Paths and cycles in graphs and hypergraphs

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Contents

Introduction

The aim of this dissertation is to present a selection of papers that may demonstrate the impact of my work on the research area. All of my papers are connected to some extent to graphs, hypergraph, algorithms, or complexity; however, I have primarily worked in four narrower areas with various coauthors. I selected two of these for the dissertation.

Part I contains the results that are connected to the generalization of Hamiltonian paths and cycles for hypergraphs. It is based on 7 papers that I wrote with different coauthors. In my opinion, this is the topic that made the most impact; many other researchers started to work on similar problems, and our papers got a substantial number of references.

In Part II is based on 5 papers that I coauthored, all connected to various generalizations of matchings in graphs. These papers also received a good amount of interest and references.

In both Parts each paper is presented in a separate Chapter since the topics are clearly distinguishable in most cases. The results of Chapters 8-10 are more closely related. Nevertheless, I decided to keep them in separate Chapters for consistency.

In the most recent years, I mostly worked in two different areas. First, with my Ph.D. student László F. Papp and other coauthors, we wrote 6 papers about pebbling of graphs. Second, with my other Ph.D. student Kitti Varga and various coauthors, we studied different problems about minimally tough graphs. The results appeared in 3 papers, and we also have some other forthcoming ones. In my opinion, these results also deserve attention and are valuable. However, since they only appeared recently, there is less evidence for their impact, so I did not include these in this dissertation.

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I would like to thank my father Gyula O.H. Katona for showing me the beauty of mathematics at an early age, and not showing me the less appealing aspects of being a mathematician. Also for his continues support and stimulation. In high-school learned a lot from László Surányi and Márta Vincze. I express my deep gratitude to my mentor András Recski who influenced my career in many ways. I thank all my 36 coauthors from 7 countries. Especially Mikio Kano whom I worked with at Ibaraki University, Japan for 2 years, and Glenn Hurlbert who invited me to Arizona State University for 1 year.

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Notation

In this section the reader finds notations and definitions that are used throughout the dissertation. More specific ones, that are only used in one or two Chapters, will be given in the corresponding Chapter.

For graphs and hypergraphs, the standard notation will be used. However, for completeness some of the basic notations and notions are collected here.

Unless stated otherwise, we assume that a graph is simple, i.e., there are no loops and multiple edges. For a graph G, we denote its vertex and edge sets by $V(G)$ and $E(G)$, respectively. For two vertices x and y of G, we write xy or yx for an edge joining x to y. The number of vertices and edges is denoted by $v(G)$ and $e(G)$. The neighborhood of a vertex $v \in V(G)$ is denoted by $N_G(v) = \{u \in V(G) \mid \exists u\ (vu \in E(G)\}$. The degree of vertex v is the size of its neighborhood, denote it with $d_G(v) := |N_G(v)|$. The minimum degree of G is denoted by $\delta(G)$. For a set of vertices $X \subseteq V(G)$ let $N_G(X) := \cup_{v \in X} N_G(v)$. In these notations, if the context is clear, we omit the graph G. For $X \subseteq V(G)$, let $G[X]$ be the subgraph of G induced by X.

A *path* is a sequence of distinct vertices with the property that each vertex in the sequence is adjacent to the vertex next to it. A *cycle* is path that begins and ends at the same vertex. A *Hamiltonian-path* and a *Hamiltonian-cycle* is a path or cycle that contains all vertices of the graph.

The complete graph on n vertices is denoted by K_n.

We also define the notion of a hypergraph, a generalization of graphs. The vertex set $\{v_1, v_2, \ldots, v_n\}$ of a hypergraph $\mathcal{H}$ is denoted by $V(\mathcal{H})$. v_{n+x} with $x \geq 0$ denotes the same vertex as v_x for simplicity of notation if not stated otherwise. The edge set $\{E_1, E_2, \ldots, E_m\}$, denoted by $\mathcal{E}(\mathcal{H})$, is a collection of distinct subsets of $V(\mathcal{H})$. We will write simply V for $V(\mathcal{H})$ and $\mathcal{E}$ for $\mathcal{E}(\mathcal{H})$ if no confusion can arise. We say that $\mathcal{H}$ is r-uniform if $\mathcal{E}(\mathcal{H})$ consists only of r-element subsets of $V(\mathcal{H})$. Thus a 2-uniform hypergraph is a simple graph. Denote by $\mathcal{H}(U)$ the subhypergraph of $\mathcal{H}$ induced by U, where $U \subseteq V(\mathcal{H})$.

The complete r–uniform hypergraph on n vertices is denoted by $\mathcal{K}_n^{(r)}$.

Finally $\lfloor x \rfloor$ denotes the largest integer which is not larger than x and $\lceil x \rceil$ is the smallest integer which is not smaller than x, as usual.

Part I

Hamiltonian chains

This part contains results connected to the generalization of Hamiltonian paths and cycles for hypergraphs. In my Candidate degree dissertation [76] I gave a new generalization of the Hamiltonian cycle for r-uniform hypergraphs: A cyclic ordering of the vertices is called a *Hamiltonian chain* iff any r consecutive vertices in this order form an edge of the hypergraph. A hypergraph is called *Hamiltonian* if it contains a Hamiltonian chain. Part I contains results connected to this notion.

In [76] I already proved a Dirac-type theorem for this new notion. Later, with Kierstead, we proved a stronger and more general version of this, which is the main result of Chapter 1: *If the minimum co-degree is at least* $(1-\frac{1}{2r})n+4-r-\frac{5}{2r}$, *then the hypergraph contains a Hamiltonian chain (Theorem 1.2.1).* The rest of Chapter 1 contains bounds on the maximum number of edges in a hypergraph containing no Hamiltonian chain and results about a different kind of generalization of Hamiltonian cycles for hypergraphs called *Hamiltonian nets.*

In Chapter 2 we define the following notion: *A hypergraph is* k-edge-Hamiltonian *if by the removal of any* k *edges a Hamiltonian hypergraph is obtained.* The main aim of this Chapter is to investigate minimum size k-edge-Hamiltonian hypergraphs. We prove that the number of edges in a 3-uniform 1-edge-Hamiltonian hypergraph is between $\frac{14}{9}n$ and $\frac{11}{6}n$ (Theorem 2.2.1 and 2.2.2). We also proved bounds for general r when $k=1$. In the proof of Theorem 2.4.1 we apply these results to give a better upper bound for the maximum number of edges in a r-uniform containing no Hamiltonian chain: $\binom{n}{r}\left(1-\frac{4r}{(4r-1)n}\right)$.

In Chapter 3 we investigate a graph theory question that arose in the previous Chapter. If the graph G has the property that removing any k edges of G, the resulting graph still contains a subgraph isomorphic to P_4, then we say that G is *k-stable.* $S(n,k)$ denotes the minimum number of edges in a k-stable graph on n vertices, and $S(k)$ denotes the minimum number of edges in any k-stable graph. The main result of this Chapter is Theorem 3.2.1, we determine the exact value of this function: $S(1)=4$, *and if* $k\geq 2$, *then* $S(k)=k+\left\lceil\sqrt{2k+\frac{9}{4}}+\frac{3}{2}\right\rceil$.

In Chapter 4 we aim to generalize a famous result of Gallai for hypergraphs: what is the maximum number of edges in graph containing no path of length k. The notion of path can be generalized in a similar way to chains. Theorem 4.1.6 gives bounds for this case. However, the main results of this Chapter are *Theorems 4.1.3 and 4.1.4 that give asymptotically tight upper bounds in case of Berge-paths. The maximum number of edges in an r-uniform hypergraph containing no Berge path of length k is* $\frac{n}{k}\binom{k}{r}$ *if* $k>r+1>3$, *and* $\frac{n(k-1)}{r+1}$ *if* $r\geq k>2$.

Chapter 5 deals with a question about Hamiltonian paths in graphs, while Chapter 6 contains results of the corresponding question for hypergraphs. *What is the maximum number of edges in a graph (or hypergraph) that contain no Hamiltonian path, but adding any new edge creates one?* The main results of Chapter 5 are Theorem 5.3.1 which gives a nearly sharp lower bound in the graph case: $\lfloor\frac{3n-1}{2}\rfloor-2$, and Theorem 5.3.1 that gives a close upper bound: $\lfloor\frac{3n-1}{2}\rfloor$. We also present results for a more general case, when instead of Hamiltonian paths, we consider m-path covers (Theorem 5.2.4).

In Chapter 6 we investigate the related question for hypergraphs. We construct a 3-uniform hypergraph with $O(n^{5/2})$ edges (Theorem 6.3.7). On the other hand, we have a general lower bound of order $\Omega(n^{r-1})$.

Chapter 1

Dirac-type theorem for Hamiltonian chains

1.1 Introduction

The aim of this Chapter is to show some hypergraph analogues of known results connected to Hamiltonian paths and cycles of graphs.

The earliest result about the Hamiltonian cycle in graphs is the following theorem of Dirac.

Theorem 1.1.1 (Dirac [37]). *If G is a graph on $n \geq 3$ vertices such that $d_G(v) \geq \frac{1}{2}n$ for every vertex v in G (i.e. $\delta(G) \geq \frac{1}{2}n$), then G contains a Hamiltonian cycle.*

This theorem is best possible in the sense that the bound $\frac{1}{2}n$ cannot be relaxed without destroying the conclusion of the theorem. This is shown by the complete bipartite graphs $K_{m,m+1}$ ($m \geq 1$).

Is there an analogue of this for hypergraphs? First of all, it is not clear how to generalize the notion of a Hamiltonian cycle for hypergraphs. Also, there is a number of different ways to define the minimum degree in hypergraph as well.

The first such definition and result is due to Bermond et al. in [17]. They defined the Hamiltonian cycle in hypergraphs the following way.

Definition 1.1.2 (Bermond et al. [17]). *A $(v_1, v_2, \ldots, v_n)$ cyclic sequence of the vertex set is called a* hypergraph Hamiltonian cycle *iff for every $1 \leq i \leq n$ there exists an edge E_j of $\mathcal{H}$ such that $\{v_i, v_{i+1}\} \subseteq E_j$.*

Since an ordinary graph is a 2-uniform hypergraph, this definition gives the definition of the Hamiltonian cycle in a graph. The *total degree* of a vertex v in a hypergraph, by definition, is the number of edges containing v. It is denoted by $\delta_{\mathcal{H}}(v)$. Bermond et al. proved a Dirac type theorem. It is worth noting that this does not give the best possible result for $r = 2$.

Theorem 1.1.3 (Bermond et al. [17]). *Let $\mathcal{H}$ be a r–uniform hypergraph on $n \geq r+1$ vertices. If $\delta_{\mathcal{H}}(v) \geq \binom{n-2}{r-1} + r - 1$ holds for every vertex v of the hypergraph then $\mathcal{H}$ contains a Hamiltonian cycle.*

In my Candidate degree dissertation [76] I gave a different definition as an other generalization of the Hamiltonian cycle.

Definition 1.1.4. *Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph. A cyclic ordering $(v_1, v_2, \dots, v_n)$ of V is called a* Hamiltonian chain *iff $\{v_i, v_{i+1}, \dots, v_{i+r-1}\} \in \mathcal{E}$ whenever $1 \le i \le n$. (Note that v_{n+x} denotes the same vertex as v_x). We call a r-uniform hypergraph r-*maximal *if it does not contain a Hamiltonian chain, but the addition of any new r-edge produces a Hamiltonian chain. An ordering $(v_1, v_2, \dots, v_{l+1})$ of a subset of the vertex set is called an* open chain of length l *between v_1 and v_{l+1} iff $\{v_i, v_{i+1}, \dots, v_{i+r-1}\} \in \mathcal{E}$ whenever $1 \le i \le l - r + 2$.*

Generalizations of path, cycle and Hamiltonian path are defined analogously. This definition was motivated by two different problems. In [36] a similar, however more complicated problem arose in connection with database theory. The other motivating problem concerns the powers of Hamiltonian cycles. The rth power of a cycle C is obtained by joining every pair of vertices with distance at most r in C. Seymour conjectured in [96] that G contains the rth power of a Hamiltonian cycle if $d(v) \ge \frac{r}{r+1}n$ holds for every vertex in the graph. Let us transform a r–uniform hypergraph to a simple graph by replacing every hyperedge by a complete graph on the same r vertices and replacing multiple edges with a single edge. In this way a Hamiltonian chain is transformed to the $(r-1)$th power of a Hamiltonian cycle. However, it is not true that if the graph contains the $(r-1)$th power of a Hamiltonian cycle then the original hypergraph contains a Hamiltonian chain. Therefore the results in [48, 49] about the square of the Hamiltonian cycle does not imply anything for Hamiltonian chains. On the contrary, our result implies a result for the square of a Hamiltonian cycle with a different degree condition.

There are many more possibilities to modify the definitions. Note that in these paths and cycles the consecutive edges must intersect in $r-1$ vertices. In a more general definition the size of the intersection is required to be another constant t, these are called *t-tight r-paths and cycles.* In the past 20 years more than 150 papers appeared involving one of these notions. See [84, 108, 115, 130] for some surveys on these results.

When $t = r - 1$, then this is the same as our definition for chains, however, the literature usually calls it *tight paths and cycles.* On the other hand, the in case of $t = 1$ it is called a *loose path or cycle.* In the present work we mostly deal with the case of chains. Since in the later Chapters some notions will be derived from this, it is more convenient to use our original name despite the more popular other name.

In this Chapter we will also use yet an other special notion.

Definition 1.1.5. *Let $\mathcal{H}$ be a 4–uniform hypergraph. A cyclic ordering $(v_1, v_2, \dots, v_n)$ of the vertex set is called a* Hamiltonian net *iff for every $1 \le i \le n$ and $1 \le j \le n$ for which $|i - j| \ge 2$ there exists an edge E_m of $\mathcal{H}$ such that $\{v_i, v_{i+1}, v_j, v_{j+1}\} = E_m$. (Again, v_{n+x} with $x \ge 0$ denotes the same vertex as v_x.)*

In order to give Dirac type results the definition of the degree must be generalized, too. It is defined now in full generality, however, only some special cases will be used.

Definition 1.1.6. *The* degree of a fixed ℓ–tuple *of distinct vertices, $\{v_1, \dots, v_\ell\}$, in a r–uniform hypergraph is the number of edges of the hypergraph containing all $\{v_1, \dots, v_\ell\}$. It is denoted by $d^{(r)}(v_1, v_2, \dots, v_\ell)$. Furthermore $\delta_\ell^{(r)}(\mathcal{H})$ denotes the minimum value of $d^{(r)}(v_1, v_2, \dots, v_\ell)$ over all ℓ-tuples in $\mathcal{H}$. (The superscript will be omitted if it clear from the context.) The* neighborhood *of a vertex v is defined by*

$$N_{\mathcal{H}}(v) := \{E - \{v\} \mid v \in E, E \in \mathcal{E}(\mathcal{H})\}.$$

A Dirac type theorem is proved for r–uniform hypergraphs in Section 1.2. In the special case $r = 3$ a smaller bound is proved to be sufficient. A natural extremal question is discussed in Section 1.4: What is the maximum number of edges in a r–uniform hypergraph which does not contain a Hamiltonian chain? Section 1.3 contains a construction which shows that Hamiltonian nets differ from Hamiltonian chains very much. These results appeared in [K11].

Rödl and Ruciński write in [108]: *"In 1952 Dirac [37] proved a celebrated theorem... In 1999, Katona and Kierstead initiated a new stream of research to studying similar questions for hypergraphs, and subsequently, for perfect matchings..."*

I mention only a few important results in this area that are more closely related to our result. Let $h^t(r,n)$ denote the smallest integer m such that every r-uniform hypergraph $\mathcal{H}$ on n vertices with $\delta_{r-1}(\mathcal{H}) \geq m$ contains a t-tight Hamiltonian r-cycle, provided that $r-t$ divides n. Thus we showed that

$$\left\lfloor \frac{n-r+3}{2} \right\rfloor \leq h^{r-1}(r,n) \leq \left(1 - \frac{1}{2r}\right) n + O(1).$$

Rödl, Ruciński and Szemerédi in [109, 111] showed that $h^{r-1}(r,n) = \frac{n}{2} + o(1)$ for all $k \geq 3$. Later [112] they were able to obtain the exact result when $k = 3$: $h^2(3,n) = \lfloor \frac{n}{2} \rfloor$. Markström and Ruciński showed [94] that if $0 \leq t \leq r-1$, $r-t \mid n$, $r-t \mid r$ then this implies the more general result: $h^t(r,n) = \frac{n}{2} + o(1)$.

When $r - t \nmid r$ then the bound is different. Kühn and Osthus [83] proved that $h^1(3,n) = \frac{n}{4} + o(n)$. This was generalized to arbitrary r and t by Kühn, Mycroft and Osthus [82]:

$$h^t(r,n) = \frac{n}{\lceil \frac{r}{r-t} \rceil (r-t)} + o(1) \quad \text{if } r-t \nmid r.$$

Czygrinow and Molla [34] showed that $h^1(3,n) = \lceil \frac{n}{4} \rceil$, independently Han and Zhao [64] proved that $h^t(r,n) = \lceil \frac{n}{2r-2t} \rceil$ for all $t < \frac{r}{2}$.

It is worth mentioning that all these proofs assume that the number of vertices in the hypergraph is huge. On the other hand, our proof works for smaller size hypergraphs, too.

1.2 Hamiltonian Chains in r–uniform Hypergraphs

In [76] I proved a Dirac-type theorem for 3-uniform hypergraphs. This result was not published apart from the dissertation, since soon after we joined forces with Hal Kierstead and proved a more general and better result.

Theorem 1.2.1 (Katona, Kierstead [K11])**.** *If $\mathcal{H} = (V, \mathcal{E})$ is a r-uniform hypergraph on n vertices with $\delta_{r-1}(\mathcal{H}) > (1 - \frac{1}{2r})n + 4 - r - \frac{5}{2r}$, then $\mathcal{H}$ contains a Hamiltonian chain.*

Proof. Let $c = 4 - r - \frac{5}{2r}$ and $\alpha = \frac{n}{2} + r^2 + (c-3)r + 2$. Then $2\alpha - n = 2r - 1$. For any $(2r-2)$–permutation $(s_1, \ldots, s_{2r-2})$ of V, let

$$R(s_1, \ldots, s_{2r-2}) = \{w \in V \mid (s_1, \ldots, s_{r-1}, w, s_r, \ldots, s_{2r-2}) \text{ is an open chain}\}.$$

Lemma 1.2.2. *For any permutation $(s_1, \ldots, s_{2r-2})$ of V, $|R(s_1, \ldots, s_{2r-2})| > \alpha$.*

Proof. Let $R = R(s_1, \ldots, s_{2r-2})$ and suppose that $w \notin R$. Then there exists $j \in \{1, \ldots, r\}$ such that $\{w\} \cup \{s_j, \ldots, s_{j+r-2}\} \notin \mathcal{E}$. Since $\delta_{r-1}(\mathcal{H}) > (1 - \frac{1}{2r})n + c$, there are less than $\frac{n}{2r} - c$ such bad w for any j. Since $\{s_i\} \cup \{s_j, \ldots, s_{j+r-2}\} \notin \mathcal{E}$ if $j \leq i \leq j + r - 2$, $r - 1$ of these bad w come from $S \cup T$. Thus there are less than $\frac{n}{2r} - c - r + 1$ such bad w in $V - S - T$. Considering all r possibilities for j, there are less than $\frac{n}{2} - r^2 - (c - 1)r$ vertices $w \in V - S - T$ for which $w \notin R$. Since $|V - S - T| = n - 2r + 2$ the lemma follows. □

Suppose for a contradiction that $\mathcal{H} = (V, \mathcal{E})$ is a r-maximal hypergraph. Then $r \geq 3$ by Dirac's Theorem and there exists a cyclic ordering $C = (v_1, \ldots, v_n)$ of V such that $\{\{v_i, \ldots, v_{i+r-1}\} \mid 1 \leq i \leq n - 1\} \subset \mathcal{E}$. (But $\{v_n, v_1, \ldots, v_{r-1}\} \notin \mathcal{E}$.) Let $R_i = R(v_{i-r+1}, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{i+r-1})$ and $D(v) = |\{i \mid v \in R_i\}|$. By the pigeon hole principle and the lemma, there exists an index i such that $D(v_i) > \alpha$.

First suppose that $i \in \{n, 1, \ldots, r - 1\}$. Then either $B = (v_i, \ldots, v_{i+n-1})$ or $B' = (v_{i+1}, \ldots, v_i)$ is an open chain. We may assume that it is B. Since $|R_i| + D(v_i) - n > 2\alpha - n \geq 0$, there exists an index r such that $v_i \in R_r$ and $v_r \in R_i$. If $i = r$ then B is a Hamiltonian chain; otherwise $(v_{i-1}, v_r, v_{i+1}, \ldots, v_{r-1}, v_i, v_{r+1}, \ldots, v_{i-2})$ is a Hamiltonian chain. In either case we have a contradiction.

Now suppose that $i \notin \{n, 1, \ldots r - 1\}$. Choose $h \in \{1, r - 1\}$so that the two segments $S = (v_h, \ldots, v_{i-1})$ and $T = (v_{i+1}, \ldots, v_{h-1})$ of $C - v_i$ each have length at least $r - 1$. By reversing the ordering if necessary, we can assume that $h = 1$. Let $Q = R(v_{n-r+2}, \ldots, v_n, v_1, \ldots, v_{r-1})$. Since $|R_i| + |Q| - n > 2\alpha - n \geq 2$, there exists an index $j \notin \{i, i - 1\}$ such that $v_j \in R_i$ and $v_{j+1} \in Q$. Without loss of generality, assume that $i < j$ and let $B = (v_{j+2}, \ldots v_n, v_{j+1}, v_1, \ldots, v_{i-1}, v_j, v_{i+1}, \ldots, v_{j-1})$. By the choice of j and the fact that $|S|, |T| \geq r - 1$, B is an open chain of length $n - 2$ (v_i is missing). Let $P = R(v_{j-r+1}, \ldots, v_{j-1}, v_{j+2}, \ldots, v_{j+r})$ and $X = \{v_p \in P \mid v_i \in R_p\}$. Then $|X| > 2\alpha - n = 2r - 1$. Suppose that $v_p \in X$. Since $v_i \in R_p$, $p \notin \{i - r + 1, \ldots, i + r - 1\} \setminus \{i\}$. Since $v_p \in P$, $p \notin \{j - r + 1, \ldots, j + r\} \setminus \{j, j + 1\}$. Moreover $p \notin \{i, j\}$, since otherwise $(v_{j+2}, \ldots v_n, v_{j+1}, v_1, \ldots v_{i-1}, v_j, v_{i+1}, \ldots v_{j-1}, v_i)$ or $(v_{j+2}, \ldots v_n, v_{j+1}, v_1, \ldots, v_j)$ is a Hamiltonian chain. Thus there exists $v_p \in X$ such that

$$p \notin \{i - r + 1, \ldots, i + r - 1\} \cup \{j - r + 1, \ldots, j + r - 1\} \cup \{1, \ldots, r - 1\} \cup \{n - r + 2, \ldots, n\}.$$

It follows that B' is a Hamiltonian chain, where B' is obtained from B by replacing v_p with v_i and then putting v_p at the end of the resulting chain. This contradiction completes the proof. □

The bound given in Theorem 1.2.1 is probably not best possible. However, it is quite easy to show that this bound cannot be lower than $\lfloor \frac{n-r+1}{2} \rfloor$.

Theorem 1.2.3 (Katona, Kierstead [K11]). *For all integers r and n such that $2 \leq r$ and $r^2 < n$, there exists a r-maximal hypergraph $\mathcal{H} = (V, \mathcal{E})$ on n vertices such that*

$$\delta_{r-1}(\mathcal{H}) \geq \left\lfloor \frac{n - r + 1}{2} \right\rfloor.$$

Proof. First we construct a graph $\mathcal{H}' = (V', \mathcal{E}')$ on $n - 1$ vertices with $\delta_{r-1}(\mathcal{H}') \geq \lfloor \frac{n-r-1}{2} \rfloor$ such that $\mathcal{H}'$ does not have an open Hamiltonian chain. Then $\mathcal{H}$ will be the result of adding a new vertex v to V' and all r-sets containing v to $\mathcal{E}'$.

Partition V into sets $X = \left\{x_1, \ldots, x_{\lfloor \frac{n-1}{2} \rfloor}\right\}$ and $Y = \left\{y_1, \ldots, y_{\lceil \frac{n-1}{2} \rceil}\right\}$. Let $\mathcal{E}$ consist of all r-subsets $E \subset V$ such that $|E \cap X| \neq \left\lfloor \frac{r-1}{2} \right\rfloor$.

Suppose that $(v_1, \ldots, v_{n-1})$ is an open Hamiltonian chain in $\mathcal{H}$. Now let $E_i = \{v_i, \ldots, v_{i+r-1}\}$ and $\xi_i = |E_i \cap X|$. It is easy to see that $|\xi_{i+1} - \xi_i| \leq 1$. Consider the following inequalities.

$$r \left\lfloor \frac{n-1}{2} \right\rfloor - \binom{r}{2} \leq \sum_{i=1}^{n-r} \xi_i \leq r \left\lfloor \frac{n-1}{2} \right\rfloor$$

The upper bound holds because each vertex of X is counted once for each edge E_i containing it and each vertex is contained in at most r edges E_i. The lower bound holds because every vertex of X is counted at least once and only the last $i \leq r$ vertices of X can be counted at most i times. Since $\xi_i \neq \left\lfloor \frac{r-1}{2} \right\rfloor$ for $1 \leq i \leq n-r$, either $\xi_i \leq \left\lfloor \frac{r}{2} \right\rfloor - 1$ for all $1 \leq i \leq n-r$ or $\xi_i \geq \left\lfloor \frac{r}{2} \right\rfloor + 1$ for all $1 \leq i \leq n-r$. In the first case

$$r\frac{n-2}{2} - \binom{r}{2} \leq r \left\lfloor \frac{n-1}{2} \right\rfloor - \binom{r}{2} \leq \sum_{i=1}^{n-r} \xi_i \leq (n-r) \left\lfloor \frac{r-2}{2} \right\rfloor \leq (n-r)\,\frac{r-2}{2}$$

This is a contradiction since $n > \frac{3}{2}r$. In the second case

$$(n-r)\,\frac{r+1}{2} \leq (n-r) \left\lfloor \frac{r+2}{2} \right\rfloor \leq \sum_{i=1}^{n-r} \xi_i \leq r \left\lfloor \frac{n-1}{2} \right\rfloor \leq r\frac{n-1}{2}.$$

This is a contradiction since $n > r^2$.

Now consider the degree condition. Let S be any $(r-1)$ -subset of V. First suppose that $|S \cap X| = \left\lfloor \frac{r}{2} \right\rfloor$. Since adding any vertex form $X \backslash S$ to S yields an edge of the hypergraph, $d(S) \geq \left\lfloor \frac{n-1}{2} \right\rfloor - \left\lfloor \frac{r}{2} \right\rfloor$. Next suppose that $|S \cap X| = \left\lfloor \frac{r}{2} \right\rfloor - 1$. Since adding any vertex from $Y \backslash S$ to S yields an edge of the hypergraph, $d(S) \geq \left\lceil \frac{n-1}{2} \right\rceil - \left\lceil \frac{r}{2} \right\rceil$. Finally suppose that $|S \cap X| \notin \left\{\left\lfloor \frac{r}{2} \right\rfloor, \left\lfloor \frac{r}{2} \right\rfloor - 1\right\}$. Since adding any vertex from $V \backslash S$ to S yields an edge, $d(S) \geq n - r$. Therefore

$$\delta_{r-1}(\mathcal{H}') \geq \min\left(\left\lfloor \frac{n-1}{2} \right\rfloor - \left\lfloor \frac{r}{2} \right\rfloor, \left\lceil \frac{n-1}{2} \right\rceil - \left\lceil \frac{r}{2} \right\rceil\right) = \left\lfloor \frac{n-r-1}{2} \right\rfloor.$$

To see that $\mathcal{H}$ is r-maximal, consider a non-edge, say

$$E = \left\{x_1, \ldots, x_{\lfloor \frac{r}{2} \rfloor}, y_1, \ldots, y_{\lceil \frac{r}{2} \rceil}\right\} \notin \mathcal{E}.$$

Then $\left(v, x_{\lfloor \frac{n-1}{2} \rfloor}, \ldots, x_1, y_1, \ldots, y_{\lceil \frac{n-1}{2} \rceil}\right)$ is a Hamiltonian chain in $\mathcal{H} + E$. □

1.3 Hamiltonian Nets

The following Theorem shows that a lower bound cn with $c < 1$ for the degrees of triples is not sufficient to force the existence of a Hamiltonian net.

Theorem 1.3.1 (Katona, Kierstead [K11]). *For any given integer $n \geq 6$ there exists a 4-uniform hypergraph $\mathcal{H}$ on n vertices which does not contain a Hamiltonian net and*

$$d(x, y, z) \geq \begin{cases} n-9 & \text{if } n \text{ is odd} \\ n-12 & \text{if } n \text{ is even} \end{cases}$$

holds for any x, y, z triple of vertices.

Proof. The construction of $\mathcal{H}$ is given first for odd n. To shorten notation let $r = \frac{n-1}{2}$. It is a well known fact that one can partition the edges of the complete graph $K_n^{(2)}$ into Hamiltonian cycles so let $E\left(K_n^{(2)}\right) = H_1 \cup \cdots \cup H_r$ where $H_1, \ldots, H_r$ are the edge sets of edge disjoint Hamiltonian cycles. $\mathcal{H}$ is defined on the vertex set of $K_n^{(2)}$. A four element subset of the vertex set is a hyperedge of $\mathcal{H}$ iff these vertices do not span two nonadjacent edges of the same H_i.

Suppose indirectly that $\mathcal{H}$ contains a Hamiltonian net. The cyclic ordering can be considered as a Hamiltonian cycle C in $K_n^{(2)}$ which has $n = 2r+1$ edges, of course. Since the edge set is partitioned into r Hamiltonian cycles there is an H_i which has at least 3 common edges with C and thus there are at least two of them which are nonadjacent. The four end vertices of these two edges do not form a hyperedge of $\mathcal{H}$ by definition, a contradiction.

We prove now that the degree condition holds for $\mathcal{H}$. Let x, y, z be arbitrary distinct vertices of $\mathcal{H}$. An upper bound will be given for the number of those vertices v for which $\{x, y, z, v\} \notin \mathcal{E}(\mathcal{H})$. Each of the three pairs $\{x, y\}, \{x, z\}$ and $\{y, z\}$ is an edge of $K_n^{(2)}$ so each of them is contained in exactly one H_i. Without loss of generality we may assume that $\{x, y\} \in H_1$. z is also a vertex of the Hamiltonian cycle formed by the edges of H_1. It is clear that $\{x, y, z, v\} \notin \mathcal{E}(\mathcal{H})$ iff v is a neighbor of z on this Hamiltonian cycle. If z is not a neighbor of x or y on this Hamiltonian cycle then there are two such vertices, otherwise only one. In the same manner one can prove that there may be at most two other choices for v along the Hamiltonian cycle containing $\{x, z\}$ and two other along the one containing $\{y, z\}$. Thus there are at most 6 "bad" vertices v out of the $n-3$ possible choices proving that $d_3^{(4)}(x, y, z) \geq n-9$ always holds in this case.

The proof is similar when n is even. Now let $r = \frac{n-2}{2}$. Obviously it is not possible to partition the edge set of $K_n^{(2)}$ into Hamiltonian cycles, however, it is also well known that it can be partitioned into r Hamiltonian cycles and a perfect matching. We define a new partition by rearranging this. Let $E\left(K_n^{(2)}\right) = H_1' \cup \cdots \cup H_r' \cup \{e\}$ where each H_i' is the union of a Hamiltonian cycle and one of the edges from the perfect matching (each H_i' is a Hamiltonian cycle with a chord) and $\{e\}$ is the remaining edge of the perfect matching. $H_1' \cup \cdots \cup H_r'$ can also be considered as a partition of $K_n^{(2)} - \{e\}$ into r edge disjoint sets. $\mathcal{H}$ is defined on the vertex set of $K_n^{(2)}$. A four element subset of the vertex set is a hyperedge of $\mathcal{H}$ iff these vertices do not span two nonadjacent edges of the same H_i'.

Suppose indirectly that $\mathcal{H}$ contains a Hamiltonian net. The cyclic ordering can be considered as a Hamiltonian cycle C in $K_n^{(2)}$ which have $n = 2r+2$ edges in this case. It may contain e but it has $2r+1$ edges different from e. Since the edge set is partitioned into r sets, there is an H_i' which has at least 3 common edges with C. There are at least two of them which are nonadjacent because otherwise they would form a triangle which is impossible, since they are edges of a Hamiltonian cycle. The four end vertices of these two edges do not form a hyperedge of $\mathcal{H}$ by definition, a contradiction.

The proof for the degree condition is similar to the previous case, too. The only difference is that if $\{x, y\}$ is an edge of H_1' then it is possible that z has three neighbors if z is an endvertex of the chord. This gives at most 9 "bad" choices of v out of the $n-3$ possibilities. $\square$

1.4 Large Maximal Hypergraphs

In this section we bound the maximum number of edges a r-uniform maximal hypergraph on n vertices can have. Ore [99] proved that there is a unique maximal graph G with $\binom{n-1}{2}+1$ edges. The graph G is formed from a $(r-1)$-clique together with one additional edge incident to the remaining vertex. The following theorem generalizes this construction to larger r, but as we shall see is not optimal when $r=3$.

Theorem 1.4.1 (Katona, Kierstead [K11]). *For all integers r and n with $2 \leq r$ and $2r-1 \leq n$, there exists a r–maximal hypergraph $\mathcal{H}=(V,\mathcal{E})$ on n vertices such that*

$$|\mathcal{E}| = \binom{n-1}{r} + \binom{n-2}{r-2} = \binom{n}{r} - \binom{n-2}{r-1}. \tag{1.1}$$

Proof. For two distinct vertices x and y of V, let $\mathcal{E}$ consist of all r-subsets of V except those which contain x, but do not contain y. It is clear that (1.1) holds.

Suppose to the contrary that $\mathcal{H}$ contains a Hamiltonian chain, say $(v_1,\dots,v_n)$ with $x=v_r$. Then both $\{v_1,\dots,v_r\}$ and $\{v_r,\dots,v_{2r-1}\}$ are edges in $\mathcal{E}$. By the hypotheses on r and n, these are distinct edges whose intersection is $\{x\}$. This is a contradiction since both contain x, but one does not contain y.

To see that $\mathcal{H}$ is maximal, consider any r-nonedge E. Since $E \notin \mathcal{E}$, $x \in E$, but $y \notin E$. Say $E=\{v_1,\dots,v_r\}$, where $x=v_r$. Then any extension of the ordering $(v_1,\dots,v_r,y)$ is a Hamiltonian chain. □

The following theorem improves this lower bound when $r=3$ and 3 divides $n-1$.

Theorem 1.4.2 (Katona, Kierstead [K11]). *For all positive integers n and $q \geq 2$ with $n=3q+1$ there exists a 3-maximal hypergraph $\mathcal{H}=(V,\mathcal{E})$ that satisfies*

$$|\mathcal{E}| = \binom{n-1}{3} + n - 1. \tag{1.2}$$

Proof. Fix $x \in V$ and partition the vertex set $V-\{x\}$ into 3-sets $V_1,\dots,V_q$. The edge set $\mathcal{E}$ consists of all 3-subsets of $V-\{x\}$ together with all subsets of the form $\{x\}\cup Y$, where Y is a 2-subset of some V_i, with $1 \leq i \leq q$. It follows immediately that (1.2) holds.

Suppose to the contrary that $\mathcal{H}$ contains a Hamiltonian chain, say $(v_1,\dots,v_n)$ with $x=v_3$. Then each of the sets $\{v_i,v_{i+1},v_{i+2}\}$, $1 \leq i \leq 3$ is an edge in $\mathcal{E}$ containing x. It follows easily that $\{v_1,v_2,v_4,v_5\}$ is a subset of some V_i with $1 \leq i \leq q$. But this is a contradiction since $|V_i|=3$.

To see that $\mathcal{H}$ is maximal, consider any nonedge E. Since $E \notin \mathcal{E}$, $x \in E$, but $E\backslash\{x\} \subsetneq V_i$, for any i with $1\leq i \leq q$. Say $E=\{v_1,\dots,v_3\}$, where $x=v_3$ and $v_2 \in V_i=\{v_2,v_4,v_5\}$. Then any extension of the ordering $(v_1,\dots,v_5)$ is a Hamiltonian chain. □

It is not hard to verify that if the above example has the most edges among all 3-maximal hypergraphs that contain $K^{(3)}_{n-1}$. Finally we prove an easy upper bound.

Theorem 1.4.3 (Katona, Kierstead [K11]). *If $\mathcal{H}=(V,\mathcal{E})$ is a r–uniform hypergraph on n vertices satisfying*

$$|\mathcal{E}| \geq \frac{n-1}{n}\binom{n}{r} \tag{1.3}$$

then $\mathcal{H}$ contains a Hamiltonian chain.

Proof. Let m denote the number of nonedges of $\mathcal{H}$. By (1.3) we obtain

$$m < \binom{n}{r} - \frac{n-1}{n}\binom{n}{r} = \frac{1}{n}\binom{n}{r}.$$

There are $\frac{1}{2}(n-1)!$ different Hamiltonian chains in $\mathcal{K}_n^{(r)}$ and any edge is contained in $\frac{1}{2}r!(n-r)!$ of these chains. If a Hamiltonian chain in $\mathcal{K}_n^{(r)}$ contains an edge which is not an edge of $\mathcal{H}$ then it is not a Hamiltonian chain in $\mathcal{H}$. Since the number of Hamiltonian chains in $\mathcal{H}$ is

$$\geq \frac{1}{2}(n-1)! - \frac{1}{2}r!(n-r)!m > 0,$$

our claim is proved. $\square$

Tuza improved the lower bound in [132] by giving a construction having $\binom{n-1}{r} + \binom{n-1}{r-2}$ edges that contains no Hamiltonian chain. With Frankl we improved the upper bound, this is presented in Section 2.4. Although the bounds are asymptotically tight, it is still open to find an exact result.

Chapter 2

Extremal k-edge-Hamiltonian hypergraphs

2.1 Introduction

We continue to use the definitions and notations of Chapter 1, but we introduce a new notion now.

Definition 2.1.1. *A hypergraph is* k-edge-Hamiltonian *if by the removal of any k edges a Hamiltonian hypergraph is obtained.*

The main aim of this Chapter is to investigate minimum size k-edge-Hamiltonian hypergraphs. In [100, 124] the authors settle this question for graphs.

Theorem 2.1.2 (Paoli, Wong, Wong, [100, 124])**.** *The number of edges in a minimum k-edge-Hamiltonian graph on $n \geq k+3$ vertices is $\lceil n(k+2)/2 \rceil$.*

Since the degree of any vertex in an r-uniform Hamiltonian chain is r, the minimum degree in a k-edge-Hamiltonian hypergraph is at least $r+k$, so the number of edges is at least $\lceil n(r+k)/r \rceil$. For $r = 2$ this shows that the constructions in the above theorem are best possible. However, for $r > 2$ this lower bound is not best possible.

In Section 2.2 we give upper and lower bounds on the number of edges in a 3-uniform k-edge-Hamiltonian hypergraph for various k values. Section 2.3 contains bounds for general r when $k = 1$. In Section 2.4 we apply these results to give an upper bound for the maximum number of edges in a r-uniform containing no Hamiltonian chain. These results appeared in [K3].

2.2 3-uniform hypergraphs

If a hypergraph contains $k+1$ edge-disjoint Hamiltonian chains, then it is clearly k-edge-Hamiltonian. This observation leads to the trivial upper bound on the minimum number of edges: $(k+1)n$. If $k = 1$ then the following slightly better upper bound is obtained.

Theorem 2.2.1 (Frankl, Katona [K3])**.** *There exists a 1-edge-Hamiltonian 3-uniform hypergraph $\mathcal{H}$ on n vertices with*

$$|\mathcal{E}(\mathcal{H})| = \frac{11}{6}n + o(n).$$

Proof. Let $\mathcal{V}(\mathcal{H}) := \{w_1, \dots, w_p, v_1, \dots, v_q\}$ where $p = \lceil n/6 \rceil$ and $q = n - p$. There are two types of edges in $\mathcal{H}$. The first kind of edges form a chain on $\{v_1, \dots, v_q\}$,

$$\mathcal{E}_1(\mathcal{H}) := \left\{ \{v_i, v_{i+1}, v_{i+2}\} \;\middle|\; 1 \le i \le q \right\}.$$

The second kind connects the rest of the vertices to this chain:

$$\mathcal{E}_2(\mathcal{H}) := \left\{ \{w_i, v_{5(i-1)+j}, v_{5(i-1)+j+1}\} \;\middle|\; 1 \le i \le p, 1 \le j \le 6 \right\}.$$

This means that the neighborhood of w_i is an ordinary graph, a path of length 6 formed by vertices $v_{5(i-1)+1}, \dots, v_{5(i-1)+7}$. The neighborhood of w_{i+1} is also a path of length 6, which begins at $v_{5(i-1)+6}$, so $v_{5(i-1)+6}, v_{5(i-1)+7} \in N(w_i) \cap N(w_{i+1})$ (except maybe for $N(w_1)$ and $N(w_p)$ where the overlap is larger if $6 \nmid n$). Let $\mathcal{E}(\mathcal{H}) := \mathcal{E}_1(\mathcal{H}) \cup \mathcal{E}_2(\mathcal{H})$, then it is clear that $|\mathcal{E}(\mathcal{H})| = q + 6p = n + 5\lceil n/6 \rceil = 11n/6 + o(n)$. (See Fig. 2.1.)

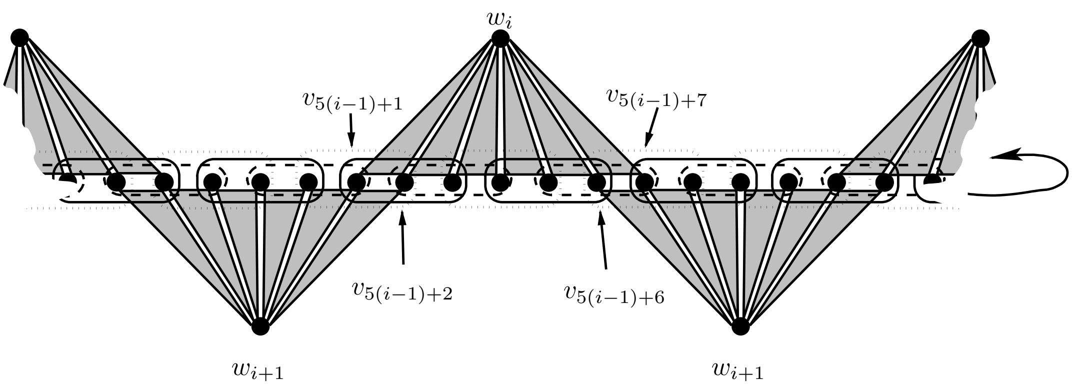

Figure 2.1: 3-uniform 1-edge-Hamiltonian hypergraph

This hypergraph contains many Hamiltonian-chains which can be obtained in the following way. Start with the chain formed by $\{v_1, \dots, v_q\}$ and extend this cycle by inserting the rest of the vertices one by one. It is obvious that we can insert w_i between any two consecutive vertices of $v_{5(i-1)+2}, v_{5(i-1)+3}, v_{5(i-1)+4}, v_{5(i-1)+5}, v_{5(i-1)+6}$ (but we cannot insert it between $v_{5(i-1)+1}$ and $v_{5(i-1)+2}$ or $v_{5(i-1)+6}$ and $v_{5(i-1)+7}$). Note that the new chain contains 3 "consecutive edges" of $N(w_i)$ but it does not contain 2 "consecutive edges" from the original chain (those which contain both neighbors of w_i in the new chain (See Fig. 2.2).

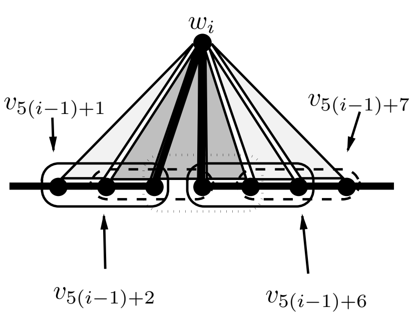

Figure 2.2: How to insert w_i?

Now we prove that $\mathcal{H}$ is 1-edge-Hamiltonian, that is, $\mathcal{H}-E$ contains a Hamiltonian-chain for any $E \in \mathcal{E}(\mathcal{H})$.

Suppose that $E = \{v_t, v_{t+1}, v_{t+2}\} \in \mathcal{E}_1(\mathcal{H})$. Then it is easy to check that there is a w_i which we can insert either between v_t and v_{t+1} or v_{t+1} and v_{t+2}, so the new chain does not contain E any more. Further, we can insert all other w vertices into suitable places, hence we obtain the desired Hamiltonian-chain (see Fig. 2.3), for example the following one

$$v_t, v_{t+1}, w_i, v_{t+2}, v_{t+3}, \dots, v_{t+5}, v_{t+6}, w_{i+1}, v_{t+7}, v_{t+8}, \dots, v_{t+5j}, v_{t+5j+1}, w_{i+j}, v_{t+5j+2}, \dots$$

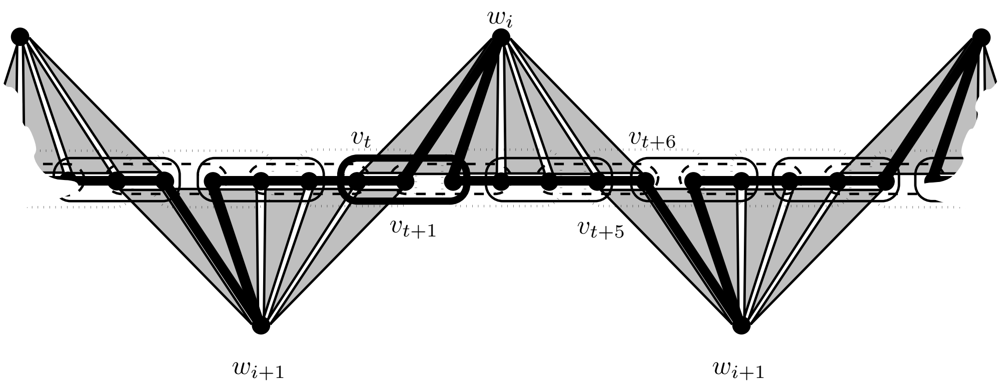

Figure 2.3: Hamiltonian chain in $\mathcal{H} - E$

On the other hand, if $w_i \in E$ for some i then it is clear that $N(w_i) - E$ always contains 3 "consecutive edges", therefore w_i can be inserted into the chain formed by $\{v_1, \dots, v_q\}$. Inserting the rest of the vertices in the same way as in the other case, we obtain a Hamiltonian-chain of $\mathcal{H} - E$. $\square$

Theorem 2.2.2 (Frankl, Katona [K3])**.** *For any 1-edge-Hamiltonian 3-uniform hypergraph $\mathcal{H}$ on $n \geq 5$ vertices*

$$|\mathcal{E}(\mathcal{H})| \geq \frac{14}{9}n$$

holds.

Proof. Observe that the neighborhood of a vertex in a Hamiltonian-chain is a path on 4 distinct vertices, a P_4. Let us call a graph *stable* if it contains a P_4 after deleting any edge of the graph. Thus, the neighborhood of every vertex of a 1-edge-Hamiltonian graph is stable. We also call a vertex of the hypergraph *stable* iff its neighborhood is stable.

It is easy to check that the only stable graph with 4 edges is the C_4, the cycle with 4 edges. All other stable graphs contain at least 5 edges. Clearly any graph which contains C_4 as a subgraph is also 1-stable. There are such graphs. On the other hand, there are 3 other 1-stable graphs with 5 edges without a C_4 (see Fig. 2.4).

Let $\mathcal{H}$ be a 1-edge-Hamiltonian 3-uniform hypergraph and let $v_1, \dots, v_n$ be a Hamiltonian chain.

Claim 2.2.3. $d(v_{i-2}) + d(v_i) + d(v_{i+2}) \geq 14$ *holds for any i.*

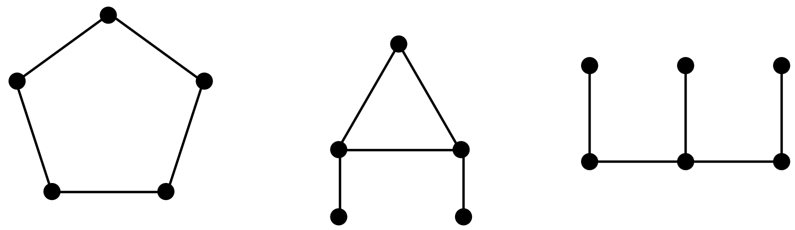

Figure 2.4: Stable graphs with 5 edges

Proof. Note that, the only way to make $|N(v_i)| = 4$ is to add the edge $\{v_i, v_{i-2}, v_{i+2}\}$ to $\mathcal{H}$, because $N(v_i)$ already contains the edges $\{v_{i-2}, v_{i-1}\}$, $\{v_{i-1}, v_{i+1}\}$ and $\{v_{i+1}, v_{i+2}\}$.

Suppose that $d(v_{i-2}) + d(v_i) + d(v_{i+2}) \leq 13$. Since $d(v_j) \geq 4$ for any j, there are only two cases.

If $d(v_{i-2}) = d(v_i) = 4 \leq d(v_{i+2})$ (or $d(v_{i-2}) \geq 4 = d(v_i) = d(v_{i+2})$) then $\{v_{i-2}, v_i, v_{i+2}\} \in \mathcal{E}(\mathcal{H})$ must hold, but this implies $d(v_{i-2}) \geq 5$, a contradiction.

The other case is when $d(v_{i-2}) = d(v_{i+2}) = 4 \leq d(v_i)$. Since v_{i-2} and v_{i+2} is stable, $\{v_{i-4}, v_{i-2}, v_i\}, \{v_i, v_{i+2}, v_{i+4}\} \in \mathcal{E}(\mathcal{H})$ holds. However, this means that $N(v_i)$ contains a path of length 5 with 6 distinct vertices. This is a contradiction, because none of the stable graphs with 5 edges contains such a subgraph, therefore $d(v_i) \geq 6$. □

Using the above claim, we obtain that

$$9|\mathcal{E}(\mathcal{H})| = 3\sum_{i=1}^{n} d(v_i) = \sum_{i=3}^{n+2} d(v_{i-2}) + d(v_i) + d(v_{i+2}) \geq 14n,$$

proving the theorem. □

Theorem 2.2.4 (Frankl, Katona [K3]). *There exists a 2-edge-Hamiltonian 3-uniform hypergraph $\mathcal{H}$ on n vertices with*

$$|\mathcal{E}(\mathcal{H})| = \frac{13}{4}n + o(n).$$

Proof. The structure of the construction is very similar to that of Theorem 2.2.1. Let $\mathcal{V}(\mathcal{H}) := \{w_1, \ldots, w_p, v_1, \ldots, v_q\}$ where $p = \lceil n/4 \rceil$ and $q = n - p$. There are two types of edges in $\mathcal{H}$. The first kind of edges form a chain on $\{v_1, \ldots, v_q\}$,

$$\mathcal{E}_1(\mathcal{H}) := \left\{\{v_i, v_{i+1}, v_{i+2}\} \;\middle|\; 1 \leq i \leq q\right\}.$$

The second kind connects the rest of the vertices to this chain:

$$\mathcal{E}_2(\mathcal{H}) := \left\{\{w_i, v_{4(i-1)+j}, v_{4(i-1)+j+1}\} \;\middle|\; 1 \leq i \leq p, 1 \leq j \leq 9\right\}.$$

This means that the neighborhood of w_i is an ordinary graph, a path of length 9 formed by vertices $v_{4(i-1)+1}, \ldots, v_{5(i-1)+10}$. The neighborhood of w_{i+1} is also a path of length 9, which begins at $v_{4(i-1)+5}$, so the neighborhood of w_i and w_{i+1} have 6 common vertices and the neighborhood of w_i and w_{i+2} have 2 common vertices (except maybe at the "end" where the overlap is larger if $4 \nmid n$). Let $\mathcal{E}(\mathcal{H}) := \mathcal{E}_1(\mathcal{H}) \cup \mathcal{E}_2(\mathcal{H})$, then it is clear that $|\mathcal{E}(\mathcal{H})| = q + 9p = n + 9\lceil n/4 \rceil = 13n/4 + o(n)$. (See Fig. 2.5.)

Using the method described in the proof of Theorem 2.2.1 it can be easily proven, that $\mathcal{H}$ is 1-edge-Hamiltonian. It is also clear that $\mathcal{H}$ remains Hamiltonian if the 2

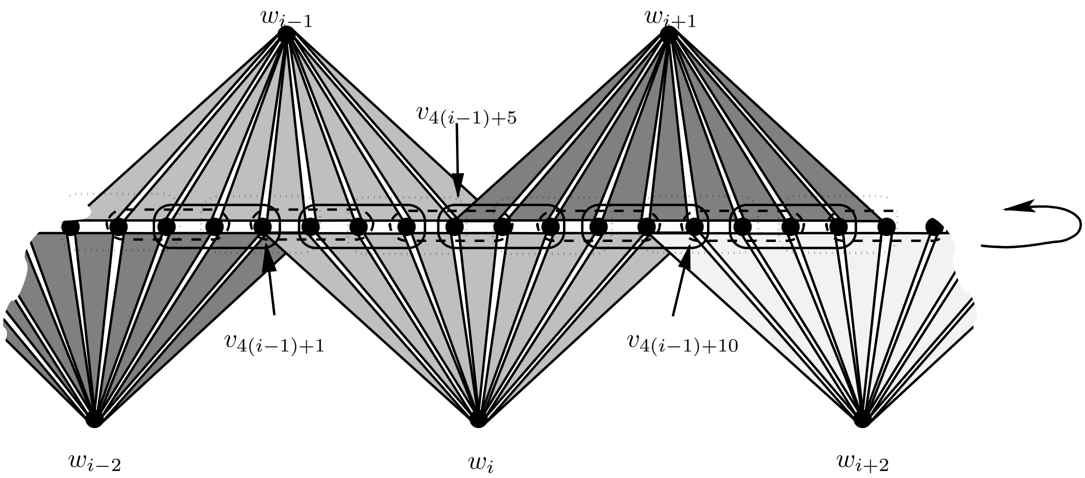

Figure 2.5: 2-edge-Hamiltonian 3-uniform hypergraph

removed edges are "far" from each other, namely if no w_i for which its neighborhood intersects both removed edges.

If both edges contains w_i then we can still insert w_i in a similar way as in Fig. 2.2, since there are 9 edges containing w_i, so after the removal of 2, we still have 3 consecutive.

The other cases can be also proved one by one, the reader may verify this with the help of a few examples in Fig. 2.6.

□

In order to obtain a lower bound for general k, one should know the minimum number of edges in a graph which contains a P_4 after removing any k edges of the graph. We will call such graphs k-*stable* and denote the minimum number of edges in a k-stable graph by $S(k)$.

Once we know $S(k)$ for a particular k, a lower bound can be proven for the number of edges in a k-edge-Hamiltonian 3-uniform hypergraph:

Theorem 2.2.5 (Frankl, Katona [K3]). *For any k-edge-Hamiltonian 3-uniform hypergraph $\mathcal{H}$ on n vertices*

$$|\mathcal{E}(\mathcal{H})| \geq \frac{S(k)}{3}n$$

holds.

Proof. If $\mathcal{H}$ is k-Hamiltonian then the neighborhood of any vertex must be k-stable, which implies that any vertex is contained in at least $S(k)$ edges. Since every edge contains exactly 3 vertices, the claim is proved. □

In [K3] we determined the exact value of $S(k)$ for $1 \leq k \leq 8$, and conjectured the formula for larger k values. With Horváth [K5] we verified this conjecture. It was proven that $S(1) = 4$, and if $k \geq 2$, then

$$S(k) = k + \left\lceil \sqrt{2k + \frac{9}{4}} + \frac{3}{2} \right\rceil.$$

This result is presented in Chapter 3.

Combining this formula with the previous Theorem, we obtain a lower bound which is considerably better than the trivial bound.

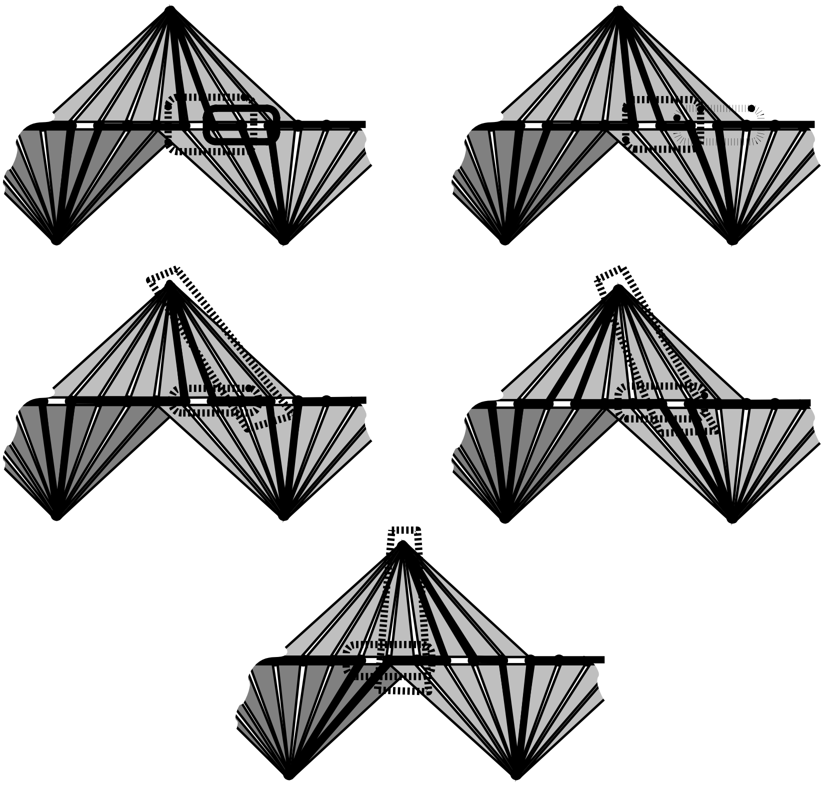

Figure 2.6: Examples of the more complicated cases

2.3 1-edge-Hamiltonian hypergraphs

Theorem 2.3.1 (Frankl, Katona [K3]). *There exists a* 1*-edge-Hamiltonian* r*-uniform hypergraph* $\mathcal{H}$ *on* n *vertices with*

$$|\mathcal{E}(\mathcal{H})| = \frac{4r-1}{2r}n + o(n).$$

Proof. The idea of the construction is similar to the one on Fig. 2.1. Let $\mathcal{V}(\mathcal{H}) := \{w_1, \dots, w_p, v_1, \dots, v_q\}$ where $p = \lceil n/2r \rceil$ and $q = n - p$. There are two types of edges in $\mathcal{H}$. The first kind of edges form a chain on $\{v_1, \dots, v_q\}$,

$$\mathcal{E}_1(\mathcal{H}) := \left\{ \{v_i, v_{i+1}, \dots v_{i+r-1}\} \;\middle|\; 1 \le i \le q \right\}.$$

The second kind connects the rest of the vertices to this chain:

$$\mathcal{E}_2(\mathcal{H}) := \left\{ \{w_i, v_{(2r-1)(i-1)+j}, \dots, v_{(2r-1)(i-1)+j+r-2}\} \;\middle|\; 1 \le i \le p, 1 \le j \le 2r \right\}.$$

This means that the neighborhood of w_i is an $(r-1)$-uniform open chain of length $2r$ formed by vertices $v_{(2r-1)(i-1)+1}, \dots, v_{(2r-1)(i-1)+3r-2}$. The neighborhood of w_{i+1} is also an open chain of length $2r$, which begins at $v_{(2r-1)(i-1)+2r}$, so

$$v_{5(i-1)+2r}, \dots, v_{5(i-1)+3r-2} \in N(w_i) \cap N(w_{i+1})$$

(except maybe for $N(w_1)$ and $N(w_p)$ where the overlap is larger if $(2r) \nmid n$). Let $\mathcal{E}(\mathcal{H}) := \mathcal{E}_1(\mathcal{H}) \cup \mathcal{E}_2(\mathcal{H})$, then it is clear that $|\mathcal{E}(\mathcal{H})| = q + 2rp = n + (2r-1)\lceil n/2r \rceil = \frac{4r-1}{2r}n + o(n)$.

One can prove that this hypergraph is 1-Hamiltonian in the same way as in Theorem 2.2.1. □

Theorem 2.3.2 (Frankl, Katona [K3]). *For any 1-edge-Hamiltonian 4-uniform hypergraph $\mathcal{H}$ on $n \geq 6$ vertices*

$$|\mathcal{E}(\mathcal{H})| \geq \frac{3}{2}n$$

holds.

Proof. Following the idea of the proof of Theorem 2.2.2 we need to know what is the minimum number of edges in a 1-stable 3-uniform hypergraph. Now 1-stable means that the hypergraph contains an open chain with 4 edges on 6 vertices $\mathcal{P}_6^{(3)}$, since the edges of a Hamiltonian-chain containing a fixed vertex form such an open chain.

It is easy to see that it is impossible to create a 1-stable hypergraph by adding only one edge to $\mathcal{P}_6^{(3)}$, therefore the minimum number of edges in a 1-stable hypergraph is 6, since the 3-uniform hyperchain on 6 vertices, $\mathcal{C}_6^{(3)}$ is a 1-stable with 6 edges.

This gives that the minimum degree is 6, completing the proof. □

Note, that the above bound is already better than the trivial one. On the other hand, by case analysis, we can also prove that $\mathcal{C}_6^{(3)}$ is the only 1-stable hypergraph with 6 vertices, which leads to a better lower bound:

$$|\mathcal{E}(\mathcal{H})| \geq \frac{11}{6}n$$

However, the proof is too long compared with the improvement, so it is omitted.

2.4 An application

In Section 1.4 we gave lower and upper bounds for the maximum number of edges in an n vertex graph containing no Hamiltonian chain. Tuza improved our lower bound in [132] by giving a construction having $\binom{n-1}{r} + \binom{n-1}{r-2}$ edges that contains no Hamiltonian chain. Now we improve the upper bound $\binom{n}{r}(1 - 1/n)$ given in Theorem 1.4.3.

Theorem 2.4.1 (Frankl, Katona [K3]). *If an r-uniform hypergraph $\mathcal{H}$ on n vertices has no Hamiltonian chain then*

$$|\mathcal{E}(\mathcal{H})| \leq \binom{n}{r}\left(1 - \frac{4r}{(4r-1)n}\right) \tag{2.1}$$

holds.

Proof. Let m denote the number of missing edges (the r-element subsets which are not edges of $\mathcal{H}$). By (2.1) we obtain

$$m < \frac{4r}{(4r-1)n}\binom{n}{r}.$$

Observe, that if a hypergraph contains a 1-edge-Hamiltonian subgraph then one must delete at least 2 edges from it to destroy all Hamiltonian chains. Therefore in $\mathcal{K}_n^{(k)}$ we count the number of occurrences of the 1-edge-Hamiltonian hypergraph constructed in Theorem 2.3.1. Let $\mathcal{G}$ denote this r-uniform hypergraph on n vertices.

It is a simple matter to prove that there are $\frac{n!}{|\operatorname{Aut}(\mathcal{G})|}$ different $\mathcal{G}$ sub-hypergraphs in $\mathcal{K}_n^{(k)}$, where $\operatorname{Aut}(\mathcal{G})$ denotes the automorphism group of $\mathcal{G}$. Since every edges of $\mathcal{K}_n^{(k)}$ is contained in the same number of $\mathcal{G}$ sub-hypergraphs, the number of $\mathcal{G}$ sub-hypergraphs which contains a specified edge is

$$\frac{|\mathcal{E}(\mathcal{G})|}{\binom{n}{r}} \cdot \frac{n!}{|\operatorname{Aut}(\mathcal{G})|}.$$

Thus the number of Hamiltonian chains in $\mathcal{H}$ is

$$\geq 2 \cdot \frac{n!}{|\operatorname{Aut}(\mathcal{G})|} - m\frac{|\mathcal{E}(\mathcal{G})|}{\binom{n}{r}} \cdot \frac{n!}{|\operatorname{Aut}(\mathcal{G})|} > 0,$$

our claim is proved, because by Theorem 2.3.1 $|\mathcal{E}(\mathcal{G})| = \frac{4r-1}{2r}n$. □

Chapter 3

Extremal P_4-stable graphs

3.1 Introduction

In Section 2.2 on page 21 we defined the notion of k-stable graphs and denoted the minimum number of edges in a k-stable graph by $S(k)$. The main result in this Chapter is a formula for $S(k)$, which appeared in Theorem 2.2.5. However, this problem fits a more general framework, and similar problems are interesting on their own.

There is a wide range of graph theoretical questions that is a special case of the following very general one: *Let Π be a graph property that is preserved by adding edges to a graph. What is the minimum number of edges in graphs on n vertices that have the following property: removing any k edges or vertices from the graph, it still has property Π?*

Note that in many models, it is important that the addition of vertices is not allowed.

A classic example is the following: what is the minimum number of edges in a k-edge-connected graph on n vertices? With the above terminology, being k-stable with respect to connectedness is equivalent to being $(k+1)$-edge-connected.

More examples concerning Hamiltonian and so called hypo-Hamiltonian graphs can be found in [K3] and [100]. While these examples are closely related to the topic of the present Chapter, they are not necessary to understand the results, and are omitted.

In the present Chapter we concentrate on the problem where Π is the property that G contains some fixed subgraph H. We regard the original problem a separate question for each different choice of H. The main result is for $H = P_4$ (the path of 3 edges on 4 distinct vertices), but we also state some general facts.

Some notations and definitions are needed.

Definition 3.1.1 (Stability)**.** *Let H be a fixed graph. If the graph G has the property that removing any k edges of G, the resulting graph still contains a subgraph isomorphic to H, then we say that G is* k-stable*.*

$S(n,k)$ denotes the minimum number of edges in a k-stable graph on n vertices, and $S(k)$ denotes the minimum number of edges in any k-stable graph (that is, $S(k) = \min_n S(n,k)$).

Remark. *The word stability is used for many different notions, but it was natural choice for this one, too. Hopefully, it does not cause any confusion.*

To keep the notation simple, we do not include H in the notation, instead just keep in mind that a graph H is always fixed. Again, we regard calculating the value of $S(k)$ a separate question for each H.

Note that for k fixed, $S(n,k)$ is decreasing in n, as we can just add isolated vertices to get k-stable graphs on $n+1, n+2, \ldots$ vertices. This implies that for any fixed k, $S(n,k) = S(k)$ if n is large enough. Also note that $S(k)$ is strictly increasing in k.

A general bound for the value of $S(k)$ holds.

Proposition 3.1.2.

(a) $k + |E(H)| \le S(k)$

(b) $S(k) \le 2(|V(H)|-1)k$ *if* k *is large enough.*

Proof. (a) Trivial.

(b) Turán's theorem states that if we delete at most $\left((|V(H)|-1)\binom{m}{2}-1\right)$ edges from the complete graph on $n_0 = m(|V(H)|-1)$ vertices then the remaining graph contains a complete graph on $|V(H)|$ vertices. Obviously, this implies that it also contains a subgraph isomorphic to H.

Therefore K_{n_0} is $\left((|V(H)|-1)\binom{m}{2}-1\right)$-stable. So, let m be the smallest integer such that $(|V(H)|-1)\binom{m}{2}-1 \ge k$. Then K_{n_0} is indeed k-stable.

By the choice of m, $\binom{m-1}{2}(|V(H)|-1) \le k$, and if $m \ge 6$, then $\binom{m-1}{2} \ge \frac{m^2}{4}$, so

$$|E(K_{n_0})| = \binom{m(|V(H)|-1)}{2} \le \frac{m^2(|V(H)|-1)^2}{2} \le$$
$$2\binom{m-1}{2}(|V(H)|-1)^2 \le 2(|V(H)|-1)k.$$

Finally note that $m \ge 6$ if we choose k large enough, say $k \ge 10|V(H)|$, and the proposition holds.

□

The main morale of the previous proposition is that for any choice of H, $S(k)$ is of a linear order. Of course, computing the exact value of $S(k)$ is a completely different matter. We are also interested in the structure of the extremal graphs, e.g. the graphs for which the minimal value in $S(k)$ is reached. For some choices of H, the problem is trivial:

Proposition 3.1.3.

(a) If $H = P_3$, *then* $S(k) = k + |E(H)| = k + 2$.

(b) If H *is the graph on 4 vertices with two nonadjacent edges, then* $S(k) = k + |E(H)| = k + 2$.

Proof (and identifying the extremal graphs) is left to the reader.

$H = P_4$ is settled in the rest of this Chapter. However, for larger choices of H, the problem is increasingly difficult.

Following our work Borowiecki, Drgas-Burchardt, Sidorowicz [21] showed a connection between stable graphs and $\mathcal{P}$-apex graphs. Later Dudek, Szymański and Zwonek introduced the vertex deletion version of stable graphs [39]. Several authors worked on determining the minimum number of edges in a (K_q, k) vertex-stable graph until it was settled: [50–52, 126].

Results concerning some other smaller graphs, complete bipartite graphs, or general bounds can also be found in [28, 39, 40, 42, 128].

3.2 The case $H = P_4$

From this point on, we fix $H = P_4$.

Theorem 3.2.1 (Horváth, Katona [K5]). *For $H = P_4$, $S(1) = 4$, and if $k \geq 2$, then $S(k) = k + \left\lceil \sqrt{2k + \frac{9}{4}} + \frac{3}{2} \right\rceil$.*

Although it is written in an explicit form above, the following alternative definition may be easier to understand and just as useful.

Proposition 3.2.2. *The above formula for $S(k)$ is equivalent with the following: $S(1) = 4, S(2) = 6$, and if $k \geq 3$, then*

$$S(k) = \begin{cases} S(k-1) + 2 & \text{if } k = \binom{l}{2} \text{ for some } l \\ S(k-1) + 1 & \text{otherwise.} \end{cases}$$

Proof. (Proposition 3.2.2) If $k = \binom{l}{2}$, then

$$\left\lceil \sqrt{2(k-1) + \frac{9}{4}} + \frac{3}{2} \right\rceil = \left\lceil \sqrt{l^2 + l - 2 + \frac{9}{4}} + \frac{3}{2} \right\rceil = l + 2$$

while

$$\left\lceil \sqrt{2k + \frac{9}{4}} + \frac{3}{2} \right\rceil = \left\lceil \sqrt{l^2 + l + \frac{9}{4}} + \frac{3}{2} \right\rceil = l + 3,$$

because $\sqrt{l^2 + l + \frac{9}{4}} > l + \frac{1}{2}$. It is also clear that if $\binom{l}{2} < k < \binom{l+1}{2}$, then

$$\left\lceil \sqrt{2k + \frac{9}{4}} + \frac{3}{2} \right\rceil = l + 2.$$

This yields exactly what we wanted. □

Graphs containing no P_4 can be identified in the following way:

Proposition 3.2.3. *If the graph G contains no P_4 as a subgraph, then all of its components are triangles and stars.*

Proof. If there are two incident edges $\{v_1, v_2\}$ and $\{v_2, v_3\}$ in a component, then either $\{v_1, v_3\}$ is an edge and there are no more edges in this component, which means that the component is a triangle, or all the other edges in this component must contain v_2, and the component is a star. If any edges different from $\{v_1, v_3\}$ is connected to v_1 or v_3, we get a P_4. □

Proposition 3.2.4. *G is a graph with e edges on n vertices. G is k-stable if and only if G can not be covered by $k + n - e$ stars and any number of disjoint triangles.*

Proof. If G is not k-stable then we can delete k edges such that the remaining graph contains no P_4, in other words, we can select $e-k$ edges of G such that they contain no P_4. Therefore we can select $e-k$ edges of G such that they are edges of disjoint triangles and stars, so G can be covered by $k+n-e$ stars and any number of triangles.

The last part may need some explanation. If there are t stars and some disjoint triangles on n vertices, they contain $n-t$ edges, because a triangle has 3 vertices and 3 edges while a star has one less edge than vertex, so we "lose" 1 edge for every star. Then just take $t = k+n-e$.

Since all implications were reversible, the reverse implication holds as well. □

In order to prove Theorem 3.2.1, we use the following theorem.

Theorem 3.2.5 (Horváth, Katona [K5]). *Let G be a graph with $e \geq 5$ edges. If $e \geq \binom{l-1}{2}+1$, then there exists a subgraph of G with $l-1$ edges that contains no P_4 as a subgraph.*

Theorem 3.2.5 is proved in Section 3.3. For now, let us see what we can make of it.

Proof. (Theorem 3.2.1) Notice the assumption $e \geq 5$ in Theorem 3.2.5. Of course, graphs with at most 4 edges may only be 1-stable. On the other hand, $S(1) \geq 4$. It turns out that C_4 is 1-stable (and it is essentially the only 1-stable graph with 4 edges), so $S(1) = 4$. From now on, we examine $k \geq 2$, which implies $S(k) \geq 5$, so the assumption in Theorem 3.2.5 will always hold.

Let us take a graph with $e = k + \left\lceil \sqrt{2k+\frac{9}{4}} + \frac{3}{2} \right\rceil - 1$ edges; we will show that it is not k-stable. Let l_0 be such that $\binom{l_0-2}{2} \leq k \leq \binom{l_0-1}{2} - 1$ (l_0 is unique to k). It is easy to check that $l_0 - \frac{5}{2} < \sqrt{2k+\frac{9}{4}} \leq l_0 - \frac{3}{2}$, so $\left\lceil \sqrt{2k+\frac{9}{4}} + \frac{3}{2} \right\rceil = l_0$, which implies $e-k = l_0 - 1$. Then

$$e = k + l_0 - 1 \geq \binom{l_0-2}{2} + l_0 - 1 = \frac{l_0^2 - 5l_0 + 6 + 2l_0 - 2}{2} = \frac{l_0^2 - 3l_0 + 4}{2} = \binom{l_0-1}{2} + 1,$$

so l_0 is a valid choice for l in Theorem 3.2.5, and thus $l_0 - 1 = e - k$ edges can be selected from the graph such that they contain no P_4. This means that $S(k) \geq k + \left\lceil \sqrt{2k+\frac{9}{4}} + \frac{3}{2} \right\rceil$.

In order to prove the reverse inequality we will exhibit a k-stable graph with $e = k + \left\lceil \sqrt{2k+\frac{9}{4}} + \frac{3}{2} \right\rceil$ edges.

Let l_0 be such that $\binom{l_0-2}{2} \leq k \leq \binom{l_0-1}{2} - 1$. Thus

$$\binom{l_0-1}{2} + 2 = \binom{l_0-2}{2} + l_0 \leq e = k + l_0 \leq \binom{l_0-1}{2} - 1 + l_0 = \binom{l_0}{2}.$$

We have two cases: either l_0 is divisible by 3 or not. First we consider the case when l_0 is not a multiple of three.

Let us take any graph G with e edges on l_0 vertices (this is an "almost complete" graph, because $e \geq \binom{l_0-1}{2} + 2$, so G can be regarded as a graph resulting from the

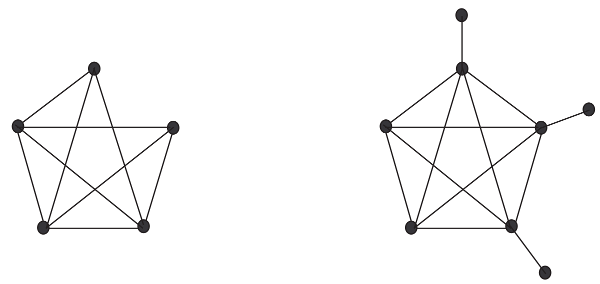

Figure 3.1: Construction for $k = 4$ and $k = 7$.

deletion of at most $l_0 - 3$ edges from K_{l_0}). We will prove that G is k-stable. According to Proposition 3.2.4, it is enough to show that the vertices of G can not be covered by $l_0 + k - e = 0$ stars and any number of disjoint triangles. This is clearly true, because G can not be covered only by triangles, as $3 \nmid l_0$. The graph on the left-hand side in the figure below shows an example for $k = 4$ (where $l_0 = 5$ and $e = 9$).

The case when $3|l_0$ needs a different example. Take a complete graph K_{l_0-1} and a set X of $e - \binom{l_0-1}{2}$ additional independent vertices, then connect each vertex in X to a different vertex of the complete graph with a new edge. Note that $e \geq \binom{l_0-1}{2} + 2$ by the choice of l_0, so it is always possible to do this. (The graph on the right-hand side in Figure 3.1 shows the example for $k = 7$; then $e = 13$ and $l_0 = 6$). It is enough to show that the vertices of this graph can not be covered by $n + k - e$ stars and any number of triangles, where n is the number of vertices of this graph.

Since $e = k + l_0$, the number of stars that can be used in the cover is $n + k - e = n - l_0$. It is clear from the construction that the vertices in X have degree one, so they cannot be covered by any triangle. Since they are connected to different vertices of the complete graph, one star cannot cover two vertices of X. However, $|X| = n - (l_0 - 1) > n - l_0$, so the vertices of this graph cannot be covered by $n + k - e$ stars and any number of triangles. $\square$

3.3 Proof of Theorem 3.2.5

From now on, the proof of Theorem 3.2.5 is presented through several lemmas. We make some remarks first.

We may suppose that $e \leq \binom{l}{2}$, otherwise the theorem can be applied with a higher l. With $\binom{l-1}{2} + 1 \leq e \leq \binom{l}{2}$, l is unique to e.

Notice that l is the least possible number of vertices such that l vertices can span e edges; if the G graph has l vertices, then it is an "almost complete" graph, with at most $l - 2$ edges missing from K_l.

Let G be a graph on n vertices of size e. Then, from $e \leq \binom{l}{2}$, $l = \left\lceil \frac{1+\sqrt{1+8e}}{2} \right\rceil$, and let $s = n - l$. Of course, $s \geq 0$. The parameter s measures how "spread out" the graph is in some sense.

It is also clear that $l - 1$ edges on $n = l + s$ vertices without a P_4 form a graph which has at most $s + 1$ star components and any number of triangle components. So we will actually prove that the vertices of the graph can be covered by at most $s + 1$ stars and any number of triangles.

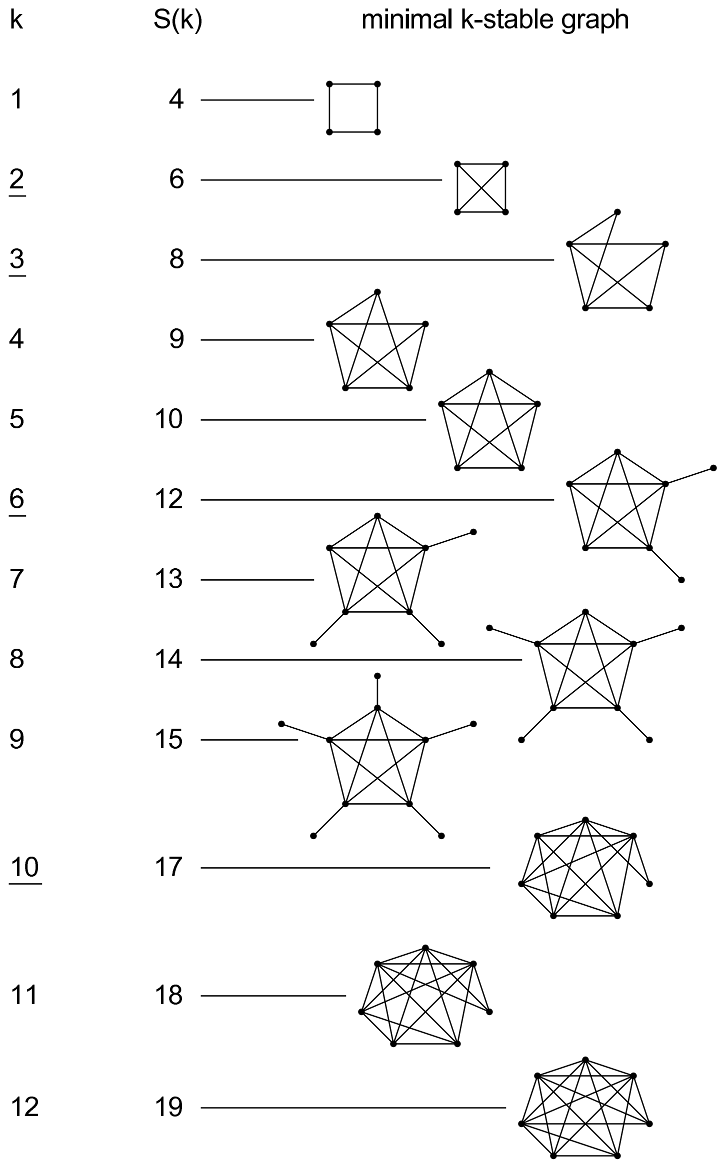

Figure 3.2: A display of the values of $S(k)$ and the minimal k-stable graphs for values of k up to 12. The values of k where $S(k)$ has a jump of two units are marked. Note that the examples are almost complete graphs except for values of k multiples of 3.

First we prove Theorem 3.2.5 for connected graphs. We prove separately for $s = 0$ and $s = 1$. For $s \geq 2$, a stronger property is true: vertices of such graphs can be covered using only stars (at most $s+1$ of course). Finally, to complete the proof of Theorem 3.2.5, we will consider non-connected graphs.

Lemma 3.3.1 (the $s = 0$ case). *Let G be a graph with $n \geq 5$ vertices and e edges, and l a positive integer such that $\binom{l-1}{2} + 1 \leq e \leq \binom{l}{2}$. If G has $n = l$ vertices, then $l - 1$ edges can be selected such that they contain no P_4.*

Proof. We will select exactly 1 star and at most one triangle. It is enough to prove the statement in the case where $e = \binom{l-1}{2} + 1$, because otherwise we may delete some edges from G to get a graph with $\binom{l-1}{2} + 1$ edges, from which we can still select 1 star and some triangles.

So, let us fix a graph on $n = l$ vertices with $\binom{l-1}{2} + 1$ edges. Sometimes we will regard the number of edges as $l-2$ less than that of the complete graph, for it is easier to count the non-edges.

The maximum degree is either $l - 1$ or $l - 2$, because the average degree of the graph is $\frac{2}{l}\left(\binom{l-1}{2} + 1\right) = \frac{1}{l}(l^2 - 3l + 4) > l - 3$. If the maximum degree is $l - 1$, then this is the center of a star that covers all vertices, and we are done.

If the maximum degree is $l - 2$, then let A be a vertex with such a degree. A is connected to every vertex except B, whose degree is b. If any two vertices among the neighbors of B are connected, then they form a triangle with B, and this triangle together with the star with center A is the configuration we need.

If none of the neighbors of B are connected, then let us count the non-edges in G. There are $l - 1 - b$ containing B, $\binom{b}{2}$ among the neighbors of B, and possibly some others, which means that $l - 1 - b + \binom{b}{2} \leq l - 2$, from which we get $b \leq 2$.

If $b = 1$, then every non-edge contains B, so its only neighbor, C, is connected with every vertex, which contradicts the maximum degree being $l - 2$. If $b = 2$, then its two neighbors, C and D are not connected. $l \geq 5$, so there is another vertex, E, which is connected with all vertices except B. Now the triangle ADE and the star with center C (covering all vertices except A, D, E) contain $l - 1$ edges together. $\square$

Lemma 3.3.2 (the $s = 1$ case). *If G is a graph with $s = 1$ and $l \geq 5$ vertices, then there are $l - 1$ edges such that they contain no P_4.*

Proof. Again we may suppose that G has $e = \binom{l-1}{2} + 1$ edges, in other words, $2l - 2$ edges are missing from the K_{l+1} complete graph. We have to cover by (at most) two stars and any number of triangles.

Let D be a vertex of the highest degree. The average degree of the graph is at least $l - 4$, so that vertex A has at most three non neighbors. If there are only 0 or 1 of them, there is a trivial covering of the graph. There can be either 2 or 3 non-neighbors of D.

If there are 2 of them (say A and B), then let us make some remarks.

- If the two vertices are connected, then we are done (a big star with center D and a star containing A and B covers the graph).

- If any of A and B is covered by a triangle, we are done (the triangle plus the other vertex as a star plus the star of center D covers the graph).

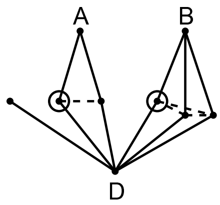

Figure 3.3: The center of the two stars if $a = 2$ and $b = 3$.

- If the two vertices have a common neighbor, they are covered by a star, and we are done.

If none of the previous hold, then let the degrees of A and B be a and b respectively. We count the non-edges. There is one non-edge going between A and B. There are $l-1-a$ and $l-1-b$ other non-edges from A, respectively, B; finally, between the neighbors of A and B there are $\binom{a}{2}$ and $\binom{b}{2}$ non-edges respectively. That is a total of $2l-2$ non-edges, so

$$1 + (l-1-a) + \binom{a}{2} + (l-1-b) + \binom{b}{2} \leq 2l-2,$$

from which we obtain

$$\binom{a}{2} - a + \binom{b}{2} - b \leq -1.$$

The possible values for $\binom{a}{2} - a$ (and $\binom{b}{2} - b$):

a	0	1	2	3	4
$\binom{a}{2} - a$	0	-1	-1	0	2

If $a > 3$, then $\binom{a}{2} - a \geq 2$. From this, we get that one of $\binom{a}{2} - a$ and $\binom{b}{2} - b$ is -1, the other is -1 or 0, so the possibilities for $\{a, b\}$ are $\{1,1\}, \{1,2\}, \{2,2\}, \{1,3\}, \{2,3\}$. In the first three cases $\binom{a}{2} - a + \binom{b}{2} - b = -2$, so there is one more non-edge, while in the last two cases there are no more non-edges. We may suppose that $a \leq b$. In each case we show that one can choose one vertex adjacent to A and one adjacent to B such that they are the center of two covering stars.

- $\{1,1\}$. Let C and E be the only neighbors of A and B respectively. The stars with center C and E cover all vertices, because they cover A, B and D and every other vertex is connected to either C or E as there is only one more non-edge.

- $\{1,2\}$. We may choose A's only neighbor and the one out of B's neighbors that is not affected by the last non-edge to get a similar solution to that of the previous case.

- $\{2,2\}$. From both neighborhoods there is a vertex not affected by the last non-edge.

- $\{1,3\}$ and $\{2,3\}$. The only neighbor of A and any neighbor of B will do. (See Fig. 3.3.)

If D has 3 non-neighbors, then the proof is quite different. We will show that the 3 vertices can be covered by one star, and this is enough, as all the other vertices can be covered by the star with center D. Let the 3 non-neighbors be A, B, C. Suppose indirectly that they can not be covered by a star, then there is at most 1 edge between them, and they don't have a common neighbor from the remaining $l-3$ vertices, so the sum of the edges going between $\{A, B, C\}$ and the rest is at most $2l-6$ (pigeonhole principle).

We count the sum of non-degrees, that is, the number of non-edges to each vertex.

- The vertices other than A, B, C have a non-degree at least 3, which gives at least $3l-6$ to the sum of the non-degrees.
- A, B and C has at least 2 non-edges among themselves, that is 4 in the non-degrees.
- Between $\{A, B, C\}$ and D, there are 3 non-edges, that is 3 in the non-degrees of A, B and C.
- Between $\{A, B, C\}$ and the rest, there are at least $3(l-3)-(2l-6)=l-3$ edges by the pigeonhole principle.

The total sum is at least $(3l-6)+4+3+(l-3)=4l-2$, which is impossible, because there are only $2l-2$ non-edges, so the sum should be $4l-4$. This is a contradiction, so A, B and C can be covered by a single star, and thus the whole graph can be covered by 2 stars. $\square$

The last case ($s \geq 2$) is a corollary of the following theorem of Vizing.

Theorem 3.3.3 (Vizing [122]). *Let $\beta(G)$ denote the minimum number of start covering the vertices of a graph G. ($\beta(G)$ is called the* domination number *of the graph.) If G is a connected graph on n vertices and e edges, then*

$$\beta(G) \leq \left\lfloor \frac{1+2n-\sqrt{8e+1}}{2} \right\rfloor, \quad \text{if } e \leq \frac{(n-2)(n-3)}{2}.$$

(See [107] for an English language proof.)

Corollary 3.3.4 (the $s \geq 2$ case). *If G is a connected graph with $s \geq 2$, then G can be covered by at most $s+1$ stars.*

Proof. In our case $e = \binom{l-1}{2}+1$ (and $l = \left\lceil \frac{1+\sqrt{1+8e}}{2} \right\rceil$) and $n = l+s \geq l+2$, so

$$\frac{(n-2)(n-3)}{2} \geq \frac{l(l-1)}{2} > e,$$

so the previous inequality states

$$\beta(G) \leq \left\lfloor \frac{1+2(l+s)-\sqrt{8e+1}}{2} \right\rfloor = \left\lfloor \frac{1+2(l+s)-2(l-\frac{1}{2})}{2} \right\rfloor = s+1,$$

which is exactly what we need. $\square$

We are done with connected graphs. Now we turn to disconnected graphs.

It is clear that any graph is the finite union of connected graphs. Now we will prove that the inequality is true for such unions, too.

Lemma 3.3.5. *Let G_1 and G_2 be two graphs which satisfy Theorem 3.2.5. Then*

(a) Their disjoint union $G = G_1 \cup G_2$ also satisfies Theorem 3.2.5, and

(b) If G_1 has at least one edge then the disjoint union $G = C_4 \cup G_2$ also satisfies Theorem 3.2.5.

Proof. (a) $n = n_1 + n_2$ and $e = e_1 + e_2$ clearly hold, where n and e are the respective parameters of G. Let us also use s as the parameter of G. We have to prove that G can be covered by $s+1$ stars and any number of triangles. Theorem 3.2.5 holds for G_1 and G_2, so they can be covered by $s_1 + 1$ and $s_2 + 1$ stars and any number of triangles respectively, and thus G can be covered by $s_1 + s_2 + 2$ stars and some triangles. We need $s_1 + s_2 + 2 \leq s + 1$. This is true because $e_1 \leq \binom{l_1}{2}$ from the definition of l_1, $e_2 \leq \binom{l_2}{2}$ similarly and $l \leq l_1 + l_2 - 1$ because

$$e = e_1 + e_2 \leq \binom{l_1}{2} + \binom{l_2}{2} \leq \binom{l_1 + l_2 - 1}{2},$$

as $(l_1 - 1)(l_2 - 1) \geq 0$ is equivalent to the right hand side inequality.

From $l \leq l_1 + l_2 - 1$ we get $s \geq s_1 + s_2 + 1$ and finally $s_1 + s_2 + 2 \leq s + 1$.

(b) $l_2 \geq 2$, and $l \leq l_2 + 2$, because

$$e = 4 + e_2 \leq 4 + \binom{l_2}{2} \leq \binom{l_2 + 2}{2},$$

which is equivalent to $4l_2 \geq 8$.

□

To complete the proof of Theorem 3.2.5, take a non-connected graph G with $e \geq 5$ edges. If none of its components are isomorphic to C_4, part (a) of Lemma 3.3.5 can be applied directly. If there is a C_4 component, first take any other component that actually contains at least one edge and apply part (b) of Lemma 3.3.5. Then apply part (a).

Chapter 4

Hypergraph extensions of the Erdős-Gallai theorem

4.1 Introduction

The aim of the present Chapter is to extend the following classical result to uniform hypergraphs.

Theorem 4.1.1 (Erdős-Gallai [46]). *Let G be a graph on n vertices containing no path of length k. Then $e(G) \leq \frac{1}{2}(k-1)n$. Equality holds iff G is the disjoint union of complete graphs on k vertices.*

The problem of finding the maximal size of a graph without a cycle of length k is much harder even for ordinary graphs. However, there are some papers containing results on the hypergraph version. Since the present paper concentrates on the path version we only cite those results that are connected to the path problem.

We consider several generalizations of Theorem 4.1.1 for hypergraphs. This is due to the fact that there are several possible ways to define paths in hypergraphs other than the ones defined in Chapter 1. One such definition of paths in hypergraphs is due to Berge.

Definition 4.1.2. *A Berge path of length k in a hypergraph is a collection of k hyperedges $h_1, \ldots, h_k$ and $k+1$ vertices $v_1, \ldots, v_{k+1}$ such that for each $1 \leq i \leq k$ we have $v_i, v_{i+1} \in h_i$.*

We find the extremal sizes of r-uniform hypergraphs avoiding Berge paths of length k. Interestingly, the size of the extremal hypergraphs depends on the relationship between r and k. In particular, the cases when $k \leq r$ and $k > r$ behave differently.

Theorem 4.1.3 (Győri, Katona, Lemons [K4]). *Fix $k > r+1 > 3$, and let $\mathcal{H}$ be an r-uniform hypergraph containing no Berge path of length k. Then $e(\mathcal{H}) \leq \frac{n}{k}\binom{k}{r}$.*

Mubayi and Verstraete independently proved this theorem if $k > 2r > 2$ or $k > r+1 > 11$ but it was not published. On the other hand, if $k \leq r$, we have a different theorem. It was very annoying at the time that the case $k = r+1$ was left open. It was later settled by Davoodi, Győri, Methuku and Tompkins [35]. We were able to prove it only when $r = 3$ and $k = 4$. The proof of this case is presented after the proof of Theorem 4.1.3.

Theorem 4.1.4 (Győri, Katona, Lemons [K4]). *Fix $r \geq k > 2$. If $\mathcal{H}$ is an r-uniform hypergraph with no Begre path of length k, then $e(\mathcal{H}) \leq \frac{n(k-1)}{r+1}$.*

Remark. *Both of the above theorems are sharp for infinitely many n as the following two examples show. In the first case, if $k > r$, suppose that k divides n and partition the n vertices into sets of size k. In each k element set, take all possible subsets of size r to be in the hypergraph. Such a hypergraph has exactly $\frac{n}{k}\binom{k}{r}$ hyperedges and clearly contains no Berge k-path.*

In the second case, $k \leq r$, suppose that $r+1$ divides $k-1$. Here we partition the vertices into sets of size $r+1$ and then in each $r+1$ element set, we select exactly $k-1$ of its subsets of size r to be in the hypergraph. This hypergraph has exactly $\frac{k-1}{r+1}n$ hyperedges and as each component is encompasses exactly $k-1$ edges, it is clear that there is no path of length k. Here we will not deal with the case when $k = 2$, as it is trivial, but it is interesting to note that the above construction is not best possible when $k = 2$.

A similar result can be found in [98], the authors give upper bounds for the maximum number of edges in a hypergraph that avoids so called minimal k-paths. These are Berge-paths that satisfy an additional constraint that two edges of the path h_i and h_j are disjoint iff $|i - j| \geq 2$.

One can further restrict the idea of a Berge path, the cycle version of the following notion first appeared in [38].

Definition 4.1.5. *Fix $r \geq 2$ and $t, 1 \leq t \leq r-1$. A t-tight path of length k in a r-uniform hypergraph is a Berge-path on $k+1$ vertices $\{v_1, v_2, \ldots, v_{k+1}\}$ and k hyperedges $\{h_1, h_2, \ldots, h_k\}$ such that consecutive hyperedges intersect in at least t points.*

Of course a 1-tight path is the same as a Berge path. In [38] t-tight paths have been studied in other settings. A similar, but more restrictive notion called ℓ-cycle appears in [82] yet in other context.

As in the case of Berge paths, we can get quite exact results regarding hypergraphs avoiding t-tight paths.

Theorem 4.1.6 (Győri, Katona, Lemons [K4]). *Fix $r \geq 2$ and $t, 1 \leq t \leq r-1$. Fix k large (and n should be large enough too). Let $\mathcal{H}$ be an extremal r-uniform hypergraph on n vertices containing no t-tight path of length k. Then*

$$(1 - o(1))\frac{\binom{n}{t}\binom{k}{r}}{\binom{k}{t}} \leq e(\mathcal{H}) \leq \frac{\binom{n}{t}\binom{k}{r}}{\binom{k}{t}}$$

The lower bound follows directly from a theorem of Rödl [110].

Theorem 4.1.7 (Rödl [110]). *The packing number $m(n, k, l)$, i.e. the size of the largest k-uniform family of subsets of an n-set such that every l-set is contained in at most 1 member of the family is $(1 + o(1))\frac{\binom{n}{l}}{\binom{k}{l}}$.*

Remark. *The error term, in fact, was improved by Kim [78] providing a slightly better lower bound for us, as well. Also, Keevash proved that if the necessary divisibility conditions are satisfied then the error term is 0 [77] implying that our bound is sharp in that case.*

Next we consider the more strict definition of a path that we called an open chain (see Definition 1.1.4). For convenience we repeat it.

Definition 4.1.8. *A open chain of length k in a r-uniform hypergraph is a collection of distinct $k+r-1$ vertices $\{v_1, v_2, \dots, v_{k+r-1}\}$ and k hyperedges $\{h_1, h_2, \dots h_k\}$ such that for each $1 \leq i \leq k$, $h_i = \{v_i, v_{i+1}, \dots, v_{i+r-1}\}$.*

It follows from the definition that an open chain is always an $(r-1)$-tight path, but not all $(r-1)$-tight paths are open chains. The difference between the two definitions will be explored later on.

For open chains our lower and upper bounds differ by a factor r.

Theorem 4.1.9 (Győri, Katona, Lemons [K4]). *Let $\mathcal{H}$ be an extremal r-uniform hypergraph containing no open chain of length k. Then*

$$(1-o(1))\frac{k-r+1}{r}\binom{n}{r-1} \leq |e(\mathcal{H})| \leq (k-1)\binom{n}{r-1}.$$

The lower bound again follows easily from Rödl's Theorem 4.1.7.

Remark. *The definition of open chains in uniform hypergraphs can be extended to tight trees in r-uniform hypergraphs. Such trees are defined inductively. A single edge forms a tight 1-tree. If a collection of edges form a tight $(k-1)$-tree, then adding a new edge which intersects a previous edge in $r-1$ vertices and contains a 'new' vertex (which is not contained in any of the previous edges) yields a tight k-tree. It is easy to see that open chains are also tight trees. Kalai [53] made the following conjecture regarding the extremal number for tight trees.*

Conjecture 4.1.10 (Kalai [53]). *If the number of edges in an r-uniform hypergraph is $> \frac{k-1}{r}\binom{n}{r-1}$ then it contains every tight k-tree.*

In [53] the authors prove a special case of this conjecture.

Note that if true, the conjecture would imply that the upper bound in Theorem 4.1.9 is $\frac{k-1}{r}\binom{n}{r-1}$.

As noted above, there is an important difference between $(r-1)$-tight paths and open chains in r-uniform hypergraphs. We can investigate this difference by considering the following.

Definition 4.1.11. *Let $k > r \geq 2$ and $1 \leq J < k$. Then a Berge path of length k in a r-uniform hypergraph $\mathcal{H}$ on hyperedges $e_1, \dots, e_k$ satisfies intersection conditions (J) if*

$$\textit{for all } 1 \leq l \leq J \textit{ and for all } i > l, \quad |e_i \cap e_{i-l}| = \max\{r-l, 0\}.$$

Of course a Berge path satisfying intersection conditions (1) is the same as an $(r-1)$-tight path. Furthermore, a Berge path satisfying intersection conditions $(k-1)$ is exactly an open chain. It is interesting that as above, an extremal hypergraph excluding $(r-1)$-tight paths of length k contains asymptotically $\frac{k-r+1}{r}\binom{n}{r-1}$ hyperedges. On the other hand, our best construction for a hypergraph excluding an open chain of length k contains $\frac{k-1}{r}\binom{n}{r-1}$ hyperedges. Supposedly, each Berge path satisfying intersection conditions (J) falls somewhere between these two. However, while there are $k-1$ different intersection conditions, there are only $r-1$ possible block sizes in the construction of Theorem 4.1.7 (which we believe form the extremal hypergraphs).

The rest of the Chapter is organized as follows. In section 4.2 we prove Theorem 4.1.3 and Theorem 4.1.4. No really new ideas are needed; our proof is very similar to the original one. In section 4.3 we look at t-tight paths. In section 4.4 we consider open chains and $(r-1)$-tight paths satisfying a certain number of intersection conditions as in Definition 4.1.11. These results appeared in [K4].

An extended abstract of our results already appeared in [63], much earlier than the full paper. Since then more than 50 papers cited our paper in various context. Allen, Böttcher, Cooley and Mycroft [13] have obtained upper and lower bounds for an analogue of Theorem 4.1.9 in the range $k = \alpha n$ for α constant, which are asymptotically almost equal when α is small. Using our result Glebov, Person and Weps [60] gave bounds for the corresponding problem in case of t-tight cycles. Győri, Methuku, Salia, Tompkins and Vizer [62] proved an asymptotic version of the Erdős-Gallai theorem for Berge-paths in connected hypergraphs whenever r is fixed and n and k tend to infinity. Many papers contain explicit bounds in different cases for Berge-cycles in connected and 2-connected hypergraphs [47, 54, 57, 58, 61, 81]. Füredi, Jiang and Sievert [56] deals with the case of linear paths using some of our methods. The authors of [45, 68] use our results to obtain interesting results on k-trees (of ordinary graphs).

4.2 Berge Paths

Proof of Theorem 4.1.3. Let P be a longest path in $\mathcal{H}$. Let $v_1, v_2, \ldots, v_{l+1}$ be the vertices of P, and $h_1, h_2, \ldots, h_l$ the hyperedges such that for each $i = 1, \ldots l$, $v_i, v_{i+1} \in h_i$. Let $\mathcal{H}'$ be the hypergraph obtained by deleting the edges of P from $\mathcal{H}$. Specifically, let $\mathcal{H}' = \mathcal{H} \setminus \{h_1, h_2, \ldots, h_l\}$. Suppose that $l < k$.

Lemma 4.2.1. *If there is a cycle of length $l+1$ on the vertices $v_1, v_2, \ldots, v_{l+1}$, then these vertices constitute a component of the hypergraph $\mathcal{H}$.*

Proof. To see this, suppose that C is such a cycle. Then if an edge h in the cycle C does not lie completely within the vertices $v_1, v_2, \ldots, v_{l+1}$, then deleting h from C we have an l-path which can be extended (by the edge h) to a path of length at least $l+1$. Thus every edge h in the cycle C must be contained within the vertices $v_1, v_2, \ldots, v_{l+1}$. In fact, something stronger is true. For each vertex in the cycle, v_i, the neighborhood of v_i lies within $v_1, v_2, \ldots, v_{l+1}$. (The neighborhood of a vertex is the set of vertices in $\mathcal{H}$ which are connected to v_i by an edge.) Indeed, suppose that for some i, the vertex v_i has a neighbor y outside of $\{v_1, v_2, \ldots, v_{l+1}\}$. Then the edge containing both v_i and y is not an edge of C (by the above argument.) Thus, removing an appropriate edge of C so that it is a path of length l with v_i as an endvertex, we can extend this to a path of length $l+1$ with y as an endvertex, a contradiction. □

Based on Lemma 4.2.1, we prove the theorem by induction on n. Clearly, for small values of n, the theorem trivially holds. Now, fix n such that the theorem holds for all $n' < n$. Then let $\mathcal{H} = (\mathcal{E}, V)$ be a r-uniform hypergraph on n vertices with $e(\mathcal{H}) > \frac{n}{k}\binom{k}{r}$. We can assume that the following holds for the minimal degree, δ, in $\mathcal{H}$.

$$\delta = \delta(\mathcal{H}) > \frac{1}{r}\binom{k-1}{r-1}. \tag{4.1}$$

Otherwise, if there is a vertex x in $\mathcal{H}$ with degree no more than $\frac{1}{r}\binom{k-1}{r-1}$, then we may delete this vertex (and all the edges incident with it) from $\mathcal{H}$. The result will be a hypergraph on $n-1$ vertices with more than $\frac{n-1}{k}\binom{k}{r}$ edges. Thus by the induction hypothesis, this hypergraph will have a path of length k.

Recall that $\mathcal{H}'$ is the hypergraph obtained by deleting the edges of P from $\mathcal{H}$. Note that by the choice of P, the neighborhoods of v_1 and v_{l+1} in $\mathcal{H}'$ must fall within the set $\{v_1, v_2, \ldots, v_{l+1}\}$.

Claim 4.2.2. *We may assume that v_{l+1} (and similarly v_1) is contained by at least one hyperedge in $\mathcal{H}'$.*

Let us explore the consequences of this claim. Clearly, if there is an edge of $\mathcal{H}'$ containing both v_1 and v_{l+1}, then the edges of P form a cycle of length $l+1$ and we may apply Lemma 1 to show this cycle forms a component. Any component on at most k vertices may then be deleted allowing us to apply the inductive hypothesis on the resulting sub-hypergraph. Furthermore, if there exist edges $g_1, g_2 \in \mathcal{H}'$ such that for some i, $1 < i < l+1$, $v_1, v_{i+1} \in g_1$ and $v_{l+1}, v_i \in g_2$ then there is an $(l+1)$-cycle again on the vertices

$$v_1, v_{i+1}, v_{i+2}, v_{i+3}, \ldots, v_{l+1}, v_i, v_{i-1}, v_{i-2}, \ldots, v_1.$$

Thus Claim 4.2.2 implies $k \geq 2r$.

Finally, by the pigeonhole principle, if in $\mathcal{H}'$ the degrees of both v_1 and v_{l+1} are greater than $\binom{\frac{k-2}{2}}{r-1}$, then there is a $(l+1)$-cycle on $v_1, v_2, \ldots, v_{l+1}$ in $\mathcal{H}$. If the degrees in $\mathcal{H}$ of both v_1 and v_{l+1} are at most $\binom{\frac{k-2}{2}}{r-1}$, then delete these vertices and the hyperedges in P and we are done since

$$\frac{2}{r}\binom{k-1}{r-1} \geq 2\binom{\frac{k-2}{2}}{r-1} + k - 1$$

if $k \geq 2r$. (We leave the details to the reader.)

On the other hand, if in $\mathcal{H}'$ the degree of v_1 is greater than $\binom{\frac{k-2}{2}}{r-1}$ but the degree of v_{l+1} in $\mathcal{H}$ is at most $\binom{\frac{k-2}{2}}{r-1}$ then the degree of v_{l+1} is at most $\binom{\frac{k-2}{2}}{r-1} + \frac{k-2}{2}$ since if v_j is contained by a hyperedge e in $\mathcal{H}'$ and h_{j-1} contains v_{l+1} then $e, h_1, \ldots h_j - 1, h_l, h_{l-1}, \ldots h_j$ constitute a cycle and we are done by induction. So the degree of v_{l+1} in $\mathcal{H}$ is at most

$$\binom{\frac{k-2}{2}}{r-1} + \frac{k-1}{2} \leq \frac{1}{r}\binom{k-1}{r-1}$$

and we are done again by induction as above if we delete the vertex v_{l+1} and the hyperedges containing it.

This completes the proof of the theorem.

□

Proof of Claim 4.2.2. Suppose v_{l+1} is not contained by any hyperedge in $\mathcal{H}'$. We consider permutations of the vertices and edges of P which map vertices to vertices and edges to edges. Such a permutation σ is called valid if for $1 \leq i \leq l$, $\sigma^{-1}(v_i), \sigma^{-1}(v_{i+1}) \in \sigma^{-1}(h_i)$. In other words, the permutation σ is valid if $\sigma^{-1}(v_1), \sigma^{-1}(h_1), \sigma^{-1}(v_2), \ldots, \sigma^{-1}(h_l), \sigma^{-1}(v_{l+1})$ constitute a path in $\mathcal{H}$. Let S be a maximal set of vertices v_i of P such that the following hold:

1. $\forall v_i \in S$, $\exists$ a valid permutation σ_i : $\sigma_i^{-1}(v_1) = v_1$ and $\sigma_i^{-1}(v_{l+1}) = v_i$

2. $\forall v_j \notin S$, if also $v_{j+1} \notin S$, then for each valid σ_i, $\sigma_i(v_j)$ and $\sigma_i(v_{j+1})$ are consecutive vertices in the path P_{σ_i} and in particular are joined in this path by the edge $\sigma_i(h_j)$.

Note that the second condition only requires $\sigma_i(v_j)$ and $\sigma_i(v_{j+1})$ to be consecutive; both of the two possible orders are allowed. We also note that $\{v_{l+1}\}$ is such a set so S is nonempty.

We prove that if h_j is a hyperedge such that $v_{j-1}, v_j \notin S$ then h_j does not contain any vertex $v \in S$. Suppose that h_j is a hyperedge such that $v_{j-1}, v_j \notin S$ but h_j contains a vertex $v \in S$. Then there exists a valid permutation σ with associated path P_σ satisfying $\sigma(v_{l+1}) = v$. Then take the starting segment of P_σ to the first vertex of v_{j-1} and v_j , then continue with h_j and the segment of P_σ backward from v to the second vertex of v_{j-1} and v_j. By definition, the last vertex of this path should belong to S, contradicting the maximality of S.

Finally we show that there are at most $2|S| - 1$ edges of P incident with S. If we delete the vertices $v_j \in S$ from $V(\mathcal{H})$ then we delete just the hyperedges h_{j-1}, h_j from $\mathcal{H}$, so at most $2|S|-1$ edges. So, delete the vertices in S and the at most $2|S|-1$ edges incident to these vertices. As $\frac{1}{r}\binom{k-1}{r-1} \geq 2$ we are done by the inductive hypothesis. □

We were not able to settle the case when $k = r+1 > 2$, but we made the following conjecture.

Conjecture 4.2.3 (Győri, Katona, Lemons [K4]). *Fix $k = r+1 > 2$, and let $\mathcal{H}$ be an r-uniform hypergraph containing no Berge path of length k. Then $e(\mathcal{H}) \leq \frac{n}{k}\binom{k}{r} = n$.*

This was later proved by Davoodi, Győri, Methuku and Tompkins [35]. However, we did prove it in the spacial case $r = 3$ and $k = 4$.

Theorem 4.2.4 (Győri, Katona, Lemons [K4]). *Let $\mathcal{H}$ be a 3-uniform hypergraph containing no Berge path of length 4. Then $e(\mathcal{H}) \leq n$.*

Proof. The following claim is an easy exercise to prove.

Claim 4.2.5. *If a connected graph on n vertices contains at least $n+1$ edges and does not contain a path of length 4 (with 5 different vertices) then it is either a K_4 or $K_4 - e$.*

To prove the theorem we use the method introduced in [59]. Suppose indirectly that there exists a 3-uniform hypergraph $\mathcal{H}$ without a path of length 4 that contains at least $n+1$ edges. If the hypergraph is not connected then at least one of the components must have more edges than vertices, so we can assume that our hypergraph is connected.

Let us construct a graph H on the ground set of $\mathcal{H}$ by embedding a (unique) edge into each hyperedge of $\mathcal{H}$. Construct H greedily, take the hyperedges of $\mathcal{H}$ in arbitrary order and for each hyperedge embed an edge that has not already been used in H.

If at some step we cannot find such an edge, then $\mathcal{H}$ contains the edge $e_0 = \{v_1, v_2, v_3\}$ so that each edge in H is already assigned to a hyperedge. Let these edges be $e_1 = \{v_1, v_2, u_1\}, e_2 = \{v_2, v_3, u_2\}, e_3 = \{v_3, v_1, u_3\}$. If $u_1 = u_2 = u_3$ then $u_1, e_1, v_1, e_0, v_2, e_2, v_3, e_3, u_1$ is a cycle of length 4, so by Lemma 4.2.1 these 4 vertices and 4 edges form a component, contradicting the assumption. On the other hand, if say $u_1 \neq u_2$, then $u_1, e_1, v_1, e_0, v_2, e_2, v_3, e_3, u_2$ is a path of length 4.

Thus we can assume that the greedy algorithm assigned an edge to each hyperedge. If H contains a path of length 4 on 5 different vertices then a hyperedges that were assigned to the edges to the path clearly form a path of length 4 in $\mathcal{H}$, a contradiction again. Thus by Claim 4.2.5 every component of H is a K_4 or $K_4 - e$. Take one such component. It clearly contains a cycle of length 4, therefore the corresponding hyperedges form a cycle of length 4. Using Lemma 4.2.1 we get a contradiction again. □

We now consider the case of r-uniform hypergraphs avoiding a Berge path of length k where $r \geq k > 2$.

Proof of Theorem 4.1.4. We will prove Theorem 4.1.4 by induction on k. In fact, we prove a bit stronger statement.

Proposition 4.2.6. *Fix k and r such that $r \geq k > 2$. Let H be a connected r-uniform hypergraph with*

$$e(H) > \frac{k-1}{r+1} n$$

where n is the number of vertices in H. Then for each edge $e \in H$, there is a Berge path of length k in H starting with e.

It is easy to see that the proposition is a strengthening of Theorem 4.1.4.

Proof of Proposition 4.2.6. By induction on k.

We first consider the case $k = 3$. Suppose the theorem does not hold and let H be a minimal (in terms n) counterexample. Then by assumption, there exists an edge $e \in H$ such that all paths starting with e in H are of length $k-1$ or less. We will show this leads to a contradiction.

If there exists an edge $f \in H$ disjoint from e then, as H is connected, there must be a 3-path starting at e. We can thus suppose that every edge of H meets e. Suppose that two edges f and g of H meet outside e. Clearly e, f, g form a 3-path. On the other hand, if there exist edges f and g such that $|e \cap f| \geq 2$ and $|e \cap f \cap g| \geq 1$ then again e, f, g form a 3-path. Thus we may assume that every edge of H meets e, no edges meet outside of e and an edge meeting e in at least 2 vertices meets no other edges of H.

We can now count the n vertices of H. First there are the r vertices in e. Then there is at most one edge, f, which intersects e in two or more points; this edge contributes at least one new vertex to the count. The remaining edges each intersect e in one point and thus each contribute $r-1$ vertices to the count: $n \geq (r-1)(e(\mathcal{H})-1)+2$. But then as $n \geq r+1$,

$$\begin{aligned} e(H) &\leq \frac{n-2}{r-1} + 1 \\ &\leq \frac{2}{r+1} n, \end{aligned}$$

contradicting the inequality in Proposition 4.2.6.

Now suppose the theorem holds for $k-1$, where $k \geq 4$ is fixed. Let H a connected r-uniform hypergraph satisfying the inequality in Proposition 4.2.6. Fix an edge e in H. The basic idea is to remove e from H and apply induction on the remaining graph to find a $k-1$ path P, such that $P + e$ forms a k path. The only difficulty arise in finding an appropriate subgraph in which to apply the inductive hypothesis. To this end, consider the components of $H - e$: $C_1, C_2, \ldots, C_m$. We claim there must be an i such that

$$e(C_i) + 1 > \frac{k-1}{r+1} v(C_i).$$

Otherwise we get the contradiction

$$e(H) \leq \sum e(C_i) + 1 \leq \sum \frac{k-1}{r+1} v(C_i) \leq \frac{k-1}{r+1} n.$$

Now pick a vertex $x \in e \cap C_i$ and let $C_i - x$ be the (possibly no longer uniform) hypergraph obtained by removing x from every edge of C_i: $C_i - x = \{g - x | g \in C_i\}$. Let $C_{i1}, C_{i2}, \ldots, C_{it}$ be the connected components of $C_i - x$. It can be checked that for $r \geq k$, $e(C_i) + 1 > \frac{k-1}{r+1} v(C_i)$ implies $e(C_i) > \frac{k-2}{r}(v(C_i) - 1)$. Thus there exists a j such that

$$e(C_{ij}) > \frac{k-2}{r} v(C_{ij}).$$

Let e^* be an edge of C_{ij} for which the edge $e^* \cup \{x\}$ belongs to C_i. To complete the proof, we will reduce the r-edges of C_{ij} each by one vertex to achieve a $(r-1)$-uniform hypergraph, H^*, connected and satisfying

$$e(H^*) > \frac{k-2}{r} v(H^*).$$

We will then use induction to find a $k-1$ path starting at e^* in H^*. To be able to apply the inductive hypothesis, we must ensure that the process of reducing the r edges of C_{ij} to $r-1$ edges neither disconnects the graph nor creates multiple edges. We claim that, one by one, for each r-edge of C_{ij}, we can pick a vertex of the edge and remove it from the edge such that the remaining graph is still connected and such that no multiple edges are created.

Suppose for some r-edge f this is not possible. If every vertex of f is a cut vertex, then no other edge meets f in more than one vertex and we simply contract the vertices of f to one vertex and delete f from the graph. The graph is still connected, there are no multiple edges created in this step (otherwise not every vertex of f would be a cut-vertex) and it can be checked that

$$\frac{(k-2)(r-1)}{r} \geq 1,$$

which holds for $r \geq k \geq 4$, implies that

$$e(C_{ij}) - 1 > \frac{k-2}{r} \left(v(C_{ij}) - (r-1) \right).$$

Thus we may assume that not every vertex of f is a cut vertex. Suppose now that the deletion of any vertex of f would lead to multiple edges in the graph. This means that every $r-1$ subset of f is already an edge of the graph. In this case there is clearly a P_{r-1} path within the edge f. This path can be extended (in the original graph H) to the edge e; such a path will have length at least k.

Finally it is clear that if removing a vertex x from f causes a multiple edge to appear in the graph then there cannot be another vertex y of f whose removal (from

f) would cut the graph. Thus we can indeed transform C_{ij} into a $(r-1)$ uniform, connected hypergraph satisfying

$$e(H^*) > \frac{k-2}{r} v(H^*).$$

In particular, by the induction hypothesis, there is a $k-1$ path in H^* starting at e^*. Let $e^*, e_2^*, \ldots, e_{k-1}^*$ be the edges of this path and consider the associated edges in H : $e_1, e_2, \ldots, e_{k-1}$. Now by definition of H^*, $x \in e \cap e_1$ and $x \notin \cup_{l>2} e_l^*$. Thus we can extend the path in H to $e, e_1, e_2, \ldots, e_{k-1}$ except in the case when $r = k$ and e consists precisely of those vertices in the intersections $e_l \cap e_{l+1}$ together with the endpoint of the path in e_{k-1}. However as e_{k-1} must be different from e, we may simply choose a different endpoint in e_{k-1}. $\square$

$\square$

4.3 t-Tight Paths

Proof of Theorem 4.1.6. First consider the lower bound. By Theorem 4.1.7, there is a family $\mathcal{B}$ of k-sets of an initial n set such that no t set is contained in more than 1 element of the family and such that

$$|\mathcal{B}| \geq (1 - o(1)) \frac{\binom{n}{t}}{\binom{k}{t}}.$$

We claim that the r-uniform hypergraph $\mathcal{H}$ obtained by replacing each member of $\mathcal{B}$ with all its $\binom{k}{r}$ r-sets contains no t-tight path. Any such path would have vertices in at least 2 different members of $\mathcal{B}$. Specifically, such a path would contain vertices u and v with $u \in B_1 \backslash B_2$ and $v \in B_2 \backslash B_1$ where B_1 and B_2 are two distinct members of $\mathcal{B}$. But $|B_1 \cap B_2| < t$ (the same holds for all distinct pairs of members of $\mathcal{B}$.) Thus there can be no t-tight path in $\mathcal{H}$ from u to v.

We now look at the upper bound. If $t = 1$ then we are done by Theorem 4.1.3. Suppose then that $t \geq 2$. Let $\mathcal{H}$ be a hypergraph on n vertices with more than $\binom{n}{t}\binom{k}{r}/\binom{k}{t}$ hyperedges. Then it is easy to see that there exists a vertex $x_1 \in V$ with degree at least

$$\frac{r}{n} \frac{\binom{n}{t}\binom{k}{r}}{\binom{k}{t}} = \frac{\binom{n-1}{t-1}\binom{k-1}{r-1}}{\binom{k-1}{t-1}}.$$

Let $\mathcal{H}_1 = \{h \backslash \{x_1\} \ : \ h \in \mathcal{H} \wedge x_1 \in h\}$ be the link of x_1. Then continuing we can clearly find vertices $x_2, \ldots, x_{t-1}$ such that for $1 < i < t$, $\mathcal{H}_i$ is the link of x_i in $\mathcal{H}_{i-1}$ and such that for $1 < i < t$, the degree of x_i in $\mathcal{H}_{i-1}$ is greater than $\binom{n-i}{t-i}\binom{k-i}{r-i}/\binom{k-i}{t-i}$. But then $\mathcal{H}_{t-1}$ is simply a $(r-t+1)$-graph on $n-t+1$ vertices with more than $\frac{n-t+1}{k-t+1}\binom{k-t+1}{r-t+1}$ edges. But then applying Theorem 4.1.3 we can find a path of length $k-t+1$ in $\mathcal{H}_{t-1}$. If the minimal degree in $\mathcal{H}_{t-1}$ is large enough, we can then extend this path using the vertices $x_1, \ldots, x_{t-1}$ to a t-tight path of length k in $\mathcal{H}$. $\square$

4.4 Even Tighter Paths

In this section we consider the relationship between open chains and t-tight paths. We prove Theorem 4.1.9 and a related theorem concerning $(r-1)$-tight paths satisfying intersection conditions (J) for fixed $1 \leq J \leq k-1$. First we will need a simple averaging argument.

Lemma 4.4.1. *Let $\mathcal{H}$ be an r-uniform hypergraph on n vertices with strictly more than $c\binom{n}{r-1}$ edges. Then there exists an nonempty sub-hypergraph, $\mathcal{H}'$, of $\mathcal{H}$ such that*

$$\forall S \in \binom{\mathcal{V}(\mathcal{H})}{r-1}, d_{\mathcal{H}'}(S) \leq c \Rightarrow d_{\mathcal{H}'}(S) = 0 \tag{4.2}$$

where $d_{\mathcal{H}'}(S)$ refers to the number of hyperedges of $\mathcal{H}'$ containing the set S.

Proof of Lemma 4.4.1. Let $\mathcal{H}$ be as in the statement of the lemma, and let $\mathcal{H}_1$ be the $(r-1)$-uniform hypergraph on $\mathcal{V}(\mathcal{H})$ with edge set $\{e \in \binom{\mathcal{V}(\mathcal{H})}{r-1} : \exists h \in \mathcal{H}, e \subset h\}$. The hypergraph $\mathcal{H}_1$ is commonly called the lower shadow of $\mathcal{H}$. Let w be a weight function on the edges of $\mathcal{H}_1$ where $w(e) = \mathrm{d}_{\mathcal{H}}(e)$. Then

$$\sum_{e \in \mathcal{H}_1} w(e) = r \cdot \mathrm{e}(\mathcal{H}) > rc\binom{n}{r-1}$$

and the average weight, $\bar{w}$, over the edges of $\mathcal{H}_1$ is strictly more than $rc\binom{n}{r-1}$. Let e be an edge of $\mathcal{H}_1$ with weight no more than c. Now let $\mathcal{H}' = \mathcal{H} \setminus \{h : e \subset h\}$ be a subgraph of $\mathcal{H}$ and let define a new weight function w' on the edges of $\mathcal{H}_1$: $w'(g) = \mathrm{d}_{\mathcal{H}'}(g)$ for each $g \in \mathcal{H}_1$. Finally let $\mathcal{H}_1' = \mathcal{H}_1 \setminus \{g : w'(g) = 0\}$. Then

$$\sum_{g \in \mathcal{H}_1'} w'(g) = \sum_{g \in \mathcal{H}_1} w'(g) \geq \sum_{g \in \mathcal{H}_1} w(g) - rc > rc \cdot \mathrm{e}(\mathcal{H}_1')$$

and in particular, $(1/\mathrm{e}(\mathcal{H}_1')) \sum_{g \in \mathcal{H}_1'} w'(g) > rc$. Replacing $\mathcal{H}$ with $\mathcal{H}'$ and $\mathcal{H}_1$ with $\mathcal{H}_1'$, we may repeat the above operation (as long as there are edges $e \in \mathcal{H}_1$ with degree no more than c and the average weight of the resulting hypergraph $\mathcal{H}_1'$ will always be bounded below by rc. In particular at some point there will be no more edges of degree less than or equal to c in $\mathcal{H}_1'$, and at that point $\mathcal{H}'$ will have the desired property. Note that $\mathcal{H}'$ will not be empty as the average degree of $(r-1)$ sets in the shadow of $\mathcal{H}'$ will be bounded below by rc. □

Proof of Theorem 4.1.9. The lower bound follows from our usual construction using Theorem 4.1.7. The upper bound follows just as easily from Lemma 4.4.1. If an r-uniform hypergraph $\mathcal{H}$ on n vertices satisfies 4.2 then it is quite clear that we can find a tight path of length k in $\mathcal{H}$: in fact every edge of $\mathcal{H}$ will be contained in such a path. □

It is thus perhaps unsurprising that the upper bound (and trivially the lower bound) of Theorem 4.1.6 for $(r-1)$-tight paths also holds for paths satisfying intersection conditions (2) if k is big enough compared to r. On the other hand it is easy to see that our construction for maximal hypergraphs containing no open chain does indeed contain Berge paths satisfying intersection conditions $(k-2)$. We give the following theorem which is clearly best possible up to a factor of r.

Theorem 4.4.2 (Győri, Katona, Lemons [K4])**.** *Let $\mathcal{H}$ be an r-uniform hypergraph containing no Berge path of length k which satisfies intersection conditions (J) in Definition 4.1.11. If $k-J > r-1$, set $a := r-1$. Otherwise set $a := k-J$. Then $e(\mathcal{H}) < (k-a)\binom{n}{r-1}$.*

Note that for $J = 1$ (i.e. for $(r-1)$-tight paths), the theorem is a weaker result than Theorem 4.1.6.

Proof of Theorem 4.4.2. This proof is a simple application of Lemma 4.4.1. □

Chapter 5

Hamiltonian path saturated graphs with small size

5.1 Introduction

In this Chapter we deal with ordinary simple graphs without loops and multiple edges. We discuss a problem that was originally posed for Hamiltonian cycles and paths, but here it is generalized. The next chapter contains results of the same problem for hypergraphs.

Let m and n be positive integers such that $n \geq m+1$ and let G be a graph of order n. A subgraph F of G is called an *m-path cover (or briefly mPC) of G* if

1. each component of F is a path, (paths of length 0 are allowed)
2. F has at most m components, and
3. $V(F) = V(G)$,

where $V(G)$ denotes the vertex set of the graph G. Clearly, a 1-path cover of G is a Hamiltonian path of G.

G is *m-path cover saturated* (mPCS for short), if G has no m-path cover, but connecting any two nonadjacent vertices by a new edge creates an mPC. We shall then write $G \in m$PCS. We write HPS (Hamiltonian path saturated) instead of 1PCS.

Graphs with mPC and mPCS graphs were investigated in several papers, see [20, 26, 116, 117]. Skupień [117] gave the maximum size of mPCS graphs of order n and characterized all such graphs. Also, this notion has some connection to Chvátal's toughness conjecture. A graph G is *t-tough* if $|S| \geq t\omega(G-S)$ for every subset S of the vertex set $V(G)$ with $\omega(G-S) > 1$. Chvátal conjectured [27] that there exists a finite constant t_0 such that every t_0-tough graph is Hamiltonian. Bauer et al. [15] presented $(\frac{9}{4}-\epsilon)$-tough graphs without a Hamiltonian cycle for arbitrary $\epsilon > 0$, proving that $t_0 \geq \frac{9}{4}$ if it exists. To improve this bound we need to construct a non-Hamiltonian graph which has high toughness. Since adding edges to a graph cannot decrease the toughness, it is easy to see that the best constructions must be Hamiltonian cycle saturated graphs.

A graph G is said to be *P_m-saturated* if G has no P_m (a path of order m) as a subgraph, but connecting any nonadjacent vertices by a new edge creates a P_m in G.

We are interested in two functions:

$$\mathrm{sat}(n,m) = \min\{e(G) \mid v(G) = n \text{ and } G \in m\mathrm{PCS}\}$$

and

$$\operatorname{sat}(n, P_m) = \min\{e(G) \mid v(G) = n,\ G \text{ is } P_m\text{-saturated}\}.$$

We shall write $\operatorname{sat}(n, \mathrm{HP})$ in place of $\operatorname{sat}(n, 1)$.
$\operatorname{Sat}(n, m)$ will denote the set of graphs

$$\operatorname{Sat}(n, m) = \{G \mid v(G) = n,\ e(G) = \operatorname{sat}(n, m) \text{ and } G \in m\mathrm{PCS}\}.$$

We shall write $\operatorname{Sat}(n,\mathrm{HP})$ in place of $\operatorname{Sat}(n, 1)$.
For a graph G and a subset of its vertices S we denote by $G\langle S\rangle$ the subgraph of G induced by S.

In Sections 5.2 and 5.3 we prove lower bounds on $\operatorname{sat}(n, m)$ and $\operatorname{sat}(n, \mathrm{HP})$. In Section 5.4 we prove that for $n \geq 54$ or $n \in \{22, 23, 30, 31, 38, 39, 40, 41, 42, 43, 46, 47, 48, 49, 50, 51\}$

$$\left\lfloor \frac{3n-1}{2} \right\rfloor - 2 \leq \operatorname{sat}(n, \mathrm{HP}) \leq \left\lfloor \frac{3n-1}{2} \right\rfloor.$$

We use this result in Section 5.5 to estimate $\operatorname{sat}(n, m)$ and $\operatorname{sat}(n, P_m)$ for certain n and m. These results appeared in [K1].

In [129] Zelinka gave a construction of a family of graphs $\mathcal{F}_n$ of order $n \in \mathbb{N}$, and conjectured that every HPS graph of order n is in $\mathcal{F}_n$. Since in Zelinska's construction $e(G) = O(n^2)$, for every graph $G \in \mathcal{F}_n$, Theorem 5.4.2 disproves this conjecture.

Following our work some results were improved. Frick and Singleton [55] improved our lower bound for $\operatorname{sat}(n, \mathrm{HP})$, and showed that the upper bound is sharp if $n \geq 54$. Burger and Singleton [25] obtained the precise values for many smaller n values. In [24] Bullock, Frick, van Aardt and Mynhardt investigated the structure of HPS graphs that are not 1-tough and they constructed several interesting new classes of HPS graphs, some of them are generalizations of our constructions. In [23] Bullock, Frick and Singleton constructed an HPS graph on 18 vertices with 30 edges which is the smallest know HPS graph at the moment.

5.2 The lower bound of $\operatorname{sat}(n, m)$

In order to determine a lower bound for $\operatorname{sat}(n, m)$ we need the following lemmas concerning the degrees of the vertices of an mPCS graph.

Lemma 5.2.1. *Let $G \in m$PCS and let u be a vertex of degree 2 in G. Then the neighbors of u are adjacent.*

Proof. Suppose to the contrary that the neighbors, w and z, of u are not adjacent. Since $G \in m$PCS, $G \cup \{wz\}$ has an mPC, F_m, which contains wz as an edge. Thus F_m cannot contain both uw and uz. Therefore, u must be an end vertex of one of the paths in F_m. So, one of the paths in F_m starts as $(uwz\dots)$ or $(uzw\dots)$. Replacing this path by $(wuz\dots)$ in the first case and $(zuw\dots)$ in the second case we obtain an mPC in G which does not contain the "new" edge wz, a contradiction. □

Lemma 5.2.2. *Let $G \in m$PCS and let F_{m+1} be an $(m+1)$PC of G. Let w, u, z be three consecutive vertices on some path $P^i = (v_{i,1}, v_{i,2}, \dots, v_{i,j_i})$ in F_{m+1}, with $j_i \geq 5$. Then the following hold:*

1. *If $d(u) = 2$ and w, u, z are internal vertices of P^i, then $d(w) \geq 4$ and $d(z) \geq 4$.*

2. *If $d(u) = 2$ and w is an end-vertex of P^i, then $d(z) \geq 4$. Similarly, if z is an end-vertex of P^i, then $d(w) \geq 4$.*

3. *If w, u, z are internal vertices of P^i and $d(u) \leq 4$, then $d(w) \geq 3$ or $d(z) \geq 3$.*

Proof. We shall prove the first part of the lemma. By Lemma 5.2.1 we have $wz \in E(G)$, so $d(w) > 2$ and $d(z) > 2$. Let us suppose that $d(w) = 3$. Let $w' := v_{i,k-2}$. It will be shown that $w'u \in E(G)$ which leads to a contradiction. Since $G \in m$PCS, $G \cup \{w'u\}$ has an mPC, F_m, which contains $w'u$ as an edge. It is obvious that (w) cannot be a trivial path of F_m. Observe that the path $(\dots zww'u)$ cannot belong to F_m since replacing it by $(\dots zuww')$ we obtain an mPC in G - a contradiction. Thus F_m must contain uw or uz. In the first case this implies that the corresponding path of F_m contains a segment $\dots w'uwz \dots$ or a segment $\dots w'uw$ (w is a terminal vertex of a path in F_m). Replacing this part by $\dots w'wuz \dots$ ($w'wu$ respectively) we obtain an mPC which does not contain the "new" edge $w'u$, a contradiction. In the second case there are two possibilities: $ww'uz \dots$ and $wzuw' \dots$, since w must be covered as well. These can be replaced by $w'wuz \dots$ and $zuww' \dots$ to obtain the same type of contradiction.
In a similar manner we can prove the second part of the lemma.
Now we prove the last part of the lemma.

Since w and z are internal vertices of a path of F_{m+1}, they have degree at least two. It will be shown that it is not possible, that both degrees are equal to 2. Let $w' =: v_{i,k-2}$ and $z' =: v_{i,k+2}$. Suppose to the contrary that $d(w) = d(z) = 2$. By Lemma 5.2.1 we have $w'u \in E(G)$ and $uz' \in E(G)$. Since $G \in m$PCS, $G \cup \{wz\}$ has an mPC, F_m, which contains wz as an edge.
Case 1: wz is a separate path in F_m.

If there is another path P in which u is an internal vertex then $P = (\dots w'uz' \dots)$. By replacing P with $P' = (\dots w'wuzz' \dots)$ we obtain an $(m-1)$PC of G, a contradiction. If u is an end vertex of a path of F_m then it is easily seen that there is an $(m-1)$PC of $G \cup \{wz\}$.
Case 2: F_m contains wzu.

If (wzu) is a separate path, then replacing this by wuz we obtain an mPC of G which does not contain the "new" edge.

If F_m contains the segment $\dots wzuz' \dots$ then replacing it by $\dots wuzz' \dots$ we get the same contradiction.

If F_m contains the end segment $wzuw' \dots$ then replacing it by $zuww' \dots$ we get the same contradiction. (The case when F_m contains zuw is symmetric.)
Case 3: F_m contains wzz'.
If F_m also contains $z'u$ then it contains either the end segment $wzz'uw' \dots$ or the end segment $uz'zww' \dots$. In each case the end segment can be replaced by $z'zuww' \dots$ to obtain an mPC of G.

If F_m contains $w'u$ then one path in F_m ends with $\dots w'u$ and another with $\dots z'zw$ which means there is a path $(\dots z'zwuw' \dots)$ in F_m which can be replaced by the path $(\dots z'zuww' \dots)$ and we get the same type of contradiction.

If u is a trivial path of F_m then this path and the edge wz in F_m may be replaced with $\dots wuz \dots$ and we obtain an mPC in G, a contradiction. (The case when F_m contains zww' is symmetric.) $\square$

Lemma 5.2.3. *Let $G \in mPCS$ and let F_{m+1} be an $(m+1)PC$ of G. If w, u, z are three consecutive internal vertices of some path in F_{m+1}, then $d(w) + d(u) + d(z) \geq 9$.*

Proof. Since w, u, z are all internal vertices on some path of F_{m+1} all of them have degree at least 2. By Lemma 5.2.1 if two of them have their degree equal to 2 then $d(w) = d(z) = 2$. On the other hand, in this case $d(u) \geq 5$ by Lemma 5.2.2. Hence, we may suppose that exactly one of vertices u, w z has its degree equal to 2. If $d(w) = 2$ then $d(u) \geq 4$ by Lemma 5.2.2 and Lemma 5.2.3 holds. If $d(z) = 2$ then we obtain the same result by symmetry.

If $d(u) = 2$ then by Lemma 5.2.2 $d(w) \geq 4$ and $d(z) \geq 4$ proving our claim. □

Theorem 5.2.4 (Dudek, Katona, Wojda [K1]). *Let n and m be positive integers, $n \geq m+1$. Then*

$$\operatorname{sat}(n, m) \geq \frac{3}{2}n - 3(m+1).$$

Proof. Let $G \in \operatorname{Sat}(n, m)$. By definition G has no mPC, however, it is easy to see that it has an $(m+1)$PC. Let $F_{m+1} = \bigcup_{i=1}^{m+1} P^i$ be an $(m+1)$PC of G, with $P^i = (v_{i,1}v_{i,2}\ldots v_{i,j_i})$; $i = 1, \ldots, m+1$. To prove the theorem we estimate the sum of the degrees in G using Lemma 5.2.3. Let $P^1, \ldots, P^{m_1}$ be the paths with exactly one vertex; $P^{m_1+1}, \ldots, P^{m_1+m_2}$ the paths with exactly two vertices; $P^{m_1+m_2+1}, \ldots, P^{m_1+m_2+m_3}$ the paths with exactly three vertices and $P^{m_1+m_2+m_3+1}, \ldots, P^{m+1}$, the paths with at least four vertices. Denote by $n_l, l = 1, 2, 3$ the number of vertices of G covered by paths of order l. Denote by n_4 the number of vertices of G covered by paths of order at least four, and by m_4 the number of paths with at least four vertices. Clearly n, m, m_1, m_2, m_3, m_4 satisfy $n \geq m+1$, $m_1 + m_2 + m_3 + m_4 = m+1$. We have $n_l = lm_l$ for $l = 1, 2, 3$. Thus $n = m_1 + 2m_2 + 3m_3 + n_4$. For the degrees of vertices $v_{i,2}$ and v_{i,j_i-1} we will use the trivial lower bound 2 and the degrees of $v_{i,1}$ and v_{i,j_i} the lower bound 1. (This part of proof could be improved by similar methods, but the proof would be too large compared to the small improvement.)

$$6e(G) = 3\sum_{v \in V(G)} d(v) = \sum_{i=m_1+m_2+m_3+1}^{m+1} \sum_{k=3}^{j_i-2} (d(v_{i,k-1}) + d(v_{i,k}) + d(v_{i,k+1})) +$$

$$+ \sum_{i=m_1+m_2+m_3+1}^{m+1} [3\,(d(v_{i,1}) + d(v_{i,j_i})) + 2\,(d(v_{i,2}) + d(v_{i,j_i-1})) +$$

$$+ (d(v_{i,3}) + d(v_{i,j_i-2}))] + 3\sum_{i=m_1+m_2+1}^{m_1+m_2+m_3} (d(v_{i,1}) + d(v_{i,2}) + d(v_{i,3})) +$$

$$+3\sum_{i=m_1+1}^{m_1+m_2} (d(v_{i,1}) + d(v_{i,2})) + 3\sum_{i=1}^{m_1} d(v_{i,1}) \geq$$

$$9(n_4 - 4m_4) + 6m_4 + 8m_4 + 4m_4 + 18m_3 + 6m_2 = 9n_4 - 18m_4 + 18m_3 + 6m_2 =$$

$$= 9n - 9m_1 - 12m_2 - 9m_3 - 18m_4 \geq 9n - 18m_1 - 18m_2 - 18m_3 - 18m_4 = 9n - 18(m+1)$$

The above inequality implies that $e(G) \geq \frac{3}{2}n - 3(m+1)$. □

5.3 The lower bound of $\mathrm{sat}(n, \mathrm{HP})$

Theorem 5.3.1 (Dudek, Katona, Wojda [K1])**.** *Let* $G \in \mathrm{Sat}(n, \mathrm{HP}), n \geq 14$. *Then* $e(G) \geq \lfloor \frac{3n-1}{2} \rfloor - 2$.

Proof. To prove the theorem we estimate the sum of degrees using Lemmas 5.2.1 and 5.2.3. Note that it is sufficient to prove that $6e(G) \geq 9n - 20$. It is clear that the vertices of G may be covered by two vertex disjoint paths P^1, P^2. We shall consider five cases.
Case 1. $|P^1| \geq 5, |P^2| \geq 5, P^1 = (a_1, a_2, \ldots, a_k), P^2 = (b_1, b_2, \ldots, b_{n-k})$.
By Lemma 5.2.3, we have

$$
\begin{aligned}
6e(G) = 3 \sum_{v \in V(G)} d(v) = \\
&= \sum_{i=3}^{k-2} (d(a_{i-1}) + d(a_i) + d(a_{i+1})) + \\
&3\,(d(a_1) + d(a_k)) + 2\,(d(a_2) + d(a_{k-1})) + d(a_3) + d(a_{k-2}) + \\
&\sum_{i=3}^{n-k-2} (d(b_{i-1}) + d(b_i) + d(b_{i+1})) + \\
&3\,(d(b_1) + d(b_{n-k})) + 2\,(d(b_2) + d(b_{n-k-1})) + d(b_3) + d(b_{n-k-2}) \\
&\geq 9(n-8) + \\
&2(d(a_1) + d(a_2) + d(a_k) + d(a_{k-1}) + d(b_1) + d(b_2) + d(b_{n-k}) + d(b_{n-k-1})) \\
&+ d(a_1) + d(a_3) + d(a_k) + d(a_{k-2}) + d(b_1) + d(b_3) + d(b_{n-k}) + d(b_{n-k-2})
\end{aligned}
$$

Since $d(a_i), d(b_j) \geq 1$, for $i = 1, k$; $j = 1, n-k$ and $d(a_i), d(b_j) \geq 2$, for $i = 2, 3, k-1, k-2$, $j = 2, 3, n-k-1, n-k-2$, we have $6e(G) \geq 9n - 36$. Observe that if $d(a_1) + d(a_3) = 3$ then $d(a_1) = 1, d(a_3) = 2$ and the addition of any edge a_2x does not create any Hamiltonian path in $G \cup \{a_2x\}$ so $d(a_2) = n-1$ and $6e(G) \geq 9n - 36 + 2(n-3) \geq 11n - 42 \geq 9n - 20$ for $n \geq 11$. Hence we may suppose that $d(a_1) + d(a_3) \geq 4$ and similarly $d(a_k) + d(a_{k-2}) \geq 4$; $d(b_1) + d(b_3) \geq 4$; $d(b_{n-k}) + d(b_{n-k-2}) \geq 4$.

It follows easily from Lemma 5.2.1 that $d(a_1) + d(a_2) \geq 4$. If, however, $d(a_1) + d(a_2) = 4$ then, clearly, for any nonadjacent pair of vertices x, y such that $x, y \notin \{a_1, a_2\}$ any Hamiltonian path in $G \cup \{xy\}$ starts either from a_1 or a_2. The same situation is for the couples $\{a_k, a_{k-1}\}$, $\{b_1, b_2\}, \{b_{n-k}, b_{n-k-1}\}$. One may check that for at most two of these four couples we may have the degrees sum equal to 4. Hence $d(a_1) + d(a_k) + d(a_2) + d(a_{k-1}) + d(b_1) + d(b_{n-k}) + d(b_2) + d(b_{n-k-1}) \geq 18$.
We conclude that $6e(G) \geq 9n - 20$.
Case 2. $|P^1| = n-1, |P^2| = 1, P^1 = (a_1, a_2, \ldots, a_{n-1}), P^2 = (a_n)$.

We have $6e(G) = 3\sum_{v\in V(G)} d(v) = s_1 + s_2 + s_3 + s_4 + s_5$ where

$$\begin{aligned}
s_1 &= \sum_{i=3}^{n-3}(d(a_{i-1}) + d(a_i) + d(a_{i+1}))\\
s_2 &= 3\,(d(a_1) + d(a_{n-1}))\\
s_3 &= 2\,(d(a_2) + d(a_{n-2}))\\
s_4 &= (d(a_3) + d(a_{n-3}))\\
s_5 &= 3d(a_n)
\end{aligned}$$

If $s_5 = 0$ then, clearly, $G - \{a_n\} = K_{n-1}$ and the theorem holds. Thus we may suppose $s_5 \geq 3$.
If $d(a_1) = d(a_{n-1}) = d(a_n) = 1$ then $G - \{a_1, a_{n-1}, a_n\} = K_{n-3}$, $e(G) = \binom{n-3}{2} + 3 \geq \frac{9n-19}{6}$ and the theorem follows. So we may assume $d(a_1) + d(a_{n-1}) + d(a_n) \geq 4$. Observe that if one of the vertices of a_1 or a_{n-1} has its degree equal to 1 then, by Lemma 5.2.1, its neighbor on P^1 has degree at least 3. So we may assume that either $d(a_1) \geq 2$ or $d(a_2) \geq 3$. Considering all the possibilities one may check that $s_1 + s_2 + s_3 + s_4 + s_5 \geq \frac{9n-19}{6}$.
Case 3. $|P^1| = n-2, |P^2| = 2, P^1 = (a_1, a_2, \ldots, a_{n-2}), P^2 = (a_{n-1}, a_n)$.
Clearly, neither a_{n-1} nor a_n is adjacent to $\{a_1, a_{n-2}\}$. If a_{n-1} or a_n is adjacent to one of the vertices $a_2, \ldots, a_{n-3}$ then we have Case 1 or Case 2. Hence $G = K_2 \cup K_{n-2}$ and $e(G) = \binom{n-2}{2}+1 \geq \frac{9n-19}{6}$ for $n \geq 14$.
Case 4. $|P^1| = n-3, |P^2| = 3, P^1 = (a_1, a_2, \ldots, a_{n-3}), P^2 = (a_{n-2}, a_{n-1}, a_n)$.
Neither a_{n-2} nor a_n is adjacent to $\{a_1, a_{n-3}\}$. If a_{n-2} or a_n is adjacent to one of the vertices $a_2, \ldots, a_{n-4}$ then we have Case 1 or 2. If $a_{n-2}a_n \in E$, then we have either Cases 1 or 2 or $G = K_3 \cup K_{n-3}$ and $e(G) = \binom{n-3}{2}+3 \geq \frac{9n-19}{6}$ for $n \geq 14$. Hence we may suppose $d(a_{n-2}) = d(a_n) = 1$. Then the graph $G\langle a_1, \ldots, a_{n-3}, a_{n-1}\rangle = K_{n-2}$ and the case follows.
Case 5. $|P^1| = n-4, |P^2| = 4, P^1 = (a_1, a_2, \ldots, a_{n-4})$,
$P^2 = (a_{n-3}, a_{n-2}, a_{n-1}, a_n)$.
Clearly, neither a_{n-3} nor a_n is adjacent to $\{a_1, a_{n-4}\}$. If a_{n-3} or a_n is adjacent to one of the vertices $a_2, \ldots, a_{n-5}$ then we have Case 1 since we are assuming $n \geq 14$.
If $a_{n-3}a_n \in E$ then for similar reasons none of the vertices $a_{n-3}, a_{n-2}, a_{n-1}, a_n$ is adjacent to any vertex of the set $\{a_1, \ldots, a_{n-4}\}$ and thus $G = K_4 \cup K_{n-4}$ and the proof follows. So we may suppose $a_{n-3}a_n \notin E$. We may also assume that one of the vertices a_{n-2} or a_{n-1} is adjacent to the path P^1, say $a_{n-1}a_{i_0} \in E$ with $i_0 \in \{1, \ldots, n-4\}$.
If $a_{n-2}a_n \in E$ then we have the path $(a_{n-1}, a_n, a_{n-2}, a_{n-3})$ with $a_{n-1}a_{i_0} \in E$ and it is easy to see that we have Case 1. So we assume $a_{n-2}a_n \notin E$.
If $a_{n-3}a_{n-1} \in E$ then we have the path $(a_{n-2}, a_{n-3}, a_{n-1}, a_n)$ and the rest of the proof runs as before, with a_{n-2} and a_{n-3} interchanged. So we may assume $a_{n-3}a_{n-1} \notin E$ and thus $d(a_n) = d(a_{n-3}) = 1$. The graph $G' = G \cup \{a_{n-2}a_n\}$ has the Hamiltonian path $P = (a_{n-3}, a_{n-2}, a_n, a_{n-1}, \ldots x)$. The paths $(a_{n-3}, a_{n-2}, a_{n-1}, \ldots x), (a_n)$ cover the vertices of G as in Case 2, which completes the proof. □

5.4 HPS graphs with small size

The following lemma is a consequence of some results (theorems 5, 7, 9 and 11) given in [30] by L.H. Clark, R.C. Entringer and H.D. Shapiro (see also [29]).

Lemma 5.4.1. *For every n even, $n \geq 52$ or $n \in \{20, 28, 36, 38, 40, 44, 46, 48\}$ there is a non Hamiltonian graph G_n of order n with the following properties:*

1. *G_n is cubic,*
2. *there is an edge $e = xy$ in G_n such that $N(x) \cap N(y) = \emptyset$ and, for every pair of nonadjacent vertices u, v, the graph $G_n \cup \{uv\}$ has a Hamiltonian cycle containing e.*

More precisely Clark, Entringer and Shapiro proved that for k odd and sufficiently big, the Isaacs' graphs J_k of order $4k$ defined in [69] and their modifications given in [30] have the property of the thesis of Lemma 5.4.1. Their results imply in particular, that for $n \geq 52$ the minimum size of a maximally non-Hamiltonian graph of order n is $\lceil \frac{3n}{2} \rceil$. Lin, Jiang, Zhang and Yang in [85] set the values of the size of smallest maximally non-Hamiltonian graphs for all remaining orders n. Note that Kalinowski and Skupień proved in [71, 72] that the Isaacs' graphs are maximally non-Hamiltonian-connected with minimum size.

Theorem 5.4.2 (Dudek, Katona, Wojda [K1])**.** *For every $n \geq 54$ or $n \in \{22, 23, 30, 31, 38, 39, 40, 41, 42, 43, 46, 47, 48, 49, 50, 51\}$ there is an HPS graph of order n and size $\lfloor \frac{3n-1}{2} \rfloor$.*

Proof. Theorem 5.4.2 follows from Lemma 5.4.1 and the three lemmas given below.

Lemma 5.4.3. *Let $G = (V; E)$ be a graph of order n which satisfies the conditions of Lemma 5.4.1. Then the graph $G - x$ is Hamiltonian.*

Proof. As in the proof of hypohamiltonicity of J_k given in [29] we take a path (xyz). Since $xz \notin E(G)$, the edge xy is contained in Hamiltonian path $P = (xy, \ldots, z)$ of G. Deleting from P the vertex x and adding the edge yz we obtain a Hamiltonian cycle of $G - x$. □

Lemma 5.4.4. *Let $G = (V; E)$ be a graph satisfying the conditions of Lemma 5.4.1. Then the graph $G' = (V'; E')$ where $V' = V \cup \{z_1, z_2\}$, $E' = E \cup \{xz_1, yz_2\}$ is HPS.*

Proof. Since G is not Hamiltonian, it is clear that G' has no Hamiltonian path. Let uv be a new edge of G'. We shall prove that $G' \cup \{uv\}$ has a Hamiltonian path. We shall consider 4 cases.

Case 1. $u = z_1, v = z_2$

Let P be a Hamiltonian path of G starting from x. Then $z_2 z_1 P$ is a Hamiltonian path of $G' \cup \{uv\}$.

Case 2. $u, v \notin \{z_1, z_2\}$

The graph $G \cup \{uv\}$ contains a Hamiltonian cycle through xy. A Hamiltonian path of $G' \cup \{uv\}$ is easy to obtain.

Case 3. $u = z_1, v = y$

Let w be a vertex of G such that $wy \notin E$. Let $P = (y, x, v_1, \ldots, v_{n-3}, w)$ be a Hamiltonian path in G through xy. $P' = (z_2, y, z_1, x, v_1, \ldots, v_{n-3}, w)$ is a Hamiltonian path

of $G' \cup \{uv\}$.
Case 4. $u = z_1, v \in V - \{x, y\}$
If $yv \notin E$ then consider a Hamiltonian path $P = (v, \dots, y)$, then $z_1 P z_2$ is a Hamiltonian path of $G' \cup \{uv\}$. So we may suppose $yv \in E$. By Lemma 5.4.3 the graph $G - \{x\}$ is Hamiltonian and since in $G - \{x\}$ the degree of y is equal to 2, there is in $G - \{x\}$ a Hamiltonian path P from y to v. Then $z_2 P z_1 x$ is a Hamiltonian path of $G' \cup \{uv\}$. □

The proof of the following result is similar to that of Lemma 5.4.4.

Lemma 5.4.5. *Let $G = (V; E)$ be a graph of order n which satisfies the assumption of Lemma 5.4.1. Then the graph $G' = (V'; E')$ where $V' = V \cup \{z_1, z_2, z_3\}$, $E' = E \cup \{xz_1, yz_2, yz_3, z_2z_3\}$ is HPS.*

□

5.5 P_q-saturated graphs and m-path cover saturated graphs

In [131] Kászonyi and Tuza gave the minimum size of a P_q-saturated graph G for q sufficiently small with respect to the order of G by proving the following theorem.

Theorem 5.5.1 (Kászonyi, Tuza [131])**.** *Let*

$$a_q = \begin{cases} 3 \cdot 2^{k-1} - 2 & \text{if } q = 2k, k > 2, \\ 2^{k+1} - 2 & \text{if } q = 2k+1, k \geq 2. \end{cases}$$

Then, for $n \geq a_q$,

$$\operatorname{sat}(n, P_q) = n - \left\lfloor \frac{n}{a_q} \right\rfloor .$$

Corollary 5.5.2. *Let $p \geq q$.*

1. *For q even, $q - 2 \in \{20, 28, 36, 38, 40, 44, 46, 48\}$ or $q - 2 \geq 52$ we have*

$$\operatorname{sat}(p, P_q) \leq p + \frac{q}{2} - 1,$$

2. *For q odd, $q - 3 \in \{20, 28, 36, 38, 40, 44, 46, 48\}$ or $q - 3 \geq 52$ we have*

$$\operatorname{sat}(p, P_q) \leq \frac{3p}{2}.$$

Proof. Let $G = (V; E)$ be a graph of order n satisfying the assumptions of Lemma 5.4.1. We shall construct three families of graphs.

1. $H^1_{n+l+1} = (V^1; E^1)$ where $V^1 = V \cup \{v_0, v_1, \dots, v_l\}$, $l \geq 1$,
   $E^1 = E \cup \{xv_0, yv_1, yv_2, \dots, yv_l\}$,
2. $H^2_{n+2l+1} = (V^2; E^2)$ where $V^2 = V \cup \{v_0, v_1, \dots, v_l, w_1, \dots, w_l\}$,
   $l \geq 1$, $E^2 = E \cup \{v_0 x, v_i w_i, y v_i, y w_i; i = 1, \dots, l\}$

3. $H^3_{n+2l+2} = (V^3; E^3)$ where $V^3 = V \cup \{v_0, v_1, \ldots, v_l, w_0, w_1, \ldots, w_l, \}$, $l \geq 1$, $E^3 = E \cup \{v_i w_i; i = 0, \ldots, l, v_0 x, w_0 x, v_i y, w_i y; i = 1, \ldots, l\}$.

The corollary follows from the following three observations.

1. the graph H^1_{n+l+1} is P_{n+2}-saturated of order $p = n + l + 1$ and size $\frac{3n}{2} + l + 1$,
2. the graph H^2_{n+2l+1} is P_{n+3}-saturated of order $p = n + 2l + 1$ and size $\frac{3n}{2} + 3l + 1$,
3. the graph H^3_{n+2l+2} is P_{n+3}-saturated of order $p = n + 2l + 2$ and size $\frac{3n}{2} + 3(l + 1)$ (note that H^3_{n+2l+2} is also P_{n+4}-saturated).

□

Observe that if $G = (V; E)$ is an HPS graph without a total vertex then the graph $G' = (V'; E')$ with $G' = G \cup (m-2)K_2 \cup K_1$ is mPCS. By Theorem 5.4.2 we have the following corollary.

Corollary 5.5.3. *For $m \geq 2, n - 2m + 3 \in \{22, 23, 30, 31, 38, 39, 40, 41, 42, 43, 46, 47, 48, 49, 50, 51\}$ or $n - 2m + 3 \geq 54$ we have*

$$\mathrm{sat}(n, m) \leq \frac{3n}{2} - 2m + 2.$$

Observe that the values of sat(n,HP), sat(p, P_q) and sat(n, m) remain undetermined.

Chapter 6

Hamilton-chain saturated hypergraphs

6.1 Introduction

In Section 5.4 we discussed Hamiltonian-path saturated graphs. In this Chapter we investigate the corresponding question for uniform hypergraphs. For this we use the generalization of Hamiltonian-paths in hypergraphs given by Definition 1.1.4. In this Chapter we consider only open chains so for simplicity *we will write chain instead of open chain.*

For clarity, let us note that by removing a vertex we mean to remove also every edge containing this vertex.

Definition 6.1.1. *We say that an r-uniform hypergraph $\mathcal{H}$ is Hamiltonian path saturated if $\mathcal{H}$ does not contain an (open) Hamiltonian chain but by adding any new r-edge we create an (open) Hamiltonian chain in $\mathcal{H}$.*

Originally, the problem of estimating the number of edges in a Hamiltonian cycle saturated graph appeared in O. Ore [99] where it is proved that a nonhamiltonian graph (and so, a Hamiltonian cycle saturated graph) of order n has at most $\binom{n-1}{2}+1$ edges. Bollobás [18] posed the problem of finding the minimum number, $\mathrm{sat}(n; C_n)$, of edges in a Hamiltonian cycle saturated graph on n vertices. In 1972 Bondy [19] proved that $\mathrm{sat}(n; C_n) \geq \lceil \frac{3n}{2} \rceil$ for $n \geq 7$. Combined results of Clark, Entrigner and Shapiro [29, 30] and Xiaohui, Wenzhou, Chengxue and Yuansheng [85] show that this bound is sharp apart from a few smaller values of n. The constructions are mostly tricky graphs based on Isaacs' snarks (see [69]) and generalized Petersen graphs. It was natural to ask the same question for Hamiltonian path saturated graphs. Such results were given in Chapter 5. In the present Chapter we study the related problem for r-uniform hypergraphs, mainly for $r = 3$, our results appeared in [K2].

Definition 6.1.2. *Let $g_r(n)$ $(r \geq 2)$ denote the minimum number of edges in a Hamiltonian path saturated r-uniform hypergraph on n vertices.*

Results mentioned in Chapter 5 imply that $g_2(n) = \left\lfloor \frac{3n-1}{2} \right\rfloor$ for $n \geq 54$. On the other hand, in [K11] a construction is given of an n-vertex Hamiltonian chain saturated r-uniform hypergraph with

$$\sim \left(\frac{1}{r!} - \frac{1}{2^r \lceil r/2 \rceil ! \lfloor r/2 \rfloor !} \right) n^r$$

edges, our upper bound for $g_r(n)$. For $r=3$, this yields $g_3(n) \le \frac{5}{48}n^3 + o(n^3)$.

In the present Chapter we improve the construction from [K11] for $r = 3$. As a result, for any $n \ge 12$ we obtain a 3-uniform hypergraph with $O(n^{5/2})$ edges. It is interesting that the existence of a Hamiltonian chain depends on the order of some sets in our construction. On the other hand, we obtain a general lower bound $g_r(n) \ge \binom{n}{r}/(r(n-r)+1)$ which is of order $\Omega(n^{r-1})$. These results appeared in [K2].

Following our work, Dudek and Żak [41] generalized our construction for all r-uniform hypergraphs, they constructed Hamiltonian chain saturated r-uniform hypergraphs having $O(n^{r-1/2})$ edges. Then Żak [127] improved the construction to have size $O(n^{r-1})$, thus obtaining an asymptotically tight bound. Ruciński and Żak [113] considered the corresponding problem for t-tight r-uniform Hamiltonian cycles, and proved that if $r - t \mid n$ and $\frac{4r}{5} \le t \le r$ then the number of edges in a saturated hypergraph is $\Theta(n^t)$, which generalizes the previous results. For other values of t the same authors [114] proved various upper bounds, but these are not known to be tight.

6.2 Lower bound

Theorem 6.2.1 (Dudek, Katona, Żak [K2])**.** *If $\mathcal{H}$ an r-uniform hypergraph is Hamiltonian chain saturated, then $|\mathcal{E}(H)| \ge \binom{n}{r}/(r(n-r)+1)$.*

Proof. We prove that every r-tuple $E_0 = \{v_1, \dots, v_r\}$ contains an $(r-1)$-element subset, which is contained by an edge of $\mathcal{H}$.

If $E_0 \in \mathcal{E}(H)$ then any $(r-1)$-element subset is contained by E_0 which is an edge, so the claim holds.

Now suppose that $E_0 \notin \mathcal{E}(H)$. Since $\mathcal{H}$ is Hamiltonian path saturated, it does not contain a Hamiltonian chain, but adding E_0 creates one. Therefore E_0 must be an edge of this Hamiltonian chain, so it has a neighboring edge in the chain (even if it is at the end of the chain). This edge satisfies the conditions of the claim.

Using the claim we obtain that for all possible r-tuples we can find an edge that intersects the r-tuple in at least $(r-1)$ elements. However, in this way every such edge is counted $r(n-r)+1$ times. □

6.3 Hamiltonian path saturated 3-uniform hypergraphs

In this section we present a construction of a family of 3-uniform Hamiltonian chain saturated hypergraphs. We start with two definitions.

Definition 6.3.1. *Let p and k be non-negative integers and $U_0, U_1, \dots, U_k$ be pairwise disjoint sets of vertices such that $|U_0| = p$ and $|U_i| \ge 2$ for $i = 1, 2, \dots, k$. Define the vertex set of the hypergraph $\mathcal{H} = \mathcal{H}(U_0, U_1, \dots, U_k)$ to be $V(\mathcal{H}) = \bigcup_{i=0}^{k} U_i$. The edge set is defined such that the induced subhypergraph $\mathcal{H}(U_0 \cup U_i)$ is complete hypergraph for all $i = 1, 2, \dots, k$. The family of all hypergraphs obtained by this construction is denoted by $\mathcal{I}(p, k)$.*

Definition 6.3.2. *Let $\mathcal{H} \in \mathcal{I}(p, k)$. An edge $E_0 = \{x, y, z\}$ where $x \in U_i$ and $y, z \in U_j$ or $x \in U_i$, $y \in U_0$ and $z \in U_j$ is called a jumping edge from U_i to U_j. The set of all jumping edges from U_i to U_j is denoted by $J_{i,j}$.*

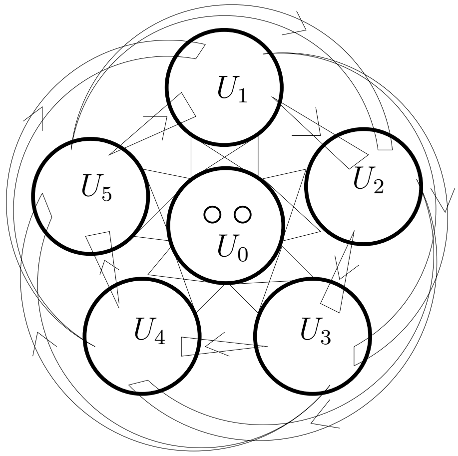

Figure 6.1: A hypergraph from the family $\mathcal{J}(2,5)$.

If $E_1 \in J_{i_1,j_1}$ and $E_2 \in J_{i_2,j_2}$ then we say that jumping edges E_1, E_2 are from different sets when $j_1 \neq j_2$.

Let K_n be a complete graph on n vertices, $n \geq 2$, with vertices labeled by natural numbers $\{1, ..., n\}$. By $\overrightarrow{K_n}$ we denote the following orientation of K_n. Namely the oriented edges in $\overrightarrow{K_n}$ are of the form $(i, i+1), ..., (i, i+\lceil n/2 \rceil - 1)$ for $i = 1, ..., n$, where the numbers are understood cyclically so $n + r = r$, if $r > 0$. The remaining edges of $\overrightarrow{K_n}$, for even n, are oriented in an arbitrary way. We write $i \prec j$ if there is an oriented edge from i to j in $\overrightarrow{K_n}$.

Definition 6.3.3. *Let $\mathcal{H}(U_0, U_1, \ldots, U_k) \in \mathcal{I}(p, k)$. Define the hypergraph $\mathcal{G} = \mathcal{G}(U_0, U_1, \ldots, U_k)$ as a hypergraph with vertex set $V(\mathcal{G}) = V(\mathcal{H})$ and edge set*

$$\mathcal{E}(\mathcal{G}) = \mathcal{E}(\mathcal{H}) \cup \{J_{i,j} : i \prec j\}.$$

The family of hypergraphs obtained by this construction is denoted by $\mathcal{J}(p, k)$.

Lemma 6.3.4. *Let $\mathcal{G} \in \mathcal{J}(p, k)$. A chain in $\mathcal{G} - U_0$ cannot contain jumping edges from two different sets.*

Proof. Suppose indirectly that a chain contains edges from two different set of jumping edges $E_1 \in J_{i_1,j_1}$ and $E_2 \in J_{i_2,j_2}$, $j_1 \neq j_2$. Without a loss of generality we can assume that there are no other jumping edges in the chain between these two edges.

By this assumption only non-jumping edges can be found between these edges on the chain. E_1 is adjacent on the chain to edges contained in U_{j_1}. These edges are adjacent to edges of the same kind and jumping edges which cannot be used now. So the chain can be continued only by such edges. However, E_2 is not adjacent to such an edge, since their intersection contain only one vertex. Therefore the chain cannot reach E_2, a contradiction. □

Theorem 6.3.5 (Dudek, Katona, Żak [K2]). *Let $\mathcal{G} \in \mathcal{J}(p, k)$ where p, k are non-negative integers such that $\lceil 2k/3 \rceil \geq p + 2$. Let $|U_i| \in \{\alpha - 1, \alpha\}$ for $i = 1, ..., k$ and $|U_j| = \alpha$ for some $j \in \{1, ..., k\}$, where α is an integer satisfying $\alpha \geq 5(p + 1) + 1$. Then $\mathcal{G}$ has no Hamiltonian chain.*

Proof. Suppose indirectly that $v_1v_2...v_n$ is a Hamiltonian chain in $\mathcal{G}$. By removing all vertices of U_0 the sequence $v_1v_2...v_n$ falls to m parts, where $m \leq p+1$. Each part induce a chain in $\mathcal{G} - U_0$ or consists of one or two vertices. If a part contains an edge $E \in \mathcal{E}(G)$ such that $|E \cap U_i| \geq 2$ for some $i \in \{1, ..., k\}$ then by Lemma 6.3.4 every edge in this part have at least two vertices from U_i. We say that the set U_i is a *dominating set* for this part. Let x_i denote the number of vertices of the i-th part which belong to its dominating set. Consequently, let y_i denote the number of remaining vertices in the i-th part. Recall that among every three consecutive vertices of some part at least two belong to its dominating set. Hence $x_i \geq 2(y_i - 1)$ if $x_i > 0$, and $x_i + y_i \leq 2$ otherwise. Thus $x_i + y_i \leq \frac{3}{2}\alpha + 1$. Therefore

$$k(\alpha-1) < |U_1| + ... + |U_k| = \sum_{i=1}^{m}(x_i+y_i) \leq \sum_{i=1}^{m}\left(\frac{3}{2}\alpha+1\right) \leq (p+1)\left(\frac{3}{2}\alpha+1\right),$$

hence

$$\frac{2}{3}k < (p+1)\frac{\alpha+2/3}{\alpha-1} = (p+1)+(p+1)\frac{5}{3(\alpha-1)}.$$

Thus

$$p+2 \leq \left\lceil \frac{2}{3}k \right\rceil \leq \frac{2}{3}k + \frac{2}{3} < (p+1)+(p+1)\frac{5}{3(\alpha-1)} + \frac{2}{3}, \text{ hence}$$

$$1 < (p+1)\frac{5}{\alpha-1}, \text{ a contradiction.}$$

□

Theorem 6.3.6 (Dudek, Katona, Żak [K2])**.** *Let t be a nonnegative integer and let $\mathcal{G} \in \mathcal{J}(2t, 3t+2)$. Let $|U_i| \in \{\alpha-1, \alpha\}$ for $i = 1, ..., 3t+2$ and $|U_j| = \alpha$ for some $j \in \{1, ..., k\}$, where α is an integer satisfying $\alpha \geq 10t+6$. Then $\mathcal{G}$ is Hamiltonian chain saturated.*

Proof. Since $\lceil \frac{2}{3}(3t+2) \rceil = 2t+2$, by Theorem 6.3.5, $\mathcal{G}$ has no Hamiltonian chain. We will show that adding any new edge E to $\mathcal{G}$ creates a Hamiltonian chain. Let $E = \{u, v, w\}$. There are two different types of E:

Case 1. $u \in U_i, v \in U_j, w \in U_k$ with $i \neq j, i \neq k, j \neq k$; in this case we may assume that $i \prec j$ and $j \prec k$,

Case 2. $u \in U_j, v \in U_j, w \in U_k$ with $j \prec k$.

We deal with both of the cases simultaneously.

Note that for $t \geq 2$ the set $V(\overrightarrow{K}_{3t+2}) \setminus \{j, k\}$ can be decomposed into triples (a_n, b_n, c_n), $n = 1, ..., t$, such that $a_n \prec b_n$ and $a_n \prec c_n$ for every n. Indeed, for the triples we can take consecutive vertices in the sequence $k+1, k+2, k+3, ..., \widehat{j}, ..., \widehat{k}$ where the symbol $\widehat{x}$ means that x is omitted in the sequence. Let

$$\mathcal{C} \sim j, j, ..., j, u, v, w, k, k, ..., k$$

denote the sequence containing vertices u, v, w and all vertices from the sets U_j – in the positions denoted by j – and U_k – in the positions denoted by k. Note that C is a chain in $\mathcal{G} + E$. Consequently let

$$\mathcal{C}_n \sim a, b, b, a, b, b, a, ..., a, b, b, a, (b), 0, a, c, c, a, c, c, a, ..., a, c, c, a, (c),$$

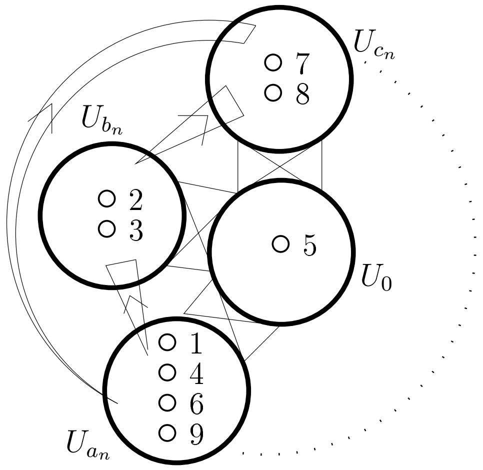

Figure 6.2: The sequence 1,2,3,4,5,6,7,8,9 realizes the fragment '$\ldots a, b, b, a, 0, a, c, c, a \ldots$' of $\mathcal{C}_n$.

$n = 1, ..., t$, denote the sequence containing one vertex from U_0 (denoted by 0) and vertices from the set $U_{a_n} \cup U_{b_n} \cup U_{c_n} \setminus \{u\}$ in the positions denoted by a, b, c, respectively. The symbol (x) means that x may or may not occur in the sequence depending on the parity of $|U_{x_n} \setminus \{u\}|$.

Note that we are always able to place all the vertices from $U_{a_n} \cup U_{b_n} \cup U_{c_n} \setminus \{u\}$ in such sequence. Indeed, let A, B, C denote the number of a's, b's, and c's in $\mathcal{C}_n$, respectively. Then $A = \left\lfloor \frac{1}{2} B \right\rfloor + 1 + \left\lfloor \frac{1}{2} C \right\rfloor + 1$. Since $|U_{b_n}|, |U_{c_n}| \geq \alpha - 1$, $A \geq \left\lfloor \frac{\alpha-1}{2} \right\rfloor + 1 + \left\lfloor \frac{\alpha-2}{2} \right\rfloor + 1 = \alpha$ because the vertex u may belong to U_{b_n} or to U_{c_n}. If $2\alpha - 3 < B + C$ $(= |U_{b_n} \cup U_{c_n} \setminus \{u\}|)$ or $|U_{a_n} \setminus \{u\}| < \alpha$ then we can delete from $\mathcal{C}_n$ an appropriate number of a's without ruining the chain. In any case we can modify $\mathcal{C}_n$ in such a way that the resulting sequence contains exactly one vertex from U_0 and all vertices from $U_{a_n} \cup U_{b_n} \cup U_{c_n} \setminus \{u\}$. We denote such modified $\mathcal{C}_n$ by $\mathcal{C}'_n$. Clearly each $\mathcal{C}'_n$ is a chain in $\mathcal{G} + E$. The following sequence is also a chain in $\mathcal{G} + E$

$$\mathcal{C}, 0, \mathcal{C}'_1, 0, \mathcal{C}'_2, 0, ..., 0, \mathcal{C}'_t,$$

(here symbols 0 denote different vertices from the set U_0). Since $\mathcal{C}$ does not contain a vertex from U_0 and each $\mathcal{C}'_n$ contains exactly one vertex from U_0, the above sequence contains all vertices of $\mathcal{G}$, hence is a Hamiltonian chain.

If $t = 1$ then, due to symmetry, we can assume that $j = 1$ and $k = 2$ or $j = 1$ and $k = 3$. In the former case we can repeat previous argument since in $V(\overrightarrow{K_5}) \setminus \{1, 2\}$, $3 \prec 4$ and $3 \prec 5$. Assume that $j = 1$ and $k = 3$. Then $i = 4$, 5 or 1 because $i \prec j$ or $i = j$. If $i = 4$ then the following sequence, or its modification resulting by deleting an appropriate number of 3's, is a Hamiltonian chain in $\mathcal{G} + E$

$$1, ..., 1, v, u, w, 4, 4, 3, 4, 4, 3, ..., 3, 4, 4, 3, (4), 0, 3, 5, 5, 3, 5, 5, 3, ..., 3, 5, 5, 3, (5), 0, 2, ..., 2$$

(as previously, symbols x different from u, v, w denote distinct vertices from the set U_x while symbol (x) denote that x may or may not appear in the sequence depending on the parity of $|U_x|$). Indeed, let A, B, C denote the number of 3's, 4's and 5's in the sequence, respectively. Then $A = \left\lceil \frac{B}{2} \right\rceil + \left\lfloor \frac{C}{|2|} \right\rfloor + 1 \geq \left\lceil \frac{\alpha-1}{2} \right\rceil + \left\lfloor \frac{\alpha-1}{2} \right\rfloor + 1 = \alpha$. If $2\alpha - 2 < B + C$ $(= |U_4| + |U_5|)$ or $|U_3| < \alpha$ then we can delete from the sequence an appropriate number of 3's without spoiling the chain. Similar argument holds when $i = 5$ or $i = 1$.

Finally, it is clear that $\mathcal{G} + E$ contains a Hamiltonian chain if $t = 0$. □

Theorem 6.3.7 (Dudek, Katona, Żak [K2]). *For every $n \geq 12$ there exists a 3-uniform Hamiltonian chain saturated hypergraph with at most $\frac{3\sqrt{30}}{25}n^{5/2} + o(n^{5/2})$ edges.*

Proof. Let $t_0 := \left\lfloor \frac{\sqrt{10}}{30}\sqrt{3n+4} - \frac{2}{3} \right\rfloor$. Hence $t_0 \geq 0$. Let $\mathcal{G} \in \mathcal{J}(2t_0, 3t_0 + 2)$ with the property that the sets U_i have equal or nearly equal size. Hence $|U_i| \in \{\alpha - 1, \alpha\}$, $i = 1, ..., 3t_0 + 2$, where $\alpha = \left\lceil \frac{n-2t_0}{3t_0+2} \right\rceil$. Moreover, at least one U_j satisfies $|U_j| = \alpha$. By simple computations

$$\frac{n-2t}{3t+2} \geq 10t + 6 \Leftrightarrow t \leq \frac{\sqrt{10}}{30}\sqrt{3n+4} - \frac{2}{3}.$$

Hence α satisfies conditions from Theorem 6.3.6. Thus $\mathcal{G}$ is Hamiltonian chain saturated. Note that the number of edges of any hypergraph $\mathcal{G}' \in \mathcal{J}(2t, 3t+2)$ with $|U_i| \in \{\alpha - 1, \alpha\}$, $i = 1, ..., 3t + 2$, satisfies

$$\begin{aligned} |\mathcal{E}(\mathcal{G}')| &\leq \binom{\alpha+2t}{3}(3t+2) + \binom{3t+2}{2}\binom{\alpha+2t}{2} \\ \alpha &\leq \frac{(\alpha+2t)^3}{6}(3t+2) + \frac{(3t+2)^2(\alpha+2t)^2\alpha}{4} \\ &\simeq (n+6t^2+2t)^2\left(\frac{n+6t^2+2t}{6(3t+2)^2} + \frac{n-2t}{4(3t+2)}\right). \end{aligned} \tag{6.1}$$

Hence for $t = t_0$

$$|\mathcal{E}(\mathcal{G})| \leq \left(n + 6\frac{3n+4}{90}\right)^2 \frac{n}{12\frac{\sqrt{10}}{30}\sqrt{3n+4}} + o(n^{5/2}) = \frac{3\sqrt{30}}{25}n^{5/2} + o(n^{5/2}).$$

□

Remarks

Note that our construction cannot be improved by taking another t. Indeed, if we take t of order different from $n^{1/2}$ then, by (6.1), $|\mathcal{E}(\mathcal{G})|$ is asymptotically greater than the value obtained in Theorem 6.3.7. Hence t of the form $a\sqrt{n}$ is best. Then

$$|\mathcal{E}(\mathcal{G})| \sim (n + 6a^2n)^2\frac{n}{12a\sqrt{n}} + o(n^{5/2}) = \frac{(1+6a^2)^2}{12a}n^{5/2} + o(n^{5/2}).$$

Recall that $t < \frac{\sqrt{10}}{30}\sqrt{3n+4} - \frac{2}{3} \sim \frac{1}{\sqrt{30}}n^{1/2}$. On the other hand it is easy to check that the function $f(a) = \frac{(1+6a^2)^2}{12a}$ is decreasing for $a \in (0, 1/\sqrt{18})$. Thus taking the largest possible value of t gives best result.

We observe that the same bounds can be obtained in case we consider closed Hamiltonian chain $v_1, v_2, ...v_n, v_1$ (Hamiltonian cycle) instead of an open one $v_1, v_2, ..., v_n$. The proof of the lower bound is very similar to the proof of Theorem 6.2.1. On the other hand the upper bound can be realized by a hypergraph $\mathcal{G} \in \mathcal{J}(2t+1, 3t+2)$ with $\alpha \geq 10t + 6$.

Part II

Problems related to matching theory

Matchings in graphs have been studied extensively. It is also a popular topic among Hungarian researchers. The famous book of Lovász and Plummer [92] is a good source of essential results in this area. This Part contains a collection of results on different generalizations and variations of the original problem.

In the classical problem, we want to cover all vertices of a graph with vertex disjoint edges. One way of generalization is when we wish to cover all vertices with another smaller graph or with a set of smaller graphs instead of a single edge. A particularly interesting case if we can use any path of length at least 2. We call such a cover a $\{P_n \mid n \geq 3\}$-*factor* (P_n denotes the path of length $n-1$ on n vertices).

Kaneko [73] proved a necessary and sufficient condition for the existence a $\{P_n \mid n \geq 3\}$-*factor*. In Chapter 7 a simpler and much shorter proof is presented for this theorem using generalizations of the techniques in standard matching theory. We also prove a Berge-formula that gives a formula for the order of a maximum $\{P_n \mid n \geq 3\}$-packing (Theorem 7.3.2).

Chapters 8-10 contain results on $(1, f)$-*odd subgraphs*. Here f is an odd integer valued function on the set of vertices, and each degree in such a subgraph must be an odd integer and must not exceed $f(v)$. If the subgraph is spanning then we call it a $(1, f)$-*odd factor*. If $f \equiv 1$ then a $(1, f)$-odd factor is a perfect matching.

The main result of Chapter 8 is a Berge-type formula for the size of maximum $(1, f)$-odd subgraphs (Theorem 8.2.3). We also prove an augmentation property for $(1, f)$-odd subgraphs (Theorem 8.3.2). In Chapter 9 a Gallai-Edmonds type structure theorem is given (Theorem 9.3.4). This also yields an algorithm that is a direct generalization of Edmonds blossom algorithm with running time $O(|V(G)|^3)$ (Theorem 9.5.4). The focus of Chapter 10 is on f-*parity subgraphs*, which is a further generalization of $(1, f)$-odd subgraphs. First, we show a reduction of the f-parity subgraph problem to matchings, which will then be used to prove the Gallai–Edmonds type structure theorem on the f-parity subgraph problem (Theorem 10.2.4). Then we prove some properties of barriers also hold in this general case. Furthermore, several results on f-*elementary graphs* are proved.

Chapter 11 involves *odd factors* (there is no upper bound on the degrees now). However, we consider a different type of problem. Theorem 11.1.10 gives a necessary and sufficient condition for a multigraph to be decomposed into two odd subgraphs. Theorem 11.1.11 gives a polynomial-time algorithm for finding such a decomposition or showing its non-existence.

Chapter 7

Packing paths of length at least two

7.1 Introduction

For a set $\{A, B, C, \ldots\}$ of connected graphs, a subgraph F of a graph G is called an $\{A, B, C, \ldots\}$*-packing* of G if each component of F is isomorphic to one of $\{A, B, C, \ldots\}$. An $\{A, B, C, \ldots\}$-packing is said to be maximum iff it covers a maximum number of vertices of G. If F is a spanning subgraph, then it is called a perfect $\{A, B, C, \ldots\}$-packing or an $\{A, B, C, \ldots\}$*-factor*. Let P_n denote the path which contains n vertices and $n-1$ edges. With this notation the well-known 1-factor (perfect matching) is a $\{P_2\}$-factor. Observe that a graph has a $\{P_3, P_4, P_5\}$-factor if and only if it has a $\{P_n \mid n \geq 3\}$-factor, which we abbreviate as $\{P_{\geq 3}\}$-factor. We will use this fact throughout this Chapter.

A graph H is said to be *factor-critical* if $H - \{v\}$ has a 1-factor for all $v \in V(H)$. Note that factor critical graphs are connected. For a factor-critical graph H with $V(H) = \{v_1, v_2, \ldots, v_n\}$, add new vertices $\{u_1, u_2, \ldots, u_n\}$ together with new edges $\{v_i u_i \mid 1 \leq i \leq n\}$ to H. Then the resulting graph is called a *sun*. Note that K_2 is a sun and by definition, we regard K_1 also as a sun (see Figure 7.1). We call a sun with one vertex a *small sun*, otherwise a *big sun*. We denote by $\mathrm{Sun}(G)$ the set of sun components of G and let $\mathrm{sun}(G) = |\mathrm{Sun}(G)|$ the number of sun components. A vertex of degree one is called a *pendant vertex*, and an edge incident with a pendant vertex is called a *pendant edge*.

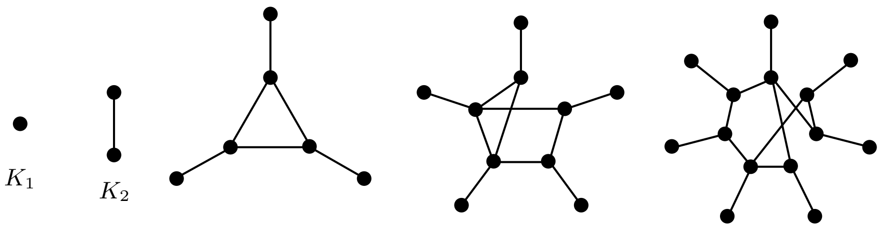

Figure 7.1: Suns

Wang [123] characterized the *bipartite* graphs having a $\{P_{\geq 3}\}$-factor. Kaneko [73] generalized this theorem to general graphs. There are many results on component factors (for example, see [12] and [86]), but besides the well known theorem of Tutte [121] about f-factors and the more general theorem of Lovász [88] about (g, f)-factors all previous positive results (i.e. that gives a good characterization) allow P_2 as a component. Hell and Kirkpatrick [79] proved that if H is a connected graph on at least 3 vertices then deciding whether a given graph G contains an $\{H\}$-factor is NP-complete. Thus, for example, we do not have a good characterization of graphs having a $\{P_3\}$-factor.

On the other hand we should mention the corresponding theorems of Hartvigsen (see [65] and [66]) about cycle-factors without short cycles.

Theorem 7.1.1 (Kaneko [73]). *A graph G has a $\{P_{\geq 3}\}$-factor if and only if*

$$\mathrm{sun}(G-S) \leq 2|S| \quad \textit{for all} \quad S \subset V(G). \tag{7.1}$$

In Section 7.2 a simpler proof is presented for this theorem. In Section 7.3 we prove a Berge-formula. These appeared in [K8].

Following our work, Hell, Hartvigsen and Szabó [67] started to work on a generalized question, the k-piece packing problem. They proved a Tutte-type characterization and a Berge-formula for k-pieces. Here a k-piece is defined to be a connected graph with maximum degree exactly k and a k-piece packing is a collection of vertex disjoint k-pieces. Thus a 1-piece packing is a matching, and a 2-piece is a $\{P_{\geq 3}\}$-packing. Janata, Loebl and Szabó [70] proved the analogue of the Gallai-Edmonds structure theorem for k-pieces. These results also imply a polynomial algorithm to find optimal k-piece packings. Kano, Lee and Suzuki [74] showed that every connected cubic bipartite graph has a $\{P_n \mid n \geq 8\}$-factor if its order is at least 8. Apart from these, there are numerous papers on closely related topics, more then 30 publications cite our results.

7.2 A Simple Proof of Theorem 7.1.1

The following lemma is an easy consequence of Hall's theorem ([92], Theorem 1.1.3).

Lemma 7.2.1. *Let B be a bipartite graph with bipartition $X \cup Y$ such that $|Y| = 2|X|$. B has a $\{P_3\}$-factor, ie. a factor H such that $d_H(x) = 2$ for all $x \in X$ and $d_H(y) = 1$ for all $y \in Y$ if and only if*

$$|N_B(S)| \geq 2|S| \quad \textit{for all} \quad S \subseteq X.$$

Important properties of suns are described in the following two lemmas.

Lemma 7.2.2. *Let D be a big sun, and let vv' be a pendant edge of D. Then $D - \{vv'\}$ has a $\{P_4\}$-factor.*

Proof. Let $V(D) = \{v, x, y, z, \dots\} \cup \{v', x', y', z', \dots\}$, where v' is the pendant vertex connected to v etc. Let F be the factor-critical graph $D[\{v, x, y, z, \dots\}]$. If M denotes the perfect matching in $F - v$, then it is clear that by extending the edges of M by the adjacent pendant edges we obtain a $\{P_4\}$-factor of $D - \{v, v'\}$. □

Lemma 7.2.3. *Let D be a big sun, and v' a pendant vertex of D. Then $D - v'$ has a $\{P_4, P_5\}$-factor.*

Proof. Using the notations of the previous proof, choose a neighbor x of v in F. Now $F - x$ has a perfect matching M, with some $vy \in M$. Take the path $\{y', y, v, x, x'\}$ and the $\{P_4\}$-s extending the other edges of M by pendant ones. □

Proof of Theorem 7.1.1. Since no sun component can have a $\{P_{\geq 3}\}$-factor, it is easy to show that if G has a $\{P_{\geq 3}\}$-factor, then (7.1) holds.

We now prove the sufficiency by induction on $||G|| = |E(G)|$. Our method is based on the ideas of Gallai's proof for Tutte's theorem. Suppose that G satisfies (7.1). By setting $S = \emptyset$, condition (7.1) implies that no component of G is a sun. We may assume that G is connected and $|G| \geq 3$. We consider some cases.

Case 1 There exists $\emptyset \neq S \subset V(G)$ such that $\mathrm{sun}(G - S) = 2|S|$.

Choose a nonempty subset S of $V(G)$ satisfying $\mathrm{sun}(G-S) = 2|S|$.

Let C be any non-sun component of $G - S$. Then for a subset $X \subset V(C)$, we have

$$2|S \cup X| \geq \mathrm{sun}(G - (S \cup X)) = \mathrm{sun}(G - S) + \mathrm{sun}(C - X) = 2|S| + \mathrm{sun}(C - X).$$

Thus $\mathrm{sun}(C - X) \leq 2|X|$. Hence C satisfies (7.1), and so C has a $\{P_{\geq 3}\}$-factor by induction.

We define the bipartite graph B with vertex set $S \cup \mathrm{Sun}(G - S)$ by contracting every sun-component into a single vertex and removing multiple edges and edges inside S. Now $|\mathrm{Sun}(G - S)| = 2|S|$, and we show that

$$|N_B(X)| \geq 2|X| \quad \text{for all} \quad X \subseteq S. \tag{7.2}$$

Suppose that $|N_B(Y)| < 2|Y|$ holds for some $Y \subseteq S$.
Then $\mathrm{Sun}(G - (S \setminus Y)) \supseteq \mathrm{Sun}(G - S) \setminus N_B(Y)$ holds, and thus

$$\mathrm{sun}(G - (S \setminus Y)) \geq \mathrm{sun}(G - S) - |N_B(Y)| > 2|S| - 2|Y| = 2|S \setminus Y|$$

is implied, which contradicts the assumption (7.1). Thus (7.2) holds.

Therefore, by Lemma 7.2.1, graph B has a factor H such that $d_H(x) = 2$ for all $x \in S$ and $d_H(C) = 1$ for all $C \in \mathrm{Sun}(G - S)$, note that it consists of $|S|$ copies of P_3. By making use of this factor, we can obtain a $\{P_{\geq 3}\}$-factor of G in the following way. First, for each edge xC of H where $x \in S$ and C is a sun, replace this edge with xc, where c is an arbitrary vertex of C connected to s. Now every P_3 of H has endvertices in two distinct suns. For every endvertex c, if it is not a small sun itself, lengthen the path with the pendant edge incident to c. Now we covered all the small suns and exactly one pendant edge in every big sun. The remaining parts of big suns have a $\{P_{\geq 3}\}$-factor by Lemma 7.2.2 and the non-sun components have a $\{P_{\geq 3}\}$-factor by induction (see Figure 7.2).

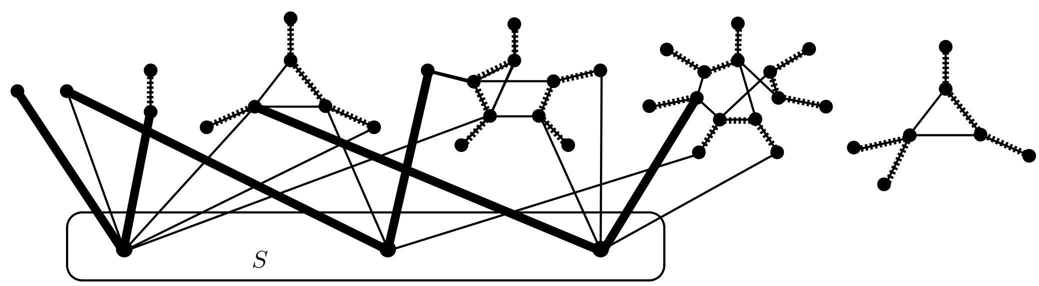

Figure 7.2: Extension of the $\{P_3\}$-factor of B to a $\{P_{\geq 3}\}$-factor in G

Case 2 $\mathrm{sun}(G-S) < 2|S|$ for all $\emptyset \neq S \subset V(G)$ and there exists $\emptyset \neq S' \subset V(G)$ for which $\mathrm{sun}(G-S') = 2|S'|-1$.

Choose a subset S so that S is maximal among all subsets S' satisfying $\mathrm{sun}(G-S') = 2|S'|-1$.

Let C be any non-sun component of $G-S$ and let $\emptyset \neq X \subset V(C)$. Using the maximality of S we obtain

$$\begin{aligned} 2|S\cup X|-2 \geq \mathrm{sun}(G-(S\cup X)) &= \mathrm{sun}(G-S)+\mathrm{sun}(C-X) \\ &= 2|S|-1+\mathrm{sun}(C-X). \end{aligned}$$

$$\text{Thus}\quad \mathrm{sun}(C-X) \leq 2|X|-1. \tag{7.3}$$

Hence C has a $\{P_{\geq 3}\}$-factor by induction.

Claim 7.2.4. *If $G-S$ has a non-sun component then the desired $\{P_{\geq 3}\}$-factor exists.*

Proof. Let C be such a component, $v \in S$ and $w \in C$ such that vw is an edge. Let w^* be a new vertex and consider the graph $H := G[C]+ww^*$. Using (7.3) it is easy to see that H satisfies (7.1) for nonempty sets: $\mathrm{sun}(H-X) \leq \mathrm{sun}(C-X)+1 \leq 2|X|-1+1$. Clearly $||H|| < ||G||$, so by induction H has a $\{P_{\geq 3}\}$-factor containing a path P ending with $\{,w,w^*\}$ or H itself is a sun. In the latter case, by Lemma 7.2.2, H has a $\{P_2,P_4\}$-factor so that the only P_2 is $P=\{w,w^*\}$.

Let G' be the graph created from G by adding a new pendant edge vw'. Then w' is a new sun component of $G-S$, that is $\mathrm{sun}(G'-S) = 2|S|$ holds. As before, G' satisfies (7.1) for nonempty sets. Construct bipartite graph B from G' as in Case 1. The method of Case 1 is used to prove the fact that the empty set satisfies (7.1). (For $Y=S$ we know that $|N_B(Y)| = 2|Y|$ by the property of S.) Thus the same argument shows that B satisfies (7.2) hence we obtain a $\{P_3\}$-factor of G' containing a path Q ending with $\{,v,w'\}$. Now take $P-w^*$ in C and $Q-w'$ in G and join them by the edge vw. Using this factor we can obtain a $\{P_{\geq 3}\}$-factor in the same way as in the previous case, except that for the remaining part of C we use the $\{P_{\geq 3}\}$-factor found in the first paragraph, but without path P. □

Claim 7.2.5. *If there exists $v \in S$ connected to no small sun, or connected to at least two small suns in* $\mathrm{Sun}(G-S)$, *then the desired $\{P_{\geq 3}\}$-factor exists.*

Proof. Construct B as before with the additional pendant edge vw'. Extend the $\{P_3\}$-factor of B as before to obtain a $\{P_{\geq 3}\}$-factor of $G' = G+w'$ and then delete w'. The path containing v becomes shorter, if it still has at least two edges then we are done, so suppose it contains only one edge vw. By the above construction, w cannot be in a big sun, because the pendant edge incident to w would also be part of this path. Therefore w is a small sun. By our assumptions another small sun w^* is also connected to v, and w^* is an endvertex of another path that can be joined to vw by adding edge w^*v. □

Claim 7.2.6. *If $\{w\} \in$* $\mathrm{Sun}(G-S)$ *is a small sun and w is not pendant in G then the desired $\{P_{\geq 3}\}$-factor exists.*

Proof. As w is not pendant, it is connected to some $v \in S$ and $v' \in S$, $v' \neq v$. Construct B as before with the additional pendant edge vw'. We claim that $B' = B - vw$ satisfies (7.2). If $X \subset S$ then either $v \notin X$ and $N_{B'}(X) = N_B(X)$ or $v \in X$ and $|N_B(X)| \geq 2|X|+1$ otherwise $S \setminus X$ would be a set with $\text{sun}(G-(S\setminus X)) \geq 2|S\setminus X|$. For $X = S$ we need to prove $w \in N_{B'}(S)$ which is true because wv' is an edge. Now take the path ending in w' in the $\{P_{\geq 3}\}$-factor obtained using the $\{P_3\}$-factor of B', delete w' and connect the remains of this path to the path ending in w by edge vw. □

Summing up, we may assume from now on that there are $|S|$ small suns in $\text{Sun}(G-S)$ and $|S|-1$ big suns. Each small sun is a pendant vertex of G and they are connected to different vertices in S. Moreover every component of $G-S$ is a sun.

Claim 7.2.7. *If every vertex of G with degree ≥ 2 has a pendant neighbor then the desired $\{P_{\geq 3}\}$-factor exists.*

Proof. Let U be the set of vertices with degree ≥ 2. If $G[U]$ has a perfect matching, we are done. Otherwise there exists $X \subset U$ such that there are more than $|X|$ factor-critical components in $G[U \setminus X]$, consequently $\text{sun}(G-X) > 2|X|$, which is a contradiction. □

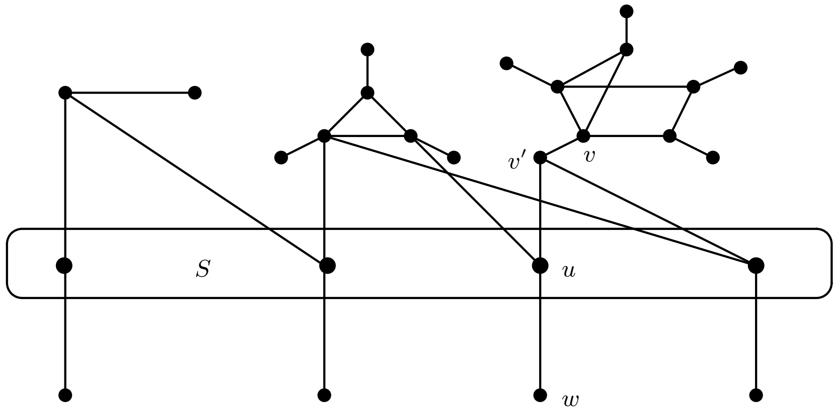

Figure 7.3: Remaining case after Claim 4

So we may assume there is a vertex with degree ≥ 2 which has no pendant neighbor in G. Clearly it is in a big sun $D \in \text{Sun}(G-S)$. This means that $G[D]$ has a pendant vertex v' with neighbor $v \in D$ so that v' is connected (in G) to some $u \in S$ and if $|D| = 2$ then v is also connected to S. (See Figure 7.3.)

Subcase 2.1 $|D| = 2$

Let G' be the graph obtained from G by deleting the edge vv'. Now $\text{sun}(G'-S) = 2|S|$. Construct B as in Case 1. It is easy to see that (7.2) is satisfied, so we obtain a $\{P_{\geq 3}\}$-factor as in Case 1.

Subcase 2.2 $|D| > 2$

Let the small sun neighbor of u be $\{w\}$. Construct B as before by adding pendant edge uw'. Take the $\{P_3\}$-factor of B, construct a $\{P_{\geq 3}\}$-factor of $G+w'$ and delete the path $\{w, u, w'\}$. Now we have a $\{P_{\geq 3}\}$-factor of $G-u-w$. If v' is an endvertex of a path of this factor then we can extend this path by adding edges $v'u$ and uw. Otherwise the path P containing v' ends with $\{, s, v', v\}$ (by our construction this is the only possibility) where $s \in S$, $s \neq u$. Observe further that, because $\mathrm{sun}(G' - S) = 2|S|$, no other path leaves D. Now delete edge $v'v$ from P as well as all the paths inside D, extend the shortened P by the edges $v'u$ and uw, and use Lemma 7.2.3 to obtain a $\{P_{\geq 3}\}$-factor of $D - v'$.

Case 3 $\mathrm{sun}(G - S) \leq 2|S| - 2$ for all $\emptyset \neq S \subset V(G)$.

If G has a pendant vertex u connected to v, then $\mathrm{sun}(G - \{v\}) \geq 1$, which contradicts the assumption of this case. Thus G is not a tree, and so we can find an edge e for which $G - e$ is connected. For every subset $\emptyset \neq S \subset V(G - e)$, we have

$$\mathrm{sun}((G - e) - S) \leq \mathrm{sun}(G - S) + 2 \leq 2|S| - 2 + 2 = 2|S|.$$

Moreover, $G - e$ is not a sun because having at least three vertices it would be a sun with at least three pendant vertices, but in this case G would have at least one pendant vertex as well. Therefore, $G - e$ has a $\{P_{\geq 3}\}$-factor by the inductive hypothesis, and so does G.

Consequently, the proof is complete. □

7.3 The Order of a Maximum $\{P_{\geq 3}\}$-packing

Lemma 7.3.1. *Let B be a bipartite graph with bipartition $X \cup Y$ and $Y^* \subseteq Y$. Define*

$$\mathrm{def}(B) := \max_{Y' \subseteq Y}(|Y'| - 2|N_B(Y')|)$$

and

$$\mathrm{def}^*(B) := \max_{Y' \subseteq Y^*}(|Y'| - 2|N_B(Y')|).$$

If $\mathrm{def}(B) = |Y| - 2|X|$ then B has a $\{P_3\}$-packing which covers $|Y| - \mathrm{def}(B) = 2|X|$ vertices of Y including $|Y^*| - \mathrm{def}^*(B)$ vertices of Y^*, and every vertex of X is the middle vertex of some P_3.

Proof. Define bipartite graph $B' = ((X \cup X') \cup Y, E')$ by adding for every vertex $x \in X$ a new vertex $x' \in X'$ and connecting x' to all neighbors of x. By Ore's theorem ([92], Theorem 1.3.1) there exists a matching M' in B' that covers

$$|Y| - \max_{Y' \subseteq Y}(|Y'| - |N_{B'}(Y')|)$$

vertices of Y. Since $\mathrm{def}(B) = |Y| - 2|X|$ and $|N_{B'}(Y')| = 2|N_B(Y')|$, we have

$$|Y| - \max_{Y' \subseteq Y}(|Y'| - |N_{B'}(Y')|) = 2|X| = |X \cup X'|$$

so M' covers all vertices of $X \cup X'$. Moreover, there exists another matching M^* that covers

$$|Y^*| - \max_{Y' \subseteq Y^*}(|Y'| - |N_{B'}(Y')|) = |Y^*| - \mathrm{def}^*(B)$$

vertices of Y^*. It is well known that this implies the existence of a matching which covers $X \cup X'$ and $|Y^*| - \mathrm{def}^*(B')$ vertices of Y^* (see [97]). This gives the desired $\{P_3\}$-packing in B if we contract all pairs x, x'. □

Let $\mathrm{k}_2(H)$ denote the number of components of H which consist of an edge, in other terms the number of sun components isomorphic to a K_2.

Theorem 7.3.2 (Kano, Katona, Király [K8])**.** *The order of a maximum $\{P_{\geq 3}\}$-packing in a graph G is*

$$\mathrm{pp}(G) := |V(G)| - \max_{T \subseteq S \subset V(G)} (\mathrm{sun}(G-S) - 2|S| + \mathrm{k}_2(G-T) - 2|T|).$$

Proof. It is proved first that the above expression is an upper bound on the order of a maximum $\{P_{\geq 3}\}$-packing. Let $T \subseteq S \subset V(G)$ such that $|V(G)| - \mathrm{pp}(G) = \mathrm{sun}(G-S) - 2|S| + \mathrm{k}_2(G-T) - 2|T|$. Clearly if F is a $\{P_{\geq 3}\}$-packing of G then there is a vertex in at least $\mathrm{sun}(G-S) - 2|S|$ components in $\mathrm{Sun}(G-S)$ which cannot be covered by F. Moreover, there are at least $\mathrm{k}_2(G-T) - 2|T|$ K_2-components of $G-T$ where none of the two vertices can be covered.

To prove the other direction choose $T \subseteq S \subset V(G)$ such that $\mathrm{sun}(G-S) - 2|S| + \mathrm{k}_2(G-T) - 2|T|$ is maximum.

Let C be a non-sun component of $G-S$. Then for a subset $X \subseteq V(C)$ we have

$$\mathrm{sun}(G-(S \cup X)) - 2|S \cup X| + \mathrm{k}_2(G-T) - 2|T| \leq \mathrm{sun}(G-S) - 2|S| + \mathrm{k}_2(G-T) - 2|T|$$

by the choice of S and T. Since $\mathrm{sun}(G-(S \cup X)) = \mathrm{sun}(G-S) + \mathrm{sun}(C-X)$ and $X \cap S = \emptyset$, it follows that $\mathrm{sun}(C-X) \leq 2|X|$ which implies that C has a $\{P_{\geq 3}\}$-factor. Hence, we may assume from now on that $G-S$ has only sun components.

Construct a bipartite graph B from G by contracting each sun component into a single vertex and removing multiple edges and edges inside S. The set of vertices which arose from the sun components in $G-S$ is denoted by Y, the set of vertices that arose from the contraction of K_2 components of $G-T$ is denoted by Q and the set of vertices which arose from the contraction of K_2 components of $G-S$ is denoted by Y^*. The K_2 components of $G-T$ are elements of $\mathrm{Sun}(G-S)$, because if there exists a K_2 component of $G-T$ such that $V(D) \cap (S-T) \neq \emptyset$ then we get a contradiction by considering $S \setminus V(D)$ and T. Thus $Q \subseteq Y^* \subseteq Y$ holds.

First we show that $\mathrm{def}^*(B) = \mathrm{k}_2(G-T) - 2|T|$. Suppose that

$$R - 2|N_B(R)| > \mathrm{k}_2(G-T) - 2|T|$$

holds for some $R \subseteq Y^*$. Then choosing $N_B(R)$ instead of T (and keeping S), obviously violates the choice of T and S.

Next we prove that $\mathrm{def}(B) = \mathrm{sun}(G-S) - 2|S|$. Suppose that

$$R - 2|N_B(R)| > \mathrm{sun}(G-S) - 2|S|$$

holds for some $R \subseteq Y$. Let $S' := N_B(R \cup Q)$ and $T' := N_B(R \cap Q)$. Since $N_B(Q) = T$, it is obvious that $S' = N_B(R) \cup T$ and $T' \subseteq N_B(R) \cap T$. In this way

$$\begin{aligned}\mathrm{sun}(G-S') - 2|S'| + \mathrm{k}_2(G-T') - 2|T'| \geq |R \cup Q| - 2|S'| + |R \cap Q| - 2|T'| = \\ |R| + |Q| - 2|S'| - 2|T'| \geq |R| + |Q| - 2|N_B(R) \cup T| - 2|N_B(R) \cap T| = \\ |R| + |Q| - 2|N_B(R)| - 2|T| > \mathrm{sun}(G-S) - 2|S| + \mathrm{k}_2(G-T) - 2|T|\end{aligned}$$

holds, which contradicts the choice of S and T.

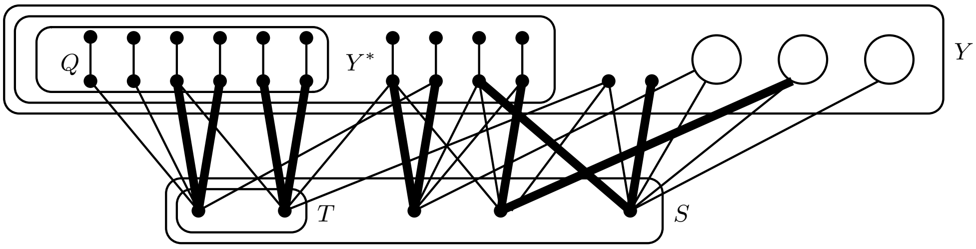

Figure 7.4: B with the desired factor

By applying Lemma 7.3.1 a $\{P_3\}$-packing is obtained in B which covers all vertices in S, $|Y| - \mathrm{sun}(G-S) + 2|S|$ vertices of Y including $|Y^*| - \mathrm{k}_2(G-T) + 2|T|$ vertices of Y^*. (See Figure 7.4.)

In the original graph G we can extend this packing in the usual way by Lemma 7.2.2 (see Figure 7.2). On the other hand, the sun-components with more than two vertices which are not covered by the P_3-s in B can be almost covered by a $\{P_{\geq 3}\}$-packing by Lemma 7.2.3. This gives a $\{P_{\geq 3}\}$-packing of size $\mathrm{pp}(G)$. □

Note, that

$$\max_{S \subset V(G)} (\mathrm{sun}(G-S) - 2|S|) = 0 \tag{7.4}$$

implies that $\mathrm{k}_2(G-T) - 2|T| \leq 0$ holds for any $T \subset V(G)$, because K_2 is a sun. This shows that

$$\max_{T \subseteq S \subset V(G)} (\mathrm{sun}(G-S) - 2|S| + \mathrm{k}_2(G-T) - 2|T|) = 0$$

holds if and only if (7.4) holds, hence Theorem 7.3.2 implies Theorem 7.1.1.

Chapter 8

Odd subgraphs and matchings

8.1 Introduction

Let

$$f : V(G) \longrightarrow \{1,\ 3,\ 5,\ 7,\ \cdots\}$$

be an odd integer valued function defined on $V(G)$, where we allow $f(v) > d_G(v)$ for some vertices v, and f always denotes this function throughout this Chapter. Then a subgraph H of G is called a $(1, f)$-*odd subgraph* if $d_H(x) \in \{1, 3, \ldots, f(x)\}$ for all $x \in V(H)$. A spanning $(1, f)$-odd subgraph is called a $(1, f)$-*odd factor* of G. If $f(x) = 1$ for all $x \in V(G)$, then a $(1, f)$-odd subgraph is a matching, and a $(1, f)$-odd factor is a 1-factor (i.e., a perfect matching). Note that for convenience, we define a *matching* as a subgraph with all degrees one. A $(1, f)$-odd subgraph H of G is said to be *maximum* if G has no $(1, f)$-odd subgraph K with $v(K) > v(H)$. A subgraph with all degrees odd is called an *odd subgraph*, and a spanning odd subgraph is called an *odd factor*. In this Chapter, we shall show some results on $(1, f)$-odd subgraphs, which are generalizations of those on matchings. Then we can expect that some other results on matchings can be generalized to those on $(1, f)$-odd subgraphs. However, there exist some theorems on matchings that cannot be directly generalized. Such an example is given in Theorem 8.3.5. Some results on $(1, f)$-odd factors, which are generalizations of results on 1-factors, can be found in [33], [75] and [120].

A component of a graph is said to be *odd* or *even* according to the parity of its order. We denote by $o(G)$ the number of odd components of G. For two graphs H and K, the *join* $H + K$ denotes the graph with vertex set $V(H) \cup V(K)$ and edge set $E(H) \cup E(K) \cup \{xy \mid x \in V(H) \text{ and } y \in V(K)\}$. Let $H \triangle K$ denote the subgraph formed by the symmetric difference of the two edge sets, so $V(H \triangle K) = V(H) \cup V(K)$ and $E(H \triangle K) = (E(H) \cup E(K)) \setminus (E(H) \cap E(K))$. Let R be a subgraph of a graph G and X a subset of $V(G)$. Then we say that R *covers* X if $V(R) \supseteq X$, and that K *avoids* X if $V(R) \cap X = \emptyset$. Other notation and definitions not defined here can be found in [18] or [92].

In Section 8.2 we give a Berge-type formula for the size of maximum $(1, f)$-odd subgraphs. In Section 8.3 we prove an augmentation property for $(1, f)$-odd subgraphs. These results appeared in [K6]. Further result on this topic are in Chapters 9 and 10.

Following our work, Yu and Zhang [125] investigated the structure and properties of a graph with a unique $(1, f)$-odd factor, and determined the maximum number of edges in a graph of a given order which has a unique $[1, k]$-odd factor. Atanasov, Petruševski and Škrekovski [14] uses some of our methods to give a characterization of graphs with given odd chromatic index. In an other paper Petruševski [104] also uses our methods to obtain results on odd edge colorings of multigraphs.

8.2 Berge-type formula

A criterion for a graph to have $(1,f)$-odd factor is given in the following theorem, which is a generalization of Tutte's 1-factor Theorem ([92] p.84).

Theorem 8.2.1 (Cui, Kano [33]). *A graph G has a $(1,f)$-odd factor if and only if*

$$o(G-S) \leq \sum_{x\in S} f(x) \quad \text{for all} \ \ S \subset V(G) \ . \tag{8.1}$$

We first give a formula for the order of a maximum $(1,f)$-odd subgraph, which is similar to the following.

Theorem 8.2.2 (Berge [16]; [92] p.90). *The order of a maximum matching M of a graph G is given by*

$$|M| = v(G) - \max_{S\subseteq V(G)} \{o(G-S) - |S|\}.$$

Theorem 8.2.3 (Kano, Katona [K6]). *The order of a maximum $(1,f)$-odd subgraph H of a graph G is given by*

$$v(H) = v(G) - \max_{S\subseteq V(G)} \{o(G-S) - \sum_{x\in S} f(x)\}.$$

In order to prove the above theorem, we need the following lemma.

Lemma 8.2.4 ([91] No.42, p.54). *Let G be a connected graph. Then the following statements hold.*
(i) If $v(G)$ is even, then G has an odd factor.
(ii) If $v(G)$ is odd, then G has an odd subgraph of order $v(G)-1$.

Proof. We give a short proof to (i). Let n be an odd integer such that $n \geq v(G)$, and define the function f by $f(x) = n$ for all $x \in V(G)$. Then by Theorem 8.2.1, G has a $(1,f)$-odd factor, which is the required odd factor.

Statement (ii) is an easy consequence of (i) since G has a vertex v such that $G-\{v\}$ is connected. □

Proof of Theorem 8.2.3. Let H be a maximum $(1,f)$-odd subgraph of G, and let $d := \max_{S\subseteq V(G)}\{o(G-S) - \sum_{x\in S} f(x)\}$. Then $d \geq 0$ as $o(G) \geq 0$, and $v(G)+d$ is even since $v(G) \equiv o(G-S) + |S| \equiv d \pmod 2$.

For every odd component C of $G-S$, if $V(C)$ is covered by H, then there exists at least one edge of H that joins C to S. Thus at least $o(G-S) - \sum_{x\in S} d_H(x)$ odd components of $G-S$ are not covered by H. This implies $v(H) \leq v(G) - d$.

We next prove the reverse inequality. Let $G' := G + K_d$ be the join of G and the complete graph K_d of order d, and define $f' : V(G') \to \{1, 3, \ldots\}$ by $f'(x) = f(x)$ for all $x \in V(G)$ and by $f'(x) = 1$ for all $x \in V(K_d)$. Then $o(G') = 0$ since $v(G) + d$ is even. Let X be a non-empty subset of $V(G')$. If $V(K_d) \not\subseteq X$, then $o(G' - X) \le 1 \le \sum_{x \in X} f'(x)$. If $V(K_d) \subseteq X$, then

$$o(G' - X) = o(G - X \cap V(G)) \le \sum_{x \in X \cap V(G)} f(x) + d = \sum_{x \in X} f'(x).$$

Hence by Theorem 8.2.1, G' has a $(1, f')$-odd factor F'. Let $M := F' - V(K_d)$. Then M is a spanning subgraph of G and has at most d vertices of even degree, some of which may be isolated vertices of M. Therefore M has at most d odd components.

By applying (i) or (ii) of Lemma 8.2.4 to each component of M according to whether its order is even or odd, we obtain an odd subgraph H of M such that $v(H) \ge |M| - d = v(G) - d$. Since H is a $(1, f)$-odd subgraph of G, the proof is complete. □

Let H be a $(1, f)$-odd subgraph of a graph G. Then H is said to be *maximal* if G has no $(1, f)$-odd subgraph H_1 such that $V(H)$ is a proper subset of $V(H_1)$. Recall that H is said to be *maximum* if G has no $(1, f)$-odd subgraph H_2 such that $v(H_2) > v(H)$. Moreover, if H has a cycle C, then $H - E(C)$ is also a $(1, f)$-odd subgraph with vertex set $V(H)$. By repeating this procedure, we can obtain a $(1, f)$-odd subgraph H' which is a forest and whose vertex set is $V(H)$.

For a subgraph K of a graph G and edge subsets $X \subset E(K)$ and $Y \subset E(G) \setminus E(K)$, we denote by $K - X + Y$ the subgraph of G induced by $(E(K) \setminus X) \cup Y$. A path in a graph G connecting two vertices x and y is denoted by $P(x, y)$ or $P_G(x, y)$.

8.3 Augmentation property

We now show that the augmentation property – which is well know from the theory of matroids – is true for $(1, f)$-odd subgraphs. More precisely it is a generalization of the following property of matchings.

Theorem 8.3.1 ([43]; [92] p.88)**.** *Let G be a graph, and B and R be subsets of $V(G)$ such that $|B| < |R|$. Then if there exists a matching which covers B and one which covers R, then there exists a matching which covers B and at least one vertex of $R \setminus B$.*

Theorem 8.3.2 (Kano, Katona [K6])**.** *Let G be a graph, and B and R be subsets of $V(G)$ such that $|B| < |R|$. Then if there exists a $(1, f)$-odd subgraph which covers B and one which covers R, then there exists a $(1, f)$-odd subgraph which covers B and at least one vertex of $R \setminus B$. In particular, every maximal $(1, f)$-odd subgraph is a maximum $(1, f)$-odd subgraph.*

Proof. Let H_B and H_R be $(1, f)$-odd subgraphs which cover B and R, respectively. We may assume that both H_B and H_R are forests. If H_B contains a vertex in $R \setminus B$, then H_B itself is the desired $(1, f)$-odd subgraph. Thus we may assume that H_B avoids $R \setminus B$ (i.e., $V(H_B) \cap (R \setminus B) = \emptyset$). If an edge e joins a vertex in $R \setminus B$ to a vertex in

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$V(G) \setminus V(H_B)$, then $H_B + e$ is the desired $(1, f)$-odd subgraph. Hence we may assume that every neighbor of a vertex $r \in R \setminus B$ is in $V(H_B)$.

For convenience, we call the edges of H_B and H_R *blue* and *red edges*, respectively. Let $F := H_B \triangle H_R - I$, where I is the set of isolated vertices of $H_B \triangle H_R$. The red and blue degrees of a vertex v in F, which are the numbers of red and blue edges of F incident to v, are denoted by $d_{Fr}(v)$ and $d_{Fb}(v)$, respectively. Note that $d_{Fr}(v) < d_{H_R}(v)$ and $d_{Fb}(v) < d_{H_B}(v)$ if and only if there is an edge in $E(H_R) \cap E(H_B)$ that covers v. Let $m(v) = \max\{d_R(v), d_B(v)\}$. Clearly $m(v)$ is a positive integer.

We now construct a new graph F' from F in the following way. First we split up some of the vertices of F contained in $V(H_R) \cap V(H_B)$. Let v be a vertex of F.

Case (i) If $m(v) = 1$ or $v \notin V(H_R) \cap V(H_B)$, then we define a vertex $v' \in V(F')$.

Case (ii) If $m(v) \geq 2$ and $v \in V(H_R) \cap V(H_B)$, then we define $m(v)$ independent vertices, $v'_1, v'_2, \ldots, v'_{m(v)}$.

In Case (i) if a blue or red edge is incident to a vertex v in F, then let it be incident to v' in F'. In Case (ii) for every blue edge incident to v in F we pick an arbitrary vertex from $v'_1, v'_2, \ldots, v'_{m(v)}$ to be its endvertex in F' but we pick a different one for each blue edge. This is possible since $m(v) \geq d_{Fb}(v)$. We apply the same procedure for the red edges. If $m(v) = d_{Fb}(v) > d_{Fr}(v)$ then we define some new red edges. In this case some vertices from $v'_1, v'_2, \ldots, v'_{m(v)}$ are not covered by red edges. We claim that $m(v) - d_{Fr}(v)$ is even. If there are k edges incident to v in $E(H_B) \cap E(H_R)$, then $d_{Fb}(v) = d_{H_B}(v) - k$ and $d_{Fr}(v) = d_{H_R}(v) - k$, which proves our claim since $d_{H_B}(v) - d_{H_R}(v)$ is even. Therefore we can cover the uncovered vertices of $v'_1, v'_2, \ldots, v'_{m(v)}$ with a new set of independent red edges, namely with a red matching (see Figure 8.1). Then every new red edge covers two of the uncovered vertices of $v'_1, v'_2, \ldots, v'_{m(v)}$. Similarly if $m(v) = d_{Fr}(v) > d_{Fb}(v)$ then an even number of vertices of $v'_1, v'_2, \ldots, v'_{m(v)}$ are not covered by the blue edges, so we cover these by a new blue matching.

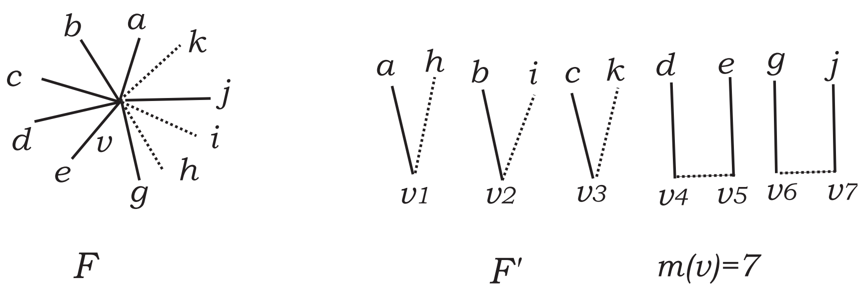

Figure 8.1: A vertex v in F and the vertices $v_1, v_2, \ldots, v_{m(v)}$ in F'

Let B' and R' be the set of those vertices of F' which correspond to a vertex of B and R, respectively. Also let F'_b and F'_r be the subgraph induced by the blue and red edges of F', respectively. It is easy to see that every vertex in $V(F'_b) \cap V(F'_r)$ is incident to exactly one blue and one red edge, vertices of $V(F'_b) \setminus V(F'_r)$ are not incident to any red edges and vertices of $V(F'_r) \setminus V(F'_b)$ are not incident to any blue edges. Recall that a *trail* is a walk such that all its edges are distinct. A trail connecting two vertices x and y is called $x - y$ trail and denoted by $T(x, y)$. A trail $T(x, y)$ of a graph G is said to be *maximal with respect to* y if $T(x, y)$ cannot be extended at y by adding a new edge of G to $T(x, y)$ (i.e., if $d_G(y) = d_{T(x,y)}(y)$).

Claim 8.3.3. *If there exists a path in F' with one endvertex in $R' \setminus V(F'_b)$ and the other in $V(F'_r) \setminus V(F'_b)$ (such that these two endvertices are distinct), then there exists an $(1, f)$-odd subgraph in G which covers B and at least one vertex of $R \setminus B$.*

Proof. If there exists a path between $R' \setminus V(F'_b)$ and $V(F'_r) \setminus V(F'_b)$, then let $P(x', y')$, where $x' \in R' \setminus V(F'_b)$ and $y' \in V(F'_r) \setminus V(F'_b)$, be the shortest path which satisfies the conditions in Claim. Then it follows immediately from the choice of $P(x', y')$ that $P(x', y')$ does not contain any vertex of $V(F'_r) \setminus V(F'_b)$ other than x', y'. It is also obvious that the degree of x' is one in the path and so is y', while the degree of any other vertex in the path is two. Moreover, the red degrees of x' and of y' are ones, since their blue degrees are zeros.

Let $G'_b := F'_b \triangle P(x', y')$. In other words, we change the color of every edge in the path from red to blue or blue to red, and take the resulting blue subgraph. The blue degree of x' and y' became one. The blue degree of each vertex in $V(F'_b) \setminus V(F'_r)$ may be decreased by 2, thus still remains odd. The blue degree of each vertex in $V(F'_b) \cap V(F'_r)$ does not change.

Now define G_b to be the subgraph formed by those edges of F which correspond to an edge of G'_b together with the edges in $E(H_B) \cap E(H_R)$. In particular, G_b contains no new edges joining two vertices of $v'_1, v'_2, \ldots, v'_{m(v)}$. We claim that G_b is the desired subgraph.

If a vertex v' is not in $P(x', y')$, then $d_{G_b}(v) = d_{H_B}(v) \le f(v)$ and it is odd. Suppose that v' is in $P(x', y')$. Clearly, $d_{G_b}(x) = 1 \le f(x)$ and $d_{G_b}(y) = 1 \le f(y)$, and both are odd. For a vertex $v \in V(H_B) \setminus V(H_R)$, we have $d_{G_b}(v) = d_{Fb}(v) - 2 = d_{H_B}(v) - 2$. If $v \in V(H_B) \cap V(H_R)$ then $d_{G_b}(v) - d_{H_B}(v)$ is even, and so $d_{G_b}(v)$ is odd, however, it is possible that $d_{G_b}(v) > d_{H_B}(v)$. This means that the path contains some new blue edges connecting two vertices of $v'_1, v'_2, \ldots, v'_{m(v)}$. If it contains k such edges then $d_{G_b}(v) \le 2k + d_{H_B}(v) \le d_{H_R}(v) \le f(v)$. Consequently, G_b is a $(1, f)$-odd subgraph, and covers $V(H_B)$ and $x \in R \setminus B$. □

Claim 8.3.4. *If there exists a trail $T(x', y')$ in F' with $x' \in R' \setminus V(F'_b)$ and $y' \in V(F'_b) \setminus (B' \cup V(F'_r))$ such that $T(x', y')$ is maximal with respect to y', then there exists an $(1, f)$-odd subgraph in G which covers B and at least one vertex of $R \setminus B$.*

Proof. Let $T(x', y')$ be a trail in F' with $x' \in R' \setminus V(F'_b)$ and $y' \in V(F'_b) \setminus (B' \cup V(F'_r))$ such that $T(x', y')$ is maximal with respect to y'.

Obviously, the degrees of x' and y' are odd in the trail, while the degree of any other vertex in the trail is even. Moreover, the red degree of x' is odd since its blue degree is zero and the blue degree of y' is odd and equal to $d_{F'_b}(y') = d_{H_B}(y')$ by the maximality of the trail.

If there exists a vertex $z' \in T(x', y')$ such that $z' \in V(F'_r) \setminus V(F'_b)$ and $z' \ne x'$, then there exists a path between x' and z'. Thus by Claim 8.3.3 we can find the desired subgraph. Therefore the red degree of each vertex of the trail, except x', is at most one.

Let $G'_b := F'_b \triangle T(x', y')$. In other words, we change the color of every edge in the trail from red to blue or blue to red, and take the resulting blue subgraph. The blue degree of x' became odd and at most $d_{H_R}(x) \le f(x)$, while the blue degree of y' is zero. The blue degree of any other vertex in $V(F'_b) \setminus V(F'_r)$ may be decreased by an even number, and thus still remains odd. The blue degree of vertices in $V(F_b') \cap V(F_r')$ does not change.

Now G_b is defined in the same way as in Claim 8.3.3. If a vertex v' is not in $T(x', y')$, then $d_{G_b}(v) = d_{H_B}(v) \leq f(v)$ and it is odd. Clearly, $d_{G_b}(x) \leq f(x)$ and it is odd. On the other hand we have $d_{G_b}(y) = 0$, and so y is not covered by G_b, but since $y \notin B$ this does not cause any problem. Since for a vertex $v \in V(H_B) \setminus V(H_R)$, $d_{H_B}(v) - d_{G_b}(v)$ is even and non-negative, we have that $d_{G_b}(v) \leq d_{H_B}(v) \leq f(v)$ and $d_{G_b}(v)$ is odd. If $v \in V(H_B) \cap V(H_R)$ then $d_{H_B}(v) - d_{G_b}(v)$ is even but it may be negative. This means that the trail contains some blue edges connecting two vertices of $v_1', v_2', \ldots, v_{m(v)}'$. If it contains k such edges then $d_{G_b}(v) \leq 2k + d_{H_B}(v) \leq d_{H_R}(v) \leq f(v)$. Consequently, G_b is a $(1, f)$-odd subgraph, and covers $H_B \setminus \{y\} \supseteq B$ and $x \in R \setminus B$. $\square$

We now turn our attention to the proof of Theorem 8.3.2. We may assume that neither the conditions of Claim 8.3.3 nor those of Claim 8.3.4 hold. This means that for every trail $T(x', y')$ with $x' \in R' \setminus V(F_b')$ which is maximal with respect to y', we have $y' \in B' \setminus V(F_r')$ or $y' \in V(F_b') \cap V(F_r')$. Since the degree of any vertex in $V(F_b') \cap V(F_r')$ is exactly two, $y' \notin\in V(F_b') \cap V(F_r')$. Therefore for each vertex $x' \in R' \setminus V(F_b')$, there exists a vertex $y' \in B' \setminus V(F_r')$ such that there exists a $T(x', y')$ trail.

By the assumption $|B| < |R|$ in the theorem, we have $|B' \setminus V(F_r')| \leq |B \setminus R| < |R \setminus B| = |R' \setminus V(F_b')|$ since $V(F_b') \cap (R' \setminus B') = \emptyset$. So there must be two distinct vertices $x', z' \in R' \setminus V(F_b')$ and a vertex $y' \in B' \setminus V(F_r')$ such that two trails $T(x', y')$ and $T(z', y')$ exist. Thus there exists a trail $T(x', z')$ and hence a path between x' and z', which is a contradiction. $\square$

We conclude this Chapter by showing a property which matchings possess but $(1, f)$-odd subgraphs do not.

Theorem 8.3.5 (Bollobás [18] p.57)**.** *Let G be a graph and W be a subset of $V(G)$. Then G has a matching which covers W if and only if*

$$o(G - S \mid W) \leq |S| \quad \textit{for all} \quad S \subseteq V(G), \tag{8.2}$$

where $o(G - S \mid W)$ denotes the number of those odd components of $G - S$ whose vertices are contained in W.

Let G be a graph with vertex $V(G) = \{a, b, c, d, e, u, x, y, z\}$ and edge set $E(G) = \{ax, bx, bz, cx, cy, cz, dy, ey, uz\}$, and define the function f by $f(a) = f(b) = f(c) = f(d) = f(e) = f(u) = 1$ and $f(x) = f(y) = f(z) = 3$. Then for a subset $W = \{a, b, c, d, e, u\}$, G and W satisfies $o(G - S \mid W) \leq \sum_{x \in S} f(x)$ for all $S \subset V(G)$, but G has no $(1, f)$-odd subgraph which covers W. Hence $(1, f)$-odd subgraphs do not have the exactly same property given in Theorem 8.3.5.

Chapter 9

Structure theorem and algorithm on $(1, f)$-odd subgraphs

9.1 Introduction

In this Chapter we present further results related to $(1, f)$-odd subgraphs, so we continue to use the definitions given in the previous Chapter. The results appeared in [K7].

Lovász [89] and Cornuéjols [32] deal with general factors which is a common generalization of all factor problems. For each vertex $v \in V(G)$, let B_v be a subset of $\{0, 1, 2, \ldots, d_G(v)\}$. The *general factor problem* asks whether there exists a spanning subgraph F of G such that for each vertex v we have $d_F(v) \in B_v$. With this terminology a $(1, f)$-odd factor is a general factor with $B_v = \{1, 3, 5, \ldots, f(v)\}$, and a $(1, f)$-odd subgraph is a general factor with $B_v = \{0, 1, 3, 5, \ldots, f(v)\}$.

An integer h is called a *gap* of $\mathcal{H} \subseteq \mathbb{N} \cup \{0\}$ if $h \notin \mathcal{H}$ but $\mathcal{H}$ contains an element less than h and an element greater than h. The general problem becomes NP-complete if B_v may have consecutive gaps [32, 89], so restrict B_v to sets without consecutive gaps. If there is no general factor in G, we can define an optimal solution in the following sense. Let the *deficiency* of a spanning subgraph F at a vertex v be defined as the distance between the degree of v in F and the set B_v. Specifically, let l_v and u_v be the smallest and largest elements of B_v. Then the deficiency of F at v is

$$\operatorname{def}_F(v) = \begin{cases} 0, & \text{if } d_F(v) \in B_v \\ l_v - d_F(v), & \text{if } d_F(v) < l_v \\ d_F(v) - u_v, & \text{if } d_F(v) > u_v \\ 1, & \text{if } d_F(v) \notin B_v \text{ and } l_v < d_F(v) < u_v, \end{cases}$$

since there are no gaps of length 2. The *total deficiency* of F is defined as

$$\operatorname{def}(F) = \sum_{v \in V(G)} \operatorname{def}_F(v).$$

We say that F is *optimal* if it has the smallest total deficiency.

Applying this definition to our special case, by setting $B_v = \{1, 3, 5, \ldots, f(v)\}$ for each v, we obtain the definition of an *optimal* $(1, f)$*-odd spanning subgraph*. Note that this definition is different from the definition of a maximum $(1, f)$-odd subgraph given in Chapter 8, i.e. a $(1, f)$-odd subgraph containing the maximum number of vertices. In fact, it is not true that the edge set of every optimal $(1, f)$-odd spanning subgraph induces a $(1, f)$-odd subgraph. On the other hand, we will show in

Section 9.2 that there exists an optimal $(1, f)$-odd spanning subgraph whose edge set induces a maximum $(1, f)$-odd subgraph. This means that the total deficiency of any optimal $(1, f)$-odd spanning subgraph is equal to the number of uncovered vertices in a maximum $(1, f)$-odd subgraph.

In Section 9.3 a Gallai-Edmonds type structure theorem is given. This result does not seem to be easily deducible from Lovász's [89] theorems, so we prove it directly.

The algorithm in [32] is an Edmonds type algorithm, but it does not seem to be a generalization of Edmonds blossom algorithm. Its running time is $O(|V(G)|^5)$. The polynomial algorithm given in Section 9.4 follows a different approach from this. Our algorithm is a direct generalization of Edmonds blossom algorithm with running time $O(|V(G)|^3)$.

9.2 Maximum is Optimal

For problems of finding a maximum $(1, f)$–odd subgraph and an optimal $(1, f)$–odd spanning subgraph, the number of edges in the solution is not determined by the number of vertices, that is, there can be several solutions with different numbers of edges. However, we can make some useful observations about the solutions with minimum number of edges. Let us call a maximum $(1, f)$–odd subgraph with the fewest edges a *smallest maximum* $(1, f)$*–odd subgraph*, and call an optimal $(1, f)$–odd spanning subgraph with the fewest edges a *smallest optimal* $(1, f)$*–odd spanning subgraph*.

Observation 9.2.1. *A smallest maximum* $(1, f)$*–odd subgraph is a forest.*

Proof. Suppose indirectly, that a smallest maximum $(1, f)$–odd subgraph F contains a cycle. By removing all edges of this cycle, the degrees of vertices in F remain odd, hence cannot decrease to 0. Thus the new subgraph covers the same vertices as F, but it has fewer edges, contradicting our assumption. □

Theorem 9.2.2 (Kano, Katona [K7])**.** *The edge set of a smallest optimal* $(1, f)$*–odd spanning subgraph induces a maximum* $(1, f)$*–odd subgraph.*

Proof. First note that the edge set of an optimal $(1, f)$–odd spanning subgraph does not necessarily induce a $(1, f)$–odd subgraph since it may contain vertices v with $d_F(v) \notin B_v \cup \{0\}$ where $B_v = \{1, 3, \dots, f(v)\}$.

Let F be a smallest optimal $(1, f)$–odd spanning subgraph for which the number of vertices v with $d_F(v) \notin B_v \cup \{0\}$ is minimum. If there is no such vertex, then $E(F)$ induces a maximum $(1, f)$–odd subgraph. Otherwise, let x be a vertex such that $d_F(x) \notin B_x \cup \{0\}$. If we remove any edge xy, then $\mathrm{def}(x)$ decreases by 1, and $\mathrm{def}(y)$ is either increased or decreased by 1. Thus $\mathrm{def}(F)$ is either decreased by 2 or does not change. So, remove some edges incident to x to obtain F' until its degree will be in B_x. If $\mathrm{def}(F') < \mathrm{def}(F)$ then we have a contradiction since F is an optimal $(1, f)$–odd spanning subgraph. Otherwise, F' is an optimal $(1, f)$–odd spanning subgraph, as well, but it has fewer edges than F, which contradicts choice of F. Therefore, the edge set of a smallest optimal $(1, f)$-odd spanning subgraph induces a $(1, f)$-odd subgraph, which does not cover exactly $\mathrm{def}(F)$ vertices and must be a maximum $(1, f)$-odd subgraph. □

9.3 Structure Theorem

For subsets A and B of a set, we denote by $A \subseteq B$ if A is a subset of B, and by $A \subset B$ if A is a proper subset of B. Let G be a graph. A component of G is called an *odd component* if it has an odd order, and the number of odd components of G is denoted by odd(G). A subset $X \subseteq V(G)$ is called a **barrier** in G for $(1, f)$-odd factor if $\mathrm{odd}(G-X) - \sum_{x \in X} f(x)$ is maximal. Let barr(G) denote this maximal value. Thus

$$\mathrm{barr}(G) = \max_{S \subseteq V(G)} \left\{ \mathrm{odd}(G-S) - \sum_{x \in S} f(x) \right\} = \mathrm{odd}(G-X) - \sum_{x \in X} f(x) \tag{9.1}$$

holds. Note that $\mathrm{barr}(G) \geq 0$ is obtained by setting $S = \emptyset$. A barrier X is said to be **minimal** if no proper subset of X is a barrier.

Theorem 9.3.1 (Cui, Kano [33]). *A graph G contains a $(1, f)$-odd factor if and only if* $\mathrm{barr}(G) = 0$, *that is, if and only if* $\mathrm{odd}(G-S) \leq \sum_{x \in S} f(x)$ *for all $S \subseteq V(G)$.*

Using the above notation Theorem 8.2.3 can be written as follows:

Theorem 9.3.2 (Kano, Katona [K6]). *The order $v(H)$ of a maximum $(1, f)$-odd subgraph H of a graph G is given by*

$$v(H) = v(G) - \mathrm{barr}(G).$$

Theorem 9.3.3 (Topp, Vestergaard [120]). *Let G be a graph having no $(1, f)$-odd factors. Let X be a minimal barrier for $(1, f)$-odd factor in G. Then every vertex $v \in X$ is adjacent to at least $f(v) + 2$ odd components of $G - X$. In particular, there exists a subset $A_v \subseteq V(G)$ such that $\langle A_v \rangle_G = K_{1,f(v)+2}$ and its center is v.*

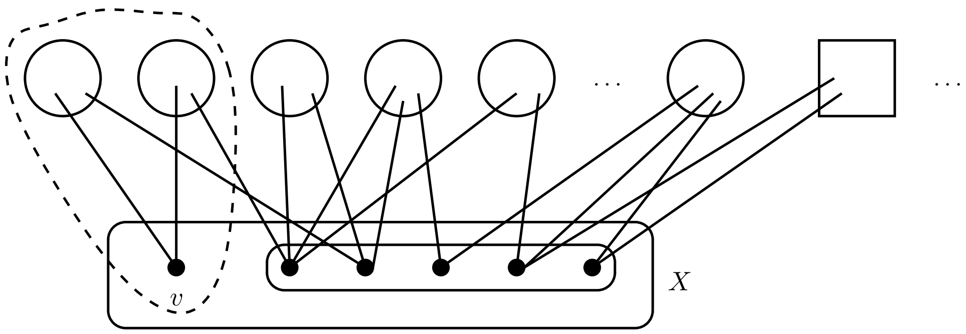

Figure 9.1: Components of $G - X$ and those of $G - (X - v)$.

Proof. Suppose that a vertex $v \in X$ is adjacent to at most $f(v)+1$ odd components of $G-X$. If v is adjacent to exactly $f(v)+1$ odd components of $G-X$, then v and these $f(v)+1$ odd components form a new odd components of $G-(X-v)$ (see Figure 9.1). Hence, in any case, we have

$$\mathrm{odd}(G-(X-v)) \geq \mathrm{odd}(G-X) - f(v).$$

On the other hand, by (9.1), we have

$$\mathrm{odd}(G-(X-v)) - \sum_{x \in X-v} f(x) \le \mathrm{odd}(G-X) - \sum_{x\in X} f(x).$$

Thus $\mathrm{odd}(G-(X-v)) = \mathrm{odd}(G-X) - f(v)$, and so

$$\mathrm{odd}(G-(X-v)) - \sum_{x \in X-v} f(x) = \mathrm{odd}(G-X) - \sum_{x\in X} f(x),$$

which implies that $X - v$ is also a barrier. This contradicts the minimality of X. Therefore v adjacent to at least $f(v) + 2$ odd components of $G - X$. The latter part follows immediately from the former part. □

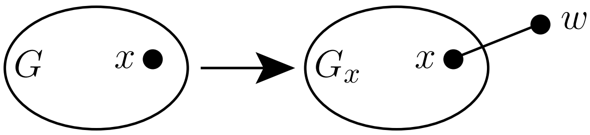

Figure 9.2: Construction of G_x.

For a graph G, define

$$\tau(G) = \text{the order of a maximum } (1,f)\text{-odd subgraph of } G.$$

For any vertex x of G, we denote by G_x the graph obtained from G by adding a new vertex w together with a new edge wx and define $f(w) = 1$ (see Figure 9.2). Let $D(G)$ denote the set of all vertices x of G such that $\tau(G_x) = \tau(G) + 2$. Let $A(G)$ be the set of vertices of $V(G) - D(G)$ that are adjacent to at least one vertex in $D(G)$. Finally, define $C(G) = V(G) - D(G) - A(G)$. Then $V(G)$ is decomposed into three disjoint subsets

$$V(G) = D(G) \cup A(G) \cup C(G). \tag{9.2}$$

Note that if $f(x) = 1$ for all vertices x of G, then a maximum $(1, f)$-odd subgraph is a maximum matching and a vertex y satisfies $\tau(G_y) = \tau(G)+2$ if and only if y is not contained in a certain maximum matching in G, and thus the above decomposition $V(G) = D(G) \cup A(G) \cup C(G)$ becomes the Gallai-Edmonds decomposition ([92] p.94).

A graph G is said to be **critical with respect to** $(1, f)$**-odd factor** if for every vertex x of G, $G_x = G + wx$ has a $(1, f)$-odd factor. It is obvious that if G is critical with respect to $(1, f)$-odd factor, then G is a connected graph of odd order.

Theorem 9.3.4 (Kano, Katona [K7]). [Structure Theorem on $(1, f)$-odd subgraphs]. *Let G be a graph, and $V(G) = D(G) \cup A(G) \cup C(G)$ the decomposition defined in (9.2). Then the following statements hold (see Figure 9.3):*

(i) Every component of $\langle D(G)\rangle_G$ is critical with respect to $(1, f)$-odd factor.

(ii) $\langle C(G)\rangle_G$ has a $(1, f)$-odd factor.

(iii) Every maximum $(1, f)$-odd subgraph H of G covers $C(G) \cup A(G)$, and for every vertex $x \in A(G)$, $d_H(x) = f(x)$ and every edge of H incident with x joins x to a vertex in $D(G)$.

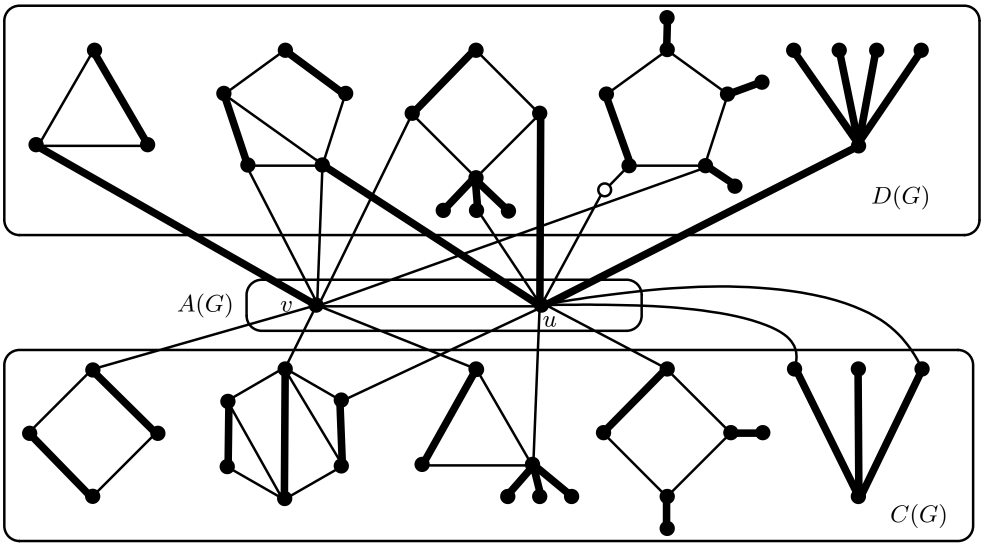

Figure 9.3: Decomposition $V(G) = D(G) \cup A(G) \cup C(G)$, with $f(v) = 1, f(u) = 3$ and one uncovered vertex.

(iv) The order $v(H)$ of a maximum $(1, f)$-odd subgraph H is given by

$$v(H) = v(G) + \omega(\langle D(G) \rangle_G) - \sum_{x \in A(G)} f(x), \tag{9.3}$$

where $\omega(\langle D(G) \rangle_G)$ denotes the number of components of $\langle D(G) \rangle_G$.

Proof. We may assume that G is connected since if the theorem holds for each component of G, then the theorem holds for G. Moreover, we may assume that G has no $(1, f)$-odd factor since otherwise $D(G) = \emptyset$, $A(G) = \emptyset$ and $C(G) = V(G)$, and thus the theorem holds.

Let S be a maximal barrier of G, that is, S is a subset of $V(G)$ such that

$$\mathrm{odd}(G - S) - \sum_{x \in S} f(x) = \max_{X \subset V(G)} \{\mathrm{odd}(G - X) - \sum_{x \in X} f(x)\} = \mathrm{barr}(G) > 0 \tag{9.4}$$

and

$$\mathrm{odd}(G - T) - \sum_{x \in T} f(x) < \mathrm{odd}(G - S) - \sum_{x \in S} f(x) \quad \text{for all} \quad S \subset T \subseteq V(G). \tag{9.5}$$

Claim 9.3.5. *Every even component of $G - S$ has a $(1, f)$-odd factor.*

Proof. Assume that an even component D of $G - S$ does not have a $(1, f)$-odd factor. Then by Theorem 9.3.1, there exists a subset $\emptyset \neq X \subset V(D)$ such that $\mathrm{odd}(D - X) >$

$\sum_{x\in X} f(x)$. Then

$$\begin{aligned}\operatorname{odd}(G-(S\cup X))-\sum_{x\in S\cup X} f(x) &= \operatorname{odd}(G-S)-\sum_{x\in S} f(x)+\operatorname{odd}(D-X)-\sum_{x\in X} f(x)\\ &> \operatorname{odd}(G-S)-\sum_{x\in S} f(x),\end{aligned}$$

contrary to (9.4). Hence D has a $(1,f)$-odd factor. □

Claim 9.3.6. *Every odd component of $G-S$ is critical with respect to $(1,f)$-odd factor.*

Proof. Suppose that an odd component C of $G-S$ is not critical with respect to $(1,f)$-odd factor. Then there exist a vertex $v\in V(C)$ such that C_v has no $(1,f)$-odd factor. Let Y be a minimal barrier of $C_v = C+vw$. Then w is not contained in Y by Theorem 9.3.3, and so $\emptyset \neq Y \subseteq V(C)$. It follows that

$$\operatorname{odd}(C-Y) \geq \operatorname{odd}(C+vw-Y)-1 \geq \sum_{x\in Y} f(x)+2-1=\sum_{x\in Y} f(x)+1.$$

Hence

$$\begin{aligned}&\operatorname{odd}(G-(S\cup Y))-\sum_{x\in S\cup Y} f(x)\\ &=\operatorname{odd}(G-S)-1+\operatorname{odd}(C-Y)-\sum_{x\in S} f(x)-\sum_{x\in Y} f(x)\\ &\geq \operatorname{odd}(G-S)-\sum_{x\in S} f(x),\end{aligned}$$

which implies that $S\cup Y$ is a barrier in G. This contradicts the maximality (9.5) of S. Hence C is critical with respect to $(1,f)$-odd factor. □

Let $\{C_1, C_2, \dots, C_m\}$, where $m=\operatorname{odd}(G-S) > \sum_{x\in S} f(x)$, be the set of odd components of $G-S$. We define the bipartite graph B with bipartite sets S and $\{C_1, C_2, \dots, C_m\}$ as follows: a vertex $x\in S$ and C_i are joined by an edge of B if and only if x and C_i are joined by at least one edge of G. In other words, each of $\{C_1, C_2, \dots, C_m\}$ is contracted to one vertex and all edges inside S are removed.

Claim 9.3.7. *$|N_B(X)| \geq \sum_{x\in X} f(x)$ for all $X\subseteq S$.*

Proof. Assume that $|N_B(Y)| < \sum_{x\in Y} f(x)$ for some $\emptyset\neq Y\subseteq S$. Then

$$\begin{aligned}\operatorname{odd}(G-(S-Y))-\sum_{x\in S-Y} f(x) &\geq |\{C_1,C_2,\dots,C_m\}-N_B(Y)|-\sum_{x\in S-Y} f(x)\\ > m-\sum_{x\in Y} f(x)-\sum_{x\in S-Y} f(x) &= m-\sum_{x\in S} f(x)=\operatorname{odd}(G-S)-\sum_{x\in S} f(x),\end{aligned}$$

contrary to (9.4). Hence Claim 9.3.7 holds. □

Claim 9.3.8. *There exists the unique maximum proper subset $S_0\subset S$ in B such that $|N_B(S_0)|=\sum_{x\in S_0} f(x)$. Furthermore, $|N_B(Y)-N_B(S_0)| > \sum_{x\in Y} f(x)$ for every $\emptyset\neq Y\subseteq S-S_0$.*

Proof. It follows from (9.4) that $|N_B(S)| = m > \sum_{x\in S} f(x)$. Suppose that $|N_B(X_1)| = \sum_{x\in X_1} f(x)$ and $|N_B(X_2)| = \sum_{x\in X_2} f(x)$ for two subsets $X_1,\ X_2 \subset S$. Then using Claim 9.3.7, we have

$$\begin{aligned}\sum_{x\in X_1\cup X_2} f(x) &\le |N_B(X_1\cup X_2)| \le |N_B(X_1)| + |N_B(X_2)| - |N_B(X_1\cap X_2)| \\ &\le \sum_{x\in X_1} f(x) + \sum_{x\in X_2} f(x) - \sum_{x\in X_1\cap X_2} f(x) = \sum_{x\in X_1\cup X_2} f(x)\end{aligned}$$

Hence $|N_B(X_1\cup X_2)| = \sum_{x\in X_1\cup X_2} f(x)$. Therefore there exists the unique maximum subset $S_0 \subset S$ such that $|N_B(S_0)| = \sum_{x\in S_0} f(x)$.

Let $\emptyset \ne Y \subseteq S - S_0$. Then it follows from the maximality of S_0 and $S_0 \subset Y \cup S_0$ that

$$|N_B(Y) - N_B(S_0)| = |N_B(Y\cup S_0) - N_B(S_0)| > \sum_{x\in Y\cup S_0} f(x) - \sum_{x\in S_0} f(x) = \sum_{x\in Y} f(x).$$

Therefore the claim is proved. ∎

Let $T = S - S_0$. Then

$$\begin{aligned}\mathrm{odd}(G-T) &\ge |\{C_1, C_2, \ldots, C_m\} - N_B(S_0)| = m - \sum_{x\in S_0} f(x) \\ &= \mathrm{odd}(G-S) - \left(\sum_{x\in S} f(x) - \sum_{x\in T} f(x)\right),\end{aligned}$$

and so

$$\mathrm{odd}(G-T) - \sum_{x\in T} f(x) = \mathrm{odd}(G-S) - \sum_{x\in S} f(x) = \mathrm{barr}(G).$$

Let $\{C_1, C_2, \ldots, C_k\}$ be the set of odd components of $G-T$, where $k = \mathrm{odd}(G-T)$, and $\{C'_1, C'_2, \ldots, C'_r\}$ the set of odd components of $G-S$ which corresponds to $N_B(S_0)$, where $r = |N_B(S_0)| = \sum_{x\in S_0} f(x)$. Then by Hall's Marriage Theorem and by Claims 9.3.7 and 9.3.8, it follows that

(i) B has a subgraph M satisfying

$$\begin{aligned}&d_M(x) = f(x) \quad \text{for all } x \in S, \\ &d_M(C') = 1 \quad \text{for all } C' \in \{C'_1, C'_2, \ldots, C'_r\}\ , \quad \text{and} \\ &d_M(C) = 1 \quad \text{for all } C \in \{C_1, C_2, \ldots, C_k\} \cap V(M)\ .\end{aligned}$$

(ii) for each C_i $(1 \le i \le k)$, there exist subgraphs M_1 and M_2 of B satisfying the above condition (i) such that C_i is covered by M_1 but not by M_2.

Let M be a subgraph of B given in (i). Then for every odd component C'_j $(1 \le j \le r)$, there exists an edge in M joining C'_j to a vertex x_j in S. Take an edge e_j of G joining x_j to a vertex v_j in C'_j, and let R'_j be a $(1,f)$-odd factor of $C'_j + v_jx_j$, whose existence is guaranteed by Claim 9.3.6.

For an odd component C_i $(1 \le i \le m)$, if M has an edge joining C_i to a vertex x_i of S, then there exists an edge e_i of G joining x_i to a vertex v_i of C_i, and take a

$(1, f)$-odd factor R_i of $C_i + v_i x_i$. If M has no such an edge, then take a maximum $(1, f)$-odd subgraph H_i of C_i, whose order is $|C_i| - 1$ by Claim 9.3.6. Define

$$\begin{aligned} K &= \bigcup \{(1,f)\text{-odd factor of even components of } G - S\} \\ &\quad + \bigcup_{1 \le j \le r} \{R'_j\} + \bigcup_{1 \le i \le k} \{R_i \text{ or } H_i\}. \end{aligned}$$

Then for each vertex $x \in S$, it follows that $d_K(x) = d_M(x) = f(x)$ by (i), and by Theorem 9.3.2, K is a maximum $(1, f)$-odd subgraph of G since the order of K is $v(G) - (k - \sum_{x \in T} f(x)) = v(G) - \mathrm{barr}(G)$.

Conversely, every maximum $(1, f)$–odd subgraph in G is obtained in this way since for any maximum $(1, f)$–odd subgraph H in G, H cannot cover at least $k - \sum_{x \in T} f(x)$ odd components in $\{C_1, C_2, \dots, C_k\}$, and at least one of whose vertices is not contained is H. Since H is a maximum $(1, f)$–odd subgraph, H does not cover exactly $k - \sum_{x \in T} f(x)$ of these vertices and covers all the other vertices. Therefore H induces a subgraph N in B which covers S and $\{C'_1, C'_2, \dots, C'_r\}$ and $k - \sum_{x \in T} f(x)$ elements in $\{C_1, C_2, \dots, C_k\}$, and thus N satisfies the condition (i) as M does. Therefore H can be constructed from a subgraph of B satisfying (i) in the same way as K is obtained.

Claim 9.3.9. *$D(G) = V(C_1) \cup V(C_2) \cup \cdots \cup V(C_k)$ and $A(G) = T$.*

Proof. It is clear that for every vertex x of any C_t $(1 \le t \le k)$, B has a subgraph that satisfies (i) and does not cover x by (ii). Since $C_t - x$ has a $(1, f)$–odd factor by Claim 2, there exists a maximum $(1, f)$–odd subgraph in G which does not cover x. Thus $V(C_1) \cup V(C_2) \cup \cdots \cup V(C_k) \subseteq D(G)$.

Since every maximum $(1, f)$–odd subgraph in G is obtained in the way mentioned above, $D(G) \subseteq V(C_1) \cup V(C_2) \cup \cdots \cup V(C_k)$. Consequently, $D(G) = V(C_1) \cup V(C_2) \cup \cdots \cup V(C_k)$.

Since $N_B(S_0) = \{C'_1, C'_2, \dots, C'_r\}$, it follows that $N_B(\{C_1, C_2, \dots, C_k\}) = S - S_0 = T$. Therefore

$$A(G) = N_G(V(C_1) \cup \cdots \cup V(C_k)) - (V(C_1) \cup \cdots \cup V(C_k)) = T.$$

Hence the claim is proved. □

It is easy to see that $\langle V(C'_1) \cup \cdots \cup V(C'_r) \cup S_0 \rangle$ has a $(1, f)$-odd factor, and forms even components of $G - T$. Consequently, (i)-(iv) are proved, and the proof of Theorem 9.3.4 is complete. □

Since every component of $\langle D(G) \rangle_G$ is factor critical, the Gallai-Edmonds Structure Theorem on matchings is an easy consequence of the above Structure Theorem.

A subgraph H of a graph G is said to *avoid* a subset $X \subseteq V(G)$ if H contains no vertex in X. The following theorem, which holds for matching ([92] p.88), was conjectured in [K6].

Theorem 9.3.10 (Kano, Katona [K7])**.** *Let G be a graph, X and Y be two subsets of $V(G)$ such that $|X| < |Y|$. If there exist a maximum $(1, f)$-odd subgraph which avoids X and one which avoids Y, then there exists a maximum $(1, f)$-odd subgraph which avoids X and at least one vertex of $Y - X$.*

Proof. Let H_X and H_Y be maximum $(1, f)$-odd subgraphs which avoid X and Y, respectively. We may assume that H_X covers all the vertices in $Y - X$ since otherwise H_X is the desired maximum $(1, f)$–odd subgraph. Let $\{C_1, C_2, \ldots, C_k\}$ be the components of $\langle D(G)\rangle_G$, and B denote the bipartite subgraph with bipartite sets T and $\{C_1, C_2, \ldots, C_k\}$ defined in the proof of Theorem 9.3.4. By Theorem 9.3.4, X and Y consist of vertices that are taken from each C_i at most one.

By Theorem 9.3.4 and by $|X| < |Y|$, there exist a vertex $v \in A(G)$ and two components C_s and C_t in $\{C_1, C_2, \ldots, C_k\}$ such that both C_s and C_t are adjacent to v in G, H_X covers C_s but not C_t, $V(C_s) \cap X = \emptyset$, $V(C_s) \cap Y \neq \emptyset$ and $V(C_t) \cap (X \cup Y) = \emptyset$. Then by removing the edge joining v to C_s of the subgraph in B corresponding to H_X, and by adding an edge of B joining v to C_t, we obtain a new subgraph of B satisfying the condition (i) given in the proof of Theorem 9.3.4, and can construct a maximum $(1, f)$–odd subgraph H of G from it as in the proof of Theorem 9.3.4, which covers C_t but not the unique vertex in $V(C_s) \cap Y$ and avoids X. Therefore H is the desired subgraph, and the theorem is proved. $\square$

9.4 Augmenting walks

For subgraphs H and K of a graph G, we denote by $H \triangle K$ the subgraph of G induced by $E(H) \triangle E(K) = (E(H) \cup E(K)) - (E(H) \cap E(K))$.

Let F be a $(1, f)$–odd subgraph in G. Edges of F will be called **blue edges** and edges of $G - F$ are called **red edges**. For a subgraph S of G and a vertex v of S, $d_S^B(v)$ ($d_S^R(v)$) denotes the number of blue (red) edges incident with v in S. In particular, $d_F(v) = d_G^B(v)$. A vertex v is **saturated** if $d_F(v) = f(v)$.

An F–**augmenting walk** between x and y is a walk W such that

(i) $d_W^B(x) = d_W^B(y) = 0$,

(ii) $d_W^R(x) = d_W^R(y) = 1$,

(iii) $d_W^R(v) - d_W^B(v) \leq f(v) - d_F(v) \qquad$ for all $v \in V(W) - \{x, y\}$.

A $(1, f)$-odd subgraph H of G is said to be **maximal** if G has no $(1, f)$-odd subgraph H' such that $V(H) \subset V(H')$. Obviously, a maximum $(1, f)$–odd subgraph is a maximal $(1, f)$-odd subgraph. The next lemma was directly proved in [K6], and can be easily shown by using Theorem 9.3.4.

Lemma 9.4.1. *A maximal $(1, f)$–odd subgraph is a maximum $(1, f)$–odd subgraph.*

The following lemma will not be used directly to prove the correctness of the algorithm, however it helps to understand the concept of the algorithm. On the other hand, the algorithm will provide a stronger result about the properties of augmenting walks in a non-maximum $(1, f)$–odd subgraph.

Lemma 9.4.2. *A $(1, f)$–odd subgraph is maximum if and only if there is no augmenting-walk.*

Proof. Suppose that F is a $(1, f)$–odd subgraph of G and there is an F–augmenting-walk W between x and y, where $x, y \notin V(F)$. Then $W \triangle F$ is a $(1, f)$-odd subgraph since for every vertex $v \in V(W) - \{x, y\}$,

$$d_{W \triangle F}(v) = d_F(v) - d_W^B(v) + d_W^R(v) \equiv d_F(v) \pmod 2$$

and $d_{W\triangle F}(v) \leq f(v)$ by (iii). Furthermore, $W\triangle F$ covers all the vertices of F and $\{x, y\}$, therefore F cannot be maximum.

Suppose now that a $(1, f)$–odd subgraph F of G is not maximum. By Lemma 9.4.1, there exists a $(1, f)$–odd subgraph F' such that $V(F) \subset V(F')$. Let $H = F\triangle F'$. We call edges in $E(H) \cap E(F')$ red edges, and edges in $E(H) \cap E(F)$ blue edges. Then H has the following properties.

(iv) $d_H^B(x) = 0 \quad$ for all $x \in V(F') - V(F)$,

(v) $d_H^R(x)$ is positive and odd for all $x \in V(F') - V(F)$,

(vi) $d_H^R(v) - d_H^B(v) \leq f(v) - d_F(v)$ and $d_H^R(v) \pm d_H^B(v)$ is even for all $v \in V(F)$.

This may not be the desired augmenting-walk. We start to build a walk from an arbitrary vertex of $V(F') - V(F)$, say x. If x has a neighbor in $V(F') - V(F)$ in H, then this red edge is an augmenting-walk. Otherwise, we choose any red edge and its endvertex in $V(F)$ to continue the walk. When the next edge is selected to go further, only one rule has to be applied. If it is possible then red edges and blue edges are used alternately, otherwise we continue on an edge of the same colour. So, if we arrive to a vertex u through red edge and there is an unused blue edge incident with u then we continue on this blue edge and vice versa.

Since all vertices of $V(F)$ have even degree in H, this path will reach a vertex of $V(F') - V(F)$. If this vertex is different from x then we obtain the desired augmenting-walk. Properties (i) and (ii) clearly hold. The validity of (iii) follows from (vi) and the rule of the construction of the walk.

If we get to x again, then deleting the edges of this closed walk keeps the properties of H, so the existence of the desired walk can be proved by induction on the number of edges incident with vertices of $V(F') - V(F)$. □

Therefore, if we can give an algorithm which finds an augmenting-walk, then we can construct an algorithm to find a maximum $(1, f)$–odd subgraph in a natural way.

9.5 The algorithm

The algorithm is a generalization of Edmonds's algorithm for finding a maximum matching. We will adopt the terminology and notation used in [31] to describe the Edmonds algorithm.

The main idea is to search for augmenting-paths (augmenting-walks with maximum degree 2) with a Breadth First Search style method. If no augmenting-path is found then try to shrink some cycles. If this fails, too, then we have a maximum $(1, f)$–odd subgraph.

We define a basic structure maintained by the algorithm. Suppose we have a $(1, f)$–odd subgraph F of G which is a forest and whose edges are called blue edges. The vertices of $V(F)$ are said to be F-*covered* and the vertices of $V(G) - V(F)$ are said to be F-*exposed*. Recall that a vertex v of F is said to be *saturated* if $d_F(v) = f(v)$. Let r be a fixed F–exposed vertex. We build up a rooted tree T with root r, where the root lies on the top and its ancestors lie below, such that red edges connect r and some components of F to each other forming a tree. This tree will satisfy the following properties (see Fig. 9.4):

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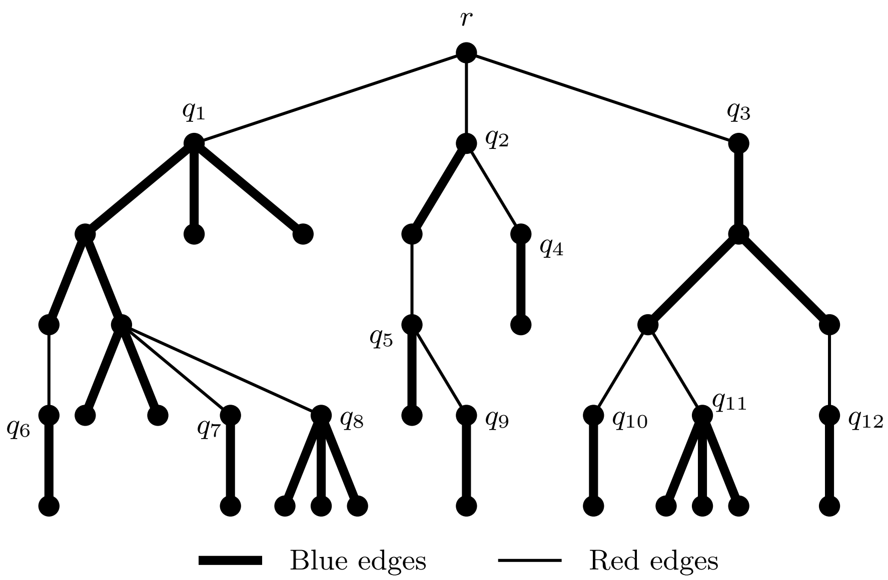

Figure 9.4: Alternating tree

a) The blue degree of r is even (initially zero).

b) The blue degree of every $v \in V(T) - \{r\}$ is odd and therefore > 0.

c) If T contains a vertex v of F, then it contains the whole component of F containing v.

d) If a vertex $v \in V(T) - \{r\}$ is saturated, then no red edge goes downward from v, but it may happen that a red edge goes upward from v.

This tree is called an F–**alternating tree** for similarity to [31]. These properties guarantee that the path from r to any vertex can be the first part of an F–augmenting-path. If T contains a component T_i of F then let q_i denote the unique vertex of T_i which is closest to r. We also maintain a variable $s(t)$ for each vertex $t \in T$:

$$s(t) = \begin{cases} SAT, & \text{if } t = q_i \text{ for some } i \text{ and } t \text{ is saturated} \\ UNSAT, & \text{if } t = q_i \text{ for some } i \text{ and } t \text{ is unsaturated} \\ NON, & \text{otherwise} \end{cases}$$

It seems that we loose information by forgetting the actual f values, but in one augmentation step we will not increase the degrees of the $(1, f)$-odd subgraph by more that 2 in any unsaturated q_i, and we will not increase it at all in other vertices. After the augmentation step the function s is recalculated from the original f.

Let $A(T)$ denote the subset of all vertices t for which $s(t) = SAT$, and let $B(T) = V(T) - A(T)$. In the algorithm, basic subroutines will be used which correspond to extension, augmentation and shrinking in the Edmonds algorithm.

Use vw to extend T

Input: A $(1, f)$–odd subgraph F of G, an F–alternating tree T, and an edge vw of G such that $v \in B(T)$, $w \notin V(T)$ and w is F–covered.

Action: Let T_i be the component of F containing w. (Note that none of its vertices are in $V(T)$.) Replace T by the tree having edge–set $E(T) \cup \{vw\} \cup E(T_i)$, set $q_i = w$ and set $s(w)$ according to the relation of $f(w)$ and $d_F(w)$. For any other vertex $u \in V(T_i)$ set $s(u) = NON$.

It is trivial that this step maintains the properties of the F–alternating tree. It is used to extend some of the alternating-paths.

Use vw to augment F

Input: A $(1, f)$–odd subgraph F of G, an F–alternating tree T, and edge vw of G such that $v \in B(T)$, $w \notin V(T)$ and w is F–exposed.

Action: Let P be the path obtained by attaching vw to the path from r to v in T. (Note that P is an F–augmenting-walk.) Replace F by $F' = F \triangle P$.

In this step a larger $(1, f)$–odd subgraph is found, we will start to build a new alternating tree after this.

For the contracting step a new notation is needed. Also some extra manipulation is necessary when the length of the cycle is even. C be a cycle and $G' = G/C$ be the graph obtained from G by contracting C, as follows. Let $V(C) = \{k_1, k_2, \ldots, k_l\}$. If the length l of C is odd, then G/C has vertex set $(V(G) - V(C)) \cup \{c\}$, the edge set is obtained by deleting all edges in $\langle V(C) \rangle_G$; all vertices of $V(C)$ are identified with the new vertex c, so all edges with one end on the cycle will have that end at c. This new vertex c is called a **pseudo-vertex**.

If the length of C is even, then first we contract C in the same way as above, then we attach an extra vertex c' to c with a new blue edge cc'. So $V(G') = (V(G) - V(C)) \cup \{c, c'\}$, $E(G') = E(G) - E(\langle V(C) \rangle_G) \cup cc'$ and $d_{G'}(c') = 1$. This extra edge is called a **dummy edge**. The variables $s(c)$ and $s(c')$ are defined in the following procedure.

Use vw to contract a cycle

Input: A $(1, f)$–odd subgraph F of G, an F–alternating tree T, and edge vw of G such that $v, w \in B(T)$.

Action: Let C be the cycle formed by vw together with the path in T from v to w, and let z be the unique vertex of C on the highest level of the tree. If the length of C is odd then replace G by $G' = G/C$, F by $F' = F - E(C)$ and T by the tree T' of G' having edge-set $E(T) - E(C)$. If the length of C is even then do the same, but add the dummy edge cc' to F' and T', too.

Finally set $s(c) := s(z)$, but reset $s(c) = UNSAT$ if both edges of C incident with z are blue, and set $s(c') = NON$.

Lemma 9.5.1. *The tree T' obtained in the above step is an F'-alternating tree in G'.*

Proof. To prove that properties a) and b) hold, it must be shown that $d^B_{T'}(c) - d^B_T(z)$ is even. Let us first deal with the case when the length of C was odd and $z \neq r$.

In this case the sum of the blue degrees of all vertices of C is odd, since each of them is odd. The blue edges of C are counted twice in this sum. Thus $d^B_{T'}(c)$ is odd since it is the number of edges incident with a vertex on C minus twice the number of blue edges of C. Hence $d^B_{T'}(c) - d^B_T(z)$ is even, because $d^B_T(z)$ is also odd.

In the other case, when C is an even cycle and $z \neq r$, a similar argument shows that $d^B_{T'}(c)$ is odd, since we have attached the dummy-edge to c. Hence $d^B_{T'}(c) - d^B_T(z)$ is even again.

If $z = r$ then the proof works in a similar way.

Property c) trivially holds. To prove that property d) holds, we show that $s(c) \neq SAT$. The only way to obtain $s(c) = SAT$ is that $s(z) = SAT$. However, in this case no red edge goes downwards from z, therefore both edges of C incident to z must be blue. But then $s(c)$ must be set to $UNSAT$. $\square$

After this step we continue with several other extension and contracting steps, until we find an augmenting-path. We need to show now, that if there is an augmenting-walk in the contracted graph then there is one in the original graph. Unfortunately, this is not true in general, only for ”elementary augmenting-walks”.

An augmenting-walk W is called *elementary* if for every vertex v of the walk $d^R_W(v) \leq 2$, $d^B_W(v) \leq 2$ and if $d^R_W(v) = 2$ then it contains the edge connecting v to its parent. This implies that for any internal vertex v of the walk either

a) $d^R_W(v) = d^B_W(v) = 1$ or

b) $d^R_W(v) = 0, d^B_W(v) = 2$ or

c) $d^R_W(v) = d^B_W(v) = 2$ or

d) $d^R_W(v) = 2, d^B_W(v) = 0$, but this is only possible if $s(v) = UNSAT$.

The algorithm might do a few contracting steps until it finds a augmenting-walk in the actual alternating-tree. Fortunately, an augmenting-walk in an alternating-tree is always elementary, so by the next Lemma, if we can augment in the contracted graph, we can "blow up" the augmenting-walk into an elementary augmenting-walk in the original, uncontracted graph.

It is enough to show the following for one contracting step.

Lemma 9.5.2. *Let F be a $(1, f)$-odd subgraph of G and let $G' = G/C$ and F' be the $(1, f)$-odd subgraph of G' obtained during the contracting procedure from F. If G' contains an elementary F'-augmenting-walk W', then G contains an elementary F-augmenting-walk W.*

Proof. We consider the following cases and subcases:

Case i) *c is not the root of G', and the length of C is odd.*

a) $d^R_{W'}(c) = 1$, $d^B_{W'}(c) = 1$
This means that W' contains one blue and one red edge incident to c in the contracted graph. In the original graph these two edges have one end on the cycle C. Let k_i and k_j be the vertices of the cycle which are the endvertices of the red and blue edges, respectively. If $i = j$ then let $W = W'$, it is clear that this is elementary, too. If $i \neq j$ then W' is broken into two parts. These two parts can be connected using the edges of C in the following way.

If at least one of the edges of C incident to k_i is blue, then take that arc of C connecting k_i to k_j which starts with this blue edge. We have to show that the resulting walk is an elementary augmenting-walk.

The conditions are surely satisfied outside of C. It is also satisfied in k_i since $d^R_W(k_i) = 1$, $d^B_W(k_i) = 1$. For k_j we have $d^B_W(k_i) \geq 1$, so the conditions hold regardless of the colour of the other end of the arc.

In the other vertices of the arc, there is only one way to violate the augmenting-walk conditions: if there are two red edges incident to a saturated vertex. However, this is not possible, since saturated vertices in the alternating tree cannot have two red edges incident with them.

If there are two red edges incident to an unsaturated vertex then it will be an augmenting-walk, but we have to show that it is elementary. The only nontrivial part is that one of the red edges must be the edge connecting the vertex to its parent. This follows from the procedure of obtaining C. All but one edge of C are edges of the augmenting tree, so they connect a vertex to its parent. The exceptional edge is the one which "closed" the cycle before contracting.

In the other case, when both edges of C incident to k_i are red, take any of the two arcs connecting k_i to k_j. By the previous argument the conditions for being elementary augmenting path are satisfied in k_i, too.

b) $d^R_W(c) = 0$, $d^B_W(c) = 2$
Using the notation and the observations of the previous case it is easy to see, that we can take any of the two arcs of C connecting k_i to k_j.

c) $d^R_W(c) = 2$, $d^B_W(c) = 2$
Let k_i and k_j be the vertices of the cycle which are the endvertices of the red edges, and k_n and k_m the ones of the blue edges. Since W' is elementary and $d^R_W(c) = 2$ one of the red edges connects c to its parent. This edge in the original

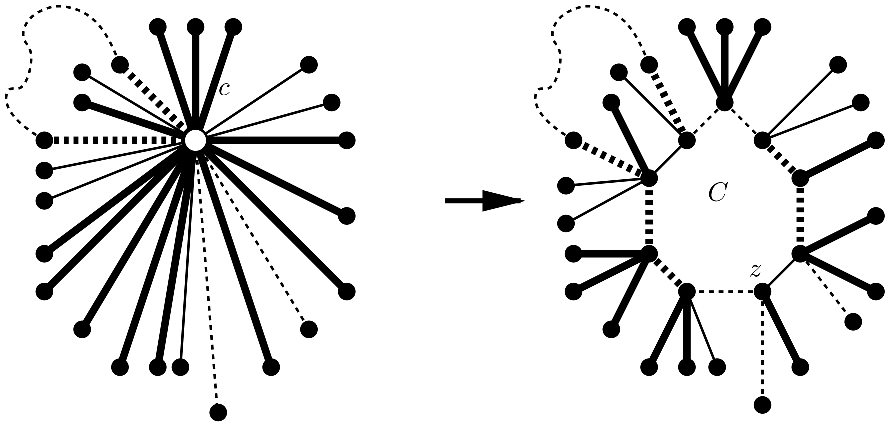

Figure 9.5: Blowing up c to C.

graph must have the same property, so w. l. o. g. we may assume that $k_i = z$. Also it may be assumed that they are four different vertices. If not, the proof is similar.

If at least one of the edges of C incident to k_j is blue, then take that arc of C connecting k_j to the next vertex on the cycle among z, k_n and k_m which starts with this blue edge. Then connect the remaining two vertices with the arc between them. (See Fig. 9.5.) Using the observations of case a) it can be proved that the resulting W is an elementary augmenting-walk, only two aspects need more attention.

Since $d_W^R(c) = 2$, we have $s(c) = UNSAT$, but it would be possible that $s(z) = SAT$ and $d_W^R(z) = 2$, violating the conditions for the augmenting-walk. However, by the algorithm, the only reason to set $s(c) = UNSAT$ is that the two edges of the cycle incident to z are blue, so $d_W^R(z) = 2$ is impossible.

The other problem is that it might happen that we connect k_i, k_j, k_n, k_m in a wrong way: we obtain a separate augmenting-walk and a cycle (or cyclic walk). In this case we only keep the augmenting-walk part, and forget about the cycle.

d) $d_W^R(c) = 2$, $d_W^B(c) = 0$
This case can be handled with similar methods to the above cases.

Case ii) *c is not the root of G', and the length of C is even.*

The argument is similar to the above. The dummy edge added to c is only for preserving parity. It is never used in the augmenting-walk, so the proof is the same as in Case i).

Case iii) *c is the root of G'.*

In this case the difference is that W' starts in c, but it is easy to construct W using the above methods. $\square$

We are now prepared to give the algorithm of building an F-alternating tree. Fix an F-exposed vertex r and start to build an F-alternating tree from it by using the extension and contracting steps. If augmentation is possible then enlarge F, choose a new F-exposed vertex r.

Build an F-alternating tree T.

Input: A $(1, f)$-odd subgraph F of G and an F-exposed vertex r.

Action:

Let $H := G$ and $T := (\{r\}, \emptyset)$;
(Later H may become a contracted version of G.)
While $\exists\, vw \in E(H)$ with $v \in B(T), w \notin A(T)$ do {
    Case: $w \notin V(T)$, w is F-exposed
        Use vw to construct an elementary F–augmenting-walk;
        Extend this into an elementary F–augmenting-walk in G using Lemma 9.5.2
and use it to augment F to obtain F' in G;
        Replace F by F', H by G;
        If $\nexists$ an F-exposed vertex then ***Stop***
            else replace T by $(\{r\}, \emptyset)$ where r is F-exposed;
    Case: $w \notin V(T)$, w is F-covered
        Use vw to extend T;
    Case: $w \in B(T)$
        Use vw to contract a cycle, update H, F and T;
    }

When the above procedure terminates then there is no vw edge with $v \in B(T), w \notin A(T)$. At this point it must be shown that we have a maximum $(1, f)$-odd subgraph. In fact, we can prove that a maximum $(1, f)$-odd subgraph cannot cover all vertices of the tree T which was obtained at the termination of the algorithm.

Lemma 9.5.3. *For the F-alternating tree T*

$$\mathrm{odd}(T - A(T)) \geq 1 + \sum_{v \in A(T)} f(v)$$

holds.

Proof. It is clear that at the termination of the algorithm every red edge of T has one end in $A(T)$, since otherwise both of its ends would be in $B(T)$ and we could use the contracting step. Hence all red edges are deleted in $T - A(T)$, so we can restrict our argument to the blue edges of T.

Next we prove that all children of all vertices in $A(T)$ are in different odd components of $T - A(T)$. By the definition of $A(T)$, it is clear that no child of $v \in A(T)$ is in $A(T)$, in other words $A(T)$ is a stable set, since red edges of T are neglected. It is clear that the only path between two distinct vertices v and u of T must contain either the parent of v or the parent of u. If v and u are both children of some vertices in $A(T)$, then this means that $T - A(T)$ does not contain a path between u and v. The blue degree of all vertices of T except the root r is odd, so if we delete all the vertices of $A(T)$, which are the roots of the blue subtrees of T, then the blue degree of the children of vertices in $A(T)$ decreases by one, but the blue degree of the other vertices does not change. So if we consider u which is a child of $v \in A(T)$, then the component containing u will contain several vertices with odd blue degree and one vertex, u, with even blue degree. Thus, by parity reasons, the number of vertices in the component of u must be odd.

By the same argument and by the definition of $A(T)$ it can be shown that r is in a different component from the above ones and that this component is also odd.

It remains to show that the number of children of a vertex $v \in A(T)$ is $f(v)$. This follows again from the definition of $A(T)$, since it is the set of saturated roots of blue trees. Therefore, the claim of the lemma follows from counting the total number of components containing a child of a vertex in $A(T)$ and the additional component containing r. □

When we have concluded that the vertices of T cannot be all covered by a maximum $(1, f)$-odd subgraph, then remove all vertices and edges of the F-alternating tree from G. Fix a new F-exposed vertex r in the remaining graph and repeat the above process until there are no F-exposed vertices. A more formal description of the algorithm is the following.

Algorithm to find a maximum $(1, f)$-odd subgraph

Input: G and a function $f : V(G) \mapsto \{1, 3, 5, \ldots\}$

Action:

Set $H = G$ and $F = F^* = A^* = U^* = \emptyset$.
While $\exists$ F-exposed vertex ***do*** {
  Choose an F-exposed vertex r of H and put $T = (\{r\}, \emptyset)$;
  Build an F-alternating tree from r
  Extend F to a $(1, f)$-odd subgraph F' of G using Lemma 9.5.2,
    replace F^* by $F^* \cup (F' \cap T)$, A^* by $A^* \cup A(T)$
    and U^* by $U^* \cup \{r\}$.
  Remove all vertices and edges of T from G, set $F = \emptyset$.
  }
Replace F^* by $F^* \cup F$.
Return F^*, a maximum $(1, f)$-odd subgraph,
  U^* the set of uncovered vertices and
  A^* which proves that (9.3) holds, so F^* is maximum.

Theorem 9.5.4 (Kano, Katona [K7]). *The above algorithm terminates in* $O(|V(G)|^3)$ *steps, and it gives a smallest maximum* $(1, f)$*-odd subgraph.*

Proof. It is clear that each augmentation step decreases the number of F-exposed vertices, so there will be $O(|V(G)|)$ augmentation steps. Between augmentations, each contracting step decreases the number of vertices in G' while not changing the number of vertices not in T, and each extension step decreases the number of vertices not in T while not changing the number of vertices in G'. Hence the total number of these steps between augmentations is $O(|V(G)|)$. On the other hand, it is easy to see, that between two augmentation step every edge of the graph is scanned at most once. Hence, we may conclude, that total number of steps is $(|V(G)|^3)$.

It was shown above that F remains a smallest $(1, f)$-odd subgraph throughout, so it remains to prove that F^* is maximal. Each time the tree T is removed from the graph the only uncovered vertex, which is removed, is the root of the tree r, all other removed vertices are covered by F. So, if the while loop is executed i times, then F^* covers $|V(G)| - i$ vertices of the graph. On the other hand we can apply Lemma 9.5.3 for each removed T. It is easy to see, that there are no edges in $G - A^*$ between different T trees, hence

$$\mathrm{odd}(G - A^*) = \sum_T \mathrm{odd}(T - A(T)) \geq \sum_T \left(1 + \sum_{v \in A(T)} f(v)\right) = i + \sum_{v \in A^*} f(v)$$

holds. By Theorem 9.3.1 this proves that F^* is maximum. □

Remark. It is not too difficult to show, that the above algorithm gives the sets $A(G)$, $C(G), D(G)$ of Theorem 9.3.4, as well. Namely, $A(G) = A^*$, odd components of $G - A(G)$ form $D(G)$ and even components of $G - A(G)$ form $C(G)$.

Chapter 10

Elementary graphs with respect to f-parity factors

10.1 Introduction

In this Chapter we deal with a special case of the *degree prescribed subgraph problem*, introduced by Lovász [87]. This is as follows. Let G be an undirected graph and let $\emptyset \neq \mathcal{H}_v \subseteq \mathbb{N} \cup \{0\}$ be a degree prescription for each $v \in V(G)$. For a spanning subgraph F of G, define $\delta^F_{\mathcal{H}}(v) = \min\{|d_F(v) - i| : i \in \mathcal{H}_v\}$, and let

$$\delta^F_{\mathcal{H}} = \sum_{v \in V(G)} \delta^F_{\mathcal{H}}(v) \qquad \text{and} \qquad \delta_{\mathcal{H}}(G) = \min_F \delta^F_{\mathcal{H}},$$

where the minimum is taken over all the spanning subgraphs F of G. A spanning subgraph F is called *$\mathcal{H}$-optimal* if $\delta^F_{\mathcal{H}} = \delta_{\mathcal{H}}(G)$, and it is an *$\mathcal{H}$-factor* if $\delta^F_{\mathcal{H}} = 0$, i.e., if $d_F(v) \in \mathcal{H}_v$ for all $v \in V(G)$. The *degree prescribed subgraph problem* is to determine the value of $\delta_{\mathcal{H}}(G)$.

Lovász [89] gave a structural description on the degree prescribed subgraph problem in the case where $\mathcal{H}_v$ has no two consecutive gaps for all $v \in V(G)$. The first polynomial time algorithm was given by Cornuéjols [32]. It is implicit in Cornuéjols [32] that this algorithm implies a Gallai–Edmonds type structure theorem for the degree prescribed subgraph problem (first stated in [118]), which is similar to – but in some respects much more compact than – that of Lovász'.

The special case when an odd value function $f : V(G) \to \mathbb{N}$ is given and $\mathcal{H}_v = \{1, 3, 5, \ldots, f(v)\}$ for all $v \in V(G)$, is the *$(1, f)$-odd subgraph problem*. We already presented results about these in the previous Chapters.

In the present Chapter we show a new approach to the $(1, f)$-odd subgraph problem. Actually, it is worth allowing f to have also even values and defining $\mathcal{H}_v$ equal to $\{1, 3, \ldots, f(v)\}$ or $\{0, 2, \ldots, f(v)\}$, according to the parity of $f(v)$. We call this the *f-parity subgraph problem*. We show an easy reduction of the f-parity subgraph problem to the matching problem, and we show that this reduction easily yields the above mentioned Gallai–Edmonds and Berge type theorems on the f-parity subgraph problem. Then we investigate barriers with respect to the f-parity subgraph problem. As another application, we explore the graphs for which the edges belonging to some f-parity factor form a connected spanning subgraph. We call such a graph an *f-elementary graph*. We generalize some results on matching elementary graphs (proved by Lovász [90]) to f-elementary graphs. An attempt putting the f-parity subgraph problem into the general context of graph packing problems can be found in [119].

The f-parity subgraph problem can be reduced to the $(1, f)$-odd subgraph problem by the following construction: for every vertex $v \in V(G)$ with $f(v)$ even, connect a new vertex w_v to v in G, define $f(w_v) = 1$ and increase $f(v)$ by 1. Now $\delta_f(G)$ remains the same.

To avoid minor technical difficulties we assume that $f > 0$. Almost all results would hold without this restriction, too. Note that if G is a nontrivial f-elementary graph then $f > 0$ always holds.

The constant function $f \equiv 1$ is simply denoted by $\mathbf{1}$. $f(X) = \sum\{f(x) : x \in X\}$ and χ_X denote the function with $\chi_X(x) = 1$ if $x \in X$ and $\chi_X(x) = 0$ otherwise. The number of components of a graph G is denoted by $\omega(G)$. Two isomorphic graphs H and K are denote by $H \simeq K$. In this Chapter the graphs are finite and undirected, have no loops, but may have multiple edges.

In Section 10.2 we show a reduction of the f-parity subgraph problem to matchings, which will then be used to prove the Gallai–Edmonds type structure theorem on the f-parity subgraph problem. Then we prove some properties of barriers. In Section 10.3 we prove several results on f-elementary graphs. These appeared in [K9].

10.2 Reduction to matchings

First we define an auxiliary graph.

Definition 10.2.1. *For a graph G and a function $f : V(G) \to \mathbb{N}$, define G^f to be the following undirected graph. Replace every vertex $v \in V(G)$ by a new complete graph on $f(v)$ vertices, denoted by $K_{f(v)}$, and for each pair of vertices $u, v \in V(G)$ adjacent in G, add all possible $f(u)f(v)$ edges between $K_{f(u)}$ and $K_{f(v)}$. Let $V_{f(v)} = V(K_{f(v)})$.*

Observe that $G^{\mathbf{1}} = G$, $|V(G^f)| = f(V(G))$ and that $V_{f(v)} \neq \emptyset$ for every $v \in V(G)$. There is a strong connection between the maximum matchings of G^f and the optimal f-parity subgraphs of G. Note that the size of a maximum matching of G is just $|V(G)| - \delta_{\mathbf{1}}(G)$.

Lemma 10.2.2. *For every optimal f-parity subgraph F of G, there exists a matching M of G^f such that $|V(M)| = f(V(G)) - \delta_f^F$. Moreover, if $d_F(w) \in \{\ldots, f(w) - 3, f(w) - 1\}$ for a vertex $w \in V(G)$ then M can be chosen to miss a prescribed vertex $x \in V_{f(w)}$. On the other hand, for every maximum matching M of G^f there exists a spanning subgraph F of G such that $\delta_f^F = f(V(G)) - |V(M)|$. Moreover, if M misses a vertex in $V_{f(w)}$ for some $w \in V(G)$ then F can be chosen such that $d_F(w) \in \{\ldots, f(w) - 3, f(w) - 1\}$. In particular, $\delta_f(G) = \delta_{\mathbf{1}}(G^f)$.*

Proof. Let F be an optimal f-parity subgraph of G. If $d_F(u) > f(u)$ for some $u \in V(G)$ then clearly $\delta_f^{F'} \leq \delta_f^F$ for the spanning subgraph F' that is obtained from F by deleting an edge e incident to u. As F is f-parity optimal, e is not adjacent to w, so $d_{F'}(w) = d_F(w)$. Hence we assume that $d_F(v) \leq f(v)$ for every vertex v. Now it is easy to construct from F a matching of G^f missing exactly δ_f^F vertices, one in each $V_{f(v)}$ for the vertices v with $d_F(v) \not\equiv f(v) \bmod 2$. If w is such a vertex then M can be chosen to miss a prescribed vertex $x \in V_{f(w)}$.

For the second part, let M be a maximum matching of G^f. If M contains two edges between $K_{f(u)}$ and $K_{f(v)}$ for some $u, v \in V(G)$, then replace them by two edges, one inside $K_{f(u)}$ and the other one inside $K_{f(v)}$. Thus we may assume that M contains at most one edge between $K_{f(u)}$ and $K_{f(v)}$ for all distinct $u, v \in V(G)$. By contracting each $K_{f(u)}$ to one vertex u, we get a spanning subgraph F of G with $\delta_f^F = f(V(G)) - |V(M)|$. Moreover, $d_F(w) \in \{\ldots, f(w)-3, f(w)-1\}$ in the case that M misses a vertex in $K_{f(w)}$. $\square$

We define critical graphs with respect to the f-parity subgraph problem as in the matching case. If $f = \mathbf{1}$ the graphs defined below are called *factor-critical.*

Definition 10.2.3. *Given a graph G and a function $f : V(G) \to \mathbb{N}$, G is called f-critical if for every $w \in V(G)$ there exists a spanning subgraph F of G such that $d_F(w) \in \{\ldots, f(w)-3, f(w)-1\}$ and $d_F(v) \in \{\ldots, f(v)-2, f(v)\}$ for all $v \in V(G) - \{w\}$.*

By Lemma 10.2.2, G is f-critical if and only if G^f is factor-critical. The Gallai–Edmonds structure theorem for the f-parity subgraph problem follows from the classical Gallai–Edmonds theorem easily.

A direct generalization of the above result is the version for the f-parity subgraph problem, which is also a generalization of Theorem 9.3.4.

Theorem 10.2.4 (Kano, Katona, Szabó [K9], [118])**.** *Let G be a graph and $f : V(G) \to \mathbb{N}$ be a function. Let $D_f \subseteq V(G)$ consist of those vertices v for which there exists an optimal f-parity subgraph F of G with $d_F(v) \in \{\ldots, f(v)-3, f(v)-1\}$. Let $A_f = N(D_f)$ and $C_f = V(G) - (D_f \cup A_f)$. Then*

1. *every component of $G[D_f]$ is f-critical,*
2. *$|\{K : K$ is a component of $G[D_f]$ adjacent to $A'\}| \geq f(A') + 1$ for all $\emptyset \neq A' \subseteq A_f$,*
3. *$\delta_f(G) = \omega(G[D_f]) - f(A_f)$,*
4. *$G[C_f]$ has an f-parity factor.*

Proof. Take the classical Gallai–Edmonds decomposition $V(G^f) = D \cup A \cup C$ of G^f. By symmetry, if $V_{f(v)}$ meets D then $V_{f(v)} \subseteq D$. These vertices $v \in V(G)$ form D_f by Lemma 10.2.2. The other properties follow from the construction and from Lemma 10.2.2. $\square$

This proof implies:

Lemma 10.2.5. *For $X = D, A, C$, it holds that $X_f(G) = \{v \in V(G) : V_{f(v)} \subseteq X(G^f)\}$.*

From Theorem 10.2.4 the Berge type minimax formula on the f-parity subgraph problem follows easily.

Definition 10.2.6. *A component K of G is called f-odd or f-even when $f(V(K))$ is odd or even. Let f-odd(G) denote the number of f-odd components of G. For $Y \subseteq V(G)$, let $\mathrm{def}_f(Y) = f\text{-odd}(G-Y) - f(Y)$.*

Theorem 10.2.7 (Kano, Katona, Szabó [K9])**.** *For a graph G and a function $f : V(G) \to \mathbb{N}$, it follows that*

$$\delta_f(G) = \max\{\mathrm{def}_f(Y) : Y \subseteq V(G)\}.$$

Proof. Let $Y \subseteq V(G)$. Since an f-odd component K of $G-Y$ has no f-parity factor, it follows that $\delta_f^F \geq f\text{-}odd(G-Y) - f(Y) = \operatorname{def}_f(Y)$ for every spanning subgraph F of G, and thus $\delta_f(G) \geq \operatorname{def}_f(Y)$.

By virtue of Theorem 10.2.4 and by the fact that every f-critical component of $G - A_f$ is f-odd, we have

$$\delta_f(G) = \omega(G[D_f]) - f(A_f) = f\text{-}odd(G - A_f) - f(A_f) = \operatorname{def}_f(A_f).$$

Hence the theorem holds. □

Now we show how to use this approach to analyze barriers.

Definition 10.2.8. *A set $Y \subseteq V(G)$ is called an* f-barrier *if* $\operatorname{def}_f(Y) = \delta_f(G)$.

As f-critical graphs are f-odd, the canonical Gallai–Edmonds set A_f is an f-barrier. A $\mathbf{1}$-barrier is just an ordinary barrier in matching theory. One can observe that if $Y \subseteq V(G^f)$ satisfies $V_{f(v)} \cap Y \neq \emptyset$ and $V_{f(v)} \setminus Y \neq \emptyset$, then $V_{f(v)} \cap Y$ is adjacent to only one component of $G^f - Y$. Moreover, if Y is a barrier in G^f then each $X \subseteq Y$ is adjacent to at least $|X|$ odd components of $G^f - Y$ since otherwise $\operatorname{def}_{\mathbf{1}}(Y - X) > \operatorname{def}_{\mathbf{1}}(Y)$, which is impossible. Hence if Y is a barrier in G^f then $|Y \cap V_{f(v)}| \in \{0, 1, f(v)\}$ for all $v \in V(G)$. It also follows that if $|Y \cap V_{f(v)}| = 1$ and $V_{f(v)} \setminus Y \neq \emptyset$ then $Y \setminus V_{f(v)}$ is a barrier of G^f. Thus if Y is a barrier of G^f then $Y' = \{v \in V(G) : V_{f(v)} \subseteq Y\}$ is an f-barrier of G. On the other hand, if Y' is an f-barrier of G then $\bigcup\{V_{f(v)} : v \in Y'\}$ is clearly a barrier of G^f. The canonical Gallai–Edmonds barrier $A(G^f)$ of G^f has this form.

Definition 10.2.9. *An f-barrier Y of G is called strong if the f-odd components of $G - Y$ are f-critical.*

It is obvious that A_f is a strong f-barrier. Since a graph K is f-critical if and only if K^f is factor-critical, we have the following.

Observation 10.2.10. *A set $Y \subseteq V(G)$ is a strong f-barrier in G if and only if $\bigcup\{V_{f(v)} : v \in Y\}$ is a strong $\mathbf{1}$-barrier in G^f.*

Király proved that the intersection of strong $\mathbf{1}$-barriers is also a strong $\mathbf{1}$-barrier [80]. This result holds for the f-parity subgraph problem as well.

Theorem 10.2.11 (Kano, Katona, Szabó [K9])**.** *The intersection of strong f-barriers is a strong f-barrier.*

Proof. Let Y_1, Y_2 be strong f-barriers of G. Then $Y_i' = \bigcup\{V_{f(v)} : v \in Y_i\}$ are strong $\mathbf{1}$-barriers of G^f, hence their intersection, which is just $\bigcup\{V_{f(v)} : v \in Y_1 \cap Y_2\}$, is also a strong $\mathbf{1}$-barrier by [80]. Thus $Y_1 \cap Y_2$ is a strong f-barrier of G. □

By Tutte's theorem, maximal barriers for matching are strong. This remains true for f-barriers, too. Indeed, let Y be a maximal f-barrier of G and K be an f-odd component of $G - Y$. Then K has no f-parity factor, and so $C_f(K) \neq V(K)$ in its canonical Gallai–Edmonds decomposition. Hence either $D_f(K) = V(K)$ or $A_f(K) \neq \emptyset$. In the first case K is f-critical by Theorem 10.2.4, and in the second case $Y \cup A_f(K)$ would be a larger f-barrier than Y, which is impossible. Thus all f-odd components of $G - Y$ are f-critical, implying that Y is strong.

In the matching case, it holds that the canonical Gallai–Edmonds barrier A is the intersection of all maximal barriers. This fails for the general case: take a triangle together with a pendant vertex w of degree 1, and define $f \equiv d$. Then this graph is of order four and has an f-parity factor, which is a whole graph, and $A_f = \emptyset$. But it has exactly one nonempty barrier $\{w\}$.

However, the fact that in matchings the canonical Gallai–Edmonds barrier A is the intersection of all strong barriers remains true for f-parity subgraphs by Observation 10.2.10 and the fact that A_f itself is strong.

10.3 f-elementary graphs

In this section we generalize some results on elementary graphs, obtained in Lovász [90], to the f-parity case.

Definition 10.3.1. *Let G be a connected graph and $f : V(G) \to \mathbb{N}$. An edge $e \in E(G)$ is said to be f-*allowed *if G has an f-parity factor containing e. Otherwise e is f-forbidden. The graph G is said to be f-elementary if the f-allowed edges induce a connected spanning subgraph of G. The graph G is weakly f-elementary if G_2 is f-elementary, where G_2 is the graph obtained from G by replacing every edge $e \in E(G)$ by two parallel edges.*

A $\mathbf{1}$-elementary graph is briefly called *elementary*. An f-elementary graph is weakly f-elementary, but not vice versa: $G = K_2$ with $f \equiv 2$ is weakly f-elementary but not f-elementary. These classes coincide if $f = \mathbf{1}$. Lemma 10.3.2 justifies why we introduced the weak version of f-elementary graphs.

Lemma 10.3.2. *G^f is elementary if and only if G is weakly f-elementary.*

Proof. Let M be a perfect matching of G^f. If M contains at least three edges between $K_{f(u)}$ and $K_{f(v)}$ for some $u, v \in V(G)$, then replace two of them by another two edges, one inside $K_{f(u)}$ and the other one inside $K_{f(v)}$. So the number of edges of M between $K_{f(u)}$ and $K_{f(v)}$ decreased by 2. This construction shows that if G^f is elementary then G is weakly f-elementary.

On the other hand, if G is weakly f-elementary then G^f is clearly elementary. □

The $f = \mathbf{1}$ special cases of the following two theorems can be found in Lovász and Plummer [92] (Theorems 5.1.3 and 5.1.6). Using our reduction, these special cases together with Lemmas 10.2.5 and 10.3.2 imply both Theorem 10.3.3 and 10.3.4.

Theorem 10.3.3 (Kano, Katona, Szabó [K9]). *A graph G is weakly f-elementary if and only if $\delta_f(G) = 0$ and $C_{f-\chi_w}(G) = \emptyset$ for all $w \in V(G)$.*

Proof. A graph G is weakly f-elementary if and only if G^f is elementary by Lemma 10.3.2, and G^f is elementary if and only if $\delta_{\mathbf{1}}(G^f) = 0$ and $C(G^f - x) = \emptyset$ for all $x \in V(G^f)$ ([92], Theorem 5.1.3). Since $\delta_f(G) = \delta_{\mathbf{1}}(G^f)$, it is enough to prove that under the assumption $\delta_f(G) = \delta_{\mathbf{1}}(G^f) = 0$, for every $w \in V(G)$ it follows that

$$C(G^f - x) = \emptyset \text{ for every } x \in V_{f(w)} \iff C_{f-\chi_w}(G) = \emptyset. \tag{10.1}$$

If $f(w) \geq 2$, then $G^f - x \simeq G^{f-\chi_w}$ and so (10.1) follows from Lemma 10.2.5. Thus assume that $f(w) = 1$. As $G^f - x \simeq (G-w)^f$, Lemma 10.2.5 implies that $C(G^f - x) = \emptyset \iff C_f(G-w) = \emptyset$. Hence it suffices to show that

$$C_f(G-w) = \emptyset \quad \iff \quad C_{f-\chi_w}(G) = \emptyset. \tag{10.2}$$

Since $\delta_f(G) = 0$ and $f(w) = 1$, it is easy to see that an optimal $(f - \chi_w)$-parity subgraph of G is either an f-parity factor of G or an optimal f-parity subgraph of $G - w$ enlarged by w as an isolated vertex, and vice versa. Since $(f - \chi_w)(w) = 0$, we have $w \notin D_{f-\chi_w}(G)$ by the definition of $D_{f-\chi_w}(G)$. Thus $D_{f-\chi_w}(G) = D_f(G-w)$. For the edge $e = wu$ of an f-parity factor F of G, $F - e$ is an optimal $(f - \chi_w)$-parity subgraph of G, and hence $u \in D_{f-\chi_w}(G)$ and $w \in A_{f-\chi_w}(G)$. It is immediate that $A_{f-\chi_w}(G) - \{w\} = A_f(G - w)$. Hence (10.2) holds. $\square$

Theorem 10.3.4 (Kano, Katona, Szabó [K9]). *A graph G is weakly f-elementary if and only if f-odd$(G - Y) \leq f(Y)$ for all $Y \subseteq V(G)$, and if equality holds for some $Y \neq \emptyset$ then $G - Y$ has no f-even components.*

Proof. Call $Y \subseteq V(G)$ *f-bad* if either f-*odd*$(G - Y) > f(Y)$ or equality holds here and $G - Y$ has an f-even component. It follows from Lemma 10.3.2 that the graph G is weakly f-elementary if and only if G^f is elementary, which is equivalent to that G^f has no $\mathbf{1}$-bad set ([92], Theorem 5.1.6). So we only have to prove that G has an f-bad set Y if and only if G^f has a $\mathbf{1}$-bad set Y'. If $Y \subseteq V(G)$ is f-bad then $Y' = \bigcup\{V_{f(v)} : v \in Y\}$ is $\mathbf{1}$-bad in G^f. On the other hand, let $Y' \subseteq V(G^f)$ be $\mathbf{1}$-bad in G^f. If $V_{f(v)} \cap Y' \neq \emptyset$ and $V_{f(v)} \setminus Y' \neq \emptyset$ for some $v \in V(G)$ then let $x \in V_v \cap Y'$. Now x is adjacent to only one component of $G^f - Y'$ hence $Y' - x$ is also $\mathbf{1}$-bad. So we can assume that Y' is a union of some $V_{f(v)}$. Now $Y = \{v \in V(G) : V_{f(v)} \subseteq Y'\}$ is f-bad in G. $\square$

In the matching case the existence of a certain canonical partition of the vertex set was revealed by Lovász [90] (Lovász, Plummer [92], Theorem 5.2.2). We cite this result.

Definition 10.3.5. *A set $X \subseteq V(G)$ is called nearly f-extreme if $\delta_{f-\chi_X}(G) = \delta_f(G) + |X|$. Besides, X is f-extreme if $\delta_f(G - X) = \delta_f(G) + f(X)$.*

It is clear that $\delta_{f-\chi_X}(G) \leq \delta_f(G) + |X|$ and $\delta_f(G - X) \leq \delta_f(G) + f(X)$ for every $X \subseteq V(G)$. Nearly $\mathbf{1}$-extreme and $\mathbf{1}$-extreme sets coincide.

Theorem 10.3.6 (Lovász [90]). *If G is elementary then the maximal barriers of G form a partition $\mathcal{S}$ of $V(G)$. Moreover, it holds that*

1. *for $u, v \in V(G)$, the graph $G - u - v$ has a perfect matching if and only if u and v are contained in different classes of $\mathcal{S}$, (hence an edge xy of G is $\mathbf{1}$-allowed in G if and only if x and y are contained in different classes of $\mathcal{S}$),*

2. *$S \subseteq V(G)$ is a class of $\mathcal{S}$ if and only if $G - S$ has $|S|$ components, each factor-critical,*

3. *$X \subseteq V(G)$ is $\mathbf{1}$-extreme if and only if $X \subseteq S$ for some $S \in \mathcal{S}$.*

Lemma 10.3.2 implies the analogue of this result.

Theorem 10.3.7 (Kano, Katona, Szabó [K9]). *If G is weakly f-elementary then its maximal f-barriers form a subpartition $\mathcal{S}'$ of $V(G)$. Call the classes of $\mathcal{S}'$ proper, and add all elements $v \in V(G)$ not in a class of $\mathcal{S}'$ as a singleton class yielding the partition $\mathcal{S}$ of $V(G)$. Now it holds that*

1. *for $u, v \in V(G)$, the graph G has an $(f - \chi_{\{u,v\}})$-parity factor if and only if u and v are contained in different classes of $\mathcal{S}$ (hence an edge xy of G is f-allowed in G_2 if and only if x and y are contained in different classes of $\mathcal{S}$),*
2. *$S \subseteq V(G)$ is a class of $\mathcal{S}'$ if and only if $G - S$ has $f(S)$ components, each f-critical,*
3. *$X \subseteq V(G)$ is nearly f-extreme (f-extreme, resp.) if and only if $X \subseteq S$ for some $S \in \mathcal{S}$ ($S \in \mathcal{S}'$, resp.).*

Proof. Suppose that G is weakly f-elementary. Then G^f is elementary. As we already observed, every barrier Y of G^f satisfies $|Y \cap V_{f(v)}| \in \{0, 1, f(v)\}$ for all $v \in V(G)$. Since G^f is elementary, its maximal barriers form a partition $\mathcal{S}^f$ of $V(G^f)$ by Theorem 10.3.6. Thus if a maximal barrier of G^f contains exactly one vertex x of $V_{f(u)}$ and whole $V_{f(v)}$, then by symmetry, another maximal barrier contains one vertex $y \in V_{f(u)} - \{x\}$ and $V_{f(v)}$, which contradicts the above fact that the maximal barriers form a partition $V(G^f)$. Hence every maximal barrier of G^f is either a union of some $V_{f(v)}$ or a singleton. If Y' is an f-barrier of G then $\bigcup\{V_{f(v)} : v \in Y'\}$ is a barrier of G^f. On the other hand, if Y is a maximal barrier of G^f of the form $\bigcup V_{f(v)}$ then $Y' = \{v \in V(G) : V_{f(v)} \subseteq Y\}$ is clearly a maximal f-barrier of G. So these barriers Y' form the proper classes of $\mathcal{S}$, and for a singleton class $\{v\} \in \mathcal{S} - \mathcal{S}'$, it holds that each vertex $x \in V_{f(v)}$ is a maximal barrier of G^f. Now statement (i) is immediate from Theorem 10.3.6 (i), and (ii) also follows from Theorem 10.3.6 (ii) since if $S \in \mathcal{S}$ and $|S| \geq 2$ then $\bigcup\{V_v : v \in S\} \in \mathcal{S}^f$, and by the fact that a graph K is f-critical if and only if K^f is factor-critical.

Finally, (iii) follows from Theorem 10.3.6 and from the observation that $X \subseteq V(G)$ is f-extreme if and only if G^f has an extreme set X' consisting of one vertex from each V_v, $v \in X$. □

Remark. *It follows from Theorem 10.3.7 that $\mathcal{S}$ could be introduced as the partition $\{X \subseteq V(G) : X$ is a maximal nearly f-extreme set of $G\}$. Besides, if $X \subseteq V(G)$, $|X| \geq 2$ is maximal nearly f-extreme, then X is an f-barrier of G.*

Corollary 10.3.8. *If G is f-elementary then an edge e is f-allowed if and only if e joins two classes of $\mathcal{S}$.*

Proof. Suppose that e joins u to v and let $g = f - \chi_{\{u,v\}}$. By Theorem 10.3.7 (i), we only have to prove that G has a g-parity factor if and only if e is f-allowed. Assume that G has a g-parity factor but e is not f-allowed. (The other direction is trivial.) If $G - e$ had a g-parity factor F then $F + e$ would be an f-parity factor of G, which is impossible. Thus by Theorem 10.2.7 there exists a set $Y \subseteq V(G)$ such that g-$odd(G - e - Y) > g(Y)$. Since G has a g-parity factor, it follows from parity reasons that g-$odd(G - e - Y) = g(Y) + 2$, and e joins two g-odd components Q_1 and Q_2 of $G - e - Y$. But then clearly no edge joining Y to $V(Q_1) \cup V(Q_2)$ is f-allowed in G. Since G is f-elementary, we have $V(G) = V(Q_1) \cup V(Q_2)$ and $Y = \emptyset$, but then e is an f-forbidden cut edge, which contradicts that G is f-elementary. □

Our last subject is generalizing bicritical graphs.

Definition 10.3.9. *Let G be a graph and $f : V(G) \to \mathbb{N}$ be a function. Then G is said to be f-bicritical if G has an $(f - \chi_{\{u,v\}})$-parity factor for all pairs $u, v \in V(G)$.*

Theorem 10.3.10 (Kano, Katona, Szabó [K9])**.** *If G is weakly f-elementary then the following statements are equivalent.*

1. *G is f-bicritical.*
2. *All classes of $\mathcal{S}$ are singletons.*
3. *If $Y \subseteq V(G)$ and $|Y| \geq 2$ then f-odd$(G - Y) \leq f(Y) - 2$.*

Proof. (i) $\Rightarrow$ (ii): Each edge in G_2 is allowed, and thus Theorem 10.3.7 (i) implies the equivalence.

(ii) $\Rightarrow$ (iii) : Assume otherwise. By parity reasons, we have a set $Y \subsetneq V(G)$ with $|Y| \geq 2$ such that f-*odd*$(G - Y) = f(Y)$. So Y is an f-barrier, and is contained in a set $S \in \mathcal{S}$ with $|S| \geq 2$ by Theorem 10.3.7, which contradicts (ii).

(iii) $\Rightarrow$ (i): Assume that G is not f-bicritical. Let $g = f - \chi_{\{u,v\}}$. Then G has no g-parity factor for some $u, v \in V(G)$. Thus there exists a set $Y \subseteq V(G)$ such that g-*odd*$(G - Y) > g(Y)$. Recall that G has an f-parity factor. If u or v belongs to a g-odd component Q of $G - Y$ then Y is an f-barrier of G and Q is an f-even component of $G - Y$, contradicting to Theorem 10.3.4. Hence both u and v belong to Y, thus $|Y| \geq 2$ and f-*odd*$(G - Y) = f(Y)$, a contradiction. $\square$

Chapter 11

Decomposition of a graph into two disjoint odd subgraphs

11.1 Introduction

In this Chapter we mainly consider *multigraphs*, which may have multiple edges but have no loops. Let $\mathrm{EvenV}(G)$ denote the set of vertices of even degree and $\mathrm{OddV}(G)$ denote the set of vertices of odd degree. Recall that for a vertex set U of G, the subgraph of G induced by U is denoted by $G[U]$. For two disjoint vertex sets U_1 and U_2 of G, the number of edges between U_1 and U_2 is denoted by $e_G(U_1, U_2)$. As usual, the degree of v in G is denoted by $d_G(v)$. Moreover, when some edges of G are colored with red and blue, for a vertex v, the number of red edges incident with v is denoted by $d_{\mathrm{red}}(v)$, and the number of red edges in G is denoted by $e_{\mathrm{red}}(G)$. Analogously, $d_{\mathrm{blue}}(v)$ and $e_{\mathrm{blue}}(G)$ are defined.

An *odd (resp. even) subgraph* of G is a subgraph in which every vertex has odd (resp. even) degree. An *odd factor* of G is a spanning odd subgraph of G. It is obvious by the handshaking lemma that every connected multigraph containing an odd factor has even order. This condition is also sufficient as follows.

Proposition 11.1.1 (Problem 42 of §7 in [91]). *A multigraph G has an odd factor if and only if every component of G has even order.*

Moreover, such an odd factor, if it exists, can be found in polynomial time (Problem 42 of §7 in [91]). Consider a connected multigraph of even order on the vertices $v_1, \dots, v_{2m}$ and for any $i \in \{1, 2, \dots, m\}$, fix a path P_i connecting v_i and v_{i+m}. Then the edges appearing odd number of times in the paths $P_1, \dots, P_m$ form an odd factor. The above proposition also follows from the fact that for a tree T of even order, the set

$$\{e \in E(T) : T - e \text{ consists of two odd components}\}$$

forms an odd factor of T.

We say that G can be *decomposed into n odd subgraphs* if its edge set can be partitioned into n (possibly empty) sets $E_1, \dots, E_n$ so that for every $i \in \{1, \dots, n\}$, E_i forms an odd subgraph.

Our main result gives a criterion for a multigraph to be decomposed into two odd subgraphs, and proposes a polynomial time algorithm for finding such a decomposition or showing its non-existence.

We begin with some known results related to ours.

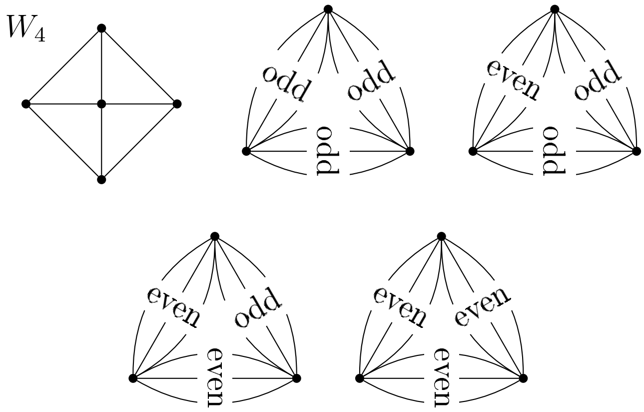

Figure 11.1: The wheel W_4 and the Shannon triangles of type (1,1,1), (2,1,1), (2,2,1), (2,2,2).

Theorem 11.1.2 (Pyber [106]). *Every simple graph can be decomposed into four odd subgraphs.*

This upper bound is sharp, for example, the wheel of four spokes (W_4, see Figure 1) cannot be decomposed into three odd subgraphs. In [95] Mátrai constructed an infinite family of graphs with the same property.

Theorem 11.1.3 (Pyber [106]). *Every forest can be decomposed into two odd subgraphs.*

Theorem 11.1.4 (Pyber [106]). *Every connected multigraph of even order can be decomposed into three odd subgraphs.*

Since every connected multigraph G of even order has an odd factor, if we take an odd factor F with maximum size, then $G - E(F)$ becomes a forest, and it can be decomposed into two odd subgraphs by Theorem 11.1.3. Thus Theorem 11.1.4 follows.

Theorem 11.1.5 (Lužar [93]). *Every connected multigraph can be decomposed into six odd subgraphs. Moreover, with the exception of the Shannon triangles of type* $(2, 2, 2)$ *(see Figure 1), a decomposition into five odd subgraphs always exists.*

The following theorem strengthens both Theorems 11.1.2 and 11.1.5.

Theorem 11.1.6 (Petruševski [105]). *Every connected multigraph can be decomposed into four odd subgraphs except for the Shannon triangles of types* $(2, 2, 2)$ *and* $(2, 2, 1)$ *(see Figure 1).*

We say that G can be *covered by n odd subgraphs* if its edge set can be covered by n sets $E_1, \ldots, E_n$ (not necessarily disjointly) so that for every $i \in \{1, \ldots, n\}$, E_i forms an odd subgraph.

Theorem 11.1.7 (Mátrai [95]). *Every connected simple graph of odd order can be covered by three odd subgraphs.*

In this Chapter we study the decomposability of a multigraph into an even subgraph and an odd subgraph, and into two odd subgraphs. We also remark that the case of decomposing into two even subgraphs is trivial.

Proposition 11.1.8. *A multigraph G can be decomposed into an even subgraph and an odd subgraph if and only if every component of $G[\mathrm{OddV}(G)]$ has even order.*

Proof. Such a decomposition exists if and only if there is an odd factor in $G[\mathrm{OddV}(G)]$, since all edges incident with any vertex of even degree must belong to the even subgraph. So by Proposition 11.1.1, the proposition follows. $\square$

Since an odd factor can be found in polynomial time, we can conclude the following.

Corollary 11.1.9. *There is a polynomial time algorithm for decomposing a multigraph into an odd subgraph and an even subgraph or showing the non-existence of such a decomposition.*

Remark. *The case of the decomposability into two even subgraphs is trivial: a multigraph can be decomposed into two even subgraphs if and only if every vertex of the multigraph has even degree.*

The following two theorems are our main results.

Theorem 11.1.10 (Kano, Katona, Varga [K10])**.** *Let G be a multigraph and let $\mathcal{X}$ denote the set of components of $G[\mathrm{OddV}(G)]$, and let $\mathcal{Y}$ and $\mathcal{Z}$ denote the sets of components of $G[\mathrm{EvenV}(G)]$ with odd order and even order, respectively. Now G can be decomposed into two odd subgraphs if and only if for every $\mathcal{S} \subseteq \mathcal{Y} \cup \mathcal{Z}$ with $|\mathcal{S} \cap \mathcal{Y}|$ odd, there exists a component $X \in \mathcal{X}$ such that $e_G(X, S)$ is odd, where S is the set of vertices that belong to the components in $\mathcal{S}$.*

Theorem 11.1.11 (Kano, Katona, Varga [K10])**.** *There is a polynomial time algorithm for decomposing a multigraph into two odd subgraphs or showing the non-existence of such a decomposition.*

Most of the above results can be reformulated in terms of edge coloring. An *odd coloring of a graph G* is an assignment of colors to the edges of G in a way that each vertex is incident to either zero or an odd number of edges of each color. The minimum number of colors needed in such a coloring is called the *odd chromatic index of G*, and it is denoted by $\chi'_o(G)$. Using these terms Theorem 11.1.2 of Pyber is equivalent to $\chi'_o(G) \leq 4$ in case of simple graphs, and Theorem 11.1.4 states that if a simple graph has even number of vertices then $\chi'_o(G) \leq 3$. Theorem 11.1.10 gives a necessary and sufficient condition in case of multigraphs for $\chi'_o(G) = 2$, while Theorem 11.1.11 provides a polynomial algorithm to decide if $\chi'_o(G) = 2$.

Following our work several results appeared related to our results. Petruševski and Škrekovski [101] generalized our results for the case when the parities of the vertex degrees of the colors can be prescribed arbitrarily. In [103] proved analogues of our results for directed graphs, and in [102] they discuss the list version of the same problem as well as some other variations. Botler, Colucci and Kohayakawa [22] proved that that almost every simple graph on an even (resp. odd) number of vertices satisfies $\chi'_o(G) = 2$ (resp. $\chi'_o(G) = 3$).

11.2 Proof of Theorem 11.1.10

We begin with a definition and a proposition on it.

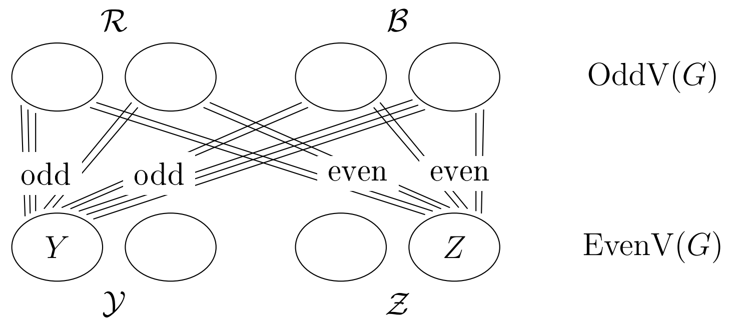

Figure 11.2: The structure of the multigraph.

Definition 11.2.1. *Let G be a multigraph and $T \subseteq V(G)$. A subgraph J of G is called a T-*join *if* $\mathrm{OddV}(J) = T$.

Proposition 11.2.2 ([44])**.** *Let G be a multigraph and $T \subseteq V(G)$. There exists a T-join in G if and only if every component of G contains an even number of vertices of T.*

The following theorem gives another necessary and sufficient condition for a multigraph to be decomposed into two odd subgraphs.

Theorem 11.2.3 (Kano, Katona, Varga [K10])**.** *Let G be a multigraph and $\mathcal{Y}$ and $\mathcal{Z}$ denote the sets of components of $G[\mathrm{EvenV}(G)]$ with odd order and even order, respectively. Then G can be decomposed into two odd subgraphs if and only if there exists a partition $\mathcal{R} \cup \mathcal{B}$ of the components of $G[\mathrm{OddV}(G)]$ such that*

(i) $e_G(R, Y)$ and $e_G(B, Y)$ are both odd for every $Y \in \mathcal{Y}$, and

(ii) $e_G(R, Z)$ and $e_G(B, Z)$ are both even for every $Z \in \mathcal{Z}$,

where R and B are the sets of vertices that belong to the components in $\mathcal{R}$ and $\mathcal{B}$, respectively. (See Figure 11.2.)

Proof. Suppose that G can be decomposed into two odd subgraphs, and color the edges of one with red and with blue the other. Obviously, if a vertex of G has odd degree, then all edges incident with it must have the same color. Consider an arbitrary component X of $G[\mathrm{OddV}(G)]$. Then all edges that have at least one endpoint in X have the same color. Let $\mathcal{R}$ and $\mathcal{B}$ denote the sets of those components of $G[\mathrm{OddV}(G)]$ in which the edges are red and blue, respectively. Let $Y \in \mathcal{Y}$. Then

$$\sum_{v \in Y} d_{\mathrm{red}}(v) = 2e_{\mathrm{red}}(G[Y]) + e_G(R, Y).$$

Since $|Y|$ is odd and $d_{\mathrm{red}}(v)$ is odd for every $v \in Y$, the left side of the equation is odd, and so $e_G(R, Y)$ must be odd. Similarly, $e_G(B, Y)$ is also odd, and $e_G(R, Z)$ and $e_G(B, Z)$ are both even. Therefore, conditions (i) and (ii) hold.

Next assume that there exists a partition $\mathcal{R} \cup \mathcal{B}$ satisfying (i) and (ii). Then color all the edges incident with any vertex of R red and all the edges incident with any vertex of B blue. Note that no edge of $G[\mathrm{EvenV}(G)]$ is colored yet, and there exist no edges between R and B. Let $T \subseteq \mathrm{EvenV}(G)$ be the set of vertices having even red-degree at this stage.

Now we show that every component of $G[\mathrm{EvenV}(G)]$ contains an even number of vertices of T. Let $Y \in \mathcal{Y}$. Then by condition (i),

$$\sum_{v\in Y} d_{\mathrm{red}}(v) = \underbrace{\sum_{v\in Y\cap T} \underbrace{d_{\mathrm{red}}(v)}_{\text{even}}}_{\text{even}} + \sum_{v\in Y\setminus T} \underbrace{d_{\mathrm{red}}(v)}_{\text{odd}} = \underbrace{e_G(R,Y)}_{\text{odd}}.$$

Hence $|Y\setminus T|$ is odd, and since $|Y|$ is odd, $|Y\cap T|$ must be even. By the same argument given above, for any $Z \in \mathcal{Z}$ since $|Z|$ is even and by condition (ii) $e_G(R,Z)$ is even, it follows that $|Z \setminus T|$ is even and $|Z \cap T|$ is even. So by Proposition 11.2.2, there exists a T-join in $G[\mathrm{EvenV}(G)]$. Color all the edges of this T-join red, and all the remaining edges blue. Now the resulting red subgraph and blue subgraph are odd subgraphs and form a partition of $E(G)$. □

11.3 Proof of Theorem 11.1.11

Proof of Theorem 11.1.11. Let $\mathcal{X}$ denote the set of components of $G[\mathrm{OddV}(G)]$, and let $\mathcal{Y}$ and $\mathcal{Z}$ denote the sets of components of $G[\mathrm{EvenV}(G)]$ with odd order and even order, respectively.

Consider the bipartite graph G^*, whose vertices correspond to the elements of $\mathcal{X}$ and $\mathcal{Y}\cup\mathcal{Z}$, and an element of $\mathcal{X}$ and that of $\mathcal{Y}\cup\mathcal{Z}$ are joined by an edge if and only if there are odd number of edges of G between the corresponding components. Then it is easy to see that every vertex $Y \in \mathcal{Y}$ and $Z \in \mathcal{Z}$ has even degree in G^*.

Our goal is to give a system of linear equations that is solvable if and only if G is decomposable into two odd subgraphs and its solutions describe partitions satisfying the properties of Theorem 11.2.3. For every $X_i \in \mathcal{X}$, we assign a binary variable x_i which decides whether $X_i \in \mathcal{R}$ or not. If $x_i = 1$, then $X_i \in \mathcal{R}$, and if $x_i = 0$, then $X_i \in \mathcal{B}$. Since we want $e_G(R,Y)$ to be odd for every $Y \in \mathcal{Y}$ and $e_G(R,Z)$ to be even for every $Z \in \mathcal{Z}$, consider the following system of linear equations over the binary field $GF(2) = \{0,1\}$:

$$\sum_{X_i\in N_{G^*}(Y)} x_i = 1 \qquad \text{for all } \; Y \in \mathcal{Y},$$
$$\sum_{X_i\in N_{G^*}(Z)} x_i = 0 \qquad \text{for all } \; Z \in \mathcal{Z}.$$

By Theorem 11.2.3, the multigraph G is decomposable into two odd subgraphs if and only if this system has a solution. The system is solvable if and only if one of the following three equivalent statements holds.

(i) There is no collection of equations such that the sum of the left-hand sides is 0 and the sum of the right-hand sides is 1 (over the binary field).

(ii) For any subset of the equations if the sum of the right-hand sides is 1, then there exists a variable x_i which appears odd number of times in these equations.

(iii) For any $\mathcal{S} \subseteq \mathcal{Y}\cup\mathcal{Z}$ for which $|\mathcal{S}\cap\mathcal{Y}|$ is odd, there exists $X \in \mathcal{X}$ such that $|N_{G^*}(X)\cap\mathcal{S}|$ is odd.

Note that statement (iii) is a graph presentation of statement (ii). □

Since a system of linear equations over the binary field can be solved in polynomial time, Theorem 11.1.11 follows.

However, it is worth translating the algorithm to the language of graphs. The steps of the Gaussian elimination can be followed in the auxiliary bipartite graph $\widehat{G}^*$ which is a slight modification of the graph G^* used in the proof of Theorem 11.1.10, see Figure 11.3 for an example. In the following, we use * as an operation that contracts components into single vertices and takes the number of resulting multiple edges modulo 2. So the color classes of G^* are the vertex sets $\mathcal{X}^*$ and $\mathcal{Y}^* \cup \mathcal{Z}^*$, and our goal is to partition $\mathcal{X}^*$ into $\mathcal{R}^*$ and $\mathcal{B}^*$. To obtain $\widehat{G}^*$ a new vertex b is added to G^* and is connected to all vertices in $\mathcal{Y}^*$. This vertex b corresponds to the constant 1 on the right sides of the linear equations.

To start the Gaussian elimination we need to select a variable that has a non-zero coefficient (i.e. 1) in at least two equations and pick one of these equations. Therefore in $\widehat{G}^*$ we choose an edge $x_i w$ with $|N_{\widehat{G}^*}(x_i)| \geq 2$, $x_i \in \mathcal{X}^*$ and $w \in \mathcal{Y}^* \cup \mathcal{Z}^*$. Now in the Gaussian elimination, we add the equation corresponding to w to all the equations corresponding to any element of $N_{\widehat{G}^*}(x_i) - \{w\}$ to make the coefficient of x_i zero in these equations. Then the resulting system of linear equations corresponds to the bipartite graph $\widehat{G}_1^*$ that is obtained from $\widehat{G}^*$ by replacing the induced subgraph $\widehat{G}^*[N_{\widehat{G}^*}(w) \cup \big(N_{\widehat{G}^*}(x_i) - \{w\}\big)]$ with its bipartite complement: $x' \in N_{\widehat{G}^*}(w)$ and $w' \in N_{\widehat{G}^*}(x_i) - \{w\}$ are adjacent in $\widehat{G}_1^*$ if and only if x' and w' are not adjacent in $\widehat{G}^*$. The other edges are not changed. Notice that the degree of x_i in $\widehat{G}_1^*$ is one.

Next we repeat this procedure by choosing another edge $x_j w'$ in $\widehat{G}_1^*$ that satisfies the same conditions. Since the degree of x_i is already one, x_j will automatically differ from the previously chosen vertices, but we also choose w' to be different from all previously chosen vertices. If there are no more such edges then the procedure stops.

Consider the graph of the final stage. At this point we can obtain the desired partition of the edge set into two odd subgraphs or show the non-existence of such a partition as follows.

- If a vertex $w \in \mathcal{Y}^* \cup \mathcal{Z}^*$ is adjacent only to the vertex b, then the graph G cannot be decomposed into two odd subgraphs, since this means that adding up some equations results in 0 on the left-hand side and 1 on the right-hand side.

So we may assume that no vertex $w \in \mathcal{Y}^* \cup \mathcal{Z}^*$ is adjacent only to the vertex b. In this case we obtain a solution as follows.

- If a vertex $x_i \in \mathcal{X}^*$ has degree at least two, then let $x_i \in \mathcal{B}^*$ and remove all the edges incident with x_i. This means that the variable x_i is a free variable, so it can be set to 0. Thus we may assume that every $x \in \mathcal{X}^*$ is adjacent to at most one vertex of $\mathcal{Y}^* \cup \mathcal{Z}^*$. Removing these edges makes x_i an isolated vertex, but note that other vertices in $\mathcal{X}^*$ cannot be isolated.

- If there is a vertex in $\mathcal{Y}^* \cup \mathcal{Z}^*$ that is adjacent to b and has more than one neighbor in $\mathcal{X}^*$ (that are all leaves), then let one of these neighbors be in $\mathcal{R}^*$ and all the others in $\mathcal{B}^*$. This means that we set one variable to 1 and all the others to 0, so their sum is equal to 1.

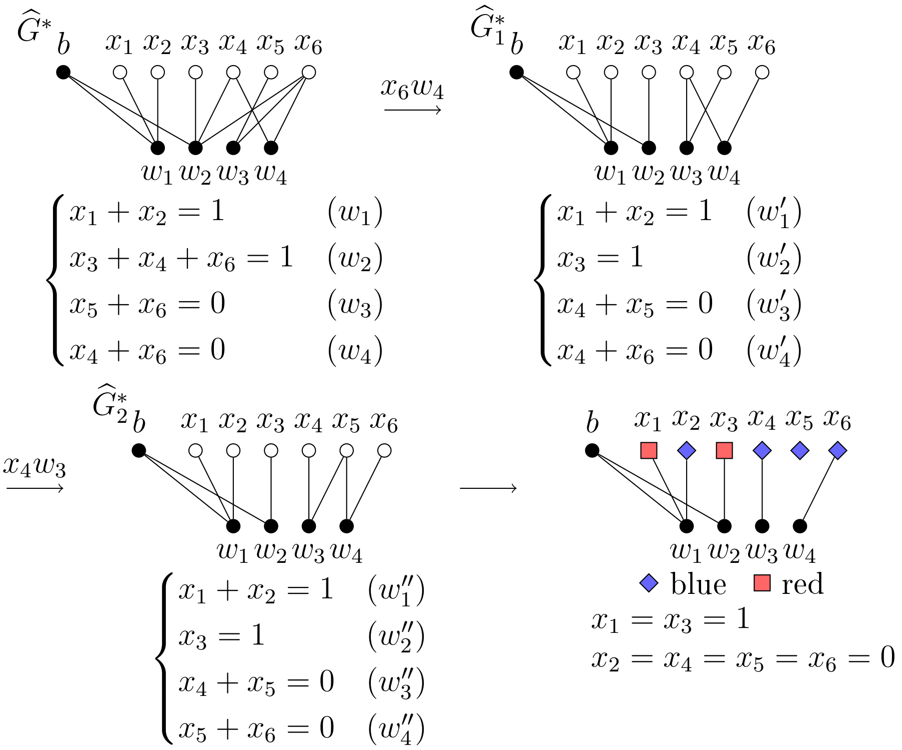

Figure 11.3: An example for the algorithm.

- Otherwise, if x_i is in the same component as b, then let $x_i \in \mathcal{R}^*$, meaning that x_i was set to 1 in the solution.
- If x_i is not in the component of b, then let $x_i \in \mathcal{B}^*$, meaning that x_i was set to 0 in the solution.

The above graph operation gives us a partition of $\mathcal{X}^*$ into $\mathcal{R}^* \cup \mathcal{B}^*$ and the corresponding partition of $\mathcal{X}$ satisfies the conditions in Theorem 11.2.3, and hence G is decomposed into two odd subgraphs. $\square$

The example on Figure 11.3: The steps of the Gaussian elimination in an auxiliary bipartite graph $\widehat{G}^*$.
First, we eliminate the variable x_6 with the help of equation (w_4), i.e. we take the bipartite complement of $\widehat{G}^*[x_4, x_6; w_2, w_3]$ to obtain $\widehat{G}_1^*$.
Second, we eliminate the variable x_4 with the help of equation (w_3'), i.e. we take the bipartite complement of $\widehat{G}_1^*[x_4, x_5; w_4]$ to obtain $\widehat{G}_2^*$.
In the graph $\widehat{G}_2^*$ there is only one vertex belonging to the color class $\mathcal{X}^*$ that has degree at least two, namely x_5, but since both of its neighbors have been already chosen before, the procedure stops.
Since in $\widehat{G}_2^*$ no vertex is adjacent only to b, there exists a decomposition of G into two odd subgraphs. Since the vertex x_5 is a free variable, i.e. it has degree (at least) two after the procedure stopped, we can set it to zero, i.e. we can put it into $\mathcal{B}^*$ (here in the figure we indicate this by coloring it blue) and remove all the edges incident with it.
In the so obtained graph the vertex w_1 is adjacent to b and has (at least) two neighbors in $\mathcal{X}^*$, so we put one of its neighbors, namely x_1, into $\mathcal{R}^*$ (which we indicate in the figure by coloring it red), and all the others, namely x_2, into $\mathcal{B}^*$. Finally, considering the still "uncolored" vertices of $\mathcal{X}^*$, since x_3 belongs to the same component as b, we put it into $\mathcal{R}^*$, and since x_4 and x_6 do not belong to the component of b, we put them into $\mathcal{B}^*$.

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