Statistical properties of some non-uniformly hyperbolic billiards

Dissertation

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Comments on colored text

The electronic version of this dissertation uses colors.
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This dissertation is dedicated to the memory of my dear father, Lajos Bálint.

Chapter 1

Introduction

1.1 Overview

This dissertation describes results of the author on three major classes of planar hyperbolic billiards: stadia, dispersing billiards with cusps and infinite horizon Lorentz gases. Although all three types of systems have their own characteristics, an important common feature is intermittent behavior. By this we mean that the dynamics is characterized by alternating chaotic and regular patterns.

If it were not for the regular components, we would expect that the dynamics has strong statistical properties. The main questions of interest are how the presence of the regular component effects statistical properties: reduces the rate of correlation decay from exponential to polynomial, along with the emergence of nonstandard probabilistic limit phenomena. As for mixing rates, we are particularly interested in the decay of correlations for the flow (in continuous time) which is more difficult than the map (discrete time), due to the presence of a neutral direction. For probabilistic limit laws, a major feature is the occurrence of the non-standard scaling $\sqrt{n \log n}$ in contrast to the standard diffusive $\sqrt{n}$.

As we will discuss in section 1.2, intermittency and its effect on statistical properties has been intensively studied earlier in simpler dynamical contexts, such as one dimensional maps. The main new challenge of the research presented here is that we investigate billiard examples, which are models of mechanical origin. This, on the one hand, makes their study relevant for applications in physics and, on the other hand, provides substantial mathematical challenges, in particular, related to handling the singularities which are intrinsic to this type of dynamics.

The main results of this dissertation are stated in ten major theorems, labelled by letters as Theorem A, Theorem B.... Theorem J in the sequel. These are based on six research papers, in particular Theorems A and B are from [BG06], Theorem G is from [BM08], Theorems D, F and E are from [BCD11], Theorem H is from [BCD17], Theorems C and J are form [BBM19] while Theorem I is from [BBT23].

These results are arranged in the three main chapters of the dissertation according to the class of systems that they concern. Chapter 3 is on stadia, containing Theorems A, B and C. Chapter 4 is on dispersing billiards with cusps, containing Theorems D, E, F, G and H. Chapter 5 is on infinite horizon Lorentz gases containing Theorems I and J.

From a different perspective, Theorems G, C and J are on the decay of correlations for the studied billiard flows. In particular, Theorem G is on rapid (superpolynomial) decay, while Theorems C and J are on polynomial decay. The other main theorems are on probabilistic limit laws in various contexts. Theorems A, B, D, F and I are on convergence in distribution for the relevant

Birkhoff sums, Theorem E is about its extension to a functional level (weak invariance principle), while Theorem H is on the convergence of moments.

It is also useful to take a quick look at these results from a methodological viewpoint. One of the most powerful approaches to statistical properties of dynamical systems is by studying the spectra of the associated transfer operators. The exposition in Chapter 3 is mostly based on such spectral methods. Theorem A in particular was the first non-standard limit theorem proved for a billiard model by such tools. The spectral techniques developed in [BG06] are frequently referred in the literature and have been extended in several directions.

The approach of Chapter 4 is somewhat different. Instead of applying perturbed transfer operators, it obtains the limit theorem by studying the characteristic functions of the relevant observables directly. This relies on classical techniques such as moment bounds, probabilistic inequalities and the Bernstein big-small block technique. When applying this method, the connection between decorrelation estimates and probabilistic limit phenomena appears more apparent and direct. This approach led, in particular, to Theorem H, which clarified the effects causing discrepancies between convergence in distribution and convergence of moments, an issue that had been previously observed in the physics literature ([CEF08]). The results of Chapter 4 have been extended in several interesting directions in the literature, let us mention in particular the emergence of stable limit laws when the order of the tangency at the cusp is higher than quadratic, see [JZ18], and the end of section 4.2 for further discussion.

The proof of Theorem I in Chapter 5 is in a sense a combination of the spectral technique of Chapter 3 and the more direct approach of Chapter 4, although it is rather spectral theory based. The peculiarity of this work is that, by considering a joint limit, these are the first results on an intermediate case between the two well-studied regimes of the periodic Lorentz gas, that is fixed infinite horizon configurations ([SV07], [DC09]) and Boltzmann-Grad type situations ([MS11a], [MT16]). Let us mention that the results on the decay rates for the flow (Theorems C, G and J) rely also on spectral techniques, yet, special care is needed to handle the difficulties connected to the presence of the neutral flow direction.

Beyond the above comments on the general framework, it is important also to mention an equally significant ingredient of the arguments, the analysis of the geometry and the dynamics of the studied billiard model. This analysis is very much specific to the system in question, and has to be carried out essentially independently in each of the three cases. In the dissertation, we will put a particular emphasis on this component as the probabilistic phenomena can be essentially understood from the geometry of the model.

The dissertation is structured as follows. In section 1.2 some models from one dimensional dynamics are mentioned, which is meant to serve as an analogy and some motivation for the work presented in the bulk of the thesis. Chapter 2 gives a brief overview of planar dispersing billiards with disjoint scatterers and finite horizon. The techniques developed for this best understood class of chaotic billiards play an important role in the investigation of the models in the subsequent Chapters 3, 4 and 5 as well. All three major chapters are organized as follows. In the first sections, a general description of the model is given with special emphasis on the core geometrical phenomena. The second sections contain the results of the chapter, with the presentation of some of the preceding and the follow-up work. It is important to emphasize that the theory of hyperbolic billiards is an utmost active research area, so when describing related work, we do not aim for completeness. Instead, our goal is to focus on the impact that the results presented in the dissertation have made on the development of the field. Finally, the third sections describe some key ingredients of the proofs of the results presented in the chapter. For complete arguments, we refer to the original research papers.

As mentioned above, the main results of the dissertation are stated in the Theorems labelled by letters. Further theorems and definitions, lemmas, propositions etc. are labelled numerically.

1.2 Motivating examples

One of the best understood paradigms of chaotic dynamical systems is the *doubling map*.

Example 1.1 (Doubling map). *Define* $T : [0, 1] \to [0, 1]$,

$$Tx = \begin{cases} 2x & \text{if } 0 \le x \le \frac{1}{2}; \\ 2x - 1 & \text{if } \frac{1}{2} < x \le 1. \end{cases}$$

Let m denote the Lebesgue measure on $[0, 1]$, which is a probability measure. For a general introduction to ergodic theory, in particular, ergodcity and mixing of a probability preserving transformation, see [Pet83] or [CM06, Appendix C]The following properties of the doubling map are well-known (see, for example, [Bal00]):

- m is invariant and *ergodic* (and actually, mixing) with respect to T. Consequently, by Birkhoff's ergodic theorem, given $f \in L^1(m)$, for m-a.e. $x \in [0, 1]$

$$\frac{1}{n} S_n f(x) = \frac{f(x) + f(Tx) + \cdots + f(T^{n-1}x)}{n} \to \int f dm \text{ as } n \to \infty.$$

- T has *exponential decay of correlations* on Hölder continuous functions. That is, for every $\eta \in (0, 1]$ there exists $\beta \in (0, 1)$ such that for every f, g Hölder continuous on $[0, 1]$ with exponent η[1] there exists $C > 0$ such that

$$|\,Corr(f, g; n)| = \left| \int f(x) g(T^n x) dm(x) - \int f dm \cdot \int g dm \right| \le C \cdot \|f\|_\eta \cdot \|g\|_\eta \beta^n, \qquad (1.1)$$

where $\|f\|_\eta$ denotes the η-Hölder norm of f ($\|f\|_\eta = \|f\|_0 + C_f$, sum of the supremum norm and the best Hölder constant.)

- T satisfies the classical *central limit theorem* on Hölder functions. That is, assume that f is Hölder continuous and centered (i.e. $\int f dm = 0$), then there exists $\sigma^2 \ge 0$ such that, for every $a \in \mathbb{R}$, we have

$$m\left(x \in [0, 1] \;\middle|\; \frac{S_n f(x)}{\sigma \sqrt{n}} \le a \right) \to \left(\sqrt{2\pi}\right)^{-1} \int_{-\infty}^{a} \exp\left(-\frac{x^2}{2}\right) dx; \text{ as } n \to \infty.$$

That is,

$$\frac{S_n f}{\sqrt{n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2)$$

where $\mathcal{N}(0, \sigma^2)$ is the normal law with variance σ^2, and $\xrightarrow{\mathcal{D}}$ denotes convergence in distribution. Moreover,

$$\sigma^2 = \int f^2 dm + 2 \sum_{n=1}^{\infty} \int (f \cdot f \circ T^n) dm \qquad (1.2)$$

and in particular $\sigma^2 = 0$ if and only if f is a coboundary, which means that there exists some measurable function χ such that $f = \chi - \chi \circ T$.

[1] That is, there exists $C_f > 0$ such that $|f(x) - f(y)| \le C_f d(x, y)^\eta$, where $d(x, y)$ is the distance of the points $x, y \in [0, 1]$.

In rough terms, these properties demonstrate that the doubling map has a quick loss of memory and thus, when x is distributed according to m, the behaviour of the sequence of random variables $f(x), f(Tx), \ldots$ resembles that of an i.i.d. sequence. This is related to the *uniform expansion* of the doubling map; when iterated by the dynamics, the lengths of small intervals grow at an exponential rate. Accordingly, probability densities initially supported on such small intervals are distributed more and more evenly on the phase space when pushed forward by the dynamics.

In contrast, consider the following family of maps, which have been extensively studied in the literature.

Example 1.2 (Intermittent interval maps)**.** *For $\alpha \in (0,1)$, let $T_\alpha : [0,1] \to [0,1]$,*

$$T_\alpha x = \begin{cases} x(1+2^\alpha x^\alpha) & \text{if } 0 \le x < \frac{1}{2}; \\ 2x-1 & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

We note that the case $\alpha = \frac{1}{2}$ is the most relevant for this dissertation.

Here, although the map is expanding, the origin is a neutral fixed point ($T'_\alpha(x) > 1$ for any $x \neq 0$ while $T'_\alpha(0) = 1$). Hence, trajectories may spend an arbitrary long time in the vicinity of the fixed point, which alternates with the effect of uniform expansion elsewhere in the interval. The length asymptotic of the time intervals spent in the vicinity of the neural fixed point is scaled by the parameter α.

We restrict to the case $\alpha = \frac{1}{2}$. The following results are proved in [LSV99], [You99] and [Gou04a].

- T_α has a unique absolutely continuous invariant probability measure μ, which is ergodic and mixing. Its density with respect to the Lebesgue measure m, to be denoted by h, is Lipschitz continuous on intervals of the form $(\varepsilon, 1]$, for any $\varepsilon > 0$.

- T_α has *polynomial decay of correlations* on Hölder continuous functions. That is, given f, g Hölder continuous on $[0,1]$, there exists $C > 0$ such that

$$|\,Corr(f,g;n)| = \left| \int f(x) g(T_\alpha^n x) d\mu(x) - \int f d\mu \cdot \int g d\mu \right| \le C n^{(\alpha^{-1}-1)} = C\,\frac{1}{n}.$$

  Actually, by [Sar02, Gou04b], for f and g taking nonzero constant values in a neighborhood of 0, this rate is optimal in the sense that $Corr(f,g;n) = cn^{\frac{1}{\alpha}-1}(1+o(1))$ where $c \neq 0$. For f and g vanishing in a neighborhood of 0 the decay is $O(n^{\frac{1}{\alpha}})$.

- Assume that f is Hölder continuous and centered (i.e. $\int f d\mu = 0$), and let $S_n f = f + f \circ T_\alpha + \cdots + f \circ T_\alpha^{n-1}$. Then

  - if $f(0) \neq 0$, then letting $D^2 = h(\frac{1}{2})(f(0))^2$

  $$\frac{S_n f}{\sqrt{n \log n}} \Rightarrow \mathcal{N}(0, D^2).$$

  - if $f(0) = 0$, $S_n(f)$ follows a classical central limit theorem, as above.

  If $\alpha \in (\frac{1}{2}, 1)$ and $f(0) \neq 0$, $\frac{S_n(f)}{n^\alpha}$ converges in distribution to a stable law of index $1/\alpha$. In the case $\alpha \in (0, \frac{1}{2})$; $S_n f$ always satisfies the classical CLT, irrespective of the value of $f(0)$.

Although the study of the maps T_α is quite challenging, they have some nice properties which makes them accessible by several methods. In particular, both T and T_α admit Markov partitions, which are inherited by appropriately defined first return maps. Let $Y = [\frac{1}{2}, 1]$ and consider

$$\begin{array}{rll} \text{the first return time} & r_Y : Y \to \mathbb{Z}^+; & r_Y(y) = \min\{k \geq 1 \,|\, T_\alpha^k y \in Y\}; \\ \text{the first return map} & T_Y : Y \to Y; & T_Y(y) = T_\alpha^{r_Y(y)} y; \\ \text{and the induced observable} & f_Y : Y \to \mathbb{R}; & f_Y(y) = f(y) + f(T_\alpha y) + \cdots + f(T_\alpha^{r_Y(y)-1} y). \end{array}$$

Then T_Y has a nice Markovian structure; it maps each of the intervals $Y_k = \{y \in Y \;|\, r_Y(y) = k\}$, $k = 1, 2, \ldots$ onto Y. The return time r_Y and (typically) the induced observables f_Y are unbounded, and actually, do not even belong to L^2 (with respect to μ_Y, i.e. μ conditioned on Y) for $\alpha \geq \frac{1}{2}$. Specifically, for $\alpha = \frac{1}{2}$, if $f(0) \neq 0$, then f_Y belongs to the non-standard domain of attraction of the normal law, in accordance with the limit theorem stated above.

The billiard systems studied in this dissertation lack such direct Markovian structures. They also have some other characteristic features, to be discussed in detail below – among others, presence of both contracting and expanding directions, intrinsic singularities associated to tangential collisions, and unbounded derivatives – which make their investigation more challenging than that of one dimensional maps. Nonetheless, it is useful to keep in mind the maps T and T_α as motivating examples. In a sense, dispersing billiards with finite horizon, discussed in chapter 2, resemble T, while Bunimovich stadia, dispersing billiards with cusps and infinite horizon Lorentz gases studied in the bulk of this dissertation resemble T_α with $\alpha = \frac{1}{2}$.

1.3 Some notations and conventions

Throughout this dissertation, positive constants which may depend on the dynamical system studied, but not on the specific point, curve, or tangent vector, will be called uniform constants. The exact value of a uniform constant is typically irrelevant, it is often denoted by C, which may change from line to line (or within a line).

Given two functions $f, g : \mathbb{Z}^+ \to \mathbb{R}^+$, the notations $f \ll g$ and $f = O(g)$ will be used interchangeably, both meaning that there exists some uniform $C > 0$ such that $f(n) \leq C g(n)$. If $f \ll g$ and $g \ll f$ simultaneously, we will write $f \asymp g$. If, additionally, $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$, we will write $f \sim g$. The same notations will be used in other contexts to compare the asymptotic behaviour of two quantities f and g, for example near 0 if $f, g : (0, \varepsilon) \to \mathbb{R}^+$ for some $\varepsilon > 0$.

Chapter 2

Dispersing billiards

A planar billiard dynamical system is the time evolution of a point particle in a domain $Q \subset \mathbb{R}^2$ of the Euclidean plane or the two dimensional flat torus $Q \subset \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. The state of the particle is described by its position $q \in Q$ and velocity $v \in \mathbb{S}^1$[1] of unit length, and it proceeds uniformly within the interior of the domain Q. The boundary ∂Q is assumed to be at least piecewise C^3-smooth, so that the normal vector is well defined in almost all its points. When the particle reaches the boundary, it proceeds according to the rule of geometric optics: the angle of incidence with respect to the normal vector equals the angle of reflection. That is, if $q \in \partial Q$, letting v^- and v^+ denote the incoming and the outgoing velocities, respectively, we have $v^+ = v^- - 2\langle v^-, n\rangle n$, where $n = n(q)$ is the normal vector of ∂Q at q, pointing inwards Q, and $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathbb{R}^2$.

Our main reference on billiards is the excellent monograph [CM06]. Here we introduce only the terminology and the notation that is needed for our discussion. The above described continuous time dynamical system will be referred to as the billiard *flow*. It acts on the phase space $\mathcal{M} = Q \times \mathbb{S}^1/\sim$, where incoming and outgoing configurations are identified, that is, $(q, v^+) \sim (q, v^-)$ for $q \in \partial Q$. For $t \in \mathbb{R}$, the time t action of the flow will be denoted as $S^t : \mathcal{M} \to \mathcal{M}$. There is a natural Poincaré section corresponding to collisions at the boundary with phase space

$$M = \{(q, v) \in \mathcal{M} \,|\, q \in \partial Q, \langle v, n(q)\rangle \geq 0\},$$

where $n(q)$ denotes the normal vector of ∂Q at the point q. Let τ denote the first return time of the flow to the section M, that is

$$\tau : M \to \mathbb{R}^+; \qquad \tau(x) = \min\{t > 0 \,|\, S^t x \in M\}.$$

That is, given $x \in M$ the value of the *free flight function* $\tau(x)$ is the time that elapses until the first collision at the boundary along a trajectory starting from the state x. Then the *billiard map* is defined as

$$T : M \to M; \qquad Tx = S^{\tau(x)}x.$$

In other words, the billiard flow can be regarded as a suspension flow with base transformation $T : M \to M$ and roof function $\tau : M \to \mathbb{R}^+$, see section 2.2.2 below. To avoid ambiguity let us point out the we consider outgoing collisions, that is, the points of M correspond to pairs (q, v) where $q \in \partial Q$ and v is the post-collision velocity. If τ is uniformly bounded, the billiard is said to have *finite horizon*, otherwise it has infinite horizon. For most of the examples in this dissertation, the domain Q is bounded, in which case there is a natural, absolutely continuous

[1] Here $\mathbb{S}^1$ is the unit circle in $\mathbb{R}^2$.

invariant probability measure for T, called the Liouville measure, and denoted by μ (see (2.1) below). When talking about the ergodic and statistical properties of the billiard, we mean those with respect to the measure μ (and the associated invariant measure μ^τ for the suspension flow, see section 2.2.2). Nonetheless, let us point out that infinite horizon may very well occur in compact $Q \subset \mathbb{T}^2$, see Chapter 5.

Depending on the shape of Q billiards may exhibit a wide range of dynamical phenomena. In this chapter, we consider *dispersing billiards*, when the domain is obtained by *removing* smooth strictly convex scatterers from the torus $\mathbb{T}^2$. That is, let $Q = \mathbb{T}^2 \setminus \left(\cup_{i=1}^I C_i\right)$ where $I < \infty$ and the scatterers C_i are closures of simply connected domains. Denote $\Gamma_i = \partial C_i$. Throughout Chapter 2 we assume that

- the C_i are pairwise disjoint (i.e. there are no corner points or cusps);
- the boundaries Γ_i are (at least) C^3 smooth curves;
- the C_i are strictly convex, that is, the curvature of Γ_i is positive at every $q \in \Gamma_i$;
- the horizon is finite.

Accordingly, we have $M = \cup_{i=1}^I M_i$ where M_i corresponds to $q \in \Gamma_i$. It is convenient to use the coordinates (r, φ) on M_i, where $r \in [0, |\Gamma_i|)$ identifies the collision point on Γ_i, parameterized by arclength[2], while $\varphi \in [-\pi/2, \pi/2]$ denotes the angle of reflection, that is, the angle of the normal vector $n(q)$ and the outgoing velocity v^+, oriented in the same way as the arclength r (either both r and φ are oriented clockwise, or both of them are oriented counterclockwise). As the coordinate r is cyclic, M can be regarded as a finite union of disjoint cylinders. In these coordinates, the density of the Liouville measure is proportional to $\cos\varphi$, that is

$$d\mu = c_\mu \cos\varphi \, dr \, d\varphi \tag{2.1}$$

where $c_\mu > 0$ is a normalizing constant.

2.1 Hyperbolic and geometric properties

In this section we consider a dispersing billiard table as defined above. See [CM06] for a more detailed description of dispersing billiards. Note that, by compactness of M, both the curvature of the scatterers and (in the finite horizon case) the length of the free flight are uniformly bounded away from 0 and infinity. Let us introduce the following notations:

$$0 < \mathcal{K}_{\min} \le \mathcal{K}_{\max} < \infty; \quad \text{minimal and maximal values of the curvatures of the scatterers } \Gamma_i; \tag{2.2}$$

$$0 < \tau_{\min} \le \tau_{\max} < \infty; \quad \text{minimal and maximal values of the free flight.} \tag{2.3}$$

Also, for $x = (r, \varphi) \in M$, let us denote $Tx = x_1 = (r_1, \varphi_1) \in M$.

Singularities. The boundary of the phase space, $\partial M = \{\varphi = \pm\pi/2\}$ corresponds to tangential collisions. Define

$$\mathcal{S}_1 = \partial M \cup T^{-1}\partial M \quad \text{and} \quad \mathcal{S}_{-1} = \partial M \cup T\partial M$$

which are the discontinuity sets for T and T^{-1}, respectively. In particular, $T : M \setminus \mathcal{S}_1 \to M \setminus \mathcal{S}_{-1}$ is a homeomorphism. Analogously,

$$\mathcal{S}_n = \partial M \cup T^{-1}\partial M \cup \ldots T^{-n+1}\partial M \quad \text{and} \quad \mathcal{S}_{-n} = \partial M \cup T\partial M \cup \ldots T^{n-1}\partial M$$

[2] r can be regarded as a cyclic coordinate, hence it is not important which point of Γ_i corresponds to the value $r = 0$.

are the discontinuity sets for T^n and T^{-n}, respectively. For each n, $\mathcal{S}_n$ is a finite collection of smooth and compact curves on M, which are non-increasing in the (r, φ) coordinates. Similarly, $\mathcal{S}_{-n}$ is a collection of non-decreasing curves in the (r, φ) coordinates. Note that both $\mathcal{S}_\infty = \cup_{n\geq 1}\mathcal{S}_n$ and $\mathcal{S}_{-\infty} = \cup_{n\geq 1}\mathcal{S}_{-n}$ are dense on the phase space M.

Hyperbolicity. At any $x \in M \setminus \mathcal{S}_1$, the map T is differentiable, and its derivative matrix DT_x can be computed explicitly, see [CM06, Formula (2.26)]. We will denote the tangent vectors, that is, the elements of the tangent space $\mathcal{T}_x M$ by $dx = (dr, d\varphi)$. Define the *unstable cone* as

$$C_x^u = \left\{ (dr, d\varphi) \in \mathcal{T}_x M \,|\, \mathcal{K}_{\min} \leq \frac{dr}{d\varphi} \leq \mathcal{K}_{\max} + \frac{1}{\tau_{\min}} \right\}.$$

Then, for $x \in M \setminus \mathcal{S}_1$

$$DT_x C_x^u \subset C_{Tx}^u$$

and there exist $c > 0$ and $\Lambda > 1$ (in fact, $\Lambda = 1 + 2\mathcal{K}_{\min}\tau_{\min}$) such that:

$$|DT_x^n dx| \geq c\Lambda^n |dx| \quad \text{for every } dx \in C_x^u, x \in M \setminus \mathcal{S}_n.$$

A C^2 smooth compact curve $W \subset M$ is an *unstable curve* if at every point $x \in W$ its tangent vector $\mathcal{T}_x W$ belongs to the unstable cone C_x^u – thus in particular, unstable curves are increasing in the (r, φ) coordinates. The invariance and the expansion of the unstable cone field can be derived from the derivative matrix, however, a more geometric interpretation corresponds to diverging wavefronts. A diverging wavefront can be obtained by considering a short smooth curve of strictly positive curvature in the billiard domain Q, and endowing it with its unit normal vectors in all points, obtaining this way a curve in the flow phase space $\mathcal{M}$. For $t > 0$, the flow S^t preserves the class of diverging wavefronts, and expands distances on them. Unstable curves are the traces of diverging wavefronts on the Poincaré phase space M.

Involution. The billiard dynamics are invertible, and the inverse can be essentially obtained by reverting the velocity directions. More precisely define

$$\begin{aligned} \mathcal{I} : \mathcal{M} \to \mathcal{M}; \qquad & \mathcal{I}(q, v) = (q, -v); \quad \text{which induces} \\ \mathcal{I} : M \to M; \qquad & \mathcal{I}(r, \varphi) = (r, -\varphi). \end{aligned}$$

Then $S^{-t} = \mathcal{I} \circ S^t \circ \mathcal{I}$, and $T^{-n} = \mathcal{I} \circ T^n \circ \mathcal{I}$. By applying the involution I on the unstable cone and the unstable curves, the stable counterparts can be obtained, which have the analogous properties. In particular, we define the *stable cone* as

$$C_x^s = \left\{ (dr, d\varphi) \in \mathcal{T}_x M \,\middle|\, -\mathcal{K}_{\max} - \frac{1}{\tau_{\min}} \leq \frac{dr}{d\varphi} \leq -\mathcal{K}_{\min} \right\}.$$

Then

$$\begin{aligned} & DT_x^{-1} C_x^s \subset C_{T^{-1}x}^s; \qquad \text{for } x \in M \setminus \mathcal{S}_{-1}; \text{ and} \\ & |DT_x^{-n} dx| \geq c\Lambda^n |dx| \quad \text{for every } dx \in C_x^s, x \in M \setminus \mathcal{S}_{-n}. \end{aligned}$$

Curves with tangent vectors in C_x^s are called stable curves. These curves are decreasing in the (r, φ) coordinates, and they correspond to the traces of converging wavefronts on the Poincaré phase space M. It is also worth noting that the stable and the unstable cones are uniformly transversal.

Growth lemmas. The above properties of the invariant cone fields resemble those in smooth uniformly hyperbolic systems. However, the singularities also play a crucial role in billiard dynamics. One of the main challenges in the theory of chaotic billiards is to understand this competition between hyperbolicity and singularities. On the long run, "expansion prevails over fractioning", which is often quantified in various growth lemmas, a quintessential example of which is formulated in Formula (2.4) below. To state this Formula, let W be an unstable curve shorter than some fixed small constant $\delta_0 > 0$. When evolved by T, W is expanded, and partitioned by the singularities: TW consists of finitely (or countably, see below) many connected components, each of which are unstable curves. Let us denote these curves by TW_j, and introduce $\Lambda_j = \frac{|TW_j|}{|W_j|}$, which is, roughly speaking, the expansion rate on the smooth component W_j of W. Then

$$\sum_j (\Lambda_j)^{-1} \leq \theta < 1, \qquad \text{where } \theta \text{ is uniform in } W. \tag{2.4}$$

Actually, the growth lemma does not hold literally the way it is stated above. However, this issue can be easily fixed in two different ways: (i) (2.4) holds true if the length of the curves is measured with respect to an adapted metric (which is equivalent, but not equal to the usual Riemannian metric). (ii) (2.4) holds true even with the original metric if, instead of T, stated for a fixed higher iterate $T' = T^{n_0}$ (here $n_0 \geq 1$ is an appropriately chosen fixed integer, uniform in the point $x \in M$).

Unbounded derivatives, Homogeneity layers. Another important feature of billiard dynamics is that the expansion rates are unbounded in the vicinity of singularities. In particular, there exists $C > 0$ such that, for $x \in M \setminus \mathcal{S}_1$ and $dx \in C^u_x$:

$$(C \cos \varphi_1)^{-1} \leq \frac{|DT dx|}{|dx|} \leq C (\cos \varphi_1)^{-1}$$

where we recall that $Tx = x_1 = (r_1, \varphi_1)$. As a consequence, the distortions are unbounded[3]. To handle this, homogeneity layers are introduced as follows. Fix some $k_0 \in \mathbb{Z}^+$, and let

$$\begin{aligned}
\mathbb{H}_k =& \{(r,\varphi) \in M \,|\, \pi/2 - k^{-2} \leq \varphi \leq \pi/2 - (k+1)^{-2}\} \quad \text{for } k \geq k_0, \\
\mathbb{H}_{-k} =& \{(r,\varphi) \in M \,|\, -\pi/2 + k^{-2} \leq \varphi \leq -\pi/2 + (k+1)^{-2}\} \quad \text{for } k \geq k_0, \\
\mathbb{H}_0 =& \{(r,\varphi) \in M \,|\, -\pi/2 + k_0^{-2} \leq \varphi \leq \pi/2 + k_0^{-2}\}.
\end{aligned}$$

Restricted to these homogeneity layers, the distortions can be controlled, hence, their boundaries are treated as additional singularities. In particular, in the formulation of the growth lemma discussed above, the components TW_j are assumed be homogeneous, that is, they are included in a single homogeneity layer. The estimate (2.4) remains true with this additional cutting taken into account.

Here we discuss some further key objects of billiard theory, the properties of which are strongly connected to the growth lemma.

The **LUM** or local unstable manifold of a point $x \in M$ is an unstable curve W with x in its interior such that, for all $n \geq 1$, T^{-n} is smooth on W and $T^{-n}W$ is an unstable curve. Consequently, the LUM of x consists of points y such that $d(T^{-n}x, T^{-n}x)$ tends to zero exponentially as $n \to \infty$. Local stable manifolds (LSM) are defined analogously. LUM-s terminate on $\mathcal{S}_n$ for some n, and thus it is a remarkable fact that μ almost every $x \in M$ has a unique LUM of positive length. Actually, even a local version of this statement holds: given an unstable curve W, LSM-s of positive length exist for almost every $x \in W$ (with respect to Lebesgue measure on W).

For $x \in M$, let $W^u(x)$ and $W^s(x)$ denote the LUM and the LSM of x, respectively (as mentioned above these are well defined for μ almost every x). A **rectangle** is a set R with the property that whenever $x, y \in R$, we also have $[x, y] = W^u(x) \cap W^s(y) \in R$. For $\delta_1, \delta_2 > 0$,

[3]By distortions we mean the fluctuations in the magnitude of $J(x)$, where $J(x)$ is the Jacobian of T (either of the two dimensional map, or along some unstable curve) for x belonging to a continuity component of T.

sufficiently small rectangles can be obtained as the intersection points $(\cup W^u) \cap (\cup W^s)$, where $\cup W^u$ and $\cup W^s$ are families of LUMs of length δ_1 and LSMs of length δ_2, respectively. As the singularities are dense in the phase space, rectangles necessarily have a Cantor structure (which is the reason why such rectangles are often called Cantor rectangles). Yet – for δ_1 and δ_2 sufficiently small – rectangles of positive μ measure can be constructed. See [CM06, section 7.11] for further details.

Standard pairs and families. A standard pair $\ell = (W, \rho)$ is an unstable curve W endowed with a Hölder continuous probability density function ρ_W. It induces a probability measure on W, and thus on M. When iterated by T, a standard pair evolves into a countable collection of standard pairs. A *standard family* $\mathcal{G}$ is a parameterized collection of standard pairs $\ell_a = (W_a, \rho_a); a \in A$, with a factor probability measure λ on (the countable or uncountable) index set A, which induces a probability measure $\mu_{\mathcal{G}}$ on $\cup_{a\in A} W_a$. To measure the regularity of $\mathcal{G}$, introduce $r_{\mathcal{G}} : \cup_{a\in A} W_a \to \mathbb{R}$, where for $x \in W_a$ $r_{\mathcal{G}}(x)$ is the distance of x to the boundary (that is, the closest endpoint) of W_a, with respect to the arclength on W_a. Then define the quantity

$$\mathcal{Z}_{\mathcal{G}} = \sup_{\varepsilon>0} \frac{\mu_{\mathcal{G}}(r_{\mathcal{G}} < \varepsilon)}{\varepsilon},$$

the Z-function, which measures (up to a uniform constant factor) the average inverse length $|W_a|^{-1}$ of the curves included in the family. Standard families have the following remarkable properties:

- The class of standard families is invariant under the time evolution. To see this, consider a standard pair (W_a, ρ_a) included in the family. As discussed above, $W = \cup W_i$ for a countable collection of subcurves such that TW_i is an unstable curve for each i. Pushing forward the restrictions of the density ρ to the W_i, a countable collection of standard pairs is obtained, supported on the curves TW_i. Now the image standard family $T\mathcal{G}$ is obtained when evolving this way each of the curves (W_a, ρ_a) included in $\mathcal{G}$, with the weights λ_a taken into account. By construction, we have $T_*(\mu_{\mathcal{G}}) = \mu_{T\mathcal{G}}$. Similarly, for any $n \geq 1$ the image under T^n, $\mathcal{G}_n = T^n\mathcal{G}$, is another standard family.

- Many important and physically relevant measures can be represented as standard families. In particular, the Liouville measure μ can be represented as a standard family.

- Standard families are regularized by the dynamics in the following sense. There exists $\theta < 1$ and $c_1, c_2 > 0$ such that for any standard family $\mathcal{G}$

  $$\mathcal{Z}_{\mathcal{G}_n} \leq c_1 \mathcal{Z}_{\mathcal{G}} \theta^n + c_2. \tag{2.5}$$

  Formula (2.5) follows from (2.4), where the term c_2 appears as the components W_j that grow long have to be chopped up into pieces of length δ_0 (where δ_0 is some fixed length scale), to ensure that the estimate of the growth lemma (2.4) can be iterated. A consequence of (2.5) is that there exists some $C_0 > 0$ such that if $\mathcal{Z}_{\mathcal{G}} \leq C_0$, then $\mathcal{Z}_{\mathcal{G}_n} \leq C_0$ for all $n \geq 1$. Standard families with $\mathcal{Z}_{\mathcal{G}} \leq C_0$ are called proper standard families.

For further details on standard families, we refer to [CM06, section 7.4].

2.2 Ergodic and statistical properties

In this section we describe some of the most important milestones in the development of the ergodic theory of dispersing billiards. In section 2.2.1 we focus on the billiard map $T : M \to M$, while in section 2.2.2 we discuss the flow $S^t : \mathcal{M} \to \mathcal{M}$. Unless otherwise stated, ergodic and statistical properties are understood with respect to the Liouville measure, and the functions f, g etc. are Hölder continuous[4].

2.2.1 The billiard map

The mathematical investigation of dispersing billiards was initiated by Sinai in the seminal article [Sin70]. In this paper he laid the foundations of the theory; constructed the local invariant manifolds, and used these to prove the ergodicity of the map $T : M \to M$ (and actually, stronger ergodic properties such as K-mixing for both the map and the flow).

Sinai and his students extended their research on billiards in the direction of statistical properties in a series of papers [BS81a], [BS81b], [BSC90], [BSC91]. Essentially, the approach of these works was to approximate the map by Markov chains with countably many states. This resulted in the proof of the Central Limit Theorem for Hölder observables, and a stretched exponential bound on the decay of correlations. This means a bound of the type

$$|\,Corr(f, g; n)| \leq Ce^{-an^{\gamma}} \quad \text{for some } \gamma \in (0, 1) \text{ (with } a > 0 \text{ and } C > 0). \tag{2.6}$$

The optimal, *exponential bounds on correlation decay* were achieved by Young in [You98]. In this paper and in the follow-up work [You99], Young developed general methods which have turned out to be very powerful for studying the statistical properties of hyperbolic dynamical systems in general, and billiards in particular. A key idea in [You98] was to model the billiard map by appropriate dynamical systems which are now called **Young towers**. There are two version of Young towers, hyperbolic and expanding, and we include here Definition 2.1 on the later, for future reference.

Definition 2.1 (Expanding Young tower). *The probability space $(\bar{\Delta}, \mu_{\bar{\Delta}})$ with the probability preserving map $\bar{F} : \bar{\Delta} \to \bar{\Delta}$ is an expanding Young tower if there exist integers $r_p \in \mathbb{N}^*$ and a partition $\{\bar{\Delta}_{k,p}\}_{p\in\mathbb{N}, k\in\{0,\dots,r_p-1\}}$ of $\bar{\Delta}$ such that*

1. *For all p and $k < r_p - 1$, $\bar{F}$ is a measurable isomorphism between $\bar{\Delta}_{k,p}$ and $\bar{\Delta}_{k+1,p}$, preserving $\mu_{\bar{\Delta}}$.*
2. *For all p, $\bar{F}$ is a measurable isomorphism between $\bar{\Delta}_{r_p-1,p}$ and $\bar{\Delta}_0 := \bigcup_m \bar{\Delta}_{0,m}$.*
3. *Let $\bar{F}_0$ be the first return map induced by $\bar{F}$ on $\bar{\Delta}_0$. For $x, y \in \bar{\Delta}_0$, define their separation time $s(x, y) = \inf\{n \in \mathbb{N} \mid \bar{F}_0^n(x) \text{ and } \bar{F}_0^n(y) \text{ are not in the same } \bar{\Delta}_{0,p}\}$. We extend this separation time to the whole tower in the following way: if x, y are not in the same element of partition, set $s(x, y) = 0$. Otherwise, $x, y \in \bar{\Delta}_{k,p}$. Let $x', y' \in \bar{\Delta}_{0,p}$ be such that $x = \bar{F}^k x'$ and $y = \bar{F}^k y'$, and set $s(x, y) = s(x', y')$.*

   For $x \in \bar{\Delta}$, let $J(x)$ be the inverse of the jacobian of $\bar{F}$ at x. We assume that there exist $\theta < 1$ and $C > 0$ such that, for all x, y in the same element of partition,

$$\left|1 - \frac{J(x)}{J(y)}\right| \leq C\theta^{s(\bar{F}x, \bar{F}y)}. \tag{2.7}$$

For further reference, we introduce some more notation. Let $\bar{\Delta}_n = \bigcup \bar{\Delta}_{n,p}$, this is the set of points at height n in the tower. By the above described isomorphisms, $y \in \bar{\Delta}_n$ can then be also represented as $y = (x, n)$ with $x \in \bar{\Delta}_0$. We define $\omega : \bar{\Delta} \to \mathbb{Z}_0^+$, $\omega(x, n) = n$ as the height of the point and $\pi_0 : \bar{\Delta} \to \bar{\Delta}_0$ as the projection to the base.

[4] The results hold also for piecewise Hölder continuous functions, as long as their discontinuity sets are included in those of the dynamics

- The *hyperbolic Young tower* is an extension (Δ, F, μ_Δ) of the billiard map (M, T, μ). The base $\Delta_0 = R$ is a rectangle in the sense of section 2.1, and the return time $r : \Delta_0 \to \mathbb{Z}^+$ is typically *not* the first return of $T : M \to M$ to R. It is constructed in such a way that there is a countable partition R_j of $R = \Delta_0$ such that r takes a constant value r_j on each R_j, the R_j are s-subrectangles of R (i.e. stretch along R in the stable direction) while the sets $T^{r_j} R_j$ are u-subrectangles of R (stretch along R in the unstable direction). We have then

$$\Delta = \{(y,k) \,|\, y \in \Delta_0, 0 \le k \le r(y) - 1\}; \qquad F : \Delta \to \Delta;$$
$$F(y,k) = (y, k+1) \quad \text{for } k < r(y) - 1, \text{ while}$$
$$F(y, r(y) - 1) = (T^{r(y)} y, 0).$$

  Then $\pi : \Delta \to M$; $\pi(y,k) = T^k y$ is a measure preserving semi-conjugacy, that is, $\pi \circ F = T \circ \pi$, $\pi_*(\mu_\Delta) = \mu$, and F preserves μ_Δ .

- The *expanding Young tower* $(\bar\Delta, \bar F, \bar\mu_\Delta)$ is obtained from the invertible (Δ, F, μ_Δ) by collapsing its stable manifolds. That is, $\bar\pi : R \to \bar R$ maps a local stable manifold of the rectangle R into a single point, and the projection $\bar\pi$ naturally extends to the higher levels of the tower. The sets $\bar R_j = \bar\pi(R_j)$ then provide a partition of the base $\bar R$. As F maps stable manifolds to stable manifolds, the map $\bar F$ is well defined on $\bar\Delta$. We will denote by $\bar r$ the first return time of $\bar F$ on $\bar R$. On each of the partition elements $\bar R_j$, $\bar r$ takes the constant value $\bar r(\bar R_j) = r(R_j)$, and $\bar F^{\bar r(\bar R_j)}(\bar R_j) = \bar R$. Hence, while $\bar F(\bar y, k) = (\bar y, k+1)$ if $k < \bar r(\bar y) - 1$, each of the sets $\bar F^{\bar r(\bar R_j)-1} \bar R_j$ are mapped onto $\bar R$, which means that $\bar F$ is highly non-invertible. Finally, let $\mu_{\bar\Delta} = \bar\pi_*(\mu_\Delta)$, which is invariant for $\bar F$.

For further details on Young towers, we refer to [You98]. See also sections 3.3.4 and 5.3.3 of this dissertation, with more detailed discussion in the corresponding papers, [BG06] and [BBM19], respectively.

The Young tower has *exponential tails* if there exist $\rho < 1$ and $C > 0$ such that

$$\mu_\Delta(\{y \in R \ | \, r(y) > n\}) = \mu_{\bar\Delta}(\{\bar y \in \bar R \ | \, \bar r(\bar y) > n\}) \le C\rho^n.$$

The strategy of [You98] was to

- prove that, if the corresponding Young tower has exponential tails, then (M, T, μ) has exponential decay of correlations on Hölder functions.[5]

- construct Young towers with exponential tails for various examples of hyperbolic maps (including dispersing billiards).

[You99] than extended this strategy to the case of polynomial tails. To prove that bounds on the tail of the tower imply bounds on the decay of correlations, [You98] and [You99] applied two different methods.

The **spectral method** used in [You98] has a long tradition in hyperbolic dynamics, see the monographs [Bal00] and [Bal18]. In the context of expanding Young towers with exponential tails, the main ingredients are as follows.

- Consider a Banach space $\mathcal{B}$ of functions on $\bar\Delta$. For the definition of the norm on $\mathcal{B}$, see [You98] or [BG06]. Here we mention that to belong in $\mathcal{B}$, a function has to be locally Hölder continuous (see also section 3.3.4), that is, Lipschitz with respect to the symbolic metric d_θ when restricted to any of the partition elements introduced in Definition 2.1.

[5] Actually, it is also needed that the greatest common divisor of the return times $\{r(R_i)\}$ is equal to 1, which corresponds to the mixing of (M, T, μ).

- Study the action of the *transfer operator* $P : \mathcal{B} \to \mathcal{B}$, $(Pf)(x) = \sum J(x_p)f(x_p)$, where $\{x_p\}_{p\in\mathbb{N}}$ are the preimages of $x \in \bar{\Delta}$ under $\bar{F}$. P describes the evolution of probability densities under $\bar{F}$.

- Show that P above has a unique fixed point and a spectral gap. This then implies exponential decay of correlations on $\mathcal{B}$ with respect to $\mu_{\bar{\Delta}}$, and thus by approximation arguments for (M, T, μ).

The **coupling method** used in [You99] has its origins in probability theory. Let ν_1 and ν_2 be two sufficiently regular measures on $\bar{\Delta}$ (say absolutely continuous w.r.t. to $\mu_{\bar{\Delta}}$, with Lipschitz densities), the method aims to obtain effective bounds on the total variation distance $|\bar{F}^n_*\nu_1 - \bar{F}^n_*\nu_2|$. To do so, $\bar{F}^n_*\nu_1$ and $\bar{F}^n_*\nu_2$ are considered as the marginals of a measure on $\bar{\Delta}\times\bar{\Delta}$. The mass on the diagonal is regarded as coupled as it does not contribute to the total variation distance. In the context of Young towers, some amount of mass can be coupled whenever simultaneous returns to $\bar{\Delta}_0$, the base of the tower are made. This way the tail of the return time, $\mu_{\Delta}(R > n)$ is related to the uncoupled mass, and thus to $|F^n_*\nu_1 - F^n_*\nu_2|$. In particular, with the choice $\nu_2 = \mu_{\bar{\Delta}}$, this provides estimates on the rate of convergence to equilibrium $|\bar{F}^n_*\nu_1 - \mu_{\bar{\Delta}}|$, which is essentially the rate of correlation decay. An important advantage of coupling over the spectral method is that it can handle polynomial, and not only exponential tails.

In [You98], Young towers with *exponential* tails were constructed among others for the class of dispersing billiard maps discussed in subsection 2.1. In [Che99] Chernov extended the construction to further classes of billiard maps (allowing in particular infinite horizon or corner points). The approach of [Che99] was to formulate a set of assumptions based on which the tower can be constructed, and then verified these assumptions for several classes of billiard maps. Since then, such axiomatic strategies have been implemented in a variety of contexts.

Beyond bounding the rate of correlation decay, Young towers were also used to obtain a multitude of further statistical properties, including large deviation estimates ([MN08], [RBY08]), the almost sure invariance principle ([MN05], [Gou10]), and convergence of moments ([MT12], [GM14]), just to mention a few.

Although Young towers have been very powerful, they have also turned out to be suboptimal for some purposes. Here we discuss briefly two other approaches, which have the common feature that, instead of modeling the dynamics with the Young tower, the spectral or the coupling methods are applied directly to the billiard map (M, T, μ).

Originating from classical results on Markov chains ([DF37], [ITM50], [Nag57]), the *spectral (or functional analytic) approach* was carried out **directly on the phase space (without any symbolic coding)** for piecewise expanding maps of the interval ([LY73], [Kel84]) and then for several classes of multidimensional expanding maps (eg. [Sau00], [Tsu01]). In these expanding situations, the authors used various function spaces, in particular, functions of bounded variation and its generalizations. A big challenge in the hyperbolic case is to accommodate for both expanding and contracting directions, which can be handled by *anisotropic Banach spaces* of distributions, rather than functions. Several variants of this technology were developed first for smooth uniformly hyperbolic maps (eg. [BKL02], [GL06], [BT07]) then for some classes of hyperbolic maps with singularities ([BG09], [BG10] [DL08]), and the approach of [DL08] was extended to dispersing billiard maps in [DZ11]. This not only reproved the result of [You98] on exponential decay of correlations (and several other statistical properties obtained earlier by Young towers), but also provided the stability of these properties with respect to small perturbations ([DZ13]). One of the most important advantages of this spectral approach was that it opened up the perspectives for an optimal (exponential) bound on the mixing rates for the billiard *flow* proved in [BDL18] – see subsection 2.2.2 for details. Further developments include a thermodynamic formalism for billiard maps ([BD20], [BD22]). It is also worth mentioning the related technique of Birkhoff cones ([Liv95]) which has been recently applied to dispersing billiard maps ([DL21]).

On the other hand, *coupling arguments applied directly to the billiard map* have also turned out to be very insightful and productive. In hyperbolic dynamics, as already discussed in [Dol00] and [BL02], a key idea is to couple mass along contracting directions. A coupling scheme for dispersing billiards was then developed by Chernov and Dolgopyat ([CD09, Appendix A], [Che06]). Below we use the terminology of subsection 2.1 to summarize, in rough terms, the main ingredients of this method.

- The class of measures to be coupled are *standard pairs*. When ℓ and ℓ' are two standard pairs, $T_*^n\ell$ and $T_*^n\ell'$ are both countable collections of standard pairs. By the growth lemma, most of the components included in these collections are supported on fairly long unstable curves.

- Coupling occurs along the LSM-s included in a fixed Cantor rectangle R (which is often called a magnet, cf. [CM06, Chapter 7]). By the above phenomena, there exists n_0 such that, for any $n \geq n_0$, a fixed percentage of the components of $T_*^n\ell$ and $T_*^n\ell'$ cross R simultaneously.

- When components of $T_*^n\ell$ and $T_*^n\ell'$ cross R simultaneously, a fixed percentage of their points x and x', respectively, are connected by some LSM. In this case we consider x and x' coupled, which implies that the distance $d(T^m x, T^m x')$ shrinks exponentially as m grows. Let us denote by $\ell_{c,n}$ and $\ell'_{c,n}$ the mass coupled this way up to time n, and let f be a Hölder function. We have that $\left|\int f dT_*^m\ell_{c,n} - \int f dT_*^m\ell'_{c,n}\right|$ shrinks exponentially as m grows.

Exponential decay of correlations and other statistical properties of dispersing billiard maps were proved in [Che06] by the above described coupling method. Furthermore, given its flexibility, coupling has turned out to be especially useful in various non-stationary situations ([CD09], [SYZ13]).

For further reference it should be noted that all of the above discussed methods (either coupling or spectral, performed either directly on the phase space, or on the Young tower) are powerful enough to extend, in the case of Sinai billiards, the exponential bounds to multiple correlations, see eg.[CM06, section 7.7]. By multiple correlations we mean

$$\begin{aligned}
&Corr(\tilde{f}, \tilde{g}; n) \qquad \text{where} \\
&\tilde{f} = f_1 \cdot f_2 \circ T^{-1} \cdots f_k \circ T^{-k+1}, \quad \tilde{g} = g_1 \cdot g_2 \circ T \cdots g_k \circ T^{k-1} \quad \text{for some} \\
&k \geq 1 \quad \text{and the functions } f_i, g_i, i = 1 \dots k \quad \text{are Hölder continuous.}
\end{aligned} \tag{2.8}$$

On the other hand, combinations of the coupling and the spectral methods may provide further strong results as eg. in [DSL18].

We close this section by mentioning three methods that can be used for proving the Central Limit Theorem (CLT) in dispersing billiards.

- A popular method for proving the CLT in dynamical systems is to construct a *martingale approximation* for $S_n f$, following the strategy of [Gor69]. This approach was applied in a variety of contexts, see eg. [Liv96], [KV86], [You98], and proved to be very powerful when the rates of correlation decay are sumable. Nevertheless, other methods turned out to extend better to cases with nonstandard scaling, which is the main interest of this dissertation.

- The *perturbed transfer operator* or, as often called, Nagaev-Guivarc'h method was first used for Markov chains in [Nag57] and then extended to hyperbolic dynamical systems, see [GH88]. A key ingredient is the transfer operator P, acting on one of the above mentioned Banach spaces $\mathcal{B}$, with a fixed point corresponding to the natural invariant measure μ, and a spectral gap. If the aim is to prove the CLT for the Birkhoff sums of an observable

$f : M \to \mathbb{R}$, then the perturbed transfer operator is defined as $P_t(\phi) = P(e^{itf}\phi)$, $\phi \in \mathcal{B}$, for real parameters t, with $|t| < \varepsilon$ for some $\varepsilon > 0$. The central quantity is $\lambda(t)$, the leading eigenvalue of P_t: the behaviour of the characteristic function $\mathbb{E}_\mu(e^{itS_nf})$ is determined by $\lambda(t)^n$. Thus if the dependence of $\lambda(t)$ on t is sufficiently smooth, then the characteristic function of f can be estimated, as in the classical proof of the CLT for the i.i.d. case. The method extends to a wide range of further probabilistic limit phenomena that can be treated by characteristic functions, including local limit laws, Berry-Esseen type bounds, almost sure invariance principles and also to nonstandard situations, just to mention a few. We refer to [Gou15] for an excellent review on this approach. See also chapters 3 and 5 of the present dissertation.

- Another fruitful approach, applied in particular to dispersing billiards in [BSC91] (for a more recent exposition, see [CM06, section 7.8]), is to estimate the characteristic function $\mathbb{E}_\mu(e^{itS_nf})$ directly, without any reference to spectral techniques. In particular, following ideas dating back to Bernstein, the terms in S_nf can be split into alternating big and small blocks of size n^p and n^q, respectively, where $0 < p < q < 1$. The method relies on fast decay of multiple correlations (cf. (2.8)), based on which the big blocks can be treated as if they were independent. Classical tools from probability theory (such as moment bounds or Chebyshev's inequality) then can be used to show that the small blocks are negligible, and to handle the contribution of the big blocks. See [CM06, section 7.8] for details, [Che06] for further limit laws and chapter 4 of this dissertation for an extension to a situation with nonstandard scaling.

2.2.2 The billiard flow

A standard approach to the billiard flow is to represent it as a suspension flow, that is, consider

$$M^\tau = \{(x,s) \in M \times \mathbb{R} \,|\, 0 \le s \le \tau(x)\} \,/\, \sim; \quad (x,\tau(x)) \sim (Tx, 0)$$
$$\Phi^t : M^\tau \to M^\tau; \qquad \Phi^t(x,s) = (x, s+t) \text{ subject to identifications}$$
$$\mu^\tau = (\bar\tau)^{-1}\,(\mu \times Leb)\,; \qquad \text{invariant with respect to } \Phi^t \qquad (\text{where } \bar\tau = \int_M \tau d\mu).$$

Given $F : M^\tau \to \mathbb{R}$ (piecewise) Hölder continuous, let

$$\bar F = \int_{M^\tau} F d\mu^\tau;$$
$$S_tF(y) = \int_0^t F(\Phi^u y) du \quad \text{for } t > 0 \text{ and } y \in M^\tau;$$
$$f(x) = \int_0^{\tau(x)} F(x,u) du \quad \text{for } x \in M.$$

Assume $\bar F = 0$, which then implies $\bar f = \int_M f d\mu = 0$. Ergodicity of μ^τ with respect to Φ^t follows immediately from the ergodicity of μ with respect to T, hence $S_tF(y)/t \to 0$ as $t \to \infty$, for μ^τ almost every y, by the Birkhoff ergodic theorem. Furthermore, the central limit theorem for the suspension flow is known to follow from the central limit theorem for the base transformation under fairly general conditions – see [MT04]. In particular

$$\text{If } \frac{S_nf}{\sqrt{n}} \overset{\mathcal{D}}{\Longrightarrow} \mathcal{N}(0,\sigma^2) \text{ for some } \sigma^2 \ge 0 \quad \text{and} \quad \frac{S_n\tau - n\bar\tau}{\sqrt{n}} \overset{\mathcal{D}}{\Longrightarrow} \mathcal{N}(0,D_\tau) \text{ for some } D_\tau \ge 0$$
$$\text{then} \quad \frac{S_tF}{\sqrt{t}} \overset{\mathcal{D}}{\Longrightarrow} \mathcal{N}(0,(\bar\tau)^{-1}\sigma^2) \quad (\text{where } \overset{\mathcal{D}}{\Longrightarrow} \text{ denotes convergence in distribution}).$$

This applies in particular to *finite horizon* dispersing billiard flows, the CLT for which was actually established already in [BS81b]. (For a discussion of the case of infinite horizon, see chapter 5.) Some further statistical limit laws similarly extend from the base transformation to the suspension flow, see [MT04]. Mixing and bounds on the rates of correlation decay for the flow are, however, more delicate and do not follow from such general considerations. This is related to the flow direction, which is an additional dimension of the phase space M^τ (when compared to M), along which the Lyapunov exponent is necessarily zero, no matter how strongly hyperbolic the base transformation $T : M \to M$ is. For a counterexample to mixing, just consider a suspension flow with *constant* roof function. Hence, to obtain mixing of the flow, some kind of *non-integrability of the roof function* is needed.

Even for smooth uniformly hyperbolic flows – such as, in increasing order of generality, geodesic flows on compact Riemannian surfaces of negative curvature with respect to volume; Anosov flows, and Axiom A flows with respect to some equilibrium measure – progress related to the mixing rates was much slower than for their discrete time analogues. In particular, there exist examples of Axiom A flows with arbitrary slow rates of correlation decay ([Pol85]), while it is still an open question if all mixing Anosov flows mix exponentially. See [Mel18] for a recent survey.

The case of geodesic flows on surfaces with negative curvature is, by now, much better understood as there is a geometric mechanism to produce non-integrability in the flow direction, and in that regard, dispersing billiards behave similarly. The mechanism is related to the preservation of a *contact structure*, which ensures that the strong stable and unstable manifolds[6] of the flow correspond to strictly converging and strictly diverging wavefronts, respectively (see section 2.1). This implies that, for $y = (x, u) \in M^\tau$, moving along a loop of stable $\to$ unstable $\to$ stable $\to$ unstable curves of x, when lifted to the respective strong unstable and stable manifolds of y, small advances along the flow direction can be obtained, see [CM06, sections 6.9–6.11] for details. It was this mechanism based on which Sinai proved mixing (actually, the stronger K property) for dispersing billiard flows in his seminal paper [Sin70].

Effective bounds on the rates of mixing for hyperbolic flows, however, turned out to be much more difficult to handle. For that purpose, it is important how the size of the above mentioned advance along the flow direction depends on the size of the loop of stable-unstable-stable-unstable curves, quantified in terms of a *temporal distance function*. A breakthrough was achieved by Chernov in [Che98] where he proved a stretched exponential bound on the decay of correlations (cf. (2.6)) for a substantial class of Anosov flows, including geodesic flows on smooth and compact surfaces of negative curvature. He extended this result to dispersing billiard flows with finite horizon in [Che07]. The method of proof used in Chernov's works was approximating the billiard flow by Markov chains with countably many states.

Combining Chernov's geometric ideas with spectral techniques, Dolgopyat proved exponential decay of correlations for a class of Anosov flows (including again geodesic flows on surfaces of negative curvature) in the major work [Dol98a]. This paper exploited a strong form of non-integrability in the flow direction – since then often called uniform non-integrability (UNI) – which essentially means a good control of the temporal distance function. There are many further developments based on this paper, let us mention in particular [Liv04] which, using the anisotropic spaces of [BKL02], extended Dolgopyat's result on exponential mixing rates to higher dimensional contact Anosov flows. As for dispersing billiard flows with finite horizon and no corner points, it was [BDL18] which solved the longstanding conjecture on exponential decay of correlations, implementing Dolgopyat's approach in the context of the anisotropic Banach spaces introduced in [DZ11].

Dolgopyat, actually, developed Chernov's work also in another direction. In [Dol98b] he intro-

[6] The strong stable manifold of $y \in \mathcal{M}$ is the curve $W^s(y)$ with the property that for any $y' \in W^s(y)$ we have $d(\Phi^t y, \Phi^t y') \to 0$ exponentially as $t \to +\infty$. Strong unstable manifolds are defined analogously.

duced the notion of rapid mixing (or superpolynomial decay of correlations) which means that, for any $q > 0$, for F and G Hölder continuous and sufficiently regular in the flow direction (with the amount of regularity depending on q), $Corr(F, G; t) = O(t^{-q})$ holds. In [Dol98b] then he used a weak form of non-integrability – essentially, it is enough to have two periodic points for the flow such that the ratio of their periods is Diophantine – to prove rapid mixing for a "typical" class of Axiom A flows. In contrast to [Dol98a], the method of [Dol98b] turned out to combine well with the Young towers constructed for dispersing billiards. In [Mel07] Melbourne proved rapid mixing for flows that can be represented as sufficiently regular suspensions of Young towers with exponential tails. This class includes in particular dispersing billiards with finite horizon. Although this result of [Mel07] on rapid mixing is suboptimal – in finite horizon dispersing billiard flows correlations are known to decay exponentially by [BDL18] – it is remarkable that it was generalized to other classes of billiards (see eg. Theorem G) to which [BDL18] has not been extended. Furthermore, using truncation techniques, this approach was adapted to treat some examples with polynomial decay rates, too. See [Mel09], [Mel18], [BM08], [BBM19] and section 5.3.3 of this dissertation on further details.

2.3 Outlook: multidimensional dispersing billiards

This dissertation focuses on two dimensional (or planar) hyperbolic billiard models. Multidimensional models are more complicated and less understood. For the sake of comparison, here we summarize some results – in part obtained by the author – on multidimensional dispersing billiards.

Sinai's work, [Sin70], on the hyperbolic and ergodic properties of dispersing billiards was extended to the multidimensional case by Chernov and Sinai in [SC87]. An important direction in which the theory was further developed is the proof of the Boltzmann-Sinai ergodic hypothesis, that is, the ergodicity of hard ball systems on the Euclidean torus, with research spanning over several decades, and completed by Simányi in [Sim09], [Sim13]; see also [Sim19] for a survey.

Let us turn back to multidimensional dispersing billiards, which also have strong motivation originating from physics, for instance associated to the higher dimensional versions of Lorentz gases, see chapter 5. Although – in contrast to the semi-dispersing billiards corresponding to hard ball systems – these models have uniform expansion rates, their geometric properties provide significant additional challenge related to the planar case, specially when finer statistical properties are considered. A systematic study was initiated in the papers [BCS02] and [BCS03], which in particular led to the discovery of pathological behaviour (the lack of curvature bounds) in the local geometry of the singularity manifolds. A further important advance was achieved in [BT08] where we managed to handle these phenomena and proved, by constructing a Young tower, the exponential decay of correlations (and the CLT) for multidimensional dispersing billiard maps with finite horizon; however, subject to a combinatorial condition, the sub-exponential (or slightly more generally, the sub-expansion) complexity of the singularity set. Although believed to hold generically, the status of the complexity condition is still unclear; actually, an example with exponentially growing complexity was constructed in [BT12].

Chapter 3

The Bunimovich stadium

3.1 Description

Let $\ell > 0$, and consider a region D in the plane delimited by two semicircles of unit radius, connected by two parallel segments of length ℓ. The billiard dynamics inside this domain is called the Bunimovich stadium billiard, see Figure 3.1. Although at first sight it seems quite different from the dispersing billiard examples discussed in the previous chapter, it shares some important properties with them. In particular, Bunimovich proved ([Bun90]) that the billiard map in the stadium is ergodic and hyperbolic. The mechanism responsible for this behaviour is described below, along with some further geometric features which characterize the dynamics of Bunimovich stadia.

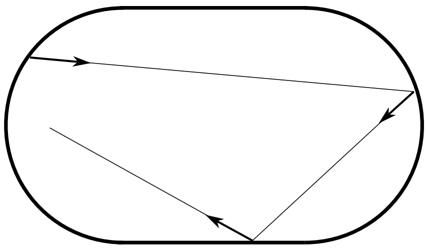


Figure 3.1: The Bunimovich stadium

We use the billiard map coordinates (r, φ) introduced earlier, the phase space M is thus a single cylinder. More precisely, the periodic position coordinate $r \in \mathbb{R}/(2\pi + 2\ell)\mathbb{Z}$ is arclength along the (inner) boundary of D, measured counterclockwise, with $r = 0$ corresponding to the lower endpoint of the right semicircle, while, as usual, $\varphi \in [-\pi/2, \pi/2]$ is the angle the post-reflection velocity makes with the (inward pointing) normal vector of the boundary at the point r. The billiard map is denoted by $T : M \to M$, and the Liouville measure μ this time takes the form

$$d\mu = \frac{\cos\varphi \, d\varphi \, dr}{2(2\pi + 2\ell)}.$$

The singularities of T correspond to corner points, where the semicircles and the straight segments meet. Accordingly, letting

$$\mathcal{S}_0 = (\{0\} \cup \{\pi\} \cup \{\pi + \ell\} \cup \{2\pi + \ell\}) \times [-\pi/2, \pi/2],$$

we have that although T is continuous, its derivative is discontinuous at $\mathcal{S}_0 \cup T^{-1}\mathcal{S}_0$.

Let us also note that, as usual, the billiard flow $S^t : \mathcal{M} \to \mathcal{M}$ in the stadium can be regarded as a suspension flow with base transformation $T : M \to M$ and roof function $\tau : M \to \mathbb{R}^+$, where $\tau(x)$ is the free flight function. For the flow, we will use the notations of section 2.2.2.

Defocusing mechanism. As discussed in the previous chapter, in dispersing billiards it is the scattering effect of the boundary that is responsible for hyperbolicity, that is, the invariance and the expansion of the cone field associated to dispersing wavefronts. In contrast, the boundary components of the stadium are either flat (the segments) or focusing (the semicircles), thus a different cone field is needed. On the semicircles, define

$$\text{for } x = (r, \varphi) \text{ with } 0 < r < \pi \text{ or } \pi + \ell < r < 2\pi + \ell, \text{let}$$
$$C_x^u = \left\{ (dr, d\varphi) \in \mathcal{T}_x M \mid -1 \leq \frac{dr}{d\varphi} \leq 0 \right\}.$$

Unstable curves, that is, curves with tangent vectors in C_x^u correspond to converging wavefronts; actually, these wavefronts are designed to converge strongly enough to ensure that, when evolved by the free flight, they defocus and continue as dispersing wavefronts before reaching the next collision – see Figure 3.2 for an illustration. Moreover, if x is on one of the semicircles, and Tx is on the other semicircle, the flight time after defocusing exceeds the flight time before defocusing. This ensures the expansion of the length of the wavefront in course of such transitions. Finally, upon collision on the semicircular arc, the arriving dispersing wavefront bounces off as a converging wavefront. This ensures that the above cone field is preserved.

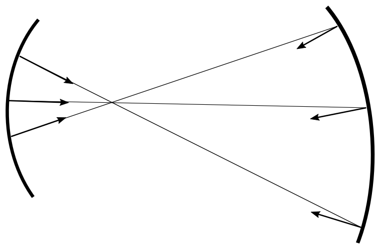


Figure 3.2: The defocusing mechanism

The cone field is extended to the flat sides as follows.

$$\text{For } x = (r, \varphi) \text{ with } \pi < r < \pi + \ell \text{ or } 2\pi + \ell < r < 2\pi + 2\ell, \text{let}$$
$$C_x^u = \left\{ (dr, d\varphi) \in \mathcal{T}_x M \mid 0 \leq \frac{dr}{d\varphi} \right\},$$

which simply means that unstable curves give rise to dispersing wavefronts when bouncing off the flat sides. With these definitions, the defocusing mechanism ensures the invariance of the cone field: for $x \in M \setminus (\mathcal{S}_0 \cup T^{-1}\mathcal{S}_0)$

$$DT_x C_x^u \subset C_{Tx}^u,$$

as in dispersing billiards. However, the uniform expansion can be only ensured at transitions from one semicircle to the other semicircle. This motivates the definition of the induced phase space.

Induced phase space. Let

$$\begin{aligned} \hat{M} &= \{x \in M \mid x \text{ belongs to a semicircle, but } T^{-1}x \text{ does not belong to that semicircle}\} \\ &= \bigcup_{r \in (0,\pi)} \{r\} \times (-r/2, \pi/2 - r/2) \cup \bigcup_{r \in (0,\pi)} \{r + \pi + \ell\} \times (-r/2, \pi/2 - r/2), \end{aligned}$$

which has the shape of two parallelograms in M. See Figure 3.3.

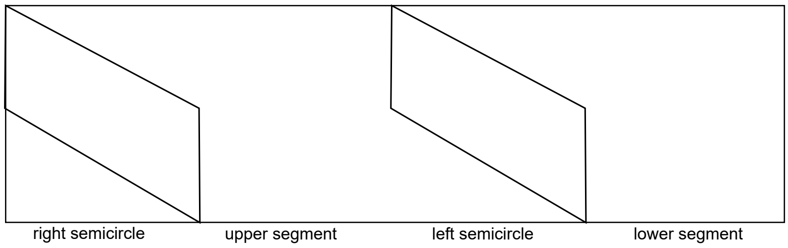


Figure 3.3: The phase space M and the induced phase space $\hat{M}$.

Let us also introduce

$$\begin{array}{rll} \text{the first return time} & r_{\hat{M}} : \hat{M} \to \mathbb{Z}^+; & r_{\hat{M}}(x) = \min\{k \geq 1 \,|\, T^k x \in \hat{M}\}; \\ \text{and the first return map} & \hat{T} : \hat{M} \to \hat{M}; & \hat{T}(x) = T^{r_{\hat{M}}(x)} x. \end{array}$$

Note that $r_{\hat{M}}$ is unbounded, and thus the singularity set of $\hat{T}$,

$$\mathcal{S}_1 = \mathcal{S}_0 \cup \hat{T}^{-1} \mathcal{S}_0$$

consists of countably infinitely many smooth curves. The connected components of $\hat{M} \setminus \mathcal{S}_1$, that is, the sets where the return time $r_{\hat{M}}$ takes some constant value, are often called cells. Cells accumulate at the corners of the parallelograms, and play an important role in the analysis of the stadium billiard, see below for further details. We also let, for $n \geq 1$, $\mathcal{S}_n = \mathcal{S}_0 \cup \hat{T}^{-1} \mathcal{S}_0 \cup \dots \hat{T}^{-n+1} \mathcal{S}_0$, the discontinuity set for $\hat{T}^n$.

The map $\hat{T} : \hat{M} \to \hat{M}$ is uniformly hyperbolic in the sense that for $x \in \hat{M} \setminus \mathcal{S}_1$

$$D\hat{T}_x C^u_x \subset C^u_{\hat{T}x}$$

and there exist $c > 0$ and $\Lambda > 1$ such that:

$$|D\hat{T}^n_x dx| \geq c\Lambda^n |dx| \quad \text{for every } dx \in C^u_x, x \in \hat{M} \setminus \mathcal{S}_n.$$

The Liouville measure μ induces a $\hat{T}$-invariant, absolutely continuous probability measure $\hat{\mu}$ on $\hat{M}$. By direct computation

$$\mu(\hat{M}) = 2 \int_{r=0}^{\pi} \int_{\varphi=-r/2}^{\pi/2-r/2} \frac{\cos\varphi \,\mathrm{d}r \,\mathrm{d}\varphi}{4\pi + 4\ell} = \frac{2}{\pi + \ell}$$

and thus

$$\mathrm{d}\hat{\mu} = \frac{\cos\varphi \,\mathrm{d}r \,\mathrm{d}\varphi}{8}.$$

Intermittency. The strategy for describing the statistical properties of the map (M, T, μ) is to

- investigate the induced map $(\hat{M}, \hat{T}, \hat{\mu})$. It turns out that in most regards, it resembles the dispersing billiard maps of Chapter 2.
- study the distribution of the return time function $r_{\hat{M}} : \hat{M} \to \mathbb{R}^+$. By the previous bullet, it is essentially the tail of this distribution that determines the statistical behaviour of (M, T, μ).

Both of these ingredients rely on the analysis of points in $\hat{M}$ with high values of $r_{\hat{M}}$; that is, from which long trajectories originate with a high number of consecutive hits in $M \setminus \hat{M}$. In the stadium, such trajectory sections belong to one of two types: series of *bouncing* almost perpendicularly on the parallel segments, and series of *sliding* almost tangentially along one of the semicircles, see Figure 3.4.

(a) Bouncing (b) Sliding

Figure 3.4: Intermittent behaviour in the stadium

Accordingly, there exists some $n_0 \geq 1$ such that for $n \geq n_0$ we have

$$\{x \in \hat{M} \,|\, r_{\hat{M}}(x) = n\} = B_n \cup D_n$$

where B_n is the set of points from which bouncing series originate, while D_n is the set of points from which sliding series originate. A more detailed description of these sets is given in section 3.3. Here we mention that the B_n and the D_n accumulate at the obtuse vertices and the acute vertices of the parallelogram $\hat{M}$, respectively, and that we have the following rough estimates on their measures:

$$\hat{\mu}(B_n) \asymp \frac{1}{n^3}; \qquad \hat{\mu}(D_n) \asymp \frac{1}{n^4}. \tag{3.1}$$

Hence, the bouncing trajectories dominate the tail asymptotics of the return time $r_{\hat{M}}$, while the role of the sliding trajectories is less significant. The lower magnitude of the measure of the sliding sets D_n (compared to the bouncing sets B_n) is related to the density $\cos\varphi$ of the invariant measure μ.

The asymptotic of $\hat{\mu}(B_n)$ from (3.1) implies that

$$\hat{\mu}(x \in \hat{M} \,|\, r_{\hat{M}}(x) \geq n) \asymp \frac{1}{n^2}. \tag{3.2}$$

This means that $r_{\hat{M}}$ does not belong to $L^2_{\hat{\mu}}(\hat{M})$, but it belongs to $L^{2-\varepsilon}_{\hat{\mu}}(\hat{M})$ for any $\varepsilon > 0$. It also indicates that $r_{\hat{M}}$, as a random variable on the probability space $(\hat{M}, \hat{\mu})$, belongs to the non-standard domain of attraction of the normal law.

3.2 Results

Preceding results. The Bunimovich stadium is one of the most popular billiard models with a very extensive literature, here we only list some important results that preceded our paper [BG06].

The mathematical investigation of the stadium was initiated by Bunimovich in [Bun79], with more detailed arguments in [Bun90], proving in particular the ergodicity of the map $T : M \to M$ (and the flow $S^t : \mathcal{M} \to \mathcal{M}$).

Concerning decay of correlations, a major step was achieved first in [Mar04], and then, with a slightly different strategy in [CZ05]. These constructed a *Young tower with exponential tails* for the *induced map* $(\hat{M}, \hat{T}, \hat{\mu})$. This, together with the tail estimates on the distributions of $r_{\hat{M}}$, provided a Young tower with *polynomial tails* for the *original map* (M, T, μ). As for bounding the rate of correlation decay, an additional challenge is to relate the tails of R (the return time to the base of the tower) to the tails of $r_{\hat{M}}$ (the return time to the induced phase space $\hat{M}$). Given $f, g : M \to \mathbb{R}$ Hölder continuous, a first rough bound $Corr(f, g; n) \ll \frac{\log^2 n}{n}$ was obtained in [Mar04] and [CZ05], which was improved to $Corr(f, g; n) \ll \frac{1}{n}$ in [CZ08]. We comment on this improvement below in section 3.3.5.

Main results of Chapter 3. As before, let us consider $f : M \to \mathbb{R}$ Hölder continuous. By the ergodicity (M, T, μ), we have $\frac{\sum_{k=0}^{n-1} f(T^k x)}{n} \to \int f \, \mathrm{d}\mu$ for μ-a.e. $x \in M$. Let us introduce

$$I_f = I = \frac{1}{2\ell} \left[\int_{r \in [\pi, \pi+\ell]} f(r, 0) \, \mathrm{d}r + \int_{r \in [2\pi+\ell, 2\pi+2\ell]} f(r, 0) \, \mathrm{d}r \right]. \tag{3.3}$$

This is the average of f along the trajectories bouncing perpendicularly to the straight segments of the stadium; which is, in a sense, the set of "infinite bouncing".

Theorem A. *Let $f : M \to \mathbb{R}$ be Hölder continuous, satisfying $\int f \, \mathrm{d}\mu = 0$ and $I_f \neq 0$. Then*

$$\frac{\sum_{k=0}^{n-1} f \circ T^k}{\sqrt{cn \log n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1),$$

where

$$c = \frac{4 + 3 \log 3}{4 - 3 \log 3} \cdot \frac{\ell^2 I^2}{4(\pi + \ell)}.$$

In very rough terms, Theorem A shows that if $I \neq 0$, then the Birkhoff sums of f are determined by the tail behaviour of $r_{\hat{M}}$, and thus follow a corresponding nonstandard limit theorem. Nonetheless, the exact form of the scaling and the asymptotic variance requires further explanation, which is given in section 3.3.

The following direct consequence of Theorem A shows that the bound $1/n$ on the correlations is optimal.

Remark 3.1. *Under the assumptions of Theorem A, $Corr(f, f; n) \neq o(1/n)$, that is, the quantity $n \int f \cdot f \circ T^n$ does not tend to zero. Indeed, we have*

$$\int \left[\sum_{k=0}^{n-1} f \circ T^k \right]^2 = n \int f^2 + 2 \sum_{i=1}^{n-1} (n - i) \int f \cdot f \circ T^i.$$

Now we argue by contradiction: assume $\int f \cdot f \circ T^i = o(1/i)$, it follows that $\int \left[\sum_{k=0}^{n-1} f \circ T^k \right]^2 = o(n \log n)$. In particular, the variance of the random variable $\frac{\sum_{k=0}^{n-1} f \circ T^k}{\sqrt{n \log n}}$ tends to zero. This implies that this random variable tends to zero in probability, which is in contradiction with Theorem A.

On the other hand, if $I = 0$, then the Birkhoff sums of f satisfy a standard Central Limit Theorem:

Theorem B. *Let $f : M \to \mathbb{R}$ be Hölder continuous, satisfying $\int f \, \mathrm{d}\mu = 0$ and $I_f = 0$. Then there exists $(\sigma_f^2 =)\sigma^2 \geq 0$ such that*

$$\frac{\sum_{k=0}^{n-1} f \circ T^k}{\sqrt{n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2).$$

Remark 3.2. *In Theorem B, $\sigma^2 = 0$ if and only if f is a coboundary, that is, there exists some measurable function $\chi : M \to \mathbb{R}$ such that $f = \chi - \chi \circ T$ almost surely. Also, in contrast to the simple geometric description for I_f, in the case of Theorem B, σ_f^2 depends on more complicated autocorrelations of f, see (1.2).*

To proceed to the billiard flow, we consider the Hölder observable $\tau : M \to \mathbb{R}$, the free path length. Its average, the mean free path $\bar{\tau}$ can be computed by the Formula [CM06, (2.32)]:

$$\bar{\tau} = \frac{\pi|D|}{|\partial D|} = \frac{\pi(\pi + 2\ell)}{2\ell + 2\pi},$$

where $|D|$ and $|\partial D|$ denote the area and the perimeter of the stadium, respectively. On the other hand we have $I_\tau = 2$, hence for the Birkhoff sums of the centralized $\tau_0(x) = \tau(x) - \bar{\tau}$ the standard limit theorem, Theorem B applies in the special case $I_{\tau_0} = 0 \Leftrightarrow \bar{\tau} = 2 \Leftrightarrow \ell = \frac{4\pi - \pi^2}{2\pi - 4} \approx 1.18$, and the nonstandard limit theorem, Theorem A applies for any other value of the parameter ℓ.

As the billiard flow can be regarded as suspension flow $(M^\tau, \Phi^t, \mu^\tau)$ with base transformation (M, T, μ) and roof function τ (see section 2.2.2), several limit theorems on the flow follow from those on the map. For $F : M^\tau \to \mathbb{R}$ Hölder continuous and centered (ie. $\int_{M^\tau} F \, d\mu^\tau = 0$), let

$$S_T F(y) = \int_0^T F(\Phi^t y) \, dt, \text{ where } y = (x, s) = (r, \varphi, s) \in M^\tau;$$

$$J_F = \frac{1}{2\ell} \left[\int_{r \in [\pi, \pi+\ell] \cup [2\pi+\ell, 2\pi+2\ell]} \int_{t \in [0,2]} F(r, 0, t) \, dt \, dr \right].$$

Corollary 3.3. *The following limit theorems for the billiard flow can be derived from Theorems A and B.*

1. *If $J_F \neq 0$, then*
$$\frac{S_T F}{\sqrt{\frac{c}{\bar{\tau}} T \log T}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1).$$
Here c is the constant from Theorem A, with I replaced by J_F.

2. *If $J_F = 0$, then*
$$\frac{S_T F}{\sqrt{T}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma_F^2)$$
for some $\sigma_F^2 \geq 0$.

As for bounds on the rate of correlation decay for the flow, a first comment is that, similarly to the map case discussed in Remark 3.1, a direct consequence of the limit law of Corollary 3.3 is that if $J_F \neq 0$, then $Corr(F, F; t) \neq o(1/t)$. Theorem C below states, at least for observables smooth along the flow direction, an optimal upper bound $O(1/t)$.

Definition 3.4. *We will say that $F : M^\tau \to \mathbb{R}$ is differentiable in the flow direction if for any $y \in M^\tau$ $F(\Phi^t y)$ is differentiable as a function of t. In this case we define $(\partial_t F)(y)$ as the derivative of $F(\Phi^t y)$ at $t = 0$. Higher order differentiability and $(\partial_t^k F)$ for $k \geq 1$ are defined analogously. Given $\eta \in (0, 1]$ and $m \geq 1$, the notation $F \in C^{m,\eta}(M^\tau)$ means that F is m times differentiable in the flow direction and the functions $F, \partial F, \ldots, \partial^m F$ are all Hölder continuous (as functions on M^τ) with exponent η. Finally, $F : M^\tau \to \mathbb{R}$ is smooth along the flow direction if there exists $\eta \in (0, 1]$ such that $F \in C^{m,\eta}(M^\tau)$ for any $m \geq 1$.*

Theorem C. *Assume $F, G : M^\tau \to \mathbb{R}$ are smooth along the flow direction. Then*

$$Corr(F, G; t) \ll \frac{1}{t}.$$

Theorems A and B are proved in [BG06], while Theorem C is proved in [BBM19]. For detailed arguments, we refer to these papers. Nonetheless, we present some important ideas from the proofs of the limit theorems, focusing mostly on the non-standard case of Theorem A in section 3.3. A similar discussion of Theorem C is given in section 5.3.3 along with correlation bounds on other billiard flows, in particular the infinite horizon Lorentz gas.

3.3 Ingredients of proofs

The main goal of this section is to describe the phenomena related to Theorem A. Let us fix some $f : M \to \mathbb{R}$, Hölder continuous with exponent $\eta \in (0, 1]$, centered ($\int_M f \, d\mu = 0$), with $I = I_f \neq 0$. Let us denote the Birkhoff sum as $S_n f(x) = \sum_{k=0}^{n-1} f(T^k x)$, for $x \in M$ and $n \geq 1$.

3.3.1 Induced limit theorem

Let us define the induced observable

$$\hat{f} : \hat{M} \to \mathbb{R}; \qquad \hat{f}(\hat{x}) = \sum_{k=0}^{r_{\hat{M}}(\hat{x})-1} f(T^k \hat{x}), \text{ for } \hat{x} \in \hat{M}$$

and its Birkhoff sum with respect to the induced map

$$\hat{S}_n \hat{f}(\hat{x}) = \hat{f}(\hat{x}) + \hat{f}(\hat{T}\hat{x}) + \cdots + \hat{f}(\hat{T}^{n-1}\hat{x}), \quad \text{for } \hat{x} \in \hat{M}.$$

Theorem A can be reduced to the following induced limit theorem.

Theorem 3.5. *Consider $f : M \to \mathbb{R}$ as in Theorem A, and let $\hat{f}$ denote the associated induced observable. Then*

$$\frac{\hat{S}_n \hat{f}}{\sqrt{\hat{c} n \log n}} \stackrel{\mathcal{D}}{\Longrightarrow} \mathcal{N}(0, 1),$$

where

$$\hat{c} = c \cdot (\mu(\hat{M}))^{-1} = \frac{4 + 3\log 3}{4 - 3\log 3} \cdot \frac{\ell^2 I^2}{8}.$$

To see that Theorem 3.5 implies Theorem A, note that, as (M, T, μ) is ergodic, the Kac lemma (see eg. [Pet83, p. 46]) applies stating $\int r_{\hat{M}} \, d\hat{\mu} = (\mu(\hat{M}))^{-1}$. Hence, on the average, a single iteration of $\hat{T}$ corresponds to $(\mu(\hat{M}))^{-1}$ iterations of T, and thus $S_n f \approx \hat{S}_{n\mu(\hat{M})} \hat{f}$, which is responsible for the scaling factor in the asymptotic variance. See [Gou07, Appendix A] or [BCD11, Section 2] for further details.

3.3.2 The tail of the distribution of $\hat{f}$

The natural approach to Theorem 3.5 is as follows. The system $(\hat{M}, \hat{T}, \hat{\mu})$ has strong statistical properties, in particular, it can be modeled by a Young tower with exponential tails, which indicates a quick loss of memory. Hence, naively, it could be expected that the stationary sequence of random variables $\hat{f}, \hat{f} \circ \hat{T}, ..., \hat{f} \circ \hat{T}^{n-1}$... behaves as an independent sequence. This heuristics is almost appropriate, yet, as argued below, there is an additional effect to be taken into account. In any case, to prove Theorem 3.5, a key ingredient is the description of the tail of the random variable $\hat{f}$, as expressed by Proposition 3.6 below. Recall the $\asymp$ and the $\sim$ notations from section 1.3.

Proposition 3.6.

$$\hat{\mu}(\hat{x} \in \hat{M} \mid |\hat{f}(\hat{x})| \geq n) \sim \sum_{k=n/|I|}^{\infty} \frac{\ell^2}{4k^3} \sim \frac{I^2\ell^2}{8n^2} \quad \textit{as } n \to \infty$$

or equivalently

$$\int_{|\hat{f}|\leq K} (\hat{f})^2 \, \mathrm{d}\hat{\mu} \sim \frac{I^2\ell^2}{4} \log K \quad \textit{as } K \to \infty.$$

The aim of this section is to clarify the following two issues:

- Proposition 3.6 requires more precise estimates than the rough bounds of (3.1).
- If we had an i.i.d. sequence $X_1, X_2, \dots$ of centered random variables with tail behaviour given by Proposition 3.6, then, by classical results from probability theory on the non-standard domain of attraction of the normal law (see, for example [Fel57, section XVII.5]), we would have
$$\frac{X_1 + \cdots + X_n}{\sqrt{n \log n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, D) \qquad \text{with } D = \frac{\ell^2 I^2}{8}$$
so we still need to argue for the additional factor $\frac{4+3\log 3}{4-3\log 3}$ in Theorem 3.5.

Recall that $r_{\hat{M}} = n$ for large n occurs on two types of orbits: sliding points ($\hat{x} \in D_n$) and bouncing points ($\hat{x} \in B_n$). As f is bounded, we have $|\hat{f}||_{B_n \cup D_n} \ll n$. Hence, by the rough tail estimate (3.1), *sliding* points make a lower order (actually, L^2) contribution to the tail of $\hat{f}$. Hence Proposition 3.6 follows from the two Lemmas below:

Lemma 3.7.

$$\hat{f}(\hat{x}) \sim nI_f; \quad \textit{for } \hat{x} \in B_n.$$

Lemma 3.8.

$$\hat{\mu}(B_n) \sim \frac{\ell^2}{4n^3}.$$

To prove Lemma 3.7, recall that the trajectory starting from $\hat{x} \in B_n$ follows the pattern on Figure 3.4a. Hence $\hat{f}(\hat{x}) = \sum_{i=1}^{n} f(x_i) + O(1)$ with $x_i = (r_i, \varphi_i)$, where

- $\varphi_i \ll \frac{1}{n}$,
- letting $m = \lceil \frac{n}{2} \rceil$, we have

$$\begin{aligned}
&\pi \leq r_1 < r_3 < \cdots < r_{2m-1} \leq \pi + \ell;\\
&\pi + 2\ell \leq r_2 < r_4 < \cdots < r_{2m} \leq 2\pi + 2\ell;\\
&\text{and } |r_i - r_{i-2}| \sim \frac{2\ell}{n}; \quad i = 3, \dots, n.
\end{aligned}$$

Hence, by the Hölder continuity of f, the value of $\hat{f}(\hat{x})$ can be approximated by a Riemannian integral

$$\hat{f}(\hat{x}) = \sum_{i=1}^{n} f(r_i, 0) + o(n) = \left(\sum_{i=1}^{n} f(r_i, 0) \cdot \frac{2\ell}{n} \right) \frac{n}{2\ell} + o(n) =$$
$$= \frac{n}{2\ell} \left(\int_{\pi}^{\pi+\ell} f(r, 0)\, \mathrm{d}r + \int_{2\pi+\ell}^{2\pi+2\ell} f(r, 0)\, \mathrm{d}r \right) + o(n) = nI + o(n)$$

which completes the proof of Lemma 3.7.

We proceed to Lemma 3.8. First note that, to handle bouncing orbits, it is useful to apply the unfolding trick: when bouncing on the straight segments, reflect, instead of the trajectory, the billiard domain; this way the trajectories can be regarded as straight lines. It is also better for visualization purposes to rotate the picture by 90 degrees, so that trajectories of long bouncing series correspond to almost horizontal lines. See Figure 3.5.

Instead of B_n, we prefer to describe the slightly simpler geometry of $\hat{T}B_n$. (By invariance of the measure, we have $\hat{\mu}(B_n) = \hat{\mu}(\hat{T}B_n)$). The cells $\hat{T}B_n$ accumulate at one of the obtuse vertices of the parallelogram as n increases, their geometry is depicted on Figure 3.7a.

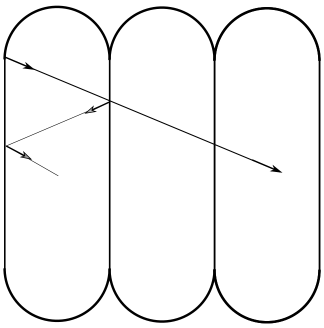


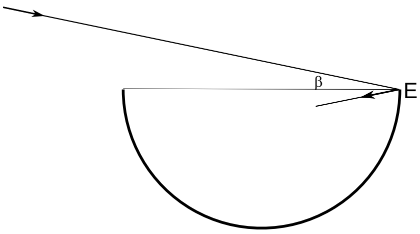
Figure 3.5: The unfolding trick.

To identify the corners of this quadrangular cell, note that $\hat{x} \in \hat{T}B_n$ corresponds to some trajectory that reaches the semicircle after a sequence of n consecutive bounces, which thus makes an angle $\beta \sim \frac{\ell}{2n}$ with the horizontal direction. The corners correspond to the extremal situations for the point of impact, for β fixed; see Figure 3.6. In particular on Figure 3.6a E represents a phase point with $\hat{x}_E = (0, \beta)$, while on Figure 3.6b C represents a phase point with $\hat{x}_C = (2\beta, -\beta)$.

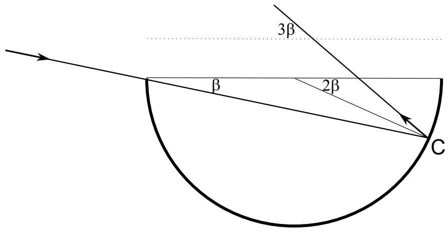
The corners of $\hat{T}B_n$ on Figure 3.7a can be identified as follows; E_n and E_{n+1} correspond to Figure 3.6a with $\beta \sim \frac{\ell}{2n}$ and $\beta \sim \frac{\ell}{2(n+1)}$, respectively, while C_n and C_{n+1} correspond to Figure 3.6b with $\beta \sim \frac{\ell}{2n}$ and $\beta \sim \frac{\ell}{2(n+1)}$, respectively. From this description, the dimensions of $\hat{T}B_n$ follow, and its measure can be computed, hence Lemma 3.8 follows. Note that on $\hat{T}B_n$ we have $\varphi \approx 0$, hence in leading order the density is $\frac{1}{8}$, and that Figure 3.7a appears four times, at the four obtuse vertices of $\hat{M}$.

3.3.3 Short range correlations for $\hat{T}$

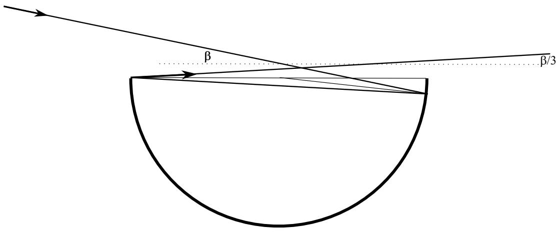
It remains to explain the factor $\frac{4+3\log 3}{4-3\log 3}$ in Theorem 3.5. We will see that it is related to the following phenomena: a long bouncing series is followed by another long bouncing series of comparable length. Hence, in contrast to an i.i.d. situation, large values of $r_{\hat{M}}$ occur in clusters of a



(a) The (upper) left corner E of $\hat{T}B_n$.



(b) The (lower) right corner C of $\hat{T}B_n$.



(c) A point with transition $n \to i \sim 3n$.

Figure 3.6: Trajectories corresponding to various points of $\hat{T}B_n$.

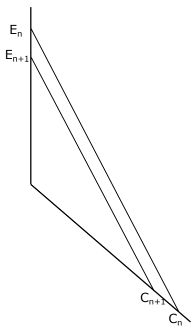
specific structure. This is determined by the transition rules between consecutive bouncing series, described in Lemma 3.9 below. For brevity, introduce $\alpha = \frac{3}{4}\log 3$ and note $\frac{4+3\log 3}{4-3\log 3} = \frac{2\alpha}{1-\alpha} + 1$.

Lemma 3.9. *We have*

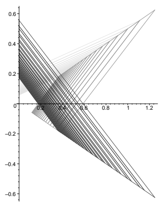
$$\hat{T}B_n \cap B_i \neq \emptyset \iff \frac{n}{3} + o(n) \leq i \leq 3n + o(n). \tag{3.4}$$

Moreover, there exists $C_1 > 0$ and a sequence ϵ_n that tends to 0 as $n \to \infty$, such that for any $i \in [n/3 + C_1, 3n - C_1]$,

$$(1-\epsilon_n)\frac{3n}{8i^2} \leq \frac{\hat{\mu}(\hat{T}B_n \cap B_i)}{\hat{\mu}(\hat{T}B_n)} \leq \frac{3n}{8i^2}(1+\epsilon_n). \tag{3.5}$$



(a) The geometry of the cell $\hat{T}B_n$.



(b) The intersections $\hat{T}B_n \cap B_i$.

Figure 3.7: The cells $\hat{T}B_n$ and B_i.

In other words, we go from n to i asymptotically with probability $\frac{3n}{8i^2}$ (note that $\sum_{i=n/3+C_1}^{3n-C_1} \frac{3n}{8i^2} \to 1$).

Let us explain first how Lemma 3.9 implies the occurrence of the factor $\frac{2\alpha}{1-\alpha}+1$ in Theorem 3.5. Note that by (3.5), given a bouncing series of length n, the expected length of the next bouncing series satisfies

$$\mathbb{E}(r_{\hat{M}} \circ \hat{T}|B_n) = \sum_{i=n/3+C_1}^{3n-C_1} i \cdot \frac{3n}{8i^2} = \frac{3n}{8} (\log(3n) - \log(n/3)) + o(n) = \alpha \cdot n + o(n).$$

Thus, indeed, large values of $r_{\hat{M}}$ (and thus of $\hat{f}$) arise in clusters. Consider a bouncing point with a large value $r_{\hat{M}} = n_{\max}$; it is followed by a sequence of further large values, which are, consecutively, $\alpha \cdot n_{\max}, \alpha^2 \cdot n_{\max}, \alpha^3 \cdot n_{\max}, \ldots$ in expectation. Similarly, $r_{\hat{M}} = n_{\max}$ is preceded by another sequence of large values of similar character. The contribution of these two geometric series rescale $n_{\max}$ to $\left(\frac{2\alpha}{1-\alpha} + 1\right) n_{\max}$, as reflected in the asymptotic variance in Theorem 3.5.

Lemma 3.9 relies on the geometry of the possible intersections of $\hat{T}B_n$ and B_i, depicted on Figure 3.7b. In fact, for i fixed, B_i consists of two connected components, B_i^1 and B_i^2. A point $\hat{x} \in B_i^1$ is shown on Figure 3.6b, here the series of bouncing on the segments starts immediately from $\hat{x}$, while for $\hat{x} \in B_i^2$ shown on Figure 3.6c, the series of bouncing on the segments is preceded by an additional collision on the semicircle. Both the B_i^1 and the B_i^2 are stripes accumulating on the obtuse vertex of $\hat{M}$. The intersections with $\hat{T}B_n$ are transversal as the $\hat{T}B_n$ have long sides which are negatively sloped, while both B_i^1 and B_i^2 have long sides which are positively sloped in the (r, φ) coordinates. To describe the possible transitions $\hat{T}B_n \cap B_i \neq \emptyset$, note that, with the unfolding trick, the bouncing series arising from $\hat{x} \in B_i$ corresponds to a flight that makes a small angle $\beta' \sim \frac{\ell}{2i}$ with the horizontal direction. Hence we need to consider the allowed $\beta \to \beta'$ transitions. The two extremal situations with $\beta' = 3\beta \Leftrightarrow i \sim \frac{n}{3}$, and $\beta' = \beta/3 \Leftrightarrow i \sim 3n$, are depicted on Figures 3.6b and 3.6c, respectively. This explains the relation (3.4). To see (3.5), note that the B_i^1 and B_i^2 have width $\asymp \frac{1}{i^2}$, while $\hat{T}B_n$ has length $\asymp \frac{1}{n}$. Hence by transversality of the intersections we have transition probabilities of the form $\hat{\mu}(B_i \,|\, \hat{T}B_n) \sim c\frac{n}{i^2}$ for some $c > 0$, and $c = \frac{3}{8}$ follows from normalization, see the statement of Lemma 3.9.

3.3.4 More comments on the proof of Theorems A and B

After describing the geometric phenomena behind Theorem A, let us mention that a key ingredient in the proof given in [BG06] is a nonstandard limit theorem in expanding Young towers. For terminology and notations on Young towers, we refer to section 2.2.1. Also, recall from section 3.2 that, by [Mar04] and [CZ05], the induced map $(\hat{M}, \hat{T}, \hat{\mu})$ can be modelled by a hyperbolic Young tower (Δ, F, μ_Δ) with exponential tails. After collapsing along stable directions, an expanding Young tower $(\bar{\Delta}, \bar{F}, \mu_{\bar{\Delta}})$ with exponential tails is obtained, in the sense of Definition 2.1.

To state our nonstandard limit theorem in expanding Young towers, we introduce some more notation. A function $\bar{g} : \bar{\Delta} \to \mathbb{R}$ is *locally Hölder continuous* if there exists $C > 0$ and $\gamma < 1$ such that $|\bar{g}(x) - \bar{g}(y)| \leq C\gamma^{s(x,y)}$ whenever x and y belong to the same partition element of $\bar{\Delta}$. Locally Hölder continuous functions can very well be unbounded. It is possible to associate to our induced observable $\hat{f} : \hat{M} \to \mathbb{R}$ a locally Hölder function $\bar{g} : \bar{\Delta} \to \mathbb{R}$ with some standard procedures (which, nonetheless, are technically involved due to the unboundedness of $\hat{f}$, see [BG06, section 2]). $\bar{g}$ then has the same tail distribution (with respect to $\mu_{\bar{\Delta}}$) as $\hat{f}$ (with respect to $\hat{\mu}$), in particular, it belongs to the non-standard domain of attraction of the normal law, in the following sense.

A function $L : \mathbb{R}_+^* \to \mathbb{R}_+^*$ is *slowly varying* if, for all $\lambda > 0$, $L(\lambda x)/L(x)$ tends to 1 when $x \to \infty$. By classical probabilistic results, a real random variable Z is in the nonstandard domain of attraction of the Gaussian distribution $\mathcal{N}(0,1)$ if and only if it satisfies one of the following equivalent conditions:

- The function $L(x) := \mathbb{E}(Z^2 1_{|Z|\le x})$ is unbounded and slowly varying.
- $\mathbb{P}(|Z| > x) \sim x^{-2} l(x)$ for some function l such that $\tilde{L}(x) := 2\int_1^x \frac{l(u)}{u}\,\mathrm{d}u$ is unbounded and slowly varying.

In this case, $\tilde{L}(x) \sim L(x)$ when $x \to \infty$, and $l(x) = o(L(x))$. We will say that l and L are the *tail functions* of Z. They are defined up to asymptotic equivalence. Choose a sequence $b_n \to \infty$ such that $\frac{n}{b_n^2} L(b_n) \to 1$. Then, if $Z_0, Z_1, \dots$ is a sequence of independent random variables distributed as Z, then

$$\frac{Z_0 + \cdots + Z_{n-1} - n\mathbb{E}(Z)}{b_n} \overset{\mathcal{D}}{\Longrightarrow} \mathcal{N}(0,1).$$

Recall from Definition 2.1 that $\omega(x)$ denotes the height of the point $x \in \bar{\Delta}$, while $\pi_0 : \bar{\Delta} \to \bar{\Delta}_0$ is the projection to the base. Given $\bar{g} : \bar{\Delta} \to \mathbb{R}$, define $\bar{G} : \bar{\Delta} \to \mathbb{R}$ as $\bar{G}(x) = \sum_{k=0}^{\omega(x)-1} \bar{g}(\bar{F}^k \pi_0 x)$. Now we can state our non-standard limit theorem in expanding Young towers (this is [BG06, Theorem 3.4]):

Theorem 3.10. *Let $(\bar{\Delta}, \bar{F}, \mu_{\bar{\Delta}})$ be an expanding Young tower with exponential tails, and let $\bar{g} : \bar{\Delta} \to \mathbb{R}$ be locally Hölder continuous. Assume that the distribution of $\bar{g}$ is in the nonstandard domain of attraction of $\mathcal{N}(0,1)$, with tail functions l and L. Assume moreover that l and L are slowly varying, and $l(x\ln x)/l(x) \to 1$, $L(x \ln x)/L(x) \to 1$ when $x \to \infty$. Finally, assume that there exists a real number $a \neq -1/2$ such that*

$$\int \bar{g}(e^{it\bar{G}} - 1) = (a + o(1))itL(1/|t|)) \text{ when } t \to 0. \tag{3.6}$$

Write $L_1(x) = (2a+1)L(x)$, and choose a sequence $b_n \to \infty$ such that $\frac{n}{b_n^2} L_1(b_n) \to 1$. Then

$$\frac{\sum_{k=0}^{n-1} \bar{g} \circ \bar{F}^k - n\int \bar{g}}{b_n} \overset{\mathcal{D}}{\Longrightarrow} \mathcal{N}(0,1).$$

Remark 3.11. *For $\bar{g}$ obtained from the observable $\hat{f} : \hat{M} \to \mathbb{R}$ of Theorem 3.5, we have $L(x) = \frac{I^2\ell^2}{4} \log x$ or equivalently $l(x) = \frac{I^2\ell^2}{8}$, and thus the assumptions of Theorem 3.10 are satisfied. Also, in this case we have $a = \frac{\alpha}{1-\alpha}$, where $\alpha = \frac{3}{4}\log 3$ from the above description of transitions between consecutive bouncing series in the stadium.*

The proof of Theorem 3.10 in [BG06, section 3] uses the perturbed transfer operator (or Nagaev-Guivarc'h) method, see section 2.2.1. Let P denote the transfer operator of $(\bar{\Delta}, \bar{F}, \mu_{\bar{\Delta}})$ acting on an appropriate Banach space $\mathcal{B}$ as in [You98], and, for real parameters t, let P_t denote its perturbation associated to $\bar{g}$, defined as $P_t(\psi) = P(e^{it\bar{g}}\psi)$, $\psi \in \mathcal{B}$. Then the main goal is to obtain an expansion of the leading eigenvalue λ_t of P_t, in particular (in the centered case $\int \bar{g} = 0$) $\lambda_t = 1 - \frac{t^2}{2} L_1(1/|t|)(1 + o(1))$. The argument is strongly inspired by the work of [AD01] on Gibbs-Makov maps, see Definition 5.7 in Chapter 5.3.3 for a definition. This is an important class of dynamical systems which resemble the first return maps of $\bar{F}$ to the base $\bar{\Delta}_0$ of the expanding tower. In [AD01] Aaronson and Denker obtain the limit law of Theorem 3.10 in the context of Gibbs-Markov maps, however, with $L_1(1/|t|)$ replaced by $L(1/|t|)$ in the expansion of λ_t, and consequently in the normalizing sequence b_n. In our case, there is an additional effect due to the presence of the higher levels of the tower, reflected by $a \neq 0$ (which replaces L by L_1). This effect is connected to potential short-range correlations of high values of $\bar{g}$ that may occur at upper levels of the tower, before returning to the base. In [BG06, section 4] then it is proved that, in the case of the stadium, such short range correlations are dominated by the transitions between consecutive long bouncing series described above in the present section. This way a can be expressed in terms of the stadium geometry, reducing Theorem 3.5 to Theorem 3.10.

Let us comment briefly also on the **proof of Theorem B**. An important ingredient here, too, is inducing, this time on the *base of the hyperbolic Young tower* $\Delta_0 \subset \Delta$. Let $F_0 : \Delta_0 \to \Delta_0$ be the first return map of $F : \Delta \to \Delta$. Equivalently, as $\Delta_0 = R$ is a subset (the Cantor rectangle) in $\hat{M}$, we have $F_0\hat{x} = \hat{T}^{\hat{r}(\hat{x})}\hat{x}$, however, $\hat{r} : R \to \mathbb{Z}^+$ is *not* the first return of $\hat{T}$ to R, see the description of hyperbolic Young towers in section 2.2.1. Accordingly, the natural F_0-invariant probability measure μ_{Δ_0} on Δ_0 is just obtained by conditioning $\hat{\mu}$ on $R = \Delta_0$. Define the induced observable $g_0 : \Delta_0 \to \mathbb{R}$ as $g_0(\hat{x}) = \sum_{j=0}^{\hat{r}(\hat{x})-1} \hat{f}(\hat{T}^j\hat{x})$. In the case of Theorem B, as $\int f = 0$ and $I_f = 0$, the induced observable g_0 belongs in $L^{2+\varepsilon}(\Delta_0)$ for some $\varepsilon > 0$. This makes it possible to apply Gordin's martingale approximation from [Gor69] to prove the following standard CLT: there exists $\sigma_0 \geq 0$ such that

$$\frac{\sum_{k=0}^{n-1} g_0 \circ F_0^k}{\sqrt{n}} \overset{\mathcal{D}}{\Longrightarrow} \mathcal{N}(0, \sigma_0^2),$$

from which Theorem B follows. To apply [Gor69] to $\sum_{k=0}^{n-1} g_0 \circ F_0^k$, one has to find an appropriate sigma algebra $\mathcal{F}_0$ on Δ_0 and check two L^2-summability conditions, namely

$$\sum_{n\geq 0} \|E(g_0 \,|\, F_0^n \mathcal{F}_0) - g_0\|_{L^2} < \infty; \qquad \sum_{n\geq 0} \|E(g_0 \,|\, F_0^{-n} \mathcal{F}_0)\|_{L^2} < \infty.$$

$\mathcal{F}_0$ is obtained as the pull-back of the standard sigma algebra of $\bar{\Delta}_0$ by $\bar{\pi} : \Delta_0 \to \bar{\Delta}_0$ (ie. collapsing stable manifolds). The summability conditions can be obtained from the expansion properties of F_0, the Hölder regularity of f, and the tails of the return times involved, see [BG06, section 5.2] for further details.

3.3.5 Comments on the rate of correlation decay

The methods and the results of [BG06] have been applied and extended in several directions, here we discuss a particularly interesting aspect, the mixing rates of the map (M, T, μ). As mentioned in section 2.2, if a map can be modelled by a Young tower, estimates on the rate of correlation decay follow from bounds on the tail of the return time to the base of the tower. In the case of the stadium, there are several return times involved, and here we summarize the notation:

- $r_{\hat{M}}$ denotes the first return time of T to $\hat{M}$. The associated first return map is $(\hat{M}, \hat{T}, \hat{\mu})$.
- Now, by [Mar04, CZ05] the induced map $(\hat{M}, \hat{T}, \hat{\mu})$ can be modelled by a Young tower with exponential tails. $\Delta_0 \subset \hat{M}$ is the base of this Young tower, $\hat{r}$ is the return time. The tower map is $(\hat{\Delta}, \hat{F}, \mu_{\hat{\Delta}})$, which thus models $(\hat{M}, \hat{T}, \hat{\mu})$. The return time $\hat{r}$ extends from Δ_0 to $\hat{\Delta}$ as a hitting time.
- This then generates another tower map (Δ, F, μ_Δ) (modelling the original map (M, T, μ)) with return time $r : \Delta_0 \to \mathbb{Z}^+$, $r(y) = \sum_{k=0}^{\hat{r}(y)-1} r_{\hat{M}}(\hat{T}^k y)$, which extends to the higher levels of Δ as a hitting time.

It is the tails of the hitting time r that is relevant for the mixing rates of (M, T, μ): [You99] essentially states that, for $f, g : M \to \mathbb{R}$ Hölder continuous, $Corr(f, g; n) \ll \mu_\Delta(r > n)$. Now in

[Mar04] and [CZ05], exponential bounds on the tails of $\hat{r}$ were combined with the bounds of (3.2) on the tails of $r_{\hat{M}}$ to give a naive bound of the form $\mu_\Delta(r > n) \ll \frac{\log^2 n}{n}$ on the tails of r. In [CZ08], a better estimate of the form $\mu_\Delta(r > n) \ll \frac{1}{n}$ was obtained with a more careful analysis, which crucially relied on the statistics of short range correlations described in subsection 3.3.3 above.

The upper bound $Corr(f, g; n) = O(\frac{1}{n})$ of [CZ08] is essentially optimal, see Remark 3.1. We mention a recent remarkable result on the optimality of this bound from [BMT21], which extends the methods of [Sar02] and [Gou04b]. Let $f, g : M \to \mathbb{R}$ be Hölder continuous and *supported on* $\hat{M}$ (that is, $f \equiv 0, g \equiv 0$ on $M \setminus \hat{M}$) with $\int f \, \mathrm{d}\,\mu \neq 0$ and $\int g \, \mathrm{d}\mu \neq 0$. Then

$$Corr(f, g; n) \sim \mu_\Delta(r > n) \cdot \int f d\mu \int g d\mu. \tag{3.7}$$

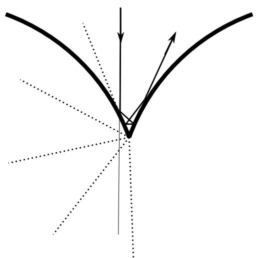
Chapter 4

Dispersing billiards with cusps

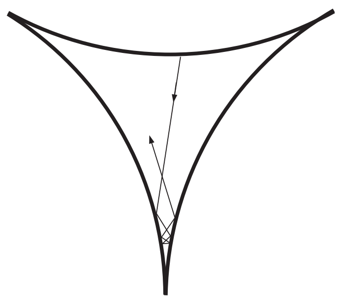
4.1 Description

In this chapter we turn back to the investigation of dispersing billiards, however, in contrast to Chapter 2, we no longer assume that the C^3-smooth boundary components Γ_i are disjoint. More precisely, the billiard domain $Q \subset \mathbb{T}^2$ is assumed to have a *piecewise* smooth boundary $\partial Q = \cup_{i=1}^{I} \Gamma_i$ where the C^3 smooth compact curves Γ_i can only intersect at their endpoints: $\Gamma_i \cap \Gamma_j = \partial\Gamma_i \cap \partial\Gamma_j$ for $i \neq j$. Throughout, the curves Γ_i are assumed to be strictly convex when viewed from the exterior of the domain, that is, from $\mathbb{T}^2 \setminus D$. We distinguish two types of intersections:

- *corner points*, when the intersections are transversal, that is, the angle of the tangent vectors of Γ_i and Γ_j at the intersection point is positive.
- *cusps*, the main interest of this chapter, when the intersections are tangential, that is, the angle of the tangent vectors of Γ_i and Γ_j at the intersection point is zero.



(a) Bounded number of iterations in the corner



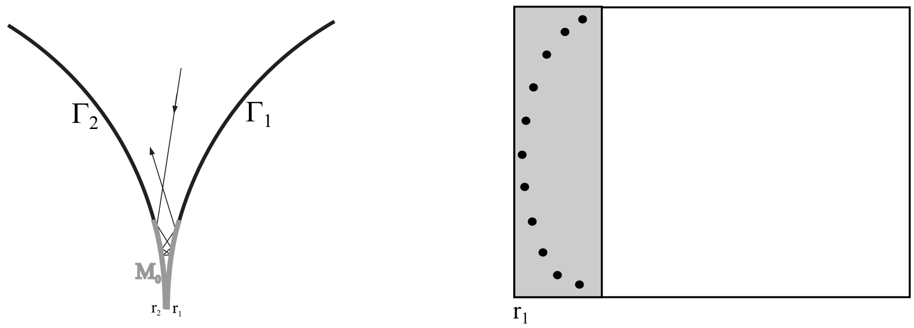
(b) Unbounded number of iterations in the cusp.

Figure 4.1: Tables with corner points and cusps.

The billiard dynamics arising in these two geometries differ significantly. In particular, as it can be seen for example by the unfolding trick mentioned in the previous chapter, the number of iterations in the vicinity of a transversal corner point is uniformly bounded from above (see Figure 4.1a). On the other hand, an unbounded number of consecutive collisions can take place in an arbitrary small neighborhood of a cusp (see Figure 4.1b). We will refer to such successive collisions as *corner series*.

Although there are some technical challenges (most notably the problem of complexity, see section 2.3, which was clarified in [DST14] for this category) many results on the strong statistical properties of dispersing billiards have been extended to domains with *transversal* corners. In particular, exponential decay of correlations and the central limit theorem (with standard $\sqrt{n}$ normalization) was proved for the map (M, T, μ) by two different methods, Young towers with exponential tails in [Che99], and anisotropic Banach spaces in [DZ14].

The statistical properties of the billiard map in the presence of cusps are, however, strikingly different. As we will see in the next section, long corner series slow down the rate of correlation decay, and cause anomalous scaling in the limit theorem. To describe these phenomena, as in Chapter 3, we introduce a set $\hat{M} \subset M$ – with associated first return time $r_{\hat{M}} : \hat{M} \to \mathbb{Z}^+$, first return map $\hat{T} : \hat{M} \to \hat{M}$, and invariant absolutely continuous probability measure $\hat{\mu}$ on $\hat{M}$ – in such a way that $(\hat{M}, \hat{T}, \hat{\mu})$ has strong statistical properties. This time $\hat{M}$ is obtained by *cutting out a fixed small neighborhood of the cusp*. That is, $\hat{M} = M \setminus M_0$, where M_0 corresponds to collisions in a small neighborhood of the cusp – or, if there are several cusps, in some fixed small neighborhoods of any of these – see Figure 4.2a. The exact size of the neighborhood is not relevant, as although it effects the form of the limit laws on $(\hat{M}, \hat{T}, \hat{\mu})$, this factor scales out when getting back to (M, T, μ).



(a) A small neighborhood of the cusp. (b) A corner series in M_0.

Figure 4.2: Inducing and corner series

Note that the cusp – made by the points r_1 and r_2 on the two boundary components Γ_1 and Γ_2, see Figure 4.2a – correspond to two lines in the phase space, spanning from $-\pi/2$ to $\pi/2$ in the collision angle coordinate φ. Accordingly, M_0 corresponds to two rectangularly shaped domains, one of which, in the vicinity of r_1, is displayed as the grey region on Figure 4.2b. There is a similar rectangle near r_2, and during the corner series, the trajectory alternates between these two rectangular regions. Consider $\hat{x} \in \hat{M}$ with a large value of $r_{\hat{M}}(\hat{x})$: the black dots on Figure 4.2b show $T\hat{x}, T^3\hat{x}, T^5\hat{x}, ...$ (while $T^2\hat{x}, T^4\hat{x}...$ follow a similar pattern in the vicinity of r_2). The longer $r_{\hat{M}}(\hat{x})$, the closer these points are to the boundary line $r = r_1$. On the other hand, for long corner series, the trajectory enters M_0 almost tangentially, that is, with φ coordinate close to $-\frac{\pi}{2}$. Then φ increases and $|r - r_1|$ decreases as the trajectory goes deeper into the cusp until $\varphi \approx 0$, the turning point, beyond which both φ and $|r - r_1|$ keeps increasing. This exiting part of the trajectory follows a pattern essentially symmetric to the entering part, until it leaves M_0 with $\varphi \approx \pi/2$, that is, with an almost tangential collision. We will show in section 4.3 that, in the asymptotics of $r_{\hat{M}} \to \infty$, the distribution of φ values occurring in course of a corner series follows a specific density. This is reflected below in the statement of Theorem D (specifically in the value of the asymptotic variance D_f, see Formula (4.1)).

We close this section by commenting briefly on the *flow* $S^t : \mathcal{M} \to \mathcal{M}$ in dispersing billiards with cusps. As discussed in section 2.2.2, the billiard flow can be represented as a suspension flow $(M^\tau, \Phi^t, \mu^\tau)$. The core phenomena are that the long corner series in the cusp, that correspond to an unbounded number of iterations of the map, take place within a uniformly bounded flow time. In particular, the roof function τ – the free flight length – vanishes in the points of M that correspond to the cusp: in the notations of Figure 4.2, we have $\tau(r_1, \varphi) = 0$ for any $\varphi \in [-\pi/2, \pi/2]$ (this can be seen by (piecewise) Hölder continuity of τ). Accordingly, the billiard flow has stronger statistical properties than the map, as stated in Theorem G below.

4.2 Results

In this section we summarize the results of the dissertation on dispersing billiards with cusps. These belong to three categories. In section 4.2.1 results on limit laws for the map $(\hat{M}, \hat{T}, \hat{\mu})$ are described from [BCD11]. Then in section 4.2.2 we move on to the results on statistical properties of the flow from [BM08]. Finally, in section 4.2.3 results on the convergence of moments are discussed from [BCD17].

4.2.1 Limit laws for the billiard map

Preceding results. It was first conjectured in [Mac83] that correlations decay with rate $O(1/n)$ for the map (M, T, μ). Back in this paper Machta supported this conjecture by heuristic arguments, in particular, he introduced an approximation of the evolution during corner series by differential equations, which also play an important role in our investigation, see section 4.3.2. Ergodicity was proved by Rehaček in [Ř95], which, by general arguments based on hyperbolicity, also implies further ergodic properties, such as mixing, K-mixing and the Bernoulli property, see [CM06, Chapter 6] and [CH96]. A polynomial bound on the rate of correlation decay was obtained in [CM07] which proved that, given $f, g : M \to \mathbb{R}$ Hölder continuous, we have $Corr(f, g; n) \ll \frac{\log^2 n}{n}$. The method of [CM07] was the one already mentioned at the beginning of section 3.2 (see also section 3.3.5): construction of a Young tower with exponential tails for the induced map $(\hat{M}, \hat{T}, \hat{\mu})$, combined with bounds on the tail of the return time $r_{\hat{M}} : \hat{M} \to \mathbb{Z}^+$. The bound on the rate of correlation decay was improved to $O(1/n)$ in [CZ08].

Results on limit laws for the billiard map. To state our results, let us introduce some notation. Recall that the Liouville measure μ on M is given by $\mathrm{d}\mu = c_\mu \cdot \cos\varphi \cdot \mathrm{d}r\, \mathrm{d}\varphi$, where the normalization constant satisfies $c_\mu = [2 \cdot \mathrm{Length}(\Gamma)]^{-1}$ so that μ is a probability measure on M. We consider the situation sketched on Figure 4.2: there is one cusp formed by two boundary components Γ_1 and Γ_2, meeting tangentially at their respective endpoints r_1 and r_2. Denote, furthermore, the curvatures of Γ_1 at r_1 and Γ_2 at r_2 by $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively, and introduce $\mathcal{K}_0 = \frac{1}{2}(\mathcal{K}_1 + \mathcal{K}_2)$. Now consider $f : M \to \mathbb{R}$ Hölder continuous, and define the nonnegative constant

$$D_f^2 = \frac{c_\mu}{8\mathcal{K}_0} \left[\int_{-\pi/2}^{\pi/2} [f(r_1, \varphi) + f(r_2, \varphi)] \sqrt{\cos\varphi}\, d\varphi \right]^2. \tag{4.1}$$

Remark 4.1. *If there is more than one cusp formed by the boundary components $\Gamma = \cup_{i=1}^{I} \Gamma_i$, then D_f^2 is obtained as a sum of the expressions (4.1) on all cusps.*

As before, let us denote the Birkhoff sum as $S_n f : M \to \mathbb{R}$; $S_n f(x) = \sum_{k=0}^{n-1} f(T^k x)$.

Theorem D. *Let $f : M \to \mathbb{R}$ be Hölder continuous, satisfying $\int f \, d\mu = 0$ and $D_f \neq 0$. Then*

$$\frac{S_n f}{\sqrt{n \log n}} \stackrel{\mathcal{D}}{\Longrightarrow} \mathcal{N}(0, D_f^2).$$

Let us note that as in the case of the stadium, Theorem D implies a lower bound on the rate of correlation decay: for f as in the statement of the Theorem, in particular, with $D_f \neq 0$, we have $Corr(f, f; n) \neq o(1/n)$, see Remark 3.1. We also note that, for f and g supported on $\hat{M}$, (3.7) applies.

Our next result concerns the extension of Theorem D to a functional limit theorem (or Weak Invariance Principle). For an integer $N \geq 1$ and $0 \leq s \leq 1$, define $S_{sN} f$ as follows: if $s = \frac{k}{N}$ for some integer $k = 1, \ldots N$, then let $S_{sN} f = S_k f$, the kth Birkhoff sum of f; and for intermediate values $\frac{k-1}{N} < s < \frac{k}{N}$, extend by linear interpolation between $S_{k-1} f$ and $S_k f$ (with the convention $S_0 f = 0$).

Theorem E. *Let $f : M \to \mathbb{R}$ satisfy the assumptions of Theorem D, in particular $D_f \neq 0$. Define $S_{sN} f$ as above. Then the process*

$$W_N(s) = \frac{S_{sN} f}{D_f \sqrt{N \log N}}; \qquad 0 < s < 1$$

converges, as $N \to \infty$, to standard Brownian motion.

In contrast, if $D_f = 0$, we have the following standard central limit theorem.

Theorem F. *Let $f : M \to \mathbb{R}$ be Hölder continuous, satisfying $\int f \, d\mu = 0$ and $D_f = 0$. Then there exists $\sigma_f^2 \geq 0$ such that*

$$\frac{\sum_{k=0}^{n-1} f \circ T^k}{\sqrt{n}} \stackrel{\mathcal{D}}{\Longrightarrow} \mathcal{N}(0, \sigma_f^2).$$

Remark 4.2. *As in the case of the stadium, for Theorem F, Remark 3.2 applies: $\sigma^2 = 0$ if and only if f is a coboundary. σ^2 depends on the time autocorrelations of f, see Theorem 4.8 for an expression.*

Stable limit laws and Levy processes in case of higher order tangency. At this point it is important to mention a very interesting direction of research which was, to considerable extent, motivated by the results of this section. For the case of billiard tables when the order of tangency is higher than quadratic, in [JZ18] Jung and Zhang proved the emergence of *stable limit laws* in a setting analogous to our Theorem D. The index of the limiting stable distribution is determined by the order of the tangency at the cusp. This result was then extended to the functional level in a series of papers, [MV20], [JPZ20] and [JMP21], proving convergence to a Levy process for a setting analogous to our Theorem E, but again with higher order tangency at the cusp. Note that this research is interesting also from the functional analytic perspective: as the limit process is discontinuous, it is a subtle question in which topology the convergence occurs. We refer to the above mentioned papers for further details.

4.2.2 Statistical properties of the billiard flow

In this section we consider the billiard flow $S^t : \mathcal{M} \to \mathcal{M}$, or equivalently, the suspension flow $\Phi^t : M^\tau \to M^\tau$ with the invariant probability measure μ^τ (where τ is the first return time). We consider the class of observables $F \in C^{m,\eta}(M^\tau)$ for $\eta \in (0, \eta]$ and $m \geq 1$, that is, η-Hölder and m times differentiable along the flow direction, see Definition 3.4. These spaces are all contained in $C^\eta(M^\tau)$, which is just the space of η-Hölder continuous functions $F : M^\tau \to \mathbb{R}$. Recall, furthermore, the notion of rapid mixing (or superpolynomial decay of correlations) from section 2.2.2.

Definition 4.3. *The flow $(M^\tau, \Phi^t, \mu^\tau)$ has superpolynomial decay of correlations (or rapid mixing) if, for any $q \ge 1$ there exists $m \ge 1$ such that for any $\eta \in (0,1]$, and any $F, G \in C^{m,\eta}(M^\tau)$ we have $Corr(F,G;t) \ll t^{-q}$.*

Before stating the results on the statistical properties of the billiard flow we make a comment. As noted earlier (see section 2.2.2, or the discussion before Corollary 3.3), under fairly general conditions, limit theorems for the suspension flow can be reduced to limit theorems for the base transformation. Now consider $F : M^\tau \to \mathbb{R}$ Hölder continuous, and define $f : M \to \mathbb{R}$, $f(x) = \int_0^{\tau(x)} F(\Phi^t x)\, \mathrm{d}t$. Let us recall the notation from (4.1) (see also Figure 4.2) and note that $\tau(x) = 0$ whenever $x = (r_i, \varphi)$ for some $\varphi \in [-\pi/2, \pi/2]$ and $i = 1, 2$, where r_1 and r_2 are the two points that make the cusp. Accordingly, if $f : M \to \mathbb{R}$ is obtained from some flow observable $F : M^\tau \to \mathbb{R}$, then f also vanishes at such points, and thus $D_f = 0$. Hence Theorem F, a central limit theorem with standard normalization applies. In fact, for the flow, we have a stronger result, the almost sure invariance principle, in the following sense.

Definition 4.4. *Consider a suspension flow $(M^\tau, \Phi^t, \mu^\tau)$ and a vector valued Hölder continuous $F : M^\tau \to \mathbb{R}^d$ (for some $d \ge 1$) and centered ($\int F \,\mathrm{d}\mu^\tau = 0$) observable. The flow satisfies the vector valued almost sure invariance principle (ASIP) if there exists some $\lambda > 0$ such that for any F as above there exists some d dimensional Brownian motion $\mathcal{W}(T)$ such that (on a possibly enlarged probability space)*

$$S_T F = \int_0^T F \circ \Phi^t dt = \mathcal{W}(T) + O(T^{\frac{1}{2}-\lambda}) \quad \textit{almost surely, as } T \to \infty.$$

We note that the almost sure invariance principle implies several other statistical limit laws, including the central limit theorem and its functional version (with the standard $\sqrt{T}$ normalization), or the law of iterated logarithm, see [PS75] for a more complete list.

Theorem G. *The billiard flow for dispersing billiards with cusps has superpolynomial decay of correlations in the sense of Definition 4.3, and satisfies the vector valued almost sure invariance principle in the sense of Definition 4.4.*

Remark 4.5. *We note that analogous results for billiards with transversal corner points were proved in [Mel07].*

The requirement of smoothness along the flow direction excludes some physically relevant observables – such as the velocity – that may change instantaneously at collisions.

The optimal rate (the value of the exponent λ) in the ASIP (Definition 4.4) is a delicate question that we do not intend to survey here, see eg. [MN09], [Gou10], [Kor18] for some discussion. Let us just mention that for the scalar case ($d = 1$) and $F : M^\tau \to \mathbb{R}$ smooth along the flow direction, any value $\lambda < \frac{1}{4}$ can be obtained and the ASIP extends to the time one map of the flow, see [Mel18, section 1.2].

4.2.3 Convergence of the second moment

Consider a dynamical system $H : N \to N$ with an ergodic invariant probability measure ν. Let the integrable and centered $A : N \to \mathbb{R}$ (with $\nu(A) = \int_N A d\nu = 0$) satisfy some limit theorem. That is, denoting as usual the Birkhoff sum as $S_n A = A + \cdots + A \circ H^{n-1}$, we have a convergence in distribution: there exists some nontrivial probability distribution function $G : \mathbb{R} \to \mathbb{R}$ and a scaling sequence b_n such that

$$\lim_{n\to\infty} \nu\left(x \in N \;\middle|\; \frac{S_n A(x)}{b_n} \le z \right) = G(z) \tag{4.2}$$

for every $z \in \mathbb{R}$ where G is continuous. Then we may study the following question: do we have for some $p > 0$

$$\lim_{n\to\infty} \nu\left(|(S_n A)/b_n|^p\right) = \int |z|^p \,\mathrm{d}G(z) \tag{4.3}$$

in which case we say that *the pth moment converges properly.* The pth moment of $(S_n A)/b_n$ may also converge to a value different from the right hand side of (4.3) or diverge altogether. We mention the following standard facts (e.g., [Dur10, Exercise 3.2.5, p. 101]):

- Suppose (4.2) holds and $\sup_n \nu\left(|(S_n A)/b_n|^p\right) < \infty$. Then the qth moment of $(S_n A)/b_n$ properly converges for every $q < p$.

- There is a *critical moment* $p_* \in [0, \infty]$ such that
  (a) the qth moment of $(S_n A)/b_n$ properly converges for all $q < p_*$
  (b) the qth moment of $(S_n A)/b_n$ diverges for all $q > p_*$.

- In case $p_* = \infty$ we have proper convergence of all moments. In case $p_* = 0$ we have divergence of all moments. The p_*th moment itself may converge (properly or improperly) or diverge. We note, however, that convergence in distribution implies that

$$\liminf_{n\to\infty} \mu\left(|(S_n A)/b_n|^p\right) \geq \int |z|^p \,\mathrm{d}G(z)$$

  therefore the limit of the p_*th moments can only be greater than (or equal to) the p_*th moment of the limit distribution.

For the case of collision maps of dispersing billiards with cusps, specifically in the setting of Theorem D, we have the following result.

Theorem H. *Let (M, T, μ) be a dispersing billiard map with cusps, and $f : M \to \mathbb{R}$ as in Theorem D, that is, Hölder continuous with $\int f d\mu = 0$ and $D_f \neq 0$. Then*

$$\lim_{n\to\infty} \frac{\mu\left([S_n f]^2\right)}{n \log n} = 2D_f^2. \tag{4.4}$$

In other words, the second moments converge to a value which is twice the second moment of the limit distribution. We will refer to this phenomenon as the "doubling effect".

Theorem H specifically implies that in dispersing billiards with cusps $p_* = 2$. Here we mention the related work of Gouëzel and Melbourne [GM14] which, following upon [MT12], studies this question in the context of Young towers. Essentially, [GM14] shows that if the tails of the tower satisfy $\mu_\Delta(r > n) \sim Cn^{-\beta}$ for some constant $C > 0$ and exponent $\beta > 0$, then the critical moment is $p_* = 2\beta$. In particular, for dispersing billiards with cusps, [GM14] shows that $p_* = 2$ and obtains bounds of the form $\mu(|S_n f|^2) \ll n \log n$, but does not prove the doubling effect. Let us also mention [Det12] which identifies the doubling effect in the context of the infinite horizon Lorentz gas, see section 5. In [BCD17, Appendix A] we argue that this effect is of entirely probabilistic origin, see also section 4.3 on further details.

4.3 Ingredients of proofs

4.3.1 Inducing and truncation

As in the case of the stadium, a first step in the proof of the results of subsection 4.2.1 is inducing on $(\hat{M}, \hat{T}, \hat{\mu})$. As in Chapter 3, given $f : M \to \mathbb{R}$ Hölder continuous, we use the notations

$$\hat{f}(\hat{x}) = \sum_{k=0}^{r_{\hat{M}}(\hat{x})-1} f(T^k \hat{x}); \quad \text{and } \hat{S}_n \hat{f}(\hat{x}) = \hat{f}(\hat{x}) + \cdots + \hat{f}(\hat{T}^{n-1}\hat{x}); \quad \text{for } \hat{x} \in \hat{M}. \tag{4.5}$$

Recall also that $\hat{\mu} = (\mu(\hat{M}))^{-1} \mu|_{\hat{M}}$, where the normalizing factor satisfies $(\mu(\hat{M}))^{-1} = \hat{\mu}(r_{\hat{M}}) = \int r_{\hat{M}} \, d\hat{\mu}$ by the Kac lemma.

Theorems D, E and F can be reduced to the following three theorems, respectively.

Theorem 4.6. *Let $f : M \to \mathbb{R}$ satisfy the assumptions of Theorem D, in particular, $\int f d\mu = 0$ and $D_f \neq 0$. Consider the induced observable $\hat{f}$ as in (4.5). Then*

$$\frac{\hat{S}_n \hat{f}}{\sqrt{n \log n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, D_{\hat{f}}^2),$$

where $D_{\hat{f}}^2 = (\mu(\hat{M}))^{-1} D_f^2$.

Theorem 4.7. *Consider $f : M \to \mathbb{R}$ as in Theorem 4.6, then it satisfies the assumptions of Theorem E. Using the notations of (4.5), define also, for $N \geq 1$ and $0 \leq s \leq 1$, $\hat{S}_{sN}\hat{f}$ as the Birkhoff sum for integer values of sN, and otherwise by linear interpolation. Then*

$$\hat{W}_N(s) = \frac{\hat{S}_{sN}\hat{f}}{D_{\hat{f}}\sqrt{N \log N}}, \quad 0 < s < 1,$$

converges, as $N \to \infty$, to the standard Brownian motion.

Theorem 4.8. *Let $f : M \to \mathbb{R}$ satisfy the assumptions of Theorem F, in particular, $\int f d\mu = 0$ and $D_f = 0$. Consider the induced observable $\hat{f}$ as in (4.5). Then*

$$\frac{\hat{S}_n \hat{f}}{\sqrt{n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma_{\hat{f}}^2),$$

with the asymptotic variance $\sigma_{\hat{f}}^2$ given by the Green-Kubo formula:

$$\sigma_{\hat{f}}^2 = \sum_{n=-\infty}^{\infty} \int \hat{f} \cdot \hat{f} \circ \hat{T}^n \, d\hat{\mu}. \tag{4.6}$$

This series converges exponentially, see also Lemma 4.10. Furthermore, $\sigma_{\hat{f}}^2 = (\mu(\hat{M}))^{-1} \sigma_f^2$ from Theorem F.

The theorems of section 4.2.1 can be reduced to these induced statements along the lines mentioned in section 3.3.1, specifically for the weak invariance principle (that is, Theorem E) see [BCD11, section 8].

Let us introduce some more notation. For integers $m \geq 1$ and $1 \leq p < q \leq \infty$, let

$$M_m = \{\hat{x} \in \hat{M} \,|\, r_{\hat{M}}(\hat{x}) = m\}; \;\; H_p = \cup_{m \geq p} M_m; \;\; M_{p,q} = H_p \setminus H_{q+1} = \cup_{p \leq m \leq q} M_m$$

and

$$\hat{f}_{p,q} = \hat{f} \cdot \mathbf{1}_{M_{p,q}}, \qquad \hat{f}_{p,q;0} = \left(\hat{f} - (\hat{\mu}(M_{p,q}))^{-1}\hat{\mu}(\hat{f}_{p,q})\right) \cdot \mathbf{1}_{M_{p,q}},$$

where $\mathbf{1}_{M_{p,q}}$ is the indicator of the set $M_{p,q}$. For brevity, we will sometimes write $\hat{f}_q = \hat{f}_{1,q}$, the truncation of $\hat{f}$ to the set of points with return time at most q. In the proof of the limit laws (Theorems 4.6, 4.8 and 4.7) we will use truncation levels q that depend on n (or N), see section 4.3.3. Note also that, although $\hat{f}$ is centered ($\hat{\mu}(\hat{f}) = 0$), that may not be the case for the truncation, which is the reason for introducing $\hat{f}_{p,q;0}$.

In section 4.3.2 below we describe the dynamics near the cusp (the corner series) along with the geometry and the dimensions of the cells M_m. In particular, we will argue that

$$\hat{\mu}(M_m) \asymp m^{-3} \tag{4.7}$$

which already indicates that $\hat{f}$ belongs to the non-standard domain of attraction of the normal law. However, as in the case of the stadium, we need more precise estimates on the tail. The analogue of Proposition 3.6 in the case of the cusp is the following Lemma:

Lemma 4.9. *Given $\hat{f}$ as in Theorem 4.6, we have*

$$\hat{\mu}(\hat{f}_p^2) = 2D_{\hat{f}}^2 \log p + O(1)$$

as $p \to \infty$.

Two other key statements stated below, Lemma 4.10 and Lemma 4.12 concern the autocorrelations of $\hat{f}$ and its Hölder regularity, respectively.

Lemma 4.10. *Given $\hat{f}$ as in Theorem 4.6, there exist $C > 0$ and $\theta \in (0, 1)$ such that for any $n \geq 1$*

$$\hat{\mu}(\hat{f} \cdot \hat{f} \circ \hat{T}^n) \leq C\theta^n. \tag{4.8}$$

Remark 4.11. *Let us make some comments.*

- *In Lemma 4.10 the requirement $n \geq 1$ is crucial. For $n = 0$ the expression gives $\hat{\mu}(\hat{f}^2) = \infty$, the induced observable does not belong to L^2 (in fact, Lemma 4.9 describes the rate of the blow up of the L^2 norm).*
- *Another important comment is that Lemma 4.10 does not hold in the stadium. Indeed, in the stadium short range correlations are more significant than in dispersing billiards with cusps, see Lemma 3.9.*
- *Lemma 4.10 extends to the correlations of the truncated observables: for any $1 \leq p < q \leq \infty$ and $1 \leq p' < q' \leq \infty$, we have*

$$\hat{\mu}(\hat{f}_{p,q;0} \cdot \hat{f}_{p',q';0} \circ \hat{T}^n) \leq C\theta^n \tag{4.9}$$

where $C > 0$ and $\theta \in (0, 1)$ may depend on $\hat{f}$, but are independent of the truncation levels p, q, p', q' and the time gap n.

To move on to the regularity properties of $\hat{f}$, let us note that the boundaries of the cells M_m are discontinuities for $\hat{T}$, and thus the natural question to consider is the Hölder regularity of $\hat{f}|_{M_m}$, the restriction of the induced observable to the cells. Also, as $\hat{f}$ is unbounded on $\hat{M}$, it can be expected that its Hölder norm blows up as $m \to \infty$ – however, in a controllable manner.

Lemma 4.12. *Consider $f : M \to \mathbb{R}$ Hölder continuous with exponent $\eta \in (0, 1]$. Then for any $m \geq 1$ $\hat{f}|_{M_m}$ is Hölder continuous with exponent $\eta/4$ and Hölder norm $O(m^2)$.*

4.3.2 Description of corner series and its consequences

In this subsection we give a rough description of the dynamics of corner series, the geometry and the dimensions of the cells M_m, and describe how these properties imply our key Lemmas 4.9, 4.10 and 4.12. For a more detailed exposition see [BCD11]. In particular, for simplicity let us assume that both boundary components Γ_1 and Γ_2 are circular arcs of unit radii. Instead of the usual (r, φ) we will use the coordinates (α, γ) where α measures the angular distance along the arc Γ_i from the endpoint r_i, $i = 1, 2$; while $\gamma = \frac{\pi}{2} - \varphi$ is the angle the outgoing trajectory makes with the tangent line of the arc. As mentioned earlier, any corner series consists of two, symmetric parts: entering the cusp and exiting the cusp, and we will focus on the former. While entering the cusp, let us denote $x = (\alpha, \gamma)$ and $Tx = x' = (\alpha', \gamma')$, which can be related by simple geometric considerations, see Figure 4.3. In particular:

$$\begin{aligned} \gamma' - \alpha' &= \gamma + \alpha; \\ 2 - \cos\alpha - \cos\alpha' &= (\sin\alpha - \sin\alpha') \cdot \tan(\alpha + \gamma) \;\Rightarrow \\ \alpha' - \alpha &\approx -\frac{\alpha^2}{\tan(\gamma)}. \end{aligned} \tag{4.10}$$

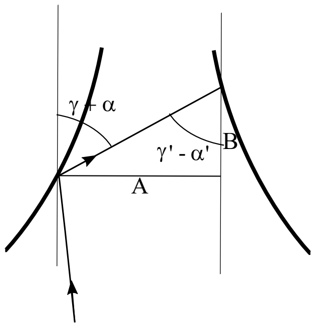


Figure 4.3: Evolution of coordinates while entering the cusp. We have $A = 2 - \cos\alpha - \cos\alpha'$ and $B = \sin\alpha - \sin\alpha'$.

We introduce the following indices: m denotes the length of the corner series, that is, we consider $\hat{x} \in M_m \subset \hat{M}$, while for $i = 1, \ldots, m-1$ we have $T^i\hat{x} = (\alpha_i, \gamma_i) \in M_0$. Then (4.10) implies, for $i < m/2$[1]:

$$\gamma_{i+1} - \gamma_i = 2\alpha_i; \qquad \alpha_{i+1} - \alpha_i \approx -\frac{\alpha_i^2}{\tan\gamma_i}$$

As already noticed by Machta in [Mac83], these relations can be regarded as discretizations of the following differential equations:

$$\dot{\gamma} = 2\alpha, \qquad \dot{\alpha} = -\frac{\alpha^2}{\tan\gamma}.$$

These differential equations have an integral

$$I = \alpha^2 \sin\gamma \tag{4.11}$$

[1]For $i \geq m/2$ we have similar relations with i replaced by a countdown index $i' = m - i$.

(it is easy to check that $\dot{I} = 0$). Accordingly, the consecutive points in the corner series on Figure 4.2b fit, with better precision as m grows, the level curves of I. In what follows we determine a relation between I and the return time m, along with a formula for $\hat{f}|_{M_m}$. To do so, let us introduce the new variable

$$s = \Psi(\gamma) = \int_0^\gamma \sqrt{\sin z}\, \mathrm{d}z,$$

for which

$$s_{i+1} - s_i \approx \sqrt{\sin \gamma_i}(\gamma_{i+1} - \gamma_i) = 2\alpha_i \sqrt{\sin \gamma_i} = 2\sqrt{I},$$

in other words, during corner series, the variable s evolves (in leading order) linearly with i. For long corner series (ie. large m), the billiard trajectory both enters and leaves M_0 almost tangentially, and during the series the coordinate γ changes from 0 to π. Accordingly, the variable s changes linearly on its range, given by the following elliptic integral

$$\kappa := \int_0^\pi \sqrt{\sin z} dz = 2\sqrt{\frac{2}{\pi}} \left(\Gamma\left(\frac{3}{4}\right)\right)^2 \approx 2.39628.$$

Thus, as $m \to \infty$

$$\begin{aligned} \kappa \sim s_{m-1} - s_1 = \sum_{i=1}^{m-2} (s_{i+1} - s_i) \sim m \cdot 2\sqrt{I}, \quad \text{that is} \\ m \sim \frac{\kappa}{2\sqrt{I}}, \quad \text{and} \\ s_i \sim \frac{i\kappa}{m} \sim 2i\sqrt{I}. \end{aligned} \tag{4.12}$$

Our next goal is to compute the large m asymptotic of $\hat{f}(\hat{x}) = \sum_{i=0}^{m-1} f(T^i \hat{x})$ for $\hat{x} \in M_m$. It has to be taken into account that $T^i \hat{x}$ alternates between Γ_1 and Γ_2; that is, in the (r, φ) coordinates, (α_i, γ_i) correspond, for even and odd i, to $(r_1 - \alpha_i, \frac{\pi}{2} - \gamma_i)$ and $(r_2 - \alpha_i, -\frac{\pi}{2} + \gamma_i)$, respectively (or vice versa). It is useful to introduce:

$$\bar{f} : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R}, \qquad \bar{f}(\varphi) = \frac{1}{2} \left(f(r_1, \varphi) + f(r_2, -\varphi) \right),$$

then, by the Hölder continuity of f:

$$\begin{aligned} \hat{f}(\hat{x}) \sim \sum_{i=0}^{m-1} \bar{f}\left(\frac{\pi}{2} - \gamma_i\right) \sim \sum_{i=0}^{m-1} \bar{f}\left(\frac{\pi}{2} - \Psi^{-1}\left(\frac{i\kappa}{m}\right)\right) \\ \sim \frac{m}{\kappa} \int_0^{\Psi^{-1}(\pi)} \bar{f}\left(\frac{\pi}{2} - \Psi^{-1}(s)\right) \mathrm{d}s = \frac{m}{\kappa} \int_0^\pi \bar{f}\left(\frac{\pi}{2} - \gamma\right) \sqrt{\sin \gamma}\, \mathrm{d}\gamma. \end{aligned}$$

So, to summarize

$$\hat{f}|_{M_m} \sim J_f \cdot m, \quad \text{with} \quad J_f = \frac{1}{2\kappa} \int_{-\pi/2}^{\pi/2} \left(f(r_1, \varphi) + f(r_2, \varphi) \right) \sqrt{\cos \varphi}\, \mathrm{d}\varphi. \tag{4.13}$$

Note that Formula (4.13) is analogous to Lemma 3.7 from Chapter 3. To proceed with the proof of Lemma 4.9, by

$$\hat{\mu}(\hat{f}_p^2) \sim \sum_{m=1}^{p} J_f^2 m^2 \hat{\mu}(M_m) \sim 2J_f^2 \sum_{m=1}^{p} m\hat{\mu}(H_m) \tag{4.14}$$

we need an estimate on $\hat{\mu}(H_m) = (\mu(\hat{M}))^{-1}\mu(\cup_{k=m}^{\infty} M_k)$. Instead of $H_m \subset \hat{M}$ let us consider $H'_m = \cup_{k=m}^{\infty} T^{[k/2]} M_k \subset M_0$, the set of points, with return time at least m, at their deepest point in the cusp in course of their corner series. We record the following facts:

- As the sets $T^{[k/2]} M_k$ do not overlap, invariance of the measure implies $\mu(H_m) = \mu(H'_m)$.
- The set H'_m has two components, one on the arc Γ_1 and another on the arc Γ_2.
- On H'_m, $\gamma \sim \frac{\pi}{2}$ (or in other words, the coordinate φ is close to 0) so the density of the measure μ is $\sin\gamma = \cos\varphi \approx 1$, μ is proportional to Lebesgue measure in leading order.
- Points of H'_m are characterized by return time at least m, which, by Formulae (4.11) and (4.12), means $\frac{\kappa}{2m} \leq \sqrt{I} \sim \alpha$ (recall again $\sin\gamma \sim 1$ on H'_m).
- The dimensions of H'_m in the γ direction are determined by the amount of change in the γ coordinate in course of one iteration of T, which, by (4.10), is equal to 2α.

Thus H'_m consists of two components (on the two arcs Γ_1 and Γ_2) both of which look like

$$\left\{(\alpha,\gamma)\,|\,0 \leq \alpha \leq \frac{\kappa}{2m};\ \gamma_1(\alpha) \leq \gamma \leq \gamma_2(\alpha)\right\}; \quad \text{with } \gamma_2(\alpha) - \gamma_1(\alpha) \sim 2\alpha.$$

This gives (recall $c_\mu = (2\text{length}(\Gamma))^{-1}$, the normalizing factor for the invariant measure μ.)

$$\mu(H'_m) \sim 2c_\mu \int_0^{\frac{\kappa}{2m}} 2\alpha\, \mathrm{d}\alpha = \frac{c_\mu \kappa^2}{2m^2}.$$

Now, as in course of the calculations we made the simplifying assumption that both of the arcs Γ_1 and Γ_2 are circular with radii 1, in the general case an additional factor $(\mathcal{K}_0)^{-1}$ appears as a scaling between the arclength and the coordinate α. As in Formula (4.1), $\mathcal{K}_0 = \frac{1}{2}(\mathcal{K}_1 + \mathcal{K}_2)$, where $\mathcal{K}_1$ and $\mathcal{K}_2$ are the curvatures of Γ_1 and Γ_2 at the points r_1 and r_2, respectively. We arrive at

$$\hat{\mu}(H_m) = (\mu(\hat{M}))^{-1} \frac{c_\mu \kappa^2}{2\mathcal{K}_0 m^2} \tag{4.15}$$

Equations (4.13), (4.14) and (4.15) together imply

$$\hat{\mu}(\hat{f}_p^2) \sim 2J_f^2 \sum_{m=1}^{p} m\hat{\mu}(H_m) \sim J_f^2 \frac{c_\mu \kappa^2}{\mathcal{K}_0 \mu(\hat{M})} \log p = 2D_{\hat{f}}^2 \log p$$

as in Lemma 4.9.

Here we record some further important facts on the properties $\hat{T} : \hat{M} \to \hat{M}$ and the cells M_m. For proofs, see [BCD11] and [CM07].

The dynamics of unstable curves under $\hat{T} : \hat{M} \to \hat{M}$ is determined by uniform expansion and fragmentation caused by singularities, out of which the former is stronger. This is quantified by the crucial growth lemma which holds for $\hat{T}$ in the form stated in (2.4). Standard pairs and standard families and their Z functions can be defined as in section 2.1. The growth lemma implies Formula (2.5) on the evolution of standard families. As discussed in section 2.2.1, further consequences are that $(\hat{M}, \hat{T}, \hat{\mu})$ can be modeled by a Young tower with exponential tails, the map has exponential decay of correlations as stated in (1.1), which extends to multiple correlations in the sense of (2.8).

The cell $M_m \subset \hat{M}$ has length $\asymp m^{-7/3}$ in the unstable direction and length $\asymp m^{-2/3}$ in the stable direction. The map $\hat{T} = T^m$ is continuous when restricted to M_m, and the image cell $\hat{T}M_m$ has length $\asymp m^{-2/3}$ in the unstable direction and length $\asymp m^{-7/3}$ in the stable direction. Let us denote by ν_m the probability measure obtained by conditioning $\hat{\mu}$ on M_m. Foliating M_m by unstable curves of length $m^{-7/3}$, ν_m can be represented as a standard family $\mathcal{G}$ with Z-function $\mathcal{Z}_{\mathcal{G}} \asymp m^{7/3}$. Introducing $\mathcal{G}_1 = \hat{T}\mathcal{G}$ for the image of this standard family – which represents the probability measure $\hat{T}_* \nu_m$ – we have $\mathcal{Z}_{\mathcal{G}_1} \ll m^{2/3}$. As the induced map satisfies the growth lemma (2.5), this estimate can be propagated to

$$\mathcal{Z}_{\mathcal{G}_n} \ll m^{2/3}, \qquad \forall n \geq 1, \qquad \text{where } \mathcal{G}_n = \hat{T}^n \mathcal{G} \text{ is the standard family representing } \hat{T}^n_* \nu_m, \tag{4.16}$$

which means that the average length of the unstable curves included in $\mathcal{G}_n$ is at least $m^{-2/3}$, see section 2.1 for further details on standard families. In particular, this gives an estimate on the conditional probabilities $\hat{T}^n_* \nu_m(M_k)$. As the size of the cells M_k in the unstable direction is $\asymp k^{-7/3}$, (4.16) implies

$$\hat{T}^n_* \nu_m(M_k) \ll \mathcal{Z}_{\mathcal{G}_n} k^{-7/3} \ll m^{2/3} k^{-7/3}; \qquad \forall n \geq 1$$

by comparing the unstable dimensions, see also (3.5) (and its explanation in section 3.3.3) for a similar estimate in the case of the stadium. This implies

$$\hat{\mu}(M_k \cap \hat{T}^n M_m) = \hat{\mu}(M_m) \cdot \hat{T}^n_* \nu_m(M_k) \ll m^{-7/3} k^{-7/3}, \qquad \forall n \geq 1. \tag{4.17}$$

This estimate expresses that $\hat{T}$ mixes rapidly (if M_k and $\hat{T}^n M_m$ were independent, we would have $\asymp m^{-3} k^{-3}$). Formula (4.17) will play an important role in the proof of Lemma 4.10 below, but before discussing this, let us consider Lemma 4.12.

Proof of Lemma 4.12. Let us consider $f : M \to \mathbb{R}$ Hölder continuous with exponent η, and $\hat{x}, \hat{y} \in M_m \subset \hat{M}$. We have

$$|\hat{f}(\hat{x}) - \hat{f}(\hat{y})| \leq \sum_{i=0}^{m-1} |f(T^i \hat{x}) - f(T^i \hat{y})| \ll \sum_{i=0}^{m-1} \text{dist}(T^i \hat{x}, T^i \hat{y})]^{\eta}. \tag{4.18}$$

The images $T^i(M_m)$, $i = 1, \ldots, m-1$, keep stretching in the unstable direction and shrinking in the stable direction, as i increases (see [CM07, pp. 750–751]), thus we can assume that $\hat{x}, \hat{y}$ lie on one unstable curve.

It was shown in [CM07, Eq. (4.5)] that unstable vectors u at points $\hat{x} \in M_m$ are expanded under $\hat{T} = T^m$ by a factor

$$\|D_{\hat{x}} T^m(u)\| / \|u\| \asymp m \lambda_1 \lambda_{m-1}, \tag{4.19}$$

where

$$\lambda_1 = \|D_{\hat{x}} T(u)\| / \|u\|, \qquad \lambda_{m-1} = \|D_{\hat{x}} T^{m-1}(u)\| / \|D_{\hat{x}} T^{m-2}(u)\|$$

are the one-step expansion factors at two“special” iterations at which the corresponding points $T(\hat{x})$ and $T^{m-1}(\hat{x})$ may come arbitrarily close to ∂M, i.e., experience almost grazing collisions. For this reason λ_1 and λ_{m-1} do not admit upper bounds, they may be arbitrarily large (see [CM07, p. 741]).

For those two iterations with unbounded expansion factors we can use the Hölder continuity (with exponent 1/2) of the original billiard map T, i.e.,

$$\text{dist}(Tx, Ty) \leq C_1 [\text{dist}(x, y)]^{1/2}$$

for some $C_1 > 0$ (see, e.g., [CM06, Exercise 4.50]). Then due to (4.19) for all $i = 2, \ldots, m-2$ we have

$$\text{dist}(T^i \hat{x}, T^i \hat{y}) \ll m \, \text{dist}(T\hat{x}, T\hat{y}) \ll m \, [\text{dist}(\hat{x}, \hat{y})]^{1/2}$$

Lastly, again by the Hölder continuity of T

$$\mathrm{dist}(T^{m-1}\hat{x}, T^{m-1}\hat{y}) \ll [\mathrm{dist}(T^{m-2}\hat{x}, T^{m-2}\hat{y})]^{1/2} \ll m^{1/2}[\mathrm{dist}(\hat{x}, \hat{y})]^{1/4}.$$

Adding it all up according to (4.18) gives

$$|\hat{f}(\hat{x}) - \hat{f}(\hat{y})| \ll m^2[\mathrm{dist}(\hat{x}, \hat{y})]^{\eta/4},$$

which completes the proof of Lemma 4.12.

Proof of Lemma 4.10. Let us introduce two truncation levels $\mathbf{p} \le \mathbf{q}$, the values of which will be determined later. Note that as $\hat{\mu}(\hat{f}) = 0$, we have

$$|\hat{\mu}(\hat{f}_{1,\mathbf{q}})| = |\hat{\mu}(\hat{f}_{\mathbf{q},\infty})| \ll \sum_{m=\mathbf{q}}^{\infty} m\hat{\mu}(M_m) \ll \sum_{m=\mathbf{q}}^{\infty} m^{-2} \ll \mathbf{q}^{-1}.$$

As $\hat{f}|_{M_m}$ has supremum norm $\ll m$ (by construction) and Hölder norm $\ll m^2$ (by Lemma 4.12), the standard correlation estimate for (piecewise) Hölder functions, (1.1) implies

$$\hat{\mu}(\hat{f}_{1,\mathbf{q}} \cdot \hat{f}_{1,\mathbf{q}} \circ T^n) = \hat{\mu}(\hat{f}^0_{1,\mathbf{q}} \cdot \hat{f}^0_{1,\mathbf{q}} \circ T^n) + (\hat{\mu}(\hat{f}_{1,\mathbf{q}}))^2 \ll \mathbf{q}^4\beta^n + \mathbf{q}^{-2}$$

for some $\beta < 1$ that depends only on the Hölder exponent η of f. Now

$$\begin{aligned}\hat{\mu}(\hat{f}_{1,\mathbf{p}} \cdot \hat{f}_{\mathbf{q},\infty} \circ T^n) &\ll \sum_{m=1}^{\mathbf{p}} \sum_{k=\mathbf{q}}^{\infty} mk\hat{\mu}(M_m \cap \hat{T}^n M_k) \ll \mathbf{p} \sum_{k=\mathbf{q}}^{\infty} k\hat{\mu}\left((\cup_{m=1}^{\mathbf{p}} M_m) \cap \hat{T}^n M_k\right) \\ &\ll \mathbf{p} \sum_{k=\mathbf{q}}^{\infty} k\hat{\mu}(M_k) \ll \mathbf{p}/\mathbf{q},\end{aligned}$$

and the same estimate applies to $\hat{\mu}(\hat{f}_{\mathbf{q},\infty} \cdot \hat{f}_{1,\mathbf{p}} \circ T^n)$. Finally, we use (4.17) to obtain

$$\hat{\mu}(\hat{f}_{\mathbf{p},\infty} \cdot \hat{f}_{\mathbf{p},\infty} \circ T^n) \ll \sum_{m=\mathbf{p}}^{\infty} \sum_{k=\mathbf{p}}^{\infty} mk\hat{\mu}(M_m \cap \hat{T}^n M_k) \ll \sum_{m=\mathbf{p}}^{\infty} \sum_{k=\mathbf{p}}^{\infty} m^{-4/3}k^{-4/3} \ll \mathbf{p}^{-2/3}.$$

Now choosing $\mathbf{q} = \beta^{-n/5}$ and $\mathbf{p} = \mathbf{q}^{1/2}$, the above estimates together imply

$$\hat{\mu}(\hat{f} \cdot \hat{f} \circ T^n) \ll \mathbf{q}^4\beta^n + \mathbf{q}^{-2} + \mathbf{p}/\mathbf{q} + \mathbf{p}^{-2/3} \ll \beta^{n/5} + \mathbf{q}^{-1/3} \ll \theta^n; \quad \text{with } \theta = \beta^{\frac{1}{15}}$$

which completes the proof of Lemma 4.10.

4.3.3 Further comments on the proofs of the limit theorems

Based on the description of the corner series and the key Lemmas 4.9, 4.10 and 4.12 from section 4.3.2, here we discuss some ideas from the proofs of the limit theorems of section 4.2. We **start with Theorem 4.6**. In contrast to the case of the stadium from chapter 3, which reduced the analogous nonstandard limit law to a theorem in Young towers, here the convergence in distribution is obtained directly by the Levy continuity theorem, that is, studying the characteristic function of $\frac{\hat{S}_n\hat{f}}{\sqrt{n\log n}}$. More precisely, as the function $\hat{f}$ is unbounded, a first step is to introduce two truncation levels:

$$p = \frac{\sqrt{n}}{(\log n)^{100}}; \qquad \text{and} \qquad q = \sqrt{n}\log\log n;$$

and decompose, using the notations of section 4.3.1, $\hat{f} = \hat{f}_{1,p} + \hat{f}_{p,q} + \hat{f}_{q,\infty}$ (here the power 100 in the denominator of p could be replaced by any sufficiently large number $\omega > 10$). Now by (4.7):

$$\hat{\mu}(\exists i \in \{0, \dots, n-1\} \,|\, \hat{f}_{q,\infty} \circ \hat{T}^i \neq 0) \ll nq^{-2} = (\log\log n)^{-2} \to 0 \quad \text{as } n \to \infty,$$

hence $\frac{\hat{S}_n \hat{f} - \hat{S}_n \hat{f}_{1,q}}{\sqrt{n \log n}}$ tends to zero in probability, so we can replace $\hat{f}$ with $\hat{f}_{1,q}$. We make one further truncation and introduce

$$\tilde{f} = \hat{f}_{1,q} - \hat{f}_{p,q;0}, \tag{4.20}$$

where we recall that $\hat{f}_{p,q;0}(\hat{x}) = \hat{f}_{p,q}(\hat{x}) - (\hat{\mu}(M_{p,q}))^{-1}\hat{\mu}(\hat{f}_{p,q})$ for $\hat{x} \in M_{p,q}$, the centered version of $\hat{f}_{p,q}$. Now, by the correlation estimate of Lemma 4.10 (see also Remark 4.11),

$$\hat{\mu}[(\hat{S}_n \hat{f}_{p,q;0})^2] = \sum_{i=0}^{n-1} \hat{\mu}[(\hat{f}_{p,q} \circ \hat{T}^i)^2] + O(n),$$

while by Lemma 4.9

$$\hat{\mu}[(\hat{f}_{p,q} \circ \hat{T}^i)^2] = \hat{\mu}(\hat{f}^2_{p,q}) \ll \log(q/p) \ll \log\log n \;\; \forall i = 0, ..., (n-1) \text{ and thus}$$
$$\hat{\mu}[(\hat{S}_n \hat{f}_{p,q;0})^2] = n \cdot \hat{\mu}(\hat{f}^2_{p,q}) + O(n) \ll n \log\log n.$$

Hence, for any $\varepsilon > 0$

$$\hat{\mu}(|\hat{S}_n \hat{f}_{p,q;0}| > \varepsilon\sqrt{n \log n}) \ll \frac{n \log\log n}{\varepsilon^2 n \log n} \to 0 \text{ as } n \to \infty$$

by Chebyshev's inequality. Thus, indeed, Theorem 4.6 can be reduced to

$$\frac{\hat{S}_n \tilde{f}}{\sqrt{n \log n}} \overset{\mathcal{D}}{\Longrightarrow} \mathcal{N}(0, D^2_{\hat{f}}) \qquad \text{with respect to } \hat{\mu}, \text{ as } n \to \infty. \tag{4.21}$$

To proceed with the proof of (4.21), the following properties of $\tilde{f}$ (defined in (4.20)) are useful. $\tilde{f}$ vanishes on $M_{q,\infty}$ and takes a constant value

$$(\hat{\mu}(M_{p,q}))^{-1}\hat{\mu}(\hat{f}_{p,q}) \ll p^2 p^{-1} \ll p$$

on $M_{p,q}$. For its moments, we have

$$\begin{aligned}
\hat{\mu}(\tilde{f}) &= \hat{\mu}(\hat{f}_{1,q}) \ll q^{-1}, \\
\hat{\mu}(\tilde{f}^3) &= O\left(\sum_{m=1}^{p} \frac{m^3}{m^3}\right) + O\left(\frac{p^3}{p^2}\right) \ll p, \\
\hat{\mu}(\tilde{f}^4) &= O\left(\sum_{m=1}^{p} \frac{m^4}{m^3}\right) + O\left(\frac{p^4}{p^2}\right) \ll p^2, \\
\hat{\mu}(\tilde{f}^2) &= \hat{\mu}(\hat{f}^2_{1,p}) + O(1) = 2D^2_{\hat{f}} \log p + O(1).
\end{aligned} \tag{4.22}$$

Remark 4.13. *By the above discussion, convergence in distribution can be studied by the Birkhoff sums of $\tilde{f}$. Now $\tilde{f}$ is obtained by truncating $\hat{f}$ at a level $\sim \sqrt{n}$. Accordingly, we have $2D^2_{\hat{f}} \log p \sim D^2_{\hat{f}} \log n$ in the leading term of $\hat{\mu}(\tilde{f}^2)$. This is reflected by the cancellation of a factor 2 in (4.22) when compared to Lemma 4.9.*

(4.21) is proved by Bernstein's big-small block technique already mentioned in section 2.2.1. We refer to [BCD11] for details and here give only a brief sketch of the argument. The interval $0, ..., (n-1)$ is split into alternating big and small blocks of size $P = [n^a]$ and $Q = [n^b]$ for some $0 < b < a < 1$. The total number of big blocks is $K \sim n^{1-a}$ and at the end there may be a leftover block of size $\ll P + Q$. Then

$$\frac{\hat{S}_n \tilde{f}}{\sqrt{n \log n}} = \frac{\hat{S}'_n \tilde{f}}{\sqrt{n \log n}} + \frac{\hat{S}''_n \tilde{f}}{\sqrt{n \log n}}$$

where $\hat{S}'_n \tilde{f}$ is the contribution from big blocks, and $\hat{S}''_n \tilde{f}$ is the contribution from small blocks and the leftover block. Now on the one hand, by Lemmas 4.9 and 4.10 the second moment of $\hat{S}''_n \tilde{f}$ can be controlled, hence it can be neglected by Chebyshev's inequality. On the other hand, the small blocks serve as gaps between big blocks, which thus, thanks to the decorrelation estimates of Lemma 4.10, can be regarded independent. Thus the study of (4.21) can be reduced to the characteristic function

$$\phi_n(t) = \hat{\mu}\left(\exp\left(\frac{it\hat{S}'_n \tilde{f}}{\sqrt{n \log n}}\right)\right) = \left[\hat{\mu}\left(\exp\left(\frac{it\hat{S}_P \tilde{f}}{\sqrt{n \log n}}\right)\right)\right]^K + o(1)$$

where $\hat{S}_P \tilde{f}$ is the contribution from the first big block. By Taylor expansion

$$\exp\left(\frac{it\hat{S}_P \tilde{f}}{\sqrt{n \log n}}\right) = 1 + \frac{it\hat{S}_P \tilde{f}}{\sqrt{n \log n}} - \frac{t^2(\hat{S}_P \tilde{f})^2}{2n \log n} + O\left(\frac{|\hat{S}_P \tilde{f}|^3}{(n \log n)^{3/2}}\right)$$

which is integrated with respect to $\hat{\mu}$, so we need bounds on the moments of $\hat{S}_P \tilde{f}$. By (4.22) and Lemma 4.10:

$\hat{\mu}(\hat{S}_P \tilde{f}) \ll P/q;$

$\hat{\mu}((\hat{S}_P \tilde{f})^2) = P\hat{\mu}(\tilde{f}^2) + O(P) = 2PD_{\tilde{f}}^2 \log p + O(P)$ which gives the leading contribution; and

$\hat{\mu}(|\hat{S}_P \tilde{f}|^3) \leq (\hat{\mu}((\hat{S}_P \tilde{f})^2)\hat{\mu}((\hat{S}_P \tilde{f})^4))^{1/2}$ by the Cauchy-Schwartz inequality.

The fourth moment can be estimated as

$$\hat{\mu}((\hat{S}_P \tilde{f})^4) \sim \sum \hat{\mu}\left(\tilde{f} \circ \hat{T}^{j_1} \cdot \tilde{f} \circ \hat{T}^{j_2} \cdot \tilde{f} \circ \hat{T}^{j_3} \cdot \tilde{f} \circ \hat{T}^{j_4}\right)$$

where the summation is on all ordered sets of indices $0 \leq j_1 \leq j_2 \leq j_3 \leq j_4 \leq P$. By the decorrelation estimates of Lemma 4.10, the main contribution comes from the case when all three time gaps $D_1 = j_2 - j_1$, $D_2 = j_3 - j_2$ and $D_4 = j_4 - j_3$ are small, in particular satisfy $D_j \leq C \log n$ for some (fixed) large constant $C > 0$. The contribution of any such term can be estimated by the Cauchy-Schwartz inequality and (4.22) as

$$\hat{\mu}\left(\tilde{f} \circ \hat{T}^{j_1} \cdot \tilde{f} \circ \hat{T}^{j_2} \cdot \tilde{f} \circ \hat{T}^{j_3} \cdot \tilde{f} \circ \hat{T}^{j_4}\right) \leq \mu(\tilde{f}^4) \ll p^2$$

and the total number of such terms is $\ll P(\log n)^3$, which gives

$$\hat{\mu}((\hat{S}_P \tilde{f})^4) \ll Pp^2 \log^3 n \qquad \Longrightarrow \qquad \hat{\mu}(|\hat{S}_P \tilde{f}|^3) \ll Pp \log^2 n \ll \frac{P\sqrt{n}}{(\log n)^{10}}. \tag{4.23}$$

Note that it is this estimate (4.23) on $\hat{\mu}(|\hat{S}_P \tilde{f}|^3)$ where the second truncation – replacing $\hat{f}_{1,q}$ with $\tilde{f}$, see (4.20) – is needed. Summarizing, we arrive at

$$\hat{\mu}\left(\exp\left(\frac{it\hat{S}_P \tilde{f}}{\sqrt{n \log n}}\right)\right) = 1 - \frac{t^2 PD_{\tilde{f}}^2}{2n} + O\left(\frac{P}{n\sqrt{\log n}}\right)$$

and thus

$$\phi_n(t) = \exp\left(-\frac{t^2 PKD_{\hat{f}}^2}{2n} + O\left(\frac{PK}{n\sqrt{\log n}}\right)\right) + o(1)$$

which, as $PK \sim n$, converges to $\exp\left(-\frac{1}{2}t^2 D_{\hat{f}}^2\right)$, as desired.

The invariance principle, Theorem 4.7 requires two main components: convergence of finite dimensional distributions and tightness. After similar truncation steps (see [BCD11, section 8] for details) as in the proof of the limit law Theorem 4.6, both properties have to be verified for the family of functions

$$\tilde{W}_N(s) = \frac{\hat{S}_{sN}\tilde{f}}{\sqrt{D_{\hat{f}}^2 N \log N}} \qquad 0 < s < 1. \tag{4.24}$$

To derive convergence of finite dimensional distributions it is enough to show that for any $0 < s_1 < \cdots < s_k \le 1$, any sequences

$$\frac{n_1}{N} \to s_1, \frac{n_2}{N} \to s_2, \dots, \frac{n_k}{N} \to s_k,$$

and any fixed $t_1, t_2, \dots, t_k$ we have

$$\hat{\mu}\left(\exp\left(\frac{i\sum_{j=1}^k t_j \hat{S}_{n_j}\tilde{f}}{\sqrt{N\log N}}\right)\right) \to \prod_{j=1}^k \exp\left(-\frac{D_{\hat{f}}^2 (s_j - s_{j-1})^2 T_j^2}{2}\right) \tag{4.25}$$

where $s_0 = 0$ and $T_j = \sum_{r=j}^k t_r$. This convergence can be proved by the same big small block technique as in the proof of Theorem 4.6 above. It remains to show that the family (4.24) is tight. By standard arguments (see eg. [Bil13, section 7]), this requires a uniform control on the modulus of continuity of the functions $\tilde{W}_N$. In particular, it is enough to show that there exists a sequence $\{\delta_k\}$ with $\sum_k \delta_k < \infty$ such that $\hat{\mu}(\tilde{M}_{K,N}) \to 0$ as $K \to \infty$ uniformly in N, where

$$\tilde{M}_{K,N} = \left\{\exists j,k\colon j < 2^k \text{ and } \left|\tilde{W}_N\left(\tfrac{j+1}{2^k}\right) - \tilde{W}_N\left(\tfrac{j}{2^k}\right)\right| > K\delta_k\right\}. \tag{4.26}$$

Let $\delta_k = 1/k^2$. First we estimate the $\hat{\mu}$-measure of

$$\tilde{M}_{K,N,k,j} = \left\{|\hat{S}_{n_1}\tilde{f} - \hat{S}_{n_1}\tilde{f}| \ge \tfrac{1}{k^2} K\sqrt{N\log N}\right\} \tag{4.27}$$

where $n_1 = [jN/2^k]$ and $n_2 = [(j+1)N/2^k]$. Recall that $\tilde{f} = O(p)$, hence

$$\hat{S}_{n_2}\tilde{f} - \hat{S}_{n_1}\tilde{f} = O\left(p(n_2 - n_1)\right) = O\left(pN/2^k\right).$$

Thus the set (4.27) is empty if $2^k/k^2 > N$, in particular if $k > 100\log N$. For $k \le 100\log N$, we use the fourth moment estimate from (4.23) and the Markov inequality to get

$$\begin{aligned}\hat{\mu}(\tilde{M}_{K,N,k,j}) &\le \frac{k^8 \hat{\mu}\left([\hat{S}_{n_2}\tilde{f} - \hat{S}_{n_1}\tilde{f}]^4\right)}{K^4 N^2 \log^2 N} \\ &= O\left(\frac{k^8 (n_2 - n_1) p^2 \log^3 N}{K^4 N^2 \log^2 N}\right) = O\left(\frac{k^8}{K^4 2^k \log^{99} N}\right).\end{aligned}$$

Summing over $j = 0, \dots, 2^k - 1$ and then over $k \le 100\log N$ gives $\hat{\mu}(\tilde{M}_{K,N}) = O(1/K^4) \to 0$ as $K \to \infty$, uniformly in N, which implies the tightness.

The proof of **the standard limit theorem 4.8** in [BCD11, section 7] follows the same line of reasoning as that of Theorem 4.6, with some modifications (in fact, simplifications) as this time $\hat{f}$ has less heavy tails. In particular, as $D_f = 0$, we have $\hat{f}|_{M_m} \ll m^{1-\eta/2}$ where η is the Hölder exponent of f, and thus $\hat{f}$ belongs to L^2, in fact even $\hat{\mu}(\hat{f}^{2+\delta}) < \infty$ for some $\delta > 0$ determined by η. It is enough to use a single truncation level $q = \sqrt{n}\log\log n$ and consider the Birkhoff sums with the standard scaling:

$$\frac{\hat{S}_n\tilde{f}}{\sqrt{n}} \quad \text{for} \quad \tilde{f} = \hat{f}_{1,q}.$$

The moment estimates (4.22) can be replaced by

$$\hat{\mu}(\tilde{f}) \ll q^{-1-\eta/2}, \qquad \hat{\mu}(\tilde{f}^4) \ll q^{2-2\eta}; \qquad \hat{\mu}(\tilde{f}^2) \sim \hat{\mu}(\hat{f}^2) < \infty.$$

Also, Lemma 4.10 remains valid, actually, it extends to the case of $n = 0$, and in particular the sum (4.6) is finite. To proceed, the same type of big-small blocks of size $P = n^a$ and $Q = n^b$ can be used as in the proof of Theorem 4.6, and it remains to consider

$$\exp\left(\frac{it\hat{S}_P\tilde{f}}{\sqrt{n}}\right) = 1 + \frac{it\hat{S}_P\tilde{f}}{\sqrt{n}} - \frac{t^2(\hat{S}_P\tilde{f})^2}{2n} + O\left(\frac{|\hat{S}_P\tilde{f}|^3}{(n)^{3/2}}\right).$$

The main contribution arises from the second moment

$$\hat{\mu}([\hat{S}_P\tilde{f}]^2) \sim P\hat{\mu}(\hat{f}^2) + 2\sum_{k=1}^{P-1}(P-k)\hat{\mu}(\hat{f}\cdot\hat{f}\circ\hat{T}^k) \sim P\sigma^2_{\hat{f}}$$

with $\sigma^2_{\hat{f}}$ from (4.6). In particular

$$\hat{\mu}\left(\exp\left(\frac{it\hat{S}_P\tilde{f}}{\sqrt{n}}\right)\right) = 1 - \frac{t^2P\sigma^2_{\hat{f}}}{2n} + O\left(\frac{P}{n^{1+\eta/4}}\right)$$

which, when raised to the power $K \sim n^{1-a}$ (the number of big blocks), gives the desired result.

The proof of **the statistical properties for the flow** in dispersing billiards with cusps stated in Theorem G can be reduced to general results on statistical properties for suspension flows of Young towers. In particular, if a suspension flow $(\Phi^t, \hat{M}^{\hat{\tau}}, \hat{\mu}^{\hat{\tau}})$ satisfies the three properties below, then it has rapid mixing (by [Mel09]) and satisfies the almost sure invariance principle (by [MN09]). The three required properties are as follows:

1. The base transformation $(\hat{M}, \hat{T}, \hat{\mu})$ can be modelled by a *Young tower with exponential tails*.

2. The roof function $\hat{\tau} : \hat{M} \to \mathbb{R}^+$ satisfies *a mild non-integrability condition*. We refer to [BBM19, Proposition 6.6] for the precise form of this condition – it suffices if there exist three periodic points x_1, x_2, x_3 such that $(\hat{\tau}(x_1) - \hat{\tau}(x_3))/(\hat{\tau}(x_2) - \hat{\tau}(x_3))$ is Diophantine, see also Proposition 5.17. An alternative sufficient condition can be formulated in terms of the range of the temporal distance function, see Lemma 5.18.

3. The roof function $\hat{\tau} : \hat{M} \to \mathbb{R}^+$ is *uniformly piecewise Hölder continuous* in the following sense. The map $\hat{T} : \hat{M} \to \hat{M}$ has countably many singularities: there exists a countable partition $\hat{M} = \bigcup_{m\geq 1} M_m$ such that $\hat{T}|_{M_m}$ is continuous for any $m \geq 1$. A function $f : \hat{M} \to \mathbb{R}$ is uniformly piecewise Hölder continuous if for some $\eta \in (0, 1]$ we have $\sup_{m\geq 1} \|f|_{M_m}\|_\eta < \infty$, where $\|\cdot\|_\eta$ denotes the η-Hölder norm.

See also section 5.3.3, in particular Remark 5.19. To proceed with the verification, it is an important observation that the flow in dispersing billiards with cusps can be regarded as a suspension flow in several different ways. In particular, as the notation in the above list of required properties suggests, instead of the original billiard map it is the induced map $(\hat{M}, \hat{T}, \hat{\mu})$ that is taken as the base transformation, which, as discussed in the preceding sections, satisfies property 1. The relevant roof function is then the induced roof function $\hat{\tau} : \hat{M} \to \mathbb{R}^+$; $\hat{\tau}(\hat{x}) = \sum_{k=0}^{r_{\hat{M}}(\hat{x})-1} \tau(T^k \hat{x})$. Property 2. holds as dispersing billiard flows preserve a contact structure, see section 2.2.2 and [BBM19, section 8.4]. It remains to prove property 3., in particular that there exists $\eta \in (0, 1]$ and $C > 0$ such that, for any $m \geq 1$ and $\hat{x}, \hat{y} \in M_m$, $|\hat{\tau}(\hat{x}) - \hat{\tau}(\hat{y})| \leq C|\hat{x} - \hat{y}|^\eta$. This is implied by the following two bounds:

$$(i)\ |\hat{\tau}(\hat{x}) - \hat{\tau}(\hat{y})| \ll m^2 |\hat{x} - \hat{y}|^{1/8};\ \forall m \geq 1, \forall \hat{x}, \hat{y} \in M_m;$$
$$(ii)\ |\hat{\tau}(\hat{x}) - \hat{\tau}(\hat{y})| \ll m^{-1/3};\ \forall m \geq 1, \forall \hat{x}, \hat{y} \in M_m.$$

Indeed, (i) and (ii) together imply

$$\begin{aligned} |\hat{\tau}(\hat{x}) - \hat{\tau}(\hat{y})| &\leq |\hat{\tau}(\hat{x}) - \hat{\tau}(\hat{y})|^{\frac{1}{7}} \cdot |\hat{\tau}(\hat{x}) - \hat{\tau}(\hat{y})|^{\frac{6}{7}} \\ &\ll m^{2/7} |\hat{x} - \hat{y}|^{\frac{1}{56}} \cdot m^{-2/7} \ll |\hat{x} - \hat{y}|^{\frac{1}{56}}. \end{aligned}$$

Now (i) is just Lemma 4.12 combined with the 1/2-Hölder continuity of the billiard flow. Property (ii) relies on the fact that the total flow time spent in $M \setminus \hat{M}$ – the vicinity of the cusp – shrinks uniformly as the length of the excursion (understood in terms of the discrete time of billiard reflections) grows. This is expressed by

$$\tau(T\hat{x}) + ... + \tau(T^{r_{\hat{M}}(\hat{x})-2}\hat{x}) \ll m^{-1} \qquad \forall \hat{x} \in M_m,$$

see [CM07, Section 3]. It is important to note that this estimate also provides uniform bounds on the supremum component of the Hölder norm of $\hat{\tau}|_{M_m}$. On the other hand the diameter of M_m satisfies $\ll m^{-2/3}$ (see the discussion preceding Formula (4.16)), which again by the 1/2-Hölder continuity of the billiard flow implies

$$|\tau(\hat{x}) - \tau(\hat{y})| \ll m^{-1/3}; \qquad \text{and} \qquad |\tau(T^{r_{\hat{M}}(\hat{x})-1}\hat{x}) - \tau(T^{r_{\hat{M}}(\hat{x})-1}\hat{y})| \ll m^{-1/3}$$

by time reversibility. These estimates together imply the bound (ii), completing the verification of property 3.

Remark 4.14. *With a similar argument, in [BM08, Section 3.1] the rapid mixing and the almost sure invariance principle are proved also for another class of billiard flows called Bunimovich flowers. The boundary components Γ_i (where $\partial Q = \cup_{i=1}^I \Gamma_i$) for these billiards are either dispersing or focusing, and the focusing components are, actually, circular arcs satisfying some further conditions that ensure the defocusing mechanism described in section 3.1, see [BM08, section 3.1] for details. As flat components, like the parallel segments in the stadium, are excluded, bouncing does not occur, and the only intermittent effect corresponds to sliding along circular arcs, see Figure 3.4. Now sliding orbit segments may consist of an arbitrary high number of collisions, but are uniformly bounded in flow time, and thus for Bunimovich flowers the flow has stronger statistical properties than the map.*

Finally, let us comment on the **convergence of the second moments**. The doubling effect expressed in the statement of Theorem H is related to the fact that, when studying convergence in distribution for the induced Birkhoff sum $\hat{S}_n \hat{f}$, $\hat{f}$ can be truncated at the level $\sim \sqrt{n}$, see Remark 4.13. However, when studying convergence of the second moment, values of $\hat{f}$ up to the level $\sim n$ are relevant, which result in a doubled value for the (truncated) second moment. For a detailed proof, see [BCD17], to give some impression, a short computation is included here. Recall that, by [CM07] and [CZ08], $\mu(f \cdot f \circ T^n) = Corr(f, f; n) \ll 1/n$. Hence

$$\mu([S_n f]^2) = n \cdot \mu\left(f \cdot \left(\sum_{k=-n}^{n} f \circ T^k\right)\right) + O(n). \tag{4.28}$$

To proceed, recall the notation that $M_m = \{\hat{x} \in \hat{M} \,|\, r_{\hat{M}}(\hat{x}) = m\}$ and introduce $L_m = \cup_{j \le m} M_j \subset \hat{M}$ and also $\mathcal{M}_m = \cup_{k=0}^{m-1} T^k M_m \subset M$, which is essentially the "column above" M_m. The sets $\mathcal{M}_m$, $m \ge 1$ give a partition of M. Let, furthermore, $\mathcal{H}_m = \cup_{j>m} \mathcal{M}_j$ and $\mathcal{L}_m = M \setminus \mathcal{H}_m = \cup_{j \le m} \mathcal{M}_j$. Note that $\hat{\mu}(M_m) \ll m^{-3}$ and thus $\mu(\mathcal{M}_m) \ll m^{-2}$ while $\mu(\mathcal{H}_m) \ll m^{-1}$. Now

$$\begin{aligned}\mu\left(f \cdot \left(\sum_{k=-n}^{n} f \circ T^k\right)\right) &= \int_{\mathcal{L}_{n/10}} \left(f(x) \cdot \left(\sum_{k=-n}^{n} f(T^k x)\right)\right) d\mu(x) + O(1) = \\ &= \sum_{m=1}^{n/10} \sum_{j=0}^{m-1} \int_{T^j M_m} \left(f(x) \cdot \left(\sum_{k=-j}^{m-j-1} f(T^k x)\right)\right) d\mu(x) + O(1) = \\ &= \mu(\hat{f}^2 \cdot \mathbf{1}|_{L_{n/10}}) + O(1) = 2(\log n) D_f^2 (1 + o(1)),\end{aligned} \tag{4.29}$$

which, when combined with (4.28) and Lemma 4.9, gives (4.4) in Theorem H.

It is the second step in (4.29) that requires some justification in the above computation. It expresses the fact that, for $x \in \mathcal{M}_m$, it is only points "in the same column" that make a leading contribution to the correlations, the interactions between distinct columns can be neglected. This essentially follows from Lemma 4.10 on the decay of correlations for the induced map $\hat{T}$. For further details [BCD17] is referred. We would also like to note that, for the special class of (Hölder) functions $f : M \to \mathbb{R}$ that vanish on $M \setminus \hat{M}$, (4.4) actually follows from the correlation asymptotics recently obtained in [BMT21], in particular from Formula (3.7).

As mentioned in section 4.2.3, the doubling effect is of entirely probabilistic origin, it is a consequence of the tail distribution of $\hat{f}$, that is, Lemma 4.9. To demonstrate this, the description of a probabilistic model from [BCD17, Appendix A] is included here. This is a stochastic process $\xi(t)$ that has continuous time $t > 0$ and takes values ± 1. Switching from one value to the other occurs at random moments $0 < T_0 < T_1 < \cdots$, and intervals between switching times, $R_k = T_k - T_{k-1}$, are independent identically distributed random variables with a polynomial tail bound $\mathbb{P}(R_k > x) \sim cx^{-2}$ for $x \to \infty$. We denote by $\mathbb{E}(R_k) = \mu$ their common mean value.

We note that T_0 can be chosen so that the sequence $\{T_k\}$ will be stationary in the following sense. For each $t > 0$, denote $m(t) = \min\{m \ge 0 \colon T_m > t\}$ and $H(t) = T_{m(t)} - t$. Then the stationarity means that

$$\mathbb{P}(H(t) > u) = \mathbb{P}(T_0 > u)$$

does not depend on t (for each $u > 0$). By [Fel57], Chapter XI, Equation (4.6) we have

$$\mathbb{P}(T_0 > t) = \frac{1}{\mu} \int_t^\infty \mathbb{P}(R_k > x)\, dx \sim \frac{c}{\mu t}.$$

Now we define our process $\xi(t)$. Let $\xi_0, \xi_1, \ldots$ be i.i.d. random variables taking values ± 1, each with probability $1/2$. We set $\xi(t) = \xi_k$ if $t \in [T_{k-1}, T_k]$ and $\xi(t) = \xi_0$ if $t < T_0$.

Now consider $S(T) = \int_0^T \xi(t)\, dt$. Denote $\mathcal{R}_k = R_k \xi_k$ and $S_m = \sum_{k=0}^{m} \mathcal{R}_k$. Then obviously $S(T) \sim S_{m(T)}$ as $T \to \infty$. By [Fel57], Section XVII.5, Theorem 2 the sequence $S_m/\sqrt{mV_m}$

converges in distribution to the standard normal law $\mathcal{N}(0,1)$, where

$$V_m = \int_1^{\sqrt{cm}} x^2 \, d\mathbb{P}(R_k < x) \sim c \ln m. \tag{4.30}$$

By the Law of Large Numbers, $m(T) \sim T/\mu$, so that

$$\frac{S(T)}{\sqrt{T \ln T}} \Rightarrow \mathcal{N}\left(0, \frac{c}{\mu}\right)$$

as $T \to \infty$. On the other hand,

$$\mathbb{E}\big(S^2(T)\big) = 2 \iint_{0<s<t<T} \mathbb{E}(\xi(s)\xi(t)) \, ds \, dt.$$

Since the ξ_k's are independent, we have

$$\mathbb{E}(\xi(s)\xi(t)) = \mathbb{P}\big(H(s) > t-s\big) = \mathbb{P}(T_0 > t-s).$$

Accordingly

$$\mathbb{E}\big(S^2(T)\big) \sim 2 \int_{0<s<T} \frac{c}{\mu} \ln(T-s) \, ds \sim \frac{2c}{\mu} T \ln T \tag{4.31}$$

hence we observe the doubling effect again. It can be traced to the upper limit $\sqrt{cm}$ in the integration (4.30). If we change it to cm, then V_m would double and would match the asymptotics of the second moment (4.31).

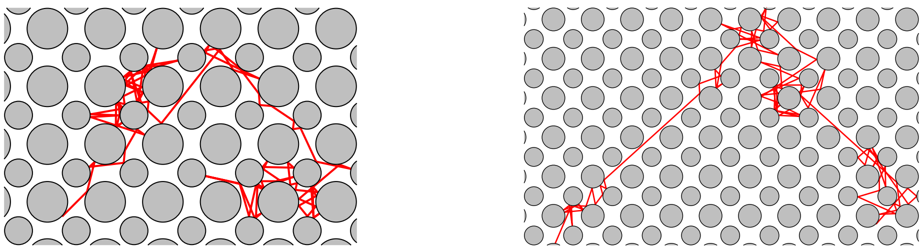
Chapter 5

Infinite horizon

5.1 Description

In this Chapter planar dispersing billiard models are considered, however, in contrast to Chapter 4, we no longer allow the boundary components to touch or intersect. Thus the strictly convex scatterers – which, for simplicity, we may think of as circular disks – are disjoint. Nevertheless, we relax the assumptions of Chapter 2 in a different direction: *infinite horizon* tables are considered, that is, there is no uniform upper bound on the length of the free flight between two consecutive collisions.

The difference between finite and infinite horizon configurations is particularly apparent if one considers the infinite table obtained by unfolding the billiard configuration from $\mathbb{T}^2$ to the plane. The obtained billiard system, corresponding to a periodic configuration of scatterers on $\mathbb{R}^2$, is called the *periodic Lorentz gas*. See Figure 5.1 for a comparison of the finite and the infinite horizon situations. Lorentz gas models were introduced by Hendrik Lorentz in 1905 ([Lor05]) to model the motion of electrons in metals. Here the scatterers correspond to the atoms and the billiard particle corresponds to the electron. As Figure 5.1 suggests, finite and infinite horizon Lorentz gases are popular mechanical models for diffusion and superdiffusion, respectively.



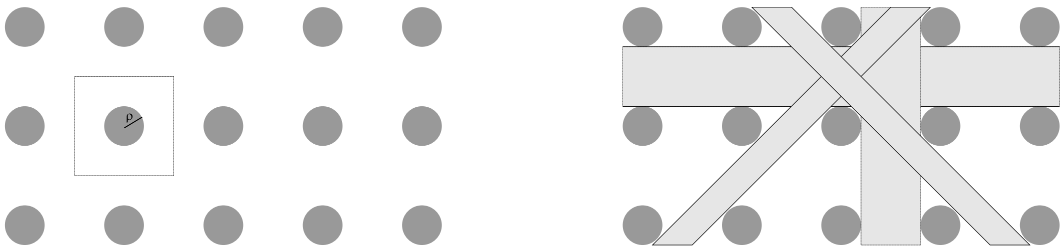
(a) Finite horizon.   (b) Infinite horizon

Figure 5.1: Finite and infinite horizon Lorentz gases

Figure 5.1b also demonstrates that long flights occur along the so-called *corridors*, that is, infinite scatterer-free stripes, which are aligned with specific directions on the plane. Corridors provide the core phenomena responsible for superdiffusive motion.

A remarkably simple way of constructing a Lorentz gas is to remove a circular scatterer of radius $\rho < \frac{1}{2}$ about each point of the Euclidean lattice $\mathbb{Z}^2$, see Figure 5.2a. The obtained billiard configuration necessarily has infinite horizon for any value of ρ, however, the corridor structure

– and thus the characteristics of superdiffusivity – do depend on the scatterer size. As ρ shrinks, more and more corridors open up parallel to rational directions $\xi = (p, q) \in \mathbb{Z}^2$, see Figure 5.2b. This particular model will be one of our main interests throughout Chapter 5.



(a) The configuration (b) The corridors

Figure 5.2: Lorentz gas with a single circular scatterer of radius $\rho < \frac{1}{2}$ on $\mathbb{Z}^2$

Some comments on the terminology. When studying the billiard model on the torus $\mathbb{T}^2$ (with compact phase space) we will use the expression "dispersing billiard (possibly with infinite horizon)" as opposed to the billiard dynamics on the infinite, periodic configuration on $\mathbb{R}^2$, which we will refer to as "the periodic Lorentz gas". The later is a $\mathbb{Z}^2$ extension of the former, see section 5.2 for details. This is the most common terminology in the literature, although there is some ambiguity.[1] A further comment is that throughout Chapter 5 we consider *periodic Lorentz gases*, as opposed to aperiodic models which cannot be reduced to billiards on the torus. In particular, one of our main concerns is to prove limit laws for the displacement of the billiard particle in the periodic Lorentz gas (see below for details), where randomness arises only from the initial conditions. In *random Lorentz gases* (see eg. [LT20]) there is additional randomness arising from the location of the scatterers (sampled, for example, according to a Poisson point process); this in particular implies that the horizon is almost surely finite.

Let us turn back to the investigation of the map $T : M \to M$ for dispersing billiards with infinite horizon, with the Liouville invariant measure μ. It turns out that in several respects it resembles the *induced maps* arising in stadia and dispersing billiards with cusps studied in the previous two Chapters. In particular, there is a competition between expansion and singularities, which is most strikingly present at the accumulating singularity structures. These occur in the vicinity of singular periodic orbits that arise at the boundaries of corridors, see Figure 5.3a, and section 5.3 for further details. However, as in the induced maps mentioned above, "expansion prevails fractioning" which can be quantified in growth lemmas. As a consequence, most of the strong statistical properties of the map (M, T, μ) discussed in Chapter 2 extend to the infinite horizon case. In particular, the map has exponential decay of correlations and the central limit theorem holds for *Hölder continuous functions*. Nonetheless, the main differences between the infinite and finite horizon cases can be captured when studying them from the following two, highly relevant perspectives.

Displacement and discrete displacement functions. Let M denote the dispersing billiard phase space with the single scatterer of radius ρ (depicted on Figure 5.2) and let us introduce the following two, particularly relevant observables (see also Figure 5.3b):

- $\Delta (= \Delta_\rho) : M \to \mathbb{R}^2$ is the displacement function defined as follows. By unfolding, a billiard trajectory starting at the phase point $x \in M$ can be lifted from the billiard on the torus

[1]Sometimes the billiard model on the infinite table with a single particle is called the Lorentz process, while the expression Lorentz gas refers to an infinite collection of billiard particles in the same configuration, see [BGS21]. On the other hand, some papers use the expression Lorentz gas even for the billiard on the torus.

to the planar configuration (the Lorentz gas). Let $Q \in \mathbb{R}^2$ denote the point from which this trajectory emerges, and let $Q_1 \in \mathbb{R}^2$ denote its first impact on the infinite scatterer configuration. Finally let $\Delta(x) = Q_1 - Q$, which can be easily shown to be independent of the lift.

- $\kappa (= \kappa_\rho) : M \to \mathbb{Z}^2$ is a discretized version of Δ. After lifting the trajectory, let $K \in \mathbb{Z}^2$ denote the index of the cell on $\mathbb{Z}^2$ from which the lifted trajectory emerges, and let K_1 denote the index of the cell where the first impact occurs. Then let $\kappa = K_1 - K$, which can be again shown to be independent of the lift.

Our interest in these functions is related to their Birkhoff sums

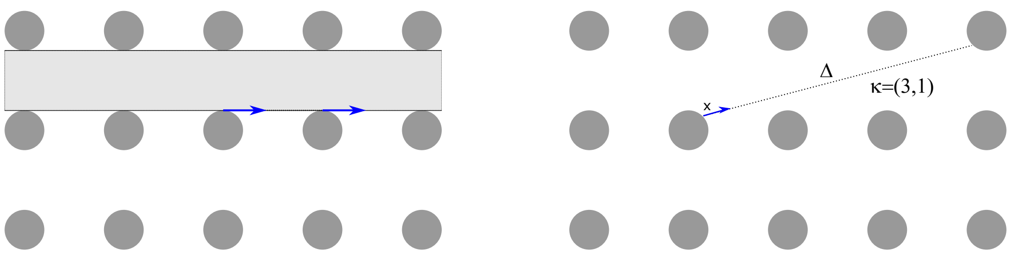
$$\Delta_n(x) = \Delta(x) + \Delta(Tx) + \ldots + \Delta(T^{n-1}x); \qquad \kappa_n(x) = \kappa(x) + \kappa(Tx) + \ldots + \kappa(T^{n-1}x).$$

$\Delta_n(x)$ is precisely the displacement after n iterations in the planar periodic configuration of the Lorentz gas. If x is chosen randomly on the starting scatterer (in particular according to μ on M), then Δ_n can be regarded as a random walk of mechanical origin on $\mathbb{R}^2$. The properties of this random walk thus can be studied via the Birkhoff sums of the particular observable Δ with respect to the billiard map (M, T, μ).

However, it is important to note that in the infinite horizon case $\Delta : M \to \mathbb{R}$ is unbounded. Accordingly, the results on the strong statistical properties of (M, T, μ) for Hölder functions (in particular, the CLT) do NOT apply. In fact, Δ does not even belong to $L^2(\mu)$: the blow-up of the second moment can be studied by some geometrical analysis (see section 5.3) and it actually turns out that Δ belongs to the non-standard domain of attraction of the normal law.

Now the discrete displacement κ is cohomologus to Δ in the following sense. Let $q : M \to \mathbb{R}^2$ denote the location of the point associated to $x \in M$ in $\mathbb{T}^2$ (which is this time identified with the unit square). Then $q(x)$ is bounded (actually smooth) as a function of $x \in M$, and $\Delta(x) = \kappa(x) + q(Tx) - q(x)$. Hence, the discrepancy $\Delta_n - \kappa_n = q(T^n x) - q(x)$ is bounded uniformly in n. Thus, the asymptotic properties of the random walk can be also studied by the Birkhoff sums κ_n.

Let us also mention that $\Delta : M \to \mathbb{R}$ (and naturally also κ) are discontinuous, with infinitely many pieces of continuity. The singularities of these functions coincide with those of the map T mentioned above, and thus accumulate on the singular periodic points of Figure 5.3a, which are also the points where the values of Δ (and κ) blow up.



(a) Singular fixed point at the corridor boundary

(b) Continuous (Δ) and discrete (κ) displacement functions

Figure 5.3: Singularities of the billiard map and the displacement function

The billiard flow. Although the map (M, T, μ) has exponential decay of correlations, the billiard flow has much slower decay rates. As usual, the billiard flow can be regarded as a suspension flow, with roof function $\tau : M \to \mathbb{R}^+$, where $\tau(x) = |\Delta(x)|$, that is, the free flight is the length of the displacement. Accordingly, τ is unbounded. The long stretches of free flight are analogous to the bouncing trajectory segments in the stadium (see Chapter 3), and cause intermittent behaviour for the billiard flow which resembles the intermittency of the billiard maps of stadia and dispersing billiards with cusps.

In accordance with the above two points, our main concerns in Chapter 5 are to

- obtain limit laws for the Birkhoff sums Δ_n and κ_n (corresponding to the superdiffusion in periodic Lorentz gases with infinite horizon); and to
- study mixing rates for the associated billiard flow.

5.2 Results

Throughout, we will use the notations introduced in section 5.1. The dynamics of the Lorentz gas can be also considered as a $\mathbb{Z}^2$-extension of the billiard map (M, T, μ) with $\kappa : M \to \mathbb{Z}^2$. That is, let

$$\tilde{M} = M \times \mathbb{Z}^2; \qquad \tilde{T} : \tilde{M} \to \tilde{M}; \quad \tilde{T}(x, m) = (Tx, m + \kappa(x)) \text{ for } x \in M, m \in \mathbb{Z}^2,$$

with the σ-finite invariant measure $\tilde{\mu} = \mu \times \ell_{\mathbb{Z}^2}$, where $\ell_{\mathbb{Z}^2}$ denotes the counting measure on $\mathbb{Z}^2$.

Preceding results. The Lorentz gas is a popular model of mathematical physics and a complete survey of results is far beyond the scope of this dissertation, we only mention works that are strongly connected to the subject of the present Chapter 5. As mentioned earlier, the billiard map (M, T, μ) on he torus associated to infinite horizon tables has strong statistical properties. Exponential decay of correlations for Hölder observables was first proved by a Young tower construction in [Che99], and then later via a spectral gap for anisotropic Banach spaces constructed on M in [DZ11].

From now on, for simplicity we restrict to the billiard table with *a single circular scatterer of radius* $\rho < \frac{1}{2}$. To point out the dependence on $qrho$, we will occasionally use the notation $T_\rho : M \to M$ instead of $T : M \to M$ for the billiard map. For fixed ρ, the following *non-standard limit theorem* was conjectured by Bleher in [Ble92], and was proved rigorously by Szász and Varjú in [SV07]. There exists a positive definite matrix Σ_ρ such that

$$\frac{\Delta_{n,\rho}}{\sqrt{n \log n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \Sigma_\rho), \text{ as } n \to \infty \qquad \text{where } \Delta_{n,\rho} = \sum_{k=0}^{n-1} \Delta(T_\rho^k x), \tag{5.1}$$

and, as Δ and κ are cohomologous, the same result holds for $\kappa_{n,\rho} = \sum_{k=0}^{n-1} \kappa(T_\rho^k x)$ instead of $\Delta_{n,\rho}$. The subscript ρ will be often omitted for brevity but this time we have included it to indicate that the statement holds for fixed ρ. It is important to note that there is an explicit formula for the covariance matrix. Let

$$X = X_\rho = \left\{ x = (r, \varphi) \in M \;\middle|\; \varphi = \pm\frac{\pi}{2},\, T_\rho x = x \right\}$$

the set of singular fixed points mentioned in the previous section, see Figure 5.3a. Recall that $\kappa(x)$ denotes the value of the discrete free flight. Any point $x \in X_\rho$ determines a corridor parallel to $\kappa(x)$, let $d(\kappa(x))$ denote the width of this corridor. Then

$$\Sigma_\rho = \frac{1}{8\rho\pi} \sum_{x \in X_\rho} \frac{d^2(\kappa(x))}{|\kappa(x)|} \cdot \kappa(x) \otimes \kappa(x). \tag{5.2}$$

One of our main concerns in this Chapter is the small ρ asymptotic of the Lorentz gas, hence it is important to note that

$$\lim_{\rho \to 0} \left(\rho^2 \Sigma_\rho\right) = \Sigma, \qquad \text{where} \quad \Sigma = \frac{1}{4\pi^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{5.3}$$

The Formulas above for Σ_ρ and Σ will be discussed in section 5.3. For the sake of comparison with further results stated below, in particular, Theorem I, note the following consequence of (5.1), (5.2) and (5.3):

$$\frac{\kappa_{n,\rho}}{\rho^{-1}\sqrt{n\log n}} \overset{\mathcal{D}}{\Longrightarrow} \mathcal{N}(0,\Sigma) \quad \text{as first } n\to\infty \text{ and then } \rho\to 0. \tag{5.4}$$

Let us mention that [SV07] also proves the following local limit theorem:

$$\text{For fixed } \rho\in(0,1/2), \quad (n\log n)\mu(\kappa_{n,\rho}=0)\to\phi_{\Sigma_\rho}(0) \text{ as } n\to\infty,$$

where ϕ_{Σ_ρ} is the density of the Gaussian random variable in (5.1). This local limit theorem implies the recurrence of the associated Lorentz process on the plane. The proofs of [SV07] use the Nagaev-Guivarc'h method with the Banach spaces of Young towers, building in several respects on [BG06], see also Chapter 3 of the present dissertation.

Actually, the limit theorem (5.1) was also proved by an alternative method in [DC09]. This is based on a direct estimation of the characteristic function, as in Chapter 4 of the present dissertation. On top of (5.1), [DC09] also proved the weak invariance principle (or functional limit law) for Δ_n, analogous to our Theorem E. Several more recent works, eg. [P̀19], [PT21] and [MPT22], study the mixing properties of the $\mathbb{Z}^2$ extension $(\tilde{M},\tilde{T},\tilde{\mu})$.

The Boltzmann-Grad limit. From a different perspective, in a series of works [MS11a, MS11b], Marklof & Strömbergsson studied the Boltzmann-Grad limit of the periodic Lorentz gas. This corresponds to letting the scatterer size $\rho\to 0$ and investigating the displacement in the rescaled continuous time $T=\rho t$ (so that the mean free path remains bounded). In particular, [MS11a] proves that, in this Boltzmann-Grad limit, the displacement of the particle converges, on any finite time interval, to an explicitly given Markov process. Marklof & Tóth [MT16] then studied the large time asymptotic of this Markov process, and obtained a limit law and a weak invariance principle with the non-standard normalization $\sqrt{T\log T}$.

These results on the Boltzmann-Grad limit scenario hold in any dimension, not just in $d=2$ as the results for fixed ρ mentioned above. For more details, we refer to the original references. What is most relevant for us is that [MT16, Theorem 1.1] and [MT16, Theorem 1.3] are reduced to discrete time statements that can be formulated in terms of the behavior of $\kappa_{n,\rho}$ in the limits $\rho\to 0$ first and then $n\to\infty$. In particular, [MT16, Theorem 1.2] states for $d=2$ that:

$$\frac{\kappa_{n,\rho}}{\rho^{-1}\sqrt{n\log n}} \overset{\mathcal{D}}{\Longrightarrow} \mathcal{N}(0,\Sigma) \qquad \text{as } \rho\to 0 \text{ followed by } n\to\infty, \tag{5.5}$$

where $\kappa_{n,\rho}$ and Σ are as in (5.4). [2]

Results on the joint limit of $\rho\to 0$ and $n\to\infty$. The limit laws (5.4) and (5.5) naturally raise the question (formulated also in [MT16]) if it is possible to handle a simultaneous scaling of $\rho\to 0$ and $n\to\infty$. Theorem I below, a result from [BBT23], makes a step in that direction.

Theorem I. *Let Σ be as in (5.3) and*

$$b_{n,\rho} = \frac{\sqrt{n\log(n/\rho^2)}}{\rho}.$$

There exists a function $M(\rho)$ with $M(\rho)\to\infty$ as $\rho\to 0$ such that

$$\frac{\kappa_{n,\rho}}{b_{n,\rho}} \overset{\mathcal{D}}{\Longrightarrow} \mathcal{N}(0,\Sigma), \text{ as } n\to\infty \text{ and } \rho\to 0 \text{ such that } M(\rho)=o(\log n).$$

[2] Actually, [MT16, Theorem 1.2] is stated for Δ instead of κ, but as mentioned above, these two functions are cohomologous.

For the explicit form of $M(\rho)$, we refer to section 5.3, at this stage we mention that this quantity depends on the rate of correlation decay for Hölder observables as $\rho \to 0$. In particular, as we discussed above, it is known that

$$\left| \int_M \psi_1 \cdot \psi_2 \circ T_\rho^n \, d\mu \right| \le C_\rho(\psi_1, \psi_2) \cdot \hat{\theta}_\rho^n \text{ for all } n \ge 0, \tag{5.6}$$

where $\psi_1 : M \to \mathbb{R}$ and $\psi_2 : M \to \mathbb{R}$ are Hölder continuous, centered, and $\hat{\theta}_\rho < 1$ may depend on the Hölder exponent, while $C_\rho = C_\rho(\psi_1, \psi_2) > 0$ on the Hölder norm of these functions, and both of these constants depend also on ρ (ie. on the billiard table). It is essentially the constant C_ρ and $\hat{\theta}_\rho$ that determine $M(\rho)$ in Theorem I. Let us make two further comments on correlation decay estimates.

- It is important to emphasize that the observable κ_ρ is unbounded, the bound (5.6) does not apply to it. Hence, even for fixed ρ, the autocorrelations of κ_ρ have to be studied directly, as in the analogous Lemma 4.10. Mixing rates of κ_ρ are discussed in more detail in section 5.3 below.
- Several powerful methods have been designed to prove bounds of the form (5.6) – in particular using quasi-compactness of the transfer operator on Young towers ([You98]) or anisotropic Banach spaces ([DZ11]), coupling of standard pairs ([CM06, Chapter 7]) or most recently, Birkhoff cones ([DL21]). However, each of these methods involve some non-constructive compactness argument which is the reason why there is no explicit information available on how the rate of decay (ie. C_ρ and $\hat{\theta}_\rho$) depend on ρ. For instance, in the framework of quasi-compact transfer operators, this corresponds to having effective bounds on the essential spectral radius, but not on the spectral gap. This is a major source of unknown dependence of the quantity $M(\rho)$ on ρ in Theorem I.

It is also important to note that under the condition $M(\rho) = o(\log n)$ we have

$$\frac{b_{n,\rho}}{(\rho)^{-1}\sqrt{n \log n}} \to 1,$$

which shows that our Theorem I is indeed a direct analogue of both (5.4) and (5.5).

Comments on the random Lorentz gas. Let us mention the work of Lutsko and Tóth, [LT20], which studies the analogous question of scatterer size tending to zero simultaneously with time tending infinity in the context of the $d = 3$ dimensional random Lorentz gas. However, the setting of [LT20] differs from ours in several respects. The random Lorentz gas, where additional randomness arises from the placement of the scatterers, has almost surely finite horizon, which results in diffusive behavior, in contrast to the superdiffusivity arising in our case. Furthermore, the starting point of Lutsko & Tóth is the Boltzmann Grad limit of the random Lorentz gas, and accordingly, [LT20] can handle situations when time tends to infinity at a *sufficiently slow* pace in relation to the scatterer size tending to 0. In contrast, the starting point of our work is the superdiffusive limit in the infinite horizon periodic Lorentz gas with fixed scatterer size, and accordingly we can handle situations when time tends to infinity at a *sufficiently fast* pace in relation to the scatterer size tending to 0.

Upper bounds on the rates of mixing of the billiard flow. The other main result of Chapter 5, stated in Theorem J below, concerns the rate of correlation decay in the billiard *flow* associated to infinite horizon Lorentz gases. This can be regarded in the usual way (cf. section 2.2.2) as a suspension flow $(M^\tau, \Phi^t, \mu^\tau)$ of the billiard map (M, T, μ), just this time the roof function τ is unbounded. It is important to point out that in theorem J below we consider the billiard on the torus $\mathbb{T}^2$, and that the result applies to arbitrary infinite horizon configurations, not just the one with a single circular scatterer. Based on the tails of the distribution of τ, it has been conjectured back in [FM88] that the rate of correlation decay for the flow is $O(t^{-1})$.

Note that in the setting of the billiard *maps*, analogous results on $O(n^{-1})$ decay rate had been obtained earlier in the papers [Mar04], [CZ05], [CZ08] for various examples, in particular for planar semi-dispersing billiard maps, which can be obtained eg. by placing a single circular scatterer on the square, instead of the torus. However, as explained in section 2.2.2, studying decay rates of the flow are more difficult than for the map. To demonstrate this, let us recall that for finite horizon dispersing billiards exponential decay rates for the map were first proved in [You98], while for the flow much later, in [BDL18]. In the case of infinite horizon, as mentioned above, the map mixes exponentially on Hölder observables, and the natural question that arises concerns the rates of correlation decay for the flow. Theorem J from [BBM19] gives an affirmative answer to this long standing conjecture (dating back, as mentioned above, to [FM88]). To state it, recall from Definition 3.4 the terminology and notations related to smoothness along the flow direction.

Theorem J. *Consider a planar dispersing billiard flow $(M^\tau, \Phi^t, \mu^\tau)$ with infinite horizon, and assume that $v : M^\tau \to \mathbb{R}$ and $w : M^\tau \to \mathbb{R}$ are smooth along the flow direction. Then $Corr(v, w; t) \ll \frac{1}{t}$.*

In fact, it is enough if one of the two observables, say v, is smooth along the flow direction, and the other is Hölder continuous. Also, it suffices if $v \in C^{m,\eta}(M)$ for some $\eta \in (0, 1]$ and for some fixed $m \geq 1$, which in principle could be estimated with difficulty, but it is likely to be quite large. Also, the above result has been preceded by several works on the decay rates of hyperbolic semiflows and flows. We give a brief overview in section 5.3 below, and refer to the papers [BBM19] and [Mel18] for further details.

5.3 Ingredients of proofs

5.3.1 Geometry of long flights

The purpose of this section is to collect some useful estimates on the distribution of the (discrete) free flight function $\kappa_\rho : M \to \mathbb{Z}^+$. Although often these estimates date back to earlier works, we will mostly refer to the exposition in [BBT23], as one of our main concerns is the dependence on ρ, which typically requires a reconsideration of the arguments.

Let us start with pointing out that throughout, we will use the coordinate $\theta \in \mathbb{S}^1$ (parameterized as $\theta \in [0, 2\pi]$ with endpoints identified) to locate the position on the boundary of the circular scatterer. The coordinate θ is then related to the usual arclength as $r = \rho\theta$. This way we may use the same phase space

$$M = \left\{ (\theta, \varphi) \,|\, \theta \in [0, 2\pi), \varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right] \right\}$$

for every value of ρ. On M, we will use a clockwise orientation for both θ and φ, and in these coordinates the Liouville measure μ takes the form

$$d\mu = \frac{1}{4\pi} \cos\varphi \cdot \mathrm{d}\theta \, \mathrm{d}\varphi \tag{5.7}$$

which is again independent of ρ. Note, however, that the function κ_ρ depends on ρ.

In what follows, first we discuss why the covariance matrix (5.2) has the limiting behaviour (5.3). Then we include Lemma 5.3 to explain why Σ_ρ in (5.2) is the covariance matrix for fixed ρ.

Corridor structure. The argument below is essentially from [BBT23], but see also [Det12] or [NSV14] for similar computations. Let O_ℓ denote the circular scatterer of radius ρ placed at the lattice point $\ell \in \mathbb{Z}^2$. The computation of the asymptotics of $\mu(x \in \partial O_0 \times [-\frac{\pi}{2}, \frac{\pi}{2}] : \kappa_\rho(x) = (p,q))$, in the limit as the norm of $(p,q) \in \mathbb{Z}^2$ tends to infinity is based on the division of the phase space according to corridors. These are infinite strips on $\mathbb{R}^2$ in rational directions given by some $\xi \in \mathbb{Z}^2 \setminus \{0\}$ that are, for ρ sufficiently small, disjoint from all scatterers (but maximal with respect to this property), and periodically repeated under integer translations. As soon as $\rho < \frac{1}{2}$, there are infinite corridors parallel to the coordinate axes. If $\rho < \frac{1}{4}\sqrt{2}$, then corridors at angles of $\pm 45°$ open up, and the smaller ρ becomes, the more corridors open up at rational angles.

Given $0 \neq \xi \in \mathbb{Z}^2$ and $\rho > 0$ sufficiently small, there are two corridors simultaneously tangent to O_0 and O_ξ, one corridor on either side of the arc connecting 0 and ξ. The widths of the corridors are denoted by $d_\rho(\xi)$ and $\tilde{d}_\rho(\xi)$, see Figure 5.4.

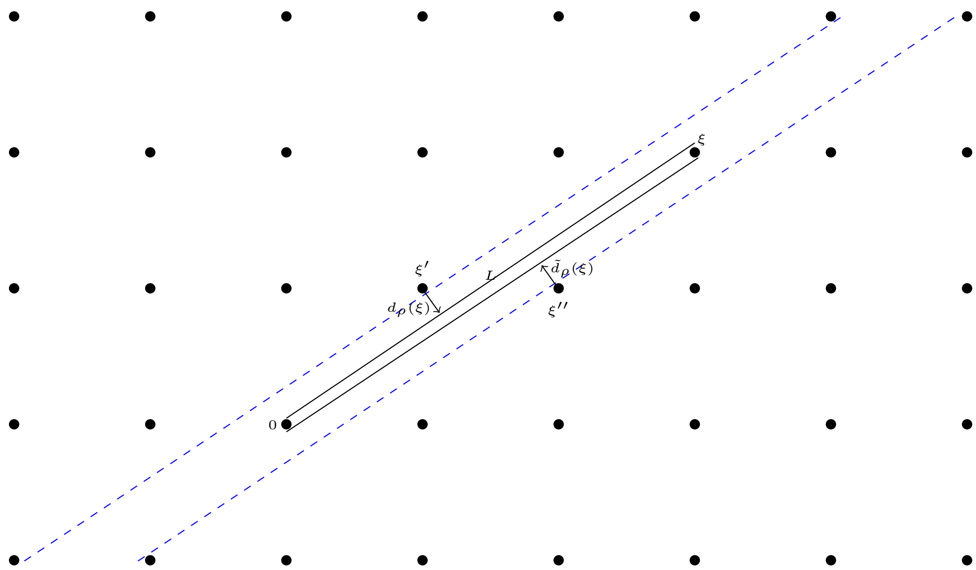


Figure 5.4: Corridors tangent to the scatterers at 0 and $\xi = (3,2)$

Lemma 5.1. *If $\rho = 0$ and $\xi = (p,q) \in \mathbb{Z}^2$ is expressed in lowest terms, then*

$$d_0(\xi) = \tilde{d}_0(\xi) = \frac{1}{|\xi|}.$$

For $\rho > 0$, the actual width of the corridor is then $d_\rho(\xi) = \tilde{d}_\rho(\xi) = \max\{0, |\xi|^{-1} - 2\rho\}$.

Remark 5.2. *Let us call these two corridors in the direction ξ the ξ-corridors. They open up only when $\rho < d_0(\xi)/2 = \tilde{d}_0(\xi)/2$. For $\rho = 0$, the common boundary (called ξ-boundary) of the two ξ-corridors is the line through 0 and ξ. The other boundaries are lines parallel to the ξ-boundary, going through lattice points that are called ξ' and ξ'' in the proof below. For $\xi = (p,q)$ (with $\gcd(p,q) = 1$), these points $\xi' = (p', q')$, $\xi'' = (p'', q'')$ are uniquely determined by ξ in the sense that p'/q' and p''/q'' are convergents preceding p/q in the continued fraction expansion of p/q. In particular $|\xi'|, |\xi''| \leq |\xi|$. In the sequel, we usually only need one of these two ξ-corridors, and we take the one with ξ' in its other boundary.*

Proof. If $(p,q) = (0, \pm 1)$ or $(\pm 1, 0)$, then clearly $d_0(\xi) = \tilde{d}_0(\xi) = 1$, so we can assume without loss of generality that $p \geq q > 0$. Let L be the arc connecting $(0,0)$ to (p,q). The corridors associated to ξ intersect $[0,p] \times [0,q]$ in diagonal strips on either side of L.

Let $\frac{q}{p} = [0; a_1, \ldots, a_n = a]$ be the standard continued fraction expansion with $a \geq 1$, and the previous two convergents are denoted by q'/p' and q''/p'', say $q''/p'' < q/p < q'/p'$ (the other inequality goes analogously). Therefore $q'p - qp' = 1$ and $q''p' - q'p'' = -1$. Also

$$\frac{(a-1)q' + q''}{(a-1)p' + p''} < \frac{q}{p} < \frac{q'}{p'}$$

are the best rational approximations of q/p, belonging to lattice points ξ' above L and ξ'' below L. The vertical distance between ξ' and the arc L is $|q' - p'\frac{q}{p}| = \frac{1}{p}|q'p - p'q| = \frac{1}{p}$. The vertical distance between L and ξ'' is

$$\begin{aligned}
((a-1)p' + p'')\frac{q}{p} - ((a-1)q' + q'') &= \frac{1}{p}((a-1)(qp' - q'p) + qp'' - q''p) \\
&= \frac{1}{p}(1 - a + (aq' + q'')p'' - (ap' + p'')q'') \\
&= \frac{1}{p}(1 - a + a(q'p'' - q''p')) = \frac{1}{p}.
\end{aligned}$$

The corridor's diameter is perpendicular to ξ, so $d_0(\xi)$ is computed from this vertical distance as the inner product of the vector $(0, 1/p)^T$ and the vector $\xi = (p, q)^T$ rotated over $90°$:

$$\frac{1}{\sqrt{p^2+q^2}} \left\langle \begin{pmatrix} 0 \\ 1/p \end{pmatrix} , \begin{pmatrix} -q \\ p \end{pmatrix} \right\rangle = \frac{1}{\sqrt{p^2+q^2}} = \frac{1}{|\xi|}.$$

The computation for $\tilde{d}_0(\xi) = |\xi|^{-1}$ is the same. $\square$

Let us introduce (see also Figure 5.5)

$$\begin{aligned}
\Psi_\rho =& \{\xi = (p, q) \in \mathbb{Z}^2 \setminus 0 \,|\, g.c.d.(|p|, |q|) = 1, p^2 + q^2 \leq (2\rho)^{-2}\}; \\
\Psi_{\rho;0} =& \{\xi = (p, q) \in \Psi_\rho \,|\, 0 \leq q \leq p\}.
\end{aligned}$$

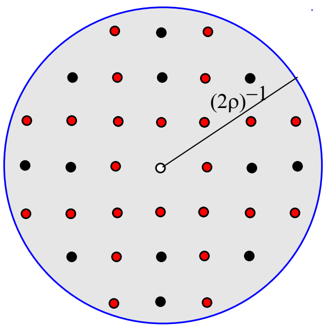


Figure 5.5: The set Ψ_ρ. Red dots are primitive lattice points in $\mathbb{Z}^2$.

To proceed to the analysis of the expression (5.2), instead of summing on points $x \in X_\rho$, we prefer to sum on corridors parallel to some $\xi \in \Psi_\rho$, with $\xi = \kappa_\rho(x)$. Note that to each ξ there correspond two corridors at the "top" and at the "bottom" of the scatterer – the ξ' and the ξ'' corridors on Figure 5.4 – which account for a factor 2 when summing on $\xi \in \Psi_\rho$ instead of $x \in X_\rho$.

Note, furthermore, that to each corridor there also correspond two singular phase points $x \in X_\rho$ pointing in opposite directions, however, this effect is already taken into account by including both (p,q) and $(-p,-q)$ in Ψ_ρ. This way we reduce (5.2) to

$$\Sigma_\rho = \frac{1}{4\rho\pi} \sum_{\xi \in \Psi_\rho} \frac{d^2(\xi)}{|\xi|} \cdot \xi \otimes \xi.$$

Now to reduce to $\Psi_{\rho;0}$, let us introduce the following equivalence relation on $\mathbb{Z}^2$.

$$\text{Let } ((p,q)=)\xi \cong \bar{\xi}(=(\bar{p},\bar{q})) \text{ if } (\bar{p},\bar{q}) = (\pm p, \pm q) \text{ or } (\bar{p},\bar{q}) = (\pm q, \pm p)$$

and note that $|\xi| = |\bar{\xi}|$ and thus $d(\xi) = d(\bar{\xi})$ for $\xi \cong \bar{\xi}$, while for $\xi \in \mathbb{Z}^2$ fixed we have

$$\sum_{\bar{\xi} \cong \xi} \bar{\xi} \otimes \bar{\xi} = 4|\xi|^2 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This implies

$$\Sigma_\rho = A_\rho \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{with} \quad A_\rho = \frac{1}{\rho\pi} \sum_{\xi \in \Psi_{\rho;0}} d^2(\xi) \cdot |\xi| = \frac{1}{8\rho\pi} \sum_{\xi \in \Psi_\rho} d^2(\xi) \cdot |\xi|. \tag{5.8}$$

Now recall that for $f, g : \mathbb{R}^+ \to \infty$, $f(R) \sim g(R)$ means that $\lim_{R\to\infty} \frac{f(R)}{g(R)} = 1$. The rest of the computation relies on the following Formula on the asymptotics of primitive lattice points in a circular disc (see for example [HN96]):

$$\text{for any } a > -2 \text{ we have: } \sum_{\xi \in \Psi_\rho} |\xi|^a \sim \frac{1}{\zeta(2)} \frac{2\pi \cdot (2\rho)^{-a-2}}{a+2} = \frac{12 \cdot (2\rho)^{-a-2}}{\pi(a+2)} \text{ as } \rho \to 0,$$

where ζ is the Riemann zeta function, hence $\zeta(2) = \frac{\pi^2}{6}$. By the above Formulae and Lemma 5.1:

$$A_\rho = \frac{1}{8\rho\pi} \sum_{\xi \in \Psi_\rho} \left(|\xi|^{-1} - 4\rho + 4\rho^2 |\xi| \right) \sim \frac{1}{8\rho\pi} \cdot (2\rho)^{-1} \left(\frac{12}{\pi} - \frac{12}{\pi} + \frac{4}{\pi} \right) = \frac{1}{4\rho^2\pi^2}$$

which implies (5.3).

Tail of κ_ρ. To complete the picture concerning the asymptotic variance in the limit law for κ_ρ, here we discuss the following Lemma.

Lemma 5.3. *For any ρ fixed, we have*

$$\mu(|\kappa_\rho|^2 \cdot \mathbf{1}_{|\kappa_\rho|<R}) \sim 2A_\rho \cdot \log(R) \tag{5.9}$$

as $R \to \infty$, where A_ρ is defined in (5.8) above.

Now by Lemma 5.3 we have that if $\kappa_\rho, \kappa_\rho \circ T_\rho, \dots$ was an i.i.d. sequence with this blow-up of the variance, then (5.4) (and thus (5.1)) would follow. Note the factor 2 in (5.9) which is in accordance with the doubling effect discussed in Theorem H.

Lemma 5.3 actually dates back to [Ble92] and it relies on the fact that large values of κ_ρ occur along corridors. Let is introduce

$$D_{\xi,m} = \{x = (\theta,\varphi) \in M \,|\, \kappa_\rho(x) = \xi' + m\xi\}; \quad \text{for some } \xi \in \Psi_\rho \text{ and } m \geq 1.$$

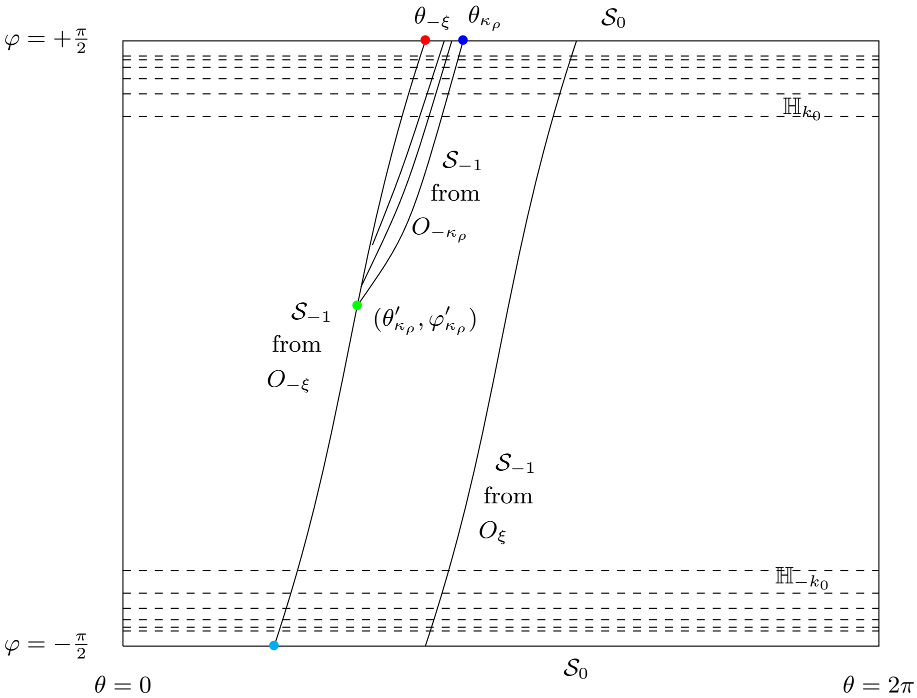


Figure 5.6: The parameter subset M_0 with singularity lines and $\kappa_\rho = \xi' - m\xi$.

Our aim is to obtain estimates on the asymptotics of $\mu(D_{\xi,m})$ as $m \to \infty$. In the computation below, which is from [BBT23], we consider the geometry of $T_\rho D_{\xi,m}$; which is suitable for our purposes as $\mu(D_{\xi,m}) = \mu(T_\rho D_{\xi,m})$ by the invariance of μ. Also, we occasionally use the notation $M_{(0,0)}$ instead of M for the phase space to indicate that we consider the scatterer $O_{(0,0)}$, at the specific position $(0,0) \in \mathbb{Z}^2$.

Recall that $\mathcal{S}_0 = \{\varphi = \pm\frac{\pi}{2}\}$ is a discontinuity of the billiard map corresponding to grazing collisions. The forward and backward discontinuities are

$$\mathcal{S}_n = \cup_{i=0}^{n} T_\rho^{-i}(\mathcal{S}_0) \quad \text{and} \quad \mathcal{S}_{-n} = \cup_{i=0}^{n} T_\rho^{i}(\mathcal{S}_0),$$

so that $T_\rho^n : M \setminus \mathcal{S}_n \to M \setminus \mathcal{S}_{-n}$ is a diffeomorphism. Let us also recall that we line the curve $\mathcal{S}_0$ with homogeneity strips

$$\begin{aligned} \mathbb{H}_k =& \{(\theta,\varphi) \in M \,|\, \pi/2 - k^{-r_0} \leq \varphi \leq \pi/2 - (k+1)^{-r_0}\} \quad \text{for } k \geq k_0, \\ \mathbb{H}_{-k} =& \{(\theta,\varphi) \in M \,|\, -\pi/2 + k^{-r_0} \leq \varphi \leq -\pi/2 + (k+1)^{-r_0}\} \quad \text{for } k \geq k_0, \\ \mathbb{H}_0 =& \{(\theta,\varphi) \in M \,|\, -\pi/2 + k_0^{-r_0} \leq \varphi \leq \pi/2 + k_0^{-r_0}\}; \end{aligned} \tag{5.10}$$

for a fixed number $r_0 > 1$. The standard value is $r_0 = 2$, but as distortion results and some other estimates improve when r_0 is larger, we choose the optimal value of r_0 later.

The set $\mathcal{S}_{-1}$ consists of multiple curves inside $M_{(0,0)}$, one for each scatterer from which a particle can reach $O_{(0,0)}$ in the next collision. In Figure 5.6 we consider the corridor in the direction of $\xi \in \mathbb{Z}^2$, and depict the parts of $\mathcal{S}_{-1}$ coming from scatterers O_ξ, $O_{-\xi}$ and $O_{-\kappa_\rho}$ for some scatterer on the other side of this corridor.

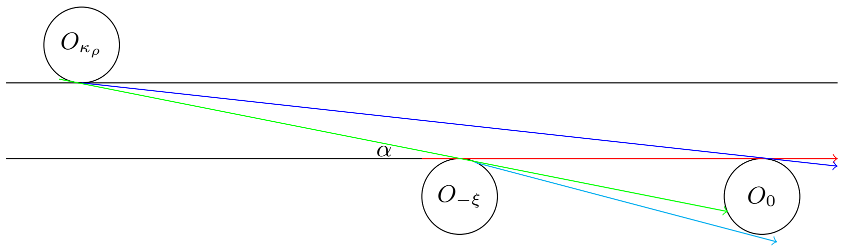


Figure 5.7: A corridor collision map from $O_{-\xi}$ and $O_{-\kappa_\rho}$ to O_0.

Lemma 5.4. *For the ξ-corridor, let $(\theta_{-\xi}, \frac{\pi}{2}) \in M_{(0,0)}$ be the point of intersection of $\mathcal{S}_0$ and the part of $\mathcal{S}_{-1}$ associated to the scatterer $O_{-\xi}$, and $(\theta_{\kappa_\rho}, \frac{\pi}{2}) \in M_{(0,0)}$, $\kappa_\rho \circ T_\rho^{-1} = \xi' - m\xi$, be the point of intersection of $\mathcal{S}_0$ and the part of $\mathcal{S}_{-1}$ associated to the scatterer $O_{\kappa_\rho} = O_{\xi' - m\xi}$ at the other side (i.e., the ξ'-boundary) of the ξ-corridor. Let $(\theta'_{\kappa_\rho}, \varphi'_{\kappa_\rho})$ be the intersection of the parts of $\mathcal{S}_{-1}$ associated to the scatterers $O_{-\xi}$ and the scatterer O_{κ_ρ}. These are the red, blue and green points, respectively, on the phase space sketch of Figure 5.6, with corresponding red, blue and green trajectories, respectively on Figure 5.7. Then*

$$|\theta_{-\xi} - \theta_{\kappa_\rho}| = \frac{d_\rho(\xi)}{|\xi| m}\left(1 + \mathcal{O}\left(\frac{\rho}{|\xi| m}\right)\right)$$

and

$$\frac{\pi}{2} - \varphi'_{\kappa_\rho} = \sqrt{\frac{2 d_\rho(\xi)}{\rho m}}\left(1 - \mathcal{O}\left(\frac{\rho}{|\xi|} - \frac{1}{m} + \frac{\sqrt{d_\rho(\xi)\rho}}{|\xi|\sqrt{m}}\right)\right).$$

Proof. The angle $\theta_{-\xi}$ refers to the point where the common tangent line of $O_{(0,0)}$ and $O_{-\xi}$ touches $O_{(0,0)}$, see the blue trajectory on Figure 5.7. For the value θ_{κ_ρ}, $\kappa_\rho \circ T_\rho = \xi' - m\xi$, we take the common tangent line to $O_{(0,0)}$ and O_{κ_ρ} which has slope $\frac{d_\rho(\xi)}{m|\xi|}\left(1 + \mathcal{O}(\frac{\rho}{|\xi| m})\right)$. This is then also $|\theta_{-\xi} - \theta_{\kappa_\rho}|$.

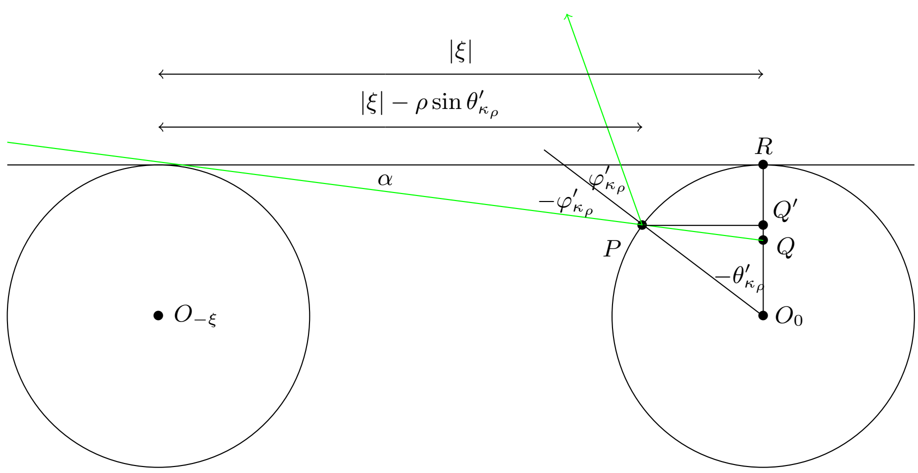


Figure 5.8: Illustration of the proof of Lemma 5.4

Now for the other endpoint of this piece of $\mathcal{S}_{-1}$, consider the common tangent line to $O_{-\xi}$ and O_{κ_ρ} which has slope $\tan\alpha := \frac{d_\rho(\xi)}{(m-1)|\xi|}(1+\mathcal{O}(\frac{\rho}{|\xi|(m-1)}))$, hitting the scatterer $O_{(0,0)}$ in point P. This is the green trajectory on Figure 5.7, with its section between scatterers O_ξ and $O_{(0,0)}$ depicted on Figure 5.8. If this line is extended inside O_0, it hits the vertical line through the center O_0 in the point Q, see Figure 5.8. Let also R be the tangent point of O_0 to the corridor, and Q' is the point on O_0R at the same horizontal height as P, indicated also on Figure 5.8. Then $|RQ| = |\xi|\sin\alpha$ whereas $|O_0Q'| = \rho - (|\xi| - \rho\sin\theta'_{\kappa_\rho})\sin\alpha = \rho\cos\theta'_{\kappa_\rho}$. The latter gives

$$\theta'_{\kappa_\rho} = \sqrt{\frac{2|\xi|}{\rho}\sin\alpha\left(1-\mathcal{O}(\frac{\rho}{|\xi|}\sin\theta)\right)} = \sqrt{\frac{2d_\rho(\xi)}{\rho m}\left(1-\mathcal{O}(\frac{\rho}{|\xi|}-\frac{1}{m})\right)}.$$

The triangle $\triangle PO_0Q$ has angles φ'_{κ_ρ}, $\alpha+\frac{\pi}{2}$ and θ'_{κ_ρ}, which add up to π. Hence

$$\frac{\pi}{2} - \varphi'_{\kappa_\rho} = \alpha + \theta'_{\kappa_\rho} = \sqrt{\frac{2d_\rho(\xi)}{\rho m}}\left(1-\mathcal{O}\left(\frac{\rho}{|\xi|}-\frac{1}{m}+\frac{\sqrt{d_\rho(\xi)\rho}}{|\xi|\sqrt{m}}\right)\right) \tag{5.11}$$

as claimed. □

To denote the m-dependence more explicitly, we will denote $\theta_{\kappa_\rho}, \varphi'_{\kappa_\rho}$ and θ'_{κ_ρ} from the proof of Lemma 5.4 by θ_m, φ'_m and θ'_m. It follows that the sets $T_\rho D_{\xi,m}$ are, in leading order in m, slanted stripes in M, with

$$\text{vertical extension: } \varphi'_m \le \varphi \le \frac{\pi}{2}; \quad \text{with } \frac{\pi}{2} - \varphi'_m \sim \sqrt{\frac{2d_\rho(\xi)}{\rho m}} \quad \text{and}$$

$$\text{horizontal extension: } \theta_{m+1}(\varphi) \le \theta \le \theta_m(\varphi); \quad \text{with } \theta_m - \theta_{m+1} \sim \frac{d_\rho(\xi)}{|\xi| m^2}.$$

Taking into account also the density of μ from (5.7), this implies

$$\mu(D_{\xi,m}) = \mu(T_\rho D_{\xi,m}) \sim \frac{1}{4\pi}\frac{(d_\rho(\xi))^2}{|\xi|\rho m^3}.$$

From this Formula, (5.9) follows by summation.

5.3.2 Further comments on the proof of Theorem I

The proof of Theorem I uses primarily the Nagaaev-Guivarc'h (or perturbed transfer operator) method, described in section 2.2.1, and applied also in the proof of Theorem A (see Chapter 3). However, the standard pairs (on which the proof of Theorem D strongly relies) also play an important role in the argument.

The Nagaaev-Guivarc'h method requires: 1) the existence of a Banach space $(\mathcal{B}, \|\,\|_\mathcal{B})$ on which the transfer operator P_ρ of T_ρ has a spectral gap; 2) the perturbed transfer operator $(P_\rho(t)\psi = P(e^{it\kappa_\rho}\cdot\psi)$ for $\psi\in\mathcal{B})$ has sufficiently good continuity estimates $\|P_\rho(t)-P_\rho(0)\|_\mathcal{B} \le C|t|^\nu$; the larger $\nu > 0$, the better.[3] Regarding 1), it turns out to be more advantageous to use, instead of the Young tower spaces applied in the stadium case (see Chapter 3) or in the paper [SV07] for fixed ρ, the Demers-Zhang anisotropic Banach spaces, see section 2.2.1. We exploit in particular that, using a Lasota-Yorke inequality on the associated strong space $\mathcal{B}$ and weak space $\mathcal{B}_w$, Demers & Zhang [DZ11, DZ13, DZ14] established the spectral gap for every fixed ρ.

[3] In [BBT23] this (perturbed) transfer operator is denoted by R_ρ, however, we prefer to use P_ρ here to keep the notation consistent with the previous chapters of the dissertation.

Regarding 2), as in Keller & Liverani [KL99], one could work with the weak space. In [BBT23] we establish continuity estimates in the strong and weak Banach spaces, neither of which have been obtained previously in the context of infinite horizon dispersing billiards. These continuity estimates are, however, $O(|t|^\nu)$ for $\nu < 1/2$ with explicit dependence on ρ, in both the strong and the weak spaces. This exponent ν is too small to obtain the asymptotics of the leading eigenvalue of $P_\rho(t)$ directly. Therefore we resort to a decomposition of the eigenvalue in several pieces, and exploit a bootstrap of the standard pairs argument from [DC09, Proposition 9.1] (an analogue of Proposition 4.10 from Chapter 4) to deal with some parts of the estimate.

The construction of the Banach spaces and the continuity estimates use the following key ingredients.

Admissible curves. The norms of $\mathcal{B}$ and $\mathcal{B}_w$ are defined by the dual action on test functions supported on a suitable class of admissible curves, $W \in \mathcal{W}^s$, the construction is recalled in [BBT23, section 4], and we refer to [DZ14] for full details. Because of this dual character, instead of evolving unstable curves by T, it is more appropriate to consider the analogous evolution of *stable* curves $W \in \mathcal{W}^s$ under T^{-1}, in particular the growth lemma (see below) can be equivalently formulated in this context.

To belong in the admissible class $\mathcal{W}^s$, as in [DZ14], W has to satisfy some criteria – W is included in a single homogeneity layer, its tangent belongs to the stable cone, and its curvature is uniformly bounded. Nonetheless, on top of that, there is an additional requirement: if $|W|$ denotes the length of the curve, then

$$|W| < \delta_0 \quad \text{where } \delta_0 = c\rho^\nu, \text{ for some uniform } c \ll 1 \text{ and } 0 \le \nu < \frac{1}{2} - \frac{1}{2r_0}. \tag{5.12}$$

We recall that r_0 is the index of the homogeneity layers in (5.10). For $\nu = 0$, the condition (5.12) is the usual uniform (also in ρ) upper bound on the length of admissible curves; the version with positive ν is needed to establish the continuity estimate for the perturbed transfer operator (with the same exponent ν) mentioned above.

Growth lemma. Consider some admissible stable curve $W \in \mathcal{W}^s$, and let V_i $(i = 1, 2, \dots)$ denote the connected and homogeneous components of $T^{-1}W$. Then $W = \bigcup_{i=1}^{\infty} TV_i$, and, by expansion of stable vectors under T^{-1}, the $|V_i|$ are larger than the $|TV_i|$. As mentioned earlier, growth lemmas express that this growth is stronger, in a statistical sense, than the partitioning caused by singularities and the boundaries of homogeneity layers. In our case, Lemma 5.5 below specifically states that this phenomenon is (i) uniform in ρ and (ii) effective even when compared to some weight associated to long flights. By construction, the discrete free flight function κ_ρ takes a constant value on V_i, which we denote by $\kappa(V_i)$. It is also important to note that, as the V_i are homogenous, the distortion bounds on $T|_{V_i}$ are uniform in $W \in \mathcal{W}^s$ and also in ρ, see [BBT23, Appendix B].

Lemma 5.5. *Assume $0 \le \nu < \frac{1}{2} - \frac{1}{2r_0}$. Then there is a constant $C > 0$, uniform in ρ, ν and r_0 such that*

$$\sum_i |\kappa_\rho(V_i)|^\nu \, \frac{|T_\rho V_i|}{|V_i|} \le C\left(\rho + \rho^{-\nu}\,\delta_0\right)$$

for every stable leaf $W \in \mathcal{W}^s$.

Remark 5.6. *(i) Since $|W| \le \delta_0 \le c\rho^\nu$, there is $\theta_* < 1$, uniform in ρ, such that*

$$\sum_{V_i} |\kappa_\rho(V_i)|^\nu \, \frac{|T_\rho V_i|}{|V_i|} \le 3C(\rho + c) \le \theta_*, \tag{5.13}$$

for ρ sufficiently small, and c chosen appropriately small in (5.12). We note that this statement is new even for $\nu = 0$, as θ_* is uniform in ρ.
(ii) As for forthcoming estimates it is useful to have $\nu > \frac{1}{3}$, we can take $r_0 = 5$ and thus $\nu = \frac{3}{8}$.

For the proof of Lemma 5.5, we refer to [BBT23, section 3]. The main challenge is that, when summing on the V_i, we have to take into account a growing number of corridors, yet obtain estimates uniform in ρ.

Banach spaces and spectral gap. As mentioned above, following the exposition of [DZ14], the Banach spaces $\mathcal{B}$ and $\mathcal{B}_w$ are constructed as distributions acting on test functions supported on admissible stable curves. Technically these spaces are obtained by closing $C^1(M)$, the space of continuously differentiable functions, with respect to carefully constructed strong and weak norms. For $h \in C^1(M)$, the action of the transfer operator P_ρ is defined by duality as

$$\int_M (P_\rho h) \cdot \psi \,\mathrm{d}\,(Leb) = \int_M h \cdot \psi \circ T_\rho \,\mathrm{d}\,(Leb)$$

where Leb denotes Lebesgue measure on M. The following properties can be proved as in [DZ14]:

- There exist $\beta > 0$ and $p > 0$ such that $C^\beta(M) \hookrightarrow \mathcal{B} \hookrightarrow \mathcal{B}_w \hookrightarrow (C^p(M))'$, where $C^\beta(M)$ and $(C^p(M))'$ denote the space of Hölder continuous functions, and its dual, respectively.
- The unit ball of $\mathcal{B}$ is relative compact in $\mathcal{B}_W$.
- The transfer operator with the weak and the strong norms satisfies a Lasota-Yorke inequality: there exist some $N \geq 1$ and $\sigma < 1$ (both uniform in ρ) and some $C_{\delta_0} > 0$ (which may depend on δ_0 from (5.12), and thus on ρ) such that

$$\|P_\rho^N h\|_{\mathcal{B}} \leq \sigma^N \|h\|_{\mathcal{B}} + C_{\delta_0} \|h\|_{\mathcal{B}_w}.$$

- By the previous two items the operator P is quasicompact, in particular its essential spectral radius is bounded by some $\sigma < 1$, uniformly in ρ. Outside a disc of radius less than 1, the spectrum of P consists of finitely many eigenvalues of finite multiplicity.
- Mixing of T_ρ with respect to μ implies that the peripheral spectrum consists of the simple eigenvalue 1. This implies that P_ρ has a spectral gap:

$$\begin{aligned} P_\rho^n =& \Pi_\rho + Q_\rho^n; \quad \forall n \geq 1, \quad \text{where} \\ & \Pi_\rho \text{ is the projection onto the 1 dimensional subspace spanned by } \mu \\ & \text{and there exist } C_\rho > 0 \text{ and } \hat{\theta}_\rho < 1 \text{ such that} \\ & \|Q_\rho^n\| \leq C_\rho (\hat{\theta}_\rho)^n. \end{aligned} \tag{5.14}$$

  Here $\gamma_\rho = 1 - \hat{\theta}_\rho$ is the spectral gap of P_ρ. It is important to note that, in contrast to the essential spectral radius σ above, the constants C_ρ and γ_ρ do depend on ρ, and actually, there is no information available on their specific form of dependence.
- The above spectral bounds imply that T_ρ has exponential decay of correlations on Hölder continuous functions, in the sense of (5.6).

Perturbed transfer operator and its leading eigenvalue. The perturbed transfer operator $P_\rho(t)$ is defined, for $t \in \mathbb{R}^2$ as

$$P_\rho(t)h = P_\rho(e^{it\kappa_\rho} h), \qquad \text{for} \quad h \in \mathcal{B}.$$

It is related to the characteristic function of $\kappa_{n,\rho}$ by

$$\mu(e^{it\kappa_{n,\rho}}) = \mu((P_\rho(t))^n \mathit{1})$$

where $\mathit{1} \in \mathcal{B}$ is the function identically equal to 1 on M, and $\mu(\cdot)$ denotes integration with respect to the Liouville measure μ. As discussed in [BBT23, sction 5], it follows from Lemma 5.5 that

$$\|P_\rho(t) - P_\rho\|_{\mathcal{B}} \ll \rho^{-\nu} \cdot |t|^\nu. \tag{5.15}$$

Recall from Remark 5.6 that, with an appropriate choice of r_0, we obtain (5.15) with some $\nu \in (\frac{1}{3}, \frac{1}{2})$, but we can never go up to $\nu = \frac{1}{2}$. This estimate implies that, for small $|t|$, the spectrum of $P_\rho(t)$ depends on t continuously: the spectral gap persists, and there is a leading simple eigenvalue $\lambda_\rho(t)$, with $\lambda_\rho(0) = 1$, see Figure 5.9.

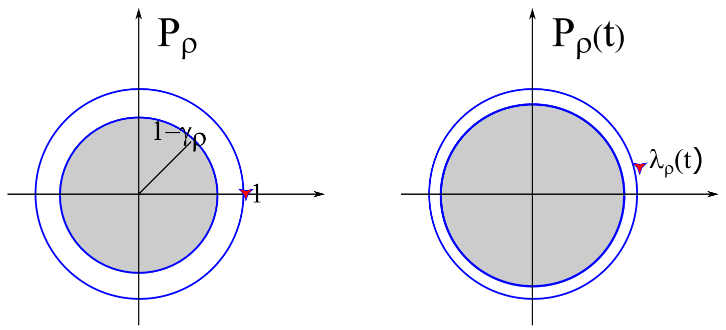


Figure 5.9: The spectra of P_ρ and $P_\rho(t)$.

The asymptotic behaviour of the leading eigenvalue $\lambda_\rho(t)$ for small $|t|$ is the key to Theorem I. As we have the estimate (5.15) only for $\nu < \frac{1}{2}$, which itself is not sufficient to obtain the required expansion, hence it has to be augmented with additional information on the decay of correlations for κ_ρ.

Decay of correlations for κ_ρ. Note that the observable κ_ρ is unbounded, hence the bound (5.6) on the decay of correlations does not apply. The following result by Chernov and Dolgopyat, see [CD09, Proposition 9.1], is analogous to our Lemma 4.10:

$$\begin{aligned}&\text{For fixed } \rho \in (0, 1/2), \text{ there exist } \hat{\vartheta}_\rho \in (0,1) \text{ and } \hat{C}_\rho > 0 \\ &\text{so that } \left|\int_M \kappa_\rho \cdot \kappa_\rho \circ T_\rho^n d\mu\right| \le \hat{C}_\rho \cdot \hat{\vartheta}_\rho^n \text{ for all } n \ge 1.\end{aligned} \tag{5.16}$$

As in the case of Lemma 4.10, the following comments are relevant: the bound does not apply to $n = 0$, as κ_ρ does not belong to $L^2(\mu)$; and the proof is based on standard pair arguments. In [BBT23, Appendix C], this proof is revisited and extended in several directions.

On the one hand, on top of the mere existence of some $\hat{C}_\rho > 0$ and $\hat{\vartheta}_\rho < 1$ in (5.16), we relate these constants to C_ρ and $\hat{\theta}_\rho$ of (5.6) (cf. also (5.14))[4], in particular, we show that (5.16) holds with

$$\hat{C}_\rho = \rho^{-\frac{352}{85}} C_\rho^{\frac{1}{17}}, \ \hat{\vartheta}_\rho = \hat{\theta}_\rho^{\frac{1}{17}}.$$

On the other hand, to exploit correlation bounds of the type (5.16) in the expansion of the leading eigenvalue $\lambda_\rho(t)$, these have to be combined with the action of the transfer operator P_ρ as follows: there exist ρ-dependent constants $\hat{\gamma}_\rho \in (0,1)$ and $\hat{C}_\rho > 0$ so that for every $j \ge 1$,

$$\left|\int_M (e^{it\kappa_\rho} - 1)\,(e^{it\kappa_\rho} - 1) \circ T_\rho^j\, d\mu - \Big(\int_M (e^{it\kappa_\rho} - 1)\, d\mu\Big)^2\right| \le \hat{C}_\rho |t|^2 (1 - \hat{\gamma}_\rho)^j. \tag{5.17}$$

[4]We also use the notations $\gamma_\rho = 1 - \hat{\theta}_\rho$ and $\hat{\gamma}_\rho = 1 - \hat{\vartheta}_\rho$.

More generally, there exist ρ-dependent constants $\bar{\gamma}_\rho \in (0,1)$ and $\bar{C}_\rho > 0$ so that for every $j \geq 1$ and every $m \geq 0$

$$\begin{aligned}
&\Big| \int_M (e^{it\kappa_\rho} - 1) \cdot P_\rho^m (e^{it\kappa_\rho} - 1)\, (e^{it\kappa_\rho} - 1) \circ T_\rho^j \, d\mu \\
&- \int_M (e^{it\kappa_\rho} - 1) P_\rho^m (e^{it\kappa_\rho} - 1)\, d\mu \int_M (e^{it\kappa_\rho} - 1)\, d\mu \\
&- C\Big(\int_M (e^{it\kappa_\rho} - 1)\, d\mu \Big) \int_M (e^{it\kappa_\rho} - 1)\, (e^{it\kappa_\rho} - 1) \circ T_\rho^j \, d\mu \\
&+ C\Big(\int_M (e^{it\kappa_\rho} - 1)\, d\mu \Big)^3 \Big| \leq \bar{C}_\rho |t|^2 (1 - \bar{\gamma}_\rho)^{m+j},
\end{aligned} \tag{5.18}$$

where $C = 0$ if $m = 0$ and $C = 1$ if $m \geq 1$, see [BBT23, Propositon C.1].

Now let

$$M(\rho) = \max\{C_\rho \gamma_\rho^{-2}, \rho^2 \bar{C}_\rho \bar{\gamma}_\rho^{-2}\}$$

with C_ρ, $\gamma_\rho = 1 - \hat{\theta}_\rho$ from (5.14), and $\bar{C}_\rho$, $\bar{\gamma}_\rho = 1 - \bar{\vartheta}_\rho$ from (5.18). Based also on other estimates, especially Lemma 5.3 and the continuity estimate (5.15), in [BBT23, Propositon 6.3] the following expansion is obtained for the leading eigenvalue of $P_\rho(t)$:

$$1 - \lambda_\rho(t) = \mu(1 - e^{it\kappa}) + O(|t|^2 M(\rho)\rho^{-2}) = \frac{1}{\rho^2} \log(1/|t|)(\Sigma t, t) + O(|t|^2 M(\rho)\rho^{-2}), \tag{5.19}$$

where Σ is defined in (5.3). Recall also from the statement of Theorem I that $b_{n,\rho} = \frac{\sqrt{n \log(n/\rho^2)}}{\rho}$. The estimate (5.19), along with the spectral description above, then implies that

$$\begin{aligned}
&\log \left(\mathbb{E}_\mu \left(e^{it \frac{\kappa_{n,\rho}}{b_{n,\rho}}} \right) \right) = \\
&- \frac{n}{\rho^2} \log \left(\frac{\sqrt{n}}{\rho} \frac{\sqrt{\log(n/\rho^2)}}{|t|} \right) \cdot \frac{(\Sigma t, t)}{(b_{n,\rho})^2} + n \cdot O \left(M(\rho)\rho^{-2} \frac{|t|^2}{(b_{n,\rho})^2} \right) = \\
&- \frac{1}{2} (\Sigma t, t) \cdot \frac{\log((\dots) n/\rho^2)}{\log(n/\rho^2)} + O \left(\frac{M(\rho)}{\log n} |t|^2 \right) \longrightarrow -\frac{1}{2} (\Sigma t, t)
\end{aligned}$$

where the limit is taken as specified in Theorem I, that is, $n \to \infty$ along with $\rho \to 0$ such that $M(\rho) = o(\log n)$. This implies that the remainder term tends to 0, while in the main term, we use that the expression $(\dots)$ is $O(\log(n/\rho^2))$. By the Lévy continuity theorem, the limit law of Theorem I follows.

5.3.3 Further comments on the proof of Theorem J

As we mentioned already in Section 2.2.2, a possible approach to study decay of correlations for hyperbolic billiard flows is to regard these as suspension flows of Young towers. This is our viewpoint in [BBM19] for the proof of Theorem J, too, some key steps of which are summarized below.

We collect notations, some of which have been already introduced in Section 2.2.2 (or elsewhere in Chapter 2). However instead of $R \subset M$ we prefer to denote the base of the (hyperbolic) Young tower by $Y \subset M$. Then $r : Y \to \mathbb{Z}^+$ is the height of the tower, that is, $\Delta = \{(y,k) \,|\, y \in Y; 0 \leq k \leq r(y) - 1\}$, and we know that r has exponential tails. The tower map (Δ, F, μ_Δ) is then an extension of the billiard map (M, T, μ). Inside the tower, F just moves up (increases the second coordinate) and thus most relevant is the action on the top level, which is essentially the return map to the base. We will denote this return map by $F_Y : Y \to Y$, which preserves the probability measure μ_Y (obtained by conditioning μ on Y).

Now to move on to the investigation of flows, recall that the billiard flow is a suspension of (M, T, μ) with the roof function $\tau : M \to \mathbb{R}^+$. τ is unbounded, actually, we have already discussed its tail properties in the previous section when studying κ. Indeed, τ and $|\kappa|$ are the length of the free flight and the discrete free flight, respectively, hence their difference is uniformly bounded. We will denote the pullback of τ to the tower as $h : \Delta \to \mathbb{R}^+$. The billiard flow thus can be modeled by its extension, which is the suspension of the tower map (Δ, F, μ_Δ) with the roof function $h : \Delta \to \mathbb{R}^+$. However, it turns out to be more advantageous to model the billiard flow by another suspension flow.

To this end, let

$$\psi : Y \to \mathbb{R}^+; \qquad \psi(y) = \sum_{k=0}^{r(y)-1} h(F^k y)$$

the induced roof function. Let us consider the suspension flow $H^t : Y^\psi \to Y^\psi$ obtained from the base transformation (Y, F_Y, μ_Y) with the roof function $\psi : Y \to \mathbb{R}^+$, which is another option for modelling the billiard flow.[5] In other words, we consider the return map F_Y to the base of the tower Y and the associated induced roof function $\psi : Y \to \mathbb{R}^+$. (Y, F_Y, μ_Y) has great hyperbolic properties, thus the main concern is to analyze $\psi : Y \to \mathbb{R}^+$ from various aspects – such as the tail distribution, the regularity and the non-integrability.

As in Chapters 2 and 3, we will denote the expanding Young towers – obtained, in particular, by collapsing (Δ, F, μ_Δ) along stable leaves – as $(\bar{\Delta}, \bar{F}, \mu_{\bar{\Delta}})$. The return map to the base of the expanding Young tower will be denoted as $(\bar{Y}, \bar{F}_{\bar{Y}}, \mu_{\bar{Y}})$, which is a Gibbs-Markov map, in the sense of [AD01], see Definition 5.7 below (these transformations were already mentioned in section 3.3.4). We note, however, that there is no immediate candidate for the construction of an induced roof function $\bar{\psi} : \bar{Y} \to \mathbb{R}^+$ from $\psi : Y \to \mathbb{R}^+$ as this later function takes variable values along the stable leaves of Y.

The first major step towards the proof of Theorem J was made by Melbourne in [Mel09] where he proved the analogous result in the context of Gibbs-Markov semiflows which are appropriate suspensions of Gibbs-Markov maps, see below for the description. [Mel09] also had results for a class of slowly mixing flows which, however, turned out to be too restrictive to include the billiard examples. Our method of proof in [BBM19] was to extend the result of [Mel09] on Gibbs-Markov semiflows gradually to various classes of further suspension flows. In each case, the base transformation is an invertible map $F_Y : Y \to Y$ which is like the first return map to the base of the (hyperbolic) Young tower, yet, more and more general classes of roof functions $\psi : Y \to \mathbb{R}^+$ are permitted, covering eventually the case of Theorem J. Below we summarize the main steps of this gradual extension.

For brevity, slightly abusing notation, we will denote the return map to the base of the Young tower by $F : Y \to Y$ (instead of $F_Y : Y \to Y$) and the Gibbs-Markov map obtained by collapsing Y along stable leaves, by $\bar{F} : \bar{Y} \to \bar{Y}$ (instead of $\bar{F}_{\bar{Y}} : \bar{Y} \to \bar{Y}$).

Gibbs-Markov maps and semiflows. Gibbs-Markov semiflows are built as suspensions over Gibbs-Markov maps. A standard reference for background material on Gibbs-Markov maps is [AD01].

Suppose that $(\bar{Y}, \mu_{\bar{Y}})$ is a probability space with an at most countable measurable partition $\{\bar{Y}_j,\, j \geq 1\}$ and let $\bar{F} : \bar{Y} \to \bar{Y}$ be a measure-preserving transformation. For $\theta \in (0, 1)$, define $d_\theta(y, y') = \theta^{s(y,y')}$ where the *separation time* $s(y, y')$ is the least integer $n \geq 0$ such that $\bar{F}^n y$ and

[5] In [BBM19] ψ is denoted as φ, and H^t is denoted as F_t, however, we prefer to use ψ and H^t to avoid confusion with other notations used in Chapter 5.

$\bar{F}^n y'$ lie in distinct partition elements in $\{\bar{Y}_j\}$. It is assumed that the partition $\{\bar{Y}_j\}$ separates trajectories, so $s(y, y') = \infty$ if and only if $y = y'$. Then d_θ is a metric, called a *symbolic metric*.

A function $v : \bar{Y} \to \mathbb{R}$ is *d_θ-Lipschitz* if $|v|_\theta = \sup_{y \neq y'} |v(y) - v(y')|/d_\theta(y, y')$ is finite. Let $\mathcal{F}_\theta(\bar{Y})$ be the Banach space of Lipschitz functions with norm $\|v\|_\theta = |v|_\infty + |v|_\theta$.

More generally (and with a slight abuse of notation), we say that a function $v : \bar{Y} \to \mathbb{R}$ is *piecewise d_θ-Lipschitz* if $|1_{\bar{Y}_j} v|_\theta = \sup_{y,y' \in \bar{Y}_j,\, y \neq y'} |v(y) - v(y')|/d_\theta(y, y')$ is finite for all j. If in addition, $\sup_j |1_{\bar{Y}_j} v|_\theta < \infty$ then we say that v is *uniformly piecewise d_θ-Lipschitz*. Note that such a function v is bounded on partition elements but need not be bounded on $\bar{Y}$.

Definition 5.7. *The map $\bar{F} : \bar{Y} \to \bar{Y}$ is called a* (full branch) Gibbs-Markov map *if*

- *$\bar{F}|_{\bar{Y}_j} : \bar{Y}_j \to \bar{Y}$ is a measurable bijection for each $j \geq 1$, and*
- *The potential function $\log(d\mu_{\bar{Y}}/d\mu_{\bar{Y}} \circ \bar{F}) : \bar{Y} \to \mathbb{R}$ is uniformly piecewise d_θ-Lipschitz for some $\theta \in (0, 1)$.*

Definition 5.8. *A suspension semiflow $\bar{H}^t : \bar{Y}^{\bar{\psi}} \to \bar{Y}^{\bar{\psi}}$ is called a* Gibbs-Markov semiflow *if there exist constants $C_1 \geq 1$, $\theta \in (0, 1)$ such that $\bar{F} : \bar{Y} \to \bar{Y}$ is a Gibbs-Markov map, $\bar{\psi} : \bar{Y} \to \mathbb{R}^+$ is an integrable roof function with $\inf \bar{\psi} > 0$, and*

$$|1_{\bar{Y}_j} \bar{\psi}|_\theta \leq C_1 \inf_{\bar{Y}_j} \bar{\psi} \quad \textit{for all } j \geq 1. \tag{5.20}$$

(Equivalently, $\log \bar{\psi}$ is uniformly piecewise d_θ-Lipschitz.) It follows that $\sup_{\bar{Y}_j} \bar{\psi} \leq 2C_1 \inf_{\bar{Y}_j} \bar{\psi}$ for all $j \geq 1$.

To have a mixing semi-flow, we still need to ensure some kind of non-integrability of the roof function. This is captured by the somewhat technical Definition 5.9 below. For $b \in \mathbb{R}$, we define the operators

$$M_b : L^\infty(\bar{Y}) \to L^\infty(\bar{Y}), \qquad M_b v = e^{ib\bar{\psi}} v \circ \bar{F}.$$

Definition 5.9. *A subset $Z_0 \subset \bar{Y}$ is a* finite subsystem *of $\bar{Y}$ if $Z_0 = \bigcap_{n \geq 0} \bar{F}^{-n} Z$ where Z is the union of finitely many elements from the partition $\{\bar{Y}_j\}$. (Note that $\bar{F}|_{Z_0} : Z_0 \to Z_0$ is a full one-sided shift on finitely many symbols.)*

We say that M_b has approximate eigenfunctions *on Z_0 if for any $\alpha_0 > 0$, there exist constants α, $\xi > \alpha_0$ and $C > 0$, and sequences $|b_k| \to \infty$, $\psi_k \in [0, 2\pi)$, $u_k \in \mathcal{F}_\theta(\bar{Y})$ with $|u_k| \equiv 1$ and $|u_k|_\theta \leq C|b_k|$, such that setting $n_k = [\xi \ln |b_k|]$,*

$$|(M_{b_k}^{n_k} u_k)(y) - e^{i\psi_k} u_k(y)| \leq C|b_k|^{-\alpha} \quad \textit{for all } y \in Z_0,\ k \geq 1. \tag{5.21}$$

Remark 5.10. *For brevity, the statement "Assume absence of approximate eigenfunctions" is the assumption that there exists at least one finite subsystem Z_0 such that M_b does not have approximate eigenfunctions on Z_0.*

Below we will state explicitly checkable conditions that guarantee absence of approximate eigenfunctions for some important applications. Most notably, for Gibbs-Markov semiflows arising from hyperbolic billiard examples, absence of approximate eigenfunctions automatically follows from the preservation of a contact structure.

Skew product Gibbs-Markov flows. Here we summarize the terminology for skew product Gibbs-Markov flows from [BBM19, section 3].

Let (Y, d) be a metric space with $\operatorname{diam} Y \leq 1$, and let $F : Y \to Y$ be a piecewise continuous map with ergodic F-invariant probability measure μ_Y. Let $\mathcal{W}^s$ be a cover of Y by disjoint measurable subsets of Y called *stable leaves*. For each $y \in Y$, let $W^s(y)$ denote the stable leaf containing y. We require that $F(W^s(y)) \subset W^s(Fy)$ for all $y \in Y$.

Let $\bar{Y}$ denote the space obtained from Y after quotienting by $\mathcal{W}^s$, with natural projection $\bar{\pi} : Y \to \bar{Y}$. We assume that the quotient map $\bar{F} : \bar{Y} \to \bar{Y}$ is a Gibbs-Markov map as in Definition 5.7, with partition $\{\bar{Y}_j\}$, separation time $s(y, y')$, and ergodic invariant probability measure $\mu_{\bar{Y}} = \bar{\pi}_*\mu_Y$.

Let $Y_j = \bar{\pi}^{-1}\bar{Y}_j$; these form a partition of Y and each Y_j is a union of stable leaves. The separation time extends to Y, setting $s(y, y') = s(\bar{\pi}y, \bar{\pi}y')$ for $y, y' \in Y$.

Next, we require that there is a measurable subset $\widetilde{Y} \subset Y$ such that for every $y \in Y$ there is a unique $\tilde{y} \in \widetilde{Y} \cap W^s(y)$. Let $\pi : Y \to \widetilde{Y}$ define the associated projection $\pi y = \tilde{y}$. $\widetilde{Y}$ can be thought of as one of the unstable leaves included in Y. (Note that $\widetilde{Y}$ can be identified with $\bar{Y}$, but in general $\pi_*\mu_Y \neq \mu_{\bar{Y}}$.)

We assume that there are constants $C_2 \geq 1$, $\gamma \in (0, 1)$ such that for all $n \geq 0$,

$$d(F^n y, F^n y') \leq C_2\gamma^n \qquad \text{for all } y, y' \in Y \text{ with } y' \in W^s(y), \tag{5.22}$$

$$d(F^n y, F^n y') \leq C_2\gamma^{s(y,y')-n} \qquad \text{for all } y, y' \in \widetilde{Y}. \tag{5.23}$$

Let $\psi : Y \to \mathbb{R}^+$ be an integrable roof function with $\inf \psi > 0$, and define the suspension flow[6] $H^t : Y^\psi \to Y^\psi$ as in section 2.2.2 with ergodic invariant probability measure μ_Y^ψ.

In the context of skew product Gibbs-Markov flows, we suppose that ψ is constant along stable leaves and hence projects to a well-defined roof function $\psi : \bar{Y} \to \mathbb{R}^+$. It follows that the suspension flow H^t projects to a quotient suspension semiflow $\bar{H}^t : \bar{Y}^\psi \to \bar{Y}^\psi$. We assume that $\bar{H}^t$ is a Gibbs-Markov semiflow (Definition 5.8). In particular, increasing $\gamma \in (0, 1)$ if necessary, (5.20) is satisfied in the form

$$|\psi(y) - \psi(y')| \leq C_1 \inf_{Y_j}\psi\, \gamma^{s(y,y')} \quad \text{for all } y, y' \in \bar{Y}_j,\ j \geq 1. \tag{5.24}$$

We call H^t a *skew product Gibbs-Markov flow*, and we say that H^t has *approximate eigenfunctions* if $\bar{H}^t$ has approximate eigenfunctions (Definition 5.9).

Fix $\eta \in (0, 1]$. For $v : Y^\psi \to \mathbb{R}$, define

$$|v|_\gamma = \sup_{(y,u),(y',u)\in Y^\psi,\, y\neq y'} \frac{|v(y,u) - v(y',u)|}{\psi(y)\{d(y,y') + \gamma^{s(y,y')}\}}, \qquad \|v\|_\gamma = |v|_\infty + |v|_\gamma,$$

$$|v|_{\infty,\eta} = \sup_{(y,u),(y,u')\in Y^\psi,\, u\neq u'} \frac{|v(y,u) - v(y,u')|}{|u-u'|^\eta}, \qquad \|v\|_{\gamma,\eta} = \|v\|_\gamma + |v|_{\infty,\eta}.$$

(Here $|u - u'|$ denotes absolute value, with u, u' regarded as elements of $[0, \infty)$.) Let $\mathcal{H}_\gamma(Y^\psi)$ and $\mathcal{H}_{\gamma,\eta}(Y^\psi)$ be the spaces of observables $v : Y^\psi \to \mathbb{R}$ with $\|v\|_\gamma < \infty$ and $\|v\|_{\gamma,\eta} < \infty$ respectively.

We say that $w : Y^\psi \to \mathbb{R}$ is *differentiable in the flow direction* if the limit $\partial_t w = \lim_{t\to 0}(w \circ H^t - w)/t$ exists pointwise. Note that $\partial_t w = \frac{\partial w}{\partial u}$ on the set $\{(y, u) : y \in Y,\ 0 < u < \psi(y)\}$. Define $\mathcal{H}_{\gamma,0,m}(Y^\psi)$ to consist of observables $w : Y^\psi \to \mathbb{R}$ that are m-times differentiable in the flow direction with derivatives in $\mathcal{H}_\gamma(Y^\psi)$, with norm $\|w\|_{\gamma,0,m} = \sum_{j=0}^m \|\partial_t^j w\|_\gamma$.

We can now state the main theoretical results for skew product Gibbs-Markov flows from [BBM19]. Theorem 5.11 is [BBM19, Theorem 3.1] while Theorem 5.12 is [BBM19, Theorem 3.2], and are proved in [BBM19, section 4] and [BBM19, section 5], respectively.

Theorem 5.11. *Suppose that $H^t : Y^\psi \to Y^\psi$ is a skew product Gibbs-Markov flow such that $\psi \in L^q(Y)$ for all $q \in \mathbb{N}$. Assume absence of approximate eigenfunctions.*

Then for any $q \in \mathbb{N}$, there exists $m \geq 1$ and $C > 0$ such that

$$|\mathit{Corr}(v, w; t)| \leq C\|v\|_\gamma \|w\|_{\gamma,0,m}\, t^{-q} \quad \text{for all } v \in \mathcal{H}_\gamma(Y^\psi),\ w \in \mathcal{H}_{\gamma,0,m}(Y^\psi),\ t > 1.$$

[6] Strictly speaking, H^t is not always a flow since F need not be invertible. However, H^t is used as a model for various flows, and it is then a flow when ψ is the first return to Y, so it is convenient to call it a flow.

Theorem 5.12. *Suppose that $H^t : Y^\psi \to Y^\psi$ is a skew product Gibbs-Markov flow such that $\mu_Y(\psi > t) = O(t^{-\beta})$ for some $\beta > 1$. Assume absence of approximate eigenfunctions.*
Then there exists $m \geq 1$ and $C > 0$ such that

$$|\,Corr(v, w; t)| \leq C\|v\|_{\gamma,\eta}\|w\|_{\gamma,0,m}\, t^{-(\beta-1)} \quad \text{for all } v \in \mathcal{H}_{\gamma,\eta}(Y^\psi),\ w \in \mathcal{H}_{\gamma,0,m}(Y^\psi),\ t > 1.$$

The proof of these theorems is based on bounding the Laplace transform of the correlation function $Corr(v, w; t)$ by spectral methods, using in particular the absence of approximate eigenfunctions condition, we refer to [BBM19] for details.

General Gibbs-Markov flows. Now we move on to more general classes of flows – in particular, drop the requirement that ψ is constant along stable leaves. This setup will be suitable to cover the billiard examples. Let $F : Y \to Y$ be a map as in the case of skew product Gibbs-Markov flows (essentially, the first return map to the base of the Young tower) with quotient Gibbs-Markov map $\bar{F} : \bar{Y} \to \bar{Y}$, and define $\widetilde{Y}_j = Y_j \cap \widetilde{Y}$. Let $\psi : Y \to \mathbb{R}^+$ be an integrable roof function with $\inf \psi > 1$ and associated suspension flow $H^t : Y^\psi \to Y^\psi$.

We no longer assume that ψ is constant along stable leaves. Instead of condition (5.24) we require that

$$|\psi(y) - \psi(y')| \leq C_1 \inf_{Y_j} \psi\, \gamma^{s(y,y')} \quad \text{for all } y, y' \in \widetilde{Y}_j,\ j \geq 1. \tag{5.25}$$

(Clearly, if ψ is constant along stable leaves, then conditions (5.24) and (5.25) are identical.)

Recall that $\pi : Y \to \widetilde{Y}$ is the projection along stable leaves. Define

$$\chi(y) = \sum_{n=0}^{\infty} (\psi(F^n \pi y) - \psi(F^n y)),$$

for all $y \in Y$ such that the series converges absolutely. We assume

(H) (a) The series converges almost surely on Y and $\chi \in L^\infty(Y)$.

(b) There are constants $C_3 \geq 1$, $\gamma \in (0, 1)$ such that

$$|\chi(y) - \chi(y')| \leq C_3(d(y, y') + \gamma^{s(y,y')}) \quad \text{for all } y, y' \in Y.$$

When conditions (5.25) and (H) are satisfied, we call H^t a *Gibbs-Markov flow.* (If ψ is constant along stable leaves then $\chi = 0$, so every skew product Gibbs-Markov flow is a Gibbs-Markov flow.)

Since $\inf \psi > 0$, it follows that $\psi_n = \sum_{j=0}^{n-1} \psi \circ F^j \geq 4|\chi|_\infty + 1$ for all n sufficiently large. For simplicity we suppose from now on that $\inf \psi \geq 4|\chi|_\infty + 1$ (otherwise, replace F by F^n).

Define

$$\tilde{\psi} = \psi + \chi - \chi \circ F. \tag{5.26}$$

Note that $\inf \tilde{\psi} \geq \inf \psi - 2|\chi|_\infty \geq 1$ and $\int_Y \tilde{\psi}\, d\mu_Y = \int_Y \psi\, d\mu_Y$, so $\tilde{\psi} : Y \to \mathbb{R}^+$ is an integrable roof function. Hence we can define the suspension flow $\widetilde{H}^t : Y^{\tilde{\psi}} \to Y^{\tilde{\psi}}$. Also, a calculation shows that $\tilde{\psi}(y) = \sum_{n=0}^{\infty}(\psi(F^n \pi y) - \psi(F^n \pi F y))$, so $\tilde{\psi}$ is constant along stable leaves and we can define the quotient roof function $\bar{\psi} : \bar{Y} \to \mathbb{R}^+$ with quotient semiflow $\bar{H}^t : \bar{Y}^{\bar{\psi}} \to \bar{Y}^{\bar{\psi}}$.

Now based on the above construction it can be shown that

- $\widetilde{H}^t : Y^{\tilde{\psi}} \to Y^{\tilde{\psi}}$ is a skew product Gibbs-Markov flow.
- the statement on polynomial mixing (Theorem 5.12) extends from the skew product Gibbs-Markov flow $(Y^{\tilde{\psi}}, \widetilde{H}^t, \mu_Y^{\tilde{\psi}})$ to the Gibbs-Markov flow $(Y^\psi, H^t, \mu_Y^\psi)$.

In fact, for the case of general Gibbs-Markov flows, the class of observables for which the bound on decay of correlations holds is slightly more restrictive than for skew product Gibbs-Markov flows, see [BBM19, Theorem 6.4]. Yet, it can be shown that if $v : M^\tau \to \mathbb{R}$ and $w : M^\tau \to \mathbb{R}$ are smooth along the flow direction as in Theorem J, then they lift to observables that belong to this class. See [BBM19, sections 6 & 7] for details.

In the remainder of this section it is briefly sketched how the required conditions are checked for infinite horizon dispersing billiard flows.

Condition (H). Checking condition (H) relies on the following two properties of infinite horizon dispersing billiards:

- The map $T: M \to M$ contracts stable manifolds uniformly: there exist $C > 0$ and $\lambda < 1$ such that for $x, x' \in M$, $x' \in W^s(x)$, $d(T^n x, T^n x') \leq C\lambda^n d(x, x')$. Similarly, T^{-1} contracts unstable manifolds.

- During free flight, the trajectories are straight lines, which implies that $|\tau(x) - \tau(x')| \leq d(x, x') + d(Tx, Tx')$ whenever $x, x' \in M$ are in the same continuity component of T. This, along with the uniform contraction mentioned above, implies the following estimate. Recall that, for $n \geq 1$, $\tau_n = \sum_{i=0}^{n-1} \tau \circ T^i$. Then there exists $C > 0$ such that, whenever $x' \in W^s(x)$, $|\tau_n(x) - \tau_n(x')| \leq C d(x, x')$, uniformly in $n \geq 1$.

These properties imply the below stated Formulas (5.27) and (5.28) for the map $F: Y \to Y$ (the return map to the base of the Young tower) – see [BBM19, section 9.1] for details. There exist $C_1 > 0$ and $\gamma < 1$ such that, for all $y, y' \in Y$,

$$|\psi(y) - \psi(y')| \leq C_1 d(y, y') \quad \text{for all } y' \in W^s(y), \tag{5.27}$$

$$|\psi(y) - \psi(y')| \leq C_1 \gamma^{s(y,y')} \quad \text{for all } y' \in W^u(y),\ s(y, y') \geq 1. \tag{5.28}$$

This ensures that $H^t: Y^\psi \to Y^\psi$ is a Gibbs-Markov flow:

Lemma 5.13. *If conditions (5.27) and (5.28) are satisfied, then both (5.25) and condition (H) holds.*

Proof. Recall that $\widetilde{Y}$ is one of the unstable leaves in Y. Hence (5.25) is implied by the following property:

There exists $C_1 > 0$ such that

$$|\psi(y) - \psi(y')| \leq C_1 C_4 (d(y, y') + \gamma^{s(y,y')}) \text{ for all } y, y' \in Y, s(y, y') \geq 1. \tag{5.29}$$

To see (5.29), let $z = W^s(y) \cap W^u(y')$. Then

$$\begin{aligned} |\psi(y) - \psi(y')| &\leq |\psi(y) - \psi(z)| + |\psi(z) - \psi(y')| \\ &\leq C_1 (d(y, z) + \gamma^{s(z,y')}) \leq C_1 C_4 (d(y, y') + \gamma^{s(y,y')}), \end{aligned}$$

where C_4 is determined by the minimum angle between the stable and the unstable direction.

Now let us proceed to the verification of condition (H). By (5.27) and uniform hyperbolicity of F, for all $y \in Y$, $n \geq 0$,

$$|\psi(F^n \pi y) - \psi(F^n y)| \leq C_1 d(F^n \pi y, F^n y) \leq C_1 C_2 \gamma^n d(\pi y, y) \leq C_1 C_2 \gamma^n$$

for some $C_2 > 0$. It follows that

$$|\chi(y)| \leq \textstyle\sum_{n=0}^{\infty} |\psi(F^n \pi y) - \psi(F^n y)| \leq C_1 C_2 (1 - \gamma)^{-1}.$$

Hence $|\chi|_\infty \leq C_1 C_2 (1 - \gamma)^{-1}$ and condition (H)(a) is satisfied.

Next, let $y, y' \in Y$, and set $N = [\frac{1}{2}s(y, y')]$, $\gamma_1 = \gamma^{1/2}$. Write

$$\chi(y) - \chi(y') = A(\pi y, \pi y') - A(y, y') + B(y) - B(y'),$$

where

$$A(p, q) = \sum_{n=0}^{N-1} (\psi(F^n p) - \psi(F^n q)), \qquad B(p) = \sum_{n=N}^{\infty} (\psi(F^n \pi p) - \psi(F^n p)).$$

By the calculation for $|\chi|_\infty$, we obtain $|B(p)| \le C_1 C_2 (1-\gamma)^{-1}\gamma^N$ for all $p \in Y$. Also, $\gamma^N \le \gamma^{-1}\gamma^{\frac{1}{2}s(y,y')} = \gamma^{-1}\gamma_1^{s(y,y')}$, so $B(p) = O(\gamma_1^{s(y,y')})$ for $p = y, y'$.

For $n \le N-1$ we have $s(F^n y, F^n y') \ge 1$, and $s(F^n y, F^n y') = s(y, y') - n$. Then by (5.29),

$$|\psi(F^n y) - \psi(F^n y')| \le C_1 C_4 (d(F^n y, F^n y') + \gamma^{s(y,y')-n}) \le C(\gamma^n d(y, y') + \gamma^{s(y,y')-n}),$$

where $C = 2C_4^2 C_1 C_2$. Hence

$$\begin{aligned} |A(y, y')| &\le \sum_{n=0}^{N-1} |\psi(F^n y) - \psi(F^n y')| \le C \sum_{n=0}^{N-1} (\gamma^n d(y, y') + \gamma^{s(y,y')-n}) \\ &\le C(1-\gamma)^{-1}(d(y, y') + \gamma^{s(y,y')-N}) \le C(1-\gamma)^{-1}(d(y, y') + \gamma_1^{s(y,y')}). \end{aligned}$$

Similarly for $A(\pi y, \pi y')$. Hence $|\chi(y) - \chi(y')| \ll d(y, y') + \gamma_1^{s(y,y')}$, so (H)(b) holds. □

Tail of ψ. To reduce Theorem J to Theorem 5.12, a crucial ingredient is the following Proposition.

Proposition 5.14. $\mu_Y(\psi > t) = O(t^{-2})$.

To prove it, let us recall that the tower is $\Delta = \{(y, \ell) \in Y \times \mathbb{Z} : 0 \le \ell \le r(y) - 1\}$ with probability measure $\mu_\Delta = \mu_Y \times \text{counting}/\bar{r}$ where $\bar{r} = \int_Y r \, d\mu_Y$. In the argument below, we switch back to our original notation $F : \Delta \to \Delta$ for the tower map, while $F_Y : Y \to Y$ denotes the return to the base. Recall also $\mu = \pi_* \mu_\Delta$ where $\pi(y, \ell) = T^\ell y$.

Since

$$\mu_\Delta(x \in \Delta : h(x) > t) = \mu(x \in M : \tau(x) > t) = O(t^{-2}), \text{ while} \tag{5.30}$$

$$\mu(y \in Y : r(y) > n) = O(e^{-cn}) \quad \text{for some } c > 0, \tag{5.31}$$

a standard argument shows that $\mu(\psi > t) = O((\log t)^2 t^{-2})$, but this has to be improved.

The crucial ingredient for proving Proposition 5.14 is due to Szász & Varjú [SV07].

Lemma 5.15 ([SV07, Lemma 16], [CZ08, Lemma 5.1])**.** *There are constants $p, q > 0$ with the following property. Define*

$$X_b(m) = \big\{x \in \Delta : [h(x)] = m \ \text{ and } \ h(F^j x) > m^{1-q} \ \text{ for some } j \in \{1, \dots, b \log m\}\big\}.$$

Then for any b sufficiently large there is a constant $C = C(b) > 0$ such that

$$\mu_\Delta(X_b(m)) \le C m^{-p} \mu_\Delta(x \in X : [h(x)] = m) \quad \text{for all } m \ge 1.$$

A core phenomenon behind Lemma 5.15 is that given $|\kappa| = m$, the conditional expectation of $|\kappa \circ T|$ is $\asymp \sqrt{m}$. This is closely related to the exponential decay of correlations for κ, cf. Formula (5.16). Let us mention that short range correlations of consecutive long excursions follow a similar pattern in dispersing billiards with cusps, but are more substantial in stadia, cf. Lemma 4.10, Remark 4.11 and Lemma 3.9.

To proceed with the proof of Proposition 5.14, define, for $b > 0$,

$$Y_b(n) = \{y \in Y : r(y) \le b \log n \ \text{ and } \max_{0 \le \ell < r(y)} h(F^\ell y) \le \tfrac{1}{2} n \ \text{ and } \ \psi(y) \ge n\}.$$

Corollary 5.16. *For b sufficiently large, $\mu_Y(Y_b(n)) = o(n^{-2})$.*

Proof. Fix p and q as in Lemma 5.15. Also fix b sufficiently large.

Let $y \in Y_b(n)$. Define $h_1(y) = \max_{0\le \ell < r(y)} h(F^\ell y)$ and choose $\ell_1(y) \in \{0, \dots, r(y)-1\}$ such that $h_1(y) = h(F^{\ell_1(y)}y)$. Define $h_2(y) = \max_{0\le \ell<r(y),\, \ell\neq \ell_1(y)} h(F^\ell y)$. Then $h_1(y)$ and $h_2(y)$ are the two largest free flights $h \circ F^\ell$ during the iterates $\ell = 0, \dots, r(y)-1$.

We begin by showing that these two flight times have comparable length. Indeed, let $m_i = [h_i]$, $i = 1, 2$. Then $n \le \psi \le h_1 + (r-1)h_2 \le n/2 + (b \log n)h_2$. Hence

$$\frac{n}{2b\log n} - 1 \le m_2 \le m_1 \le \frac{n}{2}. \tag{5.32}$$

In particular, $m_1 > m_2^{1-q}$ and $m_2 > m_1^{1-q}$ for large n.

Choose $\ell_2(y) \in \{0, \dots, r(y)-1\}$ such that $\ell_2(y) \neq \ell_1(y)$ and $h_2(y) = h(F^{\ell_2(y)}y)$. We can suppose without loss that $\ell_1(y) < \ell_2(y)$. For large n, it follows from (5.32) that $F^{\ell_1(y)}y \in X_b(m_1(y))$. Hence

$$Y_b(n) \subset F^{-\ell}X_b(m) \quad \text{for some } \ell < b\log n,\ m \ge n/(2b\log n) - 1,$$

and so

$$\mu(Y_b(n)) \ll \mu_X(Y_b(n)\times 0) \le b\log n \sum_{m\ge n/(2b\log n)-1} \mu_X(X_b(m)).$$

By Lemma 5.15 and (5.30),

$$\begin{aligned}\mu(Y_b(n)) &\ll \log n \sum_{m\ge n/(2b\log n)-1} m^{-p}\mu_X(x\in X : [h(x)] = m)\\ &\ll \log n (n/\log n)^{-(2+p)} = o(n^{-2}),\end{aligned}$$

as required. ☐

Proof of Proposition 5.14

Write $\max_{0\le\ell<r(y)} h(F^\ell y) = h(F^{\ell_1(y)}y)$ where $\ell_1(y) \in \{0, \dots, r(y)-1\}$. Then

$$\begin{aligned}\mu_Y\{y \in Y : \max_{0\le\ell<r(y)} h(F^\ell y) > n/2\} &= \bar r\mu_\Delta\{(y,0)\in\Delta : h(F^{\ell_1(y)}y) > n/2\}\\ &= \bar r\mu_\Delta\{(y,\ell_1(y)) : h(F^{\ell_1(y)}y) > n/2\} = \bar r\mu_\Delta\{(y,\ell_1(y)) : h\circ\pi(y,\ell_1(y)) > n/2\}\\ &\le \bar r\mu_\Delta\{p\in\Delta : h\circ\pi(p) > n/2\} = \bar r\mu\{x\in M : \tau(x) > n/2\},\end{aligned}$$

and so $\mu_Y\{y \in Y : \max_{0\le\ell<r(y)} h(F^\ell y) > n/2\} = O(n^{-2})$ by (5.30). Hence it follows from Corollary 5.16 that

$$\mu\{y\in Y : r(y)\le b\log n \text{ and } \psi(y)\ge n\} = O(n^{-2}).$$

Finally, by (5.31), $\mu_Y(r > b\log n) = O(n^{-bc}) = o(n^{-2})$ for any $b > 2/c$ and so $\mu_Y(\psi \ge n) = O(n^{-2})$ as required. ☐

Nonintegrability of ψ. To see that Theorem 5.12 applies in the context of infinite horizon dispersing billiards, the condition on absence of approximate eigenfunctions still has to be verified. In this regard, we refer to the following results.

Proposition 5.17 ([BBM19, Porposition 6.6])**.** *Let $y_1, y_2, y_3 \in \bigcup Y_j$ be fixed points for F_Y, and let $\mathcal{L}_i = \psi(y_i)$, $i = 1, 2, 3$, be the corresponding periods for H^t. Let $Z_0 \subset \bar Y$ be the finite subsystem corresponding to the three partition elements containing $\bar\pi y_1, \bar\pi y_2, \bar\pi y_3$.*

If $(\mathcal{L}_1 - \mathcal{L}_3)/(\mathcal{L}_2 - \mathcal{L}_3)$ is Diophantine, then there do not exist approximate eigenfunctions on Z_0.

The idea that periodic points with Diophantine ratios of their flow periods could be relevant for excluding approximate eigenfunctions is due to Dolgopyat. In [Dol98b], in the context of Axiom A flows, he assumed a condition analogous to Proposition 5.17, however, with only two periodic points. The method of [Dol98b] was then extended to suspensions of Young towers in [Mel07], where a similar condition with four periodic points was used. In more recent work ([Mel18], [BBM19]) then it was shown that three periodic points suffice, in the sense of Proposition 5.17 above. These Diophantine conditions ensure in particular that absence of approximate eigenfunctions (and thus, depending on the tails of ψ, the relevant bound on the mixing rates, as stated in Theorems 5.11 and 5.12) is a typical property, see eg.[Mel07]. Nonetheless, for hyperbolic billiard examples, it turns out to be more powerful to exploit that the flow preserves a contact structure.

Let $y_1, y_4 \in Y$ and set $y_2 = W^s(y_1) \cap W^u(y_4)$, $y_3 = W^u(y_1) \cap W^s(y_4)$. Define the *temporal distance function* $D : Y \times Y \to \mathbb{R}$,

$$D(y_1, y_4) = \sum_{n=-\infty}^{\infty} \Big(\psi(F_Y^n y_1) - \psi(F_Y^n y_2) - \psi(F_Y^n y_3) + \psi(F_Y^n y_4) \Big).$$

It follows from the construction in [Mel18, Section 5.3] that inverse branches $F_Y^n y_i$ for $n \leq -1$ can be chosen so that D is well-defined.

Lemma 5.18 ([Mel18, Theorem 5.6])**.** *Let $Z_0 = \bigcap_{n=0}^{\infty} F_Y^{-n} Z$ where Z is a union of finitely many elements of the partition $\{Y_j\}$. Let $\bar{Z}_0$ denote the corresponding finite subsystem of $\bar{Y}$. If the lower box dimension of $D(Z_0 \times Z_0)$ is positive, then there do not exist approximate eigenfunctions on $\bar{Z}_0$.*

Lemma 5.18 is particularly useful for flows with a contact structure, in which case a formula for D in [Kat94, Lemma 3.2] can be exploited and the lower box dimension of $D(Z_0 \times Z_0)$ is indeed positive, see [Mel09, Example 5.7]. In particular this applies to hyperbolic billiard flows, such as (both finite and infinite horizon) dispersing billiards.

Remark 5.19. *Above we described some key ingredients of the proof of Theorem J from [BBM19]. We note that Theorem C can be proved by similar methods, see [BBM19, section 9.3]. The main difference is that for stadia instead of $T : M \to M$ the induced map $\hat{T} : \hat{M} \to \hat{M}$ is to be considered, which is known to have a Young tower representation with exponential tails. In the stadium case, the tails of ψ can be estimated as described in [CZ08, section 3].*
Thus both Theorems J and C can be proved by reducing them, after applying several approximation steps, to Theorem 5.12. Let us also note that the proof of Theorem G requires a similar reduction of the flow in dispersing billiards with cusps to Theorem 5.11. This was initially proved in [BM08] (building on [Mel07]), and the argument is revisited in [BBM19, Mel18].

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