψ which is the graph of $z \mapsto \|z\|^2$ on $p^\perp$. For $\tau > 0$, define

$$C(\tau) := \left\{ z + tp : z \in p^\perp \ \text{ and } \ \|z\|^2 \leq t \leq \tau^{\frac{2}{d+1}} \right\},$$

that is, a cap of Ψ of height $\tau^{\frac{2}{d+1}}$. It is easy to check that $V(C(\tau)) = \tau V(C(1))$. We define

$$\tilde{s}(\beta, s) := \frac{V(\widetilde{C}(y, \beta s))}{V(C(\beta))}.$$

Then (4.2.12) implies that

$$\tilde{s}(\beta, s) = \frac{\beta s}{\mu(\beta,s)\varrho(y)\beta V(C(1))} = \frac{s}{\mu(\beta,s)\varrho(y) V(C(1))},$$

where $\mu(\beta, s) \to 1$ as $s \to 0^+$. Let A_s, $s > 0$, denote the affinity of $\mathbb{R}^d$ with $A_s(y) = y$ for which the associated linear map $\tilde{A}_s$ satisfies $\tilde{A}_s(v) = s^{\frac{1}{d+1}} v$ for $v \in u^\perp$ and $\tilde{A}_s(u) = s^{\frac{2}{d+1}} u$. Then the image under $A_{s^{-1}}$ of a cap of K at y converges in the Hausdorff metric, as $s \to 0^+$, to a cap of the osculating paraboloid of K at y. For a more explicit statement, let A be a volume preserving affinity of $\mathbb{R}^d$ such that $A(y) = o$ and $A(y-u) = p$, which maps the osculating paraboloid of K at y to Ψ. Then $\Phi_{s,\beta} := A \circ A_{\tilde{s}(\beta,s)^{-1}}$ is an affinity satisfying

$$\Phi_{s,\beta}(y) = o, \quad \det(\Phi_{s,\beta}) = \tilde{s}(\beta, s)^{-1} = \frac{V(C(\beta))}{V(\widetilde{C}(y, \beta s))},$$

and, consequently, $\Phi_{s,\beta}(\widetilde{C}(y, \beta s)) \to C(\beta)$ in the Hausdorff metric as $s \to 0^+$. Moreover, we have

$$\lim_{s \to 0^+} \Phi_{s,\beta}(\tilde{y}_s) = \lim_{s \to 0^+} \Phi_{s,1}(\tilde{y}_s) = p,$$

since $\mu(\beta, s) \to 1$ and $\mu(1, s) \to 1$ as $s \to 0^+$, $\tilde{y}_s \in \partial\widetilde{C}(y, s)$ and $\Phi_{s,1}(\tilde{y}_s) \in \partial C(1)$, and by (4.2.17). Since ϱ is continuous at y, the properties of $\Phi_{s,\beta}$ imply that, for $i = 0, \dots, k$,

$$\lim_{s \to 0^+} \mathbb{P}_{\varrho, \widetilde{C}(y,\beta s)} \left(\tilde{y}_s \notin \widetilde{C}(y, \beta s)_{(i)} \right) = \mathbb{P}_{\mathbf{1}, C(\beta)} \left(p \notin C(\beta)_{(i)} \right). \tag{4.2.29}$$

From (4.2.28) and (4.2.29) we get

$$\begin{aligned} \int_0^{n^{-1/2}} \mathbb{P}_\varrho \left(\tilde{y}_s \notin K_{(n)} \right) s^{-\frac{d-1}{d+1}} \, ds \;&=\; O\left(\frac{\varepsilon}{n^{\frac{2}{d+1}}} \right) + \sum_{i=0}^k \binom{n}{i} \int_{\varepsilon^{(d+1)/2}/n}^{\alpha/n} (\beta s)^i (1 - \beta s)^{n-i} \\ &\qquad \times \mathbb{P}_{\mathbf{1}, C(\beta)} \left(p \notin C(\beta)_{(i)} \right) s^{-\frac{d-1}{d+1}} \, ds. \end{aligned}$$

The same formula is obtained for

$$\int_0^{n^{-1/2}} \mathbb{P}_{\mathbf{1}, B^d} \left(\tilde{w}_s \notin B^d_{(n)} \right) s^{-\frac{d-1}{d+1}} \, ds,$$

since $C(\beta)$ is independent of K. Since $\varepsilon \in (0, 1)$ was arbitrary,

$$\begin{aligned} &\lim_{n \to \infty} \int_0^{n^{-1/2}} n^{\frac{2}{d+1}} \mathbb{P}_\varrho \left(\tilde{y}_s \notin K_{(n)} \right) s^{-\frac{d-1}{d+1}} \, ds \\ &\qquad = \lim_{n \to \infty} \int_0^{n^{-1/2}} n^{\frac{2}{d+1}} \mathbb{P}_{\mathbf{1}, B^d} \left(\tilde{w}_s \notin B^d_{(n)} \right) s^{-\frac{d-1}{d+1}} \, ds. \end{aligned}$$

Now (4.2.27) yields Lemma 4.2.6. □

Proof of Theorem 4.1.1. Let $y \in \partial K$ be a normal boundary point of K. Combining Lemma 4.2.4, Lemma 4.2.6 and (4.2.26), we obtain

$$J_\varrho(y) = c_d \, \varrho(y)^{\frac{-2}{d+1}} \kappa(y)^{\frac{1}{d+1}}.$$

Therefore Theorem 4.1.1 is implied by (4.2.9). □

Chapter 5

Circumscribed random polytopes

This chapter of the dissertation is based on parts of the paper [BFH10] by K.J. Böröczky, F. Fodor, and D. Hug, *The mean width of random polytopes circumscribed around a convex body*, J. Lond. Math. Soc. (2) **81** (2010), no. 2, 499–523. (DOI 10.1112/jlms/jdp077)

5.1 The probability space and the main goal

In order to state our results on random circumscribed polyhedral sets we start with describing the simplest version of probability model we use. In fact, we prove our statements in a much more general setting but that requires a lengthier introduction which is left to Section 5.2.

Let $\mathcal{H}$ denote the space of hyperplanes in $\mathbb{R}^d$ endowed with their usual topology, see [SW08, Chapter 13.2]. Let $\mathcal{H}_K \subset \mathcal{H}$ be the subspace of $\mathcal{H}$ such that for each $H \in \mathcal{H}_K$, it holds that $H \cap K_1 \neq \emptyset$ and $H \cap \operatorname{int} K = \emptyset$, where K_1 is the radius 1 parallel domain of K. Let μ denote the unique rigid motion invariant Borel measure on $\mathcal{H}$ with the property that $\mu(\{H \in \mathcal{H} : H \cap M \neq \emptyset\}) = W(M)$ for every convex body M in $\mathbb{R}^d$. Here $W(M)$ is the mean width of M, see Section 2.2 for a definition. Let $\mu_K := (1/2)\mu \llcorner \mathcal{H}_K$, that is, the restriction of $(1/2)\mu$ to $\mathcal{H}_K$. Then μ_K is a probability measure on $\mathcal{H}_K$ because

$$\mu_K(\mathcal{H}_K) = \frac{1}{2}\mu(\mathcal{H}_K) = \frac{1}{2}(W(K + B^d) - W(K)) = \frac{1}{2}W(B^d) = 1,$$

Let $H_1, \dots, H_n$ be independent random hyperplanes in $\mathbb{R}^d$ selected according to the probability distribution μ_K. If for each $1 \leq i \leq n$, H_i^- is the closed half-space bounded by H_i that contains K, then the intersection

$$K^{(n)} := \bigcap_{i=1}^{n} H_i^-$$

is a random polyhedral set containing K. Note that $K^{(n)}$ can be unbounded with positive probability.

It is our aim to investigate the geometric properties of $K^{(n)}$. In particular, in Section 5.2 we determine an asymptotic formula for the expectation $\mathbb{E}W(K^{(n)} \cap K_1)$. We consider the intersection of $K^{(n)}$ with K_1 because $K^{(n)}$ is unbounded with positive probability. We note that the use of K_1 is only a convenience, its role is not essential in the sense that we could use any other convex body containing K in its interior and it would only affect some of the constants without changing the essence of the asymptotic behaviour.

Instead of $\mathbb{E}W(K^{(n)} \cap K_1)$, we could also consider the conditional expectation $\mathbb{E}_1 W(K^{(n)})$ of $W(K^{(n)})$ under the condition that $K^{(n)}$ is contained in K_1. However, it was proved by Böröczky and Schneider [BS10] that $\mathbb{E}W(K^{(n)} \cap K_1) = \mathbb{E}_1 W(K^{(n)}) + O(\gamma^n)$ with $\gamma \in (0,1)$, so there is no difference in the asymptotic behaviours of these two quantities, as $n \to \infty$.

The main asymptotic result concerning the expected difference of the mean widths of $K^{(n)}$ and K is the following theorem.

Theorem 5.1.1 (Böröczky, Fodor, Hug [BFH10, Theorem 2.1 on page 501]). *If K is a convex body in $\mathbb{R}^d$, then*

$$\lim_{n\to\infty} n^{\frac{2}{d+1}}\, \mathbb{E}(W(K^{(n)} \cap K_1) - W(K)) = 2\, c_d\, \omega_d^{-\frac{d-1}{d+1}} \int_{\partial K} \kappa(x)^{\frac{d}{d+1}}\, \mathcal{H}^{d-1}(dx),$$

where c_d is defined in (4.1.1).

Let $f_i(P)$, $i \in \{0, \dots, d-1\}$, denote the number of i-dimensional faces of a polyhedral set P. In the statement of the following theorem, $K^{(n)}$ could be replaced by the intersection of $K^{(n)}$ with a fixed polytope containing K in its interior without changing the right-hand side. Alternatively, instead of $\mathbb{E}(f_{d-1}(K^{(n)}))$ we could consider the conditional expectation of $f_{d-1}(K^{(n)})$ under the assumption that $K^{(n)}$ is contained in K_1.

Theorem 5.1.2 (Böröczky, Fodor, Hug [BFH10, Theorem 2.2 on page 502]). *If K is a convex body in $\mathbb{R}^d$, then*

$$\lim_{n\to\infty} n^{-\frac{d-1}{d+1}}\, \mathbb{E}(f_{d-1}(K^{(n)})) = c_d\, \omega_d^{-\frac{d-1}{d+1}} \int_{\partial K} \kappa(x)^{\frac{d}{d+1}}\, \mathcal{H}^{d-1}(dx),$$

where c_d is defined in (4.1.1).

Generalizations of Theorem 5.1.1, and also of Theorem 5.1.2 below, which hold under more general distributional assumptions, are provided in Section 5.2. There we also indicate the connection to the p-affine surface area of a convex body.

Both theorems will be deduced from a “dual” result on weighted volume approximation of convex bodies by inscribed random polytopes, namely from Theorems 4.1.1 and 4.1.2. The usefulness of duality in random or best approximation has previously been observed, e.g., in [Zie70], [Kal90], [GG97], [DW96].

5.2 Polarity and the proof of Theorem 5.1.1

In this section, we deduce Theorem 5.1.1 and Theorem 5.1.2 from Theorem 4.1.1 and Corollary 4.1.2, respectively. In order to obtain more general results, for not necessarily homogeneous or isotropic hyperplane distributions, we start with a description of the basic setting.

Let $K \subset \mathbb{R}^d$ be a convex body with $o \in \operatorname{int}(K)$, let $K^* := \{z \in \mathbb{R}^d : \langle x, z\rangle \leq 1 \text{ for all } x \in K\}$ denote the polar body of K, and put $K_1 := K + B^d$. Let $\mathcal{H}_K$ denote the set of all hyperplanes H in $\mathbb{R}^d$ for which $H \cap \operatorname{int}(K) = \emptyset$ and $H \cap K_1 \neq \emptyset$. The motion invariant locally finite measure μ on the space $A(d, d-1)$ of hyperplanes, which satisfies $\mu(\mathcal{H}_K) = 2$, is explicitly given by

$$\mu = 2 \int_{S^{d-1}} \int_0^\infty \mathbf{1}\{H(u,t) \in \cdot\}\, dt\, \sigma(du),$$

where σ is the rotation invariant probability measure on the unit sphere S^{d-1}. The model of a random polytope (random polyhedral set) described in the introduction is based on random hyperplanes with distribution $\mu_K := 2^{-1}(\mu \llcorner \mathcal{H}_K)$. More generally, we now consider random hyperplanes with distribution

$$\mu_q := \int_{S^{d-1}} \int_0^\infty \mathbf{1}\{H(u,t) \in \cdot\} q(t,u)\, dt\, \sigma(du), \tag{5.2.1}$$

where $q : [0,\infty) \times S^{d-1} \to [0,\infty)$ is a measurable function which is

(q1) concentrated on $D_K := \{(t,u) \in [0,\infty) \times S^{d-1} : h(K,u) \leq t \leq h(K_1,u)\}$,

(q2) positive and continuous in a neighbourhood of $\{(t,u) \in [0,\infty) \times S^{d-1} : t = h(K,u)\}$ relative to D_K,

(q3) and satisfies $\mu_q(\mathcal{H}_K) = 1$.

The intersection of n half-spaces H_i^- containing the origin o and bounded by n independent random hyperplanes H_i with distribution μ_q is denoted by $K^{(n)} := \bigcap_{i=1}^n H_i^-$. Probabilities and expectations with respect to μ_q are denoted by $\mathbb{P}_{\mu_q}$ and $\mathbb{E}_{\mu_q}$, respectively. The special example $q \equiv \mathbf{1}_{D_K}$ (q is the characteristic function of D_K) covers the situation discussed in the introduction.

In the following, beside the support function, we will also need the radial function $\rho(L, \cdot)$ of a convex body L with $o \in \operatorname{int}(L)$. Let F be a nonnegative measurable functional on convex polyhedral sets in $\mathbb{R}^d$. Using (5.2.1) and Fubini's theorem, we get

$$\begin{aligned}\mathbb{E}_{\mu_q}(F(K^{(n)})) &= \int_{A(d,d-1)^n} F\left(\bigcap_{i=1}^n H_i^-\right) \mu_q^{\otimes n}(d(H_1,\dots,H_n)) \\ &= \int_{(S^{d-1})^n} \int_{h(K,u_1)}^{h(K_1,u_1)} \dots \int_{h(K,u_n)}^{h(K_1,u_n)} F\left(\bigcap_{i=1}^n H_i^-(u_i,t_i)\right) \prod_{i=1}^n q(t_i,u_i) \\ &\qquad \times dt_n \dots dt_1\, \sigma^{\otimes n}(d(u_1,\dots,u_n)).\end{aligned}$$

For $t_1, \dots, t_n > 0$, we have

$$\bigcap_{i=1}^n H_i^-(u_i,t_i) = \left[{t_1}^{-1}u_1, \dots, {t_n}^{-1}u_n\right]^*.$$

Using the substitution $s_i = 1/t_i$, $\rho(L^*, u_i) = h(L, u_i)^{-1}$ for $L \in \mathcal{K}^n$ with $o \in \operatorname{int}(L)$, and polar coordinates, we obtain

$$\mathbb{E}_{\mu_q}(F(K^{(n)})) = \frac{1}{\omega_d^n} \int_{(K^* \setminus K_1^*)^n} F([x_1, \dots, x_n]^*) \prod_{i=1}^n \left(\tilde{q}(x_i) \|x_i\|^{-(d+1)} \right) d(x_1, \dots, x_n)$$

with $K_1^* := (K_1)^*$ and

$$\tilde{q}(x) := q\left(\frac{1}{\|x\|}, \frac{x}{\|x\|} \right), \qquad x \in K^* \setminus \{o\}.$$

The case $n = 1$ and $F \equiv 1$ yields

$$\frac{1}{\omega_d} \int_{K^* \setminus K_1^*} \tilde{q}(x) \|x\|^{-(d+1)} \, dx = 1,$$

hence

$$\varrho(x) := \begin{cases} {\omega_d}^{-1} \tilde{q}(x) \|x\|^{-(d+1)}, & x \in K^* \setminus K_1^*, \\ 0, & x \in K_1^*, \end{cases}$$

is a probability density with respect to $\mathcal{H}^d \llcorner K^*$ which is positive and continuous in a neighbourhood of ∂K^* relative to K^*. Thus,

$$\begin{aligned} \mathbb{E}_{\mu_q}(F(K^{(n)})) &= \int_{(K^*)^n} F([x_1, \dots, x_n]^*) \prod_{i=1}^n \varrho(x_i) \, d(x_1, \dots, x_n) \\ &= \mathbb{E}_{\varrho, K^*} \left(F((K_{(n)}^*)^*) \right), \end{aligned}$$

where $K_{(n)}^* := (K^*)_{(n)}$.

Proposition 5.2.1. *Let $K \subset \mathbb{R}^d$ be a convex body with $o \in \operatorname{int}(K)$, and let q and ϱ be defined as above. Then the random polyhedral sets $K^{(n)}$ and $(K_{(n)}^*)^*$ are equal in distribution.*

For a first application, let

$$F(P) := \mathbf{1}\{P \subset K_1\} \left(W(P) - W(K) \right),$$

for a polyhedral set $P \subset \mathbb{R}^d$, with the convention $0 \cdot \infty := 0$. For $x_1, \dots, x_n \in K^* \setminus K_1^*$, we have $K \subset [x_1, \dots, x_n]^*$ and, arguing as before,

$$\begin{aligned} F([x_1, \dots, x_n]^*) &= \mathbf{1}\{[x_1, \dots, x_n]^* \subset K_1\} \left(W([x_1, \dots, x_n]^*) - W(K) \right) \\ &= 2 \cdot \mathbf{1}\{[x_1, \dots, x_n]^* \subset K_1\} \int_{K^* \setminus [x_1, \dots, x_n]} \lambda(x) \, dx, \end{aligned}$$

where

$$\lambda(x) := \begin{cases} {\omega_d}^{-1} \|x\|^{-(d+1)}, & x \in K^* \setminus K_1^*, \\ 0, & x \in K_1^*. \end{cases}$$

Note that if $[x_1,\dots,x_n]^* \subset K_1$, then the set $[x_1,\dots,x_n]^*$ is bounded, hence $o \in \operatorname{int}([x_1,\dots,x_n])$, and therefore $K_1^* \subset [x_1,\dots,x_n]^{**} = [x_1,\dots,x_n]$.

As in [BS10], it can be shown that $\mathbb{P}_{\mu_q}(K^{(n)} \not\subset K_1) \ll \alpha^n$, for some $\alpha \in (0,1)$ depending on K and q. By Proposition 5.2.1, we also get

$$\mathbb{P}_{\varrho,K^*}\left((K^*_{(n)})^* \not\subset K_1\right) = \mathbb{P}_{\mu_q}\left(K^{(n)} \not\subset K_1\right) \ll \alpha^n.$$

Hence

$$\begin{aligned}
&\mathbb{E}_{\mu_q}\left(W(K^{(n)} \cap K_1) - W(K)\right)\\
&\quad = \mathbb{E}_{\mu_q}\left(\mathbf{1}\{K^{(n)} \subset K_1\}\left(W(K^{(n)}) - W(K)\right)\right) + O(\alpha^n)\\
&\quad = 2\cdot \mathbb{E}_{\varrho,K^*}\left(\mathbf{1}\{(K^*_{(n)})^* \subset K_1\}\int_{K^*\setminus K^*_{(n)}} \lambda(x)\,dx\right) + O(\alpha^n)\\
&\quad = 2\cdot \mathbb{E}_{\varrho,K^*}\left(\int_{K^*\setminus K^*_{(n)}} \lambda(x)\,dx\right) + O(\alpha^n),
\end{aligned}$$

where we used that λ is integrable. Therefore, by Theorem 4.1.1,

$$\begin{aligned}
&\lim_{n\to\infty} n^{\frac{2}{d+1}}\,\mathbb{E}_{\mu_q}(W(K^{(n)} \cap K_1) - W(K))\\
&\quad = 2\cdot \lim_{n\to\infty} n^{\frac{2}{d+1}}\,\mathbb{E}_{\varrho,K^*}\int_{K^*\setminus K^*_{(n)}} \lambda(x)\,dx\\
&\quad = 2\,c_d \int_{\partial K^*} \varrho(x)^{-\frac{2}{d+1}}\lambda(x)\kappa^*(x)^{\frac{1}{d+1}}\,\mathcal{H}^{d-1}(dx)\\
&\quad = 2\,c_d\,\omega_d^{-\frac{d-1}{d+1}} \int_{\partial K^*} \tilde{q}(x)^{-\frac{2}{d+1}}\|x\|^{-d+1}\kappa^*(x)^{\frac{1}{d+1}}\,\mathcal{H}^{d-1}(dx),
\end{aligned}$$

where κ^* denotes the generalized Gauss curvature of K^*. In the following, for $x \in \partial K$, let $\sigma_K(x)$ denote an exterior unit normal vector of K at x. It is unique for $\mathcal{H}^{d-1}$-almost all $x \in \partial K$.

Theorem 5.2.2 (Böröczky, Fodor, Hug [BFH10, Theorem 5.2 on page 517])**.** *Let $K \subset \mathbb{R}^d$ be a convex body with $o \in \operatorname{int}(K)$, and let $q : [0,\infty) \times S^{d-1} \to [0,\infty)$ be a measurable function satisfying (q1)–(q3). Then*

$$\begin{aligned}
&\lim_{n\to\infty} n^{\frac{2}{d+1}}\,\mathbb{E}_{\mu_q}(W(K^{(n)} \cap K_1) - W(K))\\
&\qquad = 2\,c_d\,\omega_d^{-\frac{d-1}{d+1}} \int_{\partial K} q(h(K,\sigma_K(x)),\sigma_K(x))^{-\frac{2}{d+1}}\kappa(x)^{\frac{d}{d+1}}\,\mathcal{H}^{d-1}(dx),
\end{aligned} \tag{5.2.2}$$

where c_d is defined in (4.1.1).

The proof is completed in Section 5.3 by providing Lemma 5.3.2.

Example. *Observe that if $q : \{(h(K,u),u) \in (0,\infty) \times S^{d-1} : u \in S^{d-1}\} \to [0,\infty)$ is positive and continuous, then q can be extended to $[0,\infty) \times S^{d-1}$ such that (q1)–(q3) are* *satisfied. For any such extension, the right-hand side of* (5.2.2) *remains unchanged. As an example, we may choose* q_1 *such that* $q_1(t,u) = t^{(d^2-1)/2}$ *for* $t = h(K,u)$ *and* $u \in S^{d-1}$*. Then the integral in* (5.2.2) *turns into*

$$\int_{\partial K} \frac{\kappa(x)^{\frac{d}{d+1}}}{\langle x, \sigma_K(x)\rangle^{d-1}} \mathcal{H}^{d-1}(dx) = \Omega_{d^2}(K),$$

where

$$\Omega_p(K) := \int_{\partial K} \frac{\kappa(x)^{\frac{p}{d+p}}}{\langle x, \sigma_K(x)\rangle^{\frac{(p-1)d}{d+p}}} \mathcal{H}^{d-1}(dx)$$

is the p-affine surface area of K (see [Lut96], [Hug96a], [Hug96b], [Lei98], [Wer07], [WY08], [LR99], [LR10]*). It has been shown that* $\Omega_{d^2}(K) = \Omega_1(K^*)$*; see* [Hug96b]*. Moreover, for a convex body* $L \subset \mathbb{R}^d$*, the equiaffine isoperimetric inequality states that*

$$\Omega_1(L) \le d\alpha_d^{\frac{2}{d+1}} V(L)^{\frac{d-1}{d+1}}$$

with equality if and only if L is an ellipsoid (cf. [Pet85], [Lut93a], [Lut96], [Hug96a], [Bör10]*). Thus we get*

$$\lim_{n\to\infty} n^{\frac{2}{d+1}} \mathbb{E}_{\mu_{q_1}}(W(K^{(n)} \cap K_1) - W(K)) \le 2dc_d\omega_d^{-\frac{d-1}{d+1}} \alpha_d^{\frac{2}{d+1}} V(K^*)^{\frac{d-1}{d+1}}$$

with equality if and only if K^* *is an ellipsoid, that is, if and only if K is an ellipsoid. This can be interpreted as saying that among all convex bodies for which the volume of the polar body is fixed, ellipsoids are worst approximated asymptotically by circumscribed random polytopes (with respect to the density* q_1*) in the sense of the mean width.*

For another application, we define

$$F(P) := f_{d-1}(P),$$

for a convex polyhedral set $P \subset \mathbb{R}^d$. It is well known that $f_0(P) = f_{d-1}(P^*)$ for a convex polytope $P \subset \mathbb{R}^d$ with $o \in \operatorname{int}(P)$. Thus, from Proposition 5.2.1 we get

$$\begin{aligned}
\mathbb{E}_{\mu_q}\left(f_{d-1}(K^{(n)})\right) &= \mathbb{E}_{\varrho,K^*}\left(f_{d-1}((K^*_{(n)})^*)\right) \\
&= \mathbb{E}_{\varrho,K^*}\left(\mathbf{1}\{(K^*_{(n)})^* \subset K_1\} f_{d-1}((K^*_{(n)})^*)\right) \\
&\qquad + \mathbb{E}_{\varrho,K^*}\left(\mathbf{1}\{(K^*_{(n)})^* \not\subset K_1\} f_{d-1}((K^*_{(n)})^*)\right) \\
&= \mathbb{E}_{\varrho,K^*}\left(\mathbf{1}\{(K^*_{(n)})^* \subset K_1\} f_0(K^*_{(n)})\right) + O(n\cdot\alpha^n) \\
&= \mathbb{E}_{\varrho,K^*}\left(f_0(K^*_{(n)})\right) + O(n\cdot\alpha^n),
\end{aligned}$$

where $\alpha \in (0,1)$ is a suitable constant.

The following Theorem 5.2.3 generalizes Theorem 5.1.1 in the same way as Theorem 5.2.2 extends Theorem 5.1.2.

Theorem 5.2.3 (Böröczky, Fodor, Hug [BFH10, Theorem 5.3 on page 518]). *Let $K \subset \mathbb{R}^d$ be a convex body with $o \in \operatorname{int}(K)$, and let $q : [0, \infty) \times S^{d-1} \to [0, \infty)$ be a measurable function satisfying (q1)–(q3). Then*

$$\lim_{n\to\infty} n^{-\frac{d-1}{d+1}}\, \mathbb{E}_{\mu_q}(f_{d-1}(K^{(n)}))$$
$$= c_d\, \omega_d^{-\frac{d-1}{d+1}} \int_{\partial K} q(h(K, \sigma_K(x)), \sigma_K(x))^{\frac{d-1}{d+1}} \kappa(x)^{\frac{d}{d+1}}\, \mathcal{H}^{d-1}(dx),$$

where c_d is defined in (4.1.1).

The proof follows by applying Corollary 4.1.2 and Lemma 5.3.2.

5.3 Polarity and an integral transformation

In this section, we establish the required integral transformation involving the generalized Gauss curvatures of a convex body and its polar body. The main difficulty of the proof is due to the fact that we do not make any smoothness assumptions on the convex bodies that are considered.

Let $L \subset \mathbb{R}^d$ be a convex body. If the support function h_L of L is differentiable at $u \neq o$, then the gradient $\nabla h_L(u)$ of h_L at u is equal to the unique boundary point of L having u as an exterior normal vector. In particular, the gradient of h_L is a function which is homogeneous of degree zero. Note that h_L is differentiable at $\mathcal{H}^{d-1}$-almost all unit vectors. We write $D_{d-1}h_L(u)$ for the product of the principal radii of curvature of L in direction $u \in S^{d-1}$, whenever the support function h_L is twice differentiable in the generalized sense at $u \in S^{d-1}$. Note that this is the case for $\mathcal{H}^{d-1}$-almost all $u \in S^{d-1}$. The Gauss map σ_L is defined $\mathcal{H}^{d-1}$-almost everywhere on ∂L. If σ_L is differentiable in the generalized sense at $x \in \partial L$, which is the case for $\mathcal{H}^{d-1}$-almost all $x \in \partial L$, then the product of the eigenvalues of the differential is the Gauss curvature $\kappa_L(x)$. The connection to curvatures defined on the generalized normal bundle $\mathcal{N}(L)$ of L will be used in the following proof (cf. [Hug98]).

Lemma 5.3.1. *Let $L \subset \mathbb{R}^d$ be a convex body containing the origin in its interior. If $g : \partial L \to [0, \infty]$ is measurable, then*

$$\int_{\partial L} g(x)\kappa_L(x)^{\frac{1}{d+1}}\, \mathcal{H}^{d-1}(dx) = \int_{S^{d-1}} g(\nabla h_L(u)) D_{d-1}h_L(u)^{\frac{d}{d+1}}\, \mathcal{H}^{d-1}(du).$$

Proof. In the following proof, we use results and methods from [Hug98], to which we refer for additional references and detailed definitions. Let $\mathcal{N}(L)$ denote the generalized normal bundle of L, and let $k_i(x, u) \in [0, \infty]$, $i = 1, \dots, d-1$, be the generalized curvatures of L, which are defined for $\mathcal{H}^{d-1}$-almost all $(x, u) \in \mathcal{N}(L)$. Expressions such as

$$\frac{k_i(x,u)^{\frac{1}{d+1}}}{\sqrt{1 + k_i(x,u)^2}} \qquad \text{or} \qquad \frac{k_i(x,u)}{\sqrt{1 + k_i(x,u)^2}}$$

with $k_i(x, u) = \infty$ are understood as limits as $k_i(x, u) \to \infty$, and yield 0 or 1, respectively in the two given examples. As is common in measure theory, the product $0 \cdot \infty$ is defined as 0.

Our starting point is the expression

$$I := \int_{\mathcal{N}(L)} g(x) \prod_{i=1}^{d-1} \frac{k_i(x,u)^{\frac{1}{d+1}}}{\sqrt{1+k_i(x,u)^2}} \, \mathcal{H}^{d-1}(d(x,u)), \tag{5.3.1}$$

which will be evaluated in two different ways. A comparison of the resulting expressions yields the assertion of the lemma.

First, we rewrite I in the form

$$I = \int_{\mathcal{N}(L)} g(x) \left(\prod_{i=1}^{d-1} k_i(x,u) \right)^{-\frac{d}{d+1}} J_{d-1}\pi_2(x,u) \, \mathcal{H}^{d-1}(d(x,u)), \tag{5.3.2}$$

where

$$J_{d-1}\pi_2(x,u) = \prod_{i=1}^{d-1} \frac{k_i(x,u)}{\sqrt{1+k_i(x,u)^2}},$$

for $\mathcal{H}^{d-1}$-almost all $(x,u) \in \mathcal{N}(L)$, is the (approximate) Jacobian of the map $\pi_2 : \mathcal{N}(L) \to S^{d-1}$, $(x,u) \mapsto u$. To check (5.3.2), we distinguish the following cases. If $k_i(x,u) = 0$ for some i, then the integrands on the right-hand sides of (5.3.1) and of (5.3.2) are zero, since $0 \cdot \infty = 0$ and $J_{d-1}\pi_2(x,u) = 0$. If $k_i(x,u) \neq 0$ for all i and $k_j(x,u) = \infty$ for some j, then again both integrands are zero. In all other cases the assertion is clear.

For $\mathcal{H}^{d-1}$-almost all $u \in S^{d-1}$, $\nabla h_L(u) \in \partial L$ is the unique boundary point of L which has u as an exterior unit normal vector. Then the coarea formula yields

$$I = \int_{S^{d-1}} g(\nabla h_L(u)) \left(\prod_{i=1}^{d-1} k_i(\nabla h_L(u), u) \right)^{-\frac{d}{d+1}} \mathcal{H}^{d-1}(du).$$

Using Lemma 3.4 in [Hug98], we get

$$I = \int_{S^{d-1}} g(\nabla h_L(u)) D_{d-1} h_L(u)^{\frac{d}{d+1}} \, \mathcal{H}^{d-1}(du). \tag{5.3.3}$$

Now we consider also the projection $\pi_1 : \mathcal{N}(L) \to \partial L$, $(x,u) \mapsto x$, which has the (approximate) Jacobian

$$J_{d-1}\pi_1(x,u) = \prod_{i=1}^{d-1} \frac{1}{\sqrt{1+k_i(x,u)^2}},$$

for $\mathcal{H}^{d-1}$-almost all $(x,u) \in \mathcal{N}(L)$. A similar argument as before yields

$$\begin{aligned} I &= \int_{\mathcal{N}(L)} g(x) \left(\prod_{i=1}^{d-1} k_i(x,u) \right)^{\frac{1}{d+1}} J_{d-1}\pi_1(x,u) \, \mathcal{H}^{d-1}(d(x,u)) \\ &= \int_{\partial L} g(x) \left(\prod_{i=1}^{d-1} k_i(x, \sigma_L(x)) \right)^{\frac{1}{d+1}} \mathcal{H}^{d-1}(dx). \end{aligned}$$

By Lemma 3.1 in [Hug98], we also get

$$I = \int_{\partial L} g(x)\kappa_L(x)^{\frac{1}{d+1}}\,\mathcal{H}^{d-1}(dx). \tag{5.3.4}$$

A comparison of equations (5.3.3) and (5.3.4) gives the required equality. □

Lemma 5.3.2. *Let $K \subset \mathbb{R}^d$ be a convex body with $o \in \operatorname{int}(K)$. If $f : [0,\infty)\times S^{d-1} \to [0,\infty)$ is a measurable function and $\tilde{f}(x) := f\left(\|x\|^{-1}, \|x\|^{-1}x\right)$, $x \in \partial K^*$, then*

$$\int_{\partial K^*} \tilde{f}(x)\|x\|^{-d+1}\kappa^*(x)^{\frac{1}{d+1}}\,\mathcal{H}^{d-1}(dx) = \int_{\partial K} f(h(K,\sigma_K(x)),\sigma_K(x))\kappa(x)^{\frac{d}{d+1}}\,\mathcal{H}^{d-1}(dx).$$

Proof. We apply Lemma 5.3.1 with $L = K^*$ and $g(x) = \tilde{f}(x)\|x\|^{-d+1}$, $x \in \partial K^*$, and thus we get

$$\begin{aligned}&\int_{\partial K^*} \tilde{f}(x)\|x\|^{-d+1}\kappa^*(x)^{\frac{1}{d+1}}\,\mathcal{H}^{d-1}(dx)\\ &\qquad= \int_{S^{d-1}} \tilde{f}(\nabla h_{K^*}(u))\|\nabla h_{K^*}(u)\|^{-d+1} D_{d-1}h_{K^*}(u)^{\frac{d}{d+1}}\,\mathcal{H}^{d-1}(du).\end{aligned}$$

Next we apply Theorem 2.2 in [Hug96b] (or the second part of Corollary 5.1 in [Hug02]). Thus, using the fact that, for $\mathcal{H}^{d-1}$-almost all $u \in S^{d-1}$, h_{K^*} is differentiable in the generalized sense at u and $\rho(K,u)u$ is a normal boundary point of K,

$$D_{d-1}h_{K^*}(u)^{\frac{d}{d+1}} = \kappa(x)^{\frac{d}{d+1}}\langle u, \sigma_K(x)\rangle^{-d},$$

where $x = \rho(K,u)u \in \partial K$ and $u = \|x\|^{-1}x \in S^{d-1}$. Hence,

$$\begin{aligned}&\int_{\partial K^*} \tilde{f}(x)\|x\|^{-d+1}\kappa^*(x)^{\frac{1}{d+1}}\,\mathcal{H}^{d-1}(dx)\\ &\qquad= \int_{S^{d-1}} \tilde{f}(\nabla h_{K^*}(u))\frac{\|\nabla h_{K^*}(u)\|^{-d+1}}{\langle u, \sigma_K(\rho(K,u)u)\rangle^d}\,\kappa(\rho(K,u)u)^{\frac{d}{d+1}}\,\mathcal{H}^{d-1}(dx).\end{aligned}$$

The bijective and bilipschitz transformation $T : S^{d-1} \to \partial K$, $u \mapsto \rho(K,u)u$, has the Jacobian

$$JT(u) = \frac{\|\nabla h_{K^*}(u)\|}{h_{K^*}(u)^d}$$

for $\mathcal{H}^{d-1}$-almost all $u \in S^{d-1}$ (see the proof of Lemma 2.4 in [Hug96b]). Therefore,

$$\begin{aligned}&\int_{\partial K^*} \tilde{f}(x)\|x\|^{-d+1}\kappa^*(x)^{\frac{1}{d+1}}\,\mathcal{H}^{d-1}(dx)\\ &\qquad= \int_{\partial K} \tilde{f}\left(\nabla h_{K^*}\left(\frac{x}{\|x\|}\right)\right)\frac{\|\nabla h_{K^*}\left(\frac{x}{\|x\|}\right)\|^{-d}}{\langle\frac{x}{\|x\|}, \sigma_K(x)\rangle^d} h_{K^*}\left(\frac{x}{\|x\|}\right)^d \kappa(x)^{\frac{d}{d+1}}\,\mathcal{H}^{d-1}(dx)\\ &\qquad= \int_{\partial K} \tilde{f}\left(\nabla h_{K^*}(x)\right)\frac{\|\nabla h_{K^*}(x)\|^{-d}}{\langle x, \sigma_K(x)\rangle^d} h_{K^*}(x)^d\,\kappa(x)^{\frac{d}{d+1}}\,\mathcal{H}^{d-1}(dx)\end{aligned}$$

$$= \int_{\partial K} f(\|\nabla h_{K^*}(x)\|^{-1}, \nabla h_{K^*}(x)/\|\nabla h_{K^*}(x)\|)\kappa(x)^{\frac{d}{d+1}}\,\mathcal{H}^{d-1}(dx),$$
$$= \int_{\partial K} f(h_K(\sigma_K(x)), \sigma_K(x))\kappa(x)^{\frac{d}{d+1}}\,\mathcal{H}^{d-1}(dx),$$

since $h_{K^*}(x) = 1$ for $x \in \partial K$ and $x^* := \nabla h_{K^*}(x)$ satisfies $\|x^*\|^{-1} = \langle x, \sigma_K(x)\rangle$ as well as $x^*/\|x^*\| = \sigma_K(x)$, for $\mathcal{H}^{d-1}$-almost all $x \in \partial K$. $\square$

Chapter 6

Random points on the boundary

This chapter of the dissertation is based on the paper [BFH13] by K.J. Böröczky, F. Fodor, and D. Hug, *Intrinsic volumes of random polytopes with vertices on the boundary of a convex body*, Trans. Amer. Math. Soc., **365** (2013), no. 2, 785–809. (DOI 10.1090/S0002-9947-2012-05648-0)

6.1 Introduction and results

In this chapter, we shall consider the following probability model. Let K be a convex body with a rolling ball of radius r. Let ϱ be a continuous, positive probability density function defined on ∂K; throughout this chapter this density is always considered with respect to the boundary measure on ∂K. Select the points $x_1, \dots, x_n$ randomly and independently from ∂K according to the probability distribution determined by ϱ. The convex hull $K_n := [x_1, \dots, x_n]$ then is a random polytope inscribed in K. We are going to study the expectation of intrinsic volumes of K_n. In order to indicate the dependence on the probability density ϱ, we write $\mathbb{P}_\varrho$ to denote the probability of an event in this probability space and $\mathbb{E}_\varrho$ to denote the expected value. For a convex body K, the expected value $\mathbb{E}_\varrho(V_j(K_n))$ of the j-th intrinsic volume of K_n tends to $V_j(K)$ as n tends to infinity. It is clear that the asymptotic behaviour of $V_j(K) - \mathbb{E}_\varrho(V_j(K_n))$ is determined by the shape of the boundary of K. In the case when the boundary of K is a C^2_+ submanifold of $\mathbb{R}^d$, this asymptotic behaviour was described by M. Reitzner [Rei02].

Theorem 6.1.1 (Reitzner [Rei02]). *Let K be a convex body in $\mathbb{R}^d$ with C^2_+ boundary, and let ϱ be a continuous, positive probability density function on ∂K. Denote by $\mathbb{E}_\varrho(V_j(K_n)), j = 1, \dots, d$, the expected j-th intrinsic volume of the convex hull of n random points on ∂K chosen independently and according to the density function ϱ. Then*

$$V_j(K) - \mathbb{E}_\varrho(V_j(K_n)) \sim c^{(j,d)} \int_{\partial K} \varrho(x)^{-\frac{2}{d-1}} H_{d-1}(x)^{\frac{1}{d-1}} H_{d-j}(x)\, \mathcal{H}^{d-1}(dx) \cdot n^{-\frac{2}{d-1}} \quad (6.1.1)$$

as $n \to \infty$, where the constant $c^{(j,d)}$ only depends on j and the dimension d.

For $j = d$, that is in the case of the volume functional, C. Schütt and E. Werner [SW03] extended (6.1.1) to any convex body K such that a ball of radius r rolls freely in K and, in addition, K rolls freely in a ball of radius R, for some $R > r > 0$. The latter assumption of K rolling freely inside a ball implies a uniform positive lower bound for the principle curvatures of ∂K whenever they exist. They also calculated the constant $c^{(d,d)}$ explicitly, that is

$$c^{(d,d)} = \frac{(d-1)^{\frac{d+1}{d-1}}\Gamma(d+1+\frac{2}{d-1})}{2(d+1)![(d-1)\alpha_{d-1}]^{\frac{2}{d-1}}}.$$

Moreover, C. Schütt and E. Werner [SW03] showed that for fixed K, the minimum of the integral expression in (6.1.1) is attained for the probability density function

$$\varrho_0(x) = \frac{H_{d-1}(x)^{\frac{1}{d+1}}}{\int_{\partial K} H_{d-1}(x)^{\frac{1}{d+1}}\ \mathcal{H}^{d-1}(dx)}.$$

Our main goal is to extend Theorem 6.1.1 to the case where K is only assumed to have a rolling ball, for all $j = 1, \dots, d$. In particular, the Gauss curvature is allowed to be zero on a set of positive boundary measure. More explicitly, we shall prove

Theorem 6.1.2 (Böröczky, Fodor, Hug [BFH13, Theorem 1.2 on page 788])**.** *The asymptotic formula* (6.1.1) *holds if K is a convex body in $\mathbb{R}^d$ in which a ball rolls freely.*

The present method of proof for Theorem 6.1.2 is different from the one used by Reitzner [Rei02] or Schütt and Werner [SW03]. It is inspired by the arguments from the paper [BFH10] by Böröczky, Fodor and Hug (as presented in Section 4.2) concerning random points chosen from a convex body, however, the case of random points chosen from the boundary is more delicate.

Examples show that in general the condition that a ball rolls freely inside K cannot be dropped in Theorem 6.1.2. General bounds are provided in the following theorem.

Theorem 6.1.3 (Böröczky, Fodor, Hug [BFH13, Theorem 1.3 on page 788])**.** *Let K be a convex body in $\mathbb{R}^d$, and let ϱ be a continuous, positive probability density function on ∂K. Then there exist positive constants c_1, c_2, depending on K and ϱ, such that for any $n \geq d+1$,*

$$c_1 n^{-\frac{2}{d-1}} \leq \mathbb{E}_\varrho(V_1(K) - V_1(K_n)) \leq c_2 n^{-\frac{1}{d-1}}.$$

The lower bound is of optimal order if K has a rolling ball, and the upper bound is of optimal order, if K is a polytope.

Let us review the main known results about the convex hull $K_{(n)}$ of n points chosen randomly, independently and uniformly from K. In the case where a ball rolls freely inside K, the analogue of Theorem 6.1.2 is established in K. Böröczky Jr., L. M. Hoffmann and D. Hug [BHH08]. For the case of the volume functional and an arbitrary convex body K, C. Schütt [Sch94] proved (see K.J. Böröczky, F. Fodor, D. Hug [BFH10] for some corrections and an extension) that

$$\lim_{n\to\infty} n^{\frac{2}{d+1}}(V_d(K) - \mathbb{E}(V_d(K_{(n)})) = c_d V_d(K)^{\frac{2}{d+1}} \int_{\partial K} H_{d-1}(x)^{\frac{1}{d+1}}\ \mathcal{H}^{d-1}(dx),$$

where the constant $c_d > 0$ only depends on the dimension d and is explicitly known. Concerning the order of approximation, we have

$$\gamma_1 n^{-2/(d+1)} < V_1(K) - \mathbb{E}V_1(K_{(n)}) < \gamma_2 n^{-1/d}, \tag{6.1.2}$$

$$\gamma_3 n^{-1} \ln^{d-1} n < V_d(K) - \mathbb{E}V_d(K_{(n)}) < \gamma_4 n^{-2/(d+1)}, \tag{6.1.3}$$

where $\gamma_1, \ldots, \gamma_4 > 0$ are constants that may depend on K. The inequality (6.1.2) was proved by R. Schneider [Sch87], and the inequality (6.1.3) was proved by I. Bárány and D. Larman [BL88]. The left inequality of (6.1.2) and the right inequality of (6.1.3) are optimal for sufficiently smooth convex bodies. The right inequality of (6.1.2) and the left inequality of (6.1.3) are optimal for polytopes.

The proof of Theorem 6.1.2 is given in the following three sections. In Section 6.2, we rewrite the difference $V_j(K) - \mathbb{E}_\varrho(V_j(K_n))$ in an integral geometric way. The inner integral involved in this integral geometric description is extended over the projection $K|L$ of K to L, where L is a j-dimensional linear subspace. Then we show that up to an error term of lower order the main contribution comes from a neighbourhood of the (relative) boundary $\partial(K|L)$ of $K|L$ with respect to L, where this neighbourhood is shrinking at a well-defined speed $t(n)$ as $n \to \infty$. Further application of an integral geometric decomposition then shows that the proof boils down to determining the limit

$$\lim_{n\to\infty} \int_0^{t(n)} n^{\frac{2}{d-1}} \langle y, u(y)\rangle \mathbb{P}_\varrho\left(y_t \notin K_n|L\right)\, dt,$$

where $y \in \partial(K|L)$ and x is a normal boundary point of K with $y = x|L$. The case where the Gauss curvature of K at x is zero is treated directly. In Section 6.3, we deal with the case of positive Gauss curvature. In a first step, we choose a reparametrization of the integral which relates the parameter t to the probability content s of that part of the boundary of K near x that is cut off by a cap determined by the parameter t. This reparametrization has the effect of extracting the relevant geometric information from K. What remains to be shown is that the transformed integrals are essentially independent of K and yield the same value for the unit ball with the uniform probability density on its boundary. This latter step is divided into two lemmas in Section 6.3. Whereas both lemmas have analogues in our previous work [BFH10] (see Section 4.2), the present arguments are more delicate and the second lemma has to be established by a reasoning different from the one in [BFH10]. The proof is then completed in Section 6.4, where, in addition to the previous steps, a very special case of Theorem 6.1.1 is employed (K being the unit ball) as well as an integral geometric lemma from [BHH08]. The final section of this chapter is devoted to the proof of Theorem 6.1.3.

6.2 General estimates

In order to prove Theorem 6.1.2, we start by rewriting $V_j(K) - \mathbb{E}_\varrho(V_j(K_n))$ in an integral geometric form. For this, we use Kubota's formula and Fubini's theorem to obtain

$$V_j(K) - \mathbb{E}_\varrho(V_j(K_n))$$

$$
\begin{aligned}
&= \int_{\partial K} \cdots \int_{\partial K} (V_j(K) - V_j(K_n)) \prod_{i=1}^{n} \varrho(x_i)\, \mathcal{H}^{d-1}(dx_1) \ldots \mathcal{H}^{d-1}(dx_n) \\
&= \frac{\binom{d}{j}\alpha_d}{\alpha_j \alpha_{d-j}} \int_{\partial K} \cdots \int_{\partial K} \int_{\mathcal{L}_j^d} (V_j(K|L) - V_j(K_n|L)) \\
&\qquad \times \prod_{i=1}^{n} \varrho(x_i)\, \nu_j(dL)\, \mathcal{H}^{d-1}(dx_1) \ldots \mathcal{H}^{d-1}(dx_n) \\
&= \frac{\binom{d}{j}\alpha_d}{\alpha_j \alpha_{d-j}} \int_{\mathcal{L}_j^d} \int_L \int_{\partial K} \cdots \int_{\partial K} \mathbf{1}\{y \in K|L \text{ and } y \notin K_n|L\} \\
&\qquad \times \prod_{i=1}^{n} \varrho(x_i)\, \mathcal{H}^{d-1}(dx_1) \ldots \mathcal{H}^{d-1}(dx_n) \mathcal{H}^j(dy)\, \nu_j(dL) \\
&= \frac{\binom{d}{j}\alpha_d}{\alpha_j \alpha_{d-j}} \int_{\mathcal{L}_j^d} \int_{K|L} \mathbb{P}_\varrho(y \notin K_n|L)\, \mathcal{H}^j(dy)\, \nu_j(dL). \qquad (6.2.1)
\end{aligned}
$$

Now we introduce some geometric tools. If K has a rolling ball of radius r, then so does $K|L$ for any $L \in \mathcal{L}_j^d$. Furthermore, ∂K has a unique outer unit normal vector $u(x)$ at each boundary point $x \in \partial K$. If $L \in \mathcal{L}_j^d$, $y \in \partial(K|L)$ and $x \in K$ such that $y = x|L$, then $x \in \partial K$ and the outer unit normal of $\partial(K|L)$ at y is equal to $u(x)$.

Since the statement of the theorem is translation invariant, we may assume that

$$
rB^d \subset K \subset RB^d \qquad (6.2.2)
$$

for some $R > 0$. For $t \in (0, 1)$, let $K_t := (1-t)K$, and for $x \in \partial K$, let $x_t := (1-t)x$. Similarly, $(K|L)_t := (1-t)(K|L)$ and $y_t := (1-t)y$ for $y \in \partial(K|L)$.

For $x \in \partial K$ and $t \in (0, 1)$, let

$$
x_t^* := x - \langle tx, u(x) \rangle u(x).
$$

If $t \in (0, \frac{r}{R})$, then (6.2.2) implies that

$$
tr \le \langle x - x_t^*, u(x) \rangle = \langle x - x_t, u(x) \rangle < r. \qquad (6.2.3)
$$

The existence of a rolling ball at x yields that if $t \in (0, \frac{r}{R})$, then

$$
x_t^* + r\sqrt{t}(u(x)^\perp \cap B^d) \subset K. \qquad (6.2.4)
$$

On the other hand, we have

$$
\|x_t^* - x_t\| < Rt. \qquad (6.2.5)
$$

In the following, we write $\gamma_1, \gamma_2, \ldots$ for positive constants which merely depend on K and ϱ.

Let us estimate the probability that $o \notin K_n$. There exists a constant $\gamma_1 > 0$ such that the probability content of each of the parts of ∂K contained in one of the 2^d coordinate corners of $\mathbb{R}^d$ is at least γ_1. Now if $o \notin K_n$, then o can be strictly separated from K_n by a hyperplane. It follows that $\{x_1,\ldots,x_n\}$ is disjoint from one of these coordinate corners, and hence

$$\mathbb{P}(o \notin K_n) \le 2^d(1-\gamma_1)^n. \tag{6.2.6}$$

This fact will be used, for instance, in the proof of the subsequent lemma. In the following, for $x \in \mathbb{R}^d$ we use the shorthand notation $\mathbb{R}_+x := \{\lambda x : \lambda \ge 0\}$.

Lemma 6.2.1. *There exist constants $\delta, \gamma_2 \in (0,1)$, depending on K and ϱ, such that if $L \in \mathcal{L}_j^d$, $y \in \partial(K|L)$ and $t \in (0,\delta)$, then*

$$\mathbb{P}_\varrho\left(y_t \notin K_n|L\right) \ll \left(1-\gamma_2 t^{\frac{d-1}{2}}\right)^n.$$

Proof. Let $y \in \partial(K|L)$ and $x \in \partial K$ be such that $y = x|L$. Let $\Theta'_1,\ldots,\Theta'_{2^{d-1}}$ be the coordinate corners with respect to some basis vectors in $u(x)^\perp$. In addition, for $i = 1,\ldots,2^{d-1}$ and $t \in (0,1)$, let

$$\Theta_{i,t} = \partial K \cap \left(x_t + \left[\Theta'_i, \mathbb{R}_+x\right]\right).$$

Since ϱ is positive and continuous, we have

$$\int_{\Theta_{i,t}} \varrho(x)\,\mathcal{H}^{d-1}(dx) \ge \gamma_3 \mathcal{H}^{d-1}(\Theta_{i,t}).$$

If $y_t \notin K_n|L$ and $o \in K_n$, then there exists a $(j-1)$-dimensional affine plane H_L in L through y_t, bounding the half-spaces H_L^- and H_L^+ in L, for which $K_n|L \subset H_L^-$. Now, if $L^\perp$ is the orthogonal complement of L in $\mathbb{R}^d$, then $H := H_L + L^\perp$ is a hyperplane in $\mathbb{R}^d$ with the property that $x_t \in H$ and $K_n \subset H^- := H_L^- + L^\perp$. Furthermore, $\Theta_{i,t} \subset H^+ := H_L^+ + L^\perp$ for some $i \in \{1,\ldots,2^{d-1}\}$, because $o \in K_n \subset H^-$. Therefore

$$\mathbb{P}_\varrho\left(y_t \notin K_n|L, o \in K_n\right) \le \sum_{i=1}^{2^{d-1}} \left(1-\gamma_3\mathcal{H}^{d-1}(\Theta_{i,t})\right)^n.$$

Combining (6.2.4) and (6.2.5), we deduce the existence of a constant $\gamma_4 > 0$ such that if $t \le \gamma_4$, then the orthogonal projection of $\Theta_{i,t}$ into $u(x)^\perp$ contains a translate of $\Theta'_i \cap (r/2)\sqrt{t}B^d$, and therefore

$$\mathcal{H}^{d-1}(\Theta_{i,t}) \ge \gamma_5 t^{\frac{d-1}{2}}$$

for $i = 1,\ldots,2^{d-1}$. In turn, we obtain

$$\mathbb{P}_\varrho\left(y_t \notin K_n|L, o \in K_n\right) \ll \left(1-\gamma_6 t^{\frac{d-1}{2}}\right)^n. \tag{6.2.7}$$

On the other hand, if $o \notin K_n|L$, then (6.2.6) holds. Combining this with (6.2.7), we conclude the proof of the lemma. □

Subsequently, the estimate of Lemma 6.2.1 will be used, for instance, to restrict the domain of integration (cf. Lemma 6.2.3) and to justify an application of Lebesgue's dominated convergence theorem (see (6.2.12)). For these applications, we also need that if $x \in \partial K$ and $c > 0$ satisfies $\bar{\omega} := c\delta^{\frac{d-1}{2}} < 1$, then

$$\int_0^\delta \left(1-ct^{\frac{d-1}{2}}\right)^n dt = c^{\frac{-2}{d-1}}\frac{2}{d-1}\int_0^{\bar{\omega}} s^{\frac{2}{d-1}-1}(1-s)^n\,ds \ll c^{\frac{-2}{d-1}} \cdot n^{\frac{-2}{d-1}}, \tag{6.2.8}$$

where we use that $(1-s)^n \le e^{-ns}$ for $s \in [0,1]$ and $n \in \mathbb{N}$.

The next lemma will allow us to decompose integrals in a suitable way. We write $u(y)$ to denote the unique exterior unit normal to $\partial(K|L)$ at $y \in \partial(K|L)$. It will always be clear from the context whether we mean the exterior unit normal at a point $x \in \partial K$ or at a point $y \in \partial(K|L)$. In the next lemma, δ is chosen as in Lemma 6.2.1.

Lemma 6.2.2. *If* $0 \le t_0 < t_1 < \delta$ *and* $h : K|L \to [0,\infty]$ *is a measurable function, then*

$$\begin{aligned}\int_{(K|L)_{t_0}\setminus(K|L)_{t_1}} &\mathbb{P}_\varrho\left(x \notin K_n|L\right) h(x)\, \mathcal{H}^j(dx)\\ &= \int_{\partial(K|L)} \int_{t_0}^{t_1} (1-t)^{j-1}\mathbb{P}_\varrho\left(y_t \notin K_n|L\right) \langle y, u(y)\rangle h(y_t)\, dt\, \mathcal{H}^{j-1}(dy).\end{aligned}$$

Proof. The set $\partial(K|L)$ is a $(j-1)$-dimensional submanifold of L of class C^1, and the map

$$T : \partial(K|L) \times (t_0,t_1) \to \operatorname{int}(K|L)_{t_0} \setminus (K|L)_{t_1}, \quad (y,t) \mapsto y_t,$$

is a C^1 diffeomorphism with Jacobian $JT(y,t) = (1-t)^{j-1}\langle y, u(y)\rangle \ge 0$. Thus the assertion follows from Federer's area/coarea theorem (see [Fed69]). $\square$

In the following, we use the abbreviation $t(n) := n^{\frac{-1}{d-1}}$.

Lemma 6.2.3. *Let* $1 \le j \le d-1$. *Then we have*

$$\int_{\mathcal{L}_j^d} \int_{(K|L)_{t(n)}} \mathbb{P}_\varrho\left(y \notin K_n|L\right)\, \mathcal{H}^j(dy)\, \nu_j(dL) = o\left(n^{\frac{-2}{d-1}}\right).$$

Proof. Let $\delta, \gamma_2 \in (0,1)$ be chosen as in Lemma 6.2.1. We may assume that n is large enough to satisfy $t(n) < \delta$ and $n \ge (\gamma_2)^2$. First, we treat that part of the integral which extends over the subset $(K|L)_\delta$ of $(K|L)_{t(n)}$.

Let $\omega := \delta r$. Then (6.2.3) yields

$$\langle x - x_\delta, u(x)\rangle \ge \omega \quad \text{for} \quad x \in \partial K. \tag{6.2.9}$$

There exists a constant $\gamma_7 > 0$ such that the probability measure of $(x + \frac{\omega}{2}\, B^d) \cap \partial K$ is at least γ_7 for all $x \in \partial K$. We choose a maximal set $\{z_1, \dots, z_m\} \subset \partial K$ such that $\|z_i - z_l\| \ge \frac{\omega}{2}$ for $i \ne l$.

For $L \in \mathcal{L}_j^d$, let $y \in (K|L)_\delta$. If $y \notin K_n|L$, then there exist a hyperplane H in $\mathbb{R}^d$ and a half space H^- bounded by H such that $y \in H$, H is orthogonal to L, and $K_n \subset \operatorname{int}(H^-)$. Choose $x \in \partial K$ such that $u(x)$ is an exterior unit normal to H^-. Since H intersects K_δ, we have $\langle x - y, u(x)\rangle \ge \omega$ by (6.2.9). Now there exists some $i \in \{1,\dots,n\}$ with $\|x - z_i\| \le \frac{\omega}{2}$, and hence $\{x_1,\dots,x_n\} \subset \operatorname{int}(H^-)$ yields that $\{x_1,\dots,x_n\}$ is disjoint from $z_i + \frac{\omega}{2}\, B^d$. In particular, we have

$$\mathbb{P}_\varrho\left(y \notin K_n|L\right) \le m(1-\gamma_7)^n. \tag{6.2.10}$$

Next let $y \in \partial(K|L)$. If $t \in (t(n),\delta)$, then Lemma 6.2.1 yields

$$\mathbb{P}_\varrho\left(y_t \notin K_n|L\right) \ll \left(1 - \gamma_2 n^{-\frac{1}{2}}\right)^n < e^{-\gamma_2 n^{\frac{1}{2}}} \ll n^{\frac{-3}{d+1}}. \tag{6.2.11}$$

In particular, writing I to denote the integral in Lemma 6.2.3, we obtain from Lemma 6.2.2, (6.2.10) and (6.2.11) that

$$\begin{aligned} I &\ll \int_{\mathcal{L}_j^d}\int_{(K|L)_\delta} \mathbb{P}_\varrho\left(y \notin K_n|L\right)\, \mathcal{H}^j(dy)\, \nu_j(dL) + \\ &\quad + \int_{\mathcal{L}_j^d}\int_{t(n)}^{\delta}\int_{\partial(K|L)} \mathbb{P}_\varrho\left(y_t \notin K_n|L\right)\, \mathcal{H}^{j-1}(dy)\, dt\, \nu_j(dL) \\ &\ll m(1-\gamma_7)^n + \int_{\mathcal{L}_j^d}\int_{\partial(K|L)} n^{\frac{-3}{d-1}}\, \mathcal{H}^{j-1}(dy)\, \nu_j(dL) = o\left(n^{\frac{-2}{d-1}}\right), \end{aligned}$$

which is the required estimate. □

It follows by applying (6.2.1), Lemma 6.2.3 and Lemma 6.2.2, in this order, that

$$\begin{aligned} &\lim_{n\to\infty} n^{\frac{2}{d-1}}(V_j(K) - \mathbb{E}_\varrho(V_j(K_n))) \\ &= \frac{\binom{d}{j}\alpha_d}{\alpha_j\alpha_{d-j}} \lim_{n\to\infty} n^{\frac{2}{d-1}} \int_{\mathcal{L}_j^d}\int_{K|L} \mathbb{P}_\varrho(y \notin K_n|L)\, \mathcal{H}^j(dy)\, \nu_j(dL) \\ &= \frac{\binom{d}{j}\alpha_d}{\alpha_j\alpha_{d-j}} \lim_{n\to\infty} n^{\frac{2}{d-1}} \int_{\mathcal{L}_j^d}\int_{(K|L)\setminus(K|L)_{t(n)}} \mathbb{P}_\varrho(y \notin K_n|L)\, \mathcal{H}^j(dy)\, \nu_j(dL) \\ &= \frac{\binom{d}{j}\alpha_d}{\alpha_j\alpha_{d-j}} \lim_{n\to\infty} \int_{\mathcal{L}_j^d}\int_{\partial(K|L)}\int_0^{t(n)} n^{\frac{2}{d-1}}\mathbb{P}_\varrho(y_t \notin K_n|L)\times \\ &\qquad\qquad \times (1-t)^{j-1}\langle y, u(y)\rangle\, dt\, \mathcal{H}^{j-1}(dy)\, \nu_j(dL). \end{aligned}$$

We deduce from Lemma 6.2.1 and (6.2.8) that if $n > n_0$, $L \in \mathcal{L}_j^d$ and $y \in \partial(K|L)$, then

$$\int_0^{t(n)} n^{\frac{2}{d-1}}\mathbb{P}_\varrho\left(y_t \notin K_n|L\right)\langle y, u(y)\rangle(1-t)^{j-1}\, dt \ll C,$$

where n_0 and C depend on K and ϱ. Therefore, we may apply Lebesgue's dominated convergence theorem, and thus we conclude

$$\lim_{n\to\infty} n^{\frac{2}{d-1}}(V_j(K) - \mathbb{E}_\varrho(V_j(K_n))) = \frac{\binom{d}{j}\alpha_d}{\alpha_j\alpha_{d-j}} \int_{\mathcal{L}_j^d}\int_{\partial(K|L)} J_\varrho(y, L)\, \mathcal{H}^{j-1}(dy)\, \nu_j(dL), \quad (6.2.12)$$

where, for $L \in \mathcal{L}_j^d$ and $y \in \partial(K|L)$, we have

$$J_\varrho(y, L) := \lim_{n\to\infty}\int_0^{t(n)} n^{\frac{2}{d-1}}\langle y, u(y)\rangle\mathbb{P}_\varrho\left(y_t \notin K_n|L\right)\, dt. \quad (6.2.13)$$

Subsequently, we shall inspect this limit more closely. In a first step, we shall consider those points $y \in \partial(K|L)$ for which there is a normal boundary point $x \in \partial K$ with $y = x|L$ and $H_{d-1}(x) = 0$.

Lemma 6.2.4. *Let $L \in \mathcal{L}_j^d$, and let $y \in \partial(K|L)$. If $x \in \partial K$ is a normal boundary point of K with $y = x|L$ and $H_{d-1}(x) = 0$, then $J_\varrho(y, L) = 0$.*

Proof. Let $x \in \partial K$ be a normal boundary point with $y = x|L$ and $H_{d-1}(x) = 0$. First, we show the existence of a decreasing function φ on $(0, \frac{r}{R})$ with $\lim_{t\to 0^+} \varphi(t) = \infty$ satisfying

$$\mathbb{P}_\varrho (y_t \notin K_n|L) \leq 2^{d-1} \left(1 - \varphi(t)t^{\frac{d-1}{2}}\right)^n . \tag{6.2.14}$$

In the following, we always assume that $t > 0$ is sufficiently small, that is n is sufficiently large, so that all expressions that arise are well defined. Let $v_1, \ldots, v_{d-1}$ be an orthonormal basis in $u(x)^\perp$ such that these vectors are principal directions of curvature of K at x and such that the curvature is zero in the direction of v_1. In addition, let $\Theta'_1, \ldots, \Theta'_{2^{d-1}}$ be the coordinate corners in $u(x)^\perp$, and, for $i = 1, \ldots, 2^{d-1}$ and $t \in (0, 1)$, let $\Theta_{i,t} = \partial K \cap (x_t + [\Theta'_i, \mathbb{R}_+ x])$ as before. The continuity of ϱ yields that

$$\int_{\Theta_{i,t}} \varrho(x)\, \mathcal{H}^{d-1}(dx) \gg \mathcal{H}^{d-1}(\Theta_{i,t}).$$

Since the curvature is zero in the direction of v_1, there exists a function ψ on $(0, \frac{r}{R})$ with $\lim_{t\to 0^+} \psi(t) = \infty$ satisfying

$$x_t^* - \psi(t)\sqrt{t}v_1 \in K \quad \text{and} \quad x_t^* + \psi(t)\sqrt{t}v_1 \in K.$$

Combining (6.2.4) and (6.2.5), we deduce the existence of a decreasing function $\tilde{\varphi}$ on $(0, \frac{r}{R})$ with $\lim_{t\to 0^+} \tilde{\varphi}(t) = \infty$ satisfying

$$\int_{\Theta_{i,t}} \varrho(x)\, \mathcal{H}^{d-1}(dx) \geq \tilde{\varphi}(t)t^{\frac{d-1}{2}},$$

for $i = 1, \ldots, 2^{d-1}$.

First, we assume that $y_t \notin K_n|L$ and $o \in K_n$. In particular, then we also have $x_t \notin K_n$, and hence there exists a hyperplane H through x_t such that K_n lies on one side of H. Since $o \in K_n$, it follows that H separates K_n from some $\Theta_{i,t}$, and therefore

$$\mathbb{P}_\varrho (y_t \notin K_n|L, o \in K_n) \leq 2^{d-1} \left(1 - \tilde{\varphi}(t)t^{\frac{d-1}{2}}\right)^n . \tag{6.2.15}$$

On the other hand, if $o \notin K_n|L$, then (6.2.6) holds. Combining this with (6.2.15), we conclude (6.2.14). In turn, we deduce from (6.2.8) that

$$J_\varrho(y, L) \ll \lim_{n\to\infty} n^{\frac{2}{d-1}} \int_0^{t(n)} (1 - \varphi(t(n))t^{\frac{d-1}{2}})^n \, dt \ll \lim_{n\to\infty} \varphi(t(n))^{\frac{-2}{d-1}} = 0.$$

□

In the next section, we study the more difficult case of boundary points with positive Gauss curvature.

6.3 Normal boundary points and caps

Let $L \in \mathcal{L}_j^d$, and let $y \in \partial(K|L)$ be such that $y = x|L$ for some (uniquely determined) normal boundary point $x \in \partial K$ with $H_{d-1}(x) > 0$. We keep x and y fixed throughout this section. First, we reparametrize x_t and y_t in terms of the probability measure of the corresponding cap of ∂K. Using this reparametrization, we show that $J_\varrho(y, L)$ essentially depends only on the random points near x (see Lemma 6.3.1), and then in a second step we pass from the case of a general convex body K to the case of a Euclidean ball.

For $t \in (0, 1)$, we consider the hyperplane $H(x, t) := \{z \in \mathbb{R}^d : \langle u(x), z\rangle = \langle u(x), x_t\rangle\}$, the half-space $H^+(x, t) := \{z \in \mathbb{R}^d : \langle u(x), z\rangle \geq \langle u(x), x_t\rangle\}$, and the cap $C(x, t) := K \cap H^+(x, t)$ whose bounding hyperplane is $H(x, t)$. Next we reparametrize x_t in terms of the induced probability measure of the cap $C(x, t)$; namely,

$$\tilde{x}_s := x_t \qquad \text{and} \qquad \tilde{y}_s := y_t,$$

where, for a given sufficiently small $s \geq 0$, the parameter $t \geq 0$ is uniquely determined by the equation

$$s = \int_{C(x,t)\cap\partial K} \varrho(w)\, \mathcal{H}^{d-1}(dw). \tag{6.3.1}$$

Note that s is a strictly increasing and continuous function of t. We further define

$$\widetilde{C}(x, s) = C(x, t) \qquad \text{and} \qquad \widetilde{H}(x, s) = H(x, t), \tag{6.3.2}$$

where again, for given s, the parameter t is determined by (6.3.1). Observe that $\partial K \cap H^+(x, t) = \partial K \cap C(x, t)$. Subsequently, we explore the relation between s and t. Let $f : u(x)^\perp \to [0, \infty]$ be a convex function such that the restriction of the map

$$F : u(x)^\perp \to \mathbb{R}^d, \qquad z \mapsto x + z - f(z)u(x),$$

to a neighbourhood of o parametrizes ∂K in a neighbourhood of x. Moreover, we consider the transformations

$$\Pi : \mathbb{R}^d \to u(x)^\perp, \qquad y \mapsto y - x - \langle y - x, u(x)\rangle u(x),$$

and

$$T : u(x)^\perp \times \mathbb{R} \to u(x)^\perp \times \mathbb{R}, \qquad (z_1, \dots, z_{d-1}, \alpha) \mapsto (\sqrt{k_1} z_1, \dots, \sqrt{k_{d-1}} z_{d-1}, \alpha),$$

where $u(x)^\perp$ is considered to be a subset of $u(x)^\perp \times \{0\}$ and $k_i = k_i(x)$, $i = 1, \dots, d-1$, are the principle curvatures of ∂K at x. Then we obtain

$$\begin{aligned}
&\int_{\partial K\cap H^+(x,t)} \varrho(w)\, \mathcal{H}^{d-1}(dw) \\
&= \int_{\Pi(\partial K\cap H^+(x,t))} \varrho(F(z)) \sqrt{1 + \|\nabla f(z)\|^2}\, \mathcal{H}^{d-1}(dz) \\
&= \int_{T(\Pi(\partial K\cap H^+(x,t)))} \varrho(F \circ T^{-1}(z)) \sqrt{1 + \|\nabla f(T^{-1}(z))\|^2}\, H_{d-1}(x)^{-1/2}\, \mathcal{H}^{d-1}(dz).
\end{aligned}$$

Let $\overline{K} := T(K-x)+x$, and hence $T(\Pi(\partial K \cap H^+(x,t))) = \Pi(\partial\overline{K} \cap H^+(x,t))$. If $\overline{f}$ is defined for $\overline{K}$ as f is defined for K, and

$$\overline{\varrho}(w) := \varrho(F \circ T^{-1} \circ \Pi(w)), \qquad g(w) := \frac{\sqrt{1+\|\nabla f(T^{-1}(\Pi(w)))\|^2}}{\sqrt{1+\|\nabla\overline{f}(\Pi(w))\|^2}},$$

for $w \in \partial\overline{K} \cap H^+(x,t)$, then we obtain

$$\int_{\partial K\cap H^+(x,t)} \varrho(w)\,\mathcal{H}^{d-1}(dw) = H_{d-1}(x)^{-1/2}\int_{\partial\overline{K}\cap H^+(x,t)} \overline{\varrho}(w)g(w)\,\mathcal{H}^{d-1}(dw).$$

Next we put $H(r) := x - ru(x) + u(x)^\perp$ and denote by $n_{\overline{K}}(w)$ the exterior unit normal of $\overline{K}$ at $w \in \partial\overline{K}$. Since (cf. the notes for Section 1.5 (2) in [Sch14])

$$\overline{f}(z) = \frac{1}{2}\|z\|^2 + o(\|z\|^2), \quad \|\nabla\overline{f}(z)\| = \|z\| + o(\|z\|), \quad n_{\overline{K}}(w) = \frac{\nabla\overline{f}(\bar{w})+u(x)}{\sqrt{1+\|\nabla\overline{f}(\bar{w})\|^2}}$$

with $\bar{w} := \Pi(w)$ and $z \in u(x)^\perp$, we get

$$\sqrt{1-\langle n_{\overline{K}}(w),u(x)\rangle^2}^{\,-1} = \frac{\sqrt{1+(\|\bar{w}\|+o(\|\bar{w}\|))^2}}{\|\bar{w}\|+o(\|\bar{w}\|)}.$$

Thus a simple application of the coarea formula yields that, for $t>0$ sufficiently small and $d \geq 2$,

$$\begin{aligned}&\int_{\partial K\cap H^+(x,t)} \varrho(w)\,\mathcal{H}^{d-1}(dw)\\ &= H_{d-1}(x)^{-1/2}\int_0^{t\langle x,u(x)\rangle}\int_{\partial\overline{K}\cap H(r)} \overline{\varrho}(w)g(w)\sqrt{1-\langle n_{\overline{K}}(w),u(x)\rangle^2}^{\,-1}\,\mathcal{H}^{d-2}(dw)\,dr.\end{aligned}$$

Since also $\overline{K}$ has a rolling ball, the map $w \mapsto n_{\overline{K}}(w)$ is continuous, and therefore also

$$r \mapsto \int_{\partial\overline{K}\cap H(r)} \overline{\varrho}(w)g(w)\sqrt{1-\langle n_{\overline{K}}(w),u(x)\rangle^2}^{\,-1}\,\mathcal{H}^{d-2}(dw)$$

is continuous. This implies that

$$\begin{aligned}&\frac{\partial}{\partial t}\int_{\partial K\cap H^+(x,t)} \varrho(w)\,\mathcal{H}^{d-1}(dw)\\ &= \frac{\langle x,u(x)\rangle}{H_{d-1}(x)^{1/2}}\int_{\partial\overline{K}\cap H(t\langle x,u(x)\rangle)} \overline{\varrho}(w)g(w)\sqrt{1-\langle n_{\overline{K}}(w),u(x)\rangle^2}^{\,-1}\,\mathcal{H}^{d-2}(dw)\\ &= \frac{\langle x,u(x)\rangle}{H_{d-1}(x)^{1/2}}\int_{\partial\overline{K}\cap H(t\langle x,u(x)\rangle)} \overline{\varrho}(w)g(w)\frac{\sqrt{1+(\sqrt{2t\langle x,u(x)\rangle}+o(\sqrt{t}))^2}}{\sqrt{2t\langle x,u(x)\rangle}+o(\sqrt{t})}\,\mathcal{H}^{d-2}(dw).\end{aligned}$$

Clearly, we have $\overline{\varrho}(w) \to \overline{\varrho}(x) = \varrho(x)$ and $g(w) \to 1$, as $t \to 0^+$, uniformly with respect to $w \in \partial\overline{K} \cap H(t\langle x,u(x)\rangle)$. Moreover, since

$$\overline{\Gamma} := \left\{x+z-\frac{1}{2}\|z\|^2u(x) : z \in u(x)^\perp\right\}$$

is the osculating paraboloid of $\overline{K}$ and $\overline{\Gamma}$ has rotational symmetry, we obtain for $s = s(t)$ that

$$\begin{aligned}\lim_{t\to 0^+} t^{-\frac{d-3}{2}} \cdot \frac{\partial s}{\partial t}(t) &= \frac{\varrho(x)\langle x, u(x)\rangle}{H_{d-1}(x)^{1/2}} \lim_{t\to 0^+} \left(t^{-\frac{d-3}{2}} (d-1)\alpha_{d-1} \frac{\sqrt{2t\langle x, u(x)\rangle}^{d-2}}{\sqrt{2t\langle x, u(x)\rangle}} \right) \\ &= (d-1)\alpha_{d-1} H_{d-1}(x)^{-\frac{1}{2}} \varrho(x) \left(2\langle x, u(x)\rangle\right)^{\frac{d-3}{2}} \langle x, u(x)\rangle \\ &= (d-1)\alpha_{d-1}\varrho(x) 2^{\frac{d-3}{2}} \langle x, u(x)\rangle^{\frac{d-1}{2}} H_{d-1}(x)^{-\frac{1}{2}}.\end{aligned}$$

Thus we have shown that

$$\lim_{t\to 0^+} t^{-\frac{d-3}{2}} \cdot \frac{\partial s}{\partial t}(t) = (d-1) \cdot \varrho(x) 2^{\frac{d-3}{2}} \langle x, u(x)\rangle^{\frac{d-1}{2}} H_{d-1}(x)^{-\frac{1}{2}} \alpha_{d-1}. \tag{6.3.3}$$

In the same way, we also obtain

$$\lim_{t\to 0^+} t^{-\frac{d-1}{2}} \cdot s(t) = \varrho(x) 2^{\frac{d-1}{2}} \langle x, u(x)\rangle^{\frac{d-1}{2}} H_{d-1}(x)^{-\frac{1}{2}} \alpha_{d-1}. \tag{6.3.4}$$

Observe that (6.3.3) and (6.3.4) are valid also for $d = 2$. In particular, (6.3.3) and (6.3.4) imply that $J_\varrho(y, L)$ can be rewritten as (cf. (6.2.13))

$$J_\varrho(y, L) = (d-1)^{-1} G(x)^2 \lim_{n\to\infty} \int_0^{\zeta(y,n)} n^{\frac{2}{d-1}} \mathbb{P}_\varrho\left(\tilde{y}_s \notin K_n | L\right) s^{-\frac{d-3}{d-1}}\, ds, \tag{6.3.5}$$

where

$$G(x) := (\alpha_{d-1})^{\frac{-1}{d-1}} \varrho(x)^{\frac{-1}{d-1}} H_{d-1}(x)^{\frac{1}{2(d-1)}}$$

and

$$\lim_{n\to\infty} n^{\frac{1}{2}} \zeta(y, n) = \alpha_{d-1} \varrho(x) (2\langle u(x), x\rangle)^{\frac{d-1}{2}} H_{d-1}(x)^{-\frac{1}{2}}.$$

Now we show that in the domain of integration $\zeta(y, n)$ can be replaced by $n^{-1/2}$, that is

$$J_\varrho(y, L) = (d-1)^{-1} G(x)^2 \lim_{n\to\infty} \int_0^{n^{-1/2}} n^{\frac{2}{d-1}} \mathbb{P}_\varrho\left(\tilde{y}_s \notin K_n | L\right) s^{-\frac{d-3}{d-1}}\, ds. \tag{6.3.6}$$

It follows from Lemma 6.2.1 and (6.3.4) that there exist constants $c_0 > 0$ and $c_2 > c_1 > 0$ depending on y, K, L, ϱ such that if $s > 0$ is small enough, then

$$\mathbb{P}_\varrho\left(\tilde{y}_s \notin K_n | L\right) \ll (1 - c_0 s)^n,$$

and if n is large and s is between $\zeta(n, y)$ and $n^{-1/2}$, then $c_1 n^{-1/2} < s < c_2 n^{-1/2}$. In particular,

$$\begin{aligned}&\lim_{n\to\infty} \int_{c_1 n^{-1/2}}^{c_2 n^{-1/2}} n^{\frac{2}{d-1}} \mathbb{P}_\varrho\left(\tilde{y}_s \notin K_n | L\right) s^{-\frac{d-3}{d-1}}\, ds \\ &\ll \lim_{n\to\infty} n^{\frac{2}{d-1}} \int_{c_1 n^{-1/2}}^{c_2 n^{-1/2}} e^{-c_0 n s} s^{-\frac{d-3}{d-1}}\, ds \\ &\le \lim_{n\to\infty} c_2 n^{\frac{2}{d-1} - \frac{1}{2}} e^{-c_1 c_0 n^{\frac{1}{2}}} c_1^{-\frac{d-3}{d-1}} n^{\frac{d-3}{2(d-1)}} = 0,\end{aligned}$$

and hence (6.3.5) yields (6.3.6).

Let $\pi : \mathbb{R}^d \to u(x)^\perp$ denote the orthogonal projection to $u(x)^\perp$. Using (6.2.5), (6.2.3) and (6.3.4), we obtain

$$\begin{aligned} \lim_{s\to 0^+} s^{\frac{-1}{d-1}} \|\pi(x-\tilde{x}_s)\| &= 0, \\ \lim_{s\to 0^+} s^{\frac{-2}{d-1}} \langle u(x), x-\tilde{x}_s\rangle &= \frac{1}{2}G(x)^2. \end{aligned} \tag{6.3.7}$$

Let Q denote the second fundamental form of ∂K at x (cf. (2.2.1)), considered as a function on $u(x)^\perp$. Then there are an orthonormal basis $v_1,\dots,v_{d-1}$ of $u(x)^\perp$ and positive numbers $k_1,\dots,k_{d-1}>0$ such that

$$Q\left(\sum_{i=1}^{d-1} z_i v_i\right) = \sum_{i=1}^{d-1} k_i z_i^2.$$

Further, let π be the orthogonal projection to $u(x)^\perp$, and define

$$E := \{z \in u(x)^\perp : Q(z) \le 1\},$$

which is the Dupin indicatrix of K at x, whose half axes are $k_i(x)^{-1/2}$, $i=1,\dots,d-1$. In addition, let Γ be the convex hull of the osculating paraboloid of K at $x\in\partial K$, that is

$$\Gamma = \{x+z-tu(x) : z\in u(x)^\perp, t\ge \tfrac{1}{2}Q(z)\}.$$

Hence, we have

$$\Gamma \cap H(x,t) = x_t^* + \sqrt{2t\langle x, u(x)\rangle}\, E,$$

and there exists an increasing function $\tilde{\mu}(s)$ with $\lim_{s\to 0^+}\tilde{\mu}(s)=1$ such that

$$\tilde{x}_s^* + \tilde{\mu}(s)^{-1}G(x)\cdot s^{\frac{1}{d-1}} E \subset K\cap \widetilde{H}(x,s) \subset \tilde{x}_s^* + \tilde{\mu}(s)G(x)\cdot s^{\frac{1}{d-1}} E, \tag{6.3.8}$$

where $\tilde{x}_s^* := x_t^* \in (x-\mathbb{R}_+u(x))\cap \widetilde{H}(x,s)$, and s and t are related by equation (6.3.1). From (6.3.7) it follows that also

$$\tilde{x}_s + \tilde{\mu}(s)^{-1}G(x)\cdot s^{\frac{1}{d-1}} E \subset K\cap \widetilde{H}(x,s) \subset \tilde{x}_s + \tilde{\mu}(s)G(x)\cdot s^{\frac{1}{d-1}} E, \tag{6.3.9}$$

The rest of the proof is devoted to identifying the asymptotic behaviour of the integral (6.3.6). First, we adjust the domain of integration and the integrand in a suitable way. In a second step, the resulting expression is compared to the case where K is the unit ball. We recall that $x_1,\dots,x_n$ are random points in ∂K, and we put $\Xi_n := \{x_1,\dots,x_n\}$, hence $K_n = [\Xi_n]$. For a finite set $X\subset\mathbb{R}^d$, let $\#X$ denote the cardinality of X.

Lemma 6.3.1. *For $\varepsilon\in(0,1)$, there exist $\alpha,\beta>1$ and an integer $k>d$, depending only on ε and d, with the following property. If $L\in\mathcal{L}_j^d$, $y\in\partial(K|L)$, $x\in\partial K$ is a normal boundary point of K such that $y=x|L$ and $H_{d-1}(x)>0$, and if $n>n_0$, where n_0 depends on $\varepsilon, x, K, \varrho, L$, then*

$$\int_0^{n^{-1/2}} \mathbb{P}_\varrho\left(\tilde{y}_s\notin K_n|L\right) s^{-\frac{d-3}{d-1}}\,ds = \int_{\frac{\varepsilon^{(d-1)/2}}{n}}^{\frac{\alpha}{n}} \varphi(K,L,y,\varrho,\varepsilon,s) s^{-\frac{d-3}{d-1}}\,ds + O\left(\frac{\varepsilon}{n^{\frac{2}{d-1}}}\right),$$

where

$$\varphi(K,L,y,\varrho,\varepsilon,s) = \mathbb{P}_\varrho\left(\left(\tilde{y}_s\notin([\widetilde{C}(x,\beta s)\cap\Xi_n]|L)\right) \text{ and } \left(\#(\widetilde{C}(x,\beta s)\cap\Xi_n)\le k\right)\right).$$

Proof. Let $\varepsilon \in (0,1)$ be given. Then $\alpha > 1$ is chosen such that

$$2^{d-1+\frac{2d}{d-1}} \int_{2^{-d}\alpha}^{\infty} e^{-r} r^{\frac{2}{d-1}-1}\, dr < \varepsilon. \tag{6.3.10}$$

Further, we choose $\beta \geq (16^2(d-1))^{d-1}$ such that

$$2^{d-1} e^{-2^{-3d+2}\sqrt{\beta}\cdot\varepsilon^{\frac{d-1}{2}}} < \varepsilon \cdot \alpha^{\frac{-2}{d-1}}, \tag{6.3.11}$$

and then we fix an integer $k > d$ such that

$$\frac{(\alpha\beta)^k}{k!} < \frac{\varepsilon}{\alpha^{\frac{2}{d-1}}}. \tag{6.3.12}$$

Lemma 6.3.1 follows from the following three statements, which we will prove assuming that n is sufficiently large.

(i)

$$\int_0^{n^{-1/2}} \mathbb{P}_\varrho\left(\tilde{y}_s \notin K_n|L\right) s^{-\frac{d-3}{d-1}}\, ds = \int_{\frac{\varepsilon^{(d-1)/2}}{n}}^{\frac{\alpha}{n}} \mathbb{P}_\varrho\left(\tilde{y}_s \notin K_n|L\right) s^{-\frac{d-3}{d-1}}\, ds + O\left(\frac{\varepsilon}{n^{\frac{2}{d-1}}}\right).$$

(ii) If $\varepsilon^{(d-1)/2}/n < s < \alpha/n$, then

$$\mathbb{P}_\varrho\left(\#\left(\widetilde{C}(x,\beta s)\cap \Xi_n\right) \geq k\right) \leq \frac{\varepsilon}{\alpha^{\frac{2}{d-1}}}.$$

(iii) If $\varepsilon^{(d-1)/2}/n < s < \alpha/n$, then

$$\mathbb{P}_\varrho\left(\tilde{y}_s \notin K_n|L\right) = \mathbb{P}_\varrho\left(\tilde{y}_s \notin \left[(\widetilde{C}(x,\beta s)\cap \Xi_n)|L\right]\right) + O\left(\frac{\varepsilon}{\alpha^{\frac{2}{d-1}}}\right).$$

Before proving (i), (ii) and (iii), we note that they imply

$$\begin{aligned}\int_0^{n^{-1/2}} \mathbb{P}_\varrho\left(\tilde{y}_s \notin K_n|L\right) s^{-\frac{d-3}{d-1}}\, ds \;&=\; \int_{\frac{\varepsilon^{(d-1)/2}}{n}}^{\frac{\alpha}{n}} \varphi(K,L,y,\varrho,\varepsilon,s) s^{-\frac{d-3}{d-1}}\, ds + \\ &\quad + O\left(\frac{\varepsilon}{\alpha^{\frac{2}{d-1}}}\right) \int_{\frac{\varepsilon^{(d-1)/2}}{n}}^{\frac{\alpha}{n}} s^{-\frac{d-3}{d-1}}\, ds + O\left(\frac{\varepsilon}{n^{\frac{2}{d-1}}}\right),\end{aligned}$$

which in turn yields Lemma 6.3.1.

First, we introduce some notation. As before, let Q be the second fundamental form at $x \in \partial K$, and let $v_1,\ldots,v_{d-1}$ be an orthonormal basis of $u(x)^\perp$ representing the principal directions. In addition, let $\Theta_1',\ldots,\Theta_{2^{d-1}}'$ be the corresponding coordinate corners, and for $i = 1,\ldots,2^{d-1}$ and $s \in (0, n^{-1/2})$, let

$$\widetilde{\Theta}_{i,s} = \widetilde{C}(x,s) \cap \left(\tilde{x}_s + \left[\Theta_i', \mathbb{R}_+ x\right]\right).$$

Subsequently, we show that

$$\lim_{s\to 0^+} s^{-1}\int_{\widetilde{\Theta}_{i,s}\cap\partial K}\varrho(z)\,\mathcal{H}^{d-1}(dz)=2^{-(d-1)}. \tag{6.3.13}$$

In fact, since a ball rolls freely inside K, ϱ is continuous and positive at x, and by (6.3.7) we deduce that

$$\begin{aligned}
&\lim_{s\to 0^+} s^{-1}\int_{\widetilde{\Theta}_{i,s}\cap\partial K}\varrho(z)\,\mathcal{H}^{d-1}(dz)\\
&=\varrho(x)\lim_{s\to 0^+} s^{-1}\mathcal{H}^{d-1}\left(\widetilde{\Theta}_{i,s}\cap\partial K\right)\\
&=\varrho(x)\lim_{s\to 0^+} s^{-1}\mathcal{H}^{d-1}\left(\partial K\cap\widetilde{C}(x,s)\cap\left(\tilde{x}_s^*+[\Theta_i',\mathbb{R}_+u(x)]\right)\right).
\end{aligned}$$

Let $\Psi:\partial\Gamma\cap C(x,r/R)\to\partial K\cap C(x,r/R)$ be the diffeomorphism which assigns to a point $z\in\partial\Gamma\cap\widetilde{H}(x,s)$ the unique point $\Psi(z)\in\partial K\cap(\tilde{x}_s^*+\mathbb{R}_+(z-\tilde{x}_s^*))$. It follows from (6.3.7) that there exists an increasing function $\mu:\mathbb{R}_+\to\mathbb{R}_+$ with $\lim_{s\to 0^+}\mu(s)=1$ such that

$$\mu(s)^{-1}\le \mathrm{Lip}(\psi|(\partial\Gamma\cap\widetilde{C}(x,s)))\le\mu(s).$$

Thus we get

$$\begin{aligned}
&\lim_{s\to 0^+} s^{-1}\mathcal{H}^{d-1}\left(\partial K\cap\widetilde{C}(x,s)\cap\left(\tilde{x}_s^*+[\Theta_i',\mathbb{R}_+u(x)]\right)\right)\\
&=\lim_{s\to 0^+} s^{-1}\mathcal{H}^{d-1}\left(\Psi\left(\partial\Gamma\cap\widetilde{C}(x,s)\cap\left(\tilde{x}_s^*+[\Theta_i',\mathbb{R}_+u(x)]\right)\right)\right)\\
&=\lim_{s\to 0^+} s^{-1}\mathcal{H}^{d-1}\left(\partial\Gamma\cap\widetilde{C}(x,s)\cap\left(\tilde{x}_s^*+[\Theta_i',\mathbb{R}_+u(x)]\right)\right)\\
&=2^{-(d-1)}\lim_{s\to 0^+} s^{-1}\mathcal{H}^{d-1}\left(\partial\Gamma\cap\widetilde{C}(x,s)\right).
\end{aligned}$$

Now we can repeat the preceding argument in reverse order and finally use (6.3.1) to arrive at the assertion (6.3.13).

To prove (i), we observe that

$$\int_0^{\frac{\varepsilon^{(d-1)/2}}{n}}\mathbb{P}_\varrho\left(\tilde{y}_s\notin K_n|L\right)s^{-\frac{d-3}{d-1}}\,ds\le\int_0^{\frac{\varepsilon^{(d-1)/2}}{n}}s^{-\frac{d-3}{d-1}}\,ds\ll\frac{\varepsilon}{n^{\frac{2}{d-1}}}.$$

Let $\alpha/n<s<n^{-1/2}$, and let n be sufficiently large. First, (6.2.6) yields that

$$\mathbb{P}_\varrho\left(o\notin K_n,\ \tilde{y}_s\notin K_n|L\right)\le\varepsilon n^{-\frac{2}{d-1}}.$$

On the other hand, if $o\in K_n$, then $\tilde{y}_s\notin K_n|L$ implies that $\widetilde{\Theta}_{i,s}\cap K_n=\emptyset$ for some $i\in\{1,\dots,2^{d-1}\}$, and hence (6.3.13) yields

$$\mathbb{P}_\varrho\left(o\in K_n,\ \tilde{y}_s\notin K_n|L\right)\le 2^{d-1}(1-2^{-d}s)^n<2^{d-1}e^{-2^{-d}ns}. \tag{6.3.14}$$

Therefore, by (6.3.10) we get

$$\begin{aligned}\int_{\alpha/n}^{n^{-1/2}} \mathbb{P}_\varrho\left(\tilde{y}_s \notin K_n|L\right) s^{-\frac{d-3}{d-1}}\, ds &\ll 2^{d-1}\int_{\alpha/n}^{\infty} e^{-2^{-d}ns} s^{\frac{2}{d-1}-1}\, ds + \frac{\varepsilon}{n^{\frac{2}{d-1}}} \\ &= \frac{2^{d-1+\frac{2d}{d-1}}}{n^{\frac{2}{d-1}}}\int_{2^{-d}\alpha}^{\infty} e^{-r} r^{\frac{2}{d-1}-1}\, dr + \frac{\varepsilon}{n^{\frac{2}{d-1}}} \\ &\leq \frac{2\varepsilon}{n^{\frac{2}{d-1}}},\end{aligned}$$

which verifies (i).

Next (ii) simply follows from (6.3.1) and (6.3.12). In fact, if $0 < s < \alpha/n$, then

$$\mathbb{P}_\varrho\left(\#\left(\widetilde{C}(x,\beta s)\cap \Xi_n\right) \geq k\right) \leq \binom{n}{k}(\beta s)^k \leq \binom{n}{k}\left(\frac{\alpha\beta}{n}\right)^k < \frac{(\alpha\beta)^k}{k!} \leq \frac{\varepsilon}{\alpha^{\frac{2}{d-1}}}.$$

Finally, we prove (iii). To this end, if $\varepsilon^{(d-1)/2}/n < s < \alpha/n$ and $i \in \{1,\dots,2^{d-1}\}$, then we define $w_i \in \Theta_i'$ by

$$w_i := \left(\sqrt{\beta}s\right)^{\frac{1}{d-1}} \sum_{m=1}^{d-1} \frac{\eta_m G(x)}{4\sqrt{(d-1)k_m(x)}}\, v_m, \tag{6.3.15}$$

where $\eta_m = \eta_m^i \in \{-1,1\}$ for $m = 1,\dots,2^{d-1}$. Now let

$$\widetilde{\Omega}_{i,s} := \partial K \cap [\tilde{x}_s + \Theta_i', \tilde{x}_{\sqrt{\beta}\, s} + w_i + \Theta_i'].$$

We claim that for large n, if $\tilde{y}_s \in K_n|L$ but $\tilde{y}_s \notin \left[(\widetilde{C}(x,\beta s)\cap \Xi_n)|L\right]$, then there exists $i \in \{1,\dots,2^{d-1}\}$ such that

$$\Xi_n \cap \widetilde{\Omega}_{i,s} = \emptyset. \tag{6.3.16}$$

Moreover, for all $i = 1,\dots,2^{d-1}$, we have

$$\int_{\widetilde{\Omega}_{i,s}} \varrho(z)\, \mathcal{H}^{d-1}(dz) \geq 2^{-3d+2}\sqrt{\beta}s. \tag{6.3.17}$$

To justify (6.3.17), let $i \in \{1,\dots,2^{d-1}\}$ be fixed. It follows from the definition of w_i that

$$w_i \in \left(\sqrt{\beta}s\right)^{\frac{1}{d-1}} \frac{G(x)}{4}\cdot \partial E.$$

Recall that $\pi : \mathbb{R}^d \to u(x)^\perp$ denotes the orthogonal projection to $u(x)^\perp$. If n is large enough, and hence $0 < s < \alpha/n$ is sufficiently small, then (6.3.7), (6.3.9) and (6.3.15) yield that $w_i \in \pi(\widetilde{\Omega}_{i,s})$, since by assumption $\sqrt{\beta}^{1/(d-1)}/4 > 2$, and therefore

$$(w_i + \Theta_i') \cap \left(w_i + \left(\sqrt{\beta}s\right)^{\frac{1}{d-1}} \frac{G(x)}{4}\cdot E\right) \subset \pi(\widetilde{\Omega}_{i,s}).$$

In particular, (6.3.17) now follows from

$$\begin{aligned}\int_{\widetilde{\Omega}_{i,s}} \varrho(z)\,\mathcal{H}^{d-1}(dz) &\geq \frac{\varrho(x)}{2}\cdot\mathcal{H}^{d-1}(\widetilde{\Omega}_{i,s})\\ &\geq \frac{\varrho(x)}{2}\cdot\mathcal{H}^{d-1}(\pi(\widetilde{\Omega}_{i,s}))\\ &\geq \frac{\varrho(x)}{2}\cdot\frac{1}{2^{d-1}}\sqrt{\beta}s\frac{G(x)^{d-1}}{4^{d-1}}\alpha_{d-1}H_{d-1}(x)^{-1/2}\\ &= 2^{-d}4^{1-d}\sqrt{\beta}s.\end{aligned}$$

Next we verify (6.3.16). We assume that $\tilde{y}_s \in K_n|L$ but $\tilde{y}_s \notin \left[(\widetilde{C}(x,\beta s)\cap\Xi_n)|L\right]$. Then there exist $a \in \left[(\widetilde{C}(x,\beta s)\cap\Xi_n)|L\right]$ and $b \in \left(K_n\setminus\widetilde{C}(x,\beta s)\right)|L$ such that $\tilde{y}_s \in (a,b)$. Thus there exists a hyperplane H in $\mathbb{R}^d$ containing $\tilde{y}_s + L^\perp$ and bounding the half-spaces H^+ and H^- such that $\widetilde{C}(x,\beta s)\cap\Xi_n \subset \operatorname{int}(H^+)$ and $b \in \operatorname{int}(H^-)$. In addition, there exists $i \in \{1,\ldots,2^{d-1}\}$ such that

$$\tilde{x}_s + \Theta_i' \subset H^-. \tag{6.3.18}$$

Now we define points q and q' by

$$\{q\} = [\tilde{y}_s, b]\cap\widetilde{H}(x,\sqrt{\beta}s), \qquad \{q'\} = [\tilde{y}_s, b]\cap\widetilde{H}(x,\beta s).$$

Relation (6.3.7) implies that

$$\widetilde{H}(x,\beta s)\cap K \subset \tilde{x}^*_{\beta s} + 2G(x)(\beta s)^{\frac{1}{d-1}}E$$

if $s>0$ is sufficiently small. Arguing as in [BFH10], we obtain that

$$\langle u(x), \tilde{y}_s - \tilde{y}_{\beta s}\rangle < \frac{\beta^{1/(d-1)}}{\beta^{1/(d-1)}-1}\langle u(x), \tilde{y}_{\sqrt{\beta}s} - \tilde{y}_{\beta s}\rangle$$

and

$$\frac{\|q-\tilde{y}_{\sqrt{\beta}s}\|}{\|q'-\tilde{y}_{\beta s}\|} = \frac{\langle u(x), \tilde{y}_s - \tilde{y}_{\sqrt{\beta}s}\rangle}{\langle u(x), \tilde{y}_s - \tilde{y}_{\beta s}\rangle},$$

which yields (cf. [BFH10])

$$q \in \tilde{y}_{\sqrt{\beta}s} + 2s^{\frac{1}{d-1}}G(x)E.$$

Since $\beta \geq [8^2(d-1)]^{d-1}$, we thus arrive at

$$q \in \tilde{y}_{\sqrt{\beta}s} + \frac{1}{4\sqrt{d-1}}(\sqrt{\beta}s)^{\frac{1}{d-1}}G(x)E. \tag{6.3.19}$$

Now (6.3.18) implies that $q + \Theta_i' \subset H^-$. Hence it follows from (6.3.19) that $\tilde{y}_{\sqrt{\beta}s} + w_i \subset q + \Theta_i' \subset H^-$, and therefore also $\tilde{y}_{\sqrt{\beta}s} + w_i + \Theta_i' \subset H^-$. Thus $\widetilde{\Omega}_{i,s} \subset H^-$, which yields $\Xi_n \cap \widetilde{\Omega}_{i,s} = \emptyset$.

Assertion (iii) follows from (6.3.16) and (6.3.17). In fact, if $\varepsilon^{(d-1)/2}/n < s < \alpha/n$, then

$$\mathbb{P}_\varrho\left(\tilde{y}_s \notin \left[(\widetilde{C}(y,\beta s)\cap\Xi_n)|L\right]\right) - \mathbb{P}_\varrho\left(\tilde{y}_s \notin (K_n|L)\right)$$

$$\leq \sum_{i=1}^{2^{d-1}} \left(1 - \int_{\widetilde{\Omega}_{i,s}} \varrho(z)\, \mathcal{H}^{d-1}(dz)\right)^n$$
$$\leq 2^{d-1} e^{-2^{-3d+2}\sqrt{\beta}\cdot sn}$$
$$\leq \varepsilon\, \alpha^{-\frac{2}{d+1}},$$

by the choice of β. □

To actually compare the situation near the normal boundary point x of K with $H_{d-1}(x) > 0$ to the case of the unit ball, let $\sigma = (d\alpha_d)^{-1}$ be the constant density of the corresponding probability distribution on S^{d-1}. Let $w \in S^{d-1}$ be the d-th coordinate vector in $\mathbb{R}^d$, and hence $\mathbb{R}^{d-1} = w^\perp$. We write B_n to denote the convex hull of n random points distributed uniformly and independently on S^{d-1} according to σ. For $s \in (0, \frac{1}{2})$, we fix a linear subspace $L_0 \in \mathcal{L}_j^d$ with $w \in L_0$, and let $\tilde{w}_s$ be of the form λw for $\lambda \in (0,1)$ such that

$$(d\alpha_d)^{-1} \cdot \mathcal{H}^{d-1}(\{z \in S^{d-1} : \langle z, w\rangle \geq \langle \tilde{w}_s, w\rangle\}) = s.$$

In particular, $\tilde{w}_s | L_0 = \tilde{w}_s$.

Lemma 6.3.2. *If $L \in \mathcal{L}_j^d$, $y \in \partial(K|L)$ and $x \in \partial K$ is a normal boundary point such that $y = x|L$ and $H_{d-1}(x) > 0$, then*

$$\lim_{n\to\infty} \int_0^{n^{-1/2}} n^{\frac{2}{d-1}} \mathbb{P}_\varrho\left(\tilde{y}_s \notin K_n|L\right) s^{-\frac{d-3}{d-1}}\, ds = \lim_{n\to\infty} \int_0^{n^{-1/2}} n^{\frac{2}{d-1}} \mathbb{P}_\sigma\left(\tilde{w}_s \notin B_n|L_0\right) s^{-\frac{d-3}{d-1}}\, ds.$$

Proof. First, we assume $d \geq 3$. It is sufficient to prove that for any $\varepsilon \in (0,1)$ there exists $n_0 > 0$, depending on $\varepsilon, x, K, \varrho, L$, such that if $n > n_0$, then

$$\int_0^{n^{-1/2}} \mathbb{P}_\varrho\left(\tilde{y}_s \notin K_n|L\right) s^{-\frac{d-3}{d-1}}\, ds = \int_0^{n^{-1/2}} \mathbb{P}_\sigma\left(\tilde{w}_s \notin B_n|L_0\right) s^{-\frac{d-3}{d-1}}\, ds + O\left(\frac{\varepsilon}{n^{\frac{2}{d-1}}}\right). \tag{6.3.20}$$

Let α, β and k be the quantities associated with $\varepsilon, x, K, \varrho, L$ in Lemma 6.3.1, let $\widetilde{C}(x,s)$ denote the cap of K defined in (6.3.2), and let $\widetilde{C}(w,s)$ denote the corresponding cap of B^d at w. We define the densities ϱ_s on $\partial\widetilde{C}(x,\beta s)$ and σ_s on $\partial\widetilde{C}(w,\beta s)$ of probability distributions by

$$\varrho_s(z) = \begin{cases} \varrho(z)/(\beta s), & \text{if } z \in \partial K \cap \widetilde{C}(x,\beta s), \\ 0, & \text{if } z \in \partial\widetilde{C}(x,\beta s)\setminus\partial K, \end{cases}$$
$$\sigma_s(z) = \begin{cases} \sigma(z)/(\beta s), & \text{if } z \in S^{d-1} \cap \widetilde{C}(w,\beta s), \\ 0, & \text{if } z \in \partial\widetilde{C}(w,\beta s)\setminus S^{d-1}. \end{cases}$$

For $i = 0, \ldots, k$, we write $\widetilde{C}(x,\beta s)_i$ and $\widetilde{C}(w,\beta s)_i$ to denote the convex hulls of i random points distributed uniformly and independently on $\partial\widetilde{C}(x,\beta s)$ and $\partial\widetilde{C}(w,\beta s)$ according to ϱ_s and σ_s, respectively.

If n is large, then Lemma 6.3.1 yields that the left-hand and the right-hand side of (6.3.20) are

$$O\left(\frac{\varepsilon}{n^{\frac{2}{d-1}}}\right)+\sum_{i=0}^{k}\binom{n}{i}\int_{\frac{\varepsilon^{(d-1)/2}}{n}}^{\frac{\alpha}{n}}(\beta s)^{i}(1-\beta s)^{n-i}\times\mathbb{P}_{\varrho_s}\left(\tilde{y}_s\not\in\widetilde{C}(x,\beta s)_i|L\right)s^{-\frac{d-3}{d-1}}\,ds,$$

$$O\left(\frac{\varepsilon}{n^{\frac{2}{d-1}}}\right)+\sum_{i=0}^{k}\binom{n}{i}\int_{\frac{\varepsilon^{(d-1)/2}}{n}}^{\frac{\alpha}{n}}(\beta s)^{i}(1-\beta s)^{n-i}\times\mathbb{P}_{\sigma_s}\left(\tilde{w}_s\not\in\widetilde{C}(w,\beta s)_i|L_0\right)s^{-\frac{d-3}{d-1}}\,ds.$$

For each $i\leq k$, the representation of the beta function by the gamma function and the Stirling formula (see E. Artin [Art64]) imply

$$\lim_{n\to\infty}n^{\frac{2}{d-1}}\binom{n}{i}\int_{0}^{1/\beta}(\beta s)^{i}(1-\beta s)^{n-i}s^{-\frac{d-3}{d-1}}\,ds=\frac{\beta^{\frac{-2}{d-1}}\Gamma\left(i+\frac{2}{d-1}\right)}{i!}<1. \tag{6.3.21}$$

Therefore to prove (6.3.20), it is sufficient to verify that for each $i=0,\ldots,k$, if $s>0$ is small, then

$$\left|\mathbb{P}_{\varrho_s}\left(\tilde{y}_s\not\in\widetilde{C}(x,\beta s)_i|L\right)-\mathbb{P}_{\sigma_s}\left(\tilde{w}_s\not\in\widetilde{C}(w,\beta s)_i|L_0\right)\right|\ll\frac{\varepsilon}{k}. \tag{6.3.22}$$

If $i\leq j$, then (6.3.22) readily holds as its left-hand side is zero.

To prove (6.3.22) if $i\in\{j+1,\ldots,k\}$, we transform both K and B^d in such a way that their osculating paraboloid is $\Omega=\{z-\|z\|^2\,w\,:\,z\in\mathbb{R}^{d-1}\}$, and the images of the caps $\widetilde{C}(x,\beta s)$ and $\widetilde{C}(w,\beta s)$ are very close. Using these caps, we construct equivalent representations of $\mathbb{P}_{\varrho_s}\left(\tilde{y}_s\not\in\widetilde{C}(x,\beta s)_i|L\right)$ and $\mathbb{P}_{\sigma_s}\left(\tilde{w}_s\not\in\widetilde{C}(w,\beta s)_i|L_0\right)$, based on the same space Ξ_s and on comparable probability measures and random variables.

We may assume that $u(x)=w$. Let $v_1,\ldots,v_{d-1}$ be an orthonormal basis of $w^{\perp}$ in the principal directions of the fundamental form Q of K at $x\in\partial K$. We define the linear transform A_s of $\mathbb{R}^d$ by

$$\begin{aligned}A_s(w)&=2(\beta s)^{\frac{-2}{d-1}}G(x)^{-2}w,\\A_s(v_i)&=(\beta s)^{\frac{-1}{d-1}}\sqrt{k_i(x)}G(x)^{-1}v_i,\quad i=1,\ldots,d-1,\end{aligned}$$

and choose an orthonormal linear transform P_s such that $P_sw=w$, and $P_s\circ A_s(L^{\perp})=L_0^{\perp}$. Based on these linear transforms, let Φ_s be the affine transformation

$$\Phi_s(z)=P_s\circ A_s(z-x).$$

In addition, we define the linear transform R_s of $\mathbb{R}^d$ by

$$\begin{aligned}R_s(w)&=2(\beta s)^{\frac{-2}{d-1}}\left(\frac{\alpha_{d-1}}{d\alpha_d}\right)^{\frac{2}{d-1}}w,\\R_s(v_i)&=(\beta s)^{\frac{-1}{d-1}}\left(\frac{\alpha_{d-1}}{d\alpha_d}\right)^{\frac{1}{d-1}}v_i,\quad i=1,\ldots,d-1,\end{aligned}$$

and let Ψ_s be the affine transformation

$$\Psi_s(z)=R_s(z-x).$$

Subsequently, we also write $\Phi_s z$ for $\Phi_s(z)$ or $\Phi_s z|L_0$ for $\Phi_s(z)|L_0$, and similarly for Ψ_s. We observe that Ω is the osculating paraboloid of both $\Phi_s K$ and $\Psi_s B^d$ at o, and

$$\begin{aligned}
\lim_{s\to 0^+} \Phi_s \tilde{x}_s = \lim_{s\to 0^+} \Psi_s \tilde{w}_s &= -\beta^{\frac{-2}{d-1}} w =: w^*, \\
\lim_{s\to 0^+} \Phi_s \widetilde{C}(x,\beta s) = \lim_{s\to 0^+} \Psi_s \widetilde{C}(w,\beta s) &= \{z - \tau\, w : \, z \in B^{d-1} \text{ and } \|z\|^2 \le \tau \le 1\}.
\end{aligned}$$

For $p \in \widetilde{C}(x,\beta s) \cap \partial K$ and $z = \pi \circ \Phi_s(p)$, let $D(p)$ be the Jacobian of $\pi \circ \Phi_s$ at p as a map $\pi \circ \Phi_s : \widetilde{C}(x,\beta s) \cap \partial K \to \mathbb{R}^{d-1}$, and let

$$\tilde{\varrho}_s(z) = \varrho_s(p) \cdot D(p)^{-1}.$$

In addition, for $p \in \widetilde{C}(w,\beta s) \cap S^{d-1}$ and $z = \pi \circ \Psi_s(p)$, let $\widetilde{D}(p)$ be the Jacobian of $\pi \circ \Psi_s$ at p as a map $\pi \circ \Psi_s : \widetilde{C}(w,\beta s) \cap S^{d-1} \to \mathbb{R}^{d-1}$, and let

$$\tilde{\sigma}_s(z) = \sigma_s(p) \cdot \widetilde{D}(p)^{-1}.$$

We define

$$\Xi_s = \left[\pi \circ \Phi_s \widetilde{C}(x,\beta s)\right] \cup \left[\pi \circ \Psi_s \widetilde{C}(w,\beta s)\right],$$

and extend $\tilde{\varrho}_s$ and $\tilde{\sigma}_s$ to Ξ_s by

$$\begin{aligned}
\tilde{\varrho}_s(z) &= 0, \ \text{ if } \ z \in \left[\pi \circ \Psi_s \widetilde{C}(w,\beta s)\right] \setminus \left[\pi \circ \Phi_s \widetilde{C}(x,\beta s)\right], \\
\tilde{\sigma}_s(z) &= 0, \ \text{ if } \ \left[\pi \circ \Phi_s \widetilde{C}(x,\beta s)\right] \setminus \left[\pi \circ \Psi_s \widetilde{C}(w,\beta s)\right].
\end{aligned}$$

Therefore $\tilde{\varrho}_s$ and $\tilde{\sigma}_s$ are densities of probability distributions on Ξ_s. For $z \in \Xi_s$, let $\varphi_s(z) \in \Phi_s \partial K$ and $\psi_s(z) \in \Psi_s S^{d-1}$ be the points near z whose orthogonal projection into $\mathbb{R}^{d-1}$ is z. For random variables $z_1, \ldots, z_i \in \Xi_s$ either with respect to $\tilde{\varrho}_s$ or $\tilde{\sigma}_s$, the quantities above were defined so as to satisfy

$$\mathbb{P}_{\varrho_s}\left(\tilde{y}_s \notin \widetilde{C}(x,\beta s)_i | L\right) = \mathbb{P}_{\tilde{\varrho}_s}\left(\Phi_s \tilde{x}_s | L_0 \notin [\varphi_s(z_1), \ldots, \varphi_s(z_i)] | L_0\right), \tag{6.3.23}$$

$$\mathbb{P}_{\sigma_s}\left(\tilde{w}_s \notin \widetilde{C}(w,\beta s)_i | L\right) = \mathbb{P}_{\tilde{\sigma}_s}\left(\Psi_s \tilde{w}_s \notin [\psi_s(z_1), \ldots, \psi_s(z_i)] | L_0\right). \tag{6.3.24}$$

Now there exists an increasing function $s \mapsto \mu^*(s)$ with $\lim_{s\to 0^+} \mu^*(s) = 1$ such that

$$\mu^*(s)^{-1} B^{d-1} \subset \left[\pi \circ \Phi_s \widetilde{C}(x,\beta s)\right] \cap \left[\pi \circ \Psi_s \widetilde{C}(w,\beta s)\right] \subset \Xi_s \subset \mu^*(s) B^{d-1},$$

we have $\mu^*(s)^{-1} \varphi_s(z) \le \psi_s(z) \le \mu^*(s) \varphi_s(z)$ for all $z \in \Xi_s$, and

$$\begin{aligned}
\mu^*(s)^{-1} \alpha_{d-1}^{-1} &\le \tilde{\varrho}_s(z) \le \mu^*(s) \alpha_{d-1}^{-1}, && \text{if } z \in \pi \circ \Phi_s \widetilde{C}(x,\beta s), \\
\mu^*(s)^{-1} \alpha_{d-1}^{-1} &\le \tilde{\sigma}_s(z) \le \mu^*(s) \alpha_{d-1}^{-1}, && \text{if } z \in \pi \circ \Psi_s \widetilde{C}(w,\beta s).
\end{aligned}$$

Therefore

$$\lim_{s\to 0^+} \int_{\Xi_s} |\tilde{\varrho}_s(z) - \tilde{\sigma}_s(z)|\, \mathcal{H}^{d-1}(dz) = 0. \tag{6.3.25}$$

From (6.3.25) we deduce that if $s > 0$ is small, then

$$\begin{aligned} &|\mathbb{P}_{\tilde{\varrho}_s}\left(\Phi_s\tilde{x}_s|L_0 \notin [\varphi_s(z_1),\ldots,\varphi_s(z_i)]|L_0 \text{ and } \Psi_s\tilde{w}_s \notin [\psi_s(z_1),\ldots,\psi_s(z_i)]|L_0\right) \qquad (6.3.26)\\ &-\mathbb{P}_{\tilde{\sigma}_s}\left(\Phi_s\tilde{x}_s|L_0 \notin [\varphi_s(z_1),\ldots,\varphi_s(z_i)]|L_0 \text{ and } \Psi_s\tilde{w}_s \notin [\psi_s(z_1),\ldots,\psi_s(z_i)]|L_0\right)| \ \leq \ \frac{\varepsilon}{k}. \end{aligned}$$

Next, if $s > 0$ is small, then

$$\|w^* - \Phi_s\tilde{x}_s\| \leq \frac{\varepsilon}{k^{j+1}} \qquad \text{and} \qquad \|w^* - \Psi_s\tilde{w}_s\| \leq \frac{\varepsilon}{k^{j+1}},$$

and in addition

$$\|\varphi_s(z) - \psi_s(z)\| \leq \frac{\varepsilon}{k^{j+1}} \qquad \text{for all } z \in \Xi_s.$$

Assume that $\Phi_s\tilde{x}_s|L_0 \notin [\varphi_s(z_1),\ldots,\varphi_s(z_i)]|L_0$ but $\Psi_s\tilde{w}_s \in [\psi_s(z_1),\ldots,\psi_s(z_i)]|L_0$ for some $z_1,\ldots,z_i \in \Xi_s$. In this case, the point a of $[\varphi_s(z_1),\ldots,\varphi_s(z_i)]|L_0$ closest to $\Phi_s\tilde{x}_s|L_0$ is contained in some $(j-1)$-simplex $[\varphi_s(z_{m_1}),\ldots,\varphi_s(z_{m_j})]|L_0$, i.e. there are $\lambda_1,\ldots,\lambda_j \geq 0$, $\lambda_1+\ldots+\lambda_j = 1$, such that $a = \sum_{r=1}^{j}\lambda_r\varphi(z_{m_r})|L_0$. Moreover, there are $\mu_1,\ldots,\mu_i \geq 0$, $\mu_1+\ldots+\mu_i = 1$, so that $\Psi_s\tilde{w}_s = \sum_{r=1}^{i}\mu_r\psi_s(z_r)|L_0$. Then we have

$$\begin{aligned} \|\Phi_s\tilde{x}_s|L_0 - a\| \ &\leq \ \left\|\Phi_s\tilde{x}_s|L_0 - \sum_{r=1}^{i}\mu_r\varphi_s(z_r)|L_0\right\| \\ &\leq \ \|\Phi_s\tilde{x}_s|L_0 - w^*\| + \|w^* - \Psi_s\tilde{w}_s\| + \left\|\sum_{r=1}^{i}\mu_r(\psi_s(z_r) - \varphi_s(z_r))|L_0\right\| \\ &\leq \ \frac{\varepsilon}{k^{j+1}} + \frac{\varepsilon}{k^{j+1}} + \frac{\varepsilon}{k^{j+1}} = \frac{3\varepsilon}{k^{j+1}}, \end{aligned}$$

and hence

$$\|w^* - a\| \leq \frac{4\epsilon}{k^{j+1}}.$$

Choose a maximal set $v_1,\ldots,v_l \in S^{d-1}\cap L_0$ such that the distance between any two points is at least $\varepsilon k^{-(j+1)}$, in particular

$$l \ll \varepsilon^{-(j-1)}k^{(j-1)(j+1)}.$$

Since $a, \varphi_s(z_{m_1})|L_0,\ldots,\varphi_s(z_{m_j})|L_0$ lie in a $(j-1)$-dimensional affine subspace of L_0, there is a unit vector $v \in S^{d-1}\cap L_0$ such that $|\langle\varphi_s(z_{m_r}) - w^*, v\rangle| \leq 4\varepsilon k^{-(j+1)}$ for $r = 1,\ldots,j$, and thus

$$|\langle\varphi_s(z_{m_r}) - w^*, v_m\rangle| \leq \frac{6\varepsilon}{k^{j+1}}$$

for $r = 1,\ldots,j$ and a suitably chosen $m \in \{1,\ldots,l\}$. In fact, for the given vector $v \in S^{d-1}\cap L_0$, there is some $m \in \{1,\ldots,l\}$ such that $\|v - v_m\| \leq \varepsilon k^{-(j+1)}$. Since $\Phi_s\widetilde{C}(x,\beta s) \subset w^* + 2B^d$, we deduce that

$$\begin{aligned} |\langle\varphi_s(z_{m_r}) - w^*, v_m\rangle| \ &\leq \ |\langle\varphi_s(z_{m_r}) - w^*, v\rangle| + \|\varphi_s(z_{m_r}) - w^*\|\cdot\|v_m - v\| \\ &\leq \ \frac{4\epsilon}{k^{j+1}} + 2\cdot\frac{\epsilon}{k^{j+1}} = \frac{6\epsilon}{k^{j+1}}. \end{aligned}$$

Therefore, if we define, for $m = 1, \dots, l$,

$$\Pi_m := \left\{ p \in \partial \Phi_s \widetilde{C}(x, \beta s) : |\langle p - w^*, v_m \rangle| \le 6\varepsilon k^{-(j+1)} \right\},$$

we get the following: if $\Phi_s \tilde{x}_s | L_0 \notin [\varphi_s(z_1), \dots, \varphi_s(z_i)] | L_0$ but $\Psi_s \tilde{w}_s \in [\psi_s(z_1), \dots, \psi_s(z_i)] | L_0$ for some $z_1, \dots, z_i \in \Xi_s$, then there exists $m \in \{1, \dots, l\}$ such that Π_m contains some j of the points $\varphi_s(z_1), \dots, \varphi_s(z_i)$. Since $\mathcal{H}^{d-1}(\Pi_m) \ll \varepsilon k^{-(j+1)}$, we have

$$\begin{aligned}
&\mathbb{P}_{\tilde{\varrho}_s} \left(\Phi_s \tilde{x}_s | L_0 \notin [\varphi_s(z_1), \dots, \varphi_s(z_i)] | L_0 \text{ and } \Psi_s \tilde{w}_s \in [\psi_s(z_1), \dots, \psi_s(z_i)] | L_0 \right) \\
&\le \binom{i}{j} \sum_{m=1}^{l} \mathbb{P}_{\tilde{\varrho}_s} \left(\varphi_s(z_1), \dots, \varphi_s(z_j) \in \Pi_m \right) \\
&\ll \binom{i}{j} \cdot l \cdot (\varepsilon k^{-(j+1)})^j \ll \frac{\varepsilon}{k}. \qquad (6.3.27)
\end{aligned}$$

Similarly, we have

$$\mathbb{P}_{\tilde{\sigma}_s} \left(\Psi_s \tilde{w}_s \notin [\psi_s(z_1), \dots, \psi_s(z_i)] | L_0 \text{ and } \Phi_s \tilde{x}_s | L_0 \in [\varphi_s(z_1), \dots, \varphi_s(z_i)] | L_0 \right) \ll \frac{\varepsilon}{k}. \quad (6.3.28)$$

Combining (6.3.23), (6.3.24) as well as (6.3.26), (6.3.27) and (6.3.28) yields (6.3.22), and in turn Lemma 6.3.2 if $d \ge 3$.

If $d = 2$, then a similar argument works, only some of the constrains should be modified as follows. In (6.3.21), we only have $\beta^{\frac{-2}{d-1}} \Gamma \left(i + \frac{2}{d-1} \right) / i! < k + 1$, and hence in (6.3.22), we should verify an upper bound of order $\frac{\varepsilon}{k^2}$, not of order $\frac{\varepsilon}{k}$. Therefore the upper bound in (6.3.26) should be $\frac{\varepsilon}{k^2}$. $\square$

6.4 Completing the proof of Theorem 6.1.2

In order to transfer an integral over an average of projections of a convex body to a boundary integral, we are going to use the following lemma from K. Böröczky Jr., L. M. Hoffmann, D. Hug [BHH08].

For $L \in \mathcal{L}_j^d$ and $y \in \partial(K|L)$, we choose a point $x(y) \in \partial K$ such that $y = x(y)|L$. In general, $x(y)$ is not uniquely determined, but we can fix a measurable choice (cf. [BHH08, p. 152]). Recall, however, that $x(y)$ is uniquely determined for ν_j a.e. $L \in \mathcal{L}_j^d$ and $\mathcal{H}^{j-1}$ a.e. $y \in \partial(K|L)$.

Lemma 6.4.1. *Let $K \subset \mathbb{R}^d$ be a convex body in which a ball rolls freely, let $f : \partial K \to [0, \infty)$ be nonnegative and measurable, and let $j \in \{1, \dots, d-1\}$. Then*

$$\frac{j\alpha_j}{d\alpha_d} \int_{\partial K} f(x) H_{d-j}(x)\, \mathcal{H}^{d-1}(dx) = \int_{\mathcal{L}_j^d} \int_{\partial(K|L)} f(x(y))\, \mathcal{H}^{j-1}(dy)\, \nu_j(dL).$$

By the very special case $K = B^d$ of (6.1.1), due to M. Reitzner [Rei02], we have

$$\lim_{n\to\infty} n^{\frac{2}{d-1}} \left[V_j(B^d) - \mathbb{E}_\sigma V_j(B_n) \right] = c^{(j,d)} (d\alpha_d)^{\frac{d+1}{d-1}}.$$

Therefore the rotational symmetry of B^d, (6.2.12) and (6.3.6) yield

$$\begin{aligned} c^{(j,d)}(d\alpha_d)^{\frac{d+1}{d-1}} &= \frac{\binom{d}{j}\alpha_d}{\alpha_{d-j}\alpha_j}\cdot\frac{j\alpha_j(d\alpha_d)^{\frac{2}{d-1}}}{d-1}(\alpha_{d-1})^{-\frac{2}{d-1}} \\ &\qquad\times\lim_{n\to\infty}\int_0^{n^{-1/2}} n^{\frac{2}{d-1}}\mathbb{P}_\sigma\left(\tilde{w}_s\notin B_n|L_0\right)s^{-\frac{d-3}{d-1}}\,ds. \qquad (6.4.1)\end{aligned}$$

We can now transform the asymptotic formulae to K. Let $L\in\mathcal{L}_j^d$ and let $y\in\partial(K|L)$ be such that $y=x|L$ for some normal boundary point $x=x(y)\in\partial K$. If $H_{d-1}(x)=0$, then $J_\varrho(y,L)=0$ by Lemma 6.2.4. If $H_{d-1}(x)>0$, then it follows from (6.3.6), Lemma 6.3.2 and (6.4.1) that

$$\begin{aligned} J_\varrho(y,L) &= (d-1)^{-1}(\alpha_{d-1})^{-\frac{2}{d-1}}\varrho(x)^{\frac{-2}{d-1}}H_{d-1}(x)^{\frac{1}{d-1}} \\ &\qquad\times\lim_{n\to\infty}\int_0^{n^{-1/2}} n^{\frac{2}{d-1}}\mathbb{P}_\sigma\left(\tilde{w}_s\notin B_n|L_0\right)s^{-\frac{d-3}{d-1}}\,ds \\ &= c^{(j,d)}\varrho(x)^{\frac{-2}{d-1}}H_{d-1}(x)^{\frac{1}{d-1}}\left(\frac{\binom{d}{j}\alpha_d}{\alpha_{d-j}\alpha_j}\cdot\frac{j\alpha_j}{d\alpha_d}\right)^{-1},\end{aligned}$$

where $x=x(y)$. Finally, we apply first (6.2.12), and afterwards Lemma 6.4.1, to deduce

$$\begin{aligned} &\lim_{n\to\infty} n^{\frac{2}{d-1}}\left[V_j(K)-\mathbb{E}_\varrho(V_j(K_n))\right] \\ &= c^{(j,d)}\frac{d\alpha_d}{j\alpha_j}\int_{\mathcal{L}_j^d}\int_{\partial(K|L)}\varrho(x(y))^{\frac{-2}{d-1}}H_{d-1}(x(y))^{\frac{1}{d-1}}\,\mathcal{H}^{j-1}(dy)\,\nu_j(dL) \\ &= c^{(j,d)}\int_{\partial K}\varrho(x)^{\frac{-2}{d-1}}H_{d-1}(x)^{\frac{1}{d-1}}\,H_{d-j}(x)\,\mathcal{H}^{d-1}(dx),\end{aligned}$$

which concludes the proof of Theorem 6.1.2.

6.5 Proof of Theorem 6.1.3

Using the Stirling formula $\Gamma(n+1)\sim(\frac{n}{e})^n\sqrt{2\pi n}$, as $n\to\infty$ (see E. Artin [Art64]), for any $\alpha>0$ and $\gamma\in(0,1]$, we deduce

$$\begin{aligned} \lim_{n\to\infty} n^\alpha\int_0^\gamma s^{\alpha-1}(1-s)^n\,ds &= \lim_{n\to\infty} n^\alpha\int_0^1 s^{\alpha-1}(1-s)^n\,ds \\ &= \lim_{n\to\infty} n^\alpha\frac{\Gamma(\alpha)\Gamma(n+1)}{\Gamma(n+1+\alpha)}=\Gamma(\alpha). \qquad (6.5.1)\end{aligned}$$

In the following argument, $\gamma_1,\gamma_2,\dots$ again denote positive constants that may depend on K and ϱ. We can assume that $o\in\operatorname{int}(K)$. Further, let $(\partial K)_*^n$ denote the set of all $x_1,\dots,x_n\in\partial K$ such that $o\in[x_1,\dots,x_n]$. For $u\in S^{d-1}$ and $t\geq 0$, let

$$C(u,t):=\{x\in K:\,\langle x,u\rangle\geq h_K(u)-t\},$$

where h_K denotes the support function of K. To deduce the upper bound, we start with the estimates

$$\begin{aligned}
&\mathbb{E}_\varrho(V_1(K)-V_1(K_n))\\
&=\frac{1}{\alpha_{d-1}}\int_{(\partial K)^n}\int_{S^{d-1}}(h_K(u)-h_{K_n}(u))\,\mathcal{H}^{d-1}(du)\varrho(x_1)\cdots\varrho(x_n)\\
&\qquad\mathcal{H}^{d-1}(dx_1)\ldots\mathcal{H}^{d-1}(dx_n)\\
&\le\frac{1}{\alpha_{d-1}}\int_{(\partial K)^n_*}\int_{S^{d-1}}(h_K(u)-h_{K_n}(u))\,\mathcal{H}^{d-1}(du)\varrho(x_1)\cdots\varrho(x_n)\\
&\qquad\mathcal{H}^{d-1}(dx_1)\ldots\mathcal{H}^{d-1}(dx_n)\\
&\qquad+2^d(1-\gamma_1)^n\\
&\le\frac{1}{\alpha_{d-1}}\int_{S^{d-1}}\int_0^{h_K(u)}\int_{(\partial K)^n}\mathbf{1}\{x_1,\ldots,x_n\in\partial K\setminus C(u,s)\}\varrho(x_1)\cdots\varrho(x_n)\\
&\qquad\mathcal{H}^{d-1}(dx_1)\ldots\mathcal{H}^{d-1}(dx_n)\,ds\,\mathcal{H}^{d-1}(du)+2^d(1-\gamma_1)^n\\
&\le\frac{1}{\alpha_{d-1}}\int_{S^{d-1}}\int_0^{h_K(u)}\left(1-\int_{\partial K\cap C(u,t)}\varrho(x)\,\mathcal{H}^{d-1}(dx)\right)^n dt\,\mathcal{H}^{d-1}(du)\\
&\qquad+2^d(1-\gamma_1)^n. \qquad (6.5.2)
\end{aligned}$$

For suitable positive constants $\gamma_2,\gamma_3,\gamma_4$ we get, for $u\in S^{d-1}$ and $t\in(0,\gamma_2)$,

$$\int_{\partial K\cap C(u,t)}\varrho(x)\,\mathcal{H}^{d-1}(dx)\begin{cases}>&\gamma_3t^{d-1}, & \text{if } t\in(0,\gamma_2),\\ >&\gamma_4, & \text{if } t\ge\gamma_2.\end{cases} \qquad (6.5.3)$$

In particular, $\gamma_4,\gamma_3(\gamma_2)^{d-1}\in(0,1)$. We deduce from (6.5.2), (6.5.3) and (6.5.1) that, for suitable $\gamma_5,\ldots,\gamma_9$ with $\gamma_7,\gamma_9\in(0,1)$,

$$\begin{aligned}
\mathbb{E}_\varrho(V_1(K)-V_1(K_n)) &\le \gamma_5\int_0^{\gamma_2}(1-\gamma_3t^{d-1})^n\,dt+\gamma_6\gamma_7^n\\
&= \gamma_8\int_0^{\gamma_9}s^{\frac{1}{d-1}-1}\cdot(1-s)^n\,ds+\gamma_6\gamma_7^n\le\gamma_{10}n^{\frac{-1}{d-1}}.
\end{aligned}$$

To prove the lower bound for $\mathbb{E}_\varrho(V_1(K)-V_1(K_n))$, we need the following observation.

Lemma 6.5.1. *Let $K\subset\mathbb{R}^d$ be a convex body, and let h_K be twice differentiable at $u_0\in S^{d-1}$. Then there is some $R>0$ such that $K\subset x_0-Ru_0+RB^d$, where $x_0=\nabla h_K(u_0)\in\partial K$. In particular, there exist a measurable set $\Sigma\subset S^{d-1}$ with $\mathcal{H}^{d-1}(\Sigma)>0$ and some $R>0$, all depending on K, such that for any $u\in\Sigma$ there is some $x\in\partial K$such that $K\subset x-Ru+RB^d$.*

Proof. For the proof of the first assertion, we may assume that $x_0=o$, hence also $h_K(u_0)=0$. We put $h:=h_K$. By assumption, there is a function $R:\mathbb{R}_+\to[0,\infty)$ with $\lim_{t\to0^+}R(t)=0$ and

$$\left|h(u)-\frac{1}{2}\cdot d^2h(u-u_0,u-u_0)\right|\le R(\|u-u_0\|)\|u-u_0\|^2.$$

Thus there is a constant $R_1 > 0$ and $\delta > 0$ such that $h(u) \leq R_1\|u - u_0\|^2$ for all $u \in S^{d-1}$ with $\langle u, u_0\rangle \geq 1 - \delta$. But then for $R_2 := \max\{2R_1, \max\{h(u) : u \in S^{d-1}\}/(2\delta)\}$ and all $u \in S^{d-1}$, we obtain

$$h(u) \leq R_2\,(1 - \langle u_0, u\rangle) = h(-R_2u_0 + R_2B^d, u),$$

that is $K \subset -R_2u_0 + R_2B^d$.

The second assertion follows immediately from the first assertion. □

Let t_0 be the inradius of K. Now Lemma 6.5.1 yields, for $u \in \Sigma$ and $t \in (0, t_0)$, that

$$\int_{\partial K \cap C(u,t)} \varrho(x)\,\mathcal{H}^{d-1}(dx) < \gamma_{11} \cdot t^{\frac{d-1}{2}}.$$

Choosing a constant $\gamma_{12} \in (0, t_0)$ satisfying $\gamma_{11}(\gamma_{12})^{\frac{d-1}{2}} < 1$, it follows as in the derivation of (6.5.2) that, with a suitable constant $\gamma_{13} \in (0, 1)$, we have

$$\begin{aligned}\mathbb{E}_\varrho(V_1(K) - V_1(K_n)) &\geq \frac{1}{\alpha_{d-1}} \int_\Sigma \int_0^{\gamma_{12}} \left(1 - \gamma_{11}t^{\frac{d-1}{2}}\right)^n dt\,\mathcal{H}^{d-1}(dx) \\ &= \int_0^{\gamma_{13}} s^{\frac{2}{d-1}-1} \cdot (1-s)^n\,ds > \gamma_{14} \cdot n^{\frac{-2}{d-1}}.\end{aligned}$$

Theorem 6.1.2 shows that the lower bound of Theorem 6.1.3 is of optimal order if K has a rolling ball. In fact, the assumption of a rolling ball ensures that the integral on the right side of (6.1.1) is positive. This follows, for instance, from the absolute continuity of the Gauss curvature measure of a convex body which has a rolling ball (cf. [Hug99]).

On the other hand, the upper bound for $\mathbb{E}_\varrho(V_1(K) - V_1(K_n))$ is of optimal order if K is a polytope. To explain this, let $\Sigma_0 \subset S^{n-1}$ be contained in the interior of the exterior normal cone of one of the vertices of K and such that $\mathcal{H}^{d-1}(\Sigma_0) > 0$. In this case

$$\int_{\partial K \cap C(u,t)} \varrho(x)\,\mathcal{H}^{d-1}(dx) < \gamma_{15} \cdot t^{d-1},$$

for $u \in \Sigma_0$ and $t \in (0, \gamma_{16})$, and hence $\mathbb{E}_\varrho(V_1(K) - V_1(K_n)) \geq \gamma_{17} \cdot n^{\frac{-1}{d-1}}$.

Chapter 7

Approximation by random disc-polygons

This chapter of the dissertation is based on the papers:

- [FKV14] by F. Fodor, P. Kevei and V. Vígh, *On random disc polygons in smooth convex discs*, Adv. in Appl. Probab., **46** (2014), no. 4, 899–918. (DOI 10.1239/aap/1418396236)
- [FV18] by F. Fodor and V. Vígh, *Variance estimates for random disc-polygons in smooth convex discs*, J. Appl. Probab., **55**, (2018), no. 4, 1143–1157. (DOI 10.1017/jpr.2018.76)

7.1 Expectations

In their ground-braking papers, Rényi and Sulanke [RS63, RS64, RS68] investigated the geometric properties of approximations of convex discs by random convex polygons. In particular, they considered the following probability model.

Let K be a convex disc (a compact convex set with nonempty interior) in the Euclidean plane $\mathbb{R}^2$ and let $y_1, y_2, \dots$ be independent random points chosen from K according to the uniform probability distribution. Let K_n denote the convex hull of $Y_n = \{y_1, \dots, y_n\}$. The set K_n is called a uniform random convex polygon in K.

Rényi and Sulanke [RS63, RS64] proved asymptotic formulae for the expectation of the number of vertices of K_n and the expectation of the missed area of K_n under the assumption that the boundary ∂K of K is three times continuously differentiable and the curvature is strictly positive everywhere. They also proved an asymptotic formula for the expectation of the perimeter difference of K and K_n under stronger differentiability assumptions on ∂K and assuming that the curvature $\kappa(x) > 0$ for all $x \in \partial K$. For later comparison, we state their results below in a slightly modified form.

Let $f_0(K_n)$ denote the number of vertices of K_n, $A(K)$ the area of K and $\Gamma(\cdot)$ Euler's Gamma function. Then (cf. Satz 3 on page 83 in [RS63])

$$\lim_{n\to\infty} \mathbb{E}(f_0(K_n)) \cdot n^{-1/3} = \sqrt[3]{\frac{2}{3A(K)}}\Gamma\left(\frac{5}{3}\right)\int_{\partial K} \kappa(x)^{1/3}\mathrm{d}x, \tag{7.1.1}$$

where integration is with respect to the one-dimensional Hausdorff measure on ∂K. We note that with the help of Efron's identity [Efr65], (7.1.1) implies directly the following statement

$$\lim_{n\to\infty} \mathbb{E}(A(K \setminus K_n)) \cdot n^{2/3} = \sqrt[3]{\frac{2A(K)^2}{3}}\,\Gamma\left(\frac{5}{3}\right) \int_{\partial K} \kappa(x)^{1/3} \mathrm{d}x. \qquad (7.1.2)$$

Rényi and Sulanke derived (7.1.2) by direct computation, cf. formula (48) in Satz 1 on page 144 in [RS64].

Assuming that the boundary of K is sufficiently smooth and $\kappa(x) > 0$ for all $x \in \partial K$, Rényi and Sulanke proved the asymptotic formula

$$\lim_{n\to\infty} \mathbb{E}(\mathrm{Per}(K) - \mathrm{Per}(K_n)) \cdot n^{2/3} = \frac{1}{12}\Gamma\left(\frac{2}{3}\right)(12A(K))^{2/3} \int_{\partial K} \kappa(x)^{4/3} \mathrm{d}x \qquad (7.1.3)$$

for the perimeter difference of K and K_n, cf. formula (47) in Satz 1 on page 144 in [RS64].

For more information about approximations of convex bodies by random polytopes we refer the reader to the recent book by Schneider and Weil [SW08], and the survey articles by Bárány [Bár08], Schneider [Sch08, Sch18], Weil and Wieacker [WW93].

In this chapter, we investigate the *R-spindle convex* analogue of the above probability model. Let $R > 0$. R-spindle convex discs are those convex discs that are intersections of (not necessarily finitely many) closed circular discs of radius R. For a precise definition of spindle convexity, see Section 7.1.1. The intersection of finitely many closed circular discs of radius R is a closed convex R-disc-polygon. Let X be a compact set which is contained in a closed circular disc of radius R. The intersection of all R-spindle convex discs containing X is called the R-spindle convex hull of X, and it is denoted by $\mathrm{conv}_{s,R}(X)$.

Now we are ready to define our probability model. Let S be an R-spindle convex disc in $\mathbb{R}^2$. Let $x_1, x_2, \ldots$ be independent random points in S chosen according to the uniform probability distribution (the Lebesgue measure in S normalized by the area of S). The R-spindle convex hull $S_n^R = \mathrm{conv}_{s,R}(X_n)$, where $X_n = \{x_1, \ldots, x_n\}$, is called a *uniform random R-disc-polygon* in S. We prove the R-spindle convex analogues of (7.1.1), (7.1.2) and (7.1.3) in this probability model.

The concept of spindle convexity was (probably) first introduced by Mayer [May35] as a generalization of linear convexity in the wider context of Minkowski geometry. In the Euclidean plane $\mathbb{R}^2$, a closed convex set can be represented as the intersection of closed half-planes. In the definition of an R-spindle convex set, the radius R closed circular discs play the role of closed half-planes. Thus, formally, the $R = \infty$ case corresponds to linear convexity.

Early investigations of spindle convex sets were carried out in the first half of the 20th century. For a short survey of the early history of the subject and references see the paper by Danzer, Grünbaum and Klee [DGK63]. Fejes Tóth proved packing and covering theorems for R-spindle convex discs in [FT82b] and [FT82a]. More recently, Bezdek et al. [BLNP07] and Kupitz et al. [KMP05], [KMP10] investigated spindle convex sets and proved numerous results about them, many of which are analogous to those of linearly convex sets. They also considered higher dimensional R-spindle convex sets. Intersections of a finite number of radius R closed balls in $\mathbb{R}^d$ are called *ball-polyhedra* (cf. [BLNP07]).

This notion of hyperconvexity arises naturally in many questions where a convex set can be represented as the intersection of equal radius closed balls. As recent examples of such problems, we mention the Kneser-Poulsen conjecture, see, for example, Bezdek, Connelly [BC02], Bezdek [Bez18], Bezdek, Naszódi [BN18], and inequalities for intrinsic volumes by Pauris, Pivovarov [PP17]. A more complete list can be found in [BLNP07], for short overviews see also Fejes Tóth, Fodor [FTF15], Fodor, Kevei, Vígh [FKV14], and Fodor, Vígh [FV12].

Fodor and Vígh [FV12] proved asymptotic formulae for best approximations of R-spindle convex discs by R-disc-polygons generalizing some of the corresponding results of Fejes Tóth [FT53] and McClure and Vitale [MV75] about best approximations of linearly convex discs by convex polygons (see Chapter 8). For a systematic treatment of geometric properties of hyperconvex sets and further references, see, for example, the recent papers by Bezdek, Lángi, Naszódi, Papez [BLNP07], Fodor, Kurusa, Vígh [FKV16], and in a more general setting the paper by Jahn, Martini, Richter [JMR17].

There is a wealth of new information about properties of spindle convex bodies and ball-polyhedra in the recent monographs [Bez10] and [Bez13] by Bezdek.

The notion of spindle convexity is related to diametrical completeness of convex bodies through the so-called spherical intersection property. A convex body K is diametrically complete if for any point $x \notin K$, the diameter of conv $(K \cup \{x\})$ is strictly larger than that of K. It was proved by Eggleston [Egg65] that in a Banach space the diametrically complete convex bodies are exactly those which have the so-called spherical intersection property, that is, they are equal to the intersection of all closed balls whose centre is contained in K and whose radius is equal to the diameter of K. In Euclidean spaces diametrically complete convex bodies are exactly those of constant width, however, in Minkowski spaces this is not the case. Recently, much effort has been devoted to investigating the properties of diametrically complete sets in Minkowski spaces where sets that are intersections of congruent closed balls play a fundamental role (see, for example, Moreno and Schneider [MS07] and the references therein), and to investigating various properties of the ball hull, see, for example, Moreno and Schneider [MS12] for more information.

Random approximations of R-spindle convex sets by R-disc-polygons naturally appear, for example, in the so-called Diminishing Process of Tóth, see Ambrus et al. [AKV12]. Let $D_0 = B_R$ be the radius R closed circular disc in $\mathbb{R}^2$ centred at the origin. Define the random process (D_n, p_n) for $n \geq 1$ as follows. Let p_{n+1} be a uniform random point in D_n and let $D_{n+1} = D_n \cap (B_R + p_{n+1})$. Then each D_n is a (non-uniform random) R-disc-polygon, and the process converges (in the Hausdorff metric of compact sets) to a set of constant width R with probability 1. This process can be readily generalized for a general convex body $K \subset \mathbb{R}^d$, in place of B_R, that contains the origin. If the body K is symmetric with respect to the origin, then it determines a Minkowski metric and the sets K_n are all (random) spindle convex bodies with respect to K in this Minkowski space.

Finally, we remark that there are various terms used for R-spindle convex sets in the literature. Mayer introduced the word "Überkonvexität" in [May35]. Authors of early articles used the translation of Mayer's term. Fejes Tóth [FT82b, FT82a] named such sets "R-convex". Bezdek et al. [BLNP07] and Kupitz et al. [KMP05, KMP10] used the expression "spindle convex". The notion of spindle convexity arose naturally and was investigated from different points of view, which explains the various names used for these sets and it also indicates their importance.

The main results of this chapter are described in the following theorems.

Theorem 7.1.1 (Fodor, Kevei, Vígh [FKV14, Theorem 1.1 on page 901])**.** *Let $R > 0$, and let S be an R-spindle convex disc with C^2 smooth boundary and with the property that $\kappa(x) > 1/R$ for all $x \in \partial S$. Then*

$$\lim_{n\to\infty} \mathbb{E}(f_0(S_n^R)) \cdot n^{-1/3} = \sqrt[3]{\frac{2}{3A(S)}} \cdot \Gamma\left(\frac{5}{3}\right) \int_{\partial S} \left(\kappa(x) - \frac{1}{R}\right)^{1/3} \mathrm{d}x, \tag{7.1.4}$$

and

$$\lim_{n\to\infty} \mathbb{E}(A(S \setminus S_n^R)) \cdot n^{2/3} = \sqrt[3]{\frac{2A(S)^2}{3}} \Gamma\left(\frac{5}{3}\right) \int_{\partial S} \left(\kappa(x) - \frac{1}{R}\right)^{1/3} \mathrm{d}x. \tag{7.1.5}$$

We note that the two statements are connected with an Efron-type relation [Efr65], see (7.1.31) in Section 7.1.4.

Theorem 7.1.2 (Fodor, Kevei, Vígh [FKV14, Theorem 1.2 on page 902])**.** *Let $R > 0$, and let S be an R-spindle convex disc with C^5 smooth boundary and with the property that $\kappa(x) > 1/R$ for all $x \in \partial S$. Then*

$$\lim_{n\to\infty} \mathbb{E}(\mathrm{Per}(S) - \mathrm{Per}(S_n^R)) \cdot n^{2/3}$$
$$= \frac{(12A(S))^{2/3}}{36} \Gamma\left(\frac{2}{3}\right) \int_{\partial S} \left(\kappa(x) - \frac{1}{R}\right)^{1/3} \left(3\kappa(x) + \frac{1}{R}\right) \mathrm{d}x. \tag{7.1.6}$$

Theorem 7.1.3 (Fodor, Kevei, Vígh [FKV14, Theorem 1.3 on page 902])**.** *Let $R > 0$, and let $S = B_R$ be a circular disc of radius R. Then*

$$\lim_{n\to\infty} \mathbb{E}(f_0(S_n^R)) = \frac{\pi^2}{2}, \tag{7.1.7}$$

$$\lim_{n\to\infty} \mathbb{E}(A(B_R \setminus S_n^R)) \cdot n = \frac{R^2 \cdot \pi^3}{2}, \tag{7.1.8}$$

and

$$\lim_{n\to\infty} \mathbb{E}(\mathrm{Per}(B_R) - \mathrm{Per}(S_n^R)) \cdot n = \frac{R \cdot \pi^3}{2}. \tag{7.1.9}$$

It is somewhat surprising that the expectation of the number of the vertices of uniform random spindle convex polygons in circular discs tends to a (very small) constant. Roughly speaking this means that after choosing many random points from a circle, the spindle convex hull will have about 5 vertices. This is a surprising fact that has no clear analogue in the classical convex case. A similar phenomenon was recently established by Bárány, Hug, Reitzner, Schneider [BHRS17] about the expectation of the number of facets of certain spherical random polytopes in halfspheres, see [BHRS17, Theorem 3.1].

Furthermore, for a (linearly) convex disc K with C^2 smooth boundary and strictly positive curvature, the asymptotic formulae (7.1.1) and (7.1.2) of Rényi and Sulanke follow from (7.1.4) and (7.1.5), respectively. Similarly, for a convex disc with C^5 smooth boundary and strictly positive curvature, the asymptotic formula (7.1.3) of Rényi and Sulanke follows from (7.1.6). Thus, the results of Theorems 7.1.1 and 7.1.2 are generalizations of the corresponding results of Rényi and Sulanke.

The rest of this chapter is organized as follows. In Section 7.1.1, we introduce the necessary notations. In Section 7.1.2, we prove how the asymptotic formulae of Rényi and Sulanke follow from our results. In Section 7.1.3, we investigate some properties of disc-caps of spindle convex discs that are used in the subsequent arguments. We give the proofs of Theorem 7.1.1 and Theorem 7.1.2 in Section 7.1.4. Finally, in Section 7.1.5, we provide an outline of the proof of Theorem 7.1.3.

7.1.1 Spindle convex sets: definition and notations

In this chapter, the symbol B_R denotes the closed circular disc of radius R centred at the origin. We use S^1_R to denote ∂B_R. We tacitly assume that the plane is embedded in $\mathbb{R}^3$ and write $x \times y$ for the cross product of the vectors x and y.

We use the notation $\kappa(x)$ for the curvature of ∂K at x. If the boundary of K is C^2 smooth, then at every $x \in \partial K$ there exists a unique outer unit normal vector $u_x \in S^1$ to ∂K.

For a convex disc K, integration on the boundary of K with respect to the one-dimensional Hausdorff measure (the arc-length of ∂K) is denoted by $\int_{\partial K} \cdots \mathrm{d}x$. In the case that the boundary of K is C^2 smooth and $f(u)$ is a measurable function on S^1, $\int_{S^1} f(u)\mathrm{d}u = \int_{\partial K} f(u_x)\kappa(x)\mathrm{d}x$, (cf. formula 2.5.62 in [Sch14]).

Let $x, y \in \mathbb{R}^2$ be such that their distance does not exceed $2R$. We define the *closed R-spindle* $[x, y]_{s,R}$ of x and y as the intersection of all closed circular discs of radius R that contain both x and y. The closed R-spindle of two points whose distance is greater than $2R$ is defined to be the whole plane $\mathbb{R}^2$. The closed spindle of two points whose distance is less than $2R$ looks like a spindle, which explains the origin of its name. A set $S \subseteq \mathbb{R}^2$ is called *R-spindle convex* if from $x, y \in S$ it follows that $[x, y]_{s,R} \subseteq S$. Spindle convex sets are also convex in the usual linear sense. In this chapter we restrict our attention to compact spindle convex sets. We call a compact set $S \subset \mathbb{R}^2$ with nonempty interior an *R-spindle convex disc* if it has the R-spindle convex property.

Below, we list those properties of spindle convex discs that will be used in our arguments. For more detailed information about spindle convexity we refer the reader to Bezdek et al. [BLNP07].

A compact convex set S is R-spindle convex if and only if it is the intersection of (not necessarily finitely many) congruent closed circular discs of radius R (cf. Corollary 3.4 on page 205 in [BLNP07]). If the closed circular disc $B_R + p$ contains an R-spindle convex disc S and there is a point $x \in \partial S$ such that also $x \in \partial B_R + p$, then we say that $B_R + p$ *supports S at x*. Let P be a convex R-disc-polygon, and let $B_R + p$ a circle supporting P at $H = \partial P \cap (\partial B_R + p)$. Then H either consists of only one point, called a *vertex*, or it consists of the points of a closed circular arc, called a *side (or edge)* of P. The number of edges of P equals the number of vertices of P (except in the case that P is a circle of radius R); we denote this number by $f_0(P)$.

If S is an R-spindle convex disc with C^2 smooth boundary, then $\kappa(x) \geq 1/R$ for all

$x \in \partial S$, and for every unit vector $u \in S^1$, there exists a unique point $x \in \partial S$ such that $u = u_x$; we denote this point by x_u. We also note that if $x \in \partial S$, then $B_R + x - R \cdot u_x$ supports S at x.

7.1.2 The limit case

In this section we show how Theorems 7.1.1 and 7.1.2 imply the asymptotic formulae (7.1.1), (7.1.2), and (7.1.3) of Rényi and Sulanke.

Let K be a (linearly) convex disc with C^2 smooth boundary and $\kappa(x) > 0$ for all $x \in \partial K$. Let $\kappa_{\min} = \min_{\partial K} \kappa(x) > 0$. It follows from Mayer's results (cf. (Ü4) and (Ü5) on page 521 in [May35], or for a more recent and more general reference see also Theorem 2.5.4. in [Sch14]) that K is R-spindle convex for all $R \geq R_0 = 1/\kappa_{\min}$. For $R \geq R_0$ and n sufficiently large, we introduce the following notation

$$\begin{aligned}
\delta_S^R(n) &= \mathbb{E}(A(K \setminus S_n^R)) \cdot n^{\frac{2}{3}},\\
\delta(n) &= \mathbb{E}(A(K \setminus K_n)) \cdot n^{\frac{2}{3}},\\
I_S^R &= \sqrt[3]{\frac{2A^2}{3}} \cdot \Gamma\left(\frac{5}{3}\right) \int_{\partial K} \left(\kappa(x) - \frac{1}{R}\right)^{\frac{1}{3}} \mathrm{d}x,\\
I &= \sqrt[3]{\frac{2A^2}{3}} \cdot \Gamma\left(\frac{5}{3}\right) \int_{\partial K} \kappa^{\frac{1}{3}}(x) \mathrm{d}x,
\end{aligned}$$

with $A = A(K)$.

We claim that (7.1.5) implies the asymptotic formula (7.1.2) of Rényi and Sulanke.

Let $\varepsilon > 0$ be fixed. Then $\lim_{R \to \infty} I_S^R = I$ yields that there exists $R_1(\varepsilon) > R_0$ such that

$$1 - \varepsilon < \frac{I_S^R}{I} < 1 + \varepsilon \tag{7.1.10}$$

for all $R > R_1(\varepsilon)$.

Elementary calculations show that there exists $R_2(\varepsilon) \geq R_0$, depending only on K and ε such that for all $R > R_2(\varepsilon)$,

$$\frac{A([p,q]_{s,R})}{A([p,q]_{s,R_0}) - A([p,q]_{s,R})} < \varepsilon, \tag{7.1.11}$$

for any points $p, q \in K$.

Let D_m^R denote an R-disc-polygon in K with vertices $p_1, \ldots, p_m$ indexed in the cyclic order, and let P_m denote the (linear) convex hull of $p_1, \ldots, p_m$. Note that this is a polygon with vertices $p_1, \ldots, p_m$. If $R > R_2(\varepsilon)$, then (7.1.11) yields

$$\begin{aligned}
1 < \frac{\delta(n)}{\delta_S^R(n)} &= 1 + \frac{\mathbb{E}(A(S_n^R) - A(K_n))}{\mathbb{E}(A(K) - A(S_n^R))}\\
&< 1 + \sup_{\substack{D_m^R \subset K,\\ 2 \leq m \leq n}} \frac{A(D_m^R) - A(P_m)}{A(D_m^{R_0}) - A(D_m^R)} < 1 + \varepsilon.
\end{aligned} \tag{7.1.12}$$

Now assume that $R > \max\{R_1(\varepsilon), R_2(\varepsilon)\}$. It is clear that for any such R, the convergence $\lim_{n\to\infty} \delta_S^R(n)/I_S^R = 1$ yields that there exists $n(R)$ such that

$$1 - \varepsilon < \frac{\delta_S^R(n)}{I_S^R} < 1 + \varepsilon \tag{7.1.13}$$

for all $n \geq n(R)$.

Thus, from (7.1.10), (7.1.12), (7.1.13), and from

$$\frac{\delta(n)}{I} = \frac{\delta(n)}{\delta_S^R(n)} \cdot \frac{\delta_S^R(n)}{I_S^R} \cdot \frac{I_S^R}{I},$$

we obtain that

$$1 - 3\varepsilon < \frac{\delta(n)}{I} < 1 + 7\varepsilon$$

for all $R > \max\{R_1(\varepsilon), R_2(\varepsilon)\}$ and $n > n(R)$, which proves that

$$\lim_{n\to\infty} \frac{\delta(n)}{I} = 1.$$

A similar argument shows that (7.1.6) implies the asymptotic formula (7.1.3) of Rényi and Sulanke. Finally, formula (7.1.1) for the number of vertices follows by Efron's equality (7.1.31).

7.1.3 Caps of spindle convex discs

From now on we restrict our attention to the case when $R = 1$ and we omit R from the notation. We use the simpler terms *spindle convex* and *disc-polygon* in place of 1-*spindle convex* and 1-*disc polygon*, respectively. In particular, $B = B_1$ denotes the unit disc. The R-spindle convex analogues of the following lemmas can be obtained by simple scaling.

Let S be a spindle convex disc with C^2 smooth boundary and assume that $\kappa(x) > 1$ for all $x \in \partial S$. A subset D of S is a *disc-cap of S* if $D = \mathrm{cl}\,(S \cap (B + p)^C)$ for some point $p \in \mathbb{R}^2$. Note that in this case $\partial B + p$ intersects ∂S in at most two points. (This follows, for example, from Theorem 2.5.4. in [Sch14].) Thus, the boundary of a nonempty disc-cap D consists of at most two connected arcs: one arc is a subset of ∂S, and the other arc is a subset of $\partial B + p$. In order to define the vertex and the outer normal of a disc-cap we need the following claim.

Lemma 7.1.4. *Let S be a spindle convex disc with C^2 smooth boundary and assume that $\kappa(x) > 1$ for all $x \in \partial S$. Let $D = \mathrm{cl}(S \cap (B + p)^C)$ be a non-empty disc-cap of S (as above). Then there exists a unique point $x_0 \in \partial S \cap \partial D$ such that there exists a $t \geq 0$ with $B + p = B + x_0 - (1 + t)u_{x_0}$. We refer to x_0 as the* vertex *of D and to t as the* height *of D.*

Proof. Pick any $x \in \partial S \cap \partial D$, and consider the vectors $\overrightarrow{px}$ and the outer unit normal u_x. We claim that there is a unique x for which $\overrightarrow{px}$ is a positive multiple of u_x. The existence follows from a simple continuity argument since the angles formed by the two vectors have different orientations at the endpoints of $\partial S \cap \partial D$. Uniqueness is proved as follows. Suppose that both $x_1 \neq x_2$ fulfil the requirements. Let φ be the (positive) angle between u_{x_1} and u_{x_2} and denote by I the arc of ∂S between x_1 and x_2 (according to the positive orientation), and by Δs the length of I. By the spindle convexity of S, we obtain that x_1 and x_2 can be joined by a unit circular arc in S. The length of this circular arc is clearly smaller then Δs, on the other hand it is larger than φ, and thus $\Delta s > \varphi$. Using the assumption that the curvature of ∂S is strictly larger than 1, we obtain that

$$\varphi = \int_I \kappa(s)\mathrm{d}s > \int_I \mathrm{d}s = \Delta s > \varphi,$$

a contradiction. $\square$

Let $D(u,t)$ denote the disc-cap with vertex $x_u \in \partial S$ and height t. Note that for each $u \in S^1$, there exists a maximal positive constant $t^*(u)$ such that $(B + x_u - (1+t)u) \cap S \neq \emptyset$ for all $t \in [0, t^*(u)]$. Let $V(u,t) = A(D(u,t))$ and let $\ell(u,t)$ denote the arc-length of $\partial D(u,t) \cap (\partial B + x_u - (1+t)u)$.

Lemma 7.1.5. *Let S be a spindle convex disc with C^2 boundary such that $\kappa(x) > 1$ for all $x \in \partial S$. Then for a fixed $x \in \partial S$, the following hold*

$$\lim_{t \to 0^+} \ell(u_x, t) \cdot t^{-1/2} = 2\sqrt{\frac{2}{\kappa(x) - 1}}, \tag{7.1.14}$$

and

$$\lim_{t \to 0^+} V(u_x, t) \cdot t^{-3/2} = \frac{4}{3}\sqrt{\frac{2}{\kappa(x) - 1}}. \tag{7.1.15}$$

Proof. Assume that $x = (0,0)$ and $u_x = (0,-1)$. Then, in a sufficiently small open neighbourhood of the origin, ∂S is the graph of a C^2 smooth function $f(\sigma)$. Taylor's theorem yields that

$$f(\sigma) = \frac{\kappa(x)}{2}\sigma^2 + o(\sigma^2), \text{ as } \quad \sigma \to 0. \tag{7.1.16}$$

In the same open neighbourhood of the origin, the boundary of $B + x - (1+t)u_x$ is the graph of the function $g_t(\sigma) = t + 1 - \sqrt{1 - \sigma^2}$. Simple calculation yields that the positive solution of the equation $g_t(\sigma) = f(\sigma)$ is

$$\sigma_+ = \sqrt{\frac{2}{\kappa(x) - 1}} \cdot t^{1/2} + o(t^{1/2}), \text{ as } \quad t \to 0^+.$$

Clearly, $\ell(u_x, t) \sim 2\sigma_+$ as $t \to 0^+$ by the fact that the ratio of the lengths of an arc and the corresponding chord tends to 1 as the length of the arc tends to 0.

Let σ_- denote the negative solution of the equation $g_t(\sigma) = f(\sigma)$. Then

$$\begin{aligned} V(u_x, t) &= \int_{\sigma_-}^{\sigma_+} (g_t(\sigma) - f(\sigma))\mathrm{d}\sigma \\ &= 2\int_0^{\sigma_+} \left[t + \frac{\sigma^2}{2} - \frac{\kappa(u_x)}{2}\sigma^2 + o(\sigma^2) \right] \mathrm{d}\sigma \end{aligned}$$

$$= \frac{4}{3}\sqrt{\frac{2}{\kappa(x)-1}} \cdot t^{3/2} + o(t^{3/2}), \text{ as } \quad t \to 0^+.$$

This finishes the proof of Lemma 7.1.5. $\square$

Let $x_1, x_2 \in S$ be two distinct points. Then there are exactly two disc-caps of S, say $D_-(x_1, x_2) = \mathrm{cl}\,(S \cap (B + p_-)^C)$ and $D_+(x_1, x_2) = \mathrm{cl}\,(S \cap (B + p_+)^C)$ with the property that $x_1, x_2 \in \partial B + p_-$ and $x_1, x_2 \in \partial B + p_+$. Let $V_-(x_1, x_2) = A(D_-(x_1, x_2))$ and $V_+(x_1, x_2) = A(D_+(x_1, x_2))$, respectively, and assume that $V_-(x_1, x_2) \leq V_+(x_1, x_2)$.

Lemma 7.1.6. *Let S be a spindle convex disc with C^2 boundary and $\kappa(x) > 1$ for all $x \in \partial S$. Then there exists a constant $\delta > 0$, depending only on S, such that $V_+(x_1, x_2) > \delta$ for any two distinct points $x_1, x_2 \in S$.*

Proof. We note that $[x_1, x_2]_s$ cannot cover S because of the C^2 smoothness of ∂S and the assumption that $\kappa(x) > 1$ for all $x \in \partial S$. Thus, by compactness, there exists a constant $\delta > 0$, depending only on S, such that $A(S \setminus [x_1, x_2]_s) > 2\delta$ for any two distinct points $x_1, x_2 \in S$. Now, the statement of the lemma readily follows from the fact that $S = D_-(x_1, x_2) \cup D_+(x_1, x_2) \cup [x_1, x_2]_s$. $\square$

Let K be a convex disc with C^2 boundary and with the property that $\kappa(x) > 0$ for all $x \in \partial K$. Let $\kappa_0 > 0$ denote the minimum of the curvature of ∂K. Then there exists an $\varepsilon_0 > 0$, depending only on K, with the property that for any $x \in \partial K$ the (unique) circle of radius $1/\kappa_0$ that is tangent to ∂K at x supports K in a neighbourhood of radius ε_0 of x. Moreover, Mayer proved (see statement (Ü5) on page 521 in [May35], or for a more recent and more general reference see also Theorem 2.5.4. in [Sch14]) that in this case the tangent circles of radius $1/\kappa_0$ of ∂K not only locally support K but also contain K and thus they globally support K.

Let S be a spindle convex disc with C^2 smooth boundary and with the property that $\kappa(x) > 1$ for all $x \in \partial K$. Then, by the above, there exists $0 < \hat{\varrho} < 1$, depending only on S, such that S has a supporting circular disc of radius $\hat{\varrho}$ at each $x \in \partial S$. Thus, Lemma 7.1.5 yields that there exists a $0 < t_0 \leq \hat{\varrho}$ with the property that for any $u \in S^1$

$$\ell(u, t) \leq 4\sqrt{\frac{2\hat{\varrho}}{1-\hat{\varrho}}}t^{\frac{1}{2}} \quad \text{for } t \in [0, t_0]. \tag{7.1.17}$$

A convex disc K has a *rolling ball* if there exists a real number $\varrho > 0$ with the property that any $x \in \partial K$ lies in some closed circular disc of radius ϱ contained in K. Hug proved in [Hug00] that the existence of a rolling ball is equivalent to the exterior unit normal being a Lipschitz function on ∂K. This implies that if the boundary of K is C^2 smooth, then K has a rolling ball. We remark that this last fact was already observed by Blaschke [Bla56].

It follows from the assumption that the boundary of S is C^2 smooth that there exists a rolling ball for S with radius $0 < \varrho < 1$. The existence of the rolling ball and (7.1.15) yield that there exists $0 < \hat{t} < \varrho$ such that for any $u \in S^1$

$$V(u, t) \geq \frac{1}{2}\left(\frac{4}{3}\sqrt{\frac{2\varrho}{1-\varrho}}\right)t^{\frac{3}{2}} \quad \text{for } t \in [0, \hat{t}]. \tag{7.1.18}$$

Note that although the statements in Lemma 7.1.5 are not uniform in u, both (7.1.17) and (7.1.18) are uniform in u.

7.1.4 Proofs of Theorem 7.1.1 and Theorem 7.1.2

Proof of Theorem 7.1.1. We essentially use the method invented by Rényi and Sulanke [RS63]. Note that it is enough to prove the theorem for $R = 1$, from that the statement follows by a scaling argument. Thus, from now on we assume that $R = 1$, and omit R from the notation.

Let $A = A(S)$. First, observe that the pair of random points x_1, x_2 determine an edge of S_n if and only if at least one of the disc-caps $D_-(x_1, x_2)$ and $D_+(x_1, x_2)$ does not contain any other points from X_n. Thus

$$\mathbb{E}(f_0(S_n)) = \binom{n}{2} W_n,$$

where

$$W_n = \frac{1}{A^2}\int_S\int_S \left[\left(1 - \frac{V_-(x_1,x_2)}{A}\right)^{n-2} + \left(1 - \frac{V_+(x_1,x_2)}{A}\right)^{n-2}\right] \mathrm{d}x_1\mathrm{d}x_2. \qquad (7.1.19)$$

Note that if all points of X_n fall into the closed spindle spanned by x_1 and x_2, then x_1 and x_2 contribute two edges to S_n (since in this case $\mathrm{conv}_S\, X_n = [x_1, x_2]_S$), and accordingly this event is counted in both terms in the integrand of (7.1.19).

Lemma 7.1.6 yields that

$$\begin{aligned}
&\lim_{n\to\infty} n^{-\frac{1}{3}}\binom{n}{2}\frac{1}{A^2}\int_S\int_S\left(1 - \frac{V_+(x_1,x_2)}{A}\right)^{n-2}\mathrm{d}x_1\mathrm{d}x_2\\
\leq &\lim_{n\to\infty} n^{-\frac{1}{3}}\binom{n}{2}\frac{1}{A^2}\int_S\int_S e^{-\frac{\delta}{A}(n-2)}\mathrm{d}x_1\mathrm{d}x_2\\
= &\lim_{n\to\infty} n^{-\frac{1}{3}}\binom{n}{2}e^{-\frac{\delta}{A}(n-2)} = 0.
\end{aligned}$$

Thus, the contribution of the second term of (7.1.19) is negligible, hence, in what follows, we will consider only the first term. Note that a similar argument yields that in the first term of (7.1.19) it is enough to integrate over pairs of random points x_1, x_2 such that $V_-(x_1, x_2) < \delta$. Let $\mathbf{1}(\cdot)$ denote the indicator function of an event. Then

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}(f_0(S_n))n^{-\frac{1}{3}}\\
&\quad = \lim_{n\to\infty} n^{-\frac{1}{3}}\binom{n}{2}\frac{1}{A^2}\int_S\int_S\left(1 - \frac{V_-(x_1,x_2)}{A}\right)^{n-2}\mathbf{1}(V_-(x_1,x_2) < \delta)\mathrm{d}x_1\mathrm{d}x_2. \qquad (7.1.20)
\end{aligned}$$

Now, we re-parametrize the pair (x_1, x_2) as follows. Let

$$(x_1, x_2) = \Phi(u, t, u_1, u_2), \qquad (7.1.21)$$

where $u, u_1, u_2 \in S^1$ and $0 \le t \le t_0(u)$ are chosen such that

$$D(u,t) = D_-(x_1, x_2),$$

and

$$(x_1, x_2) = (x_u - (1+t)u + u_1, x_u - (1+t)u + u_2).$$

Note that u_1 and u_2 are the unique outer unit normal vectors of $\partial B + x_u - (1+t)u$ at x_1 and x_2, respectively. This yields that, for fixed u and t, both u_1 and u_2 are in the same arc of length $\ell(u,t)$ in S^1. We denote this unit circular arc by $L(u,t)$.

Note that since $V_-(x_1, x_2) < \delta$, $D_-(x_1, x_2)$ is uniquely determined by Lemma 7.1.6. Now, the uniqueness of the vertex and height of a disc-cap guarantees that Φ is well-defined, bijective, and differentiable (see Section 7.1.6) on a suitable domain of (u, t, u_1, u_2). To continue the estimate of W_n we need the Jacobian of the transformation Φ. This calculation can be found in Santaló's paper [San46], but for the sake of completeness, we give a sketch in Section 7.1.6.

We obtain that the Jacobian of Φ satisfies

$$|J\Phi| = \left(1 + t - \frac{1}{\kappa(x_u)}\right) |u_1 \times u_2|. \tag{7.1.22}$$

We note that $|u_1 \times u_2|$ equals the sine of the length of the unit circular arc between x_1 and x_2 on the boundary of $D(u,t)$. Also note that there exists $t_1 > 0$ with the property that $V(u,t) < \delta$ for all $0 \le t \le t_1$ and for all $u \in S^1$.

Now, (7.1.20) and (7.1.22) yield that

$$\begin{aligned}\lim_{n\to\infty} &\mathbb{E}(f_0(S_n)) n^{-\frac{1}{3}} \\ &= \lim_{n\to\infty} n^{-\frac{1}{3}} \binom{n}{2} \frac{1}{A^2} \int_{S^1} \int_0^{t_1} \int_{L(u,t)} \int_{L(u,t)} \left(1 - \frac{V(u,t)}{A}\right)^{n-2} \\ &\qquad \times \left(1 + t - \frac{1}{\kappa(x_u)}\right) |u_1 \times u_2| \mathrm{d}u_1 \mathrm{d}u_2 \mathrm{d}t \mathrm{d}u. \end{aligned} \tag{7.1.23}$$

Integration by u_1 and u_2 yields

$$\begin{aligned}(7.1.23) = \lim_{n\to\infty} n^{-\frac{1}{3}} \binom{n}{2} \frac{2}{A^2} \int_{S^1} \int_0^{t_1} &\left(1 - \frac{V(u,t)}{A}\right)^{n-2} \\ &\times \left(1 + t - \frac{1}{\kappa(x_u)}\right) (\ell(u,t) - \sin \ell(u,t)) \mathrm{d}t \mathrm{d}u.\end{aligned}$$

Now, we will split the domain of integration with respect to t into two parts. Let $h(n) = (c \ln n / n)^{2/3}$, where c is a positive (absolute) constant to be specified later. From (7.1.18) it follows that there exists $n_0 \in \mathbb{N}$ and $\gamma_1 > 0$, depending only on S, such that if $n > n_0$, then $h(n) < t_1$, and $V(u,t) > \gamma_1 \cdot h(n)^{3/2}$ for all $h(n) \le t \le t_1$ and for all $u \in S^1$.

Lemma 7.1.7. *Let $h(n)$ be defined as above. Then*

$$\lim_{n\to\infty} n^{-\frac{1}{3}}\binom{n}{2}\frac{2}{A^2}\int_{S^1}\int_{h(n)}^{t_1}\left(1-\frac{V(u,t)}{A}\right)^{n-2} \times\left(1+t-\frac{1}{\kappa(x_u)}\right)(\ell(u,t)-\sin\ell(u,t))\mathrm{d}t\mathrm{d}u=0.$$

Proof. Note that $t_1\le 2\pi$, and there exists a universal constant $\gamma_2>0$ such that $\ell(u,t)-\sin\ell(u,t)\le\gamma_2$ for all $0\le t\le t_1$ and $u\in S^1$. Hence, for any fixed $u\in S^1$ and any $n>n_0$, it holds that

$$\begin{aligned}
&\int_{h(n)}^{t_1}\left(1-\frac{V(u,t)}{A}\right)^{n-2}\left(1+t-\frac{1}{\kappa(x_u)}\right)(\ell(u,t)-\sin\ell(u,t))\mathrm{d}t\\
&\le 3\gamma_2\int_{h(n)}^{t_1}\left(1-\frac{\gamma_1 h(n)^{3/2}}{A}\right)^{n-2}\mathrm{d}t\\
&\le 3\gamma_2\int_{0}^{t_1}\left(1-\frac{\gamma_1 c(\ln n/n)}{A}\right)^{n-2}\mathrm{d}t\\
&\le 6\gamma_2 n^{-\frac{c\gamma_1}{A}}.
\end{aligned}$$

Now, let $c>5A/(3\gamma_1)$. Then

$$\begin{aligned}
&\lim_{n\to\infty} n^{-\frac{1}{3}}\binom{n}{2}\frac{2}{A^2}\int_{S^1}\int_{h(n)}^{t_1}\left(1-\frac{V(u,t)}{A}\right)^{n-2}\left(1+t-\frac{1}{\kappa(x_u)}\right)\times\\
&\qquad\times(\ell(u,t)-\sin\ell(u,t))\mathrm{d}t\mathrm{d}u\\
&\le\gamma_2\frac{24\pi}{A^2}\lim_{n\to\infty} n^{-\frac{1}{3}}\binom{n}{2}n^{-\frac{c\gamma_1}{A}}=0.
\end{aligned}$$

□

Now, for $n>n_0$ we define

$$\theta_n(u)=n^{-\frac{1}{3}}\binom{n}{2}\int_0^{h(n)}\left(1-\frac{V(u,t)}{A}\right)^{n-2}\left(1+t-\frac{1}{\kappa(x_u)}\right)\times \times(\ell(u,t)-\sin\ell(u,t))\mathrm{d}t \quad (7.1.24)$$

and so

$$\lim_{n\to\infty}\mathbb{E}(f_0(S_n))\cdot n^{-\frac{1}{3}}=\lim_{n\to\infty}\frac{2}{A^2}\int_{S^1}\theta_n(u)\,\mathrm{d}u. \quad (7.1.25)$$

We recall formula (11) from [BFRV09] that states the following. For any $\beta\ge 0$, $\omega>0$ and $\alpha>0$ we have that

$$\int_0^{g(n)} t^\beta\left(1-\omega t^\alpha\right)^n\mathrm{d}t\sim\frac{1}{\alpha\omega^{\frac{\beta+1}{\alpha}}}\cdot\Gamma\left(\frac{\beta+1}{\alpha}\right)\cdot n^{-\frac{\beta+1}{\alpha}}, \quad (7.1.26)$$

as $n\to\infty$, assuming

$$\left(\frac{(\beta+\alpha+1)\ln n}{\alpha\omega n}\right)^{\frac{1}{\alpha}}<g(n)<\omega^{-\frac{1}{\alpha}},$$

for sufficiently large n.

Formula (7.1.17) implies that there exists $\gamma_3 > 0$ such that $\ell(u,t) - \sin \ell(u,t) < \gamma_3 t^{3/2}$ for all $0 < t < t_0$ and $u \in S^1$. We recall that $1 + t - 1/\kappa(x_u) < 3$ for all $u \in S^1$ and $0 \leq t \leq t_1$. Now (7.1.18) and (7.1.26) with $\alpha = \beta = 3/2$ and $\omega = (2/(3A))\sqrt{2\rho/(1-\rho)}$ yield that there exists $\gamma_4 > 0$, depending only on S, such that $\theta_n(u) < \gamma_4$ for all $u \in S^1$ and sufficiently large n. Thus, Lebesgue's dominated convergence theorem implies that

$$\lim_{n\to\infty} \mathbb{E}(f_0(S_n)) \cdot n^{-\frac{1}{3}} = \frac{2}{A^2} \int_{S^1} \lim_{n\to\infty} \theta_n(u)\, \mathrm{d}u. \tag{7.1.27}$$

Let $u \in S^1$ and $\varepsilon \in (0,1)$. It follows from Lemma 7.1.5 that there exists $0 < t_\varepsilon < t_1$ such that

$$(1-\varepsilon)\frac{4}{3}\left(\frac{2}{\kappa(x_u)-1}\right)^{\frac{3}{2}} t^{\frac{3}{2}} \leq \ell(u,t) - \sin \ell(u,t) \leq (1+\varepsilon)\frac{4}{3}\left(\frac{2}{\kappa(x_u)-1}\right)^{\frac{3}{2}} t^{\frac{3}{2}} \tag{7.1.28}$$

and

$$(1-\varepsilon)\frac{4}{3}\sqrt{\frac{2}{\kappa(x_u)-1}}\,t^{\frac{3}{2}} \leq V(u,t) \leq (1+\varepsilon)\frac{4}{3}\sqrt{\frac{2}{\kappa(x_u)-1}}\,t^{\frac{3}{2}}, \tag{7.1.29}$$

for any $t \in (0, t_\varepsilon)$.

Now (7.1.28) and (7.1.29) yield that

$$\begin{aligned}\lim_{n\to\infty} \theta_n(u) = {}& \frac{4\sqrt{2}}{3}\left(\frac{1}{\kappa(x_u)-1}\right)^{\frac{3}{2}} \\ &\times \left[\frac{\kappa(x_u)-1}{\kappa(x_u)} \lim_{n\to\infty} n^{\frac{5}{3}} \int_0^{h(n)} \left(1 - \frac{4}{3A}\sqrt{\frac{2}{\kappa(x_u)-1}}\,t^{\frac{3}{2}}\right)^{n-2} t^{\frac{3}{2}}\mathrm{d}t\right. \\ &\left. + \lim_{n\to\infty} n^{\frac{5}{3}} \int_0^{h(n)} \left(1 - \frac{4}{3A}\sqrt{\frac{2}{\kappa(x_u)-1}}\,t^{\frac{3}{2}}\right)^{n-2} t^{\frac{5}{2}}\mathrm{d}t\right].\end{aligned} \tag{7.1.30}$$

Note that (7.1.26) with $\alpha = 3/2$, $\beta = 5/2$ implies that the second term of (7.1.30) is 0. Now, (7.1.26) yields that

$$\begin{aligned}\lim_{n\to\infty} n^{\frac{5}{3}} &\int_0^{h(n)} \left(1 - \frac{4}{3A}\sqrt{\frac{2}{\kappa(x_u)-1}}\,t^{\frac{3}{2}}\right)^{n-2} t^{\frac{3}{2}}\mathrm{d}t \\ &= \frac{2}{3}\left(\frac{4}{3A}\sqrt{\frac{2}{\kappa(x_u)-1}}\right)^{-\frac{5}{3}} \Gamma\left(\frac{5}{3}\right).\end{aligned}$$

Thus,

$$\lim_{n\to\infty} \theta_n(u) = \frac{8\sqrt{2}}{9}\left(\frac{1}{\kappa(x_u)-1}\right)^{\frac{3}{2}} \frac{\kappa(x_u)-1}{\kappa(x_u)} \left(\frac{4}{3A}\sqrt{\frac{2}{\kappa(x_u)-1}}\right)^{-\frac{5}{3}} \Gamma\left(\frac{5}{3}\right).$$

Therefore,

$$\lim_{n\to\infty} \mathbb{E} f_0(S_n) \cdot n^{-\frac{1}{3}} = \frac{2}{A^2} \int_{S^1} \lim_{n\to\infty} \theta_n(u) \mathrm{d}u$$
$$= \sqrt[3]{\frac{2}{3A}} \Gamma\left(\frac{5}{3}\right) \int_{S^1} \frac{1}{\kappa(x_u)} \left(\kappa(x_u) - 1\right)^{\frac{1}{3}} \mathrm{d}u$$
$$= \sqrt[3]{\frac{2}{3A}} \Gamma\left(\frac{5}{3}\right) \int_{\partial S} \left(\kappa(x) - 1\right)^{\frac{1}{3}} \mathrm{d}x.$$

To compute the expectation of the missed area by S_n, we use the following identity

$$\mathbb{E}(f_0(S_n)) = \frac{n\mathbb{E}(A(S \setminus S_{n-1}))}{A}. \tag{7.1.31}$$

(7.1.31) is the spindle convex analogue of Efron's identity [Efr65]. The proof of (7.1.31) is as follows.

$$\mathbb{E}(f_0(S_n)) = \sum_1^n \mathbb{P}(x_i \text{ is a vertex of } S_n) = n\mathbb{P}(x_1 \text{ is a vertex of } S_n)$$
$$= n\mathbb{P}(x_1 \notin \operatorname{conv_S}(x_2, \dots, x_n)) = \frac{n\mathbb{E}(A(S \setminus S_{n-1}))}{A}$$

Now, combining (7.1.4) and (7.1.31) yields (7.1.5), thus completing the proof of Theorem 7.1.1. □

Now we turn to the proof of Theorem 7.1.2. The argument is based on ideas developed by Rényi and Sulanke in [RS64], and it is similar to the argument of the proof of Theorem 7.1.1.

We start with a refinement of Lemma 7.1.5 under the hypothesis that the boundary of S is C^5 smooth and that $\kappa(x) > 1$ for all $x \in \partial S$.

Lemma 7.1.8. *Let S be a spindle convex disc with C^5 smooth boundary and with the property that $\kappa(x) > 1$ for all $x \in \partial S$. Then uniformly in $u \in S^1$*

$$\ell(u,t) = l_1 t^{1/2} + l_2 t^{3/2} + O(t^{5/2}) \quad \text{as } t \to 0^+, \text{ and} \tag{7.1.32}$$
$$V(u,t) = v_1 t^{3/2} + v_2 t^{5/2} + O(t^{7/2}) \quad \text{as } t \to 0^+, \tag{7.1.33}$$

with

$$l_1 = l_1(u) = 2\sqrt{\frac{2}{\kappa(x_u) - 1}}$$
$$l_2 = l_2(u) = \frac{2^{3/2}\left(15b(x_u)^2 - (\kappa(x_u) - 1)(1 + 6(c(x_u) - 1/8) - \kappa(x_u))\right)}{3(\kappa(x_u) - 1)^{7/2}}$$
$$v_1 = v_1(u) = \frac{4}{3}\sqrt{\frac{2}{\kappa(x_u) - 1}}$$
$$v_2 = v_2(u) = \frac{2^{5/2}\left(5b(x_u)^2 - 2(c(x_u) - 1/8)(\kappa(x_u) - 1)\right)}{5(\kappa(x_u) - 1)^{7/2}},$$

where $b(x)$ and $c(x)$ are functions depending only on S and x.

Proof. With the same notation and choice of coordinate system as in the proof of Lemma 7.1.5, Taylor's theorem and the C^5 smoothness of the boundary yield that in a sufficiently small neighbourhood of the origin

$$f(\sigma) = \frac{\kappa}{2}\sigma^2 + b\sigma^3 + c\sigma^4 + O(\sigma^5) \quad \text{as } \sigma \to 0,$$

uniformly in $u \in S^1$. We suppress the notation of dependence of the coefficients on u for brevity. Let $g_t(\sigma) = t + 1 - \sqrt{1-\sigma^2}$. From the equation $f(\sigma) = g_t(\sigma)$ we obtain

$$t = \frac{\kappa - 1}{2}\sigma^2 + b\sigma^3 + \left(c - \frac{1}{8}\right)\sigma^4 + O(\sigma^5) \quad \text{as } \sigma \to 0,$$

and routine calculations yield that the positive and negative solutions of the equation $f(\sigma) = g_t(\sigma)$ are

$$\begin{aligned} \sigma_+ &= \sigma_+(t) = d_1 t^{1/2} + d_2 t + d_3 t^{3/2} + O(t^2) \quad \text{as } t \to 0^+, \\ \sigma_- &= \sigma_-(t) = -(d_1 t^{1/2} - d_2 t + d_3 t^{3/2}) + O(t^2) \quad \text{as } t \to 0^+, \end{aligned} \tag{7.1.34}$$

where

$$\begin{aligned} d_1 &= \sqrt{\frac{2}{\kappa - 1}}, \\ d_2 &= -\frac{2b}{(\kappa-1)^2}, \\ d_3 &= \frac{\sqrt{2}\left(5b^2 - 2(c - 1/8)(\kappa - 1)\right)}{(\kappa-1)^{7/2}}. \end{aligned}$$

Now, using that $\ell(u,t) = \arcsin \sigma_+ + \arcsin|\sigma_-|$ and that $V(u,t) = \int_{\sigma_-}^{\sigma_+}[g_t(\sigma) - f(\sigma)]\mathrm{d}\sigma$, a short calculation finishes the proof. □

Proof of Theorem 7.1.2. Let $L = \mathrm{Per}(S)$ for brevity. Let $x_1, x_2 \in S$, and let $i(x_1, x_2)$ denote the length of the shorter unit circular arc joining x_1 and x_2. We define U_n with

$$\begin{aligned} &\mathbb{E}(\mathrm{Per}(S) - \mathrm{Per}(S_n)) \\ &= L - \binom{n}{2}\mathbb{E}\left[\mathbf{1}(x_1, x_2 \text{ is an edge of } S_n) \cdot i(x_1, x_2)\right] =: L - \binom{n}{2}U_n. \end{aligned}$$

Using the same notation as in the proof of Theorem 7.1.1, similar arguments show that

$$U_n = \frac{1}{A^2}\int_S\int_S \left[\left(1 - \frac{V_-(x_1,x_2)}{A}\right)^{n-2} + \left(1 - \frac{V_+(x_1,x_2)}{A}\right)^{n-2}\right] i(x_1,x_2)\,\mathrm{d}x_1\mathrm{d}x_2,$$

and

$$\lim_{n\to\infty} n^{2/3}\binom{n}{2}\frac{1}{A^2}\int_S\int_S\left(1 - \frac{V_+(x_1,x_2)}{A}\right)^{n-2} i(x_1,x_2)\mathrm{d}x_1\mathrm{d}x_2 = 0,$$

and also that

$$\lim_{n\to\infty} n^{2/3}\binom{n}{2}\frac{1}{A^2}\int_S\int_S\left(1-\frac{V_-(x_1,x_2)}{A}\right)^{n-2}\mathbf{1}(V_-(x_1,x_2)>\delta)\times$$
$$\times\, i(x_1,x_2)\mathrm{d}x_1\mathrm{d}x_2=0.$$

Now, the integral transformation Φ in (7.1.21) yields that

$$\begin{aligned}&\frac{1}{A^2}\int_S\int_S\left(1-\frac{V_-(x_1,x_2)}{A}\right)^{n-2}\mathbf{1}(V_-(x_1,x_2)\le\delta)i(x_1,x_2)\mathrm{d}x_1\mathrm{d}x_2\\&=\frac{1}{A^2}\int_{S^1}\int_0^{t_1}\int_{L(u,t)}\int_{L(u,t)}\left(1-\frac{V(u,t)}{A}\right)^{n-2}\\&\times\left(1+t-\frac{1}{\kappa(x_u)}\right)\cdot|u_1\times u_2|\arccos\langle u_1,u_2\rangle\mathrm{d}u_1\mathrm{d}u_2\,\mathrm{d}t\,\mathrm{d}u,\end{aligned}$$

where $\arccos\langle u_1,u_2\rangle$ is the length of the arc of S^1 spanned by u_1 and u_2. Routine calculations show that

$$\int_{L(u,t)}\int_{L(u,t)}|u_1\times u_2|\arccos\langle u_1,u_2\rangle\mathrm{d}u_1\mathrm{d}u_2=2\left(2-2\cos\ell(u,t)-\ell(u,t)\sin\ell(u,t)\right).$$

Let $\varepsilon>0$ be arbitrary. According to Lemma 7.1.8 we may choose $t_2>0$ such that for all $t\in(0,t_2)$ and for all $u\in S^1$

$$\begin{aligned}\left|\ell(u,t)-(l_1t^{1/2}+l_2t^{3/2})\right|&\le\frac{\varepsilon}{2}t^{3/2},\\\left|V(u,t)-(v_1t^{3/2}+v_2t^{5/2})\right|&\le\varepsilon t^{5/2}.\end{aligned}\tag{7.1.35}$$

For any $\varepsilon'>0$ for sufficiently small x it holds that

$$\left|2\left(2-2\cos x-x\sin x\right)-\left(\frac{x^4}{6}-\frac{x^6}{90}\right)\right|\le\varepsilon' x^6,$$

which, together with (7.1.35), implies that there exists $t_3>0$ with the property that for any $t\in(0,t_3)$ and for all $u\in S^1$

$$\left|2\left(2-2\cos\ell(u,t)-\ell(u,t)\sin\ell(u,t)\right)-\frac{1}{6}\left[l_1^4t^2+\left(4l_1^3l_2-\frac{l_1^6}{15}\right)t^3\right]\right|\le\frac{\varepsilon}{6}t^3.\tag{7.1.36}$$

The second order Taylor expansion of the function $\log(1-y)$ at $y=0$ yields that there exists $t_4>0$ such that for $0<y\le n\min_{u\in S^1}v_1(u)t_4^{2/3}/A$ and for any $c\in[-a_1,a_1]$, with $a_1=A^{2/3}\max_{u\in S^1}\left|v_2(u)/v_1^{5/3}(u)\right|$, and for all $u\in S^1$

$$e^{-y}e^{-(c+\varepsilon)y^{5/3}n^{-2/3}}\le\left[1-\frac{y}{n}-c\left(\frac{y}{n}\right)^{5/3}\right]^n\le e^{-y}e^{-cy^{5/3}n^{-2/3}}\tag{7.1.37}$$

and

$$e^{-(1+\varepsilon)y}\le\left[1-\frac{y}{n}-c\left(\frac{y}{n}\right)^{5/3}\right]^n\le e^{-(1-\varepsilon)y}.\tag{7.1.38}$$

Let $\delta = \delta(\varepsilon)$ be small enough such that for all $|y| \leq \delta$

$$e^{-y} \leq 1 - (1-\varepsilon)y, \tag{7.1.39}$$

and let n_0 be large enough such that

$$\max_{u \in S^1} \frac{|v_2(u)| A^{2/3}}{v_1^{5/3}(u)} \leq n_0^{1/3} \delta. \tag{7.1.40}$$

Finally, let $t' := \min\{t_2, t_3, t_4\}$. A similar argument as in the proof of Lemma 7.1.7 yields that

$$\lim_{n\to\infty} n^{2/3} \binom{n}{2} \frac{1}{A^2} \int_{S^1} \int_{t'}^{t_1} \left(1 - \frac{V(u,t)}{A}\right)^{n-2} \\ \times 2\left[2 - 2\cos\ell(u,t) - \ell(u,t)\sin\ell(u,t)\right] \left(t + 1 - \frac{1}{\kappa(x_u)}\right) \mathrm{d}t\, \mathrm{d}u = 0.$$

Thus we need to determine the limit

$$\lim_{n\to\infty} n^{2/3} \Bigg[L - \binom{n}{2} \frac{1}{A^2} \int_{S^1} \int_{0}^{t'} \left(1 - \frac{V(u,t)}{A}\right)^{n} \\ \times 2\left[2 - 2\cos\ell(u,t) - \ell(u,t)\sin\ell(u,t)\right] \left(t + 1 - \frac{1}{\kappa(x_u)}\right) \mathrm{d}t\, \mathrm{d}u\Bigg].$$

By Lemma 7.1.8, for sufficiently small t it holds uniformly in $u \in S^1$ that

$$1 \leq \left(1 - \frac{V(u,t)}{A}\right)^{-2} \leq 1 + \frac{3\max_{u\in S^1} v_1(u)}{A} t^{3/2}.$$

Therefore changing the exponent from $n-2$ to n in the inner integral above does not affect either the main or the first order term.

By (7.1.35) and (7.1.36), we have that

$$\hat{\theta}_n(u) := \frac{1}{A^2} \int_0^{t'} \left(1 - \frac{V(u,t)}{A}\right)^n \times \\ \times 2\left[2 - 2\cos\ell(u,t) - \ell(u,t)\sin\ell(u,t)\right] \left(t + 1 - \frac{1}{\kappa}\right) \mathrm{d}t \\ \leq \frac{1}{6A^2} \int_0^{t'} \left(1 - \frac{v_1}{A} t^{3/2} - \frac{v_2 - \varepsilon}{A} t^{5/2}\right)^n \\ \times \left[l_1^4 \left(1 - \frac{1}{\kappa}\right) t^2 + \left(l_1^4 + \left(1 - \frac{1}{\kappa}\right)\left(4 l_1^3 l_2 - \frac{l_1^6}{15}\right) + \varepsilon\right) t^3\right] \mathrm{d}t.$$

To shorten the notation, let

$$D_1 = l_1^4\left(1 - \kappa^{-1}\right), \quad D_1 D_2^{\varepsilon} = l_1^4 + \left(1 - \kappa^{-1}\right)\left(4l_1^3 l_2 - l_1^6/15\right) + \varepsilon, \quad \text{and } D_2 = D_2^0. \tag{7.1.41}$$

Letting $t'' = (t')^{3/2} v_1/A$, the substitution $t^{3/2} v_1/A = y/n$ yields

$$\begin{aligned}
\hat{\theta}_n(u) &\leq \frac{D_1}{6A^2} \int_0^{nt''} \left[1 - \frac{y}{n} - \frac{v_2 - \varepsilon}{A} \left(\frac{Ay}{nv_1}\right)^{5/3}\right]^n \left(\frac{Ay}{nv_1}\right)^{4/3} \\
&\quad \times \left[1 + D_2^{\varepsilon} \left(\frac{Ay}{nv_1}\right)^{2/3}\right] \frac{2}{3} y^{-1/3} \left(\frac{A}{nv_1}\right)^{2/3} \mathrm{d}y \\
&= \frac{D_1}{9n^2 v_1^2} \int_0^{nt''} \left[1 - \frac{y}{n} - \frac{(v_2 - \varepsilon)A^{2/3}}{v_1^{5/3}} \left(\frac{y}{n}\right)^{5/3}\right]^n \left[1 + D_2^{\varepsilon} \left(\frac{Ay}{nv_1}\right)^{2/3}\right] y \mathrm{d}y \\
&=: I_n + J_n,
\end{aligned}$$

where I_n stands for the integral over the interval $[0, n^{1/5}]$, and J_n stands for the integral over the interval $[n^{1/5}, t''n]$. Using (7.1.38), for J_n we obtain that

$$J_n \leq \frac{D_1}{9n^2 v_1^2} \int_{n^{1/5}}^{nt''} e^{-(1-\varepsilon)y} \, 2nt'' \mathrm{d}y \leq \frac{D_1}{9v_1^2} e^{-(1-\varepsilon)n^{1/5}},$$

which tends to 0 faster than any polynomial of n. For I_n, using (7.1.37), (7.1.39) and (7.1.40) for $n \geq n_0$ we have that

$$\begin{aligned}
I_n &\leq \frac{D_1}{9n^2 v_1^2} \int_0^{n^{1/5}} e^{-y} \exp\left\{-\frac{(v_2 - \varepsilon)A^{2/3}}{v_1^{5/3}} \frac{y^{5/3}}{n^{2/3}}\right\} \left[1 + D_2^{\varepsilon} \left(\frac{Ay}{nv_1}\right)^{2/3}\right] y \mathrm{d}y \\
&\leq \frac{D_1}{9n^2 v_1^2} \int_0^{n^{1/5}} e^{-y} \left(1 - (1-\varepsilon)\frac{(v_2 - \varepsilon)A^{2/3}}{v_1^{5/3}} \frac{y^{5/3}}{n^{2/3}}\right) \left[1 + D_2^{\varepsilon} \left(\frac{Ay}{nv_1}\right)^{2/3}\right] y \mathrm{d}y \\
&\leq \frac{D_1}{9n^2 v_1^2} \int_0^{n^{1/5}} e^{-y} \left[1 + n^{-2/3} A^{2/3} \left(\frac{D_2^{\varepsilon}}{v_1^{2/3}} y^{2/3} - (1-\varepsilon)\frac{v_2 - \varepsilon}{v_1^{5/3}} y^{5/3} + \varepsilon\right)\right] y \mathrm{d}y \\
&\leq \frac{D_1}{9n^2 v_1^2} \left[1 + n^{-2/3} A^{2/3} \left(\frac{D_2^{\varepsilon}}{v_1^{2/3}} \Gamma(8/3) - (1-\varepsilon)\frac{v_2 - \varepsilon}{v_1^{5/3}} \Gamma(11/3) + 2\varepsilon\right)\right],
\end{aligned}$$

where in the last inequality we extended the domain of the integration, and used the definition of the $\Gamma(\,\cdot\,)$ function.

We may obtain a lower estimate for $\hat{\theta}_n(u)$ in a similar way, and as $\varepsilon > 0$ was arbitrary, we have that $\hat{\theta}_n(u)$ asymptotically equals to the last upper bound with $\varepsilon = 0$. Since $D_1/(18v_1^2) = \kappa^{-1}$ and $\int_{S^1} \kappa^{-1}(x_u)\mathrm{d}u = L$, we have that

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{E}(L - \mathrm{Per}(S_n)) \cdot n^{2/3} &= \lim_{n\to\infty} n^{2/3} \left(L - \binom{n}{2} \int_{S^1} \hat{\theta}_n(u)\mathrm{d}u\right) \\
&= \int_{S^1} \frac{D_1 A^{2/3}}{18v_1^2} \left(\frac{D_2}{v_1^{2/3}} \Gamma(8/3) - \frac{v_2}{v_1^{5/3}} \Gamma(11/3)\right) \mathrm{d}u.
\end{aligned}$$

Substituting to the formula above the values of D_1, D_2 from (7.1.41) and l_1, l_2, v_1, v_2 from

Lemma 7.1.8 we obtain that

$$\frac{D_1 A^{2/3}}{18 v_1^2}\left(\frac{D_2}{v_1^{2/3}}\Gamma(8/3)-\frac{v_2}{v_1^{5/3}}\Gamma(11/3)\right)$$
$$=\frac{A^{2/3}\Gamma(8/3)}{\kappa}\frac{(3/2)^{2/3}\left[60b^2+(\kappa-1)\left(5(\kappa-1)^2+9(\kappa-1)+3-24c\right)\right]}{10(\kappa-1)^{8/3}},$$

and thus

$$\lim_{n\to\infty}\mathbb{E}(L-\mathrm{Per}(S_n))\cdot n^{2/3}$$
$$=\frac{(12A)^{2/3}\Gamma(2/3)}{36}\int_{\partial S}\frac{(\kappa-1)\left(24c-5(\kappa-1)^2-9(\kappa-1)-3\right)-60b^2}{(\kappa-1)^{8/3}}\mathrm{d}x. \qquad (7.1.42)$$

To finish the proof of Theorem 7.1.2, we must show that the constant in (7.1.42) is the same as in (7.1.6). Let $r(s)$ be the arc-length parametrization of ∂S. It is not difficult to verify that

$$b(r(s))=\frac{1}{6}\left\langle r'''(s),\frac{r''(s)}{\kappa(r(s))}\right\rangle,$$
$$c(r(s))=\frac{1}{24}\left(\left\langle r^{(4)}(s),\frac{r''(s)}{\kappa(r(s))}\right\rangle-4\kappa(r(s))\langle r'''(s),r'(s)\rangle\right).$$

After substituting these formulae into (7.1.42), some tedious but straightforward calculations yield (7.1.6). □

7.1.5 The case of the unit circular disc

In this section we discuss the case, when $S=B_R$. Note that in the hypotheses of Theorems 7.1.1 and 7.1.2 it is assumed that $\kappa(x)>1/R$ for all $x\in\partial S$. This assumption no longer holds in the case that $S=B_R$, and therefore we may not use Lemma 7.1.6. However, the arguments of the proofs of Theorems 7.1.1 and 7.1.2 can be modified slightly to yield a proof of Theorem 7.1.3. Below we provide the outline of the proof of Theorem 7.1.3 and leave the technical details to the interested reader.

Proof of Theorem 7.1.3. As in the previous section, we may and do assume that $R=1$.

First note that by Efron's identity (7.1.31), it is enough to prove (7.1.7) and (7.1.9). Also note that for any $u\in S^1$ and $0\le t\le 2$ simple calculations yield

$$\ell(u,t)=\ell(t)=2\arcsin\sqrt{1-\frac{t^2}{4}}, \qquad (7.1.43)$$

and

$$V(u,t)=V(t)=t\sqrt{1-\frac{t^2}{4}}+2\arcsin\frac{t}{2}. \qquad (7.1.44)$$

Let W_n and U_n be defined as in the proofs of Theorems 7.1.1 and 7.1.2, respectively, and let Φ and $L(t)=L(u,t)$ be defined as in the proof of Theorem 7.1.1. Then

$$W_n=\frac{1}{\pi^2}\int_{S^1}\int_0^2\int_{L(t)}\int_{L(t)}\left(1-\frac{V(t)}{\pi}\right)^{n-2}t|u_1\times u_2|\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}t\mathrm{d}u,$$

$$U_n = \frac{1}{\pi^2}\int_{S^1}\int_0^2\int_{L(t)}\int_{L(t)}\left(1-\frac{V(t)}{\pi}\right)^{n-2} t\arccos\langle u_1,u_2\rangle |u_1\times u_2| \mathrm{d}u_1\mathrm{d}u_2\mathrm{d}t\mathrm{d}u.$$

Integration by u_1, u_2 and u yields

$$\begin{aligned} W_n &= \frac{4}{\pi}\int_0^2\left(1-\frac{V(t)}{\pi}\right)^{n-2} t(\ell(t)-\sin\ell(t))\mathrm{d}t,\\ U_n &= \frac{4}{\pi}\int_0^2\left(1-\frac{V(t)}{\pi}\right)^{n-2} t(2-2\cos\ell(t)-\ell(t)\sin\ell(t))\mathrm{d}t. \end{aligned}$$

Formulas (7.1.43), (7.1.44) and the substitution $t = 2\sin(\sigma/2)$ yield

$$\begin{aligned} W_n &= \frac{4}{\pi}\int_0^\pi \sin\sigma\,(\pi-\sigma-\sin\sigma)\left(1-\frac{\sin\sigma+\sigma}{\pi}\right)^{n-2}\mathrm{d}\sigma\,,\\ U_n &= \frac{4}{\pi}\int_0^\pi \sin\sigma\,(2+2\cos\sigma-\sin\sigma(\pi-\sigma))\left(1-\frac{\sin\sigma+\sigma}{\pi}\right)^{n-2}\mathrm{d}\sigma. \end{aligned} \tag{7.1.45}$$

Now, by similar arguments as in the proofs of Theorems 7.1.1 and 7.1.2, we obtain that

$$\begin{aligned} W_n &\sim \frac{\pi^2}{n^2},\\ U_n &\sim \frac{4\pi}{(n-2)^2}\left[1-\frac{1}{n-2}\left(\frac{\pi^2}{4}+3\right)\right]+O(n^{-3}), \end{aligned}$$

which yield the statements of Theorem 7.1.3. □

7.1.6 The Jacobian of Φ

In this section we sketch the calculation of the Jacobian of the transformation Φ defined in (7.1.21). We remark that $J\Phi$ was calculated by Santaló in [San46].

Let $r\colon [0,2\pi)\to\partial S$ be a parametrization of ∂S such that the outer normal $u_{r(\alpha)} = (\cos\alpha,\sin\alpha)$. We introduce α,ϕ_1 and ϕ_2 such that $u=(\cos\alpha,\sin\alpha)$, $u_1=(\cos\phi_1,\sin\phi_1)$ and $u_2=(\cos\phi_2,\sin\phi_2)$. Clearly, $\mathrm{d}u\mathrm{d}u_1\mathrm{d}u_2=\mathrm{d}\alpha\mathrm{d}\phi_1\mathrm{d}\phi_2$.

To make the calculation more apparent, we add an extra step: let (v,w) be the centre of the unit circle that defines $D_-(x_1,x_2)$ (here $v,w\in\mathbb{R}$). Then $x_1=(v+\cos\phi_1, w+\sin\phi_1)$ and $x_2=(v+\cos\phi_2,w+\sin\phi_2)$, and by differentiation we obtain that

$$\mathrm{d}x_1\mathrm{d}x_2 = |(\sin\phi_1\cos\phi_2-\sin\phi_2\cos\phi_1)|\mathrm{d}\phi_1\mathrm{d}\phi_2\mathrm{d}v\mathrm{d}w.$$

Next, observe that $(v,w)=(r_1(\alpha)-(1+t)\cos\alpha, r_2(\alpha)-(1+t)\sin\alpha)$, thus

$$\mathrm{d}v\mathrm{d}w = |(-r_1'(\alpha)\sin\alpha+r_2'(\alpha)\cos\alpha-(1+t))|\mathrm{d}\alpha\mathrm{d}t,$$

and hence

$$\mathrm{d}x_1\mathrm{d}x_2 = |(-r_1'(\alpha)\sin\alpha+r_2'(\alpha)\cos\alpha-(1+t))\sin(\phi_1-\phi_2)|\mathrm{d}\phi_1\mathrm{d}\phi_2\mathrm{d}\alpha\mathrm{d}t.$$

Using the special choice of $r(\alpha)$ one can see that $-r_1'(\alpha)\sin\alpha + r_2'(\alpha)\cos\alpha = 1/\kappa(r(\alpha))$, and by assumption $\kappa > 1$, thus

$$|(-r_1'(\alpha)\sin\alpha + r_2'(\alpha)\cos\alpha - (1+t))\sin(\phi_1 - \phi_2)| = (1 + t - 1/\kappa(r(\alpha)))\sin(|\phi_1 - \phi_2|).$$

We note that $|u_1 \times u_2|$ equals the sine of the length of the unit circular arc between x_1 and x_2 on the boundary of $D(u,t)$, that is, $\sin(|\phi_1 - \phi_2|) = |u_1 \times u_2|$, which proves (7.1.22).

7.2 Variances

This section of the dissertation is based on the paper [FV18]. We continue to use most of the notations of the previous section, however, it is more convenient to use different symbols at certain place which we indicate...

Let K be a convex disc (compact convex set with non-empty interior) in the Euclidean plane $\mathbb{R}^2$ with C^2_+ smooth boundary. Let κ_m (κ_M) be the minimum (maximum) of the curvature over ∂K. It is known, see [Sch14, Section 3.2], that in this case a closed circular disc of radius $r_m = 1/\kappa_M$ rolls freely in K, that is, for each $x \in \partial K$, there exists a $p \in \mathbb{R}^2$ with $x \in r_m B^2 + p \subset K$. Moreover, K slides freely in a circle of radius $r_M = 1/\kappa_m$, which means that for each $x \in \partial K$ there is a vector $p \in \mathbb{R}^2$ such that $x \in r_M \partial B^2 + p$ and $K \subset r_M B^2 + p$. The latter yields that for any two points $x, y \in K$, the intersection of all closed circular discs of radius $r \geq r_M$ containing x and y, denoted by $[x,y]_r$ and called the r-spindle of x and y, is also contained in K. Furthermore, for any $X \subset K$, the intersection of all radius $r \geq r_M$ circles containing X, called the closed r-hyperconvex hull (or r-hull for short) and denoted by $\operatorname{conv}_r(X)$, is contained in K.

Here we examine the following random model. Let $r \geq r_M$, and let $K_n^r = \operatorname{conv}_r(X_n)$ be the r-hull of X_n, which is a (uniform) random disc-polygon in K. Let $f_0(K_n^r)$ denote the number of vertices (and also the number of edges) of K_n^r, and let $A(K_n^r)$ denote the area of K_n^r.

Obtaining information on the second order properties of random variables associated with random polytopes is much harder than on first order properties. It is only recently that variance estimates, laws of large numbers, and central limit theorems have been proved in various models, see, for example, Bárány, Fodor, Vígh [BFV10], Bárány, Reitzner [BR10a], Bárány, Vu [BV07], Fodor, Hug, Ziebarth [FHZ16], Böröczky, Fodor, Reitzner, Vígh [BFRV09], Reitzner [Rei03, Rei05], Schreiber, Yukich [SY08], Vu [Vu05, Vu06], and the very recent papers by Thäle, Turchi, Wespi [TTW18], Turchi, Wespi [TW18]. For an overview, we refer to Bárány [Bár08] and Schneider [Sch18].

In this section, we prove the following asymptotic estimates for the variance of $f_0(K_n^r)$ and $A(K_n^r)$ in the spirit of Reitzner [Rei03].

Theorem 7.2.1 (Fodor, Vígh [FV18], Theorem 3 on page 1145)**.** *Let K be a convex disc whose boundary is of class C^2_+. For any $r > r_M$ it holds that*

$$\operatorname{Var}(f_0(K_n^r)) \ll n^{\frac{1}{3}}, \tag{7.2.1}$$

and

$$\operatorname{Var}(A(K_n^r)) \ll n^{-\frac{5}{3}}, \tag{7.2.2}$$

where the implied constants depend only on K and r.

In the special case when K is the closed circular disc of radius r, we prove the following.

Theorem 7.2.2 (Fodor, Vígh [FV18], Theorem 4 on page 1145)**.** *It holds that*

$$\operatorname{Var}(f_0(K_n^r)) \approx const., \tag{7.2.3}$$

and

$$\operatorname{Var}(A(K_n^r))) \ll n^{-2}, \tag{7.2.4}$$

where the implied constants depend only on r.

From Theorem 7.2.1 we can conclude the following strong laws of large numbers. Since the proof follows a standard argument based on Chebysev's inequality and the Borel-Cantelli lemma, see, for example, Böröczky, Fodor, Reitzner, Vígh [BFRV09, p. 2294] or Reitzner [Rei03, Section 5], and [BS13, p. 174], we omit the details.

Theorem 7.2.3 (Fodor, Vígh [FV18], Theorem 5 on page 1145)**.** *Let K be a convex disc whose boundary is of class C^2_+. For any $r > r_M$ it holds with probability 1 that*

$$\lim_{n\to\infty} f_0(K_n^r)\cdot n^{-1/3} = \sqrt[3]{\frac{2}{3A(K)}}\Gamma\left(\frac{5}{3}\right)\int_{\partial K}\left(\kappa(x)-\frac{1}{r}\right)^{1/3}\mathrm{d}x,$$

and

$$\lim_{n\to\infty} A(K\setminus K_n^r)\cdot n^{2/3} = \sqrt[3]{\frac{2A(K)^2}{3}}\Gamma\left(\frac{5}{3}\right)\int_{\partial K}\left(\kappa(x)-\frac{1}{r}\right)^{1/3}\mathrm{d}x.$$

In the theory of random polytopes there is more information on models in which the polytopes are generated as the convex hull of random points from a convex body K than on polyhedral sets produced by random closed half-spaces containing K. For some recent results and references in this direction see, for example, Böröczky, Fodor, Hug [BFH10], Böröczky, Schneider [BS10], Fodor, Hug, Ziebarth [FHZ16] and the survey by Schneider [Sch18].

Finally, in Section 7.2.4, we consider a model of circumscribed random disc-polygons that contain a given convex disc with C^2_+ boundary. In this circumscribed probability model, we give asymptotic formulas for the expectation of the number of vertices of the random disc-polygon, the area difference and the perimeter difference of the random disc-polygon and K, see Theorem 7.2.6. Furthermore, Theorem 7.2.7 provides an asymptotic upper bound on the variance of the number of vertices of the random disc-polygons.

7.2.1 Preparations

We note that it is enough to prove Theorem 7.2.1 for the case when $r_M < 1$ and $r = 1$, and Theorem 7.2.2 for $r = 1$. The general statements then follow by a simple scaling argument. Therefore, from now on we assume that $r = 1$ and to simplify notation we write K_n for K_n^1.

Let $\overline{B}^2$ denote the open unit ball of radius 1 centred at the origin o. Let $D(u,t)$ denote the disc-cap with vertex $x_u \in \partial K$ and height t. Note that for each $u \in S^1$, there exists a maximal positive constant $t^*(u)$ such that $(B + x_u - (1 + t)u) \cap K \neq \emptyset$ for all $t \in [0, t^*(u)]$. For simplicity we let $A(u,t) = A(D(u,t))$ and let $\ell(u,t)$ denote the arc-length of $\partial D(u,t) \cap (\partial B + x_u - (1+t)u)$, similarly as before.

It is clear that Lemma 7.1.5 implies that $A(u,t)$ and $\ell(u,t)$ satisfy the following relations uniformly in u:

$$\ell(u_x, t) \approx t^{1/2}, \qquad A(u_x, t) \approx t^{3/2}, \tag{7.2.5}$$

where the implied constants depend only on K.

Let D be a disc-cap of K with vertex x. For a line $e \subset \mathbb{R}^2$ with $e \perp u_x$, let e_+ denote the closed half plane containing x. Then there exist a maximal cap $C_-(D) = K \cap e_+ \subset D$, and a minimal cap $C_+(D) = e'_+ \cap K \supset D$.

Claim 7.2.4. *There exists a constant $\hat{c}$ depending only K such that if the height of the disc-cap D is sufficiently small, then*

$$\hat{c}(C_-(D) - x) \supset (C_+(D) - x).$$

Proof. Let us denote by h_- (h_+) the height of $C_-(D)$ ($C_+(D)$ resp.), which is the distance of x and e (e' resp.). By convexity, it is enough to find a constant $\hat{c} > 0$ such that for all disc-caps of K with sufficiently small height $h_+/h_- < \hat{c}$ holds.

Choose an arbitrary $R \in (1/\kappa_m, 1)$, and consider $\hat{B} = RB^2 + x - Ru_x$, the disc of radius R that supports K in x. Clearly, $\hat{B} \supset K$ implies $D = K \cap (\overline{B}^2 + p) \subset (\hat{B} \cap (\overline{B}^2 + p) = \hat{D}$. Also, for the respective heights $\hat{h}_-$ and $\hat{h}_+$ of $C_-(\hat{D})$ and $C_+(\hat{D})$, we have $\hat{h}_- = h_-$ and $\hat{h}_+ > h_+$. Thus, it is enough to find $\hat{c}$ such that $\hat{h}_+/\hat{h}_- < \hat{c}$. The existence of such $\hat{c}$ is clear from elementary geometry. $\square$

Let x_i, x_j ($i \neq j$) be two points from X_n, and let $B(x_i, x_j)$ be one of the unit discs that contain x_i and x_j on its boundary. The shorter arc of $\partial B(x_i, x_j)$ forms an edge of K_n if the entire set X_n is contained in $B(x_i, x_j)$. Note that it may happen that the pair x_i, x_j determines two edges of K_n if the above condition holds for both unit discs that contain x_i and x_j on its boundary.

We recall that the Hausdorff distance $d_H(A, B)$ of two non-empty compact sets $A, B \subset \mathbb{R}^2$ is defined as

$$d_H(A, B) := \max\{\max_{a \in A} \min_{b \in B} d(a,b), \max_{b \in B} \min_{a \in A} d(a,b)\},$$

where $d(a,b)$ is the Euclidean distance of a and b.

First, we note that for the proof of Theorem 7.2.1, similar to Reitzner [Rei03], we may assume that the Hausdorff distance $d_H(K, K_n)$ of K and K_n is at most ε_K, where $\varepsilon_K > 0$ is a suitably chosen constant. This can be seen the following way. Assume that $d_H(K, K_n) \geq \varepsilon_K$. Then there exists a point x on the boundary of K_n such that $\varepsilon_K B^2 + x \subset K$. There exists a supporting circle of K_n through x that determines a disc-cap of height at least ε_K. By the above remark, the probability content of this disc-cap is at least $c_K > 0$, where c_K is a suitable constant depending on K and ε_K. Then

$$\mathbb{P}(d_H(K, K_n) \geq \varepsilon_K) \leq (1 - c_K)^n. \tag{7.2.6}$$

Our main tool in the variance estimates is the Efron-Stein inequality [ES81], which has previously been used to provide upper estimates on the variance of various geometric quantities associated with random polytopes in convex bodies, see Reitzner [Rei03], and for further references in this topic we recommend the recent survey articles by Bárány [Bár08] and Schneider [Sch18].

7.2.2 Proof of Theorem 7.2.1

We present the proof of the asymptotic upper bound on the variance of the vertex number in detail. Our argument is similar to the one in Reitzner [Rei03, Sections 4 and 6]. Since the proof for the variance of the missed area is very similar we omit it in this dissertation. A short outline of the argument with the key steps can be found in the last two paragraphs of Section 4 on page 1151 in [FV18]. The basic idea of the argument rests on the Efron-Stein inequality, which bounds the variance of a random variable (in our case the vertex number or the missed area) in terms of expectations. To calculate the involved expectations we use some basic geometric properties of disc caps and the integral transformation [FKV14, pp. 907-909], see also [San46]. Finally, the asymptotic estimate (11) in [BFRV09, pp. 2290] for the order of magnitude of beta integrals yields the desired asymptotic upper bound.

For the number of vertices of K_n, the Efron-Stein inequality [ES81] states the following

$$\operatorname{Var} f_0(K_n) \leq (n+1)\mathbb{E}(f_0(K_{n+1}) - f_0(K_n))^2.$$

Let x be an arbitrary point of K and let $x_i x_j$ be an edge of K_n. Following Reitzner [Rei03], we say that the edge $x_i x_j$ is visible from x if x is not contained in K_n and it is not contained in the unit disc of the edge $x_i x_j$. For a point $x \in K \setminus K_n$, let $\mathcal{F}_n(x)$ denote the set of edges of K_n that can be seen from x, and for $x \in K_n$ set $\mathcal{F}_n(x) = \emptyset$. Let $F_n(x) = |\mathcal{F}_n(x)|$.

Let x_{n+1} be a uniform random point in K chosen independently from X_n. If $x_{n+1} \in K_n$, then $f_0(K_{n+1}) = f_0(K_n)$. If, on the other hand, $x_{n+1} \notin K_n$, then

$$\begin{aligned} f_0(K_{n+1}) &= f_0(K_n) + 1 - (F_n(x_{n+1}) - 1) \\ &= f_0(K_n) - F_n(x_{n+1}) + 2. \end{aligned}$$

Therefore,

$$|f_0(K_{n+1}) - f_0(K_n)| \leq 2F_n(x_{n+1}),$$

and by the Efron–Stein jackknife inequality

$$\begin{aligned} \operatorname{Var}(f_0(K_n)) &\leq (n+1)\mathbb{E}(f_0(K_{n+1}) - f_0(K_n))^2 \\ &\leq 4(n+1)\mathbb{E}(F_n^2(x_{n+1})). \end{aligned} \tag{7.2.7}$$

Similar to Reitzner, we introduce the following notation (see [Rei03] p. 2147). Let $I = (i_1, i_2), i_1 \neq i_2,\ i_1, i_2 \in \{1, 2, \ldots\}$ be an ordered pair of indices. Denote by F_I the shorter arc of the unique unit circle incident with x_{i_1} and x_{i_2} on which x_{i_1} follows x_{i_2} in the positive cyclic ordering of the circle. Let $\mathbb{I}(A)$ denote the indicator function of the event A. For the sake of brevity, we use the notation $x_1, x_2, \ldots$ for the integration variables as well.

We wish to estimate the expectation $\mathbb{E}(F_n^2(x_{n+1}))$ under the condition that $d_H(K,K_n)<\varepsilon_K$. To compensate for the cases in which $d_H(K,K_n)\geq\varepsilon_k$, using (7.2.6), we add an error term $O((1-c_K)^n)$.

$$
\begin{aligned}
\mathbb{E}(F_n(x_{n+1})^2) &= \frac{1}{A(K)^{n+1}}\int_K\int_{K^n}\left(\sum_I \mathbb{I}(F_I\in\mathcal{F}_n(x_{n+1}))\right)^2 \mathrm{d}X_n\mathrm{d}x_{n+1}\\
&= \frac{1}{A(K)^{n+1}}\int_K\int_{K^n}\left(\sum_I \mathbb{I}(F_I\in\mathcal{F}_n(x_{n+1}))\right)\\
&\quad\times\left(\sum_J \mathbb{I}(F_J\in\mathcal{F}_n(x_{n+1}))\right)\mathrm{d}X_n\mathrm{d}x_{n+1}\\
&\leq \frac{1}{A(K)^{n+1}}\sum_I\sum_J\int_K\int_{K^n}\mathbb{I}(F_I\in\mathcal{F}_n(x_{n+1}))\mathbb{I}(F_J\in\mathcal{F}_n(x_{n+1}))\\
&\quad\times\mathbb{I}(d_H(K,K_n)\leq\varepsilon_K)\mathrm{d}X_n\mathrm{d}x_{n+1}+O((1-c_K)^n)
\end{aligned}
\tag{7.2.8}
$$

Choose ε_K so small that $A(K\setminus K_n)<\delta$. Note that with this choice of ε_K only one of the two shorter arcs determined by x_{i_1} and x_{i_2} can determine an edge of K_n.

Now we fix the number k of common elements of I and J, that is, $|I\cap J|=k$. Let F_1 denote one of the shorter arcs spanned by x_1 and x_2, and let F_2 be one of the shorter arcs determined by x_{3-k} and x_{4-k}. Since the random points are independent, we have that

$$
\begin{aligned}
(7.2.8) &\ll \frac{1}{A(K)^{n+1}}\sum_{k=0}^{2}\binom{n}{2}\binom{2}{k}\binom{n-2}{2-k}\int_K\int_{K^n}\mathbb{I}(F_1\in\mathcal{F}_n(x_{n+1}))\\
&\quad\times\mathbb{I}(F_2\in\mathcal{F}_n(x_{n+1}))\mathbb{I}(d_H(K,K_n)\leq\varepsilon_K)\mathrm{d}X_n\mathrm{d}x_{n+1}+O((1-c_K)^n)\\
&\ll \frac{1}{A(K)^{n+1}}\sum_{k=0}^{2}n^{4-k}\int_K\cdots\int_K\mathbb{I}(F_1\in\mathcal{F}_n(x_{n+1}))\\
&\quad\times\mathbb{I}(F_2\in\mathcal{F}_n(x_{n+1}))\mathbb{I}(d_H(K,K_n)\leq\varepsilon_K)\mathrm{d}X_n\mathrm{d}x_{n+1}+O((1-c_K)^n).
\end{aligned}
\tag{7.2.9}
$$

Since the roles of F_1 and F_2 are symmetric, we may assume that $\operatorname{diam}C_+(D_1)\geq\operatorname{diam}C_+(D_2)$, where $D_1=D_-(x_1,x_2)$ and $D_2=D_-(x_{3-k},x_{4-k})$ are the corresponding disc-caps, and $\operatorname{diam}(\cdot)$ denotes the diameter of a set. Thus,

$$
\begin{aligned}
(7.2.9) &\ll \frac{1}{A(K)^{n+1}}\sum_{k=0}^{2}n^{4-k}\int_K\int_{K^n}\mathbb{I}(F_1\in\mathcal{F}_n(x_{n+1}))\\
&\quad\times\mathbb{I}(F_2\in\mathcal{F}_n(x_{n+1}))\mathbb{I}(\operatorname{diam}C_+(D_1)\geq\operatorname{diam}C_+(D_2))\\
&\quad\times\mathbb{I}(d_H(K,K_n)\leq\varepsilon_K)\mathrm{d}X_n\mathrm{d}x_{n+1}+O((1-c_K)^n).
\end{aligned}
\tag{7.2.10}
$$

Clearly, x_{n+1} is a common point of the disc caps D_1 and D_2, so we may write that

$$
(7.2.10)\leq\frac{1}{A(K)^{n+1}}\sum_{k=0}^{2}n^{4-k}\int_K\int_{K^n}\mathbb{I}(F_1\in\mathcal{F}_n(x_{n+1}))
$$

$$\times \mathbb{I}(D_1 \cap D_2 \neq \emptyset)\mathbb{I}(\operatorname{diam} C_+(D_1) \geq \operatorname{diam} C_+(D_2))$$
$$\times \mathbb{I}(d_H(K, K_n) \leq \varepsilon_K)\mathrm{d}X_n \mathrm{d}x_{n+1} + O((1-c_K)^n). \tag{7.2.11}$$

In order for F_1 to be an edge of K_n, it is necessary that $x_{5-k}, \dots x_n \in K \setminus D_1$, and for $F_1 \in \mathcal{F}_n(x_{n+1})$ x_{n+1} must be in D_1. Therefore

$$\begin{aligned}
(7.2.11) \ll & \frac{1}{A(K)^{n+1}} \sum_{k=0}^{2} n^{4-k} \int_K \cdots \int_K (A(K) - A(D_1)))^{n-4+k} A(D_1) \\
& \times \mathbb{I}(D_1 \cap D_2 \neq \emptyset)\mathbb{I}(\operatorname{diam} C_+(D_1) \geq \operatorname{diam} C_+(D_2)) \\
& \times \mathbb{I}(d_H(K, K_n) \leq \varepsilon_K)\mathrm{d}x_1 \cdots \mathrm{d}x_{4-k} + O((1-c_K)^n) \\
\ll & \sum_{k=0}^{2} n^{4-k} \int_K \cdots \int_K \left(1 - \frac{A(D_1)}{A(K)}\right)^{n-4+k} \frac{A(D_1)}{A(K)} \\
& \times \mathbb{I}(D_1 \cap D_2 \neq \emptyset)\mathbb{I}(\operatorname{diam} C_+(D_1) \geq \operatorname{diam} C_+(D_2)) \\
& \times \mathbb{I}(d_H(K, K_n) \leq \varepsilon_K)\mathrm{d}x_1 \cdots \mathrm{d}x_{4-k} + O((1-c_K)^n).
\end{aligned} \tag{7.2.12}$$

Reitzner proved (see [Rei03, pp. 2149–2150]) that if $D_1 \cap D_2 \neq \emptyset$, $d_H(K, K_n) \leq \varepsilon_K$ and $\operatorname{diam} C_+(D_1) \geq \operatorname{diam} C_+(D_2)$ then there exists a constant $\bar{c}$ (depending only on K) such that $C_+(D_2) \subset \bar{c}(C_+(D_1) - x_{D_1}) + x_{D_1}$, where x_{D_1} is the vertex of D_1. Combining this with Claim 7.2.4 we obtain that there is a constant c_1 depending only on K, such that $D_2 \subset c_1(D_1 - x_{D_1}) + x_{D_1}$. Hence $A(D_2) \leq c_1^2 A(D_1)$, and therefore

$$\begin{aligned}
\int_K \cdots \int_K & \mathbb{I}(D_1 \cap D_2 \neq \emptyset)\mathbb{I}(\operatorname{diam} C_+(D_1) \geq \operatorname{diam} C_+(D_2)) \\
& \times \mathbb{I}(d_H(K, K_n) \leq \varepsilon_K)\mathrm{d}x_3 \cdots \mathrm{d}x_{4-k} \ll A(D_1)^{2-k}.
\end{aligned}$$

We continue by estimating (7.2.12) term by term (omitting the $O((1-c_K)^n)$ term).

$$\begin{aligned}
& n^{4-k} \int_K \cdots \int_K \left(1 - \frac{A(D_1)}{A(K)}\right)^{n-4+k} \frac{A(D_1)}{A(K)} \mathbb{I}(D_1 \cap D_2 \neq \emptyset) \\
& \quad \times \mathbb{I}(\operatorname{diam} C_+(D_1) \geq \operatorname{diam} C_+(D_2))\mathbb{I}(d_H(K, K_n) \leq \varepsilon_K)\mathrm{d}x_1 \cdots \mathrm{d}x_{4-k} \\
\ll & n^{4-k} \int_K \int_K \left(1 - \frac{A(D_1)}{A(K)}\right)^{n-4+k} \left(\frac{A(D_1)}{A(K)}\right)^{3-k} \mathbb{I}(d_H(K, K_n) \leq \varepsilon_K)\mathrm{d}x_1 \mathrm{d}x_2.
\end{aligned} \tag{7.2.13}$$

Now, we use the following parametrization of (x_1, x_2) the same way as in [FKV14] to transform the integral. Let

$$(x_1, x_2) = \Phi(u, t, u_1, u_2),$$

where $u, u_1, u_2 \in S^1$ and $0 \leq t \leq t_0(u)$ are chosen such that

$$D(u, t) = D_1 = D_-(x_1, x_2),$$

and

$$(x_1, x_2) = (x_u - (1+t)u + u_1, x_u - (1+t)u + u_2).$$

More information on this transformation can be found in [FKV14, pp. 907-909]. Here we just recall that the Jacobian of Φ is

$$|J\Phi| = \left(1 + t - \frac{1}{\kappa(x_u)}\right) |u_1 \times u_2|,$$

where $u_1 \times u_2$ denotes the cross product of u_1 and u_2.

Let $L(u,t) = \partial D_1 \cap \mathrm{int} K$, then we obtain that

$$\begin{aligned}
(7.2.13) \ll n^{4-k} &\int_{S^1} \int_0^{t^*(u)} \int_{L(u,t)} \int_{L(u,t)} \left(1 - \frac{A(u,t)}{A(K)}\right)^{n-4+k} \left(\frac{A(u,t)}{A(K)}\right)^{3-k} \\
&\times \left(1 + t - \frac{1}{\kappa(x_u)}\right) |u_1 \times u_2| \mathrm{d}u_1 \mathrm{d}u_2 \mathrm{d}t \mathrm{d}u \\
&= n^{4-k} \int_{S^1} \int_0^{t^*(u)} \left(1 - \frac{A(u,t)}{A(K)}\right)^{n-4+k} \left(\frac{A(u,t)}{A(K)}\right)^{3-k} \\
&\times \left(1 + t - \frac{1}{\kappa(x_u)}\right) (\ell(u,t) - \sin \ell(u,t)) \mathrm{d}t \mathrm{d}u. \qquad (7.2.14)
\end{aligned}$$

From now on the evaluation follows a standard way. First, we split the domain of integration with respect to t into two parts. Let $h(n) = (c \ln n/n)^{2/3}$, where $c > 0$ is a sufficiently large absolute constant. Using (7.2.5), we have that $A(u,t) \geq \gamma t^{3/2}$ uniformly in $u \in S^1$, hence

$$\begin{aligned}
n^{4-k} &\int_{S^1} \int_{h(n)}^{t^*(u)} \left(1 - \frac{A(u,t)}{A(K)}\right)^{n-4+k} \left(\frac{A(u,t)}{A(K)}\right)^{3-k} \\
&\times \left(1 + t - \frac{1}{\kappa(x_u)}\right) (\ell(u,t) - \sin \ell(u,t)) \mathrm{d}t \mathrm{d}u \\
&\ll n^{4-k} \int_{S^1} \int_{h(n)}^{t^*(u)} \left(1 - \frac{A(u,t)}{A(K)}\right)^{n-4+k} \mathrm{d}t \mathrm{d}u \\
&\ll n^{4-k} \int_{S^1} \int_{h(n)}^{t^*(u)} \left(1 - \frac{\gamma t^{3/2}}{A(K)}\right)^{n-4+k} \mathrm{d}t \mathrm{d}u \\
&\ll n^{4-k} \left(1 - \frac{\gamma h(n)^{3/2}}{A(K)}\right)^{n-4+k} \\
&= n^{4-k} \left(1 - \frac{\gamma(c \ln n)}{nA(K)}\right)^{n-4+k} \ll n^{-2/3},
\end{aligned}$$

if $\gamma c/A(K)$ is sufficiently large.

Therefore, it is enough to estimate the following part of (7.2.14)

$$\begin{aligned}
n^{4-k} &\int_{S^1} \int_0^{h(n)} \left(1 - \frac{A(u,t)}{A(K)}\right)^{n-4+k} \left(\frac{A(u,t)}{A(K)}\right)^{3-k} \\
&\times \left(1 + t - \frac{1}{\kappa(x_u)}\right) (\ell(u,t) - \sin \ell(u,t)) \mathrm{d}t \mathrm{d}u. \qquad (7.2.15)
\end{aligned}$$

Using (7.2.5) and the Taylor series of the sine function, we obtain that $\ell(u,t) - \sin \ell(u,t) \ll t^{3/2}$. Since $\kappa(x) > 1$ for all $x \in \partial K$, it follows that $0 < 1 + t - \kappa(x_u)^{-1} \ll 1$. We also use (7.2.5) to estimate $A(u,t)$, similarly as before. Assuming that n is large enough, we obtain that

$$\begin{aligned}(7.2.15) &\ll n^{4-k} \int_{S^1} \int_0^{h(n)} \left(1 - \frac{\gamma t^{3/2}}{A(K)}\right)^{n-4+k} \left(t^{3/2}\right)^{3-k} \cdot 1 \cdot t^{3/2} \mathrm{d}t\mathrm{d}u \\ &\ll n^{4-k} \int_0^{h(n)} \left(1 - \frac{\gamma t^{3/2}}{A(K)}\right)^{n-4+k} t^{\frac{12-3k}{2}} \mathrm{d}t \ll n^{-2/3},\end{aligned}$$

where the last inequality follows directly from formula (11) in [BFRV09, p. 2290]. Together with (7.2.7), this yields the desired upper estimate for $\operatorname{Var} f_0(K_n)$.

7.2.3 The case of the circle

In this section we prove Theorem 7.2.2. In particular, we give a detailed proof of the estimate (7.2.3) for the variance of the number of vertices of the random disc-polygon. The case of the missed area (7.2.4) is very similar.

Without loss of generality, we may assume that $K = B^2$, and that $r = 1$.

We begin by recalling from [FKV14] that for any $u \in S^1$ and $0 \leq t \leq 2$, it holds that

$$\ell(u,t) = 2 \arcsin \sqrt{1 - \frac{t^2}{2}}, \text{ and } A(u,t) = A(t) = t\sqrt{1 - \frac{t^2}{2}} + 2 \arcsin \frac{t}{2}.$$

Proof of Theorem 7.2.2 (7.2.3). From (7.1.7) and Chebyshev's inequality, it follows that

$$1 = \mathbb{P}\left(\left| f_0(K_n^1) - \frac{\pi^2}{2} \right| > 0.05\right) \leq \frac{\operatorname{Var}(f_0(K_n^1))}{0.05^2},$$

thus

$$\operatorname{Var}(f_0(K_n^1)) \geq 0.05^2.$$

This proves that $\operatorname{Var}(f_0(K_n^1)) \gg$ *const.*.

In order to prove the asymptotic upper bound in (7.2.3), we use a modified version of the argument of the previous section. With the same notation as in Section 3, the Efron-Stein inequality for the vertex number yields that

$$\operatorname{Var}(f_0(K_n^1)) \ll n\mathbb{E}(F_n(x_{n+1}))^2.$$

Following a similar line of argument as above, we obtain that

$$\begin{aligned} n\mathbb{E}(F_n(x_{n+1}))^2 = \frac{n}{\pi^{n+1}} \int_{(B^2)^{n+1}} \left(\sum_I \mathbb{I}(F_I \in \mathcal{F}_n(x_{n+1}))\right) \\ \times \left(\sum_J \mathbb{I}(F_J \in \mathcal{F}_n(x_{n+1}))\right) \mathrm{d}x_1 \cdots \mathrm{d}x_n \mathrm{d}x_{n+1}\end{aligned}$$

$$\leq \frac{n}{\pi^{n+1}} \sum_I \sum_J \int_{(B^2)^{n+1}} \mathbb{I}(F_I \in \mathcal{F}_n(x_{n+1}))\mathbb{I}(F_J \in \mathcal{F}_n(x_{n+1}))\mathrm{d}x_1 \cdots \mathrm{d}x_n \mathrm{d}x_{n+1} \quad (7.2.16)$$

Now, let $|I \cap J| = k$, where $k = 0, 1, 2$, and let $F_1 = x_1x_2$ and $F_2 = x_{3-k}x_{4-k}$. By the independence of the random points (and by also taking into account their order), we get that

$$\begin{aligned}(7.2.16) \ll \frac{n}{\pi^{n+1}} \sum_{k=0}^{2} \binom{n}{2}\binom{2}{k}\binom{n-2}{2-k} \int_{(B^2)^{n+1}} &\mathbb{I}(F_1 \in \mathcal{F}_n(x_{n+1})) \\ &\times \mathbb{I}(F_2 \in \mathcal{F}_n(x_{n+1}))\mathrm{d}x_1 \cdots \mathrm{d}x_n \mathrm{d}x_{n+1}. \\ \ll \frac{1}{\pi^{n+1}} \sum_{k=0}^{2} n^{5-k} \int_{(B^2)^{n+1}} &\mathbb{I}(F_1 \in \mathcal{F}_n(x_{n+1}))\mathbb{I}(F_2 \in \mathcal{F}_n(x_{n+1}))\mathrm{d}x_1 \cdots \mathrm{d}x_n \mathrm{d}x_{n+1}.\end{aligned} \quad (7.2.17)$$

By symmetry, we may also assume that $A(D_1) \geq A(D_2)$, therefore

$$\begin{aligned}(7.2.17) \ll \sum_{k=0}^{2} n^{5-k} \int_{(B^2)^{n+1}} &\mathbb{I}(F_1 \in \mathcal{F}_n(x_{n+1}))\mathbb{I}(F_2 \in \mathcal{F}_n(x_{n+1})) \\ &\times \mathbb{I}(A(D_1) \geq A(D_2))\mathrm{d}x_1 \cdots \mathrm{d}x_n \mathrm{d}x_{n+1}.\end{aligned} \quad (7.2.18)$$

By integrating with respect to $x_{5-k}, \ldots, x_n$ and x_{n+1} we obtain that

$$\begin{aligned}(7.2.18) \ll \sum_{k=0}^{2} n^{5-k} \int_{B^2} \cdots \int_{B^2} &\left(1 - \frac{A(D_1)}{\pi}\right)^{n-4+k} \frac{A(D_1)}{\pi} \\ &\times \mathbb{I}(A(D_1) \geq A(D_2))\mathrm{d}x_1 \cdots \mathrm{d}x_{4-k}\end{aligned} \quad (7.2.19)$$

If $A(D_1) \geq A(D_2)$, then D_2 is fully contained in the circular annulus whose width is equal to the height of the disc-cap D_1. The area of this annulus is not more than $4A(D_1)$. Therefore,

$$(7.2.19) \ll \sum_{k=0}^{2} n^{5-k} \int_{B^2} \int_{B^2} \left(1 - \frac{A(D_1)}{\pi}\right)^{n-4+k} A(D_1)^{3-k} \mathrm{d}x_1 \mathrm{d}x_2.$$

As common in these arguments, we may assume that $A(D_1)/\pi < c \log n/n$ for some suitable constant $c > 0$ that will be determined later. To see this, let $A(D_1)/\pi \geq c \log n/n$. Then

$$\begin{aligned}&\left(1 - \frac{A(D_1)}{\pi}\right)^{n-4+k} A(D_1)^{3-k} \\ &\leq \left(\frac{\pi c \log n}{n}\right)^{3-k} \cdot \exp\left(-\frac{c(n-4+k)\log n}{n}\right) \\ &\ll \left(\frac{\log n}{n}\right)^{3-k} \cdot n^{-c}\end{aligned}$$

$$\ll n^{-c}.$$

If $c > 0$ is sufficiently large, then the contribution of the case when $A(D_1)/\pi \geq c \log n/n$ is $O(n^{-1})$. Thus,

$$n\mathbb{E}(F_n(x_{n+1})) \ll \sum_{k=0}^{2} n^{5-k} \int_{B^2} \int_{B^2} \left(1 - \frac{A(D_1)}{\pi}\right)^{n-4+k} A(D_1)^{3-k} \times \mathbb{I}(A(D_1) \leq c \log n/n) \mathrm{d}x_1 \mathrm{d}x_2 + O(n^{-1}). \quad (7.2.20)$$

Now, we use the same type of reparametrization as in the previous section. Let $(x_1, x_2) = (-tu_1, -tu_2)$, $u \in S^1$ and $0 \leq t \leq c \log n/n$. Then

$$\begin{aligned}(7.2.20) &\ll \sum_{k=0}^{2} n^{5-k} \int_{S^1} \int_0^{c^* \log n/n} \int_{S^1} \int_{S^1} \left(1 - \frac{A(u,t)}{\pi}\right)^{n-4+k} A(u,t)^{3-k} \\ &\times t|u_1 \times u_2| \mathrm{d}u_1 \mathrm{d}u_2 \mathrm{d}u \mathrm{d}t + O(n^{-1}) \\ &\ll \sum_{k=0}^{2} n^{5-k} \int_0^{c^* \log n/n} \left(1 - \frac{A(u,t)}{\pi}\right)^{n-4+k} A(u,t)^{3-k} \\ &\times t(l(t) - \sin l(t)) \mathrm{d}t + O(n^{-1}). \end{aligned} \quad (7.2.21)$$

Using that $l(t) \to \pi$ as $t \to 0^+$, and the Taylor series of $V(u,t)$ at $t = 0$, we obtain that there exists a constant $\omega > 0$ such that

$$(7.2.21) \ll \sum_{k=0}^{2} n^{5-k} \int_0^{c^* \log n/n} (1 - \omega t)^{n-4+k} t^{4-k} \mathrm{d}t + O(n^{-1}) \quad (7.2.22)$$

Now, using a formula for the asymptotic order of beta integrals (see [BFRV09, p. 2290, formula (11)]), we obtain that

$$\begin{aligned}(7.2.22) &\ll \sum_{k=0}^{2} n^{5-k} n^{-(5-k)} + O(n^{-1}) \\ &\ll const,\end{aligned}$$

which finishes the proof of the upper bound in (7.2.3). □

In order to prove the asymptotic upper bound (7.2.4), only slight modifications are needed in the above argument.

7.2.4 A circumscribed model

In the section we consider circumscribed random disc-polygons. Let $K \subset \mathbb{R}^2$ be a convex disc with C^2_+ smooth boundary, and $r \geq \kappa_m^{-1}$. Consider the following set

$$K^{*,r} = \left\{x \in \mathbb{R}^2 \mid K \subset rB^2 + x\right\},$$

which is also called the r-hyperconvex dual, or r-dual for short, of K. It is known that $K^{*,r}$ is a convex disc with C^2_+ boundary, and it also has the property that the curvature is at least $1/r$ at every boundary point. For further information see [FKV16] and the references therein.

For $u \in S^1$, let $x(K,u) \in \partial K$ ($x(K^{*,r},u) \in \partial K^{*,r}$ resp.) be the unique point on ∂K ($\partial K^{*,r}$ resp.), where the outer unit normal to K ($K^{*,r}$ resp.) is u. For a convex disc $K \subset \mathbb{R}^2$ with $o \in \operatorname{int} K$, let $h_K(u) = \max_{x\in K}\langle x, u\rangle$ denote the support function of K. Let $\operatorname{Per}(\,\cdot\,)$ denote the perimeter.

The following Lemma collects some results from [FKV16, Section 2].

Lemma 7.2.5. [FKV16] *With the notation above*

1. $h_K(u) + h_{K^{*,r}}(-u) = r$ *for any* $u \in S^1$,
2. $\kappa_K^{-1}(x(u,K)) + \kappa^{-1}_{K^{*,r}}(x(-u,K^{*,r})) = r$ *for any* $u \in S^1$,
3. $\operatorname{Per}(K) + \operatorname{Per}(K^{*,r}) = 2r\pi$,
4. $A(K^{*,r}) = A(K) - r \cdot \operatorname{Per}(K) + r^2\pi$.

Now, we turn to the probability model. Let K be a convex disc with C^2_+ boundary, and let $r > \kappa_m^{-1}$ as before. Let $X_n = \{x_1, \ldots, x_n\}$ be a sample of n independent random points chosen from $K^{*,r}$ according to the uniform probability distribution, and define

$$K^{*,r}_{(n)} = \bigcap_{x\in X_n} rB^2 + x.$$

$K^{*,r}_{(n)}$ is a random disc-polygon that contains K. Observe that, by definition $K^{*,r}_{(n)} = (\operatorname{conv}_r(X_n))^{*,r}$, and consequently $f_0(K^{*,r}_{(n)}) = f_0(\operatorname{conv}_r(X_n))$. We note that this is a very natural approach to define a random disc-polygon that is circumscribed about K that has no clear analogy in classical convexity. (If one takes the limit as $r \to \infty$, the underlying probability measures do not converge.) The model is of special interest in the case $K = K^{*,r}$, which happens exactly when K is of constant width r.

Theorem 7.2.6. *Assume that K has C^2_+ boundary, and let $r > \kappa_m^{-1}$. With the notation above*

$$\lim_{n\to\infty} \mathbb{E}(f_0(K^{*,r}_{(n)})) \cdot n^{-1/3} = \sqrt[3]{\frac{2r}{3(A(K) - r\cdot \operatorname{Per}(K) + r^2\pi)}} \times \Gamma\left(\frac{5}{3}\right) \int_{\partial K} \left(\kappa(x) - \frac{1}{r}\right)^{2/3} \mathrm{d}x. \tag{7.2.23}$$

Furthermore if K has C^5_+ boundary, then

$$\lim_{n\to\infty} n^{2/3} \cdot \left(\operatorname{Per} K^{*,r}_{(n)} - \operatorname{Per} K\right) = \frac{(12(A(K) - r\cdot\operatorname{Per}(K) + r^2\pi))^{2/3}}{36} \cdot \Gamma\left(\frac{2}{3}\right) \times r^{-2/3} \int_{\partial K} \left(\kappa(x) - \frac{1}{r}\right)^{-1/3} \left(4\kappa(x) - \frac{1}{r}\right) \mathrm{d}x;$$

$$\lim_{n\to\infty} n^{2/3}\cdot A(K^{*,r}_{(n)}\setminus K) = \frac{(12(A(K)-r\cdot \operatorname{Per}(K)+r^2\pi))^{2/3}}{12}\times \Gamma\left(\frac{2}{3}\right)\cdot r^{-2/3}\int_{\partial K}\left(\kappa(x)-\frac{1}{r}\right)^{-1/3}\mathrm{d}x.$$

Proof. By Lemma 7.2.5 it follows that $K^{*,r}$ has also C^2_+ boundary. As $f_0(K^{*,r}_{(n)}) = f_0(\operatorname{conv}_r(X_n))$, we immediately get from [FKV14, Theorem 1.1] that

$$\lim_{n\to\infty} \mathbb{E}(f_0(K^{*,r}_{(n)}))\cdot n^{-1/3} = \sqrt[3]{\frac{2}{3A(K^{*,r})}}\cdot\Gamma\left(\frac{5}{3}\right)\int_{\partial K^{*,r}}\left(\kappa(x)-\frac{1}{r}\right)^{1/3}\mathrm{d}x.$$

Using Lemma 7.2.5, we proceed as follows

$$\int_{\partial K^{*,r}}\left(\kappa(x)-\frac{1}{r}\right)^{1/3}\mathrm{d}x = \int_{S^1}\frac{\left(\kappa(x(K^{*,r},u))-\frac{1}{r}\right)^{1/3}}{\kappa(x(K^{*,r},u))}\mathrm{d}u =$$

$$\int_{S^1}\frac{\left(\frac{\kappa(x(K,-u))}{r\kappa(x(K,-u))-1}-\frac{1}{r}\right)^{1/3}}{\frac{\kappa(x(K,-u))}{r\kappa(x(K,-u))-1}}\mathrm{d}u = \int_{S^1} r^{1/3}\frac{\left(\kappa(x(K,u))-\frac{1}{r}\right)^{2/3}}{\kappa(x(K,u))}\mathrm{d}u$$

$$= r^{1/3}\int_{\partial K}\left(\kappa(x)-\frac{1}{r}\right)^{2/3}\mathrm{d}x.$$

Together with Lemma 7.2.5, this proves (7.2.23).

The rest of the theorem can be proved similarly, by using [FKV14, Theorem 1.1 and Theorem 1.2], and Lemma 7.2.5.

□

As an obvious consequence of Theorem 7.2.1, Lemma 7.2.5, and the definition of $K^{*,r}_{(n)}$, we obtain the following corollary.

Corollary 7.2.7. *Assume that K has C^2_+ boundary, and let $r > \kappa_m^{-1}$. With the notation above*

$$\operatorname{Var}(f_0(K^{*,r}_{(n)})) \ll n^{1/3}.$$

Remark. We note that if K is a convex disc of constant width r, then $K^{*,r} = K$ (see e.g. [FKV16]), and similar calculations to those in the proof of Theorem 7.2.6 provide some interesting integral formulas. For example, for a real p we obtain that

$$\int_{\partial K}\left(\kappa(x)-\frac{1}{r}\right)^{p}\mathrm{d}x = r^{1-2p}\int_{\partial K}\left(\kappa(x)-\frac{1}{r}\right)^{1-p}\mathrm{d}x.$$

Chapter 8

Best approximations by disc-polygons

This chapter of the dissertation is based on the paper [FV12] by F. Fodor and V. Vígh, *Disc-polygonal approximations of planar spindle convex sets*, Acta Sci. Math. (Szeged) **78** (2012), no. 1–2, 331–350.

8.1 Introduction and results

In this chapter, we consider best approximations of a planar 1-spindle convex set by circumscribed and inscribed convex disc-polygons of unit radius with n sides. We examine best approximations with respect to the Hausdorff metric of compact convex sets, and the measures of deviation defined by perimeter and area differences. For basic definitions and facts about R-spindle convex sets, we refer to Section 7.1.1.

Let $K_1, K_2 \subset \mathbb{R}^2$ be (linearly) convex compact sets with nonempty interior containing the origin o. Let $A(\cdot)$ and $\ell(\cdot)$ denote the *area* and *perimeter* of compact convex sets in $\mathbb{R}^2$. We use the notation $\delta_A(K_1, K_2) = A(K_1 \cup K_2) - A(K_1 \cap K_2)$ for the *area deviation* of K_1 and K_2, and the notation $\delta_\ell(K_1, K_2) = \ell(K_1 \cup K_2) - \ell(K_1 \cap K_2)$ for the *perimeter deviation* of K_1 and K_2. Finally, in this chapter δ_H stands for the *Hausdorff metric* for compact sets in $\mathbb{R}^2$.

We consider R-spindle convex sets only for $R = 1$, so for the sake of simplicity, we omit R from all notations. We call a 1-spindle convex set simply spindle convex.

Let $S \subset \mathbb{R}^2$ be a compact spindle convex set with twice continuously differentiable boundary. As we will consider approximations of S by inscribed and circumscribed disc-polygons with at most n sides with respect to the Hausdorff metric, area deviation, and perimeter deviation, we have to deal with six separate problems for each fixed n. Let S_n^H, S_n^A, and S_n^ℓ ($S_{(n)}^H, S_{(n)}^A$ and $S_{(n)}^\ell$) denote a disc-polygon with at most n sides inscribed in S (circumscribed about S) that are closest to S with respect to the Hausdorff metric, area deviation, and perimeter deviation, respectively. Such a (not necessarily unique) minimizer clearly exists for each of the three measures of distance. It is also clear that in each of the six cases the distance of the minimizer and S approaches zero as n tends to infinity. The main results of this chapter are the following asymptotic formulae for the distance of the minimizers to S as n tends to infinity.

Theorem 8.1.1 (Fodor, Vígh [FV12, Theorem 1 on pages 333–334])**.** *Let S be a compact, spindle convex set in $\mathbb{R}^2$ with a twice continuously differentiable boundary. Then the following statements hold:*

$$\delta_\ell(S, S_n^\ell) \sim \frac{1}{24}\left(\int_{\partial S}(\kappa^2(s)-1)^{\frac{1}{3}}ds\right)^3 \cdot \frac{1}{n^2}, \tag{i}$$

$$\delta_A(S, S_n^A) \sim \frac{1}{12}\left(\int_{\partial S}(\kappa(s)-1)^{\frac{1}{3}}ds\right)^3 \cdot \frac{1}{n^2}, \tag{ii}$$

$$\delta_H(S, S_n^H) \sim \frac{1}{8}\left(\int_{\partial S}(\kappa(s)-1)^{\frac{1}{2}}ds\right)^2 \cdot \frac{1}{n^2}, \tag{iii}$$

$$\delta_\ell(S, S_{(n)}^\ell) \sim \frac{1}{24}\left(\int_{\partial S}(2\kappa^2(s)-3\kappa(s)+1)^{\frac{1}{3}}ds\right)^3 \cdot \frac{1}{n^2}, \tag{iv}$$

$$\delta_A(S, S_{(n)}^A) \sim \frac{1}{24}\left(\int_{\partial S}(\kappa(s)-1)^{\frac{1}{3}}ds\right)^3 \cdot \frac{1}{n^2}, \tag{v}$$

$$\delta_H(S, S_{(n)}^H) \sim \frac{1}{8}\left(\int_{\partial S}(\kappa(s)-1)^{\frac{1}{2}}ds\right)^2 \cdot \frac{1}{n^2}, \tag{vi}$$

as $n \to \infty$.

In this dissertation, we only give the detailed proofs of the inscribed cases of Theorem 8.1.1, namely, i), ii), and iii). For the proofs of the circumscribed cases, see Section 5 of [FV12] on pages 344–349.

The analogues of the formulae in Theorem 8.1.1 for linearly convex discs were first stated by L. Fejes Tóth in [FT53], and they were proved by McClure and Vitale in [MV75]. They established the order of approximation of linearly convex compact planar sets by inscribed and circumscribed convex polygons with respect to the Hausdorff metric as well as the perimeter and area deviation measures.

Approximations of (linearly) convex sets by polytopes have an extensive literature. For a detailed survey on the current state-of-the-art of this subject see the papers by Gruber [Gru93, Gru97].

To prove Theorem 8.1.1, we use, at least in part, the framework developed by McClure and Vitale in [MV75]. In the next section, we cite those results from [MV75] which will be used in our proof of Theorem 8.1.1.

8.2 Tools

In this section we summarize the relevant parts of Section 4 of the paper by McClure and Vitale [MV75], for further information see also [McC75].

Let $f : [a, b] \to \mathbb{R}$ be a function, and let $T_n = (t_0, t_1, \ldots, t_n)$ be a partition of the interval $[a, b]$ such that

$$a = t_0 < t_1 < t_2 < \ldots < t_n = b.$$

Let $E(f, T_n)$ be a functional that is additive with respect to T_n in the following sense

$$E(f, T_n) = \sum_{i=0}^{n-1} e(f, t_i, t_{i+1}).$$

Define $E_n(f) = \inf_{T_n} E(f, T_n)$.

The results will follow from the following three assumptions from [MV75].

Assumption 1. For any (α, β) satisfying $a \leq \alpha < \beta \leq b$, $e(f, \alpha, \beta) \geq 0$, and if $a \leq \alpha < \beta < \gamma \leq b$, then

$$e(f, \alpha, \beta) + e(f, \beta, \gamma) \leq e(f, \alpha, \gamma).$$

Assumption 2. There exists a function $J_f : [a, b] \to \mathbb{R}$, and a constant $m > 1$ with the following properties:

(i) J_f is nonnegative and piecewise continuous on $[a, b]$ having at most a finite number of jump discontinuities, and

(ii)

$$\lim_{h \to 0+} \frac{e(f, \alpha, \alpha + h)}{h^m} = J_f(\alpha+).$$

Moreover, $|J_f(\alpha+) - e(f, \alpha, \alpha+h)/h^m|$ can be made uniformly small when $(\alpha, \alpha+h)$ is contained in an interval where J_f is continuous.

Assumption 3. $e(f, \alpha, \beta)$ depends continuously on (α, β).

Corollary 2.1 in [MV75] states the following.

Theorem 8.2.1 (McClure, Vitale [MV75])**.** *If Assumptions 1–3 hold for $e(f, \alpha, \beta)$, then*

$$\lim_{n \to \infty} n^{m-1} E_n(f) = \left(\int_a^b (J_f(s))^{1/m} ds \right)^m.$$

We use Theorem 8.2.1 to prove parts (i), (ii), (iv), and (v) of Theorem 8.1.1. In order to prove (iii) and (vi) of Theorem 8.1.1, we need a modified form of Theorem 8.2.1, also quoted from [MV75].

Let $G(f, T_n)$ be a function that has a decomposition with respect to the partition T_n of the following form

$$G(f, T_n) = \max_{i=0,\ldots,n-1} g(f, t_i, t_{i+1}).$$

Define $G_n(f) = \min_{T_n} G(f, T_n)$.

The result will again follow from three assumptions from [MV75].

Assumption 4. $g(f, \alpha, \beta) \geq 0$ for any (α, β) satisfying $a \leq \alpha < \beta \leq b$, and if $a \leq \alpha < \beta < \gamma \leq b$, then

$$\max(g(f, \alpha, \beta), g(f, \beta, \gamma)) \leq g(f, \alpha, \gamma).$$

Assumption 5. There exists a function $J_f : [a, b] \to \mathbb{R}$, and a constant $m > 0$ with the following properties:

(i) J_f is nonnegative and piecewise continuous on $[a, b]$, having at most a finite number of jump discontinuities, and

(ii)
$$\lim_{h \to 0+} \frac{g(f, \alpha, \alpha + h)}{h^m} = J_f(\alpha+).$$
Moreover, $|J_f(\alpha+) - g(f, \alpha, \alpha+h)/h^m|$ can be made uniformly small when $(\alpha, \alpha+h)$ is contained in an interval where J_f is continuous.

Assumption 6. $g(f, \alpha, \beta)$ depends continuously on (α, β).

Lemma 5 in [MV75] states the following.

Theorem 8.2.2 (McClure, Vitale [MV75])**.** *If Assumptions 4–6 hold for $g(f, \alpha, \beta)$, then*
$$\lim_{n \to \infty} n^m G_n(f) = \left(\int_a^b (J_f(s))^{1/m} ds \right)^m.$$

In order to prove Theorem 8.1.1, in the next sections we define the functions f and e in each case and verify Assumptions 1–3 for the cases of area deviation and perimeter deviation, and we verify Assumptions 4–6 for the Hausdorff metric case.

In the proof of Theorem 8.1.1 we frequently use the following fact from the elementary differential geometry of curves. Let p and q be distinct points of the continuously differentiable curve C. Then the ratio of the length of the arc of C between p and q and the length of the segment $[p, q]$ tends to 1 as $q \to p$.

In our argument, we use the small $o(\cdot)$ notation and, in particular, the following facts. Let $n \geq 0$ and $h(t) = o(t^n) \geq 0$ (as $t \to 0$) be an n times continuously differentiable function on an open interval $(-a, a)$. Then the first derivative $h'(t) = o(t^{n-1})$ as $t \to 0$, and $H(x) = \int_0^x h(t)dt$ is $o(x^{n+1})$ as $x \to 0$.

Let $S \subset \mathbb{R}^2$ be a compact, spindle convex set, and let L denote its perimeter. Let $r : [0, L] \to \mathbb{R}^2$ be the arc-length parametrization of the boundary ∂S with a fixed $r(0) = x_0 \in \partial S$ starting point such that $r(s)$ defines the positive orientation on ∂S.

Let $f : [0, L] \to \mathbb{R}$ be defined as $f(s) = s$, that is, f is the arc-length of ∂S. The function f is fixed throughout the chapter, and it is omitted from the notation.

8.3 Proofs of the inscribed cases

We associate with a partition $T_n = (t_0, t_1, \ldots, t_n)$ of the interval $[0, L]$ the disc-polygon $P(T_n)$ with vertex set $\{x_0, x_1, \ldots, x_{n-1}\}$ such that $x_i = r(t_i)$ for $i = 0, \ldots, n-1$. We use $a(s_1, s_2)$ to denote the length of the shorter unit circular arc connecting the points $r(s_1)$ and $r(s_2)$ for $s_1, s_2 \in [0, L]$, and, in particular, $a_i = a(t_i, t_{i+1})$ for $i = 0, \ldots, n-1$.

8.3.1 Perimeter deviation

Let $E(T_n) = \delta_\ell(P(T_n), S)$. Then
$$E(T_n) = \sum_{i=0}^{n-1} e(t_i, t_{i+1}),$$

where $e(t_i, t_{i+1}) = t_{i+1} - t_i - a_i$. Furthermore,

$$\delta_\ell(S, S_n^\ell) = \inf_{T_n} E(T_n).$$

Assumptions 1 and 3 are obviously satisfied. In order to verify that Assumption 2 is satisfied with $m = 3$ and $J(s) = (\kappa^2(s) - 1)/24$, we need the following lemma.

Lemma 8.3.1. *Let K be a compact, (linearly) convex set in $\mathbb{R}^2$ with nonempty interior and twice continuously differentiable boundary. Let $r(s) : [0, L] \to \mathbb{R}^2$ be the arc-length parametrization of the boundary ∂K of K that defines the positive orientation on ∂K. Let $r(s_0) \in \partial K$ be an arbitrary fixed point. Then*

$$\lim_{\Delta s \to 0^+} \frac{\Delta s - d(r(s_0 + \Delta s), r(s_0))}{(\Delta s)^3} = \frac{\kappa(s_0)^2}{24}. \tag{8.3.1}$$

Proof. Without loss of generality, we may assume that $r(s_0) = 0$, the x-axis of the coordinate-system is tangent to K at the point $r(s_0)$, and that K lies in the upper half plane. There exists an open neighbourhood of 0 in which the boundary of K can be represented as the graph of a twice continuously differentiable convex function h with $h(0) = h'(0) = 0$.

Applying Taylor's theorem for h around 0, we obtain

$$h(x) = \frac{h''(0)}{2}x^2 + o(x^2) \quad \text{as } x \to 0^+.$$

Let $r(s_0 + \Delta s) = (x, h(x))$. Using the formula for arc-length, we obtain that

$$\lim_{\Delta s \to 0^+} \frac{\Delta s - d(r(s_0 + \Delta s), r(s_0))}{(\Delta s)^3} = \lim_{x \to 0+} \frac{\int_0^x \sqrt{1 + h'(t)^2}dt - \sqrt{x^2 + h^2(x)}}{(\int_0^x \sqrt{1 + h'(t)^2}dt)^3}.$$

From our preliminary remarks it follows that $h'(x) = h''(0)x + o(x)$ as $x \to 0^+$, and so $h'(x)^2 = h''(0)^2x^2 + o(x^2)$ as $x \to 0^+$. Using the Taylor expansion of $\sqrt{(\,\cdot\,)}$ around 1, we obtain

$$\sqrt{1 + h'(t)^2} = \sqrt{1 + h''(0)^2t^2 + o(t^2)} = 1 + \frac{h''(0)^2t^2}{2} + o(t^2) \quad \text{as } t \to 0^+,$$

and

$$\sqrt{x^2 + h^2(x)} = x(1 + \frac{h''(0)^2}{8}x^2 + o(x^2)) = x + \frac{h''(0)^2}{8}x^3 + o(x^3) \quad \text{as } x \to 0^+.$$

The above estimates yield

$$\begin{aligned} \int_0^x \sqrt{1 + h'(t)^2}dt &= \int_0^x 1 + \frac{h''(0)^2t^2}{2} + o(t^2)dt \\ &= x + \frac{h''(0)^2}{6}x^3 + o(x^3) \quad \text{as } x \to 0^+. \end{aligned} \tag{8.3.2}$$

Note that from (8.3.2) it readily follows that

$$\lim_{x\to 0+} \frac{x}{\int_0^x \sqrt{1+h'(t)^2}dt} = 1. \tag{8.3.3}$$

Finally, we obtain

$$\begin{aligned}\lim_{x\to 0+} \frac{\int_0^x \sqrt{1+h'(t)^2}dt - \sqrt{x^2+h^2(x)}}{x^3} &= \lim_{x\to 0+} \frac{\frac{h''(0)^2}{6}x^3 - \frac{h''(0)^2}{8}x^3 + o(x^3)}{x^3} \\ &= \frac{h''(0)^2}{24}.\end{aligned}$$

□

The verification of Assumption 2 goes as follows. The function $J(s) = (\kappa(s)^2 - 1)/24$ is nonnegative and continuous on the interval $[0, L]$ and thus it satisfies condition (i) of Assumption 2.

In order to check condition (ii), note that $d(r(s_0 + \Delta s), r(s_0)) < a(s_0 + \Delta s, s_0) \le \Delta s$ implies

$$\lim_{\Delta s\to 0^+} \frac{a(s_0+\Delta s, s_0)}{\Delta s} = 1. \tag{8.3.4}$$

Now, Lemma 8.3.1 and (8.3.4) yield

$$\begin{aligned}&\lim_{\Delta s\to 0^+} \frac{e(s_0, s_0+\Delta s)}{(\Delta s)^3} = \lim_{\Delta s\to 0^+} \frac{\Delta s - a(s_0, s_0+\Delta s)}{(\Delta s)^3} \qquad (8.3.5)\\ &= \lim_{\Delta s\to 0^+} \frac{[\Delta s - d(r(s_0+\Delta s), r(s_0))] - [a(s_0+\Delta s, s_0) - d(r(s_0+\Delta s, r(s_0))]}{(\Delta s)^3} \\ &= \frac{\kappa^2(s_0)}{24} - \frac{1}{24} = J(s_0).\end{aligned}$$

It is also clear that the limit (8.3.1) is uniform in $[0, L]$ in the sense of (ii) of Assumption 2.

Proof of part (i) of Theorem 8.1.1. Theorem 8.2.1 and (8.3.5) imply that

$$\lim_{n\to\infty} n^2 E_n(f) = \left(\int_0^L J(s)^{1/3} ds\right)^3 = \frac{1}{24}\left(\int_{\partial S} (\kappa(s)^2 - 1)^{1/3} ds\right)^3. \tag{8.3.6}$$

□

8.3.2 Area deviation

Let $E(T_n) = \delta_A(P(T_n), S)$. In this case,

$$E(T_n) = \sum_{i=1}^{n-1} e(t_i, t_{i+1}),$$

where $e(t_i, t_{i+1})$ is the area of the region of S enclosed by ∂S between x_i and x_{i+1} and the shorter arc of the unit circle through the points x_i and x_{i+1}. Furthermore,

$$\delta_A(S, S_n^A) = \inf_{T_n} E(T_n).$$

Assumptions 1 and 3 are clearly satisfied. We need to verify that Assumption 2 holds with $J(s) = (\kappa(s) - 1)/12$ and $m = 3$. Our argument is similar to the one we pursue in the case of perimeter deviation.

Lemma 8.3.2. *Let K be a (linearly) convex, compact set with nonempty interior and with twice continuously differentiable boundary. Let $r(s) : [0, L] \to \mathbb{R}^2$ be the arc-length parametrization of ∂K that defines the positive orientation of ∂S, and let $r(s_0) \in \partial K$ be a fixed point. Then*

$$\lim_{\Delta s \to 0+} \frac{A(s_0, s_0 + \Delta s)}{(\Delta s)^3} = \frac{\kappa(s_0)}{12},$$

where $A(s_0, s_0 + \Delta s)$ denotes the area of the smaller cap cut off from K by the straight line through the points $r(s_0)$ and $r(s_0 + \Delta s)$.

Proof. Without loss of generality, we may assume that $r(s_0) = 0$, the x-axis of the coordinate-system is tangent to K at the point $r(s_0)$, and that K lies in the upper half plane. There exists an open neighbourhood of 0 in which the boundary of K can be represented as the graph of a twice continuously differentiable convex function h with $h(0) = h'(0) = 0$ and

$$h(x) = \frac{h''(0)}{2} x^2 + o(x^2) \quad \text{as } x \to 0.$$

Let $r(s_0 + \Delta s) = (x, h(x))$. The formulae for area and arc-length and the limit (8.3.3) yield

$$\begin{aligned}\lim_{\Delta s \to 0+} \frac{A(s_0, s_0 + \Delta s)}{(\Delta s)^3} &= \lim_{x \to 0+} \frac{\frac{xh(x)}{2} - \int_0^x h(t)dt}{x^3} \\ &= \lim_{x \to 0+} \frac{h(x)}{2x^2} - \frac{h''(x)}{2x^3} \int_0^x t^2 + o(t)dt = \frac{h''(0)}{4} - \frac{h''(0)}{6} = \frac{h''(0)}{12}.\end{aligned}$$

□

The function $J(s) = (\kappa(s) - 1)/12$ is nonnegative and continuous on the entire interval $[0, L]$, and thus it satisfies condition (i) of Assumption 2. The argument to verify that condition (ii) holds for $J(s)$ is similar to (8.3.5). We leave the details to the interested reader.

In view of the above, part (ii) of Theorem 8.1.1 follows directly from Theorem 8.2.1.

8.3.3 Hausdorff distance

Let $G(T_n) = \delta_H(P(T_n), S)$. In this case

$$G(T_n) = \max_{i=0,\ldots,n-1} g(t_i, t_{i+1}),$$

where $g(t_i, t_{i+1})$ is the Hausdorff distance of the part of the curve ∂S between x_i and x_{i+1} and the shorter arc of the unit circle connecting x_i and x_{i+1}. Furthermore,

$$\delta_H(S, S_n^H) = \min_{T_n} G(T_n).$$

In order to verify that Assumptions 4, 5 and 6 are satisfied, we approximate ∂S locally by its osculating circle. The osculating circle of ∂S at $r(s)$ is the circle of radius $1/\kappa(s)$ through $r(s)$ which shares a common support line with S in $r(s)$, and which lies on the same side of this common support line as S.

Lemma 8.3.3. *Let $h_1, h_2 : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable convex functions with $h_1(0) = h_2(0) = h_1'(0) = h_2'(0) = 0$ and $h_1''(0) \geq h_2''(0) \geq 0$. Then*

(i)

$$\lim_{x \to 0+} \frac{\int_0^x \sqrt{1 + h_1'(t)^2} dt - \int_0^x \sqrt{1 + h_2'(t)^2} dt}{x^3} = \frac{h_1''(0)^2 - h_2''(0)^2}{6},$$

(ii)

$$\lim_{x \to 0+} \frac{\int_0^x h_1(t) - h_2(t) dt}{x^3} = \frac{h_1''(0) - h_2''(0)}{6}, \text{ and}$$

(iii)

$$\lim_{x \to 0+} \frac{\delta_H(h_1[0,x], h_2[0,x])}{x^2} = \lim_{x \to 0+} \frac{\max_{t \in [0,x]} |h_1(t) - h_2(t)|}{x^2} = \\ = \frac{h_1''(0) - h_2''(0)}{2},$$

where $h_i[0,x]$ denotes the graph of h_i over the closed interval $[0,x]$ for $i = 1, 2$.

Proof. Using that

$$h_i(x) = \frac{h_i''(0)}{2} x^2 + o(x^2) \quad \text{as } x \to 0^+ \text{ for } i = 1, 2,$$

part (i) of the lemma readily follows from (8.3.2).

Part (ii) can be verified as follows:

$$\lim_{x \to 0+} \frac{\int_0^x h_1(t) - h_2(t) dt}{x^3} = \lim_{x \to 0+} \frac{\int_0^x \frac{h_1''(0) - h_2''(0)}{2} t^2 + o(t^2) dt}{x^3} = \\ = \lim_{x \to 0+} \frac{\frac{h_1''(0) - h_2''(0)}{6} x^3}{x^3} + \lim_{x \to 0+} \frac{\int_0^x o(t^2) dt}{x^3} = \frac{h_1''(0) - h_2''(0)}{6}.$$

It remains to prove part (iii) of the lemma. We start by showing the first equality in (iii). Let

$$m(x) = \max_{t \in [0,x]} |h_1(t) - h_2(t)|.$$

It is clear from the definition of Hausdorff distance that

$$\delta_H(h_1[0,x], h_2[0,x]) \leq m(x).$$

Next, we prove that for any sufficiently small $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$H(h_1[0,x], h_2[0,x]) \geq m(x)(1 - \varepsilon) \quad \text{for all } 0 < x < \delta.$$

Fix an arbitrary $0 < \varepsilon < 1/4$. Then there exists a $0 < \delta < \varepsilon$ that satisfies the following conditions:

(a) $m(\delta)\varepsilon < \delta$,

(b) $h_1'(x), h_2'(x) < \varepsilon$ for all $x \in (0, 2\delta)$, and

(c) $(h_i(x + m(\delta)\varepsilon) - h_i(x))/m(\delta)\varepsilon < 2\varepsilon$, $i = 1, 2$ for all $x \in [0, \delta]$.

The existence of a $0 < \delta < \varepsilon$ that satisfies condition (a) follows from the fact that if δ is sufficiently small, then $|h_1(x) - h_2(x)| < x$ for $x \in [0, \delta]$, and so $m(\delta) < \delta$. Since $h_i(x)$, $i = 1, 2$, are twice continuously differentiable in a closed interval containing 0, therefore their difference quotients are uniformly convergent in the same interval. Thus, if δ is sufficiently small, then both (b) and (c) are satisfied.

Let $x_0 \in [0, \delta]$ where the maximum $m(\delta)$ is attained. Without loss of generality, we may assume that $h_1(x_0) > h_2(x_0)$.

The normal line of the graph of h_1 at the point $(x_0, h_1(x_0))$ intersects the graph of h_2 in $(\hat{x}, h_2(\hat{x}))$ with $\hat{x} \le x_0 + m(\delta)\varepsilon < 2\delta$.

Now, it follows from conditions (a)–(c) that

$$0 \le h_2(\hat{x}) - h_2(x_0) \le h_2(x_0 + m(\delta)\varepsilon) - h_2(x_0) < m(\delta)\varepsilon,$$

hence

$$d((x_0, h_1(x_0)), (\hat{x}, h_2(\hat{x}))) \ge h_1(x_0) - h_2(x_0) + h_2(x_0) - h_2(\hat{x}) \ge m(\delta) - m(\delta)\varepsilon.$$

This proves the first equality of part (iii) of the lemma. The second equality is an immediate consequence of Taylor's theorem. $\square$

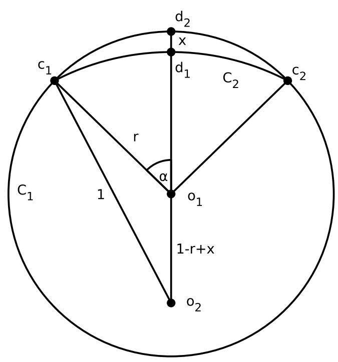

Figure 8.1:

Lemma 8.3.4. *Let C_1 be a circle of radius $r = 1/\kappa < 1$ centred at o_1, and let C_2 be a unit circle centred at o_2 which intersects C_1 in c_1 and c_2 (see Figure 8.1) such that $\angle c_1 o c_2 = 2\alpha$. The bisector of $\angle c_1 o c_2$ intersects C_1 and C_2 at d_1 and d_2, respectively. Let $x = d(d_1, d_2)$. Then*

$$\lim_{\alpha \to 0+} \frac{x}{(2\alpha r)^2} = \frac{\kappa - 1}{8}.$$

Proof. Applying the Law of Cosines to the triangle $\triangle o_1 o_2 c_1$ yields

$$1 = r^2 + (1 - r + x)^2 + 2r(1 - r + x)\cos\alpha.$$

Using that $\cos\alpha = 1 - \alpha^2/2 + o(\alpha^2)$ as $\alpha \to 0^+$, we obtain

$$0 = x^2 + 2x - r\alpha^2 + r^2\alpha^2 - rx\alpha^2 + o(\alpha^2).$$

This implies that

$$\lim_{\alpha\to 0+} \frac{x^2 + 2x}{(2\alpha r)^2} = \lim_{\alpha\to 0+} \frac{r\alpha^2 - r^2\alpha^2 + rx\alpha^2}{(2\alpha r)^2} = \frac{1}{4r} - \frac{1}{4},$$

and the statement of the lemma follows immediately. □

Lemma 8.3.5. *Let $h_1, h_2 : [-a, a] \to \mathbb{R}$ be twice continuously differentiable convex functions for some $a > 0$ such that $h_1(0) = h_2(0) = h_1'(0) = h_2'(0) = 0$ and $h_1''(0) = h_2''(0) \geq 0$. Let $C(x, h_i)$ denote the concave up shorter unit circular arcs joining $(0, 0)$ with $(x, h_i(x))$, $i = 1, 2$. Then*

$$\lim_{x\to 0+} \frac{d_H(C(x, h_1), C(x, h_2))}{x^2} = 0.$$

Proof. Note that if a is sufficiently small, then

$$\delta_H(C(x, h_1), C(x, h_2)) \leq |h_1(x) - h_2(x)| \leq m(x),$$

for all $x \in [0, a]$, and $m(x) = o(x^2)$ for h_1 and h_2 under the conditions of the lemma. □

Finally, we are going to verify Assumption 5 for $J(s) = (\kappa(s) - 1)/8$ and $m = 2$. Let $s_0 \in [0, L]$. Without loss of generality, we may assume that $r(s_0) = 0$ and the x-axis of the coordinate-system is tangent to S at $r(s_0)$ so that S is in the upper half plane. Let the real function h_1 represent the boundary of S in a suitable neighbourhood of 0, say in the interval $[-a, a]$, and let h_2 be the function that represents the osculating circle of ∂S at $r(s_0)$ in the same interval. Both h_1 and h_2 are twice continuously differentiable and convex in $[-a, a]$, and, due to the choice of the coordinate-system, $h_1(0) = h_2(0) = h_1'(0) = h_2'(0) = 0$ and $h_1''(0) = h_2''(0) \geq 0$. Let $r(s_0 + \Delta s) = (x, h_1(x))$. The triangle inequality of the Hausdorff metric implies that

$$g(s_0, s_0 + \Delta s) \leq \delta_H(h_1[0, x], h_2[0, x]) + \delta_H(h_2[0, x], C(x, h_2)) + \delta_H(C(x, h_1), C(x, h_2)),$$

and

$$g(s_0, s_0 + \Delta s) \geq -\delta_H(h_1[0, x], h_2[0, x]) + \delta_H(h_2[0, x], C(x, h_2)) - \delta_H(C(x, h_1), C(x, h_2)).$$

Now, applying Lemmas 8.3.3, 8.3.4 and 8.3.5, we obtain that

$$\lim_{\Delta s\to 0+} \frac{g(s_0, s_0 + \Delta s)}{(\Delta s)^2} = \frac{\kappa(s_0) - 1}{8} = J(s_0).$$

Part (iii) of Theorem 8.1.1 follows directly from Theorem 8.2.2.

Bibliography

[Aff91] F. Affentranger, *The convex hull of random points with spherically symmetric distributions*, Rend. Sem. Mat. Univ. Politec. Torino **49** (1991), no. 3, 359–383 (1993).

[Ale38] A. D. Aleksandrov, *On the theory of mixed volumes. III. Extension of two theorems of Minkowski on convex polyhedra to arbitrary convex bodies*, Mat. Sbornik N.S. **3** (1938), 27–46.

[Ale39] A. D. Aleksandrov, *On the surface area measure of convex bodies*, Mat. Sbornik N.S. **6** (1939), 167–174.

[AKV12] G. Ambrus, P. Kevei, and V. Vígh, *The diminishing segment process*, Statist. Probab. Lett. **82** (2012), no. 1, 191–195.

[Art64] E. Artin, *The gamma function*, Holt, Rinehart and Winston, New York-Toronto-London, 1964.

[Bár89] I. Bárány, *Intrinsic volumes and f-vectors of random polytopes*, Math. Ann. **285** (1989), no. 4, 671–699.

[Bár92] I. Bárány, *Random polytopes in smooth convex bodies*, Mathematika **39** (1992), no. 1, 81–92.

[Bár08] I. Bárány, *Random points and lattice points in convex bodies*, Bull. Amer. Math. Soc. (N.S.) **45** (2008), no. 3, 339–365.

[BFV10] I. Bárány, F. Fodor, and V. Vígh, *Intrinsic volumes of inscribed random polytopes in smooth convex bodies*, Adv. in Appl. Probab. **42** (2010), no. 3, 605–619.

[BHRS17] I. Bárány, D. Hug, M. Reitzner, and R. Schneider, *Random points in halfspheres*, Random Structures Algorithms **50** (2017), no. 1, 3–22.

[BL88] I. Bárány and D. G. Larman, *Convex bodies, economic cap coverings, random polytopes*, Mathematika **35** (1988), no. 2, 274–291.

[BR10a] I. Bárány and M. Reitzner, *On the variance of random polytopes*, Adv. Math. **225** (2010), no. 4, 1986–2001.

[BR10b] I. Bárány and M. Reitzner, *Poisson polytopes*, Ann. Probab. **38** (2010), no. 4, 1507–1531.

[BS13] I. Bárány and W. Steiger, *On the variance of random polygons*, Comput. Geom. **46** (2013), no. 2, 173–180.

[BV07] I. Bárány and V. Vu, *Central limit theorems for Gaussian polytopes*, Ann. Probab. **35** (2007), no. 4, 1593–1621.

[Bez10] K. Bezdek, *Classical topics in discrete geometry*, Springer, New York, 2010.

[Bez13] K. Bezdek, *Lectures on sphere arrangements—the discrete geometric side*, Fields Institute Monographs, vol. 32, Springer, New York; Fields Institute for Research in Mathematical Sciences, Toronto, ON, 2013.

[Bez18] K. Bezdek, *From r-dual sets to uniform contractions*, Aequationes Math. **92** (2018), no. 1, 123–134.

[BC02] K. Bezdek and R. Connelly, *Pushing disks apart—the Kneser–Poulsen conjecture in the plane*, J. Reine Angew. Math. **553** (2002), 221–236.

[BLNP07] K. Bezdek, Z. Lángi, M. Naszódi, and P. Papez, *Ball-polyhedra*, Discrete Comput. Geom. **38** (2007), no. 2, 201–230.

[BN18] K. Bezdek and M. Naszódi, *The Kneser-Poulsen conjecture for special contractions*, Discrete Comput. Geom. **60** (2018), no. 4, 967–980.

[BBCY19] G. Bianchi, K. J. Böröczky, A. Colesanti, and D. Yang, *The L_p-Minkowski problem for $-n < p < 1$*, Adv. Math. **341** (2019), 493–535.

[Bla56] W. Blaschke, *Kreis und Kugel*, Walter de Gruyter & Co., Berlin, 1956.

[Bör10] K. J. Böröczky, *Stability of the Blaschke-Santaló and the affine isoperimetric inequality*, Adv. Math. **225** (2010), no. 4, 1914–1928.

[BF19] K. J. Böröczky and F. Fodor, *The L_p dual Minkowski problem for $p > 1$ and $q > 0$*, J. Differential Equations **266** (2019), no. 12, 7980–8033.

[BFH10] K. J. Böröczky, F. Fodor, and D. Hug, *The mean width of random polytopes circumscribed around a convex body*, J. Lond. Math. Soc. (2) **81** (2010), no. 2, 499–523.

[BFH13] K. J. Böröczky, F. Fodor, and D. Hug, *Intrinsic volumes of random polytopes with vertices on the boundary of a convex body*, Trans. Amer. Math. Soc. **365** (2013), no. 2, 785–809.

[BFRV09] K. J. Böröczky, F. Fodor, M. Reitzner, and V. Vígh, *Mean width of random polytopes in a reasonably smooth convex body*, J. Multivariate Anal. **100** (2009), no. 10, 2287–2295.

[BHH08] K. J. Böröczky Jr., L. M. Hoffmann, and D. Hug, *Expectation of intrinsic volumes of random polytopes*, Period. Math. Hungar. **57** (2008), no. 2, 143–164.

[BLYZ13] K. J. Böröczky, E. Lutwak, D. Yang, and G. Zhang, *The logarithmic Minkowski problem*, J. Amer. Math. Soc. **26** (2013), 831–852.

[BLY$^+$] K. J. Böröczky, E. Lutwak, D. Yang, G. Zhang, and Y. Zhao, *The dual Minkowski problem for symmetric convex bodies*, arXiv:1703.06259.

[BR04] K. Böröczky Jr. and M. Reitzner, *Approximation of smooth convex bodies by random circumscribed polytopes*, Ann. Appl. Probab. **14** (2004), no. 1, 239–273.

[BS10] K. J. Böröczky and R. Schneider, *The mean width of circumscribed random polytopes*, Canad. Math. Bull. **53** (2010), no. 4, 614–628.

[Bro07] E. M. Bronshteĭn, *Approximation of convex sets by polyhedra*, Sovrem. Mat. Fundam. Napravl. **22** (2007), 5–37; English transl., J. Math. Sci. (N. Y.) **153** (2008), no. 6, 727–762.

[Caf90a] L. Caffarelli, *A localization property of viscosity solutions to Monge-Ampère equation and their strict convexity*, Ann. Math. **131** (1990), 129-134.

[Caf90b] L. Caffarelli, *Interior $W^{2,p}$-estimates for solutions of the Monge-Ampère equation*, Ann. Math. **131** (1990), 135-150.

[CLZ17] S. Chen, Q.-r. Li, and G. Zhu, *On the L_p Monge-Ampère equation*, J. Differential Equations **263** (2017), no. 8, 4997–5011.

[CLZ19] S. Chen, Q.-r. Li, and G. Zhu, *The logarithmic Minkowski problem for non-symmetric measures*, Trans. Amer. Math. Soc. **371** (2019), no. 4, 2623–2641.

[CW06] K.-S. Chou and X.-J. Wang, *The L_p-Minkowski problem and the Minkowski problem in centroaffine geometry*, Adv. Math. **205** (2006), 33-83.

[DGK63] L. Danzer, B. Grünbaum, and V. Klee, *Helly's theorem and its relatives*, Proc. Sympos. Pure Math., Vol. VII, Amer. Math. Soc., Providence, R.I., 1963, pp. 101–180.

[DD12] F. Demengel and G. Demengel, *Functional Spaces for the Theory of Elliptic Partial Differential Equations*, Vol. 205, Springer, Berlin, 2012.

[DW96] L. Dümbgen and G. Walther, *Rates of convergence for random approximations of convex sets*, Adv. in Appl. Probab. **28** (1996), no. 2, 384–393.

[Efr65] B. Efron, *The convex hull of a random set of points*, Biometrika **52** (1965), 331–343.

[ES81] B. Efron and C. Stein, *The jackknife estimate of variance*, Ann. Statist. **9** (1981), no. 3, 586–596.

[Egg65] H. G. Eggleston, *Sets of constant width in finite dimensional Banach spaces*, Israel J. Math. **3** (1965), 163–172.

[Fed69] H. Federer, *Geometric measure theory*, Springer-Verlag New York Inc., New York, 1969.

[FTF15] G. Fejes Tóth and F. Fodor, *Dowker-type theorems for hyperconvex discs*, Period. Math. Hungar. **70** (2015), no. 2, 131–144.

[FT53] L. Fejes Tóth, *Lagerungen in der Ebene, auf der Kugel und im Raum*, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1953.

[FT82a] L. Fejes Tóth, *Packing and covering with r-convex discs*, Studia Sci. Math. Hungar. **18** (1982), no. 1, 69–73.

[FT82b] L. Fejes Tóth, *Packing of r-convex discs*, Studia Sci. Math. Hungar. **17** (1982), no. 1-4, 449–452.

[FHZ16] F. Fodor, D. Hug, and I. Ziebarth, *The volume of random polytopes circumscribed around a convex body*, Mathematika **62** (2016), no. 1, 283–306.

[FKV14] F. Fodor, P. Kevei, and V. Vígh, *On random disc polygons in smooth convex discs*, Adv. in Appl. Probab. **46** (2014), no. 4, 899–918.

[FKV16] F. Fodor, Á. Kurusa, and V. Vígh, *Inequalities for hyperconvex sets*, Adv. Geom. **16** (2016), no. 3, 337–348.

[FV12] F. Fodor and V. Vígh, *Disc-polygonal approximations of planar spindle convex sets*, Acta Sci. Math. (Szeged) **78** (2012), no. 1-2, 331–350.

[FV18] F. Fodor and V. Vígh, *Variance estimates for random disc-polygons in smooth convex discs*, J. Appl. Probab. **55** (2018), no. 4, 1143–1157.

[GHW$^+$19] R. J. Gardner, D. Hug, W. Weil, S. Xing, and D. Ye, *General volumes in the Orlicz-Brunn-Minkowski theory and a related Minkowski problem I*, Calc. Var. Partial Differential Equations **58** (2019), no. 1, Art. 12, 35.

[GG97] S. Glasauer and P. M. Gruber, *Asymptotic estimates for best and stepwise approximation of convex bodies. III*, Forum Math. **9** (1997), no. 4, 383–404.

[Gru83] P. M. Gruber, *Approximation of convex bodies*, Convexity and its applications, Birkhäuser, Basel, 1983, pp. 131–162.

[Gru93] P. M. Gruber, *Aspects of approximation of convex bodies*, Handbook of Convex Geometry, Vol. A, North-Holland, Amsterdam, 1993, pp. 319–345.

[Gru96] P. M. Gruber, *Expectation of random polytopes*, Manuscripta Math. **91** (1996), no. 3, 393–419.

[Gru97] P. M. Gruber, *Comparisons of best and random approximation of convex bodies by polytopes*, Rend. Circ. Mat. Palermo (2) Suppl. **50** (1997), 189–216.

[Gru07] P. M. Gruber, *Convex and discrete geometry*, Springer, Berlin, 2007.

[HP18] M. Henk and H. Pollehn, *Necessary subspace concentration conditions for the even dual Minkowski problem*, Adv. Math. **323** (2018), 114–141.

[Hug96a] D. Hug, *Contributions to affine surface area*, Manuscripta Math. **91** (1996), no. 3, 283–301.

[Hug96b] D. Hug, *Curvature relations and affine surface area for a general convex body and its polar*, Results Math. **29** (1996), no. 3-4, 233–248.

[Hug98] D. Hug, *Absolute continuity for curvature measures of convex sets. I*, Math. Nachr. **195** (1998), 139–158.

[Hug99] D. Hug, *Absolute continuity for curvature measures of convex sets. II*, Math. Z. **232** (1999), no. 3, 437–485.

[Hug00] D. Hug, *Measures, Curvatures and Currents in Convex Geometry*, 2000. Habilitationsschrift, Albert-Ludwigs-Universität, Freiburg i. Br.

[Hug02] D. Hug, *Absolute continuity for curvature measures of convex sets. III*, Adv. Math. **169** (2002), no. 1, 92–117.

[Hug13] D. Hug, *Random polytopes*, Stochastic geometry, spatial statistics and random fields, Lecture Notes in Math., vol. 2068, Springer, Heidelberg, 2013, pp. 205–238.

[HLYZ05] D. Hug, E. Lutwak, D. Yang, and G. Zhang, *On the L_p Minkowski problem for polytopes*, Discrete Comput. Geom. **33** (2005), no. 4, 699–715.

[HLYZ16] Y. Huang, E. Lutwak, D. Yang, and G. Zhang, *Geometric measures in the dual Brunn-Minkowski theory and their associated Minkowski problems*, Acta Math. **216** (2016), no. 2, 325–388.

[HLYZ18] Y. Huang, E. Lutwak, D. Yang, and G. Zhang, *The L_p Alexandrov problem for the L_p integral curvature*, J. Differential Geom. **110** (2018), 1–29.

[HZ18] Y. Huang and Y. Zhao, *On the L_p dual Minkowski problem*, Adv. Math. **332** (2018), 57–84.

[JMR17] T. Jahn, H. Martini, and C. Richter, *Ball convex bodies in Minkowski spaces*, Pacific J. Math. **289** (2017), no. 2, 287–316.

[JW17] Y. Jiang and Y. Wu, *On the 2-dimensional dual Minkowski problem*, Journal of Differential Equations **263** (2017), 3230-3243.

[Kal90] F. J. Kaltenbach, *Asymptotisches Verhalten zufälliger konvexer Polyeder*, 1990. Doctoral thesis, Albert-Ludwigs-Universität, Freiburg i. Br.

[KMP05] Y. S. Kupitz, H. Martini, and M. A. Perles, *Finite sets in $\mathbb{R}^d$ with many diameters – a survey*, Conference on Mathematics and Applications (Bangkok, 2005), Mahidol University Press, Bangkok, 2005, pp. 91–112.

[KMP10] Y. S. Kupitz, H. Martini, and M. A. Perles, *Ball polytopes and the Vázsonyi problem*, Acta Math. Hungar. **126** (2010), no. 1-2, 99–163.

[Lei98] K. Leichtweiss, *Affine geometry of convex bodies*, Johann Ambrosius Barth Verlag, Heidelberg, 1998.

[LSW] Q.-R. Li, W. Sheng, and X.-J. Wang, *Flow by Gauss curvature to the Alexandrov and Minkowski problems*, JEMS, accepted.

[LR99] M. Ludwig and M. Reitzner, *A characterization of affine surface area*, Adv. Math. **147** (1999), no. 1, 138–172.

[LR10] M. Ludwig and M. Reitzner, *A classification of $\mathrm{SL}(n)$ invariant valuations*, Ann. of Math. (2) **172** (2010), no. 2, 1219–1267.

[Lut75] E. Lutwak, *Dual mixed volumes*, Pacific J. Math. **58** (1975), no. 2, 531–538.

[Lut93a] E. Lutwak, *Selected affine isoperimetric inequalities*, Handbook of convex geometry, Vol. A, B, North-Holland, Amsterdam, 1993, pp. 151–176.

[Lut93b] E. Lutwak, *The Brunn-Minkowski-Firey theory. I. Mixed volumes and the Minkowski problem.*, J. Differential Geom. **38** (1993), 131-150.

[Lut96] E. Lutwak, *The Brunn-Minkowski-Firey theory. II. Affine and geominimal surface areas*, Adv. Math. **118** (1996), no. 2, 244–294.

[LYZ18] E. Lutwak, D. Yang, and G. Zhang, *L_p-dual curvature measures*, Adv. Math. **329** (2018), 85-132.

[May35] A. E. Mayer, *Eine Überkonvexität*, Math. Z. **39** (1935), no. 1, 511–531.

[McC75] D. E. McClure, *Nonlinear segmented function approximation and analysis of line patterns*, Quart. Appl. Math. **33** (1975/76), 1–37.

[MV75] D. E. McClure and R. A. Vitale, *Polygonal approximation of plane convex bodies*, J. Math. Anal. Appl. **51** (1975), no. 2, 326–358.

[McM75] P. McMullen, *Non-linear angle-sum relations for polyhedral cones and polytopes*, Math. Proc. Cambridge Philos. Soc. **78** (1975), no. 2, 247–261.

[Min97] H. Minkowski, *Allgemeine Lehrsätze über die konvexen Polyeder*, Nachr. Ges. Wiess. Göttingen (1897), 189–219.

[Min03] H. Minkowski, *Volumen und Oberfläche*, Math. Ann. **57** (1903), no. 4, 447–495.

[MS07] J. P. Moreno and R. Schneider, *Continuity properties of the ball hull mapping*, Nonlinear Anal. **66** (2007), no. 4, 914–925.

[MS12] J. P. Moreno and R. Schneider, *Diametrically complete sets in Minkowski spaces*, Israel J. Math. **191** (2012), no. 2, 701–720.

[PP17] G. Paouris and P. Pivovarov, *Random ball-polyhedra and inequalities for intrinsic volumes*, Monatsh. Math. **182** (2017), no. 3, 709–729.

[Pet85] C. M. Petty, *Affine isoperimetric problems*, Discrete geometry and convexity (New York, 1982), Ann. New York Acad. Sci., vol. 440, New York Acad. Sci., New York, 1985, pp. 113–127.

[Rei02] M. Reitzner, *Random points on the boundary of smooth convex bodies*, Trans. Amer. Math. Soc. **354** (2002), no. 6, 2243–2278 (electronic).

[Rei03] M. Reitzner, *Random polytopes and the Efron-Stein jackknife inequality*, Ann. Probab. **31** (2003), no. 4, 2136–2166.

[Rei04] M. Reitzner, *Stochastic approximation of smooth convex bodies*, Mathematika **51** (2004), no. 1-2, 11–29 (2005).

[Rei05] M. Reitzner, *Central limit theorems for random polytopes*, Probab. Theory Related Fields **133** (2005), no. 4, 483–507.

[Rei10] M. Reitzner, *Random polytopes*, New perspectives in stochastic geometry, Oxford Univ. Press, Oxford, 2010, pp. 45–76.

[RS63] A. Rényi and R. Sulanke, *Über die konvexe Hülle von n zufällig gewählten Punkten*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **2** (1963), 75–84.

[RS64] A. Rényi and R. Sulanke, *Über die konvexe Hülle von n zufällig gewählten Punkten. II*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **3** (1964), 138–147.

[RS68] A. Rényi and R. Sulanke, *Zufällige konvexe Polygone in einem Ringgebiet*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **9** (1968), 146–157.

[San46] L. A. Santaló, *On plane hyperconvex figures*, Summa Brasil. Math. **1** (1946), 221–239 (1948) (Spanish, with English summary).

[Sch87] R. Schneider, *Approximation of convex bodies by random polytopes*, Aequationes Math. **32** (1987), no. 2-3, 304–310.

[Sch88] R. Schneider, *Random approximation of convex sets*, J. Microsc. **151** (1988), 211–227.

[Sch08] R. Schneider, *Recent results on random polytopes*, Boll. Unione Mat. Ital. (9) **1** (2008), no. 1, 17–39.

[Sch14] R. Schneider, *Convex bodies: the Brunn-Minkowski theory*, Cambridge University Press, Cambridge, 2014.

[Sch18] R. Schneider, *Discrete aspects of stochastic geometry*, Handbook of discrete and computational geometry, Discrete Mathematics and its Applications (Boca Raton), CRC Press, Boca Raton, FL. Third edition, 2018, pp. 299–329.

[SW08] R. Schneider and W. Weil, *Stochastic and integral geometry*, Springer-Verlag, Berlin, 2008.

[SW80] R. Schneider and J. A. Wieacker, *Random polytopes in a convex body*, Z. Wahrsch. Verw. Gebiete **52** (1980), no. 1, 69–73.

[SY08] T. Schreiber and J. E. Yukich, *Variance asymptotics and central limit theorems for generalized growth processes with applications to convex hulls and maximal points*, Ann. Probab. **36** (2008), no. 1, 363–396.

[Sch94] C. Schütt, *Random polytopes and affine surface area*, Math. Nachr. **170** (1994), 227–249.

[SW90] C. Schütt and E. Werner, *The convex floating body*, Math. Scand. **66** (1990), no. 2, 275–290.

[SW03] C. Schütt and E. Werner, *Polytopes with vertices chosen randomly from the boundary of a convex body*, Geometric aspects of functional analysis, Lecture Notes in Math., vol. 1807, Springer, Berlin, 2003, pp. 241–422.

[Syl64] J. J. Sylvester, *Problem 1491*, Educational Times, London (1864).

[TTW18] C. Thäle, N. Turchi, and F. Wespi, *Random polytopes: central limit theorems for intrinsic volumes*, Proc. Amer. Math. Soc. **146** (2018), no. 7, 3063–3071.

[TW08] N. S. Trudinger and X.-J. Wang, *The Monge-Ampère equation and its geometric applications*, in: Handbook of geometric analysis. No. 1, Adv. Lect. Math. (ALM), 7, Int. Press, Somerville, MA, (2008), 467-524.

[TW18] N. Turchi and F. Wespi, *Limit theorems for random polytopes with vertices on convex surfaces*, Adv. in Appl. Probab. **50** (2018), no. 4, 1227–1245.

[Vu05] V. H. Vu, *Sharp concentration of random polytopes*, Geom. Funct. Anal. **15** (2005), no. 6, 1284–1318.

[Vu06] V. Vu, *Central limit theorems for random polytopes in a smooth convex set*, Adv. Math. **207** (2006), no. 1, 221–243.

[WW93] W. Weil and J. A. Wieacker, *Stochastic geometry*, Handbook of convex geometry, Vol. A, B, North-Holland, Amsterdam, 1993, pp. 1391–1438.

[Wer07] E. Werner, *On L_p-affine surface areas*, Indiana Univ. Math. J. **56** (2007), no. 5, 2305–2323.

[WY08] E. Werner and D. Ye, *New L_p affine isoperimetric inequalities*, Adv. Math. **218** (2008), no. 3, 762–780.

[Wie78] J. A. Wieacker, *Einige Probleme der polyedrischen Approximation*, 1978. Diplomarbeit — Albert-Ludwigs-Universität, Freiburg i. Br.

[Zha17] Y. Zhao, *The dual Minkowski problem for negative indices*, Calc. Var. Partial Differential Equations **56(2)** (2017), Art. 18, 16.

[Zha18] Y. Zhao, *Existence of solutions to the even dual Minkowski problem*, J. Differential Geom. **110** (2018), no. 3, 543–572.

[Zhu15] G. Zhu, *The centro-affine Minkowski problem for polytopes*, J. Differ. Geom. **101** (2015), 159-174.

[Zhu17] G. Zhu, *The L_p Minkowski problem for polytopes for $p < 0$*, Indiana Univ. Math. J. **66** (2017), 1333-1350.

[Zie70] H. Ziezold, *Über die Eckenanzahl zufälliger konvexer Polygone*, Izv. Akad. Nauk Armjan. SSR Ser. Mat. **5** (1970), no. 3, 296–312.