barat.janos_51_23

Decomposition of graphs and applications

János Barát

Doctoral dissertation submitted to the
Hungarian Academy of Sciences

January 2024
Budapest

Contents

Acknowledgement

I am deeply indebted to the many individuals who have supported my scientific career throughout the years. I am grateful for their guidance, encouragement, and collaboration.

Péter Hajnal, my PhD supervisor at the University of Szeged, played a pivotal role in shaping my research interests. He introduced me to graph minor theory, a field that has captivated me ever since. He was always interested in my ongoing work and helped me with his insight. We co-authored five scientific papers together.

Carsten Thomassen, my next academic advisor, broadened my horizons in the realm of graph theory, particularly in the area of graph colorings. He imparted invaluable knowledge and insights, including Nash-Williams and Tutte's fundamental theorem on edge-connectivity and spanning trees, which I have found immensely useful in my research. Some of our joint findings are presented in Chapter 3.

Tamás Szőnyi, my professor at the University of Szeged, opened my eyes to the fascinating world of finite geometries. Later, he invited me to join his renowned research group in Geometric and Algebraic Combinatorics at Eötvös University, where I spent a year as a postdoctoral researcher and subsequently worked as a research fellow for several years after 2014. I am grateful for the immense knowledge and expertise he imparted, particularly the algebraic ideas in finite geometry. Through his connections in Ghent, Belgium, I had the privilege of collaborating with Leo Storme, whose generosity and mentorship have left a lasting impact on me. We co-authored four scientific papers together.

Géza Tóth, my opponent during my habilitation defense in 2008, has been a steadfast collaborator and mentor ever since. Our joint work on the crossing number, has been both intellectually stimulating and personally rewarding. Some results of which are presented in Chapter 5. Géza has always been willing to provide constructive criticism and encouragement, even when I posed seemingly dull questions. Our collaborative efforts have resulted in five scientific papers. Géza was instrumental in helping me refine the style and character of the present work.

Ian Wanless and David Wood, my hosts during my stay at Monash University in Melbourne between 2011 and 2013, played a significant role in my research carrier. Ian shared his extensive knowledge of Latin squares, which proved invaluable in our collaborative work. The results of our first joint paper are included in Chapter 4. Later we worked on rainbow matchings and Ryser's conjecture, one of my favourite open problems ever since. David and I first met in Barcelona over two decades ago, and our collaboration has focused primarily on non-repetitive colorings. Our work on the slope parameter and geometric thickness is featured in Chapter 2. We have co-authored four scientific papers together.

János Pach, my current scientific advisor, has provided me with unwavering support. He welcomed me into his research group in 2021 and generously supported my research through his ERC Grant ”Geoscape” No. 882971. He has entrusted me with several challenging problems, and I am eager to contribute to their solutions. Moreover, his

extensive body of work on crossing numbers, geometric and topological graphs, and other topics has profoundly influenced my research endeavors. I eagerly anticipate the opportunity to collaborate with him on future research projects.

I am immensely grateful to all my co-authors for the shared joy of exploring mathematical concepts and conducting research together. Some of them were longer collaborations and resulted in multiple papers. I would like to express my deepest appreciation to the following colleagues: András Gyárfás, Gábor Sárközy, Zoltán Blázsik, Péter Varjú, Dániel Gerbner, Gergely Ambrus, Vida Dujmović, Gwen Joret, Ray Hill, and Stefano Innamorati.

Several people have also left an indelible mark on my scientific journey through their encouragement, insightful discussions, and unselfish attention. While we have not co-authored any papers together, their contributions have been invaluable to my growth as a researcher. I am deeply indebted to Stephan Thomassé, Aart Blokhuis, Bjarne Toft, Bill Jackson, Reinhard Diestel, Gordon Royle and Brendan McKay for their support and guidance.

Finally, I would like to extend my sincere gratitude to Viktor Vígh for bringing the cube dismantling problem to my attention and for sharing reference [66].

The present work is based on 8 of my papers. They are cited as [A]-[H].

Chapter 1

Introduction

We usually denote a graph by G, its vertex set by V and edge set by E. The number of vertices is denoted by n. If xy is an edge, then x and y are *adjacent*, they are also neighbors. A *path* is a series of different vertices, every two consecutive being adjacent. A path with k edges is a *k-path* and we denote it by P_{k+1}. A graph is *connected*, if there exists a path between any two vertices. A set X of vertices form an *independent set* if any pair in X is non-adjacent. The *degree* of a vertex is the number of edges incident to it. Given a graph G, we may orient the edges in one of the two possible directions and derive an *oriented graph*. The number of edges going out from a vertex v is the out-degree of v denoted as $d^+(v)$. A graph is *cubic* if it is 3-regular. A graph is *subcubic* if it has maximum degree 3. Otherwise, we use standard graph theory terminology as in [35] and [42]. For any undefined terms on graph colorings, consult the excellent book by Jensen and Toft [65].

Although graphs can be defined in a completely abstract manner, a graph theorist would always imagine a graph as something drawn on paper or blackboard. Therefore, it is fair to say that graph drawing and visualisation of graphs is essential in many ways. Not the least because real-life problems often impose some practical condition on the drawing. This has led to many interesting concepts in recent decades. Planar graphs are visually the most fundamental objects of graph theory. Kuratowski‘s theorem is part of everyone‘s undergraduate curriculum. It says that planarity can be checked efficiently by looking for two particular graphs K_5 and $K_{3,3}$ as topological subgraphs. Euler‘s formula implies that any n-vertex planar graph can have at most $3n-6$ edges. Since the vast majority of graphs have more edges, it is logical to have some measures of non-planarity describing the conceptual distance from planar graphs.

In graph theory, the most basic question is the following: Given a large graph G and a small graph H, does G contain H as a subgraph? One can deviate from here to a zillion of different directions. One can seek the maximum number of edges in any n-vertex graph G without an H subgraph. This leads to Turán type problems [34]. One can modify the containment relation and seek H-minors in G. This leads to the fascinating theory of graph minors to which we contributed ever so slightly [22, 23].

Here we take the following route. We may always assume G is connected. A meta-theorem in several branches of Combinatorics states that any structure containing two copies of a basic unit R can be decomposed entirely into copies of R. In our context: Can we find many copies of H such that we tile the entire large graph G by copies of H? Chronologically, probably the first instance of this type of questions considered complete graphs G and a Hamiltonian path or cycle as H. Walecki constructed Hamiltonian decompositions of complete graphs on n vertices, where n is odd. A more general question is the famous Oberwolfach problem, posed by Ringel [77]. It is a natural continuation to seek other graphs with decomposition properties. What are the necessary conditions? What other parameters of the graph are related to the existence of a decomposition? On the other end, the simplest H we can consider is K_2. Clearly any graph G decomposes into disjoint edges. The next simplest H is the 2-path, sometimes called a cherry. It is already a small result by Kotzig that any graph G with an even number of edges decomposes into cherries. There is one more attractive result, that we recall now. Every 2-edge-connected cubic graph decomposes into paths with 3 edges. It also applies for the Petersen graph.

Historically it is necessary to mention here the following result. It was independently proved by Győri [57] and Lovász [78] answering a question of Frank posed in 1975. Let G be a k-connected graph, $v_1, \dots, v_k$ distinct vertices of G, and $n_1, \dots, n_k$ positive integers summing up to n. There exists a partition $V_1, \dots, V_k$ of the vertices of G such that $|V_i| = n_i$, the induced subgraphs $G[V_i]$ are connected, and $v_i \in V_i$ for every $i \in \{1, \dots, k\}$. Here, the connectivity is important. This is one of the motifs we carry on in this work.

Summary of Chapter 2.

Around 2002, we studied a particular way of representing graphs, where the edges are straight lines and bound to go in a prescribed set of directions called slopes [A]. This is a generalisation of the well-studied concept of queens graphs [27]. Any point set $\mathcal{A}$ of the plane defines a graph on its elements as follows: let P and Q be adjacent if and only if the slope of their connecting line ℓ_{PQ} belongs to a prescribed set $\mathcal{S}$. A graph G is k-slope if there exist a proper $\mathcal{A}$, and a set $\mathcal{S}$ of size k realizing G. The *slope parameter* $sl(G)$ is the minimal such k. We posed a few very fundamental question in [A]. Is there any graph for which $sl(G)$ is large compared to n? Do bounded degree graphs have bounded slope parameter? I gave a talk about this topic at ETH in Zürich in 2004 and luckily Jirka Matoušek was in the audience. He encouraged me to read the relevant pages of his book [85] and apply it to the slope parameter. Within 2 days we arrived to a solution. This became the starting point of our joint paper.

Theorem 4. For all $\epsilon > 0$ and for all sufficiently large $n > n(\epsilon)$, there exists an n-vertex graph G with slope parameter

$$\mathsf{sl}(G) \geq \frac{n^2}{(4+\epsilon)\log n}.$$

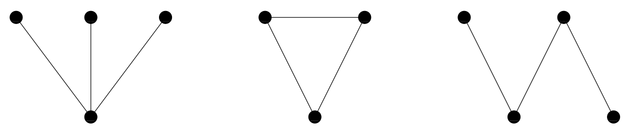
Theorem 6. For all $\Delta \geq 9$ and $\epsilon > 0$, and for all sufficiently large $n > n(\Delta, \epsilon)$, there exists a Δ-regular n-vertex graph G with slope parameter

$$\mathsf{sl}(G) > \tfrac{1}{4}((1-\epsilon)\Delta - 8)n.$$

We compare more complicated graphs to planar graphs in many ways. The crossing number or the maximum number of crossings on an edge are two of them. We mention two others, which we used in [B]. The *thickness* of a graph G is the minimum number k such that G can be decomposed into k edge-disjoint planar graphs. The book thickness of G is related to the following special drawing. A *book embedding* is a generalisation of the planar embedding of a graph to embedding into a book, a collection of half-planes all having the same line as their boundary. The vertices of the graph are required to lie on this boundary line, called the spine, and the edges are required to stay within a single half-plane. The *book thickness* of a graph is the smallest possible number of half-planes for any book embedding of the graph. The *geometric thickness* of a graph G is the minimum integer k such that there is a straight line drawing of G with its edge set partitioned into k plane subgraphs. We might think of this as k geometric graphs drawn to transparent slides and stacked upon each other. It is known that a graph with maximum degree Δ can have thickness at most $\frac{\Delta}{2}$, and this is tight. Eppstein [50] asked whether every graph of bounded maximum degree has also bounded *geometric* thickness. We answered this question in the negative, by proving that for $\Delta \geq 9$ there exist Δ-regular graphs with arbitrarily large geometric thickness. The key ingredient is to use the Milnor-Thom theorem.

Summary of Chapter 3.

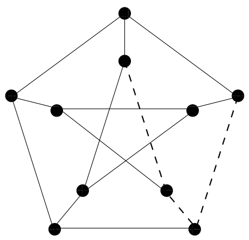
Around 1985, Jünger, Reinelt and Pulleyblank [68] studied decompositions into connected subgraphs of prescribed sizes. Among other things they show the following theorem. If G is a 4-edge-connected graph with m edges, and $m_1, \ldots, m_k$ are positive integers summing up to m, then there exists a partition $E_1, \ldots, E_k$ of the edges of G such that $|E_i| = m_i$ and each subgraph $G[E_i]$ is connected for $i \in \{1, \ldots, k\}$. The proof of this result relies on the famous and very useful theorem proved independently by Tutte [117] and Nash-Williams [93]: a graph G is $2k$-edge-connected if and only if G contains k edge-disjoint spanning trees. Jünger, Reinelt and Pulleyblank proved that every 2-edge-connected graph with $3k$ edges can be decomposed into connected graphs with three edges. These are $K_{1,3}$ (the claw), C_3 (the triangle) and P_4 (the 3-path).



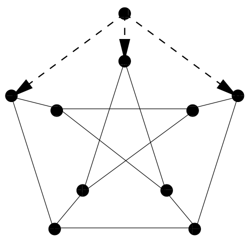
However they did not study claw-decompositions and P_4-decompositions separately. If anyone wants to test a claim in graph theory, the Petersen graph serves as a good test. It has 10 vertices each of degree 3, that makes 15 edges, see the figure below. Our test is as follows.

Is it possible to partition the edges of the Petersen graph such that each part is the same connected graph with three edges?

Clearly there is no decomposition into triangles. On the other hand, if we consider the path with dashed lines in the figure below and rotate it by 72 degrees 4 times, then we get a P_4-decomposition of the Petersen graph.



When we look for a claw-decomposition, it is convenient to orient the edges of a claw away from the center towards the leaves. Since the Petersen graph is cubic, if there was a claw-decomposition, then there would be five claw centers and five sinks (vertices of outdegree zero). On the outer 5-cycle, centers and sinks should appear alternately, which is impossible.



Let $\mathcal{H}$ be a collection of graphs. A graph G has a *$\mathcal{H}$-decomposition* if the edges of G can be divided into subgraphs each of which is isomorphic to a graph in $\mathcal{H}$. Often $\mathcal{H}$ is a single graph H. The H-decompositions are widely studied when G is a complete graph. If G is not complete, then it may be hard to find H-decompositions. Indeed, if H has at least three edges, then the problem of deciding if a graph G has an H-decomposition is NP-complete [45].

The next question we may ask is the following. Can we decompose every connected graph into paths with 3 edges? One may find graphs showing that the answer is negative, see Figure 1.1.

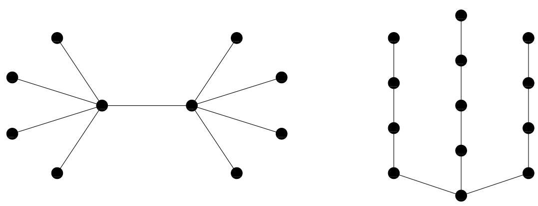


Figure 1.1: Graphs without P_4-decomposition.

These graphs have a common property, the existence of a cut-edge. The removal of such an edge makes the graph disconnected. Therefore the following definition is useful. A graph G is *k-edge-connected*, if after removing any $k-1$ edges the graph is still connected. It is natural to ask what degree of edge-connectivity implies certain decompositions. Which graphs admit a P_4-decomposition? Several people noticed the following: Every 2-edge-connected cubic graph G has a P_4-decomposition.

Let $\mathcal{H}$ be the class of 2-edge-connected graphs with s edges. Jünger, Reinelt and Pulleyblank posed the following question. Is there an edge-connectivity (depending on s) that guarantees a graph to have a $\mathcal{H}$-decomposition?

The answer to this question is negative: there is no such edge-connectivity, since there exist graphs of arbitrarily high connectivity and large girth. If we imagine ourselves sitting on a vertex v and looking around, the graph looks like a tree. Therefore, there is no 2-edge-connected subgraph covering vertex v. In view of this, we must require that a finite collection $\mathcal{H}$ contains a forest if we wish to show that large (fixed) edge-connectivity implies the existence of a $\mathcal{H}$-decomposition.

The investigations and techniques led Thomassen and myself to the following very general conjecture.

Conjecture 1. For each tree T, there exists a natural number k_T such that the following holds: If G is a k_T-edge-connected graph such that $|E(T)|$ divides $|E(G)|$, then the edges of G can be divided into parts, each of which is isomorphic to T.

We proved that for $T = K_{1,3}$ (the claw), this holds if and only if there exists a (smallest) natural number k_t such that every k_t-edge-connected graph has an orientation for which the in-degree of each vertex equals its out-degree modulo 3. Tutte's 3-flow conjecture says that $k_t = 4$. We proved the weaker statement that every $4\lceil \log n \rceil$-edge-connected graph with n vertices has an edge-decomposition into claws provided its number of edges is divisible by 3. We also proved that every triangulation of a surface has an edge-decomposition into claws.

We recall that, at the time of publication, we did not know a single instance of size at least three, for which Conjecture 1 was known to hold. The only indication was Tutte's 3-flow conjecture and related results.

We expected the Barát-Thomassen conjecture would be the source of inspiration for many years to come. It partly turned out to be true as it resulted in over 70 papers referring to it. On the other hand, we were surprised that a full solution came into light after a decade or so. One of the building blocks of the resolution was the very natural step to show that qualitatively it suffices to prove the conjecture for bipartite graphs. This idea first appeared in a paper by Thomassen for the special case, when the tree is P_4. The bipartite reduction was proved independently by Thomassen and by Gerbner and myself. Let Y be the unique tree with degree sequence (1,1,2,3). In the same paper [F], we proved that if G is a 287-edge-connected graph of size divisible by 4, then G has a Y-decomposition. This was the first instance of such a theorem, in which the tree was different from a path or a star.

In 2016, Bensmail et al. uploaded *A proof of the Barát-Thomassen conjecture* to ArXiv. This paper was the last brick of the proof. Although the original conjecture is resolved, many similar questions, unknown constants remain. There are already more than 20 new papers since the above paper appeared.

Summary of Chapter 4.

Somewhat coincidentally, around 2008, Vígh mentioned a real-life inspired geometric 3D question to us. Assume we have built a large $k \times k \times k$ cube from elementary unit cubes. Two unit cubes are neighbors if they share a common face. We may remove a cube from the stack if and only if it has precisely 3 neighbors. This process is a dismantling. How long can we go? What kind of positions may remain? To be more symmetric, we assume it all happens in zero gravity. Since this question translates to a claw-decomposition in the dual graph, we explored the problem. We found natural connections to other branches of Combinatorics, namely Latin squares, greedy algorithms and bootstrap percolation. Sometimes it is useful to view the reverse process: building. Assume that we would like to design a space module from preconstructed items. These parts are conventionally made to have cubic shape. Assume there is an initially firm-made configuration of items in the space. Later, we can add a new box to the module, if there are three fix neighbors. A move is *balanced* if the three neighbours are in three orthogonal directions. In 3 dimensions, a full dismantling results in n^2 independent vertices, that we call a *solution.* If we project the corresponding cubes to the 3 orthogonal directions, then we might get 3 full squares. In that case, there is a natural correspondence of the solution to Latin squares, and the solution is called *perfect.*

We show that it is possible to use a greedy algorithm to test whether a set of cubes forms a solution. For every $n \geq 2$, we find at least n perfect solutions. Perfect solutions turn out to be precisely those which can be reached using only balanced moves. Every perfect solution corresponds naturally to a Latin square. However, we show that almost all Latin squares do not correspond to solutions.

We construct an infinite family of imperfect solutions and show that the total size of its three orthogonal projections is asymptotic to the minimum possible value.

Summary of Chapter 5.

Instead of the edges, one may also partition the vertices of a graph into disjoint classes. We can impose various conditions on the graph induced by each class. The simplest condition is to require each class to be an independent set. This is also called vertex coloring of a graph. The chromatic number is the graph parameter minimising the number of colors. There has been an ongoing interest to understand what structural properties force large chromatic number. In a conjecture attributed to Hajós the following is formulated: every k-chromatic graph contains a complete graph on k vertices as a topological subgraph. Although this generated a lot of interest, the conjecture turned out to be false for almost all graphs. Another innocent looking and intuitive conjecture for complete graphs concerns the crossing number. That is, we would like to draw the edges of a complete n-vertex graph as topological curves in the plane. It is a long-standing conjecture posed by Hill and first published by Guy that the crossing number of K_n is $Z(n) = \frac{1}{4}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor\lfloor\frac{n-3}{2}\rfloor$. Albertson realized the subtle connection between the two areas and posed the following conjecture: If a graph G has chromatic number r, then its crossing number is at least as much as the crossing number of K_r. Albertson, Cranston, and Fox proved this for $r \leq 12$. We took one more little step and proved the conjecture for $r \leq 16$. Our method combined structural properties of critical graphs and subtle improvements of bounds on the crossing number using extra information. We also qualitatively proved the conjecture showing two results. One: if the number of edges is more than $3.57n$, then the Albertson conjecture holds. Two: if the number of edges is less than $3.57n$, then the crossing number is at least $\frac{Z(n)}{4}$.

Summary of Chapter 6.

We returned to decompositions, when an intriguing question appeared for 2-colored graphs, which puts an asymmetric and simple condition on the components induced by the monochromatic vertex classes. It has connections to a much studied conjecture by Wegner.

Thomassen formulated the following conjecture: Every 3-connected cubic graph has a red–blue vertex coloring such that the subgraph induced by blue vertices has maximum degree 1 (that is, it consists of a matching and some isolated vertices) and the red part has minimum degree at least 1 and contains no 3-edge path.

Since all monochromatic components are small in this coloring and there is a certain irregularity, we call such a coloring crumby. We proved this crumby conjecture for Generalized Petersen graphs. In the paper, we also indicated that a coloring with the same properties might exist for any subcubic graph. We confirmed this statement for all subcubic trees. This last result turned out to be useful later, in a subsequent paper.

In 2020, Bellitto et al. [28] published a construction of an infinite family refuting the crumby conjecture by Thomassen. Their prototype counterexample is 2-connected, planar, but contains a K_4-minor and also a 5-cycle. This leaves the crumby conjecture open for some important graph classes: outerplanar graphs, K_4-minor-free graphs, bipartite graphs. In this regard, we proved that 2-connected outerplanar graphs, subdivisions of K_4 and 1-subdivisions of cubic graphs all admit crumby colorings. The proof of the last result relies on the Edmonds-Gallai decomposition of graphs.

barat.janos_51_23

Chapter 2

Slope parameter and geometric thickness

If not explicitly stated, a graph has always n vertices. In the real plane, the line through points P and Q is denoted by ℓ_{PQ}. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two different points of the line ℓ on the real plane $\mathbb{R}^2$. The *slope of* ℓ is $(y_2 - y_1)/(x_2 - x_1) \in \mathbb{R} \cup \{\infty\}$, denoted by $M(\ell)$. Note that $M(\ell)$ does not depend on the choice of P and Q, and $\ell \parallel L$ if and only if $M(\ell) = M(L)$. Let $\mathcal{S} \subset \mathbb{R} \cup \{\infty\}$ be a set of slopes. Every point set $\mathcal{A}$ of the plane defines a graph in the following way: Let $P, Q \in \mathcal{A}$ be adjacent if and only if the slope of ℓ_{PQ} belongs to $\mathcal{S}$. We denote this graph by $\mathcal{G}(\mathcal{A}; \mathcal{S})$. A graph G is called $\mathcal{S}$*-slope* if and only if there exists a set of points $\mathcal{A}$ such that $G \simeq \mathcal{G}(\mathcal{A}; \mathcal{S})$. A graph G is called k*-slope* if there exist a k-element set $\mathcal{S}$ of slopes and a set of points $\mathcal{A}$ such that $G \simeq \mathcal{G}(\mathcal{A}; \mathcal{S})$. Furthermore, for a given graph G the minimal number of slopes required for this representation is called *the slope parameter* of G, denoted by $sl(G)$. The set of all k-slope graphs is denoted by $\mathcal{G}_k$. Notice, that any graph G is m-slope, where $m = |E(G)|$. Indeed, take an n-element point set $\mathcal{P}$, each point corresponding to a vertex of G such that there are no parallel lines among the lines determined by $\mathcal{P}$. If $\mathcal{S}$ consists of the slopes of the lines determined by the edges of G, then $\mathcal{G}(\mathcal{P}; \mathcal{S})$ is isomorphic to G. In all other notions we follow [35]. This chapter is devoted to the study of the slope parameter, and for this purpose we assume that all graphs are connected.

For instance, queens graphs are the graphs of $\mathcal{G}(\mathcal{P}; \{-1, 0, 1, \infty\})$, where $\mathcal{P} \subset \mathbb{Z} \times \mathbb{Z}$. The vertices are positions in the infinite chessboard and adjacency is given by 'legal queen move'. Queens graphs were studied in [18, 26].

Example. *The Petersen graph has edge-chromatic number four. Therefore, it needs at least four slopes. The next figure shows such a representation on the chessboard, that is with slopes* $(0, \infty, 1, -1)$.

Notice the following: if G is a k-slope graph and H is an induced subgraph of G, then H is k-slope as well. Such a graph property is called *hereditary.* A hereditary class can be characterized by the minimal (respect to induced subgraphs) non-members of the class, so called *obstructions.* We characterize the 2-slope graphs in this manner.

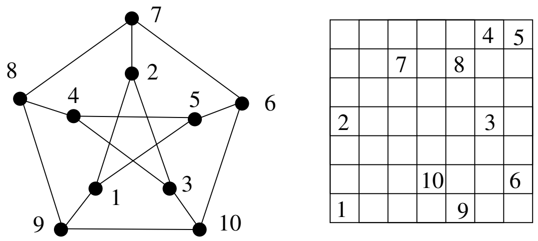


Figure 2.1: The queen representation of the Petersen graph.

2.1 The k-slope graphs for small k

A graph is 1-slope if and only if each of its connected components is a complete graph. We call these graphs *equivalence graphs*, since these are exactly the graphs where 'to be equal or to be adjacent' is an equivalence relation on the vertices.

Let $S = \{\mu, m\}$ be an arbitrary 2-element slope set. The class of graphs defined by S does not depend on the choice of the slopes. We can completely characterize these graphs.

Theorem 1. *The following statements are equivalent:*

(i) *the graph G is a 2-slope graph;*

(ii) *the edges $E(G)$ can be 2-colored (by red and blue) such that in each color class the corresponding edges define an equivalence graph;*

(iii) *the graph G does not contain $K_{1,3}$, any odd cycle of length at least 5 or K_4^- as an induced subgraph. Here K_4^- is the complete graph on four vertices minus an edge.*

Proof. (i)⇒(ii) Let $\mathcal{G}(\mathcal{P}; \{0, \infty\})$ be a representation of G as a 2-slope graph. We color an edge $e = uv$ red if and only if $\ell_{P_u P_v}$ is horizontal (and hence blue if and only if it is vertical). The coloring proves (ii).

(ii)⇒(iii) The coloring of (ii) has the property that whenever $e = uv$ and $f = vw$ are two adjacent edges of the same color, then uw is also an edge and its color is the same.

Also, every triangle is monochromatic, that is all three edges have the same color. Let us assume that $K_{1,3}$ is an induced subgraph. At least two edges of this star must have the same color. This contradicts our first observation. Similarly, we know that any two adjacent edges on an induced cycle of length at least 4 must have different colors. Hence, all the induced cycles of length at least 4 must have even length. The graph K_4^- consists of two triangles glued together along an edge. So, each occurrence of K_4^- is monochromatic, which contradicts our first observation.

(iii)⇒(ii) Take the maximal cliques of G. Each edge belongs to exactly one maximal clique, since K_4^- is not an induced subgraph. Each vertex belongs to at most two maximal cliques, since $K_{1,3}$ is not an induced subgraph. Now, the maximal cliques can be colored by red and blue such that two cliques have different color if they have a vertex in common, since there is no induced odd cycle of length longer than three. The last statement together with the previous ones define a red-blue coloring of the edges that verifies (ii).

(ii)⇒(i) Let R be the equivalence graph of the red edges on $V(G)$. For the components of R take different horizontal lines. Let B be the equivalence graph of the blue edges on $V(G)$. For the components of B take different vertical lines. Each vertex v is in one component of R and in one component of B. The corresponding horizontal and vertical lines define an intersection P_v. Let P_v be the representation of v. The point set $\mathcal{P} = \{P_v\}$ obtained in this way defines a 2-slope representation of G, namely $\mathcal{G}(\mathcal{P}, \{0, \infty\})$. □

Let $\{\mu, m, M\}$ and $\{\mu', m', M'\}$ be two arbitrary 3-element slope sets. Next, we show that the class of graphs defined by three slopes does not depend on the choice of slopes. Let e, f and g be three concurrent lines such that their slopes are $\{\mu, m, M\}$. Similarly let e', f' and g' be three concurrent lines with slopes $\{\mu', m', M'\}$. There exists an affine transformation φ of the real plane that maps $\varphi(e) = e'$, $\varphi(f) = f'$, and $\varphi(g) = g'$. In this case, if $\mathcal{P}$ is a representation of G with slopes $\{\mu, m, M\}$, then $\varphi\mathcal{P}$ is a representation of G with slopes $\{\mu', m', M'\}$.

It would be a challenging task to characterize the 3-slope graphs. We show an infinite family of graphs with slope parameter at most 3. Next, we describe a process to construct the subcubic outerplanar graphs.

Definition 1. Let us consider the cycles of length at least 3 with a designated set of independent edges being red. These are the base graphs. The red edges are sticky, that is edges with glue. Let the other edges be blue. Starting with the base graphs, we build further graphs using the following two operations:

(i) identifying a red edge of an already constructed graph with a red edge of a base cycle such that the two red edges turn into a blue edge;

(ii) identifying a 1- or 2-valent vertex of an already constructed graph with a vertex of an edge.
Let $\mathcal{O}_{\leq 3}$ be the class of graphs we can construct this way.

It is clear that $\mathcal{O}_{\leq 3}$ is the class of connected outerplanar graphs with maximum degree at most three. This view is very useful to prove the following

Theorem 2. *If G is an outerplanar graph with maximum degree at most 3, then $sl(G) \leq 3$.*

Proof. Consider a construction of G from the base graphs. We describe a 3-slope representation of G using the steps in this construction. We assume that the slopes are given by three isogonal lines, that is their pairwise angles are $2\pi/3$.

First, we describe the representation of the base graphs. The 3-slope representation of the cycle C_n can be given by a convex n-gon in such a way that any sticky edge and its two neighbors have three different slopes. The representation with these properties still has a large flexibility. We can move any edge (side of the n-gon) by a translation, still producing the same graph. In this way we are able to move the vertices away from an incidentally wrong position.

Assume that the current construction step requires the gluing of a cycle. Let us consider the representation of the already constructed part and a representation of the cycle scaled such that the two identified sides have equal length. This representation of the cycle or its reflected image can be moved in such a way that the two representations unite. This is not necessarily a representation of the glued graph. To exclude undesired relations, we must use the flexibility of the representation of the cycle. Draw each line with the given slopes through each vertex of the earlier representation. The new vertices of the cycle should not be incident to any of these lines. This can be easily achieved.

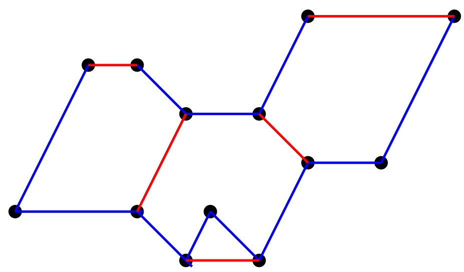


Figure 2.2: An outerplanar graph with 3 slopes.

The gluing of a vertex can be solved in the same manner. □

In [A] we posed two problems whether bounded maximum degree implies bounded slope parameter. We were convinced this to be true for maximum degree 3. This was proved by Keszegh et al.

Theorem 3 ([70])**.** *Every cubic graph has slope parameter at most* 7*. If the graph is subcubic, then* 5 *slopes suffice.*

If the maximum degree is at least 5, then the situation changes dramatically. We discuss this in the next section.

2.2 Unbounded slope parameter

We address the following two questions of [A]:
- what is the maximum slope parameter of an n-vertex graph?
- do graphs of bounded maximum degree have bounded slope parameter?

All results in the next three sections are based on the following lemma, versions of which are due to Petrovskiĭ and Oleĭnik [99], Milnor [90], Thom [109] and Warren [118]. The precise version stated here is by Pollack and Roy [100]. Let $\mathcal{P} = (P_1, P_2, \ldots, P_t)$ be a system of d-variate real polynomials. A vector $\sigma \in \{-1, 0, +1\}^t$ is a *sign pattern* of $\mathcal{P}$ if there exists an $x \in \mathbb{R}^d$ such that the sign of $P_i(x)$ is σ_i, for all $i = 1, 2, \ldots, t$.

Lemma 1 ([100])**.** *Let $\mathcal{P} = (P_1, P_2, \ldots, P_t)$ be a system of d-variate real polynomials, each of degree at most D. Then the number of sign patterns of $\mathcal{P}$ is at most*

$$\left(\frac{50\,Dt}{d}\right)^d.$$

Some of our proofs only need sign patterns that distinguish between zero and nonzero values. In this setting, Rónyai et al. [102] gave a better bound with a short proof; see [85].

Our second tool is a corollary of more precise bounds due to Bender and Canfield [29], Wormald [123], and McKay [86]; see Appendix A in [B].

Lemma 2 ([29, 86, 123])**.** *For all integers $\Delta \geq 1$ and $n \geq c\Delta$, the number of labelled Δ-regular n-vertex graphs is at least*

$$\left(\frac{n}{3\Delta}\right)^{\Delta n/2},$$

for some absolute constant c.

Now we estimate the graphs with slope parameter at most k, using Lemma 1.

Lemma 3. *The number of labelled n-vertex graphs G with slope parameter $\mathsf{sl}(G) \leq k$ is at most*

$$\left(\frac{50n^2k}{2n+k}\right)^{2n+k}.$$

Proof. Let $\mathcal{G}_{n,k}$ denote the family of labelled n-vertex graphs G with slope parameter $\mathsf{sl}(G) \leq k$. Consider $V(G) = \{1, 2, \ldots, n\}$ for every $G \in \mathcal{G}_{n,k}$. For every $G \in \mathcal{G}_{n,k}$, there is a point set $P = \{(x_i(G), y_i(G)) : 1 \leq i \leq n\}$ and slope set $S = \{s_\ell(G) : 1 \leq \ell \leq k\}$ such that $G \cong G(P, S)$, where vertex i is represented by the point $(x_i(G), y_i(G))$. Fix one such representation of G. Without loss of generality, $x_i(G) \neq x_j(G)$ for distinct i and j. Thus every $s_\ell(G) < \infty$. For all i, j, ℓ with $1 \leq i < j \leq n$ and $1 \leq \ell \leq k$, and for every graph $G \in G_{n,k}$, we define the number

$$P_{i,j,\ell}(G) := (y_j(G) - y_i(G)) - s_\ell(G) \cdot (x_j(G) - x_i(G)).$$

Consider

$$\mathcal{P} := \{P_{i,j,\ell} : 1 \leq i < j \leq n, 1 \leq \ell \leq k\}$$

to be a set of $\binom{n}{2}k$ degree-2 polynomials on the set of variables

$$\{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n, s_1, s_2, \ldots, s_k\}.$$

Observe that $P_{i,j,\ell}(G) = 0$ if and only if ij is an edge of G and ij has slope s_ℓ in the representation of G.

Consider two distinct graphs $G, H \in \mathcal{G}_{n,k}$. Without loss of generality, there is an edge ij of G that is not an edge of H. Thus $(y_j(G) - y_i(G)) - s_\ell(G) \cdot (x_j(G) - x_i(G)) = 0$

for some ℓ, and $(y_j(H)-y_i(H))-s_\ell(H)\cdot(x_j(H)-x_i(H))\neq 0$ for all ℓ. Hence $P_{i,j,\ell}(G)=0\neq P_{i,j,\ell}(H)$. That is, any two distinct graphs in $\mathcal{G}_{n,k}$ are distinguished by the sign of some polynomial in $\mathcal{P}$. Hence $|\mathcal{G}_{n,k}|$ is at most the number of sign patterns determined by $\mathcal{P}$. By Lemma 1 with $D=2$, $d=2n+k$, and $t=\binom{n}{2}k$ we have

$$|\mathcal{G}_{n,k}|\leq\left(\frac{50\cdot 2\cdot\binom{n}{2}k}{2n+k}\right)^{2n+k}<\left(\frac{50n^2k}{2n+k}\right)^{2n+k}.$$

□

In response to the first question of [A], we now prove that there exist graphs with surprisingly large slope parameter. All logarithms are binary unless stated otherwise.

Theorem 4. *For all $\epsilon>0$ and for all sufficiently large $n>n(\epsilon)$, there exists an n-vertex graph G with slope parameter*

$$\mathsf{sl}(G)\geq\frac{n^2}{(4+\epsilon)\log n}.$$

Proof. Suppose that every n-vertex graph G has slope parameter $\mathsf{sl}(G)\leq k$. There are $2^{\binom{n}{2}}$ labelled n-vertex graphs. By Lemma 3,

$$2^{\binom{n}{2}}\leq\left(\frac{50n^2k}{2n+k}\right)^{2n+k}.$$

For large $n>n(\epsilon)$,

$$2^{\binom{n}{2}}=\left(50n^2\right)^{\binom{n}{2}/\log(50n^2)}>\left(50n^2\right)^{\binom{n}{2}/(2+\epsilon/2)\log n}>\left(50n^2\right)^{2n+n^2/(4+\epsilon)\log n}.$$

We have $50n^2k\leq 50n^2(2n+k)$. Thus

$$\left(50n^2\right)^{2n+n^2/(4+\epsilon)\log n}<2^{\binom{n}{2}}\leq\left(\frac{50n^2k}{2n+k}\right)^{2n+k}<\left(50n^2\right)^{2n+k}.$$

Hence

$$k>\frac{n^2}{(4+\epsilon)\log n}.$$

The result follows. □

Now we prove that the slope parameter of degree-Δ graphs is unbounded for $\Delta\geq 5$, thus answering the second question of [A] in the negative. It remains open whether $\mathsf{sl}(G)$ is bounded for degree-4 graphs G.

Theorem 5. *For all $\Delta\in\{5,6,7,8\}$, for all ϵ with $0<\epsilon<\Delta-4$, and for all sufficiently large $n>n(\Delta,\epsilon)$, there exists a Δ-regular n-vertex graph G with*

$$\mathsf{sl}(G)>n^{(\Delta-4-\epsilon)/4}.$$

Proof. Let $k := n^{(\Delta-4-\epsilon)/4}$. Suppose that for some $\Delta \in \{5,6,7,8\}$, every Δ-regular n-vertex graph G has $\mathsf{sl}(G) \leq k$. By Lemmas 2 and 3,

$$\left(\frac{n}{3\Delta}\right)^{\Delta n/2} \leq \left(\frac{50n^2k}{2n+k}\right)^{2n+k} < (25nk)^{2n+k} < (25n)^{(\Delta-\epsilon)(2n+k)/4}.$$

For $n > (3\Delta(25)^{1-\epsilon/2})^{2/\epsilon}$,

$$(25n)^{(2\Delta-\epsilon)n/4} < \left(\frac{n}{3\Delta}\right)^{\Delta n/2} < (25n)^{(\Delta-\epsilon)(2n+k)/4}.$$

Thus $2\Delta n - \epsilon n < 2\Delta n + \Delta k - 2\epsilon n - \epsilon k$. That is, $(\Delta-\epsilon)n^{(\Delta-8-\epsilon)/4} > \epsilon$. Thus $\Delta - 8 - \epsilon \geq 0$ for large $n > n(\Delta, \epsilon)$, which is the desired contradiction for $\Delta \leq 8$. □

For $\Delta \geq 9$ there are Δ-regular graphs with linear slope parameter.

Theorem 6. *For all $\Delta \geq 9$ and $\epsilon > 0$, and for all sufficiently large $n > n(\Delta, \epsilon)$, there exists a Δ-regular n-vertex graph G with slope parameter*

$$\mathsf{sl}(G) > \tfrac{1}{4}((1-\epsilon)\Delta - 8)n.$$

Proof. Suppose that every Δ-regular n-vertex graph G has $\mathsf{sl}(G) \leq \alpha n$ for some $\alpha > 0$. By Lemmas 2 and 3,

$$\left(\frac{n}{3\Delta}\right)^{\Delta n/2} \leq \left(\frac{50\alpha\, n^2}{2+\alpha}\right)^{(2+\alpha)n}.$$

For $n > (3\Delta \cdot 8^{1-\epsilon})^{1/\epsilon}$,

$$(8\,n)^{(1-\epsilon)\Delta n/2} < \left(\frac{n}{3\Delta}\right)^{\Delta n/2} \leq \left(\frac{50\alpha\, n^2}{2+\alpha}\right)^{(2+\alpha)n} < (8\,n)^{2(2+\alpha)n}.$$

Thus

$$\alpha > \frac{(1-\epsilon)\Delta - 8}{4}.$$

Thus $\mathsf{sl}(G) \geq \frac{1}{4}((1-\epsilon)\Delta - 8)n$ for some Δ-regular n-vertex graph G. □

Note that the lower bound in Theorem 6 is within a factor of $2+\epsilon$ of the trivial upper bound $\mathsf{sl}(G) \leq \frac{1}{2}\Delta n$.

2.3 Geometric thickness

The *thickness* of an (abstract) graph G is the minimum number of planar subgraphs of G whose union is G. Thickness is a classical and widely studied graph parameter; see the survey [92]. The *thickness* of a graph drawing D is the minimum number of plane subgraphs of D whose union is D. Every planar graph can be drawn with its vertices at prespecified locations [58, 98]. It follows that a graph with thickness k has a drawing with thickness k [58]. However, in such a representation the edges might be highly curved[1].

This motivates the notion of geometric thickness, which is a central topic of this section. A drawing is *geometric*, also called a *geometric graph*, if every edge is represented by a straight line segment. The *geometric thickness* of a graph G is the minimum thickness of a geometric drawing of G. Geometric thickness was introduced by Kainen [69] under the name *real linear thickness*.

Consider the relationship between the various thickness parameters and maximum degree. A graph with maximum degree at most Δ is called *degree-*Δ. Wessel [121] and Halton [58] independently proved that the thickness of a degree-Δ graph is at most $\lceil \frac{\Delta}{2} \rceil$, and Sýkora et al. [107] proved that this bound is tight. Duncan et al. [47] proved that the geometric thickness of a degree-4 graph is at most 2. Eppstein [50] asked whether graphs of bounded degree have bounded geometric thickness. We answer this question in the negative in Theorem 7.

Eppstein [50] proved that geometric thickness is not bounded by thickness. In particular, there exists a graph with thickness 3 and arbitrarily large geometric thickness. Theorem 7 and the above result of Wessel [121] and Halton [58] imply a similar result (with a shorter proof). Namely, there exists a 9-regular graph with thickness at most 5 and with arbitrarily large geometric thickness.

A *book embedding* is a geometric drawing with the vertices in convex position. The *book thickness* of a graph G is the minimum thickness of a book embedding of G. Book thickness is also called *page-number* and *stack-number*; see [46] for over fifty references on this topic. By definition, the geometric thickness of G is at most the book thickness of G. On the other hand Eppstein [51] proved that there exists a graph with geometric thickness 2 and arbitrarily large book thickness; also see [32, 33]. Thus book thickness is not bounded by any function of geometric thickness.

Theorem 7 is analogous to a result of Malitz [84], who proved that there exists Δ-regular n-vertex graphs with book thickness at least $c\sqrt{\Delta}n^{1/2-1/\Delta}$. Malitz's proof is based on a probabilistic construction of a graph with certain expansion properties. The proof of Theorem 7 is easily adapted to prove Malitz's result for $\Delta \geq 3$. The difference in the bounds ($n^{1/2-4/\Delta}$ and $n^{1/2-1/\Delta}$) is caused by the difference between the number of order types of point sets in general and convex position ($\approx n^{4n}$ and $n!$). Malitz [84] also proved an upper bound of $\mathcal{O}(\sqrt{m}) \subseteq \mathcal{O}(\sqrt{\Delta n})$ on the book thickness, and thus the geometric thickness, of m-edge graphs.

Observe that a geometric drawing with thickness k can be perturbed so that the vertices are in *general position* (that is, no three vertices are collinear). Thus in this section we consider point sets in general position without loss of generality.

Lemma 4. *The number of labelled n-vertex graphs with geometric thickness at most k is at most $472^{kn}n^{4n+o(n)}$.*

Proof. Let P be a fixed set of n labelled points in general position in the plane. Ajtai

[1] In fact, a polyline drawing of a random perfect matching on n vertices in convex position almost certainly has $\Omega(n)$ bends on some edge [98].

et al. [11] proved that there are at most c^n plane geometric graphs with vertex set P, where $c \leq 10^{13}$. Santos and Seidel [104] proved that we can take $c = 472$. A geometric graph with vertex set P and thickness at most k consists of a k-tuple of plane geometric graphs with vertex set P. Thus P admits at most 472^{kn} geometric graphs with thickness at most k.

Let $P = (p_1, p_2, \dots, p_n)$ and $Q = (q_1, q_2, \dots, q_n)$ be two sets of n points in general position in the plane. Then P and Q have the same *order type* if for all indices $i < j < k$ we turn in the same direction (left or right) when going from p_i to p_k via p_j and when going from q_i to q_k via q_j. Say P and Q have the same order type. Then for all i, j, k, ℓ, the segments p_ip_j and p_kp_ℓ cross if and only if q_iq_j and q_kq_ℓ cross. Thus P and Q admit the same set of (at most 472^{kn}) labelled geometric graphs with thickness at most k (when considering p_i and q_i to be labelled i). Alon [15] proved (using Lemma 1) that there are at most $n^{4n+o(n)}$ sets of n points with distinct order types. The result follows. □

It is easily seen that Lemmas 2 and 4 imply a lower bound of $c(\Delta - 8) \log n$ on the geometric thickness of some Δ-regular n-vertex graph. To improve this logarithmic bound to polynomial, we now give a more precise analysis of the number of graphs with bounded geometric thickness that also accounts for the number of edges in the graph.

Lemma 5. *Let P be a set of n labelled points in general position in the plane. Let $g(P, m)$ be the number of m-edge plane geometric graphs with vertex set P. Then*

$$g(P, m) \leq \begin{cases} \binom{n}{2m} \cdot 472^{2m} & \text{, if } m \leq \frac{n}{2} \\ 472^n & \text{, if } m > \frac{n}{2}. \end{cases}$$

Proof. As in Lemma 4, $g(P, m) \leq 472^n$ regardless of m. Suppose that $m \leq \frac{n}{2}$. An m-edge graph has at most $2m$ vertices of nonzero degree. Thus every m-edge plane geometric graph with vertex set P is obtained by first choosing a $2m$-element subset $P' \subseteq P$, and then choosing a plane geometric graph on P'. The result follows. □

Lemma 6. *Let P be a set of n labelled points in general position in the plane. For every integer t such that $\frac{2m}{n} \leq t \leq m$, let $g(P, m, t)$ be the number of m-edge geometric graphs with vertex set P and thickness at most t. Then*

$$g(P, m, t) \leq \left(\frac{ctn}{m} \right)^{2m},$$

for some absolute constant c.

Proof. Fix nonnegative integers $m_1 \leq m_2 \leq \cdots \leq m_t$ such that $\sum_i m_i = m$. Let $g(P; m_1, m_2, \dots, m_t)$ be the number of geometric graphs with vertex set P and thickness t, such that there are m_i edges in the i-th subgraph. Then

$$g(P; m_1, m_2, \dots, m_t) \leq \prod_{i=1}^{t} g(P, m_i).$$

Now $m_1 \leq \frac{n}{2}$, as otherwise $m > \frac{tn}{2} \geq m$. Let j be the maximum index such that $m_j \leq \frac{n}{2}$. By Lemma 5,

$$g(P; m_1, m_2, \ldots, m_t) \leq \left(\prod_{i=1}^{j} \binom{n}{2m_i} 472^{2m_i}\right) (472^n)^{t-j}.$$

Now $\sum_{i=1}^{j} m_i \leq m - \frac{1}{2}(t-j)n$. Thus

$$g(P; m_1, m_2, \ldots, m_t) \leq \left(\prod_{i=1}^{j} \binom{n}{2m_i}\right) \left(472^{2m-(t-j)n}\right) \left(472^{(t-j)n}\right) \leq 472^{2m} \prod_{i=1}^{t} \binom{n}{2m_i}.$$

We can suppose that t divides $2m$. It follows (see Appendix B in [B]) that

$$g(P; m_1, m_2, \ldots, m_t) \leq 472^{2m} \binom{n}{2m/t}^t.$$

It is well known [67, Proposition 1.3] that $\binom{n}{k} < \left(\frac{\mathsf{e}n}{k}\right)^k$, where e is the base of the natural logarithm. Thus with $k = 2m/t$ we have

$$g(P; m_1, m_2, \ldots, m_t) < \left(\frac{236\mathsf{e}tn}{m}\right)^{2m}.$$

Clearly

$$g(P, m, t) \leq \sum_{m_1, \ldots, m_t} g(P; m_1, m_2, \ldots, m_t),$$

where the sum is taken over all nonnegative integers $m_1 \leq m_2 \leq \cdots \leq m_t$ such that $\sum_i m_i = m$. The number of such partitions [67, Proposition 1.4] is at most

$$\binom{t+m-1}{m} < \binom{2m}{m} < 2^{2m}.$$

Hence

$$g(P, m, t) \leq 2^{2m} \left(\frac{236\mathsf{e}tn}{m}\right)^{2m} \leq \left(\frac{ctn}{m}\right)^{2m}.$$

□

As in Lemma 4, we have the following corollary of Lemma 6.

Corollary 1. *For all integers t such that $\frac{2m}{n} \leq t \leq m$, the number of labelled n-vertex m-edge graphs with geometric thickness at most t is at most*

$$n^{4n+o(n)} \left(\frac{ctn}{m}\right)^{2m},$$

for some absolute constant c. □

Theorem 7. *For all $\Delta \geq 9$ and $\epsilon > 0$, for all large $n > n(\epsilon)$ and $n \geq c\Delta$, there exists a Δ-regular n-vertex graph with geometric thickness at least*

$$c\sqrt{\Delta}\, n^{1/2-4/\Delta-\epsilon},$$

for some absolute constant c.

Proof. Let t be the minimum integer such that every Δ-regular n-vertex graph has geometric thickness at most t. Thus the number of Δ-regular n-vertex graphs is at most the number of labelled graphs with $\frac{1}{2}\Delta n$ edges and geometric thickness at most t. By Lemma 2 and Corollary 1,

$$\left(\frac{n}{3\Delta}\right)^{\Delta n/2} \leq n^{4n+o(n)} \left(\frac{ct}{\Delta}\right)^{\Delta n} \leq n^{4n+\epsilon n} \left(\frac{ct}{\Delta}\right)^{\Delta n},$$

for large $n > n(\epsilon)$. Hence $t \geq \sqrt{\Delta}\, n^{1/2-4/\Delta-\epsilon}/(c\sqrt{3})$. $\square$

It remains open whether geometric thickness is bounded by a constant for graphs with $\Delta \leq 8$. The above method is easily modified to prove Malitz's lower bound on book thickness.

Theorem 8 ([84])**.** *For all $\Delta \geq 3$ and $n \geq c\Delta$, there exists a Δ-regular n-vertex graph with book thickness at least*

$$c\sqrt{\Delta}\, n^{1/2-1/\Delta},$$

for some absolute constant c.

Proof. Obviously the number of order types for point sets in convex position is $n!$. As in the proof of Theorem 7,

$$\left(\frac{n}{3\Delta}\right)^{\Delta n/2} \leq n! \left(\frac{ct}{\Delta}\right)^{\Delta n} \leq n^n \left(\frac{ct}{\Delta}\right)^{\Delta n}.$$

Hence $t \geq \sqrt{\Delta}\, n^{1/2-1/\Delta}/(c\sqrt{3})$. (The constant c can be considerably improved here; for example, we can replace 472 by 16.) $\square$

barat.janos_51_23

Chapter 3

Decomposition into copies of a tree

3.1 Introduction

Let $\mathcal{H}$ be a collection of graphs. We say that a multigraph G has a $\mathcal{H}$*-decomposition* if the edges of G can be divided into subgraphs each of which is isomorphic to a graph in $\mathcal{H}$. If $\mathcal{H} = \{H\}$, then we speak of an H*-decomposition* of G. The H-decompositions are widely studied when G is a complete graph. If H is the 3-cycle C_3, then they are the well-known Steiner triples. If G is not complete, then it may be hard to find H-decompositions. Indeed, if H has at least three edges, then the problem of deciding if a graph G has an H-decomposition is NP-complete [45].

Pak informed us that $K_{1,3}$-decompositions were studied already in 1916 in connection with rigidity of polyhedra. Dehn [40] proved (among other things) that every planar triangulation (minus a 3-cycle) has a $K_{1,3}$-decomposition.

Jünger, Reinelt and Pulleyblank [68] studied $\mathcal{H}$-decompositions, where the graphs in $\mathcal{H}$ have three edges. Among other things, they proved that every 2-edge-connected graph G with $|E(G)|$ divisible by 3, has a $\{P_4, C_3, K_{1,3}\}$-decomposition, where P_4 is the path with three edges. They proposed the following, still unsolved problem [68]:

Question 1. *Is it true that every planar* 2*-edge-connected bipartite graph* G *with* $|E(G)|$ *divisible by* 3 *has a* P_4*-decomposition?*

They also asked the following:

Question 2. *Suppose that* $\mathcal{H}$ *is the class of* 2*-edge-connected graphs with* s *vertices. Is there an edge-connectivity (depending on* s*) that guarantees a graph to have a* $\mathcal{H}$*-decomposition?*

The answer to this question is negative: there is no such edge-connectivity, because there exist graphs of arbitrarily high connectivity and girth. Erdős [52] proved that there are graphs of arbitrarily large chromatic number and girth. Mader [81] proved that any such graph has a subgraph of large connectivity. In [16], it was shown that the subgraph can also be chosen to have large chromatic number. In view of this, we must require that a finite collection $\mathcal{H}$ must contain a forest if we wish to show that large (fixed) edge-connectivity implies the existence of a $\mathcal{H}$-decomposition. This has inspired us to the following general conjecture.

Conjecture 1 (Barát-Thomassen Conjecture)**.** *For each tree T, there exists a natural number k_T such that the following holds: If G is a k_T-edge-connected graph such that $|E(T)|$ divides $|E(G)|$, then the edges of G can be divided into parts, each of which is isomorphic to T.*

A *graph* has no loops or multiple edges. A *multigraph* may have multiple edges. In order to emphasize that some of the results hold only for graphs we shall sometimes call these *simple graphs.*

3.2 Decompositions and orientations

Conjecture 2 below is a special case of Conjecture 1.

Conjecture 2. *There exists a smallest natural number k_c such that every simple k_c-edge-connected graph G has a $K_{1,3}$-decomposition, provided $|E(G)|$ is divisible by 3.*

The graph $K_{1,3}$ is also called the *claw.* Claw-decompositions can be expressed in terms of orientations. For, if a graph G has a claw-decomposition, then we can orient the edges of G as follows. Whenever there is a claw of the decomposition with center x and leaves y_1, y_2, y_3, then let the edges be oriented from x towards y_i, for $i = 1, 2, 3$. In the resulting graph, all outdegrees are congruent to 0 modulo 3. Conversely, if G has such an orientation, then it implies the existence of a claw-decomposition of G. Motivated by this connection, we now focus on orientations. If v is a vertex of an oriented graph such that $d^+(v) \equiv d^-(v) \pmod 3$, then we say that the orientation is *balanced* at $v \pmod 3$. An orientation of a graph G is called a *Tutte-orientation*, if each vertex is balanced (mod 3).

If a graph has a nowhere zero 3-flow, then we obtain a Tutte-orientation by reversing the edges of flow value 2. Tutte's 3-flow conjecture states that every multigraph with no 1-edge-cut and no 3-edge-cut has a nowhere zero 3-flow. Equivalently, every 4-edge-connected multigraph has a Tutte-orientation. For more details on Tutte's 3-flow conjecture, see e.g. [35, 42, 65]. Jaeger proposed the following weaker conjecture.

Conjecture 3 (Jaeger [63])**.** *There exists a smallest natural number k_t such that every k_t-edge-connected multigraph has a Tutte-orientation.*

Thus Tutte conjectured that $k_t = 4$, and this would imply Grötzsch's theorem that every planar triangle-free graph is 3-colorable.[1] Barát and Thomassen posed the following

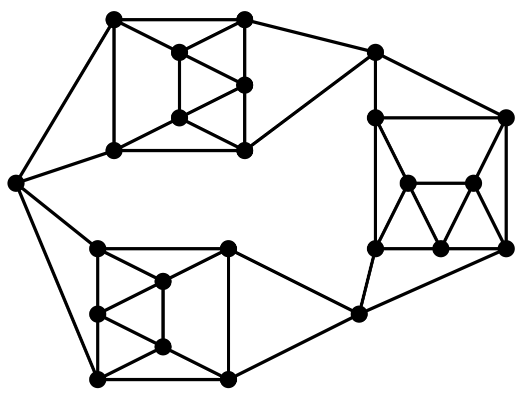
Conjecture 4. *If G is a planar, 4-edge-connected graph, and $|E(G)|$ is divisible by 3, then G has a claw-decomposition.*

[1]The dual version of Grötzsch's theorem states that every 4-edge-connected planar multigraph has a Tutte-orientation, see e.g. [65].

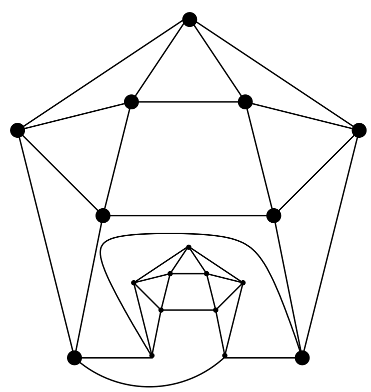
A cubic graph G has a claw-decomposition if and only if G is bipartite. For, such a graph G has $2k$ vertices and $3k$ edges. Hence a claw-decomposition of G must consist of k claws, and the centers must form an independent set. So, a 3-edge-connected, planar graph need not have a claw-decomposition, and hence Conjecture 4 is sharp.

It would be tempting to extend Conjecture 4 to the stronger statement that $k_c = k_t = 4$. But this is false. To see this, consider three copies of K_4, and add two edges between any pair such that we get a 4-regular graph G_0. This graph has 12 vertices and 24 edges. Assume that a claw-decomposition of G_0 exists. It must consist of eight claws. Orient the edges of each claw away from the center. There must be four sinks, that is, vertices of outdegree 0. By the pigeon-hole principle, two of them must be in the same K_4. This is a contradiction. Thus $k_c > 4$, and the planarity condition cannot be dropped in Conjecture 4. The construction can be iterated as follows. Take three copies of G_0 and unfold two edges between the K_4's. These altogether six edges can be used to connect each pair of the three copies of G_0 to make the graph 4-edge-connected. Now this graph has no independent set of twelve sinks by the pigeon-hole principle.

Lai [74] disproved Conjecture 4 by a family of planar graphs with edge-connectivity 4. The smallest one contains 24 vertices, see below.



In a recent computer search, Hasanvand [59] found a counterexample with 18 vertices, which is smallest possible.



Lai and Li [75] proved the existence of claw-decompositions for planar 5-edge-connected graphs.

Theorem 9 ([75])**.** *If G is a 5-edge-connected planar graph and $|E(G)|$ is divisible by 3, then G has a claw-decomposition.*

Even if we remove the planarity condition, perhaps $k_c = 5$. If so, then $k_t \leq 8$, as we prove in the theorem below.

Theorem 10. *If every 8-edge-connected simple graph with size divisible by 3 has a claw-decomposition, then every 8-edge-connected multigraph has a Tutte-orientation. In other words, if $k_c \leq 8$, then $k_t \leq 8$.*

Proof. Let us assume that every 8-edge-connected graph has a claw-decomposition. Using this, we prove that every 8-edge-connected multigraph has a Tutte-orientation. We proceed by induction on the number of vertices. The multigraph with two vertices and eight edges clearly has a Tutte-orientation. So we proceed to the induction step.

We may assume that G is 2-connected since otherwise, we apply the induction hypothesis to each block of G.

If e_1 and e_2 are parallel edges in the multigraph G under consideration, then we contract all edges parallel with e_1. The resulting multigraph is called G'. We use the induction hypothesis for G'. We orient all edges parallel with e_1 and distinct from e_1, e_2 at random. We claim that G also has a Tutte-orientation. It suffices to consider the endvertices x and y of e_1. There are three different possible orientations of the edges e_1 and e_2. Since they contribute to the outdegree of x by 0, 1 or 2, one of them will give a balanced orientation at x (mod 3). Then also y will be balanced (mod 3). We may therefore assume that G has no multiple edges.

Suppose that $v \in V(G)$ is a vertex of even degree. Using a theorem by Mader (namely Theorem 10 in [82]), there exist two edges vx and vy that we can split (that is, replace by a new edge xy) such that the edge-connectivity between any two vertices of $V(G) \setminus \{v\}$ does not change. Since the degree of v is even, we may split all edges incident with v and complete the proof by induction. (Mader's theorem allows multiple edges. That theorem only requires that v is not a cutvertex and that v has degree at least 4 and has at least two distinct neighbors.)

Assume next that $v \in V(G)$ is of odd degree, $2k+9$ say. If $k > 0$, we split two edges, and use induction for the resulting multigraph. Note that the resulting multigraph is 8-edge-connected because v has degree at least 8 (in fact, at least 9) after the splitting.

There remains only the case in which G is 8-edge-connected and 9-regular, and has no multiple edges. By assumption, G has a claw-decomposition, which corresponds to an orientation with all outdegrees divisible by 3. As all degrees are 9, such an orientation is a Tutte-orientation. □

In Theorem 10, the number 8 may be replaced by any number of the form $8 + 6k$, where k is a natural number. Thus, the existence of k_c implies the existence of k_t.

We now prove the converse, that the existence of k_t implies the existence of k_c. For this, it is convenient to study more general orientations. Let G be a multigraph, and w :

$V(G) \to \{0,1,2\}$ a prescribed weight function on the vertices such that $\sum_{v \in V(G)} w(v) \equiv |E(G)| \pmod 3$.

If there is an orientation of the edges of G with $d^+(v) \equiv w(v) \pmod 3$ for each $v \in V(G)$, then we say that G *admits the generalized Tutte-orientation prescribed by* w.

If, for every such $w : V(G) \to \{0,1,2\}$, there is an orientation of the edges of G with $d^+(v) \equiv w(v) \pmod 3$ for each $v \in V(G)$, then we say that G *admits all generalized Tutte-orientations.*

Conjecture 5. *There exists a smallest natural number k_g such that every k_g-edge-connected multigraph admits all generalized Tutte-orientations.*

Clearly $k_t \le k_g$. Also $k_c \le k_g$. Just consider the generalized Tutte orientation prescribed by the zero-function. We show that the three parameters are essentially equal. We shall use the following fundamental result by Nash-Williams [93] and Tutte [117].

Theorem 11 ([93, 117])**.** *Every $2k$-edge-connected multigraph G has k pairwise edge-disjoint spanning trees.*

Theorem 12. *If one of k_c, k_t, k_g exists, then they all exist. In this case, $k_g \le 2k_t + 2$, $k_c \le k_g$, and $k_t \le k_c + 5$.*

Proof. Assume that k_t exists. We shall prove that k_g exists and that $k_g \le 2k_t + 2$. Let G be a multigraph with edge-connectivity at least $2k_t + 2$, and let w be any prescribed weight function. By Theorem 11, G has $k_t + 1$ edge-disjoint spanning trees $T_1, \dots, T_{k_t+1}$.

Put $w^*(v) = -d_G(v) - w(v)$ for each vertex v. We orient some edges of T_{k_t+1} such that $d_F^+(v) - d_F^-(v) \equiv w^*(v) \pmod 3$ for each $v \in V(G)$, where F denotes the resulting oriented forest F. It is an easy exercise to show that such a partial orientation of T_{k_t+1} exists.

The unoriented edges of G form a k_t-edge-connected multigraph H. It has a Tutte-orientation by the assumption. That is $d_H^+(v) \equiv d_H^-(v) \pmod 3$ for each vertex v. Thus $d_H(v) = d_H^+(v) + d_H^-(v) \equiv 2d_H^+(v) \equiv -d_H^+(v) \pmod 3$. Similarly, $d_F(v) \equiv d_F^+(v) + d_F^-(v) = 2d_F^+(v) - (d_F^+(v) - d_F^-(v)) \equiv -d_F^+(v) - w^*(v) \pmod 3$ for each vertex v. Hence $d_G^+(v) = d_H^+(v) + d_F^+(v) \equiv -d_H(v) - d_F(v) - w^*(v) = -d_G(v) - w^*(v) = w(v) \pmod 3$ for each vertex v. Hence, k_g exists and $k_g \le 2k_t + 2$.

As noted after Conjecture 5, $k_c \le k_g$. The remark after the proof of Theorem 10 shows that $k_t \le k_c + 5$. □

Corollary 2. *If the 3-flow conjecture is true, then every 10-edge-connected multigraph admits all generalized Tutte-orientations. In particular, every 10-edge-connected graph has a claw-decomposition, provided its size is divisible by 3.*

Let us call a graph *mod $(2p+1)$-orientable* if it has an orientation such that each vertex is balanced mod $(2p+1)$. Jaeger also proposed the following generalization of Conjecture 3.

Conjecture 6 (Jaeger [63]). *For each $p \geq 1$, there exists a smallest natural number $k_j(p)$ such that every $k_j(p)$-edge-connected multigraph has a mod $(2p+1)$-orientation. Moreover, $k_j(p) \leq 4p$.*

A *generalized mod* $(2p+1)$*-orientation* can be defined in the obvious way.

The methods in the proof of Theorem 12 show that for each natural number $p \geq 1$ the following are equivalent:

(a) There exists a smallest natural number $k_j(p)$ such that every $k_j(p)$-edge-connected multigraph G has a mod $(2p+1)$-orientation.

(b) There exists a smallest natural number $k_c(p)$ such that every $k_c(p)$-edge-connected simple graph G, whose size is divisible by $2p+1$, has a $K_{1,2p+1}$-decomposition.

(c) There exists a smallest natural number $k_g(p)$ such that every $k_g(p)$-edge-connected multigraph G admits all generalized mod $(2p+1)$-orientations.

Clearly, (c) implies (a) and (b). The proof of Theorem 12 shows that (a) implies (c). We now indicate why also (b) implies (c). Specifically, we prove that $k_g(p) \leq 4p(k_c(p)+2p)$. Let G be a multigraph of edge-connectivity at least $4p(k_c(p)+2p)$, and let w be a function, which we shall show prescribes a generalized mod $(2p+1)$-orientation. If G has a multiple edge consisting of at least $2p$ parallel edges, then we contract them and use induction. So assume there is no such multiple edge. For every multiple edge we orient all its edges, except precisely one, at random. We delete the oriented edges and modify the function w accordingly. The resulting simple graph has edge-connectivity at least $2(k_c(p)+2p)$ and contains therefore $k_c(p)+2p$ edge-disjoint spanning trees. We use $2p$ of these spanning trees to orient some of their edges in such a way that deleting the oriented edges and modifying w accordingly, the modified w becomes the zero function.

The resulting graph has $k_c(p)$ edge-disjoint spanning trees and therefore edge-connectivity at least $k_c(p)$. Then we complete the proof using the assumption of Conjecture (b).

Note that, in this way we do not use Mader's splitting theorem. That could also be avoided in Theorem 10, but then the inequalities in Theorem 12 would become weaker.

Thomassen [113] proved the weak 3-flow conjecture by Jaeger [63]. Together with Theorem 12 this implies the following result by Thomassen.

Theorem 13 ([113]). *If G is an 8-edge-connected graph, and $|E(G)|$ is divisible by 3, then G has a claw-decomposition.*

This was later improved to 6 by Lovász et al. [79]. A graph is *essentially λ-edge-connected* if the edges of any edge cut of size strictly less than λ are incident to a common vertex.

Theorem 14 ([41, 79]). *Every 5-edge-connected, essentially 6-edge-connected graph of size divisible by 3 admits a claw-decomposition.*

Delcourt and Postle [41] proved that 4-regular random graphs do have claw-decompositions asymptotically almost surely.

Theorem 15 ([41]). *A random 4-regular n-vertex graph has a claw-decomposition asymptotically almost surely, provided that n is divisible by 3.*

Recently, Hasanvand [59] constructed planar 4-regular 4-connected, essentially 6-edge-connected graphs with no claw-decompositions.

Theorem 16 ([59]). *There are infinitely many planar* 4*-regular* 4*-connected, essentially* 6*-edge-connected graphs of size divisible by* 3 *with no claw-decompositions.*

By a computer search, Hasanvand determined there are only four 4-regular connected graphs of order 12 with no claw-decompositions using a regular generator due to Meringer. Among 4-regular connected graphs of order 15 (resp. 18) there are only 146 (resp. 15932) graphs without claw-decompositions, that is less than 0.02% (resp. 0.002%) of them. The above theorem by Delcourt and Postle [41] shows this ratio must tend to zero.

3.3 Triangulations of surfaces

If a graph on a surface is 3-colorable, then its dual graph has a 3-flow, and hence a Tutte-orientation. In particular, every triangulation of a surface, other than K_4, has a Tutte-orientation.

An n-vertex triangulation of a surface of Euler genus k has $3n-6+3k$ edges, see e.g. [89]. Hence, it is a natural candidate for having a claw-decomposition. In this section, we prove the stronger result, that every triangulation distinct from K_4 admits all generalized Tutte-orientations.

We shall use four lemmas, some of which may be of independent interest.

Lemma 7. *If the edges of a multigraph G can be acyclically oriented such that each vertex, except one, has outdegree at least* 2*, then G admits all generalized Tutte-orientations.*

Proof. Let w be any weight function on the vertices. The assumptions imply that the vertices of G can be labelled $x_1, \dots, x_n$ such that all arcs (directed edges) go from right to left. Each vertex has at least two outgoing arcs. In particular, there are at least two edges between x_1 and x_2. Contract these edges, and use induction. Orient the edges between x_1 and x_2 such that $d^+(x_2) \equiv w(x_2) \pmod 3$. The condition on w implies that also $d^+(x_1) \equiv w(x_1) \pmod 3$. □

No graph satisfies Lemma 7, as multiple edges are needed. However, any triangulation with one or more edges added, or one or more edges contracted satisfies the assumption of Lemma 7. This follows easily from the following observation. If G is a triangulation, and H is a connected subgraph containing at least two but not all vertices, then G has a vertex v that is not in H but which is joined to at least two vertices in H. (We then orient all edges between v and H from v to H, add v to H, and repeat.)

Lemma 8. *Let k be a natural number, $k \geq 3$, and let w be a weight function of the k-wheel W_k with center c such that $\sum_{x\in V(W_k)} w(x) \equiv |E(W_k)| \pmod 3$. The k-wheel admits the generalized Tutte-orientation prescribed by w, unless k is odd, and $w(x) \equiv 0 \pmod 3$ for all vertices $x \in W_k \setminus \{c\}$.*

Proof. Let $x_1x_2\dots x_kx_1$ be the cycle $W_k \setminus \{c\}$. If k is even, and $w(x_i) \equiv 0 \pmod 3$, for $i = 1, 2, \dots, k$, then we orient the edges of the wheel such that $x_1, x_2, \dots, x_k$ are sources and sinks alternately. So assume that $w(x_1) \not\equiv 0 \pmod 3$. If $w(x_1) = 1$, then we orient the edge x_1x_2 towards x_1. If $w(x_1) = 2$, then we orient it away from x_1. Then we successively orient the two unoriented edges incident with x_2, the two unoriented edges incident with x_3 etc. as prescribed by w. Clearly, it is possible to orient the last edge incident with x_1. The condition $\sum_{x\in V(W_k)} w(x) \equiv |E(W_k)| \pmod 3$ ensures that the center receives the correct prescribed outdegree. □

Lemma 9. *Let $k \geq 3$ be a natural number, and let U_k be a multigraph obtained from the k-wheel W_k by adding one or more edges. The graph U_k admits all generalized Tutte-orientations.*

Proof. Let w be any prescribed weight function of U_k. Orient all added edges, except one, at random. The last added edge can be oriented in two ways. For each of these two orientations, we modify w accordingly. At least one of these two modifications of w is not the exceptional weight function in Lemma 8. Hence, Lemma 8 implies that U_k has the desired orientation. □

Lemma 10. *Let x and y be adjacent vertices in a triangulation G such that at least one of x, y, say y, has degree at least 4 and such that $N(x) \cap N(y)$ consists of only two vertices. Let J be the subgraph induced by $N(x) \cup N(y)$. The graph J admits all generalized Tutte-orientations.*

Proof. Let w be any weight function, and let $x_1, x_2, \dots, x_{k-1}, x$ be the neighbors of y in clockwise order. The graph induced by $\{y\} \cup N(y)$ is a wheel W_y and possibly some additional edges that we first orient at random. Now we repeat the procedure from the proof of Lemma 8 with a slight modification. We orient the edge x_1x_2 arbitrarily. Next we orient successively the two unoriented edges of W_y incident with $x_2, \dots, x_{k-2}$ to achieve the prescribed outdegrees at these vertices. The remaining unoriented edges of J form a wheel W_x with center x, and possibly some additional edges that we orient at random. As the orientation of x_1x_2 can be chosen in two ways, we may assume that the exceptional case in Lemma 8 does not occur now for W_x. Hence, we can orient the edges of W_x by Lemma 8. □

Theorem 17. *Let $G \neq K_3, K_4$ be a triangulation of any surface S. The graph G admits all generalized Tutte-orientations.*

Proof. If G contains a non-facial triangle, then let x be one of the vertices of this triangle, and let J be the subgraph of G induced by x and its neighbors. The graph J is a wheel with at least one additional edge.

If all triangles of G are facial, then we let x be any vertex. As $G \neq K_3, K_4$, x has a neighbor y of degree at least 4. As all triangles containing x are facial, x and y have only two neighbors in common. In this case, we let J be the subgraph of G induced by x, y and the neighbors of x, y.

Let w be any weight function of G. We contract J into a single vertex, and we modify w accordingly. By the remark after Lemma 7, the resulting multigraph has the desired orientation. By Lemma 9 or Lemma 10, the orientation can be extended to J. □

Theorem 17 shows that every triangulation has a claw-decomposition. The decomposition may possibly be chosen such that (almost) every vertex is the center of a claw. We now prove that this holds for triangulations of surfaces of Euler genus at most 2. For this, we use the following well-known consequence of Edmonds' matroid partition theorem [48]. For completeness, we indicate a short proof.

Theorem 18. *Let G be a graph with n vertices, and let $k_1, k_2, \ldots, k_n$ be non-negative integers. The graph G has an orientation satisfying $d^+(v_i) \leq k_i$ for $i = 1, \ldots, n$ if and only if $|E(J)| \leq \sum_{i:v_i \in V(J)} k_i$ for any subgraph $J \subseteq G$.*

Proof. The necessity is obvious. For the sufficiency, let M_i be the matroid on $E(G)$ whose independent sets are the sets that consist of at most k_i edges, each of which is adjacent to v_i. The matroid partition theorem gives us a partition of $E(G)$ into sets $E_1, \ldots, E_n$ such that E_i is independent in M_i if and only if for any $S \subseteq E(G)$ we have $|S| \leq \sum_{i:S \cap M_i \neq \emptyset} k_i$. Now orient the edges in E_i away from v_i. □

Theorem 19. *Let G be a triangulation of the plane or the projective plane or the torus or the Klein bottle. The graph G has an orientation such that all outdegrees are* 3 *or* 0*, except when $G = K_4$ in the plane.*

Proof. We prescribe all outdegrees k_i to be at most 3, except for two independent vertices in the plane and one vertex in the projective plane for which we put $k_i = 0$. Next we apply Theorem 18. The required inequalities hold by Euler's formula. Since all outdegrees (except one or two) are at most 3, all of them (except one or two) are precisely 3, again by Euler's formula. □

Barát and Thomassen [C] posed the following

Conjecture 7. *If G is a triangulation of a surface of Euler genus $k \geq 2$, then G has an orientation such that each outdegree is at least* 3*, and divisible by* 3*.*

Theorem 19 shows that Conjecture 7 holds for $k = 2$. Another motivation for this conjecture is, that it can be seen as a step towards the generalization of planar Schnyder woods to higher genus surfaces. A Schnyder wood [106] of a planar triangulation is an orientation and a $\{0, 1, 2\}$-coloring of the inner edges satisfying the following local rule on every inner vertex v: going counterclockwise around v one successively crosses an outgoing 0-arc, possibly some incoming 2-arcs, an outgoing 1-arc, possibly some incoming 0-arcs, an outgoing 2-arc, and possibly some incoming 1-arcs until coming back to the outgoing 0-arc. Schnyder woods are one of the tools in the area of planar graph representations and Graph Drawing. They provide a machinery to construct space-efficient straight-line drawings, representations by touching T shapes, and are used to encode triangulations efficiently. In particular, the local rule implies that every Schnyder wood gives an orientation of the inner edges such that every inner vertex has outdegree 3 and the outer vertices are sources with respect to inner edges. Indeed, this is a one-to-one correspondence between Schnyder woods and orientations of this kind.

Albar et al. [12] proved Conjecture 7.

Theorem 20 ([12])**.** *If G is a triangulation of a surface of Euler genus $k \geq 2$, then G has an orientation such that each outdegree is at least 3, and divisible by 3.*

3.4 Dense graphs

Lai and Zhang [73] proved that every $4\lceil \log n \rceil$-edge-connected multigraph has a nowhere-zero 3-flow. We now prove that any such graph admits all generalized Tutte-orientations.

Theorem 21. *Every $4\lceil \log n \rceil$-edge-connected multigraph with n vertices admits all generalized Tutte-orientations.*

Proof. Assume for simplicity that $\log n$ is a natural number. Let w be any prescribed weight function on the vertices. We show that G can be oriented as prescribed by w. The idea is to find $\log n$ pairwise edge-disjoint spanning Eulerian subgraphs $G_1, G_2, \ldots, G_{\log n}$. We orient the remaining edges arbitrarily, and modify w accordingly. We use G_1 to give half of the vertices the prescribed outdegree modulo 3. We next use G_2 to take care of half of the remaining vertices, and so on.

We now argue formally. By Theorem 11, G has $2 \log n$ pairwise edge-disjoint spanning trees $T_1, T_2, \ldots, T_{2\log n}$. It is well-known and easy to see that the union of any two of these contains a connected spanning Eulerian subgraph. Therefore G contains $\log n$ pairwise edge-disjoint spanning Eulerian subgraphs $G_1, G_2, \ldots, G_{\log n}$.

We orient all edges not in $E(G_1) \cup E(G_2) \cup \cdots \cup E(G_{\log n})$ at random, and we modify w accordingly. Next we define w^* as in the proof of Theorem 12. In other words, we are going to orient $G_1 \cup G_2 \cup \cdots \cup G_{\log n}$ such that for each vertex v, there are $w^*(v)$ outgoing arcs, and the remaining $d(v) - w^*(v)$ arcs incident with v are balanced at v (mod 3). (We assume here that $w^*(v)$ is one of 0,1,2.) We now define the *mode* of a vertex v. Initially the mode of v is $w^*(v)$. If all vertices are in mode 0, then we just orient each G_i, for $1 \leq i \leq \log n$, such that each vertex is balanced. If some vertices are in a mode $\neq 0$, then we orient $G_1, G_2, \ldots, G_{\log n}$ successively such that we use each G_i to turn at least half of the vertices of mode $\neq 0$ into mode 0. We explain how this is done for G_1. The procedure for $G_2, \ldots, G_{\log n}$ is similar. So, we let $v_1, e_1, v_2, e_2, \ldots, e_m, v_1$ be a closed Euler walk of G_1. There may be repetition of vertices. Suppose v_2 is in mode 2. Next we orient both edges e_1, e_2 away from v_2. Suppose we have already oriented $e_1, e_2, \ldots, e_{k-1}$, and that e_{k-1} is directed towards v_k. If v_k is in mode 0, then we direct e_k away from v_k, and we say that v_k is still in mode 0. If v_k is in mode 1, then we direct e_k towards v_k, and we say that v_k is in mode 0. Now v_k is in the required mode and will remain there. Finally, we consider the case in which v_k is in mode 2. In this case, we consider the first vertex v_p in the sequence $v_{k+1}, v_{k+2}, \ldots$ which is not in mode 0. We orient the edges $e_k, e_{k+1}, \ldots, e_p$ such that v_p turns into mode 0, and $v_{k+1}, \ldots, v_{p-1}$ remain in mode 0. Vertex v_k will be in either mode 1 or mode 2. If there are k vertices in the undesired mode 1 or 2, then we change in this way at least $(k-1)/2$ of these into the desired mode 0. We repeat this argument for $G_2, \ldots, G_{\log n}$. When this procedure terminates, all vertices will be in mode 0. □

Theorem 22. *There exists a constant n_1 such that every graph G with n vertices, $n \geq n_1$, and minimum degree δ, where $\delta(G) \geq \frac{n}{2}$, admits all generalized Tutte-orientations.*

Proof. If the edge-connectivity of G is at least $4\lceil \log n \rceil$, then the claim holds by Theorem 21. Otherwise, G has an edge-cut of size smaller than $4\lceil \log n \rceil$. The minimum degree ensures that there are at least $\frac{n}{2} - 4\lceil \log n \rceil$ vertices on both sides of the cut. When n is large enough, both sides are $4\lceil \log n \rceil$-edge-connected, and hence they admit all generalized Tutte-orientations. So, for any prescribed weight function w on $V(G)$, we first orient the edges in the cut and then apply Theorem 21 to each side of the cut. We only need to make sure that the modified weight functions satisfy the congruence relation. As the cut has at least two edges, this is always possible. □

If n is even, then the graph consisting of the union of two copies of $K_{\frac{n}{2}}$ and one edge between them has neither a Tutte-orientation nor a claw-decomposition. The degree condition in Theorem 22 is therefore sharp. However, for 2-edge-connected graphs, there is a better bound.

Theorem 23. *There exists a constant n_2 such that every 2-edge-connected graph G with $n \geq n_2$ vertices and minimum degree $\delta(G) \geq \frac{n}{4}$ admits all generalized Tutte-orientations.*

The proof of Theorem 23 is similar to, but more tedious than that of Theorem 22. Theorem 23 is best possible in the following sense: If n is divisible by 4, then take the union of four copies of $K_{\frac{n}{4}}$. Add six independent edges such that there is precisely one edge between any two copies of $K_{\frac{n}{4}}$. The resulting graph has minimum degree $\frac{n}{4} - 1$ but has no Tutte-orientation.

3.5 Reduction to bipartite graphs

In a nutshell, Thomassen successfully applied the following scheme for various trees T to prove instances of Conjecture 1:

1. Remove copies of T from G such that a bipartite graph $G[A, B]$ remains, which still contains many edge-disjoint spanning trees.
2. Remove more copies of T such that each degree in A becomes divisible by k, and the rest still contains some edge-disjoint spanning trees.
3. Group the edges from A such that copies of T arise, which altogether decompose the rest.

In this section, we concentrate on step 1. We recall the following result from [112], that we make heavy use of.

Lemma 11. *Let p be any natural number, and put $m = 2^p$. If G is a multigraph with a collection of m pairwise edge-disjoint spanning trees, then G has a spanning tree T such that, for each vertex v, $d_T(v) \leq d_G(v)/m + 3p/2$.*

Using a maximum cut idea, it is easy to prove the following

Lemma 12. *If k is a natural number and G is a $2k-1$-edge-connected graph, then G has a bipartition such that $G[A,B]$ is k-edge-connected.*

In Thomassen's scheme, the first step is to delete some copies of the tree such that the remaining graph is a highly edge-connected bipartite graph. It was mentioned in [112], that perhaps this method works for every tree. In this section, we validate this hypothesis. We need the following result, which is practically a consequence of Lemma 11.

Lemma 13. *For any natural numbers k, ℓ and m, if G contains $km^{2\ell}$ edge-disjoint spanning trees, then we can choose subgraphs $M_1 \subset M_2 \subset \cdots \subset M_{\ell+1}$ such that M_1 contains k edge-disjoint spanning trees $T_1, \ldots, T_k$ and $d_{M_i}(v) \leq d_{M_{i+1}}(v)/m$ for every vertex v and for $1 \leq i \leq \ell - 1$.*

Proof. By Lemma 11, if we are given m^2 edge-disjoint spanning trees, there is one of them, T say, such that $d_T(v) \leq d_*(v)/m^2 + 3p/2 \leq d_*(v)/m$, where $d_*(v)$ is the total degree in that particular collection of m^2 spanning trees. Here in the last inequality, we implicitly use that $d_*(v) \geq m^2$. We prove the claim of the Lemma by induction on ℓ. We start with the base $\ell = 1$. If we are given km^2 edge-disjoint spanning trees, then we divide them into k equal sets and choose a spanning tree in each of these sets by Lemma 11. Summing these inequalities, we get the required result for $\ell = 1$.

Assume that we have proved the lemma for M_i, where $i \geq 1$. In the induction step, there are $m^2 km^{2i}$ edge-disjoint spanning trees given. Just as previously, we partition the set of spanning trees into sets of size m^2. We find the low-degree spanning trees by Lemma 11, as before. It gives km^{2i} edge-disjoint spanning trees. By the induction hypothesis, we can find $M_1 \subset M_2 \subset \cdots \subset M_i$ in them. Finally, M_{i+1} is the union of the $m^2 km^{2i}$ edge-disjoint spanning trees. Therefore, also the degree conditions are satisifed for M_{i+1}. □

We show that it is sufficient to prove Conjecture 1 for bipartite graphs:

Theorem 24. *Let T be a tree with t edges. The following two statements are equivalent.*
(i) There exists a natural number k_T such that for any k_T-edge-connected bipartite graph G, if t divides $|E(G)|$, then G has a T-decomposition.
(ii) There exists a natural number k'_T such that for any k'_T-edge-connected graph G, if t divides $|E(G)|$, then G has a T-decomposition.

Proof. We only prove the non-trivial implication. Let $k'_T = 8t^{2t+3} + 4k_T - 1$. By Lemma 12, we can find a partition (A, B) of the vertex set such that $G[A, B]$ is $4t^{2t+3} + 2k_T$-connected. By Theorem 11, there are at least $2t^{2t+3} + k_T$ pairwise edge-disjoint spanning trees in $G[A, B]$. In what follows, we show how to delete all edges inside A and B by removing copies of T using at most $2t^{2t+3}$ of the spanning trees. After this, the remaining k_T spanning trees provide k_T-edge-connectivity.

First, we greedily delete copies of T from $G[A]$ as long as possible. We partition the remaining edges into subgraphs of T as follows. Identify a vertex v_1 of A with a vertex x_1 of T. Connect v_1 to as many neighbors as x_1 has in T. If the degree of x_1 is too large, then connect v_1 to every possible neighbor. Continue this *copy/paste* process with the neighbors of v_1 trying to copy a subgraph of T. We skip those edges of $G[A]$, which would create a cycle in this subgraph. Once we run out of possible extensions, we finish the subgraph, which we started at v_1. We repeat this process on the set of remaining edges starting at an arbitrary vertex. In this way, we create a set $\mathcal{H}$ of subgraphs of G, which are subtrees of T.

Let H be a graph in $\mathcal{H}$. A vertex v is *incomplete* in H, if it is identified with a vertex of degree d in T, but the degree of v in H is smaller than d.

For every vertex v in A, there are at most t trees in $\mathcal{H}$, where v is incomplete. Indeed, consider the first occasion, when there were not enough edges to achieve the necessary degree at v in a subgraph H. There were at most $t-1$ edges incident to v, as they all go to vertices in H. Therefore, v appears in at most $t-1$ additional members of $\mathcal{H}$.

Claim 1. *The set $\mathcal{H}$ can be partitioned into t^2 sets $\mathcal{H}_1, \dots, \mathcal{H}_{t^2}$ such that for every vertex v in A, there is at most one tree in $\mathcal{H}_i$, for each i, $1 \le i \le t^2$, where v is incomplete.*

Proof. We can do it greedily. Select trees as long as possible into $\mathcal{H}_1$ without violating the property, and similarly for $\mathcal{H}_2, \dots, \mathcal{H}_{t^2}$. If a tree is not selected, it means that one of its at most $t+1$ vertices, v say, would violate the property, more precisely one of the at most $t-1$ other trees where v is incomplete is already in the partition class. It can happen at most $(t+1)(t-1)$ times. □

Back to the proof of the Theorem: We divide the remaining spanning trees into t^2 sets of size t^{2t+1} (each such set corresponds to a partition class $\mathcal{H}_i$) and a set of size k_T. First we add edges from the corresponding spanning trees to the members of $\mathcal{H}_1$ to get copies of T, and remove those copies. We repeat this process with the other partition classes and the other spanning trees.

Consider $\mathcal{H}_1$ and the corresponding t^{2t+1} spanning trees. We apply Lemma 13 with $t = k = \ell = m$ for the graph consisting only of these spanning trees, to get subgraphs $M_1, \dots, M_t$ and $T_1, \dots, T_t$.

Consider now the incomplete vertices in $\mathcal{H}_1$. We add the missing edges using the spanning trees $T_1, \dots, T_t$. We only use the fact that each vertex is incident to at least one edge in each spanning tree. For a vertex v, if there is a member H of $\mathcal{H}_1$, where v is incomplete, we add an edge incident to v in $T_1, \dots, T_t$, to H. If there are several incomplete vertices in H, then the edges should go to different vertices of B. Those vertices are naturally identified with vertices of T now.

More precisely, let Step 1 be a breadth-first process: we add the missing edges to the vertices of every H in $\mathcal{H}_1$, and identify the other endvertices of these edges with vertices of T. Let $H(1)$ denote the new subtree of T, that consists of H, the new edges and the new vertices, and let $D_1(H)$ denote the set of these vertices. In Step i, we add the missing edges to the elements of $D_{i-1}(H)$ for every $H \in \mathcal{H}_1$ in order to get $H(i)$, and denote the set of other endvertices by $D_i(H)$, and identify them with vertices of T.

We show that it is possible. Suppose we have finished Step i. For every vertex v in $D_i(H)$, the degree of v in M_{i+1} is at least t times more than the number of edges incident to v used in all previous Steps. In particular, this degree is at least t times more than the number of edges incident to v used in Step i. Hence for every such edge, we can choose $t-1$ edges from $M_{i+1} \setminus M_i$. Now we use those edges (independently for every $H(i)$, which contains v) to add the next level of T to $H(i)$. All we need to do is to add, say j, missing edges incident to v, and these edges should avoid those at most s vertices, which are in $H(i)$. It yields $j + s < k$, and we can choose the edges greedily.

In this way, we can find copies of T, which contain everything from $\mathcal{H}_1$. We delete them and repeat the process for $\mathcal{H}_2$, and so on. □

3.6 Y-decompositions

At the time of posing, there was no tree of size at least 3, for which Conjecture 1 was known to be true. A nice and thorough introduction to the subject is [111], where Thomassen proved that every 207-edge-connected graph G has a set E of at most 6 edges such that $G - E$ has a 4-path-decomposition. Approximately the same time, Thomassen proved

Theorem 25 ([112])**.** *If G is a 171-edge-connected graph of size divisible by 3, then G has a 3-path-decomposition.*

The proof of Theorem 25 consists of three main ingredients. In principle, the method could be applied to any tree T. Let G be a graph of sufficiently high edge-connectivity, and let T be a tree on k edges.

For any fixed tree, the above edge-connectivity condition can be largely reduced. For any such improvement, we use the same principal argument, but we can decrease the necessary number of spanning trees, by using the structure of the fixed tree. Let Y be the unique tree with degree sequence $(1, 1, 1, 2, 3)$. In particular, for the graph Y, we show the following.

Lemma 14. *If G is a $4k + 23$-edge-connected graph, then we can remove some Y-copies such that a bipartite graph with k edge-disjoint spanning trees remains.*

Proof. By Lemma 12, we can find a bipartition $G[A, B]$ of G, which is $2k + 12$-edge-connected. By Theorem 11, we find $k + 6$ edge-disjoint spanning trees in $G[A, B]$. Let T_1, T_2, T_3 be three of them. We remove Y-copies from $G[A]$ greedily as long as we can. What remains in $G[A]$ is a collection of paths, cycles, stars and subgraphs of K_4. We cut each path and each cycle into paths with three edges and a possible shorter path. We identify one of the middle vertices of such a 3-path with a 3-vertex of Y. The idea is to extend these 3-paths into Y-copies using T_1, and remove them from G. For a 2-path, we identify one end-vertex with the 3-vertex of Y. For a single edge, we identify one end-vertex with the 3-vertex of Y, and the other end-vertex with the 2-vertex of Y. We cut the stars into 3-stars and a remaining part, which is a 2-path or a single edge, as above. For a 3-star, we identify a leaf with the 2-vertex of Y. Until now, any vertex in A is identified at most once with a vertex in a Y-copy. For any subgraph H of K_4, which is different from the previous ones, we do as follows. We cut H into paths of length at most three such that after the above identifications with 3- or 2-vertices of Y, each vertex is used at most once. This is always possible with one exception, the triangle.

If a vertex of A is identified with a 3- or 2-vertex of Y, then we extend the subgraph with edges of T_1 and T_2 to achieve a Y-copy, which we remove. It works fine except for a single edge or a triangle. For a single edge, we have to add three additional edges to get a Y-copy. For the vertex identified with the 3-vertex, we use edges from T_1 and T_2. Now there exists an edge in T_1, T_2 or T_3 from the other end of the single edge, which avoids creating a cycle, hence it makes a Y-copy, which we remove. For the triangle, we cut it into a single edge and a 2-path. We do as above for the single edge, and let v be the vertex, which was identified with the 2-vertex. For the 2-path, we identify v with the 3-vertex of Y. Since we used one of $T_1 - T_3$ for the single edge, there are two edges left to use. We create a Y-copy and remove it.

We have to execute the same process for $G[B]$, where we use three more spanning trees. After all, a bipartite graph remains, which has at least k edge-disjoint spanning trees. □

Remark 1. *Even if there are only $k + 5$ spanning trees in $G[A, B]$, we can delete Y-copies using* 5 *spanning trees such that a k-edge-connected bipartite graph remains. It requires a more detailed argument, and implies an improvement by* 4 *in the statement of Lemma* 14.

We recall an implicit result from [112]. We emphasize a balanced property, which was hidden in the proof of Theorem 25.

Lemma 15. *Let G be a* 2*-edge-connected bipartite graph with classes A and B. If the degree of each vertex in A is divisible by* 3*, then G can be decomposed into paths of length* 3 *such that every vertex v of A is the endvertex of $d(v)/3$ and middle vertex of $d(v)/3$ paths of length* 3.

Our main result gives a sufficient edge-connectivity condition for Y-decompositions.

Theorem 26. *Let Y denote the tree with degree sequence $(1, 1, 1, 2, 3)$. If G is a* 287*-edge-connected graph of size divisible by* 4*, then G has a Y-decomposition.*

Proof. We first apply Lemma 14 with $k = 66$. As a result, we are given a bipartite graph $G[A, B]$ with 66 edge-disjoint spanning trees $T_1, \ldots, T_{66}$.

In the next step, we delete some copies of Y to make all degrees in A divisible by 4. In the first phase, we achieve that all degrees are even. Therefore, vertices in A of odd degree are *bad*. Let $M(1)$ be a subgraph of G, which is the union of 12 edge-disjoint spanning trees $T_1, \ldots, T_{12}$. By Lemma 11, $p = 2$, $M(1)$ has a spanning tree $T(1)$ such that for each vertex v, $d_{T(1)}(v) \leq d_{M(1)}(v)/4 + 3 \leq d_{M(1)}(v)/2$, since $d_{M(1)}(v) \geq 12$.

Similarly, the union $M(2)$ of 12 spanning trees $T_{13},\dots,T_{24}$ contains a spanning tree $T(2)$ such that for each vertex v, $d_{T(2)}(v) \le d_{M(2)}(v)/2$. The union of $T(1)$ and $T(2)$ contains a spanning Eulerian subgraph E_1.

We start a walk on E_1 at a bad vertex u_1. We construct a Y-copy, which we delete, as follows. Let e_1 be the edge adjacent to u_1 in E_1, and let e_2 be the next edge. Walking along these two edges, we are back in A in a vertex u_2. We continue this way till we arrive to another bad vertex u_r. For every i, $1 \le i \le r-1$, we consider e_{2i-1}, e_{2i} and two edges in $M(1) \cup M(2) \setminus E_1$, which are incident to u_{i+1}. We glue these four edges together to form a copy of Y and remove them. In this way, we delete an odd number of edges incident to u_1 and u_r, and an even number of edges incident to any other vertex in A. Therefore, the number of bad vertices decreases. A vertex can appear multiple times in the above sequence, but that does not change the parity of the degree.

Now we continue the walk along E_1 and do nothing until we find another pair of bad vertices. We repeat the above process of removing Y-copies between the bad vertices. Iterating these two steps, we finish the Eulerian trail, and all degrees are now even. There is a small remark, that we have to make: there are enough edges in $M(1) \cup M(2) \setminus E_1$ to use. Indeed, whenever the walk arrives to a vertex v, it means there are two incident edges in E_1. Hence we can find two more edges, as the degree of a vertex v in E_1 is at most half of the degree of v in $M(1) \cup M(2)$.

In the second phase, all degrees in A are even. Our goal is to remove some Y-copies to make all degrees divisible by 4. Therefore, vertices in A of degree $2 \mod 4$ are *bad*. As in the first phase, we need an Eulerian spanning subgraph for our purposes. Let $M(3)$ be a subgraph of G, which is the union of 15 edge-disjoint spanning trees $T_{25},\dots,T_{39}$. By Lemma 11, $p = 3$, $M(3)$ has a spanning tree $T(3)$ such that for each vertex v, $d_{T(3)}(v) \le d_{M(3)}(v)/8 + 4.5 \le d_{M(3)}(v)/2 - 1$, since $d_{M(3)}(v) \ge 15$. Similarly, the union $M(4)$ of the spanning trees $T_{40},\dots,T_{54}$ contains a spanning tree $T(4)$ such that for each vertex v, $d_{T(4)}(v) \le d_{M(4)}(v)/2 - 1$. The union of $T(3)$ and $T(4)$ contains a spanning Eulerian subgraph E_2.

On the Eulerian trail, we mark the bad vertices. We start the marking at a bad vertex b_1. Later, we only mark the bad vertices at the first appearance. We get a list $b_1,\dots,b_r$ of bad vertices, and this list reflects their order of first appearance on E_2. This direction on E_2 is fixed from now on.

In what follows, we remove Y-copies to achieve that all degrees in A are divisible by 4. If v is a bad vertex, then we remove 2 or 6 edges incident to v during the process, when we arrive to the marked copy of v. If x is an unmarked vertex, then we remove precisely 4 edges. If x is a vertex on E_2, let x^+ be the next vertex of A on E_2. There are two building bricks:

1. *remove a Y-copy at x* is a step, when two consecutive edges of E_2 starting at x, and two edges of $M(3) \cup M(4) \setminus E_2$ at x^+ are removed.
2. *remove a reversed Y-copy at x* is a step, when two consecutive edges of E_2 starting at x, and two edges of $M(3) \cup M(4) \setminus E_2$ at x are removed.

We start at b_1 and remove a Y-copy. We continue along E_2 and remove all edges of E_2 two by two. Every such pair of edges corresponds to a 2-path in a Y-copy, where one end is the 3-vertex. The only decision to make is the placement of the other two edges from $M(3) \cup M(4) \setminus E_2$. Either at the current vertex x or at the subsequent vertex x^+. This is actually automatic, according to the degree condition: we either deleted 1 or 3 edges at x due to the previous Y-copy, and our goal might be to remove 2, 4 or 6 edges in total. If we need to remove one more edge at x, we remove a Y-copy. If we need to remove three more edges at x, we remove a reversed Y-copy. Notice here, that finishing the Eulerian trail, we get back to b_1. The last condition automatically removes one more edge at b_1, since the remaining number of edges has to be divisible by 4.

After this process, bad vertices become good, and the degrees of good vertices are still divisible by 4. Here we also remark that there are enough edges in $M(3) \cup M(4) \setminus E_2$ to use, every time the walk arrives to a vertex v. This again follows from the upper bound on $d_{T(3)}$ and $d_{T(4)}$. Whenever we arrive to v, it means there are two edges incident to v in E_2, and we need two edges (or four, at most once) in $M(3) \cup M(4) \setminus E_2$. Therefore, we need the degree of v in $M(3) \cup M(4) \setminus E_2$ to be at least $d_{E_2}(v) + 2$, which is satisfied.

We are left with a bipartite graph $M[A, B]$, where all degrees in A are divisible by 4. Let $M(5)$ be the union of 6 spanning trees $T_{55}, \ldots, T_{60}$. By Lemma 11, $p = 2$, $M(5)$ contains a spanning tree $T(5)$ such that for each vertex v, $d_{T(5)}(v) \leq 3d_{M(5)}(v)/4$. We similarly define $M(6)$ and find $T(6)$. Now for every vertex v in A, the following holds: $d_{T(5)}(v) + d_{T(6)}(v) \leq 3d_M(v)/4$. For every vertex v in A, we put aside 1/4 of the edges such that $T(5)$ and $T(6)$ remains in the graph. The remaining graph M' satisfies the conditions of Lemma 15.

Therefore, we can decompose M' into paths of length 3 such that for a vertex v with degree $4d$ in M (hence degree $3d$ in the smaller graph M'), there are d paths starting from v, and d paths, where v is a middle vertex. For every vertex v, we glue the d edges, which we put aside in the beginning of the third phase, one by one to the d paths, where v is a middle vertex. This gives us a Y-decomposition. □

3.7 Resolution of the Barát-Thomassen conjecture

A necessary condition for the existence of a T-decomposition is of course that $|E(T)|$ divides $|E(G)|$. There are many theorems and conjectures in graph theory stating that this condition is also sufficient in certain cases. By a result of Wilson [122] this holds when G is a sufficiently large complete graph, and there exist more general results showing that this is also true for graphs of large minimum degree. More precisely, for every tree T there exists a constant $\varepsilon_T > 0$ such that every graph G of minimum degree $(1 - \varepsilon_T)|V(G)|$ admits a T-decomposition, provided its size is divisible by the size of T, see for instance [25].

Barát and Thomassen [C] started a different line of research, when we observed that T-decompositions are intimately related to nowhere-zero flows. Tutte conjectured that every 4-edge-connected graph admits a nowhere-zero 3-flow, but until recently it was not even known that any constant edge-connectivity suffices for this. Barát and Thomassen showed that if every 8-edge-connected graph of size divisible by 3 admits a $K_{1,3}$-decomposition, then every 8-edge-connected graph admits a nowhere-zero 3-flow.

Vice versa, we also showed that Tutte's 3-flow conjecture would imply that every 10-edge-connected graph with size divisible by 3 admits a claw-decomposition. Motivated by this intrinsic connection, we formulated Conjecture 1.

When the conjecture was made, it was only known to hold in the trivial cases, when T has less than 3 edges. After that, Conjecture 1 attracted growing attention. After a decade, it was verified for different families of trees such as stars [113], some bistars [115, F], and paths of a certain length [36, 111–114]. As the last stepping stone, breakthrough results were obtained by Merker [88], who proved Conjecture 1 for all trees of diameter at most 4, hence covering some of the results above. Botler, Mota, Oshiro, and Wakabayashi [37] proved the conjecture for all paths. This result was improved by Bensmail, Harutyunyan, Le, and Thomassé [31], who showed that, for path-decompositions, large minimum degree is a sufficient condition provided the graph is 24-edge-connected.

Finally, some of the above authors joined their forces and proved the Barát-Thomassen conjecture in full generality. They build on several previous results. We shortly summarize a few of their ideas and direct the interested reader to the full proof in [30].

It was shown by Thomassen [115], and independently by Barát and Gerbner [F], that it is sufficient to verify Conjecture 1 for bipartite graphs, see Section 3.5 for details. An important tool in the study of Conjecture 1 is the result on modulo k-orientations by Thomassen [113]. He showed that the edges of a highly edge-connected graph can be oriented so that any prescribed out-degrees modulo k are realised. An application is the following decomposition result by Thomassen [115]. A highly edge-connected bipartite graph $G(A_1, A_2)$ can be decomposed into two less-highly edge-connected graphs G_1, G_2 such that $d_{G_i}(v)$ is divisible by m for each $v \in A_i$.

By the aforementioned results, it is sufficient to prove Conjecture 1 for bipartite graphs $G(A, B)$, where all vertices in A have degree divisible by m, the size of T.

Let T_A and T_B denote the vertex classes of a bipartition of T. We may assume that B contains a leaf. The T-decompositions Bensmail et al. constructed respect the bipartitions of G and T in the sense that the vertices corresponding to T_A will lie in A for each copy of T. Vertices $v \in V(G)$ and $t \in V(T)$ are *compatible* if $v \in A$ and $t \in T_A$, or $v \in B$ and $t \in T_B$.

Assume G is (improperly) edge-colored. We denote by $d_i(v)$ the degree of vertex v in color i. For $t \in V(T)$, let $S(t)$ denote the set of edges incident with t. An edge-coloring $\phi : E(G) \to E(T)$ is T-*equitable*, if for any compatible vertices $v \in V(G)$, $t \in V(T)$ and $j, k \in S(T)$, we have $d_j(v) = d_k(v)$. Merker [88] showed highly edge-connected graphs admit T-equitable edge-colorings. Since we put no constraints on the degrees in B, necessarily the greatest common divisor of the degrees in T_B must be 1, if we want to construct a T-equitable coloring. For this reason, we chose the bipartition of T so that T_B contains a leaf.

If there exists a T-decomposition of a bipartite graph G, where all copies of T are oriented the same way (with respect to the bipartite classes), then it gives rise to a T-equitable coloring of G. Vice versa, a T-equitable coloring can also be used to construct a T-decomposition. This was done in [88], when the girth of G is at least the diameter of T, and also in general for trees of diameter at most 4.

Finally, Bensmail et al. [30] used probabilistic methods to show that a T-equitable coloring can be turned into a T-decomposition, whenever the minimum degree in each color is large enough. Combining all these results one can complete the proof of the Barát–Thomassen conjecture.

barat.janos_51_23

Chapter 4

Cube dismantling

There are n^d unit cubes in a d-dimensional hypercube of edge-length n. We identify these n^d unit cubes with the vertices of a graph, $[n]^d$. Two vertices are *neighbours* if the corresponding two unit cubes have a $(d-1)$-dimensional face in common. In this chapter, we consider sequences of induced subgraphs of $[n]^d$, sometimes specifically in three dimensions. Starting from $[n]^d$, we successively remove vertices of degree d from the current graph until no further vertices can be removed. Here the degree of a vertex is a dynamic notion referring to the degree in the current graph. We prove a number of properties of the positions that may be reached by such a *dismantling* process. In doing so, we settle several conjectures posed in [24].

We also study the reverse of our dismantling process, which we call a *build-up*. The build-up process is similar to well studied models of bootstrap percolation, see for instance [19, 20, 105]. One difference with our process is that a new site is added only when the number of neighbours exactly matches a prescribed value, whereas in bootstrap percolation "exactly matches" would be replaced by "matches or exceeds". Another issue is that we only add one vertex at a time, while in bootstrap percolation all sites satisfying the threshold condition are added simultaneously. In our setting two vertices that correspond to independent vertices of $[n]^d$ can be added simultaneously. However, if the vertices are neighbours and both have degree d, then they cannot both be added. This changes the nature of the problem. On the other hand extremal results in bootstrap percolation theory can serve as upper bounds for our problem. The size of the percolating set is fixed in our setting, provided it consists of independent sites. This is a consequence of a result mentioned in [20]. The proof uses the well-known Perimeter Lemma, which is a handy tool for several extremal properties. For bootstrap percolation it is of fundamental importance to decide the minimum and maximum sizes of percolating sets [91]. Despite these differences we will show that in an important special case, our build-up process is effectively identical to bootstrap percolation. Also, the so-called modified bootstrap percolation in $[n]^d$ requires at least one neighbour in each of the d directions [62]. This property is analogous to the idea of a balanced move in our investigation.

The well-known Perimeter Lemma [20] gives us the following

Proposition 27. *Starting from* $[n]^d$*, there are at least* n^{d-1} *unit cubes left after any sequence of moves, each move being a removal of a vertex of degree* d*. That is, at most* $n^d - n^{d-1}$ *vertices can be successively removed from* $[n]^d$*.*

In the extremal case the n^{d-1} unit cubes correspond to independent vertices of $[n]^d$. In our terminology n^{d-1} independent vertices of $[n]^d$ form a *base position.* We color the unit cubes/vertices of the base position *black*, and all other cubes/vertices *white.* Our task is the following. There is a given base position. Can we successively remove all white vertices according to the degree d rule? If the answer is yes, then the base position is a *solution.* A sequence of moves starting from $[n]^d$ leading to a solution is a *dismantling.* The reverse process is a *build-up.*

Each vertex of $[n]^d$ is identified naturally with a d-tuple $(x_1, \ldots, x_d)$. A *line* is the set of n cubes found by fixing $d-1$ coordinates and allowing one coordinate to vary. A *section* is the set of n^{d-1} cubes found by fixing one coordinate and allowing the other $(d-1)$ coordinates to vary. The intersection of a solution and a section (in three dimensions) plays an important role. A section is *facial* if its unit cubes touch a face of the original cube $[n]^d$. A *projection* of a vertex set in $[n]^d$ is a projection to $[n]^{d-1}$ in one of the fundamental directions.

Each step in a dismantling, that is, the removal of a vertex of degree d is a *move.* A move is *balanced* if the neighbours lie in different lines. A dismantling or build-up is *balanced* if all of its moves are balanced, otherwise it is unbalanced. We show in Section 4.1 that if a dismantling to a solution is balanced, then the solution corresponds to a Latin hypercube [87].

A *position* refers to any subset of the cubes in $[n]^d$, that is, an induced subgraph of $[n]^d$. The vertices in the position are black, and the remaining vertices are white. The degree of a white vertex is its number of black neighbours. Any base position with full projections in all d orthogonal directions is *perfect.* A position is *convex* if no line of $[n]^d$ contains one or more white cubes between black cubes. In particular, a perfect base position is convex. A position is *minimal* if there is no black vertex with precisely d neighbours. That is, no more dismantling moves are possible. A position is *maximal* if there is no white vertex with precisely d neighbours. That is, no more build-up moves are possible. We are mainly interested in the maximal configuration $[n]^d$, and the minimal configurations that are solutions. In Theorem 29 we show that the symmetric difference of any two minimal or any two maximal positions is non-empty.

A base position B corresponds to a two-coloring of $[n]^d$, where the vertices in B are black, and the other vertices are white. The isometry group of the cube (in 3D) has 48 elements. Two base positions B_1 and B_2 are *isometric* if and only if there exists an element of the isometry group of the cube mapping B_1 to B_2. Therefore, any base position may be isometric to at most 48 positions. If the position has some symmetry, then we get fewer isometric positions.

A dismantling gives us an edge-decomposition of $[n]^3$ into copies of the complete bipartite graph $K_{1,3}$, also known as the 3-star. Decompositions are well studied in graph theory and design theory. For example, $K_{1,3}$-decompositions are in focus in [C], where their relation to orientations and flows are described in detail. Hoffmann [61] gave a necessary and sufficient condition for star decompositions. His conditions — mainly a Hall-type criterion — imply that $[n]^d$ has a d-star decomposition. In general, the existence of a star decomposition does not imply a dismantling. It is possible that a (sub)graph has a decomposition, but that it has large minimum degree. In such a case, there is no vertex of degree d to be removed.

4.1 Greedy and balanced

If B is an imperfect solution, then any dismantling to B is unbalanced as was observed in [24] for the 3-dimensional case. Formally, we have the following

Lemma 16. *If all moves are balanced in a dismantling to a solution B, then B is perfect.*

Proof. Assume to the contrary, that two black vertices u and v have the same projection in some direction r, and there are only white vertices between u and v. Say u and v are at distance k. That is, there are $k-1$ white vertices and k edges between u and v in direction d. Every edge is removed during dismantling. However, any balanced move removes one white vertex and one edge in direction d, so it is impossible to remove the k edges between u and v with the $k-1$ available white vertices. This contradiction proves that the projection is full in all three directions. □

It is challenging to compare the importance of balanced and unbalanced moves. Note that every dismantling must use a balanced move at the start. There are dismantlings with balanced moves only, we show one particular solution in Section 4.2. On the other hand, there seem to be many more imperfect solutions than perfect ones. Theoretically, it is possible to replace a few balanced moves by a few imbalanced moves. However, the authors of [24] conjectured that the converse of Lemma 16 holds. We settle the conjecture in the next Theorem. We need the following preliminary result:

Lemma 17. *Let C be a convex position.*
(0) *Any build-up from C is balanced, and produces only convex positions.*
(1) *Any dismantling to C is balanced.*
(2) *Only convex positions can be reached by balanced dismantling from C.*

Proof. First, consider a build-up from C. Any unbalanced move requires two non-consecutive cubes in the same line. Therefore, only balanced moves are possible in a convex position.

Assume D is a non-convex position formed by adding a single vertex v to the convex position C. Since D is non-convex, there are two non-consecutive vertices in a line l such that all vertices between them are missing. Clearly one of the vertices must be v, the other one is, say, u. Vertex v was added in a balanced move, therefore there is a neighbour x of v in line l. There are no vertices between v and u in D, therefore x and u make C non-convex, a contradiction. So, by induction, only convex positions can be reached by building-up from C.

This proves (0), and (1) follows immediately since dismantling is the reverse of build-up. Part (2) is straightforward, since each move of a dismantling preserves convexity unless we remove a cube between two other cubes, and such a move is not balanced. □

A *Latin square* of order n is an $n \times n$ array of n symbols in which each symbol occurs exactly once in each row and exactly once in each column. Each position P in a dismantling can be associated with a matrix $M_P = [m_{xy}]$ of sets such that $z \in m_{xy}$ if and only if the cell (x, y, z) is included in P. It is plausible that M_P might be a Latin square with the n symbols being the n possible singleton sets. Positions that correspond to Latin squares in this way turn out to be quite special:

Proposition 28. *For a solution B the following statements are equivalent:*
(0) *B is convex.*
(1) *B is perfect.*
(2) *B corresponds to a Latin square.*
(3) *There is a balanced dismantling of Q_n to B.*
(4) *Every dismantling of Q_n to B is balanced.*
(5) *There is a balanced build-up from B to Q_n.*
(6) *Every build-up from B to Q_n is balanced.*

Proof. If B is convex, then there cannot be more than one black vertex in any line, since B consists of independent vertices. However, there has to be one vertex per line on average, which means there must be exactly one vertex in each line. So B is perfect. Conversely, if B is perfect then there is exactly one black vertex in each line, which means that B is necessarily convex. We conclude that (0) and (1) are equivalent.

Let M_B denote the matrix corresponding to the solution B. We see that the cells of M_B consist of singletons if and only if there is exactly one cell of B in each vertical line. This is the same as specifying one orthogonal projection of B to be full. The other two orthogonal projections of B are full if and only if the singletons in each row (respectively, column) of B are distinct. So (1) is equivalent to (2).

Dismantling and build-up are the reverse of each other, and balanced moves mean the same thing in both processes. Hence (3) is equivalent to (5) and (4) is equivalent to (6). We are assuming that B is a solution so there is a dismantling from Q_n to B. Thus (4) implies (3). Lemma 16 shows that (3) implies (1), and Lemma 17 shows that (0) implies (6). The theorem follows. □

We stress that Proposition 28 applies only to solutions B, not to all base positions B. We have shown in [D] that (2) is very far from implying (3) for a general base position.

When the authors of [24] made a complete search for a fixed size, most of the search time was taken up by backtracking in the search tree. They conjectured that this is unnecessary in some sense. We validate this conjecture here. The steps of a build-up can be encoded as follows: in each step, we record the vertex we are adding, and store its set $N(V)$ of neighbours for validation of the degree. We use v^* as short-hand for the pair $(v, N(v))$.

Theorem 29. *Let P be any position. There cannot be two distinct maximal positions $M_1 \subset M_2$ such that both M_1 and M_2 can be reached by build-up from P. Similarly there cannot be two distinct minimal positions $M_1 \subset M_2$ such that both M_1 and M_2 can be reached by dismantling from P.*

Proof. Assume to the contrary that some build-up $u_1^*, u_2^*, \ldots, u_f^*$ from P stops in a maximal position M_1, where $M_1 \subsetneq M_2$, while another sequence $v_1^*, v_2^*, \ldots, v_g^*$ builds up from P to M_2.

First suppose that $\{u_1^*, u_2^*, \ldots, u_f^*\} \subset \{v_1^*, v_2^*, \ldots, v_g^*\}$. Since $M_1 \neq M_2$, there is at least one vertex in the sequence $v_1, v_2, \ldots, v_g$ that does not occur in $\{u_1, u_2, \ldots, u_f\}$. Let v_j be the first such vertex in the sequence, and suppose $v_j^* = (v_j, \{x, y, z\})$. We claim that v_j^* is a possible move in position M_1, contradicting the maximality of M_1. By choice of v_j, each of x, y, z must be in P or $u_1, \ldots, u_f$ and hence each is in M_1. Thus v_j has at least 3 neighbours in M_1. Suppose v_j has a fourth neighbour w in M_1. Since $w \notin N(v_j)$ it cannot be that w is in P, so there must be some k for which $u_k = w$. By assumption, $u_k^* = v_i^*$ for some i. If $i < j$, then $w = u_k = v_i \in N(v_j)$ which contradicts the choice of w. If $i > j$, then $v_j \in N(v_i) = N(u_k) \subset M_1$ which contradicts the choice of v_j. It follows that v_j, which is not present in M_1, has exactly 3 neighbours in M_1. Thus v_j^* is a possible move in position M_1, as claimed.

It remains to consider the possibility that $u_i = v_k$ but $u_i^* \neq v_k^*$ for some i, k. We choose the first such u_i in the sequence $u_1, \ldots, u_f$. Since $u_i^* \neq v_k^*$, there is $x \in N(u_i) \setminus N(v_k)$ and this implies that $x = u_j$ for some $j < i$. By choice of i, this means that $u_j^* = v_l^*$ for some l. As $j < i$ we have $v_k = u_i \notin N(u_j) = N(v_l)$, which implies $l < k$. Hence $x = u_j = v_l \in N(v_k)$, contradicting the choice of x and proving the first claim of the theorem.

The proof of the other claim is similar. Suppose that there are two minimal positions $M_1 \subsetneq M_2$ such that M_2 can be reached from P by the dismantling $u_1^*, u_2^*, \ldots, u_f^*$, while another sequence $v_1^*, v_2^*, \ldots, v_g^*$ dismantles from P to M_1. If $\{u_1^*, u_2^*, \ldots, u_f^*\} \subset \{v_1^*, v_2^*, \ldots, v_g^*\}$ then the first move in $v_1^*, v_2^*, \ldots, v_g^*$ that is not in $u_1^*, u_2^*, \ldots, u_f^*$ will be a valid dismantling move from M_2, contradicting the minimality of M_2. Otherwise, we have $u_i = v_k$ but $u_i^* \neq v_k^*$ for some i, k. Taking the first such i yields a similar contradiction to the build-up case. □

Since any position that can be reached from P is a subset of Q_n, we have:

Corollary 3. *Let P be any position that can be reached from Q_n by dismantling. The only maximal position that can be reached by building up from P is Q_n.*

Crucially, this last result entitles us to use the following greedy algorithm to check candidate solutions. We repeatedly traverse the white vertices (in any order) and add any of degree 3, until a maximal position is reached.

Corollary 4. *A base position is a solution if and only if the greedy algorithm terminates with the full cube.*

In Corollary 4, if we end up with a maximal position other than the full cube, then we started with a position that was not a solution. The maximal position is far from being determined by the initial position. We found a set of 25 independent vertices in Q_5 from which different build-ups reach maximal positions of any size in $\{37, 38, 39, 40, 43, 46, 56, 57, 58, 59, 60, 61, 63\}$. We also found a set of 35 vertices in Q_5 such that various build-ups resulted in maximal positions of each size in $\{93, 94, \ldots, 119\}$. Figures 4.1 and 4.2 show the smallest and largest maximal positions for this example. The numbers in shaded squares show the order in which the cubes are added. Bold numbers in white squares are the degrees of white vertices in the final position. The absence of a bold 3 certifies that no more moves are possible.

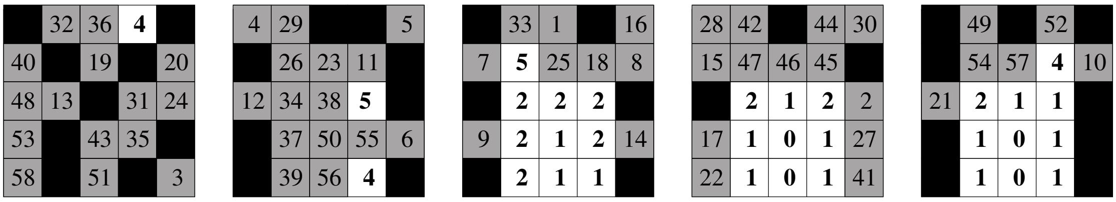


Figure 4.1: The smallest maximal position.

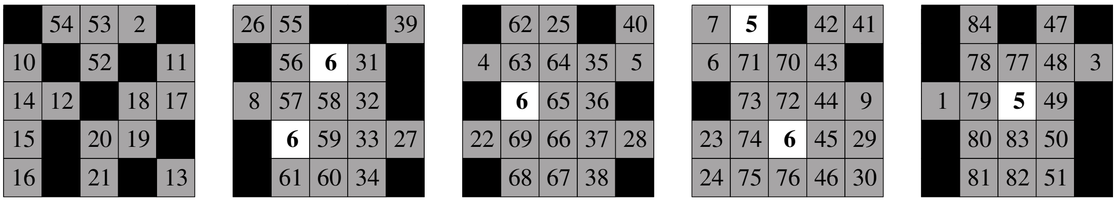


Figure 4.2: The largest maximal position.

These examples contrast sharply with Corollary 3. In Proposition 31 below, we will see another situation where only one maximal position can be reached.

Next we prove another conjecture from [24]. It states that solutions are not only stable, but in some sense the build-up is unique.

Theorem 30. *Let P_1, P_2 be any two positions, and let $u_1^*, u_2^*, \ldots, u_f^*$ and $v_1^*, v_2^*, \ldots, v_f^*$, be any two build-ups from P_1 to P_2. There exists some permutation σ of $\{1, 2, \ldots, f\}$ such that $u_{\sigma(i)}^* = v_i^*$ for $1 \leq i \leq f$.*

Proof. Assume to the contrary, there exist a j for which $u_j^* \neq v_i^*$ for $1 \leq i \leq f$. Let us fix j to be the smallest such index. Since both sequences reach the same position, $u_j = v_k$ for some k.

Suppose $u_j^* = (u_j, \{x, y, z\})$. By assumption, $u_j^* \neq v_k^*$ so at least one of x, y, z must occur after v_k in the sequence $v_1, \ldots, v_f$. Without loss of generality, suppose that z does. Now z is not in P_1, but is in $N(u_j)$, so $z = u_d$ for some $d < j$. By choice of j, it follows that $u_d^* = v_e^*$ for some e. Note that $e > k$ by choice of z. Now $v_k = u_j \notin N(u_d) = N(v_e)$ since $d < j$, but this contradicts the fact that $v_k \in N(v_e)$ as $k < e$. □

Of course, a similar statement holds for any two dismantlings from P_2 to P_1, since a dismantling sequence is just a build-up sequence in reverse.

In the introduction we mentioned several differences between our build-up process and bootstrap percolation. However, there is an important case in which they are essentially the same.

Proposition 31. *If C is any convex position, then the same maximal position will ultimately be reached from C by any sequence of moves regardless of whether they are dictated by our build-up process, by bootstrap percolation, or by modified bootstrap percolation.*

Proof. Let S be the set of white vertices that have at least 3 black neighbours in C. Since C is convex, no white vertex can have more than one black neighbour in any line of C. Hence every vertex in S has exactly 3 black neighbours, and they are in 3 orthogonal directions. Thus S is the set of sites that would be filled by a single step of either bootstrap percolation or modified bootstrap percolation. Moreover, the vertices in S are independent; for suppose that two of them, u and v were neighbours. Since they are in S, they both have a black neighbour in the line that contains u and v. But these black neighbours would necessarily have the white vertices u and v between them, which breaches the assumption that C is convex. So S is an independent set and all vertices in it can be added (in any order, and using only balanced moves) by our build-up process. By Lemma 17, $C \cup S$ is convex, so we can apply the same argument again.

Let D be the maximal position reached from C by bootstrap percolation (or modified bootstrap percolation). By induction using the above argument we know that D can be reached by build-up from C. So it only remains to show that no other maximal position D' can be reached by build-up from C. Suppose w is the first vertex that was added in the build-up to D but is not present in D'. Then w has three neighbours in D', namely the neighbours that were used to add w in the build-up to D. Also w cannot have more than three neighbours in D', since D' is convex by Lemma 17. So w is available as a move in D', contradicting the maximality of D'. We conclude that w does not exist, meaning that D is a subset of D'. By a symmetric argument D' is a subset of D, so they are in fact equal. In other words, from any convex position there is a unique maximal position that can be reached by building-up. □

4.2 The cyclic base position

The graph Q_n consists of n levels, each level being an $n \times n$ square of unit cubes. Each vertex is identified naturally with a triple (x, y, z). Here the last coordinate indexes the level, and the first two coordinates are the row and column indices. The rows are $1, 2, \ldots, n$ from top to bottom and the columns are $1, 2, \ldots, n$ from left to right as in a matrix, the levels are $1, 2, \ldots, n$ from bottom to top. We describe a specific solution, the *cyclic base position*, that can be reached after dismantling Q_n. The set of black vertices is $\{(i,j,k) : i+j-k \equiv 1 \mod n\}$. The top level contains diagonal vertices. Each consecutive level is a cyclic shift of the previous level, see Figure 4.3 for order 5.

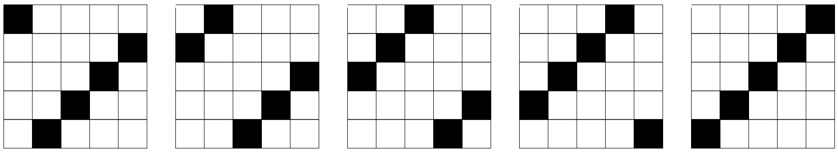


Figure 4.3: The cyclic base position of order 5.

We define a specific subroutine, which is used several times. Let S be a facial section of Q_s, that corresponds naturally to an $s \times s$ square. Let D be a diagonal of black vertices $(1,m),(2,m-1),\dots,(m,1)$, where $m \le s$. Consider the upper-left triangle T consisting of the vertices with $i+j \le m$. Now the diagonals parallel to D can be removed starting from size 1 to size $m-1$. This process is the *diagonal peeling* of T or a diagonal peeling of size $m-1$ with corner $(1,1)$. Any rotation or reflection of it is also a diagonal peeling.

Let $UTB(m) = \{(i,j,k) : i+j-k = m+1-n\}$ be the *upper triangular base* of order m, where $1 \le m \le n$. Similarly, let $LTB(m) = \{(i,j,k) : i+j-k = 2n-m\}$ be the *lower triangular base* or order m, where $1 \le m \le n-1$. The cyclic base position in Q_n is the union of $UTB(n)$ and $LTB(n-1)$. In general $TB(m)$, a triangular base of order m is a copy of $UTB(m)$ embedded and possibly rotated and reflected in Q_n, where $m \le n$. Clearly $LTB(n-1)$ is a $TB(n-1)$.

The union $\cup_{i=1}^{m} UTB(i)$ forms a *heap of oranges* of order m, $HO(m)$ for short. For any m, where $1 \le m \le n-1$, the vertices of $UTB(m)$ are independent, and any vertex of $UTB(m)$ has precisely three neighbours in $UTB(m+1)$. Therefore, $HO(n)$ can be dismantled to $UTB(n)$.

The vertices of Q_n with precisely 3 neighbours in $HO(n)$ form a $TB(n-2)$. They are on the other side of $UTB(n)$. Add these vertices to $HO(n)$. Repeating this process, we can pack a $TB(n-4), TB(n-6), \dots$ on $HO(n)$. This is the *nested heap of oranges* construction, $NHO(n)$ for short. If n is even, then $NHO(n)$ is $HO(n) \cup TB(n-2) \cup \dots \cup TB(2)$. If n is odd, then $NHO(n)$ is $NHO(n) \cup TB(n-2) \cup \dots \cup TB(1)$. Observe that $NHO(n)$ can be dismantled to $UTB(n)$, and a rotated copy of $NHO(n-1)$ can be dismantled to $LTB(n-1)$.

Theorem 32. *For each positive integer n, where $n \ge 2$, the $n \times n \times n$ cube can be dismantled to the disjoint union of $NHO(n)$ and a rotated copy of $NHO(n-1)$.*

Proof. Consider the three original corners of Q_n outside of $NHO(n)$ and $NHO(n-1)$: $(1,n,1)$, $(n,1,1)$, (n,n,n). Execute diagonal peelings of size $n-1$ with the above three corners to remove vertices of form $(1,y,z)$, $(x,1,z)$, (x,y,n). We remove $3T_{n-1}$ cubes in this step, where $T_m = m(m-1)/2$ is the triangular number. In particular, we remove $(1,2,1)$, $(2,1,1)$, $(n,1,n-1)$, $(n,2,n)$, $(1,n,n-1)$, $(2,n,n)$. Therefore, the following vertices now have three neighbours: $(2,2,1)$, $(n,2,n-1)$, $(2,n,n-1)$. Execute diagonal peelings of size $n-2$ with the above corners to remove vertices of form $(x,y,1)$, (n,y,z),

(x, n, z). In these two steps, we removed all vertices outside of $NHO(n)$ and $NHO(n-1)$ with any coordinate equal to 1 or n. In some sense, we peeled of one hull from the cube.

From now on, we repeat these two steps from corners, each of which is a diagonal step away from one of the previous six. The series of corners are: $(1+a, n-a, 1+a)$, $(n-a, 1+a, 1+a)$, $(n-a, n-a, n-a)$, $(2+b, 2+b, 1+b)$, $(n-b, 2+b, n-1-b)$, $(2+b, n-b, n-1-b)$, where $0 \le a \le \lfloor n/2 \rfloor - 1$ and $0 \le b \le \lceil n/2 \rceil - 2$. So the second iteration would use diagonal peelings from the corners $(2, n-1, 2)$, $(n-1, 2, 2)$, $(n-1, n-1, n-1)$, and $(3, 3, 2)$, $(n-1, 3, n-2)$, $(3, n-1, n-2)$ to remove vertices with any coordinate equal to 2 or $n-1$. Each new peeling is made possible by the preceding peeling, which removed the cubes that were neighbours on the "outside" of the ones we want to remove next.

We can perform $n-1$ steps of this algorithm removing $(n-1)n(n+1)/2$ cubes in total, 3 times the $(n-1)^{\text{st}}$ tetrahedral number H_{n-1}. Adding these to the number of cubes in $NHO(n)$ and $NHO(n-1)$ gives us n^3, certifying our algorithm. Indeed,

$$\begin{aligned} |NHO(n)| + |NHO(n-1)| &= H_n + T_{n-2} + T_{n-4} + \cdots \\ &\quad + H_{n-1} + T_{n-3} + T_{n-5} + \cdots \\ &= H_n + H_{n-1} + H_{n-2}. \end{aligned}$$

Therefore, we need to check $H_n + 4H_{n-1} + H_{n-2} = n^3$, which is true. □

Corollary 5. *For each positive integer n, where $n \ge 2$, the $n \times n \times n$ cube can be dismantled to the cyclic base position.*

If we are looking for more solutions, it is natural to consider variations of the cyclic base position. One option is to permute the levels, which corresponds to permuting the symbols in the corresponding latin square. Executing a computer search, we found that of the $n!$ permutations of the symbols, the number that produced a solution is as follows.

n	3	4	5	6	7	8	9	10
#solutions	6	16	40	96	200	352	552	800

We do not consider general permutations further, but rather work towards showing that cyclic permutations do produce solutions.

Starting from a corner of Q_n we can remove a line of cubes of length k, where $1 \le k \le n-1$. As a consequence, we can build-up a missing line of cubes on the edge of a cube. We use this observation in a more general context, when the line of cubes is somewhere inside Q_n, but a dismantling generated an equivalent situation.

A *staircase* of size m and depth d is the union of lines of length d starting from a set of vertices that form a diagonal peeling of size m.

Lemma 18. *We can remove a staircase of size m and depth d from Q_n, where $1 \le m < n$ and $1 \le d < n$.*

Proof. We can either say that this is the union of d diagonal peelings of size m, or refer to the repeated removal of lines in diagonal fashion. □

The cyclic base position is the intersection of Q_n with two planes. When we permute the levels cyclically, we get the set $\{(i,j,k) : i+j-k \equiv 1-s \mod n\}$, where $1 \le s \le n-1$. It is the intersection of Q_n with three planes according to whether $i+j-k$ takes the value $1-s$, $n+1-s$ or $2n+1-s$. These are two triangular and one hexagonal region. Therefore, let $HB_n^s = \{(i,j,k) : i+j-k = n+1-s\}$ be the *hexagonal board* of size s in Q_n, where $1 \le s \le n-1$. Let $HB(m+1-s,s)$ denote any rotated or reflected copy of HB_m^s embedded in Q_n, where $m \le n$. In this way $HB(1,1)$ is a single cube, $HB(2,1)$ corresponds to three cubes, $HB(2,2)$ consists of seven cubes etc. As for the triangular boards, we can put together hexagonal boards in a nested fashion. Let m be a positive integer and $P_m = 1$ if m is odd, $P_m = 2$ if m is even. Let $DNH(n,m)$ be the *doubly nested hexagon* of size m, which is the union of $HB(P_m,1),\dots,HB(m,1),\dots,HB(m,n-m+1),\dots,HB(1,n-m+1),\dots,HB(1,P_m)$. As remarked before, the nested parts can be dismantled. That is, $DNH(n,m)$ can be dismantled to $HB(m)$.

We now prove the following generalisation of Theorem 32.

Theorem 33. *Any cyclic permutation of the levels of the cyclic base position gives a solution.*

Proof. Let as assume that we shifted the levels by s. That is, level n of the cyclic base position becomes level s, and in general level i becomes level $i+s \mod n$. We denote this object by $CC(n,s)$. It is the union of three connected pieces: $UTB(n-s)$, HB_n^s, $LTB(s-1)$, see picture 1, in Figure 4.4. As in the preparation for Theorem 32 we can add cubes to these three pieces to get $NHO(n-s) + DNH(n,s) + NHO(s-1) = M$, the third picture in Figure 4.4.

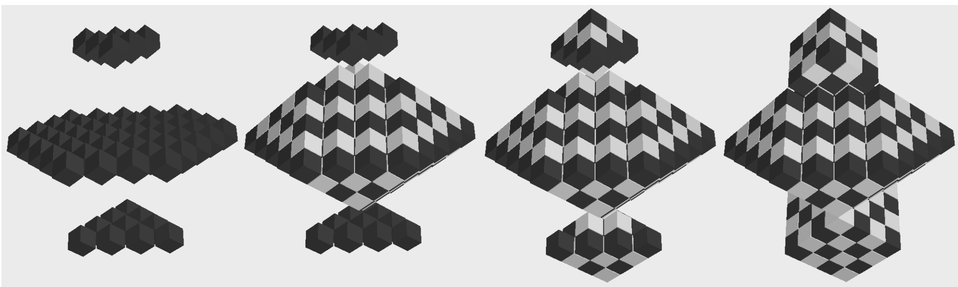


Figure 4.4: Build-up from a hexagonal board via double nesting and completion of corners.

The key observation is that the intersection of M and a Q_{n-s} placed in (n,n,n) is a copy of $NHO(n-s)$ and $NHO(n-s-1)$. Therefore, we can apply Theorem 32. Similarly for the intersection of M and a Q_{s-1} placed in $(1,1,1)$. Therefore, we can locally build-up Q_{n-s} and Q_{s-1}, see the fourth picture in Figure 4.4. Let F be the union of these two cubes with $DNH(n,s)$.

Using the above observations, we first use Lemma 18 to dismantle Q_n to F. Secondly, we use Theorem 32 to dismantle F to M. Finally, we dismantle M to $CC(n,s)$. □

4.3 Additional geometric properties

In this section we consider further properties of solutions, particularly their sections and projections.

We start by considering the number of black vertices in a section. Certainly, a section cannot consist of white vertices only, since then the last vertex removed from the section in a hypothetical dismantling would have degree at most 2, a contradiction. So each section contains at least one black vertex. For facial sections we can say something much stronger using the Perimeter Lemma.

Lemma 19. *Every facial section contains at least n black vertices.*

An analogous argument applied to a non-facial section gives no extra information. On the other hand, we get a stronger result assuming that the dismantling is balanced. By Lemma 16 the solution is then perfect. In a perfect solution the sections have an average of n black vertices each, and each section contains at least n black vertices because it has n squares in its projection. Hence, if a solution is perfect, then each section contains precisely n black vertices. We note that the converse does not hold. Figure 4.5 shows an example for $n = 4$ where each section contains precisely n black vertices but the solution is not perfect.

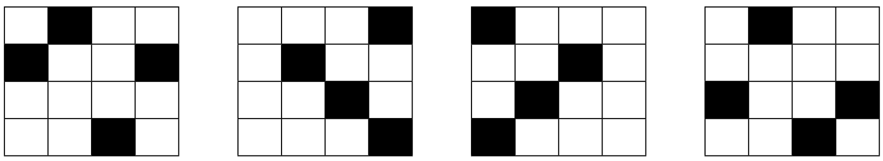


Figure 4.5: Imperfect solution with n black vertices per section.

In applications such as those mentioned in [24], there might be a condition on the location of the black cubes. One strong condition is to restrict the black cubes to being in three sections. For specific values of the form 2^t, we show a solution satisfying this extra condition. The construction relies on the following observation. If there are three mutually orthogonal sections, in which all cubes are present, then the rest of Q_n can be built-up. What is left to do is the selection of a base position in this restricted space of three sections.

Lemma 20 (The corridor idea)**.** *Let $n = 2^t$ for $t \geq 1$. There exists a solution I_n in which the black cubes are contained in three orthogonal facial sections of Q_n.*

Proof. We use induction on t. For $t = 1$, there is only one solution. For any other t, let I_{2^t} contain the vertex $(1, 1, 1)$. Let the other black cubes be contained in squares of size $(2^t - 1) \times (2^t - 1)$ in the facial sections $(1, y, z)$, $(x, 1, z)$, $(x, y, 1)$, where $x, y, z \geq 2$. The positions of the black cubes in these sections are defined recursively. Consider four copies of the example of size $(2^{t-1} - 1) \times (2^{t-1} - 1)$ in the corners plus an extra black cube in the centre, see Figure 4.6.

We can build-up Q_n from this base position as follows. The four smaller parts can be built-up by the induction hypothesis. Once they are done, the line starting from the middle black cube can be built-up. After this step the lines starting from $(1, 1, 1)$ can be built-up. We now have three orthogonal facial sections filled. We can easily build-up the rest of Q_n: for instance line by line from 2 to n on level 2, and then repeating this level by level from 3 to n. □

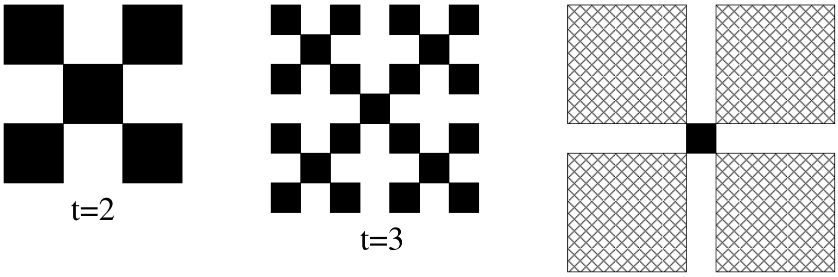


Figure 4.6: The recursive idea to create $(n-1) \times (n-1)$ squares.

We can look at this result from another point of view. In case of perfect base positions, the three orthogonal projections cover the surface of three faces. That is, an area of $3n^2$, which is clearly the maximum possible. At the other extreme, define $s(n)$ to be the minimum possible area obtained from the three orthogonal projections of a solution to the faces of Q_n. The following conjecture, posed in [24], says the above solution of three planes achieves the minimal value $s(n)$.

Conjecture 8. *$s(n) \geq n^2 + 6n - 4$ for all n.*

Using the Perimeter Lemma again, we can show that the leading term in the previous conjecture is correct.

Proposition 34. *If B is a solution in Q_n then each of its 3 orthogonal projections covers an area of at least $\frac{1}{3}n^2 + \frac{2}{3}n$. Hence $s(n) \geq n^2 + 2n$.*

Chapter 5

The Albertson conjecture

The *crossing number* $\text{CR}(G)$ of a graph G is the minimum number of edge crossings in a drawing of G in the plane. It is a natural relaxation of planarity, see [108] for a survey. The *chromatic number* $\chi(G)$ of a graph G is the minimum number of colors in a proper coloring of G. The Four Color Theorem states if $\text{CR}(G) = 0$, then $\chi(G) \leq 4$. Oporowski and Zhao [94] proved that every graph with crossing number at most two is 5-colorable. Albertson et al. [14] showed if $\text{CR}(G) \leq 6$, then $\chi(G) \leq 6$. It was observed by Schaefer that if $\text{CR}(G) = k$, then $\chi(G) = O(\sqrt[4]{k})$ and this bound cannot be improved asymptotically [13].

It is well-known that graphs with chromatic number r do not necessarily contain K_r as a subgraph, they can even have clique number 2 only [124]. The Hajós conjecture proposed that graphs with chromatic number r each contains a *subdivision* of K_r. The origin of this conjecture is unclear, but attributed to Hajós, turned out to be false for $r \geq 7$. Moreover, it was shown by Erdős and Fajtlowicz [53] that almost all graphs are counterexamples. Albertson conjectured the following.

Conjecture 9. *If $\chi(G) = r$, then $\text{CR}(G) \geq \text{CR}(K_r)$.*

If G contains a subdivision of K_r, then $\text{CR}(G) \geq \text{CR}(K_r)$. Therefore, Conjecture 9 is weaker than the Hajós conjecture, and it may be true.

For $r = 5$, the Albertson conjecture is equivalent to the Four Color Theorem. Oporowski and Zhao [94] verified it for $r = 6$, Albertson, Cranston, and Fox [13] proved it for $r \leq 12$. In [E], we proved the following.

Theorem 35. *For $r \leq 16$, if $\chi(G) = r$, then $\text{CR}(G) \geq \text{CR}(K_r)$.*

In this chapter, we discuss the tools and the proof of Theorem 35. Albertson, Cranston, and Fox combined lower bounds for the number of edges of r-critical graphs, and lower bounds on the crossing number of graphs with given number of vertices and edges. Our proof is very similar, but we use better lower bounds in both cases.

Albertson, Cranston, and Fox proved that any minimal counterexample to the Albertson conjecture should have less than $4r$ vertices. We slightly improve this result in [E] as follows. We omit the proof here.

Lemma 21. *If G is an r-critical graph with $n \geq 3.57r$ vertices, then* $\text{CR}(G) \geq \text{CR}(K_r)$.

In Section 5.1 we review lower bounds for the number of edges of r-critical graphs. In Section 5.2 we discuss lower bounds on the crossing number. In Section 5.3 we combine these bounds to obtain the proof of Theorem 35.

The letter n always denotes the number of vertices of G. In notation and terminology we follow Bondy and Murty [35]. In particular, the *join* of two disjoint graphs G and H arises by adding all edges between vertices of G and H. It is denoted by $G \vee H$. A vertex v is called *simplicial* if it has degree $n-1$. If a graph G contains a subdivision of H, then we also say that G contains a *topological* H. A vertex v is adjacent to a vertex set X means that each vertex of X is adjacent to v.

5.1 Color-critical graphs

Around 1950, Dirac introduced the concept of color criticality in order to simplify graph coloring theory, and it has since led to many beautiful theorems. A graph G is r-critical, if $\chi(G) = r$ but all proper subgraphs of G have chromatic number less than r. In what follows, let G denote an r-critical graph with n vertices and m edges.

Since G is r-critical, every vertex has degree at least $r-1$, therefore, $2m \geq (r-1)n$. The value $2m-(r-1)n$ is called the *excess* of G. Dirac [43] proved that for $r \geq 3$, if G is not complete, then $2m \geq (r-1)n+(r-3)$. For $r \geq 4$, Dirac [44] gave a characterization of r-critical graphs with excess $r-3$.

For any fixed $r \geq 3$ let Δ_r be the family of the following graphs G. The vertex set of G consists of three non-empty, pairwise disjoint sets A, B_1, B_2 with $|B_1| + |B_2| = |A| + 1 = r-1$, and two additional vertices a and b. Sets A and $B_1 \cup B_2$ both span cliques in G, and they are not connected by any edge. Vertex a is connected to $A \cup B_1$, and b is connected to $A \cup B_2$. See Figure 5.1. Graphs in Δ_r are called Hajós graphs of order $2r-1$. Observe that that these graphs have chromatic number r and they contain a topological K_r, hence they satisfy Hajós' conjecture.

Gallai [54] proved that r-critical graphs with at most $2r-2$ vertices are the join of two smaller graphs, i.e. their complement is disconnected. Based on this observation, he proved that non-complete r-critical graphs on at most $2r-2$ vertices have much larger excess than in Dirac's result.

Lemma 22. [54] *Let r, p be integers satisfying $r \geq 4$ and $2 \leq p \leq r-1$. If G is an r-critical graph with $n = r+p$ vertices, then $2m \geq (r-1)n + p(r-p) - 2$. Equality holds if and only if G is the join of K_{r-p-1} and $G \in \Delta_{p+1}$.*

Since every $G \in \Delta_{p+1}$ contains a topological K_{p+1}, the join of K_{r-p-1} and G contains a topological K_r. This yields a slight improvement for our purposes.

Corollary 6. *Let r, p be integers satisfying $r \geq 4$ and $2 \leq p \leq r-1$. If G is an r-critical graph with $n = r+p$ vertices, and G does not contain a topological K_r, then $2m \geq (r-1)n + p(r-p) - 1$.*

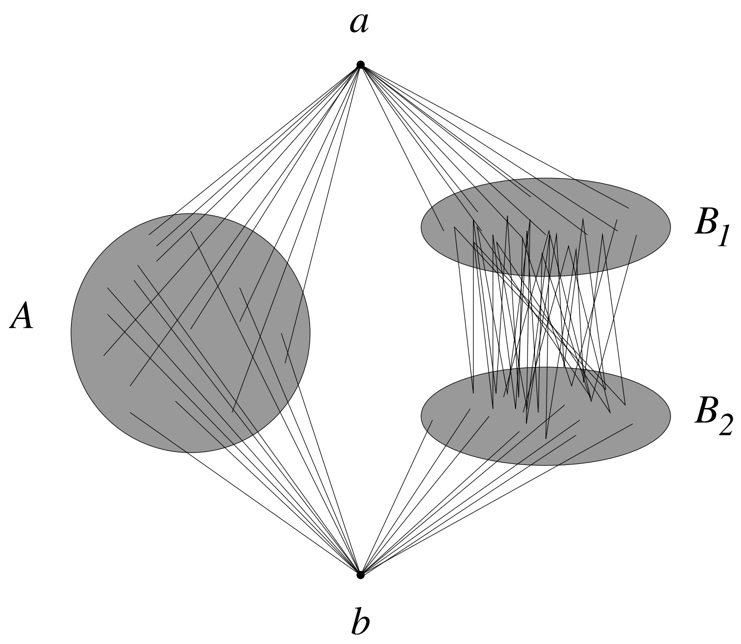


Figure 5.1: The family Δ_r.

We call the bound given by Corollary 6 the Gallai bound.

For $r \geq 3$, let $\mathcal{E}_r$ denote the family of the following graphs G. The vertex set of G consists of four non-empty pairwise disjoint sets A_1, A_2, B_1, B_2, where $|B_1| + |B_2| = |A_1| + |A_2| = r - 1$ and $|A_2| + |B_2| \leq r - 1$, and one additional vertex, c. Sets $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$ are cliques in G. Vertex c is connected to $A_1 \cup B_1$, and a vertex $a \in A$ is adjacent to a vertex $b \in B$ if and only if $a \in A_2$ and $b \in B_2$.

Clearly $\mathcal{E}_r \supset \Delta_r$, and every graph $G \in \mathcal{E}_r$ is r-critical with $2r - 1$ vertices. Kostochka and Stiebitz [72] improved the bound of Dirac as follows.

Lemma 23. [72] *Let $r \geq 4$ and G be an r-critical graph. If G is neither K_r nor a member of $\mathcal{E}_r$, then $2m \geq (r-1)n + (2r-6)$.*

Corollary 7. *Let $r \geq 4$ and G be an r-critical graph. If G does not contain a topological K_r then $2m \geq (r-1)n + (2r-6)$.*

Proof. It is not difficult to see that any member of $\mathcal{E}_r$ contains a topological K_r. Indeed, sets A and B both span a complete graph on $r - 1$ vertices. We only have to show that vertex c is connected to A_2 or B_2 by vertex-disjoint paths. To see this, we observe that $|A_2|$ or $|B_2|$ is the smallest of $\{|A_1|, |A_2|, |B_1|, |B_2|\}$. Indeed, if $|B_1|$ was the smallest, then $|A_2| > |B_1|$ implies $|A_2| + |B_2| > |B_1| + |B_2| = r - 1$ contradicting our assumption. We may assume that $|A_2|$ is the smallest. Now c is adjacent to A_1, and there is a matching of size $|A_2|$ between B_1 and B_2 and between B_2 and A_2, respectively. That is, we can find a set S of disjoint paths from c to A_2. In this way $A \cup c \cup S$ is a topological r-clique. □

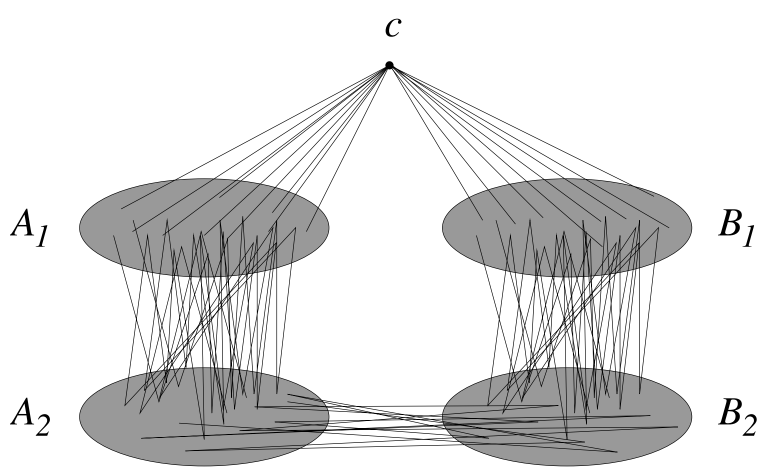


Figure 5.2: The family $\mathcal{E}_r$.

Call the bound in Corollary 7 the Kostochka, Stiebitz bound, or KS-bound for short.

In what follows, we obtain a complete characterization of r-critical graphs on $r+3$ or $r+4$ vertices.

Lemma 24. *For $r \geq 8$, there are precisely two r-critical graphs on $r+3$ vertices. They can be constructed from two 4-critical graphs on seven vertices by adding simplicial vertices.*

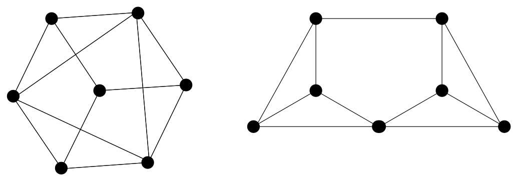


Figure 5.3: The two 4-critical graphs on seven vertices.

Proof. The proof is by induction on r. For the base case $r = 8$, there are precisely two 8-critical graphs on 11 vertices, see Royle's complete search [103].

Let G be an r-critical graph with $r \geq 9$ and $n = r+3 \geq 12$. We know that the minimum degree is at least $r-1 = n-4$. If G has a simplicial vertex v, then we use induction. So we may assume that every vertex in $\overline{G}$, the complement of G has degree 1, 2, or 3. By Gallai's theorem, $\overline{G}$ is disconnected. Observe the following: if there are at least four independent edges in $\overline{G}$, then $\chi(G) \leq n-4 = r-1$, a contradiction. That is, there are at most three independent edges in $\overline{G}$. Therefore, $\overline{G}$ has two or three components. If there is a triangle in the complement, then we can save two colors. If there were two triangles, then $\chi(G) \leq n-4 = r-1$, a contradiction.

Assume that there are three components in $\overline{G}$. Since each degree is at least one, there are at least three independent edges. Therefore, there is no triangle in $\overline{G}$ and no path with three edges. That is, the complement consists of three stars. Since the degree is at most three and there are at least 12 vertices, there is only one possibility: $\overline{G} = K_{1,3} \cup K_{1,3} \cup K_{1,3}$, see Figure 5.4.

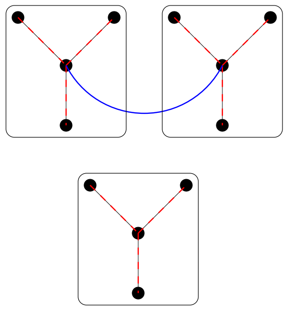


Figure 5.4: The complement and a removable edge.

We have to check whether this concrete graph is indeed critical. Observe, that if we remove the edge connecting two centers of these stars, the chromatic number remains r. Therefore, our graph is not r-critical, a contradiction.

In the remaining case, $\overline{G}$ has two components H_1 and H_2. Since there are at most three independent edges, there is one in H_1 and two in H_2. It implies that H_1 has at most four vertices. Therefore, H_2 has at least eight vertices. Consider a spanning tree T of H_2 and remove two adjacent vertices of T, one of them being a leaf. It is easy to see that the remainder of T contains a path with three edges. Therefore, in total we found three independent edges of H_2, a contradiction. □

We need the following result of Gallai.

Theorem 36. [54] *Let $r \geq 3$ and $n < \frac{5}{3}r$. Every r-critical, n-vertex graph contains at least $\left\lceil \frac{3}{2}\left(\frac{5}{3}r - n\right)\right\rceil$ simplicial vertices.*

Lemma 25. *For $r \geq 6$, there are precisely twenty-two r-critical graphs on $r+4$ vertices. They can be constructed by adding simplicial vertices to one of the following:*
- *a 3-critical graph on 7 vertices,*
- *four 4-critical graphs on 8 vertices,*
- *sixteen 5-critical graphs on 9 vertices, or*
- *a 6-critical graphs on 10 vertices.*

Proof. For the base of induction, we use Royle's table again, see [103]. The full computer search shows that there are precisely twenty-two 6-critical graphs on 10 vertices. One of them has 3 simplicial vertices, four of them has 2, sixteen have 1, and one has no simplicial vertices. For the induction step, we use Theorem 36 and see that there are at least $r - 6$ simplicial vertices. Since $r \geq 7$, there is always a simplicial vertex. We remove it and use the induction hypothesis to finish the proof. □

There is an explicite list of twenty-one 5-critical graphs on 9 vertices [103]. We have checked, partly manually, partly using Mader's extremal result [83], that each of those graphs contains a topological K_5. Also the above mentioned 6-critical graph on 10 vertices contains a topological K_6. These results imply the following

Corollary 8. *Any r-critical graph on at most $r+4$ vertices satisfy the Hajós conjecture.*

We conjectured [E] that the following slightly more general statement can be proved with similar methods.

Conjecture 10. *There is a function $g(r)$ with the following properties.*
(i) *$g(r)$ tends to infinity,*
(ii) *Every r-critical graph with at most $r+g(r)$ vertices satisfy the Hajós conjecture.*

For the next value, Luiz and Richter proved the following.

Theorem 37 ([80])**.** *Let c be a positive integer. There are numbers $n(c)$ and $r(c)$ such that for any $r > r(c)$, there are precisely $n(c)$ r-critical graphs with $r+c$ vertices. In particular, $r(5) = 7$ and $n(5) = 395$. Moreover, every r-critical graph with $r+5$ vertices has a subdivision of K_r.*

However, Luiz and Richter [80] disproved our conjecture for $c \geq 6$ as follows. They considered a family F_c of subgraphs of Catlin's graphs $L({}_kC_5)$ [38]. They proved that F_c is $(c+1)$-critical for $c \geq 6$. On the other hand, F_c does not contain a subdivision of K_{c+1}.

5.2 The crossing number

It follows from Euler's formula that a planar graph can have at most $3n-6$ edges. Suppose that G has $m \geq 3n-6$ edges. By deleting crossing edges one by one, it follows by induction that for $n \geq 3$,

$$\text{CR}(G) \geq m - 3(n-2) \tag{5.1}$$

Pach et al. [95, 96] generalized this idea and proved the following lower bounds. Each of them holds for any graph G with $n \geq 3$ vertices and m edges.

$$\text{CR}(G) \geq 7m/3 - 25(n-2)/3 \tag{5.2}$$

$$\text{CR}(G) \geq 4m - 103(n-2)/6 \tag{5.3}$$

$$\text{CR}(G) \geq 5m - 25(n-2) \tag{5.4}$$

Inequality (5.1) is the best for $m \leq 4(n-2)$, (5.2) is the best for $4(n-2) \leq m \leq 5.3(n-2)$, (5.3) is the best for $5.3(n-2) \leq m \leq 47(n-2)/6$, and (5.4) is the best for $47(n-2)/6 \leq m$.

It was also shown in [96] that (5.1) can not be improved in the range $m \leq 4(n-2)$, and (5.2) can not be improved in the range $4(n-2) \leq m \leq 5(n-2)$, apart from an additive constant. Inequalities (5.3) and (5.4) are conjectured to be far from optimal. Using the methods in [96] one can obtain an infinite family of such linear inequalities, of the form $am - b(n-2)$, for example $\text{cr}(G) \geq 3m - 35(n-2)/3$.

The most important inequality for crossing numbers is undoubtedly the *Crossing Lemma*, first proved by Ajtai, Chvátal, Newborn, Szemerédi [11], and independently by Leighton [76]. If G has n vertices and $m \geq 4n$ edges, then

$$\text{cr}(G) \geq \frac{1}{64}\frac{m^3}{n^2}. \tag{5.5}$$

The original constant was much larger, the constant $\frac{1}{64}$ comes from the well-known probabilistic proof of Chazelle, Sharir, and Welzl [10]. The basic idea is to take a random spanned subgraph and apply inequality (5.1) for that.

The order of magnitude of this bound can not be improved, see [96], the best known constant is obtained in [96]. If G has n vertices and $m \geq \frac{103}{16}n$ edges, then

$$\text{cr}(G) \geq \frac{1}{31.1}\frac{m^3}{n^2}. \tag{5.6}$$

The proof is very similar to the proof of (5.5), the main difference is that instead of (5.1), inequality (5.3) is applied for the random subgraph. The proof of the following technical lemma is based on the same idea.

Lemma 26. *Suppose that $n \geq 10$, and $0 < p \leq 1$. Let*

$$\text{cr}(n,m,p) = \frac{4m}{p^2} - \frac{103n}{6p^3} + \frac{103}{3p^4} - \frac{5n^2(1-p)^{n-2}}{p^4}.$$

Then for any graph G with n vertices and m edges

$$\text{cr}(G) \geq \text{cr}(n,m,p).$$

Proof. Observe that inequality (5.3) does not hold for graphs with at most two vertices. For any graph G, let

$$\text{cr}'(G) = \begin{cases} \text{cr}(G) & \text{if } n \geq 3 \\ 4 & \text{if } n = 2 \\ 18 & \text{if } n = 1 \\ 35 & \text{if } n = 0 \end{cases}$$

It is easy to see that for *any* graph G

$$\text{cr}'(G) \geq 4m - \frac{103}{6}(n-2). \tag{5.7}$$

Let G be a graph with n vertices and m edges. Consider a drawing of G with $\text{cr}(G)$ crossings. Choose each vertex of G independently with probability p, and let G' be a subgraph of G spanned by the selected vertices. Consider the drawing of G' *inherited* from the drawing of G, that is, each edge of G' is drawn exactly as it is drawn in G. Let

n' and m' be the number of vertices and edges of G', and let x be the number of crossings in the present drawing of G'. Using that $E(n') = pn$, $E(m') = p^2m$, $E(x) = p^4\text{CR}(G)$, by inequality 5.7, and by the linearity of expectations we have,

$$E(x) \geq E(\text{CR}(G')) \geq E(\text{CR}'(G')) - 4P(n' = 2) - 18P(n' = 1) - 35P(n' = 0) \geq$$

$$\geq 4p^2m - \frac{103}{6}pn + \frac{103}{3} - 4\binom{n}{2}p^2(1-p)^{n-2} - 18np(1-p)^{n-1} - 35(1-p)^n \geq$$

$$\geq 4p^2m - \frac{103}{6}pn + \frac{103}{3} - 5n^2(1-p)^{n-2}.$$

Dividing by p^4 we obtain the statement of the Lemma. □

Note that in our applications p is at least 1/2, n is at least 13. Therefore, the last term in the inequality $\frac{5n^2(1-p)^{n-2}}{p^4}$ is negligible.

We also need some bounds on $\text{CR}(K_r)$, the crossing number of the complete graph. It is known that

$$\text{CR}(K_r) \leq Z(r) = \frac{1}{4}\left\lfloor\frac{r}{2}\right\rfloor\left\lfloor\frac{r-1}{2}\right\rfloor\left\lfloor\frac{r-2}{2}\right\rfloor\left\lfloor\frac{r-3}{2}\right\rfloor, \tag{5.8}$$

see e.g. [101]. Hill posed and Guy [56] published first the conjecture that $\text{CR}(K_r) = Z(r)$. This conjecture has been verified for $r \leq 12$ but still open for $r > 12$. The best known lower bound is due to de Klerk et al. [71]: $\text{CR}(K_r) \geq 0.86\,Z(r)$. Balogh et al. [21] proved that asymptotically $\text{CR}(K_r) \geq 0.985\,Z(r)$.

5.3 The Albertson conjecture for $r \leq 16$

Now we show Theorem 35.

Proof. Suppose that G is an r-critical graph. If G contains a topological K_r, then clearly $\text{CR}(G) \geq \text{CR}(K_r)$. Suppose in the sequel that G does not contain a topological K_r.

Therefore, we can apply the Kostochka, Stiebitz, and the Gallai bounds on the number of edges. After that, we use Lemma 26 to get the desired lower bound on the crossing number. Albertson et al. [13] worked along the same philosophy. However, they used a weaker version of the Kostochka, Stiebitz, and the Gallai bounds, and instead of Lemma 26 they applied the weaker inequality (5.3). In the next table, we include the results of our calculations. For comparison, we also included the result Albertson et al. might have had using (5.3). We used a simple Maple program to do the calculations [E].

1. Let $r = 13$. By (5.8) we have $\text{CR}(K_{13}) \leq 225$. By Corollary 8, we only need to consider $n \geq r + 5 = 18$. If $n \geq 22$, then the KS-bound combined with (5.3) gives the desired result: $2m \geq 12n + 20 \Rightarrow \text{CR}(G) \geq 4(6n + 10) - 103/6(n - 2) \geq 224.67$.

For $18 \leq n \leq 21$ the result follows from the table below.

n	e	bound (5.3)	p	$\lceil \text{CR}(n,m,p) \rceil$
18	128	238	0.719	288
19	135	249	0.732	296
20	141	255	0.751	298
21	146	258	0.774	294

2. Let $r = 14$. By (5.8) we have $\text{CR}(K_{14}) \leq 315$. By Corollary 8, we only need to consider $n \geq r + 5 = 19$. If $n \geq 27$, then the KS-bound combined with (5.3) gives the desired result: $2m \geq 13n + 22 \Rightarrow \text{CR}(G) \geq 4(6.5n + 11) - 103/6(n - 2) \geq 316$.

For $19 \leq n \leq 26$ the result follows from the table below.

n	e	bound (5.3)	p	$\lceil \text{CR}(n,m,p) \rceil$
19	146	293	0.659	388
20	154	307	0.670	402
21	161	318	0.684	407
22	167	325	0.702	406
23	172	328	0.723	398
24	176	327	0.747	384
25	179	322	0.775	366
26	181	312	0.807	344

3. Let $r = 15$. By (5.8) we have $\text{CR}(K_{15}) \leq 441$. By Corollary 8, we only need to consider $n \geq r + 5 = 20$. Suppose now that G is 15-critical and $n \geq 28$. By the KS-bound we have $m \geq 7n+12$. Apply Lemma 26 with $p = 0.764$ and a straightforward calculation gives $\text{CR}(G) \geq \text{CR}(n, m, 0.764) \geq 441$.

For $20 \leq n \leq 27$ the result follows from the table below.

n	e	bound (5.3)	p	$\lceil \text{CR}(n,m,p) \rceil$
20	165	351	0.610	510
21	174	370	0.617	531
22	182	385	0.623	542
23	189	396	0.642	545
24	195	403	0.659	539
25	200	406	0.678	526
26	204	404	0.700	508
27	207	399	0.725	484

4. Let $r = 16$. By (5.8) we have $\text{CR}(K_{16}) \leq 588$. By Corollary 8, we only need to consider $n \geq r+5 = 21$. Suppose now that G is 16-critical and $n \geq 32$. By the KS-bound we have $m \geq 7.5n + 13$. Apply Lemma 26 with $p = 0.72$ and again a straightforward calculation gives $\text{CR}(G) \geq \text{CR}(n, m, 0.72) \geq 588$.

For $21 \leq n \leq 31$ the result follows from the table below.

n	e	bound (5.4)	p	$\lceil \mathrm{CR}(n,m,p) \rceil$
21	185	450	0.567	657
22	195	475	0.573	687
23	204	495	0.581	706
24	212	510	0.592	714
25	219	520	0.605	712
26	225	525	0.621	701
27	230	525	0.639	683
28	234	520	0.659	658
29	237	510	0.681	628
30	239	495	0.706	593
31	246	505	0.713	601

This concludes the proof of Theorem 35. □

Remarks

Our attack of the Albertson conjecture is based on the following philosophy. We calculate a lower bound for the number of edges of an r-critical n-vertex graph G. Next we substitute this into the lower bound given by Lemma 26. Finally, we compare the result and $Z(r)$. For large r, this method is not sufficient, but it gives the right order of magnitude, and the constants are roughly within a factor of 4.

Let G be an r-critical graph with n vertices, where $r \le n \le 3.57r$. Then $2m \ge (r-1)n$. We can apply (5.6):

$$\mathrm{CR}(G) \ge \frac{1}{31.1}\frac{((r-1)n/2)^3}{n^2} = \frac{(r-1)^3 n}{31.1 \cdot 8} \ge \frac{1}{250} r(r-1)^3 \ge \frac{Z(r)}{4}.$$

Let $G = G(n,p)$ be a random graph with n vertices and edge probability $p = p(n)$. It is known (see [64]) that there is a constant $C_0 > 0$ such that if $np > C_0$ then asymptotically almost surely we have

$$\chi(G) < \frac{np}{\log np}.$$

Therefore, asymptotically almost surely

$$\mathrm{CR}(K_{\chi(G)}) \le Z(\chi(G)) < \frac{n^4 p^4}{64 \log^4 np}.$$

On the other hand, by [97], if $np > 20$ then almost surely

$$\mathrm{CR}(G) \ge \frac{n^4 p^2}{20000}.$$

Consequently, almost surely we have $\mathrm{CR}(G) > \mathrm{CR}(K_{\chi(G)})$, that is, roughly speaking, unlike in the case of the Hajós conjecture, a random graph almost surely satisfies the statement of the Albertson conjecture.

Ackerman [9] improved the constant in (5.6). This automatically improves our result and the Albertson conjecture can be proved for $r \le 18$, as it was noted by Ackerman [9] in the ArXiv version of his paper.

Chapter 6

Wegner's conjecture and crumby colorings

Decomposition of vertices also appear naturally as follows. Let us color the vertices of graph G red and blue. We are only interested in the subgraphs of G spanned separately by the red vertices and the blue vertices[1]. They induce some red components and blue components. One might impose various conditions on the monochromatic components. Such a condition might be that all monochromatic components are small. For instance, Alon et al. [17] proved if G has maximum degree 4, then there exists a red-blue coloring of the vertices of G such that the monochromatic components have size at most 57. Haxell et al. [60] improved this to 6. They also showed an analogous result for graphs of maximum degree 5 with a large constant bound. Alon et al. [17] proved that such a constant does not exist for graphs of maximum degree 6. Thomassen [110] proved that the edges of every 3-regular graph can be 2-colored such that each monochromatic component is a path of length at most 5. In what follows, we study a problem similar to the above mentioned, but the condition on the red and the blue components are different.

The *square chromatic number* of a graph G is simply the chromatic number of the square of G. In notation, $\chi_\square(G) = \chi(G^2)$. Wegner [119] initiated the study of the square chromatic number of planar graphs. There has been accelerated interest in this topic due to his conjecture. We recall the case $\Delta \leq 3$.

Conjecture 11 (Wegner [119])**.** *For any subcubic planar graph G, the square of G is 7-colorable. That is, $\chi_\square(G) \leq 7$.*

Thomassen published his proof of Conjecture 11 in [116]. He formulated an attractive conjecture, which would imply Conjecture 11. This new conjecture belongs to the area of graph decompositions. It is well-known, that subcubic graphs can be 2-colored such that each color class induces only a matching and isolated vertices. To see this, one distributes the vertices arbitrarily into a red and a blue class. If inside any class, there is a vertex of induced degree at least 2, then swap the color of that vertex. At first sight, the next conjecture is very similar, but excludes the possibility of isolated vertices in the red class. Instead it relaxes the red part from a matching to a subgraph not containing a 3-edge path.

[1] We forget the edges going between any red vertex and any blue vertex

Conjecture 12 (Thomassen [116]). *If G is a 3-connected, cubic graph on at least 8 vertices, then the vertices of G can be colored blue and red such that the subgraph induced by the blue vertices has maximum degree 1 (that is, it consists of a matching and some isolated vertices) and the similar red part has minimum degree at least 1 and contains no 4-path.*

Here a 4-path is a path with 3 edges and 4 vertices. Since all monochromatic components are small in this coloring, and there is a certain irregularity, we call such a coloring *crumby*.

Remark 2. *The original conjecture was formulated for all 3-connected cubic graphs. The reviewer of [G] observed that the 3-prism does not have the required red-blue coloring. Therefore, it has to be excluded with an extra assumption.*

Thomassen gave a short and elegant argument that shows how Conjecture 12 implies Conjecture 11. We confirm Conjecture 12 for Generalized Petersen graphs. In the context of the square chromatic number of subcubic graphs, the Petersen graph plays a crucial role in the following sense. We know that $\chi_\square = 10$ for the Petersen graph. Cranston and Kim [39] proved that a dramatic drop happens in the chromatic number if we exclude only this one graph: $\chi_\square(G) \leq 8$ for any subcubic graph G different from the Petersen graph. However, Conjecture 12 holds also for the Petersen graph, as shown in Figure 6.1.

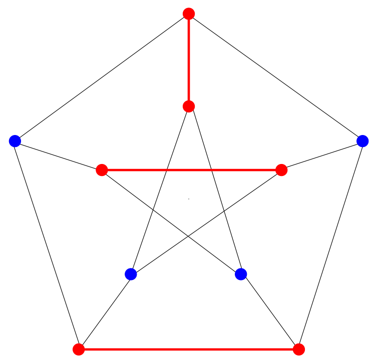


Figure 6.1: A red-blue coloring of the Petersen graph satisfying Conjecture 12.

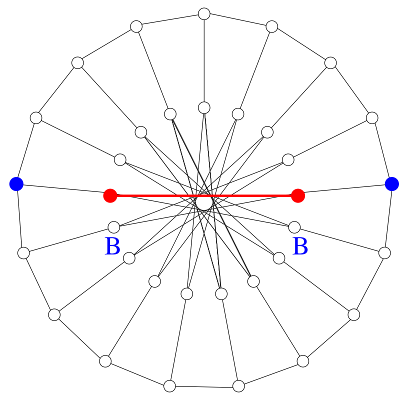


Figure 6.2: A horizontal cut.

The Generalized Petersen graph $GP(2k+1, k)$ is defined for $k \geq 2$ as follows: the vertices are $\{u_1, \ldots, u_{2k+1}\}$ and $\{v_1, \ldots, v_{2k+1}\}$ such that $\{u_1, \ldots, u_{2k+1}\}$ form a cycle (the outer cycle) in the natural order. The spoke edges are of form $u_i v_i$ for $1 \leq i \leq 2k+1$. Finally, the inner cycle is spanned by the long diagonals. That is, edges of form $v_i v_{i+k}$, where the indices are modulo $2k+1$ and $1 \leq i \leq 2k+1$.

One might define somewhat similar objects for an even number as follows: for $l \geq 3$ let $GP(2l, l-1)$ have vertices $\{u_1, \ldots, u_{2l}\}$ and $\{v_1, \ldots, v_{2l}\}$ such that $\{u_1, \ldots, u_{2l}\}$ form the outer cycle in the natural order. The spoke edges are of form $u_i v_i$ for $1 \leq i \leq 2l$.

Finally, the inner 2-factor is spanned by the second longest diagonals. That is, edges of form v_iv_{i+l-1}, where the indices are modulo $2l$ and $1 \leq i \leq 2l$. However, these edges form a cycle only if $2l$ and $l-1$ are coprime. Therefore, l needs to be even. Hence we use $l = 2k$ in the even part of the next section.

6.1 Generalized Petersen graphs

For the odd case, we can prove the following

Theorem 38. *For any $k \geq 2$,* Conjecture 12 *holds for the Generalized Petersen graph $GP(2k+1, k)$. That is, there exists a red-blue vertex coloring such that the induced blue components are vertices or edges and the red components are stars with $1, 2$ or 3 edges.*

Proof. We are going to construct an explicit coloring. There are three cases according to the number of vertices on the outer cycle modulo 6.

We use the following building blocks: horizontal cut, red wedge, blue cross and red syringe, shown in Figures 6.2-6.5.

A horizontal cut corresponds to vertices u_1, u_k and v_1, v_k, where u_1, u_k are blue and v_1, v_k are red.

A red wedge corresponds to vertices $u_i, v_i, u_{i+k-1}, u_{i+k}, v_{i+k-1}, v_{i+k}$ for any i, where the vertices u_i, u_{i+k-1}, u_{i+k} are blue and v_i, v_{i+k-1}, v_{i+k} are red.

A blue cross corresponds to vertices (u_{i-1}, u_i, u_{i+1}), (v_{i-1}, v_i, v_{i+1}), (u_{i+k}, u_{i+k+1}), (v_{i+k}, v_{i+k+1}) for any i, where $(u_{i-1}, u_i, u_{i+1}, v_i), (u_{i+k}, u_{i+k+1})$ are red and (v_{i-1}, v_{i+k}), (v_{i+1}, v_{i+k+1}) are blue.

A red syringe corresponds to vertices $u_i, v_i, u_{i+k-1}, u_{i+k}, v_{i+k-1}, v_{i+k}$ for any i, where the vertices $u_i, v_i, u_{i+k-1}, u_{i+k}$ are red and v_{i+k-1}, v_{i+k} are blue.

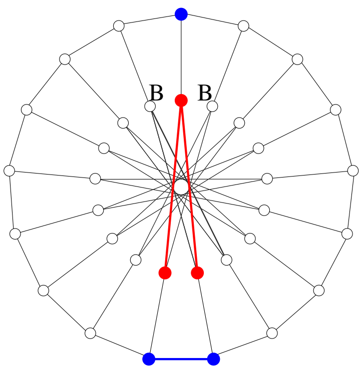


Figure 6.3: A red wedge.

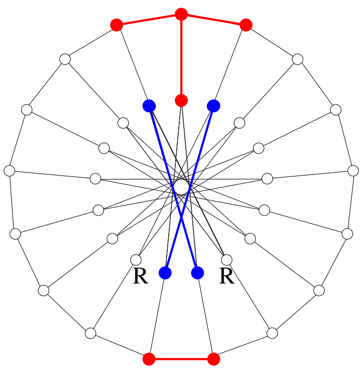


Figure 6.4: A blue cross.

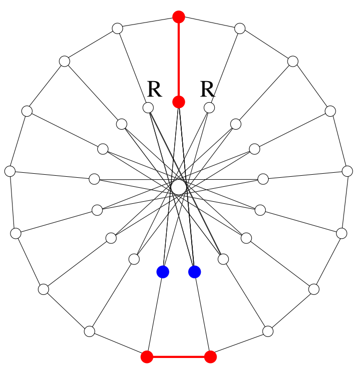


Figure 6.5: A red syringe.

Each of these colored vertex sets cut the outer cycle into two disjoint paths P and Q and divide the Generalized Petersen graphs into two halves in the figures: the vertices of P and their neighbors form one half, the vertices of Q and their neighbors the other half. Apart from the colored vertices, a building block might impose a side condition. That reflects the imposed color of some vertices in the inner cycle, see the vertices marked by R or B in Figures 6.2-6.5. It means that our construction will obey the imposed colors. (Many of the colors are necessary by the definition of the red-blue coloring.) We describe our construction by listing the colors of the vertices on the outer cycle. The horizontal cut is used first and precisely once in each case. After that, we are given an upper and a lower subgraph. We list the color sequence on the outer cycle of the upper subgraph in the clockwise order, then the color sequence on the outer cycle of the lower subgraph in the clockwise order. The two sequences are separated by a comma. For instance, RBBRRBBRRBBR,RRBRRRBRRRBRR, see Figure 6.7. One decodes this as follows: syringe, wedge, cross, wedge, cross, wedge, syringe. In each decoding step, recognising a building block, we consider the next monochromatic sequence in the color sequence (from left to right) before and after the comma. The horizontal cut is not present in the color sequence.

In the inductive steps below, we assume the existence of a coloring of a Generalized Petersen graph, which is represented by the coloring of the upper and lower path of the outer cycle, C,D say. This shorthand is used for an appropriate series of Rs and Bs such that a comma is used as explained above. After that, we consider a one size larger Generalized Petersen graph G and extend the coloring C,D by some appropriate building blocks to get the required coloring of G.

Case 1. Let the number of vertices on the outer cycle be $5 + 6s$. That is, we color the graphs $GP(5 + 6s, 2 + 3s)$, where $s \geq 0$. The initial sequence (when $s = 0$) is R,RR, which corresponds to the Petersen graph, see Figure 6.1. Now assume that the coloring C,D of $GP(5+6s, 2+3s)$ is given. We extend this coloring by adding a syringe and a wedge (such that the number of vertices increases by 3 both in the upper and the lower part) right after the horizontal cut. That is, RBBC,RRBD is the coloring of $GP(5 + 6(s + 1), 2 + 3(s + 1))$.

Case 2. Let the number of vertices on the outer cycle be $7+6s$. That is, we color the graphs $GP(7 + 6s, 3 + 3s)$, where $s \geq 0$. The initial sequence (when $s = 0$) is RR,RRR corresponding to a blue cross. Now assume that the coloring C,D of $GP(7+6s, 3+3s)$ is given. We extend this coloring by adding a syringe and a wedge right after the horizontal cut. That is, RBBC,RRBD is the coloring of $GP(7 + 6(s + 1), 3 + 3(s + 1))$.

Case 3. Let the number of vertices on the outer cycle be $15+6s$. That is, we color the graphs $GP(9 + 6s, 4 + 3s)$, where $s \geq 1$. The initial sequence is RRBBRR,RRRBRRR corresponding to cross, wedge, cross. Now assume that the coloring RRC,RRRD of $GP(9 + 6s, 4 + 3s)$ is given. We extend this coloring by adding a wedge and a syringe after the first cross. That is, RRBBRC,RRRBRRD is the coloring of $GP(9+6(s+1), 4+3(s+1))$.

The exceptional case $GP(9, 4)$ can be done as depicted in Figure 6.6.

After giving the recipe in our construction, we only have to check that the side conditions are satisfied. That means, the building blocks can be put together nicely after each other.

We always start and end the coloring sequence of the upper subgraph by a cross or a syringe. Therefore, the side condition of the horizontal cut is always satisfied, since a

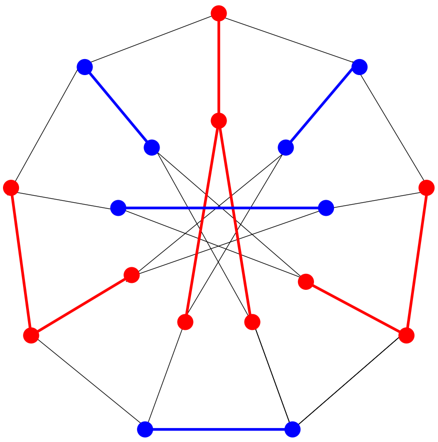


Figure 6.6: A crumby coloring of $GP(9,4)$.

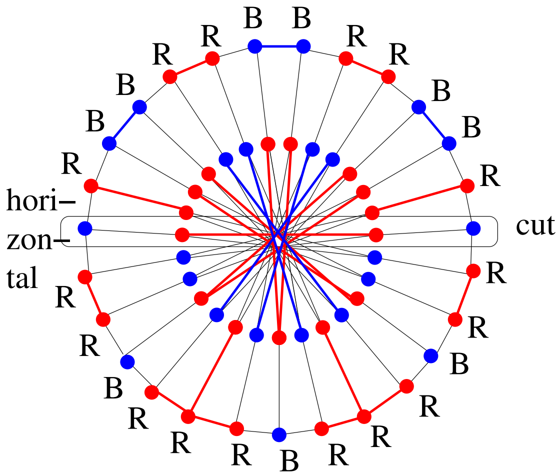


Figure 6.7: A crumby coloring of $GP(27,13)$.

cross or a syringe have blue vertices on the inner cycle, where the side condition of the horizontal cut requires that.

The side condition of a wedge is always satisfied, since we can only put cross or syringe next to a wedge. However, both a cross and an appropriately turned syringe contains blue vertices on the inner cycle, where the side condition of the wedge requires that.

Reversing the argument of the previous paragraph, we can similarly confirm that the side condition of a syringe or a cross is always satisfied. Indeed, we only put a wedge next to a syringe or a cross, and a wedge has only red vertices on the inner cycle. □

For the even case, we prove the following

Theorem 39. *For any $k \geq 2$, the Generalized Petersen graph $GP(4k, 2k-1)$ admits a crumby coloring.*

Proof. We consider two cases according to the parity of k. It is easy to find the required coloring, when k is even. In that case, the number of vertices on the outer cycle is divisible by 8. Hence we can repeatedly color the vertices RRBB..., so a colored matching is induced. Now consider two colored "diagonally opposite" edges on the outer cycle, u_1u_2 and $u_{2k+1}u_{2k+2}$ say. Suppose these vertices are red. Consider the vertices of the inner cycle next to these 4 vertices: v_1, v_2 and v_{2k+1}, v_{2k+2}. We color them blue and they induce a matching v_1v_{2k+2} and v_2v_{2k+1} on the inner cycle, since $(2k+2)+(2k-1) \equiv 1 \mod 4k$. We repeat this idea on the rest of the graph, adding 2 to all indices and interchanging the colors. This coloring satisfies the conditions.

When k is odd, we construct the coloring by giving an initial coloring and then adding the colored matchings as in the previous argument. Assume that $k \geq 5$. The initial graph is $GP(20,9)$. We use the coloring given in Figure 6.9. As before, we can identify the

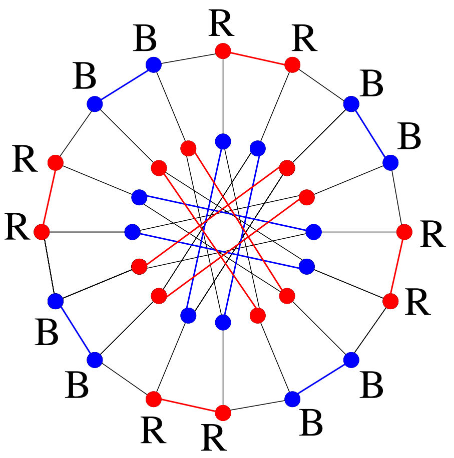


Figure 6.8: A crumby coloring of $GP(16,7)$.

coloring by listing the color sequence of the vertices on the outer cycle in clockwise order starting at u_1: RRRBBRRRBB,RRRBBRRRBB. Here the decoding is the following: three red vertices correspond to a red anchor, as indicated in Figure 6.9. For instance a red anchor represented by RRR include the 4 vertices u_1, u_2, u_3, v_2 belonging to a red claw, and its two blue neighbors v_1, v_3 on the inner cycle. Here the magic works, since the edges of the inner cycle connect vertices of opposite colors. Two consecutive blue vertices BB on the outer cycle correspond to a blue edge, for instance u_4, u_5 and its two red neighbors v_4, v_5 on the inner cycle.

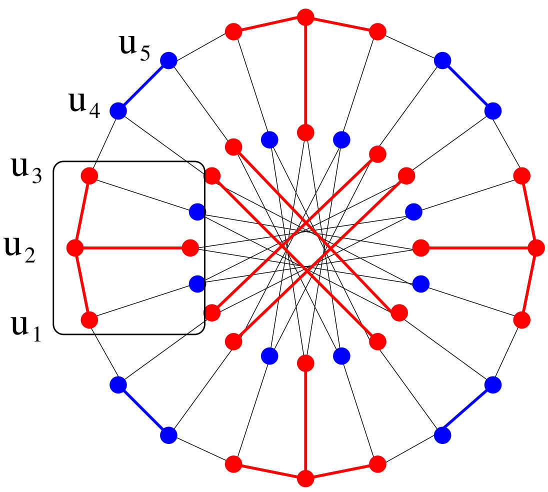


Figure 6.9: A crumby coloring of $GP(20,9)$.

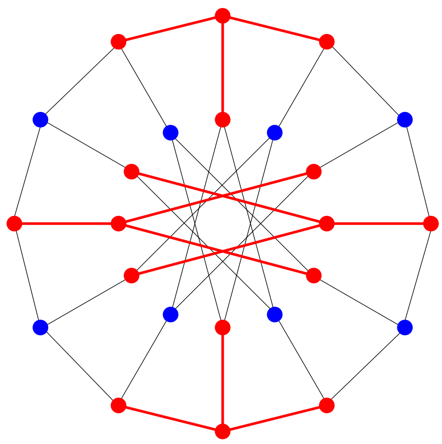


Figure 6.10: A crumby coloring of $GP(12,5)$.

When k increases by two, the number of vertices on the outer cycle increases by 8. Therefore, the following extension works: We add two red vertices and two blue vertices at the end of the color sequence. Formally, if C,C was a valid coloring of $GP(4k, 2k-1)$, where k is odd, then CRRBB,CRRBB is a valid coloring of $GP(4(k+2), 2k+3)$. This extension corresponds to inserting two colored edges on the outer cycle and two pairs of colored edges of the inner cycle. This is the same principle that we used in the first paragraph of the proof for the even case.

We excluded the case $k = 3$ from the previous argument. To complete the proof, Figure 6.10 shows the required coloring of $GP(12, 5)$. □

6.2 Subcubic trees

We had not found a counterexample to Conjecture 12 among the subcubic graphs (of any connectivity). Therefore, we posed the following slight strengthening of Conjecture 12:

Conjecture 13 ([G]). *Every subcubic graph on at least 7 vertices admits a crumby coloring.*

We confirm the statement for trees.

Theorem 40. *Every subcubic tree T posseses a red-blue vertex coloring such that the induced blue components are vertices or edges and the red components are stars with $1, 2$ or 3 edges.*

Proof. Let r be an arbitrary vertex of T. Let us draw T as a planar tree rooted at r. The planar embedding allows us to distinguish the left and right son of a vertex. The level of r is 0 and the neighbors of r lie in level 1 etc. We color the vertices of T according to a breadth-first search. In each step, we color a new vertex v, that has degree 1 in the current colored subtree. We denote the only neighbor of v by x. In certain cases, we recolor a few previously colored vertices in a small neighborhood of v. Any other vertex keeps its color. Our coloring algorithm works according to the rules listed below. Each rule describes a step, when we transform a colored subtree T_i to T_{i+1} adding the next vertex of the breadth-first search. At the end of each step, we have a coloring of T_{i+1} that satisfies the conditions of the theorem.

Case 0. We color r blue.

Case 1. Assume the degree of x is at most 1 in T_i and x is a singleton blue. Now we color v blue.

Case 2. Assume the degree of x is 1 in T_i and x is red. Now we color v blue.

Case 3. Assume the degree of x is 1 in T_i and x is blue and its parent is also blue. Now we color v red and change the color of x to red.

Case 4. Assume the degree of x is 2 in T_i and x is red. Now we set v to be blue.

Case 5. Assume the degree of x is 2 in T_i and x is blue. By the conditions, the left son l of x must be blue and the parent p of x must be red in T_i. Let g denote the parent of p in T_i, if it exists.

Case 5a. Assume the degree of p is 2 in T_i. Now we set l, v, x to be red and p to be blue. This works, unless now g would be a singleton red in T_{i+1}. In this exceptional case, we keep the colors of the vertices in T_i and only change the color of x to red and color v

blue. If g does not exist, then p is the root, and p has another son x'. Now x' plays the role of g in the previous argument and we make the same recoloring if necessary.

Case 5b. Assume the degree of p is 3 in T_i, x is the right son of p and the left son of p is blue. It implies that the parent g of p is also red. Now we set l, v, x to be red and color p blue. This yields a coloring satisfying the required conditions, unless now g is a singleton red. In this exceptional case, we do a different recoloring of T_i. We change x to red and color v blue.

Assume the degree of p is 3 in T_i, x is the right son of p and the left son of p was red in T_i. Now we set x to red and l, v to blue. Here we notice that the recoloring of x might create a red 3-star, but not a red path with 3 edges.

Case 5c. Assume the degree of p is 3 in T_i, x is the left son of p and the right son of p is blue. It implies that the parent g of p is also red. Now we set l, v, x to be red and color p blue. This yields a coloring satisfying the required conditions, unless now g is a singleton red. In this exceptional case, we do a different recoloring of T_i. We change x to red and color v blue.

Assume the degree of p is 3 in T_i, x is the left son of p and the right son of p is red. Now we set x to red and l, v to blue.

Since we covered all cases, the algorithm terminates with a crumby coloring of T. □

We remark that in many cases, the following simple idea works: we repeatedly color one level of vertices blue and the next two levels red. The failure of this coloring happens at red leaves, if their parent's color is blue. However, taking subgraphs does not keep the properties required by the theorem. Therefore, it seems difficult to make this simple idea into a full proof.

Prescribing the color of a vertex is vital for doing induction. The following result is a strengthening of Theorem 40. It is routine to check that the proof literally holds for this version.

Corollary 9. *Every subcubic tree admits a crumby coloring such that the color of a leaf is prescribed.*

We strengthen this result further prescribing the color of a vertex of degree 2.

Theorem 41. *Any subcubic tree T admits a crumby coloring such that the color of an arbitrary vertex of degree 2 is prescribed, unless $T = P_3$.*

Proof. If $T = P_3$, then the middle vertex cannot be blue in a crumby coloring. Therefore, this is an exception. From now on, we assume that T has at least 4 vertices. Every tree admits a crumby coloring by Theorem 9. Let us suppose that T is a minimal example of a tree, which has a vertex v of degree 2 such that in any crumby coloring of T, the color of v must be red. We think of v as the root, and denote the two neighbors of v by x and y.

If any of the neighbors of v is of degree 2, say x, then we can delete the edge vx and consider the two remaining trees T_v rooted at v and T_x rooted at x. We get a contradiction by using Theorem 9 with prescribed color red on x and blue on v in the respective trees.

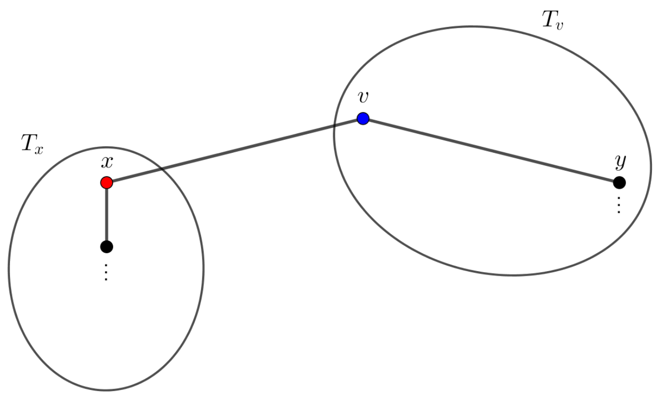


Figure 6.11: If $d_T(x) = 2$, then we get a contradiction.

Since T has at least 4 vertices, we may assume that $d_T(x) = 3$. As before, we get a contradiction if the color of x can be red in a crumby coloring of T_x, since we can color v blue and use Theorem 9 on T_v. Therefore, let us suppose that T_x is a tree, for which the degree 2 vertex x can only be colored blue in a crumby coloring. Denote the neighbors of x in T_x by z and w.

Due to the same reasons as above, the degree of z and w cannot be 2 in T_x. It cannot be 1 either, since in that case T_x has a crumby coloring in which the color of that leaf is prescribed red. Consequently x is also red, which is a contradiction. Hence $d_{T_x}(z) = d_{T_x}(w) = 3$, and by the minimality of T, we know that T_z admits a crumby coloring such that the degree 2 vertex z is blue. Now we may delete the edge xz and precolor the degree 1 vertex x red and find a crumby coloring of a subgraph of T_x. However, we can add back the edge xz giving a crumby coloring of T_x with red x, a contradiction. The same holds for T_w, but there is one exception: if both $T_z = T_w = P_3$. In Figure 6.12, we give a crumby coloring of T_x so that x is red, which concludes the proof.

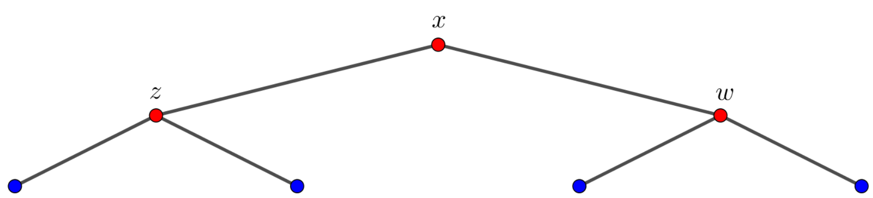


Figure 6.12: A crumby coloring of T_x such that x is red.

□

Remark 3. *If G is a graph that admits a crumby coloring, and T is an arbitrary tree with a leaf v, then let G_T denote a graph which we get by identifying v with any vertex of G. Observe that if an attachment tree T is not K_2 or $K_{1,3}$, then it is easy to get* *a crumby coloring of G_T. The key idea is to assign different colors to v and its only neighbor x inside T. Consider a crumby coloring of G, therefore the color of v is given, and color x differently. By* Theorem 9 *and* Theorem 41 *(depending on $d_T(x)$), we can extend this coloring to a crumby coloring of $T - v$ which results in a crumby coloring of G_T.*

Therefore, it is indifferent with respect to crumby colorings to attach trees, which are not isomorphic to K_2 or $K_{1,3}$. In the sequel, we assume that every attachment tree is either K_2 or $K_{1,3}$.

This allows us to significantly decrease the number of problematic attached trees.

6.3 Outerplanar graphs

In [H], we show that any 2-connected subcubic outerplanar graph admits a crumby coloring even if the color of an arbitrary vertex is prescribed.

The fact that we can prescribe the color of a vertex is useful in the following sense. We believe that crumby colorings exist for every subcubic outerplanar graph. However, there are various difficulties to extend the results on 2-connected graphs to all outerplanar graphs. In a general outerplanar graph, there might be trees attached to 2-connected blocks or between them. Since Conjecture 12 holds for trees, it gives some hope to combine these two results as building bricks, where having the extra freedom of prescribing the color of a vertex comes handy.

In the previous Section, we showed that Conjecture 12 holds for trees. A natural minor-closed class to be considered next is the class of outerplanar graphs. As the first step, we proved the following in [H]. The important assumption is the 2-connectedness. We omit the proof here since it is a bit technical.

Theorem 42. *Let G be a 2-connected subcubic outerplanar graph and let v be a vertex of G. We may prescribe the color of v and find a crumby coloring of G.*

A general outerplanar graph may not be 2-connected. It is glued together from 2-connected blocks in a tree-like manner. Some of the edges can form a tree hanging from a vertex of a block, or connecting a number of 2-connected outerplanar blocks. In our case, the maximum degree 3 condition gives some extra structural information. We are convinced that the natural extension of Theorem 42 to all subcubic outerplanar graphs holds.

Conjecture 14. *Every outerplanar graph with maximum degree 3 admits a crumby coloring.*

Considering this problem, one gets the impression that particular small trees attached to the vertices of a 2-connected outerplanar graph make the difficulty. This was described in Remark 3.

We prove a basic instance of Conjecture 14 relying on Theorems 9 and 41.

Proposition 43. *Let C be a cycle with vertices $v_1, \dots, v_k$, plus we might attach arbitrary trees $\{T_i\}$ to vertices $\{v_i\}$ of C, where $i \in I$ and $I \subseteq [k]$. The resulting graph G admits a crumby coloring.*

Proof. We may assume that each attachment tree is isomorphic to K_2 or $K_{1,3}$ by Remark 3. Our arguments slightly vary depending on some properties of G, thus we explain them separately.

Notice that some vertices of C have attachments and some do not. In the latter case, the vertex is called empty. First, let us assume that there are no empty vertices at all.

We notice that the case where k is even is simple. We color the vertices of C alternately red and blue. This gives the prescribed color of a leaf v_i in the tree T_i. We color T_i using Theorem 9 for each $i = 1, \dots, k$. These colorings together form a crumby coloring of G.

Assume now that k is odd. We try to reuse the previous strategy by cutting off two consecutive vertices v_i and v_{i+1} and the trees T_i and T_{i+1} from G. We notice that the remaining graph H admits a crumby coloring by the previous argument. In particular, the first and last vertices (v_{i+2} and v_{i-1}) on $C - \{v_i, v_{i+1}\}$ receive the same color.

For every j between 1 and k, the tree $T_j - v_j$ admits a crumby coloring. Let us record for every j the color of u_j, the neighbor of v_j in T_j. Since k is odd, there is an index ℓ such that u_ℓ and $u_{\ell+1}$ received the same color, say blue. Now we color v_ℓ and $v_{\ell+1}$ red and cut the cycle C by removing $\{v_\ell, v_{\ell+1}\}$. We color H as before such that we color the first and last vertex on $C - \{v_\ell, v_{\ell+1}\}$ blue. If u_ℓ was red, then we interchange colors accordingly. Altogether, a crumby coloring of G arises.

Unless there are no attachment trees at all (which case is easy), we can find two consecutive vertices of C, say v_1 and v_2 such that there is a tree attached to v_1, but v_2 has none. We use the following algorithm to color the vertices on C starting by coloring v_1 red and v_2 blue. Our aim is to color the vertices along C alternately, except in one case, when after a blue vertex we color an empty vertex red. In that case, the next vertex must be also red. Observe that if a red vertex is non-empty, then no matter if the tree is K_2 or $K_{1,3}$, we can color its vertices maintaining the crumby property. If v_{i-1} is blue, and v_i is an empty red, then v_{i+1} must also be red. However, it is attainable that v_{i+1} is not an end of a red P_3. Only two problems can occur during this algorithm. Both of them might happen, when we color v_k.

If v_k was blue, then v_1 might remain a red singleton. However, this cannot be the case by the existence of T_1. Otherwise if v_k is red, then we might create a large red component. If $T_1 = K_2$, then the leaf of T_1 can be blue. Hence the red component cannot contain a red P_4, since v_k was not an end of a red P_3. If $T_1 = K_{1,3}$, then the center of T_1 must be red, which causes a problem if v_{k-1} is an empty red or $T_k = K_{1,3}$. If we created a red P_4, then we recolor v_1 to blue and color the remaining vertices in T_1 red. □

6.4 Counterexamples

In 2021, Bellitto, Klimošová, Merker, Witkowski and Yuditsky [28] constructed an infinite family refuting Conjecture 12 and 13.

The gadgets of the construction are defined as follows.

Definition 2. Let H be the graph consisting of an 8-cycle $v_0v_1 \ldots v_7$ with two chords v_2v_6 and v_3v_7. Let H' be the graph consisting of two disjoint copies of H and two edges joining the two copies as in Figure 6.13. Let H'' be the graph consisting of three disjoint copies of H' and three edges joining the copies of H' as in Figure 6.13.

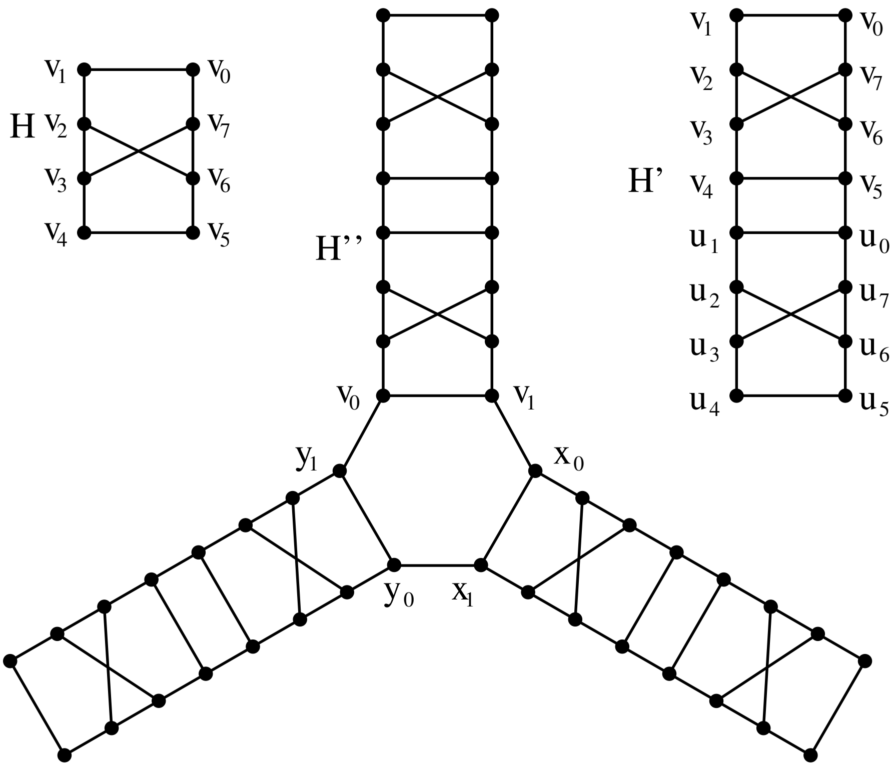


Figure 6.13: The gadgets for the counterexample.

Bellitto et al. showed that a subcubic graph containing H'' as a subgraph does not admit a crumby coloring. We will repeat the key steps below. Note that if H is a subgraph of a subcubic graph G, only the vertices that have degree 2 in H can have a neighbor in $G - H$. The same applies for H' and H''. In the following, we use the notation in Figure 6.13 to refer to the vertices of H and H'.

Lemma 27. *If H is an induced subgraph of a subcubic graph G, then in every crumby coloring of G*

- *At most one of the vertices v_0 and v_1 is colored red, and*

• *If one of v_0 and v_1 is colored red and its neighbor in H is also colored red, then both v_4 and v_5 are colored blue.*

Lemma 28. *If H' is an induced subgraph of a subcubic graph G, then in a crumby coloring of G exactly one of the following statements holds:*
• *Both v_0 and v_1 are blue, or*
• *v_0 is red, v_1 is blue, and v_0 has a red neighbor in $G - H'$, or*
• *v_1 is red, v_0 is blue, and v_1 has a red neighbor in $G - H'$.*

Theorem 44 ([28])**.** *If a subcubic graph G contains H'' as a subgraph, then G has no crumby coloring.*

The smallest 3-connected cubic graph containing H'' can be obtained from H'' by adding three edges joining the vertices of degree 2. However, there are many ways to construct 3-connected cubic graphs containing H'' as a subgraph. For instance, let G be any 3-connected cubic graph containing an induced 6-cycle C. Since H'' contains precisely six vertices of degree 2, it is possible to replace C by a copy of H'' such that the resulting graph is again 3-connected and cubic.

Finally, let us note that H'' is a 2-connected planar graph. Using H'', it is easy to construct an infinite family of 2-connected cubic planar graphs admitting no crumby coloring. It is open whether the 3-prism is the only 3-connected cubic planar graph admitting no crumby coloring.

We notice that H'' contains a K_4-minor and also a 5-cycle. This leaves Conjecture 12 open for some important graph classes: outerplanar graphs, K_4-minor-free graphs, bipartite graphs.

6.5 Bipartite graphs and subdivisions

Despite the infinite family of counterexamples in [28], we still believe that Conjecture 13 holds for most subcubic graphs. We pose the following

Conjecture 15. *Every subcubic bipartite graph admits a crumby coloring.*

We can prove this for a special class of bipartite graphs, where the degrees are all 2 in one class and 3 in the other class. In the proof, we apply the Edmonds-Gallai decomposition theorem [48, 55] that gives us information about the structure of the maximum matchings of a graph G. We recall that P_k denotes a path with k vertices and $N(X)$ denotes the set of neighbors of a vertex set X. A graph G is *hypomatchable* or *factor-critical* if for every vertex x, the graph $G - x$ has a perfect matching.

Theorem 45 (Edmonds-Gallai decomposition)**.** *Let G be a graph and let $A \subseteq V(G)$ be the collection of all vertices v such that there exists a maximum size matching which does not cover v. Set $B = N(A)$ and $C = V(G) \setminus (A \cup B)$. Now*

(*i*) *Every odd component O of $G - B$ is hypomatchable and $V(O) \subseteq A$.*

(*ii*) *Every even component Q of $G - B$ has a perfect matching and $V(Q) \subseteq C$.*

(*iii*) *For every $X \subseteq B$, the set $N(X)$ contains vertices in more than $|X|$ odd components of $G - B$.*

Next, we study subdivisions of cubic graphs. If we add precisely one new vertex on each edge, then the resulting graph is a 1-*subdivision.* We support Conjecture 15 by showing the following

Theorem 46. *Let $S(G)$ be the 1-subdivision of a cubic graph G. The bipartite graph $S(G)$ admits a crumby coloring.*

Proof. The idea of the proof is to color the original vertices (in G) red and color the subdivision vertices blue. If G admits a perfect matching M, then we recolor the subdivision vertices on M to red. This results in a crumby coloring consisting of red P_3-s and blue singletons. We refer to this idea later as the *standard process.* For instance, every 2-edge-connected graph G admits a perfect matching by Petersen's Theorem. If the graph $S(G)$ is the 1-subdivision of such G, then the standard process gives a crumby coloring of $S(G)$.

In what follows, we modify this simple idea to the general case, where G is any cubic graph. If G does not possess a perfect matching, we can still consider a maximum size matching in G and use the Edmonds-Gallai decomposition.

Let G be a cubic graph, and let B be given by the Edmonds-Gallai decomposition. Any isolated vertex in B must be connected to at least two odd components of $G - B$. The third edge might go to a third odd component, an even component or to one of the first two odd components.

Initially, let every vertex of G be red and every subdivision vertex blue. We recolor a few vertices as follows. In every even component, there exists a perfect matching and we recolor the subdivision vertices on the matching edges to red.

Consider the vertex sets A and B of the Edmonds-Gallai decomposition. Contract the components of A to vertices to get A^*. The bipartite graph (A^*, B) satisfies the Hall-condition by property (*iii*). Therefore, we find a matching M covering B. We recolor the subdivision vertices of the matching edges in M to red. We continue with the odd components corresponding to the vertices of A^* saturated by M. In these components, we use property (*i*) and find an almost perfect matching (if it is needed because the size of this component is greater than 1). The subdivision vertices on these matching edges are colored red as well. So far we only created red P_3-s separated by blue singletons. What is left to consider is the union of odd components corresponding to unsaturated vertices of A^*.

Let H be an odd component, which is a single vertex x. The G-neighbors of x are in the set B. Suppose y is a neighbor of x. There are two different types of y-vertices depending on the location of its 3 neighbors. Vertex y is *ordinary* if it has no G-neighbor in B. Vertex y is *problematic* if it has precisely one G-neighbor in B. Notice that a vertex y in B cannot have at least two G-neighbours in B by property (*iii*).

Assume y is ordinary, and x is a singleton odd component. By the above coloring, vertex y belongs to a red P_3 since B was saturated, and y has two blue subdivision vertices as neighbors in $S(G)$. We recolor the subdivision vertex v_{xy} on the edge xy red and y blue. If the third G-neighbor w of y belongs to either an even component or a saturated odd component, then it already has a red neighbor and causes no trouble. Notice that w might be a singleton odd component or it may belong to an unsaturated odd component as well. If w is another singleton odd component, then we recolor v_{wy} red and y remains a singleton blue component, which in turn further decreases the number of isolated red vertices. However, if w belongs to a larger unsaturated odd component, then again we recolor v_{wy} red and finish the coloring of this odd component as it was explained before for the larger saturated odd components using property (i).

We perform this coloring step for all of those unsaturated singleton odd components, which have an ordinary neighbor from B. Observe that at this point among the colored vertices of G the blue ones must be some ordinary vertices of B and all three G-neighbors of these vertices are red and has a red neighbor.

Continue the recoloring process by considering one-by-one the unsaturated odd singleton components, which only have problematic G-neighbors from B. Assume x is such a singleton odd component, and y is a problematic G-neighbor of x. We recolor v_{xy} red and y blue. Since y has a unique G-neighbor y' in B, we consider the third G-neighbor x' of y' (besides y and the one determined by M). If x' already has a red neighbor, then we are done and we can continue the recoloring process. Otherwise x' must belong to an odd component which still has isolated red vertices.

If x' is an isolated red vertex and belongs to a large odd component H', then denote one of its G-neighbors inside H' by z'. Use property (i) in $H' - z'$ and fix a perfect matching there and color the subdivision vertices red on these matching edges. Now recolor $v_{x'z'}$ red and x' blue. This way z' is no longer an isolated red and the blue component of x' consists of x' and $v_{x'y'}$.

If x' is an isolated red vertex and belongs to a singleton odd component which only has problematic G-neighbors from B then we continue the recoloring process with x'. Since x' has degree 3 in G, we can select a G-neighbor z different from y' and do as above. In this way, the color of y' is unchanged and therefore y remains in a blue P_2. Altogether we created a crumby coloring locally around x and y. Now we continue with x' and z playing the role of x and y in the previous argument. This process terminates and creates no loops, since every B-vertex is incident to 3 edges, one of which belongs to M. Therefore this process have to end by finding a vertex from one of the odd components which already had a red neighbor. Let us emphasize that this process cannot go back to any of the unique blue vertices of some large odd component because we cannot revisit the already visited problematic vertices of B.

At this point we either have a crumby coloring or there are some unsaturated large odd components which haven't been visited during the recoloring process.

Let H be such a large odd component and $x \in H$ be an arbitrary vertex and consider a perfect matching in $H - x$ by property (i). We recolor the subdivision vertices on these matching edges to red. Let y be a G-neighbor of x in H and recolor v_{xy} be red and

y blue. Since there was a matching edge zy and both z and v_{yz} are red, moreover on the third edge wy in G incident to y, the subdivision vertex is blue but w must be red. Indeed, since w cannot be blue if $w \in H$ and if $w \in B$ then it cannot be blue because in that case we must have already considered this edge wy and thus H cannot be an non-visited large odd component. Hence y and v_{wy} form a blue P_2 together.

We can recolor all the remaining non-visited large odd components by the same argument. After all these steps a crumby coloring of $S(G)$ arises. □

Next, we complement the previous result. Here we allow all longer subdivisions.

Lemma 29. *Let G be a cubic graph. Let H be an arbitrary subdivision of G such that every edge is subdivided at least twice. The graph H admits a crumby coloring.*

Proof. Let us color the original vertices of G blue. We find that almost any subdivided edge admits a crumby coloring such that the end-vertices are singleton blues. The only exception is the case with 4 subdivision vertices. In particular, we use the following colorings for the internal vertices (r,b stands for red and blue, respectively): rr, rrr, $rrrb$, $rrbrr$, $rrrbrr$, $rrbbrrr$, $rrbrrbrr$ etc.

Let us use these particular colorings on H. We might create some blue stars with 2 or 3 leaves. Apart from that, this coloring satisfies the crumby conditions. Now we recolor the problematic blue centers of these stars red. If the vertex c is such a center, and there was a blue 3-star at c, then we recolor the neighbor n_1 of c red and recolor the neighbor n_2 of n_1 blue. If vertex c was the center of a blue 2-star, then we have to consider two cases according to the red neighbor v of c. If v was the end-vertex of a red P_3, then we do the same recoloring as in the previous case, but also recolor v to blue. If v was the end-vertex of a red P_2, then the recoloring of c creates a red P_3 and we are done.

The process terminates with a crumby coloring of H. □

Bibliography

[A] G. Ambrus, J. Barát, P. Hajnal. The slope parameter of graphs. *Acta Sci. Math. (Szeged)* **72**:3-4 (2006), 875–889.

[B] J. Barát, J. Matoušek, D. R. Wood. Bounded-degree graphs have arbitrarily large geometric thickness. *Electron. J. Combin.* **13** (2006), Research Paper 3, 14 pp.

[C] J. Barát, C. Thomassen. Claw-decompositions and Tutte-orientations. *J. Graph Theory* **52** (2006), 135–146.

[D] J. Barát, I. M. Wanless. A Cube Dismantling Problem Related to Bootstrap Percolation. *J. Stat. Phys.* **149**:4 (2012), 754–770.

[E] J. Barát, G. Tóth. Towards the Albertson conjecture. *Electron. J. Combin.* **17** (2010), Research Paper 73, 15 pp.

[F] J. Barát, D. Gerbner. Edge-decomposition of graphs into copies of a tree with four edges. *Electron. J. Combin.* **21**:1 (2014), Paper 1.55, 11 pp.

[G] J. Barát. Decomposition of cubic graphs related to Wegner's conjecture. *Discrete Math.* **342**:5 (2019), 1520–1527.

[H] J. Barát, Z. L. Blázsik, G. Damásdi. Crumby colorings — red-blue vertex partition of subcubic graphs regarding a conjecture of Thomassen. *Discrete Math.* **346** (2023), Paper No. 113281, 15 pages.

[9] E. Ackerman. On topological graphs with at most four crossings per edge. *Comput. Geom.* **85** (2019) Paper 101574, 31pp.
Updated version `https://arxiv.org/pdf/1509.01932.pdf`

[10] M. Aigner and G. Ziegler. Proofs from the Book. Springer-Verlag, Heidelberg, (2004), viii+239 pp.

[11] M. Ajtai, V. Chvátal, M. M. Newborn, and E. Szemerédi. Crossing-free subgraphs. In *Theory and practice of combinatorics*, *North-Holland Math. Stud.* **60**, North-Holland, Amsterdam (1982), 9–12.

[12] B. Albar, D. Gonçalves, K. Knauer. Orienting triangulations. *J. Graph Theory* **83** (2016), 392–405.

[13] M. O. Albertson, D. W. Cranston and J. Fox. Crossings, Colorings and Cliques. *Electron. J. Combin.*, **16** (2009), #R45.

[14] M. O. Albertson, M. Heenehan, A. McDonough, and J. Wise. Coloring graphs with given crossing patterns. (manuscript)

[15] N. Alon. The number of polytopes, configurations and real matroids. *Mathematatika*, **33**:1 (1986), 62–71.

[16] N. Alon, J. Kahn, D. Kleitman, M. Saks, P. Seymour and C. Thomassen. Subgraphs of large connectivity and chromatic number in graphs of large chromatic number. *J. Graph Theory* **3** (1987), 367–371.

[17] N. Alon, G. Ding, B. Oporowski, D. Vertigan. Partitioning into graphs with only small components. *J. Combin. Theory Ser. B*, **87** (2003), 231–243.

[18] G. Ambrus, J. Barát. A contribution to queens graphs: a substitution method. *Discrete Math.* **306**:12 (2006), 1105–1114.

[19] H. Amini. Bootstrap percolation in living neural networks. *J. Stat. Phys.* **141**:3 (2010), 459–475.

[20] J. Balogh, G. Pete. Random disease on the square grid. *Random Structures Algorithms* **13** (1998), 409–422.

[21] J. Balogh, B. Lidický, G. Salazar. Closing in on Hill's conjecture. *SIAM J. Discrete Math.* **33** (2019), 1261–1276.

[22] J. Barát. Directed path-width and monotonicity in digraph searching. *Graphs Combin.* **22**:2 (2006), 161–172.

[23] J. Barát, G. Joret and D. R. Wood. Disproving the list Hadwiger conjecture. *Electron. J. Combin.* **18** (2011), Paper 232, 7 pp.

[24] J. Barát, M. Korondi, V. Varga. How to dismantle an atomic cube in zero gravity. *Proceedings of the* 7*th Hungarian-Japanese Symposium on Discrete Mathematics and Its Applications.* Kyoto, Japan (2011), 9 pp.

[25] B. Barber, D. Kühn, A. Lo, D. Osthus. Edge-decompositions of graphs with high minimum degree. *Adv. Math.* **288** (2016), 337–385.

[26] L. W. Beineke, I. Broere, M. A. Henning. Queens graphs. *Discrete Math.* **206** (1999), 63–75.

[27] J. Bell, B. Stevens. A survey of known results and research areas for n-queens. *Discrete Math.* **309**:1 (2009), 1–31.

[28] T. Bellitto, T. Klimošová, M. Merker, M. Witkowski, Y. Yuditsky. Counterexamples to Thomassen's conjecture on decomposition of cubic graphs. *Graphs Combin.* **37** (2021), 2595–2599.

[29] E. A. Bender and E. R. Canfield. The asymptotic number of labeled graphs with given degree sequences. *J. Combin. Theory Ser. A*, **24** (1978), 296–307.

[30] J. Bensmail, A. Harutyunyan, T.-N. Le, M. Merker, S. Thomassé. A proof of the Barát-Thomassen conjecture. *J. Combin. Theory Ser. B* **124** (2017), 39–55.

[31] J. Bensmail, A. Harutyunyan, T.-N. Le, S. Thomassé. Edge-partitioning a graph into paths: beyond the Barát–Thomassen conjecture. *Combinatorica* **39** (2019), 239–263.

[32] R. Blankenship. *Book Embeddings of Graphs.* Ph.D. thesis, Department of Mathematics, Louisiana State University, U.S.A., (2003), 73 pp.

[33] R. Blankenship and B. Oporowski. Drawing subdivisions of complete and complete bipartite graphs on books. Tech. Rep. 1999-4, Department of Mathematics, Louisiana State University, U.S.A., (1999).

[34] B. Bollobás. Extremal Graph Theory. (Reprint of the 1978 original) *Dover Publications, Inc.*, Mineola, NY, (2004), xx+488 pages.

[35] J. A. Bondy and U. S R. Murty, Graph Theory. *Grad. Texts in Math.* **244** *Springer-Verlag*, London, (2008), XII+663 pages.

[36] F. Botler, G. O. Mota, M. Oshiro, Y. Wakabayashi. Decomposing highly connected graphs into paths of length five. *Discrete Appl. Math.* **245** (2018), 128–138.

[37] F. Botler, G. O. Mota, M. Oshiro, Y. Wakabayashi. Decomposing highly edge-connected graphs into paths of any given length. *J. Combin. Theory Ser. B* **122** (2017), 508–542.

[38] P. A. Catlin. Hajós' graph-coloring conjecture: variations and counterexamples. *J. Combin. Theory Ser. B* **26** (1979), 268–274.

[39] D. W. Cranston and S-J. Kim. List-coloring the Square of a Subcubic Graph. *J. Graph Theory*, **57**:1 (2008), 65–87.

[40] M. Dehn. Über den Starrheit konvexer Polyeder (in German). *Math. Ann.* **77** (1916), 466–473.

[41] M. Delcourt, L. Postle. Random 4-regular graphs have 3-star decompositions asymptotically almost surely. *Eur. J. Combin.* **72** (2018), 97–111.

[42] R. Diestel. Graph Theory. Fifth Edition. *Grad. Texts in Math.* **173** *Springer-Verlag*, Berlin (2018), xviii+428 pp.

[43] G. A. Dirac. A theorem of R. L. Brooks and a conjecture of H. Hadwiger. *Proc. London Math. Soc.* **7** (1957), 161–195.

[44] G. A. Dirac. The number of edges in critical graphs. *J. Reine Angew. Math.* **268/269** (1974), 150–164.

[45] D. Dor and M. Tarsi. Graph decomposition is NP-complete: a complete proof of Holyer's conjecture. *SIAM J. Comput.* **26** (1997), 1166–1187.

[46] V. Dujmović and D. R. Wood. On linear layouts of graphs. *Discrete Math. Theor. Comput. Sci.*, **6**:2 (2004), 339–358.

[47] C. A. Duncan, D. Eppstein, and S. G. Kobourov. The geometric thickness of low degree graphs. In *Proc. 20th ACM Symp. on Computational Geometry* (SoCG '04), ACM Press, (2004), 340–346.

[48] J. Edmonds. Minimum partition of a matroid into independent subsets. *J. Res. Nat. Bur. Standards Sect. B* **69B** (1965), 67–72.

[49] J. Edmonds. Paths, trees, and flowers. *Can. J. Math.* **17** (1965), 449–467.

[50] D. Eppstein. Separating thickness from geometric thickness. In: Towards a Theory of Geometric Graphs. *Contemp. Math.* **342**, AMS, (2004), 75–86.

[51] D. Eppstein. Separating geometric thickness from book thickness. (2001). `http://arXiv.org/math/0109195`.

[52] P. Erdős. Graph theory and probability. *Canad. J. Math.* **11** (1959), 34–38.

[53] P. Erdős, S. Fajtlowicz. On the conjecture of Hajós. *Combinatorica* **1** (1981), 141–143.

[54] T. Gallai. Kritische Graphen. II. (German) *Magyar Tud. Akad. Mat. Kutat Int. Közl.* **8** (1963), 373–395.

[55] T. Gallai. Maximale Systeme unabhänginger Kanten. *Magyar Tud. Akad. Mat. Kutató Int. Közl.* **9** (1964), 401–413.

[56] R.K. Guy. Crossing numbers of graphs. in: *Graph theory and applications (Proc. Conf. Western Michigan Univ., Kalamazoo, Mich.,* 1972*)* Lecture Notes in Mathematics **303**, Springer-Verlag, Berlin–New York, (1972), 111–124.

[57] E. Győri. On division of graphs to connected subgraphs. Combinatorics Vol. I. *Colloq. Math. Soc. János Bolyai* **18** (1978), 485–494.

[58] J. H. Halton. On the thickness of graphs of given degree. *Inform. Sci.*, **54**:3 (1991), 219–238.

[59] M. Hasanvand. The Existence of Planar 4-Connected Essentially 6-Edge-Connected Graphs with No Claw-Decompositions. *Graphs Combin.* **39**:1 (2023), 7 pp.

[60] P. Haxell, T. Szabó, G. Tardos. Bounded size components — partitions and transversals. *J. Comb. Theory Ser. B*, **88**:2 (2003), 281–297.

[61] D. G. Hoffman. The real truth about star designs. *Discrete Math.* **284** (2004), 177–180.

[62] A. E. Holroyd. The metastability threshold for modified bootstrap percolation in d dimensions. *Electron. J. Probab.* **11**:17 (2006), 418–433.

[63] F. Jaeger. Nowhere-zero flow problems. In: Selected Topics in Graph Theory, 3 edited by L. W. Beineke and R. J. Wilson. *Academic Press*, San Diego, CA, (1988) 71–95.

[64] S. Janson, T. Łuczak, A. Ruciński. Random Graphs. *Wiley-Interscience*, New York, (2000), xii+333 pp.

[65] T. R. Jensen and B. Toft. Graph Coloring Problems. *Wiley Interscience*, New York, (1995), xxii+295 pp.

[66] N. Juhász. Makkosházi matekverseny 1989–2008. (in Hungarian) (2009), p.96.

[67] S. Jukna. *Extremal combinatorics: with applications in computer science.* Texts in Theoret. Comput. Sci. Springer-Verlag, Berlin, (2001), xviii+375 pp.

[68] M. Jünger, G. Reinelt and W. R. Pulleyblank. On partitioning the edges of graphs into connected subgraphs. *J. Graph Theory* **9** (1985), 539–549.

[69] P. C. Kainen. Thickness and coarseness of graphs. *Abh. Math. Sem. Univ. Hamburg*, **39** (1973), 88–95.

[70] B. Keszegh, J. Pach, D. Pálvölgyi, G. Tóth. Cubic graphs have bounded slope parameter. *J. Graph Algorithms Appl.* **14**:1 (2010), 5–17.

[71] E. de Klerk, J. Maharry, D. V. Pasechnik, R. B. Richter, G. Salazar. Improved bounds for the crossing numbers of $K_{m,n}$ and K_n. *SIAM J. Discrete Math.* **20** (2006), 189–202.

[72] A. V. Kostochka and M. Stiebitz. Excess in color-critical graphs. Graph theory and combinatorial biology (Balatonlelle, 1996), 87–99, *Bolyai Soc. Math. Stud.* **7** János Bolyai Math. Soc., Budapest, (1999).

[73] H.-J. Lai and C.-Q. Zhang. Nowhere-zero 3-flows of highly connected graphs. *Discrete Math.* **110** (1992), 179–183.

[74] H.-J. Lai. Mod $(2p+1)$-Orientations and $K_{1,2p+1}$-Decompositions. *SIAM J. Discrete Math.* **21**:4 (2008), 844–850.

[75] H.-J. Lai, X. Li. Group chromatic number of planar graphs of girth at least 4. *J. Graph Theory* **52** (2006), 51–72.

[76] T. Leighton. Complexity Issues in VLSI. In: Foundations of Computing Series, MIT Press, Cambridge, MA, (1983).

[77] H. Lenz, G. Ringel. A brief review on Egmont Köhler's mathematical work. *Discrete Math.*, **97**:1–3 (1991), 3–16.

[78] L. Lovász. A homology theory for spanning trees of a graph. *Acta Math. Acad. Sci. Hungar.* **30**:3-4 (1977), 241–251.

[79] L. M. Lovász, C. Thomassen, Y. Wu, C.-Q. Zhang. Nowhere-zero 3-flows and modulo k-orientations. *J. Combin. Theory Ser. B* **103** (2013) 587–598.

[80] A. G. Luiz, R. B. Richter. Remarks on a Conjecture of Barát and Tóth. *Electron. J. Combin.* **21**:1 (2014), Paper 1.57, 7 pp.

[81] W. Mader. Existenz n-fach zusammenhängender Teilgraphen in Graphen genügend grosser Kantendichte. *Abh. Math. Sem. Univ. Hamburg* **37** (1972), 86–97.

[82] W. Mader. A reduction method for edge-connectivity in graphs. *Ann. Discrete Math.* **3** (1978), 145–164.

[83] W. Mader. $3n - 5$ edges do force a subdivision of K_5. *Combinatorica* **18** (1998), 569–595.

[84] S. M. Malitz. Graphs with E edges have pagenumber $O(\sqrt{E})$. *J. Algorithms*, **17**:1 (1994), 71–84.

[85] J. Matoušek. Lectures on discrete geometry. *Grad. Texts in Math.*, **212** Springer-Verlag, New York, (2002), xvi+481 pp.

[86] B. D. McKay. Asymptotics for symmetric 0-1 matrices with prescribed row sums. *Ars Combin.*, **19**(A) (1985), 15–25.

[87] B. D. McKay and I. M. Wanless. A census of small Latin hypercubes. *SIAM J. Discrete Math.* **22**:2 (2008), 719–736.

[88] M. Merker. Decomposing highly edge-connected graphs into homomorphic copies of a fixed tree. *J. Combin. Theory Ser. B* **122** (2017), 91–108.

[89] B. Mohar and C. Thomassen. Graphs on Surfaces. *Johns Hopkins University Press*, Baltimore, MD, (2001), xii+291 pp.

[90] J. Milnor. On the Betti numbers of real varieties. *Proc. Amer. Math. Soc.*, **15** (1964) 275–280.

[91] R. Morris. Minimal percolating sets in bootstrap percolation. *Electron. J. Combin.* **16**:1 (2009), Research Paper 2, 20 pp.

[92] P. Mutzel, T. Odenthal, and M. Scharbrodt. The thickness of graphs: a survey. *Graphs Combin.*, **14**:1 (1998), 59–73.

[93] C. St. J. A. Nash-Williams. Edge-disjoint spanning trees of finite graphs. *J. London Math. Soc.* **36** (1961), 445–450.

[94] B. Oporowski, D. Zhao. Coloring graphs with crossings. *Discrete Math.* **309**:9 (2009), 2948–2951.

[95] J. Pach, G. Tóth. Graphs drawn with few crossings per edge. Lecture Notes in Computer Science **1190**, Springer–Verlag, 1997, 345–354. Also in: Combinatorica **17** (1997), 427-439.

[96] J. Pach, R. Radoičić, G. Tardos, G. Tóth. Improving the crossing lemma by finding more crossings in sparse graphs. *Discrete Comput. Geom.* **36** (2006), 527–552.

[97] J. Pach, G. Tóth. Thirteen problems on crossing numbers. *Geombinatorics* **9** (2000), 194–207.

[98] J. Pach and R. Wenger. Embedding planar graphs at fixed vertex locations. *Graphs Combin.*, **17**:4 (2001), 717–728.

[99] I. G. Petrovskiĭ and O. A. Oleĭnik. On the topology of real algebraic surfaces. *Izvestiya Akad. Nauk SSSR. Ser. Mat.*, **13** (1949), 389–402.

[100] R. Pollack and M-F. Roy. On the number of cells defined by a set of polynomials. *C. R. Acad. Sci. Paris Sér. I Math.*, **316**:6 (1993), 573–577.

[101] B. Richter and C. Thomassen. Relations between crossing numbers of complete and complete bipartite graphs. *Amer. Math. Monthly* **104** (1997), 131-137.

[102] L. Rónyai, L. Babai, and M. K. Ganapathy. On the number of zero-patterns of a sequence of polynomials. *J. Amer. Math. Soc.*, **14**:3 (2001), 717–735.

[103] Gordon Royle's small graphs. A database presently unavailable. Moving possibly to Zenodo.

[104] F. Santos and R. Seidel. A better upper bound on the number of triangulations of a planar point set. *J. Combin. Theory Ser. A*, **102**:1 (2003), 186–193.

[105] F. Sausset, C. Toninelli, G. Biroli, G. Tarjus. Bootstrap percolation and kinetically constrained models on hyperbolic lattices. *J. Stat. Phys.* **138**:1-3 (2010), 411–430.

[106] W. Schnyder. Planar graphs and poset dimension. *Order*, **5** (1989), 323–343.

[107] O. Sýkora, L. A. Székely, and I. Vrto. A note on Halton's conjecture. *Inform. Sci.*, **164**:(1-4) (2004), 61–64.

[108] L. Székely. A successful concept for measuring non-planarity of graphs: the crossing number. *Discrete Math.* **276** (2004), 331–352.

[109] R. Thom. Sur l'homologie des variétés algébriques réelles. In S. S. Cairns, ed., *Differential and Combinatorial Topology*, Princeton Univ. Press, Princeton, NJ (1965), 255–265.

[110] C. Thomassen. Two-coloring the edges of a cubic graph such that each monochromatic component is a path of length at most 5. *J. Combin. Theory Ser. B*, **75** (1999), 100–109.

[111] C. Thomassen. Edge-decompositions of highly connected graphs. *Abh. Math. Semin. Univ. Hamburg* **18** (2008), 17–26.

[112] C. Thomassen. Decompositions of highly connected graphs into paths of length 3. *J. Graph Theory* **58** (2008), 286–292.

[113] C. Thomassen. The weak 3-flow conjecture and the weak circular flow conjecture. *J. Combin. Theory Ser. B* **102** (2012), 521–529.

[114] C. Thomassen. Decomposing graphs into paths of fixed length. *Combinatorica* **33**:1 (2013), 97–123.

[115] C. Thomassen. Decomposing a graph into bistars. *J. Combin. Theory Ser. B* **103** (2013), 504–508.

[116] C. Thomassen. The square of a planar cubic graph is 7-colorable. *J. Combin. Theory Ser. B* **128** (2017), 192–218.

[117] W. T. Tutte. On the problem of decomposing a graph into n connected factors. *J. London Math. Soc.* **36** (1961), 221–230.

[118] H. E. Warren. Lower bounds for approximation by nonlinear manifolds. *Trans. Amer. Math. Soc.*, **133** (1968), 167–178.

[119] G. Wegner. Graphs with given diameter and a coloring problem. *Preprint, University of Dortmund*, (1977), 1–11.
`http://dx.doi.org/10.17877/DE290R-16496`

[120] E. W. Weisstein. Generalized Petersen Graph. From *MathWorld*–A Wolfram Web Resource. `mathworld.wolfram.com/GeneralizedPetersenGraph.html`

[121] W. Wessel. Über die Abhängigkeit der Dicke eines Graphen von seinen Knotenpunktvalenzen. In *2nd Colloquium for Geometry and Combinatorics*, Technische Hochschule Karl-Marx-Stadt, (1983), 235–238.

[122] R. M. Wilson. Decomposition of complete graphs into subgraphs isomorphic to a given graph. *Congressus Numerantium* **XV** (1975), 647–659.

[123] N. Wormald. *Some problems in the enumeration of labelled graphs.* Ph.D. thesis, Newcastle-upon-Tyne, United Kingdom, (1978).

[124] A. A. Zykov. On some properties of linear complexes (in Russian). *Mat. Sbornik N. S.* **24** (1949), 163–188. Reprinted: *Translations Series* 1, *Algebraic Topology* (1962), 418–449, AMS, Providence.