University of Szeged

Institute of Informatics

One and two-person Positional games

Dissertation submitted to The Hungarian Academy of Sciences for the title of “MTA Doktora”

András Sándor Pluhár

Szeged
2022

Contents

Abstract

The research we present here touches upon various areas of discrete mathematics: Graph theory, Combinatorics, Data Mining. One of the recurring common part in those subjects is the notion of colorings, let those be random, algebraic, based on pairings or resulted in by playing two-person games in many ways. Yet another common characteristics are the use of explicit or hidden algorithms and the guiding role of different heuristics.

We start with the generalization and deep analysis of pairing strategies for positional games, and their "dynamic version", the Chooser-Picker games. Graph games are closely connected to random graphs, but this relation can be tricky as we demonstrate in the diameter games.

One can extend games to the past or the future. More precisely, in a graph game one may interested in finding the smallest graph on which a player wins the game, i. e. the player has a zeroth step. The other direction is recycling, in which the players have limited tokens to play and it runs out they play by reusing those tokens.

In the last chapter we consider lots of different approaches and use of colorings. We discuss the use of random greedy coloring of hypergraphs, the extensions of Kőnig theorem by linear algebraic point of view. Then a model for graph clustering is given which give rise to special colorings and chromatic numbers. Finally some global discrepancy problems are considered. Here the the study of balanced colorings of spanning trees were our original motivation but the research took unexpected turns.

Chapter 1

Introduction

We have to live ahead but understand life only in retrospect. In assembling this text I had to decide what to write about, graph structures, games or data-mining? Considering the possibilities I chose the games to be the central theme, since that way I can also tell something about combinatorial and algorithmic problems that are in close connection to games.

First we pinpoint the subject of the dissertation: it is mainly about finite, two-person, zero sum, total information games that are played on different boards. In some sense, these games are simple, Zermelo's theorem states that either one player has a winning strategy, or both players have safe strategies, i.e. they can avoid losing the game.

These games also have a *normal form*, the strategies of the players (row and column) can be listed, and a matrix A can be defined, where $A_{i,j}$ is the payoff of the row player when they follow their ith and jth strategies, respectively. In this case Neumann's Minimax theorem guarantees optimal mixed solutions for the players.

However, if the games are not in normal form, but defined by other means, one might be faced to enormous combinatorial complexity. On the good side, the need to tame this complexity brought fascinating new ideas and rich theories.

Here we concentrate on those games which grew out of the ancient game of Tic-Tac-Toe, and sometimes the whole family is called as Tic-Tac-Toe type games [29]. There are many links connecting these games to classical games, graphs, hypergraph coloring, topology, complexity theory, algorithms, AI and so on. In fact, some of those problems might be considered to be *one person game* and closely related not only to two-person games, but other important part of mathematics and applied algorithmic problems. Note that the terminology has been unified only in the recent years. What we call *Positional games* earlier had been referred as Combinatorial, Achievement or Hypergraph games, and instead of the Maker-Breaker terminology the weak or strong games (and wins, draws etc) had been used.

1.1 Positional games

There is a large class of two-player games that are played on some, not necessarily physically present, board. Among these we consider those in which the players are taking turns in placing tokens to the board, and win if they achieve to reach one from the previously given configurations. Sometimes a combinatorial impossibility theorem (Ramsey-type, connectivity, bandwidth etc) can be turned into a game.

Important examples are the hex, independently invented by Piet Hein and John Nash, the 5-in-a-row, Bridgit (David Gale), or the Shannon's switching game. A common generalization of these games is the following. Let $\mathcal{F} = (V, \mathcal{H})$ be a hypergraph, and the two player taking the element of V alternately. The player who takes all elements of an $A \in \mathcal{H}$ first wins the game. The case of the hex seemingly differs since here the winning sets are not the same for the players. Before presenting earlier results, we need to discuss the possible variants/terminology shortly.

1.1.1 Maker-Maker vs Maker-Breaker games

The Maker-Maker version on $\mathcal{F} = (V, \mathcal{H})$ is just as defined before. In the Maker-Breaker version Maker wins by getting all vertices of an edge from $\mathcal{H}$, while Breaker wins otherwise, i.e. no draw is possible. Let us note that it is easy (and polynomial) to transform a Maker-Breaker game into a Maker-Maker game [43]. Using the transversal hypergraph of $\mathcal{F} = (V, \mathcal{H})$ one can exchange the role of Maker and Breaker [28].

1.1.2 Avoider-Enforcer games

Here Avoider wins by *not* occupying any edge from $\mathcal{H}$, while Enforcer wins otherwise. In this version having more vertex (move) may hurt the player. Similar problems make it difficult to prove mathematical statements for the game Go-Moku or even chess.

1.1.3 Chooser-Picker and Picker-Chooser games

In these games Picker takes two vertices of V in each round, Chooser may keep one of those, and the other goes to Picker. In the Chooser-Picker version Chooser is evaluated as Maker, i.e. wins by getting a whole edge, and Picker is the Breaker. In the Picker-Chooser version the Maker-Breaker roles are swapped. These games have intimate connection to Maker-Breaker games, see [26, 50, 51, 52, 53, 32, 33]. The Chooser-Picker game is also a kind of slow *cake cutting* that can be done in one step and turns to the problem if the underlying hypergraph has a good two-coloring.

1.2 Classical results

Here we list a few from the most basic results/definition that have relevance in the following chapters. As the goal is to connect the classical theory with the new results, we do not go into the details, most of these ideas will be explained later.

1.2.1 Strategies and strategy stealing

The careful notion of strategy has a great importance, otherwise one runs into difficulties. In general strategies are functions that map the possible states (with history included) of a game into legal steps. Strategy stealing argument in general is that assuming the existence of a winning (or drawing) strategy f to a player, we construct a strategy to the other player by the means of f.

For a Maker-Maker $\mathcal{F} = (V, \mathcal{H})$ game one gets immediately that the first player wins or the game is a draw. This was proved for the hex by Nash proved, while in general by Hales and Jewett [84].

1.2.2 Hales-Jewett games and theorems

Hales and Jewett [84] seminal paper connects Ramsey theory, Positional games and graph pairings. They study the hypergraphs defined by the d-dimensional cube, divided into n^d sub-cubes. The sub-cubes are the vertices, and the edges are the lines of the cube, shortly $HJ(n, d)$. (Actually, out of the diagonals only those are needed that contain the origin.) Note that $HJ(3, 2)$ is the Tic-Tac-Toe. The game $HJ(3, 2)$, called *Qubic* or Tic-Toc-Tac-Toe, is a Maker's win as proved by Oren Patashnik [126].

Arguably their deepest theorem states that the hypergraph outlined before is not two colorable for any fixed n, provided d is large enough. Together with the strategy stealing argument, this theorem gives a first player win in the Maker-Maker games on $HJ(n, d)$, or Maker's win in the Maker-Breaker version if d is big enough in terms of n. In the Maker-Breaker version d can be much smaller, see [29]. Hales and Jewett also develop pairing strategies that rely on the Kőnig-Hall theorem that we will be discussing later.

1.2.3 Connectivity games

In retrospective the common feature of these games that given a graph G, the players take vertices (or in other case edge) of G, while the winning requires achieving a specified subgraph.

The first game in that line is the famous hex, invented independently by Piet Hein and John Nash [34]. It can be played on hexagonal board or a simple $n \times n$ grid where a diagonals of slope -1 are added to each cell. (Note that the second form allows the n-dimensional generalization, [69].) Here the goal is to connect two opposite sides.

It turns out there is no draw in hex, in other words the hypergraph of the winning sets has no good two coloring, this is the so-called hex theorem. There are several statements equivalent to the hex theorem such as the Brouwer fix point theorem, Sperner lemma, Connector theorem, Pouzet lemma, see [69, 97, 87, 133], or chapter 5., problem 30. in [115].

Other examples for such games are the Bridgit, invented by David Gale or its generalization, the Shannon's switching game [34]. We recall the second for its significance in matroid theory and to our results. A graph G is given, Maker and Breaker take the edges, and Maker wins by getting all edges of a spanning tree. Lehman [111] showed that Maker, as a second player wins iff G has two disjoint spanning trees.

1.2.4 Erdős-Selfridge type theorems

In 1973 Erdős and Selfridge [58] introduced the use of weight functions, a tool which significance cannot be overestimated. They showed Breaker can win a Maker-Breaker game $\mathcal{F} = (V, \mathcal{H})$, as a second player, if $\Phi(\mathcal{F}) := \sum_{A \in \mathcal{H}} 2^{-|A|} < 1/2$. It turned out that practically for all versions of Positional games there are Erdős-Selfridge type theorems.

Still, the weight functions give even more, these can be combined with other ideas and these were central in our studies, too.

Note, that the sum $\Phi(\mathcal{F})$ is nothing but the expected number of one colored sets in $\mathcal{H}$ in a random coloring of V by flipping a fair coin. The standard probability method of Erdős [56] yield the *existence* of a good coloring $\mathcal{F}$. The weight function strategy gives a polynomial time deterministic algorithm, i.e. solves the problem of *derandomization*. One can also interpret this result as a random play predicts the result of the game.

1.2.5 Biased games and random intuition

Chvátal and Erdős, see [46] modified the Shannon's switching game played on K_n, the complete graph n vertices. Since Maker has a huge advantage for $n \geq 4$, in compensation Breaker may take $b > 1$ edges at each step. They looked for the value of the *breaking point* b_0, which means that Maker wins iff $b < b_0$. It turned out, $b_0 = \Theta(n/\log n)$ which suggested a powerful idea, the so-called *probabilistic intuition, or random heuristic*. To spell it out we need some preparation.

Let $\mathcal{P}$ be a monotone graph property, and edge set taken by Maker is G_M. The players, Maker and Breaker take one and b edges in each turn. (In a general biased game, one of the players make a, the other b steps in each turn. Note, that this rule may be applied for other games.) Maker wins iff $G_M \in \mathcal{P}$, and b_0 is the breaking point defined before. Then one may expect $b_0 \approx 1/p_0$, where p_0 is the threshold probability of property $\mathcal{P}$ in the random $G(n,p)$ model. This heuristic was justified in a number of cases, see e.g. [28, 26, 30, 107, 152, 89, 90]. We may rephrase it as the result of a game is the same if two perfect or two random players play it.

Sometimes even the random play is meaningful, see in [31, 149] but usually the built up of some quasi-random structure helps Maker, while Breaker utilizes some weight function. Indeed, in some cases the results belong rather to the theory of random graphs then to the games.

However, the probabilistic intuition breaks down completely in other cases which can cause great difficulty, as we shall see it.

1.2.6 Accelerated games

If $a = b$ in a biased game, we call it *accelerated version* of the original one. The acceleration may change the outcome of the game profoundly. E.g. while the status of the chess is still uncertain, a very subtle remark of Kolmogorov helped to prove the White has at least a draw in the 2-2 accelerated version. Still, the outcome of the accelerated games may differ from their 1-1 versions, see [12, 113, 129, 131, 132]. One of the most fascinating findings is that the probabilistic intuition may be restored by acceleration (similar phenomena can happen in Chooser-Picker games).

1.3 Summary of the results by Chapters

In this section we highlight the most important ideas, results and theorems that will be dealt in detail in the up-coming sections.

Chapter 2:

In that section we extend the notion of pairing strategies, show that their existence is NP-complete and prove a useful necessary condition for it, see Proposition 1.

It enables us to analyze the k-in-a-row, and similar games. First of all, there are no pairing strategies for k-in-a-row if $k \leq 8$. Furthermore we can describe *all* pairing strategies for the Harary game P_5, that is when Maker's goal is to get five consecutive cells in the grid. Lemma 1 tells that basically there are only two pairings exist, both are of domino-type, see Figure 2.4. Moreover in Theorem 2.4 we show there are no pairing strategies for Breaker in the Harary game "Snaky" which result has only a computer proof before. We managed to show that a pairing strategy of Breaker in the game "Snaky" must be also a pairing strategy in the game P_5. Then the classification in Lemma 1 yields a computer-free proof for Theorem 2.4.

In subsection 2.1.2 we continue to explore pairing strategies. For the 9-in-a-row game only one pairing strategy was known, attributed to Hales and Jewett, but can be traced back to the book of Berlekamp, Conway and Guy [34]. We managed to find new ones, and described the structure of *all* possible pairing strategies in Theorem 2.5. Briefly, these are either the extensions of the domino-pairings of the 8×8 torus or some special domino-pairings of the 16×16 torus.[1]

In subsection 2.1.8 we study the so-called Chooser-Picker (CP) and Picker-Chooser (PC) auxiliary games that were introduced by Beck [26] to model the Maker-Breaker games. He observed that Picker has easier job than the corresponding players in both the CP and PC cases. We confirmed this phenomenon for some classical games, while later Knox [103] showed that it is not true in general. We extended CP games to infinite hypergraphs giving Chooser the right to take any finite sub-hypergraph of the original one. We used this to study the CP version of the k-in-a-row and showed that Picker wins for $k \geq 8$. Later Csernenszky [51] managed to prove this for $k = 7$, too.

We showed that although it is easier to solve the CP version of concrete games, it is still NP-hard to decide about the outcome of both CP and PC games. Finally we illustrate these versions (MM, MB, CP, PC) for the 4 torus game. It turns out that the win of Chooser in the PC version is surprisingly hard to prove.

Chapter 3:

Two graph games are investigated in this section. In the *diameter game* Maker's goal is to build a spanning graph with small diameter, the most interesting case is the diameter two game. The other game is a variant of the Shannon's switching game; here the edges of Maker have to be in one connected component during the game.[2] The common theme is that in both of these games there is an easy win for Breaker, but the acceleration results in the win of Maker.

Perhaps the deepest result is Theorem 3.1. We develop the necessary tools (auxiliary games) for it, and give a sketch how these ideas lead to this theorem. Some of these auxiliary games, e.g. the *biased degree game* is interesting in its own right.

For our variant of Shannon's switching game we have to main results. Having Theorem 3.4 we are extremely lucky since a complete characterization of the $(1 : 1)$ game can be done using elementary arguments. The biased version, i.e. the $(1 : b)$ game for

[1] All torus lines must contain exactly one domino.

[2] We call the Maker of this game *PrimMaker* referring to the resemblance to Prim's celebrated algorithm for finding maximal weight trees.

$b > 1$ defies the probabilistic intuition. However, if PrimMaker can also get more than one edges in each step, the game becomes non-trivial again. Theorem 3.5 states that PrimMaker wins if $b < n/(8\log n)$, and Breaker wins if $b > n/\ln n$.

Chapter 4:

One of the extensions goes backwards in time. For a certain game Maker has to create the smallest board, usually a subgraph of K_n, which allows a Maker's wins. First D. Hefetz, M. Krivelevich, M. Stojaković and T. Szabó [91] investigated these games. If Maker wants a connected graph on n vertices, it needs exactly $2n-2$ edges; this is nothing else than the Shannon's switching game. Their other, much harder, example was the *positive minimal degree game*, that is Maker wants a subgraph having positive degree at all vertices. They showed the a graph on n vertices has to have at least $11n/8$ vertices, and $10n/7+4$ (or $10n/7$ if $n \equiv 0 \pmod 7$) is always enough.

Via a *discharging* argument we prove Theorem 4.1 that says their upper bound is basically sharp. There are some non-matching lower and upper bounds for degree two or k games.

The other extension goes ahead in time, we continue an otherwise finished game. Assume that we play a positional game by placing token, and have only limited number of tokens for each players. Then the players can replace some of their own tokens in each steps. We call this the *recycled version* of the original game. We study recycling of discrete and continuous versions of the Maker-Breaker Kaplansky's game.[3] If the board is the infinite grid, and there are only four (or constant) winning directions, Breaker wins if the $k \geq c_1 \log n + c_2$, see Theorem 4.3. Note that Maker wins in $(1:2)$ version even without recycling if $k \leq c\log_2 n$ for some $c > 0$, see Theorem 4.2.

The continuous case is much harder. We need a delicate use of weight functions and a deep discrete geometry theorem of Trotter and Szemerédi [157] to prove Theorem 4.4, which says Breaker wins if $k > 2n^{1/3}$ and n is big enough.

Chapter 5:

Here we consider various types of coloring finite objects. These include the good coloring of uniform hypergraphs, $\{0,1\}$ coloring of graphs, restrict edge structure between color classes and ± 1 coloring of graph edges.

First we investigate a greedy coloring of uniform hypergraphs that leads to Corollaries 5 and 6, Lemma 12 and Claim 4. These ideas and notions became standard tools of the area since. For sparse hypergraphs we have Theorem 5.1 and Corollary 7.

Using the elements $\mathbb{F}_2$ for two coloring connects some results of linear algebra and graph theory. We re-prove the classical theorems of Kőnig and Harary, see Theorem 8 and 5.4, and reach a new results that can be considered their dual forms.

In the next section we propose a new kind of coloring to address the clustering of certain transaction graphs. It leads to the notion of induced H-free coloring number of a graph G, $\chi_H(G)$, where the minimum of such color classes taken where there are no induced H is between two classes. In Theorem 5.11 we characterize the complexity of the computation of that, while give some bounds on $\chi_H(G)$ where G is a random graph from $G(n,p)$ in Theorem 5.13.

[3]The original Kaplansky' game is played on the plane, and a player wins having first k tokens in a line such that the other player has no token on that line.

In Section 5.6 we discuss some global discrepancy problems for spanning subgraphs. These include the discrepancy of Hamiltonian cycles in dense graphs, see Theorem 5.15 and 5.16, the discrepancy version of the Hajnal-Szemerédi theorem, Theorem 5.17. There are a number of results on sparser graphs, Theorem 5.18 for random regular graphs, Theorem 5.19 and 5.20 for planar graphs and grids.

Chapter 2

Pairs and Pairings

In this chapter we demonstrate how pairing strategies can be used to solve some games. The notion of *pairing strategies* is somehow vague, so after some examples we discuss the difference of *Hales-Jewett* and *general* pairing strategies. Having this understanding, it becomes possible to develop complexity results, give computer-free proof concerning “Snaky", describe *all* pairing strategies for some game and so on.

The other direction is the exploration of Chooser-Picker (and Picker-Chooser) games. These games were introduced by József Beck, and proved to be very useful in the understanding of the clique games. These are in close relation to pairing, and interesting to their own right. We adopt and develop some methods for their investigation, and results about the outcomes, complexity etc.

2.1 Pairing strategies revisited

Pairing strategies appear in a plethora of games, see [34]. Before stating general theories, let me add two little games to this list that I devised for educational purposes.

1. Subgroup game. Let G be an additive group such that $a + a \neq 0$ for any $a \in G$. Avoider and Enforcer play a game on the hypergraph $H = (G, E)$, where E is the set of all non-trivial subgroup. Starting the play, Avoider can win by taking 0, and in the later rounds answering the step a of Enforcer by $-a$. □

2. Bandwidth game. Let G be a simple graph on n vertices with bandwidth k, in notation $bdw(G) = k$. The players take alternately elements from the set $\{1, \ldots, n\}$ and place to an unoccupied vertex of G. The first player who places a number i to a vertex v such that it in the neighborhood of v there is a number j, and $|i - j| \geq k$, loses the game. In general not much known about this game. If $G = P_k^2$ that is the $k \times k$ grid for $k \geq 2$, then the first player wins if and only if k is odd. Indeed, if k is odd, then Avoider can place the “middle" number $\lceil k^2/2 \rceil$ to the “middle" of the board, and then if the second player places i to a vertex v, the first player places $k^2 - i + 1$ to v' where v' is the reflected image of v to the center of the grid. It is easy to check that this way the first player never introduces a distance earlier than the second player, that is never loses. On the other hand draw is not possible, since $bdw(P_k^2) = k$, so the second player loses. For an even k, the second player can steal the above described strategy. □

Certain kind of pairing strategies were introduced to the theory of Positional Games by Hales and Jewett in [84]. Based on these pairing strategies they proved the following theorem.

Theorem 2.1. *[84] Breaker wins a Maker-Breaker game on the hypergraph* (V, E) *if* $|\cup_{A\in\mathcal{G}} A| \geq 2|\mathcal{G}|$ *for all* $\mathcal{G} \subset E$.

The idea is to use the celebrated Kőnig-Hall theorem[1], and exhibit a "double" *system of distinct representatives (SDR)*, in the hypergraph (V, E). A set $X \subset V$ is an SDR if $|X| = |E|$, and there is a bijection $\phi : X \to E$ such that for all $x \in X$, $x \in \phi(x)$. If X and Y are SDR's of (V, E) with the bijections ϕ and ψ where $X \cap Y = \emptyset$, then $\rho = \psi^{-1}\phi$ is a bijection $\rho : X \to Y$. Breaker, even as a second player, wins by using ρ. That is, Breaker takes $\rho(x)$ [takes $\rho^{-1}(y)$] if Maker takes an $x \in X$ [a $y \in Y$], and an arbitrary untaken element $v \in V$ if Maker takes a $w \in V \setminus (X \cup Y)$.

While Theorem 2.1 works fine for some games, it has its drawbacks. It rarely gives sharp results, which is not surprising considering the PSPACE-completeness of those games. Another problem is that the Kőnig-Hall theorem (and consequently Theorem 2.1) applies only to finite hypergraphs. A remedy for this is a lesser known theorem of Marshall Hall Jr., that requires only the local finiteness of the hypergraph (V, E). We say that (V, E) is *locally finite* if $\deg(x) := |\{A : x \in A \in E\}| < \infty$ for all $x \in V$.

Theorem 2.2. *[85] There is a SDR in a locally finite hypergraph* (V, E) *iff* $|\cup_{A\in\mathcal{G}} A| \geq |\mathcal{G}|$ *for all* $\mathcal{G} \subset E$.

Still, Theorem 2.1 does not apply directly if $|V| < 2|E|$, for instance, one must use other ideas to tackle the k-in-a-row games in two or in higher dimensions, see [130].

Definition 1. *The bijection* $\rho : X \to Y$*, where* $X \cap Y = \emptyset$ *and* $X, Y \subset V$*, is a* **winning pairing strategy** *for Breaker in the Maker-Breaker game on hypergraph* (V, E) *if for all* $A \in E$ *there is an* $x \in X$ *such that* $\{x, \rho(x)\} \subset A$.

Although it is utterly trivial, for the record we have to spell out the Observation 1:

Observation 1 (General pairing)**.** *If for a hypergraph* $\mathcal{F}$ *there is a* ρ *winning pairing strategy, the Breaker wins the Maker-Breaker game on* $\mathcal{F}$.

Of course, we assume that both the function ρ and the decision problem that determining whether any set $Y \subset V$ has the property that $Y \subset A \in E$ are computable in polynomial time in the size of description of (V, E). (For the sake of simplicity we consider only the case when both V and E are finite.) Having the bijection ρ, Breaker wins by taking $\rho(x)$ [taking $\rho^{-1}(y)$] if Maker's last move was $x \in X$ [was $y \in Y$]. To decide the existence of ρ is not easy in general. Let us denote the class of hypergraphs for which Breaker has a winning pairing strategy by $\mathcal{B}$.

Theorem 2.3. *Determining whether a hypergraph is in* $\mathcal{B}$ *is NP-complete.*

Proof. Given a bijection ρ that witnesses a winning pairing strategy, one checks for an $A \in E$ if there is an $x \in X$ such that $\{x, \rho(x)\} \subset A$. For any pair (A, x) it can be done in polynomial time, and $|E||V|$ is an upper bound on the number of such pairs. Consequently, $\mathcal{B} \in \text{NP}$.

[1] A generalized form of this theorem will be spelled out in the next paragraph as Theorem 2.2.

To show that $\mathcal{B}$ is NP-hard one can use basically the same argument as in the proof of Theorem 2.12. There is, however, a simpler reduction. Let ϕ be an arbitrary CNF in 3-SAT. We construct a hypergraph $\mathcal{H}_\phi = (V, E)$ such that $V = \{r_i, b_i, p_i\}_{i=1}^n$ and the edge set, E, consists of all edges A such that

- A is $\{r_i, b_i, p_i\}$ for all $i \in \{1, \ldots, n\}$,
- $A = \{p_i, r_i, p_j, r_j, p_k, r_k\}$ for a clause $C = x_i \vee x_j \vee x_k$,
- $A = \{p_i, r_i, p_j, r_j, p_k, b_k\}$ for a clause $C = x_i \vee x_j \vee \bar{x}_k$,
- $A = \{p_i, r_i, p_j, b_j, p_k, b_k\}$ for a clause $C = x_i \vee \bar{x}_j \vee \bar{x}_k$,
- $A = \{p_i, b_i, p_j, b_j, p_k, b_k\}$ for a clause $C = \bar{x}_i \vee \bar{x}_j \vee \bar{x}_k$.

A winning pairing strategy for Breaker cannot contain both $\{p_i, r_i\}$ or $\{p_i, b_i\}$ for $1 \leq i \leq n$, because the strategy is a bijection. But such a strategy must contain one of $\{p_i, r_i\}$ or $\{p_i, b_i\}$ in order to have at least one pair of the form $\{x, \rho(x)\}$ in each of the edges of size 3. Let $x_i = 1$ if $\{p_i, r_i\}$ is present, while $x_i = 0$ otherwise. As a result, a clause C associated to its corresponding set A of size 6 is satisfied if and only if A contains a pair. $\square$

Remarks. If the hypergraph (V, E) is almost disjoint, then Breaker has a winning pairing strategy iff $| \cup_{A \in \mathcal{G}} A| \geq 2|\mathcal{G}|$ for all $\mathcal{G} \subset E$, that is one gets back the assumption of Theorem 2.1. This case can be decided in polynomial time in the description of (V, E). As in Theorem 2.12, $\mathcal{B}$ is NP-complete for hypergraphs (V, E), where $|A| \leq 6$ for $A \in E$. A result of Hegyháti and Tuza [95] implies that the existence of a winning pairing strategy can be decided in polynomial time if $|A| \leq 3$ for $A \in E$. The cases when $|A| \leq 4$ or $|A| \leq 5$, seems to be still open.

2.1.1 Applications for k-in-a-row and Snaky

Let d_2 be the maximum pair degree in (V, E), that is $d_2 = \max_{x \neq y} d_2(x, y)$, where $d_2(x, y) = |\{A : \{x, y\} \subset A \in E\}|$.

Proposition 1. *If Breaker has a winning pairing strategy then $d_2|X|/2 \geq |\mathcal{G}|$ must hold for all $X \subset V$, where $\mathcal{G} = \{A : A \in E, A \subset X\}$.*

Proof. Simply locate the pairs in the winning pairing strategy. There are at most $|X|/2$ such pairs, which are disjoint. Each pair will be a subset of at most d_2 edges. Since each edge of $\mathcal{G}$ must have a pair as a subset, the number of edges must be at most $d_2|X|/2$. $\square$

Before going on, let us tell a few words about k-in-a-row games.

The 5-in-a-row (amőba) is one of the most well known positional game, inspiring several deep results in this field. For a very thorough introduction for these, see Beck [29]. In the Maker-Maker version two players take the squares of a graph paper (integer lattice), alternately, and the first who achieves five in a row, i. e. five consecutive squares in a vertical, horizontal or diagonal direction, wins the game. The “strategy stealing" argument shows that in this type of games the first player either wins the game or it is a draw, so usually the *Maker-Breaker* version is investigated.

With extensive use of computers, Allis [4] solved the Maker-Maker 5-in-a-row game for the 19×19 and 15×15 boards: the first player wins. However, the case of infinite board is still open. It is natural to ask then what happens in the k-in-a-row game, where the winning condition is to get k consecutive squares in a row. The first result in that direction is due to C. Shannon and H. Pollak [34] who showed that Breaker wins the 9-in-a-row. Later A. Hales and R. Jewett gave even a winning pairing strategy for Breaker. Andries Brouwer, under the pseudonym T.G.L. Zetters in [78] published that Breaker wins the 8-in-a-row on the infinite board. The cases $k = 6, 7$ are still open, although it is widely believed that those are both draws. (Of course for $k \leq 4$ Maker wins easily.) On the other hand, we have the following

Corollary 1. *There are no pairing strategies for the k-in-a-row if $k \leq 8$.*

Proof. In the k-in-a-row game, $d_2 = k - 1$, and if X is an $n \times n$ board, then $|\mathcal{G}| = 4n^2+O(kn)$. By Proposition 1, we have $(k-1)n^2/2 \geq 4n^2+O(kn)$; that is, $k \geq 9+o(n)$. □

Another example in which we can use this ideas is the polyomino game Snaky, which were examined by Harary [93], Harborth and Seeman [94], and Sieben [145]. This game is a Maker-Breaker game in which the board consists of the cells of the infinite grid and Maker's goal is to occupy all of the cells in an isomorphic copy of the polyomino Snaky, shown in Figure 2.1.

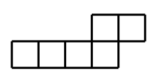


Figure 2.1: The polyomino Snaky. The "head" is the pair of cells in the upper row. The "body" is the set of four consecutive cells in the lower row.

Using a computer search, Harborth and Seeman [94] showed that there is no pairing strategy for Breaker in this game. We give a computer-free proof for their statement:

Theorem 2.4. *[94] Breaker has no pairing strategy to avoid the isomorphic copies of the polyomino "Snaky."*

Proof. Assume to the contrary that there is a winning pairing ρ for Breaker. Let P_ℓ be the polyomino which consists of ℓ consecutive squares of the table.

First we show that ρ cannot be a pairing for the polyomino P_4. Let us assume that ρ is such a pairing, and consider an $n \times n$ board X such that the edges of $\mathcal{G}$ consist of the P_4's on X. Since $d_2 = 3$, Proposition 1 gives that $3n^2/2 \geq 2n^2 + O(n)$, which is a contradiction if n is sufficiently large.

On the other hand, if ρ is a pairing for Snaky, then we will show that it must be a pairing for P_5. To see this, we assign labels to the cells such that cells receive the same label iff they are paired by ρ. Let us take the longest set of consecutive cells R such that no labels are repeated on R. We may assume that either those labels are $1, \ldots, \ell$ for some $\ell \geq 5$, or R is infinite.

We first consider the case $\ell = 5$, and in doing so let us refer to a cell of the grid by its lower left lattice point. If ρ is not a pairing for P_5, then we may assume, without loss of generality, that the set of cells $L = \{(1, 0), \ldots, (5, 0)\}$ contains no pairs. These cells

					?	?	?
L	$\diamond$	1	2	3	4	5	$\diamond$

	E	F	C				
A	A	$\diamond$	C		$\diamond$		
$\diamond$	1	2	3	4	5	6	$\diamond$
B	B	$\diamond$	D		$\diamond$		
	F	E	D				

Figure 2.2: The cases $\ell = 5$ and $\ell = 6$.

are labeled by $1, \ldots, 5$ on the left-hand side of Figure 2.2. Since $\ell = 5$, the both the cells $(0, 0)$ and $(6, 0)$ are in a pair with some cell of L. (We indicate the cells that have indices which matching with an element of L by a diamond, otherwise by capital letters.) This leaves only three elements of L that can be matched with a cell the rows above and below of L.

Consider the Snakys that have four cells in L. The head of the snake will have two cells in one of 4 disjoint sets of three consecutive cells in the row above or the row below L. Without loss of generality, we may assume that the three consecutive cells $\{(4, 1), (5, 1), (6, 1)\}$. That is, no cell of L is matched by the cells $\{(4, 1), (5, 1), (6, 1)\}$, labeled by "?" in Figure 2.2. But in that case ρ should contain, as pairs, both $\{(4, 1), (5, 1)\}$ and $\{(5, 1), (6, 1)\}$, which is impossible. So we may assume that $\ell > 5$.

Remark. In the case that $\ell > 5$, or ℓ is infinite, we again have a set L containing no pairs such that $|L| = \ell$. Every three consecutive cells in the rows above and below L must contain at least one cell whose label is matched to a cell of L, otherwise we finish the argument as in case $\ell = 5$. Here by "the rows above and below L" we mean sets that extend one cell longer than the end of L if L is finite or if L terminates in one direction.

Second is the case of $\ell = 6$ and we may assume that $\{(1, 0), \ldots, (6, 0)\}$ receive distinct labels. We will show that the only possible pattern is shown in the right-hand side of Figure 2.2. There are diamonds in the cells $(0, 0)$ and $(7, 0)$. Four diamonds remain to be placed and each set of three consecutive cells above and below L. The only possible locations do to so are $(2, \pm 1)$ and $(5, \pm 1)$. This ensures that $\{(0, 1), (1, 1)\}$ and $\{(0, -1), (1, -1)\}$ form pairs, which we label with "A" and "B", respectively.

Note that neither diamonds above and below the cell "2" can also be labeled by "2", otherwise the diamond, its right neighbor, and the cells $3, 4, 5, 6$ would be a pairing-free Snaky. The cells above and below the cell "3" are labeled "C" and "D", respectively. At this moment C could be equal to D. However, if we consider a standing Snaky on the cells $\{(1, 2), (1, 1), (2, 1), (2, 0), (2, -1), (2, -2)\}$, the only unpaired cells are those that are labeled with "E". If we consider a standing Snaky with the same body and the head towards the upper right, the only unpaired cells are those labeled "C" in the right-hand side of Figure 2.2. Symmetrically, we may assign labels "D" and "F" as shown in the figure. This, however, leads to a contradiction, since there would be a pairing-free Snaky again. In particular, the upper E and F cells make the head, and the body consists of the diamond above the cell "2", the cell of the lower C, the empty cell above "4" and the diamond above the cell "5". So, we may assume that $\ell > 6$.

The third case, where $\ell = 7$, is impossible since the rows above and below L should contain three diamonds each to avoid the snakes and two are needed to the right and left of L. This totals at least 8, more than the 7 that are available.

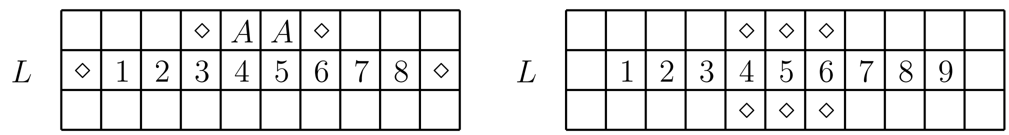


Figure 2.3: The cases $\ell = 8$ and $\ell \geq 9$.

In the fourth case, where $\ell = 8$, we have at most eight diamonds around L, two of those at the ends, and every three consecutive cells above and below L containing at least one diamond. So, there are ten cells above L and ten cells below L to receive the remaining 6 diamonds. There must be one in the three leftmost cells above L, in the three rightmost cells above L, in the three leftmost cells below L and in the three rightmost cells below L. Only two diamonds remain. One must be above one of the cells labeled "3", "4", "5" or "6". A diamond cannot be above the cell labeled "4" or "5" because for the two Snakys with heads equal to $\{(4,1),(5,1)\}$ and bodies in L, the diamond either represents one of $\{1,2,3,4\}$ or one of $\{5,6,7,8\}$. Hence, one of these Snakys must be pairing-free. As a result, the cells $\{(4,1),(5,1)\}$ must be paired with each other and so we label them with "A". See the diagram in the left-hand side of Figure 2.3. Because every three consecutive cells must contain at least one diamond, the cells above the cells labeled "3" and "6" are labeled with a diamond. This is a contradiction to the fact that only one diamond can be above these cells. So, we may assume that $\ell > 8$.

In the fifth case, where $\ell \geq 9$ and is finite, all cells above and below the cells $4, \ldots, \ell - 3$, the "critical region", must be diamonds. It is the same idea as in the previous case: If, say the cell above "4", is A, then so is the cell above "5". But the same is true for the cells above "5" and "6". Not only must the cells in the critical region be diamonds, there must be a total of at least 4 more above at below L to cover all of the triples of consecutive cells. With the additional two on the endpoints, there must be at least $2(\ell - 6) + 4 + 2$ diamonds, that is impossible, given that the total number of diamonds is at most ℓ, which is at least 9.

Finally, suppose L is infinite. Take 13 consecutive cells of L, call it L'. In the critical region of L' there must be $2(13 - 6) = 14$ cells with diamonds, but they must repeat the labels in the cells of L', a contradiction. This concludes the proof of the fact that a pairing for Snaky must be a pairing for P_5.

We exhibit two pairings for P_5. The pairing T_1 is like a chessboard, where the fields are 2×2, and alternately packed by a standing and lying pairs of dominoes as in the left-hand side of Figure 2.4. The pairing T_2 is like an infinite zipper, repeated in both directions, see the right-hand side of Figure 2.4.

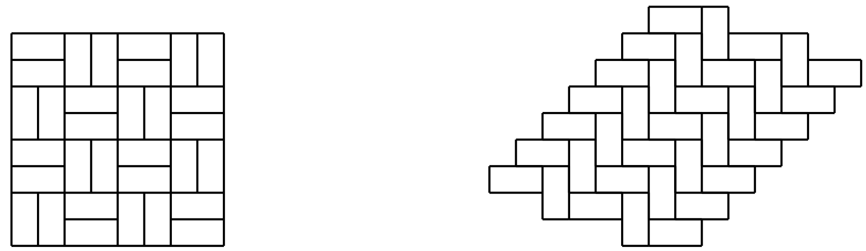


Figure 2.4: The parings T_1 and T_2.

Lemma 1. *A pairing for P_5 is either the translated and rotated copy of T_1 or T_2.*

Proof. Let us consider a pairing, ρ, for P_5. A pair $\{x, \rho(x)\}$ is *good* if x and $\rho(x)$ are neighboring cells. If $\{x, \rho(x)\}$ is good, then $d_2(x, \rho(x)) = 4$, otherwise it is smaller. The number of P_5's are $2n^2 + O(n)$ on an $n \times n$ sub-board X, so Proposition 1 implies that at all but $O(n)$ pairs on X are good. It follows that, if n is sufficiently large, then there is a $Y \subset X$, $k \times k$ square sub-board that contains only good pairs. I. e. this $k \times k$ sub-board is paired by dominoes.

There are either two dominoes meeting at their longer sides, or the two long sides meet but are offset by one unit. In these cases the immediate neighboring dominoes are forced to be in the pattern of T_1 or T_2, respectively.

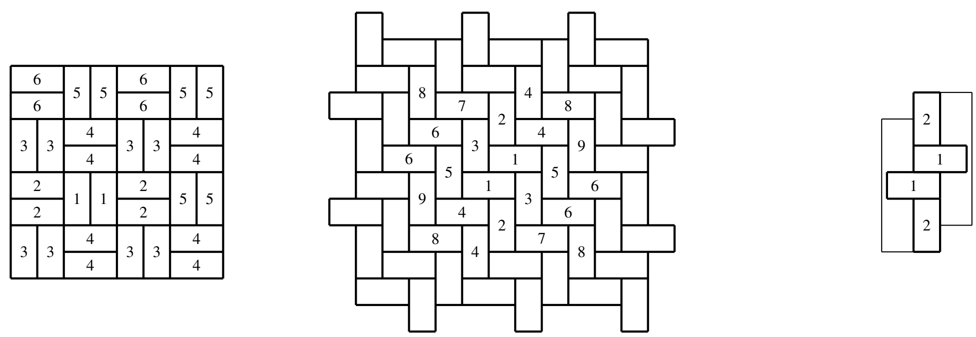


Figure 2.5: The forcing for pairs and filling.

We will show that if we have a large enough pattern of dominoes, then the pairs in the neighboring cells are forced to be in either T_1 or T_2. First suppose that, within the pattern tiled by dominoes that two dominoes share a long edge, as in the dominoes labeled with "1" in the left-hand side of Figure 2.5. Since the pairs can only occur as dominoes, we can use horizontal P_5's to ensure the pairing is oriented as in the dominoes labeled with "2". Vertical P_5's ensure the orientations of the dominoes labeled "3". We can continue in this fashion, getting the 8×8 pattern in the left-hand side of Figure 2.5. Once this is determined, one can extend the pattern to a larger rectangle, forcing not just the domino condition, but the T_1 pattern itself. This can be seen by first taking horizontal P_5's in rows 1,2,5,6 that have two cells outside of the pattern. Then taking vertical P_5's in columns 9,10, the pattern can be extended to an 8×10 rectangle. This can be continued ad infinitum, showing that the entire $n \times n$ board must be in the pattern T_1.

Next, suppose that whenever two dominoes meet at their long edge in the sub-board, that they are offset by one unit, since two dominoes meeting at their long edge will force the pattern T_1. See dominoes labeled "1" in the diagrams in the center or the right-hand side of Figure 2.5. The pairs must occur as dominoes and so vertical P_5's ensure that the dominoes labeled with "2" are placed in that location. Now, consider the right-hand side of Figure 2.5. Two P_5's are indicated by thin lines. Since the dominoes cannot share a long side, this forces the placement of the dominoes labeled with "3".

In fact, if we know that a sub-board is tiled with dominoes that do not share a long edge, then the configuration must be that of T_2. It remains to show that if we have a large enough fragment of T_2 in a sub-board, then, even if the board is not guaranteed to be tiled with dominoes, it must be completed to a T_2 pattern. The other pairs are forced even without the assumption that those are in dominoes, since the otherwise a P_5 containing no pair would arise.

To see how we can use this sub-board to extend T_2 to the whole board, we first show in the center of Figure 2.5 how enough pairs can be formed under the assumption that every pair forms a domino and no pair of dominoes can share a long edge. The numbers show the order in which dominoes can be taken. Then, in Figure 2.6 we show how, under no assumptions that the pairs occur as dominoes, that the dominoes that cover the 7×7 board can be extended to cover a 9×9 board. Again, the numbers show the order in which dominoes can be taken.

The general approach is that one can force new horizontal dominoes in every third row that touch the left and right border of the small square and vertical dominoes in every third column that touch the top and bottom border. From there, the rest of the larger square is easy to complete. This can continue *ad infinitum* until the board is filled. This concludes the proof of Lemma 1. □

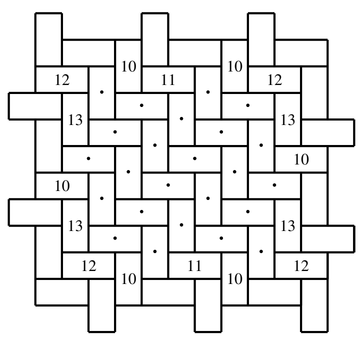


Figure 2.6: Expanding a 7×7 square to a 9×9 square. The dominoes given by the 7×7 square are marked with "·".

By Lemma 1, the pairs of ρ are either in the pattern T_1 or the pattern T_2, but none of those are pairings for Snaky. This concludes the proof of Theorem 2.4. □

2.1.2 Pairing strategies for 9-in-a-row

As we mentioned, Alfred Hales and Robert Jewett gave a winning pairing strategy for Breaker in the 9-i-a-row, see [29, 34] and also on Figure 2.7, while Corollary 1 shows there are no pairing strategies for k-in-a-row for $k \leq 8$. Since in the last about 50 years no one has given different pairings, the highly symmetric structure of the Hales-Jewett pairing, and the other examples of uniqueness or quasi uniqueness of pairings in similar problems, it is natural to think this is the only possible solution. However, it turned out there are lots of good pairings.

An other belief was that the pairing must be all extension of a pairing of the 8×8 *torus*. Somehow surprisingly, this belief is not true either; there are lot of solutions which are connected to the 16×16 torus, but are *not* extensions of the pairings of a 8×8 torus.

We will show that *all* solutions can be got either the extension of the pairings of a 8×8 torus (there are 194 543 non-isomorphic ones) or some combinations of those resulting in 16×16 toric solutions.

We prove also a special case of the conjecture of Kruczek and Sundberg [108] about the existence of pairings in higher dimensions.

Since in our discussion k-in-a-row type games play an important role, we define $\mathcal{H}_k$, the hypergraph of the k-in-a-row games.

Definition 2. *The vertices of the* ***k-in-a-row hypergraph*** $\mathcal{H}_k$ *are the squares of the infinite (chess)board, i. e. the infinite square grid. The edges of the hypergraph* $\mathcal{H}_k$ *are the* k*-element sets of consecutive squares in a row horizontally, vertically or diagonally. We refer to the whole infinite rows as* lines.

The following result is due to Hales and Jewett [34]:

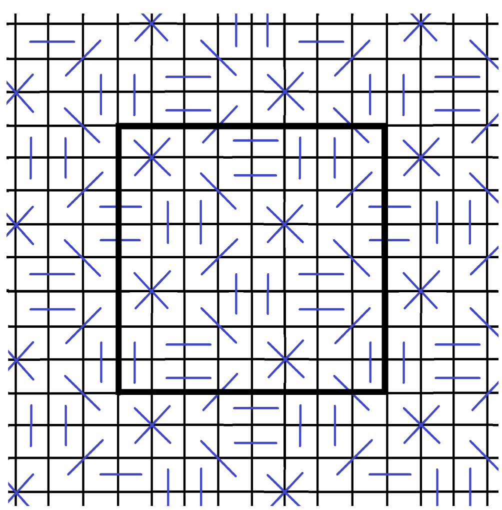


Figure 2.7: Hales-Jewett pairing blocks the 9-in-a-row

A pairing is a **domino pairing** or rather a **match(-stick) pairing** on the square grid, if all pairs consist of only neighboring cells (horizontally, vertically or diagonally). Note that the pairing on Figure 2.7 is a domino pairing.

For the case $k = 9$ Hales and Jewett gave a pairing, see [34] or Figure 2.7. However, in the literature there are neither different solutions nor claims of the uniqueness of the Hales-Jewett pairing. Our main goal is to decide about this questions.

2.1.3 Conditions for good pairings of $\mathcal{H}_9$

Considering an $n \times n$ square sub-board of the infinite board Proposition 1 gives $(k-1)n^2/2 \geq 4n^2 + O(n)$ which implies $k \geq 9 + O(1/n)$. It suggests that to block $\mathcal{H}_9$ one must use the pairs “optimally" that is a pair should block a maximum possible edges of $\mathcal{H}_9$. We make precise the notion of optimality as follows.

Definition 3. *A pairing is optimal if:*

1. *Every pair blocks exactly $k-1$ edges.*
2. *There are no over blocking, an edge is blocked by exactly one pair.*
3. *There is no empty square, i.e. a square that is no part of a pair.*

Corollary 2. *Let us consider an optimal good pairing for $\mathcal{H}_9$. Then this pairing is a domino pairing in which the dominoes are following each other by 8-periodicity in each line and all squares are covered by a pair.*

Proof. The first point of Definition 3 implies that the pairing is a domino pairing, while the second gives the 8-periodicity since otherwise it would cause either over blocking or resulting in an unblocked edge. The lack of empty squares just the repetition of the third condition. □

Definition 4. *We call a square of a pairing* **anomaly** *where the 8-periodicity is violated, a non-domino type pair or an empty square appears first in the natural geographic notation. The natural order is West to East, North to South, North-West to South-East and South-West to North-East.*

Of course the Hales-Jewett pairing is anomaly-free.

Remark 1. *There might be anomalies even in a good pairing of $\mathcal{H}_9$.*[2] *However, in subsection 2.1.6 we show that the good pairings of $\mathcal{H}_9$ are anomaly-free.*

The first step towards this is the following lemma:

Lemma 2. *For every good pairing of $\mathcal{H}_9$ there is an arbitrarily big, anomaly-free square sub-board.*

Proof. Let us take any $n \times n$ sub-board X and cut it up $\sqrt{n}/100 \times \sqrt{n}/100$ sub-boards. Applying Proposition 1, there are at most $48n - 128$ anomalies in X. Hence, among its $10000n$ sub-squares most of them must be anomaly-free. □

From now on we describe the structure of anomaly-free pairings of $\mathcal{H}_9$. Let us divide a good pairing of $\mathcal{H}_9$ into 8×8 sub-boards and designate one that we call *Central square*, shortly C. We keep only the (domino) pairs touching C and examine where should be pairs on the neighboring 8×8 sub-boards of C. In order to talk about these sub-boards Eastern (E), North-Eastern (NE) etc. while for the individual squares of the sub-boards the usual algebraic chess notation ($A1$ to $H8$) are used, see Figure 2.8.

Theorem 2.5. *Suppose we have an anomaly-free good pairing of $\mathcal{H}_9$ and we have nine 8×8 squares, $(C, E, NE, ...)$ as above. The horizontal and vertical dominoes touching the Central square C appears on the same places in all eight neighbor sub-boards of C. The diagonal dominoes also must appear on the sub-boards NE, NW, SW, SE on the same places. However, while the diagonal pattern of C may extend to the other four sub-boards, namely the E, S, W, N, it cannot be guaranteed. That is the whole plane is the periodic copies either of C or the 16×16 square consisting of the sub-boards C, S, SE, E.*

[2] A pairing with anomalies might be called “quasi-crystal" since the pairing symmetry group is trivial.

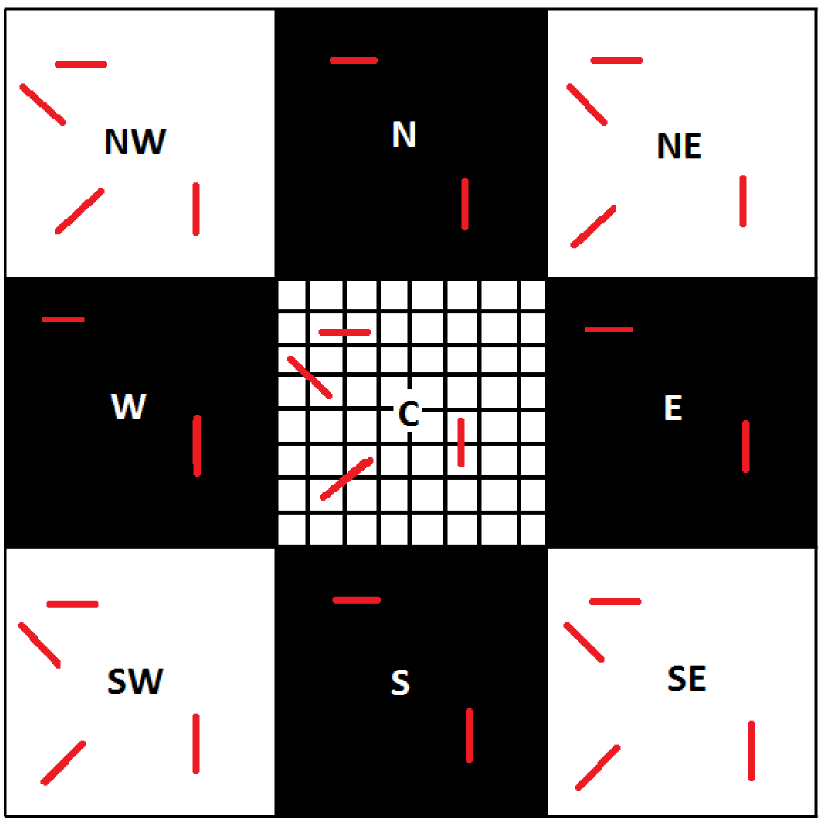


Figure 2.8: The extension of a pairing of C

Proof. It suffices to check the following five steps. We designate a general square in a 8×8 sub-board by Xi according to the chess notation. If a domino d covers the same pair of squares e. g. in the C and E square, we say that d *extends to* E *from* C.

1. Because of the 8-periodicity of the domino pairs on horizontal (vertical) lines, the pairs of C extend uniquely to the same places of W and E (N and S). The slope $+1$ diagonal dominoes extend similarly to SW and NE, while the slope -1 diagonal dominoes to SE and NW.

2. To see the horizontal (vertical) extension of dominoes on N and S (W and E) we need a little case study. We have already seen that the vertical dominoes of C extend to north and south. Suppose for example that there is a vertical domino v at the Xi square of C. If the Xi square of W is covered by a slope $+1$ (or -1) diagonal domino, then the 8-periodicity implies that the Xi square of N (or S) is also covered by a diagonal domino. This is a contradiction because we know from the previous point, that the Xi square of N is covered by a copy of the vertical domino v. The same is true for the sub-board E. If the Xi square of W (or E) is covered by a horizontal domino, then C should contain the copy of that horizontal domino at Xi by 8-periodicity, which is also a contradiction. We get that the vertical domino v in C extends to W and E, moreover, by 8-periodicity v extends to SW, NW, NE, SE, too. *So, we have seen that the vertical dominoes of* C *extend to all its eight neighboring sub-boards.* The same is true for the horizontal dominoes of C.

3. Let us check the diagonal dominoes. At the first and second step all slope $+1$ diagonal dominoes of C extend to SW and NE. Since there are no empty squares or over blocking, the remaining squares in SW and NE can be covered only by -1 slope diagonal dominoes. The same is true for $+1$ slope diagonal dominoes in SE and NW. *That is so far, all dominoes of* C *extend to the* SW, SE, NE, NW*, furthermore, the vertical and horizontal dominoes of* C *extend to* S, E, N *and* W.

4. We can see that the diagonal dominoes of C do not necessarily extend to the sub-boards S, E, N, W (colored by black on Figure 2.8). However, by 8-periodicity the diagonal pairs of E extend to S, N and W, that is the black sub-boards S, E, N, W have the exactly same structure of pairs.

5. The diagonal dominoes of C *may* extend to the sub-boards S, E, N, W, and then all 8×8 sub-boards of the infinite board are the exact copy of C. However, it is possible that there are two different diagonal structures on the whole board, one in the C, NW, NE, SE and SW types 8×8 sub-boards (colored by white on Figure 2.8) and a different diagonal structure in the sub-boards same as S, E, N and W (black ones).

□

Definition 5. *A pairing of the infinite board (or of an anomaly-free sub-board) is* ***k-toric*** *if it is an extension of a* $k \times k$ *torus, but not for a smaller value.*

We can spell out Theorem 2.5 in such a way: an anomaly-free good pairing of $\mathcal{H}_9$ is either 8-toric or 16-toric.

Observation 2. *There are 8-toric good pairings of $\mathcal{H}_9$ that are not isomorphic to the Hales-Jewett pairing.*

Proof. The extensions on Figure 2.9 result in three different 8-toric pairings. Note that the

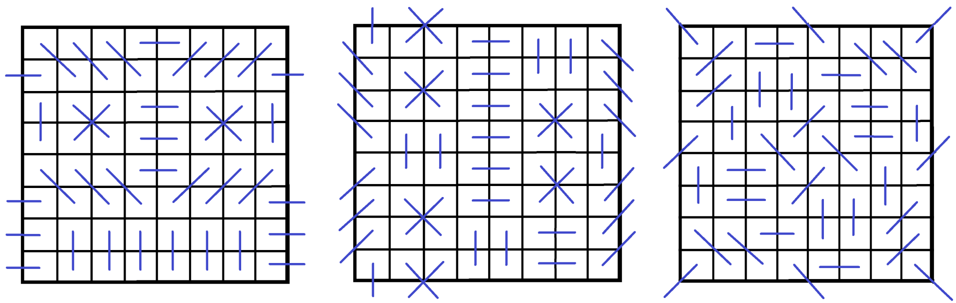


Figure 2.9: Some other pairings for 9-in-a-row

pairing on the left has reflectional symmetry, while the pairing on the right has rotational symmetry. □

It is somehow surprising that there exists also some 16-toric pairings of $\mathcal{H}_9$. To understand their structure we refine the argument of the proof of Theorem 2.5 in the next subsection.

2.1.4 Diagonal alternating cycles

The 8-toric and 16-toric good pairings of $\mathcal{H}_9$ can be considered as special perfect matchings of graphs. The vertex sets are the basic tori, and each vertex is connected to the eight neighbors of the square it represents. A domino of a pairing is an edge, and the whole pairing is not only a perfect matching but have the additional property, that is, contains exactly one edge (domino) from each torus line.

It is well-known that the union of two perfect matchings on the same vertex set consists of parallel edges and alternating cycles. So if we take the (graph theoretic) union of two good pairings (e. g. of C and W) which have the same horizontal and vertical edges, then the non-trivial alternating cycles contain only diagonal edges. (All alternating cycles are trivial - parallel edges - if and only if the two pairings are the same.) Identifying the vertices in the case of non-isomorphic G_C and G_W the system of diagonal alternating cycles gives the possible ways to get the 16-toric good pairings.

We arrive to the following simple corollary.

Corollary 3. *If there exists a 16-toric good pairing for $\mathcal{H}_9$, then we can derive it from two 8-toric good pairings (in case of non-isomorphic G_C and G_W) differ only in some diagonal cycles.*

Theorem 2.6. *An 8-toric solution C gives an 16-toric solution if and only if there exists another 8-toric solution W, differing in some diagonal dominoes, such that their union gives a system of diagonal alternating cycles. There are only two possible system of diagonal alternating cycles which are shown on Figure 2.11; the left and middle ones.*

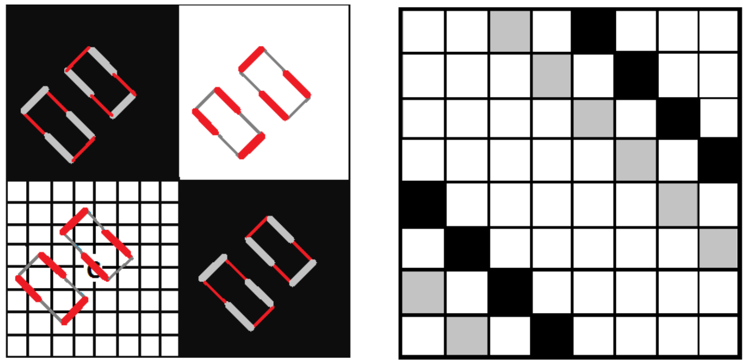


Figure 2.10: Diagonal alternating cycles give 16-toric pairing (left) and some -1 slope diagonal torus lines (right)

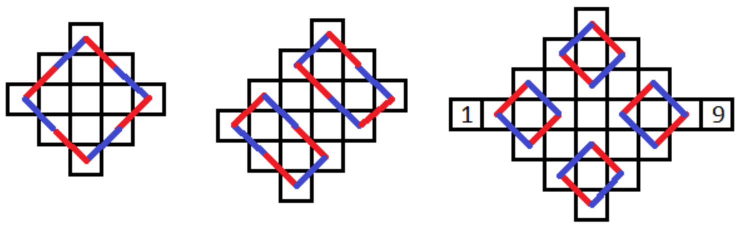


Figure 2.11: The diagonal alternating cycles

Proof. Since there is exactly one domino in each torus line of an arbitrarily chosen 8×8 square sub-board of a 8-toric solution, then the following must hold. The alternating cycles coming from the diagonal dominoes of the union of C and W must meet the torus lines either in zero or two dominoes. (If they meet only in one, then there will be an unblocked torus line in C or W. The more than two meet would mean over blocking.)

An easy case study gives that only the systems of diagonal alternating cycles of Figure 2.11 may come into consideration. However, the third one would make a horizontal line (namely the 1-9) impossible to be blocked by a domino. □

Observation 3. *There exist good pairings for $\mathcal{H}_9$ containing the first or the second type of (the systems of) diagonal alternating cycles.*

Proof. On Figure 2.12 one can see examples for the statement. Taking bold (thin) pairs of the alternating cycle for C (W) we get a 16-toric pairing. Of course this 16-toric pairing is *not* 8-toric. □

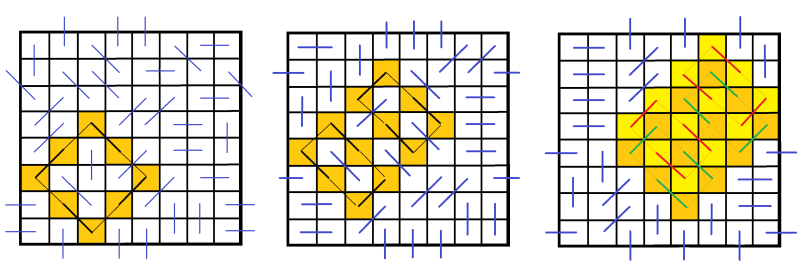


Figure 2.12: Examples of the alternating circles

2.1.5 Pairings of the 8×8 torus

We have seen that pairings on the anomaly-free sub-boards are either 8-toric or 16-toric. Since the 16-toric solutions can be reduced to 8-toric ones, we examine only the later ones in detail.

Definition 6. *The 8×8 Maker-Breaker torus game is played on the 64 squares of the discrete torus, where there are 32 winning sets; the eight rows and columns and the diagonal torus lines of slope ± 1, see the right side of Figure 2.10).*

Observation 4. *An arbitrary 8-toric pairing induces a pairing strategy for any 8×8 sub-board considering a torus game on it.*

That is, a 8-toric pairing of the whole grid (or a big sub-board) gives a good pairing for the 8×8 torus.

Remark 2. *The reverse is not true, since the 8×8 torus has good pairings which are not domino types. Of course, considering only domino pairs we can extend a good pairing of the torus into a 8-toric pairing of the whole plane.*

To find all good domino pairings for the 8×8 torus is a finite task, which is not hard by a computer. However, one also has to check for the torus symmetries to list the non-isomorphic ones. The number of non-isomorphic domino type good pairings is 194 543, which turns out to be a prime. The pairings themselves can be downloaded at the page: http://www.math.u-szeged.hu/ makay/amoba/

2.1.6 There are no quasi crystal pairings for the infinite board

We have left an open problem if there are pairings for $\mathcal{H}_9$ with anomalies? Note that on a $n \times n$ sub-board there can be $O(n)$ anomalies which might result in infinitely many (and possibly intractable) solutions. Fortunately, this is not the case as we will see.

Lemma 3. *A given anomaly-free pairing of a large enough square sub-board can be extended to the whole plane uniquely.*

Proof. We have seen that all anomaly-free pairings of a square sub-board is an extension of a domino pairing of either a 8×8-as or a 16×16 torus. Continuing the extension to the whole plane gives a good pairing. □

Lemma 4. *Let us assume that a pairing of the whole plane is an extension of an anomaly-free half-plane R. Then the whole pairing is anomaly-free.*

Proof. Assume contrary that we have an extension AL containing anomalies. Let AF be the anomaly-free extension of the half-plane pairing that exists by Lemma 3. Obviously AL is not equal to AF.

Let us take among the squares with anomaly the closest one to R, and denote it by q. As it is pictured on Figure 2.13 we may assume that the border line of the half-plane R is vertical and left to the square q the pairing AL is also anomaly-free. Let $AF(q)$ be the domino covering the square q in AF. If $AF(q)$ is horizontally placed, and q is the right half of it, then AL does not contain the domino $AF(q)$ which leaves a 9-in-a-row edge

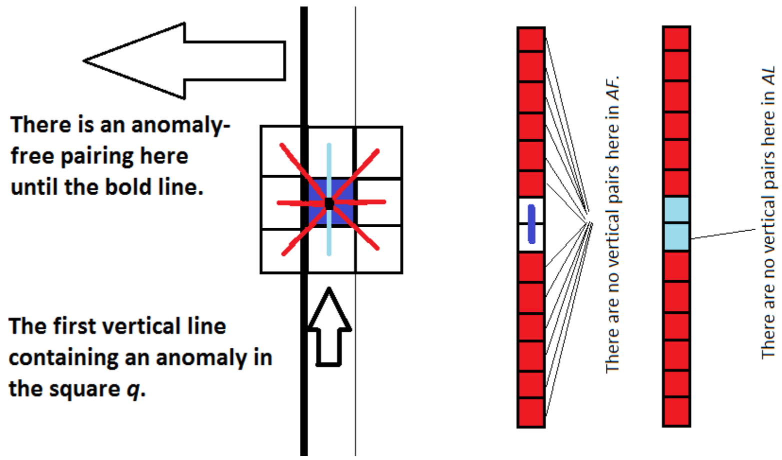


Figure 2.13: There are no quasicrystal

unblocked by AL. The similar argument shows that $AF(q)$ can be nothing but a vertical domino. Let us take the six squares above and below $AF(q)$. Because of 8-periodicity, there are no other vertical domino in AF covering these 12 squares, but there must be a half of a vertical pair s on those places in AL, because of the blocking condition. The domino $AF(s)$ is either horizontal or diagonal), and since $AF(s)$ is not in AL, it results in an unblocked horizontal or vertical edge in AL. □

In fact, we will need the ideas of the previous lemma, not the statement itself.

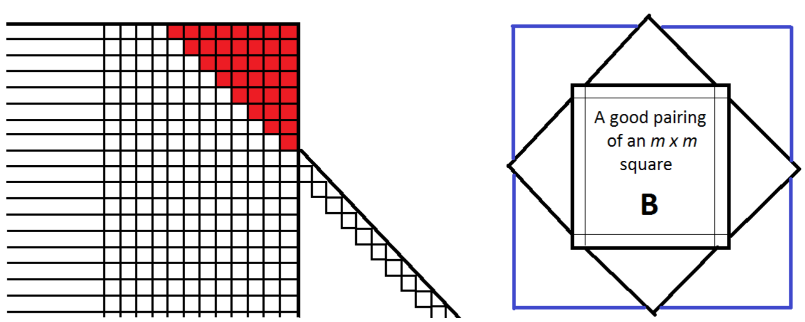


Figure 2.14: The extension of an anomaly-free pairing

Theorem 2.7. *An anomaly-free pairing of a big enough square sub-board extends uniquely and anomaly-free to the whole board.*

Proof. Fix a good pairing for $\mathcal{H}_9$ and take an $m \times m$ sub-board B that is anomaly-free; this exists by Lemma 2. The pairing on B extends anomaly-free to a large part of the right side of B, like in Lemma 4. The extension surely contains the right-angled triangle which hypotenuse length is $m - 16$, and touches the right side of B, see Figure 2.14. The argument of Lemma 4 does not work next to the top and the bottom of B, since there are no diagonal dominoes there in B which were used before.

We can do the same trick to extend the pairing on the other sides of B, which results in a bigger (about size $\sqrt{2}m - 16$) rotated square. Repeating this procedure, we can see the anomaly-free pairing of B is forced to extend to the whole plane. □

2.1.7 A pairing strategy in 3D

Kruczek and Sundberg [108] conjectured upper bounds matching with the lower bound of Proposition 1 for k-in-a-row type games in d dimension.

Conjecture 1. *[108] In the Maker-Breaker game on $\mathbb{Z}^d$ where there is a finite set $S \subset \mathbb{Z}^d$ of winning line direction-vectors, Breaker has a pairing strategy that allows him to win if the length of each winning line is at least $2|S|+1$, i.e., Breaker has a winning pairing-strategy for the game k-in-a-row if $k \geq 2|S|+1$.*

The special case on the plane just gives back that in the k-in-a-row Breaker has winning pairing strategy if and only if $k \geq 9$. The higher dimensional versions are mainly open. One possible form when the winning directions in the 3-dimensional lattice are given by the 13 vectors: $\{(0,0,1)$, $(0,1,0)$, $(1,0,0)$, $(0,1,1)$, $(1,0,1)$, $(1,1,0)$, $(0,1,-1)$, $(1,0,-1)$, $(1,-1,0)$, $(1,1,1)$, $(1,1,-1)$, $(1,-1,1)$, $(-1,1,1)\}$. Here Proposition 1 implies that to have a good pairing k should be at least 27. According to Conjecture 1, we may expect good pairings for $k = 27$.

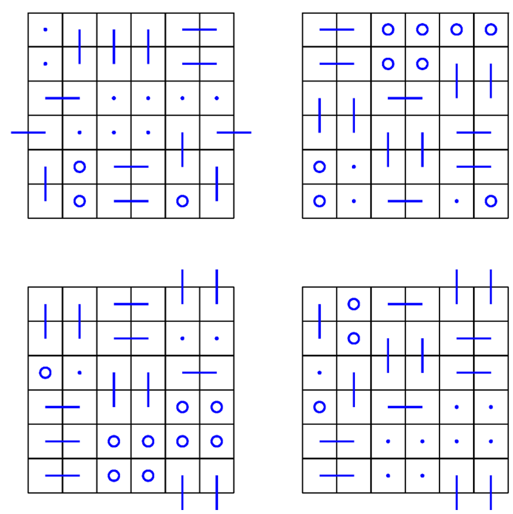


Figure 2.15: A good pairing of the 3D 7-in-a-row

We have examined a related problem in 3-dimension; the directions of winning lines are given by the three vectors: $\{(0,0,1), (0,1,0), (1,0,0)\}$. Here one expects pairing strategies if $k \geq 7$. (In other words, this is a Harary-type game [34] in 3-dimension, where the winning polyomino is the P_7, i.e. the seven connected consecutive cubes in a row.)

In fact, a computer search confirms this expectation, see on Figure 2.15. This is a domino pairing of 3-dimensional torus type, we give the pairing on the $6 \times 6 \times 6$ torus in layers. The horizontal and vertical pairs of the same layer are obvious, while the pairs between the layers are denoted by points and circles.

2.1.8 Chooser-Picker games and Beck's conjecture

Studying the very hard clique games, Beck [26] introduced a different type of heuristic, that proved to be a great success. He defined the *Picker-Chooser* or shortly P-C and the

Chooser-Picker (C-P) versions of a Maker-Breaker game that resembles fair division, (see [150]). In these versions Picker takes an unselected pair of elements and Chooser keeps one of these elements and gives back the other to Picker. In the Picker-Chooser version Picker is Maker and Chooser is Breaker, while the roles are swapped in the Chooser-Picker version. When $|V|$ is odd, the last element goes to Chooser. Beck obtained that conditions for winning a Maker-Breaker game by Maker and winning the Picker-Chooser version of that game by Picker coincide in several cases. Furthermore, Breaker's win in the Maker-Breaker and Picker's win in the Chooser-Picker version seem to occur together.

Beck [26] has another interesting remark, namely that Picker may win easily the Picker-Chooser game if Maker wins the corresponding Maker-Breaker game. He formulates this as follows:

"Note that Picker has much more control in the Picker-Chooser version than Chooser does in the Chooser-Picker version, or Maker does in the Maker-Breaker version so the Picker-Chooser game is far the simplest case. This relative simplicity explains why we start with the Picker-Chooser game instead of the perhaps more interesting Maker-Breaker game."

However, one has to be careful to spell out a good conjecture, since it is easy to check that Chooser wins the 2×2 hex.

The precise form of Beck's conjecture is:

Conjecture 2. *Picker wins a Picker-Chooser (Chooser-Picker) game on $(V, \mathcal{F})$ if Maker (Breaker) as second player wins the corresponding Maker-Breaker game.*

Remark 3. *It is enough to prove Conjecture 2 for Picker-Chooser games since the Chooser-Picker case would follow. To see this one just considers $(V, \mathcal{F}^*)$, the* transversal hypergraph *of $(V, \mathcal{F})$. That is $\mathcal{F}^*$ contains those minimal sets $B \subset V$ such that for all $A \in \mathcal{F}$, $A \cap B \neq \emptyset$. Note that Breaker as a first (second) player wins the Maker-Breaker $(V, \mathcal{F})$ iff Maker as a first (second) player wins the Maker-Breaker $(V, \mathcal{F}^*)$.*

The decision problem that if Picker wins a P-C (or C-P) game is at least NP-hard [133], but probably it is PSPACE-complete as that of the Maker-Breaker games, shown by Schaefer [146]. Still, for concrete games it can be easier to decide the outcome of the P-C (C-P) version than the Maker-Maker version. That is if Conjecture 2 is proved for a class of hypergraphs then the easier P-C (C-P) games can be used in an alpha-beta pruning algorithm for the harder Maker-Breaker game. A natural class for that is the otherwise hopeless Hales-Jewett or torus games for low dimension (see [27, 84]). We discuss some examples and useful tools for that direction in Section 2.1.11. Here we would emphasize the extension of Picker-Chooser games to infinite hypergraphs and the role of Lemma 6 and Proposition 3 in this case. These might be used in solving Harary-type of polyomino problems for Chooser-Picker games for which the Maker-Breaker versions were studied by Harary, Blass, Pluhár and Sieben [34, 129, 144].

Remark 4. *It turned out that the Beck's conjecture does not hold in general. In 2012 F. Knox [103] gave a counterexample.*

2.1.9 Chooser-Picker version of Shannon's switching game

We prove Conjecture 2 for the Picker-Chooser version of Shannon's switching game in the generalized version as Lehman did in [111]. Let $(V, \mathcal{F})$ be a matroid, where $\mathcal{F}$ is the set of bases, and Picker wins by taking an $A \in \mathcal{F}$. Note that this is equivalent with the Chooser-Picker game on $(V, \mathcal{C})$, where $\mathcal{C}$ is the collection of *cutsets* of the matroid $(V, \mathcal{F})$, that is for all $A \in \mathcal{F}$ and $B \in \mathcal{C}$, $A \cap B \neq \emptyset$.

Theorem 2.8. *Let $\mathcal{F}$ be collection of bases of a matroid on V. Picker wins the Picker-Chooser $(V, \mathcal{F})$ game, if and only if there are $A, B \in \mathcal{F}$ such that $A \cap B = \emptyset$.*

Proof.

The notation and the proof closely follow the ones given in [124] for the Maker-Breaker case.

First we show that if there are no two disjoint $A, B \in \mathcal{F}$ then Chooser wins. Let $\mathcal{M}_1 = (V, \mathcal{F})$ and $\mathcal{M} = \mathcal{M}_1 \vee \mathcal{M}_1$ be the union matroid of $\mathcal{M}_1$ with itself. The rank function $r_{\mathcal{M}}$ of the union matroid of $\mathcal{M} = M_1 \vee \cdots \vee M_k$ is the following,

$$r_{\mathcal{M}}(S) = \min_{T \subset S} \left\{ |S \setminus T| + \sum_{i=1}^{k} r_i(T) \right\},$$

where the matroids are defined on the same ground set S, and the matroid $\mathcal{M}_i$ has the rank function r_i. We have $\min_{T \subset V} \{|V \setminus T| + 2r_1(T)\} = r_{\mathcal{M}}(V) < 2r_1(V)$, since $\mathcal{M}_1$ does not have two disjoint bases. Equivalently, $|V \setminus T| < 2(r_1(V) - r_1(T))$. Receiving a pair (x, y), Chooser keeps an element of $V \setminus T$ if possible. At the end of the game Chooser owns at least $\lceil |V \setminus T|/2 \rceil$ elements of $V \setminus T$. That is Picker may own at most $\lfloor |V \setminus T|/2 \rfloor < r_1(V) - r_1(T)$ elements of $V \setminus T$ at the end of the game.

Let Y be the elements of Picker at the end of the game. Clearly,

$$r_1(Y) \leq r_1(Y \cap (V \setminus T)) + r_1(T) < r_1(V) - r_1(T) + r_1(T) = r_1(V),$$

that is Picker has lost the game.

For the other direction, we assume that $A, B \in \mathcal{F}$, $A \cap B = \emptyset$, and use induction. We consider the matroid $\mathcal{M}/y \setminus x$ given a pair (x, y) taken by Chooser and Picker, respectively. Clearly Picker wins the game for $\mathcal{M}$ if he can win it for $\mathcal{M}/y \setminus x$. (The dimension of $\mathcal{M}/y \setminus x$ is one less than that of $\mathcal{M}$, and if A' is a base of $\mathcal{M}/y \setminus x$, then $A' \cup \{y\}$ is a base of $\mathcal{M}$.)

All we need here is the *strong base exchange axiom* (or rather theorem), that says if A and B are bases of a matroid $\mathcal{M}$, then there exist $x \in A$, $y \in B$ such that both $\{A \setminus \{x\}\} \cup \{y\}$ and $\{B \setminus \{y\}\} \cup \{x\}$ are also bases of $\mathcal{M}$. Picker selects the pair (x, y) such that the above applies, and reduces the game to either $\mathcal{M}/y \setminus x$ or $\mathcal{M}/x \setminus y$. Since $A \setminus \{x\}$ and $B \setminus \{y\}$ are disjoint bases both in $\mathcal{M}/y \setminus x$ and $\mathcal{M}/x \setminus y$, we can proceed. □

2.1.10 Erdős-Selfridge-type theorems

The Erdős-Selfridge theorem [58] gives a very useful condition for Breaker's win in a Maker-Breaker $(V, \mathcal{F})$ game stating that Breaker as the second player has a winning strategy in the Maker-Breaker $(V, \mathcal{F})$ game when $\sum_{A \in \mathcal{F}} 2^{-|A|} < 1/2$.

Beck [26] considers both versions (Picker-Chooser and Chooser-Picker). Surprisingly the Picker-Chooser version is easier to prove, and the $\sum_{A \in \mathcal{F}} 2^{-|A|} < 1$ condition suffices. He used the usual weight function method, for fun, we show another approach.[3]

[3]Similar argument was used by Spencer in [149].

Proof. Chooser chooses by flipping a coin. Let us check the probability that all elements of an $A \in \mathcal{F}$ go to Picker. If $\{x_i, y_i\} \subset A$, then this probability is zero, otherwise it is $2^{-|A|}$. The expected number of edges taken by Picker $\mathbb{E} \leq \sum_{A\in\mathcal{F}} 2^{-|A|} < 1$ by *any* strategy of Picker. So Picker cannot have a winning strategy, since it would result in at least one edge taken by Picker. The game is won by one of the players, so according to Zermelo's theorem, Chooser has a winning strategy. □

For the Picker-Chooser version, Beck Using a stronger condition, Beck [26] proves Picker's win in a Chooser-Picker $(V, \mathcal{F})$ game. (For the P-C version he proved a sharp result that we include here.) Let $||\mathcal{F}|| = \max_{A\in\mathcal{F}} |A|$ be the rank of the hypergraph $(V, \mathcal{F})$.

Theorem 2.9. *[26] If*

$$T(\mathcal{F}) := \sum_{A\in\mathcal{F}} 2^{-|A|} < \frac{1}{8(||\mathcal{F}|| + 1)}, \tag{2.1}$$

then Picker has an explicit winning strategy in the Chooser-Picker game on hypergraph $(V, \mathcal{F})$. *If* $T(\mathcal{F}) < 1$, *then Chooser wins the Picker-Chooser game on* $(V, \mathcal{F})$.

We improve on his result by showing:

Theorem 2.10. *If*

$$\sum_{A\in\mathcal{F}} 2^{-|A|} < \frac{1}{3\sqrt{||\mathcal{F}|| + \frac{1}{2}}}, \tag{2.2}$$

then Picker has an explicit winning strategy in the Chooser-Picker game on hypergraph $(V, \mathcal{F})$.

Proof.

We shall modify the proof of Theorem 2.9 appropriately. The idea of the proof is to associate a weight function $T(\mathcal{F})$ to a hypergraph $(V, \mathcal{F})$ that measures the danger for Picker. The value of T becomes 1 iff Chooser wins the game, so Picker tries to keep T down. In Maker-Breaker games the greedy selection works, see the classical Erdős-Selfridge theorem in [58] or in [27]. Let $T(\mathcal{F}) = \sum_{A\in\mathcal{F}} 2^{-|A|}$, $T(\mathcal{F}; v) = \sum_{v\in A\in\mathcal{F}} 2^{-|A|}$ and $T(\mathcal{F}; v, w) = \sum_{\{v,w\}\subset A\in\mathcal{F}} 2^{-|A|}$ for an arbitrary hypergraph $(V, \mathcal{F})$.

Assume that after the ith turn Chooser already has the elements $x_1, x_2, \ldots, x_i$ and Picker has the elements $y_1, y_2, \ldots, y_i$. Now Picker picks a 2-element set $\{v, w\}$, from which Chooser will choose x_{i+1}, and the other one (i. e. y_{i+1}) will go back to Picker. Let $X_i = \{x_1, x_2, \ldots, x_i\}$ and $Y_i = \{y_1, y_2, \ldots, y_i\}$. Let $V_i = V \setminus (X_i \cup Y_i)$. Clearly $|V_i| = |V| - 2i$. Let $\mathcal{F}(i)$ be the truncated subfamily of $\mathcal{F}$ which consists of the unoccupied parts of the still dangerous winning sets:

$$\mathcal{F}(i) = \{A \setminus X_i : A \in \mathcal{F},\ |A \setminus X_i| \leq \lceil |V_i|/2 \rceil,\ A \cap Y_i = \emptyset\}.$$

Here we will deviate a little from Beck's proof, since he includes all sets $A \in \mathcal{F}$, $|A \setminus X_i| \leq |V_i|$ in $\mathcal{F}(i)$ if $A \cap Y_i = \emptyset$. But if $|A \setminus X_i| > \lceil |V_i|/2 \rceil$, then Picker *automatically* gets an element of A, so deleting these sets from $\mathcal{F}(i)$ does not change the outcome of the game.

Let $\mathcal{F}(end) = \mathcal{F}(\lceil |V|/2 \rceil)$, i. e., these are the unoccupied parts of the still dangerous sets at the end of the play. Chooser wins iff $T(\mathcal{F}(end)) \geq 1$, so to guarantee Picker's win it is enough to show that $T(\mathcal{F}(end)) < 1$. Let x_{i+1} and y_{i+1} denote the $(i+1)$th elements of Chooser and Picker, respectively. Then we have

$$T(\mathcal{F}(i+1)) = T(\mathcal{F}(i)) + T(\mathcal{F}(i); x_{i+1}) - T(\mathcal{F}(i); y_{i+1}) - T(\mathcal{F}(i); x_{i+1}, y_{i+1}).$$

It follows that

$$T(\mathcal{F}(i+1)) \leq T(\mathcal{F}(i)) + |T(\mathcal{F}(i); x_{i+1}) - T(\mathcal{F}(i); y_{i+1})|.$$

Introduce the function

$$g(v, w) = g(w, v) = |T(\mathcal{F}(i); v) - T(\mathcal{F}(i); w)|$$

which is defined for any 2-element subset $\{v, w\}$ of V_i. Picker's next move is that 2-element subset $\{v_0, w_0\}$ of V_i for which the function $g(v, w)$ achieves its minimum. Since $\{v_0, w_0\} = \{x_{i+1}, y_{i+1}\}$, we have

$$T(\mathcal{F}(i+1)) \leq T(\mathcal{F}(i)) + g(i), \tag{2.3}$$

where

$$g(i) = \min_{v,w: v \neq w, v,w \subset V_i} |T(\mathcal{F}(i); v) - T(\mathcal{F}(i); w)|. \tag{2.4}$$

To estimate $g(i)$ we take a lemma from [26]. It is an easy exercise for the reader.

Lemma 5. *If $t_1, t_2, \ldots, t_m$ are non-negative real numbers and $t_1 + t_2 + \ldots + t_m \leq s$, then*

$$\min_{1 \leq j < \ell \leq m} |t_j - t_\ell| \leq \frac{s}{\binom{m}{2}}.$$

We distinguish two phases of the play.

Phase 1: $|V_i| = |V| - 2i > 2||\mathcal{F}||$. (Note that Beck uses $|V_i| > ||\mathcal{F}||$.) Simple counting shows that

$$\sum_{v \in V_i} T(\mathcal{F}(i); v) \leq ||\mathcal{F}|| T(\mathcal{F}(i)).$$

By Lemma 5 and (2.4),

$$g(i) \leq \frac{||\mathcal{F}||}{\binom{|V_i|}{2}} T(\mathcal{F}(i)),$$

so by (2.3),

$$T(\mathcal{F}(i+1)) \leq T(\mathcal{F}(i)) \left\{ 1 + \frac{||\mathcal{F}||}{\binom{|V_i|}{2}} \right\}.$$

Since $1 + x \leq e^x = \exp(x)$, we have

$$T(\mathcal{F}(i+1)) \leq T(\mathcal{F}) \exp\left\{||\mathcal{F}|| \sum_{j=0}^{i} \frac{1}{\binom{|V_j|}{2}}\right\}.$$

It is easy to see that

$$\sum_{i:|V_i|>2||\mathcal{F}||} \frac{1}{\binom{|V_i|}{2}} < \frac{1}{2||\mathcal{F}||},$$

so if i_0 denotes the last index of the first phase then

$$T(\mathcal{F}(i_0+1)) < \sqrt{e}T(\mathcal{F}). \tag{2.5}$$

Phase 2: $|V_i| = |V| - 2i \leq 2||\mathcal{F}||$.
Then a similar counting as in *Phase 1* gives

$$\sum_{v \in V_i} T(\mathcal{F}(i); v) \leq \left\lceil \frac{|V_i|}{2} \right\rceil T(\mathcal{F}(i)).$$

One checks that $T(\mathcal{F}(i+1)) \leq T(\mathcal{F}(i))$ when $2 \leq |V_i| \leq 4$. If $|V_i| \geq 4$, then by Lemma 5 and (2.4),

$$g(i) \leq \frac{1}{|V_i|-1} T(\mathcal{F}(i)),$$

so by (2.3),

$$T(\mathcal{F}(i+1)) \leq \frac{|V_i|}{|V_i|-1} T(\mathcal{F}(i)). \tag{2.6}$$

Let us recall the well-known Wallis' formula, $\lim_{n\to\infty} \frac{1}{2n+1} \prod_{i=1}^{n} \frac{(2i)^2}{(2i-1)^2} = \frac{\pi}{2}$. Since $\frac{(2n+2)^2}{(2n+1)(2n+3)} > 1$ for all $n \in \mathbb{N}$, we have the inequality for all $n \in \mathbb{N}$

$$\prod_{i=1}^{n} \frac{2i}{2i-1} < \sqrt{\frac{\pi}{2}(2n+1)}. \tag{2.7}$$

By repeated application of (2.6) we have

$$T(\mathcal{F}(end)) \leq T(\mathcal{F}(i_0+1))2 \prod_{i:2\leq|V_i|\leq 2||\mathcal{F}||} \frac{|V_i|}{|V_i|-1} \leq T(\mathcal{F}(i_0+1))2 \prod_{j=2}^{||\mathcal{F}||} \frac{2j}{2j-1}.$$

Now using (2.7), (2.5) and (2.2), we have

$$T(\mathcal{F}(end)) < T(\mathcal{F}(i_0+1))\sqrt{\pi(||\mathcal{F}|| + \frac{1}{2})} \leq \sqrt{e\pi}T(\mathcal{F})\sqrt{||\mathcal{F}|| + \frac{1}{2}} < 1.$$

That is, Chooser cannot completely occupy a winning set, and Theorem 2.10 follows.
□

Remark 5. *In 2013 Bednarska-Bzdęga [32] proved the sharp version Erdős-Selfride-type theorem for the Chooser-Picker games. Namely if* $\sum_{A\in\mathcal{F}} 2^{-|A|} < 1/2$*, then Picker wins.*

In the next subsection we extend Chooser-Picker games to infinite hypergraph, and discuss the classical k-in-a-row games.

2.1.11 The k-in-a-row

The Picker-Chooser k-in-a-row is an easy Picker's win for all $k \in \mathbb{N}$, by Beck's argument in [27]. The Chooser-Picker is again Picker's win for $k > 1$ on the infinite board, since Picker may select elements far from each other at all time. However, the games become interesting if we restrict them to a finite board, since sooner or later all elements must be selected. (One might think that Chooser starts the game by selecting a finite part of the board.)

Proposition 2. *Picker wins the Chooser-Picker version of the game* 8*-in-a-row on any* $B \subseteq \mathbb{Z}^2$.

Proof. First we need an easy but useful lemma. Given the hypergraph $(V, \mathcal{F})$ let $(V \setminus X, \mathcal{F}(X))$ denote the hypergraph where $\mathcal{F}(X) = \{A \in \mathcal{F}, A \cap X = \emptyset\}$.

Lemma 6. *If Picker wins the Chooser-Picker game on* $(V, \mathcal{F})$*, then Picker also wins it on* $(V \setminus X, \mathcal{F}(X))$.

Proof. By induction it is enough to prove the statement for $X = \{x\}$, i. e., $|X| = 1$. Assume that p is a winning strategy for Picker in the game on $(V, \mathcal{F})$. That is, in a certain position of the game, the value of the function p is a pair of unselected elements that Picker is to give to Chooser. We can modify p in order to get a winning strategy p^* for the Chooser-Picker game on $(V \setminus \{x\}, \mathcal{F}(\{x\}))$.

Let us follow p while it does not give a pair $\{x, y\}$. Getting a pair $\{x, y\}$, we ignore it, and pretend we are playing the game on $(V, \mathcal{F})$, where Chooser has taken y and has returned x to us. If $|V|$ is odd, there is a $z \in V$ at the end of the game that would go to Chooser. Here Picker's last move is the pair $\{y, z\}$. Picker wins, since Chooser could not win from this position even getting the whole pair $\{y, z\}$. If $|V|$ is even, p^* leads to a position in which y is the last element, and it goes to Chooser. But the outcome is then the same as the outcome of the game on $(V, \mathcal{F})$, that is Picker's win. □

We shall cut up the infinite board to sub-boards in the same way as was in [78], see also Figure 2.1.11. The left tile and its mirror image are the bases of the tiling. The winning sets for the these sub-boards are the rows, the diagonals of slope one, and the two pairs indicated by the thin lines. The middle of the picture shows the tiling itself. We use one type of tile in an infinite strip, and its mirror image in the neighboring stripes. On the right side of Figure 2.1.11 the transformed tile is drawn, where the winning sets are the rows, columns and the indicated two pairs.

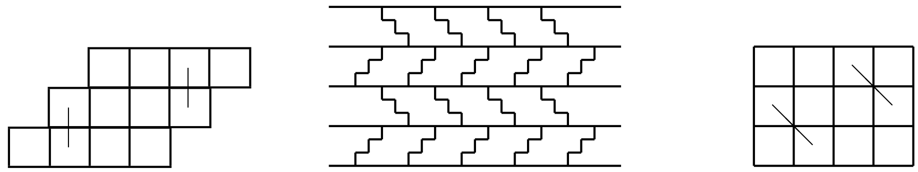


Figure 2.1.11 The subdivison of the plane.

Let $\bar{B}$ be the union of those sub-boards meeting B. We show that Picker wins the Chooser-Picker 8-in-a-row game for the board $\bar{B}$. Note that $\bar{B}$ is a union of sub-boards. Picker plays auxiliary games on the sub-boards independently of each other with the goal of preventing Chooser from getting a winning set of a sub-board.

To achieve this goal, Picker selects the two pairs first on any sub-board, that give rise to the possible positions shown in Figure subboardpairs. Then Picker uses the appropriate winning pairing strategy indicated by the thin lines. One checks easily that if Picker wins all the auxiliary games then he wins the Chooser-Picker 8-in-a-row game on playing $\bar{B}$, too. Finally, by Lemma 6, Picker wins on B. □

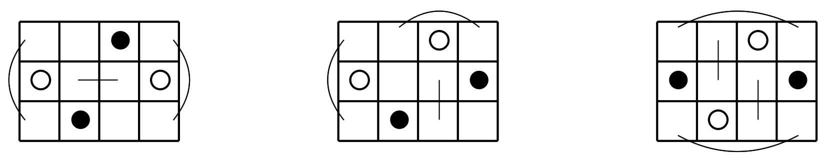


Figure 2.1.11 Pairings on a sub-board.

One might wonder how the idea of the pairings used in Proposition 2 came from. It is worthwhile to spell out the following simple fact.

Proposition 3. *In a Chooser-Picker game if a winning set contains no elements of Picker, and has only two untaken elements, x, y then Picker has an optimal strategy that starts with picking the pair $\{x, y\}$.*

Proof. We may assume that Picker has a winning strategy p, otherwise there is nothing to prove. First we show that during any optimal play of the game Picker has to offer the pair $\{x, y\}$ sometimes. If Picker offers, say, $\{x, z\}$, $z \neq y$, and y has not been taken yet, Chooser would keep x, and win taking y later. Now let us assume the Chooser has a winning strategy ρ, taking, say, x if Picker starts with $\{x, y\}$. Chooser can adapt the strategy ρ against any strategy of Picker's by pretending that the start was $\{x, y\}$. Over the course of the play Picker has to offer the pair $\{x, y\}$. Then Chooser takes x and resumes playing the strategy ρ, and Chooser wins, since the outcome of the game would be the same if Picker would have started with $\{x, y\}$. □

Remark 6. *András Csernenszky, see [51] managed to prove an even stronger result, Picker wins the Chooser-Picker version of the game 7-in-a-row on any $B \subseteq \mathbb{Z}^2$. To do this he cut the plane into 4×8 rectangular sub-boards, with appropriate winning sets such that a Picker win on a sub-board guarantees a win in the whole plane. To analyze* *a sub-board similar techniques were used as in Proposition 2, especially Proposition 3. The result, considering the connection between the Maker-Breaker and Chooser-Picker games, supports the belief that Breaker might win the Maker-Breaker 7-in-a-row game. However, the 7-in-a-row game (let alone the 6-in-a-row game) is still a mystery.*

2.1.12 From pairs to colorings

Observation 1 tells that the existence of a winning pairing strategy ρ is a sufficient condition for Breaker to win in a Maker-Breaker $\mathcal{F}$ game. On the other hand, if Breaker wins, $\chi(\mathcal{F}) \leq 2$ is necessary. (Even stronger, $\mathcal{F}$ must have a good two-coloring about same sized of color classes.)

We can fill the area between the pairings and good coloring with other objects which, thanks to the notion of accelerated Chooser-Picker games, have a game theoretical meaning.

First we generalize the pairings, and call a set $C = C^1 \cup C^2 \subset V$ of the hypergraph $\mathcal{F} = (V; \mathcal{H})$ to be a t-cake, or cake of size t if $|C^1| = p$, $|C^2| = q$ and $|C| = p + q$. It is balanced, if $p = q$. The set $\mathcal{T}$ is a t-cake placement if it consists of cakes of size at most t. When all cases are of size t, and $p = q$, we may call it p-pairing.

A t-cake C blocks an edge $A \in \mathcal{H}$ if $|A \cap C^i| > 0$ for $i = 1, 2$. A t-cake placement $\mathcal{T}$ is good t-cake placement if all edges o $\mathcal{F}$ are blocked by some cakes from $\mathcal{T}$.

Among the many possible scenarios that we examined in [81] let us restrict to ourselves to the $2p$-pairings of the k-in-a-row hypergraph.

Observation 5. *Picker wins a $p : p$ accelerated Chooser-Picker game on $\mathcal{F}$ if there is a good $2p$-placement $\mathcal{T}$.*

Complexity issues for Chooser-Picker games

The Maker-Breaker (and the Maker-Maker) games are PSPACE-complete (see [146]) so it is natural to think that Chooser-Picker or Picker-Chooser games are not easy as well. To prove PSPACE-completeness for positional games is more or less standard, see [140, 43]. Here we can prove something weaker because of the asymmetric nature of these games.

Theorem 2.11. *It is NP-hard to decide the winner in a Picker-Chooser game.*

Theorem 2.12. *It is NP-hard to decide the winner in a Chooser-Picker game.*

2.1.13 Proofs of Theorems 2.11 and 2.12

Both proofs are based on the usual reduction method. We reduce $3-\mathrm{SAT}$ to Chooser-Picker or Picker-Chooser games.

Proof of Theorem 2.11. Consider an arbitrary CNF formula $\phi(x_1, \ldots, x_n) \in 3-\mathrm{SAT}$. We denote $\phi = C_1 \wedge \cdots \wedge C_k$, where $C_i = \ell_{i_1} \vee \ell_{i_2} \vee \ell_{i_3}$ and ℓ_{i_j} is a literal for $i \in \{1, \ldots, k\}$ and $j = 1, 2, 3$. With a slight abuse of notation, we use C_i also to denote the set of literals in it. That is, if there exists a clause $C_i = x_2 \vee \bar{x}_5 \vee x_6$, then we also denote the set $C_i = \{x_2, \bar{x}_5, x_6\}$.

We will exhibit a hypergraph $\mathcal{H}_\phi = (V, E)$ such that the Picker-Chooser game is a win for Chooser if and only if ϕ is satisfiable.

The vertex set will be $V = \{x_1, \ldots, x_n, \bar{x}_1, \ldots, \bar{x}_n\}$. Let $\mathcal{B} \subset 2^V$ have the property that $B \in \mathcal{B}$ if, for all $i \in \{1, \ldots, n\}$, B contains either x_i or $\bar{x}_i$ but not both. The edge set E consists of the sets A such that $A = C_i \cup B$ for some i and some $B \in \mathcal{B}$.

Note that $\mathcal{B}$, and consequently E, has a short (polynomial in ϕ) description even though $|E| \geq |\mathcal{B}| = 2^n$.

Claim 1 allows us to restrict our attention to games in which Picker has a specific kind of strategy.

Claim 1. *If Picker fails to select pairs of the form $\{x_i, \bar{x}_i\}$ in each round, then Chooser has a winning strategy.*

Proof of Claim 1. We assume to the contrary: Let $\{x, y\}$ be the first pair selected by Picker such that $\{x, y\} \neq \{x_i, \bar{x}_i\}$ for any $i \in \{1, \ldots, n\}$. In that case, Chooser keeps, say, x, and waits until Picker offers up $\bar{x}$ in a pair. In that round, Chooser takes $\bar{x}$, and wins the game, since Picker cannot take any $B \in \mathcal{B}$. This proves Claim 1. □

First we show that if Picker-Chooser on $\mathcal{H}_\phi$ is a win for Chooser, then ϕ is satisfiable. According to Claim 1, we may assume that Picker's strategy is to select pairs of the form $\{x_i, \bar{x}_i\}$ resulting in the fact that such pairs are shared among Picker and Chooser for all i. Assume that Chooser wins the game on $\mathcal{H}_\phi$, and set $\hat{x}_i = 1$ if Chooser holds x_i, and $\hat{x}_i = 0$ otherwise. Picker holds all elements of some $B \in \mathcal{B}$, so the assumption means that Chooser has an element in each of the C_i's. That is, $\phi(\hat{x}_1, \ldots, \hat{x}_n) = 1$.

Next we show that if ϕ is satisfiable, then Picker-Chooser on $\mathcal{H}_\phi$ is a win for Chooser. Since ϕ is satisfiable, there exist $\hat{x}_1, \ldots, \hat{x}_n$, such that $\phi(\hat{x}_1, \ldots, \hat{x}_n) = 1$. Consider the Picker-Chooser game on $\mathcal{H}_\phi$. By Claim 1, we may assume that, in each round, Picker offers a pair of the form $\{x_i, \bar{x}_i\}$. In that case, Chooser takes x_i if and only if $\hat{x}_i = 1$, and wins the game. This proves Theorem 2.11. □

Proof of Theorem 2.12. Let us use the same set-up and notation for the CNF formula ϕ as in the proof of Theorem 2.11. We want to define a hypergraph $\mathcal{H}_\phi = (V, E)$ such that the Chooser-Picker game on $\mathcal{H}_\phi = (V, E)$ is a Picker's win if and only if ϕ is satisfiable.

Let the vertex set be $V = \{a_i, b_i, c_i, d_i\}_{i=1}^n$. The edge set, E, consists of all edges A such that

- $A \subset \{a_i, b_i, c_i, d_i\}$ and $|A| = 3$ for some $i \in \{1, \ldots, n\}$,
- $A = \{a_i, a_j, a_k, b_i, b_j, b_k\}$ for a clause $C = x_i \vee x_j \vee x_k$,
- $A = \{a_i, a_j, a_k, b_i, b_j, c_k\}$ for a clause $C = x_i \vee x_j \vee \bar{x}_k$,
- $A = \{a_i, a_j, a_k, b_i, c_j, c_k\}$ for a clause $C = x_i \vee \bar{x}_j \vee \bar{x}_k$,
- $A = \{a_i, a_j, a_k, c_i, c_j, c_k\}$ for a clause $C = \bar{x}_i \vee \bar{x}_j \vee \bar{x}_k$.

Claim 2 allows us to restrict our attention to games in which Chooser has a specific kind of strategy.

Claim 2:

- If Picker picks a pair (x, y) such that $\{x, y\} \not\subset \{a_i, b_i, c_i, d_i\}$ for some $i \in \{1, \ldots, n\}$, then Chooser has a winning strategy.

- Chooser has an optimal strategy that results in always choosing a_i and always giving d_i to Picker.

In particular, this means that we may assume that for all i, Picker either picks $\{(a_i, b_i), (c_i, d_i)\}$ or $\{(a_i, c_i), (b_i, d_i)\}$. Moreover, Chooser will get a_i and Picker will get d_i and each player will get exactly one of (b_i, c_i).

Proof of Claim 2. Suppose Picker offers a pair (x, y) for which $x \in \{a_i, b_i, c_i, d_i\}$ but $y \notin \{a_i, b_i, c_i, d_i\}$. Consider the first such instance. In that case, Chooser chooses x, and ultimately wins by choosing at least two more elements from $\{a_i, b_i, c_i, d_i\} \setminus \{x\}$, giving Chooser every element of some A of size 3. So, for all i, Picker will pick either $\{(a_i, d_i), (b_i, c_i)\}$ or $\{(a_i, b_i), (c_i, d_i)\}$ or $\{(a_i, c_i), (b_i, d_i)\}$. Hence, Chooser and Picker will have at least one member of each set of size 3.

However, no d_i appears in any of the sets of size 6 and so if Chooser wins by choosing d_i, then he must also win by not choosing d_i. Finally, suppose Picker picks the pair (a_i, b_i) or (a_i, c_i). Chooser will choose a_i in either case because every A of size 6 that contains either b_i or c_i will also contain a_i. So, once again, Chooser can only benefit by choosing a_i over b_i or c_i. Summarizing, if Picker plays optimally; i.e., always taking pairs with the same subscript, then for every winning strategy in which Chooser chooses d_i, there exists a winning strategy in which he does not and for every winning strategy in which Chooser does not choose a_i, there exists a winning strategy in which he does.

So, we may assume that Picker picks either $\{(a_i, b_i), (c_i, d_i)\}$ or $\{(a_i, c_i), (b_i, d_i)\}$ for all i because if Picker picks $\{(a_i, d_i), (b_i, c_i)\}$, then the outcome is the same except that he cannot control which of $\{b_i, c_i\}$ he will be given by Chooser. This proves Claim 2. □

Now let Picker's $\{(a_i, b_i), (c_i, d_i)\}$ or $\{(a_i, c_i), (b_i, d_i)\}$ moves correspond to setting the value of $x_i = 1$ or $x_i = 0$, respectively.

First we show that if Chooser-Picker on $\mathcal{H}_\phi$ is a win for Picker, then ϕ is satisfiable. We may assume that Chooser plays according to the restrictions imposed by Claim 2. At the end of the game, Picker has exactly one of $\{b_i, c_i\}$. Chooser has a_i for all $i \in \{1, \ldots, n\}$. Let $\hat{x}_i = 1$ if Picker has b_i and $\hat{x}_i = 0$ otherwise. By the construction of $\mathcal{H}_\phi$, this means that $\phi(\hat{x}_1, \ldots, \hat{x}_n) = 1$.

Next we show that if ϕ is satisfiable, then Picker-Chooser on $\mathcal{H}_\phi$ is a win for Picker. Suppose that there is some assignment that $\phi = (\hat{x}_1, \ldots, \hat{x}_n)$. Picker makes sure to get b_i (i.e., Picker picks $\{(a_i, b_i), (c_i, d_i)\}$) if $\hat{x}_i = 1$, and makes sure to get c_i (i.e., Picker picks $\{(a_i, c_i), (b_i, d_i)\}$) if $\hat{x}_i = 0$. Because of Claim 2, we may assume that Chooser will always choose a_i for all $i \in \{1, \ldots, n\}$. As a result, Picker will get at least one element from every $A \in E$, and wins the game. This proves Theorem 2.12. □

Note that this theorem implies that Chooser-Picker games are NP-hard, even in the case of hypergraphs (V, E), for which $|A| \leq 6$ for all $A \in E$.

Torus games

To test Beck's paradigm from Conjecture 2that Chooser-Picker and Picker-Chooser games are similar to Maker-Breaker games, we check the status of concrete games defined on the 4×4 torus. That is, we identify the opposite sides of the grid, and consider all lines of slopes 0 and ± 1 and size 4 to be winning sets. We denote the torus, along with those winning sets with the notation 4^2. For the general definition of torus games, see [27]. We use a chess-like notation to refer to the elements of the board. We note that the hypergraph of winning sets on 4^2 is not almost disjoint, see e. g. the two winning sets $\{a2, b1, c4, d3\}$ and $\{a4, b1, c2, d3\}$. See Figure 2.16. We consider four possible games on 4^2: Maker-Maker, Maker-Breaker, Chooser-Picker and Picker-Chooser. According to [27], the Maker-Maker version of 4^2 is a draw, and, according to [50], Picker wins the Chooser-Picker version. Here, we investigate the Maker-Breaker and the Picker-Chooser versions. In fact, the statement of the Maker-Breaker version implies the result for the Maker-Maker version, while the proof of it contains the proof of the Chooser-Picker version.

Proposition 4. *Breaker wins the Maker-Breaker version of the* 4^2 *torus game.*

Proof. Using the symmetry of 4^2, we may assume, without loss of generality, that Maker takes $a4$. Breaker's move will then be to take $d1$. Up to isomorphism, there are eight cases depending on the next move of Maker. The first element of the pair is Maker's move, while the second is Breaker's answer: 1. $(c3, b2)$, 2. $(b3, b2)$, 3. $(c2, b2)$, 4. $(b4, c3)$, 5. $(c4, b4)$, 6. $(d4, c3)$, 7. $(d2, a3)$ and 8. $(d3, b1)$.

In the first seven cases Breaker has winning pairing strategies. All eight cases are shown in the first two rows of Figure 2.16 and the pairs appear under the labels A, B, C, D, and E. We leave it to the reader to check that the pairs block all 16 winning sets.

In the eighth case Breaker does not have pairing strategy, but the game reduces to one of the seven prior cases unless Maker plays $a3$, $a2$ or $a1$ in the third step of the game. In that case, Breaker plays $b4$, $a3$ or $b2$, respectively, and wins by the pairing strategy shown in the third row of Figure 2.16. □

Note that in the Chooser-Picker version of the game 4^2, Picker can achieve a position isomorphic to Case 1. That is, Picker wins.

If Conjecture 2 were true, then Breaker has an easier job in the Maker-Breaker version than Chooser has in the Picker-Chooser game. For the 4×4 torus the outcome of these games are the same, although this is much harder to prove.

Proposition 5. *Chooser wins the Picker-Chooser version of* 4^2*, the* 4×4 *torus game.*

Proof. (Sketch.) The full proof needs a lengthy exhaustive case analysis. However, some branches of the game tree may be cut by the following result of Beck [26]: Chooser wins the Picker-Chooser game on $\mathcal{H}$ if $T(\mathcal{H}) := \sum_{A\in E(\mathcal{H})} 2^{-|A|} < 1$.

In our case, $T(\mathcal{H}) = 16 \times 2^{-4} = 1$, which just falls short. Instead we use a similar method using so-called *potential functions*. We assign weights to each edge at the i^{th} stage such that $w_i(A) = 0$ if Chooser has taken an element of A, otherwise it is $2^{-f(A)}$, where $f(A)$ is the number of untaken elements of A. The weight of a vertex x is $w_i(x) = \sum_{x\in A} w_i(A)$, while the total weight is $w_i := \sum_{A\in E(\mathcal{H})} w_i(A)$.

Note that Picker wins if and only if both $w_8 \geq 1$ and $w_0 = T(\mathcal{H}) = 1$. When a pair (x, y) is offered, Chooser can always take the one with larger weight, which results in a non-increasing total weight. In fact, if the weights of x and y differ or both x and y are elements of an A of positive weight, then the total weight strictly decreases.

In order to have any possibility of winning, Picker has to select x and y of equal weights and no edge of positive weight containing both. By the symmetries of the board, we may assume Picker gets $a4$ and Chooser gets $c3$ in the first round. After that, Picker has only pairs (x, y) that do not result in a loss for Picker: $(b4, d3)$, $(a3, c4)$, $(b3, d4)$, $(a3, b3)$, $(a3, d3)$, $(b3, d3)$, $(a1, b2)$ and $(a1, d2)$, see Figure 2.17. The letter P [C] designates the

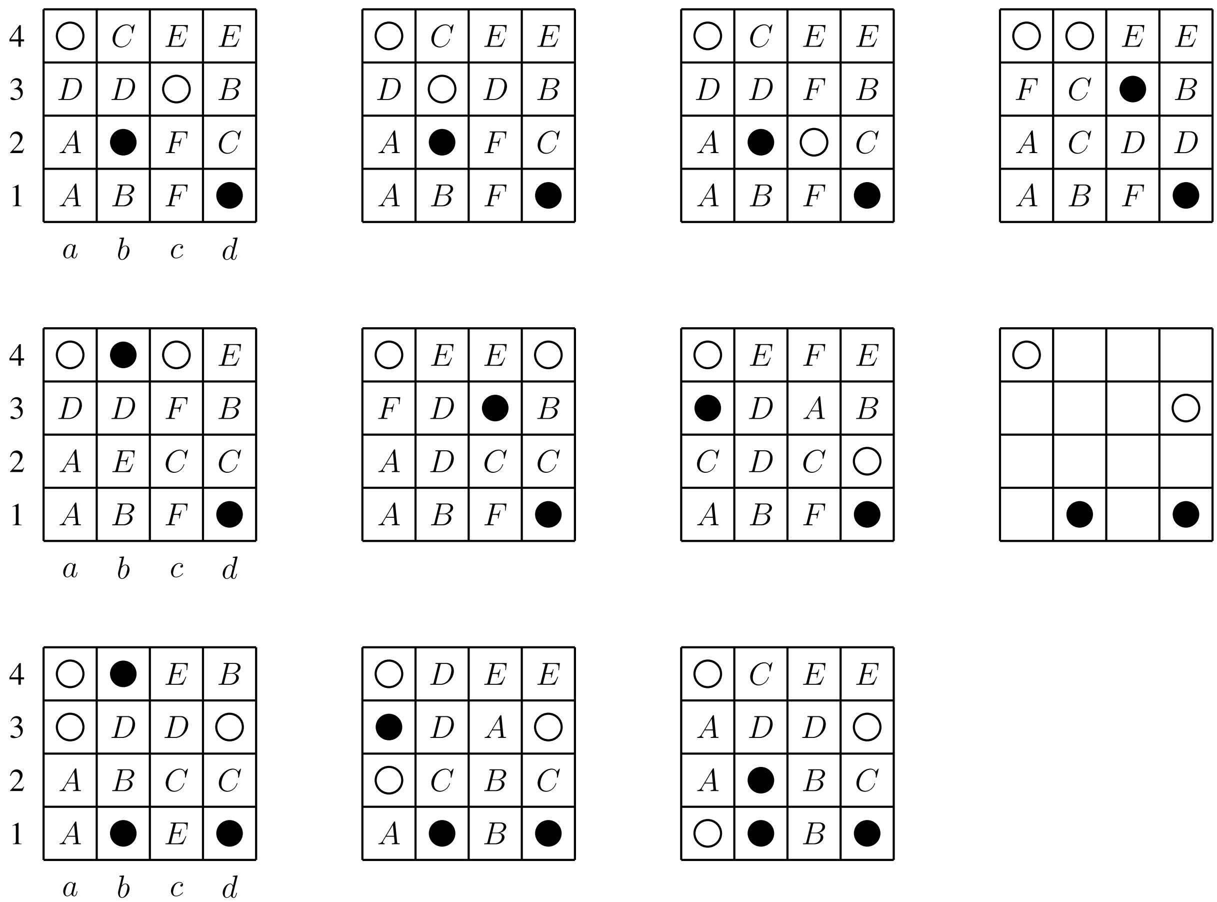


Figure 2.16: The pairings used by Picker in the game 4^2.

vertex taken by Picker [Chooser] in the first step, the numbers are the weights of the vertices.

4	P	$\frac{4}{16}$	$\frac{4}{16}$	$\frac{4}{16}$
3	$\frac{4}{16}$	$\frac{4}{16}$	C	$\frac{4}{16}$
2	$\frac{5}{16}$	$\frac{3}{16}$	$\frac{5}{16}$	$\frac{3}{16}$
1	$\frac{3}{16}$	$\frac{5}{16}$	$\frac{3}{16}$	$\frac{5}{16}$
	a	b	c	d

Figure 2.17: The beginning of the Picker-Chooser 4^2 game.

The rest of the proof is similar to that of the prior step: one needs to check that Chooser has winning strategy for each of the eight non-trivial responses of Picker. We omit the details. □

Chapter 3

Biased and accelerated games on graphs

As we mentioned before, the random heuristic is an excellent tool to guess to outcome of, possibly biased, positional games. For us the most intriguing cases when it *fails*. For the sake of completeness we need to recall some results from from the author's PhD dissertation, see [130], or in [131, 129].

The Maker-Breaker $(a:b)$-k-in-a-row behaves in a strange way. Maker wins for all $k \in \mathbb{N}$ assuming $a > b$, while if for example $b \geq 4a$, Breaker wins for $k \geq a+2$. The really hard case is when $a = b$. Because of the infinite board the expected number of one colored edges of the hypergraph is infinite for all $k \in \mathbb{N}$, and one cannot cut the game into disjoint sub-games. Still, with some effort it can be shown that Maker wins if $k \leq a + \log_2 a / \log_2 \log_2 a$ and Breaker wins if $k \geq a + 80 \log_2 a + 160$, provided $a \geq 1000$. Moreover if a restriction is added, namely that all Maker's marks should be within a distance d in each steps, then Breaker can win if $k \geq 240 \log_2 d + 480$, provided $a = 2b$ and $d \geq 4000$.

3.1 Diameter games

In diameter games are graph games played on K_n, the complete graph on n vertices. Fix a $d \in \mathbb{N}$. Taking turn, the players claim the edges of K_n, and Maker wins iff the diameter of G_M is not more than d. We denote the corresponding d-diameter game by $\mathcal{D}_d(a:b)$, or simply, by $\mathcal{D}_d$ if $a = b = 1$.

To relate this set-up with the probabilistic intuition, we use the monotone property $\mathcal{P}_d$, that is the graph has diameter at most d. Since the diameter of $G(n, 1/2)$ is 2 almost surely, one would expect that Maker wins $\mathcal{D}_2$. However, the probabilistic intuition fails completely in this case.

Proposition 6. *Assume that Maker starts the game $\mathcal{D}_2$. For $n \leq 3$ Maker has and for $n \geq 4$ Breaker has a winning.*

Proof of Proposition 6. One checks that for $n \leq 3$ the win is automatic. For $n \geq 4$ Breaker can choose an edge not incident to that chosen by Maker. Let this edge – the one Breaker chooses – be uv. The strategy of Breaker is: if Maker chooses an edge incident to u, say uw, then Breaker chooses wv. Similarly, if Maker chooses vw, then Breaker chooses wu. Otherwise Breaker may take an arbitrary edge. Clearly, at the end of the game, in Maker's graph the pair $\{u, v\}$ has distance at least three. $\square$

One thinks if Maker can take *two* edges in a move the outcome would be different. It turns out to be true, the acceleration of the game almost restores the random heuristic.

Theorem 3.1. *Maker wins the game* $\mathcal{D}_2(2 : \frac{1}{9}n^{1/8}/(\log n)^{3/8})$*, and Breaker wins the game* $\mathcal{D}_2(2 : (2+\epsilon)\sqrt{n/\ln n})$ *for any* $\epsilon > 0$*, provided* n *is large enough.*

We proved corresponding results for the game D_d for $d \geq 3$, too.

Theorem 3.2. *For any fixed* $d \geq 3$*, and* n *large enough, Maker wins the game* $\mathcal{D}_d(1 : (2d)^{-1}(n/\ln n)^{1-1/\lceil d/2 \rceil})$*.*

Furthermore, for every integer $a > 1$ *and integer* $d \geq 3$*, there exist* $c_2 = c_2(d) > 0$ *and* $c_3 = c_3(a, d) > 0$ *such that Breaker wins the games* $\mathcal{D}_d(1 : c_2 n^{1-1/(d-1)})$ *and* $\mathcal{D}_d(a : c_3 n^{1-1/d})$*, provided* n *is big enough.*

In fact Theorem 3.2 will hold even if d grows slowly, for example $d \leq c_1 \ln n/\ln\ln n$ for some positive constant c_1.

The game $\mathcal{D}_3(1 : b)$ deserves a closer look. The threshold for $\mathcal{P}_3$ is about $n^{-2/3}$; i.e., $G(n, p)$ has property $\mathcal{P}_3$ with probability close to 1 if $p = n^{-2/3+\epsilon}$, and it does not have property $\mathcal{P}_3$ if $p = n^{-2/3-\epsilon}$ for arbitrary $\epsilon > 0$ and n sufficiently large, see Bollobás [37].

As Theorem 3.2 shows, Maker wins the game $\mathcal{D}_3(1 : c_1\sqrt{n/\ln n})$, and Breaker wins the game $\mathcal{D}_3(1 : c_2\sqrt{n})$, for some $c_1, c_2 > 0$, provided n is big enough. That is, the game $\mathcal{D}_3(1 : b)$ also *defies* the probabilistic intuition. Most probably acceleration would brings the game bound closer to the probabilistic threshold. That is for $\mathcal{D}_3(3 : b)$-game the breaking point should be $b_0 \approx n^{2/3} \times \text{polylog}(n)$. Unfortunately, there has been no progress in that direction yet.

3.1.1 Auxiliary games

To prove Theorems 3.1 and 3.2 is quite long and technical. Instead we try to give a sketch of the proof and a special case of Theorem 3.1. We also introduce some of the deeper tools used in that process and to relate those to earlier results.

Namely, we define and state some results regarding minimum-degree and expansion games. However, before stating our results, let us recall and old theorem of Erdős and Chvátal [46], since it makes at least one direction of the proof of Theorem 3.1 easy.

Theorem 3.3 (Chvátal-Erdős [46])**.** *Let* $\mathcal{H}$ *be an* r*-uniform family of* k **disjoint** *winning sets. Then*
(i) Maker has a winning strategy in the $(a : 1)$*-game when*

$$r \leq (a-1)\sum_{i=1}^{k-1}\frac{1}{i}.$$

(ii) Maker has a winning strategy in the $(a : 2)$*-game when*

$$r \leq \frac{a-1}{2}\sum_{i=1}^{k-1}\frac{1}{i}.$$

Degree games

Given a graph G and a prescribed degree d, Maker and Breaker play an $(a : b)$-game on the edges of G. Maker wins by getting at least d edges incident to each vertex. For $G = K_n$ and $a = b = 1$ this game was investigated thoroughly in [155] and [25]. It was shown that Maker wins if $d < n/2 - \sqrt{n \log n}$, and Breaker wins if $d > n/2 - \sqrt{n}/12$.

For the biased case we give a condition for Maker's win first.

Lemma 7. *Let $a \leq n/(4 \ln n)$ and n be large enough. Then Maker wins the $(a : b)$-degree game on K_n if $d < \frac{a}{a+b}n - \frac{6ab}{(a+b)^{3/2}}\sqrt{n \ln n}$.*

Remark. We refer to the winning strategy of Maker as **playing as MINDEG-Maker**.

These results are in agreement with the probabilistic intuition, since in $G(n, 1/2)$ the degrees of all vertices fall into the interval $\left[n/2 - \sqrt{n \log n}, n/2 + \sqrt{n \log n}\right]$ almost surely. In the case $a \neq b$, playing on $G = K_n$ analogously one would expect that Maker wins if $d < an/(a + b) - c'\sqrt{n \log n}$, and Breaker wins if $d > an/(a + b) - c''\sqrt{n}$ for some $c', c'' > 0$.

Lemma 8. *Breaker wins the $(a : b)$-MINDEG(d) game on K_n if $d > a\left\lfloor \frac{n}{a+b} \right\rfloor$, if $n > 2a$.*

Remark. We refer to the winning strategy of Breaker as **playing as MINDEG-Breaker**.
Proof of Lemma 8. In the first round Breaker chooses a vertex, say v, which Maker has not touched and chooses all of his edges to be incident to that vertex in every round. At the end of the game, Maker has chosen at most $a\left\lfloor \frac{n-1}{a+b} \right\rfloor$ edges incident to v. $\square$

Expansion game

Different type of expansion properties of a graph are extremely useful, see Pósa [137]. These can properties might be achieved by Maker in a graph game as it was demonstrated by Beck in [28].

In the expansion game, Maker wins by achieving that for every pair of disjoint sets R and S, where $|R| = r$ and $|S| = s$, there an edge between R and S in G_M. We may assume that $s \geq r$.

Lemma 9. *Maker wins the $(a : b)$-expansion game on K_n with parameters $r \leq s$ if one of the following holds:*

a. $2b \ln n < r \ln(a + 1)$,

b. $b \ln n < r \ln(a + 1) \leq 2b \ln n$ *and* $s > \frac{rb \ln n}{r \ln(a+1) - b \ln n}$,

c. $n - s < \frac{nr \ln(a+1)}{b \ln n + r \ln(a+1)}$.

Remark. We refer to the winning strategy of Maker as **playing as EXP-Maker**.

3.1.2 About the proof of the 2-diameter game

If Maker takes more edges per move than Breaker, then a winning condition come easily from Lemma 7:

The $b < a$ case in the 2-diameter game.

First we prove that Maker wins the $\mathcal{D}_2(a : b)$-game if $b < a < (n/(72 \ln n))^{1/3}$ and n is large enough. Maker's strategy is to play the degree game with $d = \left\lceil \frac{n-1}{2} \right\rceil$ on K_n. By Lemma 7, he wins the game and it is easy to check that the diameter of Maker's graph is 2, i.e., he wins the $D_2(a : b)$-game. Indeed, if uv is not a Maker's edge for some vertices u and v, then $|N(u) \cap N(v)| \geq n - 2 - 2\left(n - 2 - \left\lceil \frac{n-1}{2} \right\rceil\right) > 0$, implying that the intersection is non-empty. That is, in Maker's graph, the distance between u and v, hence the diameter of the graph, is at most two. □

Breaker wins when b is large.

We prove that Breaker wins the $\mathcal{D}_2(2 : b)$-game for $b = (2 + \epsilon)\sqrt{n/\ln n}$. Breaker plays in two phases. In Phase I, before his first move, he picks a vertex v which has no edge in Maker's graph yet. For $r' \leq (n + b - 1)/b$ rounds, he occupies as many incident edges to v as possible. Let $u_1, \ldots, u_t$ be the list of vertices so that Maker occupied the edge vu_i before Breaker makes his $(r' + 1)^{\text{st}}$ move. Trivially $t \leq 2r' + 2$. At the end of Phase I, there is no unclaimed edge incident to v.

In Phase II, Breaker considers $n - t - 1 - 2r'$ disjoint sets of edges: For each vertex $x \notin \{v, u_1, \ldots, u_t\}$ such that neither xv nor any xu_i is occupied by Maker after round r', define $E_x := \{xu_1, \ldots, xu_t\}$. By Theorem 3.3 (ii), Breaker can occupy one of these sets, say E_x, when

$$t \leq \frac{b-1}{2} \ln(n - t - 1 - r'),$$

which is satisfied for $b = (2 + \epsilon)\sqrt{n/\ln n}$, if n is large enough. This forces v and x, in Maker's graph, to be at a distance of at least 3 from each other, i.e., Breaker won the game. □

Maker wins when b is small (sketch).

We set r, s and c such that the possible value of b is maximized. The values will be

$$b = \frac{n^{1/8}}{9(\ln n)^{3/8}}, \qquad c = \frac{1}{8}, \qquad r = \sqrt{\frac{n \ln n}{2}} \qquad \text{and} \qquad s = \frac{n^{3/4}}{\ln n}, \tag{3.1}$$

although we will not substitute these values until the end of the proof.

Maker's strategy consists of two phases. The first one, which lasts $2nr$ rounds, uses $2nr(b + 2)$ edges, and the second deals with the rest of the $\binom{n}{2}$ edges. In the first phase, Maker will play four subgames, each with a different strategy.

Denote $\deg_B^I(x)$ to be Breaker's degree and $\deg_M^I(x)$ to be Maker's degree at vertex x after Phase I. In general, $\deg_B(x)$ and $\deg_M(x)$ will denote Breaker's and Maker's degrees, respectively, in whichever round the context indicates.

Phase I. There are $2nr$ rounds in this phase. Each of the following games is played in successive rounds. That is, Maker plays game i in round j iff $i \equiv j \pmod 4$. A vertex becomes *high* if it achieves $\deg_B(x) \geq cn/b$ before the end of Phase I. Note that Maker's goals are monotone properties, i.e., if the strategy of a subgame requires Maker to occupy an edge that he already occupied in an other subgame, then he is free to use his edges in any way for this turn. The goals of Maker in the four games played in Phase I are the following:

- **Game 1. Ratio game.** If vertex x becomes high, then after this change the following relation will hold during the rest of Phase I: $\deg_B(x)/\deg_M(x) < 3b$.
- **Game 2. Degree game.** For all vertices x, $\deg_M^I(x) \geq r$.
- **Game 3. Expansion game.** For every pair of disjoint sets with $|R| \geq r$ and $|S| \geq s$ there is a Maker's edge between R and S at the end of Phase I.
- **Game 4. Connecting high vertices.** In this subgame, which lasts in the entire game not only in Phase I, the aim of Maker is to connect each pair of high vertices with a path of length at most two.

Phase II. In the odd rounds of this phase, Maker will connect with a path of length at most two each pair of vertices whose distance in Maker's graph is at least 3. As for the even turns of this game, half of them are already dedicated to continue Game 4, the other half are arbitrary moves by Maker. Because Game 4 played in the entire game, it is easier to analyze the connection of pairs of vertices by performing it only in odd rounds.

By Game 2, after Phase I is finished, $\deg_M^I(u) \geq r$ for every vertex u. By Game 3, after Phase I is finished, in Maker's graph there is an edge from the neighborhood of u into every s set of vertices, hence to all but s vertices there is a path of length at most 2 from u at the end of Phase I. The aim of Maker in Phase II to connect the remaining pairs of vertices with a path of length at most 2. This is handled in Game 4 for pairs (u, w) when both are high. So in the odd rounds in Phase II we only need to connect u and w where either u or w is not a high vertex.

3.2 All the time connected Shannon's switching game

One of the most discussed graph game is the classical Shannon's switching game, which we mentioned in the Introduction. This is a Maker-Breaker game on the edge set of a connected graph G, and Maker wins by taking the edges of a spanning tree. The outcome of this game is characterized by Lehman's theorem [111] stating that Maker wins (as a second player) if and only if the graph contains two edge-disjoint spanning trees.

To play it on K_n, Chvátal and Erdős [46] introduced the $(1:b)$ biased version. The outcome is a monotone function of b in a sense that if Maker wins for a value b, and $b' < b$ then Maker also wins the $(1:b')$-game. It turned out later that the turning point is around $b_0 = \Theta(n/\log n)$, Gebauer and Szabó [71]. Of course this is nothing else but another highly non-trivial example of the probabilistic intuition.

Epsig, Frieze, Krivelevich and Pedgen et al[62] introduced new variants of connectivity games, the so-called *Walker-Breaker game* and *PathWalker-Breaker game*. If Walker's position is a vertex x, he may claim any edge $e = (x, y)$ such that e has not been taken by Breaker before and changes his position to y. PathWalker is even more restricted; he is allowed to visit a vertex only once. For Breaker's moves, there are no restrictions.

Walker and PathWalker wants to visit as many vertices of G as possible. It was shown that Walker (and even PathWalker) reaches at least $n-2$ vertices of K_n for large n. In the $1:b$-game the number of vertices that can be visited by PathWalker falls into the interval $[n-c_1\log n, n-c_2\log n]$, where the values of $c_2<c_1$ depend on only b.

Motivated by their approach and some classic problems, we define new versions of Shannon's switching game. These are Maker-Breaker games in which Maker's goal is to get a connected spanning subgraph G_M of a graph G such that the subgraph consisting of Maker's edges is connected throughout the whole game. It is convenient to call such a game *PrimMaker-Breaker* game paying tribute to Prim's celebrated algorithm. Note that *Prim's algorithm* [138] finds a (minimal) spanning tree in a weighted undirected graph by keeping subgraph of the already selected edges connected in contrast to *Kruskal's algorithm* [109] which does not have this property.

We are lucky enough to have a complete characterization for the $(1:1)$ unbiased game. Let H_n be the graph that we get from $K_{n-2,2}$ by joining the two vertices in its two-element color class, see Figure 3.1.

Theorem 3.4. *Playing the PrimMaker-Breaker game on a graph G with n vertices, PrimMaker wins as a first player if and only if G contains H_n as a subgraph.*

Proof of Theorem 3.4. It is remarkable that in both directions of the proof of Theorem 3.4 the actual winner may utilize a pairing strategy. First we show if a graph G on n vertices contains the subgraph H_n, then PrimMaker wins the game as a first player. PrimMaker might restricts his moves the the edges of H_n, as follows. His first move is the edge $e=(u,v)$, the edge added to $K_{n-2,2}$, see Figure 3.1. The other edges of H_n are paired such that $f,g\in E(H_n)\setminus\{e\}$ is a pair if they incident and their common endpoint lies in $V(H_n)\setminus\{u,v\}$. PrimMaker plays according to this pairing, more precisely, in every turn, he takes one element of a pair. This keeps his subgraph connected and results in a spanning tree in the $(n-1)$st move.

For the other direction, let us assume that G does not contain H_n, and PrimMaker first move is an edge $e=(u,v)$. Then there must be a vertex $x\in V(G)\setminus\{u,v\}$, such that $|N(x)\cap\{u,v\}|\le 1$. Now Breaker might also use a pairing strategy: whenever PrimMaker connects a new vertex y to his subgraph, i. etakes an edge (z,y), where z had been visited earlier, then Breaker takes the edge (y,x) if $(y,x)\in E(G)$, and moves arbitrarily otherwise. Obviously, PrimMaker can never connect the vertex x to his subgraph. □

We proved a little more than was stated in Theorem 3.4. By winning PrimMaker builds a subgraph of diameter not more than three, which type of games were explored in [12]. Note the close resemblance to the proof of Theorem 6.

Breaker's strategy can be adopted to a $(1:b)$-game on K_n, and shows that Breaker wins if $b>1$, in contrast to the probabilistic intuition which predicts $b_0=\Omega(n/\log n)$. As it was observed before, acceleration of games has surprising effects, and it may restore

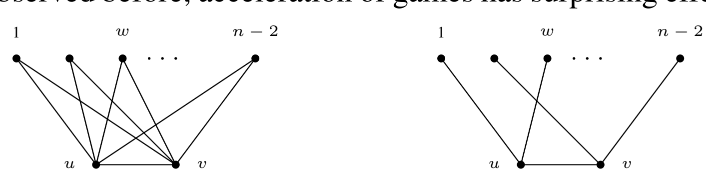


Figure 3.1: The graph H_n and a possible Maker's subgraph.

the probabilistic intuition destroyed by a pairing strategy in the $(1:1)$-game [12]. Here we can witness a, in magnitude, perfect restoration of that intuition.

Theorem 3.5. *Playing the $(2:b)$ PrimMaker-Breaker game on K_n, Maker wins if $b < n/(8\log n)$, and Breaker wins if $b > n/\ln n$.*

3.2.1 Background

The following result is not just one of the most important one in the theory of hypergraph games, but it can be used very effectively to decide the winner of biased hypergraph games. For the case $a = b = 1$ it was proved by Erdős and Selfridge in [58], the general form by Beck in [22].

Theorem 3.6. *If*

$$\sum_{A\in E(\mathcal{H})} (1+b)^{-|A|/a} < 1,$$

then Breaker has a winning strategy in the $(\mathcal{H}, a, b)$ game.

Still, several times not Theorem 3.6 but its proof techniques and corollaries are used.

For the sake of better understanding and introducing some notations, we give a sketch of the proof of the case $b = 1$, and all elements of $E(\mathcal{H})$ have the same size, a more detailed detailed proof can be found in [131].

The uniform case with $b = 1$. For any $A \in V(\mathcal{H})$ let $A_k(M)$ and $A_k(B)$ be the number of elements in A, after Maker's kth move, selected by Maker and Breaker, respectively. Now, for an $A \in E(\mathcal{H})$

$$w_k(A) = \begin{cases} \lambda^{A_k(M)} \text{ if } A_k(B) = 0 \\ 0 \text{ otherwise,} \end{cases}$$

where $\lambda = 2^{1/a}$. For any $x \in V(\mathcal{H})$ let $w_k(x) = \sum_{x\in A} w_k(A)$. The numbers $w_k(A)$ and $w_k(x)$ are called the *weight* of A and x (in the kth step), respectively.

In the kth step Breaker chooses an unselected element $y^k \in V(\mathcal{H})$ of maximum weight. Setting $w_k = \sum_{A\in E(\mathcal{H})} w_k(A)$, called the *potential*, one gets $w_k \geq w_{k+1}$, $k \geq 0$.

Particularly $w_1 \leq (\lambda^a - 1)|E(\mathcal{H})| + |E(\mathcal{H})| \leq 2|E(\mathcal{H})|$. Since $b = 1$ and the elements of $E(\mathcal{H})$ are of the same size, the inequality $\sum_{A\in E(\mathcal{H})} 2^{-|A|/a} < 1/2$ leads to the inequality $2|E(\mathcal{H})| < 2^{|A|/a}$. Let us suppose that Maker wins the game in the kth step. This would imply $w_k \geq \lambda^{|A|} = 2^{|A|/a}$, contradicting the monotonicity of the potential. □

An edge $A \in E(\mathcal{H})$ is *active* if Breaker has not taken any of its elements. On reverse, $A \in E(\mathcal{H})$ is *blocked* if Breaker has already taken an element of it. Since $w_k \leq w_1 \leq 2|E(\mathcal{H})|$ for all k, we have a bound on the "fill-in" of an active edge. Note, that this bound holds for the non-uniform hypergraphs, too.

Corollary 4. *[131] Playing the Maker-Breaker $(\mathcal{H}, a, 1)$ game, Breaker may arrange that whenever A is active, i. e$A_k(B) = 0$, then $A_k(M) \leq a + a\log_2|E(\mathcal{H})|$.*

Proof of Corollary 4. Just take the logarithm of the inequality $\lambda^{A_k(I)} = w_k(A) \leq w_k \leq w_1 \leq 2|E(\mathcal{H})|$ that holds for any active edge $A \in E(\mathcal{H})$. □

3.2.2 Proof of Theorem 3.5

PrimMaker's win. First we describe the winning strategy then show its feasibility. PrimMaker plays an equivalent auxiliary game, the *positive minimum degree game*, see Espig et al. [62], with the additional requirement that his subgraph should be connected during the game.

PrimMaker tries to get edges incident to each vertices as fast as possible. More precisely, he can guarantee an edge incident the vertex x, before Breaker takes, say $n/4$ edges incident to x. This can be done by an appropriate weight function method used before several times [129, 131, 132].

In order to utilize Corollary 4, we associate an auxiliary hypergraph game to the PrimMaker-Breaker game. To each vertex $x \in V(K_n)$ let the $A_x \in E(\mathcal{H})$ be the set of ordered pairs $\langle x, y\rangle$, where $y \in V(K_n) \setminus x$. That is $A_x \cap A_y = \emptyset$ for $x \neq y$ and $|A_x| = n-1$ for all $x \in V(K_n)$. When Maker takes the edge (x, w) in the graph game, it results in taking both $\langle x, y\rangle$ and $\langle y, x\rangle$ in the hypergraph game. Of course Breaker one move means taking $2b$ ordered pairs. Note that PrimMaker intends to play *as Breaker* in this auxiliary game.

Let us assume that PrimMaker can imitate the greedy strategy of Corollary 4 in the $(\mathcal{H}, 1, 2b)$ game. Note, that in order to do so, PrimMaker *does not have to* take the pair (edge) $\langle x, y\rangle$ of the largest weight, taking any pair from the largest weight hyperedge has the same effect on the potential function w_k.

Extending the notation of Corollary 4, we may say a vertex x is *blocked* if A_x is blocked, i. ePrimMaker has an edge that is incident to x. We shall prove by induction on the steps of the game that an arbitrary vertex can be blocked at each step. The induction hypothesis holds in the first step, and assuming it holds until the kth step, we can use the bound of Corollary 4. It gives that Breaker can take at most $b + b\log_2 n \leq n/4$ edges are incident to an unblocked vertex x. Note, that we can also assume that PrimMaker's edges form a tree T_i after the ith step, and $i \leq n/2$. Indeed, in the process of blocking we never need to create cycles, so $|V(T_i)| = 2i + 1$ if the game is not over already.

Let T_k be PrimMaker's graph and U_k be the set of unconnected (unblocked) vertices by PrimMaker after the kth round, respectively. Assume, that the blocking strategy requires to block (connect) the vertex $x \in U_k$ in the $(k+1)$st step. If there is an unoccupied edge $e = (x, y)$, $y \in T_k$, then we take it. Similarly, if there are unoccupied edges $e = (x, y)$ and $f = (y, z)$, $z \in T_k$ then we take those, and x is blocked.

Assume on contrary there is a vertex $x \in U_k$ that cannot be blocked by PrimMaker in the $(k+1)$st step, that is in the subgraph of unoccupied edges there are no paths of length at most two from x to T_k. According to the induction hypothesis, we know that Breaker has taken less than $n/4$ edges incident to x. The other endpoints of these edges cannot be in T_k and actually all edges between these endpoints and the vertices of T_k are taken by Breaker. The number of those edges is at least $(n-1-n/4)(2k+1) < 3nk/2$, since $k \leq n/2$. After round k, Breaker has claimed bk edges, therefore we would have $3nk/2 \leq bk$, which contradicts of the choice of b.

Breaker's win. One checks it is an easy consequence of Theorem 3.3. In fact Breaker wins in the original Maker-Breaker $(2 : b)$ Shannon's switching game. □

Chapter 4

Extended games

In the study of several games the question is not who (and how) wins, but how long does it take, what size of board is needed etc, see some example in [4, 29, 20, 21, 26, 28, 90].

We have already seen another idea in subsection 2.1.11. Since an infinite Chooser-Picker game has no mean, we extend the game backwards giving the opportunity to Chooser to carve out a finite board on which the game is to be played.

A general approach to create a Maker-Breaker game is to select a monotone graph property $\mathcal{P}$, let the players take the edges of K_n alternately (possibly in a biased $(a : b)$ distribution), and Maker wins iff G_M has property $\mathcal{P}$. The probabilistic intuition, or rather the larges number of examples show that Maker needs only a fraction of the edges, i.e. may win the game even if $a << b$. So it is natural to ask, what is a minimal subgraph of K_n for which Maker wins the corresponding $(1 : 1)$-game? One can make this question to be precise several ways[1], here we explore the following version. In the zeroth step Maker select a $G \subset K_n$ such that $|E(G)|$ is minimal and Maker has a win playing on G. We discuss these games in Section 4.1.

Another idea is the extend a positional game to “the other side", that is continue the play that has been ended by the rules. In fact in most of the really played board games (chess, checkers, nine-men-morris etc.) the set pieces can move according to certain rules, and this is vital part of the game. We introduce this extension to positional games and give some results in Section 4.2.

4.1 Positive minimal degree game

We consider Maker-Breaker *graph games*; that is the board is such a hypergraph $\mathcal{F} = (V, \mathcal{H})$, where $V \subset E(K_n)$ and $\mathcal{H} = \mathcal{P}$ is a graph property. Here we restrict our interest to *global properties*, that is if $A \in \mathcal{P}$ then the subgraph spanned by A has n vertices. Now, for a property $\mathcal{P} = E(\mathcal{H})$, we look for $\hat{m}(\mathcal{P})$, the smallest size of $V \subset E(K_n)$ such that Maker wins playing on V.

The idea, notations and first results due to D. Hefetz, M. Krivelevich, M. Stojaković and T. Szabó [91]. Note that in these games always Breaker starts game.

Their first obvious choice is the Shannon's switching game, in which $\mathcal{H}$ consists of the edge sets of the spanning trees in K_n, [111]. So Maker wins if and only if starting from

[1] For example Stojaković and Szabó considered a random part of K_n in [152].

an $G \subset K_n$ that contains two edge-disjoint spanning trees. That is $2n-2$ appropriately selected edges results in a Maker's win, while Breaker wins if G has fewer than $2n-2$ edges.

Another idea is to define *minimum degree games*, that is in $\mathcal{D}_k := \mathcal{D}_k(n)$ game Maker wins if the minimum degree in G_M is at least k. (We have already mentioned other degree games in subsection 3.1.1, see also in [12, 25, 29, 155].)

Hefetz et. al [91] gave the following bounds $11n/8 \leq \hat{m}(\mathcal{D}_1) \leq 10n/7 + 4$, and $\hat{m}(\mathcal{D}_1) \leq 10n/7$ if $n = 7\ell$.

They also worked on Hamiltonicity, in that case property $\mathcal{P}$ was all subgraph of K_n having Hamilton cycle.

We close the gap between the bounds and give the exact value of $\hat{m}(\mathcal{D}_1)$ in [13].

Theorem 4.1. *For* $n \geq 4$

$$\left\lceil \frac{10}{7} n \right\rceil = \hat{m}(\mathcal{D}_1), \textit{ for } n \not\equiv 2 \pmod 7,$$

and

$$\left\lceil \frac{10}{7} n \right\rceil + 1 = \hat{m}(\mathcal{D}_1), \textit{ for } n \equiv 2 \pmod 7.$$

It means that there exists graphs with $\lceil \frac{10}{7} n \rceil$ (or $\lceil \frac{10}{7} n \rceil + 1$, if $n = 7\ell + 2$) edges on n vertices on which Maker wins, and on graphs with fewer edges always Breaker wins. Similar, but much weaker results might be spelled out for $\hat{m}(\mathcal{D}_2)$ and $\hat{m}(\mathcal{D}_k)$.

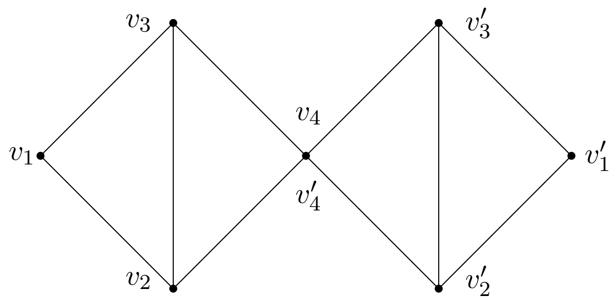
Proposition 7. *For all* $k \in \mathbb{N}$, $\hat{m}(\mathcal{D}_2) \leq \frac{20n}{7}$ *for* $n = 14k$, *and* $\hat{m}(\mathcal{D}_k) \leq \frac{10}{7} kn$ *for* $n = 7^k$.

We can show easily that at least the second part of Proposition 7 cannot be optimal. As mentioned in subsection 3.1.1, L. Székely [155] and J. Beck [25] studied degree games on the edges of $K_{n,n}$. Briefly Maker wins if $k < n/2 - \sqrt{n \log n}$, and Breaker wins if $k > n/2 - \sqrt{n}/12$. To use this bound, let us pave K_n by $n/(2\ell)$ copies of $K_{\ell,\ell}$, we get a $G \subset K_n$ with $n\ell/2$ edges. Maker plays separately on each copy of $K_{\ell,\ell}$, achieving at least $\ell/2 - \sqrt{\ell \log \ell}$ degree at each vertex. One check that $\ell \leq 2k + 4\sqrt{k}$, if ℓ is big enough, that is Maker wins $\mathcal{D}_k$ on G, which yields $\hat{\mathcal{D}}_k = (k + 2\sqrt{k})n$. Of course $(k + 2\sqrt{k})n < 10kn/7$, provided $k > 32$. In other words, while we are working on small degrees, explicit constructions, and the careful study of arising structures give the best results. However, for larger degrees the weight function method prevails, since this can take into account the current state of the game in a more flexible way.

Remark. This is not a unique phenomenon, the "elementary methods" (pairing strategies, case studies, subgame partitions etc) are always competing with the sophisticated use of weight function methods, and these are sometimes even combined, see a very subtle result of J. Beck on clique games in [26]. With the advantage of hindsight we may recognize a pattern that we word for graph games. If the graphs (the playing board G itself or G_M, the part that Maker gets eventually) are dense, then the weight function methods, if these are sparse, then elementary methods may yield better results. For the latter two nice examples are shown for biased graph games on K_n, see Hefetz, Krivelevich, Stojaković and Szabó in [89] (Hamilton cycle game) and Gebauer and Szabó [71] (Shannon's switching game). In these works the original weight function approach is (partly) substituted with more direct approach improving the game of Maker to be nearly optimal.

4.1.1 Proofs

Proof of Theorem 4.1. Observe that for $n \leq 3$ Breaker always wins, so we may assume that $n \geq 4$. First we construct graphs on which Maker can win, establishing the upper bounds. It is easy to check that Maker wins on K_4, $K_{3,3}$, and D_7, where D_7 is a “double diamond", that is we take two copies of $K_4 \setminus v_1v_4$, and identify the two vertices called “v_4." Note that this was observed already in [91].



The graph D_7. Here $v_4 = v_4'$.

For $n = 7k$, Maker wins on k vertex disjoint copies of D_7. For $n = 7k + 4$, Maker wins on k vertex disjoint copies of D_7 and one copy of K_4. Similarly, for $n = 7k + 6$, Maker wins on k vertex disjoint copies of D_7 and one copy of $K_{3,3}$. For the rest of the cases, the following observation is sufficient: Assume that Maker wins on G. Form G' by adding a vertex to G, with edges to arbitrarily chosen two other vertices. Then by playing the winning strategy on the edges of G and a simple pairing on the two newly selected edges, Maker wins on G' as well. For $n = 7k+1$ Maker wins on $(k \cdot D_7)'$, for $n = 7k+2$ on $((k \cdot D_7)')'$, for $n = 7k + 3$ on $(k - 1) \cdot D_7 + K_4 + K_{3,3}$, and for $n = 7k + 5$ on $(k \cdot D_7 + K_4)'$. □

To prove the lower bounds we need two steps. Since if Maker wins for a graph G, also wins for any G' that contains more edges than G. This allows us to consider a graph G on n vertices such that it has as few edges as possible and still Maker, as a second player, can achieve degree at least one at every vertex. We first collect some properties of G. Note the form of these properties are mainly local information bout the distribution of degrees in G. A similar approach were used in [91].

In the second step we merge together this local information into a global bound. While significant improvement could be achieved by a careful partition of G, for the optimal result we need the method of *discharging* into the realm of games.

Let us start with a definition.

A path $xz_1 \ldots z_my$ is a (k, ℓ)-(x, y)-*path*, if $d(x) = k$, $d(z_1) = \ldots = d(z_m) = 3$ and $d(y) = \ell$. Note that m can be 0, that is the path consisting of the edge xy.

Lemma 10. *The graph G has the following properties.*
(i) For every $x \in V(G)$ we have $d(x) \geq 2$.
*(ii) There is no $(2, 2)$-(x, y)-*path *in G.*
(iii) For any $k \geq 3$, there are no vertices $x, y_1, \ldots, y_{k-1}$ with a $(k, 2)$-(x, y_i)-path for every i.
(iv) There is a vertex $x \in V(G)$ with $d(x) = 2$.
(v) In a component of G if there is a vertex $x \in V(G)$ with $d(x) = 2$ then either there is a vertex $y \in V(G)$ with $d(y) \geq 4$ or the component consists of at least 7 vertices.

Proof of Lemma 10. If (i) was false then Breaker would trivially win instantly.

To prove (ii), assume for a contradiction that G contains a $(2,2)$-(x,y)-path $xz_1 \dots z_m y$, where m is chosen to be minimum possible. The minimality of m implies that this path is an induced path. Now Breaker easily wins, claiming edges $xz_1, z_1z_2, \dots, z_m y$. In order to avoid instant loss, Maker has to claim the last unclaimed edge at $x, z_1, \dots, z_m$, and finally Breaker could claim the last unclaimed edge at y.

For (iii), assume that for some $k \geq 3$ such a path system exists. We might choose one, with minimum number of vertices. Then using (ii) and the minimality, any edge spanned by the vertex set of this path system is also an edge of some of those paths. Now as in the proof of part (ii), Breaker can claim all path edges, starting from the vertices y_i, and then could claim the last edge at x.

Part (iv) follows from that if G was a counterexample for Theorem 4.1, then its density was smaller than $3/2$.

For (v) assume that G is a counterexample. Then by (ii), each component of G contains at most one vertex of degree 2. Each such component should have odd number of vertices. There is one possible graph with 5 vertices, and it is easy to see that Breaker wins on it. $\square$

We now apply the *discharging method*, see e. g. [159]:

In the charging phase a vertex $v \in V(G)$ is assigned a weight or *charge* $w(v) := d(v)$.

In the discharging phase the vertices send some of their charges to other, not necessarily neighboring vertices. The rules of the discharging are as follows:

1. A vertex of degree 2 sends no charge.
2. Only vertices of degree 2 receive any charge.
3. A 3-degree vertex x sends charge $1/7$ to a 2-degree vertex y if there is a $(3,2)$-(x,y)-path.
4. If for a $k > 3$ there is a $(k,2)$-(x,y)-path, then x sends a charge of $4/7$ to y.

In the beginning the sum of the charges is the sum of the degrees. The sum of the charges does not change during the discharging phase, so the following claim completes the proof for the case when $n \neq 7\ell + 2$.

Claim 2. *After the discharging phase every vertex has charge at least* $20/7$.

Proof of Claim 2. Observe that charges are staying within components. If every vertex of a component is of degree at least 3, then the charges do not change, and the claim is trivially true. If a component has a degree-2 vertex but has no vertex of degree at least 4, then by Lemma 10 (v) it consists of at least 7 vertices. By Lemma 10 (ii) such a component contains exactly one vertex of degree 2. Each vertex of degree 3 sends a charge of $1/7$ to the 2-degree vertex, which will have charge at least $2 + 6 \cdot 1/7 = 20/7$. The 3-degree vertices will have charge $3 - 1/7 = 20/7$.

Now consider a component, which contains vertices both of degree 2 and at least 4. By Lemma 10 (iii), a vertex x with $d(x) = 3$ sends to at most one 2-vertex a charge of

$1/7$, so it will have charge at least $3 - 1/7 = 20/7$. For any $k > 3$, by Lemma 10 (iii), a vertex x with $d(x) = k$ will have charge at least

$$k - 4(k-2)/7 = 3k/7 + 8/7 \geq 20/7. \quad (4.1)$$

Now assume that $d(x) = 2$. For some $k > 3$ there is a $(2,k)$-(x,y) path for some k-vertex y. So x will receive from y a charge of $4/7$. If $xy \notin E(G)$ then x has two neighbors of degree 3, and receives charges of $1/7$ from both. So the charge of x will be at least $2 + 4/7 + 1/7 + 1/7 = 20/7$. Otherwise, let z be the other neighbor of x. Observe, that $d(z) \geq 3$, because of Lemma 10 (ii), and z should send a charge of at least $1/7$ to x. Observe also that the vertex z must have a neighbor w that differs both from x and y, and, by Lemma 10 (ii) the degree of w is at least 3. But then w sends a charge at least $1/7$ to x, which is sufficient to achieve charge of at least $20/7$ at x. □

The proof of Theorem 4.1 is completed when $n \not\equiv 2 \pmod 7$.

Assume that $n = 7k + 2$ for some positive integer k. Then the sum of the charges in G is at least $(7k+2) \cdot 20/7 = 20k + 40/7$, yielding that $e(G) \geq 10k + 3$. If we were able to find extra $3/7$ charges, then this would be sufficient to imply $e(G) \geq 10k + 4$, and then the proof of Theorem 4.1 would have been completed.

Trivially, each component contains at least four vertices. If there is a component, with each of its vertices having degree at least 3, then each will have extra charge $1/7$ and the proof is completed. Now we can assume that each component contains a degree-2 vertex. If there is a vertex of degree at least 5, then (4.1) implies that this vertex will have at least extra $3k/7 + 8/8 \geq 20/7 + 3/7$ charge. So now we can assume that the maximum degree is at most 4. If there is a component which contains only vertices of degree 2 and 3, then Lemma 10 (iii) implies that it contains only one degree-2 vertex, and by parity reasons $2s$ degree-3 vertices for some s. One can check (we omit the details) that Breaker wins if $s \leq 4$. On the other hand, if $s \geq 5$ then we have extra charge at least $2s/7 - 6/7 \geq 4/7$. So now we can assume that each component contains vertices of both degree 2 and 4.

Since $n \equiv 2 \pmod 7$, G must have a component C, such that $|C| \not\equiv 0 \pmod 7$.

Assume now that in such a component C there are $\ell \geq 1$ vertices of degree 4. Then there should be exactly $2\ell - 1$ or 2ℓ vertices of degree 2. It cannot be fewer, otherwise either a degree-4 vertex retains charge $4/7$ or two degree-2 vertices overcharged by $2/7$ and we are done. It cannot be more, as now every degree-2 vertex receives charge from a degree-4 vertex, and a degree-4 vertex can send charges to at most two degree-2 vertices.

Case 1. There are exactly $2\ell - 1$ vertices of degree 2. Observe that each degree-4 vertex should send charge to at least two degree-2 vertices, otherwise its remaining charge would be at least $28/7 - 4/7 = 20/7 + 4/7$. So either one degree-2 vertex receives charges from at least three degree-4 vertices and becomes heavily overcharged, or at least two degree-2 vertices receive charges from two different pairs of degree-4 vertices, and the total overcharge is at least $2/7 + 2/7 = 4/7$.

Case 2. There are exactly 2ℓ vertices of degree 2. The proofs of Case 1 and Claim 2 imply that each degree-2 vertex receives charges from one degree-4 and at least two degree-3 vertices. So the number of degree-3 vertices is at least 4ℓ. Because $|C| \not\equiv 0 \pmod 7$, it is more than 4ℓ, but it must be less than $4\ell + 4$, otherwise there would be $4/7$ extra charge, So the number of degree-3 vertices is $4\ell + 2$.

Denote $\{x_1, \ldots, x_\ell\}$ the set of degree-4 vertices in C. Observe that $C - \{x_1, \ldots, x_\ell\}$ has at least 2ℓ components, and 2ℓ of them contain a degree-2 and at least two degree-

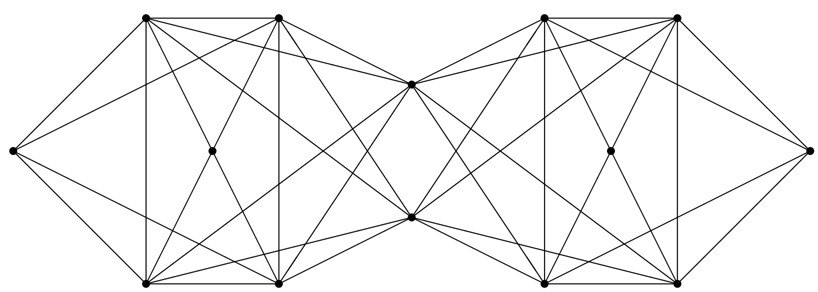
3 vertices. Assume that among these $2\ell - i$ components contains exactly two degree-3 vertices, where $i \le 2$. There at at least $4\ell - i$ edges between these 2ℓ components and the vertex set $\{x_1, \dots, x_\ell\}$, and because C is connected, $\ell - 1$ edges inside $\{x_1, \dots, x_\ell\}$. Counting the degrees within $\{x_1, \dots, x_\ell\}$, we get the inequality $4\ell - i + 2(\ell - 1) \le 4\ell$, i.e. $\ell = 1$ is the only remaining case. Then C has nine vertices, with two degree-2, six degree-3 vertices and a degree-4 cut-vertex. Up to isomorphy there are only two such graphs, and it can be checked that Breaker has a winning strategy on both, we omit the details. □

Sketch of the proof of Proposition 7.

We use the graph D_7 that was shown to be a Maker's win in the positive minimum degree game in [91]. From two copies of D_7 one can make a graph D_{14} such that $v(D_{14}) = 14$, $e(D_{14}) = 40$ and Maker wins the game $(E(D_{14}), \mathcal{D}_2)$. The construction is made in two steps.

First we take two copies of D_7, and glue them together in three vertices. The degree two and degree four vertices are associated to the same vertices in the other copy, one to each other. The resulting graph H has 11 vertices and 20 edges. Note that playing Maker's winning strategy for the positive minimum degree game separately on the edges of the D_7's, Maker gets degrees at least two at the "glued" vertices, and at least one at the others.

In the second step we glue together two copies of H, this time at the degree three vertices, taking care not creating parallel edges, this we call D_{14}. (Simply the vertices of a diamond should be glued to vertices that are not in a diamond in the other copy.) Again, playing separately on the edges of the two copies of H, Maker gets at least two degrees at all vertices of D_{14}.



The graph D_{14}.

Let $D_7^2 = D_7 \square D_7$ be the Cartesian product of D_7 with itself, and $D_7^k := D_7 \square D_7^{k-1}$ be kth power of D_7. To play the $(E(D_7^k), \mathcal{D}_k)$-game, let Maker play the winning strategy for a $(E(D_7), \mathcal{D}_1)$-game in the same projection in that Breaker has just played. This clearly gives a winning strategy for the $(E(D_7^k), \mathcal{D}_k)$-game, and $e(D_7)/v(D_7) = 10(7)^{k-1}k/7^k = 10k/7$. □

Remarks. The discharging method also gives lower bound on $\hat{m}(\mathcal{D}_2)$, alas, it is not matching with $\hat{m}(\mathcal{D}_2) \le 20n/7$, the upper bound of Proposition 7. In fact, we think the upper bound on $\hat{m}(\mathcal{D}_2)$ is not tight, but we cannot improve on it.

4.2 Recycling

Let us recall that the board of a positional game is a finite or infinite set X and the players alternately take elements of X (by marking or putting pieces onto it physically), and there is a fixed $H \subset 2^X$, the *winning sets*. Here we consider biased Maker-Breaker games that is Maker takes p and Breaker takes q elements of X in turns. If $p \neq q$, it is a *biased* game, otherwise it is called *accelerated*, see [21, 19, 20, 128, 129, 131]. Here Maker wins by occupying a winning set, while Breaker wins of preventing Maker's win.

However, this pattern does not fit for such games as the already solved Connect-4 or Nine Men's Morris, see [3], [70]. In the first case the available moves are restricted, while the whole static approach of the positional games is abandoned in the second. We shall address the issue of the second one and make an attempt to capture the idea of movements for a game. For an arbitrary positional game let us define the rules of the *recycled versions* as follows. For a natural number n the players make the first n steps as before; this is the *first phase*. Then, in the *second phase*, they just make moves with some of their earlier placed pieces in turns, instead of introducing new ones.

In order to investigate the effect of recycling, let us recall and define some games. The first is the well-known *k-in-a-row* game ($k \in N$), which is played by the two players on the infinite (chess)board, or graph paper. They alternately put their own marks to previously unmarked squares, and whoever gets k-consecutive marks first (horizontally, vertically or diagonally) of his own, wins.

An interesting way to alter the *k-in-a-row* game is to relax the consecutiveness condition. We shall call the game $L_k(p, 1; n)$ (or *line game* for short) for which:

1. Maker and Breaker mark p and 1 squares in every step, respectively.

2. Maker wins upon getting k, not necessary consecutive, marks in a line (horizontally, vertically or diagonally), which is free of Breaker's marks.

3. the game terminates after n steps

Then let $RL_k(p, 1; n)$ be the recycled version of $L_k(p, 1; n)$.

Our third subject is the Kaplansky's game, where the players put their marks on the Euclidean plane. Here the first player wins achieving k marks on a line, provided the second player has no mark on that line. Now $K_k(p, q)$ stands for the $p : q$ Maker-Breaker version of it.[2] Let $K_k(p, q; n)$ be the version which ends after n round, and $RK_k(p, q; n)$ be its recycled version.

Before stating our theorems, let us recall some earlier results on these games.

The *recycled k-in-a-row* (no matter when does the second phase start) turns out to be easy, because the decomposition methods utilized in [34] still work, and give the same bounds. That is even the Maker-Breaker version of the recycled k-in-a-row game is a draw if $k \geq 8$.

Bounds for the games $L_k(p, 1; n)$ and $RL_k(1, 1; n)$ are less obvious, we shall prove:

Theorem 4.2. *In the Maker-Breaker $L_k(p, 1; n)$ game, Breaker wins if $k \geq p \log_2 n + p \log_2 p + 3p$. On the other hand, Maker wins if $p > 1$ and $k \leq c \log_2 n$ for some $c > 0$.*

Theorem 4.3. *Breaker wins the Maker-Breaker $RL_k(1, 1; n)$ game if $k \geq 32 \log_2 n + 224$.*

[2] The original Maker-Maker version of Kaplansky's game is extremely hard and the outcome is unknown for $k \geq 4$.

Kleitman and Rothschild [102] studied a Maker-Maker version of Kaplansky's game in which the first (second) player wins by getting k (ℓ) points of a line while the opponent has none of that line, respectively. They prove that, given any $k \geq 1$, there is an $\ell(k)$ such that second player has a winning strategy whenever $\ell \geq \ell(k)$. Beck in [21] considers a $p:1$ Maker-Breaker version game in which Maker wins by getting k points on a line. He has also shown there exist constants $c_2 > c_1 > 0$, such that in $K_k(1,1;n)$: Maker wins if $k < c_1 \log_2 n$ and Breaker wins if $k > c_2 \log_2 n$. For its recycled version we have the following result.

Theorem 4.4. *Breaker wins the Maker-Breaker $RK_k(1,1;n)$ game if $k > 2n^{1/3}$.*

4.2.1 Proofs

Weight functions

In the proof of the Theorems 4.2, 4.3 and 4.4 we heavily use the *weight function method*, which was developed in [19] and developed in [20] and [58]. First let us recall some earlier definitions and results.

As we mentioned in earlier chapters, one of the most important result on positional games is the Erdős-Selfridge theorem; one of its generalization is due to József Beck.

Theorem 4.5 ([19])**.** *Breaker wins the $(p,1,H)-$ game if $\sum_{A\in H} 2^{-\frac{|A|}{p}} < \frac{1}{2}$.*

In our cases this theorem cannot be applied directly, since the hypergraphs involved are infinite, and it is not known if Theorem 4.5 holds for recycled games. The following lemma is also due to Beck, [19]. We repeat the proof in order to see the properties of the used weight function.

An edge $A \in H$ is *active* if Breaker has not taken any of its elements.

Lemma 11. *Playing a $(p,1,H)$ game, Breaker can assure that no active edge contains more than $p + p\log_2 |H|$ elements taken by Maker.*

Proof of Theorem 4.5. We may assume Maker starts the game. For any $A \in H$ let $A_k(M)$ and $A_k(B)$ be the number of elements in A, after Makers kth move, selected by Maker and Breaker, respectively. Now, for an $A \in H$

$$w_k(A) = \begin{cases} \lambda^{A_k(M)} & \text{if } A_k(B) = 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\lambda > 0$, and for any $x \in X$ let $w_k(x) = \sum_{x\in A} w_k(A)$. The numbers $w_k(A)$ and $w_k(x)$ are called the *weight* of A and x (in the kth step), respectively. When it does not cause confusion we may suppress the lower index.

Now selecting an element in the kth step Breaker uses the *greedy* algorithm, i.e. chooses an unselected element $y^k \in X$ of maximum weight. Let $x_1^{k+1}, ..., x_p^{k+1}$ be the elements selected by Maker in the $(k+1)$st step and $w_k = \sum_{A\in H} w_k(A)$ be the *total sum* or *potential*. For $k \geq 0$, following inequality holds for the potential:

$$w_k - w_k(y^k) + (\lambda^p - 1)w_k(y^k) \geq w_{k+1}.$$

Indeed, w_k decreases by $w_k(y^k)$ upon selecting y^k. The elements selected by I in the $(k+1)$st step cause the biggest increase if $w_k(x_l^{k+1})$ is maximal for $1 \leq \ell \leq p$, and for all A such that $w_k(A) \neq 0$ we have $x_\ell^{k+1} \in A$ iff $x_m^{k+1} \in A$, $1 \leq \ell,\ m \leq p$. Since the increase in this case is just $(\lambda^p - 1)w_k(y^k)$, the inequality is proved. Setting $\lambda = 2^{1/p}$, we get $w_k \geq w_{k+1}$, $k \geq 0$, which justifies that w_k is called potential.

Particularly $w_1 \leq (\lambda^p - 1)|H| + |H| \leq 2|H|$. Since $q = 1$ and the elements of H are the same size, the inequality $\sum_{A\in H} 2^{-|A|/p} < 1/2$ leads to the inequality $2|H| < 2^{|A|/p}$. Assume that Maker wins the game in the kth step. This would imply $w_k \geq \lambda^{|A|} = 2^{|A|/p}$, which contradicts the monotonicity of the potential. □

Proof of Lemma 11. Just take the logarithm of the inequality $\lambda^{A_k(M)} = w_k(A) \leq w_k \leq w_1 \leq 2|H|$ that holds for any active edge $A \in H$. □

Proof of Theorem 4.2

Let us recall that a line L means consecutive squares along an infinite line here (horizontally, vertically or diagonally). Now we have infinitely many interacting sets, so the weight function method does not seem to be helpful. The way to overcome the difficulties is to change the definition of the weights. The price of this is that the potential is no longer a decreasing function, but an increasing one. However, we can control the growth, since the game lasts only n steps.

Let H be the set of all lines, and $L_j(M)$ and $L_j(B)$ the number of squares of line L marked by Maker and Breaker after the jth step, respectively. Now the weight function of L at the jth step:

$$w_j(L) = \begin{cases} \lambda^{L_j(M)} \text{ if } L_j(M) \geq 1 \text{ and } L_j(B) = 0 \\ 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{otherwise} \end{cases}$$

where $\lambda = 2^{\frac{1}{p}}$.

For a square q,

$$w_j(q) = \sum_{L\in H, q\in L} w_j(L)$$

is the weight of q, and

$$w_j = \sum_{L\in H} w_j(L)$$

is the *total weight* at the jth step.

Breaker applies the greedy selection. For the weight functions, similarly to the proof of Theorem 4.5, we have

$$\sum_{L_j(M)\geq 1} w_{j+1}(L_{j+1}) \leq \sum_{L\in H} w_j(L_j).$$

On the other hand, in each step the number of lines whose weight becomes positive is at most $4p$, and the weight of such a line is no more than $\lambda^p = 2$. That is

$$w_{j+1} \leq w_j + 8p$$

holds for $0 \leq j \leq n$, where $w_0 = 0$. That is if the line L is unblocked at step j (i.e. $L_j(B) = 0$) and $L_j = i$ than

$$\lambda^i \leq 8pj \Leftrightarrow i \leq p(\log_2 j + \log_2 p + 3).$$

Since $0 \le j \le n$, the first part of Theorem 4.2 follows.

The second part is fairly standard, we give just the sketch of its proof. In fact, one (say vertical) winning direction is enough. Maker divides the game into phases. For the sake of simplicity we omit to write the integer parts. In the first phase Maker places $n(p-1)/p$ element in a row. Call a column *i-free* if it contains i marks of Maker, but none of Breaker. At the end of the first phase the number of 1-free columns is at least $n((p-1)/p)^2$. In the ith phase Maker uses up $n((p-1/p))^i$ new mark, each is placed to an $i-1$-free column. It is easy to check that Maker can reach the ith phase if $n((p-1)/p)^i \ge 1$, and uses up at most n marks. That is an i-free column appears if $i \le c \log_2 n$, where c is about $(\log_2 p - \log_2(p-1))^{-1}$. □

Proof of Theorem 4.3

Breaker divides the game into sub-phases. The first sub-phase is the first phase of the game, then a sub-phase consists of n pair of moves. Defining the weight function as before, but $\lambda = \sqrt{2}$, Breaker places every second mark (the *active* marks) according to the greedy strategy and deposits the others arbitrarily, i.e. *in reserve*). It may happen that one of Breakers reserved marks is already on the square q, which is to be occupied by an active mark of Breaker. In that case Breaker places the new mark arbitrarily (sends it into reserve), and the mark on the square q becomes active.

Considering only the effect of Breakers active marks, the game reduces to the game $L_k(2, 1, n)$. That is Lemma 11 applies, and for any line L if $L_j(M) = i$ and $L_j(B) = 0$, then $i \le 2(\log_2 j + 4)$ if $0 \le j \le n$.

In the other sub-phases Breaker plays a fictitious game, and keeps the status of his marks (active or reserved) strictly. The marks of Maker are indexed by the numbers $1, 2, \ldots, n$. At the beginning of a sub-phase Breaker cannot see Makers marks, and in the jth step Makers new mark and the mark indexed by j become visible for Breaker as new moves. (If Maker moved the jth mark, only one mark becomes visible.)

However Breaker responds only in every second step, using the marks from the reserve. (Breaker does nothing in the odd steps. If picking up a mark and putting back to the same place is permitted, it is easy. If it is not, Breaker designates a mark at the very beginning, which is neither active nor reserved, and moves this mark arbitrarily in the odd steps.)

Trying the previous greedy strategy another difficulty arises. Breaker may not occupy the square q of maximum weight because q has been already taken (by one of Makers invisible marks or one of Breakers own reserve). Then, Breaker blocks the lines going through q, using four marks. (See a similar idea in [131].) Now, looking only Makers visible marks, if for a line L, $L_j(M) = i$ and $L_j(B) = 0$ then $i \le 16(\log_2 j + 7)$, since after at most 16 moves of Makers, Breaker may reply, and Theorem 4.2 applies.

By the end of a sub-phase Makers all marks become visible, and a line L, which contain more than $16(\log_2 n + 7)$ of them, is blocked by Breakers reserve. Finally, Breaker starts the next sub-phase renaming his marks, the active ones become reserved and vice versa.

Since the active marks control the invisible marks during a sub-phase, if for a line L the sum of visible and invisible marks of Maker on L is i, and L is not blocked (by the active marks or by the reserve), then $i \le 32(\log_2 n + 7)$. □

Proof of Theorem 4.4

The most natural idea is to mimic the proof of Theorem 4.3.

Unfortunately it breaks down irreparably at the point where Breaker wants to occupy, or at least block the point q, which is already taken. The problem is that q can be the element of many lines, so Breaker cannot cancel the weight of q by using only constantly many points.

To overcome this difficulty, we need to change the weight function and give a more sophisticated analysis of it.

Let the weight of a line L after Maker jth move be

$$w_j(L) = \begin{cases} \lambda^{L_j(M)} \text{ if } L_j(M) \geq c_1 n^{1/3} \text{ and } L_j(B) = 0 \\ 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{otherwise} \end{cases}$$

where $\lambda = \sqrt{2}$ and $c_1 > 0$ will be specified later.

As before, for a point x of the plane, $w_j(x) = \sum_{L\in H, x\in L} w_j(L)$ is the weight of x, and $w_j = \sum_{L\in H} w_j(L)$ is the *total weight* at the jth step.

However, Breaker uses not only the greedy strategy, the recycled point also have to be designated. When Breaker removes a point y, the total weight function may grow. It grows iff there is a line L containing y such that $L_j(M) \geq c_1 n^{1/3}$ and $L_j(B) = 1$. Obviously the number of such points cannot be bigger than the number of lines containing at least $c_1 n^{1/3}$ points of Maker. To estimate this, we need a definition and a theorem of Szemerédi and Trotter.

An *incidence* of a point and a line is a pair (p, L), where p is a point, L is a line, and p lies on L.

Theorem 4.6 (Szemerédi-Trotter, [157]). *Let I denote the number of incidences of a set on n points and m lines. Then $I \leq c(n + m + (nm)^{2/3})$.*

Let us note that later László Székely published a new, more accessible proof of Theorem 4.6, see in [156].

An easy corollary of Theorem 4.6 is that there is a constant c_2 such that the number of lines containing at least k points of S is less than $c_2 n^2/k^3$ whenever $k \leq \sqrt{n}$.

That is if $c_1 > c_2^{1/3}$, then the number of lines containing at least $c_1 n^{1/3}$ points of Maker is less than n. It means Breaker can always find a mark y such that its removal does not affect the value of the total weight function. The steps of Maker and Breaker are $x_1, x_2, \ldots, x_i$ and $y_1, y_2, \ldots, y_i$, respectively.

As before, for the weight function we have

$$w_{j+2} \leq w_j - w_j(y_j) - w_{j+1}(y_{j+1}) + w_j(x_{j+1}) + w_{j+1}(x_{j+1}) + \frac{2}{c_1} n^{2/3} \lambda^{n^{1/3}} \lambda^{n^{1/3}+1}.$$

Here the term $f(n) := \frac{2}{c_1} n^{2/3} \lambda^{n^{1/3}} + \lambda^{n^{1/3}}$ bounds the growth caused by the lines that of weight becoming positive in the jth and $(j + 1)$th steps. By the argument of Theorem 4.5, $w_j(y_j) \geq w_j(x_{j+1}) + w_{j+1}(x_{j+1})$, since $\lambda = \sqrt{2}$. We also have $w_{j+1}(y_{j+1}) > w_{j+1}/n$, since the number of positive weighted lines is less than n, giving

$$w_{j+2} \leq w_j - \frac{w_{j+1}}{n} + f(n).$$

On the other hand, $w_{j+2} \leq w_{j+1} + f(n)$, or equivalently $w_{j+1} \geq w_{j+2} - f(n)$. That is the value of w_{j+2} is bounded, that is $w_{j+2} \leq w_j$ since $\frac{w_{j+1}}{n} \geq f(n)$. From here one gets that $w_{j+2} \leq (n+1)f(n)$. It means that if for a line L, $L_{j+2}(M) = s$ and $L_{j+2}(B) = 0$, then $(n+1)f(n) \geq w_{j+2} \geq \lambda^s$. Taking the logarithm of both sides, $s \leq 2\log_2 w_{j+2} \leq 2n^{1/3}$, provided n is big enough. □

Remarks and Open Questions

As we have seen, there is a large gap between the logarithmic lower and $O(n^{1/3})$ upper bound what Maker can achieve in the recycled Kaplansky's game.

Question 1. *Can the upper or lower bounds of Theorem 4.4 improved?*

Even less is known about recycled hypergraph games in general. It is easy to give example for which Breaker wins the first phase of the game, while Maker wins the recycled version.

Question 2. *Is there a hypergraph game won by Breaker, but Maker wins its recycled version?*

It is also interesting if the Erdős-Selfridge theorem extends to the recycled games.

Question 3. *Is it true if $\sum_{A\subset H} 2^{-|A|+1} < 1$, then Breaker wins the recycled version of the (X, H) game?*

Together with my MSc student, Ivánn Vrbáski, we managed to solve the recycled version of a "discrete-continuous box game" described in [92]. Here the bounds of the normal and recycled version coincide. We omit the details.

Chapter 5

Colorings

Different types of colorings the vertices (or edges) in graphs or hypergraphs are central problems in both pure combinatorics and applications. In this chapter we exhibit some research was done in the recent years.

In Section 5.1 we thoroughly discuss the use of greedy algorithm for hypergraph colorings that can be used in various circumstances. Originally it was applied for the special case of Erdős-Hajnal problem, but it turned out to be useful in various circumstances.

Considering two colorings as labellings with the zero and unit of the two element field, allows us to involve algebraic approach thus giving new proofs and suggesting new questions. In Section 5.2 how to get Kőnig's and Harary's well-known theorems and their dual forms from the classical Kronecker-Capelli theorem. Along that line one can arrive to some scheduling and cake-cutting problems.

Complementing Section 5.2, in Section 5.3 we show that colorings may be used to get algebraic results. Namely the existence of *rainbow coloring* of a hypergraph gives rise to an isomorphic embedding of an automaton to product of automata.

The research presented in Section 5.4 is completely application driven. We propose a new approach for clustering for certain real graphs, working out the algorithmic and theoretical background. In these graphs there are not many edges within the clusters, but the structure between those clusters is essential. We model this situation by introducing special graph colorings that avoid certain induced subgraphs between the clusters.

Yet another type of two coloring assigns the numbers ± 1, and adding up the labels of certain object one arrives to the problems of *discrepancy*. Although the discrepancy theory is classical and very deeply studied, the discrepancies of global structures of graphs are barely investigated. In Section 5.6 we report on that direction listing new results and including some proofs.

5.1 Greedy coloring

We use the notation of [7] and partly those of [105]. A hypergraph (V, E) is k-colorable if V can be colored by using at most k colors such that no edge $A \in E$ is monochromatic. Let $m_k(n)$ denote the minimum possible number of edges of an n-uniform hypergraph that is non-k-colorable. We suppress the lower index for $k = 2$, that is $m(n) = m_2(n)$. We list some of the significant results concerning the values of $m_k(n)$ and other colorability issues as follows.

Erdős proved lower and upper bounds on $m(n)$, namely $2^{n-1} \leq m(n) \leq cn^2 2^n$ in [56] and in [57], respectively. While the upper bound is still the best known, the lower bound on $m(n)$ was improved in a sequence of papers. Note that all subsequent works start with his idea, that is coloring the vertices randomly and independently of each other.

First Schmidt showed $(1 - 2/n)2^n < m(n)$ (see [142]), then Beck came up with the idea of *recoloring* of a random coloring, and he proved the bounds $c \log n 2^n < m(n)$ and $n^{1/3+o(1)} 2^n < m(n)$ [22, 25], respectively. The proof of the latter bound was simplified by Spencer in [148].

Twenty years later Radhakrishnan and Srinivasan modified the recoloring idea of Beck, and showed $0.7\sqrt{n/\ln n} 2^n < m(n)$ [139]. In the same paper it was shown that a hypergraph is 2-colorable if every edge meets at most $0.17\sqrt{n/\ln n} 2^n$ other edges. It is also worth noting that Erdős and Lovász guessed [59] that $m(n)$ is perhaps around $n2^n$.

The n-uniform, n-regular hypergraph is an interesting special case. The 2-colorability easily follows from the Lovász Local Lemma for $n \geq 9$, (see e.g., in [7]), for $n = 8$ it was proven by Alon and Bregman in [6], and finally Thomassen [158] showed it for $n \geq 4$.

Kostochka obtained the following lower bound on $m_k(n)$ in [105]. For every $k \in \mathbb{N}$, let $\epsilon(k) = \exp\{-4k^2\}$ and $r = \lfloor \log_2 k \rfloor$. Then for every $n > \exp\{2\epsilon^{-2}\}$, $m_k(n) \geq \epsilon(k) k^n (n/\ln n)^{r/(r+1)}$.

In followings we use a different probability space which admits easier proof, though it gives weaker bounds. The main idea is to use greedy colorings on a random order of the vertices. Note that Radhakrisnan and Srinivasan [139] also used random vertex orderings after an initial random coloring. To generate a random order we let each vertex u pick a random real x_u uniformly and independently of each other from $[0, 1]$, and order the vertices according to these values. Equivalently, one can take a uniformly selected random element among the permutations of the vertex set, although the first form is better suited for the proof of Theorem 5.1.

In the next subsection we give a simple proof for the statement $m(n) > c_1 \sqrt[4]{n} 2^n$. The analysis of a random greedy algorithm also yields $m_k(n) > c_2 k^{-1} n^{\frac{k-1}{2k}} 2^n$. The constructions lead to a “characterization" of k-colorable hypergraphs which might be of interest by its own. We conclude the text by a new proof for the 2-colorability of n-uniform, n-regular hypergraphs for $n \geq 8$.

5.1.1 Results

2-coloring

We define a random greedy coloring of a hypergraph $H = (V, E)$ as follows. Let σ be a uniformly picked random order of V. At the beginning all vertices are blue. In the i^{th} step we recolor the vertex $\sigma(i)$ to red if $\sigma(i)$ is the first element of an $A \in E$ according to the order σ.

Clearly, there are no completely blue edges in E at the end of the procedure. Let the number of completely red edges be X.

Claim 3. $\mathbb{E}X < 2\sqrt{\pi} e^{1/(6n)} n^{-\frac{1}{2}} 2^{-2n} |E|(|E| - 1)$.

Proof of Claim 3. We say that $A \in E$ *precedes* $B \in E$ if the last vertex of A becomes red because it was the first element of B. If $X_{A,B}$ is the indicator variable of the event A

precedes B then $X = \sum X_{A,B}$, where the summation runs over all ordered pairs of E. Hence

$$\mathbb{E}X = \sum \mathbb{E}X_{A,B} = \sum \Pr(A \text{ precedes } B) = \sum \frac{((n-1)!)^2}{(2n-1)!} = \sum \frac{2(n!)^2}{n(2n)!},$$

since A may precede B iff $A \cap B = \{x\}$ and x is the last element of A and the first element of B. Let us use the Stirling formula, i.e., $n! = \sqrt{2\pi n}(n/e)^n e^{\lambda_n}$, where $1/(12n+1) < \lambda_n < 1/(12n)$.

$$\mathbb{E}X = \sum \frac{2(n!)^2}{n(2n)!} \leq |E||E-1|\frac{2\sqrt{\pi}e^{2\lambda_n - \lambda_{2n}}}{\sqrt{n}}2^{-2n} \leq 2\sqrt{\pi}e^{1/(6n)}n^{-\frac{1}{2}}2^{-2n}|E|(|E|-1),$$

since $e^{2\lambda_n - \lambda_{2n}} < e^{1/(6n)}$. □

Corollary 5. *$m(n) > \frac{\sqrt{2}}{2}\pi^{-\frac{1}{4}}e^{-\frac{1}{12n}}\sqrt[4]{n}2^n$. That is $m(n) > 0.5268\sqrt[4]{n}2^n$, for $n \geq 3$.*

Proof. Just plug in $|E| = \frac{\sqrt{2}}{2}\pi^{-1/4}e^{-1/(12n)}\sqrt[4]{n}2^n$ into the formula of Claim 3. It gives $\mathbb{E}X < 1$, which means that there exists a good 2-coloring of (V, E). □

k-coloring

It is possible to get k-colorings by greedy algorithms for arbitrary $k \in \mathbb{N}$. Here greedy means that we color all the vertices with color 1, and in the i^{th} step we recolor the vertex $\sigma(i)$ if $\sigma(i)$ is a first element of an $A \in E$ according to the order σ. To recolor $\sigma(i)$ we use the smallest possible color that does not result in a monochromatic edge, otherwise we use the color k.

For an order σ of V, let $\{A_i\}_{i=1}^k$ be an *ordered k-chain* if $|A_i \cap A_{i+1}| = 1$, $A_i \cap A_j = \emptyset$ for $|i-j| > 1$ and $\sigma^{-1}(x) \leq \sigma^{-1}(y)$ for all $x \in A_i$ and $y \in A_{i+1}$, $i = 1, \ldots, k-1$. If we have a fixed order σ, let $f(A)$ and $\ell(A)$ be the first and the last vertices of an edge A, respectively.

Lemma 12. *The hypergraph (V, E) is k-colorable if and only if there is an order σ of V containing no ordered k-chains. Moreover the greedy algorithm on (V, E) in this case provides a good k-coloring.*

Proof. For the "if" part let us color the vertices of V by the greedy algorithm in order σ. By the setup of the greedy algorithm, if there is a monochromatic edge $A_{k-1} \in E$ then its color can only be k. Now $\ell(A_{k-1})$ gets the color k since there is an edge, let us call it A_k, such that $\ell(A_{k-1}) = f(A_k)$. Similarly, $f(A_{k-1})$ is colored k, since there is an edge A_{k-2} such that $\ell(A_{k-2}) = f(A_{k-1})$, and all vertices of $A_{k-2} \setminus A_{k-1}$ are colored $k-1$. Taking $f(A_{k-2})$, we can get an $A_{k-3} \in E$ such that all vertices of $A_{k-3} \setminus A_{k-2}$ are colored $k-2$. By induction there is an $A_i \in E$ such that all vertices of $A_i \setminus A_{i+1}$ are colored $i+1$ if $i \geq 1$. But then $\{A_i\}_{i=1}^k$ is an ordered k-chain. The "only if" is trivial, given a good k-coloring let σ be an order induced by the colors, breaking the ties arbitrarily. □

Claim 4. *Let X be the number of k-chains in a random order of the n-uniform hypergraph (V, E), and $s = n - 1 > 0$. Then*

$$\mathbb{E}X < |E|^k \exp\left\{\frac{k}{12s} + 1\right\}(2\pi s)^{\frac{k-1}{2}}k^{-sk-1}s^{k-1}.$$

Proof of Claim 4. The proof is almost the same as that of Claim 3. Let $\mathcal{K}$ be the set of ordered k-tuples of A. For any $H \in \mathcal{K}$,

$$\Pr(H \text{ is a } k-\text{chain}) \leq \frac{\{(n-1)!\}^2\{(n-2)!\}^{k-2}}{\{(n-1)k+1\}!} = \frac{s!^k}{(sk+1)!s^{k-2}}.$$

Using the Stirling formula, with the bounds $e^{\lambda_s k} < e^{k/(12s)}$, $1 < e^{\lambda_{sk+1}}$ we get

$$\mathbb{E}X < |E|^k \exp\{\frac{k}{12s} + 1\}(2\pi s)^{\frac{k-1}{2}} k^{-sk-1} s^{k-1}.$$

□

Corollary 6. *If* $|E| \leq (2\pi e)^{-1/2} s^{\frac{k-1}{2k}} k^s$, *then* (V, E) *is* k*-colorable. That is*

$$m_k(n) > (\sqrt{4\pi e}k)^{-1} n^{1/2-1/(2k)} k^n.$$

Proof of Corollary 6. If $|E| \leq (2\pi e)^{-1/2} s^{\frac{k-1}{2k}} k^s$ then there is an order σ of V such that (V, H) has no k-chain by Claim 4. Moreover $(\sqrt{4\pi e}k)^{-1} n^{1/2-1/(2k)} k^n < (2\pi e)^{-1/2} s^{\frac{k-1}{2k}} k^s$, and then (V, E) is k-colorable by Lemma 12. □

Remarks. One can consider Lemma 12 from yet another point of view. Given a hypergraph (V, E) and a fixed order σ on its vertices, one may construct a directed graph $G_\sigma = (V(G_\sigma), E(G_\sigma))$. Let $v \in V(G_\sigma)$ iff $v = f(A)$ or $v = \ell(A)$ for some $A \in E$, and $(u, v) \in E(G_\sigma)$ iff there is an $A \in A$ such that $u = f(A)$ and $v = \ell(A)$. Obviously if for an order σ the graph G_σ has a good k-coloring then (V, E) is also k-colorable, and if (V, E) is k-colorable, then there exists an order σ such that G_σ is k-colorable. The non trivial part of Lemma 12 says that G_σ has a good k-coloring if it has no directed paths of length k. This is nothing else but a special case of an old result attributed to T. Gallai and B. Roy, that says if a directed graph G contains no paths of length k, then G is k-colorable, see chapter 9., problem 9 in [115].

Sparse hypergraphs

If a hypergraph (V, E) is *sparse*, that is each edge meets at most D other edges, then a good 2-coloring exists if $D \leq 0.17\sqrt{n/\ln n}2^n$ and n is big enough [139]. The direct use of the random orders and the Lovász Local Lemma gives

Theorem 5.1. *Let* $H = (V, E)$ *be an* n*-uniform hypergraph in which each edge meets at most D other edges. If* $2e(2D^2 - D)((n-1)!)^2/(2n-1)! \leq 1$, *then* H *is* 2*-colorable.*

Before the proof let us recall the Lovász Local Lemma. To spell it out we need a definition. If $A_1, ..., A_n$ are events of a probability space, then a *dependence graph* $G = (V, E)$ of these events is a graph having the following properties: $V = \{1, ..., n\}$, and each event A_i is mutually independent of the events $\{A_j : (i, j) \notin E\}$. Let $\deg_G(v)$ be a degree of a vertex v in G. For details see [7] and [59].

Lemma 13. *(Lovász Local Lemma) [59] Let* $A_1, ..., A_n$ *be events of a probability space, and G be a dependence graph of these events. If* $\Pr(A_i) \leq p$ *and* $\deg_G(A_i) \leq d$ *for all* $1 \leq i \leq n$, *and* $ep(d+1) \leq 1$, *then* $\Pr(\cap_{i=1}^n \overline{A}_i) > 0$.

Proof of Theorem 5.1. Let us consider the uniform random orders of V. For $A, B \in E$ let $\mathcal{A}_{AB}$ be the bad event that either A precedes B or B precedes A. Clearly, the event $\mathcal{A}_{AB}$ is mutually independent of all the other events $\mathcal{A}_{RS}$ when $(A \cup B) \cap (R \cup S) = \emptyset$. One checks that the number of intersecting unordered pairs $(R, S) \neq (A, B)$ that also intersects $A \cup B$ is not more than $2D^2 - D - 1$. Now the Lovász Local Lemma implies, there is an order σ containing no 2-chain, if

$$e\Pr(\mathcal{A}_{AB})(2D^2 - D) = 2e(2D^2 - D)((n-1)!)^2/(2n-1)! < 1.$$

This inequality holds by assumption, so H is 2-colorable by Lemma 12. □

Remark. A quick asymptotic of Theorem 5.1 gives that such hypergraphs are 2-colorable if $D < 0.23\sqrt[4]{n}2^n$. This result is asymptotically weaker than the former $0.17\sqrt{n/\ln n}2^n$ bound, but Theorem 5.1 has better constants and works for *all* $n > 1$. It already implies the known results of the values for which an n-uniform, n-regular hypergraph is 2-colorable. Note that this follows from the Lovász Local Lemma easily if $n \geq 9$, while for the case $n = 8$, see the paper of Alon and Bregman, [6].

Corollary 7. *Every n-uniform, n-regular hypergraph is 2-colorable, for $n \geq 8$.*

Proof of Corollary 7. First we show a sharp bound on Δ_n, the number of intersecting unordered pairs $(R, S) \neq (A, B)$ that also intersects $A \cup B$. Observe that the number of pairs intersecting with the fixed (A, B) is maximum when (V, E) is *almost disjoint*, i.e., for every $R, S \in E$ we have $|R \cap S| \leq 1$ if $R \neq S$.

From the n-regularity we have

$$\Delta_n \leq 2(n-1)^4 + 2(n-1)\binom{n-1}{2} + \binom{n-2}{2} + 2(n-2).$$

Following the proof of Theorem 5.1, the Lovász Local Lemma implies that if

$$f(n) := 2e(\Delta_n + 1)((n-1)!)^2/(2n-1)! < 1,$$

then an n-uniform, n-regular hypergraph (V, E) is 2-colorable. Since $f(8) \leq 0.604$, Corollary 7 follows. □

5.2 Two-colorings of graphs

Lots of deep combinatorial statements are proved by linear algebraic tools, in many cases these are the only known proofs. The area is huge, we cannot even try to give a concise overview of it. Instead we recommend the following excellent books to the interested reader [11, 79, 116].

Without wishing to be exhaustive, we list some combinatorial statements together with the algebraic concepts that were used in their proof, details can be found in the references above:

- Fisher inequality ($r(AA^t) \leq r(A)$, where $r(A)$ is the rank of matrix A)
- Odd town theorem (the number of linearly independent vectors in an n-dimensional vector space)

- Hoffman-Singleton theorem (spectral theorem)
- Graham-Pollak theorem ($r(A+B) \leq r(A) + r(B)$)
- number of spanning trees in graph G (Laplace expansion)
- Shannon capacity (tensor product)

In practice, the question is sometimes raised in the opposite direction, if a linear algebra result is given what might be the combinatorial meaning of that result? So it is natural to ask whether there are a non-trivial combinatorial consequences of the Kronecker-Capelli theorem, which characterizes the solvable of a systems of linear equalities. Probably many people have asked this before, but I have not find any references, but some unpublished results of Zoltán Füredi. (With his consent, his result, Theorem 5.2 is reproduced here.) To understand these, we need the following concepts:

Definition 7 (Coloring)**.** *The two-coloring of a hypergraph (X, E) is a function $f : X \to \mathbb{F}_2$, where $\mathbb{F}_2$ is the Galois field with two elements.* [1] *f is a good coloring if $|f(e) \cap \mathbb{F}_2| = 2$, that is both colors occur for all $e \in E$. Furthermore f is an odd coloring if $\sum_{x \in e} f(x) = 1$ for all $e \in E$.* [2]

Observe, if for a hypergraph (X, E) the size for all $e \in E$ are even, then an odd coloring of f is a good coloring, too.

Theorem 5.2 (Füredi)**.** *[67] A hypergraph (X, E) has an odd coloring if and only if there is not such a set $H := \{e_1, \ldots, e_{2k+1}\} \subset E$, that for all $x \in X$ is contained by even number of sets from H.*

Corollary 8 (Kőnig)**.** *A graph G has a good two-coloring if and only if G does not contain odd circuit.*

Proof. Let $X = V(G)$ and $E = E(G)$, and apply Theorem 5.2 to the hypergraph (X, E). It means the graph G has a good two-coloring if and only if for all odd subset of edges $H \subset E(G)$ there must be at least one vertex of odd degree, that is H cannot be an odd walk. Of course G contains an odd walk if and only if it contains odd circuit, too. $\square$

In the next subsection we prove the generalization of Füredi's theorem and some of its consequences.

Generalized Füredi's theorem and its consequences.

As we mentioned before, we need to re-call the Kronecker-Capelli theorem:

Theorem 5.3 (Kronecker-Capelli)**.** *Let $\mathbb{F}$ be an arbitrary field, A be an $m \times n$ matrix, and b be n m-dimensional vector over $\mathbb{F}$. The equality system $Ax = b$ has a solution if and only if, there is no such a vector $y \in \mathbb{F}^m$, for which $y^t A = \mathbf{0}$ and $y^t b = 1$ hold.*

Before we spell out and prove the generalization of Theorem 5.2, we carry over the stability notion of Harary to hypergraphs, see also [88].

[1] The domain of f could be any two-element set, but the algebraic structure turns out to be very useful in $\mathbb{F}_2$.

[2] That is in all edges e odd number of vertices receive color 1.

Definition 8 (Stable coloring)**.** *Let (X, E, ϕ) be an edge-colored hypergraph, where $\phi : E \to \mathbb{F}_2$. A stable coloring of hypergraph (X, E) is such a function $f : X \to \mathbb{F}_2$, for which $\sum_{x \in e} f(x) = \phi(e)$ for all $e \in E$.*

Theorem 5.4 (Generalized Füredi)**.** *An edge-colored hypergraph (X, E, ϕ) has a stable coloring if and only if there is no such a set $H := \{e_1, \dots, e_k\} \subset E$, that for all $x \in X$ are contained in even number of sets of H, and ϕ colors odd number of elements of H by the color 1.*

Proof. Let us consider the transpose of the incidence matrix of hypergraph (X, E, ϕ), that is the rows (columns) of A are indexed by the elements of E (X), respectively, and $A_{ex} = 1$, if $x \in e$, otherwise $A_{ex} = 0$. Furthermore let $b_i := \phi(e_i)$ be the ith coordinate of the m-dimensional vector b. Then a solution of $Ax = b$ over $\mathbb{F}_2$ gives a stable coloring of the hypergraph (X, E, ϕ). If there are no stable colorings, then Ha Theorem 5.3 gives a vector $y \in \mathbb{F}_2^m$ such that $y^t A = \mathbf{0}$ and $y^t b = 1$. Let us take those edges e_i, for which $y_i = 1$. The all vertices covered even number of these edges since $y^t A = 0$, while odd number of those edges have color 1 because of the condition $y^t b = 1$. □

Theorem 5.4 is the generalization of Harary's theorem from 1954 to hypergraphs. In fact Harary's theorem can be considered as a generalization of the Theorem 8, proven by Dénes Kőnig. Its motivation came from some application in sociology, like many other problems of graph theory. The vertices of a simple graph G are entities, and the relations among those are labeled by $\{+, -\}$ signs describing friendly or hostile relations. Stability may be expected if the vertices can be partitioned into two set such that within the sets there are only edges with positive sign, while the edges between the sets has negative sign.

From Theorem 5.4 follows the original theorem of Harary in analogous way as the Corollary 8 before:

Corollary 9 (Harary)**.** *[88] A signed graph G has a stable partition if and only if all of its circuit contain even number of edges with negative sign.*

Remarks. The approach outlined above has an algorithmic meaning. Solving $Ax = b$, that is executing a Gaussian elimination, it can be decided if a graph G is bipartite (or in a signed case if it is stable). Furthermore if the answer is "yes" the color classes can be read out from the solution.

The Dual of Kőnig and Harary theorems

Since we have an algebraic formalism, we can interpret the dual of Theorem 5.4 and consequently the duals of Corollary 8 and 9. We just swap the roles of X and E and consider the matrix A^t instead of A, that is the incidence matrix of the hypergraph (X, E).

Assume that in the colored hypergraph (X, E, α) the function $\alpha : X \to \mathbb{F}_2$ a fixed coloring of the vertices and a stable edge coloring h is such a function $h : E \to \mathbb{F}_2$, for which $\sum_{x \in e} h(e) = \alpha(x)$ for all $x \in X$. Then Theorem 5.4 translated as:

Theorem 5.5 (Dual Füredi)**.** *A hypergraph (X, E, α) has a stable edge coloring if and only if, there is no such a set $Y := \{x_1, \dots, x_k\} \subset X$, such that for all $e \in E$ contain even number of elements of Y and the function α colors odd number of elements of Y by color 1.*

A special case. Let us label the vertices of a graph G by the labels $+$ and $-$. Under a stable edge partition we mean such a partition $E(G) = E_1 \cup E_2$ in which for all vertices labeled by $+$ ($-$) incident to even (odd) number of edges of E_1, respectively.

Corollary 10 (Dual Harary)**.** *A vertex signed graph G has a stable edge partition of and only if it has no component containing odd number of vertices with negative label.*

Corollary 11 (Dual Kőnig)**.** *The edge set of a graph G can be partitioned into sets E_1, E_2 such that for all vertices are incident to odd number of edges of E_1 if and only if G has no odd component.*

5.2.1 Other equalities

The structure of the solution of the system $Ax = b$ is rather complicated, we only touch upon it here.

The so-called *Necklace problem* was investigated by Alon and West in 1987, see [10]. In the original version two thief want to share the gems of a necklace. There are n types of gems on the necklace, and an even number of gems from all types. What is the minimum number of cuts they have to apply if they want the same number of gems from each types? It is easy to see, that if the necklace is modeled as $1122\ldots nn$ then at least n cuts are necessary. Alon and West proved that n cuts always suffices, using a classical result from topology, the Borsuk-Ulam theorem.

Theorem 5.6 (Borsuk-Ulam)**.** *Let f be a continuous function mapping the surface of the $n+1$-dimensional ball, $\mathbb{S}^n$, to the $\mathbb{R}^n$. Then there is an $x \in \mathbb{S}^n$ such that $f(x) = -f(-x)$.*

More precisely, they used an equivalent form, in which there is the additional requirement $f(x) = -f(-x)$. Then there is an $x \in \mathbb{S}^n$, such that $f(x) = \mathbf{0}$.

Observe, if function f is a linear function mapping $\mathbb{R}^{n+1}$ to $\mathbb{R}^n$, with an $n \times (n+1)$-dimensional matrix A, then it means the homogeneous linear system $Ax = \mathbf{0}$ has a non-trivial solution. This fact helped Seymour to prove his coloring theorem, see [147], and this was the starting point in the proof of the Beck-Fiala theorem, see [18].

Epping et al investigated the algorithmic solutions of the necklace problem, and tried to minimize the number of cuts, see [55].

Some of their results:

1. If there are n types of gems, then one can find an solution using n cuts in polynomial time.

2. Finding a solution with minimal number of cuts is NP-hard.

3. A greedy algorithm yields a solution with n cuts, provided there are exactly two gems of each types. (This special case is called *Paint shop problem.*)

4. Finding a solution with minimal number of cuts is NP-hard even in the Paint shop problem.

The Paint shop problem or the result in the 3. item can be phrased and solved by a system of linear equations. Going through on the necklace let us associate the variable x_i the the part between the ith and $i+1$st gems. The variable x_i can take the value $0, 1$, where $x_i = 1$ means an executed cut between the two gems. The sharing is good if there are an odd number of cuts between two gems of the same types, then the two gems end up at different players. We define matrix A as follows. Its rows are indexed by the gems, and if the gems of type j are in the positions $s_j < s'_j$, then $A_{ji} = 0$, provided $i < s_j$ or $s'_j < i$, otherwise $A_{ji} = 1$. It is easy to see, that a solution of $Ax = \mathbf{1}$ over $\mathbb{F}_2$ gives a good sharing of the Paint shop problem. Applying enough cuts, there are always solutions, so there is a basis solution also, in which at most n variables take non-zero value. So we re-proved the next statement:

Statement 1. *[55] If in a Paint shop problem there n different types of gems, then n cuts suffices for a good sharing.*

The above procedure can be generalized in several ways.

Intervals, clique covering

It is a classical problem to investigate the covering point set X of a set of interval $\mathbf{I} = \{I_i\}_{i=1}^n$, that is $X \subset \mathbb{R}$ is such that $X \cap I_i \neq \emptyset$ for $i = 1, \ldots, n$. Covering sets always exists, and the greedy algorithm provides a minimum size of such a set X, see [49]. What happens if we impose a stronger condition for covering, namely that $|X \cap I_i|$ is odd for all i? We call such a set X an *odd cover*.

Odd covers might not exit, e. g. $\mathbf{I} = \{[0,2], [0,1], (1,2]\}$. Still the existence of odd covers can be checked by an algebraic construction. Let us associate to a system $\mathbf{I}$ the $0-1$ matrix A of n rows that defined as follows. The ones in the rows continuous and if $I_j \subset I_k$ then $A_{ji} \leq A_{ki}$ such that in the kth row there are strictly more 1s from left (right) if I_j left (right) endpoint greater (smaller) than that of I_k.

Statement 2 (Odd cover)**.** *The system of intervals* $\mathbf{I}$ *has an odd cover if and only if* $Ax = \mathbf{1}$ *has a solution over* $\mathbb{F}_2$.

Proof. It is obvious from the construction. □

Corollary 12 (No cover)**.** *If the system* $\mathbf{I}$ *has no odd cover then it has a sub-system* $\mathbf{I}^* \subset \mathbf{I}$ *such that* $|\mathbf{I}^*|$ *is odd and for all* $x \in \cup\mathbf{I}^*$ *the number of intervals in* $\mathbf{I}^*$ *containing* x *is even.*

Proof. Apply Theorem 5.3 to the system $Ax = \mathbf{1}$ over $\mathbb{F}_2$, where A is the matrix associated to $\mathbf{I}$. □

The above method extends to the clique cover of graphs. In the first step let us note the clique-vertex incidence matrix of interval graphs are row-continuous, see [77], and the number of cliques is not more than the number of vertices. On the other hand for an interval graph G there is a *leftmost* clique (which is a simplicial vertex together with its neighbors in G), so according to Corollary 12 the cliques of G has odd cover by at most $|V(G)|$ vertices.

Of course one can use Corollary 12 to find odd covers for in general graphs, the probles is that the number of cliques can be exponential and their structure is more intricate.

5.2.2 Splitting trees

Another possibility to generalize the necklace problem if instead of a path, we lace the gems to a tree. If there are even number of gems (or of colors) of all types, then cutting enough edges we can distribute those evenly. Interestingly enough, this question were raised before the necklace problem, see Bhatt and Leiserson [35]. Let us assume a graph G is two-colored (i. e. there are two types of gems) and both colors appear on even number numbers, and partition $V(G) = A \cup B$ such that A and B contains the same number of vertices from both colors. A *two-color bisection* is the edge set of between A and B in such a partition. Among other, they proved the following theorem:

Theorem 5.7 (Bisector). *[35] Every n-vertex forest on binary trees has a two-color bisection of size $2 \log_2 n$.*

If there are two gems from each type, we have a theorem analogous the Statement 1.

Theorem 5.8 (2-gems). *If F is an n-colored tree, where all colors appear exactly twice then it has an n-color bisection of size at most n.*

Proof. Let A be the path-edge incidence matrix of the tree F, where we consider the paths connecting the vertices of the same color. That is A is an $n \times (2n-1)$ matrix. Again, $Ax = b$ has a solution over $\mathbb{F}_2$ gives a good bisection. Since any subset of path cover at least one edge in odd number of times, we can apply Theorem 5.3. On the other hand, if the is a solution, there must be a basis solution, in which the number of non-zero variables is at most n. □

One may ask if there is a direct generalization of the necklace theorem to trees? (That is if there are n types of gems, how many cuts suffice in getting good bisection?) Alas, a lots of cut is needed even for one color (one type of gems); this number is k for the star $K_{2k-1,1}$. To bound the degrees of a tree T does not help, since the size of a one-color bisection of binary tree T is $\log_2 |V(T)|$ (the observation is due to Béla Csaba).

Conjecture 3 (General bisection). *If T is an n-colored tree, in which all colors used in even times, then T has an n-color bisection of size $cdn \log_2 v(T)$, where d is the maximum degree, $v(T) = |V(T)|$ and c is an absolute constant.*

5.3 Automata isomorphisms by colorings

The notion of complete system was introduced by V. M. Glushkov in [76] where he introduced the general product and characterized the complete systems for this product under the isomorphic embedding as representation.

One can consider compositions as networks of automata. In this case, the underlying graphs are the complete graphs, and each vertex contains an automaton. If the network receives an external input sign, then simultaneously, each component automaton receives an input sign which may depend on the external input sign and all of the actual states of the ancestor automata of the considered one. On the basis of this network approach, we can define different compositions by giving the set of the available underlying graphs. Having this general definition of composition it is natural to look for conditions on the underlying graphs under which there are finite isomorphically complete systems with respect to the corresponding product. This question is studied in [72] where it is proved that if there exists a finite complete system for some composition under isomorphic embedding as representation, then for every integer k there is a graph among the underlying graphs determining the composition such that it has a connected subgraph for which the indegree of each vertex is at least k. This result immediately implies that most of the known compositions have no finite complete system under the isomorphic embedding. There are only two exceptions, namely, the general product [76] and the cube-product [98]. Paper [72] does not contain any sufficient condition for the underlying graphs which would imply the existence of a finite complete system under isomorphic embedding. Here we present a sufficient condition such that compositions determined by graphs satisfying this condition are equivalent to the general product with respect to the isomorphic completeness.

5.3.1 Automata and their products

By an *automaton*, we mean a couple $\mathbf{A} = (X, A)$, where A and X are finite nonempty sets, the set of *states* and the set of *input symbols*, respectively, and for all $x \in X$, x is realized as a unary operation on A which is denoted by $x^{\mathbf{A}}$. Since we consider an automaton as a unoid (universal algebra with unary operations), the notions such as isomorphism, homomorphism, and subautomata can be introduced in a natural way.

Let $D = (V, E)$ be a directed graph consisting of a nonempty finite set of vertices $V = \{1, \dots, n\}$ and a set $E \subseteq V \times V$ of edges, and let us consider an arbitrary nonempty set $\mathcal{D}$ of such finite directed graphs. Furthermore, let $\mathbf{A}_j = (X_j, A_j)$, $j = 1, \dots, n$, be a system of automata, X a nonempty finite set and φ a mapping of $A_1 \times \cdots \times A_n \times X \to X_1 \times \cdots \times X_n$. An automaton $\mathbf{A} = (X, A)$ is called a $\mathcal{D}$*-product* of $\mathbf{A}_j$, $j = 1, \dots, n$, if the conditions below are satisfied:

(1) $A = \prod_{j=1}^{n} A_j$.

(2) There exists a graph $D = (\{1, \dots, n\}, E)$ in $\mathcal{D}$ such that the mapping φ can be given in the following form:

$$\varphi(a_1, \dots, a_n, x) = (\varphi_1(a_1, \dots, a_n, x), \dots, \varphi_n(a_1, \dots, a_n, x))$$

for all $(a_1, \dots, a_n) \in A$, $x \in X$, moreover φ_j is independent of any a_i with $(i, j) \notin E$ for every j, $j = 1, \dots, n$.

(3) For all $(a_1, \dots, a_n) \in A$ and $x \in X$,

$$(a_1, \dots, a_n)x^{\mathbf{A}} = (a_1 x_1^{\mathbf{A}_1}, \dots, a_n x_n^{\mathbf{A}_n})$$

where $x_j = \varphi_j(a_1, \dots, a_n, x)$, $j = 1, \dots, n$.

For the product introduced above we use the notation

$$\prod_{j=1}^{n} \mathbf{A}_j(X, \varphi, D).$$

Now, let Γ be a system of automata. Γ is called an *isomorphically complete system with respect to the $\mathcal{D}$-product* if every automaton can be embedded isomorphically into a $\mathcal{D}$-product of automata from Γ.

It is worth noting that we obtain the notion of the general product if the set $\mathcal{D}$ of the underlying graphs is the set of the complete graphs of the form

$$D_n = (\{(1, \ldots, n\}, \{1, \ldots, n\} \times \{1, \ldots, n\}),$$

$n \in \mathbb{N}$.

5.3.2 Embeddings and colorings

The following statement shows that if a finite graph can be colored in an appropriate way, then the composition having this graph as its underlying graph is strong enough.

Lemma 14. *Let $D = (\{1, \ldots, n\}, E)$ be a directed graph and also let $\mathbf{A} = \prod_{j=1}^{m} \mathbf{A}_j(X, \varphi)$ be a general product of automata where $1 \leq m < n$. Moreover, let $\chi : \{1, \ldots, n\} \to \{1, \ldots, m\}$ be such a coloring of the vertices of D that for every vertex $r \in \{1, \ldots, n\}$ $\chi(S_r) = \{1, \ldots, m\}$, where S_r denotes the set of the ancestors of r in D. Then $\mathbf{A}$ can be embedded isomorphically into a $\mathcal{D}$-product of copies of $\mathbf{A}_j$, $j = 1, \ldots, m$, with the underlying graph D.*

Proof of Lemma 14. Let us denote the elements of $\prod_{j=1}^{m} \mathbf{A}_j(X, \varphi)$ by $(a_{t1}, \ldots, a_{tm})$, $t = 1, \ldots s$, where $s = |\prod_{j=1}^{m} A_j|$. Then the elements $(a_{t1}, \ldots, a_{tm})$, $t = 1, \ldots, s$, are pairwise different. Without loss of generality, it can be assumed that the copies of each automaton $\mathbf{A}_j$ occurring in the considered general product are distinguished. In this case, the sets $\{a_{t1}, \ldots, a_{tm}\}$, $t = 1, \ldots, s$, are pairwise different, and thus there exist functions φ'_j, $j = 1, \ldots, m$, such that for all $j \in \{1, \ldots, m\}$, $x \in X$ and $1 \leq t \leq s$,

$$\varphi_j(a_{t1}, \ldots, a_{tm}, x) = \varphi'_j(\{a_{t1}, \ldots, a_{tm}\}, x)$$

is valid. Let us define the mapping μ of the set $\prod_{j=1}^{m} A_j$ into the set $\prod_{i=1}^{n} A_{\chi(i)}$ as follows.

$$\mu : (a_{t1}, \ldots, a_{tm}) \to (a_{t,\chi(1)}, \ldots, a_{t,\chi(n)})$$

for all t, $t = 1, \ldots, s$ and let $C = \{(a_{t,\chi(1)}, \ldots, a_{t,\chi(n)}) : t = 1, \ldots, s\}$. Since $\chi(S_r) = \{1, \ldots, m\}$ for all $r \in \{1, \ldots, n\}$, μ is a one-to-one mapping of $\prod_{j=1}^{m} A_j$ onto C. Now, let us define the $\mathcal{D}$-product $\mathbf{B} = \prod_{i=1}^{n} \mathbf{A}_{\chi(i)}(X, \bar{\varphi}, D)$ in the following way. For all $i \in \{1, \ldots, n\}$, $x \in X$ and $(a_1, \ldots, a_n) \in \prod_{i=1}^{n} A_{\chi(i)}$, let

$$\bar{\varphi}_i(a_1, \ldots, a_n, x) = \begin{cases} \varphi'_{\chi(i)}(\{a_{t1}, \ldots, a_{tm}\}, x) & \text{if} (a_1, \ldots, a_n) = (a_{t,\chi(1)}, \ldots a_{t,\chi(n)}) \\ & \text{for some } 1 \leq t \leq s, \\ \text{an arbitrarily fixed } x \in X & \text{otherwise.} \end{cases}$$

Since the sets $\{a_{t1}, \ldots, a_{tm}\}$, $t = 1, \ldots, s$, are pairwise different, the functions $\bar{\varphi}_i$, $i = 1, \ldots, n$, are well-defined. Moreover, since $\chi(S_r) = \{1, \ldots, m\}$, for all $r \in \{1, \ldots, n\}$, the defined product is a $\mathcal{D}$-product.

Finally, we prove that C determines a subautomaton in $\prod_{i=1}^{n} \mathbf{A}_{\chi(i)}(X, \bar{\varphi}, D)$ which is an isomorphic image of $\mathbf{A}$ under the mapping μ. For this purpose, let $(a_{t1}, \ldots, a_{tm}) \in \prod_{j=1}^{m} A_j$ and $x \in X$ be arbitrary elements. Then it is enough to prove that

$$(a_{t1},\dots,a_{tm})x^{\mathbf{A}}\mu = (a_{t\chi(1)},\dots,a_{t,\chi(n)})x^{\mathbf{B}}.$$

Let us suppose that $(a_{t1},\dots,a_{tm})x^{\mathbf{A}} = (a_{q1},\dots,a_{qm})$ for some $q \in \{1,\dots,s\}$. Then the required equality is valid if $a_{t,\chi(i)}x_i^{\mathbf{A}_{\chi(i)}} = a_{q,\chi(i)}$ for all $i \in \{1,\dots,n\}$, where $x_i = \bar{\varphi}_i(a_{t,\chi(1)},\dots,a_{t,\chi(n)},x)$. Let $i \in \{1,\dots,n\}$ be arbitrary. By the definition of $\bar{\varphi}_i$,

$$x_i = \bar{\varphi}_i(a_{t,\chi(1)},\dots,a_{t,\chi(n)},x) = \varphi'_{\chi(i)}(\{a_{t1},\dots,a_{tm}\},x),$$

moreover $\varphi'_{\chi(i)}(\{a_{t1},\dots,a_{tm}\},x) = \varphi_{\chi(i)}(a_{t1},\dots,a_{tm},x)$. Let $\chi(i) = j$. Then the left side of the required equality is $a_{tj}x_i^{\mathbf{A}_j}$, where $x_i = \varphi_j(a_{t1},\dots,a_{tm},x)$. On the other hand, by the definition of the general product, $a_{tj}x_j^{\mathbf{A}_j} = a_{qj}$ where $x_j = \varphi_j(a_{t1},\dots,a_{tm},x)$. Let us observe that $x_i = x_j$, and consequently $a_{t,\chi(i)}x_i^{\mathbf{A}_{\chi(i)}} = a_{q,\chi(i)}$ for all $i = 1,\dots,n$. This completes the proof of Lemma 1. □

The next statement presents a sufficient condition for a nonempty set $\mathcal{D}$ of directed graphs for admitting the existence of a finite isomorphically complete system.

Theorem 5.9. *Let $\mathcal{D}$ be a nonempty set of finite directed graphs of the form given at the beginning of the previous section. Then, there exists a finite isomorphically complete system with respect to the $\mathcal{D}$-product, if for every positive integer m, $\mathcal{D}$ contains a graph D which has a connected subgraph $D' = (V', E')$ having a coloring $\chi : V' \to \{1,\dots,m\}$ of the vertices of D' such that, for every vertex $r \in V'$, $\chi(S_r) = \{1,\dots,m\}$ where S_r denotes the set of the ancestors of r in D'.*

Proof of Theorem 5.9. Let $\mathbf{B}_2 = (\{w,x,y,z\},\{0,1\})$ be the automaton which is defined by $0w^{\mathbf{B}_2} = 1z^{\mathbf{B}_2} = 0$ and $1y^{\mathbf{B}_2} = 0x^{\mathbf{B}_2} = 1$. To prove our statement, we shall show that an arbitrary automaton can be embedded isomorphically into a suitable $\mathcal{D}$ product of suitable copies of $\mathbf{B}_2$. For this purpose, let $\mathbf{A} = (X, A)$ be an arbitrary automaton. By the theorem of Glushkov [76], $\mathbf{A}$ can be embedded isomorphically into a general product $\prod_{j=1}^{m}\mathbf{B}_2(X,\varphi)$. Let us distinguish the copies of $\mathbf{B}_2$ as $\mathbf{A}_1,\dots,\mathbf{A}_m$. Furthermore, let $\mathbf{A}_{m+1}$ denote a further copy of $\mathbf{B}_2$. By our assumption, there is a graph $D = (V,E) \in \mathcal{D}$ containing a subgraph $D' = (V', E')$ which has a coloring $\chi : V' \to \{1,\dots,m\}$ such that for every vertex $r \in V'$, $\chi(S_r) = \{1,\dots,m\}$. Without loss of generality, it can be assumed that $V' = \{1,\dots,n\}$ for some positive integer n, furthermore, let $|V| = s$. Consider the function $\rho : V \to \{1,\dots,m+1\}$ given in the following way. For every $j \in \{1,\dots,s\}$ let

$$\rho(j) = \begin{cases} \chi(i) & \text{if} 1 \le j \le n, \\ m+1 & \text{otherwise.} \end{cases}$$

Now, let us define the $\mathcal{D}$-product $\prod_{i=1}^{s}\mathbf{A}_{\rho(i)}(X,\bar{\varphi},D)$ as follows. For the first n components, let the product be given in the same way as in the proof of Lemma 1, and for the last $s-n$ components, let us choose the state 0, and let the value of $\bar{\varphi}_j$, $n < j \le s$, be w. Then it can be seen that $\prod_{j=1}^{m}\mathbf{B}_2(X,\varphi)$, and thus also $\mathbf{A}$, can be embedded isomorphically into the $\mathcal{D}$-product $\prod_{i=1}^{s}\mathbf{A}_{\rho(i)}(X,\bar{\varphi},D)$ which ends the proof of Theorem 1. □

By the theorem of Glushkov [76], a system Γ of automata is isomorphically complete with respect to the general product if and only if there exists an $\mathbf{A} \in \Gamma$ such that $\mathbf{B}_2$ can be embedded isomorphically into a general product of $\mathbf{A}$ with a single factor. Thus, by the proof of Theorem 5.9, if Γ is isomorphically complete with respect to the general product, then Γ is isomorphically complete with respect to the $\mathcal{D}$-product given in Theorem 5.9. The converse assertion is obvious since the $\mathcal{D}$-product is a special case of the general product. Consequently, we obtain the following corollary.

Corollary 13. *If a $\mathcal{D}$-product satisfies the condition of Theorem 5.9, then it is equivalent to the general product regarding isomorphic completeness.*

Finally we spell out results that ensure the appropriate colorings. Associate the S_r in-neighborhoods for the vertex r in the graph D to the edges of a hypergraph $\mathcal{F} = (E(D), \mathcal{H})$, we get a condition for the required colorings. Namely we need a coloring $\chi : E(D) \to \{1, ..., m\}$ such that $|\chi(E)| = m$, for all $E \in \mathcal{H}$, where $m \in \mathbb{N}$ given. If such χ exists we call the hypergraph $\mathcal{F}$ to be m-colorful. (Such a coloring is also called a rainbow coloring with m colors.)

We lists some results without proofs, for the details see [73]. A standard first moment method gives the following lemma.

Lemma 15. *Let $\mathcal{F}$ be a hypergraph such that $|E| \geq k$ for all $E \in \mathcal{H}$, $|\mathcal{H}| = n$ and $k > 2m \log n$. Then $\mathcal{F}$ is m-colorful.*

Lemma 15 and Theorem 5.9 immediately give one of the results in [72], which states that a general product of automata can be embedded isomorphically into a $\mathcal{D}$-product if $\mathcal{D}$ contains large, “almost” complete graphs. Precisely, there exist a universal constant c and a sequence of graphs $D_{n_\ell} \in \mathcal{D}$ such that $|V(D_{n_\ell})| = n_\ell$ and the number of the ancestors of every vertex of D_{n_ℓ} is at least $n_\ell - c$.

Lemma 15 shows that this result can be extended in the following way. Let $f(n)$ be a function which tends to ∞ if $n \to \infty$. If every vertex has at least $f(n) \log n$ ancestors, then among the corresponding underlying graphs there exists an m-colorful for every positive integer m.

With the use of Lovász Local Lemma we get

Lemma 16. *Let $\mathcal{F}$ be a hypergraph and k a positive integer such that for all $E \in \mathcal{F}$, $|E| \geq k$. If $k > m(1 + \log d + \log m)$, then $\mathcal{F}$ is m-colorful.*

As an application of Lemma 16, let us consider the hypercubes as underlying graphs (see [98]). Let $n > 0$ be an arbitrary integer. Then, $|S_{\mathbf{r}}| = n$ for any vertex $\mathbf{r} \in \{0, 1\}^n$ of the n-dimensional hypercube. Moreover, let us observe that $S_{\mathbf{r}} \cap S_{\mathbf{r}'} \neq \emptyset$ if and only if $\mathbf{r}$ and $\mathbf{r}'$ are different exactly in two components. Consequently, $d = d(S_{\mathbf{r}}) = n(n-1)/2$, for all $\mathbf{r} \in \{0, 1\}^n$. Now, if m is a fixed positive integer, then we can choose an n such that

$$n > m(1 + \log \frac{n(n-1)}{2} + \log m)$$

is valid. Therefore, by Lemma 16, the n-dimensional hypercube is m-colorful, and consequently, by Corollary 13, the cube-product is equivalent to the general product regarding the isomorphically complete systems which is the main result in [98].

5.4 Graph Clustering via Generalized Colorings

We propose a new approach for defining and searching clusters in graphs that represent real technological or transaction networks. In contrast to the standard way of finding dense parts of a graph, we concentrate on the structure of edges between the clusters, as it is motivated by some earlier observations, e.g. in the structure of networks in ecology and economics and by applications of discrete tomography. Mathematically special colorings and chromatic numbers of graphs are studied.

5.4.1 Introduction and Results

One of the main tasks in network theory is clustering vertices, see Newman [120]. Graph clustering is a well-studied problem and has important applications in graph mining or model construction. The usual methods try to achieve many edges inside clusters and only a few between distinct clusters [141]. To measure the algorithms' efficacy the parameter known as *Newman modularity* is commonly used [120]. This approach generally works well for so-called *social graphs*, which usually contain more triangles than a random graph with similar edge density or degree properties. Note, that although the various methods are proposed to obtain clustering, the resulting clusters are usually very similar.

In contrast *technological* or *transaction graphs* contain fewer triangles and often display tree-like structures. We need to consider important examples that give us motivation to develop a completely different approach for clustering.

First is the *pollinator network* which is a bipartite graph (A, B), the vertices of one part are associated to pollinating animal species (bees, butterflies, flies, bats etc), the other to plants which these animals pollinate. An (x, y) edge is drawn in this graph iff the animal x pollinates the plant y. According to the findings, the resulting graph is not arbitrary bipartite graph, it has a strong structural property, the so-called *nestedness*. That is, the vertices of each color class can be ordered, and the smaller ranked vertex neighborhood contains the neighborhood of any higher ranked one. Formally, $A = \{x_1, \dots, x_n\}$ and $B = \{y_1, \dots, y_m\}$ such that $N(x_j) \subset N(x_i)$ and $N(y_j) \subset N(y_i)$ if $i < j$.

Uzzi [160] studied cloth making firms and fashion shops selling the products of those firms in New York City. He defined a bipartite graph of firms and shops by setting an edge (x, y) if the shop y sells the product of the firm x and observed the same nestedness of the neighborhoods (he called this *embeddedness*) if the vertices are ordered properly.

In the context of image processing, Junttila and Kaski [100] call a binary matrix A (that is, a matrix whose entries are either zero or one) *fully nested* if its rows and columns can be reordered such that the ones are in an echelon form. Let G_A be the bipartite graph whose adjacency matrix is A. Then A being fully nested is equivalent to G_A satisfying embeddedness. They main interest is to cut one partition of a bipartite graph into as few sets as possible such that the induced s are nested.

Formally, let X (the columns) and Y (the rows) be the bipartition of G_A. The matrix A and the graph G_A are each said to be *k-nested with respect to X* if X can be partitioned as $X_1, \dots, X_k$ such that all subgraphs spanned by (X_i, Y) are fully nested for $i = 1, \dots, k$. The quantity of interest for any G_A is smallest k for which G_A is k-nested.

Observe, that nestedness has a forbidden induced subgraph characterization for bipartite graphs. If neither $N(x_i) \subset N(x_j)$ nor $N(x_j) \subset N(x_i)$ hold for some indices i, j, then there are y_i and y_j vertices such that $y_i \in N(x_i)$, $y_i \in N(x_i)$ but $y_i \notin N(x_j)$ and $y_j \notin N(x_i)$. That is the set $\{x_i, x_j, y_i, y_j\}$ spans an induced $2K_2$. O the other hand, in a bipartite graph an induced $2K_2$ is obviously an obstruction for the nestedness.

Following these examples, we may try to cluster a general transaction graph similarly. The clusters are not dense subsets of vertices, just the opposite, there are no edges inside. Restricting the graph to any two clusters there must be nestedness in the neighborhoods, or equivalently, no $2K_2$ appear. In other words, the clusters are color classes of a good coloring, with the addition that the union of any two classes is an induced $2K_2$-free graph.

From the mathematical point of view it is natural to generalize this notion to an arbitrary host graph G and a forbidden bipartite subgraph H as follows.

Definition 9. *Fix a bipartite graph H. A proper coloring of a graph G is an* induced H-avoiding coloring *if the union of any two color classes spans an induced H-free graph. Let $\chi_H(G)$ be the minimum number of colors in an induced H-avoiding coloring of G.*

Remarks. There are coloring ideas in the literature that give back special cases of our notion. For a connected bipartite graph H, coloring of a graph G is H-avoiding, if the union of any two color classes do not contain H as a subgraph, see the paper of Choi, Kim and Park [45]. If e.g. $H = P_3$ then an H-avoiding coloring is automatically an induced H-avoiding coloring. However, if H is not a bipartite complete graph, then the two notions differ.

Hale [83] introduced the family of $L(h,k)$ coloring, see also at Calamoneri's survey [44]. They consider such labellings of the vertices with natural numbers in which labels of adjacent vertices apart least h and the ones with common neighbors apart at least k, while $\lambda_{h,k}(G)$ is the minimum span of labels used in such labellings. Observe if $H = P_3$ then $\lambda_{1,2}(G) = \chi_H(G)$. In fact Hale [83] proved the NP-completeness of the computation of $\lambda_{h,k}(G)$ for general h, k values, and $h = 1, k = 2$ gives the part Lemma 17.

Note that the function $\chi_H(G)$ is not necessarily monotone either in H or in G. However, we have a useful property:

Observation 6. *For any graphs H and G, $\chi(G) \leq \chi_H(G)$. If G is H-free, then $\chi(G) = \chi_H(G)$.*

To use Observation 6 we need to mention a paper of Král, Kratochvíl, Tuza and Woeginger [106] that studied the hardness of coloring H-free graphs. They gave a complete description of the problem in the theorem follows:

Theorem 5.10 (Král-Kratochvíl-Tuza-Woeginger [106])**.** *The problem H-Free Coloring is polynomial-time solvable if H is an induced subgraph of P_4 or of $P_3 \oplus K_1$, and NP-complete for any other H.*

Complexity issues

We show that that the computation of $\chi_H(G)$ is NP-hard for some graphs, and polynomially computable for others. The most interesting case, when $H = 2K_2$, gives back embeddedness as described above. For these generalized chromatic numbers we derive some theoretical extremal results as well as results on complexity.[3]

[3]Heuristics for finding induced H-avoiding colorings and case studies were presented in Gera, London and Pluhár, [74].

In the following we use K_n, P_n and C_n for the complete graph, path and cycle on n vertices, respectively. For graphs H_1 and H_2 on disjoint vertex sets, $H_1 \oplus H_2$ denotes their disjoint union. Theorem 5.11 gives a characterization of the complexity issues in computing $\chi_H(G)$ depending on the graph H.

Theorem 5.11. *The computation of $\chi_H(G)$ is polynomial-time solvable if H is $K_1 \oplus K_1$, K_2, or $K_2 \oplus K_1$ and is NP-hard for all other graphs.*

It is valuable to spell out special cases since the proofs of these are needed in proving Theorem 5.11.

Lemma 17. *It is NP-complete to decide if $\chi_{P_3}(G) \le 5$, while it is polynomial time decidable if $\chi_{P_3}(G) \le 3$. It is also NP-complete to decide if $\chi_{P_3 \oplus K_1}(G) \le 5$.*

Lemma 18. *It is NP-complete to decide if $\chi_{P_4}(G) \le 3$.*

Lemma 19. *There is a unique H-avoiding coloring of G using exactly $\chi_H(G)$ colors if $H = K_2 \oplus K_1$. One can find this coloring in polynomial time.*

Let us note that a P_3-avoiding coloring of G has a nice combinatorial meaning, it represents the edges of G as the union of independent matchings. The computation of $\chi_{P_3}(G)$ can be reduced to the normal chromatic number. Let $P_3(G)$ be a graph which made from G by adding an edge to every induced P_3, i. emaking a triangle out of these P_3.

Observation 7. $\chi(P_3(G)) = \chi_{P_3}(G)$.

Note that if a bipartite graph G_A is k-nested then it has a similar reduction as in Observation 7.

For a bipartite graph G_A with bipartition (X, Y), define the *conflict graph* $co(X)$ on X such that (x, x') is an edge in $co(G)$ for $x, x' \in X$ if there are $y, y' \in Y$ such that $\{x, x', y, y'\}$ spans a $2K_2$ in G_A.

Observation 8. *The bipartite graph G_A is exactly k-nested for X if $\chi(co(X)) = k$.*

For applications the computation of $\chi_{2K_2}(G)$ seems to be the most important case.

Theorem 5.12. *It is polynomial time decidable if $\chi_{2K_2}(G) \le 3$.*

For a fix graph H there is a linear upper bound on the value of $\chi_H(G)$. In this text, $\lg n$ is the logarithm of n in base 2 and $\log n$ is the natural logarithm of n.

Proposition 8. *Let H be a bipartite graph and let k_1 be the largest positive integer such that each bipartition of H has a part with size at least k_1. Let k_2 be the smallest positive integer such that each bipartition of H has both parts of size at least k_2. Let G be an n-vertex graph with chromatic number χ and independence number α. If $k_2 \ge 3$, then*

$$\chi_H(G) \le \min\left\{\frac{n}{k_1 - 1} + \frac{k_1 - 2}{k_1 - 1}\chi, \frac{n}{k_2 - 1}\left(1 - \frac{1}{\chi}\right) + \frac{k_2 - 2}{k_2 - 1}(\chi - 1) + 1\right\}.$$

If $k_2 = 2$, then

$$\chi_H(G) \le \min\left\{\frac{n}{k_1 - 1} + \frac{k_1 - 2}{k_1 - 1}\chi, n - \alpha + 1\right\}.$$

Random graphs

In the case where G is a random graph drawn from $G(n, p)$, the Erdős-Rényi random graph on n vertices with edge probability p, we establish tight bounds for $\chi_H(G)$. The distribution of $\alpha(G)$, where $G \sim G(n, p)$ for p fixed, was determined by Bollobás and Erdős [36]. The distribution of $\chi(G)$ was first proven by in a classic result by Bollobás [40] and the error terms have been further refined by various authors (see [8]).

To be precise, **whp** means *with high probability*, i. ea probability arbitrarily close to one, provided that the number of vertices (or other natural parameter) is large enough.

Theorem 5.13. *Let H be a bipartite graph with k_1 and k_2 defined as in Proposition 8. Fix $p \in (0, 1)$, let $d = 1/(1-p)$, and let $G \sim G(n, p)$. If $k_1 \geq 3$ and $k_2 \geq 2$, then there is a $C = C(H, p)$ such that* **whp**

$$\frac{n}{k_1 - 1} - C \log n \leq \chi_H(G) \leq \frac{n}{k_1 - 1} + O\left(\frac{n}{\log_d n}\right).$$

If $k_1 = k_2 = 2$, then there exists a $C = C(H, p)$ such that **whp**

$$n - C \log n \leq \chi_H(G) \leq n - 2 \log_d n + O\left(\log_d \log n\right).$$

In particular, if $H = 2K_2$, then **whp**

$$n - 8 \log_{1/Q} n + \Omega\left(\log_{1/Q} \log n\right) \leq \chi_H(G) \leq n - 2 \log_d n + O\left(\log_d \log n\right),$$

where $Q = 1 - 2p^2(1-p)^2$.

Finally, we mention a useful observation on the H-avoiding chromatic number, see its consequences in Section 5.5.

Observation 9. *If G is a graph such that every H-free induced subgraph has at most ℓ edges, then $\chi_H(G)$ satisfies*

$$\ell \binom{\chi_H(G)}{2} \geq e(G).$$

In the followings we provide the proofs of the main results. In Subsection 5.4.2 we prove Theorem 5.11. Subsection 5.4.3 contains the proof of Theorem 5.12, while Subsection 5.4.4 contains the proofs of Proposition 8 and of Theorem 5.13. In Subsection 5.5 we show some of the consequences of Observation 9.

5.4.2 Proof of Theorem 5.11

We start the proof with the cases in which the graph H is equal to either $K_1 \oplus K_1$, K_2 or $K_2 \oplus K_1$. The graph G has K_2-avoiding coloring if and only if G is the empty graph.

Proof of Lemma 19. If $H = K_2 \oplus K_1$ then any two color classes in an H-avoiding coloring spans either a complete or empty bipartite graph. (In the special case if $H = K_1 \oplus K_1$ then there can be only complete bipartite graphs between any two color classes.) Let us define a binary relation ρ such that for $x, y \in V(G)$ we have $x\rho y$ iff $N(x) = N(y)$. Obviously ρ is an equivalence relation, and the equivalence classes induced by ρ are exactly the color classes of G in the unique $K_2 \oplus K_1$-avoiding coloring of $\chi_{K_2 \oplus K_1}(G)$ classes. $\square$

Combining Theorem 5.10 and Observation 6, one gets immediately that the computation of $\chi_H(G)$ is NP-complete if $\chi(G)$ is also NP-complete for H-free G. On the other hand, the polynomial-time computability of χ for H-free graphs does not imply the same for χ_H. Among the polynomial cases of Theorem 5.10 we have checked already the graphs $K_1 \oplus K_1$, K_2 and $K_2 \oplus K_1$. Somehow against intuition, the computation of χ_H is NP-complete for the remaining $H = P_3$ and $H = P_4$ cases according to Lemmas 17 and 18.

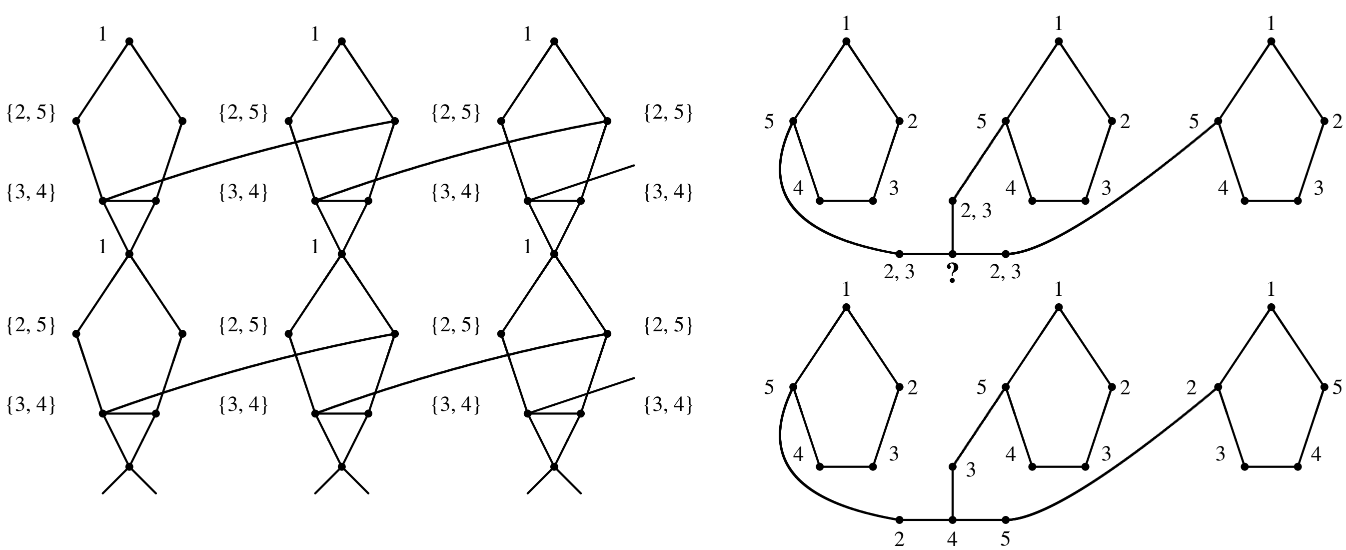
Proof of Lemma 17. We need to show $L_5 = \{G : \chi_{P_3}(G) \leq 5\}$ is NP-complete language. We use reduction from the language

$$L_{3,2} = \{T : T \text{ is a 3-uniform hypergraph}, \chi(T) \leq 2\},$$

which is a well-known NP-complete problem. Let T be an instance, that is $T \in L_{3,2}$.

We need to assign a graph G_T to T such that $\chi_{P_3}(G_T) \leq 5$ if and only if $\chi(T) \leq 2$. It turns out that the greatest difficulty is to associate the colorings of the graph G_T and the hypergraph T. The color of a vertex t of T cannot be encoded in one vertex x_t of G_T, since the gadgets constructed in G_T that enforce the good coloring of the edges of T containing t would interfere with each other. The solution is to repeat the actual color of vertex t at least as many times as the number of edges of T that contain t. For simplicity we repeat the color of any vertex t a total of m times, where m is the number of edges in T, and read out the color of t at most once from each place.

The graph G_T will consist of an $n \times m$ matrix of pentagons, in which the i-th row codes the color of the i-th vertex in T. To assess the coloring of the j-th edge of T, the j-th column of this matrix is read. The usual types of gadgets are used in G_T representing and evaluating the edges of T, see Figure 5.4.2.



5.4.2 The matrix and the colorings of gadgets.

Before examining the coloring of G_T, let us examine a P_3-avoiding good coloring of just C_5, since C_5 is the main building block of our construction. The vertices are referenced clockwise. If the first vertex is colored by 1, the second by 2, the third vertex color can be neither 2, because of adjacency, or 1 since it would create a two colored P_3.

So, without loss of generality, the first three vertices are colored 1, 2, 3 respectively. The fourth vertex needs the fourth color. It cannot be colored by 2 or 3 as before. If it would be colored by 1, the first, fifth and fourth vertices would form a two-colored P_3. Finally, the fifth vertex needs to be colored 5, since 1 and 4 are colors of adjacent vertices, while 2 or 3 would create two-colored P_3's.

Assuming that the n vertices of T are $x_1, \ldots, x_n$ and the m edges are $e_1, \ldots, e_m$, the graph G_T is first constructed by taking an $n \times m$ matrix with a C_5 in each position (i, j). The C_5 in the (i, j) position will be referred to as $C_{i,j}$. Second, connect the third and fourth vertices of $C_{i,j}$ to the first vertex of $C_{i+1,j}$ for $i = 1, \ldots, n-1$, $j = 1, \ldots n$, and similarly from $C_{n,j}$ to $C_{1,j+1}$ for $j = 1, \ldots, n-1$. Third, draw edges from the fourth vertex of $C_{i,j}$ to the second vertex of $C_{i,j+1}$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m-1$. See Figure 5.4.2.

Without loss of generality, any P_3-avoiding five-coloring should use color 1 at the first vertex of any C_5, should use the colors 2 and 5 in the second and fifth vertices (although in any order) and the colors 3 and 4 in the third and fourth vertices (again their order is arbitrary).

Furthermore, is easy to verify that a proper P_3-avoiding five-coloring must use the same order of colors 2 and 5 within a row, while the order of 2 and 5 can be arbitrary for each row. We will use the i-th row to code the color of the vertex x_i of the hypergraph T. However, when we read this “value," each C_5 is read only once.

Finally, the gadgets realizing the edges of T are m copies of $K_{1,3}$. Let e_ℓ be $\{x_p, x_q, x_r\}$ and connect the leaves of the ℓ-th $K_{1,3}$ to the fifth vertex of a yet unused C_5 in the p-th, q-th and r-th rows, respectively. The colors the vertices of e_ℓ receives are the color of the fifth vertices of C_5-s which were connected to the leaves of the representing $K_{1,3}$.

Let us check if proper five-colorings of the construction and proper two-colorings of T correspond to each other. If, for e_ℓ, the vertices in the graph coloring all receive the color, say 5, then the leaves of the representative $K_{1,3}$ can be colored 2 or 3. One of these colors appears two times, and it results in a two-colored P_3 in the graph coloring. If e_ℓ is colored properly, say 5, 5, 2, then the connected vertices in the representative $K_{1,3}$ may get the colors 2, 3, 5. Giving color 4 to the 3-degree vertex of the representative $K_{1,3}$ we get a proper P_3-avoiding five-coloring of G.

The case $\chi_{P_3 \oplus K_1}(G) \leq 5$. For instance T consider $G_T \oplus K_5$ instead of G_T. Since of five colors appear in a five-coloring of $G_T \oplus K_5$ among the vertices of K_5, $\chi_{P_3 \oplus K_1}(G_T \oplus K_5) \leq 5$ iff T is 2-colorable.

The case $\chi_{P_3}(G) \leq 3$. If G has a vertex of degree at least three, then $P_3(G)$ has a clique of size at least four, and by Observation 7, $\chi_{P_3}(G) \geq 4$. If all vertices have degree at most two, then the components of G are paths and cycles. The components can be colored independently of each other in that case, so G has a P_3-avoiding 3-coloring if and only if all components have. For all $k \in \mathbb{N}$, $\chi_{P_3}(P_k) \leq 3$, we just repeat the pattern $1, 2, 3, 1, 2, 3 \ldots$ starting from one of the ends. The same can be (and must be) done for C_k by specifying a starting vertex. However, it is successful only if $k \equiv 0 \mod 3$. □

Proof of Lemma 18. As in the proof of Lemma 17, we use a reduction from the language $L_{3,2}$, the two-coloring of 3-uniform hypergraphs. Having an instance $T \in L_{3,2}$ with vertex set $x_1, \ldots, x_n$, $n \geq 4$, the reduction to a P_4-avoiding 3-coloring of a graph G_T goes as follows. To each vertex x_i of T we create a pair of vertices x_i, x_i' and have the edge

(x_i, x_i'). An additional special vertex z is adjacent to each x_i and to each x_i'.

For each hyperedge $e_\ell = \{x_p, x_q, x_r\}$ in T, we define a gadget as follows. Take three disjoint copies of P_3, a_i, b_i, c_i for $i = 1, 2, 3$, and vertices w_1, w_2, and draw the edges $(c_1, w_1), (c_2, w_1), (c_2, w_2)$ and (c_3, w_2). Finally we set the gadget by drawing the edges (a_1, x_p), (a_2, x_q) and (a_3, x_r).

We claim that a G has a P_4-avoiding 3-coloring if and only if T has a P_4-avoiding 2-coloring. We may assume vertex z is colored by 3, so x_is are colored 1 or 2, both in the coloring of G and T.

If the vertices of an edge $e_\ell = \{x_p, x_q, x_r\}$ all receive the same color, say 2, then in the gadget associated to e_ℓ the vertices a_1, a_2, a_3 must receive the color 1. (Indeed, if say a_1 would be colored by 3, then take an $x_i \notin \{x_p, x_q, x_r\}$. Either x_i or x_i' has the color 2, say it is x_i. But a_1, x_p, z, x_i is a 2-colored P_4.) The vertices b_1, b_2 and b_3 must get color 3, since if, say b_1, is of color 2, then b_1, a_1, x_p, x_p' would induce a 2-colored P_4. If c_2 has color 1, then both w_1 and w_2 have color 2, since otherwise w_1 or w_2, c_2, b_2, a_2 would be a 2-colored P_4. But in that case, the color of c_3 could be only 1, inducing 2-colored P_4 on the vertices c_3, w_2, c_2, w_1.

If c_2 has color 2, and at least one of c_1 or c_3 has color 1, assume c_1, then w_1 must be colored 3. But then w_2, c_1, b_1, a_1 would be a 2-colored P_4. Finally, if all c_1, c_2, c_3 has color 2, then w_1 and w_2 must have different colors in order to avoid the 2-colored P_4 on c_1, w_2, c_2, w_2. But if, say w_1 has color 3, then we see a 2-colored P_4 on the vertices c_1, w_1, c_2, b_2.

For the other direction, assume that $e_\ell = \{x_p, x_q, x_r\}$ received two colors in the hypergraph coloring. Without loss of generality, we may assume two vertices are colored 1, and one with 2. The vertex colored by 2 is either on one of the side, x_p, x_r or the in the middle, x_q. Let us say x_p has color 2 and x_q, x_r received color 1. Then the coloring extends to the gadget of e_ℓ by coloring a_1, c_2, c_3 by 1, c_1, a_2, a_3, w_2 by 2, and b_1, b_2, b_3, w_1 by 3. If x_q has color 2 and x_p, x_r have color 1, then the extension is c_1, a_2, c_3 is of color 1, a_1, c_2, a_3 is of color 2, and b_1, b_2, b_3, w_1, w_2 is of color 3. Notice that in both cases all b types vertices received color 3, which “insulates" the gadgets from each other, so the defined 3-coloring is a P_4-avoiding one. □

5.4.3 Proof of Theorem 5.12

Proof of the case $H = 2K_2$**.** To see if $\chi_{2K_2}(G) \leq 3$ for a given graph G, first we check if G contains $4K_2$ as an induced subgraph. This requires no more than $O(n^4)$ time. If G does contain a $4K_2$, then $\chi_{2K_2}(G) \geq 4$, since between two color classes there can be only one of those four independent edges. Assume G does not contain $4K_4$, and recall a result of Farber, Hujter and Tuza [64]:

Theorem 5.14 (Farber-Hujter-Tuza [64])**.** *If the graph G does not contain* $(t+1)K_2$ *as an induced subgraph, then the number of maximal independent sets in G is at most* $\binom{n/t}{t}$.

The following ideas are well-known and perhaps motivated Theorem 5.14. The set $\mathcal{M}$ of all maximal independent sets can be found by, for example, a DFS tree algorithm, and can be listed in no more than $O(n^2|\mathcal{M}|)$ time. The decision problem of whether $\chi(G) \leq k$ can be solved by checking if there is k-set from $\mathcal{M}$ covering the vertex set of G. This still can be done in $O(\binom{|\mathcal{M}|}{k})$ time.

Applying Theorem 5.14 to G, $|\mathcal{M}| \leq \binom{n/3}{3} < n^3/162$, so for a possible 3-coloring we have to check a configuration of size no larger than $O(n^9)$. A configuration consists of three maximal independent sets X_1, X_2 and X_3. First, $\cup_i X_i$ should contain all vertices of G. If this holds, it readily gives a 3-coloring, however it is not necessarily $2K_2$-avoiding. Indeed we are looking for $Y_i \subset X_i$ for $i = 1, 2, 3$ such that $\cup_i Y_i$ contains all vertices of G, $Y_i \cap Y_j = \emptyset$ if $i \neq j$, and the partition $\{Y_1, Y_2, Y_3\}$ is $2K_2$-avoiding.

We can assume that $\cap_i X_i = \emptyset$, if not, these vertices are isolated, and can be assigned to any Y_i in the end. Then we start with the sets $Y_1 := X_1 \setminus (X_2 \cup X_3)$, $Y_2 := X_2 \setminus (X_1 \cup X_3)$ and $Y_3 := X_3 \setminus (X_1 \cup X_2)$ and try to put the leftover vertices into those. The triple $\{Y_1, Y_2, Y_3\}$ should be $2K_2$-avoiding, otherwise we discard the configuration. Then we have to decide, for example, if a vertex $x \in X_1 \cap X_2$ should be put in Y_1 or Y_2. If either placement would give a $2K_2$ with the set Y_3, we discard the configuration; if only one, we place it to the other; if none, we decide about it later.

At the end of this process we have disjoint sets Y_1, Y_2, Y_3 that are $2K_2$-avoiding, $Y_i \subset X_i$, and the vertices of $R_{1,2} := (X_1 \cap X_2) \setminus (Y_1 \cup Y_2)$ can be placed both Y_1 or Y_2 (and same for $R_{1,3}$ and $R_{2,3}$). Let us construct a conflict graph on $R_{1,2}$ and for other indices do similarly. For $x, y \in R_{1,2}$ there is an edge $(x, y) \in E(R_{1,2})$ if x and y cannot be placed to Y_1. (That is, they induce a $2K_2$ to Y_3. It means x and y could not be placed in Y_2 either.) It is easy to see that if all those conflict graphs $R_{i,j}$, $i \neq j$ are bipartite, then all vertices can be placed and we are ready. Otherwise the configuration is to be discarded and we have to move to the next one. If none of the configurations can be formed to be a $2K_2$-avoiding 3-coloring, then $\chi_{2K_2} > 3$. □

5.4.4 Proof of Theorem 5.13

Proof of Proposition 8.

Let G have a coloring with part sizes $s_1, s_2, \ldots, s_\chi$ and s_1 the largest. First, further partition each color class arbitrarily into subparts of size at most $k_1 - 1$. The number of parts is

$$\sum_{i=1}^{\chi} \left\lceil \frac{s_i}{k_1 - 1} \right\rceil \leq \sum_{i=1}^{\chi} \left(\frac{s_i}{k_1 - 1} + \frac{k_1 - 2}{k_1 - 1} \right) = \frac{n}{k_1 - 1} + \frac{k_1 - 2}{k_1 - 1}\chi,$$

which is an upper bound that holds regardless of the value of k_2.

Second, if $k_2 \geq 3$, partition each color class except the largest arbitrarily into subparts of size at most $k_2 - 1$. The number of parts is

$$1 + \sum_{i=2}^{\chi} \left\lceil \frac{s_i}{k_2 - 1} \right\rceil \leq 1 + \sum_{i=2}^{\chi} \left(\frac{s_i}{k_2 - 1} + \frac{k_2 - 2}{k_2 - 1} \right) = 1 + \frac{n - s_1}{k_2 - 1} + \frac{k_2 - 2}{k_2 - 1}(\chi - 1)$$
$$\leq 1 + \frac{n - n/\chi}{k_2 - 1} + \frac{k_2 - 2}{k_2 - 1}(\chi - 1).$$

Third, if $k_2 = 2$, color G by giving the largest independent set one color and every other vertex an individual color. The number of parts is $n - \alpha + 1$. Trivially, each of these partitions is an H-free coloring. All three combined bounds give the result in the proposition.

□

Proof of Theorem 5.13.

To obtain the upper bound, in the case of $k_1 \geq 3$, we use Proposition 8 together with the result from Bollobás [40] that, **whp** $\chi(G(n,p)) = (1+o(1))\frac{n}{2\log_d n}$. Hence,

$$\chi_H(G) \leq \frac{n}{k_1-1} + O\left(\frac{n}{\log_d n}\right).$$

In the case of $k_1 = 2$, the upper bound comes from Proposition 8 together with the result from Bollobás and Erdős [36] that, **whp** $\alpha(G(n,p)) = 2\log_d n - 2\log_d \log n + O(1)$. Hence,

$$\chi_H(G) \leq n - 2\log_d n + O(\log_d \log n).$$

Now we proceed to the lower bound. An $(\ell;k)$-*complex* is a family of ℓ disjoint independent sets, each of size k, $A_1,\ldots,A_\ell$ such that each pair (A_i,A_j), $1 \leq i < j \leq \ell$ induces a graph that has no induced copy of H. The key to the proof is to show that for certain values of k and $\ell = \ell(n)$, the probability that a $(\ell;k)$-complex exists goes to zero.

If no $(\ell;k)$-complex exists, then whenever there is a coloring with color classes of size $n_1,\ldots,n_t \geq k$ it is the case that $\sum_{i=1}^t \lfloor n_i/k \rfloor < \ell$. Thus,

$$\frac{1}{k}\sum_{i=1}^t n_i - \frac{k-1}{k}t \leq \sum_{i=1}^t \left\lfloor \frac{n_i}{k} \right\rfloor < \ell$$

$$\sum_{i=1}^t n_i < k\ell + (k-1)t,$$

while the leftover vertices are in color classes of size at most $k-1$. So, if there are t color classes of size at least k, then

$$\chi_H(G) \geq t + \frac{n - k\ell - (k-1)t}{k-1} = \frac{n}{k-1} - \frac{k}{k-1}\ell. \tag{5.1}$$

For the graph H, let $Q = Q(H,p)$ be the probability that a $k_1 \times k_1$ random bipartite graph has no induced copy of H. Taking the product over all $\binom{\ell}{2}$ pairs (A_i,A_j) and multiplying by the probability that each $G[A_i]$ induces an independent set, we obtain:

$$\begin{aligned}\Pr\left[\exists \text{ an } (\ell;k_1)\text{-complex}\right] &= \frac{(n)_{k_1\ell}}{\ell!(k_1!)^\ell} Q^{\binom{\ell}{2}} (1-p)^{\ell\binom{k_1}{2}} \\ &< \left[\left(\sqrt{\frac{e}{\ell}}\right) n Q^{(\ell-1)/(2k_1)} (1-p)^{(k_1-1)/2}\right]^{k_1\ell},\end{aligned} \tag{5.2}$$

which is obtained from the inequalities $(n)_{k_1\ell} \leq n^{k_1\ell}$, $\ell! \geq (\ell/e)^\ell$, and $k_1! \geq 1$.

Let $C' = C'(H,p) = \frac{2k_1}{\log(1/Q)}$. For n sufficiently large, if $\ell > C'\log n$, then the probability in (5.2) goes to zero. By (5.1), it is the case that **whp**

$$\chi_H(G) \geq \frac{n}{k_1-1} - C'\frac{k_1}{k_1-1}\log n.$$

Thus the general lower bound is satisfied.

In the special case where $H = 2K_2$, we observe that $Q(2K_2, p) = 1 - 2p^2(1-p)^2$.

$$\begin{aligned}\Pr\left[\exists \text{ an } (\ell;2)\text{-complex}\right] &= \frac{(n)_{2\ell}}{\ell! 2^\ell} Q^{\binom{\ell}{2}} (1-p)^\ell \\ &< \left[\left(\sqrt{\frac{e}{2\ell}}\right) n Q^{(\ell-1)/4} (1-p)^{1/2}\right]^{2\ell}.\end{aligned}$$

If $\ell > \frac{4\log n}{\log(1/Q)} - \frac{4\log\log n}{\log(1/Q)} + \log\log\log n$ and n is sufficiently large, then **whp** no $(\ell; 2)$-complex exists. By (5.1), it is the case that **whp**

$$\chi_{2K_2}(G) \geq n - (1 - o(1))\frac{8\log n}{\log(1/Q)},$$

where $Q = 1 - 2p^2(1-p)^2$.

□

Similar results can be obtained as long as $\min\{p, 1-p\} = \omega\left(\frac{\log n}{n}\right)$ but express our results in the case where p is a fixed constant.

5.5 Examples for Observation 9

We list without proofs some consequences of Observation 9. The details can be found in A. London, R. R. Martin and A. Pluhár [114].

Corollary 14.

$$\chi_{2K_2}(P_n) \geq \sqrt{2\left\lceil\frac{n-1}{3}\right\rceil + \frac{1}{4}} + \frac{1}{2}.$$

For $n = 2, 3, 4$, $\chi_{2K_2}(P_n) = 2$ because every proper 2-coloring is $2K_2$-avoiding. For the case $n \geq 5$, a more refined argument gives the value of $\chi_{2K_2}(P_n)$ for $n \geq 5$ as follows:

Corollary 15. *If $n \geq 5$ and k is the least integer that satisfies*

$$\left\lfloor\frac{k+1}{2}\right\rfloor (k-2) \geq \left\lceil\frac{n-1}{3}\right\rceil$$

then $\chi_{2K_2}(P_n) = k$.

Corollary 16.

$$\chi_{2K_2}\left(\frac{n}{2}\cdot K_2\right) = \left\lceil\sqrt{n + \frac{1}{4}} + \frac{1}{2}\right\rceil.$$

Bounds on some other graphs are given as follows:

Corollary 17. *Let n be odd and let T be the tree formed when each edge of $K_{1,(n-1)/2}$ is subdivided (by a vertex) exactly once.*

$$\chi_{2K_2}(T) = \left\lceil\sqrt{n - \frac{3}{4}} + \frac{1}{2}\right\rceil.$$

Corollary 18. *Let Q_d be the d-dimensional hypercube. Then*

$$\chi_{2K_2}(Q_2) = 2,$$
$$\chi_{2K_2}(Q_3) = 4,$$

and if $d \geq 4$ then

$$\chi_{2K_2}(Q_d) \geq \sqrt{\frac{d}{2d-1}2^d} + \frac{1}{2} = \sqrt{\frac{n}{2}\frac{1}{1-1/(2\lg n)}} + \frac{1}{2}.$$

Definition 10. *Let p be a prime power and let $G(p)$ be the bipartite graph on $n = 2(p^2 + p + 1)$ vertices defined by the projective plane of order $p+1$. That is, there are $p^2 + p + 1$ points and $p^2 + p + 1$ lines and a point is adjacent to a line if and only the point is in the line in the projective plane. This graph is $(p+1)$-regular with no $K_{2,2}$.*

Corollary 19. *If $G(p)$ is the graph in Definition 10 then,*

$$\chi_{2K_2}(G(p)) \geq \sqrt{\frac{2(p^2+p+1)(p+1)}{2p+1}} + \frac{1}{2} \geq \sqrt{\frac{n}{2} + \frac{\sqrt{n}}{4}} + \frac{1}{2}.$$

5.6 Discrepancies

The thorough study of discrepancy theory started with Weyl [161] and quickly gained several applications in number theory, combinatorics, ergodic theory, discrete geometry, statistics etc, see the monograph of Beck and Chen [24] or the book chapter by Alexander and Beck [1].

We touch upon only the combinatorial discrepancy of hypergraphs. Given a hypergraph (X, E), and a mapping $f : X \to \{-1, 1\}$, for an edge $A \in E$ let $f(A) := \sum_{x \in A} f(x)$. The discrepancy of f is $\mathcal{D}(X, E, f) = \max_{A \in E} |f(A)|$, while the discrepancy of the hypergraph (X, E)

$$\mathcal{D}(X, E) := \min_f \mathcal{D}(X, E, f).$$

In our case $X = E(G)$ and $E = \mathcal{S}_G \subset 2^{E(G)}$, and with a slight abuse of notation we write $\mathcal{D}(G, \mathcal{S}_G)$ for short.

Erdős, Füredi, Loebl, and Sós [61] studied the case $G = K_n$, the complete graph on n vertices, and $\mathcal{S}_G$ is the set of copies of a fixed spanning tree T_n with maximum degree Δ. They showed the existence of a constant $c > 0$, such that $\mathcal{D}(G, \mathcal{S}_G) > c(n - 1 - \Delta)$.

Erdős and Goldberg [60] defined $\mathrm{dis}(A, B) := e(A, B) - e(G)|A||B|/\binom{n}{2}$, where $A, B \subset V(G)$ and $A \cap B = \varnothing$. They showed that for every $\varepsilon > 0$ there exists an

$\varepsilon' > 0$ such that in every graph G with $e = e(G) > v(G) = n$, there are disjoint sets $A, B \subset V(G)$, $|A|, |B| \le \varepsilon n$, and $\mathrm{dis}(A, B) > \varepsilon' \sqrt{en}$.

Here we investigate the discrepancy of (spanning) trees, paths, Hamilton cycles and packing with fixed complete graphs. That is for a graph G let $\mathcal{S}_G$ be the set of spanning trees ($\mathcal{T}_n$), trees ($\mathcal{T}$), Hamiltonian paths ($\mathcal{P}_n$), paths ($\mathcal{P}$), or Hamilton cycles ($\mathcal{H}$).

Usually, one expects big discrepancy if the hypergraph has many edges. Since for every graph G, either G or $\overline{G}$ is connected, we have $\mathcal{D}(K_n, \mathcal{T}_n) = n - 1$. Beck [23] showed that there is a graph F on n vertices and $2n$ edges such that in every two-coloring of its edge set there exists a monochromatic path of length cn, that is $\mathcal{D}(F, \mathcal{P}) = cn$. Another example for this is the interpretation of the result of Burr, Erdős and Spencer [41], namely that $R(mK_3, mK_3) = 5m$. That is if $k \cdot K_3$ is the set of triangle factors in K_n, $n = 3k$ and n is divisible by 5, then $\mathcal{D}(K_n, k \cdot K_3) = n/5$.

We first consider the discrepancy of Hamilton cycles, and show that, roughly speaking, if G has sufficiently large minimum degree then for every labeling of $E(G)$ with $+1, -1$ there is a Hamilton cycle with linear discrepancy.

Theorem 5.15. *Let $c > 0$ be an arbitrarily small constant and n be sufficiently large. Let G be a graph of order n with $\delta(G) \ge (3/4 + c)n$. Then we have $\mathcal{D}(G, \mathcal{H}) \ge cn/32$.*

Figure 5.1 below shows that the minimum degree condition in Theorem 5.15 is the best possible. In this example, let $G = K_n - K_{n/4}$, i.e., $|V(G)| = n$ is divisible by 4, $|V_1| = n/4, |V_2| = 3n/4$, $\delta(G) = 3n/4$. Assign -1 to all edges incident to V_1 and $+1$ to the rest of the edges. As each Hamilton cycle in G touches V_1 exactly $n/4$ times, they all have zero discrepancy.

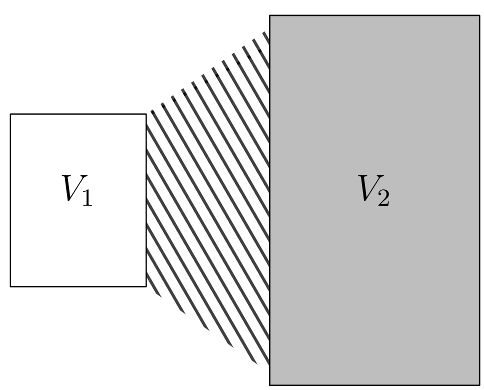


Figure 5.1: G with $\delta(G) = 3n/4$ and zero Hamilton cycle discrepancy.

For the existence of a Hamilton cycle, Dirac's Theorem requires only minimum degree $n/2$. We could also push down the minimum degree requirement for the existence of a linear discrepancy Hamilton cycle, if we have some local restriction on the coloring.

For $\nu > 0$ real number, we say a vertex is *ν-balanced* if it has at least νn edges with label $+1$, and at least νn edges of label -1, otherwise it is *ν-unbalanced*.

Theorem 5.16. *Let c, d, ν be positive numbers satisfying $c \ge 8\nu$ and $d \ge 4\nu$. Let G be a graph of order n, where $\delta(G) \ge (1/2 + c)n$. Assume that the edges of G are labelled by either $+1$ or -1, such that the number of ν-balanced vertices is at least $(3/4 + d)n$. Then there exists a Hamilton cycle in G with discrepancy at least $\nu^2 n/2000$.*

The number of the balanced vertices in the graph in Figure 5.1 is $3n/4$, hence the condition on the size of the balanced set in Theorem 5.16 is tight.

Remarks. The proofs of Theorem 5.15 and 5.16 is published in J. Balogh, B. Csaba, Y. Jing and A. Pluhár [14].

Th perfect tilings are also global structures in graphs. An H*-tiling* in a graph G is a collection of vertex-disjoint copies of H contained in G. An H-tiling is *perfect* if it covers all the vertices of G. Perfect H-tilings are also often referred to as H-factors, perfect H-packings or perfect H-matchings. H-tilings can be viewed as generalizations of both the notion of a matching (which corresponds to the case when H is a single edge) and the Turán problem (i.e. a copy of H in G is simply an H-tiling of size one). Except for the case when H contains no component of size at least 3, the decision problem of whether a graph contains a perfect H-tiling is NP-complete (see [101]). Thus, there has been substantial efforts to obtain sufficient conditions that force a graph to contain a perfect H-tiling. In particular, a cornerstone result in extremal graph theory is the Hajnal-Szemerédi theorem [82], which characterizes the minimum degree threshold that ensures a graph contains a perfect K_r-tiling. Theorem 1.3 (Hajnal and Szemerédi [82]). Every graph G whose order n is divisible by r and whose minimum degree satisfies $\delta(G) \geq (11/r)n$ contains a perfect K_r-tiling. Moreover, there are n-vertex graphs G with $\delta(G) = (11/r)n1$ that do not contain a perfect K_r-tiling. There has also been much interest in the minimum degree threshold that ensures a perfect H-tiling for an arbitrary graph H. After earlier work on this topic (see e.g. [9, 104]), Kühn and Osthus [110] determined, up to an additive constant, the minimum degree that forces a perfect H-tiling for any fixed graph H.

We have following discrepancy version of the Hajnal-Szemerédi theorem:

Theorem 5.17. *Suppose $r \geq 3$ is an integer and let $\ > 0$. Then there exists $n_0 N$ and $\gamma > 0$ such that the following holds. Let G be a graph on $n \geq n_0$ vertices where r divides n and where*

$$\delta(G) \geq (1\frac{1}{r+1} + \theta)n.$$

Given any function $f : E(G)\{1, 1\}$ there exists a perfect K_r-tiling $\mathcal{T}$ in G so that

$$|\sum_{e \in E(\mathcal{T})} f(e)| \geq \gamma n.$$

In both of the theorems above, G is dense. However, the sparsity of a graph does not imply small discrepancy, the expansion is a more important factor. Let $G \in \mathcal{G}_{n,d}$ be a randomly, uniformly selected d-regular graph on n vertices. A property $\mathcal{P}$ holds with high probability, **whp**, if for every $\varepsilon > 0$ there exist an n_ε such that $\Pr(G \in \mathcal{G}_{n,d}, G \in \mathcal{P}) \geq 1 - \varepsilon$. Similarly, property $\mathcal{P}$ holds asymptotically almost surely, a.a.s., if $\lim_{n\to\infty} \Pr(G \in \mathcal{G}_{n,d}, G \in \mathcal{P}) = 1$.

Theorem 5.18. *Let $G \in \mathcal{G}_{n,3}$. Then there exists a constant $c > 0$ such that a.a.s. we have $\mathcal{D}(G, \mathcal{T}_n) \geq cn$.*

For planar graphs, one can expect sub-linear discrepancy of spanning trees; we managed to give asymptotically sharp bounds.

Theorem 5.19. *Let G be a planar graph on n vertices. Then there exists a real number $c > 0$ such that $\mathcal{D}(G, \mathcal{T}_n) \leq c\sqrt{n}$.*

The bounds, up to the constant factor are best possible. Let $P_k^2 := P_k \,\Box\, P_k$ be the $k \times k$ grid.

Theorem 5.20. *$\mathcal{D}(P_k^2, \mathcal{T}_n) \geq ck$ for some $c > 0$, where $n = k^2$.*

If we drop the condition of spanning subgraph, then the discrepancies can be linear in the number of vertices.

Proposition 9. *Let $k, \ell > \mathbb{N}^+$. Then $\mathcal{D}(P_k \,\Box\, P_\ell, \mathcal{P}) > k\ell/8 - \max\{k, \ell\}/8 - \min\{k, \ell\}$.*

We have the following corollary since paths are also trees.

Corollary 20. $\mathcal{D}(P_k \,\Box\, P_\ell, \mathcal{T}) > k\ell/8 - \max\{k, \ell\}/8 - \min\{k, \ell\}$.

Let us make some easy observations which nevertheless give motivations for the above theorems and to those proofs. The graph $P_2 \,\Box\, P_k$ has exponentially many spanning trees, but still $\mathcal{D}(P_2 \,\Box\, P_k, \mathcal{T}_{2k}) \leq 3$. To see this, we partition the graph into a $2 \times \lceil k/2 \rceil$ grid and a $2 \times \lfloor k/2 \rfloor$ grid, and label the edges of the first grid by -1, of the second grid by $+1$. We label the edge shared by two sub-grids arbitrarily. The situation for $P_k \,\Box\, P_k$, the $k \times k$ grid, is similar: cut the grid into two halves and label $+1$ the upper, and -1 the lower region. Since any spanning tree is cut at most k times, $\mathcal{D}(P_k \,\Box\, P_k, \mathcal{T}_n) \leq k - 1$. For not necessarily spanning trees, obviously, $\mathcal{D}(G, \mathcal{T}) \geq \lceil \Delta(G)/2 \rceil$.

In what follows we give the proofs some of these results. Since the proofs of the Theorems 5.15, 5.16 and 5.17 are quite technical, we leave those out of the present text. The interested reader may find them in [14, 15].

5.6.1 Discrepancies in random 3-regular graphs

Proof Theorem 5.18

Buser [42] and later, in a much simpler paper, Bollobás [39] showed that random regular graphs have expanding properties. More precisely, let

$$i(G) := \min_{U} \frac{|\partial U|}{|U|},$$

where $U \subset V(G)$ with $|U| \leq |V(G)|/2$, and $\partial U := \{v \notin U \mid \exists u \in U, uv \in E(G)\}$.

(i) Bollobás [39] proved that $i(G) \geq 2^{-7}$ for a random 3-regular graph G with high probability. In particular, it is connected **whp**.

(ii) Bollobás [38] showed for $3 \leq j \leq k$, where k is fixed, and X_j stands for the number of cycles of length j in $G \in \mathcal{G}_{n,3}$, that $X_3, \ldots, X_k$ are asymptotically independent Poisson random variables with means $\lambda_j = 2^j/(2j)$.

(iii) Wormald proved (see [162, Lemma 2.7]) that for a fixed d and every fixed graph F with more edges than vertices, $G \in \mathcal{G}_{n,d}$ a.a.s. contains no subgraph isomorphic to F.

Fix an arbitrary $f : E(G) \to \{-1, 1\}$, denote N and P the subsets of *edges*, where f takes -1 and 1, respectively. We may assume that $|N| \leq |P|$, i.e., $|N| \leq 3n/4$.

Denote by G^+ the subgraph of G spanned by P, and let A_i be the set of components with size i in G^+, while $a_i := |A_i|$. The number of components in G^+ is $t = \sum_{i=1}^n a_i$.

Note that (i) means that G is connected **whp** so G has a spanning tree T satisfying that $|E(T) \cap N| \le t-1$. Hence if $t \le (1/2 - 2^{-12})n + o(n)$ or $t \ge (1/2 + 2^{-12})n + o(n)$ then $\mathcal{D}(G, \mathcal{T}_n) \ge 2^{-12}n - o(n)$.

Three edges of N are incident to each element of A_1, four edges to each of A_2. The number of edges incident to a component of size at least 3 could be less than four only if the component contains a cycle, i.e., w.h.p. only in $O(1)$ many components A_i for $i = 3, \dots, 2^9$. For every component larger than 2^9, and smaller than $n/2$, w.h.p. the number of incident edges is at least four by (i).

That is, w.h.p.

$$2|N| \ge 3a_1 + 4\sum_{i=2}^{n} a_i - O(1) = 4t - a_1 - O(1),$$

which gives

$$(1/2 - 2^{-12})n + O(1) \le t \le |N|/2 + a_1/4 + O(1) \le 3n/8 + a_1/4 + O(1). \tag{5.3}$$

Now we consider the number of negative edges. The number of edges in N which are incident to vertices in A_1 is $e(G[A_1]) + e(G[A_1, \overline{A}_1])$. Since $|N| \le |P|$, we have

$$\frac{3a_1}{2} \le e(G[A_1]) + e(G[A_1, \overline{A}_1]) \le |N| \le \frac{3n}{4},$$

which implies that $a_1 \le n/2$. Using the condition (i), we have $e(G[A_1, \overline{A}_1]) \ge 2^{-7}a_1$. Therefore,

$$3a_1 \le 2e(G[A_1]) + e(G[A_1, \overline{A}_1]) \le 2|N| - 2^{-7}a_1,$$

implying

$$\frac{3a_1}{2} + \frac{a_1}{2^7} \le |N| \le \frac{3n}{4},$$

which gives $a_1 \le (1/2 - 2^{-10})n$ w.h.p. With (5.3) it implies $t \le (1/2 - 2^{-12})n + o(n)$ w.h.p. That gives us $\mathcal{D}(G, \mathcal{T}_n) \ge 2^{-12}n - o(n)$. w.h.p. □

5.6.2 Discrepancies of planar graphs

Lemma 20. *Let C be a vertex cut of a connected graph G, that is $V(G) = A \cup B \cup C$ such that there are no edges between A and B, and, say, $|A| \le |B|$. Then $\mathcal{D}(G, \mathcal{T}_n) \le |B| - |A| + |C|$.*

Proof Theorem 20

Let $f(x, y) = 1$ if $(x, y) \in E(A) \cup E(A, C)$, $f(x, y) = -1$ if $(x, y) \in E(B) \cup E(B, C)$ and arbitrary in $E(C)$. Every spanning tree T of G has at most $|C|$ components restricted to $A \cup C$. It means the number of edges labeled by 1 is at least $|A| + |C| - 1 - |C| = |A| - 1$ in T, and the edges labeled by -1 at most $|B| + |C| - 1$. □

Proof Theorem 5.19

To deduce Theorem 5.19 we need to recall the celebrated planar separation theorem of Lipton and Tarjan in [112]. It says if G is a planar graph on n vertices then G has a vertex cut of size $O(\sqrt{n})$ partitioning the graph into two parts A and B, where $n/3 \le |A|, |B| \le$

$2n/3$. A well-known consequence [54, Theorem 5] of that theorem is that there exists a cut C and constants c_1, c_2, c_3 such that $n/2 - c_1\sqrt{n} \leq |A|, |B| \leq n/2 + c_2\sqrt{n}$ and $|C| = c_3\sqrt{n}$.

Having the partition above we can use Lemma 20 getting that for a planar graph G, $\mathcal{D}(G, \mathcal{T}_n) \leq |B| - |A| + |C| \leq O(\sqrt{n})$. □

Lemma 21 ([47]). *Let $S \subseteq P_k \,\square\, P_k$ such that $(k^2 - k)/2 \leq |S| \leq (k^2 + k)/2$. Then we have $|\partial S| \geq k$.*

Proof Theorem 5.20

Assume there exists an $f : E(P_k \,\square\, P_k) \to \{-1, 1\}$ such that $\mathcal{D}(P_k \,\square\, P_k, \mathcal{T}_n, f) \leq k/4$. Let P, N and M be the subset of vertices, such that $v \in P$ if all edges incident to v are positive, $v \in N$ if all edges incident to v are negative, and $M = V - N - P$. Consider an arbitrary Hamiltonian path in $P_k \,\square\, P_k$, from the assumption on f it follows that $|P|, |N| \leq k^2/2 + k/8 + 2$.

First, we show that $|M| \geq k$. If $\max\{|P|, |N|\} \leq (k^2 - k)/2$ then this follows from $|P| + |N| + |M| = k^2$. That is we may assume $(k^2 - k)/2 < |P| \leq k^2/2 + k/8 + 2$. Note that $\partial P = M$. By Lemma 21, for sets P of such size we have $|\partial P| \geq k$, which means $|M| \geq k$, too.

We identify the vertices of $P_k \,\square\, P_k$ with coordinate pairs such that $(0, 0)$ belongs to the bottom left vertex, $(k - 1, k - 1)$ to the upper right vertex. For $r, s \in \{0, 1\}$ let $X_{r,s}$ be those vertices (i, j) $(0 \leq i, j \leq k - 1)$ for which $i = r \pmod 2$ and $j = s \pmod 2$. At least one of these sets $X_{r,s}$ contains at least $k/4$ vertices of M, say $X_{0,0}$. Consider an arbitrary tree T spanned on the vertices $X_{0,1} \cup X_{1,0} \cup X_{1,1}$.

Note that we can extend T to the entire $P_k \,\square\, P_k$ such that the vertices of $X_{0,0}$ will be leaf vertices in the extension. Moreover for $(i, j) \in X_{0,0} \cap M$ we can connect (i, j) to T with either an edge labeled by -1 or 1. Fixing any extension to $X_{0,0} \setminus M$, let T^+ (T^-) be the extension where we use the edge labeled by 1 (-1) for the vertices $X_{0,0} \cap M$. Obviously, $|\sum_{e\in T^+} f(e) - \sum_{e\in T^-} f(e)| \geq k/2$, so either $|\sum_{e\in T^+} f(e)|$ or $|\sum_{e\in T^-} f(e)|$ is at least $k/4$. □

Proof Theorem 9

We show first that $\mathcal{D}(P_k \,\square\, P_2, \mathcal{P}) \geq k/2$. Let us refer to the graph $P_k \,\square\, P_2$ as a rectangle with horizontal length k in which the edges are labeled by f. Let X and Y be the set of the vertical edges labeled by $+1$ and -1 respectively. Without loss of generality, we may assume $|X| \geq |Y|$ and let $x := |X| \geq k/2$, $y := |Y|$. We consider four paths: $P(X)$ starts from the left-upper corner goes to right except when it meets an edge $e \in X$ at which point it goes down or up, depending on which one is possible. The path $P'(X)$ is almost the same, but it starts from the left-lower corner. Finally the paths $P(Y)$ and $P'(Y)$ are drawn analogously, those also start from left and go to right, but rise and fall at the edges belonging to Y. Note that $P(X)$ and $P'(X)$ each contain X, $P(Y)$ and $P'(Y)$ each contain Y. $P(X) \cup P'(X)$ and $P(Y) \cup P'(Y)$ have the same set of horizontal edges.

Let $z_1 := \sum_{e\in P(X)\setminus X} f(e)$, and $z_2 := \sum_{e\in P'(X)\setminus X} f(e)$. If $\max\{z_1, z_2\} \geq 0$, then we are done since one of $\sum_{e\in P(X)} f(e)$ or $\sum_{e\in P'(X)} f(e)$ is at least $k/2$. If both z_1 and z_2 are negative, we have $\mathcal{D}(P_k \,\square\, P_2, \mathcal{P}, f) \geq x + z_1$, and $\mathcal{D}(P_k \,\square\, P_2, \mathcal{P}, f) \geq x + z_2$. Considering the paths $P(Y)$ and $P'(Y)$ we also have $2\mathcal{D}(P_k \,\square\, P_2, \mathcal{P}, f) \geq 2y - z_1 - z_2$, since the horizontal edges in those carry exactly $z_1 + z_2$ negative surplus. Adding those up, we get $4\mathcal{D}(P_k \,\square\, P_2, \mathcal{P}, f) \geq 2x + 2y$, that is $\mathcal{D}(P_k \,\square\, P_2, \mathcal{P}, f) \geq k/2$ since $x + y = k$.

In the general case we may assume that $k \leq \ell$ and $P_k \square P_\ell$ is referred as a rectangle with k rows and ℓ columns. We cut out $\lfloor k/2 \rfloor$ non-touching stripes $P_2 \square P_\ell$. For every $f : E(P_k \square P_\ell) \to \{-1, 1\}$, applying our construction of paths above, without loss of generality, at least half of the rectangles have a path with more positive edges, and with discrepancy at least $\lceil \ell/2 \rceil$. Note also, that these paths can be joined into one path by adding at most $k-1$ edges. Thus, we create a path with discrepancy at least

$$\left\lceil \frac{1}{2} \left\lfloor \frac{k}{2} \right\rfloor \right\rceil \left\lceil \frac{\ell}{2} \right\rceil - k + 1 > \frac{k\ell}{8} - \frac{\ell}{8} - k,$$

and the result is proved. □

Bibliography

[1] J. R. Alexander, J. Beck, and W. W. L. Chen, Geometric discrepancy theory and uniform distribution. *In Handbook of Discrete and Computational Geometry.* 1997.

[2] J. D. Allen, A note on the computer solution of connect-four. pp. 134–135 in *Heuristic Programming in Artificial Intelligence: The First Computer Olympiad* (edited by D. N. L. Levy and D. F. Beal), Ellis Horwood, Chichester, England, 1989.

[3] L. V. Allis, A knowledge-based approach to connect-four. The game is solved: White wins. M.Sc. thesis, Faculty of Mathematics and Computer Science, Vrije Universiteit, Amsterdam, 1988.

[4] L. V. Allis, H. J. van den Herik and M. P. Huntjens, Go-Moku solved by new search techniques. *Proc. 1993 AAAI Fall Symposium on Games: Planning and Learning, AAAI Press Technical Report FS93-02,* pp. 1-9, Menlo Park, CA.

[5] L. V. Allis, M. van der Meulen and H. J. van den Herik, Proof-Number Search. *Artificial Intelligence* **66** (1994), 91–124.

[6] N. Alon and Z. Bregman, Every 8-uniform 8-regular hypergraph is 2-colorable. *Graphs and Combinatorics* **4** (1988), 303–305.

[7] N. Alon and J. Spencer, *The Probabilistic Method*. Wiley-Interscience, New York, 2000.

[8] N. Alon and J.H. Spencer, The probabilistic method. Fourth edition. Wiley Series in Discrete Mathematics and Optimization. *John Wiley & Sons, Hoboken, NJ*, 2016. xiv+375pp.

[9] N. Alon and R. Yuster, H-factors in dense graphs. *Journal of Combinatorial Theory*, Series **B**, **66(2)** (1996), 269–282.

[10] N. ALON AND D.B. WEST *The Borsuk-Ulam theorem and bisection of necklaces.* Proceedings of the American Mathematical Society 98.4: pp. 623–628, (1986).

[11] L. Babai and P. Frankl, *Linear Algebra Methods in Combinatorics: With Applications to Geometry and Computer Science.* Department of Computer Science, Univ. of Chicago, (1992).

[12] J. Balogh, R. Martin and A. Pluhár, The diameter game. *Random Structures and Algorithms*, Volume **35**, (2009), 369–389.

[13] J. Balogh and Pluhár, The positive minimum degree game on sparse graphs. the electronic journal of combinatorics. 2012 Jan 21; 19(1):P22

[14] J. Balogh, B. Csaba, Y. Jing and A. Pluhár, On the discrepancies of graphs. *Electron. J. Combin.* **27** (2020) P2.12

[15] J. Balogh, B. Csaba, A. Pluhár and A. Treglown, A discrepancy version of the Hajnal-Szemerédi theorem. *Combinatorics, Probability and Computing*, **30(3)** (2021), 444–459.

[16] J. Beck, On a combinatorial problem of P. Erdõs and L. Lovász. *Discrete Math.* **17** (1977), no. 2, 127–131.

[17] J. Beck, On 3-chromatic hypergraphs. *Discrete Math.* **24** (1978), no. 2, 127–137.

[18] J. Beck and T. Fiala, Integer-making theorems. *Discrete Applied Mathematics* **3.1:** (1981), 1–8.

[19] J. Beck, On positional games. *J. of Combinatorial Theory Series A* **30** (1981), 117–133.

[20] J. Beck, Van der Waerden and Ramsey games. *Combinatorica* **1** (1981), 103–116.

[21] J. Beck, On a generalization of Kaplansky's game. *Discrete Math* **42** (1982) 27-35.

[22] J. Beck, Remarks on positional games. *Acta Math Acad Sci Hungar* **40** (1982), 65–71.

[23] J. Beck, On size Ramsey number of paths, trees, and circuits. I. *Journal of Graph Theory*, **7.1** (1983), 115–129.

[24] J. Beck and W. W. L. Chen, Irregularities of Distribution. Vol. **89** of *Cambridge Tracts in Math.*, Cambridge University Press, 1987.

[25] J. Beck, Deterministic graph games and a probabilistic intuition. *Combinatorics, Probability and Computing* **3** (1994), 13–26.

[26] J. Beck, Positional games and the second moment method. *Combinatorica* **22** (2) (2002) 169–216.

[27] J. Beck, Positional Games. *Combinatorics, Probability and Computing* **14** (2005), 649–696.

[28] J. Beck, Random graphs and positional games on the complete graph. Random graphs '83 (Poznań, 1983), 7–13, *North-Holland Math. Stud.,* **118**, North-Holland, Amsterdam, 1985.

[29] J. Beck, *Combinatorial games. Tic-tac-toe theory.* Encyclopedia of Mathematics and its Applications, **114.** Cambridge University Press, Cambridge, 2008. xiv+732 pp.

[30] M. Bednarska and T. Łuczak, Biased positional games for which random strategies are nearly optimal. *Combinatorica* **20** (2000), 477–488.

[31] M. Bednarska and T. Łuczak, Biased positional games and the phase transition. *Random Structures and Algorithms* **18** (2001), 141–152.

[32] M. Bednarska-Bzdęga, On weight function methods in ChooserŰPicker games. *Theoretical Computer Science*, **475**, (2013) 21–33.

[33] M. Bednarska-Bzdęga, D. Hefetz and T. Łuczak, PickerŰChooser fixed graph games. *Journal of Combinatorial Theory*, Series B, **119**, (2016) 122–154.

[34] E. R. Berlekamp, J. H. Conway and R. K. Guy, *Winning Ways for your mathematical plays*. Academic Press, New York 1982.

[35] S. N. Bhatt and C. E. Leiserson, How to assemble tree machines. *Proceedings of the fourteenth annual ACM Symposium on Theory of Computing. ACM,* (1982).

[36] B. Bollobás and P. Erdős, Cliques in random graphs, *Math. Proc. Cambridge Phil. Soc.*, **80** (1976), pp. 419–427.

[37] B. Bollobás, *Random Graphs*. Second edition. Cambridge Studies in Advanced Mathematics, **73** xviii+498 pp.

[38] B. Bollobás, A probabilistic proof of an asymptotic formula for the number of labelled regular graphs. *European Journal of Combinatorics,* **1** (1980), 311–316.

[39] B. Bollobás, The isoperimetric number of random regular graphs. *European Journal of combinatorics*, **9.3** (1988), 241–244.

[40] B. Bollobás, The chromatic number of random graphs. *Combinatorica*, **8** (1988), pp. 49–55.

[41] S. A. Burr, P. Erdős and J. H. Spencer, Ramsey theorems for multiple copies of graphs. *Transactions of the American Mathematical Society*, **209** (1975), 87–99.

[42] P. Buser, On the bipartition of graphs, *Discrete Applied Math.* **9** (1984), 105–109.

[43] J. M. Byskov, Maker-Maker and Maker-Breaker Games are PSPACE-complete. *Technical Report, BRICS Research Series* RS-04-14, Dept. Comp. Sci., Univ. Aarhus, August 2004.

[44] T. Calamoneri, The $L(h,k)$-labelling problem: an updated survey and annotated bibliography. *The Computer Journal*, **54(8)** (2011), 1344–1371.

[45] I. Choi, R. Kim and B. Park, Characterization of forbidden subgraphs for bounded star chromatic number. *Discrete Mathematics*, **342(3)** (2019), 635–642.

[46] V. Chvátal and P. Erdős, Biased positional games. Algorithmic aspects of combinatorics (Conf., Vancouver Island, B.C., 1976). *Ann. Discrete Mathematics* **2** (1978), 221–229.

[47] J. Chvátalová, Optimal labelling of a product of two paths. *Discrete Mathematics*, **11(3)** (1975), 249–253.

[48] J. H. Conway, *On Numbers and Games*. London: Academic Press, 1976.

[49] T. H. Cormen, C. E. Leiserson, R. L. Rivest and C. Stein, *Új algoritmusok.* Scolar Kiadó, Budapest (2003).

[50] A. Csernenszky, C. I. Mándity and A. Pluhár, On Chooser-Picker Positional Games. *Discrete Mathematics* Volume **309** (2009), 5141–5146.

[51] A. Csernenszky, The Chooser-Picker 7-in-a-row-game. *Publicationes Mathematicae* **76** (2010), 431– 440

[52] A. Csernenszky, The Picker-Chooser diameter game. *Theoretical Computer Science* **411** (2010), pp. 3757–3762

[53] A. Csernenszky, R. Martin and A. Pluhár, On the Complexity of Chooser-Picker Positional Games. *Integers*, **12(3)**, (2012) 427–444.

[54] H. N. Djidjev, On the problem of partitioning planar graphs. *SIAM Journal of Algebraic and Discrete Methods*, **3** (1982), 229–241.

[55] Th Epping, W. Hochstättlerand P. Oertel, Complexity results on a paint shop problem. *Discrete Applied Mathematics* **136.2-3:** (2004), 217–226.

[56] P. Erdős, On a combinatorial problem. *Nordisk Mat. Tidskr.* **11** (1963) 5–10.

[57] P. Erdős, On a combinatorial problem, II. *Acta Math. Acad. Sci. Hungar.*, **15** (1964), 445–447.

[58] P. Erdős and J. L. Selfridge, On a combinatorial game. *J. Combinatorial Theory Series A* **14** (1973), 298–301.

[59] P. Erdős and L. Lovász, Problems and results on 3-chromatic hypergraphs and some related questions. *Infinite and finite sets, Colloq. Math. Soc. J. Bolyai, Vol.* **10** *North-Holland, Amsterdam,* (1974) pp. 609–627.

[60] P. Erdős, M. Goldberg, J. Pach and J. Spencer, Cutting a graph into two dissimilar halves. *Journal of Graph Theory*, **12** (1988), 121–131.

[61] P. Erdős, Z. Füredi, M. Loebl and V. T. Sós, Discrepancy of Trees. *Stud Sci Math*, **30** (1995), 47–57.

[62] L. Espig, A. Frieze, M. Krivelevich, and W. Pegden, (2015). Walker-Breaker games. SIAM Journal on Discrete Mathematics, **29(3)** (2015) 1476–1485.

[63] S. Even and R. E. Tarjan, A combinatorial problem which is complete in polynomial space. *J. Assoc. Comput. Mach.* **23** (1976), no. 4, 710–719.

[64] M. Farber, M. Hujter and Z. Tuza, An upper bound on the number of cliques in a graph. *Networks*, **23**(3) (1993), pp. 207–210.

[65] A. S. Fraenkel, Complexity of Games. *Combinatorial games* (Columbus, OH, 1990), 111–153, Proc. Sympos. Appl. Math., **43**, Amer. Math. Soc., Providence, RI, 1991.

[66] A. S. Fraenkel, Combinatorial games: selected bibliography with a succinct gourmet introduction. *The Electronic Journal of Combinatorics* Dynamic Surveys (2009) 88pp.

[67] Z. FÜREDI personal communication

[68] D. Gale, H. Kuhn and A. Tucker, Linear programming and the theory of games. In T. Koopmans, ed., *Activity Analysis of Production and Allocation*, John Wiley and Sons, New York, (1951) pp. 317–329.

[69] D. Gale, The game of Hex and the Brouwer fixed-point theorem. *American Mathematical Monthly* **86** (1979), no. 10, 818–827.

[70] R. Gasser, Solving Nine Men's Morris. *Computational Intelligence* **12(1)**, (1996) 24-41.

[71] H. Gebauer and T. Szabó, Asymptotic random graph intuition for the biased connectivity game. *Random Structures & Algorithms,* **35(4)** (2009), 431–443.

[72] F. Gécseg and B. Imreh, Finite isomorphically complete systems, *Discrete Applied Mathematics* **36** (1992), 307–311.

[73] F. Gécseg, B. Imreh and A. Pluhár, On the existence of Finite Isomorphically Complete Systems of Automata. *J. of Automata, Languages and Combinatorics* **3** (1998) 2, 77–84.

[74] Gera Imre, London András és Pluhár András, Mohó megközelítések gráfok egymásba ágyazottságának felderítésére. XXXIV. MAGYAR OPERÁCIÓKUTATÁSI KONFERENCIA 2021 augusztus 31-szeptember 2.

[75] J. J. Gik, *Sakk és matematika.* Gondolat, Budapest (1989).

[76] V. M. Glushkov, Abstract theory of automata. *Uspekhi Mat. Nauk*, **16:5 101** (1961), 3–62 (in Russian).

[77] M. C. Golumbic, *Algorithmic graph theory and perfect graphs* Elsevier Vol. **57.** (2004).

[78] R. K. Guy and J. L. Selfridge, Problem S.10, *Amer. Math. Monthly* **86** (1979); solution T.G.L. Zetters **87** (1980) 575–576.

[79] L. Guth, *Polynomial methods in combinatorics.* Vol. **64.** American Mathematical Soc., (2016).

[80] L. Győrffy, G. Makay and A. Pluhár, The pairing strategies of the 9-in-a-row game. *Ars Mathematica Contemporánea,* **16(1)** (2018), 97–109.

[81] L. Győrffy and A. Pluhár, Generalized pairing strategies-a bridge from pairing strategies to colorings. *Acta Universitatis Sapientiae, Mathematica,* **8(2)** (2016), 233–248.

[82] A. Hajnal and E. Szemerédi, Proof of a conjecture of Erdős. *In Combinatorial Theory and its Applications, II. Colloquia Mathematica Societatis János Bolyai, North-Holland, Amsterdam/London* (1970) .

[83] W. K. Hale, Frequency assignment: Theory and applications. *Proceedings of the IEEE,* **68(12)** (1980), 1497–1514.

[84] A. W. Hales and R. I. Jewett, Regularity and positional games. *Trans Amer. Math. Soc.* **106** (1963) 222–229; M.R. # 1265

[85] M. Hall Jr., Distinct representatives of subsets. *Bull. Amer. Math. Soc.* **54**, (1948). 922–926.

[86] P. L. Hammer, U. N. Peled and X. Sun, Difference graphs. *Discrete Applied Mathematics*, **28** (1990), pp. 35–44.

[87] C. Hartman, https://www.cs.uaf.edu/ hartman/pouzethex.pdf, downloaded on 01.02.2019

[88] Harary, F. *On the notion of balance of a signed graph.* The Michigan Mathematical Journal 2.2: pp. 143–146, (1953). DOI: 10.1307/mmj/1028989917

[89] D. Hefetz, M. Krivelevich, M. Stojaković and T. Szabó, A sharp threshold for the Hamilton cycle Maker-Breaker game. *Random Structures and Algorithms*, **34(1)**, (2009) 112–122.

[90] D. Hefetz, M. Krivelevich, M. Stojaković and T. Szabó, Fast winning strategies in Maker-Breaker games. *Journal of Combinatorial Theory, Series* **B, 99(1)** (2009), 39–47.

[91] D. Hefetz, M. Krivelevich, M. Stojaković and T. Szabó, Global MakerŰBreaker games on sparse graphs. *European Journal of Combinatorics,* **32(2)** (2011), 162–177.

[92] D. Hefetz, M. Krivelevich, M. Stojakovic and T. Szabó, T. Continuous Box game. Manuscript (2011), see http://www.cs.tau.ac.il/ krivelev/BoxGame.pdf

[93] F. Harary, Is Snaky a winner? *Geombinatorics* **2** (1993) 79–82.

[94] H. Harborth and M. Seemann, Snaky is a paving winner. *Bull. Inst. Combin. Appl.* **19** (1997), 71–78.

[95] M. Hegyháti, Z. Tuza Colorability of mixed hypergraphs and their chromatic inversions. *Journal of Combinatorial Optimization*, **25(4)**, (2013), 737–751.

[96] P. Hein, Vil de laere Polygon? *Politiken newspaper*, Denmark, 26 December 1942.

[97] R. Hochberg, C. McDiarmid and M. Saks, On the bandwidth of triangulated triangles. *Discrete Mathematics* **138** (1995) 261–265.

[98] B. Imreh, On complete systems of automata, in: *Proc. of the 2nd International Colloquium on Words, Languages and Combinatorics*, Kyoto, 1992, World Scientific, Singapore-New Jersey-London-Hong Kong, 1994, 207–216.

[99] J. P. Jones, Some undecidable determined games. *Internat. J. Game Theory* **11** (1982), no. 2, 63–70.

[100] E. Junttila and P. Kaski, Segmented nestedness in binary data. *In Proceedings of the 2011 SIAM International Conference on Data Mining* (2011), pp. 235–246.

[101] D.G. Kirkpatrick and P. Hell, On the complexity of general graph factor problems. *SIAM Journal on Computing,* **12(3)** (1983), 601–609.

[102] D.J. Kleitman and B.L. Rothschild, A generalization of Kaplansky's game. *Discrete Math* **2** (1972) 173-178.

[103] F. Knox, Two constructions relating to conjectures of Beck on positional games. (2012) *arXiv preprint* arXiv:1212.3345.

[104] J. Komlós, G. Sárközy and E. Szemerédi, Proof of the Alon-Yuster conjecture. *Discrete Mathematics*, **235(1-3)** (2001), 255–269.

[105] A. Kostochka, Coloring uniform hypergraphs with few colors. *Random Structures Algorithms* **24** (2004), no. 1, 1–10.

[106] D. Král, J. Kratochvíl, Z. Tuza and G.J. Woeginger, Complexity of coloring graphs without forbidden induced subgraphs. In *Graph-theoretic concepts in computer science (Boltenhagen, 2001)*, Springer Verlag, pp. 254–262

[107] M. Krivelevich, Positional games and probabilistic considerations. Oberwolfach Report 20/2007 16–17.

[108] K. Kruczek, E. Sundberg, A Pairing Strategy for Tic-Tac-Toe on the Integer Lattice with Numerous Directions. *Electronic J. Combinatorics* **15(1): N42,** (2008).

[109] J. B. Kruskal, On the shortest spanning subtree of a graph and the traveling salesman problem. *Proceedings of the American Mathematical society,* **7(1)**, (1956) 48–50.

[110] D. Kühn and D. Osthus, The minimum degree threshold for perfect graph packings. *Combinatorica*, **29(1)** (2009), 65–107.

[111] A. Lehman, A solution of the Shannon switching game. *J. Soc. Indust. Appl. Math.* **12** 1964 687–725.

[112] R. J. Lipton and R. E. Tarjan, A separator theorem for planar graphs. *SIAM Journal on Applied Mathematics,* **36** (1979), 177–189.

[113] A. London and A. Pluhár, Spanning Tree Game as Prim Would Have Played. *Acta Cybernetica* **23(3)**, 2018, 921–927.

[114] A. London, R. R. Martin and A. Pluhár, Graph clustering via generalized colorings. *Theoretical Computer Science,* **918**, 94–104.

[115] L. Lovász, Combinatorial problems and exercises. *North-Holland Publishing Co., Amsterdam-New York,* 1979.

[116] J. Matoušek, *Thirty-three miniatures: Mathematical and Algorithmic applications of Linear Algebra.* Providence, RI: American Mathematical Society, (2010).

[117] S. T. McCormick, Optimal approximation of sparse Hessians and its equivalence to a graph coloring problem. *Mathematical Programming,* **26(2)** (1983), 153–171.

[118] J. von Neumann and O. Morgenstern, *Theory of Games and Economic Behavior.* Princeton, NJ: Princeton University Press, 1944.

[119] Neukomm Gyula, Egy híres sakkprobléma matematikai megfejtése. *Magyar Sakkvilág* 1941 December.

[120] M.E. Newman and M. Girvan, Finding and evaluating community structure in networks. *Physical Review E*, **69 (2)** (2004), 026113.

[121] Norman Do, How to Win at TicTacToe. *The Australian Mathematical Society, Gazette*, Volume **32** Number 3, July 2005, 151–161.

[122] K. Noshita, Union-Connections and Straightforward Winning Strategies in Hex. *ICGA Journal*, 2005 28(1): cover, 3–12.

[123] R. Nowakowski eds. *Games of No Chance*, combinatorial games at MSRI, 1994.

[124] J. G. Oxley, *Matroid Theory* New York: Oxford University Press, 1992.

[125] C. H. Papadimitriou, *Computational Complexity* Addison-Wesley Publishing Company, Inc., 1994.

[126] O, Patashnik, Qubic: $4 \times 4 \times 4$ tic-tac-toe. *Math. Mag.* **53** (1980), no. 4, 202–216.

[127] Y. Peres, *Game Theory, Alive.* electronic lecture note (created 2008.01.24 2:54:38)

[128] A. Pluhár, Generalizations of the game k-in-a-row. Rutcor Research Reports 15-94 (1994).

[129] A. Pluhár, Generalized Harary Games. *Acta Cybernetica* **13** no. 1, (1997) 77–83.

[130] A. Pluhár, *Positional Games on the Infinite Chessboard.* Ph.D. dissertation, Rutgers University 1994.

[131] A. Pluhár, The accelerated k-in-a-row game. *Theoretical Computer Science* **270** (2002), 865–875.

[132] A. Pluhár, The Recycled Kaplansky's Game. *Acta Cybernetica* **16** (2004) 451–458.

[133] Pluhár András, Pozíciós játékok. *Szigma*, vol 3-4, Pécs, 2007, 111–130

[134] A. Pluhár, Greedy colorings of uniform hypergraphs. *Random Structures and Algorithms* Volume **35** (2009) 216–221.

[135] Pluhár András, Játékelmélet. *egyetemi jegyzet (2010)* illetve Typotex, (2011) ISBN 978-963-279-519-5

[136] Pluhár András, Lineáris egyenletrendszerek konzisztenciájának kombinatorikai jelentései. *Alkalmazott Matematikai Lapok* **37** (2020) 225–232.

[137] L. Pósa, L. (1976). Hamiltonian circuits in random graphs. *Discrete Mathematics,* 14(4), 359–364.

[138] R. C. Prim, Shortest connection networks and some generalizations. *Bell Labs Technical Journal*, **36(6)** (1957) 1389–1401.

[139] J. Radhakrishnan and A. Srinivasan, Improved bounds and algorithms for hypergraph 2-coloring. *Random Structures Algorithms* **16** (2000), no. 1, 4–32.

[140] S. Reisch, Hex ist PSPACE-vollständig. *Acta Inform.* **15** (1981), no. 2, 167–191.

[141] S.E. Schaeffer, Graph clustering. *Computer Science Review*, **1 (1)** (2007), pp. 27–64.

[142] W. M. Schmidt, Ein kombinatorisches Problem von P. Erdös und A. Hajnal. *Acta Math. Acad. Sci. Hungar* **15** (1964) 373–374.

[143] S. Shelah, Primitive recursive bounds for van der Waerden numbers. *J. Amer. Math. Soc.* **1** (1988), no. 3, 683–697.

[144] N. Sieben, Hexagonal polyomino weak $(1,2)$-achievement games. *Acta Cybernetica* **16** (2004), no. 4, 579–585.

[145] N. Sieben, Snaky is a 41-dimensional winner. *Integers* **4** (2004), G5, 6 pp.

[146] T. J. Schaefer, On the complexity of some two-person perfect-information games. *J. Comput. System Sci.* **16** (1978), no. 2, 185–225.

[147] P. D. Seymour, On the two-colouring of hypergraphs. *Quart. J. Math. Oxford Ser.* **(2) 25** (1974), 303–312.

[148] J. H. Spencer, Coloring n-sets red and blue. *J Combinatorial Theory, Series A*, **30** (1981), 112–113.

[149] J. Spencer, Randomization, derandomization and antirandomization: three games. *Theoretical Computer Science* **131** (1994), no. 2, 415–429.

[150] H. Steinhaus, The problem of fair division. *Econometrica* **16** (1948) 101–104.

[151] H. Steinhaus, *Matematikai kaleidoszkóp.* Gondolat Budapest (1984).

[152] M. Stojaković and T. Szabó, Positional games on random graphs. *Random Structures & Algorithms,* **26(1-2)** (2005), 204–223.

[153] B. Sudakov, Robustness of graph properties. *Surveys in combinatorics,* **440.2017** (2017), 372.

[154] Székely Gábor, Paradoxonok a véletlen matematikájában. *Műszaki Könyvkiadó*, Budapest (1982).

[155] L. A. Székely, On two concepts of discrepancy in a class of combinatorial games. Finite and Infinite Sets, Colloq. Math. Soc. János Bolyai, Vol. 37, North-Holland, 1984, 679–683.

[156] L. A. Székely, Crossing numbers and hard Erdős problems in discrete geometry. *Combin. Probab. Comput.* **6** (1997), no. 3, 353-358.

[157] E. Szemerédi and W. T. Trotter Jr., Extremal problems in discrete geometry. *Combinatorica* **3** (1983), no. 3-4, 381-392.

[158] C. Thomassen, The even cycle problem for directed graphs. *J. Amer. Math. Soc.* **5** (1992), no. 2. 217–229.

[159] R. Radoičić and G. Tóth, The discharging method in combinatorial geometry and the Pach-Sharir conjecture. *Contemporary Math.* AMS **453** (2008) 319–342.

[160] B. Uzzi, The sources and consequences of embeddedness for the economic performance of organizations: The network effect. *American Sociological Review* (1996), pp. 674–698.

[161] H. Weyl, Über die Gleichverteilung von Zahlen mod Eins. *Math. Ann.*, 77 (1916), 313–352.

[162] N. C. Wormald, Models of random regular graphs. *London Mathematical Society Lecture Note Series,* (1999), 239–298.

[163] Wu, I-C. and Huang, D.-Y. (2006) A New Family of k-in-a-row Games. *Advances in Computer Games Lecture Notes in Computer Science*, 2006, Volume 4250/2006, 180-194.

[164] *http://en.wikipedia.org/wiki/Nim*

[165] E. Zermelo, Über eine Anwendung der Mengenlehre und der Theorie des Schachspiels, *Proceedings of the Fifth International Congress of Mathematicians,* Cambridge, 501–504.

Acknowledgements

I thank the efforts of all the people without whom this work would have never had accomplished. I thank the love and support of my family, the enthusiasm and wisdom of my teachers, and the friendship and shared knowledge of my colleagues and co-authors. Their sheer number fills me with joy, and my only sorrow is that so many of them cannot see the result of their help.