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Applications of Differential Systems in Geometry

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Contents

Preface

Many geometric properties can be described and investigated in terms of differential systems: by ordinary differential equations, partial differential equations, and differential inequalities. In this dissertation, we present some of them focusing on results about the

- inverse problem of the calculus of variations,
- metrizability, and projective metrizability,
- holonomy of Finsler manifolds,
- linearizability of 3-webs.

All of them are very well motivated, classical geometric problems, and have been investigated by excellent mathematicians for decades, if not for a century. Indeed, the first results on the inverse problem of the calculus of variations, due to H. Helmholtz from 1887, who derived a differential system on the so-called variational multiplier. The metrizability and the projective metrizability appear explicitly in Hilbert's fourth problem asking for the construction of metrics for which the projective line segments are geodesics. The linearizability problem of 3-webs is also over 100 years old: T.H. Gronwall's conjecture about the linearizable but not parallelizable planar 3-webs dates back to 1912. Finally, the notion of holonomy was introduced by É. Cartan almost 100 years ago (in 1926) but the first results on the holonomy of Finsler manifold are relatively recent: Zoltán Szabó described the holonomy group of Berwald manifolds in 1981. We have been able to obtain new results in these classical fields by using new approaches and new tools.

About the structure of the dissertation: at the beginning of each chapter, one can find a section "Introduction" where we present the motivation and the description of the geometric problems with a brief overview of the works and results obtained in the corresponding fields. The introduction is followed by "Preliminaries", in which we introduce the basic notions, notations, and tools used in the chapter. These preliminary sections are constructed linearly in the sense that they contain only the additional information and materials with respect to that of the previous chapters. The sections after these introductory notes contain new results.

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Chapter 1

The inverse problem of the calculus of variations

1.1 Introduction

The inverse problem of the calculus of variations is an old problem of differential geometry consisting of the characterization of second order ordinary differential equations (SODE) or sprays derivable from a variational principle. The first results on the inverse problem, due to H. Helmholtz from 1887, showed that the problem can be formulated in terms of a system (called now Helmholtz system) containing algebraic conditions and partial differential equations on the so-called variational multiplier. The most significant contribution to this problem is the famous paper of Jesse Douglas [36] in which, using Riquier's integrability theory, he classified variational differential equations with two degrees of freedom. Generalizing his results to higher dimensional cases is a hard problem because the Helmholtz system is an over-determined partial differential system (PDE), so in general, it has no solution. We cite [5, 6, 25, 34, 59, 61, 78, 79, 80] achieving significant progress on this problem. We note that in Douglas' work and also in all the above-mentioned papers, the results were obtained by analysing the Helmholtz system.

In our work [104, 105], we adopted a completely different approach: instead of considering the Helmholtz system, we investigated the integrability of the *Euler-Lagrange partial differential system.* Here, instead of the variational multiplier, the unknown is the regular Lagrange function, and instead of algebraic and first order partial differential equations, the system is composed by second order partial differential equations. The relation between the two approaches can be given as follows: if the Lagrangian E is a regular solution of the Euler-Lagrange PDE system, then the regular matrix field (g_{ij}) with $g_{ij} = \frac{\partial^2 E}{\partial y^i \partial y^j}$ is a variational multiplier. The two approaches are essentially equivalent, but working with the Euler-Lagrange partial differential system allowed us to find and present the obstructions to the existence of a variational principle in an intrinsic, natural and coordinate-free way.

The coordinate-free classification of the variational sprays on 2-dimensional manifolds can be found in my Ph.D. dissertation [112]. It is clear from [36] and [112] that, despite the fact that the dimension of the manifold is low, the analysis is very complex. In the higher dimensional cases the situation is much more difficult since the integrability condition also involves the curvature tensor, its derivatives and the higher order elements of the graded Lie-algebra $\mathcal{A}_S$ associated to the spray (see Section 1.5). Therefore, it is not really reasonable to expect a classification of variational sprays on n-dimensional manifolds where $n \in \mathbb{N}$ is arbitrary, unless we consider a particular class of sprays. Natural restrictions can be imposed on the curvature of the canonical connection associated to the spray. We considered the flat and isotropic sprays in [104, 105]. Their geometrical meaning can be explained as follows: if they are variational, the associated Lagrangian has isotropic curvature. In this chapter we present further results. It is organized as follows.

After the preliminaries of Section 1.2, we generalize in Section 1.3 Douglas' results about algebraic conditions on the variational multiplier. Indeed, in his article [36], Douglas gives simple criteria on the existence of the variational multiplier and therefore on the existence of a variational principle for SODEs on 2-dimensional manifolds. These criteria can be carried over to the n-dimensional case [5, 78, 104]. In [79], W. Sarlet and his co-authors found a double hierarchy of algebraic conditions for the variational multiplier which is determined by the Jacobi endomorphism, the curvature tensor and their derivatives. We were able to improve these results: using a differential algebraic characterization of connections and derivations we defined a graded Lie-algebra associated in a natural way with the SODE. It contains algebraic conditions on the variational multiplier and obstructions to the existence of a variational principle (Theorem 1.3.5). This concept is of particular interest when the dimension of the base manifold is large, because we are able to obtain new information about the structure of the obstructions (Corollary 1.3.7 and 1.3.9).

The questions *how many essentially different Lagrange functions* can be associated with a SODE and *how to determine* this number in terms of geometric objects and quantities are relevant, because the answers can lead to a better understanding of the geometry of the geodesic structure. In Section 1.4 we investigate the above questions. We introduce the notion of variational freedom, denoted by $\mathcal{V}_S$, which shows how many different variational principles can be associated to a spray $\mathcal{S}$ or, in other words, how many essentially different regular Lagrange functions exist for a given spray. In general, a spray $\mathcal{S}$ is non-variational, therefore $\mathcal{V}_S = 0$. For most of the variational cases, there is an essentially unique variational principle admitting $\mathcal{S}$ as a solution, that is $\mathcal{V}_S = 1$. It may also happen that $\mathcal{V}_S > 1$, that is, there exist $\mathcal{V}_S$ essentially different Lagrange functions and variational principle associated to $\mathcal{S}$. A particularly interesting case when the Lagrange functions are 2-homogeneous. This is the case for example in general relativity, in Riemannian and Finslerian geometries. This motivates the problem to investigate the freedom of $h(2)$-variationality when the Lagrange function must be 2-homogeneous. We show that in the regular case, the holonomy distribution can be used to determine $\mathcal{V}_{S,2}$ and we give an explicit formula to calculate it.

In Section 1.5 we consider an invariant version of the inverse problem: determine whether an equation of motion possessing some symmetry property can be derived as the Euler-Lagrange equation of a regular Lagrangian having the same symmetry. This investigation is motivated by the fact that the Euler-Lagrange equation inherits the symmetries of the Lagrangian. There are interesting examples of this phenomenon in the cases of motions on Lie groups, governed by the canonical symmetric linear connection [26]. We say that there exists an invariant variational principle for the SODE if it is variational with respect to a left-invariant regular Lagrange function. Interestingly, all possible situations can occur: there are examples of Lie groups where 1) the canonical flow is not variational, 2) the canonical flow is variational but the Lagrange functions are not invariant with respect to the action of the group, 3) the canonical flow is variational and there are invariant Lagrange functions with respect to the action of the group. We give an effective necessary and sufficient condition for the existence of an invariant variational principle for the canonical flow (Theorem 1.5.3). Using this, we determine the Lie groups up to dimension four for which an invariant variational principle exists.

The results of this chapter are based on the papers [102, 113, 114, 115, 126].

1.2 Preliminaries

Throughout this paper M denotes an n-dimensional smooth manifold, $C^\infty(M)$ denotes the ring of real-valued smooth functions, $\mathfrak{X}(M)$ is the $C^\infty(M)$-module of vector fields on M, $\pi : TM \to M$ is the tangent bundle of M, $\mathcal{T}M = TM\backslash\{0\}$ is the slit tangent space. $VTM = \mathrm{Ker}\pi_*$ is the vertical sub-bundle of TTM. We denote by $\Lambda^k(M)$, $S^k(M)$ and $\Psi^k(M)$ the $C^\infty(M)$-modules of skew-symmetric, symmetric and vector valued k-forms respectively, and by $\Lambda^k_v(TM)$, $S^k_v(TM)$ and $\Psi^k_v(TM)$ the corresponding semi-basic $C^\infty(TM)$-modules. We consider $\Lambda(M) = \bigoplus_{k\in\mathbb{N}}\Lambda^k(M)$ the graded algebra of differential forms on M and $\Psi(M) = \bigoplus_{k\in\mathbb{N}}\Psi^k(M)$ for the graded algebra of vector-valued differential forms on M.

The Frölicher–Nijenhuis formalism

The Frölicher-Nijenhuis theory provides a complete description of the derivations of $\Lambda(M)$ with the help of vector-valued differential forms, for details we refer to [39]. The i_* and the d_* type derivations associated to a vector valued l-form L will be denoted by i_L and d_L. They can be introduced in the following way: if $L \in \Psi^l(M)$, then

$$i_L\omega(X_1, \dots, X_l) = \omega(L(X_1, \dots, X_l)),$$

where $X_1, \dots, X_l \in \mathfrak{X}(M)$, $\omega \in \Lambda^1(M)$. Furthermore, d_L is the commutator of the derivations i_L and d, that is

$$d_L := [i_L, d] = i_L d - (-1)^{l-1} d i_L.$$

We remark that for $X \in \mathfrak{X}(M)$ we have $d_X = \mathcal{L}_X$ the Lie derivative, and i_X is the substitution operator. The *Frölicher–Nijenhuis bracket* of $K \in \Psi^k(M)$ and $L \in \Psi^l(M)$ is the unique $[K, L] \in \Psi^{k+l}$ form, such that

$$[d_K, d_L] = d_{[K,L]}.$$

In the special case, when $K \in \Psi^1(M)$, $X, Y \in \mathfrak{X}(M)$ we have $[K, X] \in \Psi^1(M)$ defined as

$$[K, X](Y) = [KY, X] - K[Y, X].$$

Spray and associated geometric quantities

Let $J \colon TTM \longrightarrow TTM$ be the *vertical endomorphism* and $C \in \mathfrak{X}(TM)$ the *Liouville vector field*. In an induced local coordinate system (x^i, y^i) on TM we have

$$J = dx^i \otimes \frac{\partial}{\partial y^i}, \qquad C = y^i \frac{\partial}{\partial y^i}.$$

Euler's theorem for homogeneous functions implies that $L \in C^\infty(TM)$ is a k-homogeneous function in the $y = (y^1, \ldots, y^n)$ variables if and only if

$$y^i \frac{\partial L}{\partial y^i} - L^k = 0. \tag{1.1}$$

The vertical endomorphism satisfies the following properties:

$$J^2 = 0, \qquad \operatorname{Ker} J = \operatorname{Im} J = VTM, \qquad [J, C] = J. \tag{1.2}$$

A *semispray* is a vector field $\mathcal{S}$ on $\mathcal{TM}$ satisfying the relation $J\mathcal{S} = C$. A semispray is called *spray* if $[C, \mathcal{S}] = \mathcal{S}$. The coordinate representation of a spray $\mathcal{S}$ takes the form

$$\mathcal{S} = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i}, \tag{1.3}$$

where the functions $G^i(x, y)$ are homogeneous of degree 2 in y.

The *geodesics of a spray* are curves $\gamma : I \to M$ such that $\mathcal{S} \circ \dot{\gamma} = \ddot{\gamma}$. Locally, they are the solutions of the second order ordinary differential equation (SODE)

$$\ddot{x}^i = -2G^i(x, \dot{x}), \qquad i = 1, \ldots, n. \tag{1.4}$$

Sprays describe a global and coordinate free way the systems of second order differential equations.

To every spray $\mathcal{S}$ a *connection* $\Gamma := [J, \mathcal{S}]$ can be associated [46]. Γ is called the *natural connection* associated to $\mathcal{S}$. One has $\Gamma^2 = \mathrm{Id}$. The eigenspace of Γ corresponding to the eigenvalue -1 is the *vertical space* VTM, and the eigenspace corresponding to $+1$ is called the *horizontal space*. For any $z \in TM$, we have $T_zTM = H_zTM \oplus V_zTM$. The horizontal and vertical projectors are denoted by h and v. One has

$$h = \frac{1}{2}(\mathrm{Id} + \Gamma), \qquad v = \frac{1}{2}(\mathrm{Id} - \Gamma). \tag{1.5}$$

Locally, the above two projectors can be expressed as $h = \frac{\delta}{\delta x^i} \otimes dx^i$, and $v = \frac{\partial}{\partial y^i} \otimes \delta y^i$, where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - G^j_i(x, y) \frac{\partial}{\partial y^j}, \quad \delta y^i = dy^i + G^i_j(x, y) dx^j, \tag{1.6}$$

$G^j_i = \frac{\partial G^j}{\partial y^i}$. In the sequel, $\mathbb{F} \in \Psi^1(TM)$ denotes the almost complex structure associated to the connection Γ.

The *parallel translation* of a vector along curves is defined through horizontal lifts. Let $\gamma\colon [0,1] \to M$ be a curve such that $\gamma(0) = p$ and $\gamma(1) = q$. The parallel translation $\tau\colon T_pM \to T_qM$ along γ is defined as follows: if γ^h is the horizontal lift of γ (ie. $\dot{\gamma}^h(t) \in HTM$ and $\pi \circ \gamma^h = \gamma$) with $\gamma^h(0) = v$, then $\tau(v) = w$, where $\gamma^h(1) = w$. More details about the parallel translation can be found in Section 3.2.

The *Berwald connection* $\mathcal{D} : \mathfrak{X}(\mathcal{T}M) \times \mathfrak{X}(\mathcal{T}M) \to \mathfrak{X}(\mathcal{T}M)$ is a linear connection on $\mathcal{T}M$ defined as follows:

$$\mathcal{D}_X Y = v[hX, vY] + h[vX, hY] + J[vX, (\mathbb{F}+J)Y] + (\mathbb{F}+J)[hX, JY]. \tag{1.7}$$

Using formula (1.7), it follows that $\mathcal{D}h = 0$ and $\mathcal{D}v = 0$, which means that the Berwald connection preserves both the horizontal and vertical distribution. Moreover, we have $\mathcal{D}J = 0$, which implies that the Berwald connection has the same action on horizontal and vertical vector fields. Considering the (h, v, v) components of the classical curvature of the Berwald connection we obtain a tensor-field

$$\mathcal{B}(X, Y, Z) = \mathcal{D}_{hX}\mathcal{D}_{JY} JZ - \mathcal{D}_{JY}\mathcal{D}_{hX} JZ - \mathcal{D}_{[hX,JY]} JZ \tag{1.8}$$

called the *Berwald curvature.* Locally $\mathcal{B}_{(x,y)} = \mathcal{B}^i_{jkl}(x, y)dx^j \otimes dx^k \otimes dx^l \otimes \frac{\partial}{\partial x^i}$ where

$$\mathcal{B}^i_{jkl}(x, y) = \frac{\partial G^i_{jk}}{\partial y^l} = \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}. \tag{1.9}$$

It is identically zero if and only if the connection Γ is linear. The *mean Berwald curvature tensor* field $\mathcal{B}_{(x,y)} = \mathcal{B}_{jk}(x, y)dx^j \otimes dx^k$ is the trace

$$\mathcal{B}_{jk}(x, y) = \mathcal{B}^l_{jkl}(x, y) = \frac{\partial^3 G^l}{\partial y^j \partial y^k \partial y^l}. \tag{1.10}$$

The *curvature* of the nonlinear connection Γ is $R = \frac{1}{2}[h, h]$, the Nijenhuis torsion of the horizontal projection h. The curvature tensor

$$R(X, Y) = v\big[\, hX, hY \,\big], \tag{1.11}$$

for $X, Y \in \mathfrak{X}(TM)$, characterizes the integrability of the horizontal distribution HTM: it is integrable, if and only if the curvature is identically zero.

The *Jacobi endomorphism* (or Riemann curvature in [82]) is defined as

$$\Phi = i_{\mathcal{S}} R. \tag{1.12}$$

The Jacobi endomorphism determines the curvature by the formula $R = \frac{1}{3}[J, \Phi]$. The spray $\mathcal{S}$ is called *flat* if its Jacobi endomorphism has the form $\Phi = \rho J$ and *isotropic* if

$$\Phi = \rho J - \alpha \otimes C \tag{1.13}$$

with some $\lambda \in C^\infty(\mathcal{T}M)$, $\alpha \in \Lambda^1_v(\mathcal{T}M)$. We consider also the *Ricci curvature Ric*, and the *Ricci scalar* ρ, [10], [82, Def. 8.1.7], which are given by

$$\mathrm{Ric} = (n - 1)\rho = R^i_i = \mathrm{Tr}(\Phi). \tag{1.14}$$

The Euler–Lagrange partial differential equation

A *Lagrangian* is a function $E\colon TM \to \mathbb{R}$ smooth on $\mathcal{T}M$ and C^1 on the zero section. E is called *regular*, if the Euler–Poincaré 2-form $\Omega_E := dd_J E$ has maximal rank. If $\mathcal{L}_C E = 2E$, and E is $\mathcal{C}^2$ on the 0-section, then E is quadratic and it defines a *(pseudo)-Riemannian* metric on M by $g(v,v) = 2E(v)$, $v \in TM$. If $\mathcal{L}_C E = 2E$ and E is $\mathcal{C}^1$ on the null-section, then E defines a *Finsler* structure. Note that from (1.1) we find that $i_J dd_J = d_J^2 = d_{[J,J]} = 0$, so for every Lagrangian E we have

$$i_J \Omega_E = 0. \tag{1.15}$$

A regular Lagrangian E allows us to define a pseudo-Riemannian metric on the vertical bundle, by putting $g_E(JX, JY) = \Omega_E(JX, Y)$. The local expression of the 2-form Ω_E is

$$\Omega_E = \frac{1}{2}\Big(\frac{\partial^2 E}{\partial x^\alpha \partial y^\beta} - \frac{\partial^2 E}{\partial x^\beta \partial y^\alpha}\Big)\, dx^\alpha \wedge dx^\beta \; - \; \frac{\partial^2 E}{\partial y^\alpha \partial y^\beta}\, dx^\alpha \wedge \; dy^\beta, \tag{1.16}$$

and the Lagrangian E is regular if and only if

$$\det\Big(\frac{\partial^2 E}{\partial y^\alpha \partial y^\beta}\Big) \neq 0.$$

Let $E : TM \to \mathbb{R}$ be a regular Lagrangian. From [41] we know that the vector field $\mathcal{S}$ on TM defined by

$$i_{\mathcal{S}} \Omega_E = d(E - \mathcal{L}_C E) \tag{1.17}$$

is a spray and the paths of $\mathcal{S}$ are the solutions to the Euler-Lagrange equations:

$$\frac{d}{dt}\frac{\partial E}{\partial \dot{x}^i} - \frac{\partial E}{\partial x^i} = 0, \qquad \alpha = 1, \dots, n. \tag{1.18}$$

That motivates the following

Definition 1.2.1. A spray $\mathcal{S}$ is called *variational* if there exists a smooth regular Lagrangian E which satisfies (1.17), the Euler-Lagrange equation.

When a regular Lagrangian E is given, (1.17), resp. (1.18), is a second order *ordinary* differential system. On the other hand, when the spray $\mathcal{S}$ is given, then (1.17), resp. (1.18), is a second-order *partial* differential system on the Lagrange function E. We introduce the following

Definition 1.2.2. Let E be a Lagrangian and $\mathcal{S}$ a spray on the manifold M, then the *Euler-Lagrange form* associated with E and $\mathcal{S}$ is

$$\omega_E := i_{\mathcal{S}} \Omega_E + d\mathcal{L}_C E - dE. \tag{1.19}$$

It is easy to see that ω_E is semi-basic, and the local expression in the standard coordinate system on TM is $\omega_E = \sum_{i=1}^n \Big[\mathcal{S}\Big(\frac{\partial E}{\partial y^i}\Big) - \frac{\partial E}{\partial x^i}\Big]\, dx^i$. Therefore, along a curve $\gamma = (x(t))$ associated with $\mathcal{S}$ we have

$$\omega_E\big|_\gamma = \sum_{i=1}^n \left(\frac{d}{dt}\frac{\partial E}{\partial \dot{x}^i} - \frac{\partial E}{\partial x^i}\right) dx^i, \tag{1.20}$$

where d/dt denotes the derivation along γ. One can recognize that the coefficients of the form (1.20) are the left hand sides of the Euler-Lagrange system (1.18). Consequently, in order to find the solution to the inverse problem for a given second order ordinary differential system, we have to look for a regular Lagrangian such that

$$\omega_E = 0, \tag{1.21}$$

or, using local coordinate system:

$$y^j \frac{\partial^2 E}{\partial x^j \partial y^i} - 2G^j(x,y)\frac{\partial^2 E}{\partial y^j \partial y^i} - \frac{\partial E}{\partial x^i} = 0, \qquad i = 1, \dots, n. \tag{1.22}$$

Conclusion 1.2.3. To solve the inverse problem of the calculus of variations for a given system of second-order ordinary differential equations or spray, one has to find a regular Lagrangian E such that it solves the Euler–Lagrange partial differential equation (1.21).

1.3 Algebraic conditions on the variational multiplier

One of the most important contribution to the solution of the inverse problem of the calculus of variations is a paper of J. Douglas [36], where in the two-dimensional case, he classifies systems of variational differential equations of second order. He showed that the Euler-Lagrange partial differential system (1.21) associated with the system of second-order differential equations (1.4) is equivalent to the first order partial differential system

$$\begin{aligned} \frac{d}{dt} g_{ij} + \frac{\partial G^k}{\partial y^j} g_{ik} + \frac{\partial G^k}{\partial y^i} g_{jk} &= 0, \\ A^k_j g_{ik} - A^k_i g_{jk} &= 0, \\ \frac{\partial g_{ij}}{\partial y^k} - \frac{\partial g_{ik}}{\partial y^j} &= 0, \\ g_{ij} - g_{ji} &= 0, \\ \det(g_{ij}) &\neq 0, \end{aligned} \tag{1.23}$$

where the unknown functions are g_{ij}, $i, j = 1, \dots, n$, and A^i_j are the components of the Jacobi endomorphism. A solution E of (1.21) gives a solution of (1.23) by taking

$$g_{ij} = \frac{\partial^2 E}{\partial y^i \partial y^j} \tag{1.24}$$

and conversely, for every solution of (1.23) there exists a regular solution E of (1.21) so that (1.24) holds. A solution of (1.23) is called *variational multiplier*.

In this section, using a differential algebraic characterization of connections and derivations, we present a graded Lie-algebra associated in a natural way with the SODE. It contains algebraic conditions on the variational multipliers and gives information about the structure of the obstruction to the existence of a variational principle.

Proposition 1.3.1. *[113, Property 6.] Let E be a Lagrangian on the manifold M. Then $d_J\omega_E = i_\Gamma\Omega_E$. Consequently, if the spray $\mathcal{S}$ is variational and E is a Lagrangian associated to $\mathcal{S}$, then the horizontal distribution associated to the spray $\mathcal{S}$ must be Lagrangian with respect to the symplectic 2-form Ω_E.*

Proof. The Euler-Lagrange form can be written in the following form:

$$\omega_E = i_{\mathcal{S}}dd_JE + d\mathcal{L}_{\mathcal{S}}E - dE = d_J\mathcal{L}_{\mathcal{S}}E - i_{[J,\mathcal{S}]}dE = d_J\mathcal{L}_{\mathcal{S}}E - 2d_hE.$$

Since the vertical distribution is integrable, we get $[J,J]=0$ and $d_J\circ d_J = d_{[J,J]}=0$, so

$$d_J\omega_E = -2d_Jd_hE = 2d_hd_JE = 2(i_hdd_JE - di_hd_jE) = 2i_h\Omega_E - 2\Omega_E = i_\Gamma\Omega_E.$$

If the spray is variational and E is a Lagrangian associated with $\mathcal{S}$, we have $\omega_E=0$, then $i_\Gamma\Omega_E=0$, so the connection associated to the spray is Lagrangian. □

Let $\mathcal{S}$ be a spray on M, h be the horizontal projection associated to the connection $\Gamma=[J,\mathcal{S}]$, and $L\in\Psi_v(TM)$. We introduce the semi-basic derivative of L with respect to the spray $\mathcal{S}$ as

$$L' := h^*\left(v[\mathcal{S},L]\right), \tag{1.25}$$

and the semi-basic derivative of L with with respect to h:

$$d^hL := [h,L]. \tag{1.26}$$

Proposition 1.3.2. *[113, Proposition 8.] Let $\mathcal{S}$ be a spray on M and L a semi-basic vector valued 1-form. We have the formula*

$$L' = [\mathcal{S},L] + \mathbb{F}L - L\overline{\wedge}\mathbb{F}, \tag{1.27}$$

where $\mathbb{F} = h[\mathcal{S},h]-J$ is the almost complex structure associated to Γ. In particular, suppose that $\mathcal{S}$ is variational, E being a Lagrangian associated to $\mathcal{S}$. If the equation $i_L\Omega_E=0$ holds, then the equations

$$i_{L'}\Omega_E = 0, \quad i_{L''}\Omega_E = 0, \quad i_{L'''}\Omega_E = 0, \quad etc. \tag{1.28}$$

hold too.

Proof. To show the first formula, we note that

$$L'(X_1,...,X_l)=v[\mathcal{S},L](hX_1,...,hX_l)=v[\mathcal{S},L(X_1,...,X_l)]-\sum_{i=1}^{l}L(X_1,...,[\mathcal{S},hX_i],...,X_l)$$
$$=[\mathcal{S},L(X_1,...X_l)]-h[\mathcal{S},L(X_1,...X_l)]-\sum_{i=1}^{l}L(X_1,...[\mathcal{S},h]X_i,...X_l)-\sum_{i=1}^{l}L(X_1,...[\mathcal{S},X_i],...X_l)$$
$$=[\mathcal{S},L](X_1,...X_l)+\mathbb{F}L(X_1,...,X_l)-\sum_{i=1}^{l}L(X_1,...,h[\mathcal{S},h]X_i,...,X_l).$$

Using the identity $h[\mathcal{S},h]=\mathbb{F}+J$ and the hypothesis that L is semi-basic, we obtain (1.27). Secondly, by the formula (1.27) we have

$$\begin{aligned}i_{L'}\Omega &= i_{[\mathcal{S},L]}\Omega+i_{\mathbb{F}L}\Omega-i_{\mathbb{F}\overline{\wedge}}\Omega=i_{[\mathcal{S},L]}\Omega+i_{\mathbb{F}}i_L\Omega-i_Li_{\mathbb{F}}\Omega\\ &=\mathcal{L}_{\mathcal{S}}i_L\Omega-d_L\omega+i_{\mathbb{F}}i_L\Omega-i_Li_{\mathbb{F}}\Omega.\end{aligned}$$

When $\mathcal{S}$ is variational and the function E is a Lagrangian associated to $\mathcal{S}$, then $\omega_E=0$ and the connection Γ is Lagrangian, so we have $i_{\mathbb{F}}\Omega_E=0$. If the equation $i_L\Omega_E=0$ holds, we have also $i_{L'}\Omega_E=0$ and recursively we obtain (1.28).

□

Proposition 1.3.3. *[113, Proposition 10.] Let L be a semi-basic vector valued l-form. Then d^hL is semi-basic. Moreover assume that $\mathcal{S}$ is variational, and E is a Lagrangian associated to $\mathcal{S}$. If the equation $i_L\Omega_E=0$ holds, then the equation $i_{d^hL}\Omega_E=0$ holds too.*

Proof. It is not difficult to check that if L is a semi-basic vector valued l-form, then d^hL is also semi-basic. Let us show the second part of the proposition. Let us assume that $\mathcal{S}$ is variational, E is a Lagrangian associated to $\mathcal{S}$, and L is a vector-valued semi-basic l-form. By the relation

$$(-1)^l i_{[h,L]}=i_hd_L-d_Li_h-d_{L\overline{\wedge}h}$$

and taking into account that $L\overline{\wedge}h=l\,L$, because L is semi-basic, we have

$$(-1)^l i_{d^hL}\Omega_E=(-1)^l i_{[h,L]}dd_JE=i_hd_Ldd_JE-d_Li_hdd_JE-l\,d_Ldd_JE.$$

If the equation $i_L\Omega_E=0$ holds, then

$$\begin{aligned}(-1)^l i_{d^hL}\Omega_E &= i_hdi_Ldd_JE-d_Li_{\frac{1}{2}(I+\Gamma)}dd_JE-l\,di_Ldd_JE\\ &=-l\,di_Ldd_JE-\frac{1}{2}d_Li_\Gamma dd_JE=0.\end{aligned}$$

□

Using the operations (1.25) and (1.26) we introduce the following

Definition 1.3.4. [113, Definition 11.] The graded Lie algebra $\mathcal{A}_{\mathcal{S}}$ associated to the spray $\mathcal{S}$ is the graded Lie sub-algebra of the vector-valued forms spanned by the vertical endomorphism J, the Jacobi endomorphism $\Phi = v[h, \mathcal{S}]$, and generated by the action of the semi-basic derivations with respect the spray $\mathcal{S}$ and h defined by formula (1.25) and (1.26) respectively, and by the Frölicher-Nijenhuis bracket [,].

$\mathcal{A}_{\mathcal{S}}$ is a graded Lie sub-algebra of the vector-valued semi-basic forms. The gradation of $\mathcal{A}_{\mathcal{S}}$ is given by

$$\mathcal{A}_{\mathcal{S}} = \oplus_{k=1}^{n} \mathcal{A}_{\mathcal{S}}^{k} \tag{1.29}$$

where $\mathcal{A}_{\mathcal{S}}^{k} := \mathcal{A}_{\mathcal{S}} \cap \Psi^{k}(TM)$. Its importance is given by the following:

Theorem 1.3.5. *[113, Theorem 2.] Let $\mathcal{S}$ be a variational spray and E a Lagrangian associated to $\mathcal{S}$. Then for every element L of $\mathcal{A}_{\mathcal{S}}$ the equation*

$$i_L \Omega_E = 0 \tag{1.30}$$

holds. Therefore every element of $\mathcal{A}_{\mathcal{S}}$ gives an algebraic condition on the variational multiplier.

Proof. To prove the theorem we will show that in the case when $\mathcal{S}$ is variational and E is a Lagrangian associated to $\mathcal{S}$, then J and Φ satisfy the equation (1.30), and the operations generating $\mathcal{A}_{\mathcal{S}}$ preserve this property. Indeed, (1.30), that is

i.) from $[J, J] = 0$ we can easily obtain: $i_J \Omega_E = i_J dd_J E = d_J^2 E = d_{[J,J]} E = 0$,

ii.) for the Jacobi endomorphism we have

$$\begin{aligned} i_\Phi \Omega_E &= i_{[h,\mathcal{S}]} \Omega_E + i_{\mathbb{F}} \Omega_E = i_h \mathcal{L}_{\mathcal{S}} \Omega_E - \mathcal{L}_{\mathcal{S}} i_h \Omega_E + i_{\mathbb{F}} \Omega_E \\ &= i_h d\omega_E - \mathcal{L}_{\mathcal{S}}(\Omega_E + \frac{1}{2} d_J \omega_E) + i_{\mathbb{F}} \Omega_E \\ &= i_h d\omega_E - d\omega_E - \frac{1}{2} \mathcal{L}_{\mathcal{S}} d_J \omega_E + i_{\mathbb{F}} \Omega_E = d_h \omega_E - \frac{1}{2} \mathcal{L}_{\mathcal{S}} d_J \omega_E + i_{\mathbb{F}} \Omega_E \end{aligned}$$

when $\mathcal{S}$ is variational and E is a Lagrangian associated to $\mathcal{S}$, then $\omega_E = 0$ and the connection Γ is Lagrangian. Therefore every term vanishes, and the equation (1.30) also holds for $\Phi = L$.

iii.) from Propositions 1.3.2, and 1.3.3 respectively we know that if $i_L \Omega_E = 0$ holds for $L \in \mathcal{A}_{\mathcal{S}}$ then $i_{L'} \Omega_E = 0$, and $i_{d^h L} \Omega_E = 0$ hold too.

iv.) Let $K, L \in \mathcal{A}_{\mathcal{S}}^{l}(TM)$ be semi-basic vector-valued forms, such that $i_K \Omega_E = 0$ and $i_L \Omega_E = 0$. Since K and L are semi-basic, we have $L \overline{\wedge} K \equiv 0$ and hence

$$\begin{aligned} (-1)^l \, i_{[K,L]} \Omega_E &= \left(i_K d_L - (-1)^{l(m-1)} d_L i_K - d_{L \overline{\wedge} K} \right) \Omega_E \\ &= i_K (i_L d - d i_L) \, dd_J E - (-1)^{l(m-1)} d_L i_K dd_J E - d_{L \overline{\wedge} K} dd_J E \\ &= i_K d i_L \Omega_E - (-1)^{l(m-1)} d_L i_K \Omega_E = 0. \end{aligned}$$

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Moreover, it is easy to see that (1.30) gives algebraic condition on the variational multiplier. Indeed, from the local expression (1.16) of Ω_E we get that if $L \in \Psi^l(TM)$ is semi-basic, then

$$i_L\Omega_E = \frac{1}{l!}\sum_{i\in\mathfrak{S}_{l+1}} \varepsilon(i) L^j_{i_1\dots i_l}\frac{\partial^2 E}{\partial y^j \partial y^{i_{l+1}}} dx^{i_1}\wedge\cdots\wedge dx^{i_{l+1}},$$

where $\mathfrak{S}_{p+l-1}$ denotes the $(p+l-1)!$-order symmetric group and $\varepsilon(i)$ the sign of $i=(i_1,\dots,i_l)$. Then the equation $i_L\Omega_E = 0$ is an algebraic equation

$$\sum_{i\in\mathfrak{S}_{l+1}} \varepsilon(i)\, L^j_{i_1\dots i_l}\, g_{ji_{l+1}} = 0 \tag{1.31}$$

in terms of the variational multiplier $g_{jk} = \frac{\partial^2 E}{\partial y^j \partial y^k}$. □

Corollary 1.3.6. *If at $x\in TM$ one has* $\operatorname{rank}\left\{J,\, \Phi,\, \Phi', \dots, \Phi^{(k)}, \dots\right\} \geq \frac{n(n+1)}{2}$, *then $\mathcal{S}$ is not variational in the neighborhood of x.*

Proof. Let us suppose that $\mathcal{S}$ is variational, E is an associated regular Lagrangian, and $g_{ij} = \dfrac{\partial^2 E}{\partial y^i \partial y^j}$ is a variational multiplier. For every $L \in \mathcal{A}^1_{\mathcal{S}}$ the condition $i_L\Omega_E = 0$ gives

$$g_{ik}L^k_j = g_{jk}L^k_i.$$

i.e. the tensor L is symmetric with respect to g_{ij}. Since the tensors J, Φ, Φ', Φ'', ... are elements of $\mathcal{A}_{\mathcal{S}}$, we have $i_{\Phi^{(k)}}\Omega_E = 0$ for all $k\in\mathbb{N}$. Therefore, if the spray is variational, then the tensors J, Φ, Φ', Φ'', ..., are self-adjoint with respect to g_{ij}. The space of the $(1,1)$-tensors which are self-adjoint with respect to a regular matrix is $\frac{n(n+1)}{2}$–dimensional. Consequently, if the spray is variational, then J, Φ, Φ', ..., $\Phi^{(1/2)n(n+1)-1}$ are linearly dependent. □

If $\dim M = 2$, then $\mathcal{A}_{\mathcal{S}}$ only contains J, Φ and the hierarchy given by its semi-basic derivatives Φ', Φ''. However, if $\dim M > 2$, then we can find higher order derivatives and other hierarchies in $\mathcal{A}_{\mathcal{S}}$ which give, in the generic case, new necessary conditions for the variational multipliers. We arrive at the following generalization of Corollary 1.3.6:

Corollary 1.3.7. *If there exists an integer $k \leq n$ for which* $\dim \mathcal{A}^k_{\mathcal{S}}(x) \geq k\binom{n+1}{k+1}$, *then the spray is not variational.*

Definition 1.3.8. Let $\mathcal{S}$ be a spray $x\in TM$, and let us consider the system of linear equations

$$\Big\{ \sum_{i\in\mathfrak{S}_{l+1}} \varepsilon(i)\, L^j_{i_1\dots i_l} g_{ji_{l+1}} = 0 \;\Big|\; L\in\mathcal{A}_{\mathcal{S}}(x) \Big\} \tag{1.32}$$

where $L^j_{i_1\dots i_l}$ are the components of $L\in\mathcal{A}_{\mathcal{S}}(x)$ and g_{ij} are the symmetric ($g_{ij}=g_{ji}$) unknowns. The rank of the linear equations (1.32) is called the *rank of the spray* at $x\in TM$.

As equation (1.31) shows, the rank of a spray gives the number of independent equations satisfied by the variational multipliers. Consequently, if the system (1.32) does not have a solution with $\det(g_{ij}) \neq 0$, then there is no variational multiplier for $\mathcal{S}$, and therefore the spray is non-variational. Thus we arrive at

Theorem 1.3.9. *[113, Theorem 5.] If at $x \in TM$ the rank of the spray $\mathcal{S}$ is greater or equal with $\frac{n(n+1)}{2}$, then $\mathcal{S}$ cannot be variational.*

1.4 Freedom of variationality: the homogeneous case

In this section we are considering a different aspect of the inverse problem of the calculus of variations which is motivated by the fact that there are sprays for which

a) there is no regular Lagrange function, that is, the spray is not variational,
b) there is an essentially unique Lagrange function,
c) there are several essentially different regular Lagrange functions.

The questions of *how many different Lagrange functions* can be associated with a spray and *how to determine* this number in terms of geometric objects are very interesting because the answers can lead to a better understanding of the geodesic structure. In this section we investigate the above questions by considering the Euler-Lagrange partial differential system associated to sprays: We introduce the notion of *variational freedom*, denoted by $\mathcal{V}_S$, which shows how many different variational principle can be associated to the spray or in other words, how many essentially different regular Lagrange functions exist for a given spray.

To formulate properly the notion of variational freedom, we introduce the following terminology and notations. The solutions of the Euler-Lagrange partial differential equation (1.21), or locally (1.22), are called *Euler-Lagrange functions* of the spray $\mathcal{S}$. The set of Euler-Lagrange functions of $\mathcal{S}$ will be denoted by $\mathcal{E}_{\mathcal{S}}$, the subset of k-homogeneous Euler-Lagrange functions will be denoted by $\mathcal{E}_{\mathcal{S},k}$:

$$\mathcal{E}_{\mathcal{S}} = \{\, E \in C^\infty(\mathcal{T}M) \mid \omega_E = 0 \,\}, \tag{1.33}$$
$$\mathcal{E}_{\mathcal{S},k} = \{\, E \in C^\infty(\mathcal{T}M) \mid \omega_E = 0,\ \mathcal{L}_C E = kE \,\}, \quad k \in \mathbb{N}. \tag{1.34}$$

The spray $\mathcal{S}$ is variational (resp. $h(k)-$*variational*) if $\mathcal{E}_{\mathcal{S}}$ (resp. $\mathcal{E}_{\mathcal{S},k}$) contains a regular Lagrangian. Particularly interesting the $h(2)-$variational property (see for example Riemann and Finsler metrizability property, relativity theory, etc.).

If $\mathcal{S}$ is variational, then $\mathcal{V}_S := rank\,(\mathcal{E}_{\mathcal{S}})$ is called the *variational freedom*. If $\mathcal{S}$ is non-variational, then we set $\mathcal{V}_S = 0$. Remark that the notation $\mathcal{V}_S = rank\,(\mathcal{E}_{\mathcal{S}})$ means that $\mathcal{E}_{\mathcal{S}}$ can be locally generated by its $\mathcal{V}_S$ functionally independent elements. In other words, if the variational freedom of $\mathcal{S}$ is $\mathcal{V}_S \geq 1$ then for every $v_0 \in \mathcal{T}M$ there exists a neighbourhood $U \subset \mathcal{T}M$ and functionally independent $E_1, \dots, E_{\mathcal{V}_S} \in \mathcal{E}_{\mathcal{S}}$ on U such that any $E \in \mathcal{E}_{\mathcal{S}}$ can be expressed as

$$E(v) = \varphi\big(E_1(v), \dots, E_{\mathcal{V}_S}(v)\big), \qquad \forall\, v \in U,$$

with some function $\varphi\colon \mathbb{R}^{\mathcal{V}_S} \to \mathbb{R}$. An analogous way, $\mathcal{V}_{S,\,k} := rank\,(\mathcal{E}_{\mathcal{S},k})$, and we set $\mathcal{V}_{S,\,k} = 0$ if there is no regular element in $\mathcal{E}_{\mathcal{S},k}$. In particular, $\mathcal{V}_{S,\,2} = rank\,(\mathcal{E}_{\mathcal{S},2})$ shows how many different 2−homogeneous Lagrange functions, or variational principles, exist for the given spray. We are focusing our attention on this particular case.

Holonomy invariant functions

The *holonomy distribution* $\mathcal{D}_{\mathcal{H}}$ of a spray $\mathcal{S}$ is the distribution on TM generated by the horizontal vector fields and their successive Lie-brackets, that is

$$\mathcal{D}_{\mathcal{H}} := \Big\langle \mathfrak{X}^h(TM) \Big\rangle_{Lie} = \Big\{ [X_1, [\dots[X_{m-1}, X_m]...]] \;\big|\; X_i \in \mathfrak{X}^h(TM) \Big\}. \tag{1.35}$$

The holonomy distribution $\mathcal{D}_{\mathcal{H}}$ is the smallest involutive distribution containing the horizontal distribution $H\mathcal{T}M$. Using the horizontal and vertical projectors we have

$$\mathcal{D}_{\mathcal{H}} = h(\mathcal{D}_{\mathcal{H}}) \oplus v(\mathcal{D}_{\mathcal{H}}) = H\mathcal{T}M \oplus v(\mathcal{D}_{\mathcal{H}}).$$

The image of the curvature tensor is a subset of the vertical part of the holonomy distribution, that is $\operatorname{Im} R \subset v(\mathcal{D}_{\mathcal{H}})$. Moreover, we have $\mathcal{D}_{\mathcal{H}} = H\mathcal{T}M$ if and only if $R \equiv 0$. When $\mathcal{D}_{\mathcal{H}}$ is a regular distribution, then it is integrable. Using the definition of parallel translation via horizontal lifts, it is easy to see that the integral manifold through $v \in TM$ is the orbit $\mathcal{O}_\tau(v)$ of v with respect to all possible parallel translations. By the Frobenius integrability theorem one can find a coordinate system (U, z) of $\mathcal{T}M$ in a neighborhood of $v \in \mathcal{T}M$ such that the components of $\mathcal{O}_\tau \cap U$ are the sets

$$\{w \in U \mid z^i(w) = z^i_0,\, \dim \mathcal{O}_\tau + 1 \leq i \leq 2n\}, \qquad |z^i_0| < \epsilon. \tag{1.36}$$

We say that the *parallel translation is regular* if the distribution $\mathcal{D}_{\mathcal{H}}$ is regular and the orbits of the parallel translation are regular in the sense that for any $v \in \mathcal{T}M$ there is a neighbourhood $U \subset \mathcal{T}M$ such that any orbit $\mathcal{O}_\tau$ has at most one connected component in U. If the parallel translation is regular, then there exists a coordinate system (U, z) of $\mathcal{T}M$ in a neighborhood of any $v \in \mathcal{T}M$ such that in (1.36) different z^i coordinates $(\dim \mathcal{O}_\tau + 1 \leq i \leq 2n)$ correspond to different orbits of the parallel translation.

A function $E \in C^\infty(TM)$ is called *holonomy invariant*, if it is invariant with respect to parallel translation. The set of holonomy invariant functions will be denoted by $\mathcal{H}_{\mathcal{S}}$. In the case when the parallel translation is regular, the tangent spaces of its orbits are given by the holonomy distribution $\mathcal{D}_{\mathcal{H}}$, that is $T_v\big(O_\tau(v)\big) = \mathcal{D}_{\mathcal{H}}(v)$. Consequently, $E \in C^\infty(TM)$ is a holonomy invariant function if and only if we have $\mathcal{L}_X E = 0$, $X \in \mathcal{D}_{\mathcal{H}}$ that is

$$\mathcal{H}_{\mathcal{S}} = \{E \in C^\infty(\mathcal{T}M) \mid \mathcal{L}_X E = 0,\; X \in \mathcal{D}_{\mathcal{H}}\}. \tag{1.37}$$

The subset of k-homogeneous holonomy invariant functions will be denoted by $\mathcal{H}_{\mathcal{S},k}$:

$$\mathcal{H}_{\mathcal{S},k} = \{\, E \in \mathcal{H}_{\mathcal{S}} \mid \mathcal{L}_C E = kE \,\}. \tag{1.38}$$

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Proposition 1.4.1. *[102, Lemma 4.3.] A 2-homogeneous Lagrangian is an Euler-Lagrange function of a spray $\mathcal{S}$ if and only if it is a holonomy invariant function. Using the notation* (1.34) *and* (1.38) *we have*

$$\mathcal{E}_{\mathcal{S},2} = \mathcal{H}_{\mathcal{S},2}. \tag{1.39}$$

Proof. Let $\mathfrak{h}\colon TTM \to \mathcal{D}_{\mathcal{H}}$ be an arbitrary projection on $\mathcal{D}_{\mathcal{H}}$. In [115, p. 86, Theorem 1.] it was proven that a 2-homogeneous Lagrange function $E\colon \mathcal{T}M \to \mathbb{R}$ is a solution of the Euler-Lagrange PDE if and only if it satisfies the equation

$$d_{\mathfrak{h}} E = 0, \tag{1.40}$$

where the $d_{\mathfrak{h}}$ operator is defined by the formula $d_{\mathfrak{h}} E(X) = \mathfrak{h}X(E) = \mathcal{L}_{\mathfrak{h}X} E$. Consequently (1.40) is satisfied if and only if E is a holonomy invariant function. □

Considering $\mathcal{E}_{\mathcal{S}}$ and $\mathcal{E}_{\mathcal{S},k}$ ($k \in \mathbb{N}$), we can observe that both are vector spaces over $\mathbb{R}$. In particular, the linear combination of 2-homogeneous Euler-Lagrange functions of $\mathcal{S}$ are also 2-homogeneous Euler-Lagrange functions of $\mathcal{S}$. We can consider such combination as a *trivial combination.* As the next proposition shows, a much wider combination of homogeneous Euler-Lagrange functions can produce new homogeneous Euler-Lagrange functions.

Proposition 1.4.2. *[102, Proposition 4.2.] A* 1*-homogeneous functional combination of* 2*-homogeneous Euler-Lagrange functions of a spray* $\mathcal{S}$ *is also a* 2*-homogeneous Euler-Lagrange functions of* $\mathcal{S}$*.*

Proof. Let $\varphi = \varphi(z_1, \dots, z_r)$ be a smooth 1-homogeneous function and consider the functional combination

$$E := \varphi\big(E_1, \dots, E_r\big) \tag{1.41}$$

of $E_1, \dots, E_r \in \mathcal{E}_{\mathcal{S},2}$, that is, 2-homogeneous Euler-Lagrange functions of a spray $\mathcal{S}$. It is clear, that E is also $2-$homogeneous. Moreover, using (1.2) we have $E_i \in \mathcal{H}_{\mathcal{S},2}$ and from (1.37) we get $\mathcal{L}_X E_i = 0$ for any vector field $X \in \mathcal{D}_{\mathcal{H}}$ in the holonomy distribution. Consequently, for $X \in \mathcal{D}_{\mathcal{H}}$ we have

$$\mathcal{L}_X E = \frac{\partial \varphi}{\partial z^1} \cdot \mathcal{L}_X E_1 + \dots + \frac{\partial \varphi}{\partial z^r} \cdot \mathcal{L}_X E_r = 0,$$

which shows that $E \in \mathcal{H}_{\mathcal{S},2}$ and from (1.2) we get $E \in \mathcal{E}_{\mathcal{S},2}$. □

Proposition 1.4.2 shows that functional combinations of Euler-Lagrange functions can result new variational principles for the spray. The following theorem can be used to determine, in terms of geometric quantities associated to the spray, how many essentially different variational principles exist for a given spray, that is what the $h(2)$-variational freedom is.

Theorem 1.4.3. *[102, Theorem 4.4] Let $\mathcal{S}$ be a metrizable spray such that the parallel translation with respect to the associated connection is regular. Then*

$$\mathcal{V}_{S,2} = \operatorname{codim} \mathcal{D}_{\mathcal{H}}. \tag{1.42}$$

To prove the theorem we need the following lemmas:

Lemma 1.4.4. *Let $\mathcal{S}$ be a spray and $E_o \in \mathcal{E}_{\mathcal{S},2}$ non-vanishing on $\mathcal{T}M$. Then E is a $2-$homogeneous Euler-Lagrange function of $\mathcal{S}$ if and only if $\theta := E/E_o$ is a $0-$homogeneous holonomy invariant function:*

$$E \in \mathcal{E}_{\mathcal{S},2} \quad \Longleftrightarrow \quad \theta = E/E_o \in \mathcal{H}_{\mathcal{S},0}.$$

Proof. Using Lemma 1.4.1 we obtain that both E and E_o are 2-homogeneous holonomy invariant functions. Thus, $\theta := E/E_o$ is a 0-homogeneous holonomy invariant function, that is, $\theta \in \mathcal{H}_{\mathcal{S},0}$. Conversely, assume that $\theta = E/E_o \in \mathcal{H}_{\mathcal{S},0}$. Then $E = \theta E_o$ is a $2-$homogeneous holonomy invariant function. By Proposition 1.4.1, E is an Euler-Lagrange function of the spray $\mathcal{S}$. □

Lemma 1.4.5. *The smallest involutive distribution $\mathcal{D}_{\mathcal{H},C} := \langle \mathcal{D}_{\mathcal{H}}, C \rangle_{Lie}$, containing $\mathcal{D}_{\mathcal{H}}$ and the Liouville vector field C is linearly generated by $\mathcal{D}_{\mathcal{H}}$ and C, that is,*

$$\langle \mathcal{D}_{\mathcal{H}}, C \rangle_{Lie} = Span\{\mathcal{D}_{\mathcal{H}}, C\}. \tag{1.43}$$

Proof. If $C \in \mathcal{D}_{\mathcal{H}}$ then $\mathcal{D}_{\mathcal{H},C} = \mathcal{D}_{\mathcal{H}}$ and (1.43) is true. If $C \notin \mathcal{D}_{\mathcal{H}}$, then considering $X, Y \in \mathcal{D}_{\mathcal{H},C}$ and using the decomposition $X = X_{\mathcal{D}_{\mathcal{H}}} + X_C$ and $Y = Y_{\mathcal{D}_{\mathcal{H}}} + Y_C$ corresponding to the directions $\mathcal{D}_{\mathcal{H}}$ and C we get

$$[X, Y] = [X_{\mathcal{D}_{\mathcal{H}}}, Y_{\mathcal{D}_{\mathcal{H}}}] + [X_C, Y_C] + [X_C, Y_{\mathcal{D}_{\mathcal{H}}}] + [X_{\mathcal{D}_{\mathcal{H}}}, Y_C]. \tag{1.44}$$

We have $[X_C, Y_C] \in Span\{C\}$ and $[X_{\mathcal{D}_{\mathcal{H}}}, Y_{\mathcal{D}_{\mathcal{H}}}] \in \mathcal{D}_{\mathcal{H}}$. Let us consider a local basis $\mathcal{B} = \left\{\frac{\delta}{\delta x^1}, \dots, \frac{\delta}{\delta x^n}\right\}$ of the horizontal space $H\mathcal{T}M$. Then the holonomy distribution $\mathcal{D}_{\mathcal{H}}$ can be generated locally by the elements of $\mathcal{B}$ and by their successive Lie brackets. We have $[C, \frac{\delta}{\delta x^i}] = 0$, and by the Jacobi identity, this is also true for the successive brackets of the $\frac{\delta}{\delta x^i}$'s. $Y_{\mathcal{D}_{\mathcal{H}}} \in \mathcal{D}_{\mathcal{H}}$ can be written as a linear combination of the elements $Y_{\mathcal{D}_{\mathcal{H}}} = g^\alpha Y_\alpha$, where $Y_\alpha \in \mathcal{D}_{\mathcal{H}}$ can be obtained by successive brackets of the $\frac{\delta}{\delta x^i}$'s, and therefore $[C, Y_\alpha] = 0$. Hence, for the C-directional component of X we have $X_C = X^c C$ with $X^c \in C^\infty(\mathcal{T}M)$ and $[X_C, Y_{\mathcal{D}_{\mathcal{H}}}] = (X_C g^\alpha) Y_\alpha - (Y_{\mathcal{D}_{\mathcal{H}}} X^c) C$ which is an element of $Span\{\mathcal{D}_{\mathcal{H}}, C\}$. The same argument is valid for the fourth term in (1.44).

□

Lemma 1.4.6. *If the spray $\mathcal{S}$ is metrizable then C is transverse to $\mathcal{D}_{\mathcal{H}}$ on $\mathcal{T}M$, that is*

$$Span\{\mathcal{D}_{\mathcal{H}}, C\} = \mathcal{D}_{\mathcal{H}} \oplus Span\{C\}. \tag{1.45}$$

Proof. If $\mathcal{S}$ is metrizable, then there exists a Finsler energy function $E_o \in \mathcal{E}_{\mathcal{S},2}$ of $\mathcal{S}$. Because of Proposition 1.4.1 we have $E_o \in \mathcal{H}_{\mathcal{S},2}$. On the other hand, by using the homogeneity property of E_o we have $\mathcal{L}_{C_v} E = 2E(v) > 0$ at any point $v \in \mathcal{T}M$. But the derivatives of E_o with respect to the elements of $\mathcal{D}_{\mathcal{H}}$ is zero. Therefore we obtain that $C \notin \mathcal{D}_{\mathcal{H}}$ at $v \in \mathcal{T}M$. □

Proof of Theorem 1.4.3. Let us denote by $\kappa(\in \mathbb{N})$ the rank of the distribution $\mathcal{D}_{\mathcal{H}}$. We will show that in a neighbourhood of a $v \in \mathcal{T}M$ one can find exactly $\operatorname{Codim} \mathcal{D}_{\mathcal{H}} = 2n - \kappa$ locally functionally independent elements in $\mathcal{E}_{\mathcal{S},2}$.

As the spray $\mathcal{S}$ is metrizable, therefore there exists a Finsler energy function $E_o \in \mathcal{E}_{\mathcal{S},2}$ associated to $\mathcal{S}$. From (1.43) and (1.45), we have $\mathcal{D}_{\mathcal{H},C} = \mathcal{D}_{\mathcal{H}} \oplus C$, therefore $\dim \mathcal{D}_{\mathcal{H},C} = \kappa + 1$. Both $\mathcal{D}_{\mathcal{H}}$ and $\mathcal{D}_{\mathcal{H},C}$ are involutive smooth distributions on $\mathcal{T}M$. By the Frobenius integrability theorem one can find a coordinate system (U, z) of $\mathcal{T}M$ in a neighborhood of $v_0 \in \mathcal{T}M$, such that $z^i(v) = 1$, $z(U) =]1-\epsilon, 1+\epsilon[^{2n}$ and for all $z_0^{\kappa+1}, \dots, z_0^{2n}$ with $|1-z_0^i| < \epsilon$, the sets

$$\mathcal{O}_\tau = \{w \in U \mid z^i(w) = z_0^i,\ \kappa+1 \leq i \leq 2n\}, \qquad \mathcal{N} = \{w \in U \mid z^i(w) = z_0^i,\ \kappa+2 \leq i \leq 2n\}$$

are integral manifolds of the distributions $\mathcal{D}_{\mathcal{H}}$ respectively $\mathcal{D}_{\mathcal{H},C}$ over U. Moreover, by the regularity of the parallel translation, the coordinate neighbourhood U can be choosen in such a way that for any $v \in U$ the orbit $\mathcal{O}_\tau(v)$ of v has only one component in U under the parallel translations. In this case, different z^i coordinates for $\kappa+1 \leq i \leq 2n$, correspond to different orbits, hence these coordinates parametrize the orbits of the parallel translations on U. Let

$$\mathcal{D}_{\mathcal{H}} = Span\left\{\frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^\kappa}\right\}, \qquad \mathcal{D}_{\mathcal{H},C} = Span\left\{\frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^\kappa}, \frac{\partial}{\partial z^{\kappa+1}}\right\}, \tag{1.46}$$

where $Span\left\{\frac{\partial}{\partial z^{\kappa+1}}\right\} = Span\{C\}$, that is, $\frac{\partial}{\partial z^{\kappa+1}} = \lambda C$, with $\lambda(v_0) \neq 0$. Hence we get

$$\frac{\partial E_o}{\partial z^{\kappa+1}}(v_0) = \lambda (C E_o)(v_0) = 2\lambda E_o(v_0) \neq 0. \tag{1.47}$$

Considering the set of 0−homogeneous holonomy invariant functions, we have

$$\theta \in \mathcal{H}_{\mathcal{S},0} \quad \Longleftrightarrow \quad \left\{ \begin{array}{l} \mathcal{L}_X \theta = 0,\ \forall\ X \in \mathcal{D}_{\mathcal{H}} \\ \mathcal{L}_C \theta = 0, \end{array} \right\} \quad \Longleftrightarrow \quad \mathcal{L}_X \theta = 0,\ \forall\ X \in \mathcal{D}_{\mathcal{H},C}. \tag{1.48}$$

From (1.46) and from (1.48), it follows that $\theta \in \mathcal{H}_{\mathcal{S},0}$ on U if and only if it is a function of the variables $z^{\kappa+2}, \dots, z^{2n}$, that is

$$\theta = \theta(z^{\kappa+2}, \dots, z^{2n}). \tag{1.49}$$

By using a convenient bump function ψ^i in each variable z^i $(\kappa + 2 \leq i \leq 2n)$, we obtain smooth functions $\theta_i := \psi^i \cdot z^i \in C^\infty(\mathcal{T}M)$ (no summation convention is used here), such that $\theta_i(v_0) = 1$, $\frac{d\theta_i}{dz^i}(v_0) = 1$ and $\operatorname{supp}(\theta_i) \subset U$. It is clear that

$$\theta_{\kappa+2},\ \dots\ , \theta_{2n} \tag{1.50}$$

are functionally independent 0−homogeneous holonomy invariant functions on some neighbourhood $\widetilde{U} \subset U$ of v_0 and any elements of $\mathcal{H}_{\mathcal{S},0}$ can be expressed on $\widetilde{U}$ as their functional combination. The functions (1.50) can be used to "*modify*" the original Euler-Lagrange function E_o to obtain new elements of $\mathcal{E}_{\mathcal{S},2}$, functionally independent on $\widetilde{U}$.

Indeed, let $E_i := (1+\theta_i)E_o$ for $\kappa+2 \leq i \leq 2n$, and set $E_{\kappa+1} := E_o$. Since $1+\theta_i$ are $0-$homogeneous and E_o is 2-homogeneous holonomy invariant functions we get that

$$E_{\kappa+1},\, E_{\kappa+2},\, \dots\, , E_{2n}, \tag{1.51}$$

are $2-$homogeneous holonomy invariant functions. Then, by Lemma 1.4.4, the elements of (1.51) are in $\mathcal{E}_{\mathcal{S},2}$. Moreover, by the construction we have

$$dE_i = d\big((1+\theta_i)E_o\big) = \frac{d\theta_i}{dz^i}E_o dz^i + (1+\theta_i)dE_o,$$

with no summation on i. Hence,

$$(dE_i)_{v_0} = (dz^i)_{v_0} + \big(1+\theta_i(v_0)\big)(dE_o)_{v_0}.$$

and taking (1.47) into account we get

$$\begin{aligned} dE_{\kappa+1} \wedge dE_{\kappa+2} \wedge \cdots \wedge dE_{2n}(v_0) &= \big(dE_o \wedge (dz^{\kappa+2} + \theta_{\kappa+2}dE_o) \wedge \cdots \wedge (dz^{2n} + \theta_{2n}dE_o)\big)_{v_0} \\ &= (dE_o \wedge dz^{\kappa+2} \wedge \cdots \wedge dz^{2n})_{v_0} \\ &= 2(\lambda E_o\, dz^{\kappa+1} \wedge dz^{\kappa+2} \wedge \cdots \wedge dz^{2n})_{v_0} \neq 0, \end{aligned}$$

that is, the functions (1.51) are functionally independent in some neighbourhood $\widehat{U} \subset \widetilde{U}$ of $v_0 \in \mathcal{T}M$.

On the other hand, let us suppose that $E \in \mathcal{E}_{\mathcal{S},2}$ is a $2-$homogeneous Euler-Lagrange function associated to $\mathcal{S}$. Using Lemma 1.4.4, we get that $\theta = E/E_o$ is a $0-$homogeneous holonomy invariant function. Then, θ has the form (1.49) on U and it can thus be expressed as a functional combination $\theta = \Psi(\theta_{\kappa+2}, \dots, \theta_{2n})$. Since $E_o = E_{\kappa+1}$ we get

$$E = \Psi\left(\frac{E_{\kappa+2}}{E_{\kappa+1}}, \dots, \frac{E_{2n}}{E_{\kappa+1}}\right) \cdot E_{\kappa+1}$$

showing that E is locally a functional combinations of the elements (1.51).

$\square$

1.5 Invariant variational principle for canonical flows on Lie groups

In the sequel we will consider the case, where the manifold $M := G$ is a Lie group. We will denote by $L_{\hat{g}}g$ or simply by $\hat{g}g$ the left translation of $g \in G$ by $\hat{g} \in G$. Let $(x^1, \dots, x^n) = (x)$ be local coordinates on G, and let (x, y) with $(y^1, \dots, y^n) = (y)$ be the standard associated coordinate system on TG. We will also use the semi-invariant coordinates (x, α) on $TG \simeq G \times \mathfrak{g}$, where $\alpha = (L_{x^{-1}})_* y$ is the Maurer-Cartan form. The corresponding coordinates on TTG are (x, α, X, A), that is,

$$(x, \alpha, X, A) = X\frac{\partial}{\partial x}\Big|_{(x,\alpha)} + A\frac{\partial}{\partial \alpha}\Big|_{(x,\alpha)}.$$

Since the coordinates $\alpha = (\alpha_i)$ and $A = (A_i)$ are left-invariant coordinates, we find that the left translation by a group element g induces on TTG the following action

$$L_g(x, \alpha, X, A) = (gx, \alpha, gX, A) = gX\frac{\partial}{\partial x}\Big|_{(gx,\alpha)} + A\frac{\partial}{\partial \alpha}\Big|_{(gx,\alpha)}.$$

The canonical projection $\pi : TG \to G$ is $(x, \alpha) \to x$, therefore $\pi_* : TTG \to TG$ is given by $(x, \alpha, X, A) \to (x, x^{-1}X)$ and the vertical subspace on $(x, \alpha) \in TG$ is

$$V_{(x,\alpha)}TG := \operatorname{Ker} \pi_* = \big\{(x, \alpha, 0, b) \,|\, b \in \mathfrak{g}\big\}.$$

On a Lie group the geodesic flow of the canonical connection is described by the system

$$\ddot{x} = \dot{x}x^{-1}\dot{x}, \tag{1.52}$$

and the vector field on the tangent space, corresponding to the geodesic flow of the canonical connection is the spray $\mathcal{S}$ where

$$\mathcal{S}_{(x,\alpha)} = (x, \alpha, x\alpha, 0) = x\alpha\frac{\partial}{\partial x}\Big|_{(x,\alpha)}. \tag{1.53}$$

Then γ_t is a geodesic of (1.53) if and only if the equation $\mathcal{S}_{\dot{\gamma}} = \ddot{\gamma}$ holds. Moreover, with (x, α) as a local coordinate system, the vertical endomorphism and the Liouville vector fields are

$$J = (x^{-1}dx) \otimes \frac{\partial}{\partial \alpha}, \qquad C = \alpha\frac{\partial}{\partial \alpha}.$$

For more details (calculation of the horizontal and the vertical projecion, curvature etc) see [126]. We have the following

Proposition 1.5.1. *[96, Proposition 4.3] A Lagrangian $E : TG \to \mathbb{R}$ is a left-invariant solution to the Euler-Lagrange equation associated to the canonical spray of the Lie group G, if and only if the system*

$$\frac{\partial E}{\partial x^i} = 0, \qquad i = 1, \dots, n \tag{1.54}$$

$$[a, \alpha]^i\,\frac{\partial E}{\partial \alpha^i} = 0, \qquad \forall a \in \mathfrak{g}, \tag{1.55}$$

is satisfied.

Proof. The Euler-Lagrange partial differential equation associated to a spray $\mathcal{S}$ can be written as (1.19) where the unknown is the Lagrangian E. If $X = (x, a)$ denotes a left-invariant vector field on G corresponding to $a \in \mathfrak{g}$, and X^v, X^h are its vertical and horizontal lifts, then we have $\omega_E(X^v) \equiv 0$, since ω_E is semi-basic. Moreover, we have

$$\begin{aligned}\omega_E(X^h) &= (i_{\mathcal{S}}dd_JE + d\mathcal{L}_CE - dE)(X^h) = dd_JE(\mathcal{S}, X^h) + d\mathcal{L}_CE(X^h) - dE(X^h)\\ &= \mathcal{S}\big(JX^h(E)\big) - X^h\big(J\mathcal{S}(E)\big) - J[\mathcal{S}, X^h]E + X^h\big(C(E)\big) - X^hE\\ &= \mathcal{S}\big(X^v(E)\big) - J[\mathcal{S}, X^h]E - X^hE.\end{aligned}$$

and in the Lie group case, using adapted coordinate system, one can find

$$\begin{aligned}\omega_E(X^h) &= x\alpha\frac{\partial}{\partial x}\big(a\frac{\partial}{\partial \alpha}(E)\big)-\tfrac{1}{2}[a,\alpha]\frac{\partial}{\partial \alpha}(E) - \big(xa\frac{\partial}{\partial x} - \tfrac{1}{2}[a,\alpha]\frac{\partial}{\partial \alpha}\big)E \\ &= [a,\alpha]\frac{\partial E}{\partial \alpha} + x\alpha a\frac{\partial^2 E}{\partial x \partial \alpha} - xa\frac{\partial E}{\partial x}.\end{aligned}$$

If the Lagrangian is left-invariant, then $\frac{\partial E}{\partial x} = 0$ and we obtain that

$$\omega(X^h) = [a,\alpha]\frac{\partial E}{\partial \alpha} = 0,$$

which completes the proof. □

We remark that equation (1.54) expresses the fact that E is left-invariant and (1.55) expresses that E is a solution of the Euler-Lagrange equation.

Corollary 1.5.2. *The canonical flow of the Lie group G is variational with respect to a left-invariant Lagrangian if and only if there exists an ad-invariant function $\mathcal{E} : \mathfrak{g} \to \mathbb{R}$ with nondegenerate Hessian.*

Proof. Using Proposition 1.5.1 we get that the canonical flow is variational with respect to an invariant Lagrangian if and only if the Frobenius differential system (1.54) and (1.55) has a solution $E : TG \simeq G \times \mathfrak{g} \to \mathbb{R}$ satisfying the regularity condition, or equivalently, there exists a function $\mathcal{E} : \mathfrak{g} \to \mathbb{R}$ satisfying the equation

$$[a,\alpha]\,\frac{\partial \mathcal{E}}{\partial \alpha} = 0, \tag{1.56}$$

for all $a \in \mathfrak{g}$, such that the Hessian matrix $\left(\frac{\partial^2 \mathcal{E}}{\partial \alpha^i \partial \alpha^j}\right)$ is nondegenrate. The equation (1.56) is identically satisfied if and only if $\mathcal{E}$ is constant on the orbit of the *ad* representation of $\mathfrak{g}$. □

Let $\{e_1, ..., e_n\}$ be a basis of $\mathfrak{g}$. Then the structure constants $C^\gamma_{\alpha\beta}$ of the Lie algebra $\mathfrak{g}$ are defined by

$$[e_\alpha, e_\beta] = C^\gamma_{\alpha\beta} e_\gamma. \tag{1.57}$$

We have the following

Theorem 1.5.3. *[114, Theorem 3.] There exists a left-invariant variational principle for the canonical flow of the Lie group G in a neighborhood of a generic element $\alpha \in \mathfrak{g}$ if and only if the linear system*

$$C^k_{ij}\alpha_j x_k = 0, \qquad i = 1,\dots,n, \tag{1.58}$$

$$C^k_{ij}x_k + C^k_{jm}\alpha_m x_{ik} = 0, \qquad i,j = 1,\dots,n, \tag{1.59}$$

has a solution $\{x_i = \epsilon_i, x_{ij} = \epsilon_{ij}\}$ satisfying the condition $\det(\epsilon_{ij}) \neq 0$.

Proof. Let us consider the distribution Δ in the tangent space of $\mathfrak{g}$ defined as

$$\Delta_\alpha := \left\{ X_a := [a, \alpha]\frac{\partial}{\partial \alpha} \mid a \in \mathfrak{g} \right\},$$

at any $\alpha \in \mathfrak{g}$. One can easily show that Δ is involutive and the system (1.56) is integrable. Moreover, there exists a nondegenerate initial condition if and only if the conditions of the theorem are satisfied. □

Corollary 1.5.4. *The canonical connection of a commutative Lie group is variational with respect to a left-invariant Lagrangian.*

Corollary 1.5.5. *If the derived Lie algebra is one dimensional, then there is no left-invariant variational principle for the canonical flow.*

Remark 1.5.6. The Lagrangian $E(\alpha) = K(\alpha, \alpha)$, where K is the Killing form of G is always a solution to the equation (1.54) and (1.55). That way we rediscover the well-known property of semi-simple Lie groups: the canonical connection is variational with respect to a left-invariant Lagrangian.

Classification up to dimension 4

In [114], using Theorem 1.5.3, we classify Lie groups up to dimension 4 for which there exists an invariant variational principle for the canonical geodesic flow.

2-dimensional Lie groups

There are, up to isomorphism, two Lie algebras distinguished according to whether $[\,,\,]$ is trivial or not. In the former case we have the abelian Lie algebra, and according to Corollary 1.5.4 it is variational with respect to a left-invariant Lagrangian. The latter one is the Lie algebra of the affine transformation group of the line. Using Theorem 1.5.3 one can show that the solutions are non regular, and we can conclude that there is no invariant variational principle for the canonical connection of the affine group of the line.

3-dimensional Lie groups

In [93] G. Thompson proved that the canonical geodesic equations of 3-dimensional Lie groups are locally variational. Moreover, we can use Jacobson's classification of the 3-dimensional Lie algebras which depends primarily on the dimension of the first derived algebra $\mathfrak{g}^{(1)}$ where $\mathfrak{g}$ denotes the original Lie algebra. We have the following possibilities:

If $\dim(\mathfrak{g}^{(1)}) = 0$, then $\mathfrak{g}$ is abelian and, according to Corollary 1.5.4, the canonical connection of a commutative Lie group is variational with respect to a left-invariant Lagrangian.

If $\dim(\mathfrak{g}^{(1)}) = 3$, then $\mathfrak{g}$ is simple and we have $\mathfrak{g} = s\ell(2, \mathbb{R})$ or $\mathfrak{g} = so(3)$. In both cases the Killing form provides a regular invariant metric and so the connections are variational.

If $\dim(\mathfrak{g}^{(1)}) = 1$ there are, up to isomorphism, two algebras distinguished according to whether or not $\mathfrak{g}^{(1)}$ lies inside the center of $\mathfrak{g}$. In the former case $\mathfrak{g}$ is the Heisenberg algebra. Its canonical connection is a flat connection and so it is variational. In the second case $\mathfrak{g}$ is isomorphic to the Lie algebra of the group of non-singular 2×2 upper triangular matrices and so it is again variational. In both cases, however, one can show, using Theorem 1.5.3, that the solutions of the systems (1.58)–(1.59) are necessarily non-regular, and we can conclude that the corresponding invariant variational principle does not exist.

4-dimensional Lie groups

According to [73], there are 12 classes of Lie algebras in dimension 4: $A_{4,i}$, $i = 1, \dots, 12$. Several Lie algebras have parameters denoted by a or b or both. G. Thompson and his co-workers determined in [40] whether or not the canonical connections are variational. In [93] Thompson also obtained results on the existence of invariant variational principles for the canonical flows. Using Theorem 1.5.3, one can complete the classification: depending on the Lie alebra, the canonical geodesic flow of the Lie group is

- non-variational in the cases: $A_{4,7}$, $A_{4,11a}$ with $0 < a$, and $A_{4,9b}$ with $-1 < b < 1$,
- variational but invariant variational principle does not exist in the cases: $A_{4,1}$, $A_{4,2a}$, $A_{4,3}$, $A_{4,4}$, $A_{4,5ab}$, $A_{4,6ab}$, $A_{4,9}$, $A_{4,12}$,
- variational and invariant variational principle exists in the cases: $A_{4,8}$ and $A_{4,10}$.

Chapter 2

Metrizability and projective metrizability

2.1 Introduction

A special and very interesting problem, within the inverse problem of the calculus of variations, is known as the metrizability problem. Here the regular Lagrangian to search for is the energy function of a Finslerian or a Riemannian metric. If the metric exists, then the integral curves of the given SODE are the geodesics of the corresponding Finslerian or Riemannian metric. One can also consider the projective metrizability problem, where one seeks for a Finslerian or a Riemannian metric whose geodesics coincide up to an orientation preserving reparameterization with the solution of the given SODE. Although metizability and projective metizability can be considered both as a special case of the inverse problem of the calculus of variations, because of their geometric interest we dedicate an entire paragraph to these very natural and well-studied problems. We focus mainly to the Finsler metrizability and projective Finsler metrizability. Several papers are devoted to these problems (considered from the differential geometric point of view, see for example the recent papers [12, 21, 32, 56, 60, 66, 80, 81, 91]).

In Section 2.3 we investigate the Finsler and the Landsberg metrizability problems. Both can be formulated in terms of partial differential systems. Indeed, by characterizing Finsler metrics with their energy functions, the Finsler metrizability problem can be described in terms of a second order PDE system composed by the homogeneity condition (2.8) and the Euler-Lagrange partial differential equations (2.9). Completing this system with the third order partial differential equations (2.13) corresponding to the invariance of the metric with respect to the parallel translations, we can obtain the conditions of the Landsberg metrizability. In [115], using the holonomy invariance of the energy function, we proved that the Finsler metrizablity's second order PDE system can be reduced to a first order PDE system on the same unknown function (Theorem 2.3.1). We formulated in a coordinate free way the necessary and sufficient condition for metrizability in terms of a geometric object $\mathcal{D}_{\mathcal{H}}$, associated to the spray (Theorem 2.3.2 and Theorem 2.3.3). $\mathcal{D}_{\mathcal{H}}$ is called the holonomy distribution [115] or the holonomy algebra [53]. We obtained similar results on the Landsberg case: the Landsberg metrizablity's third order PDE system can be reduced to a first order PDE system (Theorem 2.3.4 and Theorem 2.3.5). Moreover, a necessary and sufficient condition for Landsberg metrizability can be formulated in a coordinate free way in terms of a distribution generated by the holonomy distribution and the image of the Berwald curvature.

In Section 2.4 we consider the Finsler metrizability with special curvature properties. The aim is to provide necessary and sufficient conditions for SODEs to be the geodesic system of a Finsler metric of constant or scalar flag curvature, respectively. We solve both problems. In the first part of Section 2.4 we focus on the constant curvature case. When the spray has zero constant curvature, then there is no obstruction to the existence of a locally defined Finsler structure that metricizes the given spray [97, 29, 115]. In Theorem 2.4.1, we solved the non-zero curvature case completely by providing a set of equations, which contains an algebraic equation and two tensorial differential equations, which have to be satisfied by the Jacobi endomorphism of the SODE. In the second part of the section we focus on the characterization of sprays that are metrizable by Finsler functions of scalar flag curvature. We provide the necessary and sufficient conditions on the Jacobi endomorphism, which can be used to decide whether or not a given homogeneous SODE represents the geodesic equations of a Finsler function of scalar curvature. In Theorem 2.4.3 we provide conditions, which together with the isotropy condition, characterizes the class of sprays that are metrizable by Finsler functions of scalar flag curvature. The proof offers an algorithm to construct the Finsler function. We also showed that our results lead to a new approach for Hilbert's fourth problem. This problem asks to construct and to investigate the geometries in which a straight line segment is the shortest connection between two points, [4]. Alternatively, one can formulate the problem as follows: "given a domain $\Omega \subset \mathbb{R}^n$, determine all (Finsler) metrics on Ω whose geodesics are straight lines", [82, page 191]. Yet another formulation of the problem requires to determine projectively flat Finsler metrics, [31]. Projectively flat Finsler functions have isotropic geodesic sprays and therefore have constant or scalar flag curvature. Such Finsler metrics were studied in [22, 86]. We used our results to investigate, when the projective deformations of a flat spray are metrizable. Using these conditions, we showed how to construct examples which are solutions to Hilbert's fourth problem.

In Section 2.5 we consider the projective metrizability problem. Recently several new results appeared about the projective Finsler metrizability [97, 31, 33, 33, 65, 68]. Various strategies can be chosen to deal with the problem: in [32] the generalized Helmholtz system was considered, in [97] a system in terms of a semi-basic 1-form was investigated, and in [33] an approach in terms of 2-forms was formulated. In [75, 76] A. Rapcsák obtained necessary and sufficient conditions for the projective metrizability in terms of a second order PDE system, now called *Rapcsák equations* [32, 91, 82]. The coordinate free formulations of these equations can be found in [54, 91]. In [110] and [111] the integrability of the *Rapcsák system* was investigated by using the Spencer version of the Cartan-Kähler theorem. The compatibility conditions can be expressed in terms of the curvature tensor. The curvature of flat and of isotropic sprays satisfies these conditions. We remark that for these classes the projective metrizability problem has been discussed in [29, 30, 97], but the approach in [110, 111] is different and can be particularly advantageous from the perspective of further investigations in the non-isotropic curvature case. We proved that if the spray is non-isotropic, then the symbol of the corresponding differential operator is not involutive and that the Cartan test fails. Therefore the PDE system is not integrable and higher integrability conditions exist. Using the Spencer technique, this level, and the number of the extra integrability conditions can be calculated.

In Section 2.6 we investigate projective deformations and the rigidity of the metrizability property. In [95] Yang shows that the projective class of a projectively flat spray of constant flag curvature contains sprays which are not Finsler metrizable. In [98] we extend Yang's example and show that for an arbitrary spray its projective class contains many sprays that are not Finsler metrizable. Considering holonomy invariant projective changes in [103], we show that only very special holonomy invariant projective factors can lead to a metrizable projective deformation. These holonomy invariant projective factors have to be related to the principal curvatures of the deformed Finsler structure.

In Section 2.7 we investigate the invariant Riemann and Finsler metrizability and projective metrizability of the canonical spray of Lie groups. Since the notion of Finslerian metric is a generalisation of the notion of Riemannian metric – as S.S. Chern said "Finsler geometry is just Riemannian geometry without the quadratic restriction" [27] – therefore every Riemann metrizable spray (necessarily quadratic) is trivially Finsler metrizable. The converse, in general, is not true; there are Finsler metrizable sprays which are not Riemann metrizable. We remark that in the class of quadratic sprays Zoltán Szabó's theorem [88] states that the notion of Finsler metrizability and Riemann metrizability coincide. The projective Finsler metrizability is, however, different from projective Riemann metrizability, even in the case of quadratic sprays. This follows since a quadratic spray can be projectively equivalent to a non-quadratic one. Therefore, the class of projective Finsler metrizable sprays is in general strictly larger then the class of Riemann metrizable sprays, even for quadratic sprays. The goal of Section 2.7 is to investigate the relationship between invariant metrizability, and the invariant projective metrizability of the canonical spray. We prove that the canonical connection of a Lie group is invariant projective Finsler metrizable if and only if it is invariant Riemann metrizable. This result shows the rigidity of the structure.

We consider also homogeneous spaces G/H with a special geodesic structure. A left-invariant geodesic structure on G/H is called *geodesic orbit structure (g.o. structure)*, if the geodesics can be derived as orbits of 1-parameter subgroups of G. In V.I. Arnold's terminology these curves are called "relative equilibria" [8]. We prove that a g.o. structure is invariant projective Riemann (resp. Finsler) metrizable if and only if it is invariant Riemann (resp. Finsler) metrizable. In the quadratic case we also obtain the rigidity property: the class of Riemann metrizable and projective Finsler metrizable g.o. sprays coincide.

The results of this chapter are based on the papers [96, 97, 98, 99, 100, 101, 102, 103, 110, 115].

2.2 Preliminaries

Finsler metric, geodesic spray, metrizability

A Finsler function on a manifold M is a continuous function $F\colon TM \to \mathbb{R}$, which is smooth and positive away from the zero section, positive homogeneous of degree one, and strictly convex on each tangent space. The pair (M, F) is called a Finsler manifold.

The energy function $E : TM \to \mathbb{R}$ associated with a Finsler function F is defined as $E := \frac{1}{2}F^2$. The symmetric bilinear form

$$g_{x,y}\colon (u,v) \ \mapsto \ g_{ij}(x,y)u^i v^j = \frac{1}{2}\frac{\partial^2 \mathcal{F}_x^2(y+su+tv)}{\partial s\,\partial t}\Big|_{t=s=0}, \quad u,v \in T_xM$$

is called the metric tensor of the Finsler manifold $(M, \mathcal{F})$. The Finsler function is called *absolutely homogeneous* at $x \in M$, if $F_x(\lambda y) = |\lambda| F_x(y)$ for all $\lambda \in \mathbb{R}$. If F is absolutely homogeneous at every $x \in M$, then the Finsler manifold $(M, \mathcal{F})$ is called reversible.

The tensor components

$$g_{ij} := \frac{\partial^2 E}{\partial y^i \partial y^j} \tag{2.1}$$

deternine a positive definite matrix $g_E = (g_{ij})$ at any point $(x,y) \in \mathcal{T}M$. The regularity condition implies that the Euler-Poincaré 2-form of E, $\Omega_E = dd_J E$, is non-degenerate and hence it is a symplectic structure. Therefore, the equation (1.17) uniquely determine a vector field $\mathcal{S}$ on $\mathcal{T}M$ that is called the *geodesic spray* of the Finsler function. The geodesics of the spray $\mathcal{S}$ are the geodesics of the Finsler manifold (M, F). The *geodesic spray* is locally given by

$$\mathcal{S} = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}, \tag{2.2}$$

where

$$G^i(x,y) := \frac{1}{4} g^{il}(x,y)\Big(2\frac{\partial g_{jl}}{\partial x^k}(x,y) - \frac{\partial g_{jk}}{\partial x^l}(x,y)\Big) y^j y^k, \tag{2.3}$$

are the *geodesic coefficients*. A spray $\mathcal{S}$ is called *Finsler metrizable* if there exists a Finsler function F that satisfies the condition (2.3).

One can reformulate the regularity condition of the energy function in terms of the Hessian of the Finsler function F as follows. Consider

$$h_{ij}(x,y) = F\frac{\partial^2 F}{\partial y^i \partial y^j} \tag{2.4}$$

the *angular metric* of the Finsler function. The metric tensor g_{ij} and the angular tensor h_{ij} are related by

$$g_{ij} = h_{ij} + \frac{\partial F}{\partial y^i}\frac{\partial F}{\partial y^j}. \tag{2.5}$$

Metric tensor g_{ij} has rank n if and only if angular tensor h_{ij} has rank $(n-1)$, see [63]. Therefore, the regularity of the Finsler function F is equivalent with the fact that the Euler-Poincaré 2-form $\omega_F = dd_J F$ has rank $2n-2$.

Consider F a Finsler function and Φ the Jacobi endomorphism (also called the Riemann curvature [82]) of its geodesic spray $\mathcal{S}$. The Finsler function F is said to be of *scalar (resp. constant) curvature* if there exists a scalar (resp. constant) function κ on $\mathcal{T}M$, such that

$$\Phi = \kappa \left(F^2 J - F d_J F \otimes \mathbb{C}\right). \tag{2.6}$$

F is called an *Einstein metric* if there exists a function $\lambda \in C^\infty(M)$ such that the Ricci scalar satisfies $\rho(x,y) = \lambda(x) F^2(x,y)$. The notion of flag curvature extends the concept of sectional curvature from the Riemannian to the Finslerian setting (see [105, Chapter 3.5]). The Jacobi endomorphism of a Finsler metric is diagonalizable in the following sense: there exist functions κ_i and horizontal vector fields $X_\alpha \in \mathfrak{X}(\mathcal{T}M)$ for $i = 1, \ldots, n$ such that $\mathcal{S} = X_n$ and

$$\Phi(X_i) = \kappa_i \, JX_i, \quad i = 1, \ldots, n. \tag{2.7}$$

(The summation convention is not applied in the above formula.) X_i is called an eigenvector field of Φ corresponding to the eigenfunction κ_i. In particular, $\mathcal{S}$ is always an eigenvector of Φ corresponding to $\lambda_n = 0$. The function $\kappa_1, \ldots, \kappa_{n-1}$ are called the principal curvatures of the Finsler metric [83].

Formal integrability of partial differential operators

Let B be a vector bundle over M. If s is a section of B, then $j_k(s)_x$ will denote the k^{th} order jet of s at the point $x \in M$. The bundle of k^{th} order jets of the sections of B is denoted by $J_k B$. In particular $J_k(\mathbb{R}_M)$ will denote the k^{th} order jet bundle of real valued functions, that is the sections of the trivial line bundle. Let B_1 and B_2 be vector bundles over M and $P\colon Sec\,(B_1) \to Sec\,(B_2)$ a differential operator. An $s \in Sec(B_1)$ is a solution to P if $Ps \equiv 0$.

If P is a linear differential operator of order k, then a morphism $p_k(P)\colon J_k(B_1) \to B_2$ can be associated to P. The l^{th} order prolongation $p_{k+l}(P)\colon J_{k+l}(B_1) \to J_l(B_2)$ can be introduced in a natural way by taking the l^{th} order derivatives. $Sol_{k+l,x}(P) := \operatorname{Ker} p_{k+l,x}(P)$ denotes the set of formal solutions of order l at $x \in M$. Obviously, we have

$$Ps \equiv 0 \qquad \Rightarrow \qquad j_{l,x}(s) \in Sol_{l,x}(P),$$

for every $l \geq k$ and $x \in M$. The differential operator P is called formally integrable if $Sol_l(P)$ is a vector bundle for all $l \geq k$, and the restriction $\overline{\pi}_{l,x}\colon Sol_{l+1,x}(P) \to Sol_{l,x}(P)$ of the natural projection is onto for every $l \geq k$. In that case any k^{th} order solution or *initial data* can be lifted into an infinite order solution. In the analytic case, formal integrability implies the existence of solutions for arbitrary initial data [23, p. 397]. To prove the formal integrability of a differential operator, one can use the Cartan-Kähler theorem. To present it, we have to introduce some notations.

Let $\sigma_k(P)$ denote the symbol of P determined by the highest order terms of the operator. It can be interpreted as a map $\sigma_k(P)\colon S^k T^* M \otimes B_1 \to B_2$. The symbol of the l^{th} order prolongation of P is denoted by $\sigma_{k+l}(P)\colon S^{k+l}T^*M\otimes B_1 \to S^lT^*M\otimes B_2$. If $\mathcal{E}=\{e_1,\dots,e_n\}$ is a basis of T_xM, we set

$$\begin{aligned} g_{k,x}(P) \quad &= \mathrm{Ker}\sigma_{k,x}(P),\\ g_{k,x}(P)_{e_1\dots e_j} &= \big\{A\in g_{k,x}(P) \mid i_{e_1}A=\cdots=i_{e_j}A=0\big\}, \quad j=1,\dots,n, \end{aligned}$$

The basis $\mathcal{E}$ is called *quasi-regular* if one has

$$\dim g_{k+1,x}(P)=\dim g_{k,x}(P)+\sum_{j=1}^{n}\dim g_{k,x}(P)_{e_1\dots e_j}.$$

A symbol is called *involutive* if there exists at any $x\in M$ a quasi-regular basis. We remark that in the works of Cartan, and more generally in the theory of exterior differential systems, "involutivity" means more than the existence of a quasi-regular basis and it refers to "integrability" (cf. [23, pages 107 and 140]). Here we follow the terminology of Goldschmidt (cf. [23, page 409]). The notion of involutivity allows us to check the formal integrability in a simple way by using the following

Theorem 2.2.1 (Cartan-Kähler). *Let P be a k^{th} order linear partial differential operator. Suppose that P is regular, that is $Sol_{k+1}(P)$ is a vector bundle over $Sol_k(P)$. If the map $\overline{\pi}_k\colon Sol_{k+1}(P)\to Sol_k(P)$ is surjective and the symbol is involutive, then P is formally integrable.*

It can be shown that the condition of the existence of a quasi-regular basis can be replaced by a weaker condition. The obstructions to the higher order successive lift of the k^{th} order solution are contained in some of the cohomology groups of a certain complex called Spencer complex. The $H^{m,i}$ Spencer cohomology group is defined as $H^{m,i}=\mathrm{Ker}\,\delta^m_i/\mathrm{Im}\,\delta^m_{i-1}$ where

$$\delta^m_i\colon g_m(P)\otimes\Lambda^iT^*M\longrightarrow g_{m-1}(P)\otimes\Lambda^{i+1}T^*M$$

is the restriction of the natural operator $\delta^m_i\colon S^{k+m}T^*M\otimes\Lambda^iT^*M\to S^{k+m-1}T^*M\otimes\Lambda^{i+1}T^*M$. The symbol of a k^{th} order linear differential operator P is *2-acyclic* if $H^{m,2}=0$ for all $m\geq k$. Using Spencer cohomology groups, a weaker version of integrability theorem can be stated:

Theorem 2.2.2 (Goldschmidt). *Let P be k^{th} order regular linear partial differential operator. If $\overline{\pi}_k : Sol_{k+1}(P)\to Sol_k(P)$ is onto and the symbol of the operator is 2-acyclic then P is formally integrable.*

Using a classical result of homological algebra, the surjectivity of $\overline{\pi}_{k+1}$ can be verified in the following way [23]: there exists a morphism $\varphi\colon Sol_k(P)\to \mathrm{Coker}\,(\sigma_{k+1}(P))$, such that the sequence

$$Sol_{k+1}(P)\xrightarrow{\overline{\pi}_k} Sol_k(P)\xrightarrow{\varphi}\mathrm{Coker}\,(\sigma_{k+1}(P))$$

is exact. Therefore $\overline{\pi}_k$ is surjective if and only if $\varphi\equiv 0$. The map φ is called obstruction map and $\mathrm{Coker}\,(\sigma_{k+1}(P))$ is called obstruction space, because a k^{th} order solution $s\in Sol_k(P)$ can be prolonged into a $(k{+}1)^{\text{st}}$ order solution if and only if $\varphi(s)=0$.

Remark 2.2.3. The map φ and therefore the integrability conditions can be computed as follows: Let $\tau\colon T^*\otimes B_2 \to K$ be a morphism such that $\operatorname{Ker}\tau = \operatorname{Im}\sigma_{k+1}(P)$. Then K is isomorphic to $\operatorname{Coker}(\sigma_{k+1}(P))$. Moreover, if $s_{k,x} = j_k(s)_x$ is a k^{th} order solution, that is $(Ps)_x = 0$, then

$$\varphi(s_{k,x}) = \tau(\nabla(Ps))_x,$$

where ∇ is an arbitrary linear connection on the bundle B_2.

2.3 Finsler metrizability

In this section we investigate the following problem: *under which conditions can a second order differential equation* (1.4) *be the geodesic equation of a Finsler metric.* The energy function of a Finsler space is necessarily a 2-homogeneous solution of the Euler-Lagrange PDE system, therefore, the second-order differential equation (1.4) is *Finsler metrizable*, if and only if, there exists a solution $E : TM \to \mathbb{R}$ to the second order differential system

$$y^i \frac{\partial E}{\partial y^i} - 2E = 0, \tag{2.8}$$

$$y^j \frac{\partial^2 E}{\partial x^j \partial y^i} + f^j(x,y)\frac{\partial^2 E}{\partial y^j \partial y^i} - \frac{\partial E}{\partial x^i} = 0, \quad i = 1, \dots, n, \tag{2.9}$$

so that the quadratic form g_E defined in (2.1) is positive definite. Since the second order PDE system describing the metrizability is composed by $n+1$ equations (and one inequality), it is an overdetermined differential system. It is not surprising then, that it has no solution in the generic case. It is more interesting however, that it can be reduced to a first order PDE system which gives a nice geometric interpretation:

Theorem 2.3.1. *[115, Theorem 1.] A Lagrangian $E : TM \to \mathbb{R}$ is a solution of the system composed by* (2.8) *and* (2.9)*, if and only if, it is a solution of the first order system*

$$\begin{cases} \mathcal{L}_C E - 2E = 0, \\ d_{\mathfrak{h}} E = 0, \end{cases} \tag{2.10}$$

where $\mathcal{D}_{\mathcal{H}} \subset TTM$ is the holonomy distribution generated by the horizontal vector fields and their successive Lie-brackets, and $\mathfrak{h}\colon TTM \to \mathcal{D}_{\mathcal{H}}$ is an arbitrary projection on $\mathcal{D}_{\mathcal{H}}$.

We note that for $X \in \mathfrak{X}(TM)$ we have $d_{\mathfrak{h}}E(X) = \mathfrak{h}X(E) = \mathcal{L}_{\mathfrak{h}X}E$, so the second equation of (2.10) means simply that the Lie-derivative of E with respect to vector fields in the holonomy distribution $\mathcal{D}_{\mathcal{H}}$ is zero.

Proof. Let us suppose that $E : TM \to \mathbb{R}$ is a solution of (2.10). Since $HTM \subset \mathcal{D}_{\mathcal{H}}$, we have $\mathfrak{h} \circ h = h$ and

$$d_h E = d_{\mathfrak{h}\circ h} E = i_h d_{\mathfrak{h}} E - d_{\mathfrak{h}} i_h E + i_{[h,\mathfrak{h}]} E = i_h d_{\mathfrak{h}} E = 0.$$

Moreover, the spray $\mathcal{S}$ is horizontal. Writing the Euler-Lagrange form in the form

$$\omega_E = i_{\mathcal{S}} dd_J E + d\mathcal{L}_C E - dE = d_J \mathcal{L}_{\mathcal{S}} E - i_{[J,\mathcal{S}]} dE = d_J \mathcal{L}_{\mathcal{S}} E - 2d_h E$$

we obtain that $\omega_E = 0$ and E is a solution of (1.21) that is of (2.9).

On the other hand, let us suppose now that $E : TM \to \mathbb{R}$ is a 2-homogeneous solution of (1.21). We have

$$i_{\mathcal{S}} \Omega_E = d(E - \mathcal{L}_C E) = -dE. \tag{2.11}$$

Since $[J, J] = 0$, we have $d_J^2 = d_J \circ d_J = d_{[J,J]} = 0$, and $i_J \Omega_E = 0$, so $i_C \Omega_E = i_J dE$. Moreover, for every $X \in \mathfrak{X}(TM)$ we have

$$i_{\mathcal{S}} \Omega_E(X) = \Omega_E(\mathcal{S}, X) = -\Omega_E(C, FX) = -i_F i_C \Omega_E(X),$$

i.e. $i_{\mathcal{S}} \Omega_E = i_F i_C \Omega_E$. Thus, we obtain

$$i_{\mathcal{S}} \Omega_E = -i_F i_C \Omega_E = -i_F i_J dE = -d_v E = -dE + d_h E. \tag{2.12}$$

Comparing (2.11) with (2.12) we obtain that $d_h E = 0$. It follows that $hX(E) = 0$, i.e. E is constant with respect to horizontal vector fields. Therefore it must be constant on the distribution generated by the horizontal sub-bundle taking the recursive Lie-bracket operations, i.e. on $\mathcal{D}_{\mathcal{H}}$. This means that we have $d_{\mathfrak{h}} E = 0$ and E is a solution of (2.10). □

Corollary 2.3.2. *If the Liouville vector field C is in $\mathcal{D}_{\mathcal{H}}$, then $\mathcal{S}$ is not metrizable.*

We have the following

Theorem 2.3.3. *[115, Theorem 3.] Let $\mathcal{S}$ be an analytical spray over the analytical manifold M. If $\mathcal{D}_{\mathcal{H}}$ has constant rank in a neighbourhood of $v \in \mathcal{T}M$, then there exists an analytical Finsler metric in a neighbourhood of v whose geodesics are given by $\mathcal{S}$, if and only if the kernel of the first prolongation of (2.10) at v contains positive definite initial data.*

Proof. The proof is based on Theorem 2.3.1. Indeed, one can show that the system (2.10) is formally integrable. In the analytic case, this formal solution gives an analytical solution in an open neighborhood of $v \in \mathcal{T}M$. Theorem 2.3.1 shows that this function is in fact a 2-homogeneous solution of the Euler-Lagrange PDE system associated to $\mathcal{S}$. □

One can also consider the Landsberg metrizability problem, where we seek for a Finsler function, such that the canonical connection $\Gamma = [J, S]$ is metric, i.e. the parallel transport preserves the metric defined by g_E. This extra condition can be described by the PDE system

$$\frac{\partial g_{jk}}{\partial x^i} - \Gamma^l_i \frac{\partial g_{jk}}{\partial y^l} - \Gamma^l_{ik} g_{lj} - \Gamma^l_{ij} g_{lk} = 0, \quad i, j, k = 1, \ldots, n, \tag{2.13}$$

which is a 3rd order PDE system in the energy function of the Finsler metric, taking into account (2.1). Then $\mathcal{S}$ is *Landsberg metrizable*, if and only if there exists a solution $E\colon TM \to \mathbb{R}$ of the third-order PDE system composed by (2.8), (2.9) and (2.13) so that the quadratic form g_E is positive definite. We have the following

Theorem 2.3.4. *[115, Theorem 4.] The third-order partial differential system composed by the equations* (2.8), (2.9) *and* (2.13) *is equivalent to the first order system*

$$\begin{cases} \mathcal{L}_C E - 2E = 0, \\ d_{\mathfrak{l}} E = 0, \end{cases} \tag{2.14}$$

where $\mathfrak{L}$ is the distribution generated by the horizontal vector fields, the image of the Berwald curvature (1.8) *and their successive Lie-brackets and $\mathfrak{l} : TTM \to \mathfrak{L}$ is an arbitrary projection of TTM onto $\mathcal{L}$.*

Using Theorem 2.3.4 one can prove the following

Theorem 2.3.5. *[115, Theorem 5.] Let $\mathcal{S}$ and M be analytical, and suppose that rank of $\mathcal{L}$ constant in a neighborhood of $v \in \mathcal{T}M$. Then there exists a Finsler metric of Landsberg type in a neighborhood of v whose geodesics are given by $\mathcal{S}$, if and only if the kernel of the first prolongation of (2.14) at v contains a positive definite initial condition.*

Freedom of the metrizability

Similarly to the notion of variational freedom, one can introduce the *metrizability freedom* of a spray $\mathcal{S}$ showing how many functionally independent Finsler energy functions and hence how many essentially different Finsler metrics exist for $\mathcal{S}$. To be more precise, let $\mathcal{E}^+_{\mathcal{S},2}$ be the set of Finsler energy functions, that is the set of regular 2−homogeneous Lagrange functions with (3.14) positive definite. The *metrizability freedom* of a spray $\mathcal{S}$ is $\mathfrak{m} := rank\,(\mathcal{E}^+_{\mathcal{S},2})$. If $\mathcal{S}$ is non-metrizable then we set $\mathfrak{m} = 0$. We have

Proposition 2.3.6. *[102, Proposition 4.9] Let $\mathcal{S}$ be a metrizable spray such that the parallel translation with respect to the associated connection is regular. Then* $\mathfrak{m} = \operatorname{codim} \mathcal{D}_{\mathcal{H}}$.

Proof. Using the reasoning of Theorem 1.4.3 one can easily prove Proposition 2.3.6. We just remark that, using the notation introduced in the proof of Theorem 1.4.3, we have $E_0 \in \mathcal{E}^+_{\mathcal{S},2}$ and for any $i = \kappa+2, \dots, 2n$, a sufficiently small nonzero constant $c_i \in \mathbb{R}$ can be chosen for $E_i = (1 + c_i\theta_i)E_o$ to remain positive definite. Hence with $E_o = E_{\kappa+1}$ we get $\{E_{\kappa+1}, E_{\kappa+2}, \dots, E_{2n}\} \subset \mathcal{E}^+_{\mathcal{S},2}$. A similar argument to that we used in the proof of Theorem 1.4.3 shows that these elements are locally functionally independent and they locally generate $\mathcal{E}^+_{\mathcal{S},2}$ which proves the Proposition.

∎

2.4 Finsler metrizability with special curvature properties

In this section we address the special case of the Finsler metrizability problem, where the Finsler function we seek for has constant curvature. When the spray has zero constant curvature, then there is no obstruction to the existence of a locally defined Finsler structure that metricizes the given spray [97, 29, 115]. Therefore, we focus on the case when the curvature is non-zero.

Metrizability by Finsler functions of constant curvature

An isotropic spray $\mathcal{S}$ (see formula (1.13)) is called *weakly Ricci-constant* if $\mathcal{L}_{\mathcal{S}}\rho = 0$, and *Ricci-constant* if $d_h\rho = 0$. It is easy to see that if a spray $\mathcal{S}$ is Ricci constant, then it is also weakly Ricci constant.

In the next theorem we provide the necessary and sufficient conditions for a spray to be metrizable by a Finsler function of non-zero constant curvature.

Theorem 2.4.1. *[99, Theorem 4.1] The spray $\mathcal{S}$ with non-vanishing Ricci curvature is metrizable by a Finsler function of non-zero constant flag curvature if and only if its Jacobi endomorphism Φ satisfies the following equations:*

$$\begin{aligned} A)&\quad \operatorname{rank} dd_J(\operatorname{Tr}\Phi) = 2n,\\ D_1)&\quad 2(n-1)\Phi - 2(\operatorname{Tr}\Phi)J + d_J(\operatorname{Tr}\Phi)\otimes C = 0,\\ D_2)&\quad d_h(\operatorname{Tr}\Phi) = 0. \end{aligned} \tag{2.15}$$

Proof. Consider a spray $\mathcal{S}$ with non-vanishing Ricci curvature. We assume that its Jacobi endomorphism, Φ, satisfies the algebraic assumption A) as well as the two tensorial equations (2.15). Since Φ satisfies D_1), it follows that the Jacobi endomorphism is given by formula (1.13), where $2(n-1)\alpha = (n-1)d_J\rho = d_J(\operatorname{Tr}\Phi)$. Therefore the spray $\mathcal{S}$ is isotropic and satisfies the condition $d_J\alpha = 0$.

Due to condition D_2), we have that $\mathcal{S}$ is Ricci constant and it follows that the spray $\mathcal{S}$ is weakly Ricci constant. Using the fact that $2\alpha = d_J\rho$ we obtain

$$2\mathcal{L}_{\mathcal{S}}\alpha = \mathcal{L}_{\mathcal{S}}d_J\rho = d_{[\mathcal{S},J]}\rho + d_J\mathcal{L}_{\mathcal{S}}\rho = d_v\rho = d\rho. \tag{2.16}$$

Within the assumption that the Ricci curvature does not vanish on $\mathcal{T}M$, we may consider the function $F > 0$ such that $F^2 = \operatorname{sign}(\rho)\rho > 0$ on $\mathcal{T}M$. Since $dd_J(\operatorname{Tr}\Phi) = (n-1)dd_J\rho = (n-1)dd_JF^2$, the assumption A) assures that F is a Finsler function. The condition $2\alpha = d_J\rho$ reads now $2\alpha = d_JF^2$ and using formula (2.16) we obtain $\mathcal{L}_{\mathcal{S}}d_JF^2 = dF^2$, which means that $\mathcal{S}$ is the geodesic spray of the Finsler function F. We replace

$$\operatorname{Tr}\Phi = (n-1)F^2 = (n-1)\rho = (n-1)i_{\mathcal{S}}\alpha$$

and

$$d_J(\operatorname{Tr}\Phi) = 2(n-1)Fd_JF = 2(n-1)\alpha = (n-1)d_J\rho$$

in the expression for Φ and obtain formula (2.6) for $\kappa = \operatorname{sign}(\rho)$. It follows that spray $\mathcal{S}$ is Finsler metrizable by the Finsler function F of constant curvature $\kappa = \operatorname{sign}(\rho)$.

Conversely, if the spray $\mathcal{S}$ is Finsler metrizable by a Finsler function of non-zero constant flag curvature then its Jacobi endomorphism is given by formula (2.6). It is a straightforward computation to see that Φ satisfies all three conditions A), D_1) and D_2) in (2.15). $\square$

Some of the conditions of Theorem 2.4.1 are related to the conditions of Theorem 7.2 in [105] as follows: both theorems use the assumption of non-vanishing Ricci curvature $\rho = i_{\mathcal{S}}\alpha$. Condition D_1), which implies that the spray $\mathcal{S}$ is isotropic and $d_J\alpha = 0$, is stronger then condition 2 of [105, Theorem 7.2]. Also condition D_2) implies that $\nabla\alpha = 0$ and therefore this implies condition 3 of [105, Theorem 7.2]. However, the conclusion in Theorem 2.4.1 is stronger as well, for a given spray, we seek for the metrizability by a Finsler function of constant flag curvature. Theorem [105, Theorem 7.2] characterizes local variational non-flat isotropic typical sprays. The differentiability assumptions are different: while here we use smoothness, in [105, Theorem 7.2] the analyticity of all geometric structures is assumed.

In Theorem 2.4.2 we will strengthen the result of Theorem 2.4.1 by limiting our discussion to the case where Finsler metrizability is equivalent to the metrizability by a Finsler function of constant curvature:

Theorem 2.4.2. *[99, Theorem 4.2] Consider $\mathcal{S}$ a spray of non-vanishing Ricci curvature on a manifold with $\dim M \geq 3$. Then, the spray is isotropic, satisfies the algebraic condition A), and the condition $d_J\alpha = 0$, if and only if the following five conditions are equivalent:*

i) $\mathcal{S}$ is Finsler metrizable;
ii) $\mathcal{S}$ is metrizable by a Finsler metric of non-vanishing scalar flag curvature;
iii) $\mathcal{S}$ is Finsler metrizable by an Einstein metric;
iv) $\mathcal{S}$ is metrizable by a Finsler metric of non-zero constant flag curvature;
v) $\mathcal{S}$ is Ricci constant.

In the proof, we use Theorem 2.4.1 and the Finslerian version of Schur's Lemma [9, Lemma 3.10.2].

Metrizability by Finsler functions of scalar curvature

The problem we want to address in this subsection is the following: provide the necessary and sufficient conditions for a spray $\mathcal{S}$ to be metrizable by a Finsler function of scalar flag curvature. Using formulae (1.13) and (2.6) it follows that for a Finsler function F of scalar flag curvature κ, its geodesic spray $\mathcal{S}$ is isotropic, with the Ricci scalar $\rho = \kappa F^2$ and the semi-basic 1-form $\alpha = \kappa F d_J F$. This fact restricts the class of relevant sprays to start with to the class of isotropic sprays. The next theorem provides an algorithm to construct the Finsler function that metricizes a given spray, in the case when it is variational.

Theorem 2.4.3. *[100, Theorem 3.1] Consider a spray $\mathcal{S}$ of non-vanishing Ricci scalar. Then $\mathcal{S}$ is metrizable by a Finsler function of non-vanishing scalar flag curvature if and only if* *i)* $\mathcal{S}$ *is isotropic;*

ii) $d_J(\alpha/\rho) = 0$*;*

iii) $D_{hX}(\alpha/\rho) = 0$*, for all* $X \in \mathfrak{X}(\mathcal{T}M)$*;*

iv) $d(\alpha/\rho) + 2i_{\mathbf{F}}\alpha/\rho \wedge \alpha/\rho$ *is a symplectic form on* $\mathcal{T}M$*.*

Proof. We assume that the spray $\mathcal{S}$ is metrizable by a Finsler function F of scalar flag curvature κ and we will prove that the four conditions *i)-iv)* are necessary. Since the Jacobi endomorphism Φ is given by formula (1.13), it follows that $\mathcal{S}$ is isotropic, and hence condition *i)* is satisfied. The semi-basic 1-form α and the Ricci scalar ρ are given by

$$\alpha = \kappa F d_J F, \quad \rho = \kappa F^2. \tag{2.17}$$

It follows that $\alpha/\rho = d_J F/F$ and therefore $d_J(\alpha/\rho) = 0$, which means that condition *ii)* is satisfied. Since $\mathcal{S}$ is the geodesic spray of the Finsler function F, it follows from the second equation of (2.10) that $d_h F = 0$. Therefore,

$$D_{hX}F = (hX)(F) = (d_h F)(X) = 0$$

and $D_{hX}d_J F = 0$. It follows that $D_{hX}(\alpha/\rho) = 0$ and hence condition *iii)* is also satisfied. We check now the regularity condition *iv)*. Using $d_h F = 0$ and $J \circ \mathbb{F} = v$, we obtain

$$i_{\mathbb{F}}\frac{\alpha}{\rho} = i_{\mathbb{F}}\frac{1}{F}d_J F = \frac{1}{F}d_v F = \frac{1}{F}dF.$$

Therefore, using the regularity of the Finsler function, it follows that

$$d\left(\frac{\alpha}{\rho}\right) + 2i_{\mathbb{F}}\frac{\alpha}{\rho} \wedge \frac{\alpha}{\rho} = d\left(\frac{d_J F}{F}\right) + \frac{2}{F^2}dF \wedge d_J F = \frac{1}{2F^2}dd_J F^2$$

is a symplectic form on $\mathcal{T}M$.

Let us prove now the sufficiency of the four conditions *i)-iv)*. Consider a spray $\mathcal{S}$ that satisfies all four conditions. First, condition *i)* says that the spray $\mathcal{S}$ is isotropic, which means that its Jacobi endomorphism Φ is given by formula (1.13). The next three conditions *ii)-iv)* refer to the semi-basic 1-form α and the Ricci scalar ρ, which enter into the expression of the Jacobi endomorphism Φ.

From condition *ii)* we have that the semi-basic 1-form α/ρ is a d_J-closed 1-form. Since the tangent structure J is integrable, it follows that $[J, J] = 0$ and hence $d_J^2 = 0$. Therefore, using a Poincaré-type Lemma for the differential operator d_J, it follows that, locally, α/ρ is a d_J-exact 1-form. It follows that there exists a function f, locally defined on $\mathcal{T}M$, such that

$$\frac{1}{\rho}\alpha = d_J f = \frac{\partial f}{\partial y^i}dx^i. \tag{2.18}$$

Note that this function f is not unique, it is given up to an arbitrary basic function $a \in C^\infty(M)$. We will prove that using this function f and a corresponding basic function a, we can construct a Finsler function $F = \exp(f - a)$, of scalar flag curvature, which metricizes the given spray $\mathcal{S}$. Using the commutation rule for $i_{\mathcal{S}}$ and d_J, see [105, Appendix A], we have

$$\mathcal{L}_C(f) = i_{\mathcal{S}} d_J f = i_{\mathcal{S}} \frac{\alpha}{\rho} = 1. \tag{2.19}$$

Using the condition *ii)* of the theorem, and the form (2.23) of the curvature tensor R, we obtain

$$3d_R f = (d_J\rho + \alpha) \wedge d_J f - C(f) d_J\alpha = (d_J\rho + \alpha) \wedge \frac{\alpha}{\rho} - d_J\alpha = -\rho d_J \left(\frac{\alpha}{\rho}\right) = 0. \tag{2.20}$$

The third condition *iii)* of the theorem can be written locally as follows

$$D_{\delta/\delta x^i} \frac{\partial f}{\partial y^i} = \frac{\partial}{\partial y^i} \left(\frac{\delta f}{\delta x^i}\right) = 0, \tag{2.21}$$

which means that the components $\omega_i = \delta f/\delta x^i$ are independent of the fibre coordinates. In other words

$$\omega = d_h f = \frac{\delta f}{\delta x^i} dx^i, \tag{2.22}$$

is a basic 1-form on $\mathcal{T}M$. Due to the homogeneity condition for isotropic sprays, the Ricci scalar is given by $\rho = i_{\mathcal{S}}\alpha$. Using formulae (1.13), it can be shown that the class of isotropic sprays can be characterized also in terms of the curvature R of the nonlinear connection, [97, Prop. 3.4],

$$3R = (d_J\rho + \alpha) \wedge J - d_J\alpha \otimes C \tag{2.23}$$

and we obtain

$$0 = d_R f = d_h^2 f = d_h(d_h f) = \frac{1}{2}\left(\frac{\partial \omega_i}{\partial x^j} - \frac{\partial \omega_j}{\partial x^i}\right) dx^i \wedge dx^j = d(d_h f). \tag{2.24}$$

It follows that the basic 1-form $d_h f \in \Lambda^1(M)$ is closed and hence it is locally exact. Therefore, there exists a function a, which is locally defined on M, such that

$$d_h f = da = d_h a. \tag{2.25}$$

We will prove now that the function

$$F = \exp(f - a), \tag{2.26}$$

locally defined on $\mathcal{T}M$, is a Finsler function of scalar flag curvature whose geodesic spray is the given spray $\mathcal{S}$. Depending on the domain of the two functions f and a, the function F might be a conic pseudo-Finsler function. From formula (2.19), we have that $C(F) = \exp(f - a)C(f) = F$, which means that F is 1-homogeneous. Using formula (2.25), we obtain that

$$d_h F = \exp(f - a) d_h(f - a) = 0. \tag{2.27}$$

The semi-basic 1-form α/ρ, which is given by formula (2.18), can be expressed in terms of the function F, given by formula (2.26), as follows

$$\frac{\alpha}{\rho} = \frac{d_J F}{F}.$$

We use now (2.27) and obtain

$$d\left(\frac{\alpha}{\rho}\right) + 2i_{\mathbf{F}}\frac{\alpha}{\rho} \wedge \frac{\alpha}{\rho} = \frac{1}{F^2} dd_J F^2. \tag{2.28}$$

The last condition of the theorem assures that $dd_J F^2$ is a symplectic form and hence F is a Finsler function. Due to formula (2.27), we obtain that $\mathcal{S}$ is the geodesic spray of the Finsler function F. To complete the proof, the last thing we have to show is that F has non-vanishing scalar flag curvature. Since the Finsler function F is given by formula (2.26), we have that $F > 0$ on $\mathcal{T}M$ and we may consider the function

$$\kappa = \frac{\rho}{F^2}. \tag{2.29}$$

It follows that the semi-basic 1-form α is given by

$$\alpha = \frac{\rho}{F} d_J F = \kappa F d_J F. \tag{2.30}$$

Since the Ricci scalar does not vanish, it follows that the function κ has the same property. The last two formulae (2.29) and (2.30) show that the Jacobi endomorphism Φ, of the geodesic spray $\mathcal{S}$ of the Finsler function F, is given by formula (2.6). Therefore, the Finsler function F has non-vanishing scalar flag curvature κ. $\square$

We remark that in the 2-dimensional case, Theorem 2.4.3 covers the Finsler metrizability problem in the most general case. This is due to the fact that any 2-dimensional spray is isotropic and therefore, the Finsler metrizability problem is equivalent to the metrizability by a Finsler function of scalar flag curvature. For the 2-dimensional case, Berwald provides the necessary and sufficient conditions in terms of the curvature scalars, such that the extremals of a Finsler space are rectilinear [15]. Dimension two is also important due to Douglas' work [36], where the inverse problem of the calculus of variation for two degrees of freedom is completely solved. In that particular situation, our Theorem 2.4.3 corresponds to case II, in Douglas' classification.

Hilbert's fourth problem

"Hilbert's fourth problem asks to construct and study the geometries in which the straight line segment is the shortest connection between two points", [4]. Alternatively, the problem can be reformulated as follows: "given a domain $\Omega \subset \mathbb{R}^n$, determine all (Finsler) metrics on Ω whose geodesics are straight lines", [82, p.191]. These Finsler metrics are projectively flat and can be studied using different techniques, [31, 33, 85]. All such Finsler functions have constant or scalar flag curvature, therefore, we can use the conditions of Theorem 2.4.1 and Theorem 2.4.3 to test when a projectively flat spray is Finsler metrizable. For such sprays we can use the algorithms provided in the proofs of the theorems to construct solutions to Hilbert's fourth problem. For working examples see [100].

2.5 Projective Finsler metrizability

Two sprays are called projectively equivalent if their geodesics coincide up to an orientation preserving reparameterization. One can easily see that the sprays $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ are projectively equivalent if and only if there exists a 1-homogeneous function $\mathcal{P} \in C^\infty(\mathcal{T}M)$, called the projective factor, such that $\widetilde{\mathcal{S}} = \mathcal{S} - 2\mathcal{P}C$, [7, 82].

A spray $\mathcal{S}$ is called projective Finsler metrizable if it is projectively equivalent to the geodesic spray of a Finsler function. Several results appeared about the projective Finsler metrizability problem [31, 32, 33, 37, 68, 97]. Various strategies can be chosen to deal with it: in [32] the generalized Helmholtz system was considered, in [33] an approach in terms of 2-forms was formulated. Here we present two approaches considered in [97] and [110, 111] respectively: the first is based on the Helmholtz system, and the second on the Rapcsák system.

Projective metrizability through the Helmholtz conditions

The projective metrizability problem, similarly to the metrizability problem, can be formulated as a particular case of the inverse problem of the calculus of variations. To solve the problem, one of the approaches seeks for the existence of a multiplier matrix that satisfies the Helmholtz conditions [61, 78]. In [25] these conditions where reformulated in terms of a semi-basic 1-form. For the particular case of the projective metrizability problem, it has been shown in [25] that only two of the four Helmholtz conditions are independent. In this subsection we discuss the formal integrability of these conditions using the Spencer-Goldschmidt version of the Cartan-Kähler Theorem.

Theorem 2.5.1. *[97, Theorem 1.] A spray $\mathcal{S}$ is projective Finsler metrizable if and only if there exists a semi-basic 1-form $\theta \in \Lambda^1_v(\mathcal{T}M)$ such that*

$$\operatorname{rank}(d\theta) = 2n-2, \quad i_{\mathcal{S}}\theta > 0, \tag{2.31}$$

$$\mathcal{L}_C\theta = 0, \quad d_J\theta = 0, \quad d_h\theta = 0. \tag{2.32}$$

Proof. We prove first that conditions (2.31) and (2.32) are necessary. We assume that $\mathcal{S}$ is projective metrizable. Therefore, there exists a Finsler function F with geodesic spray $\mathcal{S}_F$ and a 1-homogeneous projective factor $\mathcal{P}$ on $\mathcal{T}M$, such that $\mathcal{S} = \mathcal{S}_F - 2\mathcal{P}C$. Consider $\theta = d_JF$, the Euler-Poincaré 1-form of the Finsler function F. Due to the 1-homogeneity condition of F it follows that $i_{\mathcal{S}}\theta = C(F) = F > 0$. The non-degenerate property of the Finsler energy function implies that $\operatorname{rank}(d\theta) = 2n-2$. Since θ is 0-homogeneous it follows that $\mathcal{L}_C\theta = 0$. Condition $d_J\theta = 0$ is also satisfied since $d_J\theta = d_J^2F = 0$.

It remains to show that $d_h\theta = 0$. The geodesic spray $\mathcal{S}_F$ is uniquely determined by condition (1.17) which in the Finslerian case is

$$i_{\mathcal{S}} dd_J F^2 = -dF^2. \tag{2.33}$$

It follows that $\mathcal{S}_F(F^2) = 0$ and hence $\mathcal{S}_F(F) = 0$. Since $\mathcal{S}_F$ also satisfies condition (2.33) it follows that $\mathcal{L}_{\mathcal{S}_F}(F\theta) = FdF$, which implies $\mathcal{L}_{\mathcal{S}_F}\theta = dF$. Using $\mathcal{S} = \mathcal{S}_F - 2\mathcal{P}C$ we obtain that $\mathcal{L}_{\mathcal{S}}\theta - 2\mathcal{L}_{\mathcal{P}C}\theta = dF$. Using again the 0-homogeneity of the semi-basic 1-form θ it follows $\mathcal{L}_{\mathcal{P}C}\theta = \mathcal{P}\mathcal{L}_C\theta = 0$ and hence $\mathcal{L}_{\mathcal{S}}\theta = dF$. We apply now d_J to both sides of this last relation and use the commutation rules $\mathcal{L}_{\mathcal{S}} d_J - d_J\mathcal{L}_{\mathcal{S}} = d_{[\mathcal{S},J]} = -d_h + d_v$ and $dd_J + d_J d = 0$. Therefore,

$$-d_h\theta - d_v\theta = -dd_J F = d_J dF = d_J\mathcal{L}_{\mathcal{S}}\theta = \mathcal{L}_{\mathcal{S}} d_J\theta + d_h\theta - d_v\theta,$$

from where it follows that $d_h\theta = 0$.

We prove now that the conditions (2.31) and (2.32) are sufficient for the projective metrizability problem of the spray $\mathcal{S}$. Consider $\theta \in \Lambda^1(\mathcal{T}M)$ a semi-basic 1-form that satisfies conditions (2.31) and (2.32). Define the function $F = i_{\mathcal{S}}\theta$. Using the commutation rule $i_{\mathcal{S}} d_J + d_J i_{\mathcal{S}} = \mathcal{L}_C - i_{[\mathcal{S},J]}$ as well as conditions $d_J\theta = 0$ and $\mathcal{L}_C\theta = 0$ it follows that $d_J F = d_J i_{\mathcal{S}}\theta = i_h\theta = \theta$. Hence θ is the Euler-Poincaré 1-form of F. Now conditions (2.31) assure that F is a Finsler function. Consider the function $\mathcal{P} \in C^\infty(\mathcal{T}M)$ given by $2\mathcal{P} = \mathcal{S}(F)/F$, which is 1-homogeneous. We will show now that the spray $\tilde{\mathcal{S}} = \mathcal{S} - 2\mathcal{P}C$ satisfies equation (2.33) and hence it is the geodesic spray of the Finsler function F.

Using the commutation rule $i_{\mathcal{S}} d_h + d_h i_{\mathcal{S}} = \mathcal{L}_{\mathcal{S}} - i_{[\mathcal{S},h]}$ and the fact that $d_h\theta = 0$ it follows

$$0 = i_{\mathcal{S}} d_h\theta = -d_h i_{\mathcal{S}}\theta + \mathcal{L}_{\mathcal{S}}\theta - i_{[\mathcal{S},h]}\theta.$$

From the fact that $i_{[\mathcal{S},h]}\theta = dF \circ J \circ \mathcal{L}_{\mathcal{S}} h = dF \circ v = d_v F$ it follows that $\mathcal{L}_{\mathcal{S}}\theta = d_h F + d_v F = dF$. We show now that $\tilde{\mathcal{S}}$ satisfies the same equation. Indeed $\mathcal{L}_{\tilde{\mathcal{S}}}\theta = \mathcal{L}_{\mathcal{S}-2\mathcal{P}C}\theta = dF$ since $\mathcal{L}_{\mathcal{P}C}\theta = 0$. From the defining formula of function P is follows that

$$\tilde{\mathcal{S}}(F) = \mathcal{S}(F) - 2\mathcal{P}C(F) = \mathcal{S}(F) - 2\mathcal{P}F = 0.$$

Therefore

$$\mathcal{L}_{\tilde{\mathcal{S}}} d_J F^2 = 2F\mathcal{L}_{\tilde{\mathcal{S}}} d_J F = 2FdF = dF^2,$$

and hence $\tilde{\mathcal{S}}$ is the geodesic spray of the Finsler function F.

$\square$

The second part of the proof of Theorem 2.5.1 shows that if there exists a semi-basic 1-form θ on $\mathcal{T}M$ that satisfies the conditions (2.31) and (2.32) then the given spray $\mathcal{S}$ is projectively related to the spray

$$\mathcal{S}_F = \mathcal{S} - \frac{\mathcal{L}_{\mathcal{S}}(i_{\mathcal{S}}\theta)}{i_{\mathcal{S}}\theta} C,$$

which is the geodesic spray of the Finsler function $F = i_S\theta$. In this case, the semi-basic 1-form $\theta = \theta_i dx^i$ is the Poincaré-Cartan 1-form of the Finsler function F, $\theta = d_J F$. Therefore,

$$\theta_i = \frac{\partial F}{\partial y^i}, \quad h_{ij} = F\frac{\partial \theta_i}{\partial y^j}, \quad F d\theta = h_{ij}\delta y^i \wedge dx^j. \tag{2.34}$$

Formulae (2.34) show the relationship between the 1-form θ, the solution of the projective metrizability problem using Theorem 2.5.1, and the classical approach of the problem using the multiplier matrix h_{ij}.

In order to study the formal integrability of the system (2.32) we consider the first order partial differential operator $P\colon \Lambda^1_v(\mathcal{T}M) \to \Lambda^1_v(\mathcal{T}M) \oplus \Lambda^2_v(\mathcal{T}M) \oplus \Lambda^2_v(\mathcal{T}M)$, which we call *the projective metrizability operator*

$$P = (\mathcal{L}_C, d_J, d_h). \tag{2.35}$$

It induces a morphism of vector bundles

$$p^0(P) : J^1T^*_v \to F_1 := T^*_v \oplus \Lambda^2 T^*_v \oplus \Lambda^2 T^*_v,$$

where

$$p^0(P)(j^1\theta) = \left(\frac{\partial \theta_i}{\partial y^j} y^j dx^i,\ \frac{1}{2}\left(\frac{\partial \theta_i}{\partial y^j} - \frac{\partial \theta_j}{\partial y^i}\right) dx^j \wedge dx^i,\ \frac{1}{2}\left(\frac{\delta \theta_i}{\delta x^j} - \frac{\delta \theta_j}{\delta x^i}\right) dx^j \wedge dx^i\right).$$

One can show, that there exists a quasi-regular basis, therefore the symbol of the projective metrizability operator is involutive. Moreover, the integrability condition is given by the following

Proposition 2.5.2. *[97, Theorem 3.] A first order formal solution $\theta \in \Lambda^1 T^*$ of the system* (2.32) *can be lifted into a second order solution if and only if $d_R\theta = 0$, where R is the curvature tensor* (1.11).

Proof. Let us define $K = \oplus^{(2)} \Lambda^2 T^*_v \oplus^{(3)} \Lambda^3 T^*_v$ and the morphism $\tau = (\tau_1, ..., \tau_5)$ as follows

$$\begin{aligned}
\tau_1(A, B_1, B_2)(X, Y) &= A(JX, Y) - A(JY, X) - B_1(C, X, Y),\\
\tau_2(A, B_1, B_2)(X, Y) &= A(hX, Y) - A(hY, X) - B_2(C, X, Y),\\
\tau_3(A, B_1, B_2)(X, Y) &= B_1(JX, Y) - B_1(JY, X),\\
\tau_4(A, B_1, B_2)(X, Y) &= B_2(hX, Y) - B_2(hY, X),\\
\tau_5(A, B_1, B_2)(X, Y) &= B_1(hX, Y) - B_1(hY, X)B_1 + B_2(JX, Y) - B_2(JY, X),
\end{aligned}$$

for $A \in T^* \otimes T_v^*$, $B_1, B_2 \in T^* \otimes \Lambda^2 T_v^*$. We have a commutative diagram

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & g^2(P) \arrow[r] \arrow[d] & S^2T^* \otimes T_v^* \arrow[r, "\sigma^2(P)"] \arrow[d, "\varepsilon"] & T^* \otimes F_1 \arrow[r, shift left, dashed, "\tau"] \arrow[r, shift right] \arrow[d, shift left, "\varepsilon"] & K \arrow[r] & 0 \\
0 \arrow[r] & Sol_2(P) \arrow[r, "i"] \arrow[d, "\overline{\pi}_1"] & J_2T_v^* \arrow[r, shift left, dashed, "p^1(P)"] \arrow[r, shift right] \arrow[d, shift left, "\pi_1"] & J^1F_1 \arrow[u, shift left, dashed, "\nabla"] \arrow[d, "\pi"] & & \\
0 \arrow[r] & Sol_1(P) \arrow[r, shift left, dashed, "i"] \arrow[r, shift right] & J^1T_v^* \arrow[r, "p^o(P)"] \arrow[u, shift left, dashed] \arrow[d] & F_1 \arrow[d] & & \\
 & & 0 & 0 & &
\end{tikzcd}
\tag{2.36}
$$

Consider a linear connection ∇ on $\mathcal{T}M$ such that $\nabla J = 0$. It follows that the connection ∇ preserves the vertical distribution and hence it will preserve semi-basic forms. Therefore, one can view ∇ as a connection on the fibre bundle $F_1 \to \mathcal{T}M$. It follows that derivations $\mathcal{D}_C = i_C\nabla$, $\mathcal{D}_J = \tau_J\nabla$, and $\mathcal{D}_h = \tau_h\nabla$ preserve semi-basic forms. As a first order partial differential operator, we can identify a connection ∇ with the bundle morphism $p^0(\nabla) : J^1F_1 \to T^* \otimes F_1$. We will use this morphism to define the map $\varphi : Sol_1(P) \to K$ we mentioned in Remark 2.2.3. The morphism φ is represented by the dashed path in the diagram (2.36).

Consider $\theta \in \Lambda_v^1(\mathcal{T}M)$ such that $j_u^1\theta \in Sol_{1,u}(P) \subset J_u^1T_v^*$ is a first order solution of P at $u \in \mathcal{T}M$. Then, we have

$$\varphi_u\theta = \tau_u\nabla(P\theta) = \tau_u(\nabla\mathcal{L}_C\theta, \nabla d_J\theta, \nabla d_h\theta).$$

Since $\mathcal{L}_C\theta$, $d_J\theta$, and $d_h\theta$ vanish at $u \in \mathcal{T}M$, it follows that when acting on these semi-basic forms we have $\mathcal{D}_C = \mathcal{L}_C$, $\mathcal{D}_J = d_J$ and $\mathcal{D}_h = d_h$. Using the fact that $[J,C] = J$, $[h,C] = 0$, $[J,J] = 0$, and $[h,J] = 0$, it follows that

$$
\begin{aligned}
\tau_1\left(\nabla P\theta\right)_u &= \left(\tau_J\nabla\mathcal{L}_C\theta - i_C\nabla d_J\theta\right)_u = \left(d_J\mathcal{L}_C\theta - \mathcal{L}_C d_J\theta\right)_u = (d_{[J,C]}\theta)_u = 0,\\
\tau_2\left(\nabla P\theta\right)_u &= \left(\tau_h\nabla\mathcal{L}_C\theta - i_C\nabla d_h\theta\right)_u = \left(d_h\mathcal{L}_C\theta - \mathcal{L}_C d_h\theta\right)_u = (d_{[h,C]}\theta)_u = 0,\\
\tau_3\left(\nabla P\theta\right)_u &= \left(\tau_J\nabla d_J\theta\right)_u = \left(d_J^2\theta\right)_u = \tfrac{1}{2}(d_{[J,J]}\theta)_u = 0,\\
\tau_4\left(\nabla P\theta\right)_u &= \left(\tau_h\nabla d_h\theta\right)_u = \left(d_h^2\theta\right)_u = \tfrac{1}{2}(d_{[h,h]}\theta)_u = (d_R\theta)_u,\\
\tau_5\left(\nabla P\theta\right)_u &= \left(\tau_h\nabla d_J\theta + \tau_J\nabla d_h\theta\right)_u = (d_{[h,J]}\theta)_u = 0.
\end{aligned}
$$

From the above calculations follows that a first order formal solution θ of the system (2.32) can be lifted into a second order solution if and only if $d_R\theta = 0$.

□

We present now some particular classes of sprays for which the projective metrizability operator (2.35) is integrable, and hence these sprays are projective metrizable. These classes of sprays are:

i) flat sprays, $(R = 0)$;

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ii) isotropic sprays, ($R=\alpha\wedge J+\beta\otimes C$, for α a semi-basic 1-form and β a semi-basic 2-form on $\mathcal{T}M$;

iii) arbitrary sprays on 2-dimensional manifolds.

For each of these classes of sprays, one can show that the curvature obstruction is automatically satisfied and hence in the analytic case the projective metrizability problem has a solution.

Indeed, in the flat case *i)*, the obstruction is identically satisfied. The fact that flat sprays are projective metrizable was already demonstrated with other methods in [29]. Assume now *ii)*, that is, the spray $\mathcal{S}$ is isotropic. It follows that the curvature tensor has the form $R=\alpha\wedge J+\beta\otimes C$, for $\alpha\in\Lambda^1_v$ and $\beta\in\Lambda^2_v$. Then, for a semi-basic 1-form θ on $\mathcal{T}M$, we have

$$d_R\theta=\alpha\wedge d_J\theta+\beta\otimes\mathcal{L}_C\theta. \tag{2.37}$$

If θ is a solution of the differential system (2.32) it follows that $\mathcal{L}_C\theta=0$ and $d_J\theta=0$, and using formula (2.37) it follows that $d_R\theta=0$. Therefore, the obstruction for the formal integrability of P is satisfied. In [30] it has been shown that any isotropic sprays is projectively equivalent to a flat spray and hence it is projectively metrizable.

If $\dim M=2$ then for a semi-basic 1-form θ on $\mathcal{T}M$, $d_R\theta$ is a semi-basic 3-form and hence it vanishes. It has been shown by Matsumoto [64] that every spray on a surface is projectively related to a Finsler spray, using the results of Darboux about second order differential equations.

Projective metrizability through the Rapcsák system

András Rapcsák in [75] obtained necessary and sufficient conditions for the projective Finsler metrizability in terms of a second order PDE system, now called *Rapcsák equations* [32, 91, 82]. Rapcsák's approach is simple and natural: one finds conditions directly on the Finsler function that one seeks for. In this subsection, we consider the Rapcsák system composed of the 1-homogeneity condition and the Rapcsák equations.

Proposition 2.5.3. *[110, Proposition 3.1] A spray $\mathcal{S}$ is projective Finsler metrizable if and only if there exists a Finsler function $\widetilde{F}\colon TM\to\mathbb{R}$, such that*

$$i_{\mathcal{S}}dd_J\widetilde{F}=0. \tag{2.38}$$

Proof. The spray $\mathcal{S}$ is projective Finsler metrizable if and only if there exists a Finsler metrizable spray $\widetilde{\mathcal{S}}$ which is projective equivalent to $\mathcal{S}$. Because of the projective equivalence, there exists a function $\mathcal{P}$, such that $\widetilde{\mathcal{S}}=\mathcal{S}-2\mathcal{P}C$. Let us denote by $\widetilde{F}$ the Finsler function associated to $\widetilde{\mathcal{S}}$. It is well known that $\widetilde{F}$ is invariant by the parallel translation associated to the connection $\widetilde{\Gamma}=[J,\widetilde{\mathcal{S}}]$ and therefore we have $d_{\widetilde{h}}\widetilde{F}=0$. Using the relation $\widetilde{h}=h-\mathcal{P}J-d_J\mathcal{P}\otimes C$ between the horizontal projectors [98, Proposition 4.4] and the 1-homogeneity of $\widetilde{F}$, we get

$$0=d_{\widetilde{h}}\widetilde{F}=d_h\widetilde{F}-d_{\mathcal{P}J}\widetilde{F}-d_J\mathcal{P}C\widetilde{F}=d_h\widetilde{F}-d_J(\mathcal{P}\widetilde{F}). \tag{2.39}$$

Substituting $\mathcal{S}$ into (2.39), using $J\mathcal{S} = C$ and the homogeneity of $\widetilde{F}$ and $\mathcal{P}$, we get $i_{\mathcal{S}} d_{\widetilde{h}} \widetilde{F} = 0$, and we find that the projective factor is $\mathcal{P} = \frac{1}{2\widetilde{F}} \mathcal{S}\widetilde{F}$. Replacing $\mathcal{P}$ in (2.39) by the above expression we get

$$d_h \widetilde{F} - d_J \Big(\frac{1}{2\widetilde{F}} (\widetilde{F}\, d_{\mathcal{S}} \widetilde{F}) \Big) = d_h \widetilde{F} - \frac{1}{2} d_J (d_{\mathcal{S}} \widetilde{F}) = 0.$$

Using (1.5) and the relation $d_{[J,\mathcal{S}]} = d_J d_{\mathcal{S}} - d_{\mathcal{S}} d_J$ we obtain

$$0 = d_{\Gamma+I} \widetilde{F} - d_J d_{\mathcal{S}} \widetilde{F} = -i_{\mathcal{S}} d d_J \widetilde{F} - dC\widetilde{F} + d\widetilde{F} = -i_{\mathcal{S}} d d_J \widetilde{F}. \qquad \square$$

We note that a coordinate version of the above theorem was proved by A. Rapcsák in [75], and a coordinate free versions were given in [54, 91]. Here we presented a different proof.

According to Proposition 2.5.3, the projective metrizability leads to the investigation of the *Rapcsák system*, that is the partial differential system composed by the 1-homogeneity condition and the equation (2.38).

We remark that the Rapcsák system is equivalent to the PDE system composed by the Euler-Lagrange equation (1.21) and the homogeneity condition (1.1) with $k = 1$. Therefore, it is similar to the Finsler metrizability's system (see Paragraph 2.3) which is composed by the Euler-Lagrange equation (1.21) and the homogeneity condition (1.1) with $k = 2$. This is why one might think that the geometric investigations of the two systems are analogous. Unfortunately, this is not the case: a 2-homogeneous Euler-Lagrange system can be reduced to a first order partial differential system (see Theorem 2.3.1), but the method cannot be applied for the 1-homogeneous Euler-Lagrange system.

The differential operator corresponding to the Rapcsák system is $P_1 = (P_{\mathcal{S}}, P_C)$, where

$$P_{\mathcal{S}}(F) = i_{\mathcal{S}} d d_J F, \qquad P_C(F) = \mathcal{L}_C F - F,$$

that is $P_{\mathcal{S}} F = 0$ corresponds to (2.38) and $P_C F = 0$ to the 1-homogeneity property of F. The local expression of the PDE equations represented by $P_{\mathcal{S}}$ and P_C are

$$\Big(y^j \frac{\partial^2 F}{\partial x^j \partial y^i} + f^j \frac{\partial^2 F}{\partial y^j \partial y^i} - \frac{\partial F}{\partial x^j} \Big) dx^i - \Big(\frac{\partial F}{\partial y^i} + y^j \frac{\partial^2 F}{\partial y^i \partial x^j} - \frac{\partial F}{\partial x^i} \Big) dy^i = 0,$$
$$y^i \frac{\partial F}{\partial y^i} - F = 0.$$

The computation to show the formal integrability of the Rapcsák system is similar to that of the Euler-Lagrange system investigated in [105]: the first integrability condition contains new PDE equations which can be expressed in terms of the associated connection $\Gamma = [J, \mathcal{S}]$. Considering the system completed with these integrability conditions – called extended Rapcsák system [110] – its compatibility conditions can be expressed in terms of the curvature tensor. More precisely, one can show that the compatibility conditions are equivalent to the equation $i_\Phi dd_J F = 0$ or $i_W dd_J F = 0$, where Φ is the Jacobi endomorphism and

$$W = \Phi - \frac{1}{n-1}(\mathrm{Tr}\Phi)J + \frac{1}{2(n-1)} d_J(\mathrm{Tr}\Phi) \otimes C$$

is the Weyl tensor associated to $\mathcal{S}$. One obtains the following

Theorem 2.5.4. *[110, Theorem 4.6] Let $\mathcal{S}$ be a spray on a manifold M. The extended Rapcsák system is formally integrable if and only if the spray is of isotropic curvature.*

From the theorem one can get the following

Corollary 2.5.5. *[110, Corollary 4.7] Let $\mathcal{S}$ be a analytic spray on an analytic manifold M. If $\dim M = 2$, or $\dim M \geq 3$ and $\mathcal{S}$ is of isotropic curvature, then $\mathcal{S}$ is locally projective Finsler metrizable.*

To solve the projective metrizability problem in the non-isotropic case, one has to consider the extended Rapcsák system enlarged with the curvature conditions. The difficulties come from the fact that, as it was proved in [110] by using the classical Cartan-Kähler theory, the symbol of the operator is not involutive and the Cartan test fails. It follows that the system is not integrable: higher order compatibility conditions exist. Using the Spencer technique, the level where these higher order integrability conditions appear can be calculated. In [111] we obtained new results on the n-dimensional case, where the eigenvalues of the Jacobi tensor are pairwise different. We identified the higher-order compatibility conditions causing the non 2-acyclicity of the Spencer cohomology sequences. We also considered the three-dimensional case, where we have found a class of non-isotropic sprays for which the PDE system is integrable and, consequently, the corresponding SODEs are projective metrizable.

2.6 Projective rigidity of the geodesic structure

In this section, we consider projective deformations of the geodesic structure of Finsler manifolds. We note that if $\mathcal{S}$ and $\widetilde{\mathcal{S}} = \mathcal{S} - 2\mathcal{P}C$ are projectively equivalent sprays on the manifold M with projective factor $\mathcal{P} \in C^\infty(\mathcal{T}M)$, then the connections, Jacobi endomorphisms, and curvature tensors of the two sprays are related by the following formulae [98, Proposition 4.4]:

$$\begin{aligned}
\widetilde{\Gamma} &= \Gamma - 2(\mathcal{P}J + d_J\mathcal{P} \otimes C),\\
\widetilde{h} &= h - \mathcal{P}J - d_J\mathcal{P} \otimes C,\\
\widetilde{v} &= v + \mathcal{P}J + d_J\mathcal{P} \otimes C,\\
\widetilde{\Phi} &= \Phi + (\mathcal{P}^2 - \mathcal{S}(\mathcal{P}))J + (2d_h\mathcal{P} - \mathcal{P}d_J\mathcal{P} - \nabla d_J\mathcal{P}) \otimes C,\\
\widetilde{R} &= R + d_J d_h\mathcal{P} \otimes C + (\mathcal{P}d_J\mathcal{P} - d_h\mathcal{P}) \wedge J.
\end{aligned} \tag{2.40}$$

Projective deformation with the Finsler function

Let $\mathcal{S}$ be the geodesic spray of the Finsler manifold (M, F). Since the projective factor $\mathcal{P}$ has to be a 1-homogeneous function on the manifold TM, the most obvious and natural choice is $\mathcal{P} = \lambda F$, where $\lambda \in \mathbb{R}$ is a non-zero constant and F is the Finsler function determining the geodesic structure. That leads us to consider the projective deformation

$$\widetilde{\mathcal{S}} = \mathcal{S} - 2\lambda F C \tag{2.41}$$

of the geodesic spray $\mathcal{S}$. We have the following

Theorem 2.6.1. *[98, Theorem 5.1] Let $\mathcal{S}$ be the geodesic spray associated to the Finsler function F. Then the projective deformation* (2.41) *of $\mathcal{S}$ is not Finsler metrizable for almost every value of $\lambda \in \mathbb{R}$.*

The theorem shows how rigid the Finsler metrizability property is with respect to the corresponding reparameterization of the geodesics. The proof of the theorem is based on the investigation of the properties of the holonomy distribution associated to the deformed spray $\widetilde{\mathcal{S}}$. With some computations, one can show that for almost every value of $\lambda \in \mathbb{R}$, the deformed holonomy distribution $\widetilde{\mathcal{D}}_{\mathcal{H}}$ contains the Liouville vector field, and using Corollary 2.3.2 we get that the corresponding sprays are not metrizable. We obtain that the projective class of an arbitrary spray contains infinitely many sprays which are not Finsler metrizable. We remark that the particular case when the geodesic spray $\mathcal{S}$ of a Finsler function is flat and has constant flag curvature was investigated by Yang in [95] using different techniques.

Holonomy invariant projective deformation

In [103] we considered the case where the projective factor is invariant with respect to parallel translations, that is the projective factor is a holonomy invariant function. One can extend the results of Theorem 2.6.1 by proving the following

Theorem 2.6.2. *[103, Theorem 1] For any nontrivial holonomy invariant 1-homogeneous projective factor $\mathcal{P}$ and for almost any scalar $\lambda \in \mathbb{R}$, the projective deformation $\widetilde{\mathcal{S}} = \mathcal{S} - 2\lambda\mathcal{P}\, C$ of a Finsler metrizable spray $\mathcal{S}$ is not metrizable.*

Indeed, from (2.40) one can get that the geometric quantities associated to the projectively deformed spray $\widetilde{\mathcal{S}}$ are

$$\widetilde{h} = h - \lambda(\mathcal{P}J + d_J\mathcal{P} \otimes C), \tag{2.42a}$$

$$\widetilde{v} = v + \lambda(\mathcal{P}J + d_J\mathcal{P} \otimes C), \tag{2.42b}$$

$$\widetilde{\Phi} = \Phi + \lambda^2(\mathcal{P}^2 J - \mathcal{P} d_J\mathcal{P} \otimes C), \tag{2.42c}$$

For further computation and analysis, it will be very useful to introduce a decomposition of the horizontal (resp. the vertical) distributions adapted to a holonomic projective deformation associated to the projective factor $\mathcal{P}$: we introduce the endomorphsims

$$h_{\mathcal{P}} = h - \frac{d_J\mathcal{P}}{\mathcal{P}} \otimes \mathcal{S}, \qquad v_{\mathcal{P}} = v - \frac{d_v\mathcal{P}}{\mathcal{P}} \otimes C. \tag{2.43}$$

and we set $\mathcal{H}_{\mathcal{P}} := Im\, h_{\mathcal{P}}$ and $\mathcal{V}_{\mathcal{P}} := Im\, v_{\mathcal{P}}$. We have $J(\mathcal{H}_{\mathcal{P}}) = \mathcal{V}_{\mathcal{P}}$.

Lemma 2.6.3.

1. *Properties of $v_{\mathcal{P}}$ and $\mathcal{V}_{\mathcal{P}}$:*
   - *i)* $\ker(v_{\mathcal{P}}) = HTM \oplus Span\{C\}$
   - *ii)* $\mathrm{Im}(v_{\mathcal{P}}) = \mathcal{V}_{\mathcal{P}}$ *is an $(n-1)$-dimensional involutive subdistribution of VTM,*
   - *iii) any $X \in \mathcal{V}_{\mathcal{P}}$ is an infinitesimal symmetry of $\mathcal{P}$ that is $\mathcal{L}_X\mathcal{P} = 0$.*
   - *iv) the vertical distribution have the decomposition $VTM = \mathcal{V}_{\mathcal{P}} \oplus Span\{C\}$.*
2. *Properties of $h_{\mathcal{P}}$ and $\mathcal{H}_{\mathcal{P}}$:*
   - *i)* $\ker(h_{\mathcal{P}}) = VTM \oplus Span\{S\}$
   - *ii)* $\mathrm{Im}(h_{\mathcal{P}}) = \mathcal{H}_{\mathcal{P}}$ *is an $(n-1)$-dimensional subdistribution of HTM,*
   - *iii) any $X \in \mathcal{H}_{\mathcal{P}}$ is an infinitesimal symmetry of $\mathcal{P}$ that is $\mathcal{L}_X\mathcal{P} = 0$.*
   - *iv) the horizontal distribution have the decomposition $HTM = \mathcal{H}_{\mathcal{P}} \oplus Span\{\mathcal{S}\}$,*

We show *1.)* in detail. The computations for *2.)* are similar.

Proof. i) We note that $HTM = \mathrm{Ker}\, v$, therefore $HTM \subset \mathrm{Ker}\, v_{\mathcal{P}}$. Moreover, if $V \in \ker v_{\mathcal{P}}$ is vertical, then using $v(V) = V$ we get $v_{\mathcal{P}}(V) = 0$ if and only if $V = \frac{V(\mathcal{P})}{P} C$, that is $V \in \mathrm{Span}\{C\}$ and we get *i)*.

ii) We will use the simplified notation $\dot{\partial}_i := \frac{\partial}{\partial y^i}$ and $\mathcal{P}_i := \frac{\partial \mathcal{P}}{\partial y^i}$, $\mathcal{P}_{ij} := \frac{\partial^2 \mathcal{P}}{\partial y^i \partial y^j}$ etc. Let us define the vector fields

$$h_i := h_{\mathcal{P}}(\delta_i) = \delta_i - \frac{\mathcal{P}_i}{\mathcal{P}}\mathcal{S}, \quad v_i := v_{\mathcal{P}}(\dot{\partial}_i) = \dot{\partial}_i - \frac{\mathcal{P}_i}{P}C \tag{2.44}$$

for $i = 1, \ldots, n$. We get

$$\mathcal{H}_{\mathcal{P}} = \mathrm{Span}\{h_1, \ldots, h_n\}, \quad \mathcal{V}_{\mathcal{P}} = \mathrm{Span}\{v_1, \ldots, v_n\}. \tag{2.45}$$

Because the 1-homogeneity property of $\mathcal{P}$, for any $v_i, v_j \in \mathcal{V}_{\mathcal{P}}$, their Lie bracket is

$$[v_i, v_j] = \Big[\dot{\partial}_i - \frac{\mathcal{P}_i}{\mathcal{P}} y^k \dot{\partial}_k,\ \dot{\partial}_j - \frac{\mathcal{P}_j}{\mathcal{P}} y^\ell \dot{\partial}_\ell\Big] = \frac{\mathcal{P}_i}{\mathcal{P}} v_j - \frac{\mathcal{P}_j}{\mathcal{P}} v_i$$

and from (2.45) we get that $[v_i, v_j] \in \mathcal{V}_{\mathcal{P}}$, hence $\mathcal{V}_{\mathcal{P}}$ is involutive.

iii) One can check that the generators (2.45) of the distribution $\mathcal{V}_{\mathcal{P}}$ are infinitesimal symmetries of $\mathcal{P}$. Indeed, $\mathcal{L}_C\mathcal{P} = \mathcal{P}$, and therefore

$$\mathcal{L}_{v_i} P = \dot{\partial}_i(\mathcal{P}) - \frac{\mathcal{P}_i}{\mathcal{P}} C(\mathcal{P}) = \mathcal{P}_i - \frac{\mathcal{P}_i}{\mathcal{P}}\mathcal{P} = 0. \tag{2.46}$$

iv) Supposing $C \in \mathcal{V}_{\mathcal{P}}$ we get form (2.45) that $C = C^i v_i$ with some coefficients C^i. From *iii)* we get $v_i(\mathcal{P}) = 0$, $i = 1, \ldots, n$. On the other hand, because of the 1-homogeneity of $\mathcal{P}$, we have $C(\mathcal{P}) = \mathcal{P}$. Then $\mathcal{P} = C(\mathcal{P}) = C^i v_i(\mathcal{P}) = 0$ which is a contradiction.

□

In order to investigate the curvature property of the projective deformed structure, let us fix an arbitrary point $(x, y) \in \mathcal{T}M$. Then for almost any value of $\lambda \in \mathbb{R}$ the inequality

$$\kappa_i(x, y) + \lambda^2\mathcal{P}^2(x, y) \neq 0, \tag{2.47}$$

holds for any $i = 1, \ldots, n$, where $\kappa_n = 0$ and κ_i, $i = 1, \ldots, n-1$ denote the principal curvatures of the Finsler metric F. From the continuity of the eigenfunctions of Φ we get that there is an open neighbourhood $U \subset \mathcal{T}M$ of (x, y) such that the condition (2.47) is satisfied. From now on, geometric objects will be considered on this neighbourhood.

Lemma 2.6.4. *For any nonzero $\lambda \in \mathbb{R}$ such that* (2.47) *holds, the image of the Jacobi endomorphism $\widetilde{\Phi}$ of $\widetilde{S}$ is $\mathcal{V}_{\mathcal{P}}$, that is* $\operatorname{Im}\widetilde{\Phi} = \mathcal{V}_{\mathcal{P}}$.

Proof. $\widetilde{\Phi}$ is determined by (2.42c). Its image can be calculated by using horizontal vectors. We will use the horizontal eigenvecotrs $\{X_1, \ldots, X_n\}$ of Φ introduced in (2.7). We have $X_n = \mathcal{S}$ and

$$d_J\mathcal{P}(\mathcal{S}) = d_{J\mathcal{S}}\mathcal{P} = d_C\mathcal{P} = \mathcal{P},$$

hence we obtain

$$\widetilde{\Phi}(\mathcal{S}) = \Phi(\mathcal{S}) + \lambda^2\mathcal{P}^2J\mathcal{S} - \lambda^2\mathcal{P}\, d_J\mathcal{P}(\mathcal{S}) \otimes C = 0 + \lambda^2\mathcal{P}^2C - \lambda^2\mathcal{P}^2C = 0.$$

For $1 \leq i < n$ we have $X_i \in \mathcal{H}_{\mathcal{P}}$ and $JX_i \in \mathcal{V}_{\mathcal{P}}$. From (*1.iii*) of Lemma 2.6.3 we get $d_J\mathcal{P}(X_i) = \mathcal{L}_{JX_i}\mathcal{P} = 0$. It follows that

$$\widetilde{\Phi}(X_i) = \Phi(X_i) + \lambda^2(\mathcal{P}^2J - \mathcal{P}d_J\mathcal{P} \otimes C)(X_i) = (\kappa_i + \lambda^2\mathcal{P}^2)JX_i. \tag{2.48}$$

Using (2.47) we get that $JX_i \in \operatorname{Im}\widetilde{\Phi}$, and $\operatorname{Im}\widetilde{\Phi} = \operatorname{Span}\{JX_1, \ldots, JX_{n-1}\} = \mathcal{V}_{\mathcal{P}}$. □

Since the image of the Jacobi endomorphism $\widetilde{\Phi}$ is a subspace of the holonomy distribution $\widetilde{\mathcal{D}}_{\mathcal{H}}$ we get that under the hypothesis of Lemma 2.6.4:

$$\mathcal{V}_{\mathcal{P}} \subset \widetilde{\mathcal{D}}_{\mathcal{H}}. \tag{2.49}$$

Proposition 2.6.5. *If the projective factor $\mathcal{P}$ is nonlinear and $\lambda \neq 0$ satisfies* (2.47) *on $U \subset TM$, then the holonomy distribution of the non-trivial projectively deformed spray $\widetilde{S} = \mathcal{S} - 2\lambda\mathcal{P}C$ is the full second tangent space, that is $\widetilde{\mathcal{D}}_{\mathcal{H}} = TTM$.*

Proof. The holonomy distribution $\widetilde{\mathcal{D}}_{\mathcal{H}}$ of the spray $\widetilde{S}$ contains its horizontal space $\widetilde{H}TM$ and the image of the the Riemann curvature $\widetilde{\Phi}$, therefore, from Lemma 2.6.4 we get that

$$\widetilde{H}TM \oplus \mathcal{V}_{\mathcal{P}} \subset \widetilde{\mathcal{D}}_{\mathcal{H}}. \tag{2.50}$$

It follows that $\widetilde{h}_i := \widetilde{h}(h_i)$ and v_i are elements of $\widetilde{\mathcal{D}}_{\mathcal{H}}$. By the involutivity of $\widetilde{\mathcal{D}}_{\mathcal{H}}$ the Lie bracket $[\widetilde{h}_i, v_i]$ and its horizontal part are in $\widetilde{\mathcal{D}}_{\mathcal{H}}$, so its vertical part is too:

$$\widetilde{v}[\widetilde{h}_i, v_j] \in \widetilde{\mathcal{D}}_{\mathcal{H}}. \tag{2.51}$$

On the other hand, we get from (2.42a) $\widetilde{h}_i = h_i - \lambda\mathcal{P}v_i$, and hence, taking $\mathcal{L}_{v_i}\mathcal{P} = 0$ into account, we have $\widetilde{v}[\widetilde{h}_i, v_j] = \widetilde{v}[h_i, v_j] - \lambda\mathcal{P}\widetilde{v}[v_i, v_j]$. Since the distribution $\mathcal{V}_{\mathcal{P}}$ is integrable $\widetilde{v}$ is the identity on $\mathcal{V}_{\mathcal{P}}$ and we have $\widetilde{v}[v_i, v_j] = [v_i, v_j] \in \mathcal{V}_{\mathcal{P}}$. Therefore, we get that

$$\widetilde{v}[h_i, v_j] \in \widetilde{\mathcal{D}}_{\mathcal{H}}. \tag{2.52}$$

On the other hand, one can obtain that $v[h_i, v_j] = \left(G^k_{ij} - \frac{\mathcal{P}_i}{\mathcal{P}}G^k_j\right)v_k$, from which we get that $v[h_i, v_j] \in \mathcal{V}_{\mathcal{P}}$ and

$$v[h_i, v_j] \in \widetilde{\mathcal{D}}_{\mathcal{H}}. \tag{2.53}$$

By (2.42b), we have

$$\widetilde{v}[h_i, v_j] - v[h_i, v_j] = \lambda\mathcal{P}J[h_i, v_j] + \lambda\mathcal{L}_{J[h_i,v_j]}\mathcal{P}\,C \tag{2.54}$$

and because the left-hand side is in $\widetilde{\mathcal{D}}_{\mathcal{H}}$, so is the right-hand side:

$$\mathcal{P}\cdot J[h_i, v_j] + \mathcal{L}_{J[h_i,v_j]}\mathcal{P}\,C \quad \in \widetilde{\mathcal{D}}_{\mathcal{H}}. \tag{2.55}$$

Since $J[h_i, v_j] = \frac{\mathcal{P}_i}{\mathcal{P}}v_j + \frac{\mathcal{P}_{ij}}{\mathcal{P}}C$, from (*1.iii*) of Lemma 2.6.3 we get

$$\mathcal{P}\cdot J[h_i, v_j] + \mathcal{L}_{J[h_i,v_j]}\mathcal{P}\cdot C = \mathcal{P}_i v_j + 2\mathcal{P}_{ij}C. \tag{2.56}$$

Using (2.49) and (2.55) we get that $\mathcal{P}_{ij}C \in \widetilde{\mathcal{D}}_{\mathcal{H}}$, and since $\mathcal{P}$ is non linear,

$$C \in \widetilde{\mathcal{D}}_{\mathcal{H}}. \tag{2.57}$$

Completing (2.50) with $\mathrm{Span}\{C\}$ we get $\widetilde{H}TM \oplus \mathcal{V}_{\mathcal{P}} \oplus \mathrm{Span}\{C\} \subset \widetilde{\mathcal{D}}_{\mathcal{H}}$. According to (*1.iv*) of Lemma 2.6.3 we have $\mathcal{V}_{\mathcal{P}} \oplus \mathrm{Span}\{C\} = VTM = \widetilde{V}TM$, therefore

$$\widetilde{\mathcal{D}}_{\mathcal{H}} = \widetilde{H}TM \oplus \widetilde{V}TM = TTM, \tag{2.58}$$

which proves the proposition.

□

Proposition 2.6.6. *[103, Proposition 4.1] Let $\lambda \in \mathbb{R}$ be such that* (2.47) *holds. If* $\dim M > 1$ *and* $\lambda \neq 0$*, then the projectively deformed spray $\widetilde{S} = \mathcal{S} - 2\lambda\mathcal{P}C$ is not metrizable.*

Proof. Depending on the linearity of the projective factor $\mathcal{P}$ we consider two cases. If the projective factor $\mathcal{P}$ is nonlinear, from Proposition 2.6.5 we get that $\widetilde{\mathcal{D}}_{\mathcal{H}} = TTM$ and, in particular $C \in \widetilde{\mathcal{D}}_{\mathcal{H}}$. From Corollary 2.3.2 we get that the spray $\widetilde{S}$ is not metrizable.

On the other hand, if the projective factor $\mathcal{P}$ is linear, let us suppose that $\widetilde{S}$ is Finsler metrizable and $\widetilde{E}$ is a Finsler energy function associated with $\widetilde{S}$. Using (2.49) and Proposition 1.4.1 we get

$$\mathcal{L}_{v_i}\widetilde{E} = 0 \quad \Longrightarrow \quad \dot{\partial}_i\widetilde{E} - \frac{\mathcal{P}_i}{P}\mathcal{L}_C(\widetilde{E}) = 0 \quad \Longrightarrow \quad \frac{\dot{\partial}_i\widetilde{E}}{\widetilde{E}} = 2\frac{\dot{\partial}_i\mathcal{P}}{\mathcal{P}},$$

therefore, locally there exists a function $\theta(x)$ on M such that $\widetilde{E} = \mathcal{P}^2 e^{\theta(x)}$. Writing the linear projective factor in the form $\mathcal{P} = a_i(x)y^i$ we get

$$g_{ij}(x,y) = \frac{\partial^2 \widetilde{E}}{\partial y^i \partial y^j} = 2a_i(x)a_j(x)e^{\theta(x)},$$

hence g_{ij} has rank 1 and in the case $n \geq 2$, the energy function $\widetilde{E}$ is degenerate which is a contradiction. □

Proof of the Theorem 2.6.2. Let $\mathcal{P}$ be a nontrivial holonomy invariant 1-homogeneous function. Let us fix a point $x \in M$ and a direction $y \in \mathcal{T}_x M$. Then, using the eigenvalue κ_i of the Jacobi endomorphism Φ at y, the set

$$\Lambda_{(x,y)} := \left\{ \lambda \in \mathbb{R} \mid \kappa_i + \lambda^2 \mathcal{P}^2 = 0, \ i = 1, \ldots, n-1 \right\} \tag{2.59}$$

is a finite set, therefore its complement is an open dense subset of $\mathbb{R}$. For any element $\lambda \in \mathbb{R} \setminus \Lambda_{(x,y)}$ we have (2.47) and, using Theorem 2.6.6, one obtains that $\widetilde{S} = \mathcal{S} - 2\lambda\mathcal{P}\, C$ is not metrizable.

□

As the precedent results show, for a given Finsler structure (M, F), only very specific holonomy invariant projective factors can produce Finsler metrizable sprays. Such projective factor must be related to the principal curvatures of the original Finsler structure. More precisely, we have the following

Corollary 2.6.7. *Let (M, F) be a Finsler manifold, $\mathcal{S}$ its geodesic spray and let $\widetilde{\mathcal{P}}$ be a holonomy invariant nonzero function. If the projective deformation $\widetilde{S} = \mathcal{S} - 2\widetilde{\mathcal{P}}\, C$, is metrizable, then*

$$\widetilde{\mathcal{P}}^2 + \kappa_i = 0, \tag{2.60}$$

for some (nonzero) principal curvature κ_i, $i \in \{1, \ldots, n-1\}$.

In particular we obtain that if the principal curvatures are all non-negative, then there is no non-trivial holonomy invariant metrizable projective deformation of the Finsler structure.

Projective deformation with Funk function

The projective factor $\mathcal{P}$ of a projective deformation is called a Funk function, if the projective deformation preserves the curvature tensor and the Jacobi endomorphism. In [82, page 177], Zhongmin Shen asks "whether or not there always exist non-trivial Funk functions on a spray space". The existence of non-trivial Funk functions, for the general case of a non-flat SODE, is still an open problem. We proved that the answer is negative for the geodesic spray of a Finsler function of non-vanishing scalar flag curvature:

Theorem 2.6.8. *[101, Theorem 3.1] Consider a Finsler function F of non-vanishing scalar flag curvature. Then, there are no non-trivial Funk functions for the Finsler space (M, F).*

It would be very interesting to describe the necessary and sufficient conditions for a projective deformation of a metrizable spray to be metrizable. This problem is however very complex and it contains, as a particular case, Hilbert's fourth problem. Even partial results, when the projective factor possesses special geometric or analytic properties, can be interesting.

2.7 Invariant metrizability and projective metrizability

In this section, we investigate the relationship between invariant metrizability and the invariant projective metrizability of the canonical sprays of Lie groups. In the case of the invariant metrizability problem we ask if there exists a *left-invariant* Riemann (resp. Finsler) metric, such that its geodesics are the geodesics of the canonical spray. In the case of the invariant projective metrizability problem we ask if there exists a left-invariant Riemann (resp. Finsler) metric, such that its geodesics are projectively equivalent to the geodesics of the canonical spray.

Remark 2.7.1. From Section 2.3 and Section 2.5 we know that both the metrizability and projective metrizability problems can be formulated in terms of a system of partial differential equations which is composed of the appropriate homogeneity condition and the Euler-Lagrange PDE equations on the energy function: a given spray $\mathcal{S}$ is

1) *Riemann (resp. Finsler) metrizable* if and only if there exists a quadratic (resp. 2-homogeneous) function $E\colon TM \to \mathbb{R}$, such that $\left(\frac{\partial^2 E}{\partial y^i \partial y^j}\right)$ is positive definite on $\mathcal{T}M$ and the Euler-Lagrange PDE system (1.21) is satisfied.

2) *projective Riemann (resp. projective Finsler) metrizable* if and only if there exists a quadratic (resp. 2-homogeneous) function $\widetilde{E}\colon TM \to \mathbb{R}$, such that the matrix field $\left(\frac{\partial^2 \widetilde{E}}{\partial y^i \partial y^j}\right)$ is positive definite on $\mathcal{T}M$ and the Euler-Lagrange PDE system (1.21) is satisfied with $\widetilde{F} := \sqrt{2\widetilde{E}}$.

Proposition 2.7.2. *[96, Proposition 3.4] Let $\mathcal{S}$ be a spray and L be a Lagrangian. If E is a first integral for $\mathcal{S}$, then we have*

$$\omega_E = 0 \qquad \Leftrightarrow \qquad d_h E = 0. \tag{2.61}$$

Proof. Consider E a first integral for $\mathcal{S}$, that is $\mathcal{S}(E) = d_{\mathcal{S}} E = 0$. Then, using the Frölicher-Nijenhuis calculus, we get

$$\omega_E = i_{\mathcal{S}} dd_J E + dd_C E - dE = d_{\mathcal{S}} d_J E - d i_{\mathcal{S}} d_J E + dd_C E - dE = -(d_\Gamma E + dE) = -2 d_h E,$$

which shows the equivalence of the two conditions of (2.61). $\square$

Corollary 2.7.3. *Let $\mathcal{S}$ be a spray, let L be a non-zero first integral for $\mathcal{S}$ and f a smooth non-vanishing function on $\mathbb{R}$ with non-vanishing derivative. Then, E satisfies the Euler-Lagrange equation* (1.21) *associated to $\mathcal{S}$, if and only if $f \circ E$ is a solution of* (1.21).

Proof. From Proposition 2.7.2 we know that $\omega_E = 0$ is equivalent to $d_h E = 0$ and equation $\omega_{f\circ E} = 0$ is equivalent to $d_h(f \circ E) = 0$. Moreover, since $f' \neq 0$ and $d_h(f \circ E) = f' \cdot d_h E$, we have $d_h(f \circ E) = 0$ if and only if $d_h E = 0$ holds. □

Invariant metrizability and projective metrizability of the canonical geodesic structure of Lie groups

We follow the notation introduced in Section 1.4. In particular, G denotes a finite dimensional Lie group, $\lambda_g \colon G \to G$ is the left translation of G, $TG \cong G \times \mathfrak{g}$, and the corresponding *semi-invariant coordinate system* is given by $(x, \alpha) = (x^i, \alpha^i)$. Then we have the following

Proposition 2.7.4. *[96, Proposition 4.4] The canonical spray of a Lie group is left-invariant projective Riemann (resp. projective Finsler) metrizable if and only if it is left-invariant Riemann (resp. Finsler) metrizable.*

Proof. Let us denote by $\mathcal{S}$ the canonical spray of the Lie group G. It is clear that if $\mathcal{S}$ is Riemann (resp. Finsler) metrizable, then it is also projective Riemann (resp. Finsler) metrizable. Conversely, let us suppose that $\mathcal{S}$ is projective Riemann (resp. Finsler) metrizable. Then, according to Remark 2.7.1, there exists a left-invariant quadratic (resp. 2-homogeneous) function $\widehat{E} : TG \to \mathbb{R}$ such that the matrix field $(\frac{\partial^2 \widehat{E}}{\partial y^i \partial y^j})$ is positive definite on $\mathcal{T}M$ and $\widehat{F} := \sqrt{2\widehat{E}}$ satisfies the Euler-Lagrange PDE associated to $\mathcal{S}$. Because of the left-invariance condition, we have $d_{\mathcal{S}} \widehat{F} = 0$ and, using Corollary 2.7.3, we get that $\widehat{E} := \frac{1}{2}(\widehat{F})^2$ is also a solution of the Euler-Lagrange PDE associated to $\mathcal{S}$. Then, according to Remark 2.7.1, the given $\widehat{E}$ is the energy function of a Riemann (resp. Finsler) metric which implies that $\mathcal{S}$ is Riemann (resp. Finsler) metrizable. □

We have the following result.

Theorem 2.7.5. *[96, Theorem 4.5] The canonical spray of a Lie group is left-invariant projective Finsler metrizable if and only if it is left-invariant Riemann metrizable.*

Proof. In one direction the statement is trivial: if the canonical spray is Riemann metrizable, then it is trivially Finsler metrizable and also projective Finsler metrizable. Let us consider the converse statement, and suppose that the canonical spray $\mathcal{S}$ is projective Finsler metrizable. Then, according to Proposition 2.7.4, it is also Finsler metrizable. Since $\mathcal{S}$ is quadratic, it follows that the associated connection is linear. Hence, the Finsler metrizability induces the existence of a Berwald metric on the Lie group. Using Szabó's theorem which states that for every Berwald metric there exists a Riemannian metric such that the geodesics of the Berwald and Riemannian metrics are the same (cf. [88]), we get that the canonical spray is Riemann metrizable. □

We can obtain the following

Corollary 2.7.6. *[96, Corollary 4.6] The canonical spray of a Lie group G is left-invariant Riemann, Finsler, projective Riemann or projective Finsler metrizable if and only if there exists a scalar product $\langle\ ,\ \rangle$ on $\mathfrak{g}$ such that*

$$\langle [a, \alpha], \alpha \rangle = 0 \tag{2.62}$$

for every $a, \alpha \in \mathfrak{g}$.

Proof. An invariant Riemannian metric induces a scalar product $\langle\ ,\ \rangle$ on $\mathfrak{g}$. Using the coordinate system (x, α) on $TG \simeq G \times \mathfrak{g}$, the associated energy function is given by $E : G \times \mathfrak{g} \to \mathbb{R}$, where $E(x, \alpha) = \langle \alpha, \alpha \rangle$. The Euler-Lagrange equation (1.56) then implies (2.62). □

Remark 2.7.7. We want to draw attention to a few interesting phenomena. First of all, although the canonical spray of a Lie group is a very natural object, it is not true that it is always metrizable. In [40] there are several examples of Lie groups and Lie algebras where the canonical spray is non metrizable.
Secondly, despite the fact that the canonical spray is left (and also right) invariant, and the Euler-Lagrange equation inherits the symmetries of the Lagrangian, it is not true that the "*metrizability*" property means automatically "*metrizability by a left-invariant metric*". Indeed, for example the 3-dimensional Heisenberg group $\mathbb{H}_3$ is not metrizable or projective metrizable with an invariant Riemann (or Finsler) metric [114], however, since its curvature tensor vanishes identically, the canonical spray is metrizable. The corresponding (non invariant) Riemannian metric is given by $g = dx^2 + dy^2 + (dz - dx - dy)^2$, (see [40]).

Theorem 2.7.5 shows that the geometric structure associated with the canonical spray of a Lie group has a certain rigidity property: the potentially larger class of Lie groups, where the canonical spray is projective Finsler metrizable coincides with the class of Lie groups, where the canonical spray is Riemann metrizable. We note that this property relies heavily on the fact that the canonical spray of a Lie group is quadratic, and the Lie derivative of a left-invariant Lagrange function with respect to the canonical spray is identically zero. The second property is not true in general for an arbitrary left-invariant spray. An interesting generalization can be obtained by considering the class of homogeneous spaces.

Invariant metrizability and projective metrizability of geodesic orbit structures of homogeneous spaces

Let M be a connected manifold on which the Lie transformation group G acts transitively. Let us fix an origin $o \in M$ and denote by H the stabilizer of $o \in M$

in the group G and by $\pi : G \to G/H$ the projection map. H is called the isotropy group of the homogeneous space G/H. Then M is isomorphic to the factor space G/H with origin H, and its tangent space at $o \in M$ is isomorphic to $\mathfrak{g}/\mathfrak{h}$, where $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie-algebras of the Lie groups G and H respectively. The action of G on M is determined by the map

$$\lambda : (g, m) \mapsto \lambda_g m = g \cdot m : G \times M \to M.$$

Geodesic structures, sprays, metrics, Lagrangians on M are called *invariant*, if they are invariant with respect to the action of G. It is clear, that invariant sprays, metrics and Lagrangians can be characterized by their values on T_oM.

A geodesic $\gamma(t)$, emanating from the origin $o \in M$, is called *homogeneous* or *stationary* [89, 90], if there exists $X_\gamma \in \mathfrak{g}$, such that $\gamma(t)$ is the orbit of the 1-parameter subgroup $\{\exp tX_\gamma,\ t \in \mathbb{R}\}$ of G, that is

$$\gamma(t) = \lambda_{\exp tX_\gamma} o = (\exp tX_\gamma) \cdot o. \tag{2.63}$$

The Lie algebra element $X_\gamma \in \mathfrak{g}$ is called the *geodesic vector* associated to the direction $\dot\gamma(0) \in T_oM$. A left-invariant geodesic structure is called *geodesic orbit structure* (g.o. structure), if any geodesic $\gamma(t)$ emanating from the origin $o \in M$ is homogeneous. A spray is called *geodesic orbit spray* (g.o. spray), if it corresponds to a g.o. structure. Homogeneous geodesics are called in V.I. Arnold's terminology "*relative equilibria*" [8].

Definition 2.7.8. A map $\sigma \colon T_oM \to \mathfrak{g}$ is called a *homogeneous lift*[1] if the following conditions are satisfied:

1) $\pi_* \circ \sigma = id_{T_oM}$.
2) σ is 1-homogeneous, that is $\sigma(\kappa \cdot v) = \kappa\, \sigma(v)$, for every $v \in T_oM$ and $\kappa \in \mathbb{R}$.
3) σ is $\mathrm{Ad}(H)$-invariant, that is $\sigma(\lambda_{h*} v) = \mathrm{Ad}_h \sigma(v)$ for all $h \in H$ and $v \in T_oM$.

The homogeneous lift σ is called $\mathcal{C}^\infty$-differentiable if it is continuous on T_oM and $\mathcal{C}^\infty$-differentiable on $T_oM \setminus \{0\}$.

It is clear that any g.o. spray determines a $\mathcal{C}^\infty$-differentiable homogeneous lift by associating to $v \in T_oM$ its geodesic vector $X = \sigma(v)$ and vice versa, every homogeneous lift determines a g.o. spray by left translations. Moreover, it is not difficult to see that invariant functions are constant along the geodesics of a g.o. spray. Hence we can obtain the following generalisation of the Proposition 2.7.4.

Proposition 2.7.9. *[96, Proposition 5.5] A g.o. spray is invariant projective Riemann (resp. Finsler) metrizable if and only if it is invariant Riemann (resp. Finsler) metrizable.*

Proof. Let $\mathcal{S}$ be a g.o. spray. If $L \colon TM \to \mathbb{R}$ is an invariant Lagrangian, then L is constant along the geodesics, that is along the integral curves of $\mathcal{S}$. Consequently we have $d_{\mathcal{S}} L = 0$. Using Corollary 2.7.3 and similar argument that was used for Proposition 2.7.4 we can obtain the proof of the proposition. □

[1] In [118] the terminology *horizontal lift* was used, but in Finsler geometry, this terminology is widely used for a different object.

We remark that the connection $\Gamma = [J, \mathcal{S}]$ determined by a g.o. spray $\mathcal{S}$ is not necessarily linear, therefore it is not true in general that the Finsler metrizability entail the Riemann metrizability as it was the case for the canonical spray of Lie groups. However, if the g.o. spray is quadratic then the associated connection is linear. Therefore we can use Szabó's results and, similarly to Theorem 2.7.5, we can get the following

Theorem 2.7.10. *[96, Theorem 5.6] [96, Corollary 4.6] A quadratic g.o. spray is invariant projective Finsler metrizable if and only if it is invariant Riemann metrizable.*

Remark 2.7.11. A different invariant metrizability concept of the G/H structure is considered in [35] by S. Deng and Z. Hou where the G/H structure is called invariant metrizable if there exists an invariant metric on it. The invariant metrizability (and projective metrizability) of a g.o. structure or g.o. spray is however more subtle, because in this case not only the G/H homogeneous space, but also the geodesic structure is fixed and we want to metrize both. It may happen that the g.o. structure on a homogeneous space G/H is not invariant metrizable, but the G/H structure is.

Chapter 3

On the holonomy of Finsler manifolds

3.1 Introduction

The parallelism on Riemannian and Finslerian manifolds is defined through the covariant differentiation with respect to the canonical connection, that is, through a system of differential equations. The attached geometric structure is the holonomy group, which can be introduced in a very natural way: it is the group generated by parallel translations along loops. In contrast to the Finslerian case, Riemannian holonomy groups have been extensively studied. One of the earliest fundamental results is the theorem of Borel and Lichnerowicz [18] from 1952, claiming that the holonomy group of a simply connected Riemannian manifold is a closed Lie subgroup of the orthogonal group $O(n)$. By now, the complete classification of Riemannian holonomy groups is known [14, 20, 51].

Similarly to the Riemannian case, the holonomy group of a Finsler manifold is the group generated by canonical (homogeneous) parallel translations along closed loops. Despite the analogous construction, Finslerian holonomy can be much more complex than Riemannian. The holonomy groups of non-Riemannian Finsler manifolds have been described only in a few special cases: Z.I. Szabó proved that for Berwald metrics there exist Riemannian metrics with the same holonomy group (cf. [88]), and for Landsberg metrics L. Kozma showed that the holonomy groups are compact Lie groups consisting of isometries of the indicatrix with respect to an induced Riemannian metric (cf. [56, 57]).

A thorough study of holonomy groups of homogeneous (nonlinear) connections was initiated by W. Barthel in his basic work [13] in 1963; he gave a construction for a holonomy algebra of vector fields on the tangent space. In [69] P. Michor proposed a general setting for the study of infinite dimensional holonomy groups and holonomy algebras which was the motivation for us to start investigating the tangent objects to a subgroup of the diffeomorphism group [119]. Since then, we manage to obtain several new results on the Finsler holonomy structure (cf. [108, 109, 120, 121, 122, 123, 124]). In this chapter we present our results about this topic.

In Section 3.2 we collect the necessary definitions and constructions of Finsler geometry: canonical covariant derivative, parallel translation and holonomy.

In Section 3.3 we investigate the tangent structure of subgroups of the diffeomorphism group $\mathcal{D}iff^\infty(M)$ of a manifold M. We introduce the notion of *tangent Lie algebra* of subgroups: denoting by $\mathcal{T}_o\mathcal{G}$ the set of tangent vector fields to a subgroup $\mathcal{G}$ of the diffeomorphism group at the identity, we prove that $\mathcal{T}_o\mathcal{G}$ is a Lie subalgebra of the Lie algebra of smooth vector fields on M (Theorem 3.3.3). It follows that any subalgebra of $\mathcal{T}_o\mathcal{G}$ inherit the tangential property, that is the elements of a subalgebra generated by tangent vector fields to the subgroup $\mathcal{G}$ are tangent to $\mathcal{G}$ (Corollary 3.3.5). This property can be particularly interesting when the Lie bracket of two tangent vector fields to $\mathcal{G}$ generates a new direction, because the tangential property will be satisfied in the new direction as well. As we show in Theorem 3.3.7, when M is compact, the group generated by the exponential image of $\mathcal{T}_o\mathcal{G}$ is a subgroup of the closure of $\mathcal{G}$ in $\mathcal{D}iff^\infty(M)$. This fact can give important information about the group $\mathcal{G}$ itself, especially in the infinite dimensional case. The concept can be adapted for any subgroup $\mathcal{G}$ of any (finite or infinite dimensional) Lie group $\mathcal{G}_L$. In the particular case when $\mathcal{G}$ is a Lie subgroup of $\mathcal{G}_L$, then $\mathcal{T}_o\mathcal{G} = \mathfrak{g}$ is just the usual Lie subalgebra of the Lie algebra of $\mathcal{G}_L$.

In Section 3.4 and 3.5 we investigate the tangent algebra and subalgebra of the holonomy group and the fibered holonomy group of Finsler manifolds. We introduce the notion of *curvature algebra* of a Finsler manifold, which is a generalization of the matrix algebra generated by curvature operators of a Riemannian manifold. We show that the vector fields belonging to the curvature algebra are tangent to the fibered holonomy group. We also define the *infinitesimal holonomy algebra* as the smallest Lie algebra of vector fields on an indicatrix, containing the curvature vector fields and their horizontal covariant derivatives with respect to the Berwald connection. We prove the tangential property of the elements of the infinitesimal holonomy algebra to the holonomy group. Both the curvature algebra and the infinitesimal holonomy algebra can give important information about the holonomy properties of Finsler manifolds.

In Section 3.6 we prove that the dimension of the curvature algebra of a positive definite non-Riemannian Finsler manifold of non-zero constant curvature with dimension $n > 2$ is strictly greater than the dimension of the orthogonal group acting on its tangent space, hence the holonomy group can not be a compact Lie group. In addition, we provide an example of a left-invariant singular (non y-global) Finsler metric of Berwald-Moór-type on the 3-dimensional Heisenberg group which has infinite dimensional curvature algebra and hence its holonomy group is not a (finite dimensional) Lie group. These results gave a positive answer to the following problem formulated by S.S. Chern and Z. Shen: *"Is there a Finsler manifold whose holonomy group is not the holonomy group of any Riemannian manifold?"* [28, page 85].

In Section 3.7 we investigate the holonomy group of locally projectively flat Finsler manifolds of constant curvature. Our aim is to characterize all locally projectively flat Finsler manifolds with infinite dimensional holonomy group. To get such a characterization, we investigate the dimension of the infinitesimal holonomy algebra. We obtain that non-Riemannian locally projectively flat Finsler manifolds of nonzero constant curvature have infinite dimensional infinitesimal holonomy algebra. Using this general result and the tangential property of the infinitesimal holonomy algebra, we prove that the holonomy group of a locally projectively flat Finsler manifold of constant curvature is finite dimensional if and only if it is a Riemannian manifold or a flat Finsler manifold.

In Section 3.8 we show that the holonomy group of a certain class of simply connected, projectively flat Finsler 2-manifolds of constant curvature is not a finite dimensional Lie group, and we prove that its topological closure is $\mathcal{D}\mathit{iff}^{\infty}_{+}(\mathbb{S}^1)$, the connected component of the diffeomorphism group of the circle (cf. [123]). The significance of this result comes from the fact that, before our investigation, not a single infinite dimensional Finsler holonomy group has been described. This class of Finsler 2-manifolds contains the standard Funk metric (of constant negative curvature) and the Bryant-Shen spheres (of constant positive curvature) [21, 85]. In these examples the holonomy groups are maximal. In addition, we investigate the holonomy structure of the most accessible and demonstrative 2-dimensional Finsler surfaces, Randers surfaces. In the Randers case, the Finsler function is a Riemann norm deformed by a 1-form. Randers metrics describe the Zermelo navigation problem on Riemannian manifolds [11]. This fact may suggest that the holonomy structures of Randers manifolds and Riemannian manifolds are similar, but our result (Theorem 3.8.6) shows that quite the opposite is true: the holonomy group of a simply connected, locally projectively flat non-Riemannian Randers two-manifold of non-zero constant flag curvature is maximal and its closure is isomorphic to $\mathcal{D}\mathit{iff}^{\infty}_{+}(\mathbb{S}^1)$. These results are surprising because they show that even in the case when the geodesic structure is simple (the geodesics are straight lines), the holonomy group can still be very large. We also obtain the classification of the holonomy groups of projectively flat Randers surfaces (Corollary 3.8.7).

The results of this chapter are based on the papers [108, 109, 120, 122, 123, 124].

3.2 Preliminaries

Covariant derivative and parallel translation

Let (M, F) be a Finsler manifold and $\mathcal{S}$ its geodesic spray. The horizontal distribution $HTM \subset TTM$ introduced in Chapter 2.2 associated with the spray $\mathcal{S}$ can be considered as the image of the *horizontal lift* which is the vector space isomorphism $l_y \colon T_xM \to H_yTM$ for $x \in M$ and $y \in T_xM$ defined by

$$l_y\Big(\frac{\partial}{\partial x^i}\Big) = \frac{\partial}{\partial x^i} - G^k_i(x,y)\frac{\partial}{\partial y^k}, \tag{3.1}$$

in the coordinate system (x^i, y^i) of TM. For the horizontal lift of a vector field X we will often use the simplified notation X^h. The pull-back bundle of (TM, π, M) corresponding to the map $\pi : TM \to M$ is denoted by (π^*TM, π, TM). Clearly, the mapping

$$\left(x,y,\xi^i\frac{\partial}{\partial y^i}\right)\mapsto\left(x,y,\xi^i\frac{\partial}{\partial x^i}\right):\ VTM\to\pi^*TM \tag{3.2}$$

is a canonical bundle isomorphism. In the following we will use the isomorphism (3.2) for the identification of these bundles.

Let $\mathfrak{X}(M)$ be the vector space of smooth vector fields on the manifold M and $\hat{\mathfrak{X}}^\infty(TM)$ be the vector space of smooth sections of the vertical bundle $(\mathcal{V}TM,\tau,TM)$. The *horizontal Berwald covariant derivative* of a section $\xi\in\hat{\mathfrak{X}}^\infty(TM)$ by a vector field $X\in\mathfrak{X}^\infty(M)$ is given by

$$\nabla_X\xi := [X^h,\xi].$$

If $\xi(x,y)=\xi^i(x,y)\frac{\partial}{\partial y^i}$ and $X(x)=X^i(x)\frac{\partial}{\partial x^i}$ then $\nabla_X\xi$ can be expressed as

$$\nabla_X\xi=\left(\frac{\partial\xi^i(x,y)}{\partial x^j}-G^k_j(x,y)\frac{\partial\xi^i(x,y)}{\partial y^k}+G^i_{jk}(x,y)\xi^k(x,y)\right)X^j\frac{\partial}{\partial y^i}, \tag{3.3}$$

where $G^i_{jk}:=\frac{\partial G^i_j}{\partial y^k}$. Moreover, by defining the horizontal Berwald covariant derivative

$$\nabla_X\phi=l(X)\phi=\left(\frac{\partial\phi}{\partial x^j}-G^k_j(x,y)\frac{\partial\phi(x,y)}{\partial y^k}\right)X^j$$

of a smooth function $\phi:\mathcal{T}M\to\mathbb{R}$, then (3.3)) can be extended to sections of the tensor bundle over (π^*TM,π,TM) using the canonical bundle isomorphism (3.2).

Curvature

The curvature tensor of the Finsler manifold $(M,\mathcal{F})$ is the curvature tensor R associated to its geodesic spray. One can calculate it by using the formula (1.11). The local expression of the curvature tensor is given by $R_{(x,y)}=R^i_{jk}(x,y)dx^j\otimes dx^k\otimes\frac{\partial}{\partial x^i}$ where

$$R^i_{jk}(x,y)=\frac{\partial G^i_j(x,y)}{\partial x^k}-\frac{\partial G^i_k(x,y)}{\partial x^j}+G^m_j(x,y)G^i_{km}(x,y)-G^m_k(x,y)G^i_{jm}(x,y) \tag{3.4}$$

in a local coordinate system. The manifold $(M,\mathcal{F})$ has constant flag curvature $\lambda\in\mathbb{R}$, if for any $x\in M$ the local expression of the Riemannian curvature is

$$R^i_{jk}(x,y)=\lambda\left(\delta^i_k g_{jm}(x,y)y^m-\delta^i_j g_{km}(x,y)y^m\right). \tag{3.5}$$

In this case the flag curvature of the Finsler manifold (cf. [28], Section 2.1 pp. 43-46) does not depend either on the point or on the 2-flag. The Landsberg curvature tensor field is defined as

$$L_{(x,y)}(u,v,w)=g_{(x,y)}\left(\nabla_w B(u,v,w),y\right),$$

for $y,u,v,w\in T_xM$. According to [82, Lemma 6.2.2], one has $\nabla_w g_{(x,y)}(u,v)=-2L_{(x,y)}(u,v,w)$. Moreover, the horizontal Berwald covariant derivative of the tensor field

$$Q_{(x,y)}=\left(\delta^i_j g_{km}(x,y)y^m-\delta^i_k g_{jm}(x,y)y^m\right)dx^j\otimes dx^k\otimes dx^l\otimes\frac{\partial}{\partial x^i} \tag{3.6}$$

vanishes. Indeed, for any vector field $W \in \mathfrak{X}^\infty(M)$ we have $\nabla_W id_{TM} = 0$ and $\nabla_W y = 0$. Moreover, since $L_{(x,y)}(y, v, w) = 0$ (cf. [82, equation 6.28]) we get the assertion.

Parallel translation

Parallel vector fields $X(t) = X^i(t)\frac{\partial}{\partial x^i}$ along a curve $c(t)$ are defined by the solutions of the differential equation

$$D_{\dot c}X(t) := \Big(\frac{dX^i(t)}{dt} + G^i_j(c(t), X(t))\dot c^j(t)\Big)\frac{\partial}{\partial x^i} = 0. \tag{3.7}$$

Since the functions $G^i_j(x, y)$ are positive 1–homogeneous with respect to the variable y, we have $D_{\dot c}(\lambda X(t)) = \lambda D_{\dot c}X(t)$ for any $\lambda \geq 0$. The differential equation (3.7) can be expressed by the horizontal covariant derivative (3.3) using the bundle isomorphism (3.2) as follows: a vector field $X(t) = X^i(t)\frac{\partial}{\partial x^i}$ along a curve $c(t)$ is parallel if it satisfies the equation

$$D_{\dot c}X = 0. \tag{3.8}$$

Clearly, for any $X_0 \in T_{c(0)}M$ there is a unique parallel vector field $X(t)$ along the curve c such that $X_0 = X(0)$. Moreover, if $X(t)$ is a parallel vector field along c, then $\lambda X(t)$ is also parallel along c for any $\lambda \geq 0$. Then the *homogeneous (nonlinear) parallel translation*

$$\mathcal{P}_c : T_{c(0)}M \to T_{c(1)}M \tag{3.9}$$

along a curve $c(t)$ is defined by the positive homogeneous map $\mathcal{P}_c : X_0 \mapsto X_1$ given by the value $X_1 = X(1)$ at $t = 1$ of the parallel vector field with initial value $X(0) = X_0$.

The parallel translation can be introduced geometrically using the notion of the horizontal distribution. Namely, we call a curve in TM horizontal if the tangent vectors of this curve are contained in the horizontal distribution $HTM \subset TTM$. Let now $c(t)$ be a curve in the manifold M joining the points p and q. The *horizontal lift* $c^h(t) = (c(t), X^i(t)\frac{\partial}{\partial x^i})$ of $c(t)$ is the curve $c^h(t)$ in TM defined by the properties that $c^h(t)$ projects on $c(t)$ and $c^h(t)$ is horizontal that is $\dot c^h(t) \in H_{c(t)}$. This means according to equation (3.1) that

$$\dot c^i(t)\frac{\partial}{\partial x^i} + \frac{d}{dt}X^i(t)\frac{\partial}{\partial y^i} = \left(\frac{\partial}{\partial x^i} - G^k_i(x, y)\frac{\partial}{\partial y^k}\right)\dot c^i(t),$$

i.e. the tangent vector of the lifted curve $c^h(t)$ is the horizontal lift of the tangent vector $\dot c^i(t)\frac{\partial}{\partial x^i}$ of $c(t)$. It follows that a vector field $X(t)$ along a curve $c(t)$ is parallel if and only if it is a solution of the differential equation

$$\frac{d}{dt}\left(c(t), X^i(t)\frac{\partial}{\partial x^i}\right) = l_{X(t)}(\dot c(t)), \tag{3.10}$$

or equivalently $X(t)$ satisfies the differential equation (3.7). Hence the parallel translation along a curve $c(t)$ joining the points p and q is the map $\mathcal{P}_c : T_pM \to T_qM$

determined by the intersection points of the horizontal lifts of the curve $c(t)$ with the tangent spaces T_pM and T_qM.

We remark that the horizontal lift φ_t^h of the flow φ_t of a vector field $X \in \mathfrak{X}(M)$ is the flow of the horizontal lift of the vector field $X^h \in \mathfrak{X}(TM)$. Therefore the parallel translation along the integral curves of X can be calculated in terms of the horizontal lift of the flow:

$$\mathcal{P}_\varphi^t = \varphi_t^h. \tag{3.11}$$

Since the parallel translation is determined by the horizontal distribution, a Finsler manifold can be considered as a particular case of a fibered manifold equipped with an Ehresmann connection (cf. [38]). Indeed, an Ehresmann connection of a fibered manifold is given by a horizontal distribution, which is a complement to the vertical distribution consisting of the tangent spaces of the fibers. For a spray manifold the fibered manifold is the tangent bundle of M and the horizontal distribution determined by the horizontal lift $l_y \colon T_xM \to H_yTM$ expressed by equation (3.1).

Finsler holonomy

The holonomy group of a Riemannian or Finslerian manifolds is the group generated by parallel translations along loops with respect to the canonical associated connection. Despite the similarities, the holonomy properties of Finsler manifolds are essentially different from the Riemannian one, and it is far from being well understood. The main difficulty comes from the fact that in the general case, the canonical connection of a Finsler manifold is neither linear nor metrical, that is the parallel translation is not necessarily preserving the metric. Only much weaker properties are fulfilled: instead of the linearity it is only $1-$homogeneous, and instead of the metrical property it is preserving only the norm function. Nonetheless these properties allow us to consider the parallel translations as maps between the indicatrices, and therefore the holonomy group can be considered as a subgroup of the diffeomorphism group of the indicatrix. Indeed, let $(M, \mathcal{F})$ be an n-dimensional Finsler manifold. The *indicatrix* $\mathcal{I}_x$ at $x \in M$ is a hypersurface of T_xM defined by

$$\mathcal{I}_x := \{\, y \in T_xM : \mathcal{F}(y) = 1 \,\}. \tag{3.12}$$

The indicatrix (3.12) is a compact hypersurface in the tangent space T_xM, diffeomorphic to the standard $(n-1)$-dimensional sphere. In the sequel $(\mathcal{I}, \pi, M)$ will denote the *indicatrix bundle* of $(M, \mathcal{F})$ and $i : \mathcal{I} \hookrightarrow TM$ the natural embedding of the indicatrix bundle into the tangent bundle (TM, π, M).

On a Finsler manifold $(M, \mathcal{F})$, the parallel translation (3.9) along a curve $c : [0, 1] \to \mathbb{R}$ is a differentiable map between the slit tangent spaces preserving the value of the Finsler function F, therefore it induces a map

$$\mathcal{P}_c \colon \mathcal{I}_{c(0)} \longrightarrow \mathcal{I}_{c(1)} \tag{3.13}$$

between the indicatrices. Since the parallel translation is also 1-homogeneous, it is entirely characterized by the map (3.13). Indeed, we have $\mathcal{P}_c(0) = 0$ and for every non-zero vector $v \in T_{c(0)}M$ we have

$$\mathcal{P}_c(v) = |v| \cdot \mathcal{P}_c\left(\frac{v}{|v|}\right).$$

It follows from these observations that the holonomy group $\mathcal{H}ol_x(M)$ of the Finsler manifold $(M, \mathcal{F})$ at the point $x \in M$ is uniquely determined by its action on the indicatrix $\mathcal{I}_x$. Hence we can formulate the following

Definition 3.2.1. The *holonomy group* $\mathcal{H}ol_x(M)$ of a Finsler space (M, F) at $x \in M$ is the subgroup of the group of diffeomorphisms $\mathcal{D}\mathit{iff}^\infty(\mathcal{I}_x)$ of the indicatrix $\mathcal{I}_x$ determined by parallel translation of $\mathcal{I}_x$ along piece-wise differentiable closed curves initiated at the point $x \in M$.

We note that the holonomy group $\mathcal{H}ol_x(M)$ is a topological subgroup of the regular infinite dimensional Lie group $\mathcal{D}\mathit{iff}^\infty(\mathcal{I}_x)$, but its differentiable structure is not known in general.

3.3 Tangent Lie algebra of a subgroup of the diffeomorphism group

Let $\mathcal{G}$ be a subgroup of $\mathcal{D}\mathit{iff}^\infty(M)$ where M is a differentiable manifold. We do not suppose any special property on $\mathcal{G}$, in particular, we do not suppose that $\mathcal{G}$ is a Lie subgroup of $\mathcal{D}\mathit{iff}^\infty(M)$. Our aim is to introduce a tangential property and a tangent object to $\mathcal{G}$, and to show that the set of tangent elements has a Lie algebra structure. We also show that the constructed tangent Lie algebra can give information about the group $\mathcal{G}$.

A smooth curve $c\colon I \to M$ on the manifold M has a $(k-1)^{\mathrm{st}}$-order singularity at $t = 0$, if its derivatives vanish up to order $k-1$, $(k \geq 0)$. It is well known that if the curve c has a $(k-1)^{\mathrm{st}}$-order singularity at $0 \in \mathbb{R}$, then its k^{th} order derivative $c^{(k)}(0) = X_p$ is a tangent vector at $p = c(0)$. In that case, the curve c is called a k^{th}*-order integral curve* of the vector $X_p \in T_pM$. Extending this concept to vector fields, we introduce the following

Definition 3.3.1. [109, Definition 3.1] A $C^\infty-$smooth curve in the diffeomorphism group $\varphi\colon I \to \mathcal{D}\mathit{iff}^\infty(M)$, $t \to \varphi_t$ is called an *integral curve of the vector field* $X \in \mathfrak{X}(M)$ if

(1) $\varphi_0 = id_M$,

(2) there exists $k \in \mathbb{N}$ such that for any point $p \in M$ the curve $t \to \varphi_t(p)$ is a k^{th}-order integral curve of $X(p) \in T_pM$.

This $k \in \mathbb{N}$ is called the *order* of the integral curve φ_t of the vector field X.

In particular, the flow φ_t^X of $X \in \mathfrak{X}(M)$ is a 1^{st}-order integral curve of X. Moreover, if $k > 1$ and $t \to \varphi_t$ is a $k^{\mathrm{th}}-$order integral curve of the vector field X then we have

$$\varphi_0 = id_M, \qquad \frac{\partial \varphi_t}{\partial t}\Big|_{t=0} = 0, \quad \dots \quad \frac{\partial^{k-1} \varphi_t}{\partial t^{k-1}}\Big|_{t=0} = 0, \qquad \frac{\partial^k \varphi_t}{\partial t^k}\Big|_{t=0} = X. \tag{3.14}$$

Using the terminology of Definition 3.3.1 we introduce the following

Definition 3.3.2. A vector field $X \in \mathfrak{X}(M)$ is called *tangent* to a subgroup $\mathcal{G}$ of the diffeomorphism group $\mathcal{D}iff^{\infty}(M)$ if there exists an integral curve of X in $\mathcal{G}$. The set of tangent vector fields of $\mathcal{G}$ is denoted by $\mathcal{T}_o\mathcal{G}$.

We have $X \in \mathcal{T}_o\mathcal{G}$ if and only if there exists a $C^{\infty}-$smooth curve $\varphi\colon I \to Diff^{\infty}(M)$ such that $\varphi_t \in \mathcal{G}$ with $\varphi_0 = id_M$, and there exists $k \in \mathbb{N}$ such that equation (3.14) is satisfied. One can observe that in Definition 3.3.2 we do not suppose that $\mathcal{G}$ is a Lie subgroup of $\mathcal{D}iff^{\infty}(M)$. Indeed, we use the differential structure of the later to formulate the smoothness condition on the curve in $\mathcal{G}$. Nevertheless, we have the following

Theorem 3.3.3. *[109, Theorem 3.4] If $\mathcal{G}$ is a subgroup of $\mathcal{D}iff^{\infty}(M)$, then $\mathcal{T}_o\mathcal{G}$ is a Lie subalgebra of $\mathfrak{X}(M)$.*

Proof. We have to show that

$$\begin{aligned} X, Y \in \mathcal{T}_o\mathcal{G} \quad &\Rightarrow \quad [X,Y] \in \mathcal{T}_o\mathcal{G}, && (3.15a)\\ X, Y \in \mathcal{T}_o\mathcal{G} \quad &\Rightarrow \quad X+Y \in \mathcal{T}_o\mathcal{G}, && (3.15b)\\ \lambda \in \mathbb{R},\ X \in \mathcal{T}_o\mathcal{G} \quad &\Rightarrow \quad \lambda X \in \mathcal{T}_o\mathcal{G}. && (3.15c)\end{aligned}$$

Indeed, let $X, Y \in \mathcal{T}_o\mathcal{G}$, that is $X, Y \in \mathfrak{X}(M)$ tangent to G. According to Definition 3.3.1, there exist $k, l \in \mathbb{N}$ such that $\varphi_t, \psi_t \in \mathcal{G}$ are integral curves of X and Y respectively. Let us suppose that φ_t is a $k^{\text{th}}-$order integral curve of X and ψ_t is an l^{th}-order integral curve of Y $(k, l \geq 1)$. Then

$$\varphi_0 = id_M, \qquad \left\{ \frac{\partial^i \varphi_t}{\partial t^i}\Big|_{t=0} = 0 \right\}_{1 \leq i < k} \qquad \frac{\partial^k \varphi_t}{\partial t^k}\Big|_{t=0} = X, \tag{3.16}$$

and

$$\psi_0 = id_M, \qquad \left\{ \frac{\partial^j \psi_t}{\partial t^j}\Big|_{t=0} = 0 \right\}_{1 \leq j < l} \qquad \frac{\partial^l \varphi_t}{\partial t^l}\Big|_{t=0} = Y. \tag{3.17}$$

To show (3.15a) we use a computation similar to that of [67]: Considering the group theoretical commutator

$$[\varphi_t, \psi_s] := \varphi_t^{-1} \circ \psi_s^{-1} \circ \varphi_t \circ \psi_s, \tag{3.18}$$

we get a two-parameter family of diffeomorphisms such that if one of the parameters s or t is zero then (3.18) is the identity transformation. From (3.16) and (3.17) we also know that the first, potentially nonzero derivative is the $(k+l)^{\text{th}}$ order mixed derivative:

$$\begin{aligned} \frac{\partial^{(k+l)}\left[\varphi_t, \psi_s\right]}{\partial t^k \partial s^l}\Big|_{(0,0)}(p) &= \frac{\partial^l}{\partial s^l}\Big|_{s=0} \left(\frac{\partial^k \left(\varphi_s^{-1} \circ \psi_t^{-1} \circ \varphi_s \circ \psi_t(p)\right)}{\partial t^k}\Big|_{t=0} \right) \\ &= \frac{\partial^l}{\partial s^l}\Big|_{s=0} \left(d\left(\varphi_s^{-1}\right)_{\varphi_s(p)} \circ \frac{\partial^k \psi_t^{-1}}{\partial t^k}\Big|_{t=0} (\psi_s(p)) \right), \end{aligned} \tag{3.19}$$

where $d\left(\varphi_s^{-1}\right)_{\varphi_s(p)}$ denotes the tangent map (or Jacobi operator) of φ_s^{-1} at the point $\varphi_s(p)$. Since $d\left(\varphi_{s=0}^{-1}\right)_{\varphi_s(p)} = id$, the above formula can be written in the form

$$d\left(\frac{\partial^l \varphi_s^{-1}}{\partial s^l}\Big|_{s=0}\right)_p \frac{\partial^k \psi_t^{-1}(p)}{\partial t^k}\Big|_{t=0} + d\left(\frac{\partial^k \psi_t^{-1}}{\partial t^k}\Big|_{t=0}\right)_p \frac{\partial^l \varphi_s(p)}{\partial s^l}\Big|_{s=0}. \tag{3.20}$$

From $\varphi_t \circ \varphi_t^{-1} = id$ we get

$$0 = \frac{\partial^k}{\partial t^k}\Big|_{t=0} \left(\varphi_t \circ \varphi_t^{-1}\right) = X + \frac{\partial^k(\varphi_t^{-1})}{\partial t^k}\Big|_{t=0}$$

which yields

$$\frac{\partial^k(\varphi_t^{-1})}{\partial t^k}\Big|_{t=0} = -X. \tag{3.21}$$

Therefore we get that (3.20) can be written as

$$d\left(\frac{\partial^l \varphi_s}{\partial s^l}\Big|_{s=0}\right)_p \frac{\partial^k \psi_t(p)}{\partial t^k}\Big|_{t=0} - d\left(\frac{\partial^k \psi_t}{\partial t^k}\Big|_{t=0}\right)_p \frac{\partial^l \varphi_s(p)}{\partial s^l}\Big|_{s=0}, \tag{3.22}$$

which is the Lie bracket of the vector fields X and Y, that is

$$\frac{\partial^{k+l}\left[\varphi_t, \psi_s\right]}{\partial t^k \partial s^l}\Big|_{(0,0)} = [Y, X]\,. \tag{3.23}$$

From (3.23) we get that $t \to [\varphi_t, \psi_t]$ is a $(k+l)^{\text{th}}$-order integral curve of $[Y, X]$ in $\mathcal{G}$. By exchanging φ_t and ψ_t we get that $t \to [\psi_t, \varphi_t]$ is a $(k+l)^{\text{th}}$-order integral curve of $[X, Y]$ in $\mathcal{G}$. It follows that $[X, Y] \in \mathcal{T}_o\mathcal{G}$ which proves (3.15a).

Let us show (3.15b): for any $c_1, c_2, m_1, m_2 \in \mathbb{R}$, $\phi_t = \varphi_{c_1 t^{m_1}} \circ \psi_{c_2 t^{m_2}}$ is a smooth curve in $\mathcal{G}$ with $\phi_0 = \varphi_0 \circ \psi_0 = id_M$. Moreover, if r denotes the least common multiple of k and l and

$$m_1 = r/k, \quad m_2 = r/l, \quad c_1 = \left(m_1^k (r-k)!\right)^{-1/r}, \quad c_2 = \left(m_2^l (r-l)!\right)^{-1/r},$$

one gets

$$\frac{\partial^r \phi_t}{\partial t^r}\Big|_{t=0} = \frac{\partial^r}{\partial t^r}\Big|_{t=0} \left(\varphi_{c_1 t^{m_1}} \circ \psi_{c_2 t^{m_2}}\right) = X + Y, \tag{3.24}$$

showing that ψ_t is an r^{th}-order integral curve of $X + Y$ in $\mathcal{G}$, therefore $X + Y$ is tangent to $\mathcal{G}$.

To show (3.15c) we have to examine two separate cases depending on the sign of λ. It is clear that in the case when $\lambda \geq 0$, one can reparametrize the integral curve of X, and using that the lower order terms are zero, we get

$$\frac{\partial^k \varphi_{\sqrt[k]{\lambda} t}}{\partial t^k}\Big|_{t=0} = \lambda X. \tag{3.25}$$

In the case when $\lambda < 0$ one can use (3.21) and we get

$$\frac{\partial^k}{\partial t^k}\Big|_{t=0} \left(\varphi^{-1}_{\sqrt[k]{|\lambda|} t}\right) = -|\lambda| X = \lambda X \tag{3.26}$$

From *(21)* and *(22)* we get that λX is tangent to G, that is $\lambda X \in \mathcal{T}_o\mathcal{G}$, and from *11b)* and *11c)* we get that any linear combinations of X and Y are in $\mathcal{T}_o\mathcal{G}$. $\square$

Definition 3.3.4. [109, Definition 3.5] $T_0\mathcal{G}$ is called the tangent Lie algebra of the subgroup $\mathcal{G} \subset \mathcal{D}\mathit{iff}^{\infty}(M)$.

As a direct consequence of Theorem 3.3.3 we get the following

Corollary 3.3.5. *[109, Corollary 3.6] Let $\mathcal{G}$ be a subgroup of $\mathcal{D}\mathit{iff}^{\infty}(M)$ and $\Sigma \subset \mathfrak{X}(M)$ a subset such that its elements are tangent to $\mathcal{G}$. Then the Lie subalgebra $\big\langle\, \Sigma \,\big\rangle_{\mathcal{L}ie}$ of $\mathfrak{X}(M)$ generated by Σ is also tangent to $\mathcal{G}$, that is*

$$\Sigma \subset \mathcal{T}_o\mathcal{G} \qquad \Rightarrow \qquad \big\langle\, \Sigma \,\big\rangle_{Lie} \subset \mathcal{T}_o\mathcal{G}.$$

Remark 3.3.6. Slightly different tangent properties of vector fields to a subgroup $\mathcal{G}$ of the diffeomorphism group were already introduced in [119]. We will refer to the property [119, Definition 2.] as the *weak tangent property* and to [119, Definition 4.] as the *strong tangent property*. Our language is justified by the fact that any strongly tangent vector field to a subgroup $\mathcal{G}$ is also tangent to $\mathcal{G}$, and any tangent vector field to $\mathcal{G}$ is also weakly tangent to $\mathcal{G}$.

The main feature of $\mathcal{T}_o\mathcal{G}$ is that one can obtain information about the group $\mathcal{G}$. Indeed, one has the following

Theorem 3.3.7. *[109, Theorem 3.10] Let M be a compact manifold, $\mathcal{G}$ a subgroup of $\mathcal{D}\mathit{iff}^{\infty}(M)$ and $\overline{\mathcal{G}}$ its topological closure with respect to the C^{∞} topology. Then the group generated by the exponential image of the tangent Lie algebra $\mathcal{T}_o\mathcal{G}$ with respect to the exponential map* $\exp\colon \mathfrak{X}(M) \to \mathcal{D}\mathit{iff}^{\infty}(M)$ *is a subgroup of $\overline{\mathcal{G}}$.*

Proof. If $X \in \mathcal{T}_o\mathcal{G}$, then there exists a C^1-differentiable 1-parameter family $\{\psi_t\} \subset \mathcal{G}$ of diffeomorphisms of M such that $\psi_0 = id_M$ and $X = \frac{\partial \psi_t}{\partial t}\big|_{t=0}$. Then, using the argument of [72, Corollary 5.4, p. 84] on ψ_t we get that

$$\psi^n\left(\tfrac{t}{n}\right) = \psi\left(\tfrac{t}{n}\right) \circ \dots \circ \psi\left(\tfrac{t}{n}\right)$$

in $\mathcal{G}$, as a sequence of $\mathcal{D}\mathit{iff}^{\infty}(M)$, converges uniformly in all derivatives to $\exp(tX)$. It follows that

$$\{\, \exp(tX) \mid t \in \mathbb{R} \,\} \subset \overline{\mathcal{G}},$$

for any $X \in \mathcal{T}_o\mathcal{G}$. Therefore, one has $\exp(\mathcal{T}_o\mathcal{G}) \subset \overline{\mathcal{G}}$ and if we denote by $\big\langle \exp(\mathcal{T}_o\mathcal{G})\big\rangle$ the group generated by the exponential image of $\mathcal{T}_o\mathcal{G}$ we get

$$\big\langle \exp(\mathcal{T}_o\mathcal{G})\big\rangle \subset \overline{\mathcal{G}},$$

which proves Theorem 3.3.7. □

The concept worked out in Definition 3.3.2 and Theorem 3.3.3 can be adapted not only for subgroups of the diffeomorphism group but for any subgroup of any (finite or infinite dimensional) Lie group (see Definition 3.11 and Theorem 3.12 of [109]). In the particular case when $\mathcal{G}$ is a Lie subgroup of $\mathcal{G}_L$, then $\mathcal{T}_o\mathcal{G} = \mathfrak{g}$ is just the usual Lie subalgebra of the Lie algebra $\mathfrak{g}_L$ of $\mathcal{G}_L$, associated to the Lie subgroup $\mathcal{G}$. Hence this construction generalizes the classical notion of the Lie subalgebra associated to a Lie subgroup.

3.4 Fibered holonomy algebra and its Lie subalgebras

The notion of *fibered holonomy group* $\mathcal{H}ol_f(M)$ of a Finsler manifold (M,F) was introduced in [119]: it is the group generated by fiber preserving diffeomorphisms Φ of the indicatrix bundle $(\mathcal{I}M,\pi,M)$, such that for any $p\in M$ the restriction $\Phi_p=\Phi|_{\mathcal{I}_p}$ is an element of the holonomy group $\mathcal{H}ol_p(M)$. It is obvious that

$$\mathcal{H}ol_f(M)\subset\mathcal{D}\mathit{iff}^\infty(\mathcal{I}M), \tag{3.27}$$

where $\mathcal{H}ol_f(M)$ is actually a subgroup of the diffeomorphism group of the indicatrix bundle. We remark that it is not known whether or not $\mathcal{H}ol_f(M)$ is a Lie subgroup of $\mathcal{D}\mathit{iff}^\infty(\mathcal{I}M)$. The set of tangent vector fields to the group $\mathcal{H}ol_f(M)$ denoted as $\mathfrak{hol}_f(M)$ that is

$$\mathfrak{hol}_f(M):=\mathcal{T}_0\big(\mathcal{H}ol_f(M)\big), \tag{3.28}$$

and called the *fibered holonomy algebra* of the Finsler manifold (M,F). From Theorem 3.3.3 one can obtain the following

Theorem 3.4.1. *[109, Theorem 4.2] The fibered holonomy algebra $\mathfrak{hol}_f(M)$ is a Lie subalgebra of the Lie algebra of smooth vector fields $\mathfrak{X}(\mathcal{I}M)$.*

Remark 3.4.2. A vector field $X\in\mathfrak{X}(\mathcal{I})$ is tangent to the fibered holonomy group $\mathcal{H}ol_f(M)$ if and only if there exists a family of fiber preserving diffeomorphisms φ_t of the bundle $(\mathcal{I},\pi,M)$ such that for any indicatrix $\mathcal{I}_p$, $p\in M$, the induced family of diffeomorphisms $\varphi_t\big|_{\mathcal{I}_p}$ is contained in the holonomy group $\mathcal{H}ol_p(M)$ and $\varphi_t\big|_{\mathcal{I}_p}$ is an integral curve of $X\big|_{\mathcal{I}_p}$. It follows that $X\big|_{\mathcal{I}_p}$ is tangent to the holonomy group $\mathcal{H}ol_p(M)$. Furthermore, since $\pi(\varphi_t\big|_{\mathcal{I}_p})\equiv p$ we get $\pi_*(X)=0$ for every $p\in M$, that is tangent vector fields to the fibered holonomy group $\mathcal{H}ol_f(M)$ are vertical vector fields.

In the sequel we will investigate the two most important Lie subalgebras of $\mathfrak{hol}_f(M)$ which can be introduced with the help of the curvature tensor (1.11) of a Finsler manifold: the curvature algebra and the infinitesimal holonomy algebra.

Curvature algebra

Definition 3.4.3. A vector field $\xi\in\mathfrak{X}(\mathcal{I}M)$ is called a *curvature vector field* if there exist vector fields $X,Y\in\mathfrak{X}(M)$ such that $\xi=R(X^h,Y^h)$. The Lie subalgebra $\mathfrak{R}$ of vector fields generated by curvature vector fields is called the *curvature algebra*.

It is easy to see from the definition of the curvature tensor that a curvature vector field can be calculated as

$$\xi=R(X^h,Y^h)=\big[X^h,Y^h\big]-\big[X,Y\big]^h, \tag{3.29}$$

and we have $\mathfrak{R}\subset\mathfrak{X}(\mathcal{I}M)$. Moreover, we have the following

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Proposition 3.4.4. *[109, Proposition 4.4]*

1. *The elements of the curvature algebra are tangent to the group $\mathcal{H}ol_f(M)$.*
2. *The curvature algebra $\mathfrak{R}$ is a Lie subalgebra of $\mathfrak{hol}_f(M)$.*

To prove the first part of the proposition, let $\xi \in \mathfrak{X}(\mathcal{I}M)$ be a curvature vector field and $X, Y \in \mathfrak{X}(M)$ such that $\xi = R(X^h, Y^h)$. We have to show that $\xi \in \mathfrak{hol}_f(M)$. We denote by φ and ψ the integral curves of X and Y respectively. Define

$$\alpha_{t,s} := \begin{cases} \varphi_s, & 0 \leq s \leq t, \\ \psi_{s-t}\varphi_t, & t \leq s \leq 2t, \\ \varphi_{2t-s}\psi_t\varphi_t, & 2t \leq s \leq 3t, \\ \psi_{3t-s}\varphi_{-t}\psi_t\varphi_t, & 3t \leq s \leq 4t. \end{cases}$$

and

$$\beta_{t,s} := \psi_{-s}\varphi_{-s}\psi_s\varphi_s, \qquad 0 \leq s \leq t,$$

for sufficiently small $t \in \mathbb{R}$. Then, for every $p \in M$ and fixed t the map $\alpha_t(p) \colon s \to \alpha_{t,s}(p)$ and $\beta_t(p) \colon s \to \beta_{t,s}(p)$ are parametrized curves: $\alpha_t(p) \colon s \to \alpha_{t,s}(p)$ is a (not necessarily closed) parallelogram and $\beta_t(p)$ joins the endpoints of $\alpha_t(p)$. Indeed, for every $p \in M$ and fixed t the endpoint of $\alpha_t(p)$ coincides with the endpoint of $\beta_t(p)$ and consequently the curve $\alpha_t(p) * \beta_t^{-1}(p)$ defined as going along the curve $\alpha_t(p)$ then continuing along $\beta_t^{-1}(p)$ (which is the curve $\beta_t(p)$ with opposed orientation) is a closed curve that starts and ends at $p \in M$. Let us consider

$$h_{t,p} := \mathcal{P}_{\alpha_t(p)*\beta_t^{-1}(p)} = \mathcal{P}_{\alpha_t(p)} \circ \mathcal{P}^{-1}_{\beta_t(p)}, \tag{3.30}$$

the parallel translation along $\alpha_t(p) * \beta_t^{-1}(p)$. We have the following

Lemma 3.4.5. *For any $p \in M$*

(1) $h_{t,p} \in \mathcal{H}ol_p(M)$,

(2) $t \to h_{t/\sqrt{2},p}$ is a second order integral curve of $\xi_p := \xi\big|_{\mathcal{I}_p} \big(\in \mathfrak{X}(\mathcal{I}_p)\big)$.

Proof. For every $p \in M$ and sufficiently small t, the curve $\alpha_t(p) * \beta_t^{-1}(p)$ is a loop starting and ending at p. Therefore, the parallel transport $h_{t,p} \colon \mathcal{I}_p \to \mathcal{I}_p$ is a holonomy transformation at p, and we get *(1)* of the lemma.

To show *(2)* we first remark that $\alpha_0(p)$ and $\beta_0(p)$ are the trivial curves ($s \to \alpha_{0,s}(p) = \beta_{0,s}(p) \equiv p$), therefore the parallel translation along them is the identity transformation and

$$h_{0,p} = id_{\mathcal{I}_p}. \tag{3.31}$$

On the other hand, the parallel transport along a curve is determined by the horizontal lift of the curve, and along the integral curves of the vector fields X and Y, it can be expressed with the flows of the horizontal lifts X^h and Y^h. Let us consider first the curve $\alpha_t(p)$: the parallel transport of a vector $v \in \mathcal{I}_p$ along the curve $\alpha_t(p)$ is

$$\mathcal{P}_{\alpha_t(p)}(v) = \begin{cases} \varphi_s^{X^h}(v), & 0 \leq s \leq t, \\ \varphi_{s-t}^{Y^h}\varphi_t^{X^h}(v), & t \leq s \leq 2t, \\ \varphi_{-(s-2t)}^{X^h}\varphi_t^{Y^h}\varphi_t^{X^h}(v), & 2t \leq s \leq 3t, \\ \varphi_{-(s-3t)}^{Y^h}\varphi_{-t}^{X^h}\varphi_t^{Y^h}\varphi_t^{X^h}(v), & 3t \leq s \leq 4t. \end{cases}$$

Therefore, $\mathcal{P}_{\alpha_t(p)}$ corresponds to the infinitesimal (not necessarily closed) parallelogram having as sides the integral curves of the horizontal lifts X^h and Y^h. From the well known properties of the Lie bracket (see for example [87, p.162]) we get that

$$\frac{d}{dt}\Big|_{t=0}\mathcal{P}_{\alpha_t}(v) = 0, \qquad \text{and} \qquad \frac{d^2}{dt^2}\Big|_{t=0}\mathcal{P}_{\alpha_t}(v) = 2\left[X^h, Y^h\right]_v. \tag{3.32}$$

On the other hand, the parallel transport of a vector $w \in \mathcal{I}_{\alpha_t(p)}$ along $\beta_t^{-1}(p)$ can be calculated with the help of its horizontal lift $\mathcal{P}_{\beta_t^{-1}}(w) = \mathcal{P}_{\beta_t}^{-1}(w) = ((\beta)^h(t))^{-1}(w)$, where by the definition of the horizontal lift $\pi \circ (\beta)^h(t) = \beta(t)$ and $(\beta^{-1})^h(0) = w$ are fulfilled. Since $\frac{d}{dt}\big|_{t=0}\beta_t(p) = 0$, and $\frac{d^2}{dt^2}\big|_{t=0}\beta_t(p)(v) = 2\left[X, Y\right]_p$, we obtain

$$\frac{d}{dt}\Big|_{t=0}\mathcal{P}_{\beta_t}^{-1} = 0 \qquad \text{and} \qquad \frac{d^2}{dt^2}\Big|_{t=0}\mathcal{P}_{\beta_t}^{-1}(v) = -\big(2\left[X, Y\right]^h\big)_v, \tag{3.33}$$

thus, from the two equations of (3.32) and the two equations of (3.33) we get

$$\frac{d}{dt}\Big|_{t=0}h_t(v) = 0 \qquad \text{and} \qquad \frac{d^2}{dt^2}\Big|_{t=0}h_t(v) = 2\left(\left[X^h, Y^h\right] - \left[X, Y\right]^h\right)_v = 2\xi_p, \tag{3.34}$$

where we also used (3.29). To summarize, we get from (3.31) and (3.34):

$$h_{0,p} = \mathrm{id}\big|_{\mathcal{I}_p}, \qquad \frac{d}{dt}\Big|_{t=0}h_{t,p} = 0, \qquad \frac{1}{2}\frac{d^2}{dt^2}\Big|_{t=0}h_{t,p} = \xi_p, \tag{3.35}$$

which means that the reparametrized map $t \to h_{t/\sqrt{2},p}$ is a second order integral curve of the curvature vector field $\xi_p \in \mathfrak{X}(\mathcal{I}_p)$ and proves point *(2)* of the lemma. $\square$

Proof of Proposition 3.4.4. Let us consider the map $h_t : \mathcal{I}M \to \mathcal{I}M$ on the indicatrix bundle, where $h_t\big|_{\mathcal{I}_p} := h_{t,p}$ and $\tilde{h}_t := h_{t/\sqrt{2}}$. From Lemma 3.4.5 we get

(1) $\tilde{h}_t \in \mathcal{H}ol_f(M)$,

(2) $t \to \tilde{h}_t$ is a second order integral curve of the vector field $\xi \in \mathfrak{X}(\mathcal{I})$,

which shows that the curvature vector field ξ is tangent to $\mathcal{H}ol_f(M)$ and proves the first part of the proposition. Applying Corollary 3.3.5, we get that the Lie algebra generated by the curvature vector field is tangent to $\mathcal{H}ol_f(M)$ which proves the second part of the proposition. $\square$

We remark that *(1)* of Proposition 3.4.4 is an improvement of Proposition 3 and Corollary 2 of [119]. Indeed, the tangent property proved in [119] is weaker: C^1 instead of C^∞ smoothness. Moreover, [119] uses a very strong topological restriction on the manifold M supposing it is diffeomorphic to the n-dimensional euclidean space. In Proposition (3.4.4) we presented a natural geometric construction without constraint on the topology of the manifold M.

Infinitesimal holonomy algebra

Definition 3.4.6. The *infinitesimal holonomy algebra* $\mathfrak{hol}^*(M)$ of a Finsler manifold (M, F) is the smallest Lie algebra on the indicatrix bundle which satisfies the following properties:

1) curvature vector fields are element of $\mathfrak{hol}^*(M)$,
2) if $\xi, \eta \in \mathfrak{hol}^*(M)$, then $[\xi, \eta] \in \mathfrak{hol}^*(M)$,
3) if $\xi \in \mathfrak{hol}^*(M)$ and $X \in \mathfrak{X}(M)$, then the horizontal Berwald covariant derivative $\nabla_X \xi$ is also an element of $\mathfrak{hol}^*(M)$.

We have the following

Proposition 3.4.7. *[109, Proposition 4.7]*

1. *The elements of the infinitesimal holonomy algebra are tangent to $\mathcal{H}ol_f(M)$.*
2. *The infinitesimal holonomy algebra $\mathfrak{hol}^*(M)$ is a Lie subalgebra of $\mathfrak{hol}_f(M)$.*

Proof. It is not difficult to show that if $\xi \in \mathfrak{X}(\mathcal{I})$ is tangent to the fibered holonomy group $\mathcal{H}ol_f(M)$, then its horizontal covariant derivative $\nabla_X \xi$ along any vector field $X \in \mathfrak{X}(M)$ is also tangent to $\mathcal{H}ol_f(M)$. Indeed, let $\mathcal{P}$ be the (nonlinear) parallel translation along the flow φ of the vector field X, i.e. for every $p \in M$ and $t \in (-\varepsilon_p, \varepsilon_p)$ the map $\mathcal{P}_t(p) \colon \mathcal{I}_p M \to \mathcal{I}_{\varphi t(p)} M$ is the (nonlinear) parallel translation along the integral curve of X. If $\{\Phi_t\}_{t \in (-\varepsilon, \varepsilon)}$ is a $\mathcal{C}^\infty$-differentiable k^{th} order integral curve of ξ in $\mathcal{H}ol_f(M)$, then the commutator

$$[\Phi_t, \mathcal{P}_t] := \Phi_t^{-1} \circ \mathcal{P}_t^{-1} \circ \Phi_t \circ \mathcal{P}_t$$

is a $k + 1^{\text{st}}$ order integral curve of $\left[X^h, \xi\right] = \nabla_X \xi$ at any point of M, which shows that the vector field $\nabla_X \xi$ is tangent to $\mathcal{H}ol_f(M)$.
Moreover, we know from 3.4.4 that the curvature vector fields are tangent to $\mathcal{H}ol_f(M)$. As a consequence, the infinitesimal holonomy algebra is generated by vector fields, tangent to $\mathcal{H}ol_f(M)$ and, according to Corollary 3.3.5, any element of the generated Lie algebra is also tangent to $\mathcal{H}ol_f(M)$ proving the second part of the proposition. □

We remark that the first part of Proposition 3.4.7 is an improvement of [119, Theorem 2], because the strong topology condition on the manifold M being diffeomorphic to $\mathbb{R}^n$ is dropped.

3.5 Holonomy algebra and its Lie subalgebras

Let (M, F) be an $n-$dimensional Finsler manifold. At any points $p \in M$ the indicatrix defined in (3.12) is an $(n-1)$-dimensional compact manifold in T_pM. Therefore, the diffeomorphism group $\mathcal{D}i\!f\!f^\infty(\mathcal{I}_p)$ is an infinite dimensional Fréchet Lie group whose Lie algebra is $\mathfrak{X}(\mathcal{I}_p)$, the Lie algebra of smooth vector fields on $\mathcal{I}_p$. As it was introduced in Section 3.2, the holonomy group

$$\mathcal{H}ol_p(M) \subset \mathcal{D}i\!f\!f^\infty(\mathcal{I}_p M), \tag{3.36}$$

is a subgroup of the diffeomorphism group $\mathcal{D}\mathit{iff}^{\infty}(\mathcal{I}_pM)$. The set of tangent vector fields to the group $\mathcal{H}ol_p(M)$ will be denoted as

$$\mathfrak{hol}_p(M) := \mathcal{T}_0\big(\mathcal{H}ol_p(M)\big). \tag{3.37}$$

$\mathfrak{hol}_p(M)$ is called the *holonomy algebra* of the Finsler manifold (M, F) at $p \in M$. From Theorem 3.3.3 one obtains

Theorem 3.5.1. *[109, Theorem 4.9] The holonomy algebra $\mathfrak{hol}_p(M)$ of a Finsler manifold (M, F) at $p \in M$ is a Lie subalgebra of $\mathfrak{X}(\mathcal{I}_p)$.*

The Berwald translate

Let $\gamma := x(t)$, $0 \leq t \leq a$ be a smooth curve joining the points $q = x(0)$ and $p = x(a)$ on the Finsler manifold $(M, \mathcal{F})$. Considering a vector field ξ on the indicatrix $\mathcal{I}_q$, the map $\mathcal{P}_{a*} \circ \xi \circ \mathcal{P}_a^{-1}$ gives a vector field

$$\mathbf{B}_\gamma \xi = \mathcal{P}_{a*}\xi \circ (\mathcal{P}_a)^{-1} \tag{3.38}$$

on the indicatrix $\mathcal{I}_pM$. Moreover, for any vector field $\xi \in \mathfrak{hol}^*_q(M) \subset \mathfrak{X}(\mathcal{I}_q)$ in the infinitesimal holonomy algebra at q, the vector field (3.38) is tangent to the holonomy group $\mathcal{H}ol_p(M)$. Indeed, if $\phi_t \in \mathcal{H}ol_q(M)$ is an integral curve of ξ in $\mathcal{H}ol_q(M)$, then

$$\tau_a \circ \phi_t \circ \tau_a^{-1} \in \mathcal{H}ol_p(M)$$

is an integral curve of (3.38). The vector field $\mathbf{B}_\gamma\xi$ will be called *the Berwald translate* of the vector field $\xi \in \mathfrak{X}(\mathcal{I}_q)$ along the curve γ.

Corollary 3.5.2. *The holonomy algebra $\mathfrak{hol}_p(M)$ of the Finsler manifold $(M, \mathcal{F})$ at the point $p \in M$ contains the Berwald translates of all infinitesimal holonomy algebras along arbitrary curves γ, joining any points q with the point p.*

Remark 3.5.3. Clearly, the smallest Lie algebra of vector fields on the indicatrix $\mathcal{I}_pM$, containing the Berwald translates of all infinitesimal holonomy algebras along arbitrary curves is a Lie subalgebra of the holonomy algebra. It is still an open question whether or not these two Lie algebras coincide.

In the sequel we identify two important Lie subalgebras of the holonomy algebra of Finsler manifolds.

Curvature algebra at a point of the Finsler manifold

Let $(M, \mathcal{F})$ be a Finsler manifold and $p \in M$.

Definition 3.5.4. A vector field $\xi_p \in \mathfrak{X}(\mathcal{I}_p)$ on the indicatrix $\mathcal{I}_p \subset T_pM$ is called a curvature vector field at p if there exist tangent vectors $X_p, Y_p \in T_pM$ such that $\xi_p = R(X_p^h, Y_p^h)$. The Lie subalgebra $\mathfrak{R}_p$ of vector fields generated by curvature vector fields at p is called the *curvature algebra at p*.

The relationship between the curvature algebra $\mathfrak{R}_p$ at $p \in M$ and the curvature algebra $\mathfrak{R}$ introduced in Definition 3.4.3 is:

$$\mathfrak{R}_p = \left\{ \xi_p = \xi|_{\mathcal{I}_p} \;\middle|\; \xi \in \mathfrak{R} \right\},$$

that is $\mathfrak{R}_p$ is the restriction of $\mathfrak{R}$ to the indicatrix $\mathcal{I}_p$. We have

Proposition 3.5.5. *[109, Proposition 4.4] The elements of the curvature algebra $\mathfrak{R}_p$ at $p \in M$ are tangent to the the holonomy group $\mathcal{H}ol_p(M)$. Furthermore, the curvature algebra $\mathfrak{R}_p$ is a Lie subalgebra of the holonomy algebra $\mathfrak{hol}_p(M)$.*

To prove the proposition, the argument is analogous to that of Proposition 3.4.4.

Infinitesimal holonomy algebra at a point of the Finsler manifold

Let $(M, \mathcal{F})$ be a Finsler manifold and $p \in M$. By restricting the infinitesimal holonomy algebra at a point we can obtain

Definition 3.5.6. The Lie algebra $\mathfrak{hol}^*_p(M) := \left\{\xi|_{\mathcal{I}_p} \;\middle|\; \xi \in \mathfrak{hol}^*(M)\right\}$ of vector fields on the indicatrix $\mathcal{I}_p$ is called the infinitesimal holonomy algebra at the point $p \in M$.

From Proposition 3.4.7 we get

Proposition 3.5.7. *[109, Proposition 4.7] The elements of the infinitesimal holonomy algebra $\mathfrak{hol}^*_p(M)$ are tangent to the group $\mathcal{H}ol_p(M)$. Furthermore, the infinitesimal holonomy algebra $\mathfrak{hol}^*_p(M)$ is a Lie subalgebra of the holonomy algebra $\mathfrak{hol}_p(M)$.*

We note that by the construction of the infinitesimal holonomy algebra, the curvature vector fields are elements of $\mathfrak{hol}^*_p(M)$, therefore we have the sequence of the Lie algebras

$$\mathfrak{R}_p(M) \;\subset\; \mathfrak{hol}^*_p(M) \;\subset\; \mathfrak{hol}_p(M) \;\subset\; \mathfrak{X}\left(\mathcal{I}_p\right). \tag{3.39}$$

We also remark that the first parts of the statement of Proposition 3.5.5 and 3.5.7 are improvements of the results of [119] because the tangential property of the Lie algebra is improved: we can guaranty C^∞-smoothness instead of C^1-smoothness.

Remark 3.5.8. In general, the curvature algebra is strictly smaller than the infinitesimal holonomy algebra. Nevertheless, one can find examples, where the curvature algebra and the infinitesimal holonomy algebra coincide. Indeed, one can show that if (M, F) is a Finsler surface with non-zero constant flag curvature, then $\mathfrak{hol}^*_p(M) = \mathfrak{R}_p(M)$ if and only if the mean Berwald curvature (1.10) of (M, F) vanishes on $\mathcal{I}_p$.

Z. Shen constructed in [84] two families of Randers surfaces depending on the real parameter ϵ, which are of constant flag curvature $\kappa = 1$ on the unit sphere $S^2 \subset \mathbb{R}^3$, and of constant flag curvature $\kappa = -1$ on a disk $\mathbb{D}^2 \subset \mathbb{R}^2$. These Finsler surfaces are not projectively flat and have vanishing S-curvature (c.f. [84, Theorems 1.1 and 1.2]). Their Finsler function is defined by

$$\alpha = \frac{\sqrt{\epsilon^2 h(\nu, y)^2 + h(y, y)\,(1 - \epsilon^2 h(\nu, \nu))}}{1 - \epsilon^2 h(\nu, \nu)}, \quad \beta = \frac{\epsilon h(\nu, y)}{1 - \epsilon^2 h(\nu, \nu)}, \tag{3.40}$$

where h is the standard metric of the sphere $\mathbb{S}^2$, (resp. the standard Klein metric on the unit disk $\mathbb{D}^2$) and ν denotes the vector field defined by $(-x_2, x_1, 0)$ at $(x_1, x_2, x_3) \in \mathbb{S}^2$, and by $(-x_2, x_1)$ at $(x_1, x_2) \in \mathbb{D}^2$ respectively. At any point of the Randers-type Finsler surfaces defined by (3.40) the infinitesimal holonomy algebra and the curvature algebra coincide.

3.6 Finsler manifolds of constant curvature

We consider a Finsler manifold (M, F) of non-zero constant curvature. In this case for any $x \in M$ the curvature vector field $R_x(X, Y)$ at $y \in \mathcal{I}_x$ has the form (cf. (3.5))

$$R(X, Y)(y) = c\left(\delta^i_j g_{km}(y) y^m - \delta^i_k g_{jm}(y) y^m\right) X^j Y^k \frac{\partial}{\partial y^i}, \quad c \in \mathbb{R}, \quad c \neq 0.$$

Putting $y_j = g_{jm}(y) y^m$ we can write $R(X, Y)(y) = c\left(\delta^i_j y_k - \delta^i_k y_j\right) X^j Y^k \frac{\partial}{\partial y^i}$. Any linear combination of curvature vector fields has the form

$$r(A)(y) = A^{jk}\left(\delta^i_j y_k - \delta^i_k y_j\right) \frac{\partial}{\partial y^i},$$

where $A = A^{jk} \frac{\partial}{\partial x^j} \wedge \frac{\partial}{\partial x^k} \in T_x M \wedge T_x M$ is arbitrary bivector at $x \in M$.

Lemma 3.6.1. *[120, Lemma 10] Let (M, F) be a Finsler manifold of non-zero constant curvature. The curvature algebra $\mathfrak{R}_x$ at any point $x \in M$ satisfies*

$$\dim \mathfrak{R}_x \geq \frac{n(n-1)}{2}, \tag{3.41}$$

where $n = \dim M$.

Indeed, let us consider the curvature vector fields $r_{jk} = r_x\big(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\big)(y)$ at a fixed point $x \in M$. If a linear combination

$$A^{jk} r_{jk} = A^{jk}(\delta^i_j y_k - \delta^i_k y_j) \frac{\partial}{\partial y^i} = 2A^{ik} y_k \frac{\partial}{\partial y^i}$$

of curvature vector fields r_{jk} with constant coefficients $A^{jk} = -A^{kj}$ satisfies $A^{jk} r_{jk} = 0$ for any $y \in T_x M$ then one has the linear equation $A^{ik} y_k = 0$ for any fixed index i. Since the covector fields $y_1, \ldots, y_n$ are linearly independent we obtain $A^{jk} = 0$ for all $j, k \in \{1, \ldots, n\}$. It follows that the curvature vector fields r_{jk} are linearly independent for any $j < k$ and hence $\dim \mathfrak{R}_x \geq \frac{n(n-1)}{2}$.

Corollary 3.6.2. *Let (M, g) be a Riemannian manifold of non-zero constant curvature with $n = \dim M$. The curvature algebra $\mathfrak{R}_x$ at any point $x \in M$ is isomorphic to the orthogonal Lie algebra $\mathfrak{o}(n)$.*

Proof. The holonomy group of a Riemannian manifold is a subgroup of the orthogonal group $O(n)$ of the tangent space T_xM and hence the curvature algebra $\mathfrak{R}_x$ is a subalgebra of the orthogonal Lie algebra $\mathfrak{o}(n)$. Hence the previous assertion implies the corollary. □

Theorem 3.6.3. *[120, Theorem 12.] Let (M, F) be a Finsler manifold of non-zero constant curvature with $\dim M = n$, $n > 2$. If the point $x \in M$ is not (semi-) Riemannian then*

$$\dim \mathfrak{R}_x > \frac{n(n-1)}{2}. \tag{3.42}$$

Proof. We assume that the point $x \in M$ is non-Riemannian but $\dim \mathfrak{R}_x = \frac{n(n-1)}{2}$. For any constant skew-symmetric matrices $\{A^{jk}\}$ and $\{B^{jk}\}$ the Lie bracket of vector fields $A^{ik} y_k \frac{\partial}{\partial y^i}$ and $B^{ik} y_k \frac{\partial}{\partial y^i}$ has the form $C^{ik} y_k \frac{\partial}{\partial y^i}$, where $\{C^{ik}\}$ is a constant skew-symmetric matrix. Using the homogeneity of g_{hl} we obtain

$$\frac{\partial y_h}{\partial y^m} = \frac{\partial g_{hl}}{\partial y^m}\, y^l + g_{hm} = g_{hm} \tag{3.43}$$

and hence

$$\begin{aligned}\left[A^{mk}\, y_k \frac{\partial}{\partial y^m}, B^{ih}\, y_h \frac{\partial}{\partial y^i}\right] &= \left(A^{mk}\, B^{ih} \frac{\partial y_h}{\partial y^m} - B^{mk}\, A^{ih} \frac{\partial y_h}{\partial y^m}\right) y_k \frac{\partial}{\partial y^i} \\ &= \left(B^{ih}\, g_{hm}\, A^{mk} - A^{ih}\, g_{hm}\, B^{mk}\right) y_k \frac{\partial}{\partial y^i} = C^{ik}\, y_k \frac{\partial}{\partial y^i}.\end{aligned}$$

In particular, for the skew-symmetric matrices $E^{ij}_{ab} = \delta^i_a \delta^j_b - \delta^i_b \delta^j_a$, $a, b \in \{1, \dots, n\}$, we have

$$\left[E^{ij}_{ab}\, y_j \frac{\partial}{\partial y^i}, E^{kl}_{cd}\, y_l \frac{\partial}{\partial y^k}\right] = \left(E^{ih}_{cd}\, g_{hm}\, E^{mk}_{ab} - E^{ih}_{ab}\, g_{hm}\, E^{mk}_{cd}\right) y_k \frac{\partial}{\partial y^i} = \Lambda^{im}_{ab,cd}\, y_m \frac{\partial}{\partial y^i},$$

where the constants $\Lambda^{ij}_{ab,cd}$ satisfy $\Lambda^{ij}_{ab,cd} = -\Lambda^{ji}_{ab,cd} = -\Lambda^{ij}_{ba,cd} = -\Lambda^{ij}_{ab,dc} = -\Lambda^{ij}_{cd,ab}$. Putting $i = a$ and computing the trace for these indices we obtain

$$(n-2)(g_{bd}\, y_c - g_{bc}\, y_d) = \Lambda^l_{b,cd}\, y_l, \tag{3.44}$$

where $\Lambda^l_{b,cd} := \Lambda^{il}_{ib,cd}$. The right hand side is a linear form in the variables $y_1, \dots, y_n$. According to the identity (3.44) this linear form vanishes for $y_c = y_d = 0$, hence $\Lambda^l_{b,cd} = 0$ for $l \neq c, d$. Denoting $\lambda^{(c)}_{bd} := \frac{1}{n-2} \Lambda^c_{b,cd}$ (no summation for the index c) we get the identities

$$g_{bd}\, y_c - g_{bc}\, y_d = \lambda^{(c)}_{bd}\, y_c - \lambda^{(d)}_{bc}\, y_d.$$

Putting $y_d = 0$ we obtain $g_{bd}\big|_{y_d=0} = \lambda^{(c)}_{bd}$ for any $c \neq d$. It follows $\lambda^{(c)}_{bd}$ is independent of the index c ($\neq d$). Defining $\lambda_{bd} := \lambda^{(c)}_{bd}$ with some c ($\neq d$) we obtain from (3.44) the identity

$$g_{bd}\, y_c - g_{bc}\, y_d = \lambda_{bd}\, y_c - \lambda_{bc}\, y_d \tag{3.45}$$

for any $b, c, d \in \{1, \dots, n\}$. We have

$$\lambda_{cd}\, y_b - \lambda_{cb}\, y_d = (g_{bd}\, y_c - g_{bc}\, y_d) - (g_{db}\, y_c - g_{dc}\, y_b) = (\lambda_{bd}\, y_c - \lambda_{bc}\, y_d) - (\lambda_{db}\, y_c - \lambda_{dc}\, y_b).$$

which implies the identity

$$\begin{aligned}(\lambda_{cd}\, y_b - \lambda_{cb}\, y_d) + (\lambda_{db}\, y_c - \lambda_{dc}\, y_b) + (\lambda_{bc}\, y_d - \lambda_{bd}\, y_c) = \\ = (\lambda_{cd} - \lambda_{dc})\, y_b + (\lambda_{db} - \lambda_{bd})\, y_c + (\lambda_{bc} - \lambda_{cb})\, y_d = 0.\end{aligned} \tag{3.46}$$

Since $\dim M > 2$, we can consider 3 different indices b, c, d and we obtain from the identity (3.46) that $\lambda_{bc} = \lambda_{cb}$ for any $b, c \in \{1, \dots, n\}$. By differentiating the identity (3.45) we get

$$\frac{\partial g_{bd}}{\partial y_a}\, y_c - \frac{\partial g_{bc}}{\partial y_a}\, y_d + g_{bd}\, \delta^a_c - g_{bc}\, \delta^a_d = \lambda_{bd}\, \delta^a_c - \lambda_{bc}\, \delta^a_d.$$

Using (3.43) we obtain

$$\frac{\partial y_a}{\partial y^q}\left(\frac{\partial g_{bd}}{\partial y_a}\, y_c - \frac{\partial g_{bc}}{\partial y_a}\, y_d\right) + g_{bd}\, g_{cq} - g_{bc}\, g_{dq} = \lambda_{bd}\, g_{cq} - \lambda_{bc}\, g_{dq},$$

since $\left(\frac{\partial g_{bd}}{\partial y^q}\, y_c - \frac{\partial g_{bc}}{\partial y^q}\, y_d\right) y^b = 0$ we get the identity

$$y_d\, g_{cq} - y_c\, g_{dq} = \lambda_{bd}\, y^b\, g_{cq} - \lambda_{bc}\, y^b\, g_{dq}.$$

Multiplying both sides of this identity by the inverse (g^{qr}) of the matrix (g_{cq}) and taking the trace with respect to the indices c, r we obtain the identity $(n-1)\, y_d = (n-1)\lambda_{bd}\, y^b$. Hence $g_{bd}\, y^b = \lambda_{bd}\, y^b$ and $g_{bd} = \lambda_{bd}$, which means that the point $x \in M$ is (semi-) Riemannian. From this contradiction follows the assertion. □

Corollary 3.6.4. *The curvature algebra $\mathfrak{R}_x$ at a point $x \in M$ of a Finsler manifold (M, F) of non-zero constant curvature satisfies*

$$\dim \mathfrak{R}_x = \frac{n(n-1)}{2}, \quad \textit{where} \quad n = \dim M, \tag{3.47}$$

if and only if $n = 2$ or the point $x \in M$ is (semi-) Riemannian.

Theorem 3.6.5. *[120, Theorem 13.] Let (M, F) be a positive definite n-dimensional Finsler manifold of non-zero constant curvature with $n > 2$. The holonomy group of (M, F) is a compact Lie group if and only if (M, F) is a Riemannian manifold.*

Proof. We assume that the holonomy group of a Finsler manifold (M, F) of non-zero constant curvature with $\dim M \geq 3$ is a compact Lie transformation group on the indicatrix $\mathcal{I}_x$. The curvature algebra $\mathfrak{R}_x$ at a point $x \in M$ is tangent to the holonomy group $\mathcal{H}ol_x(M)$ and hence $\dim \mathcal{H}ol_x(M) \geq \dim \mathfrak{R}_x$. If there exists a not (semi-) Riemannian point $x \in M$, then $\dim \mathfrak{R}_x > \frac{n(n-1)}{2}$. The $(n-1)$-dimensional indicatrix $\mathcal{I}_x$ at x can be equipped with a Riemannian metric which is invariant with respect to the compact Lie transformation group $\mathcal{H}ol_x(M)$. Since the group of isometries of an $n-1$-dimensional Riemannian manifold is of dimension at most $\frac{n(n-1)}{2}$ (cf. Kobayashi [55, p. 46]) we obtain a contradiction, which proves the assertion. □

Since the holonomy group of a Landsberg manifold is a subgroup of the isometry group of the indicatrix, we obtain that any Landsberg manifold of non-zero constant curvature with dimension > 2 is Riemannian (cf. Numata [71]).

We can summarize our results as follows:

Theorem 3.6.6. *[120, Theorem 14.] The holonomy group of any non-Riemannian positive definite Finsler manifold of non-zero constant curvature with dimension > 2 does not occur as the holonomy group of any Riemannian manifold.*

Infinite dimensional curvature algebra

Let us consider the singular (non y-global) Finsler manifold $(\mathbb{H}_3, F)$, where $\mathbb{H}_3$ is the 3-dimensional Heisenberg group and F is a left-invariant Berwald-Moór metric (cf. [82, Example 1.1.5, p. 8]). The group $\mathbb{H}_3$ can be realized as the Lie group of matrices of the form $\left[\begin{smallmatrix} 1 & x^1 & x^2 \\ 0 & 1 & x^3 \\ 0 & 0 & 1 \end{smallmatrix}\right]$, where $x = (x^1, x^2, x^3) \in \mathbb{R}^3$ and hence the multiplication can be written as

$$(x^1, x^2, x^3) \cdot (y^1, y^2, y^3) = (x^1 + y^1, x^2 + y^2 + x^1 y^3, x^3 + y^3).$$

The vector $0 = (0, 0, 0) \in \mathbb{R}^3$ gives the unit element of $\mathbb{H}_3$. The Lie algebra $\mathfrak{h}_3 = T_0\mathbb{H}_3$ consists of matrices of the form $\left[\begin{smallmatrix} 0 & a^1 & a^2 \\ 0 & 0 & a^3 \\ 0 & 0 & 0 \end{smallmatrix}\right]$, corresponding to the tangent vector $a = a^1 \frac{\partial}{\partial x^1} + a^2 \frac{\partial}{\partial x^2} + a^3 \frac{\partial}{\partial x^3}$ at the unit element $0 \in \mathbb{H}_3$. A left-invariant Berwald-Moór Finsler metric F is induced by the (singular) Minkowski functional $F_0 \colon \mathfrak{h}_3 \to \mathbb{R}$:

$$F_0(a) := \left(a^1 a^2 a^3\right)^{\frac{2}{3}}$$

of the Lie algebra in the following way: if $y = (y^1, y^2, y^3)$ is a tangent vector at $x \in \mathbb{H}_3$, then

$$F(x, y) := F_0(x^{-1} y).$$

The coordinate expression of the singular (non y-global) Finsler metric F is

$$F(x, y) = \left(y^1 \left(y^2 - x^1 y^3\right) y^3\right)^{\frac{2}{3}}.$$

Since F is left-invariant, the associated geometric structures (connection, geodesics, curvature) are also left-invariant and the curvature algebras at different points are isomorphic. Let us denote $Y^{k,m} := {y^1}^k {y^3}^m {y^2}^{1-k-m-1}$, $k, m \in \mathbb{N}$, and consider the vector fields

$$A^{k,m}(a^1, a^2, a^3) = a^1 Y^{k+1,m} \frac{\partial}{\partial y^1}\Big|_0 + a^2 Y^{k,m} \frac{\partial}{\partial y^2}\Big|_0 + a^3 Y^{k,m+1} \frac{\partial}{\partial y^3}\Big|_0, \qquad (3.48)$$

with $(a^1, a^2, a^3) \in \mathbb{R}^3$ and $k, m \in \mathbb{N}$. Then the curvature vector fields $r_0(i, j) = r_0\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$ at $0 \in \mathbb{H}_3$ can be written as

$$r_0(1, 2) = \tfrac{1}{4} A^{1,2}(-5, 1, 4), \quad r_0(1, 3) = \tfrac{1}{4} A^{1,1}(11, 0, -11), \quad r_0(2, 3) = \tfrac{1}{4} A^{2,1}(-4, -1, 5).$$

A direct computation shows that for any $(a^1, a^2, a^3), (b^1, b^2, b^3) \in \mathbb{R}^3$ one has

$$\left[A^{k,m}(a^1, a^2, a^3), A^{p,q}(b^1, b^2, b^3)\right] = A^{k+p,m+q}(c^1, c^2, c^3)$$

with some $(c^1, c^2, c^3) \in \mathbb{R}^3$ and the successive Lie brackets generate infinitely many linearly independent elements. We get the

Proposition 3.6.7. *[120, Proposition 15.] The curvature algebra $\mathfrak{R}_x$ at any point $x \in \mathbb{H}_3$ is a Lie algebra of infinite dimension.*

3.7 Projective Finsler manifolds of constant curvature

In this chapter we are investigating the holonomy property of projective Finsler manifolds of constant curvature. Our aim is to characterise the classes of such manifolds where the holonomy group is finite (resp. infinite) dimensional. One of the key tools is the infinitesimal holonomy algebra. According to Proposition 3.5.7, the infinitesimal holonomy algebra $\mathfrak{hol}^*_x(M)$ is tangent to the holonomy group $\mathcal{H}ol_x(M)$. Therefore, the group generated by the exponential image of the infinitesimal holonomy algebra at $x \in M$ with respect to the exponential map $\exp_x : \mathfrak{X}(\mathcal{I}_x) \to \mathcal{D}iff^\infty(\mathcal{I}_x)$ is a subgroup of the closed holonomy group $\overline{\mathcal{H}ol_x(M)}$. Consequently, we have the following estimation on the dimensions:

$$\dim \mathfrak{hol}^*_x(M) \leq \dim \mathcal{H}ol_x(M). \tag{3.49}$$

Projectively flat Finsler manifolds

A Finsler manifold on an open subset $D \subset \mathbb{R}^n$ is said to be *projectively flat*, if all geodesics of (D, F) are contained in straight lines of the affine space associated to $\mathbb{R}^n$. A Finsler manifold (M, F) is said to be *locally projectively flat*, if for any point in $p \in M$ there exists a local coordinate map $x : U \to \mathbb{R}^n$ of a neighbourhood $U \subset M$ of p such that the Finsler manifold induced by the Finsler function F on the image $x(U) = D$ is projectively flat. The space $\mathbb{R}^n$ containing D is called *projectively related to* (M, F).

Let (M, F) be a locally projectively flat Finsler manifold and $(x^1, \dots, x^n) : U \to D$ a local coordinate map corresponding to canonical coordinates of the space $\mathbb{R}^n$ which is projectively related to (M, F). Then the geodesic coefficients (2.3) are of the form

$$G^i(x, y) = \mathcal{P}(x, y) y^i, \quad G^i_k = \frac{\partial \mathcal{P}}{\partial y^k} y^i + \mathcal{P}\delta^i_k, \quad G^i_{kl} = \frac{\partial^2 \mathcal{P}}{\partial y^k \partial y^l} y^i + \frac{\partial \mathcal{P}}{\partial y^k}\delta^i_l + \frac{\partial \mathcal{P}}{\partial y^l}\delta^i_k, \tag{3.50}$$

where $\mathcal{P}$ is a 1-homogeneous function in y, called the *projective factor* (cf. [28, p. 63]) of (M, F). Clearly, the intersections of 2-planes of $\mathbb{R}^n$ with the image D of the coordinate map $(x^1, \dots, x^n) : U \to D$ are images of totally geodesic submanifolds of (M, F).

Remark 3.7.1. The canonical homogeneous parallel translation in a locally projectively flat Finsler manifold (M, F) along curves $c(t)$ contained in the domain of the coordinate system $(x^1, \dots, x^n)$ are linear maps if and only if the projective factor $\mathcal{P}(x, y)$ is a linear function in y. Hence the non-linearity in y of the projective factor implies that the locally projectively flat Finsler manifold is non-Riemannian.

Holonomy of projective Finsler surfaces of constant curvature

A Finsler manifold (M, F) of dimension 2 is called *Finsler surface.* In this case the indicatrix is 1-dimensional at any point $x \in M$, hence the curvature vector fields at $x \in M$ are proportional to any given non-vanishing curvature vector field. It follows that the curvature algebra $\mathfrak{R}_x(M)$ has a simple structure: it is at most 1-dimensional and commutative. Even in this case, the infinitesimal holonomy algebra $\mathfrak{hol}^*_x(M)$ can be higher dimensional, potentially infinite dimensional. For the investigation of such examples we use a classical result of S. Lie claiming that the dimension of a finite-dimensional Lie algebra of vector fields on a connected 1-dimensional manifold is less than 4 (cf. [1, Theorem 4.3.4]). We obtain the following

Lemma 3.7.2. *If the infinitesimal holonomy algebra $\mathfrak{hol}^*_x(M)$ of a Finsler surface (M, F) contains 4 simultaneously non-vanishing $\mathbb{R}$-linearly independent vector fields, then $\mathfrak{hol}^*_x(M)$, and therefore $\mathfrak{hol}_x(M)$, is infinite dimensional.*

Proof. If the infinitesimal holonomy algebra is finite dimensional, then the dimension of the corresponding Lie group acting locally effectively on the 1-dimensional indicatrix would be at least 4, which is a contradiction. □

Proposition 3.7.3. *[122, Proposition 3.2] The infinitesimal holonomy algebra $\mathfrak{hol}^*_x(M)$ of any locally projectively flat non-Riemannian Finsler surface (M, F) of constant curvature $\lambda \neq 0$ is infinite dimensional.*

Proof. Let us suppose that (M, F) is a locally projectively flat Finsler surface of non-zero constant curvature λ and the point $x \in M$ is non-Riemannian. Moreover, we assume that the infinitesimal holonomy algebra is finite dimensional at x. We will show that this assumption leads to contradiction which will prove then, that the infinitesimal holonomy algebra is actually infinite dimensional.

Let (x^1, x^2) be a local coordinate system centered at x, corresponding to the canonical coordinates of the Euclidean plane which is projectively related to (M, F), and let (y^1, y^2) be the induced coordinate system in the tangent planes T_xM.

Consider the curvature vector field $\xi \in \mathfrak{X}(I_xM)$ at the point $x \in M$ defined as

$$y \;\to\; \xi(x, y) := R\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)(x, y) = \lambda\big(\delta^i_2 g_{1m}(x, y) y^m - \delta^i_1 g_{2m}(x, y) y^m\big)\frac{\partial}{\partial y^i}.$$

Since $\lambda \neq 0$, the vector field ξ is non-vanishing. Moreover, since (M, F) is of constant flag curvature, the horizontal Berwald covariant derivative $\nabla_W R$ of the curvature tensor field R vanishes and one has

$$\nabla_W \xi = R\left(\nabla_k\left(\frac{\partial}{\partial x^1}\wedge\frac{\partial}{\partial x^2}\right)\right)W^k.$$

Since

$$\nabla_k\left(\frac{\partial}{\partial x^1}\wedge\frac{\partial}{\partial x^2}\right) = \left(G^1_{k1}+G^2_{k2}\right)\frac{\partial}{\partial x^1}\wedge\frac{\partial}{\partial x^2}$$

we obtain $\nabla_W\xi = (G^1_{k1}+G^2_{k2})\,W^k\xi$. According to (3.50) we have $G^m_{km} = 3\frac{\partial\mathcal{P}}{\partial y^k}$ and hence $\nabla_k\xi = 3\frac{\partial\mathcal{P}}{\partial y^k}\xi$, where $\nabla_k = \nabla_{\frac{\partial}{\partial x^k}}$. Moreover, we have

$$\nabla_j\left(\frac{\partial\mathcal{P}}{\partial y^k}\right) = \frac{\partial^2\mathcal{P}}{\partial x^j\partial y^k} - G^m_j\frac{\partial^2\mathcal{P}}{\partial y^m\partial y^k} = \frac{\partial^2\mathcal{P}}{\partial x^j\partial y^k} - \mathcal{P}\frac{\partial^2\mathcal{P}}{\partial y^k\partial y^j},$$

and hence

$$\nabla_j(\nabla_k\xi) = 3\left(\frac{\partial^2\mathcal{P}}{\partial x^j\partial y^k} - \mathcal{P}\frac{\partial^2\mathcal{P}}{\partial y^k\partial y^j} + 3\frac{\partial\mathcal{P}}{\partial y^k}\frac{\partial\mathcal{P}}{\partial y^j}\right)\xi.$$

According to [28, Lemma 8.2.1, equation (8.25)], we have

$$\frac{\partial^2\mathcal{P}}{\partial x^j\partial y^k} = \frac{\partial\mathcal{P}}{\partial y^j}\frac{\partial\mathcal{P}}{\partial y^k} + \frac{\partial^2\mathcal{P}}{\partial y^j\partial y^k} - \lambda\, g_{jk}, \tag{3.51}$$

hence $\nabla_j(\nabla_k\xi) = 3\left(4\frac{\partial\mathcal{P}}{\partial y^j}\frac{\partial\mathcal{P}}{\partial y^k} - \lambda\, g_{jk}\right)\xi$. It follows that for any fixed $1\le j,k\le 2$, the vector field on $\mathcal{I}_x$ defined as

$$y\to\xi(x,y),\quad y\to\nabla_1\xi(x,y),\quad y\to\nabla_2\xi(x,y),\quad y\to\nabla_j(\nabla_k\xi)\,(x,y), \tag{3.52}$$

are $\mathbb{R}$-linearly independent if and only if the functions

$$1,\qquad \frac{\partial\mathcal{P}}{\partial y^1},\qquad \frac{\partial\mathcal{P}}{\partial y^2},\qquad \frac{\partial\mathcal{P}}{\partial y^j}\frac{\partial\mathcal{P}}{\partial y^k} - \frac{\lambda}{4}\,g_{jk} \tag{3.53}$$

are $\mathbb{R}$-linearly independent. Indeed, since we assumed that the Finsler function F is non-Riemannian at the point x, then $F^2(x,y)$ is non-quadratic in y and according to Remark 3.7.1, the function $\mathcal{P}(x,y)$ is non-linear in y on T_xM. Let us choose a direction $y_0=(y^1_0,y^2_0)\in T_xM$ with $y^1_0\neq 0$, $y^2_0\neq 0$. By restricting U if it is necessary we can suppose that for any $y\in U$ we have $y^1\neq 0$, $y^2\neq 0$. To avoid confusion between coordinate indexes and exponents, we rename the fiber coordinates of vectors belonging to U by $(u,v)=(y^1,y^2)$. Using the values of $\mathcal{P}$ on U we can define a 1-variable function $f=f(t)$ on an interval $I\subset\mathbb{R}$ by

$$f(t) := \frac{1}{v}\mathcal{P}(x_1,x_2,tv,v). \tag{3.54}$$

Then we can express $\mathcal{P}$ and its derivatives with f as

$$\begin{aligned}
&\mathcal{P}=v\,f(u/v), && \mathcal{P}_1=f'(u/v), && \mathcal{P}_2=f(u/v)-\frac{u}{v}f'(u/v),\\
&\mathcal{P}_{11}=\frac{1}{v}f''(u/v), && \mathcal{P}_{12}=-\frac{u}{v^2}f''(u/v), && \mathcal{P}_{22}=\frac{u^2}{v^3}f''(u/v),
\end{aligned} \tag{3.55}$$

where we are using the simplified notation $\mathcal{P}_i = \frac{\partial \mathcal{P}}{\partial y^i}$, $\mathcal{P}_{jk} = \frac{\partial^2 \mathcal{P}}{\partial y^j \partial y^k}$. One can show that, because of the non-linearity of $\mathcal{P}$, the functions 1, $\frac{\partial \mathcal{P}}{\partial y^1}$, $\frac{\partial \mathcal{P}}{\partial y^2}$ are $\mathbb{R}$-linearly independent. Moreover, since $\mathcal{I}_x$ is 1-dimensional and we assumed that the holonomy group is finite dimensional, according to the Lemma 3.7.2, we obtain that the 4 vector fields in (3.52) are $\mathbb{R}$-linearly dependent for any $j,k \in \{1,2\}$. Therefore, the four functions (3.53) are $\mathbb{R}$-linearly dependent for any $j,k \in \{1,2\}$. Being the first three functions in (3.53) $\mathbb{R}$-linearly independent, the fourth function must be a linear combination of the first three: there exist constants $a_i, b_i, c_i \in \mathbb{R}$, $i = 1,2,3$, such that

$$\begin{aligned}
\frac{\lambda}{4}\, g_{11} &= \mathcal{P}_1\mathcal{P}_1 + a_1 + b_1\mathcal{P}_1 + c_1\mathcal{P}_2,\\
\frac{\lambda}{4}\, g_{12} &= \mathcal{P}_1\mathcal{P}_2 + a_2 + b_2\mathcal{P}_1 + c_2\mathcal{P}_2,\\
\frac{\lambda}{4}\, g_{22} &= \mathcal{P}_2\mathcal{P}_2 + a_3 + b_3\mathcal{P}_1 + c_3\mathcal{P}_2.
\end{aligned} \tag{3.56}$$

The integrability conditions $\partial_1 g_{21} - \partial_2 g_{11} = 0$ and $\partial_1 g_{22} - \partial_2 g_{12} = 0$ obtained form (2.1) yield

$$\begin{aligned}
\mathcal{P}_2\mathcal{P}_{11} - \mathcal{P}_1\mathcal{P}_{12} + b_2\mathcal{P}_{11} + (c_2 - b_1)\mathcal{P}_{12} - c_1\mathcal{P}_{22} = 0,\\
\mathcal{P}_1\mathcal{P}_{22} - \mathcal{P}_2\mathcal{P}_{12} - b_3\mathcal{P}_{11} + (b_2 - c_3)\mathcal{P}_{12} + c_2\mathcal{P}_{22} = 0,
\end{aligned} \tag{3.57}$$

and from (3.55) we obtain the equations

$$\begin{aligned}
(f - \frac{u}{v}f')\frac{1}{v}f'' + f'\frac{u}{v^2}f'' + b_2\frac{1}{v}f'' - (c_2 - b_1)\frac{u}{v^2}f'' - c_1\frac{u^2}{v^3}f'' = 0,\\
f'\frac{u^2}{v^3}f'' + (f - \frac{u}{v}f')\frac{u}{v^2}f'' - b_3\frac{1}{v}f'' - (b_2 - c_3)\frac{u}{v^2}f'' + c_2\frac{u^2}{v^3}f'' = 0.
\end{aligned} \tag{3.58}$$

Since by the non-linearity of $\mathcal{P}$ on U we have $f'' \neq 0$ and we get

$$f + b_2 + (b_1 - c_2)\frac{u}{v} - \frac{c_1 u^2}{v^2} = 0, \qquad \frac{u}{v}f - b_3 + (c_3 - b_2)\frac{u}{v} + \frac{c_2 u^2}{v^2} = 0, \tag{3.59}$$

for any $t = u/v$ in an interval $I \subset \mathbb{R}$. The solution of this system of quadratic equations for the function f is given by $f(t) = -c_2\, t - b_2$ with $c_1 = b_3 = 0$, $b_1 = 2c_2$, $c_3 = 2b_2$. But this is a contradiction, since we supposed that by the non-linearity of P we have $f'' \neq 0$ on this interval. Hence the functions 1, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_j\mathcal{P}_k - \frac{\lambda}{4}g_{jk}$ can not be linearly dependent for any $j,k \in \{1,2\}$, from which follows the assertion. □

Remark 3.7.4. From Proposition 3.7.3 we get that if (M,F) is non-Riemannian and $\lambda \neq 0$, then the holonomy group has an infinite dimensional tangent algebra.

Indeed, according to [119, Theorem 6.3] the infinitesimal holonomy algebra $\mathfrak{hol}^*_x(M)$ is tangent to the holonomy group $\mathcal{H}ol_x(M)$, from which follows the assertion.

Now, we can prove our main result:

Theorem 3.7.5. *[122, Theorem 3.7] The holonomy group of a locally projectively flat simply connected Finsler surface (M,F) of constant curvature λ is*

– *finite dimensional if (M, F) is Riemannian or $\lambda = 0$,*
– *infinite dimensional if it is non-Riemannian with nonzero curvature.*

Proof. If (M, F) is Riemannian then its holonomy group is a Lie subgroup of the orthogonal group and therefore it is a finite dimensional compact Lie group.

If (M, F) has zero curvature, then the horizontal distribution associated to the canonical connection in the tangent bundle is integrable and hence the holonomy group is trivial.

If (M, F) is non-Riemannian with non-zero curvature λ, then from Proposition 3.7.3 we get that $\mathfrak{hol}^*_x(M)$ is infinite dimensional. Using the inequality (3.49) we get that $\mathcal{H}ol_x(M)$ cannot be finite dimensional. □

Holonomy of projective Finsler manifolds of constant curvature

We consider now the n-dimensional case. First we prove that the infinitesimal holonomy algebra of a totally geodesic submanifold of a Finsler manifold can be embedded into the infinitesimal holonomy algebra of the entire manifold. This result can yield lower estimate for the dimension of the holonomy group.

A submanifold $\bar{M}$ in a Finsler manifold (M, F) is called *totally geodesic* if any geodesic which is tangent to $\bar{M}$ at some point is contained in $\bar{M}$. A totally geodesic submanifold $\bar{M}$ of (M, F) is called *auto-parallel* if the homogeneous (nonlinear) parallel translations along curves in the submanifold $\bar{M}$ leave invariant the tangent bundle $T\bar{M}$ and for every $\xi \in \mathfrak{X}\left(\bar{M}\right)$ the horizontal Berwald covariant derivative $\nabla_X \xi$ belongs to $\mathfrak{X}\left(\bar{M}\right)$.

Lemma 3.7.6. *Let $\bar{M}$ be a totally geodesic submanifold in a Finsler manifold (M, F). The following assertions hold:*

(a) the geodesic spray $\mathcal{S}$ induces a spray $\bar{\mathcal{S}}$ on the submanifold $\bar{M}$,
(b) $\bar{M}$ is an auto-parallel submanifold,

Proof. Assume that the manifolds $\bar{M}$ and M are k, respectively $n = k + p$ dimensional. Let $(x^1, \dots, x^k, x^{k+1}, \dots, x^n)$ be an adapted coordinate system, i.e. the submanifold $\bar{M}$ is locally given by the equations $x^{k+1} = \cdots = x^n = 0$. We denote the indices running on the values $\{1, \dots, k\}$ or $\{k + 1, \dots, n\}$ by α, β, γ or σ, τ, respectively. The differential equation (1.4) of geodesics yields that the geodesic coefficients $G^\sigma(x, y)$ satisfy

$$G^\sigma(x^1, \dots, x^k, 0, \dots, 0; y^1, \dots, y^k, 0, \dots, 0) = 0$$

identically, hence their derivatives with respect to $y^1, \dots, y^k$ are also vanishing. It follows that $G^\sigma_\alpha = 0$ and $G^\sigma_{\alpha\,\beta} = 0$ at any $(x^1, \dots, x^k, 0, \dots, 0; y^1, \dots, y^k, 0, \dots, 0)$. Hence the induced spray $\bar{\mathcal{S}}$ on $\bar{M}$ is defined by the geodesic coefficients

$$\bar{G}^\beta(x^1, \dots, x^k; y^1, \dots, y^k) = G^\beta(x^1, \dots, x^k, 0, \dots, 0; y^1, \dots, y^k, 0, \dots, 0). \tag{3.60}$$

The homogeneous (nonlinear) parallel translation $\tau_c : T_{c(0)}M \to T_{c(1)}M$ along curves in the submanifold $\bar{M}$ and the horizontal covariant derivative on $\bar{M}$ with respect to the spray $\mathcal{S}$ coincide with the parallel translation and horizontal covariant derivative on $\bar{M}$ with respect to the spray $\bar{\mathcal{S}}$. Hence the assertions are true.

$\square$

Totally geodesic and auto-parallel submanifolds

Lemma 3.7.7. *Let $\bar{M}$ be a totally geodesic submanifold in a spray manifold $(M, \mathcal{S})$. The curvature vector fields at any point of $\bar{M}$ can be extended to a curvature vector field of M.*

Proof. Assume that the manifolds $\bar{M}$ and M are k, respectively $n = k + p$ dimensional. Let $(x^1, \ldots, x^k, x^{k+1}, \ldots, x^n)$ be an adapted coordinate system, that is the submanifold $\bar{M}$ is locally given by the equations $x^{k+1} = \cdots = x^n = 0$. Using the notation of the proof of Lemma 3.7.6 we get from equation (3.60) that $G^\sigma_\alpha = 0$ and $G^\sigma_{\alpha\,\beta} = 0$ for any $(x^1, \ldots, x^k, 0, \ldots, 0; y^1, \ldots, y^k, 0, \ldots, 0)$ we have

$$\frac{\partial G^\sigma_\alpha}{\partial x^\beta} - \frac{\partial G^\sigma_\beta}{\partial x^\alpha} + G^\tau_\alpha G^\sigma_{\beta\tau} - G^\tau_\beta G^\sigma_{\alpha\tau} + G^\gamma_\alpha G^\sigma_{\beta\gamma} - G^\gamma_\beta G^\sigma_{\alpha\gamma} = 0$$

at $(x^1, \ldots, x^k, 0, \ldots, 0; y^1, \ldots, y^k, 0, \ldots, 0)$. Hence the curvature tensors $\bar{R}$ and R, corresponding to the spray $\bar{\mathcal{S}}$, respectively to the spray $\mathcal{S}$ satisfy

$$\bar{R}(X, Y)(x, y) = R(X, Y)(x, y) \quad \text{if} \quad x \in \bar{M} \quad \text{and} \quad y, X, Y \in T_x\bar{M}.$$

It follows that for any given $X, Y \in T_x\bar{M}$ the curvature vector field $\bar{\xi}(y) = \bar{R}(X, Y)(x, y)$ at $x \in \bar{M}$ defined on $T_x\bar{M}$ can be extended to the curvature vector field $\xi(y) = R(X, Y)(x, y)$ at $x \in \bar{M}$ defined on T_xM.

$\square$

Proposition 3.7.8. *[124, Theorem 4.3] Let $\bar{M}$ be a totally geodesic 2-dimensional submanifold of a Finsler manifold (M, F) such that the infinitesimal holonomy algebra $\mathfrak{hol}^*_x(\bar{M})$ of $\bar{M}$ is infinite dimensional. Then the infinitesimal holonomy algebra $\mathfrak{hol}^*_x(M)$ of M is infinite dimensional.*

Proof. According to Lemma 3.7.7, any curvature vector field of $\bar{M}$ at $x \in \bar{M} \subset M$ defined on $\mathcal{I}_x\bar{M}$ can be extended to a curvature vector field on the indicatrix $\mathcal{I}_xM$. Hence the curvature algebra $\mathfrak{R}_x(\bar{M})$ of the submanifold $\bar{M}$ can be embedded into the curvature algebra $\mathfrak{R}_x(M)$ of the manifold (M, F). Assume that $\bar{\xi}$ is a vector field belonging to the infinitesimal holonomy algebra $\mathfrak{hol}^*_x(\bar{M})$ which can be extended to the vector field ξ belonging to the infinitesimal holonomy algebra $\mathfrak{hol}^*_x(M)$. Any vector field $\bar{X} \in \mathfrak{X}\left(\bar{M}\right)$ can be extended to a vector field $X \in \mathfrak{X}\left(M\right)$, hence the horizontal Berwald covariant derivative along $\bar{X} \in \mathfrak{X}\left(\bar{M}\right)$ of $\bar{\xi}$ can be extended to the Berwald horizontal covariant derivative along $X \in \mathfrak{X}\left(M\right)$ of the vector field ξ. It follows that the infinitesimal holonomy algebra $\mathfrak{hol}^*_x(\bar{M})$ of the submanifold $\bar{M}$ can be embedded into the infinitesimal holonomy algebra $\mathfrak{hol}^*_x(M)$ of the Finsler manifold (M, F). Consequently, $\mathfrak{hol}^*_x(M)$ is infinite dimensional and hence the holonomy group $\mathcal{H}ol_x(M)$ is an infinite dimensional subgroup of $\mathcal{D}iff^\infty(\mathcal{I}_x)$.

$\square$

This result can be applied to locally projectively flat Finsler manifolds, as they have for each tangent 2-plane a totally geodesic submanifold which is tangent to this 2-plane.

Theorem 3.7.9. *[122, Theorem 3.6] The holonomy group of a locally projectively flat simply connected n-dimensional Finsler manifold (M, F) of constant curvature λ is*

- *finite dimensional if (M, F) is Riemannian or $\lambda = 0$,*
- *infinite dimensional if (M, F) is non-Riemannian and $\lambda \neq 0$.*

Proof. If (M, F) is Riemannian then its holonomy group is a Lie subgroup of the orthogonal group and therefore it is a finite dimensional compact Lie group.

If (M, F) has zero curvature, then the horizontal distribution associated to the canonical connection in the tangent bundle is integrable and hence the holonomy group is trivial.

If (M, F) is non-Riemannian having non-zero curvature λ, then for each tangent 2-plane $\mathcal{T} \subset T_xM$ the manifold M has a totally geodesic submanifold $\widetilde{M} \subset M$ such that $T_x\widetilde{M} = \mathcal{T}$. This $\widetilde{M}$ with the induced metric is a locally projectively flat Finsler surface of constant curvature λ. Therefore from Proposition 3.7.3 we get that $\mathfrak{hol}^*_x(\widetilde{M})$ is infinite dimensional. Moreover, according to Proposition 3.7.8 , if a Finsler manifold (M, F) has a totally geodesic 2-dimensional submanifold $\widetilde{M}$ such that the infinitesimal holonomy algebra of $\widetilde{M}$ is infinite dimensional, then the infinitesimal holonomy algebra $\mathfrak{hol}^*_x(M)$ of the containing manifold is also infinite dimensional. Using (3.49) we get that $\mathcal{H}ol_x(M)$ cannot be finite dimensional. Hence the assertion is true.

□

We note that there are examples of non-Riemannian type locally projectively flat Finsler manifolds with $\lambda = 0$ and with $\lambda \neq 0$ curvature (cf. [62, 85]).

Remark 3.7.10. In the previous theorem, the key condition for the Finsler metric tensor was not the positive definiteness but its non-degenerate property. Therefore Theorem 3.7.9 can be generalized as follows.

A pair (M, F) is called *semi-Finsler manifold* if in the definition of Finsler manifolds the positive definiteness of the Finsler metric tensor is replaced by the non-degenerate property. Then we have

Corollary 3.7.11. *The holonomy group of a locally projectively flat simply connected semi-Finsler manifold (M, F) of constant curvature λ is finite dimensional if and only if (M, F) is semi-Riemannian or $\lambda = 0$.*

3.8 Finsler surfaces with maximal holonomy

The group $\mathcal{D}iff^{\infty}_{+}(\mathbb{S}^1)$ and the Fourier algebra

Let $\mathbb{S}^1 = \mathbb{R} \mod 2\pi$ be the unit circle with the standard counterclockwise orientation. The group $\mathcal{D}iff^{\infty}_{+}(\mathbb{S}^1)$ of orientation preserving diffeomorphisms of $\mathbb{S}^1$ is the connected component of $\mathcal{D}iff^{\infty}(\mathbb{S}^1)$. The Lie algebra of $\mathcal{D}iff^{\infty}_{+}(\mathbb{S}^1)$ is $\mathfrak{X}(\mathbb{S}^1)$ – denoted also by $\mathsf{Vect}(\mathbb{S}^1)$ in the literature – can be written in the form $f(t)\frac{d}{dt}$, where f is a 2π-periodic smooth function on the real line $\mathbb{R}$. A sequence $\{f_j\frac{d}{dt}\}_{j\in\mathbb{N}} \subset \mathsf{Vect}(\mathbb{S}^1)$ converges to $f\frac{d}{dt}$ in the Fréchet topology of $\mathsf{Vect}(\mathbb{S}^1)$ if and only if the functions f_j and all their derivatives converge uniformly to f, respectively to the corresponding derivatives of f. The Lie bracket on $\mathsf{Vect}(\mathbb{S}^1)$ is given by

$$\Big[f\frac{d}{dt}, g\frac{d}{dt}\Big] = \Big(g\frac{df}{dt} - \frac{dg}{dt}f\Big)\frac{d}{dt}.$$

The *Fourier algebra* $\mathsf{F}(\mathbb{S}^1)$ on $\mathbb{S}^1$ is the Lie subalgebra of $\mathsf{Vect}(\mathbb{S}^1)$ consisting of vector fields $f\frac{d}{dt}$ such that $f(t)$ has finite Fourier series, i.e. $f(t)$ is a Fourier polynomial. The vector fields $\big\{\frac{d}{dt}, \cos nt\frac{d}{dt}, \sin nt\frac{d}{dt}\big\}_{n\in\mathbb{N}}$ provide a basis for $\mathsf{F}(\mathbb{S}^1)$. A direct computation shows that the vector fields

$$\frac{d}{dt}, \quad \cos t\frac{d}{dt}, \quad \sin t\frac{d}{dt}, \quad \cos 2t\frac{d}{dt}, \quad \sin 2t\frac{d}{dt} \tag{3.61}$$

generate the Lie algebra $\mathsf{F}(\mathbb{S}^1)$. The complexification $\mathsf{F}(\mathbb{S}^1)\otimes_{\mathbb{R}}\mathbb{C}$ of $\mathsf{F}(\mathbb{S}^1)$ is called the *Witt algebra* $\mathsf{W}(\mathbb{S}^1)$ on $\mathbb{S}^1$ having the natural basis $\big\{ie^{int}\frac{d}{dt}\big\}_{n\in\mathbb{Z}}$, with the Lie bracket $[ie^{imt}\frac{d}{dt}, ie^{int}\frac{d}{dt}] = i(m-n)e^{i(n+m)t}\frac{d}{dt}$.

Lemma 3.8.1. *The group $\Big\langle \overline{\exp\left(\mathsf{F}(\mathbb{S}^1)\right)} \Big\rangle$ generated by the topological closure of the exponential image of the Fourier algebra $\mathsf{F}(\mathbb{S}^1)$ is the orientation preserving diffeomorphism group $\mathcal{D}iff^{\infty}_{+}(\mathbb{S}^1)$.*

Proof. The Fourier algebra $\mathsf{F}(\mathbb{S}^1)$ is a dense subalgebra of $\mathsf{Vect}(\mathbb{S}^1)$ with respect to the Fréchet topology, i.e. $\overline{\mathsf{F}(\mathbb{S}^1)} = \mathsf{Vect}(\mathbb{S}^1)$. This assertion follows from the fact that any r-times continuously differentiable function can be approximated uniformly by the arithmetical means of the partial sums of its Fourier series (cf. [52, Theorem 2.12]). The exponential mapping is continuous (c.f. in [72, Lemma 4.1, p. 79]), hence we have

$$\exp\big(\mathsf{Vect}(\mathbb{S}^1)\big) = \exp\big(\overline{\mathsf{F}(\mathbb{S}^1)}\big) \subset \overline{\exp\big(\mathsf{F}(\mathbb{S}^1)\big)} \subset \mathcal{D}iff^{\infty}_{+}(\mathbb{S}^1) \tag{3.62}$$

which gives, for the generated groups, the relations

$$\big\langle\exp\big(\mathsf{Vect}(\mathbb{S}^1)\big)\big\rangle \subset \Big\langle \overline{\exp\big(\mathsf{F}(\mathbb{S}^1)\big)} \Big\rangle \subset \mathcal{D}iff^{\infty}_{+}(\mathbb{S}^1). \tag{3.63}$$

Moreover, the conjugation map $\mathsf{Ad} : \mathsf{Diff}^{\infty}_{+}(\mathbb{S}^1) \times \mathsf{Vect}(\mathbb{S}^1)$ satisfies the relation

$$h \exp s\xi\, h^{-1} = \exp s\mathsf{Ad}(h)\xi$$

for every $h \in \mathcal{D}\mathit{iff}^{\infty}_{+}(\mathbb{S}^1)$ and $\xi \in \mathsf{Vect}(\mathbb{S}^1)$. Clearly, the Lie algebra $\mathsf{Vect}(\mathbb{S}^1)$ is invariant under conjugation and hence the group $\left\langle \exp\left(\mathsf{Vect}(\mathbb{S}^1)\right)\right\rangle$ is also invariant under conjugation. Therefore $\left\langle \exp\left(\mathsf{Vect}(\mathbb{S}^1)\right)\right\rangle$ is a normal subgroup of $\mathcal{D}\mathit{iff}^{\infty}_{+}(\mathbb{S}^1)$. On the other hand $\mathcal{D}\mathit{iff}^{\infty}_{+}(\mathbb{S}^1)$ is a simple group (cf. [48]) which means that its only non-trivial normal subgroup is itself. Therefore, we have $\left\langle \exp\left(\mathsf{Vect}(\mathbb{S}^1)\right)\right\rangle = \mathcal{D}\mathit{iff}^{\infty}_{+}(\mathbb{S}^1)$, and using (3.63) we get

$$\left\langle \overline{\exp(\mathsf{F}(\mathbb{S}^1))} \right\rangle = \mathcal{D}\mathit{iff}^{\infty}_{+}(\mathbb{S}^1).$$

□

Holonomy of the standard Funk plane and the Bryant-Shen 2-spheres

Using the results of the preceding section we can prove the following statement, which provides a useful tool to investigate the closed holonomy group of Finsler 2-manifolds.

Proposition 3.8.2. *[123, Proposition 5.1] If the infinitesimal holonomy algebra $\mathfrak{hol}^*_x(M)$ at a point $x \in M$ of a simply connected Finsler 2-manifold (M, F) contains the Fourier algebra $\mathsf{F}(\mathbb{S}^1)$ on the indicatrix at x, then $\overline{\mathcal{H}ol_x(M)}$ is isomorphic to $\mathcal{D}\mathit{iff}^{\infty}_{+}(\mathbb{S}^1)$.*

Proof. Since M is simply connected we have

$$\overline{\mathcal{H}ol_x(M)} \subset \mathcal{D}\mathit{iff}^{\infty}_{+}(\mathbb{S}^1). \tag{3.64}$$

On the other hand, using Theorem 3.3.7, we get

$$\exp\left(\mathsf{F}(\mathbb{S}^1)\right) \subset \overline{\mathcal{H}ol_x(M)} \;\Rightarrow\; \overline{\exp\left(\mathsf{F}(\mathbb{S}^1)\right)} \subset \overline{\mathcal{H}ol_x(M)} \;\Rightarrow\; \left\langle \overline{\exp\left(\mathsf{F}(\mathbb{S}^1)\right)} \right\rangle \subset \overline{\mathcal{H}ol_x(M)},$$

and from the last relation, using Lemma 3.8.1, we can obtain that

$$\mathcal{D}\mathit{iff}^{\infty}_{+}(\mathbb{S}^1) \subset \overline{\mathcal{H}ol_x(M)}. \tag{3.65}$$

Comparing (3.64) and (3.65) we get the assertion.

□

Using this proposition we can prove our main result:

Theorem 3.8.3. *[123, Theorem 5.2] Let $(M, \mathcal{F})$ be a simply connected projectively flat Finsler manifold of constant curvature $\lambda \neq 0$. Assume that there exists a point $x_0 \in M$ such that on $T_{x_0}M$ the induced Minkowski norm is the Euclidean norm, that is $\mathcal{F}(x_0, y) = \|y\|$, and the projective factor at x_0 satisfies $\mathcal{P}(x_0, y) = c \cdot \|y\|$ with $c \in \mathbb{R}$, $c \neq 0$. Then the closed holonomy group $\overline{\mathcal{H}ol_{x_0}(M)}$ at x_0 is isomorphic to $\mathcal{D}\mathit{iff}^{\infty}_{+}(\mathbb{S}^1)$.*

Proof. Since (M,F) is a locally projectively flat Finsler manifold of non-zero constant curvature, we can use an (x^1,x^2) local coordinate system centered at $x_0 \in M$, corresponding to the canonical coordinates of the Euclidean space which is projectively related to (M,F). Let (y^1,y^2) be the induced coordinate system in the tangent plane T_xM. In the sequel we identify the tangent plane $T_{x_0}M$ with $\mathbb{R}^2$ by using the coordinate system (y^1,y^2). We will use the Euclidean norm $\|(y^1,y^2)\| = \sqrt{(y^1)^2+(y^2)^2}$ of $\mathbb{R}^2$ and the corresponding polar coordinate system (e^r,t) too.

Let us consider the curvature vector field ξ at $x_0=0$ defined by

$$\xi = R\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2}\right)\Big|_{x=0} = \lambda\big(\delta^i_2 g_{1m}(0,y)y^m - \delta^i_1 g_{2m}(0,y)y^m\big)\frac{\partial}{\partial x^i}$$

Since (M,F) is of constant flag curvature, the horizontal Berwald covariant derivative $\nabla_W R$ of the tensor field R vanishes, c.f. Lemma 3.2. Therefore the covariant derivative of ξ can be written in the form

$$\nabla_W\xi = R\left(\nabla_k\left(\frac{\partial}{\partial x^1}\wedge\frac{\partial}{\partial x^2}\right)\right)W^k.$$

Since

$$\nabla_k\left(\frac{\partial}{\partial x^1}\wedge\frac{\partial}{\partial x^2}\right) = \left(G^1_{k1}+G^2_{k2}\right)\frac{\partial}{\partial x^1}\wedge\frac{\partial}{\partial x^2}$$

we obtain $\nabla_W\xi = (G^1_{k1}+G^2_{k2})\,W^k\xi$. Using (3.50) we can express $G^m_{km} = 3\frac{\partial P}{\partial y^k} = 3c\frac{y^k}{\|y\|}$ and hence

$$\nabla_k\xi = 3\frac{\partial P}{\partial y^k}\xi = 3c\frac{y^k}{\|y\|}\xi,$$

where we use the notation $\nabla_k = \nabla_{\frac{\partial}{\partial x^k}}$. Moreover we have

$$\nabla_j\left(\frac{\partial\mathcal{P}}{\partial y^k}\right) = \frac{\partial^2\mathcal{P}}{\partial x^j\partial y^k} - G^m_j\frac{\partial^2\mathcal{P}}{\partial y^m\partial y^k} = \frac{\partial^2\mathcal{P}}{\partial x^j\partial y^k} - \mathcal{P}\frac{\partial^2\mathcal{P}}{\partial y^k\partial y^j},$$

and hence

$$\nabla_j\left(\nabla_k\xi\right) = 3\left\{\frac{\partial^2\mathcal{P}}{\partial x^j\partial y^k} - \mathcal{P}\frac{\partial^2\mathcal{P}}{\partial y^k\partial y^j} + 3\frac{\partial\mathcal{P}}{\partial y^k}\frac{\partial\mathcal{P}}{\partial y^j}\right\}\xi.$$

According to Lemma 8.2.1, in [28, equation (8.25), p. 155], we obtain

$$\frac{\partial^2\mathcal{P}}{\partial x^j\partial y^k} = \frac{\partial\mathcal{P}}{\partial y^j}\frac{\partial\mathcal{P}}{\partial y^k} + \mathcal{P}\frac{\partial^2\mathcal{P}}{\partial y^j\partial y^k} - \frac{\lambda}{2}\frac{\partial^2F^2}{\partial y^j\partial y^k}.$$

Using the assumptions on $\mathcal{F}$ and on the projective factor $\mathcal{P}$ we can get at x_0

$$\nabla_j\left(\nabla_k\xi\right) = 3\left(4c^2\frac{\partial F}{\partial y^j}\frac{\partial F}{\partial y^k} - \frac{\lambda}{2}\frac{\partial^2F^2}{\partial y^j\partial y^k}\right)\xi$$

and hence

$$\nabla_j\left(\nabla_k\xi\right) = 3\left(4\,c^2\frac{y^jy^k}{\|y\|^2} - \lambda\,\delta^{jk}\right)\xi,$$

where $\delta^{jk} \in \{0,1\}$ such that $\delta^{jk} = 1$ if and only if $j = k$.
Let us introduce polar coordinates $y^1 = r\cos t$, $y^2 = r\sin t$ in the tangent space $T_{x_0}M$. We can express the curvature vector field, its first and second covariant derivatives along the indicatrix curve $\{(\cos t, \sin t);\ 0 \leq t < 2\pi\}$ as follows:

$$\xi = \lambda\frac{d}{dt}, \quad \nabla_1\xi = 3c\lambda\cos t\frac{d}{dt}, \quad \nabla_2\xi = -3c\lambda\sin t\frac{d}{dt}, \quad \nabla_1(\nabla_2\xi) = 12\,c^2\lambda\sin 2t\frac{d}{dt},$$

$$\nabla_1(\nabla_1\xi) = \lambda\left(12\,c^2\cos^2 t - \lambda\right)\frac{d}{dt}, \quad \nabla_2(\nabla_2\xi) = \lambda\left(12\,c^2\sin^2 t - \lambda\right)\frac{d}{dt}.$$

Since $c\,\lambda \neq 0$, the vector fields

$$\frac{d}{dt}, \quad \cos t\,\frac{d}{dt}, \quad \sin t\,\frac{d}{dt}, \quad \cos t\sin t\,\frac{d}{dt}, \quad \cos^2 t\,\frac{d}{dt}, \quad \sin^2 t\,\frac{d}{dt}$$

are contained in the infinitesimal holonomy algebra $\mathfrak{hol}^*_{x_0}(M)$. It follows that the generator system

$$\left\{\frac{d}{dt}, \quad \cos t\,\frac{d}{dt}, \quad \sin t\,\frac{d}{dt}, \quad \cos 2t\,\frac{d}{dt}, \quad \sin 2t\,\frac{d}{dt}\right\}$$

of the Fourier algebra $\mathsf{F}(\mathbb{S}^1)$ (c.f. equation (3.61)) is contained in the infinitesimal holonomy algebra $\mathfrak{hol}^*_{x_0}(M)$. Hence the assertion follows from Proposition 3.8.2.

□

We remark, that the standard Funk plane and the Bryant-Shen 2-spheres are connected, projectively flat Finsler manifolds of nonzero constant curvature. Moreover, in each of them, there exists a point $x_0 \in M$ and an adapted local coordinate system centered at x_0 with the following properties: the Finsler norm $\mathcal{F}(x_0, y)$ and the projective factor $\mathcal{P}(x_0, y)$ at x_0 are given by $\mathcal{F}(x_0, y) = \|y\|$ and by $\mathcal{P}(x_0, y) = c{\cdot}\|y\|$ with some constant $c \in \mathbb{R}$, $c \neq 0$, where $\|y\|$ is an Euclidean norm in the tangent space at x_0. Using Theorem 3.8.3 we can obtain

Theorem 3.8.4. *[123, Theorem 5.3] The closed holonomy groups of the standard Funk plane and of the Bryant-Shen* 2*-spheres are maximal, that is diffeomorphic to the orientation preserving diffeomorphism group of* $\mathbb{S}^1$*.*

Holonomy of projectively flat Randers surfaces

Projectively flat non-Riemannian Randers manifolds with non-zero constant flag curvature were classified by Z. Shen in [86]. He proved that any projectively flat Randers manifold (M, F) with non-zero constant flag curvature has negative curvature. Moreover, these metrics can be normalized by a constant factor so that the curvature is $\lambda = -1/4$. In this case (M, F) is isometric to the Finsler manifold defined by the Finsler function

$$F_a(x,y) = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)}}{1-|x|^2} + \epsilon\left(\frac{\langle x,y\rangle}{1-|x|^2} + \frac{\langle a,y\rangle}{1+\langle a,x\rangle}\right) \qquad (3.66)$$

on the unit ball $\mathbb{D}^n \subset \mathbb{R}^n$, where $a \in \mathbb{R}^n$ is a constant vector with $|a| < 1$ and $\epsilon = \pm 1$ [86, Theorem 1.1]. We note that the restriction of any orthogonal transformation $\phi \in O(n, \mathbb{R}^n)$ on $\mathbb{D}^n$ does not change the Finsler function (3.66), therefore one can assume that $a \in \mathbb{R}^n$ has the form $a = (a_1, 0, \dots, 0)$. We can consider $(\mathbb{D}^n, F_a)$ as the *standard model* of projectively flat Randers manifolds with non-zero constant flag curvature.

According to [28, Lemma 8.2.1], if $(M \subset \mathbb{R}^n, F)$ is a projectively flat manifold, then its projective factor can be computed using the formula

$$\mathcal{P}(x,y) = \frac{1}{2F}\frac{\partial F}{\partial x^i}y^i. \tag{3.67}$$

It follows that the computation of the coefficients of the associated connection is relatively easy: in the case (3.66) it gives

$$2\mathcal{P}(x,y) = \frac{\epsilon\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)} + \langle x,y\rangle}{1-|x|^2} - \frac{\langle a,y\rangle}{1+\langle a,x\rangle}. \tag{3.68}$$

The geodesic coefficients and the connection coefficients can be computed from (3.68) by using (3.50).

Proposition 3.8.5. *[108, Proposition 1.] The closed holonomy group of $(\mathbb{D}^2, F_a)$ is diffeomorphic to $\mathcal{D}iff^{\infty}_{+}(\mathbb{S}^1)$.*

We consider here the case when $\epsilon = 1$ in the expression (3.66) of F_a. The computation when $\epsilon = -1$ is analogous. The curvature vector field $\xi = R(\partial_{x_1}, \partial_{x_2})$ at the point $0 \in \mathbb{R}^2$ is

$$\xi = R(\partial_{x_1}, \partial_{x_2}) = \frac{1}{4}\frac{y_2\,(a_1y_1 + \|y\|)}{\|y\|}\frac{\partial}{\partial y_1} - \frac{1}{4}\left(y_1 + y_1a_1^2 + 2\,a_1\,\|y\|\right)\frac{\partial}{\partial y_2}. \tag{3.69}$$

Since the Minkowski norm at $0 \in \mathbb{D}^2$ is $\mathcal{F}_a(0,y) = \|y\| + \langle a,y\rangle$, the indicatrix $\mathcal{I}_0 \subset T_0M$ at 0 is defined by the equation $\sqrt{y_1^2 + y_2^2} + a_1y_1 = 1$. Using polar coordinates (r,t) a parametrization of $\mathcal{I}_0$ is given by

$$\phi(t) = \left(\frac{\cos t}{1 + a_1\cos t}, \frac{\sin t}{1 + a_1\cos t}\right). \tag{3.70}$$

The restriction $\xi_0 := \xi\big|_{\mathcal{I}_0}$ of the curvature vector field (3.69) on $\mathcal{I}_0$ is $\xi_0 := \omega(t)\frac{d}{dt}$ where $\omega(t) := -\frac{1}{4}(1 + a_1\cos t)^2$. Let us introduce the notation

$$\Sigma_n := \mathrm{Span}_{\mathbb{R}}\left\{\, \xi_0^{l,m} \mid 0 \le l + m \le n \,\right\}, \tag{3.71}$$

where $\xi_0^{l,m} = \sin^l t\cos^m t\,\xi_0$. On can show by induction that for any $n \in \mathbb{N}$ we have $\Sigma_n \subset \mathfrak{hol}^*_0(\mathbb{D}^2, F_a)$ and consequently, the infinitesimal holonomy algebra contains the Fourier algebra. Hence from Proposition 3.8.2 we get that the holonomy group $\mathcal{H}ol_0(\mathbb{D}^2, F_a)$ is maximal, and its closure is diffeomorphic to $\mathcal{D}iff^{\infty}_{+}(\mathbb{S}^1)$.

Using Z. Shen's classification theorem of Randers manifolds we can get the following

Theorem 3.8.6. *[108, Theorem 3.] The holonomy group of a simply connected non-Riemannian projectively flat Randers surface of constant non-zero flag curvature is maximal and its closure is diffeomorphic to the orientation preserving diffeomorphism group of* $\mathbb{S}^1$*, that is*

$$\overline{\mathcal{H}ol_x(M)} \cong \mathcal{D}\mathit{iff}^{\infty}_{+}(\mathbb{S}^1).$$

Proof. Let (M, F) be a simply connected non-Riemannian projectively flat Randers two-manifold of constant non-zero flag curvature and $x \in M$. Rescaling the metric by a constant factor does not change the parallel translation and the holonomy group. Hence we can suppose that the metric is normalized so that the curvature is $\lambda = -\frac{1}{4}$. Using Shen's results, F can be locally expressed in the form F_a given in (3.66) where $a = (a_1, 0) \in \mathbb{R}^2$ is a nonzero constant vector with $|a_1| < 1$. From Proposition 3.8.5 we get, that the closed holonomy group is maximal and diffeomorphic to $\mathcal{D}\mathit{iff}^{\infty}_{+}(\mathbb{S}^1)$.

□

We can obtain the following

Corollary 3.8.7. *[108, Corollary 4.] The closure of the holonomy group* $\mathcal{H}ol(M)$ *of a simply connected, locally projectively flat Randers two-manifold of constant flag curvature* λ *is*

1. *the trivial group* $\{id\}$*, when* $\lambda = 0$*;*
2. *the rotation group* $SO(2)$*, when* $\lambda \neq 0$ *and the metric is Riemannian;*
3. *diffeomorphic to* $\mathcal{D}\mathit{iff}^{\infty}_{+}(\mathbb{S}^1)$*, when* $\lambda \neq 0$ *and the metric is non-Riemannian.*

Proof. The holonomy structures listed in *1.)* and *2.)* correspond to the (already well known) finite dimensional holonomy cases. When $\lambda \neq 0$ and the metric is non-Riemannian we get *3.)* from Theorem 3.8.6.

□

Chapter 4

Linearizability of planar 3-webs

4.1 Introduction

Let M be a two-dimensional real or complex differentiable manifold. A 3-web is given in an open domain D of M by three foliations of smooth curves in general position. Two webs $\mathcal{W}$ and $\widetilde{\mathcal{W}}$ are locally equivalent at $p \in M$, if there exists a local diffeomorphism on a neighborhood of p which transforms $\mathcal{W}$ into $\widetilde{\mathcal{W}}$. A 3-web is called *linear* (resp. *parallel*) if it is given by 3 foliations of straight lines (resp. of parallel lines). A 3-web which is equivalent to a linear (resp. parallel) web is called linearizable (resp. parallelizable).

Basic examples of planar 3-webs come from complex projective algebraic geometry. If $\mathcal{C} \subset \mathbb{P}^2$ is a reduced algebraic curve of degree 3, by duality in $\check{\mathbb{P}}^2$, one can obtain a 3-web called the algebraic web associated with $\mathcal{C} \subset \mathbb{P}^2$ (cf. [50, 74]). H. Graf and W. Sauer proved in 1924 a theorem, which in web geometry language can be stated as follows: a linear web is parallelizable if and only if it is associated with an algebraic curve of degree 3, i.e. its leaves are tangent lines to an algebraic curve of degree 3 [16, p. 24].

Although the problem of finding a linearizability criterion is a very natural one, it is far from being trivial. T.H. Gronwall conjectured that if a non-parallelizable 3-web $\mathcal{W}$ is linearizable, then up to a projective transformation there is a unique diffeomorphism which maps $\mathcal{W}$ into a linear 3-web. G. Bol suggested a method in [17] how to find a criterion of linearizability, but he was unable to carry out the computation. He showed that the number of projectively different linear 3-webs in the plane which are equivalent to a non-parallelizable 3-web is finite and less than 17. The formulation of the linearizability problem in terms of the Chern connection was suggested by M.A. Akivis in a lecture given in Moscow in 1973. In his approach the linearizability problem is reduced to the solvability of a system of nonlinear partial differential equations on the components of the affine deformation tensor. Using Akivis' idea V.V. Goldberg determined in [42] the first integrability conditions of the partial differential system.

In 2001, [106] solved the linearizability problem by determining the integrability condition of the PDE system. It was proved that, in the non-parallelizable case, there exists an algebraic submanifold $\mathcal{A}$ of the space of vector valued symmetric tensors ($\mathcal{A} \subset S^2T^* \otimes T$) on a neighborhood of any point $p \in M$, expressed in terms of the curvature of the Chern connection and its covariant derivatives up to order 6, so that the affine deformation tensor is a section of $S^2T^* \otimes T$ with values in $\mathcal{A}$. In particular: the web is linearizable if and only if $\mathcal{A} \neq \emptyset$ and there exists at most 15 projectively nonequivalent linearizations of a nonparallelizable 3-web. The expressions of the polynomials and their coefficients which define $\mathcal{A}$ can be found in [107]. The criteria of linearizability provide the possibility to make explicit computation on concrete examples to decide whether or not they are linearizable.

The controversy. In 2006 V.V. Goldberg (expert in web theory, author of several books and many papers on web theory) and V.V. Lychagin (well-known expert on PDE systems, member of the Russian Academy of Sciences) found results on the linearizability [43, 44]. Their results were different from that of [106] and they qualified them "incomplete because they do not contain all conditions" (see [43, page 171] and [44, page 70]) without pointing out any missing integrability condition or developing any further justification. The results of [106] and [44] *cannot be both correct* because there are cases where the two theories *contradict*. Hence the small but dedicated scientific community working on the problems related to web geometry was in suspense (see for example [2, page 2], [3, page 2], [19, page 30], [94, page 40]).

Decisive example. The direct comparison of the two theories is not straightforward since the formulas in both cases are long and complex containing the curvature tensor and its higher-order different (covariant resp. partial) derivatives. There is, however, a very specific case, where the two theories show clearly opposite results: Using [106] one gets that the web $\mathcal{W}$ given by the system (4.21) is linearizable (see [106, page 2653]) while [44] states the opposite (cf. page 171, line 7–10). Evidently, the correct theory should give the correct answer in that specific case. Finally, the linearizability problem of $\mathcal{W}$ has been considered in [116] and it has been proven that $\mathcal{W}$ is indeed linearizable by showing the existence of the affine deformation tensor. More explicitly, as Remark 4.4.1 shows, the web $\mathcal{W}$ is linearizable, therefore the prediction of [106] is correct and the statement of [44] is wrong.

4.2 Preliminaries

Let $\mathcal{W}$ be a differential 3-web on a manifold M given by a triplet of mutually transversal foliations $\{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3\}$. From the definitions it follows that M is even dimensional and that the dimension of the tangent distributions of the foliations $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$ is the half of the dimension of M. The foliations $\{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3\}$ are called horizontal, vertical and transversal, and their tangent spaces are denoted by T^h, T^v and T^t. The 3-web $\mathcal{W}$ is linearizable if and only if there exists a flat linear connection ∇^L preserving the web.

A 3$-$web is equivalent to a pair $\{h, j\}$ of (1,1)-tensor fields on the manifold, satisfying the following conditions:

1. $h^2 = h$, $j^2 = id$,

2. $jh = vj$, where $v = id - h$,
3. $\operatorname{Ker} h$, $\operatorname{Im} h$, and $\operatorname{Ker}(h + id)$ are integrable distributions.

Moreover, for any 3−web, there exists a unique linear connection ∇ on M which satisfies $\nabla h = 0$, $\nabla j = 0$, and $\mathcal{T}(hX, vY) = 0$, $\forall X, Y \in TM$, $\mathcal{T}$ being the torsion tensor of ∇. ∇ is called the *Chern connection* (see [70]).

A symmetrical (1,2)-tensor field L is called a *pre-linearization* if the connection

$$\nabla^L_X Y = \nabla_X Y + L(X, Y)$$

preserves the web, that is the leaves are auto-parallel curves with respect to ∇^L. A pre-linearization is a linearization if the connection ∇^L is flat i.e. its curvature vanishes. Two pre-linearizations L and L' are projectively equivalent if the connections ∇^L and $\nabla^{L'}$ are projectively related, that is there exists $\omega \in \Lambda^1(M)$, such that

$$\nabla^L_X Y = \nabla^{L'}_X Y + \omega(X)Y + \omega(Y)X.$$

Proposition 4.2.1. *A tensor field L in $S^2T^* \otimes T$ is a* linearization *if and only if*

1) $vL(hX, hY) = 0$,
2) $hL(vX, vY) = 0$,
3) $L(hX, hY) + jL(jhX, jhY) - hL(jhX, hY)$
$\qquad - hL(hX, jhY) - jvL(jhX, hY) - jvL(hX, jhY) = 0$,
4) $\nabla_X L(Y, Z) - \nabla_Y L(X, Z) + L(X, L(Y, Z)) - L(Y, L(X, Z)) + R(X, Y)Z = 0$,

holds, for any $X, Y, Z \in T$, where R denotes the curvature of the Chern connection.

The proof is a straightforward verification. Properties 1), 2) and 3) mean that L is a pre-linearization and follows from the fact that ∇^L preserves the web, while property 4) expresses, that the curvature of ∇^L vanishes.

In the sequel, we suppose that the dimension of M is two. Let $\mathcal{W}$ be a web on M and $\{e_1, e_2\}$ a frame at $p \in M$ adapted to the web, i.e. $e_1 \in T^h_p$, $e_2 = je_1 \in T^v_p$. Let L be a pre-linearization at p, whose components are L^k_{ij}, that is: $L(e_i, e_j) = L^k_{ij} e_k$, and let us set the tensor-field s to be represented by the components $2L^1_{12} - L^2_{22}$. The tensor s will be called *the base* of L. The following proposition is elementary, but it is the key for the proof of our main theorem.

Proposition 4.2.2. *Two pre-linearizations L and L' are projectively equivalent if and only if they have the same base, i.e. $s = s'$.*

Indeed, if L and L' are two projectively equivalent pre-linearizations, then there exists $\omega \in T^*$ such that $L' = L + \omega \odot id$, i.e. in the frame $\{e_1, e_2\}$:

$$L'^1_{11} = L^1_{11} + 2\omega_1, \qquad L'^2_{22} = L^2_{22} + 2\omega_2, \qquad L'^1_{12} = L^1_{12} + \omega_2$$

where ω_1 and ω_2 are the components of ω. This system is consistent if and only if $L'^1_{12} - L^1_{12} = \frac{1}{2}(L'^2_{22} - L^2_{22})$, i.e. $s = s'$.

Let $L \in E$ be a pre-linearization. We introduce the tensors x, y, $z : T^h \otimes T^h \to T^h$ defined by

$$\begin{cases} x\,(hX,\, hY) := L\,(hX,\, hY), \\ y\,(hX,\, hY) := jL\,(jhX,\, jhY), \\ z\,(hX,\, hY) := hL\,(hX\,,jhY). \end{cases} \tag{4.1}$$

One denotes by x^2 the (1,3) tensor defined by

$$x^2\,(hX, hY, hZ) := x\,(x\,(hX,\, hY),\, hZ).$$

Similarly, we define the product xy, x^3 (which is a (1,4) tensor field), etc.

The space of pre-linearizations E is a 3-dimensional vector bundle over M, and x, y, z can be used to parameterize it. However, taking into account some symmetries of the problem and the Proposition 4.2.2, it is better to introduce the tensors $s, t : T^h \oplus T^h \to T^h$ defined by

$$\begin{aligned} s &:= 2z - y \\ t &:= \tfrac{1}{2}\,(x + y - 2z) \end{aligned} \tag{4.2}$$

and parameterize E by s, t, z where s is the base of the web (see Definition 4.2).

In order to simplify the notation, we denote by C_1 and C_2 the tensor fields $\left(\bigotimes^{p+1} T^{h^*}\right) \otimes T^h$ defined by

$$\begin{aligned} C_1\,(hX, hX_1, ..., hX_p) &= (\nabla_{hX} C)\,(hX_1, ..., hX_p), \\ C_2\,(hX, hX_1, ..., hX_p) &= (\nabla_{jhX} C)\,(hX_1, ..., hX_p), \end{aligned} \tag{4.3}$$

where C is a tensor field in $\left(\bigotimes^{p} T^{h^*}\right) \otimes T^h$. By recursion, we introduce the successive covariant derivatives with the convention that $C_{i_1 i_2} := (C_{i_2})_{i_1}$. Thus, $x_{i_1,...,i_p}$ is the $(1, p+2)$ tensor defined in an adapted frame by

$$x_{i_1,...,i_p}\,(\underbrace{e_1, ..., e_1}_{p\,\text{times}}, hX, hY) = (\underbrace{\nabla\nabla\ \cdots \nabla}_{p\,\text{times}} x)\,(e_{i_1}, ..., e_{i_p}, hX, hY).$$

We denote $\mathcal{R}$ the tensor $\mathcal{R} : T^h \oplus T^h \oplus T^h \to T^h$ defined by

$$\mathcal{R}(hX,\, hY)\, hZ = R(jhX,\, hY)\, hZ \tag{4.4}$$

where R is the curvature of the Chern connection. With the above notation we have

$$(\nabla_i \nabla_j L^l_{i_1,...,i_m}) - (\nabla_j \nabla_i L^l_{i_1,...,i_m}) = R^l_{ijk} L^k_{i_1,...,i_m} - R^k_{iji_1} L^l_{k,...,i_m} - \cdots - R^k_{iji_m} L^l_{i_1,...,k}.$$

In particular

$$C_{12} - C_{21} = (p-1)\, \mathcal{R}\, C \tag{4.5}$$

for a tensor field $C \in \left(\bigotimes^{p} T^{h^*}\right) \otimes T^h$.

4.3 The linearization theorem

In the sequel E denotes the bundle of the pre-linearizations and $F := \Lambda^2 T^* \otimes T$. In order to study the linearizability of $\mathcal{W}$, we will consider the differential operator $P_1 : E \to F$ and study the integrability of the differential system

$$P_1(L) = 0, \tag{4.6}$$

where

$$\begin{aligned}\big(P_1(L)\big)(X,Y,Z) &= (\nabla_X L)(Y,Z) - (\nabla_Y L)(X,Z) \\ &\quad + L(X, L(Y,Z)) - L(Y, L(X,Z)) + R(X,Y)Z\end{aligned} \tag{4.7}$$

for every $X, Y, Z \in T$. Using the tensors (4.2) and resolving two equations in z_1 and t_2 the system (4.6) can be written as:

$$\begin{cases} t_1 = st + t^2, \\ t_2 = \frac{1}{3}s_1 - \frac{2}{3}s_2 + zt - \frac{1}{3}\mathcal{R}, \\ z_1 = \frac{2}{3}s_1 - \frac{1}{3}s_2 + zt + \frac{1}{3}\mathcal{R}, \\ z_2 = -zs + z^2. \end{cases} \tag{4.8}$$

Note that P_1 is regular because the symbol and its prolongation are regular maps. The system (4.8) can be seen as a Frobenius system on the variables t and z with s a parameter. By formula (4.5), the integrability conditions are

$$\begin{cases} z_{12} - z_{21} = \mathcal{R}z, \\ t_{12} - t_{21} = \mathcal{R}t, \\ s_{12} - s_{21} = \mathcal{R}s, \end{cases}$$

and thus from (4.8) we can arrive at the system

$$P_2 = \begin{cases} s_{22} = 2s_{21} - ss_2 + 2ss_1 + \mathcal{R}s + \mathcal{R}_2, \\ s_{11} = 2s_{21} - 2ss_2 + ss_1 + \mathcal{R}s + \mathcal{R}_1, \end{cases} \tag{4.9}$$

(see also [49, equation (* bis)]). The operator $P_2 : Sec\,(E_2) \to F_2$ corresponding to the system (4.9) is a quasi-linear second order differential operator, where $E_2 = T^{h^*} \otimes T^{h^*} \otimes T^h$, and $F_2 := F' \oplus F'$ with $F' := T^{h^*} \otimes T^{h^*} \otimes E_2$. The linearizability of the web is equivalent to the integrability of the operator P_2. In the sequel we will examine the integrability of P_2.

Proposition 4.3.1. *[107, Proposition 4.1.] At every $p \in M$ all $2^{nd}-$order solution at p of P_2 can be lifted into a $3^{rh}-$order solution.*

Indeed, fixing an adapted base $\{e_1, e_2 = je_1\}$, the symbol of P_2 is the map

$$\sigma_2 : S^2T \otimes E_2 \to F_2, \qquad \sigma_2(A) = (A_{22} - 2A_{21},\ A_{11} - 2A_{21}),$$

where $A_{ij} = A(e_i, e_j)$. So $g_2 := \operatorname{Ker} \sigma_2$ is defined by the equations

$$A_{22} - 2A_{21} = 0 \quad \text{and} \quad A_{11} - 2A_{21} = 0.$$

Since these equations are independent, we have $\operatorname{rank} \sigma_2 = 2$, $\dim g_2 = 1$. On the other hand, for the symbol of the first prolongation

$$\sigma_3 : S^3 T^* \otimes E_2 \to T^* \otimes F_2, \quad \sigma_3(B) = (B_{k22} - 2B_{k21},\, B_{k11} - 2B_{k21}) \tag{4.10}$$

where $B_{ijk} = B(e_i, e_j, e_k)$, we find that $g_3 = \operatorname{Ker} \sigma_3$ is defined by the equations

$$B_{k22} - 2B_{k21} = 0, \quad B_{k11} - 2B_{k21} = 0,$$

$k = 1, 2$. It is easy to verify that these equations are also independent. Therefore $\operatorname{rank} \sigma_3 = 4 = \dim(T^* \otimes F_2)$ and $\dim g_3 = 0$, thus σ_3 is an isomorphism and $\operatorname{Coker} \sigma_3 = 0$. We have the following exact diagram:

$$\begin{tikzcd}
 & S^2 T^* \otimes E_2 \arrow[r, "\sigma_3"] \arrow[d, "\varepsilon"'] & T^* \otimes F \arrow[r] \arrow[d, "\varepsilon"'] & \operatorname{Coker} \sigma_3 = 0 \\
R_3 \arrow[r] \arrow[d, "\bar{\pi}_3"'] & J_3(E_2) \arrow[r, "p_1(P_2)"] \arrow[d, "\pi_3"'] & J_1 F \arrow[d, "\pi_1"'] & \\
R_2 \arrow[r] & J_2(E_2) \arrow[r, "p_0(P_2)"] & F &
\end{tikzcd}$$

Using a homological algebraic argument it can be shown, that $\bar{\pi}_3$ is onto, i.e. every $2^{nd}-$ order solution of P_2 can be lifted into a $3^{rd}-$order solution. $\square$

Proposition 4.3.2. *[107, Proposition 4.2.] The operator P_2 is not $2-$acyclic, i.e. there is a higher order obstruction which arises for the integrability of P_2.*

Indeed, the sequence

$$0 \longrightarrow g_{\ell+1}(P_2) \xrightarrow{\;i\;} g_\ell(P_2) \otimes T^* \xrightarrow{\delta_\ell(P_2)} g_{\ell-1}(P_2) \otimes \Lambda^2 T^* \longrightarrow 0$$

is not exact for all $l \geq 2$, where δ_ℓ denotes the skew-symmetrization in the corresponding variables: for $\ell = 3$ we have $\operatorname{rank} \delta_3 = 0$ which is strictly smaller than $\dim(g_2 \otimes \Lambda^2 T^*) = 1$. $\square$

Obstructions to linearizability

In order to find the higher order obstruction we consider the prolongation of P_2, i.e. the operator $P_3 := (P_2, \nabla P_2)$, where $\nabla P_2 : T^* \otimes E_2 \longrightarrow T^* \otimes F_2$ is the covariant derivative of P_2 with respect to the Chern connection. Explicitly, this system is formed by the system (4.9) and by the following four equations:

$$\begin{cases} s_{212} = & ss_{21} - \frac{1}{3}s_1s_2 + \frac{4}{3}s_2^2 - \frac{2}{3}s_1^2 + \frac{4}{3}\mathcal{R}s_2 + 2s^2s_1 \\ & + \mathcal{R}s^2 + (2\mathcal{R}_2 - \mathcal{R}_1)s - \frac{2}{3}\mathcal{R}_{21} - \frac{1}{3}\mathcal{R}_{12} \\ s_{211} = & -ss_{21} + \frac{1}{3}s_1s_2 + \frac{2}{3}s_2^2 - \frac{4}{3}s_1^2 + (\frac{5}{3}\mathcal{R} + 2s^2)s_2 \\ & -10\mathcal{R}s_1 + (\mathcal{R}_2 - 2\mathcal{R}_1)s - \frac{1}{3}\mathcal{R}_{21} - \frac{2}{3}\mathcal{R}_{12} \\ s_{111} = & -2ss_{21} - \frac{4}{3}s_1s_2 + \frac{4}{3}s_2^2 - \frac{5}{3}s_1^2 + (\frac{10}{3}\mathcal{R} + 2s^2)s_2 - (\frac{5}{3}\mathcal{R} - s^2)s_1 \\ & -\mathcal{R}s^2 + (2\mathcal{R}_2 - 2\mathcal{R}_1)s - \frac{2}{3}\mathcal{R}_{21} - \frac{4}{3}\mathcal{R}_{12} + \mathcal{R}_{11} \\ s_{222} = & 2ss_{21} + \frac{4}{3}s_1s_2 + \frac{5}{3}s_2^2 - \frac{4}{3}s_1^2 + (\frac{5}{3}\mathcal{R} + s^2)s_2 - (\frac{10}{3}\mathcal{R} - 2s^2)s_1 \\ & + \mathcal{R}s^2 + (2\mathcal{R}_2 - 2\mathcal{R}_1)s - \frac{4}{3}\mathcal{R}_{21} - \frac{2}{3}\mathcal{R}_{12} + \mathcal{R}_{22} \end{cases} \tag{4.11}$$

Since (4.11) can be solved with respect to the $3^{rd}-$order derivatives, the existence of a 2^{nd}-order formal solution implies the existence of $3^{rd}-$order solutions. One can show that the symbol of P_3 is involutive. Moreover, any $3^{rd}-$order solution of P_3 can be lifted into a $4^{th}-$order solution if and only if $\varphi = 0$, where

$$\begin{aligned} \varphi(s) = & -24\mathcal{R}s_{21} - (24\mathcal{R}s + 12\mathcal{R}_1 - 6\mathcal{R}_2)s_1 + (24\mathcal{R}s + 6\mathcal{R}_1 - 8\mathcal{R}_2)s_2 \\ & + 3\mathcal{R}s^3 + (-4\mathcal{R}_2 - 3\mathcal{R}_{22} + \mathcal{R}_{21} + 2\mathcal{R}_{12} - 13\mathcal{R}^2 - 3\mathcal{R}_{11})s \\ & + 2\mathcal{R}_{122} - \mathcal{R}_{221} - \mathcal{R}_{112} - 5\mathcal{R}\mathcal{R}_1 - 2\mathcal{R}_{121} - 11\mathcal{R}\mathcal{R}_2 \end{aligned} \tag{4.12}$$

Indeed, using the equations (4.9) and (4.11), we obtain

$$\begin{aligned} \varphi(s) = & \nabla_{11}[2s_{21} - s_{22} - ss_2 + 2ss_1 + \mathcal{R}s + \mathcal{R}_2] - 2\nabla_{12}[2s_{21} - s_{22} - ss_2 + 2ss_1 + \mathcal{R}s + \mathcal{R}_2] \\ & + 2\nabla_{12}[2s_{21} - s_{11} - 2ss_2 + ss_1 + \mathcal{R}s + \mathcal{R}_1] - \nabla_{22}[2s_{21} - s_{11} - 2ss_2 + ss_1 + \mathcal{R}s + \mathcal{R}_1] \end{aligned}$$

By formula (4.5) we can eliminate the $4^{th}-$order derivatives and find (4.12). Moreover, we can remark that $\dim g_{3,p} = 0$ and therefore $\dim g_{k,p} = 0$ for every $k > 3$. It follows that the symbol of P_3 is involutive. (We use the terminology introduced in Chapter 2.2, see also the monograph [23, p.121].) □

If $\mathcal{R} = 0$, then $\varphi = 0$, therefore, all $3^{rd}-$order solution of P_3 can be lifted into a $4^{th}-$order solution. Since its symbol is involutive, P_3 is formally integrable and consequently, it is integrable in the analytical case. We have the following

Corollary 4.3.3. *[106, Corollary 4.2] If $\mathcal{W}$ is a parallelizable 3-web on the plane, then for all $L_0 \in E_p$ there exists a germ of linearizations L which prolongs L_0.*

In accord of the Graf-Sauer Theorem, one can deduce that for a parallelizable web, there exist non projectively equivalent linearizations. Indeed, it is sufficient to consider $L_0, L_0' \in E_p$ with $s_p \neq s_p'$ and to prolong them in germs of linearizations to obtain two non projectively equivalent germs of linearization.

Second obstruction to linearizability

In the sequel we will suppose that $\mathcal{R} \neq 0$. In this case the compatibility condition (4.12) is not satisfied, so we have to add it to our differential system and consider the enlarged second order quasi-linear system $P_\varphi = 0$:

$$P_\varphi := (P_2, \varphi),$$

where P_2 is defined by (4.9) and φ is given by (4.12).

Lemma 4.3.4. *A $2^{nd}-$order formal solution $j_{2,p}s$ of P_φ at $p \in M$, can be lifted into a $3^{rd}-$order solution if and only if:*

$$\begin{cases} \psi^1_p := 24\mathcal{R}(s_2)^2 - 48\mathcal{R}s_1s_2 + \alpha(s)s_1 + \beta(s)s_2 + \gamma(s) = 0 \\ \psi^2_p := -24\mathcal{R}(s_1)^2 + 48\mathcal{R}s_1s_2 + \hat{\alpha}(s)s_1 + \hat{\beta}(s)s_2 + \hat{\gamma}(s) = 0. \end{cases} \tag{4.13}$$

where α, β, $\hat{\alpha}$, $\hat{\beta}$ are polynomials in s of degree 2 with coefficients $\mathcal{R}$ and its derivatives up to order 2, γ and $\hat{\gamma}$ are polynomials in s of degree 3 with coefficients $\mathcal{R}$ and its derivatives up to order 4. Their explicit expressions are given in [107].

Proof. One can show that there are exactly two relations between the highest order terms of the prolongation, and a 2^{nd} order solution $(j_2s)_p$ of P_φ can be lifted into a 3^{rd} order solution if and only if $(\psi^1, \psi^2)_p = 0$ where

$$(\psi^1, \psi^2)_p := \tau_3\nabla(P_\varphi(s))_p$$

We have:

$$\psi^1 = 24\mathcal{R}[\nabla(P_2(s))]^1_1 + [\nabla(\varphi)]_2 - 2[\nabla(\varphi)]_1$$
$$\psi_2 = 24\mathcal{R}[\nabla(P_2(s))]^2_2 + [\nabla(\varphi)]_1 - 2[\nabla(\varphi)]_2$$

Using the equations $P_2(s)_p = 0$, $\varphi(s)_p = 0$ and the permutation formula (4.5), we find that $\psi^1(p)$ and $\psi^2(p)$ can be written as a function of s and its derivatives at $p \in M$, up to order 3. Nevertheless, using formula (4.5) we can also eliminate the 3^{rd} order derivatives of s at p. On the other hand, with the help of the equation $P_2 = 0$ and $\varphi = 0$ we can express the 2^{nd} order derivatives of s with the 1^{st} order derivatives of s. The calculation gives the formulas. $\square$

The linearization

Since the compatibility conditions $\psi^1 = 0$ and $\psi^2 = 0$ found in the previous section are not identically satisfied, we have to introduce them into the system P_φ. We arrive at the extended system:

$$P_\psi = (P_2,\, \varphi,\, \psi^1,\, \psi^2). \tag{4.14}$$

Differentiating the equations $\psi^1 = 0$ and $\psi^2 = 0$ with respect to e_1 and e_2 we find 4 equations: $\psi^i_j = 0$, $i, j = 1, 2$, where

$$\begin{cases} \psi^1_1 = 24\mathcal{R}_1s_2^2 + \tilde{\beta}s_{12} - 48\mathcal{R}_1s_1s_2 + (\alpha - 48\mathcal{R}s_2)s_{11} + \alpha_1s_1 + \beta_1s_2 + \gamma_1 \\ \psi^1_2 = 24\mathcal{R}_2s_2^2 + \tilde{\beta}s_{22} - 48\mathcal{R}_2s_1s_2 + (\alpha - 48\mathcal{R}s_2)s_{21} + \alpha_2s_1 + \beta_2s_2 + \gamma_2 \\ \psi^2_2 = -24\mathcal{R}_1s_1^2 + \tilde{\alpha}s_{11} + 48\mathcal{R}_1s_1s_2 + (48\mathcal{R}s_1 + \hat{\beta})s_{12} + \hat{\alpha}_1s_1 + \hat{\beta}_1s_2 + \hat{\gamma}_1 \\ \psi^2_2 = -24\mathcal{R}_2s_1^2 + \tilde{\alpha}s_{21} + 48\mathcal{R}_2s_1s_2 + (48\mathcal{R}s_1 + \hat{\beta})s_{22} + \hat{\alpha}_2s_1 + \hat{\beta}_2s_2 + \hat{\gamma}_2 \end{cases}$$

In this expression, we can eliminate the second order derivatives using the equation (4.9) and $\varphi = 0$, and with the help of the equations of (4.13), we can express the terms s_1^2 and s_2^2 as a function of s_1, s_2 and the product $s_1 s_2$. Therefore the system

$$P_\psi = 0, \quad \nabla P_{\psi^1} = 0, \quad \nabla P_{\psi^2} = 0 \tag{4.15}$$

is equivalent to the system formed by equation (4.14) and the four *linear* equations in s_1, s_2 and $s_1 s_2$:

$$\mathcal{L}(\mathcal{W}) = \begin{cases} a^1 s_1 + b^1 s_2 + c^1 s_1 s_2 = d^1, \\ a^2 s_1 + b^2 s_2 + c^2 s_1 s_2 = d^2, \\ a^3 s_1 + b^3 s_2 + c^3 s_1 s_2 = d^3, \\ a^4 s_1 + b^4 s_2 + c^4 s_1 s_2 = d^4, \end{cases} \tag{4.16}$$

where a^i, b^i, $i = 1, \ldots, 4$ are polynomials in s of degree 3, whose coefficients are $\mathcal{R}$ and its derivatives up to order 3, c^1 and c^4 are polynomials in s of degree 1 with coefficients $\mathcal{R}$, $\mathcal{R}_1$ and $\mathcal{R}_2$, c^2 and c^3 can be expressed as a function of $\mathcal{R}$, $\mathcal{R}_1$ and $\mathcal{R}_2$, and d^1, d^4 (resp. d^2 and d^3) are polynomials in s of degree 5 (resp. 4), with coefficients $\mathbb{R}$ and its derivatives up to order 5. Their explicit expression can be found in [107, Appendix]. The direct computation shows that the determinant

$$\det \begin{pmatrix} a^1 & b^1 & c^1 & d^1 \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \\ a^4 & b^4 & c^4 & d^4 \end{pmatrix} = 0 \tag{4.17}$$

so that the system (4.16) is compatible. On the other hand, the $3^{rd}-$order minors of the system (4.16) are polynomials in s of degree 7 which are not identically zero. Therefore, there is an open dense $\mathcal{U} \subset C^2$ on which,

$$D(s) = \begin{vmatrix} a^1 & b^1 & c^1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} \neq 0.$$

Solving on $\mathcal{U}$ the system $\mathcal{S}$ for s_1, s_2 and $s_1 s_2$ we obtain:

$$s_1 = F(s) = \frac{A(s)}{D(s)}, \qquad s_2 = G(s) = \frac{B(s)}{D(s)} \tag{4.18}$$

and

$$s_1 s_2 = H(s) = \frac{C(s)}{D(s)}, \tag{4.19}$$

where $A = A(s)$, $B = B(s)$ and $C = C(s)$ are polynomials in s of degrees 8, 8, and 11 respectively:

$$A = \begin{vmatrix} -d^1 & b^1 & c^1 \\ -d^2 & b^2 & c^2 \\ -d^3 & b^3 & c^3 \end{vmatrix}, \quad B = \begin{vmatrix} a^1 & -d^1 & c^1 \\ a^2 & -d^2 & c^2 \\ a^3 & -d^3 & c^3 \end{vmatrix}, \quad C = \begin{vmatrix} a^1 & b^1 & -d^1 \\ a^2 & b^2 & -d^2 \\ a^3 & b^3 & -d^3 \end{vmatrix}.$$

By (4.19) we must find $F(s)\cdot G(s) = H(s)$. Thus, the solution of s for the linearization system must be in the algebraic manifold defined by

$$Q_1(s) := AB - CD = 0. \tag{4.20}$$

On the other hand, the compatibility condition of the system (4.18) is

$$s_{12} - s_{21} = \mathcal{R}s.$$

Computing it explicitly we find that s must be in the algebraic manifold defined by

$$Q_2(s) = 0,$$

where Q_2 is polynomial in s of degree 15. Indeed, if $A(s) = \sum_{i=1}^{8} A_i s^i$, $B(s) = \sum_{i=1}^{8} B_i s^i$, and $D(s) = \sum_{i=1}^{7} D_i s^i$ where A_i, B_i and C_i are functions on M, then using (4.18) we obtain

$$\begin{aligned} Q_2(s) = &\Big(\sum_{i=1}^{8}(\nabla_2 B_i)s^i\Big)\Big(\sum_{i=1}^{7} D_i s^i\Big) - \Big(\sum_{i=1}^{8} B_i s^i\Big)\Big(\sum_{i=1}^{7}(\nabla_2 D_i)s^i\Big)\\ &- \Big(\sum_{i=1}^{8}(\nabla_1 A_i)s^i\Big)\Big(\sum_{i=1}^{7} D_i s^i\Big) - \Big(\sum_{i=1}^{8} A_i s^i\Big)\Big(\sum_{i=1}^{7}(\nabla_1 D_i)s^i\Big)\\ &+ \Big(\sum_{i=1}^{8} B_i s^{i-1}\Big)\Big(\sum_{i=1}^{8} A_i s^i\Big) - \Big(\sum_{i=1}^{8} B_i s^i\Big)\Big(\sum_{i=1}^{8} A_i s^{i-1}\Big) - \mathcal{R}sD^2. \end{aligned}$$

Moreover, we must impose that s_1 and s_2 given by (4.18) verify the 5 equations of P_ψ, this implies 5 polynomial equations $Q_i = 0$, $i = 3, ..., 7$. Finally, we arrive at the conclusion that if the web is linearizable then s must be in the algebraic manifold $\mathcal{A}$, where $\mathcal{A}$ is defined by the equations $Q_i = 0$, $i = 1, ..., 7$:

$$\mathcal{A} := \{Q_i = 0 \mid i = 1, \dots, 7\}.$$

So the compatibility system has a solution in the neighborhood of a point $p \in M$ if and only if the algebraic variety $\mathcal{A}$ is not empty. If $\mathcal{A} \neq \emptyset$, then for all $s_0 \in \mathcal{A}$, there exists a neighborhood U of s_0 so that all $s \in U$ can be prolonged in a germ $\tilde{s}$ as a basis of linearization. The explicit expressions of the polynomials Q_i can be computed. The degree of these polynomials Q_i, $i = 1, \dots, 7$ are 18, 15, 23, 23, 24, 17 and 17 respectively. One obtains the following results:

Theorem 4.3.5. *[106, Theorem 5.1] A non-parallelizable* $3-$*web* $\mathcal{W}$ *is linearizable if and only if there is an open set* U *of* M *on which the polynomials* $Q_1, \dots, Q_7$ *have common roots. Moreover, if this condition is satisfied, then for all* $p \in U$ *and all pre-linearization* $L_0 \in E_p$ *whose base is in* $\mathcal{A} = \{Q_i = 0 \,|\, i = 1, ..., 7\}$*, there exists a unique linearization* L *so that* $L_p = L_0$.

Since the lowest degree of the polynomials defining $\mathcal{A}$ is 15 we arrive at the

Theorem 4.3.6. *[106, Theorem 5.2] For a non parallelizable* $3-$*web, there exist at most* 15 *projectively non equivalent linearizations.*

An old problem related to the linearizability of webs is the **Gronwall Conjecture** (1912) [47]: *If a non-parallelizable* 3*-web* $\mathcal{W}$ *in the (real or complex) plane is linearizable, then, up to a projective transformation, there is a unique diffeomorphism which maps* $\mathcal{W}$ *into a linear* 3*-web.* Using Theorem 4.3.6, the Gronwall conjecture can be expressed in the following way: for any non parallelizable 3-web in the (real or complex) plane

$$\deg\left\{\gcd(Q_1, ..., Q_7)\right\} \leq 1,$$

where gcd denotes the greatest common divisor of the corresponding polynomials and deg is its degree.

4.4 The controversial web and its linearization

In 2006, V.V. Goldberg and V.V. Lychagin investigated the linearizability of 3-webs in [44]. Their results were different from that of [106]. The direct comparison of the two theories is not straightforward, since the formulas in both cases are long and complex containing the curvature tensor and its different derivatives. There is, however, a very specific case, where the two theories show clearly opposite results. This is given by an explicit example of a 3-web $\mathcal{W}$ determined by the web function $f(x,y) := (x+y)e^{-x}$, i.e. the 3-web given by the foliations

$$x = const, \qquad y = const, \qquad (x+y)\,e^{-x} = const. \tag{4.21}$$

From the theory developed in [106] it follows that (4.21) is linearizable (see [106, page 2653]) but the theory worked out by Goldberg and Lychagin leads to the non-linearizability of $\mathcal{W}$ (see [43, page 38] and [44, page 171, line 7–10]). Consequently, example (4.21) can be used for testing, which theory describes the linearizabilty condition.

We proved in [116] that the results of [106] are correct. Indeed, the Chern connection of the web $\mathcal{W}$ is determined by:

$$\nabla_{\partial_1}\partial_1 = \frac{1}{x_1+x_2-1}\,\partial_1, \quad \nabla_{\partial_2}\partial_2 = \frac{1}{1-x_1-x_2}\,\partial_2, \quad \nabla_{\partial_2}\partial_1 = \nabla_{\partial_1}\partial_2 = 0.$$

and its curvature is determined by $R^{\nabla}(\partial_1,\partial_2)\partial_i = (x_1+x_2-1)^{-2}\,\partial_i$, for $i=1,2$. Therefore, the Chern connection is non-flat and the web $\mathcal{W}$ is not parallelizable. Following step by step the computation described in the previous chapter one can find, that the polynomial Q_1 defined in (4.20) can be written as

$$Q_1(s) = (1+s)\,\tilde{Q}_1(s), \tag{4.22}$$

where $\tilde{Q}_1$ is polynomial in s of degree 18 and has the form

$$\tilde{Q}_1(s) = \sum_{k=0}^{17}\left(\sum_{i+j=0}^{\delta(k)}\alpha_{ijk}(x_1^i x_2^j + x_1^j x_2^i)\right)s^k$$

where $\delta(0) = 15$, $\delta(1) = 16$ and $\delta(k) = 17$ for $3 \leq k \leq 17$. The coefficients $\alpha_{ijk} \in \mathbb{R}$ can be computed easily with a computer algebra system like Maple. From (4.22) it is obvious that

$$s(x_1, x_2) \equiv -1 \tag{4.23}$$

is a solution of the polynomial $Q_1(s)$. Moreover, further calculation shows that (4.23) is a solution of all the polynomials $Q_i(s)$ for $i = 2, \ldots, 7$. This shows that the web is linearizable. Let us go further and find the linearization explicitly. By substituting $s(x_1, x_2) \equiv -1$ into (4.8) one can obtain the system

$$\begin{aligned} t_1 &= t^2 - t + \frac{t}{x_1 + x_2 - 1}, \\ t_2 &= tz, \\ z_1 &= tz - \frac{1}{(x_1 + x_2 - 1)^2}, \\ z_2 &= z^2 - \frac{2z}{x_1 + x_2 - 1}. \end{aligned} \tag{4.24}$$

There are two solutions of the differential system (4.24):

$$\text{Solution 1.} \qquad \begin{cases} t(x_1, x_2) = 0, \\ z(x_1, x_2) = \dfrac{1 - x_1 - a}{(-1 + x_1 + x_2)(x_2 - a)}, \end{cases} \tag{4.25}$$

$$\text{Solution 2.} \qquad \begin{cases} t(x_1, x_2) = \dfrac{(-1 + x_1 + x_2)\, e^{-x_1}}{(x_1 + x_2)e^{-x_1} + a x_2 + b}, \\ z(x_1, x_2) = \dfrac{e^{-x_1} + a - x_1 a + b}{\big((x_1 + x_2)\, e^{-x_1} + a x_2 + b\big)(x_1 + x_2 - 1)} \end{cases} \tag{4.26}$$

where a and b are arbitrary constants.

Solution 1. Here we consider the solution (4.25) of (4.24). Rewriting the expression of $t(x_1, x_2)$ and $z(x_1, x_2)$ with the help of (4.23) we can determine the components of the affine deformation tensor L:

$$L^1_{11} = -1, \quad L^2_{12} = 0, \quad L^2_{22} = -\frac{x_2 - 2 + 2x_1 + a}{(x_1 + x_2 - 1)(x_2 - a)}, \quad L^1_{12} = \frac{1 - x_1 - a}{(-1 + x_1 + x_2)(x_2 - a)}.$$

The deformed connection ∇^L in the standard base is given by the following equations:

$$\nabla^L_{\partial_1} \partial_1 = \nabla_{\partial_1} \partial_1 + L(\partial_1, \partial_1) = \frac{\kappa_1}{\kappa} \partial_1 + L^1_{11} \partial_1 = \frac{x_1 + x_2 - 2}{1 - x_1 - x_2}\, \partial_1, \tag{4.27a}$$

$$\nabla^L_{\partial_1} \partial_2 = \nabla_{\partial_1} \partial_2 + L(\partial_1, \partial_2) = L^1_{12} \partial_1 + L^2_{12} \partial_2 = \frac{1 - x_1 - a}{(-1 + x_1 + x_2)(x_2 - a)}\, \partial_1, \tag{4.27b}$$

$$\nabla^L_{\partial_2} \partial_1 = \nabla_{\partial_2} \partial_1 + L(\partial_2, \partial_1) = L^1_{12} \partial_1 + L^2_{12} \partial_2 = \frac{1 - x_1 - a}{(-1 + x_1 + x_2)(x_2 - a)}\, \partial_1, \tag{4.27c}$$

$$\nabla^L_{\partial_2} \partial_2 = \nabla_{\partial_2} \partial_2 + L(\partial_2, \partial_2) = -\frac{\kappa_2}{\kappa} \partial_2 + L^2_{22} \partial_2 = \frac{2}{a - x_2}\, \partial_2, \tag{4.27d}$$

where $\kappa = \partial_1 f/\partial_2 f = 1 - x_2 - x_2$. It is obvious that $\nabla^L_{\partial_i}\partial_j - \nabla^L_{\partial_j}\partial_i = 0$ and therefore the torsion of ∇^L is zero. The equation (4.27a)) (resp. (4.27d)) shows that the covariant derivative of a horizontal (resp. vertical) vector field with respect to a horizontal (resp. vertical) vector field is horizontal (resp. vertical). One can easily show that the covariant derivative of a transversal vector field with respect to a transversal vector field is transversal. Direct calculation shows that ∇^L is flat, that is its curvature tensor is identically zero.

Solution 2. Here we consider the solution (4.26) of (4.24)). Completing the expression of $t(x_1, x_2)$ and $z(x_1, x_2)$ with (4.23) we can find that the components of L, the affine deformation tensor, are:

$$L^1_{11} = \frac{(x_1 + x_2 - 2)e^{-x_1} - ax_2 - b}{(x_1 + x_2)e^{-x_1} + ax_2 + b}, \quad L^2_{22} = \frac{(2 - x_1 - x_2)e^{-x_1} - a(2x_1 + x_2 - 2) + b}{(x_1 + x_2 - 1)((x_1 + x_2)e^{-x_1} + ax_2 + b)},$$
$$L^2_{12} = \frac{(x_1 + x_2 - 1)\, e^{-x_1}}{(x_1 + x_2)e^{-x_1} + ax_2 + b}, \qquad L^1_{12} = \frac{e^{-x_1} - ax_1 + a + b}{(x_1 + x_2 - 1)\big((x_1 + x_2)\, e^{-x_1} + ax_2 + b\big)}.$$

The deformed connection ∇^L in the standard base is given by the following equations:

$$\nabla^L_{\partial_1}\partial_1 = \left(\frac{1}{x_1 + x_2 - 1} + \frac{(x_1 + x_2 - 2)e^{-x_1} - ax_2 - b}{(x_1 + x_2)e^{-x_1} + ax_2 + b}\right) \partial_1 \tag{4.28a}$$

$$\nabla^L_{\partial_1}\partial_2 = \frac{e^{-x_1} + a - x_1 a + b}{\big((x_1 + x_2)\, e^{-x_1} + ax_2 + b\big)(x_1 + x_2 - 1)}\, \partial_1 + \frac{(x_1 + x_2 - 1)\, e^{-x_1}}{(x_1 + x_2)e^{-x_1} + ax_2 + b}\, \partial_2, \tag{4.28b}$$

$$\nabla^L_{\partial_2}\partial_1 = \frac{e^{-x_1} + a - x_1 a + b}{\big((x_1 + x_2)\, e^{-x_1} + ax_2 + b\big)(x_1 + x_2 - 1)}\, \partial_1 + \frac{(x_1 + x_2 - 1)\, e^{-x_1}}{(x_1 + x_2)e^{-x_1} + ax_2 + b}\, \partial_2, \tag{4.28c}$$

$$\nabla^L_{\partial_2}\partial_2 = \frac{-2(e^{-x_1} + a)}{(x_1 + x_2)e^{-x_1} + ax_2 + b}\, \partial_2 \tag{4.28d}$$

As in the previous case $\nabla^L_{\partial_i}\partial_j - \nabla^L_{\partial_j}\partial_i = 0$ and the torsion of ∇^L is zero. Equation (4.28a) (resp. (4.28d)) shows that the covariant derivative of a horizontal (resp. vertical) vector field with respect to a horizontal (resp. vertical) vector field is horizontal (resp. vertical). We have

$$\begin{aligned}\nabla_{(\partial_1 - \kappa\partial_2)}(\partial_1 - \kappa\partial_2) &= \nabla_{\partial_1}\partial_1 - \kappa\, \nabla_{\partial_2}\partial_1 - \kappa\nabla_{\partial_1}\partial_2 - (\partial_1\kappa)\, \partial_2 + \kappa(\partial_2\kappa)\, \partial_2 + \kappa^2\nabla_{\partial_2}\partial_2 \\ &= \frac{(x_1 + x_2)^2 e^{-x_1} + (4a + b)(x_1 + x_2) - a(2x_1^2 + x_2^2 + 3x_2 x_1 + 2)}{(x_1 + x_2 - 1)((x_1 + x_2)e^{-x_1} + ax_2 + b)}\, (\partial_1 - \kappa\partial_2)\end{aligned}$$

which shows that the covariant derivative of a transversal vector field with respect to a transversal vector field is transversal. As in the previous case, ∇^L is flat i.e. its curvature tensor is identically zero.

The above computation shows that in both cases the corresponding affine deformation tensors define linearizations of the web $\mathcal{W}$ and the solutions correspond to different linearizations. However, these linearizations are projectively equivalent. Indeed, the parameter s, called the base of the linearization, is a projective invariant of the linearizations: two linearizations are projectively equivalent if and only if they have the same base. Here the two linearizations have the same base ($s(x_1, x_2) \equiv -1$) which shows that they are projectively equivalent.

Remark 4.4.1. [117, page 98]
There is a more direct proof of the linearizability of the web $\mathcal{W}$ given by the system (4.21). Indeed, one can consider the diffeomorphism $\bar{x} = f(x, y)$, $\bar{y} = y$ which clearly transforms the foliations $y = const$ and $f(x, y) = const$ into linear foliations. The line $x = c$ of the first foliation becomes the line $\bar{x} = (c + \bar{y})e^{-c}$. This method can be used also in the more general setting for the webs defined by the web function $f(x, y) = a(x)x + b(x)y$. The linearizing diffeomorphism was indicated to me by J.-P. Dufour in a personal communication.

Symbols

Symbol	Explanation	page
M	smooth manifold	4
$C^\infty(M)$	real-valued smooth functions	4
$\mathfrak{X}(M)$	smooth vector fields on M	4
TM	tangent bundle of M	4
$\mathcal{T}M$	slit tangent bundle	4
$\Lambda^k(M)$	skew-symmetric forms	4
$S^k(M)$	symmetric forms	4
$\Psi^k(M)$	vector valued k-forms	4
$\mathcal{L}_X$	Lie derivative with respect to X	4
d_L	d_* derivation associated to L	4
i_L	i_* derivation associated to L	4
$[K, L]$	Frölicher–Nijenhuis bracket of K, L	4
VTM	vertical bundle	4
HTM	horizontal bundle	4
J	vertical endomorphism	5
S	spray	5
G^i	spray coefficients	5
G^i_j	$\frac{\partial G^i}{\partial y^j}$	5
G^i_{jk}	$\frac{\partial G^i}{\partial y^j \partial y^k}$	6
v	vertical projector	5
h	horizontal projector	5
$\frac{\delta}{\delta x^i}$	$\frac{\partial}{\partial x^i} - G^j_i(x,y)\frac{\partial}{\partial y^j}$	5
Γ	nonlinear connection	5
C	Liouville vector field	5
$\mathcal{D}_X Y$	Berwald connection	6
R	curvature tensor	6
Φ	Jacobi endomorphism	6
ρ	Ricci scalar	6
ω_E	Euler–Lagrange form	7
Ω_E	Euler-Poincaré 2-form	7
g_{ij}	variational multiplier	8
$\mathbb{F}$	almost complex structure	9

Symbol	Explanation	page
$\mathcal{D}_{\mathcal{H}}$	holonomy distribution	14
$\mathcal{H}_{\mathcal{S}}$	set of holonomy invariant functions	14
$\mathcal{H}_{\mathcal{S},k}$	set of k-homogeneous holonomy invariant functions	14
F	Finsler norm function	26
κ_i	principal curvatures	27
$j_k(s)_x$	k^{th} order jet of s at x	27
J_kB	bundle of k^{th} order jets of B	27
$\sigma_k(P)$	symbol of P	27
$H^{m,i}$	Spencer cohomology group	28
l_y	horizontal lift	56
R^i_{jk}	component of the curvature tensor	57
$\mathcal{P}_c$	parallel translation along c	58
$\mathcal{I}_p$	indicatrix at p	59
$\mathcal{H}ol_p(M)$	holonomy group at p	60
$\mathfrak{hol}_p(M)$	holonomy algebra at p	68
$\mathcal{G}$	subgroup of $\mathcal{D}iff^{\infty}(M)$	60
$\mathcal{T}_o\mathcal{G}$	tangent Lie algebra of $\mathcal{G}$	61
$\overline{\mathcal{G}}$	topological closure of $\mathcal{G}$	63
exp	exponential map	63
$\mathcal{H}ol_f(M)$	fibered holonomy group	64
$\mathfrak{hol}_f(M)$	fibered holonomy algebra	64
$\mathfrak{R}$	curvature algebra	64
$\mathfrak{R}_p$	curvature algebra at p	68
$\mathfrak{hol}^*(M)$	infinitesimal holonomy algebra	67
$\mathfrak{hol}^*_p(M)$	infinitesimal holonomy algebra at p	69
$\mathbb{S}^1$	unite circle	81
$\mathsf{F}(\mathbb{S}^1)$	Fourier algebra	81
$\mathcal{D}iff^{\infty}(\mathbb{S}^1)$	diffeomorphism group of $\mathbb{S}^1$	81
$\mathcal{D}iff^{\infty}_{+}(\mathbb{S}^1)$	orientation preserving diffeomorphism group of $\mathbb{S}^1$	81

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