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Exact methods for the dynamics of integrable models

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The content of this thesis

This thesis reviews the scientific work of the author between 2013 and 2020, and it is based on the papers [1–29]. The focus is on the dynamics of integrable models, involving both equilibrium and out-of-equilibrium situations. Almost all concrete problems discussed here involve integrable spin chains; other types of integrable models such as quantum gas models or Quantum Field Theories are mentioned only in passing. Naturally, as we discuss the contributions of the author, all the key developments in the field are also reviewed. A few papers of the author from this period, which are only tangentially related to the main topics of the thesis are not included in this review.

The field of integrability has a rather rich history, and it is very specialized. The physical questions that we investigate are rather general, but the mathematical tools used in this field are specifically designed for these models. We made an attempt to focus on the physics behind the mathematical statements, and in the first Section below we also explain all the key ideas and common concepts behind integrability. In later Sections we tried to keep the amount of technical details to a minimum, nevertheless highlighting the main results of the author and explaining the key mathematical structures involved. We will present less and less technical details as we progress towards more and more complicated computations.

The reader should not expect to use the thesis as a stand-alone introduction to integrability. We added many references to textbooks, lecture notes, and reviews written by leading experts, which might be helpful to the interested reader.

Ackowledgements

(in hungarian)

Itt szeretnék köszönetet mondani azoknak az embereknek, akik nagy hatással voltak szakmai pályámra, s akiknek mindezért nagyon hálás vagyok.

Először is a régi témavezetőmnek, Takács Gábornak szeretném megköszönni a sok éves közös munkát, a megszámlálhatatlan diszkussziót, tanakodást, a kutatás és a tanulás iránti lelkesedés átadását. Nagyrészt a közös munka örömének és sikerének köszönhető, hogy az elméleti fizikán belül az eredeti területnél, az integrálhatóság témakörénél maradtam. Gábor több alkalommal a személyes életemben is támogatott, amiért különösen hálás vagyok.

Kiemelt köszönettel tartozom továbbá Jean-Sébastien (J-S) Caux amsterdami témavezetőmnek, akitől szintén nagyon sokat tanultam. Az ebben a dolgozatban ismertetett eredmények jelentős része olyan módszerekkel született, melyeket J-S kutatócsoportjában sajátítottam el.

Szeretnék továbbá köszönetet mondani mindazoknak az embereknek, akikkel együtt dolgoztam, illetve akiktől konferenciákon vagy egyéb helyeken különösen sokat tanultam: Bajnok Zoltán, Pasquale Calabrese, Patrick Dorey, Eisler Viktor, Fehér György, Gombor Tamás, Frank Göhmann, Arthur Hutsalyuk, Yunfeng Jiang, Karol Kozlowski, Charlotte Kristjansen, Marius de Leeuw, Georgios Linardopoulos, Kormos Márton, Chihiro Matsui, Giuseppe Mussardo, Lorenzo Piroli, Hubert Saleur, Keiichi Shigechi, Szécsényi István, Roberto Tateo, Eric Vernier, Werner Miklós, Andreas Wipf, Zaránd Gergely.

Hálás vagyok a szüleimnek, akik úgy neveltek, hogy a tanulásnak, a tudásnak, és a szorgalomnak becsülete van, és hogy a tanulmányaim során végig támogattak.

Hálás vagyok továbbá M. Lászlónak és a feleségemnek, Juliannának, kik bíztattak és támogattak abban, hogy többször is pályázzam meg az MTA Lendület pályázatát. Végül 2020-ban nyertem is, és ezzel teljesült a nagy álmom, a saját kutatócsoport létrehozása. Remélem sok szép és fontos eredményt sikerül majd elérnünk az ebben a dolgozatban ismertetett eredményekre építve.

Contents

"I got really fascinated by these (1+1)-dimensional models that are solved by the Bethe ansatz and how mysteriously they jump out at you and work and you don't know why. I am trying to understand all this better."

Richard P. Feynman, 1988 [30]

"Because of its underlying mathematical structure and the richness of its results, the Bethe ansatz has had a remarkable impact on several fields, with many surprises along the way. Given the recent advances in the manipulation of atoms in optical lattices, no doubt many more surprises lie ahead."

Murray T. Batchelor, 2007 [31]

"While there are many important questions in the field of integrable models, one of the most fundamental questions is the following: Why integrable models exist? This is a highly non-trivial question."

Masahito Yamazaki, 2018 [32]

1 Introduction

1.1 The structure of this thesis

This first Section is a general introduction to integrable models, and integrable spin chains in particular, and to the questions investigated recently by the author and other researchers working in this field. Afterwards Sections 2-6 are devoted to 5 closely related topics where the author contributed: Generalized Gibbs Ensemble (GGE), Integrable initial states, Exact overlaps, Correlation functions, and Generalized Hydrodynamics (GHD). Each Section starts with a separate introduction, and ends with a short sub-section discussing a few open problems. We did not add a separate summary to the document.

Throughout this work we use physical units such that

$$\begin{aligned} \hbar &= 1 \qquad \text{(reduced Planck constant)} \\ c &= 1 \qquad \text{(speed of light)} \\ k &= 1 \qquad \text{(Boltzmann constant).} \end{aligned} \tag{1.1}$$

1.2 History and motivation

Integrable models are special many body systems that allow for an exact solution. Examples of integrable systems can be found in both classical and quantum physics. In this work we focus on the quantum mechanical models.

Furthermore, we are interested in models that exist in one spatial dimension. It is important that there are physical models which can be considered "solvable" and which exist in higher dimensions: for example free models or certain Conformal Field Theories in dimensions greater than one. However, our focus lies in 1D.

Adding the time dimension as well, these models are sometimes called 1+1 dimensional integrable models. It is known that a one dimensional quantum system is equivalent to a 2D statistical physical system: $1 + 1 = 2$. Following this picture, it is natural that our integrable quantum systems are very closely related to certain integrable 2D statistical physical models. We will focus on the quantum systems, but the connections to the 2D models will be mentioned whenever they are relevant.

Let us give a very brief historical overview of this field, omitting the precise definitions of the models; in many cases the definitions are given later in the text. Here we just present the key historical developments.

Perhaps the most famous example for a one dimensional integrable model is the Heisenberg spin chain, which was solved by Hans Bethe in 1931 [33]. His method is known today as the (coordinate-) Bethe Ansatz. The historical significance of this work can not be under-estimated: Even today, almost 90 years later a whole community of researchers is still working on various aspects of the Bethe Ansatz. Naturally, the method has advanced and it acquired many extensions and re-formulations. And the theory still provides open questions to the researchers.

An other early and very important example for an integrable system is the 2D classical Ising model, which was solved by Lars Onsager in 1944 [34], and which was shown to be equivalent to free fermions in [35].

Exact solutions of further 1+1 dimensional integrable systems appeared in the following decades. They appeared in three rather different areas.

- Further non-relativistic condensed matter models were found that were solvable by Bethe's method in coordinate space. Important examples include the solution of the XXZ spin chain [36, 37] and the δ-function interacting Bose gas [38, 39].

- Further 2D classical statistical physical models were solved by Rodney Baxter, see his book [40]. The main examples are the so-called 6-vertex and 8-vertex models on a square lattice; they are closely connected to the XXZ and XYZ spin chains, respectively. The 6-vertex model was solved by the Bethe Ansatz in [41], but the more complicated case of the 8-vertex model and Baxter's solution thereof seemed unrelated at that time.

- In high energy physics, exactly solvable 1+1 dimensional Quantum Field Theories were found, see for example [42] and references therein. Perhaps the most famous example is the sine-Gordon model. The methods used here are extensions of the "analytic S-matrix" program from 3+1 dimensional field theory.

A further class of theories are the classical integrable systems, which also have a rich history, with many connections to the theories mentioned above.

The three different types of models listed above seemed unrelated for some time, until it was realized that there are common algebraic structures underlying their solution. The unification of the framework of integrability can be attributed to the Leningrad group led by Ludvig Faddeev. Around 1980 they pioneered the theory known today as Quantum Inverse Scattering Method [43], based on the methods from classical integrability, and adapting them to the quantum case. The work [44] contains a historical summary of the key developments, written by Faddeev himself; we strongly recommend this article to the interested reader.

Perhaps the most important algebraic relation in this theory is the so-called Yang-Baxter (YB) relation, which underlies the solvability of all the models mentioned above. The relation is named after C. N. Yang and Baxter who discovered it independently (for the work of Yang see [45]). In this thesis it is not possible to discuss all the applications of the YB-relation; various monographs serve this purpose, see for example [46]. Let us just mention that the YB-relation and its solutions served as inspiration to V. G. Drinfel'd and M. Jimbo, who discovered the quantum groups in two independent papers [47, 48]. The quantum groups are special Hopf algebras, and their study grew into a separate field in mathematics.

After the birth of the canonical framework for quantum integrability there was a steady progress in multiple directions, which lasts even to the present day. The acquired knowledge in this area is huge. Important results concerning integrable spin chains include:

- The exact computation of the thermodynamics of the models using the so-called Thermodynamic Bethe Ansatz [49] and Quantum Transfer Matrix (QTM) methods [50].

- Analytic computation of correlation functions using vertex operator algebras [51], the Algebraic Bethe Ansatz [52, 53], including computations with the QTM method [50]. By studying the large distance limit of equilibrium correlations it became possible to confirm the predictions of Conformal Field Theory (CFT) [54].

- The exact computation of short distance correlations in a relatively simple, factorized form. This will be reviewed in Section 5.2.

- Numerical computation of the correlation functions at intermediate distances [55, 56].

There are many other important results, which concern the theoretical foundations; the list above focuses on developments which concern measurable quantities.

The reader might wonder: after so much effort devoted to the study of these models, what kinds of open questions remain? What gives motivation for their further study?

We can give the following answers to these questions.

- **Fundamental developments.** It might seem suprising, but there have been new results regarding the fundamental solution of these models even in the last couple of years. As examples we can mention new developments in the Separation of Variables approach [57,58], or the computation of (factorized) correlation functions in various models, including the work of the author discussed in Sections 5 and 6. These research areas still have many exciting open questions.

- **Experiments.** It was known for a long time that the integrable spin chains accurately describe the magnetism in certain chrystals, where the effective interactions are one-dimensional with a good approximation. Examples include the compound $KCuF_3$, which is described by the Heisenberg spin chain [59]. These materials motivated the research on equlibrium correlation functions, which were measured typically with neutron scattering experiments. However, recent technologies with ultracold atoms made it possible to artificially realize the integrable models (see the reviews [60, 61], or the concrete examples [62–64]). It became possible to measure the thermodynamical state functions of these models, and to compare them to exact computations from Bethe Ansatz [65, 66]. Even more recently the non-equlibrium dynamics was also investigated, see for example [63, 67–71]. These experimtal advances motivated the study of the:

- **Non-equilibrium dynamics.** For a long time the community focused only the equilibrium properties. This only changed in the last $\sim$10 years, while many key results appeared only in the last 5-6 years. Quite surprisingly, the nature of equilibration/thermalization in integrable models was only understood in these recent years. Furthermore, there were very few results for correlation functions in highly excited states, relevant to the experiments mentioned above. This motivated a whole community of researchers to work on *quantum quenches* and related non-equilbrium problems. Most of the present thesis is devoted to this topic, and many open problems remain.

- **AdS/CFT.** The discovery of the AdS/CFT correspondence led to an unexpected cross-fertilization between the fields of theoretical high energy physics and integrable models [72, 73]. It turned out that in the so-called planar limit the CFT in question can be solved by mapping certain local operators to states of certain integrable (long range interacting) spin chains. The interaction between the researchers from the two communities led to unexpected advances in both fields. Eventually

the spectral problem of AdS/CFT was solved in the planar limit [74–76]. Certain canonical methods of integrability were also developed further, leading sometimes to unexpected results, see for example [77, 78]. The AdS/CFT correspondence also provided new *integrable initial states* for the quantum quench problems; this is discussed in Section 3.5.

These 4 points constitute ample motivation to continue research on integrable models.

In this brief introduction we did not discuss the various connections to other fields in mathematics and mathematical physics. A list of such connections is presented later in Section 1.10.

1.3 Integrable spin chains – examples

In this work we focus on the dynamics of local integrable spin chains. They are defined by a Hamiltonian H acting on the Hilbert space $\mathcal{H} = \otimes_{x=1}^{L} \mathbb{C}^D$. Here L is the length of the chain and D is the dimension of the local spaces. The Hamiltonian is given by

$$H = \sum_{x=1}^{L} h(x), \tag{1.2}$$

where $h(x)$ is a Hamiltonian density, which is typically acting on sites x and $x+1$. We assume periodic boundary conditions in almost all cases.

Perhaps the most famous example is the $SU(2)$-invariant Heisenberg spin chain, also known as the XXX model. It is given by the Hamiltonian density

$$h(j) = \boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_{j+1} - 1, \tag{1.3}$$

where $\boldsymbol{\sigma} = \{\sigma^x, \sigma^y, \sigma^z\}$.

The XXZ model is the anisotropic deformation of the XXX chain, which still retains rotational symmetry around the direction of the chain, and it is given by

$$h(j) = \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta(\sigma_j^z \sigma_{j+1}^z - 1). \tag{1.4}$$

Here Δ is the so-called anisotropy parameter; often we will use the parametrization $\Delta = \cosh(\eta)$. The Hamiltonian densities above are normalized such that the ferromagnetic state with all spins up (or down) is an eigenstate with zero eigenvalue.

The completely anisotropic deformation is the XYZ spin chains, whose Hamiltonian density is conventionally written as

$$h(j) = J_x \sigma_j^x \sigma_{j+1}^x + J_y \sigma_j^y \sigma_{j+1}^y + J_z \sigma_j^z \sigma_{j+1}^z. \tag{1.5}$$

It can be shown that (up to basis transformations) this is the most general two-site Hamiltonian which respects the symmetries of space reflection and spin reflection in three orthogonal directions. In this model there is no ferromagnetic vacuum state, which makes its solution much more complicated.

An other class of models with two dimensional local spaces are the so-called XYh models, where

$$h(j) = J_x \sigma_j^x \sigma_{j+1}^x + J_y \sigma_j^y \sigma_{j+1}^y + h_z \sigma_j^z. \tag{1.6}$$

They can be mapped to free fermionic operators using the Jordan-Wigner transformation and a further Bogoliubov transformation [79]. Special cases of this model include the transverse field quantum Ising model, where $J_y = 0$, and the $U(1)$-invariant XX model, where $J_x = J_y$. The XY model with $h_z = 0$ can be considered as a special case of the XYZ family as well; note however that adding a non-zero h_z leaves the integrable XYZ family, because the XYZ model with a magnetic field is not integrable.

Models with higher dimensional local spaces include the higher spin $SU(2)$-related models such as the integrable spin-1 chain given by [80, 81]

$$h(j) = \mathbf{S}_j \cdot \mathbf{S}_{j+1} - \left(\mathbf{S}_j \cdot \mathbf{S}_{j+1}\right)^2, \tag{1.7}$$

where now $\mathbf{S}$ denotes the collection of the $SU(2)$-generators in the spin-1 representation.

It is important that the actual spin-1 Heisenberg chain given by $h(j) = \mathbf{S}_j \cdot \mathbf{S}_{j+1}$ is not integrable, but its low energy physics is described by an integrable Quantum Field Theory, the $O(3)$-σ model [82].

Examples for models with higher rank symmetry groups are the so-called $SU(N)$-symmetric fundamental models, where the two-site Hamiltonian is

$$h(j) = P_{j,j+1} - 1, \tag{1.8}$$

where P is the permutation operator acting on the tensor product $\mathbb{C}^N \otimes \mathbb{C}^N$. In the case $N = 2$ this model coincides with the Heisenberg spin chain (the corresponding Hamiltonians are proportional to each other). For $N = 3$ an alternative representation of the Hamiltonian can be written using the $SU(2)$ generators as

$$h(j) = \mathbf{S}_j \cdot \mathbf{S}_{j+1} + \left(\mathbf{S}_j \cdot \mathbf{S}_{j+1}\right)^2 - 1. \tag{1.9}$$

This model is also called the Uimin-Lai-Sutherland spin chain [83–85]. The $SU(3)$-invariance of the representation (1.9) is not evident, but the equality of (1.8) and (1.9) follows from a straightforward computation.

Other important examples are the integrable $SO(N)$-symmetric fundamental models for any $N \geq 3$, which are given by the two-site Hamiltonian [86]

$$h(j) = \frac{N-2}{2} P_{j,j+1} - K_{j,j+1}, \tag{1.10}$$

where K is the so-called trace operator acting on neighbouring spaces. Its matrix elements are

$$K_{ab}^{cd} = \delta_{ab}\delta_{cd}. \tag{1.11}$$

For $N = 6$ this is relevant to the AdS/CFT correspondence, see [72, 87].

Finally we also define the Lieb-Liniger model, also known as the 1D Bose gas, or δ-function interacting Bose gas [38]. This model is not a spin chain, it is a continuum model, but its solution is rather similar to that of the Heisenberg spin chains, and we often refer to it in the text. The Hamiltonian of the model (in secon quantized form) is

$$H_{\mathrm{LL}} = \int_0^L dx \ \left(\partial_x \Psi^\dagger \partial_x \Psi + c\Psi^\dagger \Psi^\dagger \Psi \Psi\right). \tag{1.12}$$

Here L is the size of the system, periodic boundary conditions are understood and $\Psi(x,t)$ and $\Psi^\dagger(x,t)$ are canonical non-relativistic Bose fields satisfying

$$[\Psi(x,t), \Psi^\dagger(y,t)] = \delta(x-y). \tag{1.13}$$

The mass of the particles is set to $m = 1/2$, and $c > 0$ is the coupling constant.

1.4 Dynamics of integrable models

A common property of the integrable models is the existence of an infinite family of local conservation laws[1]. This means that there exists (at least one) family of charges Q_α, $\alpha = 1, 2, \ldots, \infty$, such that:

1. The charges are extensive. On an infinite spin chain

$$Q_\alpha = \sum_{x=-\infty}^{\infty} q_\alpha(x), \tag{1.14}$$

   such that the operator density $q_\alpha(x)$ is of compact support. In continuous models analogous expressions hold with operator densities that are strictly local.

2. The charges commute:

$$[Q_\alpha, Q_\beta] = 0. \tag{1.15}$$

3. The physical Hamiltonian is a member of the series. In spin chains the charges are chosen typically such that $q_\alpha(x)$ is an α-site operator, and in this case $Q_2 \sim H$.

In recent years it became clear that there exist also the so-called quasi-local charges, and they also play an important role in the dynamics of these models [89]. In these cases the support of $q_\alpha(x)$ is not compact, and the operator density includes operator contributions with arbitrary length, although with decreasing amplitudes. The role of these quasi-local charges is discussed later in Section 2.7.

The most important effect of the conservation laws is the non-diffractive and factorized scattering in these models. Now we explain these two properties using the language of S-matrix theory. Our discussion is rather intuitive; we do not attempt to reproduce a rigorous framework. For a more complete discussion in relativistic integrable QFT see [90].

Let us consider the simple cases when in infinite volume the finite energy excitations above a vacuum can be parametrized by a finite set of momenta or rapidities $\boldsymbol{\lambda}_N = \{\lambda_1, \ldots, \lambda_N\}$. Examples include the XXX and XXZ Heisenberg spin chains. In scattering theory we can define the asymptotic *in* and *out* states as the eigenstates that arise in the $t \to \mp\infty$ limits. If the rapidities are different, then in these $t \to \mp\infty$ limits the centers of the wave packets associated with the particles become separated from each other. The asymptotic states are eigenstates of the commuting charge operators. All of these charges are given by sums (or integrals) of local operator densities, therefore we can argue that the charges have to act additively on the asymptotic states:

$$Q_\alpha |\boldsymbol{\lambda}_N\rangle_{in/out} = \Lambda_\alpha(\boldsymbol{\lambda}_N) |\boldsymbol{\lambda}_N\rangle_{in/out}, \qquad \Lambda_\alpha(\boldsymbol{\lambda}_N) = \sum_{j=1}^{N} h_\alpha(\lambda_j). \tag{1.16}$$

Here $h_\alpha(\lambda)$ is the one-particle eigenvalue function of the charge Q_α. Later we will see that (1.14) holds even in finite volume situation, where $|\boldsymbol{\lambda}_N\rangle$ are the finite volume Bethe states. Nevertheless here $|\boldsymbol{\lambda}_N\rangle_{in/out}$ refers to the *in*- or *out*-states.

[1] Actually this property is often used to *define* the integrability of a system. However, it seems that in the quantum mechanical case there is no all-encompassing definition of integrability, see the detailed discussion in [88].

Consider now a scattering event of a finite number of particles. In a non-integrable theory the scattering event typically produces a linear combination of states

$$|\boldsymbol{\lambda}_N\rangle_{in} \quad \to \quad \sum_j c_j \Big|\boldsymbol{\mu}_{n_j}\Big\rangle_{out} \tag{1.17}$$

such that the c_j coefficients are non-zero whenever the finite set of conservation laws allow it, and they depend on the precise details of the scattering event (relative position of the incoming particles, etc). In the general case there is no restriction on the set of outgoing momenta, except the requirement that the total energy and total momentum needs to be conserved.

In contrast, in integrable models we have an infinite number of charges, and all of their eigenvalues need to be conserved. Typically the $h_\alpha(\lambda)$ functions are algebraically independent, thus the set of conservation laws results in an infinite number of non-linear equations for the outgoing rapidities. In the generic case these conservation laws can be satisfied only if **the set of outgoing rapidities is identical to the set of incoming rapidities**. This property is known as non-diffractive scattering [88].

In these cases the S-matrix can be defined simply as the scattering phase (or in more complicated situations as an actual matrix acting in some inner space) that connects the *in*- and *out*- states with the same set of rapidities. This S-matrix could still depend on some details of the scattering event, such as the original spatial ordering of the particles. However, it can be argued that **the full S-matrix factorizes into a product of two-body S-matrices, irrespective of the actual order of the scattering events**. This is a striking consequence of integrability, which was observed in a number of integrable models whose explicit solution was known. It was later shown in [42, 91, 92] in a rather general fashion that this property follows from the existence of the extra conservation laws.

1.5 Consequences for the observable processes

The existence of the charges, and the resulting non-diffractive and factorized scattering has important consequences for the measurable physical processes. Here we outline three main effects. Two of them will be discussed later in more detail.

1. **Non-decaying oscillations in a concrete experiment.** A famous experiment is the Quantum Newton's Cradle [63]: here a finite number of particles were trapped, such that they could move only in one dimension, and even along this dimension there was an applied trapping potential. When the particle cloud is released from a position far from the minimum of the trapping potential, the cloud starts to oscillate as a pendulum would in a harmonic potential. In the experiment two clouds were released on opposite positions, and started oscillations such that they would scatter on each other during each period twice, reminding us of the classical Newton's Cradle. One would expect that the scattering of the atoms would result in momentum transfer that would thermalize the system. However, no equilibration was observed over the lifetime of the clouds in the experiment. This striking phenomenon was attributed to the fact that this 1D system is well described by the Lieb-Liniger model (1D δ-function interacting Bose gas), which is integrable. As a result of the

conservation laws the kinetic energy of the colliding atom clouds could not dissipate into heat. Eventually the system would thermalize due to the integrability breaking interactions in the real world experiment, but these effects were rather small in the concrete setup. We note that a few years later the movement of the clouds was modelled accurately by Generalized Hydrodynamics (see later) [70].

2. **Equilibration in the thermodynamic limit: Emergence of the Generalized Gibbs Ensemble.** In recent years the problem of the equilibration of closed systems (which are sufficiently separated from their environment) received a lot of attention. In spatially homogeneous situations the systems are expected to equilibrate to steady states. Here it is important to mention that equilibration is expected on the level of local correlation functions or other few body observables. The unitary time evolution dictated by the Schrödinger equation is reversible and it keeps all details of the initial state, nevertheless if we look at the reduced density matrix of a small subsystem, then the rest of the system can act as a thermal bath, leading to equilibration. For generic quantum systems it is expected that such reduced density matrices can be described by the Gibbs Ensemble [93]. On the other hand, for integrable models one should also take into account the infinite family of additional conservation laws. This leads to the concept of the Generalized Gibbs Ensemble (GGE) [94,95], whose density matrix is of the form

$$\rho = \frac{e^{-\sum_\alpha \beta_\alpha Q_\alpha}}{\mathrm{Tr}\ e^{-\sum_\alpha \beta_\alpha Q_\alpha}}. \tag{1.18}$$

   Today it is now generally believed, that isolated integrable models equilibrate to a properly chosen GGE. It is important that the GGE is not a mere theoretical concept, becuase it has been already measured in an experimental setup [68] and it underlies Generalized Hydrodynamics which also has experimental confirmations [69–71]. In Section 2 we discuss the GGE in more detail.

3. **Persistent currents and Generalized Hydrodynamics (GHD).** The transport properties of the intergable models are rather special due to the non-diffractive scattering of the quasi-particles. The most important consequence is that in the transport processes there is no dissipation, therefore these models support ballistic transport and persistent currents. Furthermore, the effects of the interactions can be described by theoretical methods, and in 2016 this lead to the development of the GHD [96, 97]. The basics of the GHD and the contributions of the author are discussed in Section 6.

1.6 The Bethe Ansatz wave function

Here we give a short introduction to the Bethe Ansatz wave function, which describes the eigenstates of a certain class of integrable spin chains. This class includes those models where there exists a reference state (vacuum state) which is an eigenstate, and where the one-particle excitations above it do not have an internal degree of freedom. Examples include the Heisenberg spin chains and the Lieb-Liniger model. There are other integrable models where the eigenstates have a more complicated structure. One class of such models are the multi-component systems solvable by the so-called nested Bethe Ansatz (see later

in 2.9 and some comments at the end of this sub-section). An other class of models are those without particle number conservation, such as the XYZ model; these systems will not be discussed in this thesis.

Turning now to the simplest version of the Bethe Ansatz, let us consider the infinite volume situation. Let $\Psi(x_1, \ldots, x_N)$ be the wave function which gives the amplitude to find N particles at positions $x_1, \ldots, x_N$. In spin chains a particle corresponds to a local spin excitation: if the vacuum is chosen as the ferromagnetic state with all spins up then the particles are given by the down spins.

We restrict ourselves to bosonic models (this includes the spin chains), and we treat the wave function in the so-called fundamental domain

$$x_1 < x_2 < \cdots < x_N. \tag{1.19}$$

For simplicity here we exclude the points $x_j = x_{j+1}$. This is a valid step in the spin-1/2 Heisenberg spin chain, where only one excitation can occupy a single site. The coordinate Bethe Ansatz wave function can be constructed even in those models where two or more particles can be at the same position, for example the higher spin XXX model, see [98]. However, for the simple treatment here we exlude these cases.

The general form of the Bethe Ansatz wave function is [33, 52]

$$\Psi(x_1, \ldots, x_N) = \sum_{\mathcal{P} \in S_N} \left[\prod_{j=1}^{N} e^{i x_j p_{\mathcal{P}_j}} \prod_{\substack{j>i \\ \mathcal{P}_j < \mathcal{P}_i}} S(p_i, p_j) \right]. \tag{1.20}$$

The interpretation of the formula is as follows. There are N particles in the system with one-particle momenta p_j, $j = 1, \ldots, N$. They move freely as long as any two particles are well separated from each other. The wave function is a sum over permutations of N particles, and each term corresponds to a specific spatial ordering of the particles. For each permutation $\mathcal{P} \in S_N$ the number $\mathcal{P}_j$ denotes the final position of the number j. The interaction between the particles is taken into accound by the S-matrix $S(u, v)$ which in these simple models is actually a phase. In the wave function there is an S-matrix factor for each two-body exchange. The dependence on the physical parameters of the model comes by solving the two-body problem and extracting the two-body S-matrix. The fact that the wave function is so simple even for $N \geq 3$ is a consequence of the integrability and the factorized scattering described above. The wave function is sometimes called *two-body irreducible*.

Depending on the situation more convenient representations can be found. For example, typically it is possible to find a new parametrization of the momentum $p(\lambda)$ with λ being the rapidity parameter such that the S-matrix becomes a function of the rapidity differences only:

$$S(p(\lambda), p(\mu)) = S(\lambda - \mu). \tag{1.21}$$

This is reminiscent of the Lorentz invariant S-matrix in relativistic QFT.

A further alternative form can be obtained by introducing a function $f(u)$ satisfying

$$\frac{f(u)}{f(-u)} = S(u). \tag{1.22}$$

Such a function is reminiscent of the two-particle minimal form factor in integrable QFT, see for example [90]. Using such a function and the rapidity variables an alternative representation can be written down as

$$\Psi(x_1,\dots,x_N)=\sum_{\mathcal{P}\in S_N}\left[\prod_{j=1}^{N}e^{ix_jp(\lambda_{\mathcal{P}_j})}\prod_{j>i}f(\lambda_i-\lambda_j)\right]. \tag{1.23}$$

Even though we used the same letter Ψ this wave function is just proportional to (1.18) and not identical to it; we just did not want to burden the reader with a multitude of notations. In any case the advantage of this representation is that it is manifestly symmetric with respect to an exchange of the rapidities, whereas the wave function (1.18) explicitly depends on the ordering within $\boldsymbol{\lambda}_N$.

The model dependence of the wave function comes through the functions $p(\lambda)$ and $f(\lambda)$. In the XXX case we have

$$e^{ip(\lambda)}=\frac{\lambda-i/2}{\lambda+i/2}\qquad e^{i\delta(\lambda)}=\frac{\lambda+i}{\lambda-i}\qquad f(\lambda)=\frac{\lambda+i}{\lambda}. \tag{1.24}$$

In the XXZ case with $\Delta>1$ a typical parametrization is

$$e^{ip(\lambda)}=\frac{\sin(\lambda-i\eta/2)}{\sin(\lambda+i\eta/2)}\qquad e^{i\delta(\lambda)}=\frac{\sin(\lambda+i\eta)}{\sin(\lambda-i\eta)}\qquad f(\lambda)=\frac{\sin(\lambda+i)}{\sin(\lambda)}. \tag{1.25}$$

Note that here the fundamental domain for the one-particle rapidities is the segment $[-\pi/2,\pi/2]$.

Putting the model into a finite volume and imposing periodic boundary conditions on the above wave functions we obtain the Bethe equations

$$e^{ip(\lambda_j)}\prod_{k\neq j}S(\lambda_j-\lambda_k)=1,\qquad j=1,\dots,N. \tag{1.26}$$

They are quantization conditions for the rapidities, and typically they are highly coupled non-linear (but algebraic) equations for the rapidities. The physical interpretation of the Bethe equations is the following: if we take one particle "around the volume", then the total phase acquired has to be a multiple of 2π. This total phase consists of the one-particle propagation and the two-body scattering phases accumulated as the particle is scattered on all the other particles.

In models with internal degrees of freedom the S-matrix is an actual matrix. In such models formulas like (1.18) only make sense if the S-matrices act on some vector. Finding an appropriate vector, such that an analogue of (1.24) can be satisfied is the goal of the "nested Bethe Ansatz". The nested Bethe Ansatz will not be reviewed in this thesis, we refer the reader to the pedagogical book [99] on the Hubbard model, whose solution involves the same techniques.

1.7 Integrable spin chains and Algebraic Bethe Ansatz

Here we give a brief introduction into an alternative method which allows the solution of the spin chains: the Algebraic Bethe Ansatz. Although the construction is general, we restrict ourselves mostly to the spin-1/2 XXZ and XXX models.

The solution of these models depends on the existence of certain algebraic structures, satisfying very specific relations. A fundamental object is the so-called R-matrix. Let us take two vector spaces $V_{1,2} \simeq \mathbb{C}^D$ and let us associate rapidity parameters $\lambda_{1,2}$ to the two vector spaces. The R-matrix is a linear operator $R_{12}(\lambda_1, \lambda_2)$ acting on $V_1 \otimes V_2$, which is an analytic function of the two rapidities. It satisfies the so-called Yang-Baxter relation, which is constructed as follows. Let us take 3 vector spaces $V_{1,2,3}$, and three associated rapidities $\lambda_{1,2,3}$. Then the YB relation is

$$\begin{aligned} R_{12}(\lambda_1, \lambda_2) R_{13}(\lambda_1, \lambda_3) R_{23}(\lambda_2, \lambda_3) = \\ = R_{23}(\lambda_2, \lambda_3) R_{13}(\lambda_1, \lambda_3) R_{12}(\lambda_1, \lambda_2). \end{aligned} \tag{1.27}$$

Here it is understood that each R_{jk} acts only on the corresponding vector spaces V_j and V_k.

It is useful to give a pictorial representation of this relation and the other algebraic relations that follow from it.

The pictorial representation for the R-matrix can be drawn as a crossing of two lines, see Fig. 1. Here it is understood that both spaces carry a rapidity parameter, and the corresponding action is given by $R(\mu, \nu)$, where μ is the rapidity "coming from the left". The action of the R-matrix can be considered as a "scattering" of two spins.

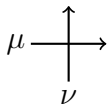


Figure 1: Graphical notaiton for the R-matrix. The crossing denotes the action of $R(\mu, \nu)$ on the two vector spaces.

A graphical representation of the Yang-Baxter equation is given by 2. In these pictures and in the others below it is understood that each crossing signals the action of the corresponding R-matrix, and concatenation of lines means matrix multiplication (summing over the possible states of the internal lines).

Figure 2: The Yang-Baxter equation for the R-matrix.

It can be shown that if certain analytic conditions are met, then the R-matrix satisfies the inversion relation

$$R_{1,2}(\lambda_1, \lambda_2) R_{2,1}(\lambda_2, \lambda_1) = g(\lambda_1, \lambda_2), \tag{1.28}$$

where $g(u, v)$ is a symmetric function of u, v. Using a proper normalization of the R-matrix it can always be chosen as 1. A pictorial representation of this inversion relation is given in Fig. 3.

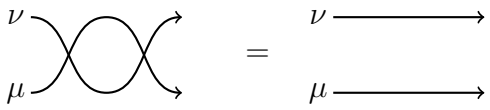


Figure 3: The inversion relation for the R-matrix.

Let us now consider a finite volume situation with the Hilbert space $\mathcal{H} = \otimes_{j=1}^{L} \mathbb{C}^D$. Let us also take a further vector space V_a called the auxiliary space. Depending on the situation this may or may not be isomorphic to the physical spaces.

The idea behind the Algebraic Bethe Ansatz is to use this auxiliary spin as a probe, and "scatter" it on the physical spins of the chain. This way we construct the so-called monodromy matrix as

$$T_a(\mu) = \mathcal{L}_{a,L}(\mu) \dots \mathcal{L}_{a,1}(\mu). \tag{1.29}$$

This is an operator acting on $V_a \otimes \mathcal{H}$. Here μ is a spectral parameter (rapidity) and the $\mathcal{L}_{a,j}(\mu)$ are the so-called Lax operators, which act on $V_a \otimes V_j$. We require that they satisfy the so-called RLL relations:

$$R_{b,a}(\nu,\mu)\mathcal{L}_{b,j}(\nu)\mathcal{L}_{a,j}(\mu) = \mathcal{L}_{a,j}(\mu)\mathcal{L}_{b,j}(\nu)R_{b,a}(\nu,\mu), \tag{1.30}$$

with a, b referring to two different auxiliary spaces.

It follows from (1.25) that

$$\mathcal{L}_{a,j}(\mu) = R_{a,j}(\mu,\xi_0) \tag{1.31}$$

is a solution to (1.28), where ξ_0 can be a fixed parameter of the model. Typically $\xi_0 = 0$.

The transfer matrix (TM) is its partial trace over the auxiliary space:

$$t(\mu) = \mathrm{Tr}_a T_a(\mu). \tag{1.32}$$

The graphical representation for $T(u)$ and $t(u)$ is given in Fig. 4.

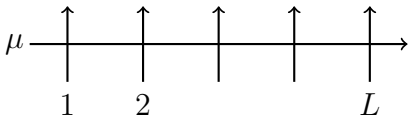


Figure 4: Construction of the monodromy matrix $T(\mu)$. The vertical lines correspond to the physical spaces, whereas the horizontal line represents the auxiliary space. The matrix $T(\mu)$ depends on the rapidity parameter μ associated to the auxiliary space. In this picture the numbers $1, \dots, L$ stand for the individual local spaces.

It can be shown that the fundamental exchange relation (1.28) leads to the so-called RTT relations

$$R_{b,a}(\nu,\mu)T_b(\nu)T_a(\mu) = T_a(\mu)T_b(\nu)R_{b,a}(\nu,\mu), \tag{1.33}$$

which are exchange relations between the different monodromy matrix elements. Multiplying with the inverse of one of the R-matrices and taking a trace in the auxiliary spaces we obtain the commutativity of transfer matrices:

$$[t(\mu), t(\nu)] = 0. \tag{1.34}$$

We define the set of conserved charges as

$$Q_\alpha = \left(\frac{d}{d\mu}\right)^{\alpha-1} \log(t(\mu))\Big|_{\mu=0}. \tag{1.35}$$

This definition is meaningful, and it results in commuting charges:

$$[Q_\alpha, Q_\beta] = 0. \tag{1.36}$$

The extensivity of the charges follows from the formal logarithm in the definition.

In this work we deal with models where the R-matrix is of difference form:

$$R(\mu, \nu) = R(\mu - \nu). \tag{1.37}$$

A famous counter-example is the Hubbard-model, where the R-matrix is not of difference form [99].

Furthermore, we will focus on the so-called fundamental models, where the Lax operator is identical to the R-matrix, see (1.29). If the difference property is satisfied then we are free to set $\xi_0 = 0$.

In the models under consideration the R-matrix satisfies the regularity condition

$$R(0) = P, \tag{1.38}$$

where P is the permutation operator. This property is needed to prove that using (1.33) the charge density for Q_α will indeed span α sites [100]. In particular, the operator density of Q_2 is given by

$$q_2(x) = P\dot{R}_{j,j+1}(0). \tag{1.39}$$

and can be identified with the Hamiltonian density, up to additive and multiplicative normalization.

The monodromy matrix is written with respect to the auxiliary space as

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}, \tag{1.40}$$

where $A(u), B(u), C(u), D(u)$ are operators acting on the physical Hilbert space of the chain.

Let us now discuss the example of the XXZ and XXX spin chains. In these models the R-matrix is of the form

$$R(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(u) & c(u) & 0 \\ 0 & c(u) & b(u) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.41}$$

In the XXX case we have

$$b(u) = \frac{u}{u+i}, \qquad c(u) = \frac{i}{u+i}, \tag{1.42}$$

In this normalization of the R-matrix the canonical construction for the charges gives $Q_2 = (H - L)/2$, where H is given by (1.3) and L is the volume.

For the XXZ spin chain we use the conventional notations

$$H = \sum_{j=1}^{L} (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta(\sigma_j^z \sigma_{j+1}^z - 1)). \tag{1.43}$$

For this model the functions in the R-matrix are

$$b(u) = \frac{\sin(u)}{\sin(u + i\eta)}, \qquad c(u) = \frac{\sin(i\eta)}{\sin(u + i\eta)}, \tag{1.44}$$

where $\Delta = \cosh(\eta)$. In this normalization of the R-matrix the we have $Q_2 = H/(2\sinh\eta)$.

Later we also use the function $f(\lambda)$ which is defined as

$$f(\lambda) = \frac{1}{b(\lambda)} \tag{1.45}$$

for both the XXZ and XXX models.

In these particular models, and in many other models with $U(1)$-symmetry there exists a reference state $|\emptyset\rangle$ which has the properties that

- It is an eigenvector of the diagonal monodromy matrix elements:

$$A(u)|\emptyset\rangle = a(u)|\emptyset\rangle, \qquad D(u)|\emptyset\rangle = d(u)|\emptyset\rangle. \tag{1.46}$$

  Here $a(u)$ and $d(u)$ are model-specific vacuum eigenvalue functions.

- It is annihilated by $C(u)$ for any u:

$$C(u)|\emptyset\rangle = 0. \tag{1.47}$$

These conditions hold in the XXX and XXZ chains, where the reference state is the ferromagnetic product state with all spins up. Then the eigenvalue functions can be determined from the fundamental Lax operators, in particular

$$a(u) = 1, \qquad d(u) = b^L(u). \tag{1.48}$$

The un-normalized Bethe states can then be created as

$$|\boldsymbol{\lambda}_N\rangle = \prod_{j=1}^{N} B(\lambda_j - \sigma)|\emptyset\rangle, \tag{1.49}$$

where we introduced the σ shift for later convenience. It is related to the so-called crossing symmetry of the R-matrix, and for the XXX and XXZ cases it is given by $\sigma = i/2$ and $\sigma = i\eta/2$, respectively. Clearly, the B-operators act as particle creation operators, but in contrast with the free case they create states made of interacting spin waves. In particular, it can be shown that the states (1.47) are actually proportional to the states defined in (1.21).

It can be shown using the algebraic methods that these states are eigenstates of the commuting set of transfer matrices if the rapidities satisfy the Bethe equations (1.24). The eigenvalue functions of the transfer matrices defined by

$$t(u)|\boldsymbol{\lambda}_N\rangle = \Lambda(u|\boldsymbol{\lambda}_N)|\boldsymbol{\lambda}_N\rangle \tag{1.50}$$

are found to be

$$\Lambda(u|\boldsymbol{\lambda}_N) = a(u)\prod_{j=1}^{N} f(\lambda_j - \sigma, u) + d(u)\prod_{j=1}^{N} f(u, \lambda_j - \sigma). \tag{1.51}$$

For practical computations it is often needed to know the norm of the vectors (1.47). Unlike free models, this norm is not determined by the definition of the B-operators only, instead it is a complicated function of all the rapidities. It was conjectured first by Gaudin, that the norm is always proportional to the determinant of a matrix, which is basically the Jacobian of the Bethe equations [101–103]. The statement for the norm is

$$\langle\boldsymbol{\lambda}_N|\boldsymbol{\lambda}_N\rangle = \prod_{j=1}^{N} F_1(\lambda_j)\prod_{j<k} F_2(\lambda_j, \lambda_k) \times \det_N G, \tag{1.52}$$

where F_1 and F_2 are some known functions and G is the $N \times N$ matrix, which is today called the Gaudin matrix. The function F_1 depends on the normalization of the B-operators, but the function F_2 is completely fixed by the underlying R-matrix. In fact, it is given by

$$F_2(\lambda_j, \lambda_k) = f(\lambda_j, \lambda_k)f(\lambda_k, \lambda_j). \tag{1.53}$$

Finally, the matrix elements of G are given by

$$G_{jk} = \frac{\partial(2\pi I_j)}{\partial\lambda_k}, \tag{1.54}$$

where we defined the functions I_j through the Bethe equations as

$$e^{i2\pi I_j(\lambda_j)} = e^{ip(\lambda_j)}\prod_{k\neq j} S(\lambda_j - \lambda_k). \tag{1.55}$$

For concrete eigenstates the I_j are integer numbers, but it is useful to regard them as functions of the rapidities. We can use the known explicit functions of the XXZ and XXX models given above to derive more explicit formulas for the matrix elements of G.

Note that G is the Jacobian of the Bethe equations, which can be written in logarythmic form as

$$J_j(\lambda_j) = 2\pi I_j, \qquad I_j \in \mathbb{Z}. \tag{1.56}$$

The Bethe states are distributed uniformly in the space of momentum quantum numbers I_j, thus the Gaudin determinant $\det G$ describes the density of states in rapidity space.

A simple argument for the validity of the norm conjecture was given in [104] and a proof using ABA was found in [105]. Similar formulas for the norm hold also in more complicated nested systems (see below), and their proof is rather involved (for a relatively early proof see [106], and for recent reviews see [107, 108]).

For future use we introduce the concept of the *inhomogeneous spin chain*. Let $\boldsymbol{\xi}_L = \{\xi_1, \dots, \xi_L\}$ be a set of complex numbers called inhomogeneities and let us define a modified monodromy matrix as

$$T_a(\mu) = \mathcal{L}_{a,L}(\mu - \xi_L) \dots \mathcal{L}_{a,1}(\mu - \xi_1). \tag{1.57}$$

The transfer matrix is still given by the trace, as in (1.30). The physical case producing the desired Hamiltonian is restored in the limit $\xi_j \to 0$. If the ξ_j are non vanishing, then it is not possible to define local Hamiltonians from the transfer matrix (except in further special cases such as $\boldsymbol{\xi}_L = \{\xi, -\xi, \xi, -\xi, \dots\}$). However, it is very useful to treat the $\boldsymbol{\xi}_L$ as arbitrary parameters: in many cases the physical value of a quantity can be found by investigating the ξ-dependence of the general case. The inhomogeneities are also crucial for the so-called Quantum Transfer Matrix method, see Section 2.6.

1.8 Spectrum and thermodynamic limit

For comparison with experiments and sometimes with other theoretical methods we need to investigate the thermodynamic limit (TDL). The physically most interesting situation is when the volume L is taken to infinity such that the particle density $n = N/L$ remains finite. This limit is required to obtain the interesting phenomena in the equilibrium and non-equlibrium dynamics, such as algebraically decaying correlations, emergence of the GHD, ballistic transport in the presence of interactions, superdiffusion, etc.

In order to understand the TDL in Bethe Ansatz solvable models we need to discuss the spectrum, in other words the solutions to the Bethe equations and their behaviour in this limit.

Depending on the model the solutions to the Bethe equations can show quite different behaviour, with different physical meaning. There are the following possibilities:

- **Purely real Bethe roots.** In some models it is known that all finite volume solutions consist of purely real roots. The most important example for such behaviour is the repulsive Lieb-Liniger model [38].

- **Bethe roots form strings.** In some models the typical configuration of a set of Bethe roots consists of strings. Each string describes a bound state of the fundamental particles. Let us consider the bound state of n particles, then the roots within the string are of the form

$$\lambda_j = \lambda_c + i\frac{2j - n - 1}{2}\kappa + i\delta_j, \qquad j = 1, \dots n. \tag{1.58}$$

  Here λ_c is the string center, and $\kappa > 0$ is a fixed parameter of the model, which is determined by the analytic properties of the S-matrix: if $S(u)$ has a pole at $u = i\kappa$ then two fundamental particles can form a bound state as above. The δ_j parameters are the so-called string deviations; these numbers become exponentiall small in L in the large volume limit. The string solutions and the corresponding wave functions can be understood in an explicit way simply by looking at the two-body wave function (1.18), which in this case indeed describes the wave function of a bound state.

Models which support string solutions include the XXX and XXZ spin chains, and the attractive Lieb-Liniger model. In all cases relevant to our discussion the S-matrix of the fundamental particles has at most one pole, and there is only one allowed κ parameter[2]. Nevertheless the question of which values of n are allowed is rather delicate. In the XXZ model it depends strongly on the anisotropy parameter Δ, this is discussed in length in the book [49]. In this thesis we focus on the regimes $\Delta > 1$ and $\Delta = 1$ (XXX model), where each $n \in \mathbb{N}$ is allowed.

For $\Delta < 1$ there are a number of additional subtleties that lead to interesting physical consequences, such as a fractal structure in the spin Drude weight (discussed for example in [109,110]). However, in this work we do not discuss the regime $\Delta < 1$.

- **Nested Bethe Ansatz.** In nested Bethe Ansatz there are various types of rapidities corresponding to the different "levels" of the nesting procedure. Typically all of them can form strings. This issue is discussed (without proofs) in Section 2.9.

In the models under consideration the *string hypothesis* is a statement about the thermodynamic limit of the behaviour of the Bethe roots. There is no strict or rigorous formulation of this hypothesis, therefore we present it here in our own interpretation. The string hypothesis states that

- In the TDL *almost all* solutions to the Bethe equations will consist of strings of the type (1.56), where the allowed types of strings are model dependent and can be determined from an analysis with a finite number of particles.
- The string deviations δ_j decay exponentially with the volume.
- The "outlier" states, i.e. those which do not belong to this classification are rare, and they do not influence the dynamics in the TDL.

The classification of the Bethe roots, and thus the string hyptohesis itself goes back to the work of Bethe [33]. He attempted to prove the completeness of the Bethe Ansatz by comparing the number of solutions of the Bethe equations with a fixed particle number to the dimension of the appropriate section of the Hilbert space. Today it is believed that Bethe's counting of the eigenstates was correct, but some simple assumptions about the nature of the solutions were wrong. Bethe assumed that all solutions satisfy the string hypothesis, and that the number of the various string solutions can be computed by combinatorics, for each volume. These assumptions hold for *almost all states*: it is today known that there are rare configurations not obeying the string hypothesis exist, see for example [111, 112]. Also, it is known that there are various types of singular solutions to the Bethe equations [113–115], which need to be discussed separately. The self-conjugate property of the set of Bethe roots was already proven in [116], but there are no further theorems for the classification of the solutions. In defence of Bethe let us add, that all his key ideas in his 1931 paper were correct, whereas it took many years for many researchers to find and characterize all the fine details about the various possibilities for the Bethe roots.

[2]This is to be contrasted with integrable QFT models, where the S-matrices typically have more than one pole in the so-called physical strip [90].

Let us now discuss the behaviour of the Bethe roots in the thermodynamic limit. As a first example we consider a simple model where the rapidities do not form strings and the Bethe equations are of the form (1.24). In logarythmic form this is written as

$$p(\lambda_j)L + \sum_{k\neq j} \delta(\lambda_j - \lambda_k) = 2\pi I_j, \qquad I_j \in \mathbb{Z}, \qquad j = 1, \dots, N, \tag{1.59}$$

where $S(u) = e^{i\delta(u)}$. Our goal is to take the thermodynamic limit of this equation, without specifying the physical situation. We derive a simple equation describing the distribution of Bethe roots, which will be applicable both to the ground states, to finite temperature states, or any steady state arising in a non-equilibrium situation.

As a first step we introduce the differential root density per volume and per rapidity $\rho(\lambda)$. We choose a normalization such that in a volume L and within a window $[\lambda, \lambda + d\lambda]$ the number of Bethe roots is $\rho(\lambda)d\lambda L$. Accordingly, the total particle density is

$$n = \int d\lambda\ \rho(\lambda). \tag{1.60}$$

Let us also introduce the concept of Bethe holes: a rapidity λ is a "hole" if it satisfies

$$e^{ip(\lambda)} \prod_{k=1}^{N} S(\lambda - \lambda_k) = -1, \tag{1.61}$$

but it is not an actual Bethe root, it is not present in the set $\boldsymbol{\lambda}_N$[3]. A Bethe hole can be regarded as the analogue of an unoccupied level in a free theory. In the TDL the Bethe holes also become dense and we introduce their density $\rho_h(\lambda)$. It follows from (1.57) that the two types of densities satisfy the linear integral equation [49]

$$\rho(\lambda) + \rho_h(\lambda) = \frac{1}{2\pi} p'(\lambda) + \int \frac{d\omega}{2\pi} \varphi(\lambda - \omega)\rho(\omega), \tag{1.62}$$

where we introduced the scattering kernel

$$\varphi(\lambda) = \delta'(\lambda). \tag{1.63}$$

The equation (1.60) does not tell anything about the physical nature of the state in question; it should be regarded simply as the thermodynamic representation of the fundamental Bethe equation (1.57). The physical properties depend on the *filling fractions*, which are defined as the ratio

$$f(\lambda) = \frac{\rho(\lambda)}{\rho(\lambda) + \rho_h(\lambda)}. \tag{1.64}$$

Specifying $f(\lambda)$ characterizes the content of the state in question. For example, for the ground state of the Lieb-Liniger model with a certain total density $f(\lambda) = 1$ within the Fermi zone $[-\Lambda, \Lambda]$ and zero elsewhere. For finite temperature states this Fermi distribution is smoothened, and it depends on the temperature and chemical potentials. It will be

[3]Here we used $S(0) = -1$.

explained in detail in Section 5 that in equilibrium $f(\lambda)$ completely determines the local physical observables. Sometimes it is useful to write (1.60) as

$$\rho(\lambda)+\rho_h(\lambda)=\frac{1}{2\pi}p'(\lambda)+\int\frac{d\omega}{2\pi}\varphi(\lambda-\omega)f(\omega)(\rho(\omega)+\rho_h(\omega)). \tag{1.65}$$

In this form we get a linear equation for the total density $\rho(\lambda)+\rho_h(\lambda)$, and the filling fraction is used as an input to the equation.

Let us comment on the nomenclature. In recent literature dealing with GHD the equation (1.60) and analogous relations for more complicated systems are called "Thermodynamic Bethe Ansatz" (TBA) equations. This name faithfully represents the origin and meaning of these equations. However, originally the name TBA referred to a different concept, namely to the computation of the finite temperature ensembles [49, 117, 118]. In these cases the filling fraction $f(\lambda)$ is also fixed by a non-linear integral equation, whereas in (1.63) it is a free function. In the present work we use the TBA acronym without making an explicit distinction between the two meanings; we hope the meaning will be clear from the context.

In models with string solutions we introduce a root density and hole density function for each string; they describe the distribution of the string centers, and the corresponding holes. The resulting integral equations are simple analogues of (1.60). There are standard steps to bring these equations to the simplest form possible [49]; here we do not review these computations, we just present some of the final results. For example, in the XXX model each n-string is allowed for $n=1,2,\dots$. Thus we have the infinite family of functions $\rho_n(\lambda)$, $\rho_{n,h}(\lambda)$ for the string centers and the holes. It can be shown that in this particular model they satisfy the linear set of equations [49]

$$\rho_n(\lambda)+\rho_{n,h}(\lambda)=\delta_{n,1}s(\lambda)+\int_{\omega=-\infty}^{\infty}\frac{d\omega}{2\pi}s(\lambda-\omega)(\rho_{n-1,h}(\omega)+\rho_{n+1,h}(\omega)). \tag{1.66}$$

Here $\delta_{n,1}$ is the Kronecker-delta, it is understood that $\rho_0(\lambda)=\rho_{0,h}(\lambda)=0$, and

$$s(\lambda)=\frac{1}{2\cosh(\pi\lambda)}. \tag{1.67}$$

Analogous equations can be obtained also in the XXZ model and for other integrable models.

It is also useful to give a graphical interpretation of this equation. We can see that the root and hole densities for the n-strings are coupled through convolution integrals to the quantities of the $n-1$ and $n+1$ strings. This is depicted on Fig. 5. Here each node represents a string type, the links stand for the convolution terms, and the fulll circle denotes the source term for the 1-strings.

It is very common to represent TBA equations in this form, and the resulting pictures have an algebraic origin. In all known cases they are given by simply laced Dynkin diagrams. The current case can be considered as the infinite dimensional A_∞ diagram. For the discussion of these diagrams in Quantum Field Theory models we refer to [119, 120].

Regarding the eigenvalues of the charges the finite volume formulas (1.14) can be transformed into the integrals

$$\lim_{TDL}\frac{\Lambda_\alpha}{L}=\int\rho(\lambda)h_\alpha(\lambda). \tag{1.68}$$

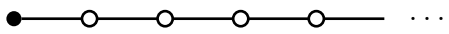


Figure 5: Graphical representation of the TBA equations for the XXX chain.

Once again this formula refers to the simple cases with a single particle type; in more complicated cases a summation is need for the different paricle types.

1.9 Exactly solvable or exactly solved?

In the previous subsections we showed that the exact eigenstates of these models can be found with either the coordinate or the Algebraic Bethe Ansatz. The reader might wonder: Does this mean that the models could be regarded as "solved"? What is then the purpose of the ongoing research in this direction? What are the difficulties?

Let us shed some light on these questions. Consider the time evolution in an isolated integrable model, and let $|\Psi_0\rangle$ be the state of the system at $t = 0$. We can choose this state to be the ground state of some other Hamiltonian, or any other state which is physically relevant (it could be realistically constructed in an experiment). Let us further look at the real time evolution of an observable $\mathcal{O}$. We can choose $\mathcal{O}$ to be a local operator, or a two-point function, or a more complicated operator product. The standard way to compute its exact real time evolution is through the spectral series:

$$\langle \mathcal{O}(t) \rangle = \sum_{n,m} \langle \Psi_0 | n \rangle \langle n | \mathcal{O} | m \rangle \langle m | \Psi_0 \rangle e^{i(E_n - E_m)t}. \tag{1.69}$$

Here the double sum runs over the eigenstates of the model; for simplicity we consider here a finite system with a finite number of states.

Let us now discuss the ingredients in this sum.

- **Form factors.** The objects $\langle n|\mathcal{O}|m\rangle$ are called the form factors of the operator. In integrable models a lot of information is known about them, both in relativistic QFT [90,121] and in the non-relativistic settings [52]. In particular, the form factors are analytic functions of the rapidities, whose explicit form is known in many cases. However, computing the thermodynamic limit of these form factors is far from trivial [54], and considerable effort was spent to study the diagonal matrix elements; for the contributions of the author see Section 5.

  Let us explain the difficulty behind the computations of the form factors. Considering formula (1.18) for the Bethe Ansatz wave function it is clear that the form factors should be given by some analytic function of the sets of rapidities. However, directly applying the exact wave functions we obtain large summations over permutations, with no immediate clues about any resummations. Thus one needs alternative approaches, which eventually simplify the resulting sum, such that the final formulas become tractable, even in the thermodynamic limit.

- **Overlaps.** The classical works on quantum integrability only focused on ground state or finite temperature mean values, and not on the non-equilibirum dynamics. Therefore, practically no results were available for overlaps before 2013. If the initial

state is a product state or it can be represented in a sufficiently simple form, then the overlaps can also be expressed as rational functions of the Bethe rapidities. However, in parallel with the form factors, these functions are given by summations over permutations, and a priori it is not clear how to get simple and convenient representations. The contributions of the author to this problem are summarized in Section 4.

- **Summation of the spectral sum.** In principle all Bethe states appearing in the double summation are known: the rapidities are given by solutions of the Bethe equations (1.24) (putting aside the singular solutions discussed above). However, the Bethe equations are non-linear equations and the solutions can not be produced in an explicit way.

  In certain situations the summation can be transformed into multiple integrals using multi-dimensional residue relations, and inserting the Bethe equations as poles of the integrands, see for example [20, 122]. However, such representations are often not very useful to find the actual values of the observables.

  There are now methods to perform the spectral sum analytically, at least for some partition functions [123, 124], but so far these methods are also limited to relatively small system sizes.

These problems show that the actual computation of the real-time dynamics is very far from having been solved, even though the Bethe Ansatz itself is well understood.

We put forward, that the spectral expansion discussed above is not the only way to treat the real time dynamics. There are exact theoretical methods that operate directly in infinite volume, see the BQTM method in Section 3. However, the computation of the actual time dependence of physical quantities is an open problem even in those methods.

1.10 Unexpected connections

We present here a list of unexpected connections between integrable lattice models and other fields of mathematics and/or theoretical physics. These topics are not discussed further in the thesis, because they are not related to the dynamics of the spin chains, which is our main interest. However, we felt it is worthwhile to compile such a list. The often cited connection to AdS/CFT is not included in this list, because it is relevant to the thesis and it is discussed for example in Section 3.5.

Unexpected connections include:

- In knot theory the Yang-Baxter relation can be used to generate knot invariants [125].

- In combinatorics the number of so-called alternating sign matrices can be found using the partition functions of the integrable 6-vertex model [126, 127].

- There is a correspondence between solutions to the Yang-Baxter equation and partition functions of so-called supersymmetric quiver gauge theories (see the reviews [32, 128]).

- There is a correspondecne between eigenstates of integrable spin chains and the vacua of certain supersymmetric gauge theories [129, 130].
- Algebro-geometric aspects of the Bethe Ansatz equations and in particular the connections to Langlands dualities were discussed in [131, 132].
- In algebraic geometry there is a connection between Schubert calculus and quantum integrable models, see for example [133].
- The twisted Yangians (originating in the Boundary Yang-Baxter relation, discussed briefly in Section 3.4) found applications in the representation theory of classical Lie algebras, see the review [134].

This list is by no means exhaustive, and most likely it can be extended in the future.

2 Generalized Gibbs Ensemble

2.1 Introduction

Thermalization in a closed system is a macroscopic phenomenon which is known from everyday life. If we pour cold milk into our hot coffee, then the two liquids will mix, the milk will warm up and the coffee will cool down a bit. This happens on the macroscopic scale, nevertheless we can ask ourselves: can we explain it using the fundamental laws of nature? We know that the physics on the atomic scale is described by Quantum Mechanics (QM), and this raises the question whether the macroscopic laws of classical physics, in particular those of statistical physics and thermodynamics, could be derived from QM. This question is of course not new, and its discussion goes back to John von Neumann [135–137].

In this work we treat equilibration and thermalization of closed quantum systems and we focus on the paradigm of the quantum quench. The usual definition of the quench is the following: we take a closed quantum system, we prepare it in the ground state of a certain Hamiltonian H_0, and at $t = 0$ we suddenly change certain parameters of the Hamiltonian. Thus for $t > 0$ the system evolves according to a new Hamiltonian H, which does not commute with H_0 and thus leads to non-trivial real time dynamics. The focus on quantum quenches is motivated by its simplicity both on the experimental and the theoretical side [138, 139]. Other often studied protocols include the quantum ramp (a slower real-time tuning of some parameters) and the Floquet systems (periodic driving), but these situations will not be considered here.

Let us therefore specify the actual theoretical setup. We consider an integrable Hamiltonian H in finite volume L and an initial state $|\Psi_0\rangle$, which is the state of the system at $t = 0$. This can be the ground state of an other Hamiltonian (not necessarily integrable), or any other state relevant to experiments or computer simulations. In our concrete examples $|\Psi_0\rangle$ will be a state with small real-space entanglement, such as a real space product state or a Matrix Product State (MPS) [140] with small bond dimension (for definitions see later). It is known that such states are always ground states of a local Hamiltonian [140], but for our purposes it is not important to specify this original Hamiltonian H_0. Instead we will focus on the properties of our initial states.

It is important that in the first instance we focus on spatially homogeneous initial states. To narrow it down, we require that $|\Psi_0\rangle$ has to be either one-site or two-site shift invariant. Examples for one-site invariant states are for example the ferromagnetic product states

$$|\Psi_0\rangle = \otimes_{j=1}^{L} |n\rangle, \qquad |n\rangle \in \mathbb{C}^2 \tag{2.1}$$

or MPS' with site-independent matrices. An example for two-site invariant state is the dimerized (or in short: dimer) state

$$|\Psi_0\rangle = |D\rangle = \otimes_{j=1}^{L/2} \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}. \tag{2.2}$$

The reason to focus on such states is motivated by their special integrability properties to be discussed in Sec. 3. But such states are also ground states of local Hamiltonians, for example the dimer state is the ground state of the Majumdar-Ghosh model [141]. In these cases we could take the zero momentum combination of $|\Psi_0\rangle$ and its one-site shifted version, but we prefer to keep the original two-site products, because they have strictly zero entanglement between the two-site blocks.

It is our goal to study the time evolution of local observables $\mathcal{O}$. As examples we will consider short range spin-spin correlation functions such as $\mathcal{O} = \sigma_j^z \sigma_{j+n}^z$ with $n = 1, 2, 3$. The explicit time dependence is given by the formal expression

$$\langle \mathcal{O}(t, L) \rangle = \langle \Psi_0 | e^{iHt} \mathcal{O} e^{-iHt} | \Psi_0 \rangle. \tag{2.3}$$

In order to study the phenomenon of equilibration we need to take the $t \to \infty$ limit. In finite systems strict equilibration does not happen, and this is related to the well-known effect of quantum revivals. One possibility would be to define an averaging over t in order to obtain equilibrated values. An other possibility is to take the limit of large system sizes *before* taking the long time limit. Thus we study

$$\lim_{t\to\infty} \lim_{L\to\infty} \langle \mathcal{O}(t, L) \rangle \,, \tag{2.4}$$

and we ask ourselves whether the second limit approaches constant values, and how to compute them.

If the system thermalizes, then the long time limit is described by the Gibbs Ensemble. Let us consider now a generic non-integrable quantum spin chain, and let us assume for simplicity that there is just one relevant conserved quantity, which is the Hamiltonian itself. In this case the Gibbs averages are given by

$$\langle \mathcal{O} \rangle_G = \frac{\text{Tr } e^{-\beta H} \mathcal{O}}{\text{Tr } e^{-\beta H}}. \tag{2.5}$$

These averages depend on the temperature $T = 1/\beta$, but also on the volume L. We say that a system thermalizes to temperature T if there is a parameter $\beta = 1/T$ such that for all local observables

$$\lim_{t\to\infty} \lim_{L\to\infty} \langle \mathcal{O}(t, L) \rangle = \lim_{L\to\infty} \langle \mathcal{O} \rangle_G \,. \tag{2.6}$$

If this relation holds then we can apply it to the Hamiltonian density, which is a constant of motion. Then we also obtain

$$\lim_{L\to\infty} \frac{\langle \Psi_0 | H | \Psi_0 \rangle}{L} = \lim_{L\to\infty} \frac{\langle H \rangle_G}{L}. \tag{2.7}$$

The mean energy on the r.h.s. is typically a monotonous function of β, so that this relation uniquely determines β. In simple terms: the energy stored in the initial state determines the resulting temperature after equilibration.

If there are other conserved quantities such as particle numbers, etc, then they have to be added to the Gibbs ensemble. This leads to a finite number of additional intensive parameters, such as chemical potentials.

In contrast, the integrable models possess an infinite family of conserved charges, which need to be taken into account. This leads to the:

2.2 Generalized Gibbs Ensemble in integrable models

As discussed in Section 1 integrable models possess an infinite family of conservation laws. This forbids thermalization in the usual sence. Instead it was proposed in [94] that integrable models equilibrate to Generalized Gibbs Ensembles (GGE's). A GGE average is given by

$$\langle \mathcal{O} \rangle_{GGE} = \frac{\mathrm{Tr}\ e^{-\sum_j \beta_j Q_j} \mathcal{O}}{\mathrm{Tr}\ e^{-\sum_j \beta_j Q_j}}, \tag{2.8}$$

where β_j are Lagrange multipliers, wich can be considered as generalized temperatures associated to the charges. Note that we are free to choose a basis in the linear space spanned by the charges, thus the β_j strongly depend on these definitions.

If the equilibration to the GGE holds, then similar to eq. (2.7) we find that

$$\lim_{L\to\infty} \frac{\langle \Psi_0 | Q_j | \Psi_0 \rangle}{L} = \lim_{L\to\infty} \frac{\langle Q_j \rangle_{GGE}}{L}, \qquad j = 1, 2, \dots . \tag{2.9}$$

In principle this infinite set of relations should fix the Lagrange multipliers β_j.

The reader might wonder what is the predictive value of the GGE if an infinite set of parameters are fitted using the above relation. The answer is simple: If the GGE holds, then the long time limit of all other local observables should be given by (2.8). A further argument can be given if we go back to finite volume. Then it can be argued that the number of parameters should scale as a polynomial in L (typically linear or quadratic), but the number of local operators for which the GGE assumption should hold scales exponentially with L. We believe that this argument can be made precise, although a more rigorous treatment has not yet been given in the literature, except in the case of the free models.

The emergence of the GGE was first investigated and proven in the case of free models (or models equivalent to free fermions, such as the XX and Ising models) [94, 95, 142–149]. In such cases it is convenient to choose mode occupation operators as the independent charges, and then the β_j play the role of mode dependent temperatures. The emergence of such a GGE was measured in an experimental setup [68].

On the other hand, the question proved to be more involved in the interacting spin chains. The following main difficulties arose:

- What is a complete set of charges that needs to be added to the GGE? Is it generally true that such a complete set exists for each integrable model?

- Given a complete set, how can we actually compute the parameters of the emerging GGE?

- Given the parameters of the GGE, how can we actually compute the GGE averages of local observables? Exact values are needed for comparisons with experiments or numerical simulations.

- Are there alternative theoretical methods, at least for a subset of initial states, that lead to exact results? Do these exact results agree with the GGE predictions?

In the following we discuss these questions in detail.

2.3 Mechanism of equilibration

First let us start with the mechanism of equilibration towards the GGE. It is now understood that the key steps are dephasing and the so-called Generalized Eigenstate Thermalization Hypothesis (GETH). Consider the long time limit of the observables in finite volume, and let us apply the averaging step mentioned above. The formal manipulation of the double sum in (1.67) gives

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T dt \;\; \langle \mathcal{O}(t) \rangle = \sum_n |\langle \Psi_0 | n \rangle|^2 \, \langle n | \mathcal{O} | n \rangle. \tag{2.10}$$

The collapse of the double sum onto the single sum is the effect of the dephasing; the r.h.s. above is called the Diagonal Ensemble (DE) of the quantum quench. Here the squared overlaps play the role of the weight functions.

The reader might notice that we neglected the contributions of possible degenerate states, and this is indeed an issue to be discussed. Degenerate or nearly degenerate states can lead to additional effects, such as preserving certain order parameters which would otherwise vanish in the above sum. Nevertheless let us focus first on the actual DE.

Conservation of the charges implies that the DE will be dominated by states for which the charge mean values are close to those of the initial state. To be more precise: a simple computation shows that if the initial state $|\Psi_0\rangle$ satisfies the cluster property, then for the relevant states with large overlaps

$$|\langle \Psi_0 | Q_j | \Psi_0 \rangle - \langle n | Q_j | n \rangle|^2 \sim L. \tag{2.11}$$

Note that the relation is written for the global extensive charges, which means that the variance of the charge densities will scale as $\mathcal{O}(1/\sqrt{L})$, for each Q_j. This argument was given for example in [7].

Standard arguments of statistical physics also show that the GGE averages as given by (2.8) are also dominated by states for which the charge mean values are close to those of $|\Psi_0\rangle$. This is in fact enforced by (2.9).

However, there is no guarantee that the summation in (2.8) and in (2.10) will involve the same weights. In fact, it is known that in most cases the weights are different (see for example the pair structure for the integrable overlaps in Section 2.5). Thus in order to show to equality of (2.8) and (2.10) we need one more ingredient.

The missing piece is the Generalized Eigenstate Thermalization (GETH), which states that in the thermodynamic limit the mean values of local observables only depend on the mean values of the charges, and not on the other details of the eigenstates. If the GETH

holds, then the DE and the GGE give the answer for the local observables, because both summations involve states with the same macroscopic mean values of the charges.

The GETH was first proposed in [150] and it can be considered as a straightforard generalization of the Eigenstate Thermalization Hypothesis [151, 152], which is thought to be responsible for the emergence of the Gibbs Ensemble in non-integrable models [93].

We should add that the ETH and the GETH include an additional statement about the off-diagonal matrix elements of the operators, namely that they should be much smaller than the diagonal ones and that they should decay exponentially with the system size. This condition is needed to ensure that the fluctuations around the time averaged values are also small in the TDL [153].

2.4 Completeness of charges

Based on the previous arguments, a set $\{Q_\alpha\}_{\alpha=1,2,\dots}$ of conserved charges is sufficient for a GGE, if it guarantees that the GETH holds. Let us discuss the diagonal matrix elemts, i.e. we look at the mean values

$$\lim_{TDL} \langle n|\mathcal{O}|n\rangle, \tag{2.12}$$

where it is understood that $|n\rangle$ is an exact eigenstate, but we consider the thermodynamic limit in the given situation. As discussed in 1.8 in large volumes the Bethe states can be described by root density functions. It is generally believed that in the TDL the mean values of local observables only depend on the set of root densities, thus we can write

$$\lim_{TDL} \langle n|\mathcal{O}|n\rangle = \mathcal{O}\left(\{\rho(\lambda)\}\right). \tag{2.13}$$

Here it is understood that if there are multiple particle species then all root densities have to be treated as separate dynamical quantities.

On top of (2.13) it is also believed that in the TDL the infinite set of local observables perfectly distinguishes the root density functions. In other words, selecting two different root densities, corresponding to two different physical situations, there is always a local observable whose mean value is different in the two cases.

These two statements are plausible and they were assumed since the early days of the Algebraic Bethe Ansatz [52]. In specific models (2.13) can be proven on a case by case basis; for the Heisenberg spin chains the author also contributed and the results are presented in 5. It becomes clear from the construction that the second statement also holds quite generally.

Regarding the charges the eigenvalues in the TDL are given by (1.66). We can thus conclude that the GETH is satisfied with a particular set of charges, if the Bethe root densities can be reconstructed from the infinite set of equations given by (1.66). This question has to be investigated on a case by case basis, and below we treat the XXZ and XXX models, where the search for the complete set proved to be highly non-trivial.

In thoses cases when the complete set is found, the Bethe Ansatz computation of the GGE proceeds as follows:

- The mean values of the charges (or their densities) are computed in $|\Psi_0\rangle$. Following (2.11) they will be equal to the mean values of the densities in the Bethe states.

- The Bethe root densities are reconstructed using (1.66).
- The steady state correlation functions are computed using exact methods, to be discussed for example in Sec. 5.

Notice that this method completely avoids the explicit determination of the Lagrange multipliers β_j in the GGE.

The reader might wonder about the mathematical rigor behind the construction above. After all the GGE denisty matrix involves a formal sum over an infinite number of operators, and we have just claimed that it is not necessary to compute the β_j parameters. We thus completely bypass the problem of whether the GGE density matrix is actually well defined. This question is studied among others in the rigorous work [154].

2.5 The Quench Action method

Before discussing the GGE in concrete models, we also introduce an alternative theoretical tool: the Quench Action (QA) method. This can be used in certain cases to find the Bethe root distributions of the steady states. It was pioneered in [155]; for a longer discussion see [156].

The starting point of the QA is the knowledge of the exact overlaps $\langle\Psi_0|n\rangle$ which are the weights in the Diagonal Ensemble (2.10). It turns out that in specific cases the overlaps can be found using analytical tools, moreover they take a special form. These cases are now called *"integrable initial states"* [15] and were studied in detail by the author and collaborators. Closely related methods dealing with these initial states are discussed in Sec. 3, and the exact overlap formulas are the topic of Sec. 4. Here we just present a brief overview of their structure, such that the QA method can be introduced.

For simplicity let us again consider a theory with a single particle species (one type of Bethe roots). For integrable initial states the overlaps are non-zero only if the set of momenta $\mathbf{p}_N$ is space reflection invariant. This means that the momenta have to come in pairs $p, -p$, or they can include a zero momentum particle as well. On the level of rapidities this is also means that the set $\boldsymbol{\lambda}_N$ have to consist of pairs $\lambda, -\lambda$, or they can include special rapidities with $p(\lambda) = 0$ (or $p(\lambda) = \pi$ in spin chains). We will focus on the case with strict pairs; the cases with zero momentum particles can be treated with simple modifications.

For such paired rapidities the overlap with the integrable initial state takes the form

$$|\langle\Psi_0|\boldsymbol{\lambda}_N\rangle|^2 = \prod_{j=1}^{N/2} u(\lambda_j^+) \times C(\boldsymbol{\lambda}_N), \tag{2.14}$$

where we write

$$\boldsymbol{\lambda}_N = \{\lambda_1^+, -\lambda_1^+, \ldots, \lambda_{N/2}^+, -\lambda_{N/2}^+\}. \tag{2.15}$$

In (2.14) the function $u(\lambda)$ is called the single pair overlap function, and the function $C(\boldsymbol{\lambda}_N)$ is such that it remains $\mathcal{O}(L^0)$ in the TDL. This function is also explicitly known, it will be discussed in 4. Quite interestingly, it does not depend on the initial state, and it is related to the pair structure only. The dependence on $|\Psi_0\rangle$ is carried only by $u(\lambda)$.

The physical meaning of the special overlap above is the following: In these cases the initial state "emits" pairs of particles with opposite momenta. The particles within each

pair are correlated, because they are always emitted together. The factorization of the overlap means that the emission amplitudes are independent from each other, thus the pairs are initially not correlated.

If the overlap takes the above special form, then the quantum quench can be solved: Assuming that in the long time limit the system will be populated with states with a well defined root densities, these densities can be determined from a saddle point evaluation of the Diagonal Ensemble (DE) (2.10). The idea is to transform the summation over individual states into a functional integral over root densities:

$$\sum_n |\langle \Psi_0 | n \rangle|^2 [\cdots] \quad \to \quad \int D[\rho(\lambda)] e^{-S_{QA}[\rho(\lambda)]} [\cdots]. \tag{2.16}$$

Here S_{QA} is the Quench Action, which is a functional of the root densities. It includes two terms: the extensive part of the logarythm of the overlap expressed using the root densities, and an entropy term which describes the microcanonical entropy of a given root distribution. This entropy is normalized such that the requirement of the pair structure is correctly taken into account. The Quench Action can be regarded as a generalization of the free energy functional, adapted to the quench setup; its precise form and its detailed derivation can be found for example in [156]. In the formula above $[\dots]$ denotes an insertion of local operators, which do not shift the saddle point of the functional integral.

The Bethe root distribution in the steady state is then found by minimizing the Quench Action. In the simplest cases (with one particle type only) the resulting equation reads [157, 158]

$$\varepsilon(\lambda) = \mu + \log(u(\lambda)) - \int \frac{d\omega}{2\pi} \varphi(\lambda - \omega) \log(1 + e^{-\varepsilon(\omega)}). \tag{2.17}$$

Here $\varepsilon(\lambda)$ is the so-called pseudoenergy function, which is related to the filling fraction (1.62) through

$$f(\lambda) = \frac{1}{1 + e^{\varepsilon(\lambda)}}, \tag{2.18}$$

and μ is a chemical potential which might be needed to fix the total particle number to the desired value. The overlaps are valid for arbitrary total particle number, and μ ensures that we go from the grand-canonical ensemble to the correct final total particle density.

Eq. (2.17) together with (1.63) completely determine the root densities given that the exact overlaps are known in factorized form. This can serve as a check of the GGE: If the root densities computed from a GGE fail to reproduce those of the QA approach then the charges used in the GGE can not be complete. Such a case will be discussed below.

The equation (2.17) and its generalizations can be called the overlap based TBA (o-TBA) equations. They have the same structural form as the TBA equations describing finite temperature situations [49, 117, 118]. We put forward that in some cases already the form of the o-TBA can give some information about the completeness of the charges. The arguments leading to (2.17) can be repeated also for the Generalized Gibbs ensemble (2.8); this leads to the GGE-TBA equations

$$\varepsilon(\lambda) = \sum_j \beta_\alpha h_\alpha(\lambda) - \int \frac{d\omega}{2\pi} \varphi(\lambda - \omega) \log(1 + e^{-\varepsilon(\omega)}). \tag{2.19}$$

Comparing (2.19) and (2.17) we see that the two equations can have the same solution only if the source terms are equal. This means that the GGE is complete, if the source term

$\log(u(\lambda))$ can be reproduced by a linear combination of the one-particle charge eigenvalue functions.

We note that the specific form (2.14) of the overlap is rather different from the weights in a GGE. Looking at the GGE density matrix there is no need to assume the pair structure. Thus if the GGE correctly describes the quantum quench, it does so *despite* having different weights than those in the Diagonal Ensemble (2.10). The entropic consequences of this mismatch are discussed for example in [159].

2.6 Generalized Gibbs Ensemble for the Heisenberg spin chain

The XXZ spin chain is one of the simplest and most studied integrable models. It has some special features which motivate its importance:

- It is experimentally relevant [59, 160–162].
- It is interacting and solvable.
- Its Bethe Ansatz solution is relatively simple compared to other models (for example the multi-component models which require the nested Bethe Ansatz).
- It is a well defined lattice model. This makes it easier to study than for example the Lieb-Liniger model, which poses difficulties both for a theoretical and a numerical treatment because it is defined in the continuum (see for example [163]).

These properties motivated the study of the GGE in the XXZ spin chain. The main question was whether a properly defined GGE can be correct in predicting the steady states after the quantum quenches.

Research in this direction started around 2013, and eventually the problem was settled in 2015. The author and the local research group led by Gábor Takács played a decisive role in this.

The first works to treat a GGE in the XXZ model where [1] and [164] which appeared around the same time, and which have an almost identical content. Here a construction was given for the GGE, using the canonical local charges, which are defined from the transfer matrix through (1.33). The main idea is to construct the GGE ensemble, and compute mean values of local observables using the Quantum Transfer Matrix (QTM) formalism [165–168] (see also the review [169] and the lecture notes [50]).

Originally the QTM was devised to treat finite temperature ensembles. Now we review the key steps.

We start with the following simple relation, which uses the expansion of the transfer matrix using the charges (1.33):

$$e^{-\beta H} = \lim_{N\to\infty} \left(1 - \frac{\beta}{N} H\right)^N = \lim_{N\to\infty} \left(t^{-1}(0) t(-c\beta/N)\right)^N . \tag{2.20}$$

Here c is some constant which might be needed to relate the canonical Q_2 defined in (1.33) to the conventional normalization of H. In the QTM approach one uses a finite N (also called Trotter number), and builds a 2D classical partition function for the combination

$$Z = \mathrm{Tr}\left(t^{-1}(0) t(-c\beta/N)\right)^N \tag{2.21}$$

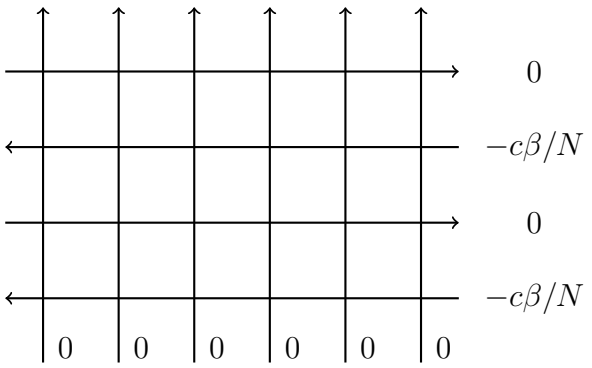


Figure 6: An example for a partition function (2.21) with $L = 6$ and $N = 2$. The horizontal lines correspond to the action of the original transfer matrices in (2.21); each crossing depicts the action of a local Lax operator, see Fig. 1. Periodic boundary conditions are understood in both directions. The QTM is built as the transfer matrix which acts in the horizontal direction, thus it is built from the Lax operators along one column.

using the Lax operators that enter the definition of the transfer matrix. The operator $t^{-1}(0)$ is actually a shift operator, which can also be represented in this framework. The Trotter limit $N \to \infty$ will be taken at the end of the computation to actually generate the thermal operator $e^{-\beta H}$.

In a finite volume L and at a finite N the trace given by (2.21) can be interpreted as a partition function of the six-vertex model on a lattice of size $L \times N$. The statistical weights of the 6-vertex model are given by the matrix elements of the local Lax operators, which in this case are identical to those of the R-matrix given by (1.39). This relation between the quantum spin chain and the vertex model is explained in detail for example in [40, 170]. It can considered as a concrete demonstration of the general statement that a D dimensional quantum system is always equivalent to a $D + 1$ dimensional classical statistical model, where the connection is given by Feynman's path integral. For the present situation sometimes the term "lattice path integral" is also used.

As a second step, the resulting partition function is evaluated by constructing a new transfer matrix $\tau(u)$ acting in the orthogonal direction: this is the Quantum Transfer Matrix (QTM) acting on a total number of $2N$ sites (corresponding to the product of $2N$ operators in (2.21)), such that the partition function in question is expressed as

$$Z = \mathrm{Tr}\ \tau^L(0). \tag{2.22}$$

The QTM $\tau(u)$ is a transfer matrix of an inhomogeneous chain, where the inhomogeneities are determined by the temperature. For a pictorial interpretation see 6.

As a third step one investigates the large L limit at finite N; the commutativity of the $N \to \infty$ and $L \to \infty$ limit is a separate problem which needs to be proven, but all existing numerical data and the experience with analytical computations confirm that this is a valid step and the $L \to \infty$ can be taken even at finite N. Furthermore, in all relevant cases it was found that the QTM is gapped, with a finite gap that remains for

all large N. Therefore, the trace above can be computed with exponential accuracy as

$$Z \approx \Lambda^L(0), \tag{2.23}$$

where now $\Lambda(0)$ is the largest eigenvalue of the QTM, still slightly depending on N. This eigenvalue determines the free energy of the model, and the corresponding eigenvector can be used to compute the finite temperature mean values of local observables.

Here we do not give the details of the computations behind the QTM method, because there are a large number of technical steps which should be discussed. Instead we refer to [50, 169] and here just give a brief overview of the results. Furthermore we focus on the massive regime with $\Delta = \cosh(\eta) > 1$.

In large volumes the partition function (2.20) can be expressed as

$$\log Z = -fL + \dots,$$

where the dots denote exponentially small corrections in L. The free energy density is given by

$$f = -\int_C \frac{d\omega}{2\pi i} \frac{\sinh\eta \log(1+\mathfrak{a}(\omega))}{\sinh\omega \sinh(\omega+\eta)}. \tag{2.24}$$

Here $\mathfrak{a}(\lambda)$ is an auxiliary function defined on the complex plane which satisfies the non-linear integral equation (NLIE), which in the Trotter limit reads

$$\log \mathfrak{a}(\lambda) = -\beta q_2(\lambda) - \int_C \frac{d\omega}{2\pi i} \frac{\sinh 2\eta \log(1+\mathfrak{a}(\omega))}{\sinh(\lambda-\omega+\eta)\sinh(\lambda-\omega-\eta)}. \tag{2.25}$$

The contour C in the equations above depends on Δ. For the $\Delta > 1$ regime it can be chosen as a union of two straight line segments:

$$C = [-i\pi/2 + \alpha, i\pi/2 - \alpha] \cup [i\pi/2 - \alpha, -i\pi/2 + \alpha],$$

where $\alpha < \eta/2$ is an arbitrary parameter. Note that the first line segment runs upwards and the second runs downwards. The function $q_2(\lambda)$ is related to the one-particle energy and it is given by

$$q_2(\mu) = \frac{\cosh(\mu)}{\sinh(\mu)} - \frac{\cosh(\mu+\eta)}{\sinh(\mu+\eta)}.$$

Finite temperature correlation functions can be computed within this formalism, and the auxiliary function $\mathfrak{a}(\lambda)$ specifies the "physical data" for the computations. This is reviewed in Sec. 5.

The main new result of the works [1, 164] is the evaluation of the GGE partition functions

$$Z = \mathrm{Tr}\ e^{-\sum_j \beta_j Q_j} \tag{2.26}$$

by choosing appropriate spectral parameters in the Trotter expansion. The key idea is to use the expansion (1.33) of the tranfer matrices to generate the charges using some products of the form

$$t(\xi_1)\dots t(\xi_m) \approx 1 + cQ_j, \tag{2.27}$$

where c_j are some constants depending on the inhomogeneities. Once such a formula is obtained, the combination $e^{-\beta_j Q_j}$ can be generated by the Trotterization. The key idea of this method appeared earlier in [171].

Eventually one obtains the NLIE for the GGE

$$\log \mathfrak{a}(\lambda) = -\sum_j \beta_j q_j(\lambda) - \int_C \frac{d\omega}{2\pi i} \frac{\sinh 2\eta \log(1+\mathfrak{a}(\omega))}{\sinh(\lambda-\omega+\eta)\sinh(\lambda-\omega-\eta)}, \tag{2.28}$$

where

$$q_j(\mu) = \left(\left(\frac{\partial}{\partial u} \right)^{j-1} \log \frac{\sinh(u-\mu-\eta)}{\sinh(u-\mu)} \right) \Bigg|_{u=0}. \tag{2.29}$$

Note that compared to (2.25) only the source term is replaced in (2.28), such that it mirrors the structure of the GGE density matrix.

In [1] this NLIE was used to give predictions for steady state correlations after a specific quantum quench. The example was the quench from the Néel state. The work [1] actually treated the so-called truncated GGE, where only a finite number of charges were added, and the dependence on the truncation level was also investigated. At this point the complete picture explained in Section 2.4 was not yet understood, and [1] used the direct approach of fitting the β_j parameters through (2.9). This proved to be possible for a small number of charges. However, at that time it was not possible to compare these results to other theoretical or numerical (or perhaps experimental) results, so the nature of the GGE was not yet clarified in these works.

A further important step was taken in the parallel works [6,172], which appeared side by side in Phys. Rev. Lett. In these papers alternative numerical approaches were applied to test the GGE predictions for steady state correlators. Both works reached the conclusion that the GGE built only on the local charges fails to describe the physical properties of the steady state. This was an unexpected result at that time, which contributed to the eventual understanding of the GGE.

The methods and results of [6,172] are almost identical, and they only differ in certain points. Here we describe our work [6] in more detail. Here one of the the key ideas was to apply a Matrix Product State (MPS) algorithm to simulate the real time dynamics after a quench, in the hope of obtaining the equilibrated correlation functions with sufficient precision. This turned out to be possible. The actual algorithm used was the so-called iTEBD (infinite Time Evolving Block Decimation) [173,174] method; the numerical computations were carried out by Miklós Werner, who was a PhD student at that time. The focus was on two particular initial states, namely the Néel state and the dimer state given by (2.2); the time evolution was generated by the XXZ Hamiltonian with $\Delta > 1$. For the Néel state the GGE predictions were taken from [1], whereas for the dimer state they were taken from [175], which applied methods similar to [164]. A direct comparison revealed that the GGE predictions are clearly wrong for the dimer state, and they appeared to be slightly wrong also for the Néel state, although in that particular case the difference was smaller. An example for the mismatch between the GGE and the o-TBA predictions is presented in Figure 7.

The papers [6,172] did not find a solution to the observed failure of the GGE. Some possible explanations at that time were:

1. The set of the conserved charges used in the GGE is not complete, and new charges could complete is.

2. Some other ingredient in the derivation is faulty.

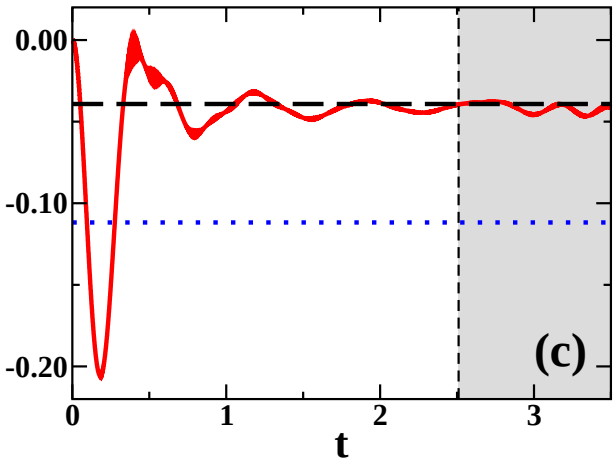


Figure 7: An example for the mismatch between the prediction of the strictly local GGE and the actual real time evolution. The data shows the time dependence of the correlator $\langle \sigma_1^z \sigma_4^z \rangle$ in the quench where the initial state is the dimer state. The model is the XXZ chain with $\Delta = 4$. The numerical data is obtained with the iTEBD algorythm. The dotted line shows the GGE prediction. The dashed line shows the Quench Action (o-TBA) prediction. The numerical data of the iTEBD program is to be trusted until the entanglement in the system is not too large; this can be monitored within the program. The grey are shows those times when the iTEBD data is not to be trusted anymore. We can see a relaxation towards the QA prediction within the white area. This confirms that the QA method is correct and the GGE built on the local charges is not complete. Figure taken from [6].

3. The main idea of the GGE is simply wrong.

Later it was found out that the first explanation is correct.

On a technical level the failure of the local GGE is manifested in the TBA equations. We explain this point in more detail here.

For a GGE of the form (1.16) where the Q_α are the canonical charges the resulting GGE-TBA equations take the form [6, 172]

$$\begin{aligned} \log(\eta_n(\lambda)) = &- \delta_{n,1} \sum_{\alpha=2}^{\infty} \beta_\alpha \left(\frac{d}{d\lambda}\right)^{\alpha-2} s(\lambda) + \\ &+ \int_{\omega=-\infty}^{\infty} \frac{d\omega}{2\pi} s(\lambda-\omega)(\log(1+\eta_{n-1}(\omega)) + \log(1+\eta_{n+1}(\omega))), \end{aligned} \tag{2.30}$$

where we defined

$$\eta_n(\lambda) = \frac{\rho_n(\omega)}{\rho_{n,h}(\omega)}. \tag{2.31}$$

It is important that in these equations only the first component $n = 1$ has a source term.

This is in contrast with the result of the Quench Action method, described in Section 2.5. In the QA method the resulting o-TBA equations are of the form

$$\log(\eta_n(\lambda)) = d_n(\lambda) + \int_{\omega=-\infty}^{\infty} \frac{d\omega}{2\pi} s(\lambda-\omega)(\log(1+\eta_{n-1}(\omega)) + \log(1+\eta_{n+1}(\omega))), \tag{2.32}$$

where the $d_n(\lambda)$ source terms are typically non-zero functions that can be derived from the factorized overlaps. For later use we present the main formulas for them. We have

$$d_n = -g_n + s \star (g_{n-1} + g_{n+1}), \text{ with } g_0 = 0, \tag{2.33}$$

where $g_n(u)$ is the overlap pair amplitude for the n-strings, given by

$$g_n(\lambda) = -\sum_{k=1}^{n} \log\big(u(\lambda + i\eta(n+1-2k)/2)\big), \tag{2.34}$$

where $u(\lambda)$ is the amplitude for a single rapidity pair, see (2.14).

This construction for the d_n might seem complicated, but contour deformations of the convolutions in (2.33) result in relatively simple final functions, which depend only on the analytic structure of the single particle overlap $u(\lambda)$. In some concrete cases explicit formulas were given in [9, 172], but they are not relevant for our discussion. We just note that all $d_n(\lambda)$ are non-zero for both the Néel and Dimer states, which were investigated in [6, 172].

The apparent contradiction between the two sets of equations was resolved in the work [176]. Here it was shown that a complete GGE can be built using the so-called quasi-local charges of the model, and that such a complete GGE indeed reproduces the TBA equations (2.32). In order to explain the main results we first discuss the quasi-local charges.

2.7 Quasi-local charges and the complete GGE

The quasi-local charges were discovered in a series of works by Tomaž Prosen and his collaborators [177–179]; for a pedagogical review see [89]. An independent work from the same time was [180].

The essence of quasi-locality is that the new charges are still extensive in the volume, but the condition that the charge densities should be strictly local operators is relaxed. Instead, the new charge densities can include operator contributions with arbitrarily increasing length, although with strongly decreasing amplitudes. The precise definition is the following:

Let $A(L)$ be a series of operators acting on spin chains of varying length L. We call the series quasi-local if it satisfies the following conditions:

1. $A(L)$ is traceless for every L.

2. In large volumes the Hilbert-Schmidt norm squared increases linearly with L:

$$\text{Tr } A^\dagger(L)A(L) \quad \sim \quad L \tag{2.35}$$

3. For every strictly local operator B which does not depend on L the operator overlap defined as

$$\text{Tr } A^\dagger(L)B \tag{2.36}$$

has a finite $L \to \infty$ limit.

The first condition is required for technical reasons, in order to avoid divergent contributions proportional to the identity operator. The second condition clearly holds for strictly local charges, and it also holds in those cases when the operator densities have long tails with quickly decreasing amplitudes. The third condition reflects the requirement that the local action of the charges should not depend on the thermodynamic limit.

The quasi-local charges that were found in the integrable spin chains are of the form

$$A(L) = \sum_{x=1}^{L} a(x, L), \tag{2.37}$$

with a translationally invariant operator density $a(x, L)$ that has a finite $L \to \infty$ limit such that the formal expression

$$A = \sum_{x=-\infty}^{\infty} a(x), \qquad a(x) = \lim_{L\to\infty} a(x, L) \tag{2.38}$$

acting on the infinite chain is meaningful. Furthermore, the limiting density can be written as

$$a(x) = \sum_{r=1}^{\infty} a_r(x), \tag{2.39}$$

where $a_r(x)$ are strictly local operators acting on r neighbouring sites, and their norm scales as

$$\|a_r(x)\| \sim e^{-\xi r}, \tag{2.40}$$

where $\xi > 0$ is a characteristic decay length of the operator in question. These decay lengths can depend on the particular charge, but it is important that each charge has exponentially decaying tails.

It was first realized in [179] that the quasi-local charges can be simply obtained by the so-called higher-spin or fused transfer matrices. These TM's were known since the early days of algebraic Bethe Ansatz, but it was not realized that they yield the quasi-local charges and that they are so crucial for the GGE. Now we give a very brief introduction into the construction of these charges, without presenting all technical details. We focus on the XXX case.

The key idea is to construct transfer matrices which carry higher dimensional representations of the $SU(2)$ group. To this order we introduce Lax operators $\mathcal{L}^{\Lambda}(\mu)$ that act on the tensor product $\mathbb{C}^{s+1} \otimes \mathbb{C}^2$, where $\mathbb{C}^{s+1}$ is the new auxiliary space, $s = 1, 2, \ldots$ is the representation index and Λ signals the $SU(2)$ representation of dimension $s + 1$. The physical spaces of the spin chain (the sites) are not modified. We require that the Lax operators satisfy the higher dimensional RLL relations:

$$R_{b,a}^{\Lambda_b,\Lambda_a}(\nu, \mu)\mathcal{L}_{b,j}^{\Lambda_b}(\nu)\mathcal{L}_{a,j}^{\Lambda_a}(\mu) = \mathcal{L}_{a,j}^{\Lambda_a}(\mu)\mathcal{L}_{b,j}^{\Lambda_b}(\nu)R_{b,a}^{\Lambda_b,\Lambda_a}(\nu, \mu). \tag{2.41}$$

Here a and b stand for two auxiliary spaces and j signals a physical space. Furthermore, $R_{b,a}^{\Lambda_b,\Lambda_a}(\nu, \mu)$ is the R-matrix acting on the tensor product of two auxiliary spaces. Consistency requires that for three arbitrary representations the following Yang-Baxter relation is satisfied:

$$\begin{aligned} R_{12}^{\Lambda_1,\Lambda_2}(\lambda_1, \lambda_2)R_{13}^{\Lambda_1,\Lambda_3}(\lambda_1, \lambda_3)R_{23}^{\Lambda_2,\Lambda_3}(\lambda_2, \lambda_3) = \\ = R_{23}^{\Lambda_2,\Lambda_3}(\lambda_2, \lambda_3)R_{13}^{\Lambda_1,\Lambda_3}(\lambda_1, \lambda_3)R_{12}^{\Lambda_1,\Lambda_2}(\lambda_1, \lambda_2). \end{aligned} \tag{2.42}$$

It is possible to construct such R-matrices and Lax operators using the so-called fusing procedure [181, 182].

Then we define the higher dimensional monodromy matrices as

$$T_a^\Lambda(\mu) = \mathcal{L}_{a,L}^\Lambda(\mu) \dots \mathcal{L}_{a,1}^\Lambda(\mu). \tag{2.43}$$

This is an operator acting on $V_a \otimes \mathcal{H}$. The higher dimensional transfer matrix is its partial trace over the auxiliary space:

$$t^\Lambda(\mu) = \mathrm{Tr}_a T_a^\Lambda(\mu). \tag{2.44}$$

It follows from the RLL relations that all these transfer matrices commute:

$$\left[t^{\Lambda_1}(\mu_1), t^{\Lambda_2}(\mu_2)\right] = 0. \tag{2.45}$$

It is customary to denote the transfer matrices as $t_s(\mu)$, where $s = 1, 2, \dots$ is the representation index.

The TM's obtained this way are not independent, they satisfy a set of functional equations called the Hirota equation or T-system [183–188]. For example in the $SU(2)$ case discussed here we have

$$t_s(\mu + i/2)t_s(\mu - i/2) = t_{s+1}(\mu)t_{s-1}(\mu) + \Phi_s(\mu), \tag{2.46}$$

where $t_0(\mu) = 1$ by definition and $\Phi_s(\mu)$ is a scalar function which depends on the precise multiplicative normalization of the Lax operators. As a result of these equations, each $t_s(\mu)$ can be expressed using the fundamental $t_1(\mu)$, see for example the determinant formulas found in [189].

Relation (2.46) can be considered the "quantum deformation" of a simple fusion rule from the representation theory of the group $SU(2)$. If we label the irreducible representations in the same way as Λ_s, such that Λ_1 is the defining representation and the dimensionality of Λ_s is $s + 1$, then the following holds for the representations:

$$\Lambda_s \otimes \Lambda_s = \Lambda_{s+1} \otimes \Lambda_{s-1} \oplus \Lambda_0. \tag{2.47}$$

The structure of eq. (2.46) mirrors (2.47) with the extra changes regarding the spectral parameters. This a generic feature of the T-system: even for models with higher rank symmetries they always mirror some simple fusion rules from the corresponding representation theory. This is why the set of the higher transfer matrices is often called the *fusion hierarchy* of the model.

Returning to the GGE, it was a crucial observation of [179] that a family of quasi-local charges can be defined as

$$X_s(\mu) = (-i)\partial_\mu \log(t_s(\mu)). \tag{2.48}$$

Here the pre-factor of $(-i)$ is just a matter of convention. The μ-derivative acts on the logarythm of the transfer matrices; this is only a formal definition. A constructive definition can be given as

$$X_s(\mu) = (-i)\bar{t}_s(-\mu)\partial_\mu t_s(\mu), \tag{2.49}$$

where $\bar{t}_s(\mu)$ is an asymptotic inverse satisfying

$$\lim_{L\to\infty} \|\bar{t}_s(-\mu)t_s(\mu) - 1\| = 0 \tag{2.50}$$

for μ within some neighbourhood of $\mu = 0$. It can be argued that this asymptotic inverse is the space reflected transfer matrix (see later in (3.7)). For concrete proofs of the asymptotic inversion relation see [179, 190]; such relations go back to the early days of integrability, see for example [191].

It is relatively easy to show that the definition (2.48) yields an extensive operator for every s and μ. However, the proof of the conditions for quasi-locality is not trivial. It was proposed in [179] that the operators $X_s(\mu)$ are quasi-local for every s if μ is inside the so-called physical strip with $|\Im(\mu)| < 1/2$. However, this was proven only for small values of s. For a discussion of these issues we refer to [89, 179].

There are two ways to see that the set $\{X_s(\mu)\}$ produces a complete GGE. First, it is possible to derive the GGE-TBA equations for a density matrix of the form

$$\rho \sim \exp\left(-\sum_{s=1}^{\infty}\int \frac{d\mu}{2\pi}\beta_s(\mu)X_s(\mu)\right), \tag{2.51}$$

where now the Lagrange mutpliers $\beta_s(\mu)$ depend both on the discrete spin index and on the continuous rapidity parameter. It is relatively easy to show that the resulting TBA-equation does indeed reproduce any QA-TBA of the form (2.32) with arbitrary source terms for each node.

The second way to see that this set of charges is complete is perhaps more enlightening: It is possible to show that the GETH holds. To be more precise, it can be shown that the initial values of the charge densities defined as

$$\Omega_s(\mu) = \lim_{L\to\infty}\frac{\langle\Psi_0|X_s(\lambda)|\Psi_0\rangle}{L} \tag{2.52}$$

completely specify the set of Bethe root densities. The mean values of $X_s(\mu)$ are conserved during time evolution, thus the functions $\Omega_s(\mu)$ have to coincide with the same mean values evaluated on those Bethe states that populate the system after the quench. It was shown in [176] that the following relation holds:

$$\rho_{s,h}(\mu) = a_{s+1}(\mu) + \frac{1}{2\pi}\lim_{\varepsilon\to 0}\left[\Omega_s(\mu + i/2 - \varepsilon) + \Omega_s(\mu - i/2 + \varepsilon)\right], \tag{2.53}$$

where $\rho_{s,h}(\mu)$ is the density of holes for the s-strings and $a_{s+1}(\mu)$ are some known functions that do not depend on the Bethe state. After specifying all hole densities the actual root densities can be computed using the integral equations (1.64). The resulting relation is

$$\rho_s(\mu) = \frac{1}{2\pi}\left\{\lim_{\varepsilon\to 0}\left[\Omega_s(\mu + i/2 - \varepsilon) + \Omega_s(\mu - i/2 + \varepsilon)\right] - \Omega_{s+1}(\mu) - \Omega_{s-1}(\mu)\right\}. \tag{2.54}$$

Thus the mean values of $X_s(\mu)$ indeed completely specify the particle content of the Bethe states.

The second part of the GETH, namely that the Bethe root densities completely specify the mean values of other local correlations is a separate issue and it is discussed in Section 5.

The reader might wonder why the operators X_s are needed if the functional equations (2.46) connect all transfer matrices to each other. The Hirota equations imply that every

$t_s(\mu)$ is completely determined by $t_1(\mu)$, which means that in finite volume the eigenvalues of the fundamental transfer matrix completely determine all other TM's. Furthermore they contain enough information to extract the values of the Bethe roots [192, 193].

The reason why we need all X_s separately lies in the thermodynamic limit. It turns out that in the TDL a lot of information is lost if we only look at $t_1(\mu)$, but this information is retained in the other members of the hierarchy. In more mathematical terms: the eigenvalue of $t_1(\mu)$ is given by (1.49), and it is a sum of two terms. In the TDL one term becomes dominant, and considerable information is lost when the other term is eventually scaled to zero. This information is then retained in the other members of the fusion hierarchy, because the other members are constructed from $t_1(\mu)$ after adding shifts of $\pm i/2$ to the argument μ, and these shifts influence which term in $t_1(\mu)$ becomes dominant. Looking at this more carefully it turns out that eventually the complete fusion hierarchy is needed to retain all the information, which was stored in $t_1(\mu)$ in a finite volume.

In [194] a slightly different point of view was suggested for the construction of the GGE. In this work root density operators $\hat{\rho}_s(\mu)$ were introduced whose eigenvalues coincide with the root density functions. It was shown in [194] that $\hat{\rho}_s(\mu)$ can be expressed using $X_s(\mu)$, the relation is essentially the combination of (2.53) and (1.64). In this approach the operators $\hat{\rho}_s(\mu)$ are added into the GGE density matrix, giving a transparent physical meaning: we have a separate Lagrange multiplier for each particle type and each rapidity (or momentum) value. However, the operators $\hat{\rho}_s(\mu)$ are not quasi-local anymore, so the statistical physical interpretation of the GGE obtained this way is less clear.

One more alternative framework is given in [195], where the Quantum Transfer Matrix formulation of the complete GGE is given. This construction can be cosidered as a generalization of the work [1] of the author so that the quasi-local charges are also included.

The relation (1.64) is an example for the so-called "string-charge duality" (the term was coined in [196]). This duality means that for each particle type (which are the various strings in the Heisenberg chains) there should be a particular conserved charge which fixes its value. This correspondence was already mentioned in Section 2.3 above. Whereas the first works [6, 172] showed that the original set of local charges does not give a complete correspondence, the papers [176, 196] argued that the "duality" between the descriptions holds when the quasi-local charges are included.

Above we investigated the XXX model in more detail, but the XXZ case can be treated using similar steps. In the case of $\Delta > 1$ analogous equations hold with slightly different concrete functions involved. However, for the case of $\Delta < 1$ some extra steps are needed. The original paper [196] only treated spin-flip invariant states. In the more general case the string-charge duality only holds if a spin-flip non-invariant family of conserved operators is also added to the GGE. This problem was eventually solved in [197], which showed how to complete the string charge duality for states that are not spin-flip invariant.

We should also note that more formal aspects of the GGE were investigated using rigorous mathematical steps in [154].

2.8 Truncated GGE

It is important to discuss the physical meaning of the GGE. We have shown that starting from the relatively simple concept given by formula (1.16) the technical details

lead to rather involved constructions. The connections to the original statistical physical ideas behind the Gibbs or Generalized Gibbs Ensembles are not so clear anymore. It is our opinion that the "string-charge duality" should be regarded as the final answer to this question, and it is more adequate to refer to the "GETH" instead of the "GGE". The reason is simply that the relations like (2.53) completely fix the root densities using information from the conserved charges, and there is no statistical physical reasoning involved: there is no "maximum entropy principle" in this construction.

On the other hand, it is an interesting (and for experiments highly relevant) question whether a *truncated* GGE with a finite number of charges can well approximate the emerging steady states. This was investigated in the case of the Ising model in [149] and in the case of the XXZ spin chain in [14].

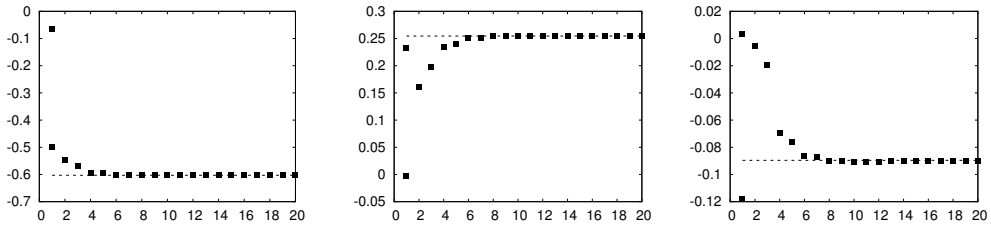
In the latter work [14] the author and his collaborators showed that in the XXZ chain the exact predictions of the string-charge duality can always be obtained by a sequence of truncated GGE's (tGGE's), if the number of charges added is gradually increased towards infinity and if the Lagrange multipliers are chosen accordingly (possibly also depending on the truncation number). From a mathematical point of view this boils down to proving that every source term in (2.32) can be reproduced by the corresponding functions appearing in the tGGE construction. Furthermore, the statement was also demonstrated on concrete examples.

Generically it was observed that the spatial locality plays a very important role in the tGGE. Both works [14,149] concluded that the spatially localized (short range) correlation functions strongly depend on the Lagrange multipliers of the short range charges. In contrast, the dependence is much weaker when we also consider the longer range charges, or the quasi-local charges with higher spin indices. It is important that this observation was made by focusing on concrete examples, therefore it should be not considered as rigorous statement. Nevertheless it agrees with the common lore that in the Heisenberg chain the higher strings (bound states with a big number of constituents) do not affect correlation functions in the most relevant physical situations.

Examples for the correlation functions within a truncated GGE are given in Figure 8. Here the idea is to take a truncation number n and consider the first n charges from each family $X_s(u)$ with $s = 1, \dots, n$. Thus at each n a total number of n^2 discrete charges are taken into account. The particular quench from the Dimer initial state is considered. The Lagrange multipliers for these charges are found numerically by requiring that the truncated GGE gives the correct values for the charges, equal to the mean values in the initial state. Then the local correlation functions are computed within this truncated GGE, following the methods described in Sec. 5. The data shows that as n is increased, the numerical values for these local correlation functions indeed converge, and the asymptotic value is equal to the one obtained from the Quench Action. Thus the truncated GGE indeed converges to the exact solution of the quench. We can see that good fits are obtained with a total number of $6^2 = 36$ charges; similar convergence was observed in all cases.

2.9 GGE in models with higher rank symmetry groups

The success of the GGE in the paradigmatic XXZ chain propagated the belief that some version of the GGE should indeed exist in every integrable model. However, there



(a) $\langle \sigma_1^z \sigma_2^z \rangle$ (b) $\langle \sigma_1^z \sigma_3^z \rangle$ (c) $\langle \sigma_1^z \sigma_4^z \rangle$

Figure 8: Evaluation of a few local observables within the truncated GGE for the dimer quench. The correlators $\sigma_1^z \sigma_a^z$, $a = 2, 3, 4$ are plotted as a function of the truncation index n. At each truncation step a total of n^2 charges are included in the truncated GGE. The horizontal line shows the correlators computed using the Quench Action solution. The value of the anisotropy is $\Delta = 3$.

is no canonical construction for the GGE which would work for every integrable model. The reason for this is that the class of integrable spin chains is actually wider that the class of models considered above.

We focused on models with nearest neighbour interactions, and in these cases the QISM method gives all the charges and eventually also the GGE. However, there are other types of models, where such a construction is not known. One such class is made of the long-range models (such as the Haldane-Shastry or the Inozemtsev chains [198–200]), where the construction of the conserved charges is more involved [201] and far from understood. The GGE has not yet been established in these models.

An other class of models where the GGE was not known are the nearest neighbour interacting models related to higher rank symmetries. A famous example is the 1D Hubbard model [99] or the $SU(N)$-symmetric spin chains. These models can be solved by the *nested Bethe Ansatz*, which is generally much more complicated than the simple one used in the one-component models. Therefore, much less is known about the correlation functions and the dynamics of these systems.

The work [202] treated the transport properties of the Hubbard model within Generalized Hydrodynamics (for an introduction into this theory see Section 6). However, for this model the complete GGE is not yet established, therefore [202] made some assumptions, and the justification is simply based on the success of the GGE for the Heisenberg model. Other works considered the real time dynamics in nested models [21,22,203,204]; all these works focused on the so-called integrable quenches (see next Section), and the question of the complete GGE was not elaborated upon.

Motivated by these works, the author and a collaborator considered the $SU(3)$-symmetric local spin chain in [190] and showed that the complete GGE of the Heisenberg chain can be generalized also to this more complicated model.

The Hamiltonian of the general $SU(N)$-symmetric local chain is given by

$$H = \sum_j P_{j,j+1} - 1, \tag{2.55}$$

where $P_{j,j+1}$ is the permutation operator acting on the tensor product $\mathbb{C}^N \otimes \mathbb{C}^N$ of the local Hilbert spaces of two neighbouring spins.

The model can be solved by the nested Bethe Ansatz for every $N > 2$ (for the algebraic treatment see for example [205], or the recent introduction [108]). The eigenstates are described by different sets of rapidities

$$\boldsymbol{\lambda}_{N_a}^{(a)} = \{\lambda_1^{(a)}, \dots, \lambda_{N_a}^{(a)}\}, \tag{2.56}$$

such that each set corresponds to a different "nesting level", denoted by the index $a = 1, \dots, N-1$, and N_a is the number of elements of the particular set. The first level corresponds to the physical, momentum carrying degrees of freedom (particles above a vacuum state), whereas the higher levels describe the internal degrees of freedom of the multi-particle spin waves. The corresponding Bethe equations can be found for example in [190].

It is known that generally the rapidities of all nesting levels can form strings. Therefore the "particles" of the model can be characterized by two discrete indices (the nesting level a and a string length s) and a continuous rapidity parameter μ. In the TDL the Bethe root distribution can thus be described by the densities $\rho_s^{(a)}(\mu)$ with $a = 1, \dots, N-1$ and $s = 1, 2, \dots$.

In [190] we considered the question whether there are quasi-local operators that fix the complete set $\{\rho_s^{(a)}(\mu)\}$ of root densities. In the case of $N = 3$ we constructed these charges and provided proof for the quasi-locality in a few concrete cases. For the general N we made analogous conjectures, but were not able to provide a rigorous proof. The actual statements are rather technical, so here we just describe the main results.

In the $SU(N)$-symmetric models there exists a transfer matrix $t^{\Lambda}(\mu)$ (2.44) for each irreducible representation of the symmetry group. These representations are characterized by Young diagrams. It is known that the rectangular Young diagrams play a special role in the algebraic description: they satisfy the T-system or Hirota equations mentioned above. For a rectangular diagram with a rows and s columns let $t_s^{(a)}(\mu)$ denote the corresponding transfer matrix. The Hirota equation reads

$$t_s^{(a)}(\mu + i/2)t_s^{(a)}(\mu - i/2) = t_{s+1}^{(a)}(\mu)t_{s-1}^{(a)}(\mu) + t_s^{(a+1)}(\mu)t_s^{(a-1)}(\mu), \qquad a = 1, \dots, N-1, \quad s = 1, 2, \dots. \tag{2.57}$$

The boundary conditions to these functional relations depend on some arbitrary choices, as discussed in detail in [183–185]. In our work we used the conventions $t_0^{(a)} = 1$, $t_s^{(N)} = 1$ and $t_s^{(0)}$ was set to a fixed function.

In analogy with (2.48) we defined the charge generating functions

$$X_s^{(a)}(\mu) = (-i)\partial_\mu \log(t_s^{(a)}(\mu)). \tag{2.58}$$

In the case of $N = 3$ we proved that two of these charges are quasi-local if μ is in the physical strip $|\Im(\mu)| < 1/2$. It is conjectured that the same statement holds in all the other cases.

The main result in [190] is the string-charge relation

$$\rho_s^{(a)}(\mu) = \lim_{L\to\infty} \frac{1}{2\pi L}\left(X_s^{(a)}(\mu + i/2) + X_s^{(a)}(\mu - i/2) - X_{s+1}^{(a)}(\mu) - X_{s-1}^{(a)}(\mu)\right), \tag{2.59}$$

where the shifts of $\pm i/2$ should be understood as limiting values from within the physical strip. This relation was proven in detail for $N = 3$ and conjectured for generic N based on the algebraic similarities between the different models. It is a generalization of (2.54).

Let us add a comment about this result, in relation with the history of the field. The string-charge duality is a recent result, which was motivated by the research on quantum quenches. In contrast, the T-system equations had been studied for a long time. They are ubiquitous tools in the theory of integrable models, and they are relevant in numerous aspects, see the giant review [188]. The authors of [188] write in the introduction: "*We therefore look forward to the next review to come, hopefully someday by some author, bringing a delightful renewal.*". We believe that the string-charge relations are such new results, which deserve to be mentioned in any future review of the T-system.

2.10 Open problems

As discussed above, the questions about the GGE have been completely solved in the Heisenberg spin chains and also in the $SU(3)$-symmetric fundamental model. At present it is widely believed that for each integrable model there is a corresponding GGE, even though there is no general construction for this. As an example we can mention the case of the Hubbard model, or the long range spin chains, where this issue has not yet been solved. Nevertheless, currently there is little motivation to work out the particular details in the remaining models, and the interest of the community shifted to the transport phenomena.

We believe that there are still some interesting open problems. For example in the repulsive Lieb-Liniger model it is not known how to make a direct invertible relation between the root density and transfer matrix eigenvalues, and it is not known hot to solve generic quench problems in that model. Also, as far as we know the emergence of the GGE has not yet been rigorously proven: the arguments we presented above contain a number of assumptions. Very important progress was made in [154], where the ill-defined concept of the GGE density matrix was placed on firm footing, and the statement of "equilibration to a GGE" was proven given some physically sound conditions are satisfied. Then the missing step is just to prove these conditions; for the details we refer to [154].

3 Integrable initial states

3.1 Introduction

In the previous Section we showed that in integrable models the GGE completely describes the steady states that emerge after the quantum quenches. Nevertheless the question remains: how can we actually compute the physical quantities in these quench situations? Objects of interest could be the time evolution and the long time limit of local observables, or the entanglement entropies.

The long time limit of local observables is in principle fixed by the GGE, but their actual determination is still quite complicated. Above we have argued that the complete set of conserved charges fixes the Bethe root distributions through the string-charge relations (see eq. (2.54) or (2.59)). These root distributions completely determine the correlation functions, see Section 5. However, the full set of Bethe root distributions can be obtained only if the mean values of all charges are known in the initial state, see for example the $\Omega_s(\mu)$ defined in (2.52). These quantities are independent from each other and need to be determined individually for every s. If the initial state is a product state, then there are simple Matrix Product State based methods to compute them [164, 175]. But it is not

possible to determine all of them in analytic form. A practical way out of this situation is to compute the first few root densities, and to leave the rest unspecified; such a procedure would be analogous to the truncated GGE. An other possibility is to focus on the so-called integrable quenches.

The term *integrable quench* was coined by the author and collaborators in [15]. There are multiple technical definitions of such quenches (or initial states), which will be explained below. In simple words **we call an initial state $|\Psi_0\rangle$ integrable if it has special properties which makes the quench problem solvable**. Of course we need to specify what *solvable* means in this context.

Here we list a few properties of these states, which characterize the solvability:

- There are methods to compute the exact overlaps between the initial state and the Bethe states (see Section 4). These overlaps take the special factorized form given by (2.14).

- The (density of the) Loschmidt echo can be computed using analytic tools. Let us define
$$G(t) = \log \left|\langle\Psi_0|e^{-iHt}|\Psi_0\rangle\right|. \tag{3.1}$$
In large volumes L this quantity behaves as $G(t) \sim Lg(t)$, where $g(t)$ is a quantity analogous to the free energy density. For integrable quenches it can be computed analytically (see Section 3.7).

- The Bethe root densities characterising the steady states emerging after the quenches can be computed analytically. The actual calculation is recursive in nature, and can be performed on a computer rather easily (see Section 3.7).

- The time evolution of entanglement entropy can be computed in a simple semi-classical picture, yielding the correct asymptotic behaviour [206] (see Section 3.9).

The properties above were observed somewhat independently by a couple of authors and they were scattered in the literature. It was largely due to the author and collaborators that all of them were linked to the integrability of the initial state, which seems to be the basis for all of these properties. This will be explained in the sub-sections below.

It is important that the content of this Section is restricted to interacting theories. The same structures also exist in free theories, but in those cases a wider class of initial states can be regarded as "solvable", see for example [207].

3.2 Integrable boundaries in time and space

Here we explain the main physical ideas behind the integrable initial states. The basic picture goes back to the seminal work of Ghoshal and Zamolodchikov [208] from '93, which considered equilibrium properties of integrable QFT with boundaries. The implementation of these ideas in spin chains and making the connection to the quench problems was done by the author and his collaborators (see [15] and some other articles that will be discussed below).

Let us consider finite time evolution in a finite volume situation. For the moment we do not specify the nature of our model, it can be a spin chain or a QFT. We consider the

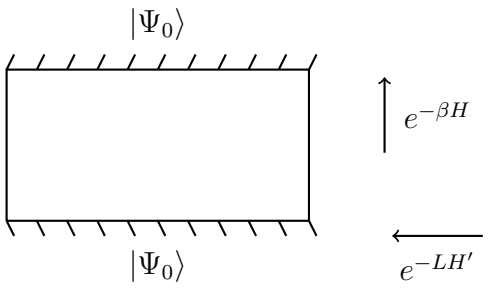


Figure 9: Pictorial representation of the partition function (3.2). The state $|\Psi_0\rangle$ plays the role of boundaries in time, and the (Euclidean) time evolution is generated by H. Periodic boundary conditions are understood in the space direction. The same partition function can be evaluated by a different evolution operator H' which acts in the horizontal direction; this system will have two integrable boundary conditions and in this channel the partition function will be given by a trace, due to the original periodic boundaries.

Loschmidt amplitude in Euclidean time:

$$Z(\beta, L) = \langle \Psi_0 | e^{-\beta H} | \Psi_0 \rangle. \tag{3.2}$$

Here the same (unspecified) state $|\Psi_0\rangle$ is chosen as the initial and final state, and $Z(\beta, L)$ is a quantity which can be understood as a partition function of a 2D classical system. We will see that in the spin chains this classical model is actually known, it is the six-vertex model or one of its generalizations. If we view (3.2) in the physical picture as it is written, then $|\Psi_0\rangle$ plays the role of boundaries in time, and H generates the time evolution between the initial and final (Euclidean) times 0 and β.

The same partition function can be evaluated alternatively if we build a transfer matrix or Hamiltonian which acts in the space direction. In a pictorial representation this corresponds to rotating the model by 90 degrees. In this new picture the state $|\Psi_0\rangle$ plays the role of boundaries in space, and a new transfer matrix or Hamiltonian H' generates time evolution. In this new time direction we have a trace, corresponding to the periodic boundary conditions in the original model. The new rotated channel is sometimes called "crossed" or "mirror" channel (the latter term is used commonly in the AdS/CFT literature). If the original model was Lorentz-invariant, then in imaginary time it is Euclidean invariant, and then H' is identical to the original H up to boundary terms. However, in the generic situation H' is different from H. For a pictorial interpretation see Fig. 9.

Having established this correspondence between the two channels the key observation is that the initial state $|\Psi_0\rangle$ is integrable iff the boundary conditions for H' are also integrable. In other words, the same boundary has to be "integrable" irrespective whether it is used as a boundary in space or in time.

The relatively simple case of integrable relativistic QFT was considered in [208], where this connection between the two channels was made explicit. In order to explain the results of this paper we should review the theory of boundary QFT. This is beyond the scope of the present work, therefore we refer the reader to [208, 209]. Below we will just use some central elements of this theory.

In the case of integrable boundary conditions there is a reflection matrix $\mathcal{R}$ which describes the reflection amplitudes off the boundary, and this reflection is completely

elastic (dissipationless) and factorized, in complete analogy with the S-matrix. In the simplest case with only one particle type in the spectrum this reflection matrix is descibed by a rapidity dependent phase $R(\theta)$. In more complicated cases it is actually a matrix and it has to satisfy the so-called boundary Yang-Baxter relation, which (in some other form) will be reviewed later.

Let us consider the local conserved charges of the original model. Typically we can divide the charges into two sets according to their space reflection properties. In the most typical case the Hamiltonian and the charges Q_{2s}, $s \geq 1$ are even under space reflection, whereas the charges Q_{2s+1} are odd. In [208] it was shown using arguments from Conformal Field Theory that any state $|\Psi_0\rangle$ which corresponds to an integrable boundary in the crossed channel satisfies the conditions

$$Q_{2s+1}|\Psi_0\rangle = 0. \tag{3.3}$$

To be more precise, this has to hold for all charges which are kept intact by the boundary in the crossed channel; we will see that in the spin chains the condition holds for every odd charge. Eq. (3.3) can be considered as a definition of an integrable state, as it was suggested by us in [15].

The first simple consequence of (3.3) is the "pair-structure" for the overlaps. Let us compute the finite volume overlap between the state $|\Psi_0\rangle$ and a Bethe state $|\boldsymbol{\lambda}_N\rangle$. Inserting an odd charge Q_{2s+1} we get

$$0 = \langle\Psi_0|Q_{2s+1}|\boldsymbol{\lambda}_N\rangle = \left(\sum_{j=1}^{N} h_{2s+1}(\lambda_j)\right)\langle\Psi_0|\boldsymbol{\lambda}_N\rangle. \tag{3.4}$$

Here $h_{2s+1}(\lambda)$ are the one-particle eigenvalues of the charges, which are odd functions of λ. We can see that the overlap can be non-vanishing only if the sum of these eigenvalues is zero, for all odd charges. These eigenvalue functions are linearly independent from each other, thus the relation can be satisfied only for very special configurations. It is easy to see that the only possibility is that the set of rapidities consists of pairs $\lambda, -\lambda$ and possibly it can also include zero momentum particles (in spin chains a further possibility is a lattice momentum π).

It was also shown in [208] using QFT arguments that in infinite volume the initial state can be written formally as

$$|\Psi_0\rangle = \mathcal{N}\exp\left(\int_0^\infty \frac{d\lambda}{4\pi}K(\lambda)A(-\lambda)A(\lambda)\right), \tag{3.5}$$

where $A(\lambda)$ are the so-called Fadeev-Zamolodchikov creation operators, $\mathcal{N}$ is an irrelevant normalization factor and $K(\lambda)$ as the pair amplitude for the overlaps. This form is identical to a "squeezed state", and most importantly it results in factorized overlaps with states respecting the pair structure. The issue of zero momentum particles has to be treated separately [208, 210].

A further key result of [208] was a direct connection between the pair amplitude and the reflection factors in the crossed channel. It was found that

$$K(\lambda) = R(i\pi/2 - \lambda). \tag{3.6}$$

The interpretation of this relation is the following: There are two different processes in the two different channels which share the same microscopic origin, therefore their amplitudes must be the same. One such process is the pair creation amplitude in the original picture, and the other process is the reflection of a particle off the boundary in the crossed channel. The shift of $i\pi/2$ simply reflects the Euclidean rotation of 90 degrees.

Even though the main results (3.5)-(3.6) were derived in '93, the connection to quench problems and the spin chain overlaps was not recognized for a long time. Below we discuss the integrable initial states in the spin chains; for a recent treatment of integrable quenches in QFT we refer to [211, 212].

3.3 Integrable states of integrable spin chains

As it was mentioned above, the first papers dealing with quantum quenches in the interacting Heisenberg chain [6,9,172,176,213] considered initial states which were product states of local two-site blocks. The main examples were the Néel and the dimer states. At that time it was not known that both of these states are actually integrable, and many of the observed properties of these states follow from their integrability.

The first remarkable property was the pair structure for the overlaps, which was observed for the Néel state in [214] (for an earlier result in the Lieb-Liniger model see [157]). Furthermore it was found that the overlaps have the factorized form (2.14) and the pair amplitude was also computed. The papers [196, 213] also assumed that the so-called Y-system equation holds for the solution of the overlap-TBA equations; this property will also be discussed below. At that time there was no deeper understanding of these properties.

The first paper to discuss the underlying mathematical structure was [15], which was built on the preceding works [3, 13]. Now we summarize the key ideas.

The first step is to relate the condition (3.3) to the action of transfer matrices, so that the special algebraic properties of the states and the Lax operators can be exploited. To this order we introduce the *space reflected transfer matrix* as

$$\bar{t}(\mu) = \mathrm{Tr}_a \bar{T}_a(\mu), \qquad \bar{T}_a(\mu) = \mathcal{L}_{a,1}(\mu) \dots \mathcal{L}_{a,L}(\mu). \tag{3.7}$$

Here we focused on the fundamental representation, but the same construction can be repeated for the higher dimensional TM's as well.

Let Π be the space reflection operator on the spin chain. Then clearly we have

$$\bar{t}(\mu) = \Pi t(\mu) \Pi. \tag{3.8}$$

If the Lax operators have an appropriate normalization then the canonical charges defined as (1.33) satisfy the space reflection properties

$$\Pi Q_s \Pi = (-1)^s Q_s. \tag{3.9}$$

It is then easy to see that the condition (3.3) is compatible with

$$t(\mu)|\Psi_0\rangle = \bar{t}(\mu)|\Psi_0\rangle. \tag{3.10}$$

The complete equivalence between (3.3) and (3.10) is not yet established; (3.10) implies two-site shift invariance, whereas this property is not immediately clear from (3.3). Nevertheless all known cases satisfy both conditions.

As far as we know (3.10) is specific to spin chains; it was not present in the work [208]. Its physical meaning is less transparent, nevertheless it is a very convenient starting point for the further computations.

The simplest possible cases for the integrable initial states are the two-site product states of the form

$$|\Psi_0\rangle = \otimes_{j=1}^{L/2} |\psi\rangle, \qquad |\psi\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2. \tag{3.11}$$

The two-site invariance is motivated by the known example and the algebraic structure to be discussed below. Note that the one-site invariant states are naturally included in this class.

The work [15] established a concrete connection between states of the form (3.11) and the theory of boundary integrability. It was shown there that a two-site product state satisfies the integrability condition for each L if the two-site block is derived from a so-called boundary K-matrix. The boundary K-matrices are fundamental building blocks for integrable lattice models with boundaries. They were introduced by E. Sklyanin in [215]. The key idea here is that in systems with boundaries (in space) the transfer matrix has to be constructed from two "rows" of Lax operators, with two boundary insertions describing the interactions with the boundary. An example for such a double row TM will be given later.

The K-matrices in question are matrices acting on the auxiliary space $\mathbb{C}^2$ and they satisfy the so-called Boundary Yang-Baxter equation, which is a relation for operators acting on $\mathbb{C}^2 \otimes \mathbb{C}^2$:

$$R_{1,2}(u-w)K_1(u)R_{1,2}(u+w)K_2(w) = K_2(w)R_{1,2}(u+w)K_1(u)R_{1,2}(u-w). \tag{3.12}$$

It was shown in [15] that a state of the form (3.11) is integrable, if the two-site block is given by

$$\psi_{jk} = (CK(-\eta/2))^j_k, \tag{3.13}$$

where C is a crossing matrix characteristic of the model, and η is the crossing parameter. In the Heisenberg chains $C = \sigma^y$ and η is i for the XXX case and $\mathrm{acosh}(\Delta)$ for the XXZ case.

The proof that (3.10) holds for such a two site state is a generalization of the famous train-argument which shows the commutativity of the transfer matrices using the local exchange relations. Here the relation (3.12) is used consecutively to obtain the desired result. For the concrete steps of the proof we refer to [15].

The connection (3.13) is somewhat analogous to the result (3.6), although (3.13) concerns the building blocks of the states and not the overlap/reflection amplitudes. Nevertheless it is similar in spirit, because it connects objects from the two different channels, as explained above.

In the Heisenberg chains all solutions of the BYB are known [216]. We do not give the formulas here, we merely state that the generic solution has three complex free parameters. In accordance with this there is a K-matrix solution for every two-site block. This means that **every two-site block produces an integrable initial state**. This explains why the earlier studies with the Néel and Dimer states were so successful: because the states themselves are rather special.

In [15] the higher spin Heisenberg chains were also treated. In these cases the K-matrices still form a three parameter family: they can be obtained by direct solution of

the corresponding BYB equations, or by the so-called fusion procedure [217,218]. In these cases the dimension of the local Hilbert spaces is larger, therefore not all two-site states are integrable. Solutions of quench problems in some integrable cases were found earlier in [219], and exact formulas for the integrable two-site states of the spin-1 XXZ chain can be found in [15].

The sub-sequent works [21, 22] also treated the $SU(3)$- and more generally $SU(N)$-symmetric fundamental models, given by Hamiltonian (2.55). Here it was found that the integrable two-site states form two families: either they are symmetric, or anti-symmetric with respect to space reflection, but there are no further conditions for them. A central and important example (within the symmetric ones) is the so-called delta-state, where the un-normalized two-site block is

$$\psi_{jk} = \delta_{jk}, \qquad j, k = 1, \dots N, \tag{3.14}$$

with δ_{jk} being the Kronecker-delta. This two-site block gives an integrable initial state in every $SU(N)$-symmetric model.

Having reviewed these basic examples the question remains: what are the most general translationally invariant integrable initial states? In the next Section we show that the this class is much wider.

3.4 Integrable Matrix Product States

Let us consider a spin chain with local dimension N and periodic Matrix Product States (MPS') given by

$$|\Psi_\omega\rangle = \sum_{j_1,\dots,j_L=1}^{N} \mathrm{tr}_A \left[\omega_{j_L} \dots \omega_{j_2}\omega_{j_1}\right] |j_L, \dots, j_2, j_1\rangle. \tag{3.15}$$

Here $|j_L, \dots, j_2, j_1\rangle$ are the real space basis vectors with $j_k = 1, \dots, N$ and $k = 1 \dots L$ with L being the length of the chain. The matrices ω_j act on some auxiliary space denoted by A. For a pictorial representation see Fig. 10. We note that two sets of matrices ω_j and ω'_j describe the same state if they are connected by a similarity transformation S as

$$\omega_j = S\omega'_j S^{-1}, \qquad j = 1, \dots, N. \tag{3.16}$$

This is as essential observation which is used in the proofs of the integrability condition.

MPS' were developed to describe and approximate 1D many body states with relatively low entanglement, such as ground states of gapped models [140, 220]. Here we do not attempt to review the huge literature of MPS', instead we focus on the concrete applications to integrable models.

It was observed in a series of works [221–225] that certain concrete MPS' are integrable initial states in the fundamental $SU(N)$ and $SO(N)$ symmetric model. These works concern certain correlation functions in the AdS/CFT correspondence, and it is quite remarkable that the integrable MPS were discovered using field theory computations. This is one more example of the cross-fertilization bewteen these related fields. In these works the integrability structure of the states was not understood, instead they just found that the overlaps have the pair structure and the factorized form, which is a clear

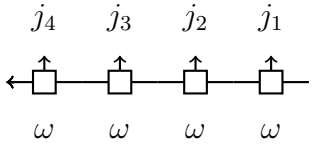


Figure 10: A pictorial representation of the Matrix Product State (3.15) built from the one-site block ω. Here the outgoing indices $j_k = 1, \ldots, N$, $k = 1, \ldots, L$ represent the physical degrees of freedom, and the horizontal lines denote the action of the matrices ω_{j_k}. The matrices are assumed to act from the right to the left; the arrow on the leftmost horizontal link signals this convention. The trace in the definition (3.15) implies periodic boundary conditions.

sign of integrability. The actual integrability condition was proven in one case in [226], but the relation to the general theory was not known at that time.

The systematic treatment of the integrable MPS was given in [23] and subsequently in [27]. Here we summarize the main findings of [23]; a few concrete examples for integrable MPS are given in Section 3.5.

Let us denote the collection of matrices $\{\omega_j\}_{j=1,\ldots,N}$ acting on V_A simply as ω. Motivated by the two-site product states we will also consider two-site invariant MPS. Furthermore, we will consider inhomogeneous (rapidity dependent) cases. Based on experience from the two-site states we can expect to find integrable two-site invariant MPS. Therefore we introduce the collection of matrices $\{\psi_{jk}(u)\}_{j,k=1,\ldots,N}$ (also called two-site blocks) acting on V_A. We will use the simplified notation $\psi(u)$.

It was shown in [23] that the following is a sufficient condition for the integrability of the MPS:

$$\check{R}_{23}(u)(\omega \otimes \psi(u)) = \check{R}_{12}(u)(\psi(u) \otimes \omega). \tag{3.17}$$

This is a relation in the triple tensor product of physical spaces $V_3 \otimes V_2 \otimes V_1$, and the elements are matrices acting on V_A. A pictorial representation is given in Fig. 11. The matrix $\check{R}(u)$ is defined simply as

$$\check{R}(u) = PR(u), \tag{3.18}$$

where $R(u)$ is the fundamental R-matrix of the model and P is the permutation operator.

Relation (3.17) was termed the "square root relation", because it involves half of the steps of the full Boundary Yang-Baxter (BYB) relation. It was a new result of [23]. The key steps of the proof of the integrability condition (3.10) are the following:

1. The two states on the two sides of (3.10) are written as a "dressed" MPS (the original matrices ω_j and the Lax operators from the transfer matrices are glued into an extended MPS with a tensor product auxiliary space).

2. For these "dressed MPS" the eq. (3.17) is essentially a similarity transformation like (3.16). Thus the two sides of (3.10) are equal.

It was also shown in [23] that under some reasonable conditions the initial condition for $\psi(u)$ at $u = 0$ turns out to be the original MPS itself:

$$\psi_{ab}(0) \sim \omega_a \omega_b. \tag{3.19}$$

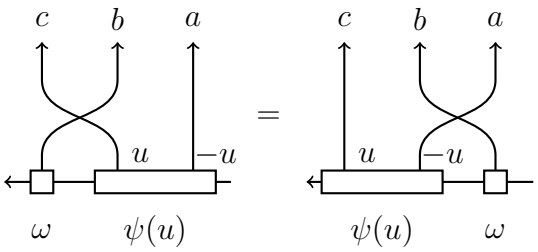


Figure 11: A pictorial interpretation of the "square root relation" (3.17), which describes the exchange of the two-site block $\psi(u)$ and the one-site block ω. This is a relation in $V_3 \otimes V_2 \otimes V_1 \otimes End(V_A)$ and the outgoing indices c, b, a describe the basis states in $V_3 \otimes V_2 \otimes V_1$. Fixing c, b, a we obtain matrices acting on V_A. The local $\check{R}$ matrices acting at the crossings are defined such that their argument is always the rapidity coming from the right minus the rapidity coming from the left. Thus we get an action of $\check{R}(u)$ on both sides, but on different vector spaces.

Thus the solution $\psi(u)$ of (3.17) can be regarded as the "Baxterization" of the original set of matrices.

At present it is not known, whether the square root relation is useful for any other purpose, but it turned out to be a very convenient tool to perform the baxterization, because the relation itself is only linear in $\psi(u)$ and has less components than the full BYB.

In contrast, the relevant full BYB can be written as

$$\check{R}_{12}(v-u)\check{R}_{23}(u+v)(\psi(v)\otimes\psi(u)) = \check{R}_{34}(v-u)\check{R}_{23}(u+v)(\psi(u)\otimes\psi(v)), \qquad (3.20)$$

which is satisfied by the two-site block $\psi(u)$. This is a relation in $V_4 \otimes V_3 \otimes V_2 \otimes V_1 \otimes End(V_A)$, and it is understood that the $\check{R}$ matrices act on the respective components in the tensor product. For a graphical interpretation of (3.20) see Fig. 12. We note that this particular form corresponds to the so-called *twisted Boundary Yang-Baxter relation*, which was also called the KYBE in a subsequent work [227].

Working out the components we can see that it is similar to the original BYB (3.12), but there are certain differences in the arrangement of the R-matrices which are actually rather important. Such twisted BYB equations were already studied for systems with boundaries (in space) [228, 229].

It was shown in [23] that if some irreducibility conditions are satisfied, then the solution of the square root relation also solves the KYBE. In all known cases it was observed that if the integrable MPS is one-site invariant, then there is an object $\psi(u)$ which satisfies both the square-root relation and the KBYE.

It is important to discuss the symmetry properties of the MPS. A certain integrable MPS is said to belong to the class (G, G') with Lie groups G and G', if the spin chain in question is G-symmetric and the MPS enjoys residual symmetry G'. All known solutions to the KYBE are such that (G, G') is a so-called symmetric pair (also called a Gelfand-pair). Examples include $(SU(N), SO(N))$ or $(SO(N), SO(D) \times SO(N-D)$. For a discussion of this observation see [230].

If the symmetric pair is specified, then the abstract algebraic structure behind the twisted BYB is called the *twisted Yangian* [231, 232]. Having found a solution to the

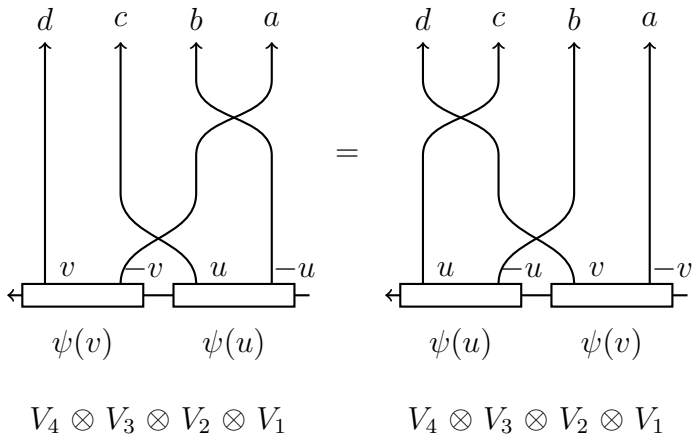


Figure 12: A pictorial representation for the BYB relation for the two-site MPS. $V_{1,2,3,4}$ denote the physical vector spaces, and a, b, c, d are the physical indices. The matrices in the MPS act in the auxiliary space from the right to left. The local $\check{R}$ matrices acting at the crossings are defined such that their argument is always the rapidity coming from the right minus the rapidity coming from the left.

twisted BYB with the given symmetry properties means that a certain representation of the abstract algebra was found. Thus the classification of all integrable MPS is in principle given by the representation theory of the twisted Yangians, which is well studied [233]. However, the problem of having one-site invariant MPS (corresponding to the factorization condition (3.19)) was not yet studied in the literature, simply because it did not come up in the abstract representation theory. Let us add that this is an area still in development: a number of open questions were treated only in a series of recent works, see for example [234, 235].

3.5 Examples and applications to AdS/CFT

The AdS/CFT correspondence [72, 236] connects two very different theories which are believed to be the description of the same physical system: a 3+1 dimensional conformal QFT on one side, and a string theory on a curved anti-de Sitter space on the other side. Integrable models play a special role in this correspondence: in the so-called 't Hooft limit the scaling dimensions of gauge invariant composite operators are given by the eigenvalues of an integrable spin chain. This was observed first at the tree-level in the perturbative expansion of the 't Hooft coupling, and it was later extended to higher orders. At tree level the corresponding Hamiltonian is completely local with a two-site interaction, whereas at higher orders it recieves long range interaction terms. For reviews we refer to [72, 73].

In this correspondence the simplest setting is when all multi-point functions are translationally invariant. However, it is also important to study situations when there are branes present in the AdS spacetime. In this case the CFT side is described by a boundary CFT. It is very natural to study the properties of such boundary field theories. In these cases the simplest objects are the one-point functions of scaling operators, which are not zero due to the breaking of the conformal symmetry. If an operator $O(x)$ has scaling dimension κ such that the two-point function behaves in translationally invariant

Euclidean space-time as

$$\langle O(x)O(0)\rangle \sim \frac{1}{x^{2\kappa}}, \tag{3.21}$$

then in the boundary setting the one-point functions have the form

$$\langle O(x)\rangle \sim \frac{1}{z^{\kappa}}, \tag{3.22}$$

where x is the full space-time coordinate and z is the Euclidean distance from the boundary. Usually the operators are normalized such that the proportionality coefficient in (3.21) is set to unity, and in this case the amplitude of the one-point function in (3.22) is a non-trivial quantity which carries information about the operator and also about the interactions with the boundary. In the studies of these brane situations it is the goal to determine this amplitude using exact methods. For a recent review (which also discusses some of the results below) see [237].

Quite remarkably it was found in [221–225] that the amplitude for the one-point functions is essentially an overlap in the integrable spin chains that describe the scaling operators. To be more precise, one has to compute an overlap between a certain MPS which is determined by the field theory computations involving the boundary, and an eigenstate of the corresponding spin chain, which describes the operator in question. This picture holds at the tree level, and it can be extended to include higher order corrections, which describe long range interactions both for the spin chain and the MPS.

It was observed in [221–225] that all the MPS that naturally come up lead to integrable overlaps. This was eventually explained in [23, 27], where the connections to the KYBE and the twisted Yangian were worked out in detail. On top of the rigorous proof of the integrability condition this led to the computation of new overlap formulas: in the most recent paper [27] the author and his collaborators (Tamás Gombor and an international team) solved a particular case which was beyond the reach of the earlier brute force methods, see for example [225].

The computations are rather technical, especially for the case treated in [27]. Here we just list some of the integrable MPS' found, together with their baxterization. The actual AdS/CFT situations (with the different brane scenarios), which led to the first discoveries of these MPS' can be found in the works [221–225]; here we just focus on the integrability properties and the formulas relevant to the spin chains. We stress once more that the concrete formulas for the MPS were derived by the international researchers of the AdS/CFT correspondence, and the new addition of the author and his local collaborator Tamás Gombor was finding the Baxterization (thus proving the integrability of the MPS) and eventually deriving some new overlap formulas.

- **Case** $(SU(3), SO(3))$

  The MPS is given by the Pauli matrices:

  $$\omega_a = \sigma_a, \qquad a = x, y, z. \tag{3.23}$$

  This MPS was discovered in the works [221–223] which studied it first after the restriction to an $SU(2)$ subgroup, and then eventually also in $SU(3)$ case. The corresponding Baxterization is found to be [23]

  $$\psi_{ab}(u) = \sigma_a\sigma_b - 2u\delta_{ab}. \tag{3.24}$$

It is interesting that this MPS is the ground state of the famous AKLT model [238], but here it shows up quite independently.

- **Case $(SU(3), SO(3))$, higher dimensional cases**

  The MPS is given by the spin-s representation of the $SU(2)$-algebra:

$$\omega_a = S_a, \qquad [S_a, S_b] = i\varepsilon_{abc}S_c, \qquad a, b, c = 1 \dots 3. \tag{3.25}$$

  The Baxterization is

$$\psi_{ab}(u) = S_a S_b + u[S_a, S_b] - u^2\delta_{ab}. \tag{3.26}$$

  Going back to $s = 1/2$, i.e. setting $S_a = \sigma_a/2$ and using the special properties of the Pauli matrices we obtain the previous solution up to some normalization factor.

- **Case $(SU(N), SO(N))$, Dirac matrices**

  The MPS is given by

$$\omega_a = \gamma_a, \qquad \{\gamma_a, \gamma_b\} = 2\delta_{ab}, \qquad a, b = 1 \dots N, \tag{3.27}$$

  The Baxterization is

$$\psi_{ab}(u) = \gamma_a\gamma_b - 2u\delta_{ab}. \tag{3.28}$$

  Clearly this is a generalization of (3.24).

- **Case $(SO(N), SO(D) \times SO(N-D))$, Dirac matrices**

  The MPS is given by

$$\omega_a = \begin{cases} \gamma_a & \text{for } a = 1 \dots D \\ 0 & \text{for } a = D+1 \dots N. \end{cases} \tag{3.29}$$

  For $N = 6$ and $D = 5$ this MPS was studied in [225]. The exact overlaps were not found at that time, and they were only derived later in [27].

  In this case let us divide the components into two subsets $1 \dots D$ and $D+1 \dots N$. The solution has a block form, which means that $\psi_{ab}(u)$ is zero if a and b are from different subsets. Furthermore

$$\psi_{ab}(u) = \begin{cases} (-2u + C)\gamma_a\gamma_b + (2u^2 + (D-2C)u)\delta_{ab} \text{ for } a = 1 \dots D \\ u(D-2u)\delta_{ab} \text{ for } a = D+1 \dots N. \end{cases} \tag{3.30}$$

It can be checked that all of these solutions satisfy both the BYB and the square root relation. They were found by solving the latter.

The above formulas were used to clarify the representation theoretical origin of the solutions, and to find the exact overlap formulas. Furthermore they can be used to construct the Loschmidt amplitude, as explained below.

3.6 The Loschmidt echo and the Boundary QTM

In this Section we show how the Baxterization can be used to compute the Loschmidt amplitude and to give a solution to the quantum quenches.

We consider the evaluaton of the partition function (3.2) for some $\beta \in \mathbb{R}$. We will focus on the large volume behaviour, which is expected to be exponentially decaying in the volume:

$$Z(\beta, L) = \langle \Psi_0 | e^{-\beta H} | \Psi_0 \rangle \quad \sim \quad e^{-g(\beta) L}. \tag{3.31}$$

Here $g(\beta)$ is a generalization of the free energy, which depends on both the initial state and the Euclidean time s. In the literature it is often called *dynamical free energy*.

This quantity can show non-analytic behaviour with respect to the variable β. This happens because even though $Z(\beta, L)$ is analytic in β for any finite L, the $L \to \infty$ limit typically does not commute with analytic continuation in β. The non-analytic points of the dynamical free energy are called *dynamical phase transitions*. In certain cases they are connected to underlying quantum phase transitions, when the quantum quench connects models from topologically different quantum phases [239, 240]. However, sometimes they can appear even "accidentally". Dynamical phase transitions were observed experimentally in [241].

In integrable spin chains (3.31) can be evaluated by the same methods that we described in Section 2.6 for the thermal partition functions. The action of the thermal operator $e^{-\beta H}$ is computed in a Trotter approximation, and the approximate partition functions

$$Z_N(\beta, L) = \langle \Psi_0 | \left(t^{-1}(0) t(-c\beta/N) \right)^N | \Psi_0 \rangle \tag{3.32}$$

are then evaluated in the crossed channel. We remind that here $t(u)$ is the fundamental transfer matrix, N is the Trotter number, and c is just a model-dependent constant.

The new addition here is the presence of the initial and final states, which in the rotated channel become boundaries in space, as explained in 3.2. It was first shown in the work [3] by the author that boundary states constructed from K-matrices as given by (3.13) lead to the double row transfer matrices known in boundary integrability. Thus the work [3] laid the foundation for understanding the integrability of the initial states. It is interesting that this work appeared 4 years before the integrability conditions were understood and 2 years before the GGE for the Heisenberg chain was clarified. The work [3] focused on the Néel and Dimer initial states.

Now we give the main formulas for this construction, already allowing for integrable MPS, and not just the two-site product states considered in [3]. Instead of the homogeneous chain it is useful to introduce an alternating sequence of inhomogeneities $(-u_1, u_1, -u_2, u_2, \dots, -u_{L/2}, u_{L/2})$. The parameters u_j will play the role of spectral parameters for double-row transfer matrices.

We define inhomogeneous transfer matrices as

$$\begin{aligned} t(v|u_1, \dots, u_{L/2}) &= \mathrm{Tr}\ T(v|u_1, \dots, u_{L/2}), \\ T(v|u_1, \dots, u_{L/2}) &= R_{0L}(v - u_{L/2}) R_{0,L-1}(v + u_{L/2}) \dots R_{02}(v - u_1) R_{01}(v + u_1). \end{aligned} \tag{3.33}$$

Here the L inhomogeneities are of alternating signs, and in the notation we write them as $(u_1, \dots, u_{L/2})$.

We also define inhomogeneous initial states as

$$|\Psi(u_1,u_2,\dots,u_{L/2})\rangle = \sum_{i_1,\dots,i_L=1}^{N} \mathrm{tr}_0\left[\psi_{i_L,i_{L-1}}(u_{L/2})\dots\psi_{i_2,i_1}(u_1)\right]|i_L,\dots,i_1\rangle, \tag{3.34}$$

where $\psi_{jk}(u)$ is the Baxterization of the original, physical MPS, which are reproduced in the homogeneous limit $u_j \to 0$ due to the initial condition (3.19).

We define the inhomogeneous dual MPS vectors as

$$\langle\Psi(u_1,u_2,\dots,u_{L/2})| = \sum_{i_1,\dots,i_L=1}^{N} \mathrm{tr}_0\left[\psi_{i_L,i_{L-1}}(-u_{L/2})\dots\psi_{i_2,i_1}(-u_1)\right]\langle i_L,\dots,i_1|. \tag{3.35}$$

It is an important technical detail that the rapidity parameters are taken with a sign difference.

Let us consider the partition functions

$$\begin{aligned} &Z(v_1,\dots,v_m|u_1,\dots,u_{L/2}) = \\ &\qquad \langle\Psi(u_1,\dots,u_{L/2})|\prod_{j=1}^{m} t(v_j|u_1,\dots,u_{L/2})|\Psi(u_1,\dots,u_{L/2})\rangle. \end{aligned} \tag{3.36}$$

It can be shown that these are completely symmetric in both the u- and the v-parameters. Symmetry with respect to v_j, $j = 1\dots m$ follows from the commutativity of the transfer matrices, whereas symmetry with respect to u_j, $j = 1\dots L/2$ follows from the BYB equations [23].

A pictorial representation for these partition functions is given in Fig. 13. Notice that this construction is the lattice analogoue of the partition function drawn in Fig. 9.

The partition functions (3.36) allow for an alternative evaluation, which leads to the introduction of the double row Quantum Transfer Matrices, which act in the horizontal direction, see the pictorial representation Fig. 13. It can be read off Fig. 13 (or it can be established by purely algebraic means) that their explicit form is

$$\begin{aligned} &\mathcal{T}(u|v_1,\dots,v_m) = \\ &\sum_{a_1,a_2,b_1,b_2=1}^{N} \psi_{b_2,b_1}(u)\otimes\Big[T_{a_2b_2}(-u|-v_1,\dots,-v_m)T_{a_1b_1}(u|-v_1,\dots,-v_m)\Big]\otimes\psi_{a_2,a_1}(-u). \end{aligned} \tag{3.37}$$

Alternatively this can be computed as

$$\mathcal{T}(u) = \mathrm{Tr}\left(M(u)K^T(-u)\right), \tag{3.38}$$

where

$$M(u) = T(-u)K(u)T^T(u) \tag{3.39}$$

is the "quantum monodromy matrix", and the products and traces above are to be understood in the indices $a_{1,2}$, $b_{1,2}$ which originally label the states of the physical spin chain.

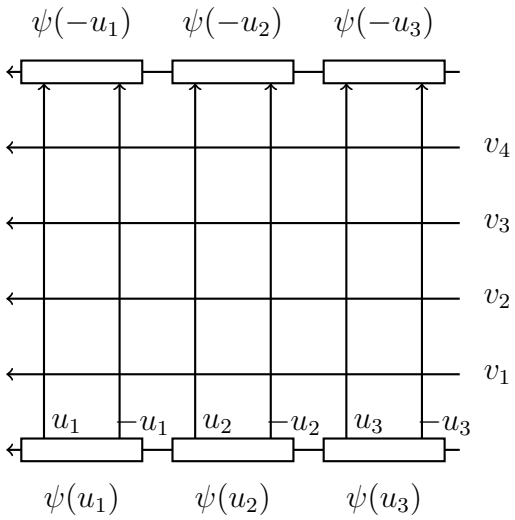


Figure 13: An example for a partition function with integrable boundaries. In the original physical picture the bottom and top rows are interpreted as the MPS which serve as the initial and final states for some discrete time evolution. Alternatively, the partition function can be evaluated by the Quantum Transfer Matrix, which acts in the horizontal direction: in this picture the two-site blocks play the role of integrable boundaries, with an additional degree of freedom at the boundary. The physical case of the original MPS (3.15) is obtained by setting $u_j = 1$, $j = 1, \ldots, L/2$, after using the factorization (3.19).

The partition function is then evaluated as

$$Z(v_1, \ldots, v_m | u_1, \ldots, u_{L/2}) = \mathrm{Tr}\left[\prod_{j=1}^{L/2} \mathcal{T}(u_j | v_1, \ldots, v_m)\right]. \tag{3.40}$$

The symmetry of Z with respect to an exchange of the u variables is equivalent to the commutativity condition

$$[\mathcal{T}(u_1), \mathcal{T}(u_2)] = 0, \tag{3.41}$$

which can be proven directly using the various Yang-Baxter relations.

The object $\mathcal{T}(u)$ can be called Boundary Quantum Transfer Matrix (BQTM) or simply just the crossed channel TM. It is evident from (3.40) that for fixed N the large L behaviour of the partition function is determined by the largest eigenvalues of the BQTM. Thus we need to understand how to compute these eigenvalues and investigate their $N \to \infty$ limits.

It was already observed in the thermal case that such Quantum Transfer Matrices are typically gapped in the strong sence, namely that they have a single leading eigenvalue, such that the gap remains finite even in the $N \to \infty$ Trotter limit. Furthermore, it was assumed that the $N \to \infty$ and $L \to \infty$ limits can be exchanged [50]. There is no general proof that this is always a valid step, but concrete computations confirm it in certain cases, for example it is rigorously proven the in the thermal case the correct result is obtained [242].

Similar behaviour was also observed in the case of the QTM. Numerical computations showed that the QTM remains gapped in some neighbourhood of $\beta = 0$ even if we allow complex values of β. Then the remaining task is to find exact formulas for the leading eigenvalue $\Lambda(s)$ of the QTM. In a proper normalization we have

$$g(\beta) = -\log(\Lambda(\beta)). \tag{3.42}$$

The diagonalization of Boundary TM's of the form (3.38) is a challenging problem for generic boundary K-matrices. There are a number of special cases when known methods can be applied, such cases will be reviewed below. However, there is no general recipe available. It is known that in the Heisenberg chain the diagonalization is easy if the K-matrix has scalar entries and is diagonal, see the original work [215] or [243] for a modern treatment. On the other hand, the case of generic off-diagonal K-matrices is more involved, see for example [244]. The situation is even more complicated for nested systems, already for the diagonal cases, see for example [245, 246].

Below we list a few examples where the BQTM could be solved with exact methods.

3.7 The Loschmidt echo and the solution of the quenches

The paper [3] considered the following special class of two-site states:

$$|\Psi_0\rangle = \otimes_{j=1}^{L/2} |\psi\rangle, \qquad |\psi\rangle = \frac{|\uparrow\downarrow\rangle - \alpha|\downarrow\uparrow\rangle}{\sqrt{1+|\alpha|^2}}, \qquad \alpha \in \mathbb{C}. \tag{3.43}$$

These correspond to diagonal K-matrices, if the connection (3.13) is used with the appropriate crossing matrix $C = \sigma_y$. The parameter α is arbitrary; the choices $\alpha = 0$ and $\alpha = 1$ produce the Néel and Dimer states, respectively.

In these cases the standard tools of boundary integrability could be used due to the diagonal nature of the K-matrices. We do not review the computation, instead we just cite the main result for the dynamical free energy. It was found that $g(\beta)$ can be expressed using a certain auxiliary function $\mathfrak{a}(u)$, which is a solution of a single non-linear integral equation (NLIE). This equation is a generalization of the corresponding NLIE from the thermal case, and it reads

$$\begin{aligned} \log \mathfrak{a}(u) = \log(k(u)) - 4\beta \frac{\sinh^2 \eta}{\sinh(u)\sinh(u+\eta)} - \\ - \int_{\mathcal{C}} \frac{d\omega}{2\pi i} \frac{\sinh(2\eta)}{\sinh(u-\omega+\eta)\sinh(u-\omega-\eta)} \log(A(\omega)), \end{aligned} \tag{3.44}$$

where the integral runs over a certain contour $\mathcal{C}$ which encirles all Bethe roots of the QTM; it depends on the anisotropy Δ, precise details can be found in [3]. Furthermore we defined

$$A(u) = \frac{1+\mathfrak{a}(u)}{1+k(u)}, \tag{3.45}$$

where $k(u)$ is a complex function which carries informaiton about the initial state. Finally the dynamical free energy is expressed as

$$g(\beta) = \frac{1}{2}\log\Lambda = \frac{1}{2}\int_{\mathcal{C}} \frac{d\omega}{2\pi i} \frac{\sinh\eta}{\sinh(\omega)\sinh(\omega+\eta)} \log(A(\omega)). \tag{3.46}$$

For the Néel state it was found that

$$k(u) = \frac{\sinh(\lambda_j + \eta)}{\sinh(\lambda_j - \eta)} \frac{\sinh(2\lambda_j - \eta)}{\sinh(2\lambda_j + \eta)}.$$

We refrain from giving the general formula for arbitrary α.

It was shown in [3] that this system of equations allows for an effective numerical treatment and they reproduce some known results in certain limits.

Around the same time a similar work appeared as an e-print [247], which proposed different equations for the same quantity, yielding different results. As far as we know, the work [247] was eventually not published in a journal.

An alternative method for finding $g(\beta)$ was developed in the sub-sequent work [13], written in collaboration with international researchers. Here the idea was to use to fusion hierarchy (T-system) of the Boundary QTM's to derive TBA-type equations for $g(\beta)$, which would eventually become generalizations of (3.49). The idea of using the T-system to derive TBA equations was not new, this goes back to [248]. However, earlier work did not consider the same methods for the boundary cases.

The first step is to show that there is a fusion hierarchy of the BQTM's, a set of higher TM's denoted as $\mathcal{T}_s(u)$, satisfying the Hirota equations (2.46) (the work [13] only focused on the XXZ chain with $\Delta > 1$; the $\Delta < 1$ case needs to be treated separately, leading to slighlty different Hirota equations). This T-system for boundary situations was already considered earlier in [249], and it was shown there that if the Lax operators use the same normalization as in the bulk cases, then the scalar functions $\Phi_j(u)$ entering the Hirota equations carry information about the boundary K-matrices.

The second step is establishing the so-called Y-system, which is also well known step; the Y-system is a set of functional relations encoding the structure of the TBA equations [188, 248]. For the Heisenberg chains in the cases mentioned above it reads

$$Y_j(u + i\eta/2)Y_j(u - i\eta/2) = (1 + Y_{j+1}(u))(1 + Y_{j-1}(u)) \tag{3.47}$$

with the initial condition $Y_0(u) = 0$. The Y-functions are constructed from the T-functions as

$$Y_j(u) = \frac{T_{j+1}(u)T_{j-1}(u)}{\Phi_j(u)}, \tag{3.48}$$

where $\Phi_j(u)$ is the scalar function mentioned above. The equivalence between (2.46) and (3.47) is easy to prove, and is discussed at length in the review [188]. In our case the T-functions are given by the eigenvalues of the fused BQTM's $\mathcal{T}_j(u)$.

The third step is transforming the Y-system into integral equations of the TBA type. This step was first performed in [248] for the thermal case. Similar ideas were also present in [196], and the connection between the Y-system and the TBA is discussed in many papers dealing with integrable QFT [119] and with the AdS/CFT correspondence [72]. The key idea is that the functional connections of the Y-system correspond to the convolution terms in the TBA, and the analytic properties (presence of certain poles or zeros, and possibly the asymptotic behaviour for large rapidities) determine the source terms of the TBA.

In [13] we performed the same analysis for the BQTM's relevant to the Loschmidt amplitude. It was found that in some neighbourhood of $\beta = 0$ the resulting TBA equations

are of the form

$$\log(Y_n(\lambda)) = -2\beta s(\lambda)\delta_{n,1} + d_n(\lambda) + \\ + \int_{\omega=-\infty}^{\infty} \frac{d\omega}{2\pi} s(\lambda-\omega)(\log(1+Y_{n-1}(\omega)) + \log(1+Y_{n+1}(\omega))). \tag{3.49}$$

Here the first source term is only present for the first node, in accordance with the TBA source term in the thermal case, see (2.30). For the XXX case the function $s(\lambda)$ is given by (1.65). In the XXZ case an analogous equation holds, and the s-function can be found in [13]. The additional source terms $d_n(\lambda)$ above depend on the initial state, some concrete examples (including new cases) were given in [13]. Finally, the eigenvalue of the BQTM is given by

$$\log(\Lambda) = \int \frac{du}{2\pi} s(u) \left[2\beta a(u) + \log\left(\frac{1+Y_1(u)}{1+Y_1^{(0)}(u)} \right) \right]. \tag{3.50}$$

Here $a(u)$ is some known function, $Y_1(u)$ comes from the solution of (3.49), and $Y_1^{(0)}(u)$ corresponds to the solution at $\beta = 0$ (see below).

The structural form of (3.49) is rather similar to the o-TBA equations (2.32); the only difference is the addition of the thermal term. In fact, if the overlaps are known, then (3.49) can be derived with the Quench Action method. However, we stress that the work [13] did not use the TBA reasoning, only the rigorous T-system and Y-system equations (and some numerical analysis about the poles/zeroes of the functions involved). The source terms $d_n(\lambda)$ are determined from the analytic structure, and not from the overlaps. The identification of the Y-functions of (3.49) (which come from an algebraic set of functional relations) and the η-functions of (2.32) (which come from the Quench Action method, using statistical reasoning) is made only afterwards. For the thermal case this correspondence was first observed in [248] , and our work explained the same connection for the quench problem.

It is very important to notice that the $\beta \to 0$ limit we reproduce the Quench Action TBA equations. Thus in this trivial limit, which corresponds to the a very complicated evaluation of the simple identity

$$1 = \langle \Psi_0 | \Psi_0 \rangle, \tag{3.51}$$

we get back to the solution of the quantum quench. This was exploited in our work [13]. The trivial limit can be achieved with setting the Trotter number to $N = 0$ (thus having no transfer matrices in (3.32)); in this patological case the T-functions and Y-functions of the BQTM can be computed exactly without the need for any Bethe Ansatz solution of the BQTM. **Thus the exact solution of the quench can be given simply based on the T-system and Y-system**.

The same ideas were used in [196, 213] to find exact solutions for the quenches from the Néel state, and these works *assumed* the T-system and Y-system equations. However, it was only our work which clarified, that these equations hold only because the initial state in question is integrable. The Y-system equations are explicitly broken in all other cases, as it is discussed in [194].

Here we refrain from giving the explicit formulas for the Y-functions in terms of the initial state, we simply state that once the initial state is specified and the corresponding K-matrix is known, then the Y-functions can be computed in a straightforward way.

Concrete examples can be found in [13]. The method also works for integrable MPS, and for the more complicated nested systems, although the concrete computations can become lengthy. We will return to the importance of the Y-system equations and (3.49) in Section 4: we will show there that (3.49) carry enough information to determine the overlaps themselves.

It is important that the equations (3.49) do not hold for arbitrary β, because analytic properties of the functions involved can change with β. For example, singularities crossing the integration contours lead to additional source terms; this is a phenomenon known from iQFT, see for example [250]. This was also observed in our sub-sequent work [18], where explicit examples for this were given. Furthermore, the phenomenon of dynamical phase transition (DPT) was also studied there. It was shown, that level crossings of the BQTM can be observed with these techniques. As explained above, a level crossing leads to a non-analytic behaviour in $g(\beta)$, because this quantity always follows the larges eigenvalue of the BQTM. Regarding the physical interpretation of the DPT we did not find a general conclusion.

In the works [21, 22] we also considered the integrable states of the $SU(3)$-symmetric fundamental model. As explained above, in this case the integrable product states are either symmetric or anti-symmetric with respect to space reflection. The anti-symmetric case is somewhat trivial, so we focused on the symmetric case with the two-site block given by

$$|\psi\rangle \quad \sim \quad \alpha|11\rangle + \beta|22\rangle + \gamma|33\rangle, \qquad \alpha, \beta, \gamma \in \mathbb{C}, \tag{3.52}$$

where $|n\rangle$ with $n = 1, 2, 3$ is a basis of $\mathbb{C}^3$.

We constructed the fusion hierarchy of the BTQM's in this model, and many of the technical details were new. We also computed the exact solution for the quenches. It was already explained in Sec. 2.9 that in this model the fusion hierarchy involves two families of fused transfer matrices $t_s^{(1)}(u)$ and $t_s^{(2)}(u)$. In accordance with this there are two families of Y-functions, which we denote as $Y_s^{(1)}(u)$ and $Y_s^{(2)}(u)$. The precise relations between the T- and Y-functions can be found in [21, 22]. Instead of going into the many technical details, we just give a simple result for the case $\alpha = \beta = \gamma = 1$, which is the so-called δ-state. It was found that in this case

$$1 + Y_1^{(1)}(u) = 1 + Y_1^{(2)}(u) = 3\frac{u^2 + 1/4}{u^2}. \tag{3.53}$$

A new technical step of the works [21,22] was showing that even the root distributions $\rho_s^{(1,2)}(u)$ can be computed directly from the BQTM. First we can notice that in each model there is a relation of the form

$$\rho_s^{(a)}(u) \sim \frac{\partial Y_s^{(a)}(u)}{\partial \beta}. \tag{3.54}$$

The proportionality factor depends on some conventions. For the XXZ case this relation is seen directly by comparing (3.49) with (1.64).

As a second step it was shown the derivative above can be computed from a BQTM's with Trotter number $N = 1$, whose diagonalization is relatively easy even with exact diagonalization, without using integrability methods. This method was worked out for the δ-state and exact formulas were given for the first two root distributions.

With this we showed that **for integrable initial states the quantum quench can be solved completely, without using the concept of the GGE, or the string-charge relations, or the Quench Action method**. In fact, the results of [21, 22] were a central motivation behind our work [190] discussed in Section 2.9. We believe that the concept of integrable initial states is more fundamental than the Quench Action itself, because the applicability of the QA hinges on the existence of factorized overlap formulas, which only exist for integrable states (see Section 4).

3.8 Further possibilities for integrable states

Following the work of the author and collaborators the concept of integrable states was generalized even further by other researchers. In [251] an interesting overlap was studied in the $SO(6)$-symmetric chain, where the pair structure for the rapidities is even more "twisted": rapidities are paired from different levels of the nesting procedure. The explanation for such a behaviour was given in [227]. It was found that in models with higher rank symmetries the possibilities for integrability conditions are wider than simply requiring (3.10). For different transfer matrices of the fusion hierarchy different types of conditions can be required, switching between the "twisted" and "original" BYB. Here we do not discuss these possibilities in detail, we refer the reader to [227].

3.9 Entanglement evolution

The pair structure for the overlaps is one of the most important consequences of integrability of the initial state. From a physical point of view it means that the initial state only emits particle pairs, which start to propagate in opposite direction. The particles within the pairs are correlated with each other, but at $t = 0$ the pairs themselves are not correlated. This follows from the fact that the overlap consists of pair amplitudes.

These simple observations were used to derive a semi-classical result for the entanglement evolution after the quantum quenches [206]. The idea is to consider the mutual entanglement between a segment of length l and the rest of the system. In the simplest case of $l = \infty$ we consider the mutual entanglement between two half-infinite parts of the chain. At $t = 0$ the two halves are typically unentangled, and for large times the entanglement will grow linearly with time. It was shown in [206] that the slope of this linear growth can be computed exactly using a simple semi-classical picture, based on the pair structure. The idea goes back to similar computations in Conformal Field Theory (CFT) [252]: it is argued that the entanglement between two systems A and B at some time t is caused by those particle pairs, which are created somewhere in the system at $t = 0$ and from which one member propagates to sub-system A and the other one to sub-system B at time t. This reasoning uses a semi-classical picture for the propagation of quasi-particles. If the speed of propagation is known, then concrete formulas can be derived for the entanglement production. For the details see the works [206, 253]; we note that even nested spin chains were treated with this method, see [204].

Here we do not discuss the formulas for the entanglement, we simply just stress that it is the integrability condition which underlies these successful computations. As far as we know the entanglement evolution of non-integrable states has not yet been considered in

the literature. Naturally we expect the same qualitative behaviour, but it is not obvious whether exact formulas exist for the asymptotic growth.

3.10 Open problems

All the methods that we discussed in this Section contribute to the solution of the quantum quenches. However, we did not find observables which would show a qualitative difference between the integrable and non-integrable initial states. On a qualitative level the behaviour of the correlation functions is rather similar for all types of quenches. The special techniques discussed in this Section *make it possible* to find analytic solutions, but there is no marked difference in the qualitative behaviour of the local observables. It is an open question whether there are any measurable quantities which would make a clear distinction between integrable and non-integrable quenches.

A further open problem in this topic is the complete classification of one-site invariant integrable MPS with a given symmetry class (G, G'), where the Lie groups G and $G' \subset G$ form a symmetric pair. As discussed above, it is now known that they correspond to representations of the twisted Yangian. In principle these representations are known: they are obtained from a dressing of some fundamental solutions, so that the dressing is given by local Lax operators (for the details see our work [23]). However, these solutions are generally two-site invariant and it is generally not known which ones produce one-site invariant MPS in the homogeneous limit. But one might argue that finding a classification is not so important, given that the relevant examples are provided by AdS/CFT anyway, and at present there is no other motivation to study these more complicated integrable states.

An other open problem is that so far the understanding of integrable initial states is limited to local spin chains with nearest neghbour interactions. It is known that long range spin chains are relevant to AdS/CFT and there are other long range models (such as the Haldane-Shastry or the Inozemtsev chains [198–200]) which have been studied thoroughly. At present it is not known how to extend our construction to these models. This problem deserves further investigation, especially in light of the interest from the AdS/CFT community (see the references also in the next Section).

4 Exact overlaps

4.1 Introduction

In this Section we consider exact overlaps. First we focus on the simple cases when the Bethe states can be characterized by a single set of rapidities $\boldsymbol{\lambda}_N$. It is our goal to find exact formulas for the overlaps

$$\langle \Psi_0 | \boldsymbol{\lambda}_N \rangle, \tag{4.1}$$

where $|\Psi_0\rangle$ is an integrable initial state.

The reader might wonder what is the reason for setting this goal. After all, at the end of the last Section we explained that the quantum quenches can be solved with the BQTM methods, without any information about the overlaps. We can give the following two motivations for the overlap computations.

First of all, there are naturally the historical reasons: Before 2014-2015 the only way to give an exact solution was the Quench Action method, which relied on the knowledge of the overlaps. This motivated the first work of the author on the topic [4], and also the excellent work [157] of the Amsterdam group, which solved an interaction quench in the Lieb-Liniger model. Furthermore, the overlaps of the spin chain played a central role in the study of the GGE, as discussed in Section 2.6.

Soon afterwards most of the experts working with non-equlibrium dynamics lost their interest in the exact overlaps once the status of the GGE was clarified. The interest of the community turned to the transport phenomena, and Generalized Hydrodynamics (GHD) was born (see Section 6). Exact overlaps were not used in GHD.

However, around the same time (2015-2016) came the new source of inspiration from AdS/CFT. As discussed in Section 3.5 it turned out that in this correspondence the one-point functions of boundary CFT are described by exact overlaps with integrable initial states. This line of research was continued until the present day, and rather involved overlap formulas were derived along the way. The eventual goal is to compare these one-point functions to string theory predictions. This explains the interest of the researchers in the full exact solutions in the very complicated, long range spin chains, that describe the operators of AdS/CFT at higher loops; for very recent papers on the subject see [227, 254–257]. We should also note that certain three point functions are also described by exact overlaps, see [251, 258].

In the overlap computations to be discussed the starting point is the state $|\Psi_0\rangle$, where we can assume that the integrability condition is proven and the corresponding K-matrix (solution to the BYB equations) is known. Then the task of computing the overlaps from the K-matrices is very similar to what was achieved by Ghoshal and Zamolodchikov, see Section 3.2. In particular, it is desirable to find something similar to eq. (3.6) which connects the boundary data (K-matrix in our case) to the overlap data (pair amplitudes). It turns out that in the spin chains these relations are much more involved, and can not be summarized in a simple equation like (3.6).

4.2 Exact overlaps - first results

Up to our knowledge the first paper to compute the overlaps in interacting spin chains was [4] by the author [4] (examples for overlaps in free fermionic models were known even earlier, but we do not review these papers here). This paper predates the integrability condition laid down in [15], nevertheless the connection between K-matrices and the initial states was already given here. The particular object considered here was the overlap between the Bethe states of the XXZ chains and the special states given by (3.43) which correspond to diagonal K-matrices.

In this work the *off-shell overlaps* were considered. An off-shell state is an element of the Hilbert space of the form (1.47) where the rapidities do not satisfy the Bethe equations. Thus such states are not eigenstates of the transfer matrices. It follows that the integrability condition can not be applied to the states, and the overlaps will not display the pair structure.

[4] Personal comment: From all my papers this is the only one which was mentioned as reference [1] in an other work. See [259].

In [4] an exact determinant formula was derived for such off-shell overlaps. The result is basically the so-called Tsushiya-determinant, first derived in [260] in the framework of boundary integrability. This result was only an intermediate step for the follow-up works, therefore we do not give the corresponding formulas here. Nevertheless we note the determinant in question was transformed into a more convenient form in [261], which treated the boundary free energy of the XXZ chains.

Afterwards a factorized overlap formula appeared in [157], which described overlaps with the Bose Einstein condensate (BEC) in the Lieb-Liniger model. This result was based on coordinate Bethe Ansatz computations. Finally the work [214] (also by the Amsterdam group) derived a factorized overlap formula in the XXZ chain, building on the results of our work [261]. Now we describe this overlap formula; we put forward that it already displays almost all features of the more complicated cases.

The paper [214] considered the on-shell states, where the rapidities satisfy the Bethe equations. For the first time it was proven (withouth the integrability condition) that the non-zero overlaps have the pair structure. For simplicity we focus on states without the special rapidities, thus we assume that the set of rapidities consists of pairs only. We will denote

$$\{\boldsymbol{\lambda}_N\} = \{\boldsymbol{\lambda}^+_{N/2}\} \cup \{-\boldsymbol{\lambda}^+_{N/2}\}. \tag{4.2}$$

The paper [214] considered the Néel state $|\Psi_0\rangle = |N\rangle$ and the overlap was found to be

$$\frac{|\langle\Psi_0|\{\pm\lambda^+\}_{N/2}\rangle|^2}{\langle\{\pm\lambda^+\}_{N/2}|\{\pm\lambda^+\}_{N/2}\rangle} = \prod_{j=1}^{N/2} u_{\text{Néel}}(\lambda^+_j) \times \frac{\det_{N/2} G^+_{jk}}{\det_{N/2} G^-_{jk}}\,, \tag{4.3}$$

where $u_{\text{Néel}}(\lambda)$ is the pair amplitude given by

$$u_{\text{Néel}}(\lambda) = \frac{\tan(\lambda_j + i\eta/2)\tan(\lambda_j - i\eta/2)}{4\sin^2(2\lambda_j)} \tag{4.4}$$

and the $N/2 \times N/2$ matrices $G^\pm$ are called Gaudin-like determinants and are given by

$$G^\pm_{jk} = \delta_{jk}\left(-L\varphi_{\eta/2}(\lambda^+_j) + \sum_{l=1}^{L/4}\varphi^+_\eta(\lambda^+_j,\lambda^+_l)\right) - \varphi^\pm_\eta(\lambda^+_j,\lambda^+_k) \tag{4.5}$$

with

$$\begin{aligned}\varphi^\pm_\eta(\lambda,\mu) &= \varphi_\eta(\lambda-\mu) \pm \varphi_\eta(\lambda+\mu)\\ \varphi_x(\lambda) &= \frac{\sinh(2x)}{\sinh(\lambda+ix)\sinh(\lambda-ix)}\,.\end{aligned} \tag{4.6}$$

For states with the pair structute the full Gaudin determinant factorizes as

$$\det G = \det G^+ \det G^-. \tag{4.7}$$

Some important remarks are in order. First, it can be shown that the ratio of the determinants remains finite even in the TDL; thus they only contribute an $\mathcal{O}(1)$ piece to $\log(|\langle\Psi_0|\boldsymbol{\lambda}_N\rangle|^2)$, while the product of the pair amplidutes is extensive in the volume and thus most relevant.

Second, this form is reminiscent of the results of Ghoshal and Zamolodchikov, except these determinants. In fact, the formal squeezed state (3.5) implies that for multi-particle states the total amplitude will be given by the product of the pair amplitudes.

Quite interestingly, these determinants appeared already in [262], where the overlaps with finite volume boundary states were considered, focusing only on integrable QFT. In other words, [262] constructed the finite volume version of the squeezed state (3.5) of Ghoshal and Zamolodchikov. The paper [262] does not include explicit matrices for all N, but the concrete formulas up to $N = 4$ and the reasoning behind them makes it clear that [262] had the same results. Unfortunately, that work only treated QFT situations and the results about the determinants were forgotten, even by the author. Therefore, the paper [262] from 2010 did not have any influence on the overlap computations.

Despite the lack of historical continuity, here we summarize the key argument of [262], which gives a fairly straightforward (but not rigorous) derivation of the Gaudin-like determinants. Consider the boundary one-point function in integrable QFT

$$\langle \Psi_0 | O(x,t) | 0 \rangle, \tag{4.8}$$

where $O(x,t)$ is any local operator, the boundary state $|\Psi_0\rangle$ is given by (3.5), and the system is directly in infinite volume. We can also consider the same quantity in the finite volume situation, where (3.5) is replaced by a finite volume boundary state with unknown overlaps.

The one-point function can be expanded into a spectral series by inserting a complete set of states between $|\Psi_0\rangle$ and $O(x,t)$:

$$\langle \Psi_0 | O(x,t) | 0 \rangle = \sum_{\boldsymbol{\lambda}_N} \langle \Psi_0 | \boldsymbol{\lambda}_N \rangle \langle \boldsymbol{\lambda}_N | O(x,t) | 0 \rangle, \tag{4.9}$$

where the sum runs over all N and all Bethe states. We observe the emergence of the overlaps (in both finite and infinite volumes) and of the form factors

$$\langle \boldsymbol{\lambda}_N | O(x,t) | 0 \rangle, \tag{4.10}$$

once again, both in finite and infinite volumes. The space and time dependence can be separated using the energy and momentum of the multi-particle states, but this is not important at the moment.

It can be argued that for any massive theory the finite size effects can be only exponentially small in L. This should hold separately for the contribution of each N-article sector in the spectral sum. Eventually this will give a relation between the finite volume and infinite volume overlaps. It is known that the ratio of the finite and infinite volume form factors of the type (4.10) is given by the square root of the Gaudin determinant (see eq. (6.19) below, or [263]), and this factor has to be present also in the finite volume overlap in order to have it canceled. But one should also take into account that in finite volume we have a sum over the discrete Bethe states, while in infinite volume we have some fixed single particle integration measures. If the volume is large enough, the summation over Bethe states can be replaced by integrals, given that we take into account the density of states up to all orders in $1/L$, leaving only exponentially small corrections to the integrals. For the overlaps in question only the states with the pair structure contribute, thus we need their density of states. Similar to (1.52) this is given by a Jacobian, which

is actually $\det G^-$. Putting everything together, the finite volume overlap formula should have a factor of

$$\frac{\sqrt{\det G}}{\det G^-} \tag{4.11}$$

on top of the product of pair amplitudes. For the squared overlap we obtain an extra factor of

$$\frac{\det G}{(\det G^-)^2} = \frac{\det G^+}{\det G^-}, \tag{4.12}$$

and this is indeed present in (4.3). We remind that (4.3) was found from a direct and rigorous computation in the spin chains, whereas the argument above is not completely rigorous. Nevertheless the same result is obtained.

This argument shows that the overlaps with an arbitrary integrable state $|\Psi_0\rangle$ should alwasy have a form

$$\frac{|\langle\Psi_0|\{\pm\lambda^+\}_{N/2}\rangle|^2}{\langle\{\pm\lambda^+\}_{N/2}|\{\pm\lambda^+\}_{N/2}\rangle} = Z(N,L) \times \prod_{j=1}^{N/2} u(\lambda_j^+) \times \frac{\det_{N/2} G^+_{jk}}{\det_{N/2} G^-_{jk}}\,, \tag{4.13}$$

where $u(\lambda)$ is the pair amplitude and $Z(N,L)$ is a factor that does not depend on the rapidities. These ingredients depend on the initial state, whereas the ratio of Gaudin-like determinants should be universal.

We will see that this statement holds for all two-site product states in the XXZ model, and many other cases even in more complicated models; in nested Bethe Ansatz the $G^\pm$ matrices have to be replaced by the appropriate generalizations. However, in the case of integrable MPS' it is not guaranteed that the exact formula is so simple: in the most general case we can have formulas of the type

$$\frac{|\langle\Psi_0|\{\pm\lambda^+\}_{N/2}\rangle|^2}{\langle\{\pm\lambda^+\}_{N/2}|\{\pm\lambda^+\}_{N/2}\rangle} = P(\boldsymbol{\lambda}_N) \times \frac{\det_{N/2} G^+_{jk}}{\det_{N/2} G^-_{jk}}\,, \tag{4.14}$$

where $P(\boldsymbol{\lambda}_N)$ can be sums of products of the type

$$\prod_{j=1}^{N/2} u(\lambda_j^+) \tag{4.15}$$

for rapidities from various levels of the nesting. Concrete examples will be given below.

Having understood the general structure of the integrable overlaps, the remaining task is to determine the pair amplitudes using the initial state (or the corresponding K-matrices). This will generalize the classical result (3.6). There are two possible types of derivations: the first (and easier) way is to assume that a formula like (4.14) holds, and then the task is to fix its ingredients. The second, much more involved way is to actually prove the full determinant formula from scratch. As far as we know this has only been done in models related to the $SU(2)$-group, and there is not a single case in nested Bethe Ansatz when the overlap formula was actually proven for all particle numbers. Existing computations involve the coordinate Bethe Ansatz, with which only low particle numbers can be reached. Most of the works of the author also follow the first way, although in [28] we also made a step towards rigorous proofs in more complicated cases. This line of research is still active.

4.3 Exact overlaps from TBA

The paper [214] derived the overlaps with the Néel state, and this result was easily extended to states of the form (3.43) in [9] using a trick of [4]. These states correspond to diagonal K-matrices, and at that time it was not clear, whether a formula like (4.13) could hold also for the non-diagonal case. This was later settled in [16], where the factorized overlaps were computed for an arbitrary two-site product state.

The paper [16] *assumed* that a formula like (4.13) exists and it fixed the functions involved using a combination of the overlap based TBA and BQTM methods. The results were then checked by comparing it to numerical data from exact diagonalization. As mentioned above, the sub-sequent work [28] also presented rigorous proofs in some new cases with non-diagonal K-matrices, but first we review the techniques of [16].

The central tool of [16] is the set of TBA equations for the solution of the quench. It was explained in the Sections above, that this TBA can be derived in two ways:

- Eq. (2.32) can be derived using the Quench Action method, given that the overlap has a factorized structure.
- Eq. (3.49) can be derived for the Loschmidt amplitude using the BQTM, and the $\beta \to 0$ limit corresponds to the solution of the quench, when we can identify $\eta_n(\lambda) = Y_n(\lambda)$.

The BQTM method implies the Y-system equations, and the identification also means that the Y-system equations hold for the η-functions of the Quench Action method. This was also proven in a more direct way in [16]: it was shown that if the source terms come from factorized overlaps, then the Y-system is automatically satisfied. Thus **factorized overlaps can only exist for those intial states where the η-functions satisfy the Y-system**. This seemingly very technical statement is actually important: it tells us that we can not expect to find *algebraically factorized* overlaps for states which are not integrable. This was not clear in the literature, and there was an expectation in the community that overlap formulas might exist even in the non-integrable cases, see for example the unsuccessful attempt [264] to find such formulas.

The idea of [16] is to have a direct comparison of the source terms $d_n(\lambda)$. On the one hand side they can be obtained from the pair amplitude $u(\lambda)$ through (2.33)-(2.34). On the other hand, they can be derived from (3.49) as well, using the exact solution for $Y_n(\lambda)$ from the BQTM method. This comparison results in a relatively simple condition: It turns out that the two sets of equations (2.32) and (3.49) are identical, if the functions

$$h_j(\lambda) = u_j(\lambda)Y_j(\lambda), \qquad u_j(\lambda) = \prod_{k=1}^{j} u(\lambda + i\eta(n+1-2k)/2) \tag{4.16}$$

are free of singularities on the physical strip. Note that here $u_j(\lambda)$ is the pair amplitude for the j-strings.

The Y-functions can be computed from the BTQM; it is actually enough to compute $Y_1(\lambda)$, the remaining ones are determined recursively by the Y-system equations. Then the conditions for $h_j(\lambda)$ completely fix the poles and zeroes of $u(\lambda)$. The important point is that each $h_j(\lambda)$ only tests a region $|\Im(\lambda)| < \eta/2$, but they depend on the original u-functions with shifted variables, therefore they scan the whole rapidity space. In [16] only

the function $h_1(\lambda)$ was considered, while higher functions were needed in the sub-sequent work [27].

Let us present here the main result of [16]. We parametrize the un-normalized two-site block with three parameters α, β, θ as

$$\begin{aligned}
\psi_{11}(\alpha,\beta,\theta) =& -e^{\theta}\sinh(\eta)\\
\psi_{12}(\alpha,\beta,\theta) =& 2(-\sinh(\alpha)\cosh(\beta)\cosh(\eta/2)+\cosh(\alpha)\sinh(\beta)\sinh(\eta/2))\\
\psi_{21}(\alpha,\beta,\theta) =& 2(\sinh(\alpha)\cosh(\beta)\cosh(\eta/2)+\cosh(\alpha)\sinh(\beta)\sinh(\eta/2))\\
\psi_{22}(\alpha,\beta,\theta) =& e^{-\theta}\sinh(\eta).
\end{aligned} \tag{4.17}$$

This is not an ad hoc parametrization, it follows from (3.13) and the characterization of K-matrices [216]. The normalized overlap is found to be

$$\frac{|\langle\Psi_0|\{\pm\lambda^+\}_{N/2}\rangle|^2}{\langle\{\pm\lambda^+\}_{N/2}|\{\pm\lambda^+\}_{N/2}\rangle} = \frac{|e^{\theta(L-2N)}|\sinh^L(\eta)}{|\mathcal{N}|^{L/2}}\prod_{j=1}^{N/2}u(\lambda_j)\times\frac{\det_{N/2}G^+_{jk}}{\det_{N/2}G^-_{jk}} \tag{4.18}$$

with $u(\lambda)$ given by

$$u(\lambda) = \frac{v^s_\alpha v^s_{\alpha^*} v^c_\beta v^c_{\beta^*}}{v^s_{\eta/2}v^c_{\eta/2}v^s_0 v^c_0}, \tag{4.19}$$

where we introduced the short-hand notation

$$v^s_\kappa(\lambda) = \sin(\lambda+i\kappa)\sin(\lambda-i\kappa) \qquad v^c_\kappa(\lambda) = \cos(\lambda+i\kappa)\cos(\lambda-i\kappa). \tag{4.20}$$

The normalization factor $\mathcal{N}$ is simply

$$\mathcal{N} = \sum_{j,k=1}^{2}|\psi_{jk}|^2. \tag{4.21}$$

The formula (4.19) connects the initial state (or equivalently, the parameters of the boundary K-matrix) to the pair amplitude in the overlap. It can be considered a generalization of (3.6). The relation is more complicated, because in the spin chain the K-matrix does not immediately describe reflection factors, moreover the non-diagonal K-matrices correspond to boundaries that break particle number conservation.

4.4 Overlaps for AdS/CFT

As mentioned above, there are a number of cases when the spin chain overlaps are relevant to AdS/CFT. In lots of cases coordinate Bethe Ansatz computations were used by other researchers to try to find the overlaps [221–225]. The idea is to assume a formula like (4.14) and try to find the function $P(\boldsymbol{\lambda}_N)$, which is assumed to be a polynomial in the products of the form (4.15). If this function is simple, then there is a good chance that explicit computations with low particle numbers can determine it. However, for complicated cases this can turn out to be impossible.

One such case was encountered in [225], where coordinate Bethe Ansatz did not lead to any compact formula, despite considerable effort of the researchers. Nevertheless, after joining forces we were able to find the compact and exact formula in [27]. This required a

combination of the BQTM method described above and the representation theory of the twisted Yangian. Now we describe this result, but we restrict the technical details to a minimum.

The overlap in question involves the $SO(6)$-symmetric fundamental model, with the Hamiltonian given by (1.10). It is known that this model is solvable by the nested Bethe Ansatz with two levels of nesting. This structure is a result of the identification of Lie algebras $\mathfrak{so}(6) \simeq \mathfrak{sl}(4)$; the defining representation of $SO(6)$ can be identified with the anti-symmetric tensor representation of $SU(4)$. Then the standard formulas of the $SU(N)$-symmetric models can be used. In accordance, there are three types of rapidities, which can be denoted as

$$\{\mathbf{u}_{N_0}, \mathbf{v}_{N_+}, \mathbf{w}_{N_-}\}. \tag{4.22}$$

Bethe equations for them can be found in the paper [27].

The overlaps in question involve states with residual symmetry $SO(5)$, thus they belong to the class $(SO(6), SO(5))$. The one-site invariant MPS are given by

$$\omega_j = \begin{cases} G_j & \text{if } j < 6 \\ 0 & \text{if } j = 6, \end{cases} \tag{4.23}$$

where the matrices G_j, $j = 1, \dots, 5$ are symmetrically fused gamma matrices in dimension 5. In the simplest case with no fusion $G_j = \gamma_j$, $j = 1, \dots, N$, and this corresponds to the case given earlier in (3.29).

We constructed the Baxterization for these MPS and computed the corresponding Y-functions. However, it was known from coordinate Bethe Ansatz, that the overlaps are rather complicated: they involve some unknown polynomial P like in (4.14) and are not of a simple product form. In these cases the BTQM and TBA methods can only fix the leading term in the polynomial. We performed this computation, we found the leading term, which was confirmed by coordinate BA to be present in the formula. But this did not help finding the remaining terms.

The solution was to use the representation theory of the twisted Yangian, which is the abstract algebraic structure behind the twisted BYB, and to relate the complicated cases to more simple, fundamental overlaps. This part of the computation was the work of Tamás Gombor. The method is an extension of the highest weight method to the Yangian, where the weights are not just numbers but polynomials in the rapidity variable. We do not review this computation here, we merely cite the main result.

Let $|\Psi_\gamma\rangle$ be the MPS constructed from the 5 dimensional Gamma matrices, and let $|\Psi_0\rangle$ be a one-site invariant product state given by the vector

$$\omega_j = \begin{cases} 0 & \text{if } j < 6 \\ 1 & \text{if } j = 6, \end{cases} \tag{4.24}$$

Note that quite naturally this vector also enjoys $SO(5)$-symmetry. It was found by Tamás Gombor is our work [27] that in some appropriate normalization

$$\langle \Psi_\gamma | \{\mathbf{u}_{N_0}, \mathbf{v}_{N_+}, \mathbf{w}_{N_-}\} \rangle = \Lambda \times \langle \Psi_0 | \{\mathbf{u}_{N_0}, \mathbf{v}_{N_+}, \mathbf{w}_{N_-}\} \rangle, \tag{4.25}$$

where Λ is a known eigenvalue of a specific transfer matrix from the fusion hierarchy of the model. Precise details are found in the paper.

This formula tells us that it is enough to find the overlap with the simple scalar state, and then the complications with the MPS are all included in the eigenvalue function Λ, which is actually a sum of products of the form (3.29).

Luckily, the BTQM-TBA method described above was sufficient to find this simple overlap. We computed the Baxterization, the corresponding Y-functions, and the combinations like (4.16). Investigating the poles and zeroes of these functions was enough to find the overlap with this simple state, and eventually (4.25) produced the overlaps relevant to the AdS/CFT one-point functions.

For all the details of this computation we refer to the original work. Let us however note that even the published paper [27] is very brief and many details are left out due to length reasons. This is common practice in those cases when the model is very complicated, due to the higher rank symmetry groups involved.

We note that we also performed similar computations for overlaps with other symmetry classes, for example in the class $(SO(6), SO(3) \times SO(3))$. However, in those cases we did not find new relevant overlap formulas, our contribution is just the representation theoretical understanding of the results derived earlier by the coordinate Bethe Ansatz. A large part of these computations remained unpublished.

4.5 Rigorous proofs

As it was mentioned above, the methods we described so far fix the pre-factors in front of the Gaudin-like determinants, once we accepted that a formula like (4.14) should hold. Recently there have been new attemts to rigorously prove the complete formula. We remind that in the case of the Néel state (and the states of the form (3.43)) an exact proof is available, but this is restricted to diagonal K-matrices and it is not clear how to generalize it to other cases.

A completely different method was introduced in [28]. Here the idea is to use the *coordinate Bethe Ansatz* formulas to derive a rigorous proof for all particle numbers. Earlier attempts also used the coordinate BA wave function, but they only treated cases with fixed low particle numbers. It was a completely new addition of [28] that recursive formulas were found which eventually prove the exact formulas. Now we summarize this method.

Let us write the coordinate BA wave function (1.21) as

$$\Psi(x_1, \dots, x_N) = \sum_{\mathcal{P} \in S_N} \mathcal{F}(P) \Psi_{\text{free}}(x_1, \dots, x_N), \tag{4.26}$$

where the sum runs over the permutations of the rapidities, Ψ_{free} is a free wave function given by

$$\Psi_{\text{free}} = \prod_{j=1}^{N} e^{i x_j p(\lambda_{\mathcal{P}_j})} \tag{4.27}$$

and $\mathcal{F}(P)$ is the factor which takes into account the interaction between the particles:

$$\mathcal{F}(P) = \prod_{j>i} f(\lambda_i - \lambda_j). \tag{4.28}$$

For technical reasons it is useful to introduce the variables $l_j = e^{ip_j} = e^{ip(\lambda_j)}$ which are the one-particle eigenvalues of the one-site shift operator. Clearly, $\Psi_{\text{free}}(x_1, \ldots, x_N)$ is the product of the powers of these variables. Furthermore, in the concrete cases it can be shown that the f-function is a rational function of two l-variables, thus the overall factor $\mathcal{F}(P)$ is also a rational function of the permuted l-variables.

The first step is to compute the overlaps with the free wave functions:

$$\langle \Psi_0 | \Psi_{\text{free}} \rangle. \tag{4.29}$$

If the initial state is a one-site or two-site invariant product state, then such overlaps are relatively easy to compute in closed form. Summations of the type

$$\sum_{x=x_1}^{x_2} l^x \tag{4.30}$$

can be performed using known formulas for geometric series. In the final result the volume L will enter only through the exponents, in combinations similar to

$$\frac{l^L - 1}{l - 1}. \tag{4.31}$$

Thus the resulting formula for (4.29) will be a rational function of the l-variables and the variables $a_j = l_j^L$. It is useful to treat the a-variables as independent.

Turning to the full overlap, it is still true that the final formula will be a rational function of the l- and a-variables. However, the summation over the permutations is not tractable in the generic case, therefore we can not expect to find compact formulas.

The computation described so far is off-shell: we have not yet used the Bethe equations. We know from the general proofs regarding the integrability of the state $|\Psi_0\rangle$ that for on-shell states only those Bethe states will have non-zero overlap, which satisfy the pair requirement (possibly also including some exceptional rapidities). This information is used to find the factorized overlap formulas.

Let us denote by $\mathcal{S}_N(\boldsymbol{l}_N, \boldsymbol{a}_N)$ the full overlap function. Let us further denote by $\tilde{\mathcal{S}}_N(\boldsymbol{l}_N)$ the function which is obtained from $\mathcal{S}_N$ by the formal substitution of the Bethe equations. This means that for each a_j we substitute the "right hand side" of the Bethe equations. It turns out that this function $\tilde{\mathcal{S}}_N(\boldsymbol{l}_N)$ is identically zero! The reason is that the overlaps have to vanish for almost all states, and it is easy to prove that the corresponding rational function has to be zero (it can be shown with some generalization that we are actually showing the vanishing of the rational function at infinitely many points).

Then the question remains: how to get the final non-zero overlaps? The trick is to focus on combinations of the type

$$\frac{a_1 a_2 - 1}{l_1 l_2 - 1}, \tag{4.32}$$

which can be shown to appear as factors in various terms in $\mathcal{S}_N(\boldsymbol{l}_N, \boldsymbol{a}_N)$. Now we focus on the case of only two particles. In this case if we substitute the Bethe equations for a_1 and a_2 and we also assume the pair structure then we get identically zero in the numerator. However, the actual combination is of the form $0/0$, which gives a finite answer: for two particles the above formula leads to the finite value L.

We can see that the vanishing of $\tilde{\mathcal{S}}_N(\boldsymbol{l}_N)$ is due to our artificial way of treating the l- and a-variables as separate. However, it is actually these terms of the type 0/0 which give a finite contribution to the final on-shell overlap, so we need to determine them by treating the variables as separate.

Let us consider the completely paired limit:

$$l_{2j-1}l_{2j} \to 1, \quad a_{2j-1}a_{2j} \to 1, \qquad j = 1, \dots, N/2, \tag{4.33}$$

and let us further investigate the apparent pole at say $l_1 l_2 = 1$. It was shown in [28] in concrete cases that the formal pole of $\mathcal{S}_N$ around the point $l_1 l_2 = 1$ is of the form

$$\mathcal{S}_N(L) \sim \frac{a_1 a_2 - 1}{l_1 l_2 - 1} F(\lambda_1) \prod_{j=3}^{N} f(\lambda_1 - \lambda_j) f(-\lambda_1 - \lambda_j) \mathcal{S}^{\mathrm{mod}}_{N-2}(\not{1}, \not{2}, L), \tag{4.34}$$

where $\mathcal{S}^{\mathrm{mod}}_{N-2}$ is the formal overlap for $N-2$ particles not including 1 and 2, evaluated with the following modified a-variables:

$$a_j^{\mathrm{mod}} = \frac{f(l_j, l_1)}{f(l_1, l_j)} \frac{f(l_j, 1/l_1)}{f(1/l_1, l_j)} a_j. \tag{4.35}$$

In (4.34) $F(\lambda)$ is a rational function of the l-variable which carries the dependence on the initial state. Unfortunately the proof of (4.34) is not available in the general case, only particular states were considered. Nevertheless these include new cases with non-diagonal K-matrices.

It can be shown that (4.34) contains enough information to completely prove the factorized overlap formulas of the type (4.13). For the pair amplitude the result is

$$u(\lambda) = \frac{F^2(\lambda)}{f(2\lambda) f(-2\lambda)}, \tag{4.36}$$

where $F(\lambda)$ stems from (4.34) and $f(\lambda)$ is a characteristic function of the model defined in (1.43).

The proof of the final overlap formula goes back to the ideas of Korepin used to prove the norm formula with the original Gaudin determinant [105]. In fact, the whole method was motivated by this old work, it was merely just adapted to the case of the overlaps. Perhaps it is not surprising that the results of similar final formulas require proofs with similar methods. This also means that we do not expect that a really simple rigorous proof will be found: the complexity of the proofs of the overlaps have to be at least as complicated as the easiest rigorous proof of the original norm formula.

This method of [28] is rather complicated but it can give a good starting point for later works. We expect that at least some partial simplification of the method will be possible. For example, it should be possible to prove (4.34) with algebraic methods, without an explicit summation of the wave function in coordinate space. Research in this direction is in progress in collaboration with Tamás Gombor.

Finally we note that [28] did not only consider the XXZ spin chain, but also the so-called $SL(2, \mathbb{R})$ chain, where the local Hilbert space is infinite dimensional. This model is also important for AdS/CFT. The proof described above is easily adapted to that model as well.

4.6 Open problems

As it was described above, there are now multiple methods available to compute the exact overlaps. However, a lot remains to be done.

First of all, the method of the last Section needs to be simplified, and also it needs to be adapted to the nested cases. Going further, it would be important to find methods that work also for the long range deformed models, which are important to AdS/CFT. The long range interaction terms correspond to higher loop corrections in the CFT, and comparison with string theory requires to have exact formulas with all such contributions included. At present little is known about the integrable states of the long range deformed chains, although some preliminary results appeared in [255, 256] (see also [227, 257]). Perhaps the method of the Separation of Variables (SoV) [265] could help in this situation.

We also note that there are a number of models where overlaps have not yet been computed, such as the XYZ chain or the Hubbard model. However, at present there is little motivation to pursue this direction.

5 Correlation functions

5.1 Introduction

The computation of correlation functions of the integrable spin chains has a very rich history, and it is practically impossible to review it in this thesis. Unfortunately there is no single review article or book which includes all the various methods and recent developments. Therefore we refer the interested reader to the following works:

- The book [52] summarizes the basics of Algebraic Bethe Ansatz and the early approaches towards the correlation functions. These approaches are barely used anymore.

- The habilitation thesis of Karol Kozlowski [54] is a recent work, which summarizes almost all the modern methods, and the works of him and his collaborators on the correlation functions. Here the focus is on the derivation of the asymptotic (large distance and large time limit) behaviour of the equilibrium correlations.

- The short paper [168] can be used as a friendly introduction to the theory of factorized correlation functions. The practical results of this theory will be explained below, because this is the method which was used in the solution of the quantum quenches and this is where the author also contributed.

- Recent developments in the so-called Separation of Variables (SoV) method can be found in the paper [266], where the history of the field is thouroughly reviewed. The author has not yet used the (SoV), therefore this method will not be considered below.

- Recent results about scalar products and form factors in nested systems are reviewed in [267].

The difficulty of computing the correlation functions lies in the Bethe Ansatz itself. The wave function is explicitly known as a sum over permutations, and the local correlators of the type

$$\langle \boldsymbol{\lambda}_N | \mathcal{O} | \boldsymbol{\lambda}_N \rangle \tag{5.1}$$

with $\mathcal{O}$ being any short range operator can be written down using a double sum over permutations. The individual terms can be evaluated explicitly, but performing the summation is very difficult. We note that alternative representations using summations over bipartite partitions were also derived [52], but this was only a slight improvement. More effective methods involving scalar products and form factors (based on the famous Slavnov-determinant [268]) were developed starting from the beginning of the 90's. This led to multiple integral formulas for the short range correlations. Later it was understood that these multiple integrals can be expressed as sums of products of simple integrals, and this led to the theory of factorized correlation functions, also known as the Hidden Grassmann Structure. Before turning to this theory let us explain why these theories were needed for quench problems, and what was missing before the work of the author.

Basically all of the works mentioned above focused on the correlation functions in the ground state or in finite temperature ensembles. This was motivated by both theory and experiments. In materials whose magnetism is well described by the 1D spin chains the neutron scattering experiments naturally probe the ground state, or perhaps the finite T cases [59, 160]. Furthermore, in gapless models the ground state correlations are special, they decay algebraically with non-integer powers, and direct comparison can be made to the predictions of Conformal Field Theory [54].

However, more recently both the experiments and the theory shifted towards the non-equilibrium setup, and this motivated the study of correlation functions in highly excited states. Computing the full time dependence of physical quantities is very difficult, and a simpler goal is to study the properties of the emerging steady-states after the quenches, as explained in Section 2. For the correlations in these states practically no results were available before 2014. In most situations it is the correlation of local observables which is measured in an experiment, therefore simply just finding the root densities after the quench is not enough to compare to experimental data. Exceptions to this are the time-of-flight measurements which give information about the root densities in quantum gases (see for example the recent work [71]), but naturally such methods do not exist for spin chains.

Let us now go back to the questions of GGE and GETH discussed in Section 2. The GETH includes the statement that the mean values of the complete set of charges determine all local correlation functions in the model. In 2 we explained that the string-charge duality fixes the Bethe root distributions, once the mean values of the charges are known. Clearly, this is only half of the story, and it needs to be shown that the root densities fix all local correlations. Furthermore, in integrable models it is expected that eventually some concrete exact formulas will be found for the correlation functions, and in the ideal case they will be simple enough that they can be used in practical computations.

This question was especially timely in 2014, when the status of the GGE was not yet clear and it was required to have hard tests of the GGE predictions [6, 172]. At that time there were no methods that could treat the correlations of the emergent steady states. Both works [6, 172] employed the Hellmann-Feynman theorem to find the mean value of the nearest neighbour $z - z$ correlation in the XXZ chain. On the other hand, our

papers [5,6,12] went further and we presented formulas for basically all local correlations. This was based on the immense work of the leading scientists of the field who developed the theory of factorized correlation functions (to be reviewed below).

Our results on correlation functions are important also for Generalized Hydrodynamics; the connection will be explained in Section 6.

5.2 Factorized correlation functions

The theory of factorized correlation functions was developed in a series of works with contributions from many researchers [269–276]. The history of this theory started with the work [277] of Boos and Korepin who observed that some concrete multiple integral formulas for correlations of the XXX chain can be factorized (see also [278]). Afterwards this was developed into a full algebraic theory. In our opinion this is one of the most advanced theories in integrability, and it is fair to say that only a handful of researchers understand it in all details. Unfortunately the author is not one of them. However, it is relatively easy to understand the main results, and to contribute to certain parts of the theory. Now we summarize these main results.

The theory deals with the normalized mean values of the form

$$\langle \Psi | \mathcal{O} | \Psi \rangle, \tag{5.2}$$

where $\mathcal{O}$ is a short range operator of the Heisenberg chain (the XXZ and XXX cases need to be treated separately), and $|\Psi\rangle$ is an eigenvector of the commuting set of transfer matrices. It is important that there is no restriction on the size of the system (other than it has to be bigger than the length of $\mathcal{O}$), and both the finite and infinite size cases can be considered. The works cited above only treated the ground state and finite temperature cases, and the extension to arbitrary excited states was the contribution of the author.

The main statement of the theory is that each mean value can be expressed using just a few functions. To be more precise, the mean values are expressed as combinations of the Taylor coefficients of some functions with one or two auxiliary variables. The number of functions depend on the model: In the XXZ chain it is enough to consider just two functions (commonly denoted as $\omega(\mu,\nu)$ and $\omega'(\mu,\nu)$) for operators that are symmetric with respect to spin-flip. For generic operators an additional function $\varphi(\mu)$ is also needed. In the XXX case the situation is more involved: for operators that are $SU(2)$-symmetric (or for states with zero magnetization) it is enough to consider just a single function $\omega(\mu,\nu)$, whereas the full solution in the generic case is not yet known. We note that even the XYZ model was treated in [279] and it is conjectured that three functions with two variables are enough to describe all correlation functions in that model.

From a practical point of view the theory of factorized correlations consists of two parts:

- **The algebraic part.** This part of the theory uses the so-called hidden Grassmann structure, an algebraic construction on the space of the operators, to express the mean values of given operators using the functions mentioned above. This part is completely independent of the physical situation: it concerns only the operators themselves. Nevertheless the factorization only holds for the mean values of the operators *in the eigenstates*, and clearly not for the operators themselves.

- **The physical part.** This part specifies the functions mentioned above depending on the concrete physical situation. For example it gives specific values to $\omega(\mu,\nu)$ depending on the temperature, magnetic field, etc. It is this part where the author also contributed.

The functions mentioned above have their origin in the two-site density matrix of an *inhomogeneous* spin chain, where the parameters μ,ν are the inhomogeneities (see (1.55)). This is why it is sometimes said in the literature that all correlations can be expressed using the two-site density matrix. It should be noted that the factorized correlations can also be found with a set of recursive equations satisfied by the inhomogeneous multi-site density matrix [280].

We now give a flavour for the results of the algebraic part. In the XXX chain let us consider states with zero total magnetization. Then short range $z-z$ correlations can be expressed as [276] [5]

$$\langle\sigma_1^z\sigma_2^z\rangle = \frac{1}{3}(1-\Psi_{0,0}) \tag{5.3}$$

$$\langle\sigma_1^z\sigma_3^z\rangle = \frac{1}{3}(1-4\Psi_{0,0}+\Psi_{1,1}-\frac{1}{2}\Psi_{2,0}) \tag{5.4}$$

$$\begin{aligned}\langle\sigma_1^z\sigma_4^z\rangle =& \frac{1}{108}(288\Psi_{1,1}-15\Psi_{2,2}+10\Psi_{3,1}+2\Psi_{0,0}(-162-42\Psi_{1,1}+3\Psi_{2,2}-2\Psi_{3,1})+\\ &+\Psi_{2,0}(-156+12\Psi_{1,1}-6\Psi_{2,0})+\Psi_{1,0}(84\Psi_{1,0}-12\Psi_{2,1}+4\Psi_{3,0})+36),\end{aligned} \tag{5.5}$$

where

$$\Psi_{n,m}=\partial_\mu^n\partial_\nu^m\Psi(x_1,x_2)|_{\mu,\nu=0}. \tag{5.6}$$

and $\Psi(\mu,\nu)$ is a function which is related to the above mentioned $\omega(\mu,\nu)$ in a linear way. For our purposes the function $\Psi(\mu,\nu)$ is more convenient.

Let us give now the formulas for the physical part. Prior to the work of the author this was known for the finite temperature cases, or the ground states in finite or infinite volume. We focus again on the XXX case and follow the notations of [168]. Let $\mathfrak{a}(\lambda)$ be an auxiliary function defined by the non-linear integral equation

$$\log(\mathfrak{a}(\lambda))=a_0(\lambda)-\int_C\frac{d\omega}{2\pi}\frac{2\log(1+\mathfrak{a}(\omega))}{1+(\lambda-\omega)^2}, \tag{5.7}$$

where C is a closed contour encircling the real line. The physical situation is encoded in the source term $a_0(\lambda)$.

For the ground state in finite volume L we have

$$a_0(\lambda)=L\log\frac{\lambda-i/2}{\lambda+i/2}. \tag{5.8}$$

In this case the auxiliary function encodes the position of the Bethe roots $\boldsymbol{\lambda}_N$ through the conditions

$$1+\mathfrak{a}(\lambda_j)=0. \tag{5.9}$$

[5] The paper [276] has a misprint: In formula (11) the coefficient of the term $(1,0)(3,0)$ for $\langle\sigma_1^z\sigma_4^z\rangle$ is written as -4/27, whereas correctly it is 4/27. This misprint was discovered by the author after some numerical tests, and at that time it caused some confusion and one or two weeks of delay. The correct formula was later confirmed by Frank Göhmann (Wuppertal).

It can be shown that if the contour C encircles all Bethe roots then (5.9) together with (5.7) is equivalent to the Bethe equations (1.24) with (1.23).

For finite temperature T we have

$$a_0(\lambda) = \frac{2}{T}\frac{1}{\lambda(\lambda+i)}. \tag{5.10}$$

In this case the conditions (5.9) encode the Bethe roots of the Quantum Transfer Matrix (see Section 2.6 and [50, 169]).

We require $T > 0$ or $L < \infty$; in this approach the ground state of the infinite volume system is singular which needs separate treatment.

The function $\Psi(\mu, \nu)$ is then expressed as [6]

$$\Psi(\mu,\nu) = \int_C \frac{d\omega}{2\pi}\frac{1}{1+\mathfrak{a}(\omega)}\frac{G(\omega|\nu)}{(\omega-\mu)(\omega-\mu-i)}, \tag{5.11}$$

where $G(\omega|\nu)$ is a further auxiliary function with parameter ν satisfying the linear equation

$$G(\omega|\nu) = -\frac{1}{(\omega-\nu)(\omega-\nu-i)} + \int_C \frac{d\kappa}{2\pi}\frac{1}{1+\mathfrak{a}(\kappa)}\frac{2G(\kappa|\nu)}{1+(\kappa-\omega)^2}. \tag{5.12}$$

In the XXZ model there are analogous formulas for the function $\Psi(\mu, \nu)$ (or $\omega(\mu, \nu)$), whereas $\omega'(\mu, \nu)$ is given by a similar definition, although with different source terms for the linear integral equations. We refer the reader to [281, 282] for the practical formulas and examples of the numerical data.

The formulas above might seem complicated, but in fact this system of equations is very convenient for numerical computations. The auxiliary function $\mathfrak{a}(\lambda)$ can be found easily, simple iterations of the convolution converge very quickly. Afterwards one usually computes the Taylor coefficients of $\Psi(\mu, \nu)$ directly, by expanding the functions involved in ω and ν. This is straightforward and the resulting linear integral equations are also easily solved. Finally the formulas for the algebraic part are already available, and simple substitutions into formulas like (5.3)-(5.5) immediately give the final numerical results. It is very rare even for integrable models that numerically exact correlation functions can be computed in such an effective way.

5.3 The physical part in excited states

Before 2014 the physical part of the theory was only known for the ground states and the finite temperature cases. Motivated by the quench problems the author and his MSc student Márton Mestyák extended the physical part to arbitrary excited states in [5]. This work was based on a simple explicit computation and the observation of some analogies between various formulas. Therefore the main results of [5] were only conjectures at that time. This work considers the infinite volume limit directly.

The only rigorous computation of [5] was the application of the Hellmann-Feynman (HF) theorem to the nearest neighbour correlator. It follows from the theorem and the expression (1.41) for the Hamiltonian density that

$$\langle \boldsymbol{\lambda}_N | \sigma_j^z \sigma_{j+1}^z | \boldsymbol{\lambda}_N \rangle = \frac{1}{L}\frac{\partial E(\boldsymbol{\lambda}_N)}{\partial \Delta}, \tag{5.13}$$

[6] Our Ψ-function is one half of the Ψ-function of [168].

where $E(\boldsymbol{\lambda}_N)$ is the exact energy eigenvalue of the given state. The derivative on the r.h.s. above is easy to evaluate, and the resulting equations turn out to have similar structure as the equations for the physical part in the QTM formalism. However, the HF theorem can be applied for any excited state, and all intermediate computations use the Bethe root densities. Thus it was possible to conjecture the formulas for the functions $\omega(\mu,\nu)$ and $\omega'(\mu,\nu)$ based only on this single non-trivial example. As the formulas only involve the Bethe root densities, we can call the results the TBA form of the physical part. We summarize it now in the XXX case.

Let $\rho_n(\lambda)$ and $\rho_{n,h}(\lambda)$ be the densities for the n-strings and their holes. Then we define parameter dependent auxiliary functions $\tilde{\rho}_n(\lambda|\nu)$ through the equations

$$\tilde{\rho}_n(\lambda|\nu) = \delta_{n,1}s(\lambda-\nu) + \int_{\omega=-\infty}^{\infty} \frac{d\omega}{2\pi} s(\lambda-\omega)\left(\frac{\tilde{\rho}_{n-1}(\lambda|\nu)}{1+\eta_{n-1}(\omega)} + \frac{\tilde{\rho}_{n+1}(\lambda|\nu)}{1+\eta_{n+1}(\omega)}\right), \tag{5.14}$$

where $\eta_n(\lambda)$ is defined in (2.31). Comparing with (1.64) we can see that for $\nu = 0$

$$\tilde{\rho}_n(\lambda|0) = \rho_n(\lambda) + \rho_{n,h}(\lambda). \tag{5.15}$$

Thus $\tilde{\rho}_n(\lambda|\nu)$ should be considered as a parameter-dependent generalization of the root density functions. The information about the actual physical densities enters (5.14) through the functions $\eta_n(\lambda)$.

Finally

$$\Psi(\mu,\nu) = G(\mu,\nu) + \int \frac{d\lambda}{2\pi} s(\lambda-\mu)\frac{\tilde{\rho}_1(\lambda|\nu)}{1+\eta_1(\lambda)}, \tag{5.16}$$

where $G(\mu,\nu)$ is a known function independent of the physical situation. We note that the specific formula (5.16) is unpublished, and [5] included a slightly more complicated form. We plan to publish this version in an upcoming work.

In [5] similar formulas were given for $\omega(\mu,\nu)$ and $\omega'(\mu,\nu)$ also in the XXZ case. Afterwards, the conjectures were tested against numerics: more complicated correlators were computed in the thermal case using both the established QTM and the conjectured TBA input for the physical part, and exact agreement was observed. Afterwards, the formulas were used in the paper [6] in combination with the Quench Action method, and this proved to essential in the investigation of the GGE.

The follow-up paper of the author [12] took a different path: it considered the finite volume case only and it aimed at deriving algebraic formulas instead of the integral representations. This has various advantages as we will see.

The starting point was the paper [283] which dealt with the correlation functions of finite chains in detail. The idea of [12] was to transform the various integrals in [283] into summation over the Bethe roots. If a contour integral involves the factor $1/(1+\mathfrak{a}(\lambda))$ then there is a pole for each Bethe root. The contour integral can be expressed as a sum over its residues, which will thus include a sum over the Bethe roots, on top of some additional poles of the integrand. This is a known and straightforward procedure. In fact, the derivation of the integral equation (5.7) proceeds through the same steps, just in the opposite order. The important observation of [12] was that for the physical part there is no need to use the integral representations. This had historical reasons, but switching back to algebraic formulas is very useful.

For the function $\Psi(\mu,\nu)$ in a Bethe state $|\boldsymbol{\lambda}_N\rangle$ the following algebraic result was found in [12]:

$$\Psi(\mu,\nu) = \mathbf{h}(\mu)\cdot G^{-1}\cdot \mathbf{h}(\nu). \tag{5.17}$$

Here G is the $N\times N$ Gaudin matrix defined by (1.52), and $\mathbf{h}(x)$ with $x=\mu,\nu$ is a vector of length N with components

$$(\mathbf{h}(x))_j = h(x-\lambda_j), \tag{5.18}$$

where $h(x)$ is a characteristic function of the model describing the one-particle charge eigenvalues. Explicitly it is given by

$$h(x) = p'(x), \tag{5.19}$$

where the one-particle momentum function $p(x)$ is given by (1.22) and (1.23).

A similar formula can be derived also for the other function $\omega'(\mu,\nu)$, for the details we refer to [12].

Strictly speaking the result (5.17) was derived only for the ground state, because the starting point of [283] was also the finite volume ground state. However, in these algebraic and analytic computations no information is used about the nature of the state in question. The only input is that the contours encircle all Bethe roots, and no extra roots (holes) other than known special poles of the integrand. Therefore, all the steps of [283] and [12] can be repeated for all excited states, thus (5.17) can be considered a rigorous general result. However, in order to avoid potential small mistakes, formula (5.17) was also tested numerically in [12] for a large number of excited states in small volumes, and complete agreement was found.

Formula (5.17) concerns the exact Bethe roots in finite volume. In order to prove the conjectured formulas (5.16)-(5.14) one needs to deal with the string solutions, and take the thermodynamic limit using the string hypothesis. It is known that in the TDL the Gaudin matrix describes a convolution of the type (5.14), thus multiplying with G^{-1} corresponds to solving such a linear integral equation. The precise details of taking the TDL can be found in the MSc thesis of Márton Borsi, and will be published in early 2021 in a paper written in collaboration with Márton Borsi and Levente Pristyák.

After the appearance of the paper [12] the formula (5.17) remained relatively unknown in the community. Perhaps the reason for this is that two groups of researchers, namely those working on the algebraic aspects of correlations and those working on the non-equilibrium problems (like the questions regarding the GGE) had different background and different interests, and the technical details about excited state correlations seemed too remote to both groups. Furthermore, the physical meaning behind the function $\Psi(\mu,\nu)$ or its Taylor coefficients was not known at that time. This situation changed when the author realized that the same formula describes the mean values of current operators. This was a central question in the new theory of Generalized Hydrodynamics (GHD), and announcing this connection raised interest in the formula. This will be explained in Section 6.

5.4 Open problems

Perhaps the most important open question in this topic is whether there are factorized correlation functions in other, more complicated integrable spin chains or other integrable

models. The existence of the Hidden Grassmann Structure discussed above seems to be a special feature of the XXZ and XXX models, with an interesting extension to the XYZ model discussed in [279]. A similar factorization procedure was developed in [284] by the author for the local correlators in the Lieb-Liniger model, but there the algebraic structures behind the factorization are not known.

At present there are clear signs that in models related to higher rank symmetries such a general factorization scheme can not exist. One the one hand side, explicit computation of few-site operators shows the absence of factorization with a finite number of functions [285, 286]. Nevertheless these works were able to compute at least a few short range correlators; for similar computations on the $O(N)$-symmetric models see [287]. On the other hand, even some computations based on classical integrability confirm that the higher rank case is fundamentally different [288].

The absence of a general factorization scheme does not mean that there are no practically useful results in higher rank spin chains. It is still possible that there is a subset of $SU(N)$-symmetric operators in the $SU(N)$-symmetric chains, whose mean values can be computed effectively. In Section 6 it is shown that the current operators belong to this class, with mean values given by formulas similar to (5.17). At present the class of operators whose mean values factorize into combinations of a single basic function $\Psi(\mu, \nu)$ is not yet known, and it is one of the future directions to explore.

6 Current operators and Generalized Hydrodynamics

6.1 Introduction

As we discussed in Section 2 the main questions around the GGE for interacting integrable models were settled in 2014-2015. Afterwards the attention of the community turned to spatially inhomogeneous situations, to quantum transport. In 2016 the theory of Generalized Hydrodynamics (GHD) was initiated in the independent papers [96, 97]. The theory was originally devised to treat the ballistic modes in the large time and large distance limit, but very soon it turned out that even the diffusive corrections can be treated within the theory [289–292]. Recent works [293–298] also treated the phenomenon of super-diffusion. We should note that the tranport properties of integrable models, and in particular the Drude weights were already considered in earlier works (see for example [299–304]), but the treatment of the full real time dynamics (in the large scale limit) became only possible with GHD. The theory was already applied successfully to describe real world experiments, see the recent works [69–71].

The contribution of the author and his students to GHD concerns the proof of a conjectured formula for the current mean values in spin chains [24, 25, 29]. Now we first describe this conjecture and in the following sub-sections we present three different proofs.

We focus on the large scale limit of the transport phenomena in the integrable models: we consider physical situations where the variations of local observables are very small on the characteristic length and time scales of the model. In these cases we expect that some kind of hydrodynamic approximation can describe the dynamics of the model. To be more precise, GHD assumes the existence of *fluid cells*: regions in space that are much larger than the microscopic length scales, but small enough that we can assume

local equilibration within the fluid cell. Then the physical parameters of the fluid cells become space and time dependent, such that their variation is slow, and the condition of local equlibration can hold throughout the dynamics. It is important that this is just an approximation, and at present there is no rigorous derivation of GHD. However, once local equilibration is assumed, the hydrodynamic equations for the real time dynamics can be derived rigorously within this approximation. It is this part where the author also contributed.

In Section 2 it was explained that in integrable models each equilibrium situation is characterized by some kind of GGE. If there are local equilibria with time dependent parameters, then this means that the parameters of the GGE become space and time dependent as well. Then the task is to find hydrodynamic equations that govern the flow of these parameters. The GGE's can be parametrized either by the Lagrange multipliers, or the mean values of the charges, or the Bethe Ansatz root densities. Flow equations can be derived by concentrating on the charges.

For simplicity let us now assume that there is a complete set of charges $\{Q_\alpha\}$ with a single index α. As discussed in 2.7 the quasi-local charges are also needed in the Heisenberg chain; but let us now not burden ourselves with the various notations regarding the complete set. The operator densities will be denoted as $q_\alpha(x)$.

In spatially inhomogeneous situations the mean values of the charge densities $\langle q_\alpha \rangle$ become space and time dependent. Local charge conservation means the existence of *current operators* $J_\alpha(x)$ that satisfy the continuity equation for operators

$$\partial_t q_\alpha(x) + \partial_x J_\alpha(x) = 0. \tag{6.1}$$

For simplicity we wrote here the formulas relevant to continuum cases; the precise relations in the lattice situation will be given below.

Local equilibration implies that it is enough to focus on the mean values of the charges, because they characterize the GGE. Thus it is enough to focus on the equation

$$\partial_t \langle q_\alpha(x,t) \rangle + \partial_x \langle J_\alpha(x,t) \rangle = 0. \tag{6.2}$$

Here the time dependence means that the operator equation is evaluated in the state of the system at time t.

If local equilibration holds for local operators, then the mean values of the currents have to be function of the GGE parameters. In other words the current mean values are given by a complicated functional of the charge mean values:

$$\langle J_\alpha \rangle = \mathcal{J}_\alpha(\{\langle q_\beta \rangle\}_{\beta=1,\dots,\infty}). \tag{6.3}$$

If these functionals are known, then substituting them back to (6.2) we obtain a closed set of flow equations for $\langle q_\alpha(x,t) \rangle$. These will be first order differential equations, with a complicated coupling between the different charges. They can be considered a generalization of the Euler equation known from classical hydrodynamics.

It is important that the assumption (6.3) refers to local equilibration in a strong sense, where currents are determined only by the local mean values of the charges. This approximation leads to the description of the ballistic part of the transport, inlcuding the Drude weights. However, better approximations can be given when the current mean value is expressed as a derivative expansion: the dependence of the currents on the first

spatial derivatives $\partial_x q_\beta(x)$ describes the diffusive corrections. We do not discuss these corrections here, instead we refer to [289–292].

The functional form of the current mean values was not known before 2016. Both papers [96, 97] conjectured a general results, which expresses the current mean values using the Bethe root densities as

$$\langle J_\alpha \rangle = \int d\lambda \; v_{\text{eff}}(\lambda) h_\alpha(\lambda) \rho(\lambda). \tag{6.4}$$

Here we assumed for simplicity that there is one particle species in the spectrum; the extension to multiple particle types such as string solutions is straightforward. Above $h_\alpha(\lambda)$ is the one-particle eigenvalue of the charge in question (see (1.14)) and $v_{\text{eff}}(\lambda)$ is an effective velocity which describes the propagation of a particle with rapidity λ in the presence of the other particles. The precise form of this effective velocity can be found in [96, 97]; for our purpose it is important that

$$v_{\text{eff}}(\lambda) = \frac{dE_{\text{dr}}(\lambda)}{dP_{\text{dr}}(\lambda)}, \tag{6.5}$$

where E_{dr} and P_{dr} are the "dressed energy" and "dressed momentum" obtained after the addition of a particle with rapidity λ into the local equilibrium.

We can thus see that (6.4) is an essentially semi-classical formula: the flow of each charge consists of the one-particle charges carried by the individual particles, multiplied by a semi-classical propagation speed. This speed takes into account the scattering of the particles, and this is the only constituent of the formula which depends on the interactions.

Let us discuss this semi-classical pciture in more detail, following the reasoning in [305]. In a free model the particles propagate with the group velocity $v = dE/dp$. However, in a non-integrable interacting model we can not expect to get a formula like (6.4), because the quasi-particles with given energy typically decay into low energy modes. Such decays are forbidden in integrable models due to the dissipationless scattering. Thus, if a quasi-particle is excited in the system, then it will propagate without dissipation in the presence of the other particles, even if there are interactions. The only effect of the interactions is the addition of phase shifts to the wave function; these phase shifts are additive due to the factorized scattering. It is known that in every two-body scattering the phase factors cause a displacement (or equivalently, a time delay) of the propagating wave packet [306–308]. The displacement is proportional to $d\delta/dp$, where $\delta(p)$ is the scattering phase shift. As the particle propagates in the sea of the other particles, these phase shifts and thus the displacements accumulate, modifying the average velocity of the wave packet. It was shown in [305] that this argument exactly reproduces the $v_{\text{eff}}(\lambda)$ found in [96, 97].

The flow equations of GHD are found if we also use the expression for the charge mean values:

$$\langle q_\alpha \rangle = \int d\lambda \; h_\alpha(\lambda) \rho(\lambda). \tag{6.6}$$

Assuming that each fluid cell is large enough so that it has a well defined Bethe root density $\rho(\lambda)$, we can make these function space and time dependent and substitute (6.6) and (6.4) back to (6.2):

$$\int d\lambda \; h_\alpha(\lambda) \partial_t \rho(\lambda|x,t) + \int d\lambda \; h_\alpha(\lambda) \partial_x \left(v_{\text{eff}}(\lambda) \rho(\lambda|x,t) \right). \tag{6.7}$$

Note that $v_{\text{eff}}(\lambda)$ depends on the root densities, thus the differential with respect to x also acts on it.

Let us write the above expression as

$$\int d\lambda\ h_\alpha(\lambda)\Big[\partial_t\rho(\lambda|x,t)+\partial_x\left(v_{\text{eff}}(\lambda)\rho(\lambda|x,t)\right)\Big]=0. \tag{6.8}$$

If the set of charges is complete, then the only way that all such expressions are zero is if

$$\partial_t\rho(\lambda|x,t)+\partial_x\left(v_{\text{eff}}(\lambda)\rho(\lambda|x,t)\right)=0. \tag{6.9}$$

This is the fundamental flow equation of GHD, which can be considered as a generalization of the Euler equation from classical HD. In concrete cases it can be solved numerically, so that the full complexity of quantum mechanics is replaced by a coupled set of ordinary first order differential equations. In certain situations even analytical solutions can be found [97, 309].

Despite all the successes of the theory (including the discussion of diffusion and super-diffusion) the complete and rigorous proof of (6.4) remained elusive for a long time. First of all, it was known that the statement holds in models equivalent to free bosons or free fermions [310]. In interacting cases proofs were given in various settings. The original paper included a proof for relativistic QFT [96]; see also [311] and the recent work [312]. Regarding spin chains the statement was proven for the spin-current in [110]. In the case of the Toda chain it was discussed in [313], and [314] includes a rather general proof based on the existence of a conserved current (a current operator which is itself a conserved charge). However, there was no microscopic proof for the interacting spin chains, which would start from first principles.

A new contribution of the author was to consider the current mean values in finite volume; all previous proofs worked basically in the thermodynamic limit. It was conjectured by the author in 2017 that a formula like (5.17) should describe the finite volume mean values. However, the publication of the actual proof was only achieved in collaboration with two students in 2019-2020 [24]. Afterwards two independent proofs of the same statements were given in [25] and [29]; the latter work includes a new algebraic construction for the current operators. Below we present the exact finite volume formula valid in the spin chains and other related models, and we also sketch the three different proofs of it.

6.2 Current mean values in finite volume

Here we discuss the results of [24], which concern the mean values of the current and the so-called generalized current operators in integrable spin chains. The focus is on the Heisenberg chains; extension to other models can be considered with the methods of the follow-up work [29].

In spin chains the current operator $J_\alpha(x)$ describing the flow of charge Q_β is defined through the continuity equation

$$i\left[H,q_\alpha(x)\right]=J_\alpha(x)-J_\alpha(x+1). \tag{6.10}$$

This operator definition makes sense in every volume where $q_\alpha(x)$ is well defined. It follows from the commutativity of H and Q_α, and from the locality of the densities, that the operator equation always has a solution in the form of a local operator.

For the mean values of these currents the following exact result was found in [24]:

$$\langle \boldsymbol{\lambda}_N | J_\alpha(x) | \boldsymbol{\lambda}_N \rangle = \mathbf{e}' \cdot G^{-1} \cdot \mathbf{h}_\alpha. \tag{6.11}$$

Here the quantities $\mathbf{e}'$ and $\mathbf{h}_\alpha$ are N-dimensional vectors with elements

$$(\mathbf{e}')_j = \frac{\partial e(\lambda_j)}{\partial \lambda}, \qquad (\mathbf{h}_\alpha)_j = h_\alpha(\lambda_j), \tag{6.12}$$

where $e(\lambda)$ is the one-particle energy eigenvalue function. Furthermore, G^{-1} is the inverse of the Gaudin matrix. Let us remind that the Gaudin matrix is defined by (1.52), where the I_j are the momentum quantum numbers of the states. Notice the similarity of equations (6.11) and (5.17).

It is possible to give a semi-classical interpretation to (6.11). Let us write it as

$$\langle \boldsymbol{\lambda}_N | J_\alpha(x) | \boldsymbol{\lambda}_N \rangle = \frac{1}{L} \sum_{j=1}^{N} v_{\text{eff}}(\lambda_j) h_\alpha(\lambda_j), \tag{6.13}$$

where we defined the quantities

$$v_{\text{eff}}(\lambda_j) = \frac{L}{2\pi} \frac{\partial E}{\partial I_j}, \tag{6.14}$$

where E is the total energy of the state.

Now we argue that (6.14) is the finite volume version of (6.5). This is true because the small changes in the dressed momentum and dressed energy can be traced back to small changes in the momentum quantum numbers and the overall finite volume energy, respectively:

$$\delta E_{\text{dr}}(\lambda_j) \sim \delta E, \qquad \delta P_{\text{dr}}(\lambda_j) \sim \delta \left(\frac{2\pi I_j}{L} \right). \tag{6.15}$$

Also, it can be proven rigorously that in the TDL the formula (6.5) is reproduced; this will be published soon with Márton Borsi and Levente Pristyák.

Besides showing the equivalence of the two formulas for v_{eff}, it is also possible to construct a semi-classical picture for the finite volume current mean values. To this order consider now N particles moving in a finite volume L with periodic boundary conditions. Let us make the semi-classical approximation that each particle has a well defined position and let us ignore the issue of wave packet dispersion. It follows from the Bethe Ansatz wave function that as long as particles are far away they don't interact and they propagate freely with the group velocity. However, every two-body scattering event leads to a displacement (time delay). These displacements accumulate over a long time, as the particles move around, leading to a well defined average propagation speed. It was shown by Márton Borsi in his TDK (Scientific Students' Association) work in 2019 that the average velocity computed form this semi-classical picture is exactly equal to (6.14); this computation was included in the paper [24].

The reader might argue that this semi-classical picture is problematic, because the spreading of the wave packet is completely ignored even though we are considering the large time limit in a finite volume. This objection is actually relevant, and at present there

is no known solution to this. The problem of the diffusion of the wave packet is already present in the derivation of [305]. Perhaps the more rigorous methods of [315] could solve this problem. However, it is known that (6.11) is the exact result in Quantum Mechanics, therefore there was no more activity devoted to the semi-classical interpretation of it.

Before turning to the proof of (6.11) let us consider two generalizations. First, we can also consider the generalized current operators, that describe the flow of the charge Q_α under the time evolution generated by Q_β. They are defined through the operator equation

$$i\,[Q_\beta, q_\alpha(x)] = J_{\alpha,\beta}(x) - J_{\alpha,\beta}(x+1). \tag{6.16}$$

For the mean values of these generalized current operators the following result was derived in [24]:

$$\langle \boldsymbol{\lambda}_N | J_{\alpha,\beta}(x) | \boldsymbol{\lambda}_N \rangle = \mathbf{h}'_\beta \cdot G^{-1} \cdot \mathbf{h}_\alpha. \tag{6.17}$$

Here $\mathbf{h}'_\beta$ is an N-element vector with components $h'_\beta(\lambda_j)$, and prime denotes differentiation. The analogy with (6.11) and (5.17) is evident.

Second, the same problem of finite volume mean values was also considered for integrable QFT in [312]. There a different derivation was presented, leading to a result that has basically the same structural form as (6.11), but it also includes field theoretical correction terms.

6.3 Proof using a form factor expansion

The main difficulty in proving (6.11) and (6.17) was that there were no explicit formulas for the current operators. The continutity equations are rather implicit. Taking the mean value on the two sides of (6.10) or (6.16) gives indentically zero. Thus it was not clear how to get information about the mean values. We put forward the an explicit construction for the current operators was eventually found in the work [29] of the author, but first we discuss the first two proofs of the main statement. We focus on proving (6.11); the proof of (6.17) is completely analogous.

The idea of [24] is to use the special properties of form factors (matrix elements of local operators) in integrable models. It is known that the finite volume mean values are not independent from the off-diagonal matrix elements. The continuity equations yield information about these off-diagonal elements, and this can be used to construct the mean values.

In any finite volume and for any two Bethe states with the same particle numbers we have

$$i\left(\sum_{j=1}^{N} e(\lambda_j) - e(\mu_j)\right) \langle \boldsymbol{\lambda}_N | Q_\alpha(x) | \boldsymbol{\mu}_N \rangle = \left(1 - \prod_{j=1}^{N} e^{i(p(\mu_j) - p(\lambda_j))}\right) \langle \boldsymbol{\lambda}_N | J_\alpha(x) | \boldsymbol{\mu}_N \rangle. \tag{6.18}$$

For the infinite volume form factors of an operator $\mathcal{O}$ we use the notation $F^{\mathcal{O}}(\boldsymbol{\lambda}_N | \boldsymbol{\mu}_N)$; it is known that the normalized finite volume matrix elements are [263]

$$\langle \boldsymbol{\lambda}_N | \mathcal{O} | \boldsymbol{\mu}_N \rangle = \frac{F^{\mathcal{O}}(\boldsymbol{\lambda}_N | \boldsymbol{\mu}_N)}{\sqrt{\det_N G(\boldsymbol{\lambda}_N) \det_N G(\boldsymbol{\mu}_N)}}, \tag{6.19}$$

where the two Gaudin determinants stem basically from the norm of the Bethe Ansatz wave functions.

Even though the relation (6.18) stems from finite volume, it can be extended by analytic continuation also to the form factors, where there is no restriction on the Bethe roots.

In order to describe the finite volume mean values, let us define the so-called symmetric evaluation of the (infinite volume) diagonal form factors as

$$F_s^{\mathcal{O}}(\boldsymbol{\lambda}_N) = \lim_{\varepsilon\to 0} F^{\mathcal{O}}(\lambda_1+\varepsilon,\dots,\lambda_N+\varepsilon|\lambda_N,\dots,\lambda_1). \tag{6.20}$$

Furthermore, it is useful to define the functions $\rho_N(\lambda_1,\dots,\lambda_N)$ as the $N\times N$ Gaudin determinants evaluated at the set of rapidities $\{\lambda_1,\dots,\lambda_N\}$. In the notations we suppress the index N and write simply

$$\rho(\boldsymbol{\lambda}_N) = \det G(\boldsymbol{\lambda}_N). \tag{6.21}$$

We remind that the Gaudin determinants describe the norms of Bethe wave functions for eigenstates, ie. for sets of rapidities satisfying the Bethe equations. On the other hand, the functions $\rho(\boldsymbol{\lambda}_N)$ are defined for arbitrary sets of rapidities.

The finite volume mean values of local operators can be computed through the expansion

$$\langle\boldsymbol{\lambda}_N|\mathcal{O}(0)|\boldsymbol{\lambda}_N\rangle = \frac{\sum_{\{\lambda^+\}\cup\{\lambda^-\}} F_s^{\mathcal{O}}(\{\lambda^+\})\rho(\{\lambda^-\})}{\rho(\boldsymbol{\lambda}_N)}, \tag{6.22}$$

where the summation runs over all partitionings of the set of the rapidities into $\{\lambda^+\}\cup\{\lambda^-\}$. The partitionings include those cases when either subset is the empty set, and in these cases it is understood that $\rho(\emptyset)=1$ and $F_s^{\mathcal{O}}(\emptyset)=\langle\mathcal{O}\rangle$ is the v.e.v. The relation (6.22) is exact when the Bethe Ansatz wave functions are exact eigenstates of the model. It was first proposed in [316] in integrable QFT, and for the integrable spin chains it was proven first in [317] and then independently in [24].

The key idea of [24] was to use the above expansion theorem twice. First it can be used to extract the symmetric form factors of the charge density operators, because the chage mean values are known. Then relation (6.18) gives the symmetric form factors of the current operators. Finally, the expansion theorem can be used a second time to find the explicit formula for the current mean values.

We do not reproduce this computation here; the details are quite lengthy and they involve somewhat cumbersome graph theoretical arguments. Also, the proof of the expansion theorem is rather lengthy and it is not immediately clear why it holds. Therefore it seemed that this proof can not be the final answer to the question, and it prompted the author to look for alternative, more transparent proofs.

6.4 Proof using long range deformations

It was realized in [25] that the theory of long range deformed spin chains can be used to give a proof of the current mean values. However, this proof can not be considered entirely rigorous, because certain statements regarding the long range chains have not yet been rigorously proven either. Nevertheless the connection is rather interesting and it might lead to new developments.

Long range deformed spin chains were discovered in the context of the AdS/CFT conjecture [72,73]. As described above in Section 3.5, the scaling dimensions of certain local operators on the CFT side are given by the solution of an integrable spi chains. At tree level this spin chain is local with a two-site interaction term [87,318,319]. However, higher loop corrections induce longer range interactions in the Hamiltonian. Quite interestingly, the full finite volume Hamiltonian is not known in this theory, but its solution is: it is believed to be given by the so-called Quantum Spectral Curve method (see [74, 75] and the more recent introduction [76]).

Despite the fact that the full Hamiltonian is not known, the first few long range terms were studied in detail, and a systematic theory for the long range deformations was developed in the series of works [320–322].

The idea of the long range deformation can be summarized as follows. Let us assume that κ is a deformation parameter, such that $\kappa = 0$ corresponds to a strictly local spin chain. Let us further assume that in infinite volume we can deform the set of commuting conserved charges

$$\{Q_\alpha\} \quad \to \quad \{Q_\alpha^\kappa\}, \tag{6.23}$$

such that commutativity is preserved at each κ:

$$[Q_\alpha^\kappa, Q_\beta^\kappa] = 0. \tag{6.24}$$

Obviously there are many ways to perform such a deformation, and many cases yield trivial results. For example it is possible to perform a linear mixing of the charges, or we can also continuously change one free parameter of the integrable model in question, such as the Δ parameter of the XXZ chain. Such deformations do not yield new physical theories.

However, two new classes of deformations were identified in [321] which actually change the model, and yield new physical systems. One such class is the so-called boost type of deformation, and the other class is the deformation by the so-called bi-local operators. Here we do not discuss the details of these deformations; instead we explain the role of the current operators in this procedure.

It was realized in [24] that the boost-type of deformations are actually triggered by the current operators. In other words, for each α there is a class of models with deformation parameter κ, such that the Hamiltonian

$$H^\kappa = H_0 + \kappa \sum_x J_\alpha(x) + \dots \tag{6.25}$$

belongs to a commuting family of charges. The dots signal higher order corrections in κ.

It is important that in most cases the J_α do not commute with the original Hamiltonian, thus the deformation yields a genuinely new model.

It is known from [321] that the effect of such a deformation is to change the dispersion relation of the one-particle states. To be more precise, to linear order we have

$$p(\lambda) \quad \to \quad p^\kappa(\lambda) = p(\lambda) + \kappa h_\alpha(\lambda) + \dots, \tag{6.26}$$

while the one-particle energy eigenvalues (as a function of the rapidity) are kept intact. This phenomenon can be actually understood from the form (6.25) without referring to

integrability. However, it was also shown in [321] that the deformation does not affect the S-matrix of the theory, thus the only effect is a change in the propagation factors.

The idea of [25] is to use the Hellmann-Feynman theorem to find the current mean values. To be more precise, if E^κ is the exact eigenenergy of a state $|\Psi\rangle$ with respect to the Hamiltonian (6.25) in a fixed volume, then

$$\langle\Psi|J_\alpha(x)|\Psi\rangle = \frac{1}{L}\,\frac{dE^\kappa}{d\kappa}\bigg|_{\kappa=0}. \tag{6.27}$$

The method known as *asymptotic Bethe Ansatz* states that the finite volume energies are still determined by the formula

$$E = \sum_j e(\lambda_j), \tag{6.28}$$

but we need to substitute the deformed rapidity parameters that are solutions of the deformed Bethe equations

$$e^{ip^\kappa(\lambda_j)}\prod_{k\neq j} S(\lambda_j-\lambda_k) = 1, \qquad j=1,\dots,N. \tag{6.29}$$

Combining (6.28), (6.29) and (6.26) and substituting the deformation of the energy back into (6.27) we obtain the statement (6.11) that we intended to prove.

The advantage of this method is that it can be applied in a variety of models. In [25] it was also applied to the $SU(3)$-symmetric fundamental model, where prior to this work basically nothing was known about the current operators. However, a disadvantage of the method is that the statements of the asymptotic Bethe Ansatz are proven only order by order in κ, and thus they can not be considered completely rigorous.

It is important that the deformation of the Bethe equations and the spectrum was already known in [321], but it was the work [25] that showed that the deforming operators are actually the currents. This also means that if there are new results for current operators (see next Section), then they could also advance the topic of the long range chains and thus AdS/CFT.

Before turning to our final proof, let us mention that the second class of long range deformations (involving the so-called bi-local operators) were discussed in the parallel papers [26, 323], where it was shown that they are the lattice analogues of the famous $T\bar{T}$-deformations known from CFT and integrable QFT [324–326]. This line of research is currently in progress.

6.5 Algebraic proof

As mentioned above, the continuity equations (6.10) and (6.16) are rather implicit and it is not evident how to construct the current operators themselves. This problem was solved in the work [29]. Here a completely new construction was worked out which goes back to the fundamentals of Algebraic Bethe Ansatz: the local Lax operators. We remind that the charges themselves are derived from the transfer matrices as given in eq. (1.33). However, there was no such definition for the currents. Now we summarize the new construction of [29].

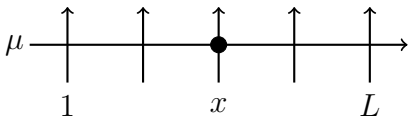


Figure 14: The generating function $q(\mu, x)$ for the charge densities is obtained by taking a derivative with respect to the rapidity at a single site. Here periodic boundary conditions are understood. The crossing with the dot denotes the action of $-i\partial_\mu R(\mu, 0)$.

The first step is to consider the generating function for the charges, instead of the individual Q_α with the discrete indices α. This is rather standard: we define

$$Q(\mu) \equiv (-i)t^{-1}(\mu)\frac{d}{d\nu}t(\mu). \tag{6.30}$$

In practical computations the inverse of the transfer matrix can be replaced by the space reflected TM (3.7), which acts as an asymptotic inverse. However, this is not important for the discussion.

The Taylor coefficients of the operator above will produce the Q_α as given by (1.33).

The next step is to find the density for the generating function $Q(\mu)$. Writing $Q(\mu) = \sum_{x=1}^{L} q(\mu, x)$ we can identify

$$q(\mu, x) \equiv (-i)t^{-1}(\mu)\mathrm{Tr}_a \left[T_a^{[L,x+1]}(\mu)\partial_\mu \mathcal{L}_{a,x}(\mu)T_a^{[x-1,1]}(\mu)\right]. \tag{6.31}$$

Here we defined the partial monodromy matrices acting on a segment $[x_1 \dots x_2]$ as

$$T_a^{[x_2,x_1]}(\mu) = \mathcal{L}_{a,x_2}(\mu)\dots\mathcal{L}_{a,x_1}(\mu). \tag{6.32}$$

A graphical representation of the generating function is given in Fig. 14.

We also define a generating function for the current operators. Naturally this will be a function of two auxiliary variables and a space coordinate. It is defined implicitly through

$$i\left[Q(\nu), q(\mu, x)\right] = J(\mu, \nu, x) - J(\mu, \nu, x+1). \tag{6.33}$$

It is our goal to derive explicit formulas for $J(\mu, \nu, x)$.

First it can be shown using a repeated action of the Yang-Baxter equation that the solution of the analogous operator equation

$$t^{-1}(\nu)\left[t(\nu), q(\mu, x)\right] = \Omega(\mu, \nu, x) - \Omega(\mu, \nu, x-1), \tag{6.34}$$

is given by

$$\Omega(\mu, \nu, x) = t^{-1}(\nu)t^{-1}(\mu)\mathrm{Tr}_{ab}\left[T_a^{[L,x+1]}(\mu)T_b^{[L,x+1]}(\nu)\Theta_{a,b}(\mu, \nu)T_a^{[x,1]}(\mu)T_b^{[x,1]}(\nu)\right]. \tag{6.35}$$

where a and b are two different auxiliary spaces and

$$\Theta_{a,b}(\mu, \nu) = (-i)R_{b,a}(\nu, \mu)\partial_\mu R_{a,b}(\mu, \nu) \tag{6.36}$$

is an operator insertion acting only on the auxiliary spaces. A pictorial representation of $\Omega(\mu, \nu, x)$ is given in Fig. 15. Note that apart from the t^{-1} operators Ω is a "double row"

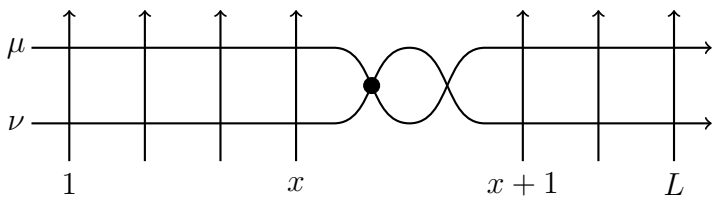


Figure 15: The operator $\Omega(\mu, \nu, x)$ positioned at site x. Here periodic boundary conditions are assumed in the horizontal direction. As before, the dot denotes derivative with respect to a rapidity parameter. Altogether the operator insertion in the middle is given by eq. (6.36).

matrix product operator, and the only difference compared to a product of two transfer matrices is the insertion $\Theta_{a,b}(\mu, \nu)$, which couples two monodromy matrices.

Taking a ν-derivative on the l.h.s. of (6.34) we recognize the continuity equation (6.33) and identify

$$J(\mu, \nu, x) = -t(\nu)\partial_\nu \Omega(\mu, \nu, x-1)t^{-1}(\nu). \tag{6.37}$$

Let $|\Psi\rangle$ be an arbitrary eigenstate of the commuting transfer matrices. For the mean values we get:

$$\langle\Psi|J(\mu, \nu, x)|\Psi\rangle = -\partial_\nu \langle\Psi|\Omega(\mu, \nu, x-1)|\Psi\rangle. \tag{6.38}$$

This connects the ν-derivatives of $\Omega(\mu, \nu, x)$ to the current mean values. To complete the picture, it can be shown that the initial value at $\nu = 0$ is given by $\Omega(\mu, 0, x) = q(\mu, x)$. Thus Ω not only describes all (generalized) currents, but also all charge densities.

The construction of the operator $\Omega(\mu, \nu)$ is the central result of [29]. Even though all its constitutents (the Lax operators and their derivatives) are well known, this particular combination did not appear earlier in the literature.

It was shown in [29] that $\Omega(\mu, \nu)$ is symmetric with respect to its variables iff the R-matrix is of the difference form:

$$R(\mu, \nu) = R(\mu - \nu). \tag{6.39}$$

Examples for this are the fundamental $SU(N)$-invariant models and the XXZ and XYZ chains. A famous counter-example is the one dimensional Hubbard model.

The mean values of $\Omega(\mu, \nu)$ can be obtained with a trick that was originally developed in [327]: it can be shown that the mean values are related to a transfer matrix eigenvalue in an auxiliary problem, namely in an enlarged spin chain with two extra sites. This is a rather technical computation, and we do not discuss its details. Let us however present the end result: it was shown in [29] that

$$\langle \boldsymbol{\lambda}_N|\Omega(\mu, \nu)|\boldsymbol{\lambda}_N\rangle = \Psi(\mu, \nu) + \ldots, \tag{6.40}$$

where $\Psi(\mu, \nu)$ is the function defined in (5.17) and the dots signal correction terms that decay for small μ, ν at least as μ^L or ν^L.

Looking at the connection (6.38), and considering the expansion of $\Psi(\mu, \nu)$ we can see that the statement (6.11) is indeed reproduced by this computation. The correction terms mentioned above do not influence this computation, because for both the charges

and the currents we are only considering cases that fit into a given volume, thus we do not consider terms that are higher order in μ or ν.

The relation (6.40) might seem somewhat misterious at first. After all the operator $\Omega(\mu,\nu)$ was constructed to describe all the current operators, which are local operators with varying range. On the other hand, it was mentioned in Section 5.2 that the function $\Psi(\mu,\nu)$ originates in the two-site density matrix of the inhomogeneous spin chain. This seems to be a rather different situation.

In the XXX case the Taylor coefficients of $\Psi(\mu,\nu)$ are the building blocks for the factorized correlation functions (in the XXZ case an other function $\omega'(\mu,\nu)$ is also needed). The mean value of a generic short range operator is expressed as sums of products of these Taylor coefficients. Our computation shows that **the generalized current operators are those special operators whose mean values are linear in** $\Psi(\mu,\nu)$.

It is possible to give a more direct connection between $\Omega(\mu,\nu)$ and the theory of factorized correlation functions. It was shown in [29] that if we define $\Omega(\mu,\nu)$ for an *inhomogeneous chain* with an analogous formula, and then specify the parameters μ,ν to two selected inhomogeneities, then actually we obtain a certain component of the corresponding two-site reduced density matrix. To be more precise, the following can be proven in a straightforward way:

$$\Omega(\xi_1,\xi_2,x=2)=\Theta_{1,2}(\xi_1,\xi_2). \tag{6.41}$$

This means that for these special values $\Omega(\mu,\nu,x)$ becomes an ultra-local operator acting only on the first two sites. This is the direct bridge to the theory of factorized correlation functions. The result (6.41) is analogous to the "solution of the inverse problem" [328,329], where the monodromy matrix elements can be specialized such that they become ultra-local operators acting on single sites only.

The great advantage of the construction of [29] is that it uses basic and microscopic objects such as fundamental R-matrices, and that the constuction is largely model independent. It can be considered as a natural extension of the basics of Algebraic Bethe Ansatz to accomodate the current operators as well. The direct connection to the factorized correlation functions is very promising, and it might advance the theory of correlation functions in more complicated models such as the nested Bethe Ansatz systems.

6.6 Open problems

Generalized Hydrodynamics is a recent but already quite advanced theory. In this thesis we did not review the tremendous progress within GHD. Therefore the list of open problems that we discuss here are also limited to the topics where the author contibuted. However, we would like to draw the attention of the reader to a Special Issue of the journal JSTAT, which is expected to appear in 2021 (after the completion of this thesis), and which is planned to give a complete review of the topic.

Regarding the current mean values it would be interesting to consider models without particle conservation (such as the XYZ model) or some more complicated nested systems such as the Hubbard model. Investigating the currents is certainly possible with the extension of the methods discussed here. Looking further, it would be important to derive more rigorous proofs for the Drude weights of generic charges, see [109,330]. Perhaps this

could be achieved by some generalization of the method of [299], in combination with the insights of our works [25, 29].

And looking even further, it would be interesting to derive rigorous conditions for the existence of fluid cells, which underlies all computations in GHD. At present this is just a well justified assumption.

The reader might argue that deriving actual proofs is not that important, once the physical processes are well understood. GHD has already shown many examples, where the actual statements were found quite soon and the proofs were supplied only much later, if they have been found at all. However, the field of integrability is special, because generally we have many computations under control, and the author believes that it is worthwhile to push forward the rigorous computations. Sometimes new connections to other fields are uncovered along the way, as the recent papers [25, 29] have clearly shown.

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