Hyperbolic Wavelet Transforms and Applications

DSc dissertation

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Contents

Chapter 1

Introduction

1.1 Notations

- $\mathbb{C}$ the set of complex numbers
- $\mathbb{D} = \{z \in \mathbb{C} \ : |z| < 1 \ \}$ the unit disc
- $\mathbb{T} = \{z \in \mathbb{C} \ : |z| = 1 \ \}$ the unit circle
- $\mathbb{C}_+ = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ the upper half plane
- $\mathcal{A}(\mathbb{D})$ the set of analytic functions on the unit disc $\mathbb{D}$
- $\mathcal{A}(\mathbb{C}_+)$ the set of analytic functions on $\mathbb{C}_+$
- $\|f_r\|_2 := \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^2 \, dt\right)^{1/2}$
- $H^2(\mathbb{D})$ the Hardy space of the unit disc
- $H^2(\mathbb{T})$ the Hardy space of the unit circle
- $\mathrm{H}^2(\mathbb{C}_+) = \left\{h \in \mathcal{A}(\mathbb{C}_+) : \sup\left\{\int_{\mathbb{R}} |h(x+iy)|^2 \, dx : y > 0\right\} < \infty\right\}$, the Hardy space of the upper half plane
- $H^2(\mathbb{R}) = \{f \in L^2(\mathbb{R}), \, sup\hat{f} \subset [0, +\infty)\}$
- $A(\mathbb{D})$ the disc algebra of the unit disc
- $K(z, w) = k_w(z) = \frac{1}{1-\overline{w}z} \quad z, w \in \mathbb{D}$ the Szegő or Cauchy kernel on the disc
- $C : \mathbb{C}_+ \to \mathbb{D}, C(\omega) = \frac{i-\omega}{i+\omega}$ the Cayley transform

- $dA_\alpha(z) := \frac{\alpha+1}{\pi}(1-|z|^2)^\alpha \, dxdy$, $z = x+iy$, the weighted area measure on $\mathbb{D}$
- $A^p_\alpha := \{f \in \mathcal{A}(\mathbb{D}) : \int_{\mathbb{D}} |f(z)|^p dA_\alpha(z) < \infty\}$, $\alpha > -1$, $0 < p < \infty$ the weighted Bergman spaces
- $A^2 = A^2_0$ the Bergman space
- $K_\alpha(\xi, z) = \frac{1}{(1-\bar{z}\xi)^{\alpha+2}}$ the reproducing kernel in the weighted Bergman space A^2_α
- $P_\alpha : L^2(\mathbb{D}, dA_\alpha) \to A^2_\alpha$, $P_\alpha f(z) = \int_{\mathbb{D}} f(\xi)\frac{1}{(1-\bar{\xi}z)^{\alpha+2}} dA_\alpha(\xi)$ the weighted Bergman projection
- The affine group: $\mathbb{A} = \{(a,b) : a \in (0,+\infty), b \in \mathbb{R}\}$ with the following operation $(a_1, b_1) \circ (a_2, b_2) = (a_1a_2, a_1b_2 + b_1)$
- $U_{(a,b)}f(x) = |a|^{-1/2}f(a^{-1}x - b)$ representation of the affine group on $L^2(\mathbb{R})$
- $W_\psi f(a,b) = \langle f, U_{(a,b)}\psi\rangle$ the continuous affine wavelet transform
- $(\mathbb{G}, \cdot)$ a locally compact topological group
- $(H, \langle \cdot, \cdot \rangle)$ a Hilbert-space
- $U_x : H \to H$ $(x \in \mathbb{G})$ a unitary representation of the group on some Hilbert space H
- $(V_g f)(x) := \langle f, U_x g\rangle$ $(x \in \mathbb{G}, f, g \in H)$ the voice transform of $f \in H$ generated by the representation U and by the parameter $g \in H$
- $\mathcal{A}_w = \{g \in H : V_g g \in L^1_w(G)\} \neq \{0\}$ the set of analyzing vectors for an integrable representation associated to the weight w
- $\mathcal{H}^1_w = \{f \in H : V_g f \in L^1_w(G)\}$ the simplest Banach space where atomic decompositions can be obtained
- $B_a(z) := \epsilon\frac{z-b}{1-\bar{b}z}$ $(z \in \mathbb{C}, \bar{b}z \neq 1)$ the Blaschke function
- $\mathbb{B} := \mathbb{D} \times \mathbb{T}$ and $a = (b, \epsilon) \in \mathbb{B}$ the set of the parameters
- $\rho(z_1, z_2) := \frac{|z_1 - z_2|}{|1 - z_1\bar{z}_2|} = |B_{(z_2,1)}(z_1)|$ $(z_1, z_2 \in \mathbb{D})$ the pseudohyperbolic metric
- $\mathbb{B} := \mathbb{D} \times \mathbb{T}$ let us define the operation induced by the function composition in the following way: $B_{a_1} \circ B_{a_2} = B_{a_1 \circ a_2}$. The set of the parameters $\mathbb{B}$ with the induced operation is called the *Blaschke group*
- $(U_{a^{-1}}f)(z) := \frac{\sqrt{e^{i\theta}(1-|b|^2)}}{(1-\bar{b}z)} f\left(\frac{e^{i\theta}(z-b)}{1-\bar{b}z}\right)$ $\left(z = e^{it} \in \mathbb{T}, a = (b, e^{i\theta}) \in \mathbf{B}\right)$ representation of the Blaschke group on $H^2(\mathbb{T})$.

- $Z_n^\ell(\rho,\theta) := \sqrt{2n+|\ell|+1}\, R_{|\ell|+2n}^{|\ell|}(\rho) e^{i\ell\theta}$, $\ell \in \mathbb{Z}$, $n \in \mathbb{N}$ the complex Zernike polynomials in polar coordinates
- $R_{|\ell|+2n}^{|\ell|}(\rho) = \rho^{|\ell|} P_n^{(0,|\ell|)}(2\rho^2-1)$ the radial terms $R_{|\ell|+2n}^{|\ell|}(\rho)$ expressed by Jacobi polynomials
- $(U_{a^{-1}}^\alpha f)(z) := e^{i\frac{\alpha+2}{2}\psi} \frac{(1-|b|^2)^{\frac{\alpha+2}{2}}}{(1-\bar{b}z)^{\alpha+2}} f\left(e^{i\psi}\frac{z-b}{1-\bar{b}z}\right) \quad (a=(b,e^{i\psi}) \in \mathbb{B})$ the representation of the Blaschke group on the weighted Bergman space A_α^2
- $\Phi_n = \Phi_n^a\,(n \in \mathbb{N}^*)$ where $\Phi_1(z) = \frac{\sqrt{1-|a_1|^2}}{1-\overline{a_1}z}, \Phi_n(z) = \frac{\sqrt{1-|a_n|^2}}{1-\overline{a_n}z}\prod_{k=1}^{n-1} B_{a_k}(z),\ n \geqslant 2$ the Malmquist-Takenaka system (M-T) in $H^2(\mathbb{T})$
- $\Psi_1(z) = \frac{1}{\sqrt{\pi}}\frac{\Phi_1(-1)}{z-\bar{\lambda}_{a_1}}, \quad \Psi_n(z) = \frac{1}{\sqrt{\pi}}\frac{\Phi_n(-1)}{z-\bar{\lambda}_{a_n}}\prod_{k=1}^{n-1}\frac{z-\lambda_{a_k}}{z-\bar{\lambda}_{a_k}},\ \lambda_a := C^{-1}(a) = i\frac{1-a}{1+a}$ the Malmquist-Takenaka system (M-T) in $H^2(\mathbb{C}_+)$

1.2 Affine wavelets and multiresolution in $L^2(\mathbb{R})$

In 1982 Morlet, a French geophysicist was the first who introduced the concept of a 'wavelet' [113]. Wavelet means a small wave. The wavelet transform was at that time new tool for seismic signal analysis. Immediately, Grossmann, a theoretical physicists, studied the inverse formula for the wavelet transform. The joint collaboration of Morlet and Grossmann [86] yielded a detailed mathematical study of the continuous wavelet transforms and their various applications, of course without the realization that similar results had already been obtained in 1950's by Calderon, Littlewood, Paley and Franklin. However, the rediscovery of the old concepts provided a new method for decomposing a function or a signal. For details one can see Morlet et al. [113], Debnath [54].

First let us recall the basic concepts of the classical one dimensional continuous affine wavelet transform and multiresolution. The affine wavelet multiresolution comes from the discretization of the continuous affine wavelet transform, defined by

$$W_\psi f(a,b) = |a|^{-1/2} \int_{\mathbb{R}} f(t)\overline{\psi(a^{-1}t - b)}dt, \quad f, \psi \in L^2(\mathbb{R}),\ (a,b) \in \mathbb{R}_+ \times \mathbb{R}. \tag{1.1}$$

There is a rich bibliography of the affine wavelet theory (see for example Grossmann, Morlet [86], Grossmann, Morlet, Paul [87], Daubechies [45], Meyer [112], Chui [35] etc.). One important question is the construction of the discrete version, i.e., to find ψ so that the discrete translates and dilates

$$\psi_{n,k} = 2^{-n/2}\psi(2^{-n}x - k) \tag{1.2}$$

form a (orthonormal) basis in $L^2(\mathbb{R})$, which generate a multiresolution (see Daubechies [45], Heil, Walnut [94], Mallat[109] etc.). The coefficients of a function f with respect to the system (1.2) are the values of the continuous wavelet transform in a discrete subset of the parameter domain $\langle f, \psi_{n,k}\rangle = W_\psi f(2^n, k)$. Knowing the coefficients with respect to the orthonormal basis (1.2), the function f can be reconstructed from the values of the continuous wavelet transform, if we know that on a discrete subset of the set of the parameters. This is why we call the reconstruction of the functions using the wavelet coefficients with respect to the system (1.2) the discretization of the continuous transform (1.1).

It turned out that multiresolution representations are very effective for analyzing the information content of images. A multiresolution representation provides a simple hierarchical framework for interpreting the image information. In the classical theory of affine wavelets the definition of multiresolution analysis is the following:

Definition 1.2.1. *Let V_j, $j \in \mathbb{Z}$ be a sequence of subspaces of $L^2(\mathbb{R})$. The collections of spaces $\{V_j,\ j \in \mathbb{Z}\}$ is called a multiresolution analysis with wavelet function ψ if the following conditions hold:*

1. (nested) $V_j \subset V_{j+1}$

2. (density) $\overline{\cup V_j} = L^2(R)$
3. (separation) $\cap V_j = \{0\}$
4. (basis) The function ψ *belongs to* V_0 *and the set* $\{2^{n/2}\psi(2^n x - k),\ k \in \mathbb{Z}\}$ *is a (orthonormal) bases in* V_n.

Applying the dilatation we arrive to higher resolution level ($f(x) \in V_n \Leftrightarrow f(2x) \in V_{n+1}$), and applying the translation we remain on the same level of the resolution.

The simplest example is the multiresolution generated by the Haar wavelets due to Alfréd Haar (1909)(see [91]). The Haar wavelets can be derived from the following function using the dilation and translation:

$$h(x) := \begin{cases} 1 & (x \in [0, 1/2)) \\ -1 & (x \in [1/2, 1)) \\ 0 & (x \in \mathbb{R}\backslash[0, 1)), \end{cases}$$

$$h_0(x) = h(x),\ h_{nk}(x) := 2^{-n/2}h(2^n x - k)$$
$$(x \in [0, 1),\ n, k \in \mathbb{N}).$$

The Haar system is orthogonal in the Hilbert space $L^2 := L^2([0, 1))$ with respect to the usual scalar product, and the Haar-Fourier series of a function $f \in L^1([0, 1))$ converges to the function in both norm and almost everywhere.

Haar introduced this system to show that the problem formulated by Hilbert has a solution. The Fourier series expansion with respect to $(h_n, n \in \mathbb{N})$ of a continuous function is convergent uniformly to the function, although $(h_n, n \in \mathbb{N})$ are not continuous. In this respect the Haar wavelet system is essentially different from the trigonometric system. It turned out that the Haar system is a very important example in many respects. Using the Paley inequality, Marczinkiewicz proved that that the Haar system is unconditionally basis in L^p for $p > 1$.

Faber in 1910 (see [62]), Schauder in 1927, (see [136]), independently took the integral of the Haar system and introduced a new system, the so called Faber-Schauder system. Franklin in 1928 applying the Gram-Schmidt orthogonalization to the Faber-Schauder system obtained the Franklin system (see [74]), which is a basis not only in $L^2[0, 1]$ but also in $C[0; 1]$. Marcinkiewicz showed that the Haar system is a basis also for $L^p[0, 1]$, $1 < p < 1$. Starting from 1960 Ciesielski and Uljanov showed that the Haar system is very important also in functional analysis.

In 1974 Bockarev (see [16]) applying the analytic extension to the unit disc of Franklin system constructed the first basis in the disc algebra $A(\mathbb{D})$. In this way he gave an answer to a problem formulated by Banach 40 years earlier. A survey paper containing all these properties of the Haar system is written by Schipp in 2015 (see[140] and the references therein).

The fact that the members of the Haar system are not continuous, make them inappropriate for approximating smooth functions. Starting from 1980 Meyer and Daubechies -

among others - started to construct smooth orthonormal systems, so called wavelets from a single function ψ called mother wavelet, of the form

$$\psi_{n,k}(x) = 2^{n/2}\psi(2^n x - k) \quad (x \in \mathbb{R}, \psi \in L^2(\mathbb{R}), \|\psi\|_2 = 1).$$

Except from the Haar system, the construction of such systems is a hard task. Taking the Fourier transform $\widehat{\psi}$ instead of the mother wavelet ψ itself turned to be a good starting point. Despite of the fact that ψ cannot be given in an explicit form generally the wavelet Fourier series enjoy nice convergence and approximation properties. The kernel functions of the partial sums can be well estimated and the wavelet Fourier coefficients can be calculated by a fast algorithm.

Since 1980 the wavelet analysis is flourishing and has many applications in practical problems. It turned out that wavelet analysis is an exciting new method for solving difficult problems in mathematics, physics, and engineering, with modern applications as diverse as wave propagation, data compression, signal processing, image processing, pattern recognition, computer graphics, the detection of aircraft and submarines and other medical image technology. Wavelets allow complex information such as music, speech, images and patterns to be decomposed into elementary forms at different positions and scales and subsequently reconstructed with high precision. Signal transmission is based on transmission of a series of numbers. The series representation of a function is important in all types of signal transmission. In many cases the wavelet transform of a function appeared as an improved version of Fourier transform. Fourier transform is a powerful tool for analyzing the components of a stationary signal. But it is failed for analyzing the non stationary signal where as wavelet transform allows the components of a non-stationary signal to be analyzed. For example Sifuzzaman, Islam, Ali in [148] showed in many cases the advantages of wavelet transform compared to Fourier transform.

Meyer and others, formulated the following question: Is it any "regular" wavelet orthonormal bases of the form

$$\psi_0(x) = \psi(x), \ \psi_{n,k}(x) := 2^{n/2}\psi(2^n x - k)$$

and multiresolution generated by this bases in $H^2(\mathbb{R}) = \{f \in L^2(\mathbb{R}), \, sup\hat{f} \subset [0, +\infty)\}$? Auscher in 1995 published results connected to this question in [11]. The word regular includes smoothness, localization, and cancellation of ψ, see the exact conditions in [11]. He showed the nonexistence of a regular wavelet that generates a wavelet basis in space $H^2(\mathbb{R})$, i.e., applying dilation and translation to a single function, or discretizing the continuous affine wavelet transform, leads to negative answer if we impose some strong "regularity" conditions.

As we will see later we will approach the construction of multiresolution in $H^2(\mathbb{R})$ by taking the analytic extension of these functions to the Hardy space of the upper half plane $\mathrm{H}^2(\mathbb{C}_+)$, because if $f \in \mathrm{H}^2(\mathbb{C}_+)$ then its non tangential boundary limit function exists almost everywhere and the limit function f belongs to $H^2(\mathbb{R})$.

Questions: Is it any other way to construct analytic (very regular) wavelets? Is it possible to generate muliresolution type decomposition in Hardy spaces of the unit disc and respectively in the Hardy space of the upper half plane or in other sub spaces of analytic functions? Is it any other continuous transform whose discretization leads to answer the previous questions?

In this thesis we collect the results obtained by the author in the last years connected to wavelets and multiresolution in analytic function spaces, like Hardy spaces and weighted Bergman spaces.

1.3 Analytic function spaces

1.3.1 Hardy spaces

Let us denote by $\mathbb{D} = \{z \in \mathbb{C} \ : |z| < 1 \ \}$ the unit disc and by $\mathbb{T} = \{z \in \mathbb{C} \ : |z| = 1 \ \}$ the unit circle. Let us denote by $\mathcal{A}(\mathbb{D})$ the set of analytic functions on the unit disc $\mathbb{D}$. Taking the integral means

$$\|f_r\|_2 := \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^2 \, dt \right)^{1/2}$$

of a function $f \in \mathcal{A}(\mathbb{D})$ we define the *Hardy space of the unit disc* $H^2(\mathbb{D})$ as the class of functions in $\mathcal{A}(\mathbb{D})$ for which $\|f\|_{H^2} := \sup_{0<r<1} \|f_r\|_2 < \infty$. It is known that the boundary function $f(e^{it}) := \lim_{r \to 1} f(re^{it})$ exists a.e. for every $f \in H^2(\mathbb{D})$ and f belongs to $L^2(\mathbb{T})$ on $\mathbb{T}$. The *Hardy space of the unit circle* $H^2(\mathbb{T})$ is a Hilbert space and contains the boundary values of the functions from $H^2(\mathbb{D})$. The linear space $H^2(\mathbb{T})$ is a Hilbert space with the scalar product

$$\langle f, g \rangle := \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) \overline{g(e^{it})} \, dt.$$

The norm induced by this scalar product satisfies the following relation: $\|f\|_{H^2} = \|f\|_{L^2(\mathbb{T})}$. The space $H^\infty(\mathbb{D})$ is the collection of functions $f \in \mathcal{A}(\mathbb{D})$ for which $\|f\|_{H^\infty} := \sup_{z \in \mathbb{D}} |f(z| < \infty$. The *disc algebra* $A(\mathbb{D})$, i.e., the set of functions analytic on $\mathbb{D}$ and continuous on its closure is a closed subspace of $H^\infty(\mathbb{D})$.

The Hardy spaces of the unit disc are applied intensively not only in the theories of complex functions and Fourier series but, as it turned out in the 1960's, in particular $H^2(\mathbb{D})$, and $H^\infty(\mathbb{D})$, are the proper Banach spaces for mathematical modeling of problems in control and operator theories (see for ex. Chui, Chen [36], Ward, Partington [172], Partington [134], Bokor, Athans [17]).

The transfer function f of a discrete linear time invariant system belongs to $H^2(\mathbb{T})$ or to $H^\infty(\mathbb{D})$. The main problem is to give a good approximation of f from some measurements made on the unit circle or in the unit disc.

The reproducing kernel in $H^2(\mathbb{T})$ or $H^2(\mathbb{D})$ is given by

$$K(z,w) = k_w(z) = \sum_{n=0}^{\infty} \overline{w^n} z^n = \frac{1}{1-\overline{w}z} \quad z, w \in \mathbb{D}.$$

This function is called the Szegő or Cauchy kernel on the disk. Every function $f \in H^1(\mathbb{D})$ can be recovered from its boundary limit function using the Cauchy reproducing kernel, i.e.,

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{it})}{e^{it}-z} e^{it} dt = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(\xi)}{1-z\overline{\xi}} d\xi.$$

Let us denote by $\mathbb{C}_+ = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ the upper half plane, and let us consider $\mathcal{A}(\mathbb{C}_+)$ the set of analytic functions on $\mathbb{C}_+$. The *Hardy space of the upper half plane* is defined by

$$\mathrm{H}^2(\mathbb{C}_+) = \left\{ h \in \mathcal{A}(\mathbb{C}_+) : \sup \left\{ \int_{\mathbb{R}} |h(x+iy)|^2 \, dx : y > 0 \right\} < \infty \right\}.$$

If $f \in \mathrm{H}^2(\mathbb{C}_+)$ then its non tangential boundary limit function exists almost everywhere and the limit function f satisfies

$$f \in H^2(\mathbb{R}) = \{f \in L^2(\mathbb{R}),\ sup\hat{f} \subset [0, +\infty)\}.$$

For more detailed description of the Hardy spaces see for example Cima, Ross [37], Mashregi [111].

The unit disk $\mathbb{D}$ and the upper half-plane $\mathbb{C}_+$ can be mapped to one-another by means of Möbius transformations, i.e., by the Cayley transform, which maps $\mathbb{C}_+$ to $\mathbb{D}$ and is defined by

$$C(\omega) = \frac{i-\omega}{i+\omega},\ \omega \in \mathbb{C}_+. \tag{1.3}$$

The correspondence between the boundaries is given by

$$e^{is} = C(t) = \frac{i-t}{i+t},\ t \in \mathbb{R},\ s \in (-\pi, \pi),$$

which implies that $s = 2\arctan(t),\ t \in \mathbb{R}$.

With the Cayley transform, the linear transformation from $\mathrm{H}^2(\mathbb{D})$ to $\mathrm{H}^2(\mathbb{C}_+)$ is defined for $f \in \mathrm{H}^2(\mathbb{D})$ by

$$Tf := \frac{1}{\sqrt{\pi}} \frac{1}{\omega+i} (f \circ C) \tag{1.4}$$

and is an isomorphism between these spaces. Consequently, the theory of the real line is a close analogy with what we have for the circle. But there are major differences too. For example in the Hardy space of the unit disc the polynomials are dense, however dense subsets in the Hardy space of the upper half plane are harder to find.

In the case of the unit disc a main tool in the proofs is the Cauchy formula for the unit disc. In the case of the upper half plane the analogue is the Cauchy formula for the upper half plane, which is the following: for any function $F \in \mathbb{H}^p(\mathbb{C}^+)$, $1 \leqslant p < +\infty$, if $F(s)$ is its non-tangential boundary limit, then

$$F(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{F(s)}{s-z} ds,\ z \in \mathbb{C}_+. \tag{1.5}$$

1.3.2 Weighted Bergman spaces

In this section we summarize the basic results connected to the weighted Bergman spaces (see [56, 93, 187]). Denote the weighted area measure on $\mathbb{D}$ by

$$dA_\alpha(z) := \frac{\alpha+1}{\pi}(1-|z|^2)^\alpha\, dxdy\ , z = x+iy.$$

For all $\alpha > -1$ the weighted Bergman spaces A^p_α are subsets of analytic functions with the following property

$$A^p_\alpha := \{f \in \mathcal{A}(\mathbb{D}) : \int_{\mathbb{D}} |f(z)|^p dA_\alpha(z) < \infty\}.$$

For $p = 2$, the set A^2_α is a Hilbert space, with the following scalar product:

$$\langle f, g\rangle_\alpha := \int_{\mathbb{D}} f(z)\overline{g(z)} dA_\alpha(z).$$

For $\alpha = 0$ we get back the unweighted case, $A^2 = A^2_0$, which is called the Bergman space (see [56, 93]). For $0 < p < \infty$, and $-1 < \alpha < \infty$ the weighted Bergman space A^p_α is a closed subspace of $L^p(\mathbb{D}, dA_\alpha) = L^p$. For any function $f \in A^p_\alpha$, and for any compact subset E of $\mathbb{D}$, there exists a positive constant $C = C(n, E, p, \alpha)$, such that

$$\sup\{|f^{(n)}(z)| : z \in K\} \leqq C\|f\|_{A^p_\alpha}.$$

This inequality implies, that the point-evaluation map is a bounded linear functional on A^p_α, and the norm convergence in A^p_α implies the locally uniform convergence on $\mathbb{D}$.

The weighted Bergman space A^2_α is a reproducing kernel Hilbert space, and the reproducing kernel, the weighted Bergman kernel, is given by the following formula

$$K_\alpha(\xi, z) = \frac{1}{(1-\overline{z}\xi)^{\alpha+2}}\,.$$

For $-1 < \alpha < +\infty$, the weighted Bergman projection, defined by

$$P_\alpha : L^2(\mathbb{D}, dA_\alpha) \to A^2_\alpha, \quad P_\alpha f(z) = \int_{\mathbb{D}} f(\xi)\frac{1}{(1-\bar{\xi}z)^{\alpha+2}} dA_\alpha(\xi),$$

is an orthogonal projection operator, which satisfies $P_\alpha f = f$ for every $f \in A^2_\alpha$. The projection operator can be extended to $L^1(\mathbb{D}, dA_\alpha)$, by mapping each $f \in L^1(\mathbb{D}, dA_\alpha)$ to an analytic function, and $P_\alpha f = f$, for every $f \in A^1_\alpha$ (see [93] p. 6).

1.4 The voice transform

In order to construct wavelet transforms and multiresolution in the analytic function spaces let us consider a general approach of the continuous affine wavelet transform. Grossmann, Morlet, Paul in [87] observed that the properties of the continuous affine wavelet transform are related to the properties of a representation of the affine group.

Let us consider the set of affine functions

$$\{\ell_{(a,b)}(x) = ax + b : \mathbb{R} \to \mathbb{R} : (a, b) \in (0, +\infty) \times \mathbb{R}\}.$$

The composition operation $\ell_1 \circ \ell_2(x) = a_1 a_2 x + a_1 b_2 + b_1$ will induce in the set of the parameters

$$\mathbb{A} = \{(a, b) : \ a \in (0, +\infty), \ b \in \mathbb{R}\}$$

the following operation: $(a_1, b_1) \circ (a_2, b_2) = (a_1 a_2, a_1 b_2 + b_1)$. The set of the parameters $\mathbb{A}$ with the induced operation is a group, namely the affine group. Then

$$U_{(a,b)} f(x) = |a|^{-1/2} f(a^{-1}x - b)$$

defines a representation of the affine group on $L^2(\mathbb{R})$. The continuous affine wavelet transform given by (1.1) can be expressed in terms of the representation as follows:

$$W_\psi f(a, b) = \langle f, U_{(a,b)}\psi \rangle.$$

We can observe also that the wavelet coefficients with respect to the wavelet system (1.2) can be expressed by the values of the continuous wavelet transform

$$\langle f, \psi_{n,k} \rangle = W_\psi f(2^{-n}, k),$$

i.e., we take the values of the continuous wavelet transform on the following discrete subset of the affine group:

$$\Lambda = \{(2^{-n}, k) : n \in \mathbb{Z}, \ k \in \mathbb{Z}\}.$$

Inversion formulas for the continuous wavelet transform and the properties of this transform in different function spaces were studied by Weisz and Szarvas in [178, 179, 180, 182].

If instead of affine group we consider a locally compact topological group $(\mathbb{G}, \cdot)$ (with left invariant Haar measure m) and a unitary representation $U_x : H \to H$ $(x \in \mathbb{G})$ of the group on some Hilbert space H, we can define a very general continuous transform: the *voice transform*.

First let us revise the definition of the *unitary representation.* Let us consider a Hilbert-space $(H, \langle\cdot,\cdot\rangle)$ and let us $\mathcal{U}$ denote the set of unitary bijections $U : H \to H$. Namely, the elements of $\mathcal{U}$ are bounded linear operators which satisfy $\langle Uf, Ug\rangle = \langle f, g\rangle$ $(f, g \in H)$. The set $\mathcal{U}$ with the composition operation $(U \circ V)f := U(Vf)$ $(f \in H)$ is a group, the neutral element of which is I the identity operator on H and the inverse element of $U \in \mathcal{U}$ is the operator U^{-1} which is equal to the adjoint operator of U : $U^{-1} = U^*$. The homomorphism of the group $(G, \cdot)$ on the group $(\mathcal{U}, \circ)$ satisfying

i) $U_{x\cdot y} = U_x \circ U_y \quad (x, y \in G)$,

ii) $G \ni x \to U_x f \in H$ is continuous for all $f \in H$

is called the unitary representation of $(G, \cdot)$ on H.

The *voice transform* of $f \in H$ generated by the representation U and by the parameter $g \in H$ is the (complex-valued) function on G defined by

$$(V_g f)(x) := \langle f, U_x g\rangle \quad (x \in \mathbb{G}, f, g \in H). \tag{1.6}$$

The name "voice transform" for this very general transform was used by Grossmann, Morlet, Paul in [87].

For any representation $U : G \to \mathcal{U}$ and for each $f, g \in H$ the voice transform $V_g f$ is a continuous and bounded function on G and $V_g : H \to C(G)$ is a bounded linear operator.

The set of continuous bounded functions defined on the group G with the norm defined by $\|F\| := \sup\{|F(x)| : x \in G\}$ form a Banach space. From the unitarity of $U_x : H \to H$ follows that, for all $x \in G$

$$|(V_g f)(x)| = |\langle f, U_x g\rangle| \leqslant \|f\|\|U_x g\| = \|f\|\|g\|,$$

consequently, $\|V_g\| \leqslant \|g\|$.

The Gábor transform (Short-time Fourier transform) is also a special voice transform generated by a representation of the Heisenberg group (see for ex. Heil, Walnut[94], Gröchenig [90], etc.). In [69, 176] Feichtinger and Weisz proved inversion formulas for the short-time Fourier transform, and in [70, 177] they studied the properties of this transform in Wiener amalgams and Hardy spaces.

Another special voice transform which is important from the point of view of the applications is the shearlet transform studied by Labate, Lim, Kutyniok, Weiss, Sauer for ex. in [105, 102, 103] etc..

Analyzing the question of discretization of these special voice transforms it turned out that different techniques are required. In the case of the affine wavelet transform one possibility is the construction of multiresolution analysis, for detailed description see for example Mallat [109].

H.G. Feichtinger and K.H. Gröchenig have established a rather general approach, giving an attempt to describe in a unified fashion the properties of the continuous affine wavelet transform and the STFT (Short-time Fourier transform) by taking a group theoretical view-point. They described a general discretization technique for the voice transforms induced by irreducible, square integrable and integrable group representations, giving atomic decompositions for large families of Banach spaces, the so called coorbit spaces (see papers of Fiechtinger, Gröchenig [64, 66, 65, 89]).

A voice transform V_g generated by an unitary representation U is one-to-one for all $g \in H\backslash\{0\}$ if U is irreducible. Consequently the invertibility of V_g it is connected to the irreducibility of the representation U which generate the a voice transform.

A representation U is called *irreducible* if the only closed invariant subspaces of H, i.e., closed subspaces H_0 which satisfy $U_x H_0 \subset H_0$ $(x \in G)$, are $\{0\}$ and H. Since the closure of the linear span of the set $\{U_x g : x \in G\}$ is always a closed invariant subspace of H, it follows that U is irreducible if and only if the collection $\{U_x g : x \in G\}$ is a closed system for any $g \in H, g \neq 0$.

The function $V_g f$ is continuous on G but in general is not square integrable. If there exist $g \in H, g \neq 0$ such that $V_g g \in L^2_m(G)$, then the representation U is *square integrable* and the g is called admissible for U. For a fixed square integrable U the collection of admissible elements of H will be denoted by H^2. If the representation is unitary, irreducible and square integrable, normalizing the vector $g \in H^2$ if necessary, the voice transform $V_g : H \to L^2_m(G)$ will be isometric, i.e.,

$$[V_g f, V_g h] = \langle f, h \rangle\,, \qquad (f, h \in H), \tag{1.7}$$

where the left hand side is the scalar product generated by the left Haar measure of the group G. For proof see for example Weil, Walnut [94] or Schipp, Wade [142]. Formula (1.7) is the analogue of the Plancherel formula for the Fourier transform, which can be interpreted as the low of energy conservation of signals.

An important consequence of this is the following reproducing formula: For convenient normalized $g \in H^2$ we have the following convolution relation (on G):

$$V_g f = V_g f * V_g g, \quad f \in H. \tag{1.8}$$

Atomic decomposition results were proved for one and n-dimensional classical Hardy spaces $H^p(\mathbb{R}), H^p(\mathbb{R}^d)$. These spaces are very important in harmonic analysis and summability theory. The first atomic decomposition results of the Hardy spaces can be found in Coifman and Weiss [41]. An atom is a simple, easy to handle function. The tempered distribution of the Hardy spaces is decomposed into a sum of atoms. The advantage of this decomposition is that many theorems need to be proved for atoms, only. Beyond these, the Hardy spaces have been introduced for martingales as well (see e.g.Weisz [174]). In Weisz [175, 182] detailed proofs for the atomic decomposition of the one and n-dimensional Hardy spaces are presented.

Feichtinger, Gröchenig in [64, 66, 65, 89] described a unified approach to atomic decomposition through integrable group representations. By a specific choice of a group and a suitable group representations formula (1.8) and its extensions permit non-orthogonal wavelet expansion for Besov-Triebel-Lizorkin spaces on $\mathbb{R}^n$, the Gábor-type expansions for modulation spaces and atomic decomposition results. The atoms for all these spaces are transforms of a single function, where the transformations are given by a certain unitary group representation. Formula (1.8) and its extensions are the very reasons for the unification of all different examples mentioned before.

Feichtinger, Gröchenig in [64, 66, 65, 89], in order to obtain atomic decomposition by discretization of the voice transform, imposed stronger integrability condition on the representation U which generates the transform.

Let us consider a positive, continuous *submultiplicative weight* w on G, i.e., $w(xy) \leqslant w(x)w(y)$, $w(x) \geqslant 1$, $\forall x, y \in G$. Assume that the representation is *integrable* i.e., the set of analyzing vectors is not trivial:

$$\mathcal{A}_w = \{g \in H : V_g g \in L^1_w(G)\} \neq \{0\}. \tag{1.9}$$

With this assumption the reproducing formula given by the convolution (1.8) can be discretized. Let us define the simplest Banach space where atomic decompositions can be obtained:

$$\mathcal{H}^1_w = \{f \in H : V_g f \in L^1_w(G)\}. \tag{1.10}$$

The definition of $\mathcal{H}^1_w$ is independent of the choice of $g \in \mathcal{A}_w$.

The simplest atomic decomposition result is for the space $\mathcal{H}^1_w$ which tells us that:

For any $g \in \mathcal{A}_w \backslash \{0\}$ there exists a collection of points $\{x_i\} \subset G$ such that any $f \in \mathcal{H}^1_w$ can be written as

$$f = \sum \lambda_i(f) U_{x_i} g, \quad \textit{with} \quad \sum_i |\lambda_i(f)| w(x_i) \leqslant C_0 \|f\|_{\mathcal{H}^1_w} \tag{1.11}$$

where the sum is absolutely convergent in $\mathcal{H}^1_w$.

This atomic decomposition result was extended also for more general Banach spaces, namely for the coorbit spaces in Feichtinger, Gröchenig [66, 65, 89]. In section 3.4 we will present more details connected to these results and applying this atomic decomposition results we will present new atomic decomposition results obtained by the author in [127] for weighted Bergman spaces.

I would like to mention that this research field is still flourishing. In the last period the coorbit theory was developed for non-integrable representations satisfying some $L^p(G), p > 1$ integrability condition (see the work of Dahlke, Steidel, Teschke, Kutyniok, Fornasier, Rauhut, Christensen, Olafson, De Mari, De Vito, Vigogna [34, 48, 49, 50, 52, 53]).

1.5 The main contribution of the thesis

In this thesis we present the main results of the author (and her coauthors) connected to the voice transforms of the Blaschke group. Because the Blaschke function plays an important role in the theory of analytic functions (see for example the factorization theorem in the Hardy spaces) it occurred naturally to use the group generated by the composition of the Blaschke functions, instead of the affine group, in order to construct analytic wavelets.

The first results in this direction were obtained by Pap and Schipp in [122, 123], where it was introduced the Blaschke group and the voice transform of the Blaschke group generated by a representation of the group on the Hardy space of unit circle.

The congruence transformation and the pseudo-hyperbolic distance in the Poincare model of the hyperbolic geometry can be described using the Blaschke functions. This motivated to call the introduced new transform hyperbolic wavelet transform. In the next chapters we present results obtained by the author in this direction.

In section 2.1 we introduce the Blaschke group and we study the main properties of this group.

In section 2.2 we present results connected to the properties of the continuous voice transforms, so called hyperbolic wavelet transform, generated by a representation of the Blaschke group on the Hardy space [122, 123]. Analyzing the question of discretization of this voice transforms it turned out that the Feichtinger-Gröchenig theory cannot be applied, because the square integrability and the integrability conditions are not satisfied, but it was showed that it is possible to construct an adapted version of multiresolution.

In subsection 2.2.1 it is presented the construction of an adapted multiresolution (MRA) and analytic wavelets in the Hardy space of the unity disc proposed by Pap in [126]. It turned out that the introduced analytic wavelet system has many advantages, and can be applied efficiently in the approximation of the transfer functions of the systems (see Pap [126, 129]). In the construction of the multiresolution in the Hardy space of the unit circle, we avoid the classical Fourier technique, instead we use complex technique, the localized Cauchy kernels corresponding to a discrete countable subset A of the unit disc, in order to construct multiresolution analysis in $H^2(\mathbb{T})$ and the Cauchy formula in the proofs. It has been showed that the resolution levels are spanned by a special rational analytic orthonormal wavelet system, i.e., by the Malmquist-Takenaka system with a special localization of the poles. In [129] it was proved that the levels of the multiresolution form a complete model set for the disc algebra of the unit disc, and it was given an estimation of the error therm for the proposed approximation process for a family of analytic functions.

In subsection 2.2.2 are presented results connected to the projection operator (P_nf, $n \in \mathbb{N}$) to the n-th resolution level of the multiresolution. These results were published by Pap in [126, 129].

Subsection 2.2.3 contains an algorithm for the computation of the wavelet coefficients based on measurements. In [126] it has been shown that the coefficients of the projection operator P_nf can be computed exactly if we know the values of the functions on $\bigcup_{k=0}^{n} A_k$.

Comparing the presented adapted multiresolution with the classical affine multiresolution we observe that this has the following advantages:

1. The levels of the multiresolution are finite dimensional, which makes easier to find a basis on every level, but in the same time the density condition remains valid.

2. We have constructed analytic orthonormal rational wavelet bases on the resolution levels given by an explicit formula.

3. We can compute the wavelet coefficients exactly measuring the values of the function f at the points of the set $A = \bigcup_{k=0}^{n} A_k \subset \mathbb{D}$. Based on these measurements we can write exactly the projection operator $(P_n f,\, n \in \mathbb{N})$ which is convergent in $H^2(\mathbb{T})$ norm on the unit circle to f, and $P_n f(z) \to f(z)$ uniformly on every compact subset of the unit disc.

4. At the same time $P_n f(z)$ is the best approximant interpolation operator on the set the $\bigcup_{k=0}^{n} A_k$ inside the unit circle for the analytic continuation of f. Based on these properties it can be used for $H^2(\mathbb{D})$ identification if we measure the values of function on this set.

The question of recovery of analytic functions from values measured in the open unit disc was also studied by Totik, see [164], where it has been proved that in H^p or in the disc algebra if we can measure the values of a function f on a non Blaschke sequence, say $(z_k)_{k\in\mathbb{N}}$, then there are polynomials $p_{n,j}$ such that $\sum_{j=1}^{n} f(z_j)p_{n,j}$ tends to f in norm. This is a beautiful theoretical result. From practical point of view the only difficulty is that we can not determine exactly the coefficients of these polynomials from the values of the measurements $f(z_k)_{k\in\mathbb{N}}$. For this reason we consider that, our method is suitable from the point of view of applications.

In subsection 2.2.4 another new property of the hyperbolic wavelet basis is presented, which is not characteristic to the affine wavelets, namely the discrete orthogonality. In [126] it has been proved also that it is possible to construct wavelets using the reproducing kernels of the multiresolution levels, similar as in Bultheel, González-Vera [25].

In section 2.3 are included results of Feichtinger and Pap. They showed that it is possible to construct similar analytic wavelets and adapted multiresolution in the Hardy space of the upper half plane (see Feichtinger, Pap [67]). The transition to the upper half plane is made using the Cayley-transform. The obtained adapted multiresolution in the Hardy space of the upper half plane inherits all the nice properties of the multiresolution presented in the section 2.2.

As long as the theory of wavelet constructions on the Hardy space of the unit disc presented in [126] is suitable for time frequency-domain description of discrete-time-invariant dynamical systems, the adaptation to the upper half plane can be used in system theory to describe the spectral behavior of continuous-time-invariant systems. In the same time the constructed wavelet system is a new example of "very regular" wavelet system in $H^2(\mathbb{R})$ with analytic continuation to the upper half plane. It has been also studied the approximation and identification of transfer functions of a continuous-time-invariant systems.

In the case of the Hardy space of the unit disc the construction of dense subsets is not so difficult. For example, the polynomials are dense. In the Hardy space of the upper half plane dense subsets are harder to find. Applying the Daubechies theory it can be shown that choosing as mother wavelet $\psi(y) = (1 + iy)^{-p}$ for $p \geqslant 2$ we can generate a frame for the Hardy space of the upper half plane. For $p = 3$ Ward, Partington in [171] described a rational wavelet decomposition of the Hardy–Sobolev class of the half plane. The case $p = 1$, the Cauchy kernel case, dos not fall under the Daubechies theory since does not have vanishing mean value, but Ward and Partington have shown that the system $\psi_{j,k} = 2^{j/2}\psi(2^j y - b_0 k)$, $j, k \in \mathbb{Z}$ does constitute a fundamental set for the upper half plan algebra.

The multiresolution analysis in the Hardy space of the upper half plane introduced by Feichtinger, Pap in [67] is generated by localized Cauchy kernels, consequently dos not fall under the Daubechies theory. But it gives an example of analytic wavelets and an adapted multiresolution also for the boundary limits of the functions from the Hardy space of the upper half plane, i.e., in $H^2(\mathbb{R})$. In this sense it is connected to the problem formulated by Meyer of construction of regular wavelets, presented in the introduction.

In section 2.4 is presented the relation between the Zernike functions and the hyperbolic wavelet transform. More exactly the matrix elements of the representation of the Blaschke group on the Hardy space of the unit disc U given by (2.3) can be expressed by the Zernike functions. An important consequence of this connection is the addition formula for Zernike functions. The addition formula for Zernike functions was one of open problems formulated in connection to Zernike functions. These results were published in [123] by Pap, Schipp and in the survey paper by Pap [129].

Zernike polynomials are often used to express wave-front data in optical tests, since they are made up of terms that are of the same form as the types of aberrations often observed. The first order wave-front aberrations coefficients can be obtained as the coefficients of the Zernike polynomials expansion of the wave-front, and they are called Zernike moments of the wave-front. The orthogonal system of Zernike functions was introduced by Fritz Zernike a Dutch physicist and winner of the Nobel prize for Physics in [185], to model symmetries and aberrations of optical systems (e.g., telescopes).

Although, the approximation of Zernike coefficients were obtained from measurements at discrete corneal points and via discrete computations, the developers of the corneal measurement devices and shape-evaluation programs could not rely on the discrete orthogonality before the discrete orthogonality of Zernike functions was not proved. Not surprisingly, the discrete orthogonality of Zernike functions was a target of research for some time see for example by Wyant, Creath in [173]. In subsection 2.4.3 it is proved the discrete orthogonality of Zernike functions. This result was published in [121] by Pap, Schipp.

Subsection 2.4.4 it is shown how can be applied the discrete orthogonlity of Zernike functions in corneal topography. As we presented in the previous subsection, Pap and Schipp introduced a discrete subset of points of the unit disc and a discrete measure on it ensuring discrete orthogonality of the Zernike functions. These points can be used as places of measurements in order to calculate the Zernike-based representations. Using computer implementations it was studied their precision for some test surfaces, including three "cornea-like" test surfaces, as well. These results were published and analyzed by Soumelidis, Fazekas, Schipp, Pap in [155, 156, 157]. The test surfaces considered herein include centrally positioned and shifted cones, pyramids, and some cornea-like surfaces. With these spatial points as input points, discrete Zernike transformation was carried out. The resulting Zernike coefficients were then used to geometrically reconstruction of the optically smooth corneal surface. The error-surfaces were compared to the ones resulting from the Zernike-based reconstructions of a cornea-like mathematical surface that had been properly fitted to the input data.

The numerical computations, reconstructions and experiments are based on the approximation of the continuous Zernike moments of the corneal surface G, This is a consequence of Theorem 2.4.2, which implies that the continuous moments are the limit of the discrete Zernike moments, computed based on the measurements on the set X of the discretization defined by (2.30).

If we take T_N an arbitrary linear combination of Zernike polynomials of degree less than $2N$, using the discrete orthogonality and the continuous orthogonality property we obtain that the coefficients A_{mn} can be expressed exactly by the discrete Zernike coeffitients. This means that we can determine the exact value of the Zernike coefficients (moments) of T_N if we can measure the values of T_N on the points of the set X defined by (2.30). This means that with the construction of the set X we give answer to the question where the so called Placido ring system is worth situated. In this case we can reconstruct exactly T_N if we measure its values on the discretization mash X.

The hyperbolic wavelet transform presented in Chapter 2 can be applied also for determining the poles of rational functions (Schipp, Soumelidies [143]), the eigenvalues of matrices (Schipp, Soumelidies [144]) and for system identification (Bokor, Schipp, Soumelidis [21, 22, 23]).

In Chapter 3 we consider the case $m = \alpha + 2$, when formula (2.1) defines a representation of the Blaschke group on the weighted Bergman space.

In section 3.1 it is proved that the representation is unitary irreducible representation of the Blaschke group on weighted Bergman space A^2_α (see Pap [125]).

Section 3.2 contains properties of the continuous hyperbolic wavelet transform generated by representation (2.1). These results follow from the general theory of the voice transform.

In section 3.3 for $\alpha \geqslant 0, m = \alpha + 2 \in \mathbb{N}$ we give an orthogonal rational wavelet system, and we show that the Bergman projection operator can be expressed with this system. These results were published by Pap in [125, 127]. For $\alpha \in \mathbb{N}$ in [125] Pap computed the matrix elements of the representation (3.1). In the computation we use the Cauchy formula, for this reason the condition $\alpha \in \mathbb{N}$ is important. The computations follow the same line as in the case of the Hardy space presented in the previous chapter. In this case the matrix elements can be expressed using the Jacobi polynomials (see [125]). The computation of the matrix elements for $\alpha \notin \mathbb{N}$ would involve the fractional Cauchy formula, which is a recent topic of the research. For this case the determination of the matrix elements of the representation is an open problem.

In section 3.4 it is analyzed the question of discretization of the hyperbolic wavelet transform defined by (3.3). It turned out that depending on the value of α different techniques are required. In the introduction of the section we presented a short summary of the theory introduced by Feichtinger and Gröchenig, the so called a unified approach to atomic decomposition through integrable group representations in Banach spaces and coorbit theory.

In the unified approach of the atomic decomposition a useful tool is the Q-density, the V-separated property and the bounded uniform partitions of the unity of the locally compact group.

In subsection 3.4.1, using the hyperbolic metric, we describe the Q density from right, and the separation from right in the Blaschke-group. Using this we can give an example of bounded uniform partitions of the unity from right. In the general theory of atomic decomposition it is used the Q-density from the left, but in Blaschke group it is easier the geometrical interpretation of Q density from right. This is the reason why we made a small modification in the discretizing operator which corresponds to the Q–density from the right in order to obtain atomic decomposition in the weighted Bergman spaces.

In subsection 3.4.2 we study the integrability of the hyperbolic wavelet transform defined by formula (3.3). For certain weighted Bergman spaces both square integrability and integrability conditions are satisfied .

Consequently in these cases it can be applied the Feichtinger-Gröchenig theory, and in this way we can obtain new atomic decomposition results in these weighted Bergman spaces (see Pap [127]). These results are presented in subsection 3.4.3. We obtain that every function from the minimal Möbius invariant space $\mathcal{B}_1$ of the analytic functions will generate an atomic decomposition in some weighted Bergman spaces. More exactly we get atomic decomposition of $f \in \mathcal{H}^1$, with atoms of the form $U^{\alpha}_{x_i^{-1}} g$, $g \in \mathcal{B}_1 \cup \{1\}$. From Theorem 3.4.7 follows that for $p > 2 + \frac{4}{\alpha}$ we have $A^p_\alpha \subset \mathcal{H}^1$, consequently the previous atomic decomposition is true also for A^p_α under the mentioned restrictions to the parameters. For the special case $g = 1$ we obtain the following atomic decomposition: if $f \in A^p_\alpha$, $\alpha > 0$, and $p > 2 + \frac{4}{\alpha}$, $f = \sum \lambda_i(f) U^{\alpha}_{x_i^{-1}} 1 = \sum \lambda_i(f) \frac{(1-|b_i|^2)^{\frac{\alpha+2}{2}}}{(1-\overline{b_i}z)^{\alpha+2}}$, holds, which is very similar to the atomic decomposition obtained with complex analysis techniques (see [187], pp. 69). The difference is that in our case we have ℓ^1 information about the coefficients instead of ℓ^p information and the convergence is in $\mathcal{H}^1$ norm instead of A^p_α. Using the classical techniques of the complex analysis in the atomic decomposition of a function $f \in A^p_\beta$, the atoms are of form (see [187], pp. 69) $\frac{(1-|x_i|^2)^a}{(1-\overline{x_i}z)^b}$. Applying the Feichtinger-Gröchenig theory we obtain more general atoms for the weighted Bergman spaces, i.e., every function $g \in \mathcal{B}_1$ generates an atomic decomposition for $f \in A^p_\alpha$ with atoms of the form $U^{\alpha}_{x_i^{-1}} g$.

In the unweighted case and in some weighted Bergman spaces the Feichtinger-Gröchenig theory can be not applied, because the the integrability condition is not satisfied. In these cases it is shown, that analogously to the case of the Hardy spaces, is possible to construct multiresolution and analytic wavelets (see Pap [128, 133]). These results are presented in section 3.5. In [128] Pap showed that, it is possible to construct a multiresolution analysis (MRA), using localized Bergman kernels in special sampling points. Later in [133] the result was extended for weighted Bergman spaces.

Although the main idea is the same as in the case of the Hardy space, the construction of the MRA in the weighted Bergman space is more complicated, than in the Hardy space. The first step of the construction is the construction of a new example of sampling set for the weighted Bergman space, which is related to the Blaschke group operation. The construction of sampling sets in weighted Bergman space is difficult in general. If once we have this the constructed discretization scheme, the construction of the multiresolution levels are similar to the case of Hardy space. The next difficulty is to describe the orthogonal wavelets on the resolution levels, because in the case of the weighted Bergman space, they can be not given explicitly in closed form. But we can give an algorithm to generate them, and using this we can prove that the projection to the resolution levels has similar interpolation properties like in the case of Hardy space. This projection operator gives opportunity of practical realization of the hyperbolic wavelet representation of a function belonging to the weighted Bergman space, if we can measure the values of the function on a given set of points inside the unit disc. We also studied the convergence properties of the hyperbolic wavelet representation.

In the construction of the MRA in weighted Bergman spaces we use frames obtained by localization of the weighted Bergman kernel in a set of sampling points connected to the Blaschke group, so called hypebolic wavelet frames. Recently, tight affine wavelet frames derived by the multiresolution analysis are used to open a few new areas of applications of frames. The application of tight wavelet frames in image restorations is one of them that includes image inpainting, image denoising, image deblurring and blind deburring, and image decompositions. [10, 47, 158]. An up to date monograph in this domain is [104], where are collected the most important ones and multivariate results connected to affine wavelet frames (framelets) and the related MRA-s, and their application in the image recovery from incomplete observed data, including the tasks of inpainting and image/video enhancement. In the recovery of missing data from incomplete and/or damaged and noisy samples, application of wavelet methods based on frames is more advanced due to the redundancy of frame systems. In the context of the introduced hyperbolic wavelet frames would be interesting to study similar properties.

The plan of this section is as follows. In subsection 3.5.1 we introduce a discrete subset of the Blaschke group, which is a sampling set for the weighted Bergman space, see (3.22). In subsection 3.5.2, using this special sampling set, we consider hyperbolic wavelet frames (see (3.24) and and using them we construct an analogue of MRA decomposition in the weighted Bergman space. First the different resolution spaces will be defined using the introduced non-orthogonal hyperbolic wavelet frames. Applying the Gram-Schmidt orthogonalization we consider the rational orthogonal basis on the n-th multiresolution level V_n. This system is the analogue of the Malmquist-Takenaka system in the Hardy spaces, possesses similar properties and is connected to the contractive zero divisors of a finite set in Bergman space.

In subsection 3.5.3 we prove that the projection operator $P_n f(z)$ on the resolution level V_n is convergent in A^2_α norm to f, and is also an interpolation operator on the set the $\bigcup_{k=0}^{n} \mathcal{A}_k$, where $\mathcal{A}_k$ is defined by (3.23) with minimal norm and $P_n f(z) \to f(z)$ uniformly on every compact subset of the unit disc (see Theorem 3.5.2).

In subsection 3.5.4 it is presented the algorithm for computation of the wavelet coefficients measuring the values of the function f on the constructed sampling set (3.22). Based on this we can write exactly the projection operator ($P_n f$, $n \in \mathbb{N}$) on the n-th resolution level.

Based on the MRA constructions in the Hardy and weighted Bergman spaces (see [126, 128, 133]) Nowak and Pap in [116] summarized the main idea of these constructions, describing in general the new method of construction of analytic wavelets which is applied in both, Hardy and weighted Bergman spaces. This method should be applied in more general setting in reproducing kernel Hilbert spaces.

The analytic wavelet constructed in Chapter 2 is, in fact, a Malquist -Takenaka system with a special localization of poles. In Chapter 4 we present results connected to Malquist -Takenaka systems in generality. We give an overview of the discretization results connected to Malmquist–Takenaka systems for the unit disc and upper half plane. We prove that the discretization nodes on the real line have similar properties like the discretization nodes on the unit circle: they satisfy some equilibrium conditions and they are stationary points of some logarithmic potential. The problem whether they are the minimum of a logarithmic potential was formulated and solved in a special case. These results were published by Pap and Schipp in [137, 118, 119, 130]. The formulated problem was solved in generality recently in [79] by Marcell Gaál, Béla Nagy, Zsuzsanna Nagy-Csiha, Szilárd Révész.

In Chapter 5 we present quaternionic extension of some results connected to the Blaschke group and Malmquist-Takenaka system.

Quaternions play an important role in modeling the time and space dependent problems in physics and engineering. For example in engineering applications unit quaternions are used to describe the three dimensional rotations. In the last years quaternions gained a new life due to their applicability in signal processing. This is due to both, the applicability of quaternion-valued functions to color-coded images as well as the link to new concepts of higher-dimensional phases, like the hypercomplex signal of Bülow or the monogenic signal by Larkin and Felsberg (see [27, 71, 106]). Another important field, where quaternions play an important role is the quantum theory. Adler, a world-renowned theoretical physicist, in his book Quaternionic Quantum Mechanics and Quantum Fields [2], provides an introduction to the problem of formulating quantum field theories in quaternionic Hilbert space. This well-written treatise is a very significant contribution to theoretical physics. Bernardo Vargas in the review of this book mentioned that the quaternionic formalism is to improve some treatments of theoretical physics. But the full power of quaternions would be even more important by using quaternionic analysis.

This motivates to extend the results of modern harmonic analysis, like the wavelet theory, to quaternion variable function spaces.

A first step in this direction is to give the quaternionic analogue of the Blaschke group. The main obstacle in the study of quaternion-valued matrices and functions, as expected, comes from the non-commutative nature of quaternionic multiplication.

Our work was inspired also by the paper of P. Cerejeiras, M. Ferreira and U. Kähler [30], where the monogenic wavelet transform for quaternion valued functions on the three dimensional unit ball in $\mathbb{R}^3$ was introduced. The construction is based on representations of the group of Möbius transformations which maps the three dimensional unit ball onto itself.

In section 5.2 we introduce the quaternionic analogue of the Blaschke group and we list the main subgroups of this groups. The results were published by Pap and Schipp in [131].

Beside the monogenic quaternionic function theory, where many difficulties appear when we want to make analysis, the theory of slice regular functions (introduced in 2006 by Gentili, Stoppato, Struppa [80, 81, 82]) and the analysis on this field, would be an alternative tool for the quantum theory. To introduce new orthonormal systems in the slice regular Hardy space, is therefore an interesting topic that is worthwhile to be studied.

In [120, 135] Pap, Schipp and Qian, Sprossig, Wang respectively, following two different ways, introduced two analogues of the M-T systems in the set of quaternions. While in the complex case both ways give the same M-T system, in the quaternionic setting this is not anymore true. The drawback of both constructions is that these extensions will not inherit all the nice properties of the before mentioned complex M-T system, e.g., the system introduced by Pap and Schipp is not analytic in the quaternionic setting. The system introduced by Qian, Sprossig, Wang, is monogenic but can not be written in closed form.

Pap in [132] introduced a new generalization of the complex Malmquist-Takenaka system in the quaternionic slice regular Hardy space, which is slice regular and in same time can be given in closed form. In Section 5.3 results connected to slice regular Malmquist-Takenaka system are presented.

Chapter 2

Hyperbolic wavelet transform and multiresolution in the Hardy spaces

2.1 The Blaschke group

The Blaschke group was introduced by Pap and Schipp in [122, 123]. Let us denote the unit disc and the unit circle by $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. Instead of linear functions let us consider the following rational linear functions:

$$B_a(z) := \epsilon \frac{z - b}{1 - \bar{b}z} \quad (z \in \mathbb{C},\, \bar{b}z \neq 1)$$

the so called *Blaschke functions*. Let us denote the set of the parameters $\mathbb{B} := \mathbb{D} \times \mathbb{T}$ and $a = (b, \epsilon) \in \mathbb{B}$. If $a \in \mathbb{B}$, then B_a is an 1-1 map on $\mathbb{T}$, and $\mathbb{D}$, respectively.

The disc $\mathbb{D}$ with the *pseudohyperbolic metric*

$$\rho(z_1, z_2) := \frac{|z_1 - z_2|}{|1 - z_1\overline{z}_2|} = |B_{(z_2,1)}(z_1)| \quad (z_1, z_2 \in \mathbb{D})$$

is a complete metric space. This metric is invariant with respect to Blaschke functions:

$$\rho(B_{(b,1)}(z_1), B_{(b,1)}(z_2)) = \rho(z_1, z_2) \quad (z_1, z_2 \in \mathbb{D},\, b \in \mathbb{D}).$$

This property characterizes the Blaschke functions. Namely, for every f which is analytic and bounded in $\mathbb{D}$ with $\|f\|_\infty \leqslant 1$ we have $\rho(f(z_1), f(z_2)) \leqslant \rho(z_1, z_2)$, and equality holds in a point $z \in \mathbb{D}$ if and only if f is a Blaschke function.

The restrictions of the Blaschke functions on the set $\mathbb{D}$ or on $\mathbb{T}$ with the operation $(B_{a_1} \circ B_{a_2})(z) := B_{a_1}(B_{a_2}(z))$ form a group. In the set of the parameters $\mathbb{B} := \mathbb{D} \times \mathbb{T}$ let us define the operation induced by the function composition in the following way: $B_{a_1} \circ B_{a_2} = B_{a_1 \circ a_2}$. The set of the parameters $\mathbb{B}$ with the induced operation is called the *Blaschke group*. The Blaschke group $(\mathbb{B}, \circ)$ will be isomorphic with the group $(\{B_a, a \in$

$\mathbb{B}\}, \circ)$. If we use the notations $a_j := (b_j, \epsilon_j)$, $j \in \{1,2\}$ and $a := (b, \epsilon) =: a_1 \circ a_2$, then the components of a are given by

$$b = \frac{b_1\overline{\epsilon}_2 + b_2}{1 + b_1\overline{b}_2\overline{\epsilon}_2} = B_{(-b_2,1)}(b_1\overline{\epsilon}_2), \quad \epsilon = \epsilon_1 \frac{\epsilon_2 + b_1\overline{b}_2}{1 + \epsilon_2\overline{b}_1 b_2} = B_{(-b_1\overline{b}_2,\epsilon_1)}(\epsilon_2).$$

The neutral element of the group $(\mathbb{B}, \circ)$ is $e := (0,1) \in \mathbb{B}$ and the inverse element of $a = (b, \epsilon) \in \mathbb{B}$ is $a^{-1} = (-b\epsilon, \overline{\epsilon})$.

Since $B_a : \mathbb{T} \to \mathbb{T}$ is bijection there exists a function $\beta_a : \mathbb{R} \to \mathbb{R}$ such that $B_a(e^{it}) = e^{i\beta_a(t)} \quad (t \in \mathbb{R})$, where β_a can be expressed in an explicit form. Let us introduce the function

$$\gamma_r(t) := \int_0^t \frac{1 - r^2}{1 - 2r\cos s + r^2}\, ds \quad (t \in \mathbb{R}, 0 \leqq r \leqslant 1).$$

Then

$$\beta_a(t) := \theta + \varphi + \gamma_r(t - \varphi), \quad (a = (re^{i\varphi}, e^{i\theta}) \in \mathbb{B}, t \in \mathbb{R}, \theta, \varphi \in \mathbb{I} := [-\pi, \pi)).$$

For the derivatives one has

$$\beta_a'(t) = \frac{1 - r^2}{|1 - \overline{b}e^{it}|^2} = \frac{1 - r^2}{1 - 2r\cos(t - \varphi) + r^2}.$$

Hence it follows that $\beta : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function.

The integral of the function $f : \mathbb{B} \to \mathbb{C}$, with respect to the left invariant Haar measure m of the group $(\mathbb{B}, \circ)$ (see [122]) can be expressed as

$$\int_{\mathbb{B}} f(a)\, dm(a) = \frac{1}{2\pi} \int_{\mathbb{I}} \int_{\mathbb{D}} \frac{f(b, e^{it})}{(1 - |b|^2)^2}\, db_1 db_2 dt,$$

where $a = (b, e^{it}) = (b_1 + ib_2, e^{it}) \in \mathbb{D} \times \mathbb{T}$.

It can be shown that this integral is invariant with respect to the left translation $a \to a_0 \circ a$ and under the inverse transformation $a \to a^{-1}$, consequently this group is *unimodular*.

The Blaschke functions play an important role not only in system identification, in factorization of functions belonging to Hardy spaces. They can be used also to represent the congruence's in the Poincaré model of the Bolyai-Lobachevsky geometry. On this basis in the construction of wavelets we take them instead of the affine transforms in $\mathbb{R}$ and we introduce the called *hyperbolic or analytic wavelets*. More exactly, we considered the voice transforms of the Blaschke group generated by a representation of this group on the Hardy space of the unit circle, on Bergman and weighted Bergman space respectively and we studied the properties of these transforms.

The representations on the Hardy space of the unit circle and weighted Bergman space respectively can be given by a single formula:

$$(U^m_{a^{-1}} f)(z) := \left(e^{i\frac{1}{2}\psi} \frac{(1 - |b|^2)^{\frac{1}{2}}}{(1 - \overline{b}z)} \right)^m f\left(e^{i\psi} \frac{z - b}{1 - \overline{b}z} \right), (a = (b, e^{i\psi}) \in \mathbb{B}). \tag{2.1}$$

For $m = 1$ if $f \in H^2(\mathbb{T})$ then (2.1) is a representation of the Blaschke group on the Hardy space $H^2(\mathbb{T})$. For $m = \alpha + 2$ if $f \in A^2_\alpha$ then (2.1) is a representation of the Blaschke group on the weighted Bergman space A^2_α.

In what follows we will present the most important results connected to the continuous hyperbolic wavelet transforms induced by the representations (2.1), i.e.,

$$(V^m_\rho f)(a^{-1}) := \langle f, U^m_{a^{-1}}\rho\rangle. \tag{2.2}$$

The name hyperbolic wavelet transform was used first time by Schipp in [140] and refers to the connection to the hyperbolic geometry.

2.2 The hyperbolic wavelet transform on the Hardy space of the unit circle

2.2.1 Multiresolution in the Hardy space of the unit circle

In this section we consider the case $m = 1$. First results connected to this case were published jointly with Schipp in [122, 123]. It was proved that formula

$$(U_{a^{-1}}f)(z) := \frac{\sqrt{e^{i\theta}(1-|b|^2)}}{(1-\bar{b}z)} f\left(\frac{e^{i\theta}(z-b)}{1-\bar{b}z}\right) \quad \big(z = e^{it} \in \mathbb{T}, a = (b, e^{i\theta}) \in \mathbf{B}\big), \tag{2.3}$$

(we take the principal rank of the square root) defines a representation of the Blaschke group on $H^2(\mathbb{T})$. Let us consider the induced voice transform, the so called *hyperbolic wavelet transform*:

$$(V_g f)(a^{-1}) := \langle f, U_{a^{-1}}g\rangle \quad (f, g \in H^2(\mathbb{T})). \tag{2.4}$$

First we studied the properties of the continuous voice transforms generated by representations of the Blaschke group on the Hardy space [122, 123]. Analyzing the question of discretization of this voice transforms it turned out that the Feichtinger-Gröchenig theory can be not applied, because the square integrability and the integrability conditions are not satisfied. But it is possible to construct an adapted multiresolution and analytic wavelets in the Hardy space of the unity disc. It turned out that the introduced analytic wavelet system has many advantages, and can be applied efficiently in the approximation of the transfer functions of the systems (see Pap [126, 129]). Comparing this with the classical affine multiresolution we observe that this has the following advantages:

1. The levels of the multiresolution are finite dimensional, which makes easier to find a basis on every level, but in the same time the density condition remains valid.

2. We have constructed analytic orthonormal rational wavelet bases on the resolution levels given by an explicit formula.

3. We can compute the wavelet coefficients exactly measuring the values of the function f at the points of the set $A = \bigcup_{k=0}^n A_k \subset \mathbb{D}$. We can write exactly the projection operator

$(P_n f,\, n \in \mathbb{N})$ which is convergent to f in $H^2(\mathbb{T})$ norm on the unit circle, and $P_n f(z) \to f(z)$ uniformly on every compact subset of the unit disc.

4. At same time $P_n f(z)$ is the best approximant interpolation operator on the set the $\bigcup_{k=0}^{n} A_k$ inside the unit circle for the analytic continuation of f.

It was showed also that the matrix elements of the representation can be given by the Zernike functions which play an important role in expressing the wave front data in optical tests. An important consequence of this connection is the addition formula for Zernike functions (see Pap, Schipp [123]).

The hyperbolic wavelet transform can be applied also for determining the poles of rational functions (Schipp, Soumelidies [143]), the eigenvalues of matrices (Schipp, Soumelidies [144] and for system identification (Bokor, Schipp, Soumelidis [21, 22, 23]).

Theorem 2.2.1 (**Pap, Schipp** [122])**.** *The mapping $(U_a)_{a\in\mathbb{B}}$ defined by (2.3) defines an irreducible unitary representation of the Blaschke group on $H^2(\mathbb{T})$ with respect to the inner product*

$$\langle f, g\rangle := \frac{1}{2\pi}\int_{\mathbb{I}} f(e^{is})\overline{g(e^{is})}\, ds \quad (f, g \in H^2(\mathbb{T})),$$

namely

$$\langle U_a f, U_a g\rangle = \langle f, g\rangle \quad (f, g \in H^2(\mathbb{T}), a \in \mathbb{B}).$$

For the proof see [122]. The representation can be extended unitarily to $L^2(\mathbb{T})$, but the irreducibility condition will not be anymore true, because $H^2(\mathbb{T})$ is a nontrivial invariant subspace of $L^2(\mathbb{T})$ of the representation.

In [123, 129] the square integrability of the representation was studied. It can be proved that $\rho = 1$ is not admissible. In order to compute $V_\rho\rho(a^{-1})$ for $\rho = 1$ we can use the Cauchy formula and we get that $V_\rho\rho(a^{-1}) = \sqrt{e^{i\theta}(1-|b|^2)}$. It is easy to prove that $V_\rho\rho \notin L^2(\mathbb{B})$. Indeed we have:

$$\int_{\mathbb{B}} |V_\rho\rho(a)|^2 dm(a) = \int_{\mathbb{D}} \frac{1}{1-|b|^2} db_1 db_2 = \infty.$$

It can be also proved that for every $\rho \in H^2(\mathbb{T})$ we have $V_\rho\rho \notin L^2(\mathbb{B})$, consequently the representation is not square integrable. But for every $p > 2$ we have L^p integrability conditions.

Lemma 2.2.2. *For $\rho = 1$ we have $V_\rho\rho \in L^p(\mathbb{B})$ for every $p > 2$.*

Also some weighted square integrability are still satisfied. Unfortunately, the weight function does not satisfy the condition $w(a) \geqslant 1$, imposed for the weights in the coorbit-theory.

Lemma 2.2.3. *Let us consider the radial weight function $w(a) = (1-|b|^2)^\alpha$ with $\alpha > 0$, then $V_\rho\rho \in L^2(\mathbb{B}, w)$, and $V_\rho\rho \in L^p(\mathbb{B}, w)$ for every $p > 2 - 2\alpha$.*

These results show that for the hyperbolic wavelet transform (2.4) the square integrability, or weighted square integrability required for the discretization theory developed by Feichtinger, Gröchenig is not satisfied (see [129]). In order to solve the discretisation problem, Pap in [126] introduced an adapted multiresolution in $H^2(\mathbb{T})$. Avoiding the classical Fourier technique, using the localized Cauchy kernels corresponding to a discrete countable subset A of the unit disc, a multiresolution analysis in $H^2(\mathbb{T})$ was constructed. This can be used for $H^2(\mathbb{D})$ identification if we measure the values of function on this set. It has been showed that the resolution levels are spanned by a special rational analytic orthonormal wavelet system, i.e., by the Malmquist-Takenaka system with a special localization of the poles. In this subsection we give an overview of the construction given in [126]. In [129] it was proved that the levels of the multiresolution form a complete model set for the disc algebra of the unit disc, and it was given an estimation of the error term for the proposed approximation process.

Let us remind that in the construction of affine wavelet multiresolutions the dilatation is used to obtain a higher resolution level ($f(x) \in V_n \Leftrightarrow f(2x) \in V_{n+1}$), and applying the translation we remain on the same level of resolution. If we want to construct a multiresolution in $H^2(\mathbb{T})$ we have to find the analogue of dilation by 2 and the analogue of translation. The analogue of dilation will be the action of the representation trough a discrete subgroup $\mathbf{B}_1$ of the Blaschke group.

Let us consider the following discrete subgroup of the Blaschke group:

$$\mathbf{B}_1 = \left\{(r_k, 1) : \ r_k = \frac{2^k - 2^{-k}}{2^k + 2^{-k}},\ k \in \mathbb{Z}\right\}. \tag{2.5}$$

It can be proved that $(r_k, 1) \circ (r_n, 1) = (r_{k+n}, 1)$ and

$$\rho(r_k, r_n) := \frac{|r_k - r_n|}{|1 - r_k \overline{r_n}|} = \left| \frac{\frac{2^k-2^{-k}}{2^k+2^{-k}} - \frac{2^n-2^{-n}}{2^n+2^{-n}}}{1 - \frac{2^k-2^{-k}}{2^k+2^{-k}} \frac{2^n-2^{-n}}{2^n+2^{-n}}} \right| = |r_{k-n}|.$$

As a consequence we get that the sequence $(r_k,\ k \in \mathbb{N})$ forms an equidistant division of the interval $[0, 1)$ in the pseudo hyperbolic metric.

Let us consider the following discrete subset in the unit disc:

$$A = \{z_{k\ell} = r_k e^{i\frac{2\pi\ell}{2^{2k}}}, \quad \ell = 0, 1, \cdots, 2^{2k} - 1, \ \ k = 0, 1, 2, \cdots, \infty\} \tag{2.6}$$

and for a fixed $k \in \mathbb{N}$ let the level k be the collection of the points from circle with radius r_k:

$$A_k = \{z_{k\ell} = r_k e^{i\frac{2\pi\ell}{2^{2k}}},\ \ell \in \{0, 1, \cdots, 2^{2k} - 1\}\}. \tag{2.7}$$

The points of A determine a similar decomposition to the Whitney cube decomposition of the unit disc (see for ex. [Partington, 1997] [134], pp.80). For our purpose it is more convenient to choose $(r_k,\ n \in \mathbb{N})$ as radius of the concentric circles because they are related to the Blaschke group operation, i.e., $(r_k, 1) \circ (r_n, 1) = (r_{k+n}, 1)$, and form this property we can derive the analogue property of the dilatation.

Let us consider the scaling function $\varphi = 1$. Let us define the 0-th resolution level by: $V_0 = \{c\varphi,\ c \in \mathbb{C}\}$. Let us consider the non-orthogonal wavelets on the n-th level the localized and normalized Cauchy kernels corresponding to points $\cup_{k=0}^{n} A_k$, given by

$$\varphi_{k\ell}(z) = \frac{\sqrt{(1 - r_k^2)}}{(1 - \overline{z_{k\ell}} z)}, \quad k = 0, \cdots, n, \quad \ell = 0, 1, \cdots, 2^{2k} - 1, \tag{2.8}$$

which can be obtained from φ using the representation $U_{(r_n,1)^{-1}}$, and the translations

$$\varphi_{k\ell}(e^{it}) = (U_{((r_k,1)^{-1}}\varphi)(e^{i(t - \frac{2\pi\ell}{2^{2k}})}).$$

Let us define the n-th resolution level by the linear span of all these localized Cauchy kernels:

$$V_n = \{f : \mathbb{D} \to \mathbb{C},\ f(z) = \sum_{k=0}^{n} \sum_{\ell=0}^{2^{2k}-1} c_{k\ell}\varphi_{k\ell},\ c_{k,\ell} \in \mathbb{C}\ \}. \tag{2.9}$$

In [126] it has been proved that the collections of spaces $\{V_j,\ j \in \mathbb{N}\}$ satisfy analogue conditions of the affine multiresolution, i.e.,

1. (nested) $V_j \subset V_{j+1}$,
2. (density) $\overline{\cup V_j} = H^2(T)$
3. (analog of dilatation) $U_{(r_1,1)^{-1}}(V_j) \subset V_{j+1}$
4. (basis) There exist $\{\psi_{k\ell},\ k = 0, \cdots, n,\ \ell = 0, 1, \cdots, 2^{2k} - 1\}$ (orthonormal) bases in V_n.

This is the adapted multiresolution (MRA) in $H^2(\mathbb{T})$.

In order to construct the orthonormal bases $\{\psi_{k\ell},\ k = 0, \cdots, n,\ \ell = 0, 1, \cdots, 2^{2k} - 1\}$ in V_n we apply the Gram-Schmidt orthogonalization to the following non-orthogonal basis in V_n:

$$\left\{\frac{1}{1 - \overline{z_{k\ell}} z},\ \ell = 0, 1, \cdots, 2^{2k} - 1,\ k = 0, 1, \cdots, n.\right\}.$$

The result of the Gram-Schmidt orthogonalization for this set of analytic linearly independent functions can be written in closed form. As a result we obtain the Malmquist–Takenaka system corresponding to the set $\cup_{k=0}^{n} A_k$ (see [110], [163]):

$$\psi_{m\ell}(z) = \frac{\sqrt{1 - r_m^2}}{1 - \overline{z_{m\ell}} z} \prod_{k=0}^{m-1} \prod_{j=0}^{2^{2k}-1} \frac{z - z_{kj}}{1 - \overline{z_{kj}} z} \prod_{j'=0}^{\ell-1} \frac{z - z_{mj'}}{1 - \overline{z_{mj'}} z} \tag{2.10}$$

$$(\ell = 0, 1, \cdots, 2^{2m} - 1,\ \ m = 0, 1, 2, \cdots, n).$$

In this way we have constructed an analytic rational orthonormal wavelet system on the resolution level V_n, i.e.,

$$\langle \psi_{m\ell}, \psi_{m'\ell'} \rangle = \frac{1}{2\pi} \int_0^{2\pi} \psi_{m\ell}(e^{it}) \overline{\psi_{m'\ell'}(e^{it})} dt = \delta_{mm'} \delta_{\ell\ell'}, \tag{2.11}$$

$$(\ell = 0, 1, \cdots, 2^{2k} - 1, \;\; k = 0, 1, 2, \cdots, \infty).$$

From the Gram-Schmidt orthogonalization process it follows that:

$$V_n = span\{\psi_{k\ell}, \;\ell = 0, 1, \cdots, 2^{2k} - 1, \; k = 0, \cdots, n\}.$$

Because the points of the set A satisfy the non-Blaschke condition

$$\sum_{k,\ell} (1 - |z_{k\ell}|) = \sum_k 2^{2k} \left(1 - \frac{2^k - 2^{-k}}{2^k + 2^{-k}} \right) = \sum_k \frac{2 \cdot 2^k}{2^k + 2^{-k}} = \infty, \tag{2.12}$$

the Malmquist–Takenaka system corresponding to the set A is a basis in $H^2(\mathbb{T})$, i.e.,

$$\overline{\bigcup_{n \in \mathbb{N}} V_n} = H^2(\mathbb{T})$$

in $H^2(\mathbb{T})$ norm, consequently the density property is also satisfied.

In signal processing and system identification the Malmquist–Takenaka system is more efficient then the trigonometric system in the determination of the transfer functions. This field has also a rich bibliography (see for example Akcay, Ninness [3], Akcay, Ninness, [4], Ninness, Gustafsson [115], Soumelidies, Bokor, Schipp [151, 153] etc.).

The wavelet space W_n is the orthogonal complement of V_n in V_{n+1}. In our case, as it was shown in [126], W_n is given as

$$W_n = span\{\psi_{n+1\ell}, \;\; \ell = 0, 1, \cdots, 2^{2n+2} - 1\}.$$

To prove this we will use the Cauchy integral formula: every function $f \in H^1$ can be recovered from its boundary function, i.e.,

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{it})}{e^{it} - z} e^{it} dt.$$

If $f \in V_n$, one has $f(z) = \sum_{k=0}^{n} \sum_{\ell=0}^{2^{2k}-1} c_{k\ell} \varphi_{k\ell}$, then

$$\langle \psi_{n+1j}, f \rangle = \sum_{k=1}^{n} \sum_{\ell=0}^{2^{2k}-1} c_{k\ell} \langle \psi_{n+1j}, \varphi_{k\ell} \rangle =$$

$$\sum_{k=1}^{n} \sum_{\ell=0}^{2^{2k}-1} c_{k\ell} \sqrt{1 - r_k^2} \psi_{n+1\ell}(z_{k\ell}) = 0, \; j = 0, 1, \cdots, 2^{2n+2} - 1.$$

Consequently

$$\langle f, \psi_{n+1j} \rangle = 0,$$

and

$$\psi_{n+1j} \perp V_n, \quad (j = 0, 1, \cdots, 2^{2n+2} - 1).$$

From

$$V_{n+1} = V_n \bigoplus span\{\varphi_{n+1,j}, \ j = 0, 1, \cdots, 2^{2n+2} - 1\}$$

it follows that W_n is an $2^{2(n+1)}$ dimensional space and

$$W_n = span\{\psi_{n+1\ell}, \quad \ell = 0, 1, \cdots, 2^{2n+2} - 1\}.$$

Consequently we have generated a multiresolution in $H^2(\mathbb{T})$ and we have constructed an analytic rational orthogonal wavelet system.

In what follows we show that the levels of the constructed multiresolution form a CMS for the disc algebra of the unit disc and we give some estimation for the convergence error.

The discrete lattice A has also certain near-density property, i.e. no point in the unit disc is too far in the pseudo-hyperbolic metric from a point of A. This leaves open the question if the resolution levels form a complete model set (CMS) for $A(\mathbb{D})$, which seems to be a harder question. In order to prove this we will show that A satisfies the Hayman-Lyons condition.

Let us consider the Whitney cube division of $\mathbb{D}$. For $n = 1, 2, \cdots$ and $k = 0, 1, \cdots, 2^n - 1$, we define

$$Q_{n,k} = \left\{ z : 1 - \frac{1}{2^n} \leqslant |z| \leqslant 1 - \frac{1}{2^{n+1}}, \ \frac{2k\pi}{2^n} \leqslant arg z \leqslant \frac{2(k+1)\pi}{2^n} \right\}.$$

If $A \in \mathbb{D}$ we set $A_{n,k} = A \cap Q_{n,k}$, $y_{n,k} = (1 - \frac{1}{2^n})e^{\frac{2\pi k}{2^n}}$, and we define

$$s(\theta) = s(\theta, A) = \sum_{A_{n,k} \neq \varnothing} \left(\frac{1 - |y_{n,k}|}{|y_{n,k} - e^{i\theta}|} \right)^2.$$

We say that A satisfies the Hayman-Lyons condition if and only if, $s(\theta) = +\infty$ for all $\theta \in [0, \ 2\pi]$.

Theorem 2.2.4 (**Pap** [129])**.** *The points of the lattice A satisfy the Hayman-Lyons condition, which implies that the corresponding localized Cauchy kernels form a fundamental set and $\{V_n\}_{n \in \mathbb{N}}$ form a CMS in the disc algebra $A(\mathbb{D})$.*

From the Hayman-Lyons condition and Theorem [2] of Ward, Partington [172] it follows that

$$\mathcal{W} = \left\{ \frac{1}{1 - \overline{z_{k\ell}} z}, \ z_{k\ell} \in A \right\}$$

is a fundamental set for $A(\mathbb{D})$. From this it follows that $\cup_{k=1}^{\infty} V_k$ is dense in $A(\mathbb{D})$, consequently the multiresolution levels form a CMS for the disc algebra. This means that if $f \in A(\mathbb{D})$ for arbitrary $\epsilon > 0$, there exists $\lambda_{k,\ell} \in \mathbb{C}$ and such that

$$\left\| f(z) - \sum_{k=1}^{N} \lambda_{k,\ell} \frac{1}{1 - \overline{z_{k,\ell}} z} \right\|_{\infty} < \epsilon, \tag{2.13}$$

which implies that the set $\mathcal{W}$ is fundamental in the disc algebra of the unit disc.

2.2.2 The properties of the projection operator corresponding to the n-th resolution level

Let us consider the orthogonal projection operator of an arbitrary function $f \in H^2(\mathbb{T})$ on the subspace V_n given by

$$P_n f(z) = \sum_{k=0}^{n} \sum_{\ell=0}^{2^{2k}-1} \langle f, \psi_{k,\ell} \rangle \psi_{k,\ell}(z), \tag{2.14}$$

called also the projection of f at scale or resolution level n.

In [126] Pap proved that the analytic continuation in the unit disc of the projection $P_n f$ on the n-th resolution level is at the same time an interpolation operator in the unit disc until the n-th level. This interpolation property is not true for the projections on the classical affine multiresolution levels.

Theorem 2.2.5 (**Pap** [126]). *For every $f \in H^2(\mathbb{T})$ the projection of f on V_n converges in norm to f, and $P_n f(z) \to f(z)$ uniformly on every compact subset of the unit disc. Moreover for every $f \in H^2(\mathbb{T})$ the projection operator $P_n f$ is an interpolation operator at the points $z_{mj} = r_m e^{i \frac{2\pi j}{2^{2m}}}$, $(j = 0, \cdots, 2^{2m} - 1, \quad m = 0, \cdots, n)$ for the analytic continuation of f in the unit disc.*

Proof. The non-Blaschke condition (2.12) implies that for every $f \in H^2(\mathbb{T})$ the projection of f on V_n converges in norm to f, i.e., we have

$$\|f - P_n f\|_{H^2(\mathbb{T})} \to 0, \quad n \to \infty.$$

Since convergence in $H^2(\mathbb{T})$ implies uniform convergence for the analytic continuation of f inside the unit disc on every compact subset, we conclude that $P_n f(z) \to f(z)$ uniformly on every compact subset of the unit disc.

In order to prove the interpolation property of $P_n f$ let us consider the kernel function of this projection operator given by

$$K_N(z, \xi) = \sum_{k=0}^{n} \sum_{\ell=0}^{2^{2k}-1} \overline{\psi_{k\ell}(\xi)} \psi_{k\ell}(z), \; N = \sum_{k=0}^{n} \sum_{\ell=0}^{2^{2k}-1} 1 = \frac{4^{n+1} - 1}{3}. \tag{2.15}$$

According to the Christoffel-Darboux formula the kernel function can be written in closed form

$$K_N(z,\xi) = (1-z\overline{\xi})^{-1}\left(1-\prod_{k=0}^{n}\prod_{\ell=0}^{2^{2k}-1}\frac{z-z_{k\ell}}{1-\overline{z_{k\ell}}z}\overline{\prod_{k=0}^{n}\prod_{\ell=0}^{2^{2k}-1}\frac{\xi-z_{k\ell}}{1-\overline{z_{k\ell}}\xi}}\right) = \tag{2.16}$$

$$= (1-z\overline{\xi})^{-1}\left(1-\prod_{k=0}^{n}\frac{z^{2^{2k}}-r_k^{2^{2k}}}{1-r_k^{2^{2k}}z^{2^{2k}}}\overline{\prod_{k=0}^{n}\frac{\xi^{2^{2k}}-r_k^{2^{2k}}}{1-r_k^{2^{2k}}\xi^{2^{2k}}}}\right).$$

From this relation it follows that the values of the kernel-function at the points $z_{mj},\ (j = 0, ..., 2^{2m}-1,\quad m = 0, ..., n)$ are equal to localized Cauchy kernels

$$K(z_{mj},\xi) = \frac{1}{1-z_{mj}\overline{\xi}}.$$

From this property and the Cauchy integral formula we get that the interpolation property holds, i.e.,

$$P_n f(z_{mj}) = \langle f, K_N(.,z_{mj})\rangle = \frac{1}{2\pi}\int_0^{2\pi}\frac{f(e^{it})}{1-z_{mj}e^{-it}}dt = f(z_{mj})$$

$$(j = 0,\cdots,2^{2m}-1,\quad m = 0,\cdots,n).$$

We are interested to know the behavior of P_n on the unit circle and the convergence in H^∞ norm. An estimation of the rate of the convergence would also be interesting. In what follows we will concentrate our attention to these questions.

Theorem 2.2.6 (**Pap** [129])**.** *If $f \in A(\mathbb{D})$ is a rational function of the form*

$$f(z) = \sum_{\ell=1}^{M}\frac{a_m}{1-\overline{\gamma_m}z},\quad \gamma_m = r_m e^{i\alpha_m} \in \mathbb{D}, \tag{2.17}$$

then $\|f - P_n f\|_{H^\infty} \to 0$.

Proof. For

$$f(z) = \sum_{\ell=1}^{M}\frac{a_m}{1-\overline{\gamma_m}z},\quad \gamma_m \in \mathbb{D},\quad \gamma_m = \alpha_m + i\beta_m \in \mathbb{D}$$

using the Cauchy formula we can compute the wavelet coefficients:

$$\langle f, \psi_{k\ell}\rangle = \sum_{m=1}^{M} a_m\overline{\psi_{k\ell}(\gamma_m)}. \tag{2.18}$$

Using the Christophel–Darboux formula for the Malmquist–Takenaka system $P_nf(z)$ can be written as

$$P_nf(z) = \sum_{k=0}^{n}\sum_{\ell=0}^{2^{2k}-1}\langle f,\psi_{k\ell}\rangle\psi_{k\ell}(z) = \sum_{k=0}^{n}\sum_{\ell=0}^{2^{2k}-1}\sum_{m=1}^{M} a_m\overline{\psi_{k\ell}(\gamma_m)}\psi_{k\ell}(z) =$$
$$= \sum_{m=1}^{M} a_m\sum_{k=0}^{n}\sum_{\ell=0}^{2^{2k}-1}\overline{\psi_{k\ell}(\gamma_m)}\psi_{k\ell}(z) = \sum_{\ell=1}^{M} a_m\frac{1-\overline{B_N(\gamma_m)}B_N(z)}{(1-\overline{\gamma_m}z)},$$

where

$$B_N(z) = \prod_{k=0}^{n}\prod_{\ell=0}^{2^{2k}-1}\frac{z-z_{k\ell}}{1-\overline{z_{k\ell}}z}, \quad N = \sum_{k=0}^{n}\sum_{\ell=0}^{2^{2k}-1}1 = \frac{4^{n+1}-1}{3}.$$

Now we are ready to estimate the error $|f(z)-P_nf(z)|$ for $|z|\leqslant 1$:

$$|f(z)-P_nf(z)| = \left|\sum_{m=1}^{M} a_m\frac{\overline{B_N(\gamma_m)}B_N(z)}{(1-\overline{\gamma_m}z)}\right| = \left|B_N(z)\sum_{m=1}^{M}\frac{a_m\overline{B_N(\gamma_m)}}{(1-\overline{\gamma_m}z)}\right| =$$
$$= |B_N(z)|\left|\sum_{m=1}^{M}\frac{a_m\overline{B_N(\gamma_m)}}{(1-\overline{\gamma_m}z)}\right| \leqslant \sum_{m=1}^{M}\frac{|a_m||\overline{B_N(\gamma_m)}|}{|1-\overline{\gamma_m}z|} \leqslant \sum_{\ell=1}^{M}\frac{|a_m||\overline{B_N(\gamma_m)}|}{1-r_m}.$$

Because the points of the set A form a non-Blaschke sequence we have $\lim_{N\to\infty}|B_N(\gamma_m)| = 0$, which implies that this last sum tends to zero if $N\to\infty$, consequently $\|f-P_nf\|_{H^\infty}\to 0$ on the closed unit disc. Using similar estimates as in Akcay, Ninness [4] for the error term we get

$$|B_N(\gamma_m)| = \left|\prod_{k=0}^{n}\prod_{\ell=0}^{2^{2k}-1}\frac{\gamma_m-z_{k\ell}}{1-\overline{z_{k\ell}}\gamma_m}\right| \leqslant \exp\left(-\frac{1}{2}(1-|\gamma_m|)\sum_{k,\ell}(1-|z_{k\ell}|)\right),$$

$$\sum_{k,\ell}(1-|z_{k\ell}|) = 2\sum_{k=0}^{n}\frac{2^k}{2^k+2^{-k}} \geqslant n+1.$$

From here we get that the error term has an exponential decay:

$$|f(z)-P_nf(z)| \leqslant \exp\left(-\frac{1}{2}\min_m(1-|\gamma_m|)(n+1)\right)\sum_{m=1}^{M}\frac{|a_m|}{1-r_m}.$$

For analytic functions on a disc $D(0,R)$ with radius $R>1$ and bounded magnitude $|f(z)|<K$, Akcay, Ninness [4] proved the following error estimation:

$$\|f-P_nf\|_\infty \leqslant \frac{KR}{R-1}\exp\left(\frac{R-1}{2R}\sum_{k,\ell}(1-|z_{k\ell}|)\right).$$

From here we get that

$$\|f-P_nf\|_\infty \leqslant \frac{KR}{R-1}\exp\left(\frac{R-1}{2R}(n+1)\right).$$

2.2.3 Reconstruction algorithm using the wavelet base

Due to a result of Walsh (see for example in Chui, Chen [36] pp. 93), for every $f \in H^2(\mathbb{D})$ there exists a unique $\hat{f}_n \in V_n$ such that

$$\|\hat{f}_n - f\| = \inf_{f_n \in V_n} \|f_n - f\|,$$

and $\hat{f}_n$ is uniquely determined by the interpolation conditions

$$\hat{f}_n(z_{mj}) = f(z_{mj}), \quad (j = 0, \cdots, 2^{2m} - 1, \quad m = 0, \cdots, n).$$

From Theorem 2.2.5 it follows that the best approximant is given by (2.14) i.e., $\hat{f}_n(z) = P_n f(z)$. Chui and Chen also proposed a computational scheme for the expression of the best approximant in the base $\{\varphi_{k\ell},\ \ell = 0, 1, \cdots, 2^{2k} - 1,\ k = 0, ..., n\}$.

In [126] Pap introduced a new computational scheme for the best approximant in the wavelet base $\{\psi_{k\ell},\ \ell = 0, 1, \cdots, 2^{2k} - 1,\ k = 0, \cdots, n\}$.

For the best approximant $P_n f$ the set of coefficients

$$\{b_{k\ell} = \langle f, \psi_{k\ell} \rangle,\ \ell = 0.1, \cdots, 2^{2k} - 1 \quad k = 0, 1, \cdots, n\}$$

is called the discrete hyperbolic wavelet transform of the function f. Thus it is important to have an efficient algorithm for the computation of the coefficients.

In [126] it has been shown that the coefficients of the projection operator $P_n f$ can be computed exactly if we know the values of the functions on $\bigcup_{k=0}^n A_k$. For this reason we express first the function $\psi_{k\ell}$ using the bases $\{\varphi_{k'\ell'}\ \ell' = 0, 1, \cdots, 2^{2k'} - 1,\ k' = 0, \cdots, k\}$, i.e., we write the partial fraction decomposition of $\psi_{k\ell}$:

$$\psi_{k\ell}(\xi) = \sum_{k'=0}^{k-1} \sum_{\ell'=0}^{2^{2k'}-1} c_{k'\ell'} \frac{1}{1 - \overline{z_{k'\ell'}}\xi} + \sum_{j=0}^{\ell} c_{kj} \frac{1}{1 - \overline{z_{kj}}\xi}.$$

Using the orthogonality of the functions $\{\psi_{k'\ell'}\ \ell' = 0, 1, \cdots, 2^{2k'} - 1,\ k' = 0, \cdots, k\}$ and the Cauchy formula we get that

$$\delta_{kn}\delta_{\ell m} = \langle \psi_{nm}, \psi_{k\ell} \rangle = \sum_{k'=0}^{k-1} \sum_{\ell'=0}^{2^{2k'}-1} \overline{c_{k'\ell'}} \psi_{nm}(z_{k'\ell'}) + \sum_{j=0}^{\ell} \overline{c_{kj}} \psi_{nm}(z_{kj}),$$

$$(m = 0, 1, \cdots, 2^{2n} - 1,\ n = 0, \cdots, k).$$

If we order these equalities so that we write first the relations for $n = k$ and $m = \ell, \ell - 1, \cdots, 0$ respectively, then for $n = k - 1$ and $m = 2^{2(k-1)} - 1,\ 2^{2(k-1)} - 2, \cdots, 0$, etc., this is equivalent to

$$\begin{pmatrix}1\\0\\0\\.\\.\\.\\0\end{pmatrix}=\begin{pmatrix}\psi_{k\ell}(z_{k\ell}) & 0 & 0 & \dots & 0\\ \psi_{k\ell-1}(z_{k\ell}) & \psi_{k\ell-1}(z_{k\ell-1}) & 0 & \dots & 0\\ \psi_{k\ell-2}(z_{k\ell}) & \psi_{k\ell-2}(z_{k\ell-1}) & 0 & \dots & 0\\ \vdots & & & \vdots & \\ \psi_{00}(z_{k\ell}) & \psi_{00}(z_{k\ell-1}) & \psi_{00}(z_{k\ell-2}) & \dots & \psi_{00}(z_{00})\end{pmatrix}\begin{pmatrix}\overline{c_{k\ell}}\\ \overline{c_{k\ell-1}}\\ \overline{c_{k\ell-2}}\\ \vdots\\ \overline{c_{00}}\end{pmatrix}.$$

Because of the elements from the main diagonal are different from zero, this system has a unique solution $(\overline{c_{k\ell}},\overline{c_{k\ell-1}},\overline{c_{k\ell-2}},...,\overline{c_{00}})^T$. If we determine this vector, then we can compute the exact value of $\langle f,\psi_{k,\ell}\rangle$ knowing the values of f on the set $\bigcup_{k=0}^n A_k$.

Indeed, using again the partial fraction decomposition of $\psi_{k\ell}$ and the Cauchy integral formula we get that

$$\langle f,\psi_{k\ell}\rangle=\sum_{k'=0}^{k-1}\sum_{\ell'=0}^{2^{2k'}-1}\overline{c_{k'\ell'}}\langle f(\xi),\frac{1}{1-\overline{z_{k'\ell'}}\xi}\rangle+\sum_{j=0}^{\ell}\overline{c_{kj}}\langle f(\xi),\frac{1}{1-\overline{z_{kj}}\xi}\rangle=$$

$$=\sum_{k'=0}^{k-1}\sum_{\ell'=0}^{2^{2k'}-1}\overline{c_{k'\ell'}}f(z_{k'\ell'})+\sum_{j=0}^{\ell}\overline{c_{kj}}f(z_{kj}).$$

The question of recovery of analytic functions from values measured in the open unit disc was also studied by Totik, see [164], where it has been proved that in H^p or in the disc algebra if we can measure the values of a function f on a non Blaschke sequence, say $(z_k)_{k\in\mathbb{N}}$, then there are polynomials $p_{n,j}$ such that $\sum_{j=1}^n f(z_j)p_{n,j}$ tends to f in norm. From practical point of view the only difficulty is that we can not determine exactly the coefficients of these polynomials from the values of the measurements$f(z_k)_{k\in\mathbb{N}}$.

Fridli, Gilián and Schipp in [77] introduced the planar version of the Malmquist-Takenaka system, i.e., when the Hardy space of the unit circle is replaced by the Bergman space of the unit disc, in order to develop an effective method for approximating surfaces. They generalized the reconstruction method presented before for the planar MT system. The coefficients of the projection operator with respect to the planar Malmquist-Takenaka system can be computed also using the values of the function on inverse poles and using them it can be written the projection operator exactly.

In [31] Cerejeiras, Chen, Gomes and Hartmann, used a compressed sensing approach to the reconstruction of a given signal in terms of Takenaka-Malmquist systems. They presented some numerical experiments using TakenakaMalmquist systems in the study of transfer functions in systems identification. For the numerical calculations they used the Matlab toolbox -1-Magic which adopts a Linear Programming to minimize the -norm of our coefficients x subject t o$y=Ax$ using the primal-dual interior point method with A

being our sampling matrix. They need to choose our points a_i for the Blaschke products. For this they choose the grid (2.6) introduced by Pap in [126]. From this grid they took N randomly chosen points, i.e., a vector $a = (a_1, ..., a_N)$. They made the simulation using Matlab 8.5.0(R2015a). They still could get a decent approximation with a dramatically smaller running time. From this example they redraw the following observations:

1. Within the same number of measurements, when $|z_0|$ is near to zero we have the best reconstruction in the least time.

2. When the modulus of the parameter z_0 is close to 1 it requires more samples to reconstruct the signal.

3. The reconstruction is better in case when $|a_j - r| < \epsilon$ with ϵ relatively small and the parameter a_j being randomly chosen.

Taking into account these two last examples they formulated the following observations:

1. Using the same number of measurements their method provides a better approximation than the approach in the thesis of Shuang [147];

2. Moreover, the same relative error is attained with their method by using a smaller number of sampling points.

Recently Abdollahi and Rahimi using the affine group constructed another example of adapted multiresolution and orthonormal wavelet on $H^2(D)$ (see [1]).

In [32] Cerejeiras , Kähler, Legatiuk published results on interpolation of monogenic functions in the higher dimensional unit ball of $\mathbb{R}^{d+1}$ using reproducing kernels and randomly chosen interpolation points. The main theoretical results are proved based on the concept of uniformly discrete sequences. In addition to the classical difficulties of hypercomplex interpolation, the problem of the choice of the nodes is an additional obstacle in practical applications of monogenic interpolation. They point out that for example, in the classic case of the unit disc in $\mathbb{C}$, there exist several ways of choosing uniform interpolation points, since it is easy to create a uniform grid on the unit circle see. One example is exactly the greed (2.6) introduced in [126]. This is not any more true for the sphere in higher dimensions.

2.2.4 Discrete orthogonality of the hyperbolic wavelet basis

Another new property of the hyperbolic wavelet basis, which is not characteristic to the affine wavelets, is the discrete orthogonality. In [126] it has been proved also that it is possible to construct wavelets using the reproducing kernels of the multiresolution levels, similar as Bultheel, González-Vera in [25]. In this subsection we will give an overview of these results.

The reproducing kernel $K : \mathbb{T} \times \mathbb{T} \to \mathbb{C}$ of a subspace $V \subset \mathbb{H}^2(\mathbb{T})$ is defined by its reproducing property, i.e.,

$$\forall\, f \in V \quad f(w) = \langle f, K(., w) \rangle, \quad w \in \mathbb{T}.$$

Let us consider the special subspace $V_n = \text{span}\{\psi_{k,\ell},\ \ell = 0, 1, \cdots, 2^{2k}-1,\ k = 0, 1, \cdots, n\ \}$ (with dimension $N = \sum_{k=0}^{n}\sum_{\ell=0}^{2^{2k}-1} 1 = \frac{4^{n+1}-1}{3}$). We recall that if an orthonormal bases is considered in V_n, then the reproducing kernel or Dirichlet kernel of the system is given by

$$K_N(\xi, w) := \sum_{k=0}^{n}\sum_{\ell=0}^{2^{2k}-1} \overline{\psi_{k,\ell}(w)}\psi_{k,\ell}(\xi)$$

and it is independent of the choice of the orthonormal system. For a fixed w such reproducing kernels are known to be localized in the neighborhood of $\xi = w$.

The orthogonal projection operator onto V_n, defined by (2.14) can be expressed with the reproducing kernel as follows:

$$P_n f(w) = \langle f, K_N(., w)\rangle \quad f \in H^2(\mathbb{T}).$$

For a set of distinct points $\mathbf{w_N} = \{w_1, w_2, ..., w_N\}$ on $\mathbb{T}$ among the points of analyticity of K_N one has

$$\langle K_N(., w_i), K_N(., w_j)\rangle = K_N(w_j, w_i).$$

We remind that the reproducing kernel of the multiresolution level V_n can be written in closed form for $z \neq \xi$, $z, \xi \in \mathbb{T}$:

$$K_N(z, \xi) = (1 - z\overline{\xi})^{-1}\left(1 - \prod_{k=1}^{n}\prod_{\ell=0}^{2^{2k}-1}\frac{z - z_{k\ell}}{1 - \overline{z_{k\ell}}z}\overline{\prod_{k-1}^{n}\prod_{\ell=0}^{2^{2k}-1}\frac{\xi - z_{k\ell}}{1 - \overline{z_{k\ell}}\xi}}\right) =$$

$$= (1 - z\overline{\xi})^{-1}\left(1 - \prod_{k=1}^{n}\frac{z^{2^{2k}} - r_k^{2^{2k}}}{1 - r_k^{2^{2k}}z^{2^{2k}}}\overline{\prod_{k-1}^{n}\frac{\xi^{2^{2k}} - r_k^{2^{2k}}}{1 - r_k^{2^{2k}}\xi^{2^{2k}}}}\right).$$

From the definition of K_N for $z = \xi = e^{it}$ we get that:

$$K_N(e^{it}, e^{it}) = \sum_{k-1}^{n}\sum_{\ell=0}^{2^{2k}-1}\overline{\psi_{k\ell}(e^{it})}\psi_{k\ell}(e^{it}) = \sum_{k-1}^{n}\sum_{\ell=0}^{2^{2k}-1}\frac{1 - r_k^2}{|1 - \overline{z_{k\ell}}e^{it}|^2}$$

$$= \sum_{k-1}^{n}\sum_{\ell=0}^{2^{2k}-1}\beta'_{k\ell}(t).$$

The finite Blaschke product which appears in the expression of the kernel function for $z = e^{it}$ can be expressed also with the beta functions as follows:

$$\prod_{k=1}^{n}\prod_{\ell=0}^{2^{2k}-1}\frac{z - z_{k\ell}}{1 - \overline{z_{k\ell}}z} = e^{iN\beta_{(N)}(t)},$$

where

$$\beta_{(N)}(t); = \frac{1}{N}\sum_{k=1}^{n}\sum_{\ell=0}^{2^{2k}-1}\beta_{k\ell}(t),$$

$$\beta_{k\ell}(t) := \beta_{(z_{k\ell},1)} = \frac{2\pi\ell}{2^{2k}} + 2\text{arctan}2^{2k}\tan\frac{(t-\frac{2\pi\ell}{2^{2k}})}{2}.$$

The function $\beta_{(N)}(t)$ is a monotonically increasing, invertible and differentiable function mapping of $\mathbb{R}$ onto itself. Using the fact that

$$\prod_{k=1}^{n}\prod_{\ell=0}^{2^{2k}-1}\frac{z-z_{k\ell}}{1-\overline{z_{k\ell}}z} = \prod_{k=1}^{n}\frac{z^{2^{2k}}-r_k^{2^{2k}}}{1-r_k^{2^{2k}}z^{2^{2k}}},$$

we obtain that the expression of the $\beta_{(N)}(t)$ can be expressed by a single sum as follows:

$$\beta_{(N)}(t); = \frac{1}{N}\sum_{k=1}^{n}\sum_{\ell=0}^{2^{2k}-1}\beta_{k\ell}(t) = \frac{1}{N}\sum_{k=1}^{n}2\text{arctan}\frac{1-r_k^{2^{2k}}}{1+r_k^{2^{2k}}}\tan 2^{2k-1}t.$$

Consequently, we have:

$$K_N(e^{it},e^{i\theta}) = \frac{\sin N\frac{\beta_N(t)-\beta_N(\theta)}{2}}{\sin\frac{t-\theta}{2}}e^{i\frac{N(\beta_N(t)-\beta_N(\theta))-(t-\theta)}{2}}.$$

Let us denote the set of equidistant nodes on the unit circle, i.e., the N^{th} roots of the unity by

$$\mathbf{U_N} = \{e^{i\nu_j} : j = 0,\cdots,N-1\}.$$

and let

$$\mathbf{W_N} := \{w_j = e^{i\gamma_j} : \gamma_j = \beta_N^{-1}(\nu_j),\ j = 0,\cdots,N-1\}.$$

Let us define the discrete scalar product

$$[f,g]_N := \sum_{\xi\in\mathbf{W_N}}\frac{f(\xi)\overline{g(\xi)}}{K_N(\xi,\xi)} = \sum_{\xi\in\mathbf{W_N}}\frac{f(\xi)\overline{g(\xi)}}{N\beta'_{(N)}(\gamma)}$$

where $\xi = e^{i\gamma}$.

It can be shown that the finite collection of the orthonormal functions $\{\psi_{k\ell},\ \ell = 0,1,\cdots,2^{2k}-1,\ k = 0,1,\cdots,n\ \}$ will be discrete orthogonal regarding the discrete scalar product. This is a special case of the result obtained by Pap, Schipp in [118].

Theorem 2.2.7 (**Pap** [126])**.** *The finite collection of analytic wavelets* $\{\psi_{k\ell},\ \ell = 0,1,\cdots,2^{2k}-1,\ k = 0,1,\cdots,n\ \}$ *forms a discrete orthonormal system with respect to the scalar product* $[.,.]_N$*, namely*

$$[\psi_{k\ell},\psi_{k'\ell'}]_N = \delta_{kk'}\delta_{\ell\ell'}.$$

It is true that for any set of distinct points $\mathbf{w_N} = \{w_1, w_2, \cdots, w_N\}$ on $\mathbb{T}$ the system

$$\{K_N(w_i, \xi),\ i = 1, \cdots, N\}$$

forms a basis for V_n.

The question is whether it is possible to choose the points of $\mathbf{w_N}$ such that this basis is orthogonal on T. In that case we would have a bases of orthogonal rational kernels at every level of the multiresolution.

Theorem 2.2.8 (**Pap** [126])**.** *The set*

$$\left\{\frac{K_N(\xi, w_i)}{\sqrt{K_N(w_i, w_i)}},\ w_i \in \mathbf{W_N}, i = 0, 1, \cdots, N-1\right\}$$

forms an orthonormal and a discrete orthonormal basis regarding to the discrete scalar product $[.,.]_N$ *for* V_n*.*

It is possible also to give a reproducing kernel basis for the 2^{2n+2} dimensional wavelet space W_n analogue. Obviously the reproducing kernel for W_n is

$$k_n(z, w) = K_{N+1}(z, w) - K_N(z, w) = \sum_{\ell=0}^{2^{2n+2}-1} \psi_{n+1\ell}(z)\overline{\psi_{n+1\ell}(w)}.$$

One interesting question is the following: Can we find 2^{2n+2} numbers on the unit circle such that the functions $\{k_n(z, w_{nj}) : j = 0, 1, 2, \cdots, 2^{2n+2} - 1\}$ form an orthogonal basis for W_n? The following theorem is providing a positive answer.

Let us denote by

$$\mathbf{w_n} = \{w_{n+1j},\ j = 0, 1, 2, \cdots, 2^{2n+2} - 1\}$$

the roots of order 2^{2n+2} of the unity.

Let us consider the discrete scalar product defined over $\mathbf{w_n}$ defined by

$$[f, g]_n := \sum_{\xi \in \mathbf{w_n}} \frac{f(\xi)\overline{g(\xi)}}{k_n(\xi, \xi)}. \tag{2.19}$$

Theorem 2.2.9 (**Pap** [126])**.** *The set*

$$\left\{\frac{k_n(w_{n+1j}, \xi)}{\sqrt{k_n(w_{n+1j}, w_{n+1j})}},\ w_{n+1j} \in \mathbf{w_n},\ i = 0, 1, \cdots, 2^{2n+2} - 1\right\}$$

forms an orthogonal basis and a discrete orthonormal basis regarding to the discrete scalar product $[.,.]_n$ *for* W_n*, i.e.,*

$$\langle k_n(z, w_{n+1j}), k_n(z, w_{n+1j'})\rangle = 0, \quad j \neq j',\ j, j' = 0, 1, 2, \cdots, 2^{2n+2} - 1.$$

$$\left[\frac{k_n(w_{n+1i}, \xi)}{\sqrt{k_n(w_{n+1i}, w_{n+1i})}}, \frac{k_n(w_{n+1j}, \xi)}{\sqrt{k_n(w_{n+1j}, w_{n+1j})}}\right]_n = \delta_{ij}, \quad i, j = 0, 1, \cdots, 2^{2n+2} - 1.$$

2.3 Multiresolution in the Hardy space of the upper half plane

2.3.1 Transition to the upper half plane, motivation

Let us denote by $\mathbb{C}_+ = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ the upper half plane, and let us consider $\mathcal{A}(\mathbb{C}_+)$ the set of analytic functions on $\mathbb{C}_+$. The *Hardy space of the upper half plane* is defined by

$$\mathrm{H}^p(\mathbb{C}_+) = \left\{ h \in \mathcal{A}(\mathbb{C}_+) : \sup \left\{ \int_{\mathbb{R}} |h(x+iy)|^p \, dx : y > 0 \right\} < \infty \right\}.$$

If $f \in \mathrm{H}^2(\mathbb{C}_+)$, then its non tangential boundary limit function exists almost everywhere and

$$f \in H^2(\mathbb{R}) = \{ f \in L^2(\mathbb{R}),\ sup\hat{f} \subset [0, +\infty) \}.$$

For more detailed description of the Hardy spaces see for example Cima, Ross [37], Mashregi [111].

As we have referred in the introduction, Meyer between others, formulated the following question: Is it any "regular" wavelet orthonormal bases of the form

$$\psi_0(x) = \psi(x),\ \psi_{n,k}(x) := 2^{n/2}\psi(2^n x - k)$$

and multiresolution generated by this bases in $H^2(\mathbb{R})$. Auscher in 1995 published results connected to this question in [11]. The word regular includes smoothness, localization, and cancellation of ψ, see the exact conditions in [11]. He showed the nonexistence of a regular wavelet that generates a wavelet basis in space $H^2(\mathbb{R})$, i.e., in this space applying dilation and translation to a single function, or discretizing the continuous affine wavelet transform, leads to negative answer if we impose some "regularity" conditions.

As we will see later we will approach the construction of multiresolution in $H^2(\mathbb{R})$ by taking the analytic extension of this functions to the Hardy space of the upper half plane $\mathrm{H}^2(\mathbb{C}_+)$, because if $f \in \mathrm{H}^2(\mathbb{C}_+)$ then its non tangential boundary limit function exists almost everywhere and the limit function f satisfies $f \in H^2(\mathbb{R})$.

Making the transition to the upper half plane of the results presented in the previous section we show that it is possible to generate multiresolution type decomposition in Hardy spaces of the upper half plane too. These results were published by in [67]. In this section we present these results: how it can be construct a rational analytic orthogonal wavelet system in the Hardy space of the upperhalf plane which generates an adapted multiresolution. All the advantages enumerated in the previous section are valid in this case too. Measuring the values of the function f at the points of the set $B = \bigcup_{k=0}^{n} B_k \subset \mathbb{C}_+$ we can write exactly the projection operator on the n-th resolution level ($P_n' f$, $n \in \mathbb{N}$). This is convergent to f in $H^2(\mathbb{C}_+)$ norm, is the best approximant interpolation operator on the set the $\bigcup_{k=0}^{n} B_k$ and $P_n' f(z) \to f(z)$ uniformly on every compact subset of the upper half plane.

The restriction to the real line of the introduced hyperbolic analytic wavelet basis given explicitly by

$$\{\Psi_{k\ell},\ \ell = 0, 1, \cdots, 2^{2k} - 1,\ k = 0, \cdots, n, \cdots\}$$

generates an adapted multiresolution in $H^2(\mathbb{R})$.

As long as the theory of wavelet constructions on the Hardy space of the unit disc presented in [126] are suitable for time frequency-domain description of discrete-time-invariant dynamical systems, the adaptation to the upper half plane can be used in system theory to describe the spectral behavior of continuous-time-invariant systems. It has been also studied the approximation and identification of transfer functions of a continuous-time-invariant systems.

In the case of the Hardy space of the unit disc where the polynomials are dense, however dense subsets in the Hardy space of the upper half plane are harder to find. Applying the Daubechies theory it can be shown that choosing as mother wavelet $\psi(y) = (1 + iy)^{-p}$ for $p \geqslant 2$ we can generate a frame for the Hardy space of the upper half plane. For $p = 3$ Ward, Partington in [171] described a rational wavelet decomposition of the Hardy–Sobolev class of the half plane. The case $p = 1$, the Cauchy kernel case, dos not fall under the Daubechies theory since does not have vanishing mean value, but Ward and Partington have shown that the system $\psi_{j,k} = 2^{j/2}\psi(2^j y - b_0 k),\ j, k \in \mathbb{Z}$ does constitute a fundamental set for the upper half plan algebra. The multiresolution introduced by Feichtinger, Pap in [67] uses localized Cauchy kernels for Hardy space of the upper half plane and uses complex techniques in the proofs.

The unit disk $\mathbb{D}$ and the upper half-plane $\mathbb{C}_+$ can be mapped to one-another by means of Möbius transformations, i.e., by the Cayley transform, which maps $\mathbb{C}_+$ to $\mathbb{D}$ and is defined by

$$C(\omega) = \frac{i - \omega}{i + \omega},\ \omega \in \mathbb{C}_+. \tag{2.20}$$

The correspondence between the boundaries is given by

$$e^{is} = C(t) = \frac{i - t}{i + t},\ t \in \mathbb{R},\ s \in (-\pi, \pi),$$

which implies that $s = 2\arctan(t),\ t \in \mathbb{R}$.

With the Cayley transform, the linear transformation from $\mathrm{H}^2(\mathbb{D})$ to $\mathrm{H}^2(\mathbb{C}_+)$ is defined for $f \in \mathrm{H}^2(\mathbb{D})$ by

$$Tf := \frac{1}{\sqrt{\pi}} \frac{1}{\omega + i}(f \circ C) \tag{2.21}$$

and is an isomorphism between these spaces. Consequently the theory of the real line is a close analogy with what we have for the circle.

In the case of the unit disc a main tool in the proofs was the Cauchy formula for the unit disc. In the case of the upper half plane the analogue is the Cauchy formula for the upper half plane, which is the following: for any function $F \in \mathbb{H}^p(\mathbb{C}^+)$, $1 \leqslant p < +\infty$, if $F(s)$ is its non-tangential boundary limit, then

$$F(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{F(s)}{s-z}ds, \; z \in \mathbb{C}_+. \tag{2.22}$$

2.3.2 A special discrete subset in the upper half plane

In the case of the Hardy space of the unit circle the analogue of the dilation by 2 was the action of the representation of the Blaschke group U corresponding to the discrete subgroup defined by

$$\mathbb{B}_1 = \left\{(r_k, 1):\; r_k = \frac{2^k - 2^{-k}}{2^k + 2^{-k}},\; k \in \mathbb{Z}\right\},$$

and the multiresolution was constructed using the localized Cauchy kernels corresponding to the set

$$A = \{z_{k\ell} = r_k e^{i\frac{2\pi\ell}{2^{2k}}}, \quad \ell = 0, 1, \cdots, 2^{2k} - 1, \;\; k = 0, 1, 2, \cdots, \infty\}$$

and the k-th resolution levels, $k \in \mathbb{N}$, were associated to

$$A_k = \{z_{k\ell} = r_k e^{i\frac{2\pi\ell}{2^{2k}}},\; \ell \in \{0, 1, \cdots, 2^{2k} - 1\}\}.$$

The inverse Cayley transform $C^{-1}(z) = i\frac{1-z}{1+z}$ takes the unit circle in the real axis and the unit disc in the upper half plane. Let us consider the image of the set A trough the inverse Cayley transform, in this way we obtain the following points of the upper half plane:

$$a_{k\ell} = C^{-1}(z_{k\ell}) = \frac{2r_k \sin\frac{2\pi\ell}{2^{2k}}}{1 - 2r_k\cos\frac{2\pi\ell}{2^{2k}} + r_k^2} + i\frac{1 - r_k^2}{1 - 2r_k\cos\frac{2\pi\ell}{2^{2k}} + r_k^2} = \alpha_{k\ell} + i\beta_{k\ell}, \tag{2.23}$$

$$B_k = \{a_{k\ell},\; \ell \in \{0, 1, \cdots, 2^{2k} - 1\}\}, \tag{2.24}$$

$$B = \{a_{k\ell}, \quad \ell = 0, 1, \cdots, 2^{2k} - 1, \;\; k = 0, 1, 2, \cdots, \infty\}. \tag{2.25}$$

The points from B are in the upper half plane, and every point from B_k is on the circle with center $(0, \frac{1+r_k^2}{1-r_k^2})$ and radius $R_k = \frac{2r_k}{1-r_k^2}$. It is easy to show that the points from B do not satisfy the Blaschke condition for the upper half plane. Indeed,

$$\sum_{k=0}^{\infty}\sum_{\ell=0}^{2^{2k}-1} \frac{\beta_{k\ell}}{1 + |a_{k\ell}|^2} = \sum_{k=0}^{\infty}\sum_{\ell=0}^{2^{2k}-1} \frac{1 - r_k^2}{2(1 + r_k^2)} = \sum_{k=0}^{\infty} \frac{2^{2k}}{2^{2k} + 2^{-2k}} = \infty. \tag{2.26}$$

2.3.3 Multiresolution in the Hardy space of the the upper half plane

Using the lattice B we introduce an adapted multiresolution in the space $H^2(\mathbb{C}_+)$.

Definition 2. A sequence $\{V_j',\ j \in \mathbb{N}\}$ of subspaces of $H^2(\mathbb{C}_+)$ is called a multiresolution in $H^2(\mathbb{C}_+)$ if the following conditions hold:

1. (nested) $V_j' \subset V_{j+1}'$,
2. (density) $\overline{\cup V_j'} = H^2(\mathbb{C}_+)$,
3. (analogue of dilatation) $(TU_{(r_1,1)^{-1}}T^{-1})V_n' \subset V_{n+1}'$,
4. (basis) There exist $\Psi_{n,\ell}$ (orthonormal) bases in V_n'.

In order to construct a multiresolution in $H^2(\mathbb{C}_+)$ let us consider the function $\phi = \frac{1}{\sqrt{\pi}(z+i)}$ and let consider $V_0' = \{c\phi,\ c \in \mathbb{C}\}$. Let us consider the nonorthogonal hyperbolic wavelets at the n-th level, the localized Cauchy kernels for the upper half plane corresponding to the set $\cup_{k=1}^n B_k$:

$$\phi_{k\ell}(z) = \sqrt{\frac{\beta_{k\ell}}{\pi}}\frac{1}{z-\overline{a_{k\ell}}} \quad k = 0, \cdots, n, \quad \ell = 0, 1, \cdots, 2^{2k}-1,$$

and let us define the n-th resolution level by

$$V_n' = \{f : \mathbb{D} \to \mathbb{C},\ f(z) = \sum_{k=0}^{n}\sum_{\ell=0}^{2^{2k}-1} c_{k\ell}\phi_{k\ell},\ c_{k\ell} \in \mathbb{C}\ \}.$$

The closed subset V_n' is spanned by

$$\{\phi_{k\ell},\ \ell = 0, 1, \cdots ., 2^{2k}-1,\ k = 0, \cdots, n\}.$$

In this way we have obtained a sequence of closed, nested subspaces of $H^2(\mathbb{C}_+)$ for $z \in \mathbb{C}_+$,

$$V_0' \subset V_1' \subset V_2' \subset \cdots V_n' \subset \cdots\ H^2(\mathbb{C}_+).$$

The elements of B are different complex numbers, consequently the corresponding finite subset of localized Cauchy kernels

$$\left\{\frac{1}{z-\overline{a_{k\ell}}},\ \ell = 0, 1, \cdots 2^{2k}-1,\ k = 0, 1, \cdots, n.\right\}$$

are linearly independent and they form a nonorthogonal basis in V_n'. Applying the Gram-Schmidt orthogonalization for this set of analytic linearly independent functions we obtain the Malmquist -Takenaka system corresponding to upper half plane and the set $\cup_{k=0}^n B_k$:

$$\Psi_{m\ell}(z) = \sqrt{\frac{\beta_{m\ell}}{\pi}}\frac{1}{z-\overline{a_{m\ell}}}\prod_{k=0}^{m-1}\prod_{j=0}^{2^{2k}-1}\frac{z-a_{kj}}{z-\overline{a_{kj}}}\prod_{j'=0}^{\ell-1}\frac{z-a_{mj'}}{z-\overline{a_{mj'}}},$$

$$(m = 0, 1, \cdots, n,\ \ell = 0, 1, \cdots, 2^{2m} - 1).$$

From the Gram-Schmidt orthogonalization process it follows that

$$V_n' = span\{\Psi_{k\ell},\ \ell = 0, 1, \cdots, 2^{2k} - 1,\ k = 0, \cdots, n\}.$$

From (2.26) it follows that the Malmquist–Takenaka system corresponding to the set B is a complete orthonormal system in $H^2(\mathbb{C}_+)$.

From the completeness of the system $\{\Psi_{k\ell},\ \ell = 0, 1, \cdots, 2^{2k} - 1,\ \ k = \overline{0, \infty}\}$ in the Hilbert space $H^2(\mathbb{C}_+)$, it follows that this system is also a closed system, consequently the density property it is valid in norm, i.e.,

$$\overline{\bigcup_{n \in \mathbb{N}} V_n'} = H^2(\mathbb{C}_+).$$

From the previous section we have seen that the multiresolution in the Hardy space of the unit disc is defined by a single function $\varphi = 1$ and the analogue of the dilatation and translation as follows

$$V_n = span\{\varphi_{k\ell},\ \ell = 0, 1, \cdots, 2^{2k} - 1,\ k = 0, \cdots, n\},$$

where

$$\varphi_{n\ell}(z) = \frac{\sqrt{(1 - r_n^2)}}{(1 - \overline{z_{n\ell}} z)} = (U_{(z_{n\ell}, 1)^{-1}} \varphi)(z) = (U_{(r_n, 1)^{-1}} \varphi)(e^{i(t - \frac{2\pi\ell}{2^{2n}})}), \quad \ell = 0, 1, \cdots, 2^{2n} - 1.$$

We observe that taking the image of $\varphi_{n,\ell}$ through the Cayley function

$$T(\varphi_{n,\ell})(\omega) = \frac{1}{\sqrt{\pi}(i + \omega)} \frac{\sqrt{1 - r_n^2}}{1 - \overline{z_{n,\ell}} \frac{i - \omega}{i + \omega}} = \sqrt{\frac{\beta_{n,\ell}}{\pi}} \frac{i\overline{(i + a_{n,\ell})}}{\sqrt{2}|i + a_{n,\ell}|} \frac{1}{\omega - \overline{a_{n,\ell}}} = B_{k,\ell} \phi_{n,\ell}(\omega),$$

where $B_{k\ell} = \frac{i\overline{(i + a_{n,\ell})}}{\sqrt{2}|i + a_{n,\ell}|}$ is a constant. From this we get that $V_n' = T(V_n)$. We have seen that if a function $f \in V_n$, then $U_{(r_1,1)^{-1}} f \in V_{n+1}$, because

$$U_{(r_1,1)^{-1}}(\varphi_{k,\ell})(e^{it}) = U_{(r_1,1)^{-1}}[(U_{(r_k,1)^{-1}}\varphi)](e^{i(t - \frac{2\pi\ell}{2^{2k}})}) =$$

$$[(U_{(r_{k+1},1)^{-1}}\varphi)](e^{i(t - \frac{2\pi 4\ell}{2^{2(k+1)}})}) \in V_{n+1}, \quad k = 1, \cdots, n,\ \ell = 1, \cdots, 2^{2k} - 1.$$

Consequently we have that

$$T U_{(r_1,1)^{-1}} T^{-1} V_n' \subset V_{n+1}'.$$

The wavelet space W_n' is the orthogonal complement of V_n' in V_{n+1}'. Analogously as in the previous section it can be proved that

$$W_n' = span\{\Psi_{n+1\ell},\ \ \ell = 0, 1, \cdots, 2^{2n+2} - 1\}.$$

For an arbitrary $f(z) = \sum_{k=0}^{n} \sum_{\ell=0}^{2^{2k}-1} c_{k,\ell}\phi_{k\ell} \in V'_n$ using the Cauchy formula for the upper half plane we obtain that

$$\langle \Psi_{n+1j}, f \rangle = \sum_{k=1}^{n} \sum_{\ell=0}^{2^{2k}-1} c_{k,\ell} \langle \Psi_{n+1j}, \phi_{k,\ell} \rangle =$$

$$\sum_{k=1}^{n} \sum_{\ell=0}^{2^{2k}-1} c_{k,\ell} \sqrt{\frac{\beta_{k\ell}}{\pi}} 2\pi i \Psi_{n+1\ell}(z_{k\ell}) = 0, \;\; j = 0, 1, \cdots, 2^{2n+2} - 1.$$

Consequently,

$$\langle f, \Psi_{n+1,j} \rangle = 0, \quad f \in V'_n$$

which implies that

$$\Psi_{n+1,j} \perp V'_n, \quad (j = 0, 1, \cdots, 2^{2n+2} - 1).$$

From

$$V'_{n+1} = V'_n \bigoplus span\{\phi_{n+1j}, \;\; j = 0, 1, \cdots, 2^{2n+2} - 1\}$$

it follows that W'_n is an $2^{2(n+1)}$ dimensional space and

$$W'_n = span\{\Psi_{n+1\ell}, \quad \ell = 0, 1, \cdots, 2^{2n+2} - 1\}.$$

2.3.4 The projection operator corresponding to the n-th resolution level

Let us consider the orthogonal projection operator of an arbitrary function $f \in H^2(\mathbb{C}_+)$ on the subspace V'_n given by

$$P'_n f(z) = \sum_{k=0}^{n} \sum_{\ell=0}^{2^{2k}-1} \langle f, \Psi_{k\ell} \rangle \Psi_{k\ell}(z).$$

This operator is called the projection of f at resolution level n.

Theorem 2.3.1 (**Feichtinger, Pap** [67])**.** *For $f \in H^2(\mathbb{C}_+)$ the projection operator $P'_n f$ is an interpolation operator at the points*

$$a_{mj} \; (j = 0, \cdots, 2^{2m} - 1, \quad m = 0, \cdots, n).$$

In [67] it has been shown that the projection $P'_n f$ is also the solution of a minimal norm interpolation problem and

$$\|f - P'_n f\|_{H^2(\mathbb{C}_+)} \to 0, \quad n \to \infty.$$

The computation of the wavelet coefficients in the wavelet basis $\{\Psi_{k\ell},\ \ell = 0, 1, \cdots, 2^{2k}-1,\ k = 0, \cdots, n\}$ and of the best approximant $P'_n f$ can be made similarly as in the case of the unit disc presented in the previous section (see Pap, Feichtinger [67]). Based on the the results of Eisner, Pap obtained in [59] it can be proved the discrete orthogonality of the obtained wavelet system (see [60]). It has been proved also that it is possible to construct wavelets using the reproducing kernels of the multiresolution levels.

In [38] Coifman and Peyriére in the same spirit as the studies of hyperbolic wavelets presented here considered orthogonal decompositions of invariant subspaces of Hardy spaces, these relate to the Blaschke based phase unwinding decompositions. They proved convergence in L^p. In particular they build an explicit multiscale wavelet basis. They also discuss the relation to various generalizations of the Takenaka–Malmquist bases, both for the torus and the upper half plane. In particular they show that there is a multiscale analysis of $H^2(R)$, and that, at each level, there is a function whose translates make an orthonormal basis. The main difference is that they use different grids, which allows to get a formalism very close to wavelets.

Soumelidis in [149] developed a new idea to find the poles of a linear dynamical system without using further assumptions on system structure. He used the hyperbolic wavelet transform. It has been shown that the Laguerre representations play significant role in this theory, as the wavelets generated by them can analytically be expressed, hence special attention was paid to them hereafter.

2.4 Connection between the hyperbolic wavelet transform and Zernike polynomials. Applications

Zernike functions play an important role in expressing the wavefront data in optical tests. In what follows we will present the relation between the Zernike functions and the hyperbolic wavelet transform. More exactly the matrix elements of the representation representation U of the Blaschke group on the Hardy space of the unit disc given by (2.3) can be expressed by the Zernike functions. An important consequence of this connection is the addition formula for Zernike functions. In this section we present these results published in [123] by Pap, Schipp and in the survey paper by Pap [129]. We also include the discrete orthogonality of Zernike functions published in [121] by Pap, Schipp and connections with corneal topography.

2.4.1 The Zernike polynomials

The orthogonal system of Zernike functions was introduced by Fritz Zernike (a Dutch physicist, winner of the Nobel prize for Physics) in [185] to model symmetries and aberrations of optical systems (e.g., telescopes). Zernike polynomials are used to express wavefront data in optical tests, since they are made up of terms that are of the same form as the types of abberations often observed. The first order wavefront abberations coefficients can be obtained as the coefficients of the Zernike polynomials expansion of the wavefront, and they are called Zernike moments of the wavefront.

There exist an infinity of complete sets of polynomials in two real variables x, y which are orthogonal regarding to the area measure of the unit disc. The circle polynomials of Zernike are distinguished from the other sets by their invariance with respect to rotations of axes about origin. A pure mathematical point of view, is better to consider the complex Zernike polynomials in polar coordinates given by

$$Z_n^\ell(\rho,\theta) := \sqrt{2n+|\ell|+1}\, R_{|\ell|+2n}^{|\ell|}(\rho)e^{i\ell\theta},\ \ell \in \mathbb{Z},\ n \in \mathbb{N}. \tag{2.27}$$

The radial terms $R_{|\ell|+2n}^{|\ell|}(\rho)$ are related to the Jacobi polynomials in the following way:

$$R_{|\ell|+2n}^{|\ell|}(\rho) = \rho^{|\ell|} P_n^{(0,|\ell|)}(2\rho^2 - 1).$$

The orthogonality relation for radial terms and complex Zernike polynomials are given by:

$$\int_0^1 R_{|\ell|+2n}^{|\ell|}(\rho) R_{|\ell|+2n'}^{|\ell|}(\rho)\rho d\rho = \frac{1}{2(|\ell|+2n+1)}\delta_{nn'}, \tag{2.28}$$

$$\frac{1}{\pi}\int_0^{2\pi}\int_0^1 Z_n^\ell(\rho,\phi)\overline{Z_{n'}^{\ell'}(\rho,\phi)}\rho d\rho d\phi = \delta_{nn'}\delta_{\ell\ell'}. \tag{2.29}$$

To compute the wave-front data the real and imaginary part of the complex Zernike functions are used, see for example in [173]. For our purpose we prefer the complex Zernike functions, because the addition formula and the discrete orthogonality of this function can be proved using this form. This is similar to the real and complex trigonometric system: we can take the real trigonometric system $\{1, \cos nx, \sin nx, n \in \mathbb{N}^*\}$ or the complex trigonometric system $\{e^{inx} = \cos nx + i \sin nx,\ n \in \mathbb{Z}\}$. In the complex form we can view them as the characters of the group $(\mathbb{R}, +)$. The addition formula for the trigonometric functions is a consequence of the properties of the characters $e^{in(x+y)} = e^{inx}e^{iny}$. Also the discrete orthogonality of the complex trigonometric system, the base of the discrete Fourier transform, is a consequence of the properties of the complex roots of the unity. We will see that something similar happens in the case of the complex Zernike polynomials too.

2.4.2 The matrix elements of the representation of the Blaschke group

The matrix elements of the representation U with respect to the basis $\{\epsilon_n : n \in \mathbb{N}\}$ are by definition $v_{mn}(a^{-1}) := \langle \epsilon_n, U_{a^{-1}}\epsilon_m\rangle$. They can be expressed using the trigonometric system $\epsilon_n(\varphi) := e^{in\varphi}$ $(n \in \mathbb{Z}, \varphi \in \mathbb{I} = [-\pi, \pi])$ and the associated Legendre polynomials given by

$$P_n^\ell(x) := \frac{x^{-\ell}}{n!}[(1-x)^n x^{n+\ell}]^{(n)}, \quad P_n^{-\ell}(x) := (-1)^\ell P_n^\ell(x) \quad (x \in [0,1], n, \ell \in \mathbb{N}),$$

which are orthogonal on $[0,1]$ with respect to the weight function x^ℓ for a fix ℓ:

$$\int_0^1 P_m^\ell(x) P_n^\ell(x) x^\ell \, dx = \delta_{mn} \frac{1}{2n + |\ell| + 1} \quad (n, m \in \mathbb{N}, \ell \in \mathbb{Z}).$$

For $a = (re^{i\varphi}, e^{i\psi})$ we have

$$v_{mn}(a^{-1}) := \langle \epsilon_n, U_{a^{-1}} \epsilon_m \rangle =$$

$$= \frac{e^{-i(m+1/2)\psi}\sqrt{1-r^2}}{2\pi} \int_{-\pi}^{\pi} \frac{(e^{-it} - re^{-i\varphi})^m}{(1 - re^{i(-t+\varphi)})^{m+1}} e^{int} \, dt.$$

Performing the change of variables $t = s + \varphi$, we obtain that

$$v_{mn}(a^{-1}) = \frac{e^{-i(m+1/2)\psi} e^{i(n-m)\varphi}\sqrt{1-r^2}}{2\pi} \int_{-\pi}^{\pi} \frac{e^{ins}(e^{-is} - r)^m}{(1 - re^{-is})^{m+1}} \, dt =$$

$$= \sqrt{1-r^2} e^{-i(m+1/2)\psi} e^{i(n-m)\varphi} \alpha_{mn}(r),$$

where

$$\alpha_{mn}(r) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(e^{-is} - r)^m}{(1 - re^{-is})^{m+1}} e^{ins} \, ds = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(1 - re^{is})^m}{(e^{is} - r)^{m+1}} e^{i(n+1)s} \, ds.$$

In this last integral performing the change of variables $\zeta = e^{is}$ and applying the Cauchy integral formula we get that

$$\alpha_{mn}(r) := \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{(1 - r\zeta)^m}{(\zeta - r)^{m+1}} \zeta^n \, d\zeta =$$

$$= \frac{r^{-n}}{m!} \frac{d^m}{dz^m}[(1 - rz)^m (rz)^n]_{z=r} = \frac{r^{-n+m}}{m!} \frac{d^m}{dx^m}[(1-x)^m x^n]_{x=r^2}.$$

If $n \geqslant m$ let us denote $n = m + \ell$, then $\alpha_{mn}(r)$ can be expressed by the associated Legendre polynomials, namely:

$$\alpha_{mn}(r) = P_m^\ell(r^2) = (-1)^m r^\ell P_m^{(0,\ell)}(2r^2 - 1).$$

Consequently,

$$v_{mn}(a^{-1}) = \sqrt{1-r^2} e^{-i(m+1/2)\psi} e^{i(n-m)\varphi} (-1)^m r^\ell P_m^{(0,\ell)}(2r^2 - 1) =$$

$$= \frac{\sqrt{1-r^2}}{\sqrt{m+n+1}} e^{-i(m+1/2)\psi}(-1)^m Z_m^{n-m}(r,\varphi),$$

where $Z_m^{n-m}(r,\varphi)$ are the complex Zernike polynomials. If $n < m$, then

$$v_{mn}(a^{-1}) := \langle \epsilon_n, U_{a^{-1}}\epsilon_m\rangle = \langle U_a\epsilon_n, \epsilon_m\rangle = \overline{\langle \epsilon_m, U_a\epsilon_n\rangle} = \overline{v_{nm}(a)} =$$

$$= \frac{\sqrt{1-r^2}}{\sqrt{m+n+1}} e^{-i(m+1/2)\psi}(-1)^m Z_n^{m-n}(r,\varphi).$$

Analyzing these two cases we have that the matrix elements of the representation U are given by the following formula:

$$v_{mn}(a^{-1}) = \frac{\sqrt{1-r^2}}{\sqrt{m+n+1}} e^{-i(m+1/2)\psi}(-1)^m Z_{min\{n,m\}}^{|m-n|}(r,\varphi).$$

It is known that in general the matrix elements of any representation satisfy the following so called addition formula:

$$v_{mn}(a_1 \circ a_2) = \sum_k v_{mk}(a_1)v_{kn}(a_2) \ (a_1, a_2 \in \mathbb{B}).$$

From this relation we obtain the following addition formula for Zernike functions:

$$\frac{\sqrt{1-r^2}}{\sqrt{(n+m+1)(1-r_1^2)(1-r_2^2)}} e^{-i(m+1/2)\psi} Z_{min\{m,n\}}^{|n-m|}(r,\varphi) =$$

$$\sum_k \frac{(-1)^k e^{-i(m+1/2)\psi_1} e^{-i(k+1/2)\psi_2}}{\sqrt{(m+k+1)(n+k+1)}} Z_{min\{m,k\}}^{|k-m|}(r_1,\varphi_1) Z_{min\{k,n\}}^{|n-k|}(r_2,\varphi_2),$$

where $a_j := (r_j e^{i\varphi_j}, e^{i\psi_j})$, $j \in \{1,2\}$ and $a := (re^{i\varphi}, e^{i\psi}) = a_1 \circ a_2$.

It is not as simple like the addition formula for the trigonometric system, but we can discover the analogies replacing the group $(\mathbb{R}, +)$ by the Blaschke group $(\mathbb{B}, \circ)$, the characters by the representation U, the addition formula is a consequence of the properties of the representation.

Starting from the Zernike functions and considering the congruence transformations on the Poincare or Cayley-Klein models, in [108] Lócsi and Schipp constracted a more general orthonormal system on the disc, the so called rational Zernike functions. In the construction it is used the representation U_a (2.3).

2.4.3 Discrete orthogonality of complex Zernike functions

Although the approximation of Zernike coefficients $A_{n\ell} = \frac{1}{\pi}\int_0^{2\pi}\int_0^1 f(\rho,\phi)\overline{Z_n^\ell(\rho,\phi)}\rho d\rho d\phi$ of the wave front function f were obtained from measurements at discrete corneal points and via discrete computations, the developers of the corneal measurement devices and shape-evaluation programs could not rely on the discrete of Zernike functions before it was not proved. Not surprisingly, the discrete orthogonality of Zernike functions was a target of research for some time. See for example the question formulated by Wyant, Creath in [173]. In this subsection we prove the discrete orthogonality of Zernike functions. This result was published in [121] by Pap, Schipp.

Let us consider the set of complex Zernike functions of degree less then $2N$.

$$\{Z_n^\ell(\rho,\theta) := \sqrt{2n+|\ell|+1}\, R_{|\ell|+2n}^{|\ell|}(\rho)e^{i\ell\theta},\ \ell \in Z,\ n \in \mathbb{N},\ |\ell|+2n < 2N\}.$$

This set contains $N(2N+1)$ linearly independent two variables complex valued polynomials of degree less than $2N$.

Pap, Schipp in [121] introduced a set of points in the unit disc and correspondingly a discrete measure and proved that regarding to the discrete measure the complex Zernike functions of degree less than $2N$ are discrete orthogonal. In order to present this property we need the following notations and quadrature formula.

Let us denote by $\lambda_k^N \in (-1,1), k \in \{1,...,N\}$ the roots of Legendre polynomials P_N of order N, and for $j = 1,...,N$, let

$$\ell_j^N(x) := \frac{(x-\lambda_1^N)...(x-\lambda_{j-1}^N)(x-\lambda_{j+1}^N)...(x-\lambda_N^N)}{(\lambda_j^N-\lambda_1^N)...(\lambda_j^N-\lambda_{j-1}^N)(\lambda_j^N-\lambda_{j+1}^N)...(\lambda_j^N-\lambda_N^N)},$$

be the corresponding fundamental polynomials of Lagrange interpolation. Denote by

$$\mathcal{A}_k^N := \int_{-1}^{1} \ell_k^N(x)dx, \quad (1 \leqslant k \leqslant N)$$

the corresponding Cristoffel-numbers. Then for every polynomial f of order less then $2N$ the following quadrature formula holds (see Szegő [162])

$$\int_{-1}^{1} f(x)dx = \sum_{k=1}^{N} f(\lambda_k^N)\mathcal{A}_k^N.$$

In order to prove the discrete orthogonality of Zernike functions let us define the following numbers with the help of the roots of Legendre polynomials of order N,

$$\rho_k^N := \sqrt{\frac{1+\lambda_k^N}{2}}, \quad k = \overline{1,N}.$$

Let us consider the set of nodal points in the unit circle on which the discrete orthogonality holds:

$$X := \left\{ z_{jk} := \left(\rho_k^N, \frac{2\pi j}{4N+1}\right), \quad k = \overline{1,N}, \quad j = \overline{0,4N} \right\}, \tag{2.30}$$

and let us define a weight correspondingly to each nodal point:

$$\nu(z_{jk}) := \frac{\mathcal{A}_k^N}{2(4N+1)}.$$

On the set of nodal points X let us consider the following discrete integral:

$$\int_X f(\rho,\phi)d\nu_N := \sum_{k=1}^{N}\sum_{j=0}^{4N} f\left(\rho_k^N, \frac{2\pi j}{4N+1}\right)\frac{\mathcal{A}_k^N}{2(4N+1)}. \tag{2.31}$$

Theorem 2.4.1 (**Pap, Schipp** [121]). *The Zernike functions with order less then $2N$ are discrete orthogonal regarding to the discrete scalar product induced by the discrete measure, i.e.,*

$$\int_X Z_n^m(\rho,\phi)\overline{Z_{n'}^{m'}(\rho,\phi)}d\nu_N = \delta_{nn'}\delta_{mm'},$$

if $n+n'+|m| \leqslant 2N-1$,$n+n'+|m'| \leqslant 2N-1$, $n,n' \in \mathbb{N}$, $m,m' \in \mathbb{Z}$.

Proof. Writing explicitly the orthogonality of the radial terms we get

$$\frac{1}{2(2n+|m|+1)}\delta_{nn'} = \int_0^1 R_{2n+|m|}^{|m|}(\rho)R_{2n'+|m|}^{|m|}(\rho)\rho d\rho =$$

$$\int_0^1 \rho^{2|m|}P_n^{(0,|m|)}(2\rho^2-1)P_{n'}^{(0,|m|)}(2\rho^2-1)\rho d\rho.$$

If in this last integral we perform the change of variables $u := 2\rho^2-1$, then we obtain the following:

$$\frac{1}{2(2n+|m|+1)}\delta_{nn'} = \frac{1}{4}\int_{-1}^1 \left(\frac{1+u}{2}\right)^{|m|} P_n^{(0,|m|)}(u)P_{n'}^{(0,|m|)}(u)du.$$

Let us denote by $f(\rho) := \left(\frac{1+u}{2}\right)^{|m|} P_n^{(0,|m|)}(u)P_{n'}^{(0,|m|)}(u)$ and $\rho_k^N := \sqrt{\frac{1+\lambda_k^N}{2}}$, $k = \overline{1,N}$. Then the order of f is $n+n'+|m|$. We observe that $Z_N^0(\rho_k^N,\phi) = P_N^{(0,0)}(2(\rho_k^N)^2-1) = P_N(\lambda_k^N) = 0$. If $n+n'+|m| \leqslant 2N-1$, then it can be applied the quadrature formula presented before:

$$\frac{1}{2(2n+|m|+1)}\delta_{nn'} = \int_0^1 R_{2n+|m|}^{|m|}(\rho)R_{2n'+|m|}^{|m|}(\rho)\rho d\rho = \frac{1}{4}\sum_{k=1}^N f(\lambda_k^N)\mathcal{A}_k^N =$$

$$= \frac{1}{4}\sum_{k=1}^n \mathcal{A}_k^N R_{2n+|m|}^{|m|}(\rho_k^N)R_{2n'+|m|}^{|m|}(\rho_k^N).$$

We obtain that

$$\int_X Z_n^m(\rho,\phi)\overline{Z_{n'}^{m'}(\rho,\phi)}d\nu_N = \sum_{k=1}^{N}\sum_{j=0}^{4N} Z_n^m(\rho_k^N, \frac{2\pi j}{4N+1})\overline{Z_{n'}^{m'}(\rho_k^N, \frac{2\pi j}{4N+1})}\frac{\mathcal{A}_k^N}{2(4N+1)} =$$

$$\frac{\sqrt{2n+|m|+1}\sqrt{2n'+|m'|+1}}{2(4N+1)}\sum_{k=1}^{N}\mathcal{A}_k^N R_{2n+|m|}^{|m|}(\rho_k^N)R_{2n'+|m|}^{|m|}(\rho_k^N)\sum_{j=0}^{4N} e^{i(m-m')\frac{2\pi j}{4N+1}}.$$

If $m \neq m'$, the first sum it is equal to 0, and if $m = m'$, then it is equal to $4N+1$. Consequently

$$\int_X Z_n^m(\rho,\phi)\overline{Z_{n'}^{m'}(\rho,\phi)}d\nu_N =$$

$$\delta_{mm'}\frac{\sqrt{2n+|m|+1}\sqrt{2n'+|m|+1}}{2}\sum_{k=1}^{N}\mathcal{A}_k^N R_{2n+|m|}^{|m|}(\rho_k^N)R_{2n'+|m|}^{|m|}(\rho_k^N) =$$

$$\delta_{mm'}2\sqrt{2n+|m|+1}\sqrt{2n'+|m|+1}\int_0^1 R_{2n+|m|}^{|m|}(\rho)R_{2n'+|m|}^{|m|}(\rho)\rho d\rho = \delta_{mm'}\delta_{nn'}.$$

Theorem 2.4.2 (**Pap, Schipp** [121]). *For all $f \in C(\overline{D})$,*

$$\lim_{N\to\infty}\int_X f d\nu_N = \frac{1}{\pi}\int_0^{2\pi}\int_0^1 f(\rho,\phi)\rho d\rho d\phi.$$

Proof. This theorem is a consequence of the Banach-Steinhaus theorem. Let us denote by $C(\overline{D})$ the set of continuous functions on the closure of the unit disc and introduce the bounded linear functionals $A_N(f) = \int_X f d\nu_N$, $A(f) = \frac{1}{\pi}\int_0^{2\pi}\int_0^1 f(\rho,\phi)\,\rho\, d\rho\, d\phi$. We will check that all conditions of the Banach-Steinhaus theorem are satisfied for the functionals $A_N : C(\overline{D}) \to \mathbb{C}$ and $A : C(\overline{D}) \to \mathbb{C}$. Let us denote by Z the set of all Zernike circle polynomials. It can be proved that Z is a dense subset of $C(\overline{D})$ on the base of Stone-Weierstrass theorem, because of the points of $C(\overline{D})$ are separated by the functions in Z. Namely, if $(\rho,\phi) \neq (\rho',\phi')$, $\rho,\rho' \in [0,1]$, $\phi,\phi' \in [0,2\phi]$, then $Z_0^1(\rho,\phi) \neq Z_0^1(\rho',\phi')$. As we have mentioned in introduction, the product of two Zernike functions can be expressed as a finite linear combination of Zernike functions. From [Szegő [162] pp. 48 (3.4.5)] it follows that A_N is a bounded linear operator, namely

$$||A_N|| = \sum_{k=1}^{N}\sum_{j=0}^{4N}\frac{|\mathcal{A}_k^N|}{2(4N+1)} = \sum_{k=1}^{N}\frac{|\mathcal{A}_k^N|}{2} = 1 < \infty.$$

From the orthonormality property it follows that for all $z = Z_n^m \in Z$ and for all N so that $2n+|m| < 2N-1$ we have $A_N(z) - A(z) = 0$, consequently $\lim_{N\to\infty}|A_N(z) - A(z)| = 0, z \in Z$. Applying the Banach-Stainhaus theorem we get that

$$|A_N(f) - A(f)| \to 0, \quad \text{for all} \quad f \in C(\overline{D}), N \to \infty.$$

In fact this theorem means that the limit of the $(0,0)-$th discrete Zernike coefficient is equal by the $(0,0)-$th continuous Zernike coefficient. In an analogous way can be proved that in general the discrete Zernike coefficients of the function f from $C(\overline{D})$ tend to the corresponding continuous Zernike coefficients. Based on this theoretical results it can be given a very efficient approximation algorithm for the Zernike moments, which has not only good convergence properties, but in some cases gives the exact values of them.

2.4.4 Zernike moments, applications

The purpose of a cornea topographic examination is to determine and display the shape and the optical power of the living cornea. Due to the high refractive power of the human cornea, the knowledge of its detailed topography is of great diagnostic importance. The corneal surface can be modeled as a surface over the unit disk and can be described by a two variable function $g(x,y)$. The application of the polar transform to variables x and y results in $x = \rho\cos\phi$, $y = \rho\cos\phi$, where $\rho \in [0,1]$ and $\phi \in [0, 2\phi]$ are the radial and azimuthal variables over the unit disc. Using the polar coordinates for the description of the corneal surface we have the function $G(\rho,\phi) = g(\rho\cos\phi, \rho\cos\phi)$. Nowadays, the ophthalmologists are quite familiar with the "smoothly waving" Zernike-surfaces. They use these surfaces to characterize various symmetries and aberrations of an optical system: those of human eyes. In case of corneal topography, the symmetries and the aberrations of the corneal surfaces are examined with and computationally reconstructed by corneal topographer devices. In case of wavefront analysis, the optical features of the eye-ball is measured with a Shack-Hartmann wavefront-sensor. These characterizations are given partly in the form of Zernike coefficients. As the optical aberrations may cause serious accuracy problems, and are significant factors to be considered in planning of sight-correcting operations, wide range of statistical data concerning the eyes of various groups of people is available concerning the most important Zernike coefficients. This is the reason why elaboration of measurement patterns are important. Although, Zernike coefficients were obtained from measurements at discrete corneal points and via discrete computations, the developers of the corneal measurement devices and shape-evaluation programs could not rely on the discrete orthogonality before the discrete orthogonality of Zernike functions was not proved. Not surprisingly, the discrete orthogonality of Zernike functions was a target of research for some time. The meshes of points ensuring discrete orthogonality of the Zernike functions presented in the previous subsection where used to calculate the Zernike-based representations and their precisions for some test surfaces, including three "cornea-like" test surfaces, as well. These results were published and analyzed by Soumelidis, Fazekas, Schipp, Pap in [155, 156, 157]. Experimental results were reported concerning the precision of the Zernike-based surface representation over the unit disk. The test surfaces considered herein include centrally positioned and shifted cones, pyramids, and some cornea-like surfaces. With these spatial points as input points, discrete Zernike transformation was carried out. The resulting Zernike coefficients were then used to geometrically reconstruction of the optically smooth corneal surface. Then, the error-surfaces were compared to the ones resulting from the Zernike-based reconstructions of a cornea-like mathematical surface that had been properly fitted to the input data.

The numerical computations, reconstructions and experiments are based on the approximation of the continuous Zernike moments of the corneal surface G. This is a consequence of Theorem 2.4.2, which implies that the continuous moments

$$A_{mn} = \frac{1}{\pi}\int_0^{2\pi}\int_0^1 G(\rho',\phi')\overline{Z_n^m(\rho',\phi')}\rho' d\rho' d\phi'$$

are the limit of the discrete Zernike moments, computed based on the measurements on the set X of the discretization defined by (2.30):

$$A'_{mn} = \int_X G(\rho',\phi')\overline{Z_n^m(\rho',\phi')} d\nu_N(\rho',\phi').$$

If instead of $G(\rho,\phi)$ we take

$$T_N(\rho,\phi) = \sum_{2n+|m|\leqq 2N-1} A_{mn} Z_n^m(\rho,\phi),$$

an arbitrary linear combination of Zernike polynomials of degree less than $2N$, then using the discrete orthogonality and the continuous orthogonality property we obtain that the coefficients A_{mn} can be expressed exactly by the discrete Zernike coeffitients:

$$A_{mn} = \frac{1}{\pi}\int_0^{2\pi}\int_0^1 T_N(\rho',\phi')\overline{Z_n^m(\rho',\phi')}\rho' d\rho' d\phi',$$

$$A_{mn} = \int_X T_N(\rho',\phi')\overline{Z_n^m(\rho',\phi')} d\nu_N(\rho',\phi').$$

This means that we can determine the exact value of the Zernike coefficients (moments) of T_N if we can measure the values of T_N on the points of the set X. This means that with the construction of the set X we give an answer to the question where the Placido ring system is worth situated.

In this case we can reconstruct T_N exactly if we measure its values on the discretization mash X:

$$T_N(\rho,\phi) = \sum_{2n+|m|\leqq 2N-1} \frac{1}{\pi}\int_0^{2\pi}\int_0^1 T_N(\rho',\phi')\overline{Z_n^m(\rho',\phi')}\rho' d\rho' d\phi' Z_n^m(\rho,\phi) =$$

$$\frac{1}{\pi}\int_0^{2\pi}\int_0^1 T_N(\rho',\phi') \sum_{2n+|m|\leqq 2N-1} \overline{Z_n^m(\rho',\phi')} Z_n^m(\rho,\phi)\rho' d\rho' d\phi'$$

and

$$T_N(\rho,\phi) = \sum_{2n+|m|\leqq 2N-1} \int_X T_N(\rho',\phi')\overline{Z_n^m(\rho',\phi')}d\nu_N(\rho',\phi')Z_n^m(\rho,\phi) =$$

$$\int_X T_N(\rho',\phi') \sum_{2n+|m|\leqq 2N-1} \overline{Z_n^m(\rho',\phi')}Z_n^m(\rho,\phi)d\nu_N(\rho',\phi').$$

Navarro and Arines in [114] studied three different aspects of complete modal representation with discrete Zernike polynomials, critical sampling in non redundant grids, including also the greed (2.30) where the discrete orthogonality holds. They concluded that the type of sampling pattern has decisive influence on the quality of the reconstructions. For instance, orthogonal discrete ZPs are efficient for wavefront fitting. They formulated that there are three different problems that one has to face when implementing practical applications (either numerical or experimental): (1) Lack of completeness of ZPs; (2) Lack of orthogonality of ZPs and (3) Lack of orthogonality of ZP derivatives. To overcome these limitations, the general standard procedure is to apply a strong oversampling (redundancy) and reconstruct the wavefront by standard least squares fit. The advantage of a strong redundancy is to minimize the reconstruction noise, but it has two main disadvantages. When one reconstructs fewer modes than measures, then there is a high probability of having cross coupling and aliasing in the modal wavefront estimation . They studied these three problems and provide practical solutions, which are tested and validated through realistic numerical simulations.

In [28] Carnicer and Godes analyzed the interpolation problem arising in critical sampling, that is, using a minimal sample. The interpolant is expressed as a linear combination of Zernike polynomials, whose coefficients represent relevant optical features of the wavefront. They studied the propagation of errors of the polynomial values and their coefficients, obtaining bounds for the Lebesgue constants and condition numbers. They proposed a node distribution leading to low Lebesgue constants and condition numbers for degrees up to 20 is proposed. The weights of the quadrature rule can be determined by imposing exactness for polynomials up to a given degree. The exactness condition leads to a linear system. Unfortunately, the solutions of the linear system need not be positive. Only well distributed points in the circle will lead to nonnegative sets of weights. They concluded that if the nodes have a particular distribution, for example it is the set of nodes (2.30), proposed by us, it is possible to obtain samples leading to good approximation properties. If $N \geqslant (3n+1)/2$, then all polynomials of degree n can be reconstructed from the corresponding discrete data by a formula based on the discrete orthogonality of the Zernike polynomials on the given set of nodes and convergence of the estimates of the coefficients can be ensured if $N \to \infty$. They remark that in this case the number of samples is at least about 18 times the dimension of the space of polynomials $(n+1)(n+2)/2$. Another choice with about n^2 points on the unit disk is $r_{l,j} = cos(l/n), \theta_{l,j} = j\pi/(n+1), l = 0, ..., n, j = 0, ..., n$ and good approximation properties has been proposed. Critical sampling, that is, using a minimal number of samples

$\#I = (n+1)(n+2)/2$, has been recently proposed. A reason for proposing critical sampling is that measuring too many data of a wavefront can be expensive. A critical sample allows to reduce the least squares problem to a polynomial interpolation problem. If the solution exists and is unique, it is independent of the choice of the weights. They pose the polynomial interpolation problem corresponding to critical sampling and analyze the stability of the solution and the propagation of errors through the introduction of condition numbers and Lebesgue constants. They obtain bounds for the condition of the Zernike basis and apply these bounds to relate the condition number of the problem of finding the Zernike coefficients in terms of the data with the Lebesgue constant of the interpolation problem. They also propose a choice of the nodes leading to low Lebesgue constants and low condition numbers. The choice of the nodes is given explicitly by a formula. They compare their results with choices suggested by the previous other authors. They obtained also the Lebesgue constant for the sample (2.30) proposed by us and weights associated to the discrete orthogonality formula. Finally they compare the values of the Lebesgue constants for different choices. Oversampling tends to reduce the Lebesgue constant. However, using too many data may lead to an increasing computational cost without a significative reduction of the Lebesgue constant. The advantage of sample (2.30) is that it is associated to a discrete orthogonality formula, giving rise to an explicit formula for the approximations to the Zernike coefficients.

Shi, Sui, Liu, Peng, and Yang in [146] studied the mathematical construction and perturbation analysis of Zernike discrete orthogonal points (2.30). As they formulated the Zernike polynomials are discrete orthogonal over the constructed set (2.30) mathematically, which can be used to deal with the engineering problems. But we not analyzed the locating tolerance of sampling points, since the actual sampling points will not coincide with the ideal ones exactly in practice. They studied the locating errors by perturbation analysis, and the requirements of the positioning precision are not very strict. Using computer simulations they show that this approach provides a very accurate wavefront reconstruction with the proposed sampling set.

Gray in [85] investigated the field dependence of the aberration functions of rotationally nonsymmetric optical imaging systems. He pointed out that our result published in [121] refers to complex number form of the Zernike polynomials and a finite set of complex number on which the Zernike polynomials are orthogonal over a finite set of discrete points across a unit radius disk. In Appendix III of [85] he provides a derivation of the discrete orthogonality properties and equations for the real number form of the Zernike polynomials used in this dissertation. The results were used as part of a Gaussian quadrature (GQ) method for obtaining the Zernike expansion coefficients of the wavefront aberration function expansion.

With the our result from [121] it is possible to select a finite number of data points over the unit radius disk such that all the Zernike functions of order max n or less remain orthogonal over these discrete data points, provided that the functions data values can be exactly represented by a sum of Zernike polynomials of order less than or equal to max n

. The number of data points needed is dependent on the maximum radial order max n of the Zernike polynomials needed to exactly define the function over the unit radius disk. One drawback to this result is that the value of max n that exactly defines the function of interest over the disk is not in general known. With the discrete sampling and finite subset of the Zernike polynomials, including the next higher order Zernike polynomial will change all the lower order coefficients. However, the change is of the order of the coefficient of the next highest order Zernike polynomial included. Then, assuming the function converges for low values of max n, only a small number of Zernike polynomials need to be considered for an acceptable approximation of the function expressed as an expansion in low order Zernike polynomials. Another potential drawback is that the highest order Zernike needed to accurately represent the function (to expand a given function) may be so large that the number of data points across the unit disk is too large to be practicable. Additionally, the higher the value of max n the more concern there is for the numeric accuracy of the calculated Zernike polynomial values. On the other hand, a significant advantage of our method from [121] is that there is no data fitting operation involved. The coefficients are calculated directly from the equations by use of the Gaussian Quadrature (GQ) technique. Gray in [85], using our result and incorporating an improvement pointed out by Shi, et al. [146], derived for the real number Zernike form polynomials. Therefore, this derivation, in terms of real number Zernike polynomials, was necessary in order to obtain the equations needed for his research.

Kaye, Personen in [96] developed novel MRI tools for visualization of the focal spot and for adaptive focusing of ultrasound. In this work, it is shown how using Zernike polynomials, actively utilized in optics, can increase the efficiency of MR-ARFI-based adaptive focusing, making it a more suitable technique for clinical applications. They construct a simulation of non-iterative adaptive focusing algorithm based on Zernike Polynomials. Discrete Zernike polynomials were calculated using the Matlab Zernike function (zernfun.m) (P. Fricker, MATLAB Central File Exchange, 2005) The novel adaptive algorithm was sampled at the (x, y), taking in consideration also the discrete sampling (2.31) proposed by us.

Chapter 3

Hyperbolic wavelet transform, atomic decomposition and multiresolution in weighted Bergman spaces

In this chapter we consider the case $m = \alpha + 2$, when formula (2.1) defines a representation of the Blaschke group on the weighted Bergman space. The properties of the continuous voice transforms generated by representations (2.1) were studied in [125, 127]. Analyzing the question of discretization of these voice transforms it turned out that different techniques are required. In the first chapter we presented a short summary of the theory introduced by Feichtinger and Gröchenig, the so called unified approach to atomic decomposition through integrable group representations in Banach spaces. For certain weighted Bergman spaces, both square integrability and integrability conditions are satisfied . Consequently, it can be applied the Feichtinger-Gröchenig theory, and in this way we can be obtained new atomic decomposition results is certain weighted Bergman spaces (see Pap [127]). In the unweighted case and also in some weighted Bergman spaces the Feichtinger-Gröchenig theory cannot be applied, because the integrability condition is not satisfied. In this case it is shown that, analogously to the case of the Hardy spaces, it is possible to construct a multiresolution and analytic wavelets in weighted Bergman spaces (see Pap [128, 133]).

3.1 The representation of the Blaschke group on the weighted Bergman space A^2_α

In [125], [124] the voice transform induced by a representation of the Blaschke group on the weighted Bergman spaces was studied. Let us consider the following set of functions

$$F_a(z) := \frac{\sqrt{\epsilon(1-|b|^2)}}{1-\bar{b}z} \quad (a = (b,\epsilon) \in \mathbb{B}, z \in \overline{\mathbb{D}}).$$

For every power α ($\alpha \geqslant 0$), F_a induces a unitary representation of Blaschke group on the space A^2_α. Namely, let define

$$U_a^\alpha f := [F_{a^{-1}}]^{\alpha+2} f \circ B_a^{-1} \quad (a \in \mathbb{B}, \alpha \geqslant 0, f \in A^2_\alpha). \tag{3.1}$$

The representation (3.1) has the following explicit form

$$(U^\alpha_{a^{-1}} f)(z) := e^{i\frac{\alpha+2}{2}\psi} \frac{(1-|b|^2)^{\frac{\alpha+2}{2}}}{(1-\bar{b}z)^{\alpha+2}} f\left(e^{i\psi}\frac{z-b}{1-\bar{b}z}\right) \quad (a = (b, e^{i\psi}) \in \mathbb{B}).$$

Let us consider the scalar product in the weighted Bergman space

$$\langle f, g\rangle = \langle f, g\rangle_\alpha := \int_{\mathbb{D}} f(z)\overline{g(z)}\, dA_\alpha(z). \tag{3.2}$$

Theorem 3.1.1 (**Pap** [125])**.** *For all $\alpha \geqslant 0$, $U_a^\alpha (a \in \mathbb{B})$ defined by (3.1) is a unitary representation of the Blaschke group $\mathbb{B}$ on the weighted Bergman space A^2_α.*

For $\alpha \in \mathbb{N}$ in [125] Pap computed the matrix elements of the representation (3.1). The computations follow the same line as in the case of the Hardy space presented in the previous chapter. In this case the matrix elements can be expressed using the Jacobi polynomials (see [125]).

Theorem 3.1.2 (**Pap** [125])**.** *The representation U_a $(a \in \mathbb{B})$ is irreducible on the weighted Bergman space A^2_α, $(\alpha \geqslant 0)$.*

3.2 Properties of the hyperbolic wavelet transform induced by representation U_a^α

It is simpler to take the expression of the representation (3.1) for $a^{-1} \in \mathbb{B}$, correspondingly it is easier to study the induced voice transform, the so called hyperbolic wavelet transform in the weighted Bergman space in $a^{-1} \in \mathbb{B}, (a = (b, e^{i\psi}) \in \mathbb{B},\ f, \rho \in A^2_\alpha)$:

$$(V_g f)(a^{-1}) = (V_g f)(-b\epsilon, \bar{\epsilon}) := \langle f, U^\alpha_{a^{-1}} g\rangle_\alpha. \tag{3.3}$$

Based on Theorem 3.1.1 and Theorem 3.1.2, the irreducibility and unitarity of the representation, and the the general theory of the voice transform, we obtain the analogue of the Plancherel formula for the hyperbolic wavelet transform defined by (3.3) and the invertibility of this transform (see [94, 142]). In what follows we present these results and we give a class of admissible elements.

Theorem 3.2.1 (**Pap** [125])**.** *If $(A^2_\alpha)^*$ denotes the set of admissible elements from A^2_α, then there is a symmetric positive bilinear map $B : (A^2_\alpha)^* \times (A^2_\alpha)^* \to \mathbb{R}$ such that*

$$[V_{\rho_1} f_1, V_{\rho_2} f_2] = B(\rho_1, \rho_2)\langle f_1, f_2\rangle_\alpha \quad (f_1, f_2 \in A^2_\alpha,\ \rho_1, \rho_2 \in (A^2_\alpha)^*), \tag{3.4}$$

where

$$[F, G] := \int_{\mathbb{B}} F(a)\overline{G(a)}\, dm(a)$$

and $dm(a)$ is the Haar measure of the group $\mathbb{B}$.

In unweighted case ($\alpha = 0$), Pap, Schipp in [124] gave a direct proof of this result, from which it turns out that every ρ form the Bergman space is admissible and the voice transform induced by $U_a = U^2_a$ satisfies

$$[V_{\rho_1} f, V_{\rho_2} g] = 4\pi\langle \rho_1, \rho_2\rangle\, \langle f, g\rangle \quad (f, g, \rho_1, \rho_2 \in A^2_0(\mathbb{D})).$$

For the unweighted case ($\alpha = 0$) Pap, Schipp in [124] proved also the following two admissible criteria:

Theorem 3.2.2 (**Pap, Schipp** [124])**.** *Every $\rho_n = z^n$ ($n \in \mathbb{N}$) is admissible, namely:*

$$\int_{\mathbb{B}} |V_{\rho_n}\rho_n(a)|^2 dm(a) < \infty.$$

Theorem 3.2.3 (**Pap, Schipp** [124])**.** *Every element $\rho \in \mathcal{H}^\infty(\mathbb{D})$ is admissible, namely:*

$$\int_{\mathbb{B}} |V_\rho \rho(a)|^2 dm(a) < \infty.$$

From the general theory (see [142], [94]) and Theorem 3.2.1 it follows that:

Theorem 3.2.4 (**Pap** [125])**.** *The voice transform generated by representation U^α_a ($a \in \mathbb{B}$) is one to one in A^2_α.*

The function $V_\rho f$ is continuous and bounded on $\mathbb{B}$. Theorem 3.2.1 implies that for $\alpha \geqslant 0$ every element from A^2_α is admissible. Moreover, taking in consideration that the Blaschke group is unimodular Theorem 3.2.1 implies that for $f, g \in A^2_\alpha$ such that $g \neq 0$ and $B(g, g) = \|Cg\|^2 = 1$ the following reproducing formula is valid:

$$V_g f = V_g f * V_g g, \quad \text{i.e.,} \quad V_g f(y^{-1}) = \int_{\mathbb{B}} V_g f(x^{-1}) V_g g(x \circ y^{-1}) dm(x). \tag{3.5}$$

3.3 Construction of orthogonal rational wavelets in the weighted Bergman spaces

In this section we give an orthogonal rational wavelet system for $\alpha \geqslant 0, m = \alpha+2 \in \mathbb{N}$, and we show that the Bergman projection operator can be expressed with this system and the voice transforms as it was shown in [124, 125]. Let us consider the shift operator

$$(S\varphi)(z) = z\varphi(z) \quad (\varphi \in A^2_\alpha).$$

Denote by

$$\varphi_{a,n}(z) := \sqrt{\frac{\Gamma(n+m)}{n!\Gamma(m)}}(U_{a^{-1}}S^n\varphi)(z)$$

$$\big(a = (b,\epsilon) \in \mathbb{B}, m \in \mathbb{N}, m \geqslant 2,\ \varphi \in A^2_\alpha, n \in \mathbb{N}.\big)$$

If we consider as *mother wavelet* $\varphi = 1 \in A^2_\alpha$, then the corresponding rational wavelets are :

$$\varphi_{a,n}(z) = \sqrt{\frac{\Gamma(n+m)}{n!\Gamma(m)}}\frac{[\epsilon(1-|b|^2)]^{\frac{m}{2}}}{(1-\bar{b}z)^m}\left(\frac{\epsilon(z-b)}{1-\bar{b}z}\right)^n, \quad n \in \mathbb{N}.$$

Taking into account the unitarity of the representation U_a it follows that they form an orthonormal system in A^2_α, for every $a \in \mathbb{B}$.

We observe that if we consider the neutral element of the group $a = e = (0,1) \in \mathbb{B}$, then we reobtain the classical orthonormal basis in A^2_α, namely

$$\varphi_n(z) = \varphi_{e,n}(z) = \sqrt{\frac{\Gamma(n+m)}{n!\Gamma(m)}}z^n, \quad n \in \mathbb{N}.$$

Theorem 3.3.1 (**Pap** [125])**.** *For all $z \in \mathbb{D}$ and $a \in \mathbb{B}$, $m = \alpha+2 \geqslant 2, \alpha \in \mathbb{N}$ the weighted Bergman projection operator $P_\alpha : L^2(\mathbb{D}, dA_\alpha) \to A^2_\alpha$ can be represented as*

$$P_\alpha f(z) = \sum_{n=0}^{\infty} V_{\varphi_n} f(a^{-1})\varphi_{a,n}(z) \quad (a \in \mathbb{B}).$$

Consequences.

1. Every f from A^2_α can be represented as

$$f(z) = \sum_{n=0}^{\infty} V_{\varphi_n} f(a^{-1})\varphi_{a,n}(z) \quad (a \in \mathbb{B},\ z \in \mathbb{D}).$$

2. For every $a \in \mathbb{B}$ for $m = \alpha+2, \alpha \in \mathbb{N}$ the functions

$$\varphi_{a,n}(z) = \sqrt{\frac{\Gamma(n+m)}{n!\Gamma(m)}}\frac{[\epsilon(1-|b|^2)]^{\frac{m}{2}}}{(1-\bar{b}z)^m}\left(\frac{\epsilon(z-b)}{1-\bar{b}z}\right)^n, \quad (z \in \mathbb{D},\ n \in \mathbb{N})$$

form an orthonormal rational basis in A^2_α.

3. We can deduce the following characterization of the poles. Let function $f \in A^2_\alpha$. Then f has n-tuple pole at $\frac{1}{b}$ outside of the unit disc if and only if for $a = (b, \epsilon) \in \mathbb{B}$,

$$V_{\varphi_n} f(a^{-1}) \neq 0, \text{ and for all } \ k, \ k > n, \ V_{\varphi_k} f(a^{-1}) = 0.$$

3.4 The hyperbolic wavelet transform on the weighted Bergman spaces and the coorbit theory

We have already mentioned in the introduction that H. G. Feichtinger and K. H. Gröchenig described a unified approach to atomic decomposition through integrable group representations in Banach spaces generated by the voice transforms of certain groups. They described a general discretization technique for the voice transforms induced by irreducible, square integrable and integrable group representations, giving atomic decompositions for large families of Banach spaces, the so called coorbit spaces (see papers of Feichtinger, Gröchenig [64, 66, 65, 89]).

Studying the properties of a hyperbolic wavelet transform of the Blaschke group generated by the representation of this group on the weighted Bergman space, outlined by the general theory, developed by Feichtinger and Gröchenig, we obtain that every function from the minimal Möbius invariant space will generate an atomic decomposition in the weighted Bergman spaces. These results were obtained by Pap in [127]. In order to present these results, first we summarize their technique.

In the unified approach of the atomic decomposition a useful tool is the Q-density, the V-separated property and the bounded uniform partitions of the unity of the locally compact group.

Using the hyperbolic metric we can describe the Q density from right, and the separation from right in the Blaschke-group. Using this we can give an example of bounded uniform partitions of the unity from right. In the general theory of atomic decomposition it is used the Q-density from the left, this is the reason why we will make a small modification in the discretizing operator which corresponds to the Q–density from the right in order to obtain atomic decomposition in the weighted Bergman spaces.

In what follows we will outline how it can be obtained atomic decomposition results in $\mathcal{H}^1$, defined by (1.10), following the exposition published in [64] for the case when the weight function $w = 1$. Assume that U is an irreducible unitary representation of the group G on the Hilbert space H which is integrable, i.e., there is a $g \in H\backslash\{0\}$ such that $\int_G |V_g g(a)| dm(a) < \infty$, and which is continuous, i.e., $U_a g$ is a continuous map of G into H for all $a \in G$. For certain spaces Y of functions on G for which the convolution operator is defined and is continuous for $g \in \mathcal{A}$, the coorbit spaces are defined in the following way:

$$Co(Y) = \{f \in \mathcal{H}^{1*} : V_g f \in Y\}, \tag{3.6}$$

and this is independent of the choice of $g \in \mathcal{A}$. Place on $Co(Y)$ the norm $\|f\|_{Co(Y)} = \|V_g f\|_Y$. For example

$$H = Co(L^2(G)), \quad \mathcal{H}^1 = Co(L^1(G)).$$

At the same time it is defined an appropriate sequence space Y_d corresponding to Y (for example if $Y = L^p(G)$ then $Y_d = \ell^p(\mathbb{Z})$). Let us consider

$$S = \{F \in Y : \; F = V_g f \;\text{ for some }\; f \in Co(Y)\}. \tag{3.7}$$

The convolution operator defined by (3.5), which is the identity on S, can be approximated by a discrete operator, similar to a Riemann sum using the so called *bounded uniform partition of the unity.*

Definition 3.4.1. *Given a compact set Q with non-void interior, a countable family $X = (x_i)$ in G is said to be Q-dense if $\bigcup x_i Q = G$. It is separated, if for some compact neighborhood V of the unity we have $x_j V \cap x_i V = \varnothing$, $j \neq i$. We say that $\Psi = \{\psi_k\}_{k\in\mathbb{N}}$ is a bounded uniform partition of unity of size Q (Q-BUPU) if for an open neighborhood Q of unity in G with compact closure there exist points in x_i in G such that*

- $0 \leqslant \psi_i(x) \leqslant 1$,
- $\operatorname{supp} \psi_i \subset x_i Q$,
- $\sum_i \psi_i(x) = 1$,
- $\sup_{z\in\mathbb{B}} \#\{i \in \mathbb{N} : z \in x_i Q'\} < \infty$ *for any $Q' \subset G$ compact.*

In order to approximate by a discrete sum $V_g f$ let write the reproducing formula (3.5) in the form

$$\int_G V_g f(x) V_g g(x^{-1}y)\, dm(x) = V_g f(y),$$

which is a convolution operator on G, namely $F = V_g f$, and $F = F * V_g g$. Define the operators $TF = F * V_g g$ and T_Ψ on Y, associated to a particular bounded uniform partition of unity Ψ, by

$$T_\Psi(y) = \sum_i \langle F, \psi_i\rangle V_g g(x_i^{-1} y). \tag{3.8}$$

From Lemma 4.3 of [64] it follows that if $F \in L^1(G)$ the sequence of coefficients $\Lambda = (\lambda_i)_{i\in\mathbb{N}}$, given by $\lambda_i = \langle F, \psi_i\rangle$ belongs to ℓ^1, more precisely, given a fixed compact neighborhood Q of unity there exists a constant C_0 such that the norms of the linear operators $F \to \Lambda$ are uniformly bounded by C_0 for all Q-BUPUs. Conversely, if $g \in \mathcal{A}$ and $\Lambda = (\lambda_i)_{i\in\mathbb{N}} \in \ell^1$ then $F := \sum_i \lambda_1 V_g g(x_i^{-1} y) \in L^1(G)$, the sum being absolutely convergent in $L^1(G)$ and there is a universal constant C_1 such that $\|F\|_1 \leqslant C_1 \|\Lambda\|_{\ell^1}$. As a consequence, the set of operators $\{T_\Psi\}$, where Ψ runs through the family of Q-BUPUs acts uniformly bounded on $L^1(G)$.

Lemma 4.5 of [64] says that the net $\{T_\Psi\}$ of Q-BUPUs directed according to inclusions of the neighborhoods Q of unity is norm convergent to T as operators on $L^1(G)$. As a consequence, it can be obtained the following atomic decomposition result for $\mathcal{H}^1$

Theorem 3.4.2. *(see [64]) For any $g \in \mathcal{A}\backslash\{0\}$, normalized by $\|Cg\|^2 = 1$, there exist a small neighborhood Q of identity and a constant C_0 (both only dependent of g), such that for any collection of points $\{x_i\} \subset G$ which is Q-dense and and V-separated and any bounded uniform partition of unity Ψ associated to $\{x_i\}$ any $f \in \mathcal{H}^1$ can be written as*

$$f = \sum \lambda_i(f) U_{x_i} g, \quad \textit{with} \quad \sum_i |\lambda_i(f)| \leqslant C_0 \|f\|_{\mathcal{H}^1},$$

where the sum is absolutely convergent in $\mathcal{H}^1$. The coefficients $\lambda_i(f) = \langle T_\Psi^{-1} V_g f, \psi_i \rangle$ depend linearly on f.

Thus this gives an atomic decomposition of $f \in \mathcal{H}^1$ with atoms $U_{x_i} g$ which can be viewed as generalizations of the frames to Banach spaces, other than Hilbert spaces. This result has extension to coorbit spaces (see papers of Fiechtinger, Gröchenig [64, 66, 65, 89]).

3.4.1 Bounded uniform partition on Blaschke-group

In what follows we present results published by Pap in [127], where it was shown that in the Blaschke group there exist right bounded uniform partitions of the unity and the question of the integrability of the hyperbolic wavelet transform given by (3.3) was studied. It turned out that the constant function $f = 1$ and every function from the minimal Möbius invariant space B_1 (defined by (3.12)) satisfy the integrability condition. It is shown that in the case of the weighted Bergman spaces, where the weight is generated by $\alpha > 0$, the general theory of atomic decomposition can be applied and in this way we can find new atoms for these spaces.

As we have seen before, in the unified approach of the atomic decomposition the Q density, the V-separated property and the bounded uniform partitions of the unity are the basic starting points.

Our aim is to construct a Q-dense and V-separated sequences in the Blaschke group. As we will see, it is easier to show and see the geometrical interpretation, of the Q-density from right, i.e., there is a sequence $(x_i)_{i\in\mathbb{N}}$ in $\mathbb{B}$ such that $\bigcup Qx_i = \mathbb{B}$, and separated from right (for some compact neighborhood V of the unity we have $Vx_j \cap Vx_i = \emptyset$, $j \neq i$) and there exist also bounded uniform partitions of the unity.

The Q density from the left is, in general, not the same as the Q-density from right, if the group is non commutative, as it is the case of the Blaschke group.

Recall that the hyperbolic and pseudo-hyperbolic distance of two points from the unit disc is given by

$$\beta(z, w) = \frac{1}{2} \log \frac{1 + \rho(z, w)}{1 - \rho(z, w)},\ \rho(z, w) = \left| \frac{z - w}{1 - \overline{w} z} \right| = \left| B_{(w,1)}(z) \right|, \tag{3.9}$$

and the hyperbolic disc or Bergman disc of radius $r > 0$ and center b is

$$D(b,r) = \{z \in \mathbb{D} : \beta(z,b) < r\}. \tag{3.10}$$

Lemma 3.4.3 (**Pap** [127])**.** *Let consider $r > 0$ and $Q = Q_1 \times \mathbb{T}$, where $Q_1 = \{z \in \mathbb{D} : |z| < \tanh r\}$. Then there exists a sequence $x_n = (b_n, -1) \in \mathbb{B}$ which is Q-dense from the right, i.e., $\bigcup Qx_n = \mathbb{B}$ and V-separated from right, i.e., $Vx_n \cap Vx_m = \varnothing, n \neq m$, and there is also a corresponding right bounded uniform partition of the unity corresponding to $\{x_n\}$.*

Proof of Lemma 3.4.3. Due to Lemma 2.13 from [93] pp. 39, for every fix r, $0 < r < +\infty$, and N positive integer there exists a sequence $\{b_n\}_{n\in\mathbb{N}} \in \mathbb{D}$ such that the disc is covered by the hyperbolic discs $\{D(b_n, r)\}_{n\in\mathbb{N}}$, and if $m \neq n$ then $\beta(b_n, b_m) \geqslant \frac{r}{2}$ and every $z \in \mathbb{D}$ belongs to at most N hyperbolic discs $D(b_n, r)$. We observe that $z \in D(b,r)$ is equivalent with $z \in \{z \in \mathbb{D} : \rho(z,b) < \tanh r < 1\} = B_{(b,-1)}(\{z \in \mathbb{D} : |z| < \tanh r\}) = B_{(b,-1)}(Q_1)$ (see [56] pp. 40). Then for

$$\begin{aligned} Qx_n = \{x \circ x_n : x = (b, \epsilon) \in Q\} &= \{(B_{(b_n,-1)}(b), B_{(-b\overline{b_n},\epsilon)}(-1)) : b \in Q_1,\, \epsilon \in \mathbb{T}\} \\ &= \{D(b_n, r)\} \times \mathbb{T}, \end{aligned}$$

from this we obtain that $\bigcup Qx_n = \mathbb{B}$. If we take $V = V_1 \times \mathbb{T}$ with $V_1 = \{z \in \mathbb{D} : |z| < \tanh \frac{r}{4}\}$, then $Vx_n \cap Vx_m = \phi$ for $m \neq n$. Now we are ready to give an example of right bounded uniform partition of unity. Due to Lemma 2.28 from [187] pp. 63, there exists a Borel set D_k satisfying the following conditions:

- $D(b_k, \frac{r}{4}) \subset D_k \subset D(b_k, r)$,
- $D_m \cap D_n = \phi$,
- $\mathbb{D} = \bigcup D_k$.

Then $B_{(b_k,-1)}(\{z \in \mathbb{D} : |z| < \tanh \frac{r}{4}\}) \subset D_k \subset B_{(b_k,-1)}(\{z \in \mathbb{D} : |z| < \tanh r\})$. Let consider $\psi_k = \chi_{D_k \times \mathbb{T}}$ the characteristic function of the set $D_k \times \mathbb{T}$. Then $\Psi = \{\psi_k\}_{k\in\mathbb{N}}$ is a bounded uniform partition of unity from right of size Q. Indeed, for all $i \in \mathbb{N}$,

- $0 \leqslant \psi_i(x) \leqslant 1$,
- $\operatorname{supp} \psi_i \subset Qx_i$,
- $\sum_i \psi_i(x) = 1,\ x \in \mathbb{B}$.
- $\sup_{z\in\mathbb{B}} \#\{i \in \mathbb{N} : z \in Q'x_i\} < \infty$ for any $Q' \subset \mathbb{B}$ compact.

We shall consider the set of Q-bounded uniform partitions of unity from right (Q-RBUPUs) as a net directed by inclusion of the associated neighborhoods, and write $\Psi \to \infty$ if these neighborhoods run trough a neighborhood base of identity. In the general theory of atomic decomposition it is used the Q-density from the left, this is the reason why in the next subsection we will make a small modification in the discretizing operator which corresponds to the Q–density from the right in order to obtain atomic decomposition in the weighted Bergman spaces.

3.4.2 Integrability of the hyperbolic wavelet transform induced by representation U_a^α

We observe that the hyperbolic wavelet transform given by formula (3.3) can be expressed by the weighted Bergman projection operator in the following way:

$$V_g f(a^{-1}) = \langle f, U_{a^{-1}} g\rangle_\alpha = e^{\frac{\alpha+2}{2}\psi}(1-|b|^2)^{\frac{\alpha+2}{2}} P_\alpha(f \cdot \overline{g(B_a)}), \tag{3.11}$$

$$(a = (b, e^{i\psi}) \in \mathbb{B},\ f, g \in A^2_\alpha).$$

First we will study the integrability of the voice transform, i.e., we show that there exists an element $g \in A^2_\alpha$, $g \neq 0$ such that

$$\int_{\mathbb{B}} |V_g g(a^{-1})| dm(a) < \infty.$$

Theorem 3.4.4 (**Pap** [127]). *If $\alpha > 0$, then the representation $U^\alpha_{a^{-1}}$ is integrable.*

Proof of 3.4.4. Let us consider $g = 1 \in A^2_\alpha$. Using (3.11) we get:

$$V_g g(a^{-1}) = e^{\frac{\alpha+2}{2}\psi}(1-|b|^2)^{\frac{\alpha+2}{2}} P_\alpha(g \cdot \overline{g(B_a)})$$

$$= e^{\frac{\alpha+2}{2}\psi}(1-|b|^2)^{\frac{\alpha+2}{2}} \int_{\mathbb{D}} \frac{1}{(1-\overline{z}b)^{\alpha+2}} dA_\alpha(z) = e^{\frac{\alpha+2}{2}\psi}(1-|b|^2)^{\frac{\alpha+2}{2}}.$$

Then

$$\int_{\mathbb{B}} |V_g g(a^{-1})| m(a) = \int_{\mathbb{D}} (1-|b|^2)^{\frac{\alpha+2}{2}} \frac{1}{(1-|b|^2)^2} dA(b) =$$

$$= \int_0^1 (1-r)^{\frac{\alpha}{2}-1} dr = \frac{2}{\alpha} < \infty.$$

Thus we have that

$$\mathcal{A}^1 = \{g \in A^2_\alpha :\ V_g g \in L^1(\mathbb{B})\} \neq \{0\}.$$

From Theorem 3.2.1, (3.5) and the connection with the weighted Bergman projection it follows that in $B(g,g) = C\|g\|^2$ the value of the constant C is $\sqrt{\pi/(\alpha+1)}$.

We will show that the integrability condition is also satisfied by every g from the minimal Möbius invariant space of analytic functions (see [8], [9]), denoted by $\mathcal{B}_1$, which contains exactly the analytic functions on the unit disc which admit the representation

$$g(z) = \sum_{j=0}^{\infty} \lambda_j \frac{z - b_j}{1 - \overline{b_j} z}, \quad |b_j| \leqslant 1, \quad \sum_{j=0}^{\infty} |\lambda_j| < \infty. \tag{3.12}$$

It is easy to prove that for $1 \leqslant p$ and $-1 < \alpha$ the space B_1 is included in A^p_α.

Theorem 3.4.5 (Pap [127]). *For $\alpha > 0$ every g from the minimal Möbius invariant space of analytic functions satisfies the integrability condition, i.e., the space $\mathcal{B}_1$ is a subset of $\mathcal{A}^1$.*

Proof of Theorem 3.4.5.

In order to prove this theorem we will use the following result (see [93]). For any $-1 < \alpha < +\infty$ and any real β, let

$$I_{\alpha,\beta}(z) = \int_{\mathbb{D}} \frac{(1 - |w|^2)^\alpha}{|1 - z\overline{w}|^{2+\alpha+\beta}} dA(w), \quad z \in \mathbb{D}.$$

Then we have the estimates

$$I_{\alpha,\beta}(z) \sim \begin{cases} 1, & \beta < 0, \\ \log \frac{1}{1-|z|^2}, & \beta = 0, \\ \frac{1}{(1-|z|^2)^\beta}, & \beta > 0 \end{cases}$$

as $|z| \to 1^-$.

For $g \in B_1$ we have the following estimate:

$$|V_g g(a^{-1})| = \left| e^{\frac{\alpha+2}{2}\psi} (1 - |b|^2)^{\frac{\alpha+2}{2}} P_\alpha(g \cdot \overline{g(B_a)}) \right| \leqslant$$

$$\leqslant (1 - |b|^2)^{\frac{\alpha+2}{2}} \int_{\mathbb{D}} \left(\sum_j^{\infty} |\lambda_j| \right)^2 \frac{1}{|1 - \overline{z}b|^{\alpha+2}} dA_\alpha(z) =$$

$$= (1 - |b|^2)^{\frac{\alpha+2}{2}} \left(\sum_j^{\infty} |\lambda_j| \right)^2 I_{\alpha,0}(b).$$

When $|b| \to 1^-$ we have $I_{\alpha,0}(b) \sim \log \frac{1}{1-|b|^2}$. For $\alpha > 0$,

$$\int_{\mathbb{D}} (1 - |b|^2)^{\frac{\alpha+2}{2}} \log \frac{1}{1 - |b|^2} \frac{1}{(1 - |b|^2)^2} dA(b) = - \int_0^1 (1 - r)^{\frac{\alpha-2}{2}} \log(1 - r) dr =$$

$$= -\left[\frac{2}{\alpha}(1-y)^{\frac{\alpha}{2}}\log(1-y) - \left(\frac{2}{\alpha}\right)^2 (1-y)^{\frac{\alpha}{2}}\right]_0^1 = \frac{4}{\alpha^2}.$$

From this it follows that

$$\int_{\mathbb{B}} |V_g g(a^{-1})| dm(a) < +\infty.$$

From now on we choose the parameter function g always from the space $\mathcal{B}_1 \cup \{1\}$, we also restrict the domain of the definition of the voice transform for $a = (b, 1) \in \mathbb{B}$. We show that the voice transform $V_g f$ can be defined not only for f belonging to A^2_α but under some assumptions on the parameters $V_g f$ has sense for $f \in A^p_\beta$, and we will study some growth properties of the voice transform.

Theorem 3.4.6 (**Pap** [127])**.** *Let fix the function g from $\mathcal{B}_1 \cup \{1\}$. If $-1 < \alpha, \beta < +\infty$, $1 \leqslant p$, $(\beta+1) < (\alpha+1)p$, then for every $f \in A^p_\beta$ the voice transform is well defined. If $a = (b, 1) \in \mathbb{B}$, then*

$$V_g f(a^{-1}) = V_g f(-b, 1) = (1 - |b|^2)^{\frac{\alpha+2}{2}} F_1(b),$$

where $F_1(b) \in A^p_\beta$, and

$$\lim_{|b| \to 1^-} (1 - |b|^2)^{\frac{\beta+2}{p} - \frac{\alpha+2}{2}} |V_g f(b)| = 0.$$

Proof of Theorem 3.4.6. In the proof we will use the following result (see for example in [93]): suppose $-1 < \alpha$, $\beta < +\infty$ and $1 \leqslant p < +\infty$. Then P_α is a bounded projection from $L^p(\mathbb{D}, dA_\beta)$ onto A^p_β if and only if $(\beta+1) < (\alpha+1)p$.

From this result and the connection of the voice transform with the weighted Bergman projection (3.11), for $g = 1$ the proof is immediately.

If $g \in \mathcal{B}_1$, then

$$g(z) = \sum_{j=0}^{\infty} \lambda_j \frac{z - b_j}{1 - \overline{b_j} z} = \sum_{j=0}^{\infty} \lambda_j B_{(b_j,1)}(z), \quad |b_j| \leqslant 1, \quad \sum_{j=0}^{\infty} |\lambda_j| < \infty.$$

This implies that

$$g(B_a(z)) = \sum_{j=0}^{\infty} \lambda_j B_{(b_j,1)\circ a}(z) \in B_1, \quad \sum_{j=0}^{\infty} |\lambda_j| < \infty.$$

We show that if $f \in A^p_\beta$, then $f \cdot \overline{g(B_a)} \in L^p(\mathbb{D}, dA_\beta)$. This follows immediately from the following inequality:

$$|f(z)\overline{g(B_a(z))}|^p \leqslant |f(z)|^p \left(\sum_{j=1}^{+\infty} |\lambda_j B_{(b_j,1)\circ a}(z)|\right)^p \leqslant |f(z)|^p \left(\sum_{j=1}^{+\infty} |\lambda_j|\right)^p.$$

From here it follows that if $-1 < \alpha, \beta < +\infty$, $1 \leqslant p$ and $(\beta+1) < (\alpha+1)p$, then P_α is a bounded projection from $L^p(\mathbb{D}, dA_\beta)$ onto A^p_β, which implies that, for every $g \in \mathcal{B}_1$ and $f \in A^p_\beta$ the voice transform

$$V_g f(a^{-1}) = e^{\frac{\alpha+2}{2}\psi}(1-|b|^2)^{\frac{\alpha+2}{2}} P_\alpha(f \cdot \overline{g(B_a)}),$$

is well defined. If we consider $a = (b, 1)$ and denote by

$$F_1(b) = P_\alpha(f \cdot \overline{g(B_{a^{-1}})}),$$

then $F_1 \in A^p_\beta$. For all $F_1 \in A^p_\beta$, if $-1 < \beta < +\infty$, $p > 0$, we have (see [187])

$$|F_1(b)| \leqslant \frac{\|F_1\|_{A^p_\beta}}{(1-|b|^2)^{\frac{\beta+2}{p}}}, \quad b \in \mathbb{D}, \tag{3.13}$$

the exponent of $(1-|b|^2)$ is best possible, and it can be obtained the following improved behavior of F_1 near the boundary:

$$\lim_{|b|\to 1^-} |F_1(b)|(1-|b|^2)^{\frac{\beta+2}{p}} = 0.$$

This implies that

$$\lim_{|b|\to 1^-} (1-|b|^2)^{\frac{\beta+2}{p} - \frac{\alpha+2}{2}} |V_g f(b)| = 0.$$

For $\alpha = \beta$ and $p = 2$ it follows that, if $f \in A^2_\alpha$, then

$$\lim_{|b|\to 1^-} |V_g f(b)| = 0.$$

The next theorem gives information about the simplest Banach space where the Feichtinger-Gröchenig theory can be applied in order to obtain new atomic decomposition results for the set defined by

$$\mathcal{H}^1 = \{f \in A^2_\alpha : V_g f \in L^1(\mathbb{B})\}. \tag{3.14}$$

Theorem 3.4.7 (**Pap** [127])**.** *Let $g \in \mathcal{B}_1 \cup \{1\}$, $\alpha > 0$, $p \geqslant 1$ and $p > \max\left\{\frac{\beta+1}{\alpha+1}, \frac{4+2\beta}{\alpha}\right\}$. Then for every $f \in A^p_\beta$ the voice transform $V_g f$ is integrable, i.e., $V_g f \in L^1(\mathbb{B})$.*

As an immediate consequence we get that for $\alpha = \beta > 0$, $p > 2 + \frac{4}{\alpha}$ we have that $A^p_\alpha \subset \mathcal{H}^1$.

Proof of Theorem 3.4.7 We have to show that if the assumptions of the theorem are satisfied, then

$$\int_{\mathbb{B}} |V_g f(a^{-1})| dm(a) < +\infty.$$

Using Theorem 3.4.6 and (3.13) we obtain that

$$\int_{\mathbb{B}} |V_g f(a^{-1})| dm(a) = \int_{\mathbb{D}} (1-|b|^2)^{\frac{\alpha-2}{2}} |F_1(b)| dA(b) \leqslant$$

$$\leqslant \|F_1\|_{A^p_\beta} \int_{\mathbb{D}} (1-|b|^2)^{\frac{\alpha-2}{2}-\frac{2+\beta}{p}} dA(b) =$$

$$= \|F_1\|_{A^p_\beta} \int_{\mathbb{D}} (1-r^2)^{\frac{\alpha-2}{2}-\frac{2+\beta}{p}} 2r dr = \|F_1\|_{A^p_\beta} \frac{1}{\frac{\alpha}{2}-\frac{2+\beta}{p}} < +\infty.$$

3.4.3 New atomic decomposition results in weighted Bergman spaces

Now we are ready to apply the general theory of Feichtinger and Gröchenig in order to obtain atomic decompositions in weighted Bergman spaces. From this result, as a special case, we reobtain some well known atomic decompositions in the weighted Bergman spaces obtained by complex techniques, but also we get new atomic decompositions for these spaces. As we have mentioned earlier in the Blaschke group, it is easier to give Q-RBUPU, it is more convenient to compute the voice transform given by (3.3) in $a^{-1} \in \mathbb{B}$. Taking into account that the Blaschke group is unimodular, the reproducing formula (3.5), can be written as follows

$$V_g f(y^{-1}) = \int_{\mathbb{B}} V_g f(x^{-1}) V_g g(x \circ y^{-1}) dm(x), \quad f, g \in A^2_\alpha,\ g \neq 0,\ \|Cg\| = 1. \tag{3.15}$$

From Theorem 3.4.7 for $\alpha = \beta > 0$, $p > 2 + \frac{4}{\alpha}$, $g \in \mathcal{B}_1 \cup \{1\}$ we have the inclusion $A^p_\alpha \subset \mathcal{H}^1$, where $\mathcal{H}^1 = \{f \in A^2_\alpha : V_g f \in L^1(\mathbb{B})\}$, and $\|f\|_{\mathcal{H}^1} = \|V_g f\|_{L^1(\mathbb{B})} \leqslant C_2 \|F_1\|_{A^p_\beta}$. Let denote $F(y^{-1}) = V_g f(y^{-1})$, $G(y^{-1}) = V_g g(y^{-1})$, then the reproducing formula (3.15) is a convolution operator T, $TF = F \star G$. To discretize this for $F, G \in L^1(\mathbb{B})$ by means of Q-RBUPU we will use the modified version of the operator (3.8) given by

$$T_\Psi F(y^{-1}) = \sum_i \langle F, \psi_i \rangle L_{x_i^{-1}} G(y^{-1}), \quad F, G \in L^1(\mathbb{B}), \tag{3.16}$$

which is composed of a coefficients mapping $F \to (\lambda_i)_{i \in \mathbb{N}}$ with

$$\lambda_i = \langle F, \psi_i \rangle = \int_{\mathbb{B}} F(y^{-1}) \psi_i(y) dm(y)$$

and a convolution operator

$$(\lambda_i)_{i \in \mathbb{N}} \to \sum_i \lambda_i L_{x_i^{-1}} G = (\sum_i \lambda_i \delta_{x_i^{-1}}) \star G.$$

Our aim is to approximate the convolution operator $TF = F \star G$ by the modified operator (3.16). Analogous to Lemma 4.3 from [64] it can be proved that:

i) For $F \in L^1(\mathbb{B})$ the sequence of coefficients $(\lambda_i)_{i\in\mathbb{N}}$ given by $\lambda_i = \langle F, \psi_i\rangle$ belongs to ℓ^1, and the norms of the linear operators $F \to (\lambda_i)_{i\in\mathbb{N}}$ are uniformly bounded.

ii) Given $G \in L^1(\mathbb{B})$, $(\lambda_i)_{i\in\mathbb{N}} \in \ell^1$ and any family $X = (x_i)_{i\in\mathbb{N}}$ in the group one has

$$F(y^{-1}) = \sum_i \lambda_i L_{x_i^{-1}} G(y^{-1}) \in L^1(\mathbb{B}),$$

the sum being absolutely convergent in $L^1(\mathbb{B})$, and there is a universal constant C_1 such that $\|F\|_1 \leqslant C_1 \|(\lambda_i)_{i\in\mathbb{N}}\|_1$.

There is valid also the analogue of Lemma 4.5 from [64], the only differences in the proof arise because of Q-RBUPU.

Lemma 3.4.8 (**Pap** [127]). *The net set $\{T_\Psi\}$ of Q-RBUPU, directed according to inclusions of the neighborhoods Q to $\{e = (0,1)\}$, is norm convergent as operators on $L^1(\mathbb{B})$: $\lim_{\Psi\to 0} |||T_\Psi - T|||_1 = 0$.*

Proof. The proof follows the steps of the proof of Lemma 4.5 from [64], the only difference occurs when we decompose the integral over the group using the R-BUPUs. For a given $F \in L^1(\mathbb{B})$ we can give the following estimate:

$$\|TF - T_\Psi F\|_1 = \left\| \left(\sum_i (F\psi_i - \langle F, \psi_i\rangle \delta_{x_i^{-1}}) \star G \right) \right\|_1 \leqslant$$

$$\leqslant \sum_i \left\| \int_{Qx_i} F(y^{-1})\psi_i (L_{y^{-1}}G - L_{x_i^{-1}}G) dy \right\|_1 \leqslant$$

$$\leqslant \sum_i \int_{Qx_i} |F(y^{-1})| \psi_i \|(L_{y^{-1}}G - L_{x_i^{-1}}G)\| dy \leqslant$$

$$\leqslant \sum_i \sup_{u\in Q} \|(L_{x_i^{-1}u^{-1}}G - L_{x_i^{-1}}G)\| \langle |F|, \psi_i\rangle \leqslant$$

$$\leqslant \sup_{u\in Q} \|(L_{u^{-1}}G - G)\|_1 \sum_i \langle |F|, \psi_i\rangle \leqslant \omega_Q(G) C_0 \|F\|_1,$$

where $\omega_Q(G) = \sup_{u\in Q} \|(L_{u^{-1}}G - G)\|_1$. Since $Q = Q_1 \times \mathbb{T}$ is invariant under the inverse operation i.e., $u \in Q$ if and only if $u^{-1} \in Q$, we have that $\omega_Q(G) = \sup_{u\in Q} \|(L_{u^{-1}}G - G)\|_1 = \sup_{u\in Q} \|(L_u G - G)\|_1$ is the modulus of continuity of G with respect to $\|.\|_1$. Thus from $G \in L^1(\mathbb{B})$ we have that

$$|||T_\Psi - T|||_1 \leqslant C_0 \omega_Q(G) \to 0 \quad \text{for} \quad Q \to \{e\}.$$

Now, taking in consideration that

$$V_g(U^{\alpha}_{a^{-1}}f) = L_{a^{-1}}V_g f,$$

from Lemma 3.4.8 we get in analogous way as in [64] the Theorem 4.7 that T_Ψ has an inverse and we get the following atomic decomposition result:

Theorem 3.4.9 (**Pap** [127]). *For any $g \in \mathcal{A}^1$, $g \neq 0$ and $\|Cg\| = 1$ there exist a neighborhood Q of the identity and a constant $C_1 > 0$, both depending only on g such that for every Q-dense family $(x_i)_{i\in\mathbb{N}}$ from right of the Blaschke group any $f \in \mathcal{H}^1$ can be written as*

$$f(z) = \sum_i \lambda_i (U^{\alpha}_{x_i^{-1}} g)(z) \quad \text{with} \quad \sum_i |\lambda_i| \leqslant C_1 \|f\|_{\mathcal{H}^1}, \tag{3.17}$$

the series is absolutely convergent in $\mathcal{H}^1$. The coefficients depend linearly on f, namely $\lambda_i = \int_{\mathbb{D}} T_\Psi^{-1}(V_g f(y^{-1}))\psi_i(y)dA(y)$.

Thus this gives an atomic decomposition of $f \in \mathcal{H}^1$ with atoms $U^{\alpha}_{x_i^{-1}} g$, $g \in \mathcal{A}^1$. For example one good choice is $g \in \mathcal{B}_1 \cup \{1\} \subset \mathcal{A}^1$. From Theorem 3.4.7 it follows that for $p > 2 + \frac{4}{\alpha}$ we have $A^p_\alpha \subset \mathcal{H}^1$, consequently the previous atomic decomposition is true also for A^p_α under the mentioned restrictions to the parameters.

The Q-density from right of the set $\{x_i = (b_i, -1)\}_{i\in\mathbb{N}}$ in the language of the complex analysis is equivalent to the ϵ-net property of $\{b_i\}_{i\in\mathbb{N}}$, with $\epsilon = \tanh r$ (see [93] pp. 172). From Lemma 8 ([56] pp. 188) for the lower density of the set $\{b_i\}$ we have

$$D^-(\{b_i\}) \geqslant \frac{(1-\tanh r)^2}{2\tanh^2 r}.$$

Using Theorem 5.23 from [93] pp. 161, we have that a separated sequence $\{b_i\}$ is a sampling sequence for A^p_α if and only if

$$D^-(\{b_i\}) > \frac{\alpha+1}{p}.$$

Let choose r so small that

$$\frac{(1-\tanh r)^2}{2\tanh^2 r} > \frac{\alpha+1}{p},$$

then $\{b_i\}$ is a sampling sequence for A^p_α.

Then for the special case $g = 1$ we obtain the following atomic decomposition: if $f \in A^p_\alpha$, $\alpha > 0$, and $p > 2 + \frac{4}{\alpha}$,

$$f = \sum \lambda_i(f) U^{\alpha}_{x_i^{-1}} 1 = \sum \lambda_i(f) \frac{(1-|b_i|^2)^{\frac{\alpha+2}{2}}}{(1-\overline{b_i}z)^{\alpha+2}}, \tag{3.18}$$

holds, which is very similar to the atomic decompositions obtained with complex analysis techniques (see [187], pp. 69). The difference is that in our case we have ℓ^1 information about the coefficients instead of ℓ^p information and the convergence is in $\mathcal{H}^1$ norm instead of A^p_α. Using the classical techniques of the complex analysis in the atomic decomposition of a function $f \in A^p_\beta$, the atoms are of form (see [187], pp. 69)

$$\frac{(1-|x_i|^2)^a}{(1-\overline{x_i}z)^b}.$$

Applying the Feichtinger – Gröchenig theory we obtain more general atoms for the weighted Bergman spaces, for example every function $g \in \mathcal{B}_1$ generates an atomic decomposition for $f \in A^p_\alpha$ with atoms of the form

$$U^\alpha_{x_i^{-1}} g.$$

3.5 Multiresolution in weighted Bergman spaces

In the previous subsection we presented atomic decomposition results in weighted Bergman spaces. It turned out that in the proofs of the results it was essential the integrability of the hyperbolic wavelet transform defined by (3.3). Consequently, these results are valid only for some weighted Bergman spaces. But what does happen when the integrability condition is not satisfied?

For example the unweighted case $\alpha = 0$, in the Bergman space, the integrability condition of the representation is not satisfied. Consequently, in this case the presented atomic decomposition results are not valid. In [128] Pap showed that, it is possible to construct a multiresolution analysis, using localized Bergman kernels in special sampling points. Later in [133] the result was extended for weighted Bergman spaces. In this subsection we present these discretization results. Pap showed that, as in the case of the Hardy spaces presented in the second chapter, an analogue of MRA decomposition can be constructed also in the weighted Bergman spaces.

Based on the MRA constructions in the Hardy and weighted Bergman spaces (see [126, 128, 133]) Nowak and Pap in [116] summarized the main idea of these constructions, describing a new method of construction of analytic wavelets which is applied in both of Hardy and weighted Bergman spaces. This method should be applied in the more general setting of reproducing kernel Hilbert spaces.

Although the main idea is similar to the case of the Hardy space, the construction of the MRA in the weighted Bergman space is more complicated than in the Hardy space. The first step is the construction of a new example of sampling set for the weighted Bergman space, which is related to the Blaschke group operation. This step is difficult in general. If once we have this, then the construction of the multiresolution levels are similar to the case of Hardy space. The next difficulty is to describe the orthogonal wavelets on the resolution levels, because in the case of the weighted Bergman space, they cannot be given explicitly in closed form. But we can give an algorithm to generate them, and using this we can prove that the projection to the resolution levels has similar interpolation properties like in the case of Hardy space. This projection operator gives opportunity of practical realization of the hyperbolic wavelet representation of a function belonging to the weighted Bergman space, if we can measure the values of the function on a given set of points inside the unit disc. We also studied the convergence properties of the hyperbolic wavelet representation.

In the construction of the MRA in weighted Bergman spaces we use frames obtained by localization of the weighted Bergman kernel. The localization is made in a set of sampling points connected to the Blaschke group. In this way we obtain so called hyperbolic wavelet frames. Recently, tight affine wavelet frames derived by the multiresolution analysis are used to open a few new areas of applications of frames. The application of tight wavelet frames in image restorations is one of them that includes image inpainting, image denoising, image deblurring and blind deburring, and image decompositions [10, 47, 158]. An up to date monograph in this domain is [104], where are collected the most important one- and multivariate results connected to affine wavelet frames (framelets) and the related MRA-s and their applications. In the recovery of missing data from incomplete and/or damaged and noisy samples, application of wavelet methods based on frames is more advanced due to the redundancy of frame systems. In the context of the introduced hyperbolic wavelet frames it would be interesting to study similar properties.

The plan of this section is as follows. We introduce a discrete subset of the Blaschke group, which is a sampling set for the weighted Bergman space. Using this special sampling set, we consider hyperbolic wavelet frames and we construct an analogue of MRA decomposition in the weighted Bergman space. First the different resolution spaces will be defined using the introduced non-orthogonal hyperbolic wavelet frames. Applying the Gram-Schmidt orthogonalization we consider the rational orthogonal basis on the n-th multiresolution level V_n. This system is the analogue of the Malmquist-Takenaka system in the Hardy spaces, possesses similar properties and is connected to the contractive zero divisors of a finite set in Bergman space. We prove that the projection operator $P_nf(z)$ on the resolution level V_n is convergent to f in A^2_α norm, and is also interpolation operator on the set the $\bigcup_{k=0}^{n} \mathcal{A}_k$, where $\mathcal{A}_k$ is defined by (3.23) with minimal norm and $P_nf(z) \to f(z)$ uniformly on every compact subset of the unit disc.

Compared with the classical affine multiresolution, according to the obtained results, we can conclude the following advantages of the constructed hyperbolic multiresolution in A^p_α.

1. The levels of the multiresolution are finite dimensional, which makes easier to find a basis on every level, but at the same time the density condition remains valid.

2. We can compute the wavelet coefficients exactly measuring the values of the function f at the points of the set $\mathcal{A} = \bigcup_{k=0}^{\infty} \mathcal{A}_k \subset \mathbb{D}$ defined by (3.23). We can write exactly the projection operator $(P_nf,\, n \in \mathbb{N})$ on the n-th resolution level.

3. At the same time $P_n f(z)$ is the best approximate interpolation operator on the set the $\bigcup_{k=0}^{n} \mathcal{A}_k$ inside the unit circle for the analytic continuation of f.

3.5.1 Special discrete subsets in $\mathbb{B}$ and their sampling property

Let us start with the definition of the main concepts. For $0 < p < \infty$, a sequence of points $\Gamma = \{z_k : k \in \mathbb{N}\}$ in the unit disc is sampling sequence for A^p_α, if there exist positive constants A and B such that

$$A||f||^p \leqslant \sum_{k=1}^{\infty} |f(z_k)|^p (1-|z_k|^2)^{2+\alpha} \leqslant B||f||^p, \quad f \in A^p_\alpha.$$

For $p = 2$, this inequality can be expressed in an equivalent form, using the localized weighted Bergman kernels in z_k. If

$$\varphi_k(z) = K(z, z_k)/\|K(z, z_k)\| = \frac{(1-|z_k|^2)^{\frac{\alpha+2}{2}}}{(1-\overline{z_k}z)^{\alpha+2}}$$

is the localized and normalized weighted Bergmen kernel, then the previous inequality is equivalent with the following

$$A||f||^2 \leqslant \sum_{k=1}^{\infty} |\langle f, \varphi_k \rangle|^2 \leqslant B||f||^2, \quad f \in A^2.$$

However, this last inequality shows that, $\{\varphi_k(z), k \in \mathbb{N}\}$ will constitute a frame for A^2_α, if and only if $\Gamma = \{z_k : k \in \mathbb{N}\}$ is a sampling set for A^2_α. The Bergman spaces A^p_α do have sampling sequences, but their construction is a difficult task. Some explicit examples are due to Seip, Duren, Schuster, Horowitz, Luecking (see for ex in [56]). An A^p_α sampling sequence is never an A^p_α zero-set, consequently a sampling set is a set of uniqueness (the values of the function in the sampling set determine uniquely the function). A total characterization of sampling sequences can be given with the uniformly discrete property and upper and lower Seip density of the set (see [56]). But the computation of the upper and lower density of a set is, in general, difficult. Duren, Schuster and Vukotic in [57] gave sufficient conditions based on the pseudo-hyperbolic metric which can be applied in the construction of the sampling sets without computing the Seip density. Using this sufficient condition it is easier to verify, if a set of points from the unit disc is sampling set. We remind that he pseudo-hyperbolic metric in the unit disc is defined by the formula

$$\rho(z, y) = \left| \frac{y - z}{1 - \overline{y}z} \right| \ (y, z \in \mathbb{D}).$$

A sequence of points $\Gamma = \{z_k\}$ in the unit disc is uniformly discrete (separated), if

$$\delta(\Gamma) = \inf_{j \neq k} \rho(z_j, z_k) = \delta > 0.$$

For $0 < \epsilon < 1$, a sequence of points $\Gamma = \{z_k : k \in \mathbb{N}\}$ in the unit disc is said to be ϵ-net, if each point $z \in \mathbb{D}$ has the property $\rho(z, z_k) < \epsilon$ for some z_k in Γ. An equivalent statement is, that $\mathbb{D} = \bigcup_{k=1}^{\infty} \Delta(z_k, \epsilon)$, where $\Delta(z_k, \epsilon)$ denotes a pseudo-hyperbolic disc.

In [57] it is shown that, if Γ is ϵ-net, then its lower density satisfies the following inequality

$$D^-(\Gamma) \geqslant \frac{(1-\epsilon)^2}{2\epsilon^2}.$$

If Γ is separated (uniformly discrete), and $D^-(\Gamma) > (\alpha+1)/p$, then is a sampling set for A^p_α (Theorem 5.23 of [93]). We will use this last sufficient condition in order to construct a sampling sequence in A^p_α.

Question: Is it possible to find a discrete subset $\{a_{k\ell} = (z_{k\ell}, 1) \in \mathbb{B}\}$ of the Blaschke group, a function $\varphi_{00} \in A^2_\alpha$, and to generate an adapted version of the multiresolution in the weighted Bergman space A^2_α using the images of this single function $\{U^\alpha_{a_{k\ell}}\varphi_{00}\}$ trough the representation?

In order to answer the formulated question first we construct a sampling set in the weighted Bergman space $A^2_\alpha(\mathbb{D})$, which is a discrete subset of the Blaschke group. Let us consider the following one parameter subgroups of the Blaschke group:

$$\mathbb{B}_1 := \{(r, 1) : r \in (-1, 1)\}, \quad \mathbb{B}_2 := \{(0, \epsilon) : \epsilon \in \mathbb{T}\}. \tag{3.19}$$

These subgroups generate $\mathbb{B}$, i.e.,

$$\mathbf{a} = (0, \epsilon_2) \circ (0, \epsilon_1) \circ (r, 1) \circ (0, \bar{\epsilon}_1) \quad (\mathbf{a} = (r\epsilon_1, \epsilon_2) \in \mathbb{B},\ r \in [0, 1), \epsilon_1, \epsilon_2 \in \mathbb{T}).$$

The subgroup $\mathbb{B}_1$ is the analogue of the group of dilation, $\mathbb{B}_2$ is the analogue of the group of translation (see Schipp [139]).

The group operation $(r, 1) = (r_1, 1) \circ (r_2, 1)$ in $\mathbb{B}_1$ can be expressed using the tangent hyperbolic and its inverse (ath) in the following way

$$r = \frac{r_1 + r_2}{1 + r_1 r_2} = \mathrm{th}(\mathrm{ath}\ r_1 + \mathrm{ath}\ r_2) \quad (r_1, r_2 \in (-1, 1)). \tag{3.20}$$

Let denote $r = \mathrm{th}\,\alpha, r_i = \mathrm{th}\,\alpha_i,\ i = 1, 2$. Then from

$$(r_1, 1) \circ (r_2, 1) = (\mathrm{th}\,\alpha_1, 1) \circ (\mathrm{th}\,\alpha_2, 1) = (\mathrm{th}\,(\alpha_1 + \alpha_2), 1),$$

it follows that $(\mathbb{B}_1, \circ)$ is isomorphic to $(\mathbb{R}, +)$. It is known that $(\mathbb{Z}, +)$ is a subgroup of $(\mathbb{R}, +)$, hence $\overline{\mathbb{B}_1} = \{(\mathrm{th}\,k, 1),\ k \in \mathbb{Z}\}$ is an one parameter subgroup of $(\mathbb{B}_1, \circ)$ (see [154]).

Let $a > 1$, and let us consider the following subset of $(\mathbb{B}, \circ)$:

$$\mathbb{B}_3 = \left\{(r_k, 1) :\ r_k = \frac{a^k - a^{-k}}{a^k + a^{-k}},\ k \in \mathbb{Z}\right\}. \tag{3.21}$$

It can be proved that $(\mathbb{B}_3, \circ)$ is a discrete subgroup of $(\mathbb{B}, \circ)$, where we have the following composition rule: $(r_k, 1) \circ (r_n, 1) = (r_{k+n}, 1)$. The pseudo-hyperbolic distance of the points $r_k,\ r_n$ has the property:

$$\rho(r_k, r_n) := \frac{|r_k - r_n|}{|1 - r_k \overline{r_n}|} = \left| \frac{\frac{a^k - a^{-k}}{a^k + a^{-k}} - \frac{a^n - a^{-n}}{a^n + a^{-n}}}{1 - \frac{a^k - a^{-k}}{a^k + a^{-k}} \frac{a^n - a^{-n}}{a^n + a^{-n}}} \right| = |r_{k-n}|.$$

This property implies that the sequence $(r_k, k \in \mathbb{N})$ forms an equidistant division of the interval $[0, 1)$ in the pseudo-hyperbolic metric.

Let $N(a, k), k \geqslant 1,\ N(a, 0) := 1$, be an increasing sequence of natural numbers. Let us consider the following set of points $z_{00} := 0$,

$$\mathcal{A} = \{z_{k\ell} = r_k e^{i\frac{2\pi\ell}{N}}, \quad \ell = 0, 1, ..., N(a, k) - 1, \quad k = 0, 1, 2, ...\}. \tag{3.22}$$

For a fixed $k \in \mathbb{N}$, let

$$\mathcal{A}_k = \{z_{k\ell} = r_k e^{i\frac{2\pi\ell}{N(a,k)}}, \ \ell \in \{0, 1, ..., N(a, k) - 1\}\}. \tag{3.23}$$

be the set of uniformly distributed points on the circle with radius r_k. This set of points will generate the level k of the multiresolution.

The question is how to choose a and $N = N(a, k)$ such that $\mathcal{A}$ to be a sampling set in the weighted Bergman space $A^p_\alpha(\mathbb{D})$. The question was answered by Pap first for the unweighted case, for $\alpha = 0$ in [128], then extended in general in [133]. It was proved that for a convenient choice of a and $N(a, k)$ one has

1. $\mathcal{A}$ is uniformly discrete,
2. $\mathcal{A}$ is an ϵ-net set for some $0 < \epsilon < 1$.

Theorem 3.5.1 (**Pap** [133]). *Let $a > 1$ and let $(N(a, k) = a^{2k} b, k \geqslant 1)$. Choose $0 < b < \infty$ such that $N(a, k) \in \mathbb{N}$, and consider the set of points $\mathcal{A}$ defined by (3.22). Let us denote by $K := 1 + \frac{(a - a^{-1})^2}{4} + \frac{a^2}{4b^2}\pi^2$. If*

$$\sqrt{1 - 1/K} < \frac{1}{1 + \sqrt{\frac{2(\alpha+1)}{p}}},$$

then $\mathcal{A}$ is a sampling set for A^p_α.

Proof of Theorem 3.5.1.

In [128] Pap proved that, if there exists $b = \lim_{k \to \infty} N(a, k) a^{-2k}$, and if $(N(a, k) a^{-2k}, k \geqslant 1)$ is increasing sequence and b is finite, then $\mathcal{A}$ is uniformly discrete and the separation constant satisfies

$$\delta \geqslant \min \left\{ r_1, \frac{1}{\sqrt{1 + b^2}} \right\}.$$

In [128] it was also proved, that if $(N(a,k)a^{-2k}, k \geqslant 1)$ is decreasing and $0 < b < \infty$, then the set A is ϵ_0-net, where $\epsilon_0 = \sqrt{1 - 1/K}$, with $K := 1 + \frac{(a-a^{-1})^2}{4} + \frac{a^2}{4b^2}\pi^2$.

Indeed, for given $z = re^{i\theta} \in \mathbb{D}$ we take k and $j \in \{0, 1, \cdots N(a,k) - 1\}$ such that $r_k < r \leqslant r_{k+1}$, $\theta \in \left[\frac{2\pi j}{N(a,k)}, \frac{2\pi(j+1)}{N(a,k)}\right)$, $\theta_{kj} = \frac{2\pi j}{N(a,k)}$. Then

$$\frac{1}{1-\rho^2(z,z_{kj})} = \frac{(1-rr_k)^2 + 4rr_k \sin^2\frac{\theta-\theta_{kj}}{2}}{(1-r^2)(1-r_k^2)} = 1 + \frac{(r-r_k)^2 + 4rr_k \sin^2\frac{\theta-\theta_{kj}}{2}}{(1-r^2)(1-r_k^2)} \leqslant$$

$$1 + \frac{(r-r_k)^2 + 4rr_k\frac{\pi^2}{N^2(a,k)}}{(1-r^2)(1-r_k^2)} = 1 + \frac{(a-a^{-1})^2}{4} + \frac{(a^{2k+2} - a^{-2k-2})(a^{2k} - a^{-2k})}{4}\frac{\pi^2}{N^2(a,k)}.$$

If $(N(a,k)a^{-2k}, k \geqslant 1)$ is decreasing and $b = \lim_{k\to\infty} N(a,k)a^{-2k} \in (0,\infty)$, then the last term in the previous inequality is upper bounded by

$$K := 1 + \frac{(a-a^{-1})^2}{4} + \frac{a^2}{4b^2}\pi^2.$$

Then for $\epsilon_0 = \sqrt{1-1/K}$, we have $\rho(z, z_{kj}) < \epsilon_0$.

If $N(a,k)a^{-2k} = b$, for $k \geqslant 1$, and $0 < b < \infty$, then $\mathcal{A}$ is in the same time uniformly discrete and ϵ_0-net. In [57] it is shown that if $\mathcal{A}$ is ϵ_0-net, then the lower density of the set satisfies

$$D^-(\mathcal{A}) \geqslant \frac{(1-\epsilon_0)^2}{2\epsilon_0^2}.$$

If $\mathcal{A}$ is separated (is a uniformly discrete) and $D^-(\mathcal{A}) > (\alpha+1)/p$ then it is a sampling set for A^p_α (see Theorem 5.23 of [93]).

Using this results we get that if

$$\epsilon_0 = \sqrt{1-1/K} < \frac{1}{1+\sqrt{\frac{2(\alpha+1)}{p}}},$$

then

$$D^-(\mathcal{A}) \geqslant \frac{(1-\epsilon_0)^2}{2\epsilon_0^2} > (\alpha+1)/p,$$

which implies that $\mathcal{A}$ is a sampling set for A^p_α.

Remarks

1. As it was showed in [128], for $\alpha = 0$, from this theorem we obtain that if $\mathcal{A}$ is a sampling set for the Bergman space A^p, then

$$(a-a^{-1})^2 < 2p,$$

therefore a must be in the interval $(1, \frac{\sqrt{2p}+\sqrt{2p+4}}{2})$. Then we can always choose $N = N(a,k)$ big enough, such that the sampling condition to be satisfied.

2. From the point of view of computations and to have on every circle the less possible points, for $p = 2, \alpha = 0$ a convenient choice is $a = 2$, and $N(2,k) = 2^{2k+\beta}$ for $k \geqslant 1$ with β a fixed integer. Then $b = 2^\beta$, and the smallest value for β for which the sampling condition is satisfied is $\beta = 3$. Also, on the k-th circle we will have $N_1(2,k) = 2^{2k+3}$ equidistant points corresponding to the roots of order 2^{2k+3} of the unity. If $a = \sqrt{2}$, then for sampling we need $N_1(\sqrt{2}, k) = 2^{k+2}$ points.

3. For $p = 2, \alpha > -1$ in order to have $\mathcal{A}$ a sampling set for A^2_α we have to choose a and the number of the points $N(a,k) = a^{2k}b$ on the level k such that

$$\frac{(a - a^{-1})^2}{4} + \frac{a^2}{4b^2}\pi^2 < \frac{1}{\sqrt{\alpha + 1}}.$$

From now on we will concentrate on this case and using this special sampling set we will construct multiresolution analysis in the A^2_α.

3.5.2 Multiresolution analysis in the weighted Bergman space A^2_α

For $p = 2$ using the set (3.22) satisfying the conditions of the Theorem 3.5.1 we define multiresolution in the weighted Bergman space. To show the analogy with the affine wavelet multiresolution, we first represent the levels V_n by non-orthogonal frames associated to the set (3.22), then we construct an orthonormal bases on the V_n. We give also an orthogonal basis in W_n which is orthogonal to V_n. We will show that the analogue of the Malmquist-Takenaka systems for weighted Bergman space will span the resolution spaces, and the density property will be fulfilled, i.e., $\overline{\bigcup_{k=1}^\infty V_k} = A^2_\alpha$ in norm.

We show that the projection $P_n f$ on the n-th resolution level is an interpolation operator in the unit disc until the n-th level, which converges in A^2_α norm to f.

Let us consider $a > 1$, denote by $r_k = \frac{a^k - a^{-k}}{a^k + a^{-k}}$, $k \in \mathbb{N}$, and the concentric circles with radius r_k. On the circle with radius r_k let us consider $N_k = N(a,k)$ equidistantly situated points $z_{k\ell} = r_k e^{i\frac{2\pi\ell}{N(a,k)}}$, such that $N(a,k) = a^{2k}b \in \mathbb{N}$ satisfies

$$0 < b < \infty, \quad (a - a^{-1})^2 + \pi^2\frac{a^2}{b^2} < 4\frac{1}{\sqrt{\alpha + 1}}.$$

If these conditions are satisfied then, due to Theorem 3.5.1 the set $\mathcal{A}$ given by (3.22) is a sampling set for A^2_α. This implies that the set of normalized and localized weighted Bergman kernels in these points

$$\left\{ \varphi_{k\ell}(z) = \frac{(1 - r_k^2)^{\frac{\alpha+2}{2}}}{(1 - \overline{z_{k\ell}}z)^{2+\alpha}},\ \varphi_{00} = 1,\ k = 0, 1, \cdots, \ \ell = 0, 1, \cdots N(a,k) - 1 \right\}$$

will constitute a frame system for A_α^2. This frame system can be derived from a single function using the representation and the discrete subset $\mathcal{A}$ of the Blaschke group in the following way:

$$\varphi_{k\ell}(z) = (U^\alpha_{(z_{k\ell},1)^{-1}}\varphi_{00})(z).$$

Due to this observation, we can consider them as an analogue of affine wavelet frames, and we call them hyperbolic wavelet frames.

From the frame theory (see for example in [90]), it follows that every function f from A_α^2 can be represented as

$$f(z) = \sum_{(k,\ell)} c_{k\ell}\varphi_{kl}(z)$$

for some $\{c_{k\ell}\} \in \ell^2$, with the series converging in A_α^2 norm. The determination of the coefficients is related to the construction of the inverse frame operator (see [90]), which is not an easy task in general. In [128, 133] Pap constructed a new approximation process for $f \in A_\alpha^2$, and gave an exactly defined algorithmic scheme for the determination of the coefficients.

Let us consider the function $\varphi_{00} = 1$ and let define $V_0 := \{c\varphi_{00},\ c \in \mathbb{C}\}$. Let us consider the non-orthogonal hyperbolic wavelets at the first level

$$\varphi_{1\ell}(z) = (U^\alpha_{(z_{1\ell},1)^{-1}}\varphi_{00})(z) = \frac{(1-r_1^2)^{\frac{\alpha+2}{2}}}{(1-\overline{z_{1\ell}}z)^{2+\alpha}}, \quad \ell = 0, 1, \cdots, N(a,1)-1.$$

They can be obtained from φ_{10} using the analogue of translation operator which in the unit disc is a multiplication by a unimodular complex number, and from φ_{00} using first the representation operator $U_{(r_1,1)^{-1}}$ followed by the translation operator

$$\varphi_{1\ell}(z) = \varphi_{10}(ze^{-\frac{2\pi i\ell}{N(a,1)}}) = (U^\alpha_{(r_1,1)^{-1}}\varphi_{00})(ze^{-\frac{2\pi i\ell}{N(a,1)}}).$$

In order to define the levels of the muliresolution let us define the first resolution level as follows:

$$V_1 := \left\{ f : \mathbb{D} \to \mathbb{C},\ f(z) = \sum_{k=0}^{1}\sum_{\ell=0}^{N(a,k)-1} c_{k\ell}\varphi_{k\ell},\ c_{k\ell} \in \mathbb{C} \right\}.$$

Let us consider the nonorthogonal wavelets on the n-th level

$$\varphi_{n\ell}(z) = (U^\alpha_{(z_{n\ell},1)^{-1}}\varphi_{00})(z) = \frac{(1-r_n^2)^{\frac{\alpha+2}{2}}}{(1-\overline{z_{n\ell}}z)^{\alpha+2}}, \quad \ell = 0, 1, ..., N(a,n)-1, \tag{3.24}$$

which can be obtained from φ_{n0} using the translation operator, and from φ_{00} using the representation $U^\alpha_{((r_{n-1},1)\circ(r_1,1))^{-1}}$, and the translations

$$\varphi_{n,\ell}(z) = (U^\alpha_{((r_{n-1},1)\circ(r_1,1))^{-1}}\varphi_{00})(ze^{-i\frac{2\pi\ell}{N(a,n)}}).$$

Let us define the n-th resolution level by

$$V_n := \left\{ f : \mathbb{D} \to \mathbb{C},\ f(z) = \sum_{k=0}^{n} \sum_{\ell=0}^{N(a,k)-1} c_{k\ell}\varphi_{k\ell},\ c_{k\ell} \in \mathbb{C} \right\}. \tag{3.25}$$

The closed subset V_n is spanned by

$$\{\varphi_{k\ell},\ \ell = 0, 1, ..., N(a,k) - 1,\ k = 0, ..., n\}.$$

Continuing this procedure we obtain a sequence of closed, nested subspaces of A^2_α for $z \in \mathbb{D}$,

$$V_0 \subset V_1 \subset V_2 \subset .....V_n \subset ....\ A^2_\alpha.$$

Due to Theorem 3.5.1 the normalized kernels

$$\{\varphi_{kl}(z),\ k = 0, 1, \cdots,\ \ell = 0, 1, \cdots N(a,k) - 1\}$$

form a frame system for A^2_α. This implies, that this is a complete and closed set in norm, consequently the density property is satisfied, i.e.,

$$\overline{\bigcup_{n\in\mathbb{N}} V_n} = A^2_\alpha.$$

From now on, for simplicity, we consider $a = 2$ and $N(2,k) = 2^{2k}b, b \in \mathbb{N}$ satisfies the following conditions:

$$0 < b < \infty, \quad (2 - 2^{-1})^2 + \pi^2 \frac{2^2}{b^2} < 4\frac{1}{\sqrt{\alpha + 1}}.$$

For $\alpha = 0$ a good choice is $N(2,k) = 2^{2k+3}$. In general on the circle k-th we will have $N(2,k) = 2^{2k}b$ points.

We show that, if $f \in V_n$, then $U^\alpha_{(r_1,1)^{-1}} f \in V_{n+1}$. This is the analogue of the dilation. For this it is sufficient to show that, for $k = 0, 1, .., n,\ \ell = 0, 1, ..., 2^{2k}b - 1$, we have

$$U^\alpha_{(r_1,1)^{-1}}(\varphi_{k\ell})(z) = U^\alpha_{(r_1,1)^{-1}}[(U^\alpha_{(r_k,1)^{-1}}\varphi_{00})](ze^{-i\frac{2\pi\ell}{2^{2k}b}}) =$$

$$= [(U^\alpha_{(r_{k+1},1)^{-1}}\varphi_{00})](ze^{-i\frac{2\pi 4\ell}{2^{2(k+1)}b}}) = \varphi_{k+1\ell'} \in V_{n+1},$$

for $\ell' = 4\ell \in \{0, 1, ..., 2^{2(k+1)}b - 1\}$.

Summarizing we have constructed a sequence of subspaces $(V_j,\ j \in \mathbb{N})$ of A^2_α with following properties:

1. (nested) $V_j \subset V_{j+1} \subset A^2_\alpha$,
2. (density) $\overline{\cup V_j} = A^2_\alpha$,
3. (analog of dilatation) $U^\alpha_{(r_1,1)^{-1}}(V_j) \subset V_{j+1}$,

4. (basis) There exist $\{\varphi_{k\ell},\ k=0,1,..,n,\ \ell=0,1,...,2^{2k}b-1\}$ (orthonormal or frame) bases in V_j.

This is the adapted definition of the multiresolution analysis in the weighted Bergman spaces. These four properties are required for $(V_j,\ j\in\mathbb{N})$ to form a hyperbolic wavelet multiresolution analysis (MRA) in the weighted Bergman spaces.

Because $\mathcal{A}$ is a sampling set, it follows that it is a set of uniqueness for A^2_α. This means, that every function $f\in A^2_\alpha$ is uniquely determined by the values $\{f(z_{k\ell}),\ z_{k\ell}\in\mathcal{A}\}$. In [186] Zhu described in general, how can be recaptured a function from a Hilbert space, when the values of the function on a set of uniqueness are known, and developed in details this process in the Hardy space. At the beginning we will follow the steps of the recapturation process, and we will combine this with the multiresolutin analysis. The elements of the set $\mathcal{A}$ are different numbers, which implies that the localized weighted Bergman kernels

$$\left\{\frac{1}{(1-\overline{z_{k\ell}}z)^{2+\alpha}},\ \ell=0,1,...,N(2,k)-1,\ k=0,1,...,n.\right\}$$

are linearly independent, and constitute a non-orthogonal basis in V_n.

Using Gram-Schmidt orthogonalization process they can be orthogonalized. Denote by $\psi_{k\ell}$ the resulting functions. They form a system, which can be viewed as the analogue of the Malmquist–Takenaka system in the Hardy space. These functions can be obtained using the following two methods. The first method arises from the orthogonalization procedure. To describe this, let reindex the points of the set $\mathcal{A}$ as follows: $a_1=z_{00}, a_2=z_{10}, a_3=z_{11},\cdots,a_{N(2,1)+1}=z_{1N(2,1)-1},\cdots,a_m=z_{k\ell}\cdots,k=0,1,....,\ell=0,1,...,N(2,k)-1$, and denote by $K_\alpha(z,z_{k\ell})=\frac{1}{(1-\overline{z_{k\ell}}z)^{2+\alpha}}:=K(z,a_m)$. The resulted orthonormal system is

$$\phi_{00}(z)=\phi(a_1,z)=\frac{K_\alpha(z,a_1)}{\|K_\alpha(.,a_1)\|},$$

$$\phi_{k\ell}(z)=\phi(a_1,a_2,...,a_m,z)=$$

$$=K(z,a_m)-\sum_{i=1}^{m-1}\phi(a_1,a_2,...,a_i,z)\frac{\langle K(.,a_m),\phi(a_1,a_2,...,a_i,.)\rangle}{\|\phi(a_1,a_2,...,a_i,.)\|^2}.$$

Thus the normalized functions

$$\left\{\psi_{k\ell}(z)=\frac{\phi_{k\ell}(z)}{\|\phi_{k\ell}\|},\ k=1,2,\cdots,\ell=0,1,\cdots,N(2,k)-1\right\}$$

become an orthonormal system. Applying similar construction in Hardy space, with the Cauchy kernel as reproducing kernel, the result of the orthogonalization process can be written in closed form using the Blaschke products, and in this way we get the Malmquist-Takenaka system. Unfortunately, in our situation the result of the orthogonalization cannot be written in closed form, and the properties of the system cannot be seen from the previous construction.

Another approach for the construction is given by Zhu in [186]. He proved that, the result of the Gram-Schmidt process is connected to some reproducing kernels, and the contractive zero divisors. Let denote $A_m = \{a_1, a_2, \cdots a_m\}$ a set of distinct points in the unit disc. Let H_{A_m} be the subspace of A^2_α consisting of all functions in A^2_α which vanish on A_m. H_{A_m} is a closed subspace of A^2_α and denote by K_{A_m} the reproducing kernel of H_{A_m}. These reproducing kernels satisfies the recursion formula

$$K_{A_{m+1}}(z,w) = K_{A_m}(z,w) - \frac{K_{A_m}(z,a_{m+1})K_{A_m}(a_{m+1},w)}{K_{A_m}(a_{m+1},a_{m+1})}, m \geqslant 0, \tag{3.26}$$

$$K_{A_0} := K_\alpha(z,w) = \frac{1}{(1-\overline{w}z)^{2+\alpha}}.$$

The result of the Gram-Schmidt process gives the orthonormal hyperbolic wavelet system in the weighted Bergman space and can be expressed as

$$\frac{K(z,a_1)}{\sqrt{K(a_1,a_1)}}, \frac{K_{A_1}(z,a_2)}{\sqrt{K_{A_1}(a_2,a_2)}}, \cdots \frac{K_{A_{m-1}}(z,a_m)}{\sqrt{K_{A_{m-1}}(a_m,a_m)}}, \cdots . \tag{3.27}$$

One element of the constructed orthonormal system corresponding to $z_{k\ell} = a_m$ and denoted by

$$\psi_{k\ell}(z) = \frac{K_{A_{m-1}}(z,a_m)}{\sqrt{K_{A_{m-1}}(a_m,a_m)}}, \tag{3.28}$$

is the solution of the problem

$$\sup\{Ref(a_m) : f \in H_{A_{m-1}}, \|f\| \leqslant 1\}.$$

These extremal functions in the context of the Bergman spaces have been studied by Hedenmalm [92]. The main result in [92] is that the function

$$\frac{K_{A_{m-1}}(z,a_m)}{\sqrt{K_{A_{m-1}}(a_m,a_m)}}$$

is a contractive divisor on the Bergman space, vanishes on A_{m-1}, and if $\mathcal{A}$ is not a zero set for A^2, as is in our case, the functions converge to 0 as $m \to \infty$. In Hardy space the partial products of a Blaschke product corresponding to a nonzero set own all these nice properties.

From the Gram-Schmidt orthogonalization process it follows, that

$$V_n = span\{\psi_{k\ell},\ \ell = 0,1,...,N(2,k)-1,\ k = 0,\cdots,n\}.$$

The wavelet space W_n is the orthogonal complement of V_n in V_{n+1}. We will prove that

$$W_n = span\{\psi_{n+1\ell},\quad \ell = 0,1,...,N(2,n+1)-1\}. \tag{3.29}$$

Indeed if $f \in V_n$, one has $f(z) = \sum_{k=0}^{n} \sum_{\ell=0}^{N(2,k)-1} c_{k\ell}\varphi_{k\ell} \subset A_\alpha^2$, then using that $\varphi_{k\ell}$ is a localized property reproducing kernel and

$$\psi_{n+1\ell}(z_{k\ell}) = \frac{K_{A_m}(z_{k\ell}, a_{m+1})}{\sqrt{K_{A_m}(a_{m+1}, a_{m+1})}} = 0$$

(because K_{A_m} is the reproducing kernel of H_{A_m} consisting of all functions in A_α^2 which vanish on A_m). We obtain that

$$\langle \psi_{n+1j}, f \rangle = \sum_{k=0}^{n} \sum_{\ell=0}^{N(2,k)-1} c_{k\ell} \langle \psi_{n+1j}, \varphi_{k\ell} \rangle =$$

$$\sum_{k=0}^{n} \sum_{\ell=0}^{N(2,k)-1} c_{k\ell}(1 - r_k^2)^{\frac{\alpha+2}{2}} \psi_{n+1\ell}(z_{k\ell}) = 0, \;\; j = 0, 1, ..., N(2, n+1) - 1.$$

We have proved that for $f \in V_n$

$$\langle f, \psi_{n+1j} \rangle = 0,$$

which is equivalent with

$$\psi_{n+1j} \perp V_n, \quad (j = 0, 1, ..., N(2, n+1) - 1).$$

From

$$V_{n+1} = V_n \bigoplus span\{\varphi_{n+1,j}, \;\; j = 0, 1, ..., N(2, n+1) - 1\}$$

it follows that, W_n is an $N(2, n+1)$ dimensional space and

$$W_n = span\{\psi_{n+1\ell}, \quad \ell = 0, 1, ..., N(2, n+1) - 1\}.$$

3.5.3 The projection operator corresponding to the n-th resolution level

Let us consider the orthogonal projection operator of an arbitrary function $f \in A_\alpha^2$ on the multiresolution level V_n defined by (3.25), given by

$$P_n f(z) = \sum_{k=0}^{n} \sum_{\ell=0}^{N(2,n)-1} \langle f, \psi_{k\ell} \rangle \psi_{k\ell}(z). \tag{3.30}$$

This operator is called the projection of f at nth scale or resolution level.

Theorem 3.5.2 (**Pap** [133])**.** *For any $f \in A^2_\alpha$ the projection operator $P_n f$ is at the same time an interpolation operator in the points*

$$z_{k\ell} = r_k e^{i\frac{2\pi\ell}{N(2,k)}}, \ (\ell = 0, ..., N(2,k) - 1, \quad k = 0, ..., n).$$

For any $f \in A^2_\alpha$ the projection operator $P_n f$ is norm convergent in A^2_α to f i.e.,

$$\|f - P_n f\| \to 0, \quad n \to \infty,$$

and uniformly convergent inside the unit disc on every compact subset.

Proof of Theorem 3.5.2. Let us consider $N = 1 + N(2,1) + \cdots + N(2,n)$, and the corresponding kernel function of the projection operator

$$\mathbf{K}_N(z,\xi) = \sum_{k=0}^{n} \sum_{\ell=0}^{N(2,k)-1} \overline{\psi_{k\ell}(\xi)}\psi_{k\ell}(z) =$$

$$\sum_{m=1}^{N} \frac{K_{A_{m-1}}(z,a_m)}{\sqrt{K_{A_{m-1}}(a_m,a_m)}} \overline{\left(\frac{K_{A_{m-1}}(\xi,a_m)}{\sqrt{K_{A_{m-1}}(a_m,a_m)}}\right)} = \sum_{m=1}^{N} \frac{K_{A_{m-1}}(z,a_m)K_{A_{m-1}}(a_m,\xi)}{K_{A_{m-1}}(a_m,a_m)}.$$

From the recursion relation (3.26) it follows that

$$\mathbf{K}_N(z,\xi) = \sum_{m=1}^{N} (K_{A_{m-1}}(z,\xi) - K_{A_m}(z,\xi)) = K(z,\xi) - K_{A_N}(z,\xi). \tag{3.31}$$

From this relation and $K_{A_N}(z_{k\ell},\xi) = 0$ for $z_{k\ell}$, $(\ell = 0, ..., N(2,k)-1, \quad k = 0, ..., n)$ it follows that the values of the kernel-function $\mathbf{K}_N(z,\xi)$ in these points $z_{k\ell}$ $(\ell = 0, ..., N(2,k)-1, \quad k = 0, ..., n)$ are equal to the localized weighted Bergman kernels

$$\mathbf{K}_N(z_{kl},\xi) = \frac{1}{(1 - z_{k\ell}\overline{\xi})^{2+\alpha}}.$$

As a consequence we get that

$$P_n f(z_{k\ell}) = \int_{\mathbb{D}} f(w)\mathbf{K}_N(z_{k\ell},w)dA_\alpha(w) = \int_{\mathbb{D}} \frac{f(w)}{(1 - \overline{w}z_{k\ell})^{2+\alpha}} dA_\alpha(w) = f(z_{k\ell})$$

for $\ell = 0, ..., N(2,k) - 1, \quad k = 0, ..., n$. We obtain that $P_n f$ is an interpolation operator for every $f \in A^2_\alpha$ on the set $\cup_{k=0}^{n} \mathcal{A}_k$.

Because $\{\psi_{k\ell}, \ k = 0, \cdots, \infty, \ \ell = 0, 1, ..., N(2,k) - 1\}$ is a closed set in the Hilbert space A^2_α, we have that $\|f - P_n f\| \to 0$ as $n \to \infty$. Since convergence in A^2_α norm implies uniform convergence on every compact subset inside the unit disc, we conclude that $P_n f(z) \to f(z)$ uniformly on every compact subset of the unit disc. From Theorem 5.3.1 of [134] there exists a unique $\hat{f}_n \in V_n$ with minimal norm such that

$$\hat{f}_n(z_{k\ell}) = f(z_{k\ell}), \quad (\ell = 0, ..., N(2,k) - 1, \quad k = 0, ..., n), \tag{2.32}$$

$\hat{f}_n$ is uniquely determined by the interpolation conditions and is equal to the orthogonal projection of f on V_n, thus $\hat{f}_n(z) = P_n f(z)$.

3.5.4 Reconstruction algorithm

In what follows we propose a computational scheme for the best approximant in the wavelet base $\{\psi_{k\ell},\ \ell = 0, 1, ..., N(2, k) - 1,\ k = 0, ..., n\}$.

The set of coefficients of the best approximant $P_n f$

$$\{b_{k\ell} = \langle f, \psi_{k\ell}\rangle,\ \ell = 0.1, \cdots, N(2, k) - 1 \quad k = 0, 1, \cdots, n\} \tag{3.32}$$

is the (discrete) hyperbolic wavelet transform of the function $f \in A^2_\alpha$. Thus it is important to have an efficient algorithm for the computation of the coefficients.

The coefficients of the projection operator $P_n f$ can be computed, if we know the values of the functions on $\bigcup_{k=0}^n \mathcal{A}_k$. For this reason we express first the function $\psi_{k\ell}$ using the bases $\{\varphi_{k'\ell'}\ \ell' = 0, 1, \cdots, N(2, k') - 1,\ k' = 0, \cdots, k\}$, i.e., we write the partial fraction decomposition of $\psi_{k\ell}$:

$$\psi_{k\ell}(\xi) = \sum_{k'=0}^{k-1} \sum_{\ell'=0}^{N(2,k')-1} c_{k'\ell'} \frac{1}{(1 - \overline{z_{k'\ell'}}\xi)^{2+\alpha}} + \sum_{j=0}^{\ell} c_{kj} \frac{1}{(1 - \overline{z_{kj}}\xi)^{2+\alpha}}. \tag{3.33}$$

Using the orthogonality of the functions $\{\psi_{k'\ell'}\ \ell' = 0, 1, \cdots, N(2, k')-1,\ k' = 0, \cdots, k\}$ and the properties of the reproducing kernel we obtain that the coefficients of the projection operators can be computed exactly if we know the values of the function on the sampling set, i.e.,

$$\begin{aligned}\langle f, \psi_{k\ell}\rangle &= \sum_{k'=0}^{k-1} \sum_{\ell'=0}^{N(2,k')-1} \overline{c_{k'\ell'}} \left\langle f(\xi), \frac{1}{(1 - \overline{z_{k'\ell'}}\xi)^{2+\alpha}} \right\rangle + \sum_{j=0}^{\ell} \overline{c_{kj}} \left\langle f(\xi), \frac{1}{(1 - \overline{z_{kj}}\xi)^{2+\alpha}} \right\rangle = \\ &= \sum_{k'=0}^{k-1} \sum_{\ell'=0}^{N(2,k')-1} \overline{c_{k'\ell'}} f(z_{k'\ell'}) + \sum_{j=0}^{\ell} \overline{c_{kj}} f(z_{kj}).\end{aligned} \tag{3.34}$$

In [33] Christensen, Gröchening, Olafsson derived atomic decomposition and frames for weighted Bergman spaces of several complex variables on the unit ball. Theorem 1.2. of this paper is a generalization for n dimensional weighted Bergman spaces of the theorem obtained by Pap in [127]. They also obtain frame expansions for the n dimensional Bergman spaces, which is related to the existence of sampling, analogous tho the results presented in this section for the one dimensional unit disc.

Chapter 4

Equilibrium conditions for the Malmquist–Takenaka systems

In Chapter 2 in the construction of analytic wavelets appeared the Malquist -Takenaka system with a special localization of poles. In this chapter we will present results connected to these systems in generality. We give an overview of the discretization results connected to Malmquist–Takenaka systems for the unit disc and upper half plane. We prove that the discretization nodes on the real line have similar properties like the discretization nodes on the unit circle: they satisfy some equilibrium conditions and they are stationary points of some logarithmic potential. The problems whether they are the minimum of a logarithmic potential was formulated and solved in a special case. These results were published by Pap and Schipp in [137, 118, 119, 130]. The formulated problem was solved in generality recently in [79] by Marcell Gaál, Béla Nagy, Zsuzsanna Nagy-Csiha, Szilárd Révész.

4.1 Malmquist–Takenaka systems

The first mention of rational orthonormal systems in the Hardy space of complex variable functions seems to have occurred in the work of Takenaka and Malmquist [110, 163]. These systems can be viewed as extensions of the trigonometric system on the unit circle, that corresponds to the special choice when all poles are located at the origin.

This orthonormal system is generated by a sequence $a = (a_1, a_2, ...)$ of complex numbers, $a_n \in \mathbb{D}$ of the unit disc $\mathbb{D}$ and can be expressed by the Blaschke functions

$$B_a(z) := \frac{z-a}{1-\bar{a}z} \qquad (a \in \mathbb{D},\ z \in \mathbb{C},\ 1-\bar{b}z \neq 0).$$

On the unit circle, B_a can be written in the form

$$B_a(e^{it}) = e^{i\beta_a(t)} \quad (t \in \mathbb{R}, a = re^{i\tau} \in \mathbb{D}),$$

where

$$\beta_a(t) := \tau + \gamma_s(t-\tau), \quad \gamma_s(t) := 2\arctan\left(s\tan\frac{t}{2}\right) \ \left(t \in [-\pi,\pi),\ s := \frac{1+r}{1-r}\right)$$

and γ_s is extended to $\mathbb{R}$ by $\gamma_s(t+2\pi) = 2\pi + \gamma_s(t)$ $(t \in \mathbb{R})$. For the detailed description of the β functions see [20].

The Malmquist-Takenaka system (M-T) $\Phi_n = \Phi_n^a$ $(n \in \mathbb{N}^*)$ is defined by

$$\Phi_1(z) = \frac{\sqrt{1-|a_1|^2}}{1-\overline{a_1}z}, \Phi_n(z) = \frac{\sqrt{1-|a_n|^2}}{1-\overline{a_n}z}\prod_{k=1}^{n-1} B_{a_k}(z),\ n \geqslant 2. \tag{4.1}$$

When all the parameters are equal, i.e., $a_n = a$, $n \in \mathbb{N}^*$, we obtain the so called discrete Laguerre system and particularly, when $a_n = 0$, $n \in \mathbb{N}^*$ we obtain the trigonometric system. These functions form an orthonormal system on the unit circle i.e.,

$$\langle \Phi_n, \Phi_m \rangle = \frac{1}{2\pi}\int_0^{2\pi} \Phi_n(e^{it})\overline{\Phi_m(e^{it})}dt = \delta_{mn} \quad (m,n \in \mathbb{N}^*),$$

where δ_{nm} is the Kronecker symbol. If the sequence $a = (a_1, a_2, ...)$ satisfies the non-Blaschke condition

$$\sum_{n\geqslant 1}(1-|a_n|) = +\infty,$$

then the corresponding M-T system is complete in the Hardy space of the unit disc.

In system theory the M-T systems are often used to identify the transfer function of the system. Fridli, Lócsi and Schipp in [76] used the M-T system in ECG signal processing. Recently Fridli, Gilian and Schipp in [77] introduced the analogue of the M-T system which is orthogonal on the unit disc regarding to the area measure. In [75] the construction of biorthogonal rational systems was studied.

Using the transforms given by (2.20) and (2.21) we can make the transition to the upper half plane. The system

$$\Psi_n(z) := (T\Phi_n)(z) = (Tf)(z) := \frac{1}{\sqrt{\pi}}\frac{1}{i+z}\Phi_n(C(z)) \ (\Im z \geqslant 0,\ n \in \mathbb{N}^*),$$

which is an analogue of the Malmquist-Takeneka system for the upper half plane, is orthonormal in $L^2(\mathbb{R})$. It is easy to check that for $a \in \mathbb{D}$ with $a^* := 1/\overline{a}$,

$$\lambda_a := C^{-1}(a) = i\frac{1-a}{1+a} \in \mathbb{C}_+,\ \lambda_{a^*} = \overline{\lambda}_a, \quad \frac{\sqrt{1-|a|^2}}{|1+\overline{a}|} = \sqrt{\Im\lambda_a}, \tag{4.2}$$

and

$$\tilde{b}_a(z) = b_a(-1)\frac{z-\lambda_a}{z-\overline{\lambda}_a}, \quad \tilde{r}_a(z) = r_a(-1)\frac{z+i}{z-\overline{\lambda}_a} \quad (z \in \overline{\mathbb{C}}_+). \tag{4.3}$$

This implies that the functions $\Psi_n = T\Phi_n$ $(n \in \mathbb{N}^*)$ are of the form

$$\Psi_1(z) = \frac{1}{\sqrt{\pi}}\frac{\Phi_1(-1)}{z-\overline{\lambda}_{a_1}}, \quad \Psi_n(z) = \frac{1}{\sqrt{\pi}}\frac{\Phi_n(-1)}{z-\overline{\lambda}_{a_n}}\prod_{k=1}^{n-1}\frac{z-\lambda_{a_k}}{z-\overline{\lambda}_{a_k}}. \tag{4.4}$$

Moreover, if the following non-Blaschke condition for the upper half plane is satisfied

$$\sum_{k=1}^{\infty}\frac{\Im\lambda_k}{1+|\lambda_k|^2} = \infty,$$

then $(\Psi_n,\, n \in \mathbb{N}^*)$ is a complete orthonormal system for $H^2(\mathbb{C}_+)$.

4.2 Discrete orthogonality of the Malmquist-Takenaka systems

First let us recall the discrete orthogonality of the Malmquist–Takenaka system for the unit disc. The Blaschke product $B_N = \prod_{j=1}^{N} B_{a_j}$ on the unit circle can be written as

$$B_N(e^{it}) = \prod_{j=1}^{N} B_{a_j}(e^{it}) = e^{i(\beta_{a_1}(t)+\cdots+\beta_{a_N}(t))} \quad (t \in \mathbb{R}, N = 1, 2...).$$

This implies that the solutions of the equation

$$\frac{w-a_1}{1-\overline{a}_1 w}\cdot\frac{w-a_2}{1-\overline{a}_2 w}\cdots\frac{w-a_N}{1-\overline{a}_N w} = e^{2\pi i\delta} \quad (\delta \in \mathbb{R}) \tag{4.5}$$

are given by

$$w_k := e^{i\tau_k}, \quad \tau_k := \theta_N^{-1}(2\pi((k-1)+\delta)/N) \quad (k = 1, 2, ...N), \tag{4.6}$$

where θ_N^{-1} is the inverse of the function

$$\theta_N(t) := \frac{1}{N}(\beta_{a_1}(t)+\cdots+\beta_{a_N}(t)) \quad (t \in \mathbb{R}).$$

Let us consider

$$\mathbb{T}_N := \mathbb{T}_N^{\mathfrak{a},\delta} := \{w_k = e^{i\tau_k} : k = 1, 2, ..., N\} \;\; (N = 1, 2, ...), \tag{4.7}$$

the set of solutions of the equation (4.5). We name $\mathbb{T}_N$ the set of discretization nodes on the unit circle. Let us consider the weight function ρ_N given for $w \in \mathbb{T}$ by

$$\frac{1}{\rho_N(w)} := \sum_{k=1}^{N}\frac{1-|a_k|^2}{|1-\overline{a}_k w|^2} \quad (w \in \mathbb{T}, N = 1, 2, ...).$$

The discrete orthogonality of the Laguerre, Kautz was investigated by Schipp [137], and later for Malmquist–Takenaka systems was investigated by Pap and Schipp in [118], where it was proved the following theorem:

Theorem 4.2.1 (**Pap, Schipp** [118]). *The finite collection of functions Φ_n $(1 \leqslant n \leqslant N)$ forms a discrete orthonormal system with respect to the scalar product*

$$[F, G]_N := \sum_{w \in \mathbb{T}_N^{\mathfrak{a},\delta}} F(w)\overline{G(w)}\rho_N(w),$$

namely

$$[\Phi_n, \Phi_m]_N = \delta_{mn} \quad (1 \leqslant m, n \leqslant N).$$

Lócsi in [107] analyzed the efficiency of numerical methods, as the bisection method and Newtons method, in the case of calculating non-equidistant discretizations (4.7) generated by Blaschke products. In the case of Blaschke products, the inverse of the argument function has no explicit form, numerical methods are needed to solve the arising non-linear equations. We have as many equations as the number of points in the discretization to generate. He made several experiments concerning the calculation of non-equidistant discretizations generated by Blaschke products and the associated argument functions. He managed to reduce the time needed for the calculation (using the bisection method) to about 50–70 percent of the original time by introducing a better order of the discrete points to calculate. Further improvements have been measured by applying Newtons algorithm combined with the above.

Kovács, in his PhD thesis [98], provides subroutines for evaluating Blaschke functions, MT and biorthogonal rational systems along with continuous and discrete implementations, implementations of the real valued MT systems, of the continuous and of the discrete biorthogonal systems, and signal processing applications with experiments on ECG recordings. These results were published also in journals [99, 100] by Kovács and Lócsi. They introduced a fast algorithm to compute the non-uniform discretization points for discrete rational orthogonal and biorthogonal systems. In order to do that, they needed new concepts for constructing an effective numerical solution. Namely, a good estimation process is developed for the sampling points based on the monotonic behavior of the argument function. Then, a sequence of fixed-point iteration is executed starting with appropriate initial values. Furthermore, they perform tests for the convergence of some root-finding algorithms in order to achieve the best accuracy. These methods are compared in terms of evaluation time and step number. A new Matlab toolbox has been introduced together with signal processing methods which can be useful in a wide range of applications. For instance, these systems are capable of representing different types of discrete-time series. Both equidistant and non-uniform discretizations can be used. Four types of signal representations are available due to different classes of rational function systems. Moreover, two Matlab GUIs were implemented for educational purposes.

The first GUI is called blaschke-tool. It can be used to visualize the connection between the position of the inverse pole and the values of the argument function. In order to demonstrate the properties of the MT systems, they build up the GUI called malmquist-tool. Here, the user can change the positions, the number and the multiplicities of inverse poles, interactively. The argument function is also displayed on the unit disc. Furthermore, all the members of the MT system can be visualized according to the selected inverse poles. It is not only the complex case, but the real valued MT-Fourier expansions are implemented as well. Moreover, both the real and the complex discretizations are available for interpolation purposes.

The analogue of the previous theorem for the Malmquist–Takenaka system of the upper half plane was proved by Eisner and Pap in [59]. The transformation formulas to the upper half plane imply that

$$\tilde{B}_N(z) := B_N(C(z)) = B_N(-1)\prod_{k=1}^{N}\frac{z-\lambda_{a_k}}{z-\overline{\lambda}_{a_k}}.$$

Let us consider the set of discretization points and the weight for the Malmquist-Takenaka system on the upper half plane

$$\mathbb{R}_N^{\mathfrak{a},\delta} := \{C^{-1}(w) : w \in \mathbb{T}_N^{\mathfrak{a},\delta}\} = C^{-1}(\mathbb{T}_n^{\mathfrak{a},\delta}), \quad \tilde{\rho}_N(t) := \pi(1+t^2)\rho_N(C(t)) \ (t \in \mathbb{R}).$$

Then by Theorem 4.2.1 we get that

$$\delta_{mn} = \sum_{z\in C^{-1}(\mathbb{T}_N^{\mathfrak{a},\delta})}\Phi_n(C(z))\overline{\Phi}_m(C(z))\rho_N(C(z)) = \pi\sum_{z\in\mathbb{R}_N^{\mathfrak{a},\delta}}\Psi(z)\overline{\Psi}_m(z)|i+z|^2\rho_N(C(z)).$$

Thus we get that the Malmquist-Takenaka system of the upper half plane is also discrete orthogonal, i.e.,

Theorem 4.2.2 (**Eisner, Pap** [59])**.** *The finite collection of functions Ψ_n $(1 \leqslant n \leqslant N)$ forms a discrete orthonormal system with respect to the weight function $\tilde{\rho}_N$:*

$$\sum_{z\in\mathbb{R}_N^{\mathfrak{a},\delta}}\Psi(z)\overline{\Psi}_m(z)\tilde{\rho}_N(z) = \delta_{mn} \quad (1 \leqslant m, n \leqslant N).$$

We mention that, based on the discrete orthogonality of the Malmquist–Takenaka systems, Szabó, Eisner, Király, Pilgermájer, Pap in [161, 59, 97] introduced new rational interpolation operators and studied their properties. In the case of the upper half plane the introduced interpolation operator gives an exact interpolation for the Runge test function.

4.3 Equilibrium condition on the unit circle

The set of the discretization nodes on the unit circle satisfies an equilibrium condition and is a stationary point for a logarithmic potential function. These results were published first for the discrete Laguarre case by Schipp in [137], and in more generality for M-T system by Pap and Schipp in [119, 118]. Here we present a short overview of them.

Let us consider the following notations: for any complex number $z \in \mathbb{C}\backslash\{0\}$ set $z^* := 1/\overline{z}$ and introduce the polynomials

$$\omega_1(w) := \prod_{k=1}^{N}(w - a_k), \quad \omega_2(w) := \prod_{k=1}^{N}(1 - \overline{a}_k w),$$

$$\omega(w) := \omega_1'(w)\omega_2(w) - \omega_2'(w)\omega_1(w) \quad (z \in \mathbb{C}).$$

It is clear that ω is a polynomial of degree $2N - 2$. It can be proved (see [118]) that if c is a root of ω with multiplicity m then c^* is also a root of ω with the same multiplicity. Denote by $c_1, c_1^*, ..., c_{N-1}, c_{N-1}^*$ the roots of ω.

Theorem 4.3.1 (**Pap, Schipp** [118]). *For every $\delta \in \mathbb{R}$ the numbers $w_n := e^{i\tau_n} \in \mathbb{T}_N^{\mathfrak{a},\delta}$, $\tau_n := \theta_N^{-1}(2\pi((n-1)+\delta)/N) \quad (n = 1, 2, ..., N)$ are the solutions of the equilibrium equations*

$$\sum_{k=1,k\neq n}^{N} \frac{1}{w_n - w_k} = \frac{1}{2}\sum_{j=1}^{N-1}\left(\frac{1}{w_n - c_j} + \frac{1}{w_n - c_j^*}\right) \quad (n = 1, ..., N). \tag{4.8}$$

The electrostatic interpretation of (4.8) is the following: if negative unit charges are placed to the points c_k and c_k^*, then n positive unit charges places to the points w_j will be in equilibrium in the external field generated by the negative charges.

In the case of discrete Laguerre functions $a_1 = \cdots = a_N = a$ and $\omega(w) = N(1 - |a|^2)(w-a)^{N-1}(1-\overline{a}w)^{N-1}$. Thus the roots of ω are a and a^* with multiplicity $N-1$, i.e., $c_1 = \cdots = c_{N-1} = a$. In the case of Kautz system $N = 2M$, $a_1 = a_3 = \cdots = a_{2M-1} = a$, $a_2 = a_4 = \cdots = a_{2M} = b := \overline{a}$ and consequently

$$\omega(w) = \Omega(w)[(w-a)(w-b)(1-\overline{a}w)(1-\overline{b}w)]^{M-1},$$

where $\Omega(w) = M[(1-|a|^2)(w-b)(1-\overline{b}w) + (1-|b|^2)(w-a)(1-\overline{a}w)]$. Thus in this case c_1, c_1^* are the roots of Ω, $c_2 = \cdots = c_M = a$, $c_{M+1} = \cdots = c_{2M-1} = b$. In the special case $a_1 = a_2 = ... = 0$ from the Malmquist– Takenaka system we reobtain the trigonometric system, the corresponding sets of discretization are

$$\mathbb{T}_N^{\delta} := \{e^{2\pi i(n-1+\delta)/N} : n = 1, 2, \ldots, N\} \ (0 \leqslant \delta < 1).$$

In this case the equilibrium condition becomes:

$$\sum_{k=1,k\neq n}^{N} \frac{1}{w_n - w_k} = \frac{N-1}{2} \cdot \frac{1}{w_n} \quad (n = 1, ..., N).$$

This special case can be found for example in the book [7] p. 425, moreover for this case the following minimum property is true: the potential energy

$$W(v_1, \cdots, v_N) = -\log \prod_{1\leqslant j<k\leqslant N} |v_k - v_j| \ \ (v_1, \cdots, v_N \in \mathbb{T})$$

attains its minimum when $v_k = w_k \in \mathbb{T}_n^\delta$ ($k = 1, ..., N$).

This special case is the so called *Stieltjes problem on the unit circle* and has the following interpretation: if N freely moving unit charges lie on thin circular conductor of unit radius, then the potential energy of the system is $W(v_1, \cdots, v_N)$ and this is minimal if the charges are located in $\mathbb{T}_N^\delta$, and this minimum is equal to $-\frac{N}{2}\log N$.

This motivates the interest in the examination of similar minimum property in general for the discretization points of M-T system given by (4.7):

$$w_n = e^{i\tau_n} \in \mathbb{T}_N^{\mathfrak{a},\delta}, \tau_n := \theta_N^{-1}(2\pi((n-1)+\delta)/N) \quad (n = 1, 2, ..., N).$$

In [119] the following theorem was proved:

Theorem 4.3.2 (**Pap** [119]). *The point $(w_1, w_2, ..., w_N) \in \mathbb{T}_N^{\mathfrak{a},\delta}$ is a stationary point of the logarithmic potential*

$$W(v_1, ..., v_N) = -\log\left(\prod_{1\leqslant j<k\leqslant N} |v_j - v_k| \prod_{k=1}^{N-1}\prod_{j=1}^{N}(|v_j - c_k||v_j - c_k^*|)^{-1/2}\right) \tag{4.9}$$

$$(v_1 = e^{it_1}, ..., v_N = e^{it_N} \in \mathbb{T}),$$

i.e.,

$$\frac{\partial W(e^{i\tau_1}, ..., e^{i\tau_N})}{\partial t_n} = 0 \quad (n = 1, ..., N). \tag{4.10}$$

For $v_j \in \mathbb{T}$ we have

$$|v_j - c_k^*| = |\overline{v}_j - 1/c_k| = |1/v_j - 1/c_k| = |v_j - c_k|/|c_k|,$$

and consequently replacing in (4.9) c_k by c_k^* we get a function which differs from W by an additive constant. Thus if negative unit charges are placed to the points c_k, the n positive unit charges places to the points w_j, will be in equilibrium in the external field generated by the negative charges.

In a natural way the following question arises for the general case: is the stationary point $(w_1, w_2, ..., w_N)$ a minimum point for the potential function W? The transition to the upper half plane of these results permitted to answer this question.

4.4 Equilibrium condition on the real line

In this section we present the transition of the equilibrium condition to the upper half plane. In what follows we prove that the discretization nodes on the real line satisfy an analogue equilibrium property. For $\lambda_1, \cdots, \lambda_N \in \mathbb{C}_+$ let us consider the polynomials

$$\phi_1(z) := \prod_{k=1}^{N}(z - \lambda_k), \quad \phi_2(z) := \prod_{k=1}^{N}(z - \overline{\lambda}_k),$$

$$\phi(z) := \phi_1'(z)\phi_2(z) - \phi_2'(z)\phi_1(z) \quad (z \in \mathbb{C}).$$

It is clear that ϕ is a polynomial of degree $2N-2$. It is easy to prove that if d is a root of ϕ with multiplicity m then $\overline{d}$ is also a root of ϕ with the same multiplicity. Let us denote by $d_1, \overline{d}_1, \ldots, d_{N-1}, \overline{d}_{N-1}$ the roots of ϕ, i.e.,

$$\phi(z) = \prod_{j=1}^{N-1}(z-d_j)(z-\overline{d}_j) \ \ (z \in \mathbb{C}). \tag{4.11}$$

The numbers $a_k := C(\lambda_k)$ $(k = 1, \cdots, N)$ are in $\mathbb{D}$ and by (4.3)

$$B_N(C(z)) = B_N(-1)\prod_{k=1}^{N}\frac{z-\lambda_k}{z-\overline{\lambda}_k} = B_N(-1)\frac{\phi_1(z)}{\phi_2(z)}.$$

The functions $\omega_1, \phi_1, \omega_2, \phi_2$ and ω, ϕ can be expressed by each others:

$$(i+z)^N\omega_1(C(z)) = \omega_1(-1)\phi_1(z), \ (i+z)^N\omega_2(C(z)) = \omega_2(-1)\phi_2(z)$$

$$(i+z)^{2N-2}\omega(C(z)) = -\omega_1(-1)\omega_2(-1)\phi(z) \tag{4.12}$$

and consequently $\omega(C(d_j)) = 0$, if $d_j \neq -i$.

Denote $w_1^\delta, \cdots, w_N^\delta \in \mathbb{T}$ the N (pairwise distinct) solutions of the equation $B_N(w) = e^{2\pi i\delta}$. Then the numbers $t_k := t_k^\delta := C^{-1}(w_k^\delta) \in \mathbb{R}$ $(k = 1, \cdots, N)$ are the solutions of

$$\frac{\phi_1(z)}{\phi_2(z)} = q := e^{2\pi i\delta}/B_N(-1) \in \mathbb{T} \tag{4.13}$$

and we have the following equilibrium condition for the discretization points of the Malmquist-Takenaka system of the upper half plane:

Theorem 4.4.1 (**Pap, Schipp** [130])**.** *Let $q \in \mathbb{T}$ and denote by $t_n = t_n^\delta \in \mathbb{R}$ $(n = 1, 2, \ldots, N)$ the solutions of (4.13), where $\delta \in [0, 1)$. Then the following equilibrium conditions are satisfied:*

$$\sum_{k=1,k\neq n}^{N}\frac{1}{t_n-t_k} = \frac{1}{2}\sum_{j=1}^{N-1}\left(\frac{1}{t_n-d_j}+\frac{1}{t_n-\overline{d}_j}\right) \quad (n = 1, \cdots, N). \tag{4.14}$$

Proof of Theorem 4.4.1. By the definition of t_k it follows that $g(z) := \phi_1(z) - q\phi_2(z) = 0$ if and only if $z = t_k$ $(k = 2, \ldots, N)$. Set

$$f(z) = \prod_{k=1}^{N}(z-t_k).$$

The polynomials f and g have the same degree and roots, therefore $f = \lambda g$ with $\lambda \in \mathbb{C}$, a constant.

It is easy to see that

$$\frac{g''(t_n)}{2g'(t_n)} = \frac{f''(t_n)}{2f'(t_n)} = \sum_{k=1,k\neq n}^{N} \frac{1}{t_n - t_k} \ \ (n = 1, 2, \ldots, N).$$

By the definition of t_n,

$$\frac{\phi_1(t_n)}{\phi_2(t_n)} = q \ \ (n = 1, 2, \ldots, N).$$

On the other hand we get that:

$$\frac{g''(t_n)}{g'(t_n)} = \frac{\phi_1''(t_n) - q\phi_2''(t_n)}{\phi_1'(t_n) - q\phi_2'(t_n)} = \frac{\phi_2(t_n)\phi_1''(t_n) - \phi_1(t_n)\phi_2''(t_n)}{\phi_2(t_n)\phi_1'(t_n) - \phi_1(t_n)\phi_2'(t_n)} = \frac{\phi'(t_n)}{\phi(t_n)}.$$

From (4.11) we have

$$\frac{\phi'(t)}{\phi(t)} = \sum_{k=1}^{N-1} \left(\frac{1}{t - d_k} + \frac{1}{t - \overline{d}_k} \right)$$

and our claim is proved.

In the special case, when $a_1 = \cdots = a_N = a = r \in [0, 1)$, we have by (4.2),

$$\lambda_j := C^{-1}(a) = i\frac{1-r}{1+r} = ip, \ p := \frac{1-r}{1+r} > 0 \ (j = 1, 2, \ldots, N).$$

By the definition of β_a and θ_N in this case we have

$$\theta_N(t) = \beta_r(t) = \gamma_s(t) = 2\arctan(s\tan(t/2)), \ \theta_N^{-1}(t) = \gamma_{1/s}(t), \ s = \frac{1+r}{1-r}.$$

By (4.6) the solution of $B_N(w) = e^{2\pi i\delta}$ are $w_k = e^{i\tau_k}$, where

$$\tau_k = \gamma_{1/s}(2\pi(k - 1 + \delta)/N) \ (k = 1, 2, \ldots, N),$$

consequently for $t_k = C^{-1}(e^{i\tau_k})$ we get

$$t_k = \tan(\tau_k/2) = \frac{1}{s}\tan\left(\frac{\pi}{N}(k - 1 + \delta)\right) = p\tan\left(\frac{\pi}{N}(k - 1 + \delta)\right).$$

Thus in the case $\lambda_1 = \lambda_2 = ... = \lambda_N = ip$ $(p > 0)$ the corresponding nodal points are $t_n = p\tan((n - 1 + \delta)\pi/N)$ $(0 \leqslant \delta < 1)$. In this case the equilibrium condition becomes:

$$\sum_{k=1,k\neq n}^{N} \frac{1}{t_n - t_k} = \frac{N-1}{2}\left(\frac{1}{t_n - ip} + \frac{1}{t_n + ip}\right) \ \ (n = 1, ..., N).$$

Let us introduce the analogue for the upper half plane of the potential function W given by (4.9) :

$$V(x_1,...,x_N) = -\log\left(\prod_{1\leqslant j<k\leqslant N}|x_j-x_k|\prod_{k=1}^{N-1}\prod_{j=1}^{N}(|x_j-d_k||x_j-\overline{d}_k|)^{-1/2}\right) \tag{4.15}$$

$$(x_1,...,x_N)\in\mathbb{R}^N,\quad (d_1,...,d_{N-1})\in\mathbb{C}_+^{N-1}.$$

Obviously (4.14) is equivalent to

$$\frac{\partial V(t_1,...,t_N)}{\partial x_n} = 0 \quad (n=1,...,N), \tag{4.16}$$

i.e., $(t_1,t_2,...,t_N)$ is a stationary point of the potential function V. The function V can be expressed by

$$V_d(s,t) := \frac{|s-t|}{|(s-d)(t-d)|} \quad (s,t\in\mathbb{R}, d\in\mathbb{C}_+).$$

Namely, using the identity $(|x-d||x-\overline{d}|)^{-1/2} = |x-d|^{-1}$ $(x\in\mathbb{R}, d\in\mathbb{C}_+)$ we get

$$V(x_1,x_2,\cdots,x_N) = -\frac{1}{N-1}\log\left(\prod_{1\leqslant j<k\leqslant N}\prod_{\ell=1}^{N-1}V_{d_\ell}(x_j,x_k)\right). \tag{4.17}$$

In a similar way the potential W in (4.9) can by expressed by the functions

$$W_c(v,w) := \frac{|v-w|}{|(v-c)(w-c)|} \quad (c\in\mathbb{D}, v,w\in\mathbb{T}).$$

Namely,

$$W(v_1,\cdots,v_N) = -\frac{1}{N-1}\log\left(\prod_{1\leqslant j<k\leqslant N}\prod_{\ell=1}^{N-1}W_{c_\ell}(v_j,v_k)\right).$$

The functions V and W are closely connected. It is easy to see that

$$W_c(K(x_1),K(x_2)) = \frac{2}{|1+c|^2}V_{K^{-1}(c)}(x_1,x_2) \quad (x_1,x_2\in\mathbb{R}, c\in\mathbb{D}).$$

This implies

Theorem 4.4.2 (**Pap, Schipp** [130]). *If* $d_k\in\mathbb{C}_+,\ c_k=C(d_k)$ $(k=1,\dots,N-1)$ *then*

$$W(C(x_1),\dots,C(x_N)) = V(x_1,\dots,x_N)+\mathit{const}\quad ((x_1,\cdots,x_N)\in\mathbb{R}^N),$$

consequently $(t_1,\cdots,t_N)\in\mathbb{R}^N$ *is a minimum position for* V, *if and only if*

$$w_1=C(t_1),\dots,w_N=C(t_N)\in\mathbb{T}$$

is a minimum position for W.

For the function V we can give a geometrical interpretation. Let us consider the triangle in the complex plan with vertices $s, t \in \mathbb{R}, d \in \mathbb{C}_+$ and denote by α the angle at the vertex d. Then by the area formula $|s-t|\Im d = |d-s||d-t|\sin\alpha$,

$$V_d(s,t) = \frac{\sin\alpha}{\Im d}.$$

Instead of V we investigate the maximum of the function

$$T(x) := T(x_1, x_2, \cdots, x_N) := \prod_{1\leqslant j<k\leqslant N} \prod_{\ell=1}^{N-1} V_{d_\ell}(x_j, x_k) \ (x \in \mathbb{R}^N).$$

Theorem 4.4.3 (**Pap, Schipp** [130]). *In the special case of discrete Laguerre-system i.e., if*

$$\lambda_1 = \cdots = \lambda_N = ip \ (p > 0) \tag{4.18}$$

the function T attains its maximum at $t_n^\delta = p\tan((n-1+\delta)\pi/N)$, where $(0 \leqslant \delta < 1, n = 1, \ldots, N)$.

Proof of Theorem 4.4.3. Let $x_1 < x_2 < \cdots < x_N$, $P = ip$ and denote the angle $x_j P x_{j+1}\sphericalangle$ by α_j $(1 \leqslant j < N)$. Then the function T is constant times of

$$\begin{aligned} S(\alpha) &:= \prod_{1\leqslant j<k\leqslant N} \sin\alpha_{jk}, \quad \alpha_{jk} := \alpha_j + \cdots + \alpha_{k-1}, \\ \alpha &= (\alpha_1, \ldots, \alpha_{N-1}) \in A := \{\alpha \in \mathbb{R}^{N-1} : 0 < \alpha_j, \quad \alpha_1 + \cdots + \alpha_{N-1} < \pi\}. \end{aligned}$$

The function S does not depend on the position of x_1. In addition S is continuous and non negative and vanishes at the boundary of A. Thus S has a maximum position in A. In addition, this position is uniquely determined. To show this, let us suppose that $\alpha'_1, \ldots, \alpha'_{N-1}$ and $\alpha''_1, \ldots, \alpha''_{N-1}$ are two position of this kind. Denote $\alpha_j := (\alpha'_j + \alpha''_j)/2$ $(j = 1, \ldots, N-1)$. Since

$$\sin s \sin t \leqslant \sin^2\left(\frac{s+t}{2}\right) \ (0 \leqslant s, t \leqslant \pi),$$

we have

$$S(\alpha')S(\alpha") \leqslant S^2(\alpha),$$

the equality sign being taken if and only if $\alpha'_j = \alpha''_j = \alpha_j$ $(j = 1, \ldots, N-1)$. This establishes the uniqueness. The maximum position $\tilde{\alpha}$ is the solution of

$$\frac{\partial S(\tilde{\alpha})}{\partial \alpha_j} = 0 \ \ (j = 1, \ldots, N-1).$$

It is easy to see that $\tilde{\alpha}_j = \pi/N$ $(j = 1,\dots,N-1)$. Fix x_1 and denote by δ_j the angle at vertices x_j of the triangle Px_j0. Then in the maximum position of S we have $\delta_j = \delta_{j-1} + \pi/N$ $(j = 2,\dots,N)$, i.e., $\delta_j = \delta_1 + (j-1)\pi/N$ $(j = 1,2,\dots,N)$. Consequently,

$$\begin{aligned} x_{N-j+1} &= p\tan(\pi/2 - \delta_{N-j+1}) = p\tan(\pi/2 - \delta_1 - (N-j)\pi/N) = \\ &= p\tan(-\pi/2 - \delta_1 + \pi/N + (j-1)\pi/N) = p\tan(\delta\pi/N + (j-1)\pi/N) \\ &\qquad (j = 1,2,\dots,N), \end{aligned}$$

where $\delta\pi/N := -\pi/2 - \delta_1 + \pi/N$. Thus $x_{N-j+1} = t_j^{\delta}$ $(j = 1,2,\dots.N)$ and we get the equilibrium positions in the Theorem.

We note that Totik in [165] following the ideas of the presented proof gave an elementary proof for the transfinite diameter of the unit circle.

In [130] Pap and Schipp solved the question connected to minimum for a special case of M-T system, for the discrete Laguarre system, and formulated the problem in general for M-T systems as an open problem.

Problem. Does the function V defined before in the position $t_1, t_2, ..., t_N \in \mathbb{R}$ attain its minimum ?

The formulated problem was solved in generality recently in [79] by Marcell Gaál, Béla Nagy, Zsuzsanna Nagy-Csiha, Szilárd Révész. The question was answered positively using a recent result given by Semmler and Weger [145]. They showed that the equilibrium condition satisfied by the discretization nodes (see Theorem 4.3.1 and Theorem 4.4.1) are equivalent that they arise from critical points of a logarithmic potential energy. In [79] the authors first studied on the unit circle a quite general logarithmic energy which is determined by a signed measure, and prove that after inverse Cayley transform the transformed energy on the real line differs only in an additive constant. Next using the result of Semmler and Wegert they could give an affirmative answer to the question posed by Pap and Schipp. The a positive answer given to this question in general is proved with different method than the proof of the presented special case.

Chapter 5

Quaternionic extension of some results

Quaternions play an important role in modeling the time and space dependent problems in physics and engineering. For example in engineering applications unit quaternions are used to describe the three dimensional rotations. In the last years quaternions gained a new life due to their applicability in signal processing. This is due to the applicability of quaternion-valued functions to color-coded images as well as the link to new concepts of higher-dimensional phases, like the hypercomplex signals of Bülow or the monogenic signals by Larkin and Felsberg (see [27, 71, 106]). Another important field, where quaternions play an important role is quantum theory. Adler, a world-renowned theoretical physicist, in his book Quaternionic Quantum Mechanics and Quantum Fields [2], provides an introduction to the problem of formulating quantum field theories in quaternionic Hilbert space. This well-written treatise is a very significant contribution to theoretical physics. Bernardo Vargas in the review of this book mentioned that the quaternionic formalism is to improve some treatments of theoretical physics. But the full power of quaternions would be even more important by using quaternionic analysis.

This motivates to extend the results of modern harmonic analysis, like the wavelet theory, to quaternion variable function spaces.

A first step in this direction is to give the quaternionic analogue of the Blaschke group. The main obstacle in the study of quaternion-valued matrices and functions, as expected, comes from the non-commutative nature of quaternionic multiplication.

Cerejeiras, Ferreira and Kähler [30] constructed monogenic wavelet transforms for quaternion valued functions on the three dimensional unit ball in $\mathbb{R}^3$. The construction is based on representations of the group of Möbius transformations which maps the three dimensional unit ball onto itself.

In section 5.2 we introduce the four dimensional quaternionic analogue of the Blaschke group and we list the main subgroups of this groups. The results were published by Pap and Schipp in [131].

Beside the monogenic quaternionic function theory, where many difficulties appear when we want to make analysis, the theory of slice regular functions and the analysis on this field would be an alternative tool for the quantum theory. To introduce new orthonormal systems in the slice regular Hardy space, is therefore an interesting topic that is worthwhile to be studied.

In [120, 135] Pap, Schipp and Qian, Sprossig, Wang respectively, following two different ways, introduced two analogues of the M-T systems in the set of quaternions. While in the complex case both ways give the same M-T system, in the quaternionic setting this is not anymore true. The drawback of both constructions is that these extensions will not inherit all the nice properties of the before mentioned complex M-T system, e.g., the system introduced by Pap and Schipp is not analytic in the quaternionic setting. The system introduced by Qian, Sprossig, Wang, is monogenic but can not be written in closed form.

Pap in [132] introduced a new generalization of the complex Malmquist-Takenaka system in the quaternionic slice regular Hardy space, which is slice regular and in same time can be given in closed form. In Section 5.3 results connected to slice regular Malmquist-Takenaka system are presented. We proved that, similar to the complex case, under certain conditions of the parameters of the system this is a complete orthonormal system in the slice regular Hardy space of the unit ball. We also proved that the associated projection operator $(P_n f,\, n \in \mathbb{N})$ is convergent in $H^2(\mathbf{D})$ norm to f, and $P_n f(z) \to f(z)$ uniformly on every compact subset of the unit ball. In the same time the restriction of $P_n f(z)$ to a slice $\mathbf{D}_I$ is an interpolation operator of f.

5.1 Quaternions

Quaternions are extensions of complex numbers. In order to introduce the quaternionic analogue of the Blaschke group it is convenient to use the matrix representation of the quaternions, because it makes possible to use the properties of the matrices at different computations.

Let us denote by

$$E := E_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ E_1 := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},\ E_2 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ E_3 := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \tag{5.1}$$

the quaternion units, where $i \in \mathbb{C}$ is the complex imaginary unit. Similar to the property $i^2 = -1$ of the complex unit, the quaternion units satisfy the following equations: $E_j^2 = -E$ $(j = 1, 2, 3)$. Since $E_1E_2 = -E_2E_1 = E_3$, $E_2E_3 = -E_3E_2 = E_1$,$E_3E_1 = -E_1E_3 = E_2$, the set $\{\pm E_j : j = 0, 1, 2, 3\}$ is closed with respect to multiplication. Let us denote by

$$\mathbf{Q} := \left\{ Z := \sum_{j=0}^{3} z_j E_j : z = (z_0, z_1, z_2, z_3) \in \mathbb{R}^4 \right\} \tag{5.2}$$

the set of quaternions, which is a non-commutative field with the unit element E and null element the null-matrix $\Theta \in \mathbb{C}^{2\times 2}$. Let us denote by

$$\overline{Z} := z_0 E_0 - \sum_{j=1}^{3} z_j E_j = Z^*,\ |Z| := \Big(\sum_{j=0}^{3} z_j^2\Big)^{1/2},\ ZZ^* = |Z|^2 E,$$

the analogue of the conjugate which in matrix representation is Z^*, the adjoint matrix of $Z \in \mathbb{C}^{2\times 2}$, and the absolute value of the $Z = \sum_{j=0}^{3} z_j E_j \in \mathbf{Q}$. The map $Z \to |Z|$ defines a multiplicative norm:

$$|Z_1 + Z_2| \leqslant |Z_1| + |Z_2|,\quad |Z_1 \cdot Z_2| = |Z_1|\,|Z_2| \quad (Z_1, Z_2 \in \mathbf{Q}).$$

The multiplicative inverse of a nonzero quaternion $Z \in \mathbf{Q}$ in matrix representation is $Z^{-1} = Z^*/|Z|^2$. The analogue of the complex torus and unit disc in the set of the quaternions are defined by $\mathbf{T} := \{Z \in \mathbf{Q} : |Z| = 1\}$, and $\mathbf{D} := \{Z \in \mathbf{Q} : |Z| < 1\}$ respectively. From the property of the norm it follows that $\mathbf{T}$ is a multiplicative subgroup of the multiplicative group of $\mathbf{Q}$, which can be identified by the matrix group $\mathbb{SU}_2$.

Taking into account that $\mathbb{R}E$ and $\mathbb{R}$ are isomorphic ($\mathbb{R}E \approxeq \mathbb{R}$) and $\mathbb{C}E \approxeq \mathbb{C}$, the field $(\mathbf{Q}, +, \cdot)$ can be considered as an extension of $\mathbb{R}$ and $\mathbb{C}$, respectively. The purely imaginary quaternion $I_{\mathbf{c}} := \sum_{j=1}^{3} c_j E_j$ ($\mathbf{c} = (c_1, c_2, c_3) \in \mathbb{R}^3$) satisfies the equation $I_{\mathbf{c}}^2 = -|\mathbf{c}|^2 E$. The map $\mathbf{c} \to I_{\mathbf{c}}$ is a linear isomorphism between $\mathbb{R}^3$ and the set of purely imaginary quaternion $\mathbf{J} := \{Z = I_{\mathbf{c}} : \mathbf{c} \in \mathbb{R}^3\} = \{Z \in \mathbf{Q} : \mathrm{spur}(Z) = 0\}$, consequently $\mathbb{R}^3$ and $\mathbf{J}$ can be identified.

The two dimensional subspace

$$\mathbf{Q}_{\mathbf{c}} := \{Q_{\mathbf{c}}(z) := xE + yI_{\mathbf{c}} : z = x + \imath y \in \mathbb{C}\} \subset \mathbf{Q} \quad (\mathbf{c} \in \mathbb{R}^3, |\mathbf{c}| = 1) \tag{5.3}$$

of $\mathbf{Q}$ is called *the slice of* $\mathbf{Q}$ *in the direction of the vector* $\mathbf{c}$. The map $Q_{\mathbf{c}} : \mathbb{C} \to \mathbf{Q}_{\mathbf{c}}$ is a linear isomorphism. From $I_{\mathbf{c}}^2 = -E$ ($|\mathbf{c}| = 1$) it follows that

$$Q_{\mathbf{c}}(z_1 + z_2) = Q_{\mathbf{c}}(z_1) + Q_{\mathbf{c}}(z_2),\ Q_{\mathbf{c}}(z_1\, z_2) = Q_{\mathbf{c}}(z_1)\, Q_{\mathbf{c}}(z_2)\ (z_1, z_2 \in \mathbb{C}), \tag{5.4}$$

and obviously $Q_{\mathbf{c}}(\overline{z}) = Q_{\mathbf{c}}^*(z)$ ($z \in \mathbb{C}$). This implies that the map $Q_{\mathbf{c}}$ is an isometric isomorphism between the fields $\mathbb{C}$ and $\mathbf{Q}_{\mathbf{c}}$.

In the literature it is used also the algebraic representation of the quaternions. Let i, j and k satisfy the following identities: $i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad ki = -ik = j$, named as Hamilton's rules. A quaternion q can be represented as $q = z_0 + z_1 i + z_2 j + z_3 k, \quad (z_n \in \mathbb{R},\ n = 0, 1, 2, 3)$. Let the set of quaternions be denoted by

$$\mathbb{H} := \{q = z_0 + z_1 i + z_2 j + z_3 k : \quad z_n \in \mathbb{R},\ n = 0, 1, 2, 3\}.$$

The conjugate of a quaternion q is given by $\overline{q} = z_0 - z_1 i - z_2 j - z_3 k$, and the quaternion norm is $\|q\| = \sqrt{q.\overline{q}} = \sqrt{\overline{q}.q} = \sqrt{z_0^2 + z_1^2 + z_2^2 + z_3^2}$. We mention that the product of quaternions is not commutative in general and $\overline{a.b} = \overline{b}.\overline{a}$. The multiplicative inverse of q is $q^{-1} = \overline{q}/q\overline{q}$ and $(\mathbb{H}, +, .)$ is a noncommutative field (skew field).

Comparing the algebraic representation with the matrix representation, E_0 corresponds to 1, E_1 to i, E_2 to j, E_3 to k, Z to q and Z^* to $\overline{q}$ respectively. The two representations are equivalent. We use both representations, depending on which is more convenient for our purpose.

As we have mentioned the complex numbers and their extensions, the quaternions are very useful in the description of many problems in geometry and physics. For example the rotations in the Euclidian plane $\mathbb{C}$ can be described using the map $z \to \epsilon z$ where $\epsilon, z \in \mathbb{C}$ are complex numbers and $\epsilon = e^{i\alpha} \in \mathbb{T}$ $(\alpha \in \mathbb{R})$. In this case α is the angle of the rotation.

If instead of complex numbers we use quaternions, we can describe the rotations in $\mathbb{R}^3$ with a relatively simple map. In order to illustrate this, we use the analogue of the Euler formula $e^{it} = \cos t + i \sin t$ $(t \in \mathbb{R})$:

$$e^{tI_c} = E\cos t + I_{\mathbf{c}} \sin t \quad (t \in \mathbb{R}, \mathbf{c} \in \mathbb{R}^3, |\mathbf{c}| = 1). \tag{5.5}$$

From this it follows that, similarly to unit complex numbers, every unit quaternion $S = z_0 E + I_{\mathbf{z}}$ $(\mathbf{z} = (z_1, z_2, z_3) \in \mathbb{R}^3), |S| = 1$ can be represented as $S = e^{tI_{\mathbf{c}}}$, where $\cos t = z_0, \mathbf{c} = \mathbf{z}/|\mathbf{z}|$.

The relation $\mathrm{spur}(SZS^*) = \mathrm{spur}(Z)$ $(S \in \mathbf{T}, Z \in \mathbf{Q})$ implies that the map $Z \to SZS^*$ takes the subspace $\mathbf{J}$, which is isomorphic with $\mathbb{R}^3$, in itself and can be interpreted as a rotation around the axis $\mathbf{c}$ of the space $\mathbb{R}^3$ with angle $2t$. The image of the slice $\mathbf{Q_c}$ trough this rotation will be the slice $\mathbf{Q_b}$ for which $I_{\mathbf{b}} = SI_{\mathbf{c}}S^*$ $(S \in \mathbf{T})$, i.e., $\mathbf{Q_b} = S\mathbf{Q_c}S^*$. The polar representation of the quaternion $Z \in \mathbf{Q}$ can be written as

$$Z = \rho e^{tI_{\mathbf{c}}} \quad (\rho = |Z|, t \in \mathbb{R}, I_{\mathbf{c}} \in \mathbf{J}). \tag{5.6}$$

5.2 The Blaschke group over the set of quaternions.

Pap and Schipp in [131] introduced the quaternionic analogue of the Blaschke group.

The Blaschke functions can be defined also among quaternions. The formulas are very similar to the complex case:

$$B_A(Z) := (Z - A)(E - A^* Z)^{-1} \quad (A \in \mathbf{D}, Z \in \overline{\mathbf{D}} := \{Z \in \mathbf{Q} : |Z| \leqslant 1\}). \tag{5.7}$$

It can be proved that these quaternionic Blaschke functions have many analogous properties of the complex Blaschke functions (see [13]). One of this is:

$$1 - |B_A(Z)|^2 = \frac{(1 - |A|^2)(1 - |Z|^2)}{|E - A^* Z|^2} \quad (A \in \mathbf{D}, Z \in \overline{\mathbf{D}}). \tag{5.8}$$

From this it follows that, similar to the complex case, for any $A \in \mathbf{D}$ the function B_A takes the quaternion unit disc $\mathbf{D}$ into $\mathbf{D}$, and the quaternion unit torus $\mathbf{T}$ into $\mathbf{T}$.

Because of the non commutativity of the product operation in $\mathbf{Q}$, in order to generate the quaternion analogue of the complex Blaschke group, we have to introduce a right and left unit quaternion factor from $\mathbf{T}$ in (5.7) instead of the multiplication by complex $\epsilon \in \mathbb{T}$. We consider in $\mathbf{Q}$ the following function:

$$C_A(Z) := (E - ZA^*)_0 := \frac{E - ZA^*}{|E - ZA^*|} \quad (A \in \mathbf{D}, Z \in \overline{\mathbf{D}}). \tag{5.9}$$

It is obvious that C_A takes $\overline{\mathbf{D}}$ into $\mathbf{T}$, and $C_Z(A) = C_A^*(Z)$ $(A, Z \in \mathbf{D})$.

First we show that for the extended quaternion Blaschke functions, given by (5.7), an analogous rule of composition holds.

Theorem 5.2.1 (Pap, Schipp [131]**).** *For every $A_1, A_2 \in \mathbf{D}$ and $Z \in \overline{\mathbf{D}}$ we have*

$$B_{A_1}(B_{A_2}(Z)) = UB_A(Z)V^*,$$

where

$$A = B_{-A_2}(A_1), \quad U = C_{-A_2}(A_1), \; V = C_{-A_2^*}(A_1^*). \tag{5.10}$$

We observe that for the complex unit parameter ϵ, in the quaternion case it corresponds a right and left unit quaternion. The product of these two factors in the complex case, where we can interchange the order of the terms, gives the analogue of the ϵ factor.

To get a collection of functions closed with respect to the composition operation $\circ$ it is convenient to introduce the parameter set $\mathbf{B} := \mathbf{T} \times \mathbf{D} \times \mathbf{T}$ and the function set

$$\mathfrak{B} := \{\mathcal{B}_{\mathfrak{a}} := UB_AV^* : \mathfrak{a} = (U, A, V) \in \mathbf{B}\}. \tag{5.11}$$

For the extended quaternion Blaschke functions we have

$$|\mathcal{B}_{\mathfrak{a}}(Z)| = |B_A(Z)| \leqslant \frac{|A| + |Z|}{1 + |A||Z|} \quad (A \in \mathbf{D}, Z \in \overline{\mathbf{D}}) \tag{5.12}$$

and $\mathcal{B}_{\mathfrak{a}}$ takes $\overline{\mathbf{D}}$ into $\overline{\mathbf{D}}$. Applying formula (5.10) for $A_1 = A$, $A_2 = -A$ we get $U = V = E$ and

$$B_A(B_{-A}(Z)) = B_{-A}(B_A(Z)) = Z \quad (Z \in \overline{\mathbf{D}}, A \in \mathbf{T}).$$

This implies that $B_A : \mathbf{D} \to \mathbf{D}, B_A : \mathbf{T} \to \mathbf{T}$ is bijective and $B_A^{-1} = B_{-A}$.

The set of functions $\mathfrak{B}$ is closed with respect to the inverse operation. In order to prove this we will use the formula

$$U^*B_A(UZV^*)V = B_{U^*AV}(Z) \quad (A \in \mathbf{D}, U, V \in \mathbf{T}). \tag{5.13}$$

Let us introduce the map $\mathfrak{a} = (U, A, V) \to \widehat{\mathfrak{a}} := UAV^*$ from $\mathbf{B}$ to $\mathbf{D}$. Based on the previous relation it follows that any function of the form

$$\mathcal{B}_{\mathfrak{a}} = UB_AV^* \quad (\mathfrak{a} = (U, A, V) \in \mathbf{B})$$

has an inverse given by

$$\mathcal{B}_{\mathfrak{a}}^{-1}(Z) = U^* B_{-UAV^*}(Z)V = U^* B_{-\widehat{\mathfrak{a}}}(Z)V. \tag{5.14}$$

Indeed, $\mathcal{B}_{\mathfrak{a}}(X) = UB_A(X)V^* = Z$ is equivalent to, $\mathcal{B}_{\mathfrak{a}}^{-1}(Z) = X = B_{-A}(U^*ZV)$. From this we get

$$\mathcal{B}_{\mathfrak{a}}^{-1}(Z) = U^* B_{-UAV^*}(Z)V = U^* B_{-\widehat{\mathfrak{a}}}(Z)V.$$

It can be proved that the set of functions $\mathfrak{B}$ is closed with respect to function composition, consequently $(\mathfrak{B}, \circ)$ is a transformation group on $\mathbf{D}$ and $\mathbf{T}$ respectively, called the *quaternion Blascke transformation group.*

Theorem 5.2.2 (**Pap, Schipp** [131])**.** *For any two functions* $\mathcal{B}_{\mathfrak{a}_1}, \mathcal{B}_{\mathfrak{a}_2} \in \mathfrak{B}$

$$(\mathfrak{a}_j = (U_j, A_j, V_j) \in \mathbf{B}, j = 1, 2),$$

we have

$$\mathcal{B}_{\mathfrak{a}_1} \circ \mathcal{B}_{\mathfrak{a}_2} = \mathcal{B}_{\mathfrak{a}} \quad (\mathfrak{a} = (U, A, V) \in \mathbf{B}),$$

where

$$A = \mathcal{B}_{\mathfrak{a}_2}^{-1}(A_1), \quad U = U_1 C_{-\widehat{\mathfrak{a}}_2}(A_1) U_2, \quad V = V_1 C_{-(\widehat{\mathfrak{a}}_2)^*}(A_1^*) V_2. \tag{5.15}$$

The unit element of this group is $\mathcal{B}_{\mathfrak{e}}$, where $\mathfrak{e} = (E, \Theta, E)$.

The bijection $\mathbf{B} \ni \mathfrak{a} \to \mathcal{B}_{\mathfrak{a}} \in \mathfrak{B}$ induces in the set of the parameters $\mathbf{B}$ an operation, $\mathfrak{a}_1 \odot \mathfrak{a}_2 = \mathfrak{a}$ for which $\mathcal{B}_{\mathfrak{a}_1} \circ \mathcal{B}_{\mathfrak{a}_2} = \mathcal{B}_{\mathfrak{a}}$. The set of the parameters with the induced operation $(\mathbf{B}, \odot)$ is *the quaternionic Blaschke group.* In the set of the parameters the inverse $\mathfrak{a}^-$ of an element $\mathfrak{a} = (U, A, V)$ is the element for which $\mathcal{B}_{\mathfrak{a}^-} = \mathcal{B}_{\mathfrak{a}}^{-1}$ where $\mathfrak{a}^- = (U^*, -\widehat{\mathfrak{a}}, V^*)$.

If instead of $\mathfrak{a}_1$ we put $\mathfrak{z}$ and instead of $\mathfrak{a}_2$ we put $\mathfrak{a}^-$, then $\widehat{\mathfrak{a}_2^-} = -U_2^* U_2 A V_2^* V_2 = -A$, and in the set of the parameters the right translations $\mathfrak{z} \to \mathfrak{z} \odot \mathfrak{a}^-$ can be described as follows:

$$\mathfrak{z} \odot \mathfrak{a}^- = (U_1 C_A(Z) U_2^*, U_2 B_A(Z) V_2^*,\ V_1 C_{A^*}(Z^*) V_2^*). \tag{5.16}$$

Proof of Theorem 5.2.2. We use that $B_{A_3}(U_3 Z V_3^*) = U_3 B_{U_3^* A_3 V_3}(Z) V_3^*$ with the parameters $U_3 = U_2, V_3 = V_2, U_3^* A_3 V_3 = A_2$. Then $A_3 = U_3 A_2 V_3^* = U_2 A_2 V_2^*$, and the following relation is true:

$$\mathcal{B}_{\mathfrak{a}_2}(Z) = U_2 B_{A_2}(Z) V_2^* = B_{U_2 A_2 V_2^*}(U_2 Z V_2^*) = B_{\widehat{\mathfrak{a}}_2}(\widehat{\mathfrak{z}}),$$

where $\widehat{\mathfrak{a}}_2 = U_2 A_2 V_2^*, \widehat{\mathfrak{z}} = U_2 Z V_2^*$. Using the previous relation and Theorem 5.2.1 we get

$$\mathcal{B}_{\mathfrak{a}_1}(\mathcal{B}_{\mathfrak{a}_2}(Z)) = U_1 B_{A_1}(B_{\widehat{\mathfrak{a}}_2}(\widehat{\mathfrak{z}})) V_1^*.$$

Applying again Theorem 5.2.1 for the parameters $A_1, \widehat{\mathfrak{a}}_2, \widehat{\mathfrak{z}}$:

$$B_{A_1}(B_{\widehat{\mathfrak{a}}_2}(\widehat{\mathfrak{z}})) = U' B_{A'}(U_2 Z V_2^*) V'^* = U' U_2 B_{U_2^* A' V_2}(Z) V_2^* V'^*,$$

where

$$A' = B_{-\hat{\mathfrak{a}}_2}(A_1), U' = C_{-\hat{\mathfrak{a}}_2}(A_1), V' = C_{-\hat{\mathfrak{a}}_2^*}(A_1^*).$$

From here we get the formula

$$\begin{aligned}
&\mathcal{B}_{\mathfrak{a}_1} \circ \mathcal{B}_{\mathfrak{a}_2} = \mathcal{B}_{\mathfrak{a}} = U B_A V^*, \\
&A = U_2^* B_{-\hat{\mathfrak{a}}_2}(A_1) V_2, \quad U = U_1 C_{-\hat{\mathfrak{a}}_2}(A_1) U_2, \quad V = V_1 C_{-\hat{\mathfrak{a}}_2^*}(A_1^*) V_2.
\end{aligned}$$

In papers [168, 169, 170, 72, 73], for the complex case, and also for higher dimension, using $C_{-a}(z), B_a(z)$ were introduced and studied the operations

$$a \oplus z = B_{-a}(z),\ gyr[a, z] = C_{-a}(z),$$

and using them it was defined the gyro group. Our description makes possible to avoid the complicated gyro group description, and is more useful also from the point of view of the extensions for higher dimension.

In what follows we list some subgroups of $\mathfrak{B}$. The set $\{B_{\rho E} : \rho \in \mathbb{I} := (-1, 1)\}$ is a subgroup of $\mathfrak{B}$, satisfying $B_{\rho_1 E} \circ B_{\rho_2 E} = B_{\rho_1 \circ \rho_2 E}$, where

$$\rho_1 \circ \rho_2 = \frac{\rho_1 + \rho_2}{1 + \rho_1 \rho_2} \quad (\rho_1, \rho_2 \in \mathbb{I}) \tag{5.17}$$

is the real Blaschke group operation on $\mathbb{I}$.

Another subgroup can be generated if we choose the parameters and variable Z on the same slice. First let us observe that if $A_j = Q_{\mathbf{c}}(a_j)$ $(j = 1, 2)$ and $Z = Q_{\mathbf{c}}(z)$ belong to the same slice, then

$$B_{A_j}(Z) = Q_{\mathbf{c}}(B_{a_j}(z)),\ B_{A_1}(B_{A_2}(Z)) = Q_{\mathbf{c}}(B_{a_1}(B_{a_2}(z)),\ B_{A_1}^{-1}(Z) = Q_{\mathbf{c}}(B_{a_1}^{-1}(z)). \tag{5.18}$$

These imply

$$\begin{aligned}
&B_{A_1}(B_{A_2}(Z)) = Q_{\mathbf{c}}(B_{a_1}(B_{a_2}(z))),\ B_A^{-1}(Z) = Q_{\mathbf{c}}(B_a^{-1}(z)), \\
&A = Q_{\mathbf{c}}(a), A_j = Q_{\mathbf{c}}(a_j),\ Z = Q_{\mathbf{c}}(z)\ (a, a_j \in \mathbb{D}, z \in \overline{\mathbb{D}}, j = 1, 2).
\end{aligned}$$

Set $\mathbf{D_c} = \mathbf{D} \cap \mathbf{Q_c}$, $\mathbf{T_c} = \mathbf{T} \cap \mathbf{Q_c}$. Then it follows that the collection

$$\mathfrak{B}_{\mathbf{c}} := \{U B_A V^* : A \in \mathbf{D_c}, U, V \in \mathbf{T_c}\}$$

is a transformation group on $\mathbf{D_c}$ and $\mathbf{T_c}$ respectively, isomorphic to the complex Blaschke transformation group.

Another interesting subgroup of the quaternion Blaschke group is induced by the following subset.

Theorem 5.2.3 (**Pap, Schipp** [131])**.** *Let $\epsilon(A) := (E - A^*)/|E - A|$ $(A \in \mathbf{D})$. Then the subset*

$$\mathfrak{A} := \{\mathcal{A}_A = \epsilon(A) B_A \epsilon(A) : \ A \in \mathbf{D}\} \subset \mathfrak{B} \tag{5.19}$$

is an one parameter subgroup of $\mathfrak{B}$. Moreover,

$$\begin{aligned} &i) \quad \mathcal{A}_A(E) = E \quad (A \in \mathbf{D}), \\ &ii) \quad \mathcal{A}_A^{-1} = \mathcal{A}_{A^-}, \ A^- = -\epsilon(A) A \epsilon(A), \\ &iii) \quad \mathcal{A}_{A_1} \circ \mathcal{A}_{A_2} = \mathcal{A}_A, \ A = \mathcal{A}_{A_2^-}(A_1), \end{aligned} \tag{5.20}$$

5.3 Slice regular Malmquist-Takenaka system in the quaternionic Hardy spaces

Pap in [132] introduced a new generalization of the complex Malmquist-Takenaka system in the quaternionic slice regular Hardy space, which is slice regular and in same time can be given in closed form. In this section we present results connected to this system.

5.3.1 Slice regular functions

The theory of slice regular functions of a quaternionic variable (often simply called regular functions) was introduced in 2006 by Gentili, Stoppato, Struppa [80, 81] and represents a natural quaternionic counterpart of the theory of complex holomorphic functions. This recent theory has been growing very fast and was developed in a series of papers, including in particular [40, 43, 44, 160], where most of the recent advances are discussed. The detailed up-to-date theory appears in the monograph [82]. The theory of regular functions is presently expanding in many directions.

Set $\mathbb{S} = \{q \in \mathbb{H} : q^2 = -1\}$ to be the 2-sphere of purely imaginary units in $\mathbb{H}$, and for $I \in \mathbb{S}$ let L_I be the complex plane $\mathbb{R} + \mathbb{R}I$. We have

$$\mathbb{H} = \cup_{I \in \mathbb{S}} L_I.$$

To recall the definition of slice regular functions we will first describe the natural domains of definition for such functions (for the definitions and main results see the monograph [82] and the reference list therein).

Definition 5.3.1. *Let Ω be a domain in $\mathbb{H}$ that intersects the real axis. Then:*

1. Ω is called a slice domain if, for all $I \in \mathbb{S}$, the intersection Ω_I with the complex plane L_I is a domain of L_I ;

2. Ω is called a symmetric domain if for all $x, y \in \mathbb{R}, x + yI \in \Omega$ implies $x + y\mathbb{S} \subset \Omega$.

Definition 5.3.2. *Let $\Omega \subset \mathbb{H}$ be a slice domain. A function $f : \Omega \to \mathbb{H}$ is said to be (slice) regular if, for all $I \in \mathbb{S}$, its restriction f_I to Ω_I is holomorphic, i.e., it has continuous partial derivatives and satisfies*

$$\overline{\partial}_I f(x+yI) := \frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f_I(x+yI) = 0. \tag{5.21}$$

Lemma 5.3.3. *(Splitting Lemma). If f is a regular function on a slice domain, then for every $I \in \mathbb{S}$ and for every $J \in \mathbb{S}$, J orthogonal to I, there exist two holomorphic functions $F, G : \Omega_I \to L_I$, such that for every $z = x + yI \in I$, we have*

$$f_I(z) = F(z) + G(z)J. \tag{5.22}$$

As shown in [81], if we consider the open unit ball $\mathbf{D}$ of $\mathbb{H}$, the class of regular functions coincides with the class of convergent power series of type $\sum_{n\geqslant 0} q^n a_n$, with all $a_n \in \mathbb{H}$.

The direct extension of the Blaschke function, presented before, is not slice regular. In general the product of two slice regular functions is not slice regular.

Definition 5.3.4. *Let $f, g : \mathbf{D} \to \mathbb{H}$ be regular functions and let $f(q) = \sum_{n\in\mathbb{N}} q^n a_n$ and $g(q) = \sum_{n\in\mathbb{N}} q^n b_n$ be their power series expansions. The regular product of f and g (referred as their $*$-product) is the regular function defined by*

$$f * g(q) = \sum_{n\in\mathbb{N}} q^n \sum_{k=0}^{n} a_k b_{n-k} \tag{5.23}$$

on the same ball $\mathbf{D}$.

We can define two additional operations on regular functions.

Definition 5.3.5. *Let $f : \mathbf{D} \to \mathbb{H}$ be a regular function and let $f(q) = \sum_{n\in\mathbb{N}} q^n a_n$ be its power series expansion. The regular conjugate of f is the regular function defined by $f^c(q) = \sum_{n\in\mathbb{N}} q^n \overline{a_n}$ on the same ball $\mathbf{D}$. The symmetrization of f is the function $f^s = f * f^c = f^c * f$.*

Definition 5.3.6. *Let f be a regular function on a symmetric slice domain Ω. If $f \neq 0$ on Ω, the regular reciprocal of f is the function*

$$f^{-*} = (f^s)^{-1} f^c.$$

One way to generalize Blaschke functions over the set of quaternions would be the direct extension of the before mentioned formula over the quaternions, i.e., let define the quaternionic variable Blaschke function like in the matrix representation by

$$B_a(q) := (1 - q\overline{a})^{-1}(q - a) = \frac{1}{\lambda_{a,q}}\overline{(1 - q\overline{a})}(q - a) = \frac{1}{\lambda_{a,q}}(1 - a\overline{q})(q - a), \tag{5.24}$$

$$\lambda_{a,q} = \overline{(1-q\overline{a})}(1-q\overline{a}) = (1-a\overline{q})(1-q\overline{a}), a \in \mathbf{D}, q \in \overline{\mathbf{D}},$$

where $\mathbf{D}$ denotes the quaternionic unit ball. This function is not slice regular.

Recently in [5, 14, 160] it was introduced and studied another quaternionic extension of the Blaschke functions, the so called slice regular Blaschke functions.

Definition 5.3.7. *The slice regular Blaschke function is defined by*

$$\mathcal{B}_a(q) = (1-q\overline{a})^{-*} * (q-a),\ a \in \mathbf{D}, q \in \overline{\mathbf{D}}. \tag{5.25}$$

This function inherits all the nice properties of the complex Blaschke functions, i.e., is a regular fractional transformations (regular Möbius transformation) that maps the open quaternionic unit ball $\mathbf{D}$ onto itself and the boundary of unit ball $\mathbf{T}$ onto itself bijectively (see [5, 6, 14, 160]).

The classical and regular Blaschke functions are related in the following way:

$$\mathcal{B}_a(q) = (1-q\overline{a})^{-*} * (q-a) = B_a(T_a(q)),$$

where $T_a(q) = (1-qa)^{-1}q(1-qa)$ is a diffeomorphism of $\mathbf{D}$.

It can also be proved that the factors in the definition of the regular Blaschke product commute, that is

$$\mathcal{B}_a(q) = (1-q\overline{a})^{-*} * (q-a) = (q-a) * (1-q\overline{a})^{-*}.$$

One of the most fertile chapters of the theory of complex holomorphic functions consists of the theory of Hardy spaces. In the papers [5, 6, 63] the quaternionic counterpart of complex Hardy spaces was introduced, and their basic and fundamental properties were investigated.

Definition 5.3.8. *Let $f : \mathbf{D} \to \mathbb{H}$ be a regular function and let $0 < p < +\infty$. Set*

$$\|f\|_p = \sup_{I\in\mathbb{S}} \lim_{r\to 1_-} \frac{1}{2\pi}\left(\int_0^{2\pi} |f(re^{I\theta})|^p d\theta\right)^{1/p}, \tag{5.26}$$

and set

$$\|f\|_\infty = \sup_{q\in\mathbf{D}} |f(q)|. \tag{5.27}$$

Then, for any $0 < p \leqslant +\infty$, we define the quaternionic Hardy space $H^p(\mathbf{D})$ as

$$H^p(\mathbf{D}) = \{f : \mathbf{D} \to \mathbb{H} | f \textit{ is slice regular and } \|f\|p < +\infty\}. \tag{5.28}$$

In [5, 6, 63] the main properties and features of the quaternionic H^p-norms were studied, and the initial properties of the quaternionic H^p spaces were established. It turned out that many properties of the complex Hardy spaces have their quaternionic analogs. The boundary behavior of functions f in $H^p(\mathbf{D})$ is very similar to the complex case, i.e., for almost every $\theta \in [0, 2\pi)$, the limit

$$\lim_{r\to 1_-} f(re^{I\theta}) = \widetilde{f}_I(e^{I\theta}) \tag{5.29}$$

exists for all $I \in \mathbb{S}$ and belongs to $L^p(\partial D_I)$. The properties of the boundary values of the $*$-product of two functions, each belonging to some $H^p(\mathbf{D})$ space was investigated. The classical H^p kernels, and in particular the Poisson kernel, to the quaternionic setting was extended and the Poisson-type and Cauchy-type representation formulas for all $f \in H^p(\mathbf{D})$ were deduced. Analogues of outer and inner functions and singular factors on $\mathbf{D}$ were given, whose definitions (when compared with those used in the complex case) clearly resent of the peculiarities of the non commutative quaternionic setting. Factorization properties of H^p functions were established. The Blaschke factors of a function f in $H^p(\mathbf{D})$ are built from the zero set of f using the regular Blaschke functions; it can be obtained also a complete factorization result, in terms of an outer, a singular and a Blaschke factor, for a subclass of regular functions, namely for the one-slice-preserving functions.

It is valid the following:

Theorem 5.3.9. *(Splitting Formula) If $f \in H^p(\mathbf{D})$ for some $p \in (0, +\infty]$, then for any $I \in \mathbb{S}$, the splitting of f on L_I with respect to $J \in \mathbb{S}$, J orthogonal to I, is $f_I(z) = F(z) + G(z)J$, and the holomorphic functions F and G are both in $H^p(\mathbf{D}_I)$.*

In analogy with the complex case, the space $H^2(\mathbf{D})$ is special. Indeed the 2-norm turns out to be induced by an inner product. Let $f \in H^2(\mathbf{D})$ and let $f(q) = \sum_{n\geqslant 0} q^n a_n$ be its power series expansion. Then the square of the 2-norm of f coincides with

$$\|f\|^2 = \sum_{n\geqslant 0} |a_n|^2. \tag{5.30}$$

This result permits a way to define an inner product on the space $H^2(\mathbf{D})$. In fact, if $f, g \in H^2(\mathbf{D})$, let $f(q) = \sum_{n\geqslant 0} q^n a_n$, $g(q) = \sum_{n\geqslant 0} q^n b_n$ be their power series expansions, based on previous result, then their inner product is defined by

$$\langle f, g\rangle = \lim_{r\to 1} \frac{1}{2\pi} \int_0^{2\pi} \overline{g(re^{I\theta})} f(re^{I\theta}) d\theta = \sum_{n\geqslant 0} \overline{b_n} a_n, \tag{5.31}$$

for any $I \in \mathbb{S}$.

Thanks to the existence of the radial limit it is possible to obtain integral representations for functions in $H^p(\mathbf{D})$ for $p \in [1, +\infty]$ (see [82, 42]).

Theorem 5.3.10. *If $f \in H^p(\mathbf{D})$ for $p \in [1, +\infty]$, then for any $I \in \mathbb{S}$, f_I is the Poisson integral and the Cauchy integral of its radial limit $\tilde{f}_I$, i.e.,*

$$f_I(re^{I\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r cos(\theta - t) + r^2} \tilde{f}_I(e^{It}) dt \tag{5.32}$$

and

$$f_I(z) = \frac{1}{2\pi I} \int_{\partial D_I} \frac{d\xi}{\xi - z} \tilde{f}_I(\xi). \tag{5.33}$$

The next result, on the other hand, is a more powerful Cauchy Formula, which allows the reconstruction of f on the entire open set of definition, by using its values on a single slice.

Theorem 5.3.11. *(Cauchy Formula). Let f be a regular function on a symmetric slice domain Ω. If U is a bounded symmetric open set with $U \subset \Omega$, $I \in \mathbb{S}$, and if ∂U_I is a finite union of disjoint rectifiable Jordan curves, then, for $q \in U$,*

$$f(q) = \frac{1}{2\pi} \int_{\partial U_I} (s-q)^{-*} ds_I f_I(s), \tag{5.34}$$

where $ds_I = -I ds$ and $(s-q)^{-*}$ denotes the regular reciprocal of $(s-q)$.

Theorem 5.3.12. *(Zero set structure). Let f be a regular function on a symmetric slice domain. If f does not vanish identically, then its zero set consists of the union of isolated points and isolated 2-spheres of the form $x + y\mathbb{S}$ with $x, y \in \mathbb{R}, y \neq 0$.*

Spheres of zeros of real dimension 2 are a peculiarity of regular functions.

Let f be a regular function on a symmetric slice domain. A 2-dimensional sphere $x + y\mathbb{S} \subset \mathcal{Z}_f$ of zeros of f is called a spherical zero of f and is represented by an element $x + yI$ of such a sphere, called a generator of the spherical zero $x + y\mathbb{S}$. Any zero of f that is not a generator of a spherical zero is called an isolated zero (or a non spherical zero or simply a zero) of f.

Theorem 5.3.13. *Let $p \in (0, +\infty]$, $f \in H^p(\mathbf{D})$, $f \neq 0$ and let $\{b_n\}_{n\in\mathbb{N}}$ be its sequence of zeros. Then $\{b_n\}_{n\in\mathbb{N}}$ satisfies the Blaschke condition*

$$\sum_{n \geqslant 0} (1 - |b_n|) < +\infty.$$

Holomorphic functions defined on a domain Ω_I, symmetric with respect to the real axis in the complex plane L_I, extend uniquely to the smallest symmetric slice domain of $\mathbb{H}$ containing Ω_I.

Theorem 5.3.14. *(Extension Lemma). Let Ω be a symmetric slice domain and choose $I \in \mathbb{S}$. If $f_I : \Omega_I \to \mathbb{H}$ is holomorphic, then setting*

$$f(x + yJ) = \frac{1}{2}[f_I(x + yI) + f_I(x - yI)] + \frac{JI}{2}[f_I(x - yI) - f_I(x + yI)] \tag{5.35}$$

extends f_I to a regular function $f : \Omega \to \mathbb{H}$. Moreover f is the unique extension and it is denoted by $ext(f_I)$.

5.3.2 The regular quaternionic Malmquist-Takenaka system

The extension of the M-T systems for quaternions (see Pap [120]) was described as follows: let us consider a sequence $a = (a_1, a_2, \cdots)$ of quaternions, $|a_n| < 1$, $(n \in \mathbb{N}^*)$ and the classical quaternionic extension of Blaschke-functions B_a. The functions $\Phi_n = \Phi_n^a\,(n \in \mathbb{N}^*)$ are defined very similarly to the complex case by the quaternionic product

$$\Phi_1(z) = \sqrt{1-|a_1|^2}(1-z\overline{a_1})^{-1},$$
$$\Phi_n(z) = \sqrt{1-|a_n|^2}\left(\prod_{k=1}^{n-1} B_{a_k}(z)\right)(1-z\overline{a_n})^{-1} \;\; (z \in \overline{\mathbb{B}},\, n = 2, 3, ...). \tag{5.36}$$

Unfortunately, they are not regular functions anymore. But still for their Dirichlet kernel it was possible to prove the analogue of the Darboux-Christoffel formula (see Pap [120]). When all the parameters are equal, that is $a_n = a = re^{\theta\mathbf{I}} = r(\cos\theta + \mathbf{I}\sin\theta)\,(n \in \mathbb{N}^*)$, we obtain the quaternionic analogue of the discrete Laguerre system. Even in this special case the orthogonality is not yet proved, but a discrete orthogonality property of this particular case can be proved(see [120]).

In [135] Qian, Sprossig, Wang studied the decompositions of functions in the quaternionic monogenic Hardy spaces into linear combinations of the basic functions in the orthogonal rational systems, which can be obtained in the respective contexts through Gram-Schmidt orthogonalization process on shifted Cauchy kernels. While in the complex case, following these two ways we get the same system, here in the quaternionic case it has not been proved yet, that the two methods give the same.

In this section we will consider the slice regular analog of the Malmquist-Takenaka system and we will investigate the properties of this system. The results were published in Pap [132].

Let us consider a sequence $a = (a_1, a_2, ...)$ of quaternions in the unit ball, i.e., $|a_n| < 1$, $(n \in \mathbb{N}^*)$. The slice regular analogue of the Malmquist-Takenaka system can be expressed by the slice regular quaternionic Blaschke-functions $\mathcal{B}_{a_n}(q) = (1-q\overline{a_n})^{-*} * (q - a_n) = (q-a_n) * (1-q\overline{a_n})^{-*}$. Namely, the functions $\mathbf{\Phi_n} = \mathbf{\Phi_n^a}\,(\mathbf{n} \in \mathbb{N}^*)$ are defined very similarly to the complex case, but here we use the slice regular product of the factors:

$$\mathbf{\Phi}_1(z) = \sqrt{1-|a_1|^2}(1-z\overline{a_1})^{-*},$$
$$\mathbf{\Phi}_n(z) = \sqrt{1-|a_n|^2}\left(*\prod_{k=1}^{n-1} \mathcal{B}_{a_k}(z)\right) * (1-z\overline{a_n})^{-*} \;\; (z \in \overline{\mathbb{B}},\, n = 2, 3, ...), \tag{5.37}$$

where $*\prod$ means the $*$-product of the factors. Because $\mathcal{B}_a(q)$ is a slice regular function and the $*$-product of two slice regular functions is slice regular, in this way we generate a slice regular system.

When all the parameters are equal, namely $a_n = a = re^{\theta\mathbf{I}} = r(\cos\theta + \mathbf{I}\sin\theta)\,(n \in \mathbb{N}^*)$, then we get the slice regular analogue of the discrete Laguerre system,

$$\mathbf{L}_1(z) = \sqrt{1-|a|^2}(1-z\overline{a})^{-*},$$

$$\mathbf{L}_n(z) = \sqrt{1-|a|^2}\left(* \prod_{k=1}^{n-1} \mathcal{B}_a(z) \right) * (1-z\overline{a})^{-*} \ \ (z \in \overline{\mathbb{B}},\, n=2,3,...),$$

When all the parameters are 0, i.e., $a_n = 0\,(n \in \mathbb{N}^*)$ we obtain $\mathbf{\Phi}_n(z) = z^n$, the quaternionic analogue of the trigonometric system.

Lemma 5.3.15 (**Pap**, [132])**.** *The slice regular analogue of the discrete Laguerre system can be written in the following form:*

$$\mathbf{L}_n(z) = \sqrt{1-|a|^2}(z-a)^{*n} * (1-z\overline{a})^{-*(n+1)}(z \in \overline{\mathbb{B}},\, n=2,3,...).$$

This can be proved by induction, using the commutativity property of the factors in $\mathcal{B}_a(z) = (1-z\overline{a})^{-*} * (z-a) = (z-a) * (1-z\overline{a})^{-*}$.

Theorem 5.3.16 (**Pap** [132])**.** *If all the parameters of the slice regular Malmquist - Takenaka system given by (5.37) are on the same slice, i.e., there exists $I \in \mathbb{S}$ such that $a_n = r_n e^{\theta_n I} = r_n(\cos\theta_n + I\sin\theta_n)\,(r_n < 1,\, n \in \mathbb{N}^*)$, then $\mathbf{\Phi}_n, (n \in \mathbb{N}^*)$ is a slice regular orthonormal system in $H^2(\mathbf{D})$.*

Proof of Theorem 5.3.16. Recall the definition of the inner product on the space $H^2(\mathbf{D})$: if $f(q) = \sum_{n\geqslant 0} q^n a_n$ and $g(q) = \sum_{n\geqslant 0} q^n b_n$, then their inner product is

$$\langle f, g\rangle = \sum_{n\geqslant 0} \overline{b_n} a_n = \lim_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} \overline{g(re^{I\theta})} f(re^{I\theta}) d\theta$$

for any $I \in \mathbb{S}$. Let I be the direction fixed by $a_n = r_n e^{\theta_n I} = r_n(\cos\theta_n + I\sin\theta_n)$. On $\mathbf{D}_I = \mathbf{D} \cap L_I$ the regular Blaschke product is slice preserving i.e., if $a \in \mathbf{D}_I$ we have

$$\mathcal{B}_a(\overline{\mathbf{D}_I}) \subset \overline{\mathbf{D}_I}.$$

Moreover, an easy computation shows that for $a, q \in \overline{\mathbf{D}_I}$:

$$(1-q\overline{a})^{-*} = (1-q\overline{a})^{-1},\ \mathcal{B}_a(q) = B_a(q).$$

Indeed, if $a \in \mathbf{D}_I$ and $q \in \overline{\mathbf{D}_I}$ taking in consideration the commutativity of the product on the slice $\overline{\mathbf{D}_I}$ it follows that:

$$(1-q\overline{a}) * (1-qa) = (1-q\overline{a})(1-qa),$$

which implies that

$$(1-q\overline{a})^{-*} = [(1-q\overline{a}) * (1-qa)]^{-1}(1-qa) = [(1-q\overline{a})(1-qa)]^{-1}(1-qa) = (1-q\overline{a})^{-1}.$$

The classical and regular Blaschke functions are related in the following way:

$$\mathcal{B}_a(q) = (1-q\overline{a})^{-*} * (q-a) = B_a(T_a(q)),$$

where $T_a(q) = (1-qa)^{-1}q(1-qa)$. If $a \in \mathbf{D}_I$ and $q \in \overline{\mathbf{D}_I}$ taking again in consideration the commutativity of the product on the slice $\overline{\mathbf{D}_I}$ it follows that $T_a(q) = (1-qa)^{-1}q(1-qa) = q$ and $\mathcal{B}_a(q) = B_a(T_a(q)) = B_a(q)$.

From the slice preserving property and the Splitting Lemma of slice regular functions it follows that for every $z = x+Iy \in \overline{\mathbf{D}_I}$ we have $\mathcal{B}_a(z) = F(z)$, where $F(z)$ is holomorphic in $\overline{\mathbf{D}_I}$. This implies that the slice regular $*$-product on this slice is equal to the point-wise product of the factors, moreover on the slice $\overline{\mathbf{D}_I}$ we have:

$$\mathbf{\Phi}_n(z) = \sqrt{1-|a_n|^2}\left(*\prod_{k=1}^{n-1}\mathcal{B}_{a_k}(z)\right)*(1-z\overline{a_n})^{-*} = \sqrt{1-|a_n|^2}\left(\prod_{k=1}^{n-1}B_{a_k}(z)\right)(1-z\overline{a_n})^{-1},$$

and the order of the factors, can be interchanged, because the pointwise product is commutative on the slice $\overline{\mathbf{D}_I}$.

Slice regular Blaschke functions and classical Blaschke functions map the unit ball into the unit ball, and the boundary into itself, consequently $|B_a(e^{I\theta})\overline{B_a(e^{I\theta})}| = 1$. Taking into account these properties, the commutativity of the product on the slice $\overline{\mathbf{D}_I}$, and the Cauchy formula, in the proof we can follow the same line as in the complex case:

$$\langle \mathbf{\Phi}_n, \mathbf{\Phi}_n\rangle = \lim_{r\to 1_-}\frac{1}{2\pi}\int_0^{2\pi}\overline{\mathbf{\Phi}_n(re^{I\theta})}\mathbf{\Phi}_n(re^{I\theta})d\theta =$$

$$(1-|a_n|^2)\frac{1}{2\pi}\int_0^{2\pi}\overline{(1-e^{I\theta}\overline{a_n})^{-1}}(1-e^{I\theta}\overline{a_n})^{-1}d\theta = 1.$$

For $m > n$ we have:

$$\langle \mathbf{\Phi}_n, \mathbf{\Phi}_m\rangle = \lim_{r\to 1_-}\frac{1}{2\pi}\int_0^{2\pi}\overline{\mathbf{\Phi}_n(re^{I\theta})}\mathbf{\Phi}_m(re^{I\theta})d\theta =$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\overline{\mathbf{\Phi}_n(e^{I\theta})}\mathbf{\Phi}_m(e^{I\theta})d\theta =$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\overline{\sqrt{1-|a_n|^2}\left(\prod_{k=1}^{n-1}B_{a_k}(e^{I\theta})\right)(1-e^{I\theta}\overline{a_n})^{-1}}\sqrt{1-|a_m|^2}\left(\prod_{k=1}^{m-1}B_{a_k}(e^{I\theta})\right)(1-e^{I\theta}\overline{a_m})^{-1}d\theta$$

$$= \sqrt{1-|a_n|^2}\sqrt{1-|a_m|^2}\prod_{k=n}^{m-1}B_{a_k}(a_n)(1-a_n\overline{a_m})^{-1} = 0.$$

Theorem 5.3.17 (**Pap** [132]). *If all the parameters of the slice regular Malmquist - Takenaka system are on the same slice satisfying the non-Blaschke condition* $\sum_{n\geqslant 0}(1-|a_n|) = +\infty$*, then the system* $\mathbf{\Phi}_n, (n \in \mathbb{N}^*)$ *is complete in* $H^2(\mathbf{D})$.

Proof of Theorem 5.3.17. To prove that the system is complete in $H^2(\mathbf{D})$ we need to prove the following implication: if for an $f \in H^2(\mathbf{D})$ we have that $\langle f, \mathbf{\Phi}_n \rangle = 0$, $n \in \mathbb{N}^*$, then $f \equiv 0$. According to the Splitting Lemma there exist two holomorphic functions $F, G : \mathbf{D}_I \to L_I$, such that for every $z = x+yI \in \mathbf{D}_I$, we have $f_I(z) = F(z)+G(z)J$, where J is orthogonal to I. Moreover $F, G \in H^2(\mathbf{D}_I)$. Then from $\langle f, \mathbf{\Phi_n} \rangle = \mathbf{0}$, $\mathbf{n} \in \mathbb{N}$ we get that $\langle F, \mathbf{\Phi}_n \rangle = 0$ and $\langle G, \mathbf{\Phi}_n \rangle = 0$, $n \in \mathbb{N}$. Because on the slice $\mathbf{D}_I$ the functions F, G are slice preserving holomorphic functions, analogue as in the case of the complex M-T, which is complete under the assumption of the theorem, we get that $F(z) = G(z) = 0$, $z \in \mathbf{D}_I$. Consequently $f_I(z) = 0$, $z \in \mathbf{D}_I$.

According to the Extension Lemma holomorphic functions defined on a domain Ω_I, symmetric with respect to the real axis in the complex plane L_I, extend uniquely to the smallest symmetric slice domain of $\mathbb{H}$ containing Ω_I. We apply this to the unit ball $\mathbf{D}$ and his slice $\mathbf{D}_I$. Then for $f \in H^2(\mathbf{D})$ we have

$$f(x+yJ) = \frac{1}{2}[f_I(x+yI) + f_I(x-yI)] + \frac{JI}{2}[f_I(x-yI) - f_I(x+yI)]$$

extends f_I uniquely to a regular function $f : \mathbf{D} \to \mathbb{H}$. Taking into consideration that $f_I(z) = 0$, $z \in \mathbf{D}_I$, we have $f(z) = 0$, $z \in \mathbf{D}$.

The theorem says that although we consider all the parameters of the slice regular Malmquist-Takenaka system on the same slice, i.e., if there exists $I \in \mathbb{S}$ such that $a_n = r_n e^{\theta_n I} = r_n(\cos\theta_n + I \sin\theta_n)$ $(r_n < 1,\, n \in \mathbb{N}^*)$, and they satisfy the non-Blaschke condition, then the function $f \in H^2(\mathbf{D})$ is determined uniquely by the coefficients $(\langle f, \mathbf{\Phi}_n \rangle,\, n \in \mathbb{N}^*)$.

5.3.3 The properties of the projection operator

For $f \in H^2(\mathbf{D})$ according to the Splitting Lemma, there exist two holomorphic functions $F, G : \mathbf{D}_I \to L_I$, such that for every $z = x+yI \in \mathbf{D}_I$, we have $f_I(z) = F(z)+G(z)J$, where J is orthogonal to I. Moreover $F, G \in H^2(\mathbf{D}_I)$. Let us consider the boundary limit of functions f in $H^2(\mathbf{D})$. Similarly to the complex case, for almost every $\theta \in [0, 2\pi)$, the limit

$$\lim_{r \to 1_-} f(re^{I\theta}) = f_I(e^{I\theta}) = F(e^{I\theta}) + G(e^{I\theta})J$$

exists for all $I \in \mathbb{S}$ and in this case $f_I(e^{I\theta}), F(e^{I\theta}), G(e^{I\theta})$ belong to $L^2(\partial\mathbb{B}_I)$.

If all the parameters of the slice regular Malmquist-Takenaka system are on the same slice, i.e., there exists $I \in \mathbb{S}$ such that $a_n = r_n e^{\theta_n I} = r_n(\cos\theta_n + I\sin\theta_n)$ $(r_n < 1,\, n \in \mathbb{N}^*)$, let us consider the orthogonal projection operator of an arbitrary function $f \in H^2(\mathbf{D})$ on the subspace V_n spanned by the functions $\{\mathbf{\Phi}_k,\, k = 1, \cdots, n\}$

$$P_n f(z) = \sum_{k=1}^{n} \mathbf{\Phi}_k(z) \langle f, \mathbf{\Phi}_k \rangle, \tag{5.38}$$

where the value of the scalar product $\langle f, \mathbf{\Phi}_k\rangle$ is

$$\langle f, \mathbf{\Phi}_k\rangle = \lim_{r\to 1_-} \frac{1}{2\pi}\int_0^{2\pi} \overline{\mathbf{\Phi}_k(re^{I\theta})} f(re^{I\theta})d\theta =$$

$$\lim_{r\to 1_-} \frac{1}{2\pi}\int_0^{2\pi} \overline{\mathbf{\Phi}_k(re^{I\theta})} F(re^{I\theta})d\theta + \lim_{r\to 1_-} \frac{1}{2\pi}\int_0^{2\pi} \overline{\mathbf{\Phi}_k(re^{I\theta})} G(re^{I\theta})d\theta J.$$

On the slice $\overline{\mathbf{D}_I}$ we have:

$$\mathbf{\Phi}_n(z) = \frac{1}{\sqrt{1-|a_n|^2}}\left(*\prod_{k=1}^{n-1} \mathcal{B}_{a_k}(z)\right) * (1 - z\overline{a_n})^{-*} =$$

$$\frac{1}{\sqrt{1-|a_n|^2}}\left(\prod_{k=1}^{n-1} B_{a_k}(z)\right) (1 - z\overline{a_n})^{-1} = \Phi_n(z),$$

consequently the coefficients of the projection operator can be expressed by

$$\langle f, \mathbf{\Phi}_k\rangle = \frac{1}{2\pi}\int_0^{2\pi} \overline{\Phi_k(e^{I\theta})} F(e^{I\theta})d\theta + \frac{1}{2\pi}\int_0^{2\pi} \overline{\Phi_k(e^{I\theta})} G(e^{I\theta})d\theta J.$$

If $\sum_{n\geqslant 0}(1-|a_n|) = +\infty$, then the system $\mathbf{\Phi}_n, (n \in \mathbb{N}^*)$ is complete in $H^2(\mathbf{D})$, this implies that for every $f \in H^2(\mathbf{D})$ the projection of f on V_n converges in norm to f, i.e., we have

$$\|f - P_n f\| \to 0, \quad n \to \infty.$$

Since convergence in norm implies uniform convergence inside the unit ball $\mathbf{D}$ on every compact subset, we conclude that $P_n f(z) \to f(z)$ uniformly on every compact subset of the unit ball.

Theorem 5.3.18. *If the parameters of the slice regular Malmquist-Takenaka system are on the same slice, i.e., there exists $I \in \mathbb{S}$ such that $a_n = r_n e^{\theta_n I}$ $(r_n < 1,\ n \in \mathbb{N}^*)$, then for all $f \in H^2(\mathbf{D})$ the restriction of the projection operator $P_n f$ to the slice $\mathbf{D}_I$ of the unit ball is an interpolation operator in the points $a_\ell = r_\ell e^{\theta_\ell I}$ $(\ell \in \{1, \cdots, n\})$.*

Proof of Theorem 5.3.18. The restriction of the projection $P_n f$ to the slice $\overline{\mathbf{D}_I}$ can be written in closed form as follows:

$$(P_n)_I f(z) = \sum_{k=0}^{n} (\mathbf{\Phi}_k)_I(z)\langle f, \mathbf{\Phi}_k\rangle =$$

$$\frac{1}{2\pi}\int_0^{2\pi} \sum_{k=0}^{n} (\Phi_k)_I(z)\overline{\Phi_k(e^{I\theta})} F(e^{I\theta})d\theta + \frac{1}{2\pi}\int_0^{2\pi} \sum_{k=0}^{n} (\Phi_k)_I(z)\overline{\Phi_k(e^{I\theta})} G(e^{I\theta})d\theta J.$$

For the Dirichlet kernel of the classical extension over the set of the quaternions of the M-T system we have an analogue of Darboux-Christoffel formula (see [120]):

$$D_n^*(z,w) := \sum_{\ell=1}^{n} \Phi_\ell(z)(1-z\overline{w})\overline{\Phi_\ell(w)} = 1 - \prod_{\ell=1}^{n} B_{a_\ell}(z) \prod_{\ell=1}^{N} \overline{B_{a_{n-\ell+1}}(w)},$$

for all $z, w \in \overline{\mathbf{D}}$, $z \neq w$. Taking the restriction of the Dirichlet kernel to the slice $\overline{\mathbf{D_I}}$, where the product is commutative, and using the slice preserving property of $\Phi_\ell(z)$ on $\overline{\mathbf{D_I}}$ we get that the restriction of the projection operator on the slice $\overline{\mathbf{D_I}}$ can be expressed very similar to the complex case:

$$(P_n)_I f(z) =$$

$$= \frac{1}{2\pi} \int_0^{2\pi} (1-ze^{-I\theta})^{-1} \left(1 - \prod_{\ell=1}^{n} B_{a_\ell}(z) \prod_{\ell=1}^{n} \overline{B_{a_{n-\ell+1}}(e^{I\theta})}\right) F(e^{I\theta})d\theta +$$

$$+ \frac{1}{2\pi} \int_0^{2\pi} (1-ze^{-I\theta})^{-1} \left(1 - \prod_{\ell=1}^{n} B_{a_\ell}(z) \prod_{\ell=1}^{n} \overline{B_{a_{n-\ell+1}}(e^{I\theta})}\right) G(e^{I\theta})d\theta J.$$

From here using the Cauchy formula we get that the restriction of the projection $P_n f$ to the slice $\overline{\mathbf{D_I}}$ is an interpolation operator for the points $z = a_\ell$, $\ell = 1, \cdots, n$. Indeed we have:

$$(P_n)_I f(a_\ell) = \frac{1}{2\pi} \int_0^{2\pi} (1-a_\ell e^{-I\theta})^{-1} F(e^{I\theta})d\theta + \frac{1}{2\pi} \int_0^{2\pi} (1-a_\ell e^{-I\theta})^{-1} G(e^{I\theta})d\theta J =$$

$$F(a_\ell) + G(a_\ell)J = f_I(a_\ell).$$

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