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Covering, Approximation and Quantitative Helly-type Questions – A Transversal

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Contents

Introduction

At the age of 19, Gauss proved the constructablility of the regular heptadecagon, that is, 17-gon, an achievement that he was so proud of that he requested in his will that this figure be inscribed on his tombstone. After his death, the stonemason Howaldt refused, explaining that it was technically impossible to carve a regular heptadecagon, since *it is essentially a circle.*[1]

The topic of this dissertation is the validity of Howaldt's claim — as long as we interpret it as one of the early examples of a geometric approximation problem. Approximation and discretization concepts appear in several fields of mathematics and computer science. Their use in applied sciences is as old as mathematics itself, and when recently, the need to process large data emerged, their importance only grew further.

We will discuss three families of problems, forming parts I, II and III of this dissertation. The first one is *approximation of a convex body by a polytope* of low complexity, eg. one with few vertices. A sample problem that is easy to state is estimating the minimum number of vertices of a d-dimensional polytope which contains the ball of radius $1/2$, and is contained in the unit radius ball.

The second family of problems is *covering* a convex set by few translates of another set. As a famous example, we mention the Illummination Conjecture which can be stated as claiming that every d-dimensional convex body can be covered by 2^d translates of its own interior. Another exaple is the problem of finding a low density covering of the whole space by translates of a given set.

Finally, we will discuss *approximating the intersection* of a family of convex sets by the intersection of a small subfamily. This question, in the case when the sets are all half-spaces, may be rephrased as follows. How do we select few facet hyperplanes of a polyhedron in d-dimensional Euclidean space such that the larger polyhedron that these hyperplanes bound is not too much larger than the original one.

This dissertation focuses on problems and methods that fall into the broad

[1] See [WD04, p. 28].

categories of combinatorial questions regarding hypergraphs, analytical volumetric arguments and the study of convex sets in finite dimensional real spaces.

The modern study of the best approximation of convex bodies by polytopes goes back to at least the 1950s. Extending a planar result of Sas, Macbeath [Mac51] showed in 1951 by a symmetrization technique that among all convex bodies in $\mathbb{R}^d$, the Euclidean ball is the worst approximable *in volume* by an inscribed polytope of a given number of vertices. Bronshtein and Ivanov [BI75] proved in 1975 that any convex body K of diameter 1 can be approximated *in the Hausdorff distance* by an inscribed polytope of $e^{O(d)}O(\varepsilon^{-\frac{d-1}{2}})$ vertices, while Dudley [Dud10] independently showed the same bound for the number of facets. Their results were strengthened in Arya, Fonseca and Mount [AdFM12] in the consant dimensional case, where an algorithmic proof is given. Bárány and Larman [BL88, Bár89] studied the relationship between the volume of the floating body and the efficiency of polytopal approximation. The surveys of Bronshtein [Bro07], Gruber [Gru93] and Bárány [Bár07] provide more details of the history of the field.

The area studying approximation of convex bodies by polytopes consists of several families of problems that may be classified according to a few properties. As a first classifying property, we need to fix a notion of distance on the set of convex bodies. The two most widely used notions are the Hausdorff distance and the Banach–Mazur distance. The former is the natural choice to measure the quality of approximation for problems with a fixed Euclidean structure. The latter is often a better choice for affine invariant questions. Results for one notion of distance can usually be translated to the setting of the other, under some assumptions on the diameter of the sets in question, or the largest ball (or ellipsoid) contained in them. Since we are primarily interested in affine invariant questions, we will mostly consider a relative of the Banach–Mazur distance, the *homothetic distance* defined as

$$d(K, L) = \inf\{\lambda \geq 1 \ : \ \alpha K \subseteq L \subseteq \lambda\alpha K \text{ for some } \alpha > 0\},$$

for two convex bodies (compact, convex sets with non-empty interior) K and L in $\mathbb{R}^d$, both containing the origin in their interior. Note that $d(K, L)$ is sensitive to the choice of the origin (that is, it is not translation invariant). Moreover, the multiplicative triangle inequality holds for it, and thus, one can say that $\ln(d(K, L))$ is a metric on the set of equivalence classes of convex bodies containing the origin in their interior, where two convex bodies are considered equivalent, if one is a positive magnified copy of the other with the origin as the center of magnification.

The second classifying property of polytopal approximation questions is the choice of a notion of complexity for polytopes that we aim to minimize.

We will consider the two most natural measures of complexity, the number of vertices and that of facets (that is, $(d-1)$-dimensional faces).

The third classifying property is the role of the dimension: in some studies, d is fixed, while in others, it tends to infinity. We study problems of the second kind, and thus, in summary, we study the dependence of three quantities: the dimension, the number of vertices of the polytope and the error of the approximation measured in terms of the (homothetic) distance of the polytope to the convex body.

Part I consists of two main results, both concerning the following question. Given a convex body K in $\mathbb{R}^d$ whose center of mass is the origin (we call it *centered*), a positive integer $t \geq d+1$, and $\lambda > 1$. Our goal is to find a convex polytope P with at most t vertices satisfying

$$P \subseteq K \subseteq \lambda K.$$

In Chapter 1, based on [Nas19], the polytope is obtained as the convex hull of a random set of points from K chosen according to the *uniform distribution on* K.

Brazitikos, Chasapis, and Hioni [BCH16] studied the case of very rough approximation, that is, where the number t of chosen points is linear in the dimension d. It states that the convex hull of $t = \alpha d$ random points in a centered convex body K is a convex polytope P which satisfies $\frac{c_1}{d} K \subseteq P$, with probability $1 - \delta = 1 - e^{-c_2 d}$, where $c_1, c_2 > 0$ and $\alpha > 1$ are absolute constants. In our first result, we obtain explicit constants.

Theorem 1.1 (Naszódi [Nas19]). *Let K be a centered convex body in $\mathbb{R}^d$, and choose $t = 60(d+1)$ points $X_1, \ldots, X_t$ of K randomly, independently and uniformly. Then*

$$\frac{1}{d} K \subseteq \operatorname{conv}(X_1, \ldots, X_t) \subseteq K.$$

with probability at least $1 - 4e^{-d-1}$.

Another instance of our general problem is Theorem 5.2 of [GM00] by Giannopoulos and V. Milman, which concerns fine approximation, that is, where the number t of chosen points is exponential in the dimension d. It states that *for any $\delta, \gamma \in (0,1)$, if we choose $t = e^{\gamma d}$ random points in any centered convex body K in $\mathbb{R}^d$, then, for sufficiently large d, the convex polytope P thus obtained satisfies $c(\delta)\gamma K \subseteq P$, with probability $1 - \delta$.*

Proposition 5.3 of [GM00] follows from the same argument as Theorem 5.2 therein. It states that *for any $\delta, \vartheta \in (0,1)$, if we choose $t = c(\delta)\left(\frac{c}{1-\vartheta}\right)^d$*

random points in any centered convex body K in $\mathbb{R}^d$, then the convex polytope P thus obtained satisfies $\vartheta K \subseteq P$, with probability $1-\delta$.

The main result of Chapter 1 is the following common generalization of the results above.

Theorem 1.2 (Naszódi [Nas19]). *Let $\vartheta \in (0,1), C \geq 2$. Set*

$$t := \left\lceil C\frac{(d+1)e}{(1-\vartheta)^d} \ln \frac{e}{(1-\vartheta)^d} \right\rceil .$$

Then for any centered convex body K in $\mathbb{R}^d$, if t points $X_1, \dots, X_t$ of K are chosen randomly, independently and uniformly, then

$$\vartheta K \subseteq \operatorname{conv}(X_1, \dots, X_t) \subseteq K$$

with probability at least $1-\delta$, where

$$\delta := 4\left[11C^2\left(\frac{(1-\vartheta)^d}{e}\right)^{C-2}\right]^{d+1}.$$

Chapter 2 is based on joint work with Fedor Nazarov and Dmitry Ryabogin [NNR20]. The problem, as in Chapter 1 is to approximate a convex body K in $\mathbb{R}^d$ by a polytope obtained as the convex hull of a randomly chosen set of points. However in Chapter 2, *the probability distribution that yields the points is not prescribed*, we are free to choose it smartly. The main result is the following.

Theorem 2.1 (Nazarov, Ryabogin, Naszódi, [NNR20]). *Let K be a convex body in $\mathbb{R}^d$ with the center of mass at the origin, and let $\varepsilon \in \left(0, \frac{1}{2}\right)$. Then there exists a convex polytope P with at most $e^{O(d)}\varepsilon^{-\frac{d-1}{2}}$ vertices such that $(1-\varepsilon)K \subset P \subset K$.*

It improves the 2012 theorem of Barvinok [Bar14] by removing the symmetry assumption on K and the extraneous $(\log \frac{1}{\varepsilon})^d$ factor appearing therein.

As a **follow-up of the material discussed in Part I**, we mention that both theorems 1.2 and 2.1 are discussed in the book chapter [Mil23] by V. Milman, and a wonderful algorithmic proof for Theorem 2.1 is presented in [AdFM23] by Arya, Fonseca and Mount.

Ergür [Erg19] used the results in [Nas19] both as inspiration and for the proof of a result therein. Mustafa included Theorem 1.2 and its proof in his monograph [Mus22] on sampling.

Part II contains our results on covering. The general problem is the following.

Given two sets K and L in $\mathbb{R}^d$ (resp. $\mathbb{S}^d$), our goal is to cover K by as few translates (resp., rotated copies) of L as possible.

A landmark result in the theory of coverings in geometry is the following theorem of Rogers [Rog57]. For a definition of the covering density, cf. [Rog64].

Theorem 3.6 (Rogers, [Rog57]). *Let K be a bounded convex set in $\mathbb{R}^d$ with non-empty interior. Then $\mathbb{R}^d$ can be covered by translates of K with density at most*

$$d \ln d + d \ln \ln d + 5d.$$

Earlier, exponential upper bounds for the translative covering density were obtained by Rogers, Bambah and Roth, and for the special case of the Euclidean ball by Davenport and Watson (cf. [Rog57] for references). The best bound is due to G. Fejes Tóth [FT09], who replaced the last term $5d$ by $d + o(d)$.

Another classical example of a geometric covering problem is the following. Estimate the minimum number of spherical caps of radius φ needed to cover the sphere $\mathbb{S}^d$ in $\mathbb{R}^{d+1}$.

Theorem 3.7 (Böröczky Jr. and Wintsche, [BW03]). *Let $0 < \varphi < \frac{\pi}{2}$. Then there is a covering of $\mathbb{S}^d$ by spherical caps of radius φ with density at most $d \ln d + d \ln \ln d + 5d$.*

This estimate was proved in [BW03] improving an earlier result of Rogers [Rog63]. The current best bound is better, when $\varphi < \frac{\pi}{3}$: Dumer [Dum07] (see [Dum18] for a corrected version) gave a covering in this case of density at most $\frac{d \ln d}{2}$.

For two Borel measurable sets K and L in $\mathbb{R}^d$, let $N(K, L)$ denote the *translative covering number* of K by L ie. the minimum number of translates of L that cover K. The fractional version $N^*(K, L)$ of $N(K, L)$ in the special case when $L = \operatorname{int}(K)$ (see Definition 3.8 on p. 44) first appeared in [Nas09] and in general for $N(K, L)$ in [AAR11] and [AAS15].

The main result, Theorem 3.2, of **Chapter 3**, which is based on [Nas16a], bounds from above the covering number $N(K, L)$ in terms of its fractional version, $N^*(K, L)$. Since it is somewhat technical, here we state a corollary first.

Theorem 3.3 (Naszódi [Nas16a]). *Let $K \subseteq \mathbb{R}^d$ be a bounded measurable set. Then there is a covering of $\mathbb{R}^d$ by translated copies of K of density at most*

$$\inf_{\delta > 0} \left[\frac{\operatorname{vol}(K)}{\operatorname{vol}(K_{-\delta})} \left(1 + \ln \frac{\operatorname{vol}\left(K_{-\delta/2}\right)}{\operatorname{vol}\left(\frac{\delta}{2}\mathbf{B}^d\right)} \right) \right],$$

where $\mathbf{B}^d$ *denotes the closed Euclidean unit ball in* $\mathbb{R}^d$ *centered at the origin, and* $K_{-\delta}$ *is the inner parallel body of* K *with parameter* δ *(see p. 42 for the definition).*

We prove a similar general statement for covering a set on the sphere by rotated copies of another set.

For two sets $K, T \subset \mathbb{R}^d$, we denote their *Minkowski difference* by $K \sim T = \{x \in \mathbb{R}^d \ : \ T + x \subseteq K\}$. The main result of the section follows.

Theorem 3.2 (Naszódi [Nas16a]). *Let* K, L *and* T *be bounded Borel measurable sets in* $\mathbb{R}^d$ *and let* $\Lambda \subset \mathbb{R}^d$ *be a finite set with* $K \subseteq \Lambda + T$*. Then*

$$N(K, L) \leq$$

$$(1 + \ln(\max_{x \in K-L} \operatorname{card}((x + (L \sim T)) \cap \Lambda))) \cdot N^*(K - T, L \sim T).$$

If $\Lambda \subset K$*, then we have*

$$N(K, L) \leq$$

$$(1 + \ln(\max_{x \in K-L} \operatorname{card}((x + (L \sim T)) \cap \Lambda))) \cdot N^*(K, L \sim T).$$

A result very similar to our Theorem 3.2 appeared as Theorem 1.6 in the paper [AAS15] by Artstein-Avidan and Slomka. The main differences are the following. Quantitatively, our result is somewhat stronger by having $\max \operatorname{card}(\dots)$ in the logarithm as opposed to $\operatorname{card}(\Lambda)$. This allows us to obtain Theorem 3.6 of Rogers and Theorem 3.3 as corollaries to Theorem 3.2. Furthermore, we have no minor term of order $\sqrt{\ln(\operatorname{card}(\Lambda))(N^* + 1)}$.

The method of the proof in [AAS15] consist of two parts. One is to reduce the problem to a finite covering problem by replacing K by a sufficiently dense finite set (a δ-net). Next, a *probabilistic argument* is used to solve the finite covering problem. A similar route is followed in [FK08] where a variant of Theorem 3.6 (previously obtained by Erdős and Rogers [ER61]) is proved (using Lovász's Local Lemma) according to which such low density covering of $\mathbb{R}^d$ by translates of K exists with the additional requirement that no point is covered too many times. An even earlier appearance of this method in the context of the illumination problem can be found in [Sch88] by Schramm. A major contribution of [AAS15] is that they used this method to bridge the gap between N and N^*, that is, they noticed that the method works with any measure, not just the volume.

In Chapter 3, we also use the first part of the method (taking a δ-net), but then replace the second, quite technical, probabilistic part by a *simple application of a deterministic (non-probabilistic) algorithmic result*, Lemma 3.9, due to Lovász [Lov75], D. Johnson [Joh74] and Stein [Ste74], independently.

This yields proofs that are considerably simpler than previous ones, we obtain somewhat more general results, and we can separate very clearly the volumetric and the combinatorial arguments from each other.

Chapter 4, based on [Nas16c], considers the opposite problem: giving a *lower bound* for a certain covering number.

Following Hadwiger [Had60], we say that a *point* $p \in \mathbb{R}^d \setminus K$ *illuminates* a boundary point $b \in \mathrm{bd}\,(K)$ of the convex body K, if the ray $\{p + \lambda(b-p) : \lambda > 0\}$ emanating from p and passing through b intersects the interior of K. Boltyanski [Bol60] gave the following slightly different definition. A *direction* $u \in \mathbb{S}^{d-1}$ is said to *illuminate* K at a boundary point $b \in \mathrm{bd}\,(K)$ if the ray $\{b + \lambda u : \lambda > 0\}$ intersects the interior of K. It is easy to see that the minimum number of directions that illuminate each boundary point of K is equal to the minimum number of points that illuminate each boundary point of K. This number is called the *illumination number* $\mathrm{i}(K)$ of K.

We call a set of the form $\lambda K + v$ a *smaller positive homothet of* K if $0 < \lambda < 1$ and $v \in \mathbb{R}^d$. Gohberg and Markus asked how large the minimum number of smaller positive homothets of K covering K can be. It is not hard to see that this number is equal to $N(K, \mathrm{int}\,(K))$. It is also easy to see that $\mathrm{i}(K) = N(K, \mathrm{int}\,(K))$.

Any smooth convex body (ie., a convex body with a unique support hyperplane at each boundary point) in $\mathbb{R}^d$ is illuminated by $d+1$ directions. Indeed, for a smooth body, the set of directions illuminating a given boundary point is an open hemisphere of $\mathbb{S}^{d-1}$, and one can find $d+1$ points (eg., the vertices of a regular simplex inscribed in the sphere) in $\mathbb{S}^{d-1}$ with the property that every open hemisphere contains at least one of the points. Thus, these $d+1$ points in $\mathbb{S}^{d-1}$ (ie., directions) illuminate any smooth convex body in $\mathbb{R}^d$ (cf. [BMS97] for details).

On the other hand, the illumination number of the cube is 2^d, since no two vertices of the cube share an illumination direction. Even though we do not discuss it any further in the present work, it would be difficult to avoid mentioning the famous *Gohberg–Markus–Levi–Boltyanski–Hadwiger Conjecture* (or, Illumination Conjecture), according to which for any convex body K in $\mathbb{R}^d$, we have $\mathrm{i}(K) \leq 2^d$, where equality is attained only when K is an affine image of the cube.

For more background on the problem of illumination, see [Bez06, Bez10, BMP05, MS99]. In Chapter VI. of [BMS97], one can find a proof of the equivalence of the four definitions of $\mathrm{i}(K)$ given above.

The Euclidean ball is a smooth convex body, and hence, is of illumination number $d+1$. The main result of Chapter 4 shows that, arbitrarily close to the Euclidean ball, there is a convex body of much larger illumination number.

Theorem 4.1 (Naszódi [Nas16c]). *Let* $1 < D < 1.116$ *be given. Then for any sufficiently large dimension* d*, there is an origin symmetric convex body* K *in* $\mathbb{R}^d$*, with illumination number*

$$\mathrm{i}(K) = N(K, \mathrm{int}\,(K)) \geq .05D^d, \tag{1}$$

for which

$$\frac{1}{D}\mathbf{B}^d \subset K \subset \mathbf{B}^d.$$

We will use a probabilistic construction to find K. We are not aware of any lower bound for the illumination number that was obtained by a probabilistic argument.

In Theorem 4.2, we give an upper bound for the illumination number for bodies close to the Euclidean ball. It follows from [BK09] but, for the sake of completeness, we will sketch an argument showing that it easily follows from the results of Chapter 3.

Theorem 4.2 (Naszódi [Nas16c]). *Let* K *be a convex body in* $\mathbb{R}^d$ *such that* $\frac{1}{D}\mathbf{B}^d \subset K \subset \mathbf{B}^d$ *for some* $D > 1$*. Then the illumination number of* K *is at most*

$$\mathrm{i}(K) \leq \frac{d \ln d + d \ln \ln d + 5d}{\Omega_{d-1}(\alpha)},$$

where $\alpha = \arcsin(1/D)$*, and* $\Omega_{d-1}(\alpha)$ *denotes the probability measure on the sphere* $\mathbb{S}^{d-1}$ *of a cap of spherical radius* α*.*

By combining Theorem 4.2 with the estimate (3.8) on Ω_{d-1}, one can see that (1) is asymptotically sharp, that is, the base D cannot be improved.

Next, we discuss an application of Theorem 4.1. Let K be an origin-symmetric convex body in $\mathbb{R}^d$, and denote its gauge function (also called distance function) by $\|\cdot\|_K$ (that is, $\|p\|_K = \inf\{\lambda > 0 \;:\; p \in \lambda K\}$, for any $p \in \mathbb{R}^d$). The *illumination parameter*, introduced by K. Bezdek [Bez06], is defined as

$$\mathrm{ill}(K) =$$

$$\inf\left\{\sum_{p\in \mathrm{vert}\,P} \|p\|_K \mid P \text{ a polytope whose vertices illuminate } K\right\}.$$

The *vertex index* of K, introduced by K. Bezdek and Litvak [BL07], is

$$\mathrm{vein}(K) = \inf\left\{\sum_{p\in \mathrm{vert}\,P} \|p\|_K \mid P \text{ a polytope such that } K \subseteq P\right\}.$$

Clearly, $\mathrm{ill}(K) \geq \mathrm{vein}(K)$ for any centrally symmetric body K, and they are equal for smooth bodies. It is shown in [BL07] that $\mathrm{vein}(\mathbf{B}^d)$ is of order $d^{3/2}$ (see also [GL12]).

By (4.2), for the body K constructed in Theorem 4.1 we have that $\mathrm{vein}(K)$ is of order $d^{3/2}$, while $\mathrm{ill}(K) \geq \mathrm{i}(K)$ is exponentially large in d.

Thus, Theorem 4.1 yileds that $\mathrm{ill}(K)$ and $\mathrm{vein}(K)$ are very far from each other for some K.

As a **follow-up of the material discussed in Part II**, the method developed in [Nas16a] was applied for example in [Pro20] and [Pro18] by Prosanov, and in [EV22] by Eisenbrand and Venzin.

Part III is a collection of quantitative Helly-type results. According to Helly's Theorem, *if the intersection of any $d+1$ members of a finite family of convex sets in $\mathbb{R}^d$ is non-empty, then the intersection of all members of the family is non-empty.*

A quantitative variant was introduced by Bárány, Katchalski and Pach [BKP82], whose *Quantitative Volume Theorem* states the following. *Assume that the intersection of any $2d$ members of a finite family of convex sets in $\mathbb{R}^d$ is of volume at least 1. Then the volume of the intersection of all members of the family is of volume at least c_d, a constant depending only on d.*

In [BKP82], it is shown that one may take $c_d = d^{-2d^2}$, and conjectured that it should hold with $c_d = d^{-cd}$ for an absolute constant $c > 0$.

The main result of **Chapter 5** is the confirmation of this conjecture with $c_d \approx d^{-2d}$.

Theorem 5.1 (Naszódi [Nas16b]). *Let $\mathcal{F}$ be a family of convex sets in $\mathbb{R}^d$ such that the volume of its intersection* $\mathrm{vol}\,(\cap\mathcal{F})$ *is positive. Then there is a subfamily $\mathcal{G}$ of $\mathcal{F}$ with* $\mathrm{card}\,(\mathcal{G}) \leq 2d$ *and* $\mathrm{vol}\,(\cap\mathcal{G}) \leq e^{d+1}d^{2d+\frac{1}{2}}\,\mathrm{vol}\,(\cap\mathcal{F})$.

According to a remark from [BKP82] (see also [BKP84]), the number $2d$ is optimal, as shown by the $2d$ half-spaces supporting the cube at its facets.

The order of magnitude d^{cd} in the Theorem (and in the conjecture in [BKP82]) is sharp: the statement does not hold with any constant $0 < c < 1/2$ in the expression d^{cd}, as we show in Section 5.3.

In Chapter 6, which is based on joint work [DFN20] with Damásdi and Földvári, who were Msc students under my supervision, we discuss a generalization of Helly's Theorem, known as the *Colorful Helly Theorem*, proved by Lovász, and later by Bárány [Bár82]. It states that *if $\mathcal{C}_1, \ldots, \mathcal{C}_{d+1}$ are finite families (color classes) of convex sets in $\mathbb{R}^d$, such that for any colorful selection $C_1 \in \mathcal{C}_1, \ldots, C_{d+1} \in \mathcal{C}_{d+1}$, the intersection $\bigcap_{i=1}^{d+1} C_i$ is non-empty, then for some j, the intersection $\bigcap_{C\in\mathcal{C}_j} C$ is also non-empty.*

We present quantitative variants of this colorful theorem. The main result of Chapter 6 is the following.

Theorem 6.3 (Damásdi, Földvári, Naszódi [DFN20]). *Let $\mathcal{C}_1, \dots, \mathcal{C}_{3d}$ be finite families of convex bodies in $\mathbb{R}^d$. Assume that for any colorful selection of $2d$ sets, $C_{i_k} \in \mathcal{C}_{i_k}$ for each $1 \leq k \leq 2d$ with $1 \leq i_1 < \dots < i_{2d} \leq 3d$, the intersection $\bigcap_{k=1}^{2d} C_{i_k}$ is of volume at least 1.*

Then, there exists an $1 \leq i \leq 3d$ such that $\operatorname{vol}\left(\bigcap_{C \in \mathcal{C}_i} C\right) \geq c^{d^2} d^{-7d^2/2}$ *with an absolute constant $c \geq 0$.*

For the historical context, we recall that in 1937, Behrend [Beh37] (see also Section 6.17 of the survey [DGK63] by Danzer, Grünbaum and Klee) proved a planar quantitative Helly-type result: *If the intersection of any 5 members of a finite family of convex sets in $\mathbb{R}^2$ contains an ellipse of area 1, then the intersection of all members of the family contains an ellipse of area 1.* We note that, since every convex set in $\mathbb{R}^2$ is the intersection of the half-planes containing it, the result is equivalent to the formally weaker statement where the family consists of half-planes only. This is the form in which it is stated in [DGK63].

In [DGK63, Section 6.17], it is mentioned that John's Theorem (Lemma 5.3) should be applicable to extend Behrend's result to higher dimensions. In Proposition 6.4, we spell out this argument, and present a straightforward proof.

Proposition 6.4 (Helly-type Theorem with Ellipsoids). *Let $\mathcal{C}$ be a finite family of at least $d(d+3)/2$ convex sets in $\mathbb{R}^d$, and assume that for any selection $C_1, \dots, C_{d(d+3)/2} \in \mathcal{C}$, the intersection $\bigcap_{i=1}^{d(d+3)/2} C_i$ contains an ellipsoid of volume 1. Then $\bigcap_{C \in \mathcal{C}} C$ also contains an ellipsoid of volume 1.*

The number $d(d+3)/2$ is best possible. Indeed, for every dimension d, there exists a family of $d(d+3)/2$ half-spaces such that the unit ball $\mathbf{B}^d$ is the maximum volume ellipsoid contained in their intersection, but $\mathbf{B}^d$ is not the maximum volume ellipsoid contained in the intersection of any proper subfamily of them. That is, the intersection of any subfamily of $d(d+3)/2-1$ members contains an ellipsoid of larger volume than the volume of $\mathbf{B}^d$ (which we denote by $\omega_d = \operatorname{vol}(\mathbf{B}^d)$), and yet, the intersection of all members of the family does not contain an ellipsoid of volume larger than ω_d. This follows from the much stronger result, Theorem 4 in [Gru88] by Gruber.

Our Proposition 6.5 is a colorful variant of the result above.

Sarkar, Xue and Soberón [SXS21, Corollary 1.0.5], using matroids, recently obtained a result involving $d(d+3)/2$ color classes, but with the number of selected sets being $2d$.

Theorem (Sarkar, Xue and Soberón [SXS21]). *Let $\mathcal{C}_1, \dots, \mathcal{C}_{d(d+3)/2}$ be finite families of convex bodies in $\mathbb{R}^d$. Assume that for any colorful selection of $2d$ sets, $C_{i_k} \in \mathcal{C}_{i_k}$ for each $1 \le k \le 2d$ with $1 \le i_1 < \dots < i_{2d} \le d(d+3)/2$, the intersection $\bigcap_{k=1}^{2d} C_{i_k}$ contains an ellipsoid of volume at least 1. Then, there exists an $1 \le i \le d(d+3)/2$ such that $\bigcap_{C \in \mathcal{C}_i} C$ has volume at least $d^{-O(d)}$.*

Observe that the smaller the number of color classes in a colorful Helly-type theorem, the stronger the theorem. For example, the Colorful Helly Theorem (see p. 9) is stated with $d+1$ color classes, but it is easy to see that it implies the same result with $\ell \ge d+2$ color classes, as the last $\ell-(d+1)$ color classes make the assumption of the theorem stronger and the conclusion weaker. We note also that the Colorful Helly Theorem does not hold with less than $d+1$ color classes, as the number $d+1$ cannot be replaced by any smaller number in Helly's Theorem.

The above theorem by Sarkar, Xue and Soberón, and similar statements can be obtained by taking the Quantitative Volume Theorem as a “basic” Helly-type theorem, and combining it with John's Theorem and a combinatorial argument. The combinatorial argument heavily relies on the fact that if $\mathcal{E}$ is the largest volume ellipsoid contained in the intersection of a finite family of sets, then there is a subfamily of size $d(d+3)/2$, for whose intersection, $\mathcal{E}$ is the largest volume ellipsoid as well. Furthermore, this does not hold in general with any number below $d(d+3)/2$. Thus, this approach yields results with $d(d+3)/2$ color classes, but does not seem to be applicable in the case of fewer color classes. Our Theorem 6.3 is the only result to date with number of color classes linear in d.

We find it an intriguing question whether one can decrease the number of color classes to $2d$ (possibly with an even weaker bound on the volume of the ellipsoid obtained), and whether an order d^{-cd} lower bound on the volume of the ellipsoid can be shown.

In **Chapter 7**, which is based on joint work with G. Ivanov [IN24], we establish a quantitative version of the following classical result of E. Steinitz [Ste13]. *Let the origin belong to the interior of the convex hull of a set $Q \subset \mathbb{R}^d$. Then there is a subset of at most $2d$ points of Q whose convex hull contains the origin in the interior.*

The first quantitative version of this result was obtained in [BKP82], where the following statement was proven.

Theorem (Quantitative Steinitz theorem). *There exists a constant $r = r(d) > 0$ such that for any subset Q of $\mathbb{R}^d$ whose convex hull contains the*

Euclidean unit ball $\mathbf{B}^d$*, there exists a subset* F *of* Q *of size at most* $2d$ *whose convex hull contains the ball* $r\mathbf{B}^d$*.*

It was also shown that $r(d) > d^{-2d}$.

With the exception of the planar case $d = 2$ [KMY92, Bra97, BH94], no significant improvement on $r(d)$ has been obtained (see also [DLLHRS17]).

The main result of Chapter 7 is a polynomial bound on $r(d)$.

Theorem 7.3 (G. Ivanov, Naszódi [IN24]). *Let* Q *be a subset of* $\mathbb{R}^d$ *whose convex hull contains the Euclidean unit ball* $\mathbf{B}^d$*. Then there exists a set of at most* $2d$ *points of* Q *whose convex hull* Q' *satisfies*

$$\frac{1}{6d^2}\mathbf{B}^d \subset Q'.$$

We conjecture the following.

Conjecture 7.4. *There is a constant* $c > 0$ *such that in any subset* Q *of* $\mathbb{R}^d$ *whose convex hull contains the Euclidean unit ball* $\mathbf{B}^d$*, there are at most* $2d$ *points whose convex hull* Q' *satisfies*

$$\frac{c}{\sqrt{d}}\mathbf{B}^d \subset Q'.$$

We provide an upper bound on $r(d)$ applying the following, more general theorem.

Theorem 7.5 (G. Ivanov, Naszódi [IN24]). *Let* $u_1, \ldots, u_n$ *be unit vectors in* $\mathbb{R}^d$*. Then their absolute convex hull, that is, the convex hull of* $\{\pm u_1, \ldots, \pm u_n\}$ *does not contain the ball* $\left(\frac{\sqrt{n}}{d} + \varepsilon\right)\mathbf{B}^d$ *for any positive* ε*.*

It follows that if $u_1, \ldots, u_m$ form a sufficiently dense subset of the unit sphere (with a large m), then their convex hull is almost the unit ball, while for any n of them with $n \leq 2d$, we have that their convex hull does not contain the ball $\frac{2}{\sqrt{d}}\mathbf{B}^d$, which shows that the order of magnitude of $r(d)$ in Conjecture 7.4 is sharp if the conjecture holds.

Finally, in **Chapter 8**, which is based on joint work with G. Ivanov [IN22b], we extend the notion of the largest volume ellipsoid contained in a convex body in $\mathbb{R}^d$ (that is, the *John ellipsoid*) to the setting of logarithmically concave functions, and present applications, most notably, a quantitative Helly-type result for the integral of the pointwise minimum of a family of logarithmically concave functions. The John ellipsoid plays a central role throughout this work, and thus, a significant portion of the ideas from earlier chapters re-appears here.

Understanding logarithmically concave functions on $\mathbb{R}^d$ (log-concave functions in short) is a natural extensions of the study of convex sets in $\mathbb{R}^d$. Indeed, if K is a convex body in $\mathbb{R}^d$, then its indicator function is log-concave.

More interestingly, if K is origin symmetric, then the function $f(x) = e^{-\|x\|_K}$ is log-concave, where $\|\cdot\|_K$ denotes the norm on $\mathbb{R}^d$ whose unit ball is K. The density function of the normal distribution, $f(x) = c_d e^{-|x|^2/2}$ is again log-concave. And, perhaps most importantly, if $\hat{K}$ is a convex body in $\mathbb{R}^{d+k}$, then the marginal on $\mathbb{R}^d$ (considered as the subspace spanned by the first d coordinate axes) of the uniform measure on $\hat{K}$ is again a log-concave function on $\mathbb{R}^d$.

The rough idea of the John ellipsoid of a logarithmically concave function f on $\mathbb{R}^d$ is the following. Consider the graph of the function $f^{1/s}$, which is a set in $\mathbb{R}^{d+1}$, and turn it into a not necessarily convex body in $\mathbb{R}^{d+1}$, which we call the *s-lifting* of f. We define also a measure-like quantity, the *s-volume* of sets in $\mathbb{R}^{d+1}$. Then we look for the ellipsoid in $\mathbb{R}^{d+1}$ which is contained in the s-lifting of f and is of maximal s-volume. We call this ellipsoid in $\mathbb{R}^{d+1}$ the John s-ellipsoid of f. This ellipsoid defines a function on $\mathbb{R}^d$, which is the John s-ellipsoid function of f. This function is pointwise less than or equal to f.

Alonso-Gutiérrez, Merino, Jiménez and Villa [AGMJV18] were the first to associate an ellipsoid with a logarithmically concave function, which is an ellipsoid in $\mathbb{R}^d$, which we will call *the AMJV ellipsoid*. It is defined as follows.

For every $\beta \geq 0$, consider the superlevel set $\{x \in \mathbb{R}^d : f(x) \geq \beta\}$ of f. This is a bounded convex set with non-empty interior, we take its largest volume ellipsoid, and multiply the volume of this ellipsoid by β. As shown in [AGMJV18], there is a unique "height" $\beta_0 \in [0, \|f\|]$ such that this product is maximal, where $\|f\|$ denotes the L_∞ norm of f. The AMJV ellipsoid is the ellipsoid E in $\mathbb{R}^d$ obtained for this β_0.

In Theorem 8.28, we show that $\beta_0 \chi_E$ is *the limit* (in a rather strong sense) of our John s-ellipsoid functions *as s tends to* 0.

On the other end of the range of s, we study our John s-ellipsoid functions *as s tends to infinity.* We show that the limit may only be a Gaussian distribution, see Theorem 8.38. What is perhaps surprising is that the limit may be one Gaussian for a certain sequence $s_1, s_2, \dots \to \infty$, while it may be a different Gaussian for another sequence. We show however, that in this case, the two Gaussians are translates of each other, see Theorem 8.33.

The first main result of Chapter 8 is a *necessary and sufficient condition* for the $(d+1)$-dimensional Euclidean unit ball $\mathbf{B}^{d+1}$ to be the John s-ellipsoid of a log-concave function f on $\mathbb{R}^d$, see Theorem 8.14. Here, we phrase a simple version of it, stated specifically for the $s = 1$ case, which is an analogue of John's celebrated theorem (Lemma 5.3) for convex bodies [Joh48].

Theorem 8.1 (G. Ivanov, Naszódi [IN22b]). *Let $\overline{K} = \{(x, \xi) \in \mathbb{R}^{d+1} : |\xi| \leq f(x)/2\} \subseteq \mathbb{R}^{d+1}$ denote the symmetrized subgraph of an upper semi-*

continuous log-concave function f on $\mathbb{R}^d$ of positive integral. Assume that the $(d+1)$-dimensional Euclidean unit ball $\mathbf{B}^{d+1}$ is contained in $\overline{K}$. Then the following are equivalent.

1. *The ball $\mathbf{B}^{d+1}$ is the unique maximum volume ellipsoid contained in $\overline{K}$.*

2. *There are contact points $\overline{u}_1, \ldots, \overline{u}_k \in \operatorname{bd}\left(\mathbf{B}^{d+1}\right) \cap \operatorname{bd}\left(\overline{K}\right)$, and weights $c_1, \ldots, c_k > 0$ such that*

$$\sum_{i=1}^{k} c_i \overline{u}_i \otimes \overline{u}_i = \overline{I} \quad \text{and} \quad \sum_{i=1}^{k} c_i u_i = 0,$$

where u_i is the orthogonal projection of $\overline{u}_i$ to $\mathbb{R}^d$ and $\overline{I}$ is the $(d+1) \times (d+1)$ identity matrix.

The implication from (1) to (2) is proved in more or less the same way as John's fundamental theorem about convex bodies, there are hardly any additional difficulties. The converse however, is not straightforward, since $\overline{K}$ is not a convex body in general.

Note that in [AGMJV18] no such characterization is achieved for the AMJV ellipsoid.

The second main result of Chapter 8, a quantitative Helly-type result for log-concave functions, is the following non-trivial application of the previous result.

Observe that the pointwise minimum of a family of log-concave functions is again log-concave.

Theorem 8.2 (G. Ivanov, Naszódi [IN22b]). *Let $f_1, \ldots, f_n$ be upper semi-continuous log-concave functions on $\mathbb{R}^d$. For every $\sigma \subseteq \{1, \ldots, n\}$, let f_σ denote the pointwise minimum:*

$$f_\sigma(x) = \min\{f_i(x) \ : \ i \in \sigma\}.$$

Then there is a set $\sigma \subset [n]$ of at most $3d+2$ indices such that, with the notation $f = f_{[n]}$, we have

$$\int_{\mathbb{R}^d} f_\sigma \leq 100^d d^{2d} \int_{\mathbb{R}^d} f. \tag{2}$$

We note that at the expense of obtaining a much worse bound in place of the multiplicative constant d^{2d}, we can show a similar result with Helly number $2d+1$ instead of $3d+2$. This is a joint unpublished result of G. Ivanov and Naszódi. On the other hand, in Subsection 8.6.6, we show that the number $2d+1$ cannot be decreased to $2d$.

As a **follow-up of the material discussed in Part III**, using the method of the proof of the Bárány-Katchalski-Pach conjecture (Theorem 5.1), Brazitikos [Bra17] improved the bound d^{2d} to $d^{3d/2}$. Galicer, Merzbacher and Pinasco claim their paper [GMP19] was inspired by [Nas16b].

Almendra-Hernández, Ambrus and Kendall [AHAK22] partly use the method of [IN22c], a joint paper with G. Ivanov laying the grounds for our [IN24], and improves some of its results.

Together with A. Jung, who was an MSc. student under my supervision, in [JN22] we developed further some of the ideas in [DFN20], which A. Jung took even further in [Jun22].

With G. Ivanov, in [IN23] we generalized our results in [IN22b] to arbitrary pairs of log-concave functions, that is, when the role of the ball in $\mathbb{R}^{d+1}$ is played by (essentially) any other log-concave function pointwise below f.

Notation

We call a compact, convex subset of $\mathbb{R}^d$ with non-empty interior a *convex body*.

Throughout this work, we use the following notation. The convex hull of a set $Y \subset \mathbb{R}^d$ is $\operatorname{conv} Y$; the standard scalar product of $x, y \in \mathbb{R}^d$ is $\langle x, y\rangle$, and $|x| = \sqrt{\langle x, x\rangle}$ denotes the length of $x \in \mathbb{R}^d$. The origin centered closed unit radius ball is $\mathbf{B}^d = \{x \in \mathbb{R}^d \ : \ |x| \leq 1\}$. For the boundary, the interior and the d-dimensional volume of a set K, we use $\operatorname{bd}(K)$, $\operatorname{int}(K)$ and $\operatorname{vol}_d(K)$, respectively. We use the standard notation $[N] = \{1, \ldots, N\}$, and $\operatorname{card}(A)$ denotes the cardinality of a set A.

Part I

Approximation of convex bodies by polytopes

Chapter 1

Approximation obtained from the Epsilon-net theorem

1.1 Introduction

A convex body K (that is, a compact, convex set with non-empty interior) in $\mathbb{R}^d$ is called *centered*, if its center of mass is the origin. In other words, the expectation of a random vector chosen uniformly in K is the origin.

In this chapter, which is based on [Nas19], we study the following problem. Given a centered convex body K in $\mathbb{R}^d$, a positive integer $t \geq d+1$, and $\delta, \vartheta \in (0,1)$. Our goal is to show that under some assumptions on the parameters d, t, δ, ϑ (and without further assumptions on K), the convex hull P of t randomly, uniformly and independently chosen points of K contains ϑK with probability at least $1-\delta$.

Theorem 1.1 of [BCH16] by Brazitikos, Chasapis, and Hioni concerns the case of very rough approximation, that is, where the number t of chosen points is linear in the dimension d. It states that the convex hull of $t = \alpha d$ random points in a centered convex body K is a convex polytope P which satisfies $\frac{c_1}{d} K \subseteq P$, with probability $1-\delta = 1-e^{-c_2 d}$, where $c_1, c_2 > 0$ and $\alpha > 1$ are absolute constants. In our first result, we obtain explicit constants.

Theorem 1.1. *Let K be a centered convex body in $\mathbb{R}^d$. Choose $t = 60(d+1)$ points $X_1, \ldots, X_t$ of K randomly, independently and uniformly. Then*

$$\frac{1}{d} K \subseteq \operatorname{conv}(X_1, \ldots, X_t) \subseteq K.$$

with probability at least $1-4e^{-d-1}$.

Another instance of our general problem are Theorem 5.2 and Proposition 5.3 of [GM00] by Giannopoulos and V. Milman, which concerns fine approximation, that is, where the number t of chosen points is exponential in the dimension d, see the Introduction of this dissertation.

Our main result of the chapter is the following joint generalization of these results.

Theorem 1.2. *Let* $\vartheta \in (0,1), C \geq 2$. *Set*

$$t := \left\lceil C\frac{(d+1)e}{(1-\vartheta)^d}\ln\frac{e}{(1-\vartheta)^d}\right\rceil.$$

Then for any centered convex body K *in* $\mathbb{R}^d$, *if* t *points* $X_1,\dots,X_t$ *of* K *are chosen randomly, independently and uniformly, then*

$$\vartheta K \subseteq \operatorname{conv}(X_1,\dots,X_t) \subseteq K \tag{1.1}$$

with probability at least $1-\delta$, *where*

$$\delta := 4\left[11C^2\left(\frac{(1-\vartheta)^d}{e}\right)^{C-2}\right]^{d+1}.$$

By substituting $\vartheta = \frac{1}{d}, C = 6$, we obtain Theorem 1.1.

In order to recover Theorem 5.2 of [GM00], substitute $C = 3$ and $\vartheta = c(\delta)\gamma$ in our Theorem 1.2. Then $t \leq e^{3c(\delta)\gamma d}$, when d is large, and δ is roughly $e^{-c(\delta)\gamma d^2}$. Fixing $c(\delta) = 1/3$ independently of δ yields the result.

We recover Proposition 5.3 of [GM00] in a form which is slightly weaker if ϑ is close to 1, as follows. In our Theorem 1.2, $t \leq \frac{10Cd^2}{(1-\vartheta)^{d+1}}$ (note the exponent $d+1$ instead of d) and $\delta \leq 11C^2/e^{C-2}$. By setting C sufficiently large (depending on the desired δ only), we can make the latter as small as required.

We compare our Theorem 1.2 with the main result, Theorem 1.2 of [BCH16], which states the following for any fixed $\beta \in (0,1)$. There exist a constant $\alpha = \alpha(\beta) > 1$ depending only on β and an absolute constant $c > 0$ with the following property. Let K be a centered convex body in $\mathbb{R}^d$, $\alpha d \leq t \leq e^d$, and choose t points uniformly distributed in K. Then the convex polytope thus obtained contains ϑK, where $\vartheta = \frac{c\beta\ln(t/d)}{d}$ with probability $1-\delta$, where $\delta \leq \exp(-t^{1-\beta}d^{\beta})$.

When ϑ is of order $1/d$, the two results are the same up to the constants involved, see our Theorem 1.1 and the discussion preceding it. For fine approximation, that is, when ϑ is a constant, by setting $C = \frac{1}{(1-\vartheta)^{d/2}}$, we obtain roughly $t \approx \exp(\vartheta d/2)$ and $\delta \approx \exp[-\vartheta d^2\exp(\vartheta d/2)]$. In the mean time, Theorem 1.2 of [BCH16] gives roughly $t \approx \exp(\vartheta d/(c\beta))$ and $\delta \approx \exp[-\exp((1-\beta)\vartheta d/(c\beta))d^{\beta}]$.

In Section 1.2, we present a generalization of a classical result of Grünbaum [Grü60], according to which any half-space containing the center of mass of a convex body contains at least a $1/e$ fraction of its volume. In Section 1.3, we state a specific form of the ε-net theorem, a result from combinatorics obtained by Haussler and Welzl [HW87] building on ideas of Vapnik and Chervonenkis [VČ68], and then refined by Komlós, Pach and Woeginger [KPW92]. In Section 1.4, we combine these two to obtain Theorem 1.2. Finally, in Section 1.5, using a recent result of Fradelizi, Meyer and Yaskin [FMY17], we extend our main result to approximating a linear section of a centered convex body.

For surveys on the topic of approximation of convex bodies by polytopes, cf. [Bár07, Bro07, Gru93], and for some further recent results on approximation in the homothetic distance, when the vertices are not necessarily picked randomly and uniformly from the body, see [Bar14] and [NNR20].

We note that, in a similar vein, Gordon, Litvak, Pajor and Tomczak-Jaegermann [GLPTJ07, Theorem 3.1] showed that if K is an origin-symmetric convex body in $\mathbb{R}^d$ and $t = (4/\varepsilon)^{2d}$ random points $X_1, \ldots, X_t$ are chosen from it uniformly and independently, then, with probability larger than $1 - \exp(-(8/\varepsilon)^d/2)$, these t points form a *metric ε-net* of K with respect to K, that is, $K \subseteq \cup_{i=1}^t (X_i + \varepsilon K)$. We will use the term '$\varepsilon$-net' in a different, combinatorial sense, to be defined in Section 1.3.

The main ideas

First, we observe that (1.1) is equivalent to the following condition. For any half-space F whose bounding hyperplane supports ϑK, and which does not contain the origin, we have that F contains at least one point of $\{X_1, \ldots, X_t\}$. Calling the intersection of K with a half-space as above a 'cap', we need to show that the random set $\{X_1, \ldots, X_t\}$ is a hitting set of the family of caps. Second, generalizing a fundamental result of Grünbaum, we give a lower bound on the volume of caps, that is, we show that for each X_i, the probability for X_i to hit a fixed cap (that is, to be contained in the cap) is not too small. Finally, using the fact that the family of caps is a combinatorially simple set family (its VC-dimension is $d+1$), we can apply the ε-net theorem, which yields that a random set of t points is a hitting set of this set family with high probability.

1.2 Convexity: A stability version of a theorem of Grünbaum

Grünbaum's theorem [Grü60] states that for any centered convex body K in $\mathbb{R}^d$, and any half-space F_0 that contains the origin we have

$$\operatorname{vol}(K)/e \leq \operatorname{vol}(K \cap F_0), \tag{1.2}$$

where $\operatorname{vol}(\cdot)$ denotes volume.

We say that a half-space F *supports* K *from outside* if the boundary of the half-space intersects $\operatorname{bd}(K)$, but F does not intersect the interior of K. Lemma 1.3, is a stability version of Grünbaum's theorem.

Lemma 1.3. *Let K be a convex body in $\mathbb{R}^d$ with centroid at the origin. Let $0 < \vartheta < 1$, and F be a half-space that supports ϑK from outside. Then*

$$\operatorname{vol}(K) \frac{(1-\vartheta)^d}{e} \leq \operatorname{vol}(K \cap F). \tag{1.3}$$

Proof of Lemma 1.3. Let F_0 be a translate of F containing o on its boundary, and let F_1 be a translate of F that supports K from outside. Finally, let $p \in \operatorname{bd}(F)_1 \cap K$. Then $\vartheta p + (1-\vartheta)(K \cap F_0)$ (that is, the homothetic copy of $K \cap F_0$ with homothety center p and ratio $1-\vartheta$) is in $K \cap F$. Its volume is $(1-\vartheta)^d \operatorname{vol}(K \cap F_0)$, which by (1.2), is at least $(1-\vartheta)^d \operatorname{vol}(K)/e$, finishing the proof. □

1.3 Combinatorics: The ε-net Theorem of Haussler and Welzl

Definition 1.4. Let $\mathcal{F}$ be a family of subsets of some set U. The *Vapnik–Chervonenkis dimension* (*VC-dimension*, in short) of $\mathcal{F}$ is the maximal cardinality of a subset V of U such that V is shattered by $\mathcal{F}$, that is, $\{F \cap V \;:\; F \in \mathcal{F}\} = 2^V$.

A *transversal* of the set family $\mathcal{F}$ is a subset Q of U that intersects each member of $\mathcal{F}$.

Let $\varepsilon \in (0,1)$ be given. When U is equipped with a probability measure for which each member of $\mathcal{F}$ is measurable, then a transversal of those members of $\mathcal{F}$ that are of measure at least ε is called an *ε-net*.

It follows from Radon's lemma (cf. Theorem 1.3.1 in [Mat02], or Theorem 1.1.5 in [Sch14]) that if U is any subset of $\mathbb{R}^d$, and $\mathcal{F}$ is a family of half-spaces of $\mathbb{R}^d$, then the VC-dimension of $\mathcal{F}$ is at most $d+1$.

The ε-net Theorem was first proved by Haussler and Welzl [HW87], and then improved by Komlós, Pach and Woeginger [KPW92]. We state a slightly weaker form of Theorem 3.1 of [KPW92] than the original, in order to have an explicit bound on the probability δ of failure.

Lemma 1.5 (ε-net Theorem)**.** *Let $0 < \varepsilon < 1/e, C \geq 2$, and let D be a positive integer. Let $\mathcal{F}$ be a family of some measurable subsets of a probability space (U, μ), where the probability of each member F of $\mathcal{F}$ is $\mu(F) \geq \varepsilon$. Assume that the VC-dimension of $\mathcal{F}$ is at most D. Set*

$$t := \left\lceil C \frac{D}{\varepsilon} \ln \frac{1}{\varepsilon} \right\rceil .$$

Choose t elements $X_1, \ldots, X_t$ of V randomly, independently according to μ. Then $\{X_1, \ldots, X_t\}$ is a transversal of $\mathcal{F}$ with probability at least $1-\delta$, where

$$\delta := 4\left[11 C^2 \varepsilon^{C-2}\right]^D .$$

Proof. We provide an outline of the first, conceptual part of the proof closely following [PA95] (Theorem 15.5 therein). Then, we continue with a detailed computation to obtain the bound on the probability stated in Lemma 1.5.

Let $T > t$ be an integer, to be set later. We select (with repetition) independently t random elements of U with respect to μ, call it the first sample, and denote it by x. Then, we choose another $T-t$ elements, call it the second sample, and denote it by y. For any $F \in \mathcal{F}$, and any finite sequence w of elements of U, let $I(F, w)$ denote the number of elements of w in F with multiplicity. Let m_F denote the median of $I(F, y)$.

Note that $I(F, y)$ is a binomial variable, and hence, its mean and median are close to each other. More precisely,

$$m_F \geq (T-t)\varepsilon - 1. \tag{1.4}$$

It is not hard to see that

$$\mu(\exists F \in \mathcal{F} : I(F, x) = 0) \leq 2\mu(\exists F \in \mathcal{F} : I(F, x) = 0 \text{ and } I(F, y) \geq m_F). \tag{1.5}$$

Denote the concatenation of the two sequences x and y by $\overline{xy}$. Fix any length T sequence z of elements of U.

It is simple to obtain a bound on the following conditional probability:

$$\mu(\exists F \in \mathcal{F} : I(F, x) = 0 \text{ and } I(F, y) \geq m_F | \overline{xy} = z) \leq \tag{1.6}$$

$$\chi[I(F, z) \geq m_F] \left(1 - \frac{t}{T}\right)^{m_F},$$

where χ denotes the indicator function of an event, that is, it is one if the event holds, and zero otherwise.

The key idea follows. Consider z as a set. Then, by the Shatter function lemma (cf. Theorem 15.4 of [PA95] or Lemma 10.2.5 of [Mat02]) proved independently by Shelah [She72], Sauer [Sau72] and Vapnik and Chervonenkis [VČ68], z has at most

$$\sum_{i=0}^{D}\binom{T}{i}$$

distinct intersections with members of $\mathcal{F}$. Thus by (1.4) and (1.6), we have

$$\mu(\exists F\in\mathcal{F} : I(F,x)=0 \text{ and } I(F,y)\geq m_F|\overline{xy}=z)\leq \tag{1.7}$$
$$\sum_{i=0}^{D}\binom{T}{i}\left(1-\frac{t}{T}\right)^{(T-t)\varepsilon-1}.$$

Let E be the 'bad' event, that is, when $\{X_1,\dots,X_t\}$ is not a transversal of $\mathcal{F}$. So far, by (1.5) and (1.7), we obtained that for any integer $T>t$, we have that the probability of the event E is

$$\mu(E)<2\sum_{i=0}^{D}\binom{T}{i}\left(1-\frac{t}{T}\right)^{(T-t)\varepsilon-1}.$$

From this point on, we describe the computations in detail, in order to obtain the bound on the probability stated in Lemma 1.5.

We set $T=\left\lfloor\frac{\varepsilon t^2}{D}\right\rfloor$ and use $\sum_{i=0}^{D}\binom{T}{i}\leq\left(\frac{eT}{D}\right)^D$, to obtain that

$$\mu(E)<2\left(\frac{eT}{D}\right)^D\left(1-\frac{t}{T}\right)^{(T-t)\varepsilon-1}<2\left(\frac{e\varepsilon t^2}{D^2}\right)^D\left(1-\frac{D}{\varepsilon t}\right)^{\left(\frac{\varepsilon t^2}{D}-t-1\right)\varepsilon-1}$$
$$<2\left(\frac{e\varepsilon t^2}{D^2}\right)^D e^{-\varepsilon t+D+D/t+D/(\varepsilon t)},$$

which, after substituting the expression for t in some places and using $\varepsilon<1/e$, is at most

$$\left(2e^{1/C+1/(eC)}\right)\left(\frac{e^2\varepsilon t^2}{D^2}\right)^D\varepsilon^{CD},$$

which, using $C\geq 2$ is at most

$$\left(2e^{1/C+1/(eC)}\right)\left(\frac{e^2(1+1/(2e))^2C^2\ln^2(1/\varepsilon)}{\varepsilon}\right)^D\varepsilon^{CD}<4\left(11C^2\varepsilon^{C-2}\right)^D,$$

completing the proof of Lemma 1.5. □

For more on the theory of ε-nets, see [PA95, Mat02, AS16, MV18] and the wonderful new textbook [Mus22].

1.4 Proof of Theorem 1.2

We consider the following set system on the base set K:

$$\mathcal{F} := \{K \cap F \ : \ F \text{ is a half space that supports } \vartheta K \text{ from outside}\}.$$

Clearly, the VC-dimension of $\mathcal{F}$ is at most $D := d+1$. Let μ be the Lebesgue measure restricted to K, and assume that $\operatorname{vol}(K) = 1$, that is, that μ is a probability measure. By (1.3), we have that each set in $\mathcal{F}$ is of measure at least $\varepsilon := \frac{(1-\vartheta)^d}{e}$. Lemma 1.5 yields that if we choose t points of K independently with respect to μ (that is, uniformly), then with probability at least $1-\delta$, we obtain a set $Q \subseteq K$ that intersects every member of $\mathcal{F}$. The latter is equivalent to $\vartheta K \subseteq \operatorname{conv}(Q)$, completing the proof of Theorem 1.2.

1.5 Approximating a section of a convex body

Let K be a centered convex body in $\mathbb{R}^d$, and V a linear subspace of $\mathbb{R}^d$. Now, $K \cap V$ may not be centered however, we may still want to approximate $K \cap V$ with a polytope $P \subset K \cap V$ such that $\vartheta(K \cap V) \subset P$ for some not too small ϑ.

The main result of [FMY17] (for further results, see also [MNRY18]) states that there is an absolute constant $c > 0$ such that for every centered convex body K in $\mathbb{R}^d$, every $(d-k)$-dimensional linear subspace V of $\mathbb{R}^d$, $0 \le k \le d-1$, and any $u \in V$ unit vector, we have

$$\operatorname{vol}_{d-k}\left(K \cap V \cap u^+\right) \ge \frac{c}{(k+1)^2}\left(1+\frac{k+1}{d-k}\right)^{-(d-k-2)} \operatorname{vol}_{d-k}(K \cap V), \tag{1.8}$$

where $u^+ = \{x \in \mathbb{R}^d \ : \ \langle u, x\rangle \ge 0\}$ is the half space with inner normal vector u.

Using this result, our proof of Theorem 1.2 immediately yields the following.

Theorem 1.6. *Let $\vartheta \in (0,1)$, $C \ge 2$. Let K be a centered convex body in $\mathbb{R}^d$ and V be $(d-k)$-dimensional linear subspace of $\mathbb{R}^d$ with $0 \le k \le d-1$. Set*

$$t := \left\lceil C \frac{(d-k+1)(k+1)^2}{c\left(1+\frac{k+1}{d-k}\right)^{d-k-2}(1-\vartheta)^{d-k}} \ln \frac{(k+1)^2}{c\left(1+\frac{k+1}{d-k}\right)^{d-k-2}(1-\vartheta)^{d-k}} \right\rceil,$$

where c is the universal constant from (1.8). Choose t points $X_1, \ldots, X_t$ of $K \cap V$ randomly, independently and uniformly with respect to the $(d-k)$-dimensional Lebesgue measure on V. Then

$$\vartheta(K \cap V) \subseteq \operatorname{conv}(X_1, \ldots, X_t) \subseteq K \cap V$$

with probability at least $1-\delta$, where

$$\delta := 4\left[11C^2\left(\frac{c\left(1+\frac{k+1}{d-k}\right)^{d-k-2}(1-\vartheta)^{d-k}}{(k+1)^2}\right)^{C-2}\right]^{d-k+1}.$$

Chapter 2

Approximation obtaiend from a measure using polarity

2.1 Introduction and main result

This chapter is based on joint work with Fedor Nazarov and Dmitry Ryabogin [NNR20]. The problem, as in the previous chapter is to approximate a convex body K in $\mathbb{R}^d$ by a polytope obtained as the convex hull of a randomly chosen set of points. However, while in the previous chapter, the probability distribution that yields the points was prescribed to be the uniform measure on the body, in this chapter, we are free to choose a distribution. The main result of this chapter is the following.

Theorem 2.1. *Let K be a convex body in $\mathbb{R}^d$ with the center of mass at the origin, and let $\varepsilon \in \left(0, \frac{1}{2}\right)$. Then there exists a convex polytope P with at most $e^{O(d)}\varepsilon^{-\frac{d-1}{2}}$ vertices such that $(1-\varepsilon)K \subset P \subset K$.*

This result improves the 2012 theorem of Barvinok [Bar14] by removing the symmetry assumption and the extraneous $(\log \frac{1}{\varepsilon})^d$ factor.

Our approach uses a mixture of geometric and probabilistic tools similar to that in [AdFM12]. The main difference is that, since we work with the Banach-Mazur distance instead of the Hausdorff distance, we need to make all our constructions invariant under linear transformations. We will rely on two non-trivial classical results (Blaschke-Santaló inequality and its reverse).

For a vector $e \in \mathbb{R}^d$, we denote by $e^{\perp}$ the linear hyperplane orthogonal to e.

The main ideas

We define a probability distribution on $\mathbb{R}^d$ using the uniform measure on K and the uniform measure on the polar of K. The first measure ensures that large flat parts of the boundary ('flat' caps) have high probability, the second measure is responsible for a lower bound on the probability of hitting a 'spiky' cap of K. Thus, by picking points according this probability distribution, all caps will be hit with large probability. At this point, we could apply the same method as in Chapter 1 and use the Epsilon-net theorem. Instead, by a rather technical argument concerning the boundary of K, we manage to obtain a bound on the number of points that we need to pick, which does not involve a $\left(\ln \frac{1}{\varepsilon}\right)^d$ factor.

2.2 Outline of the proof

Without loss of generality, we may assume that K has smooth boundary, in particular, K has a unique supporting hyperplane at each boundary point. Our task is to find a finite set of points $Y \subset \mathrm{bd}\,(K)$ such that $P = \mathrm{conv}\, Y$ satisfies $(1-\varepsilon)K \subset P$. Switching to the support functions, we see that this is equivalent to the requirement that every cap $S(x, \varepsilon) = \{y \in \mathrm{bd}\,(K) : \langle y, \nu_x \rangle \geq (1-\varepsilon)\langle x, \nu_x \rangle\}$, where $x \in \mathrm{bd}\,(K)$ and ν_x is the outer unit normal to $\mathrm{bd}\,(K)$ at x, contains at least one point of Y.

The key idea is to construct a probability measure μ on $\mathrm{bd}\,(K)$ such that for every $x \in \mathrm{bd}\,(K)\,, \varepsilon \in \left(0, \frac{1}{2}\right)$, we have $\mu(S(x,\varepsilon)) \geq p\varepsilon^{\frac{d-1}{2}}$ with some $p = e^{O(d)}$ depending on d only.

Since there are infinitely many caps, our next aim is to choose an appropriate finite net $X \subset \mathrm{bd}\,(K)$ of cardinality $C(d)\varepsilon^{-\frac{d-1}{2}}$ such that for every $Y \subset \mathrm{bd}\,(K)$, the condition $S(x, \frac{\varepsilon}{2}) \cap Y \neq \varnothing$ for all $x \in X$ implies that $S(x,\varepsilon) \cap Y \neq \varnothing$ for all $x \in \mathrm{bd}\,(K)$. Given such a net, we will be able to apply a general combinatorial result (also used by Rogers) to construct the desired set Y of cardinality approximately $\log C(d) p^{-1} \varepsilon^{-\frac{d-1}{2}}$, which will be still $e^{O(d)}\varepsilon^{-\frac{d-1}{2}}$ as long as $C(d)$ is at most double exponential in d.

A natural net to try is the Bronshtein–Ivanov net (see [BI75]), which allows one to approximate a point $x \in \mathrm{bd}\,(K)$ and the corresponding outer unit normal ν_x by a point in the net and its outer unit normal simultaneously. Unfortunately, it works only for uniformly 2-convex bodies, i.e., bodies whose every boundary point b has a supporting ball (that is, a ball containing the body and having b on its boundary sphere) of fixed controllable radius.

So, the last step will be to show that the task of approximating an arbitrary convex body K can be reduced to that of approximating a certain uniformly 2-convex body associated with K.

In the exposition, these steps are presented in reverse. We start with constructing the associated uniformly 2-convex body (Sections 2.3, 2.4, 2.5). Then we build the Bronshtein-Ivanov net X of appropriate mesh and cardinality, and check that it is, indeed, enough to consider the caps $S(x, \frac{\varepsilon}{2}), x \in X$ (Sections 2.6, 2.7, 2.8). Finally, we construct the probability measure μ and complete the proof of the theorem (Sections 2.9, 2.10).

2.3 Standard position

Since the problem is invariant under linear transformations, we can always assume that our body K is in some "standard position". The exact notion of the standard position to use is not very important as long as it guarantees that $\mathbf{B}^d \subset K \subset d^2\mathbf{B}^d$.

One possibility is to make a linear transformation such that $\mathbf{B}^d$ is John's ellipsoid (see [Bal97], Lecture 3) of the centrally-symmetric convex body $L = K \cap -K$, so $\mathbf{B}^d \subset L \subset \sqrt{d}\mathbf{B}^d$. Since the origin is the centroid of K, we have $K \supset -\frac{1}{d}K$ (see [BF87], page 57), so it follows that $\mathbf{B}^d \subset K \subset d\sqrt{d}\mathbf{B}^d$.

2.4 The function φ_δ and the mapping Φ_δ

Fix $\delta \in \left(0, \frac{1}{2}\right)$. For $r \geq 0$, define $\varphi_\delta(r)$ as the positive root of the equation $\varphi + \delta r^2\varphi^2 = 1$. Put $\Phi_\delta(x) = x\varphi_\delta(|x|)$, $x \in \mathbb{R}^d$.

Lemma 2.2. *The function φ_δ is a decreasing smooth function on $[0, +\infty)$; $r \mapsto r\varphi_\delta(r)$ is an increasing function mapping $[0, +\infty)$ to $[0, \delta^{-\frac{1}{2}})$; Φ_δ is a diffeomorphism of $\mathbb{R}^d$ onto the open ball $\delta^{-\frac{1}{2}} \operatorname{int} \mathbf{B}^d$; if ν is a unit vector and $h > 0$, then the image $\Phi_\delta(H_{\nu,h})$ of the half-space $H_{\nu,h} = \{x \,:\, \langle x, \nu\rangle \leq h\}$ is the intersection of $\delta^{-\frac{1}{2}} \operatorname{int} \mathbf{B}^d$ and the ball of radius $\sqrt{\frac{1}{4\delta^2h^2} + \frac{1}{\delta}}$ centered at $-\frac{1}{2\delta h}\nu$ (see Figure 2.1).*

Proof. The first statement is obvious. To show the second one, just notice that $r\varphi_\delta(r)$ is the positive root of $\frac{\psi}{r} + \delta\psi^2 = 1$ and, as $r \to \infty$, this root increases to $\delta^{-\frac{1}{2}}$. The third claim follows from the observation that the derivative of the mapping $r \mapsto r\varphi_\delta(r)$ is strictly positive and continuous on

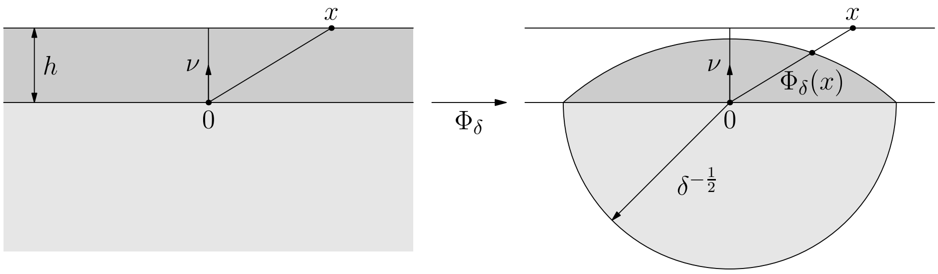

Figure 2.1: The mapping Φ_δ

$[0, +\infty)$. To prove the last claim, observe that if $\langle x, \nu\rangle = h$, then

$$\left|\Phi_\delta(x) + \frac{1}{2\delta h}\nu\right|^2 = \left|x\varphi_\delta(|x|) + \frac{1}{2\delta h}\nu\right|^2 = \\ |x|^2\varphi_\delta(|x|)^2 + \frac{\varphi_\delta(|x|)}{\delta} + \frac{1}{4\delta^2 h^2} = \frac{1}{4\delta^2 h^2} + \frac{1}{\delta}$$

by the definition of φ_δ. $\square$

It follows that for every convex body K containing the origin, $\Phi_\delta(K)$ is also convex. Since for every interval $I_x = \{rx \ : \ 0 \le r \le 1\}$, $x \in \mathbb{R}^d$, we have $\Phi_\delta(I_x) \subset I_x$, the image $\Phi_\delta(K)$ is contained in K. Moreover, if $\mathbf{B}^d \subset K$, then $\Phi_\delta(K)$ is the intersection of balls of radii not exceeding $\sqrt{\frac{1}{4\delta^2} + \frac{1}{\delta}} \le \frac{1}{\delta}$. In particular, for every boundary point $x \in \operatorname{bd}(\Phi_\delta(K))$, we can find a ball of radius $\frac{1}{\delta}$ containing $\Phi_\delta(K)$ whose boundary sphere touches $\Phi_\delta(K)$ at x.

At last, observe that since $\varphi \le 1$, we have $\delta r^2\varphi^2 \le \delta r^2$, so $\varphi \ge 1 - \delta r^2$. Thus, if $K \subset r\mathbf{B}^d$ and $\delta r^2 < 1$, we have $\Phi_\delta(K) \subset (1 - \delta r^2)K$.

2.5 From the approximation of $\Phi_\delta(K)$ to the approximation of K

Lemma 2.3. *Let $\varepsilon \in \left(0, \frac{1}{2}\right)$. Suppose that a convex body K satisfies $0 \in K \subset d^2\mathbf{B}^d$ and that $\delta < \frac{1}{4d^4}$. If $Y \subset \operatorname{bd}(K)$ is a finite set such that $\left(1 - \frac{\varepsilon}{2}\right)\Phi_\delta(K) \subset \operatorname{conv}(\Phi_\delta(Y))$, then $(1-\varepsilon)K \subset \operatorname{conv}(Y)$.*

Proof. Note that the conditions of the lemma imply that $0 \in \operatorname{conv}(\Phi_\delta(Y))$. Since for every $y \in \mathbb{R}^d$, $\Phi_\delta(y)$ is a positive multiple of y, we conclude that $0 \in P = \operatorname{conv}(Y)$ as well, so $\Phi_\delta(P)$ is convex. Suppose that there exists $x \in K$ such that $(1-\varepsilon)x \notin P$. Then,

$$\Phi_\delta((1-\varepsilon)x) \notin \Phi_\delta(P) \supset \operatorname{conv}(\Phi_\delta(Y)).$$

However,

$$\Phi_\delta((1-\varepsilon)x) = (1-\varepsilon)\frac{\varphi_\delta((1-\varepsilon)|x|)}{\varphi_\delta(|x|)}\Phi_\delta(x).$$

Denoting $\eta_t = \varphi_\delta((1-t)|x|)$, $t \in [0,1]$, we have

$$\eta_\varepsilon + \delta(1-\varepsilon)^2|x|^2\eta_\varepsilon^2 = \eta_0 + \delta|x|^2\eta_0^2 = 1.$$

Since $\delta|x|^2\eta_\varepsilon^2 \ge \delta|x|^2\eta_0^2$ and $\delta\varepsilon^2|x|^2\eta_\varepsilon^2 \ge 0$, it follows that

$$\eta_\varepsilon(1-2\delta\varepsilon|x|^2\eta_\varepsilon) \le \eta_0, \quad \text{so} \quad \frac{\eta_\varepsilon}{\eta_0} \le \frac{1}{1-2\delta\varepsilon|x|^2\eta_\varepsilon}.$$

Since $\eta_\varepsilon \le 1$ and $2\delta|x|^2 \le 2\delta d^4 \le \frac{1}{2}$, we get

$$(1-\varepsilon)\frac{\eta_\varepsilon}{\eta_0} \le \frac{1-\varepsilon}{1-\frac{\varepsilon}{2}} \le 1-\frac{\varepsilon}{2},$$

so $\left(1-\frac{\varepsilon}{2}\right)\Phi_\delta(x)$ cannot be contained in $\mathrm{conv}(\Phi_\delta(Y))$, which contradicts our assumption. □

This lemma implies that an $\frac{\varepsilon}{2}$-approximation of $\Phi_\delta(K)$ yields an ε-approximation of K. Note also that $\Phi_\delta(K)$ is rather close to K. More precisely, if $0 \in K \subset d^2\mathbf{B}^d$ and $\delta d^4 < 1$, we have $(1-\delta d^4)K \subset \Phi_\delta(K) \subset K$. The center of mass of $\Phi_\delta(K)$ may no longer be at the origin, of course, but the only non-trivial property of K we shall really use is the Santaló bound $\mathrm{vol}_d(K)\mathrm{vol}_d(K^\circ) \le e^{O(d)}d^{-d}$, where

$$K^\circ = \{y \in \mathbb{R}^n \;:\; \langle x,y\rangle \le 1 \quad \text{for all} \quad x \in K\}$$

is the polar body of the convex body K. This bound holds for K because 0, being the center of mass of K, is, thereby, the Santaló point of K° (see Section 2.10 for details). For sufficiently small $\delta > 0$, it is inherited by $\Phi_\delta(K)$ just because $(\Phi_\delta(K))^\circ \subset (1-\delta d^4)^{-1}K^\circ$ and, thereby,

$$\mathrm{vol}_d(\Phi_\delta(K))\mathrm{vol}_d((\Phi_\delta(K))^\circ) \le (1-\delta d^4)^{-d}\mathrm{vol}_d(K)\mathrm{vol}_d(K^\circ).$$

Choosing $\delta = \frac{1}{4d^5}$, we see that the body $\Phi_\delta(K)$ also satisfies the Santaló bound with only marginally worse constant. At last, if $\mathbf{B}^d \subset K$, we have $\frac{1}{2}\mathbf{B}^d \subset (1-\delta)\mathbf{B}^d \subset \Phi_\delta(K)$.

Thus, replacing K by $\Phi_\delta(K)$ (and ε by $\frac{\varepsilon}{2}$) if necessary, from now on we can restrict ourselves to the class $\mathcal{K}_R$ of convex bodies K with smooth boundary such that $\frac{1}{2}\mathbf{B}^d \subset K \subset d^2\mathbf{B}^d$ and for every boundary point $x \in \mathrm{bd}\,(K)$, there exists a ball of fixed radius $R = 4d^5$ containing K whose boundary sphere touches K at x. Moreover, we can also assume that $\mathrm{vol}_d(K)\mathrm{vol}_d(K^\circ) \le e^{O(d)}d^{-d}$.

2.6 The Bronshtein–Ivanov net

Let $\rho \in \left(0, \frac{1}{2}\right)$. Let K be a convex body with smooth boundary containing the origin and contained in $d^2\mathbf{B}^d$. Consider the set S of points $\{x+\nu_x : x \in \mathrm{bd}(K)\}$, where ν_x is the outer unit normal to $\mathrm{bd}(K)$ at x. Let $\{x_j + \nu_{x_j} : 1 \le j \le N\}$ be a maximal ρ-separated set in S, i.e., a set such that any two of its members are at distance at least ρ (see Figure 2.2). We will call the corresponding set $\{x_j : 1 \le j \le N\}$ a Bronshtein-Ivanov net of mesh ρ for the body K.

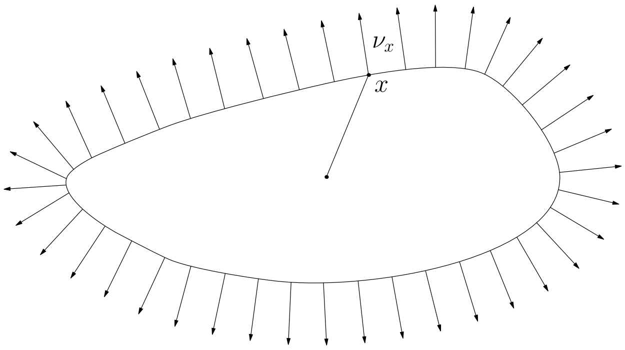

Figure 2.2: The Bronshtein-Ivanov net

Lemma 2.4. *We have $N \le 2^d(d^2+3)^d\rho^{-d+1}$, and for every $x \in \mathrm{bd}(K)$, we can find j such that $|x - x_j|^2 + |\nu_x - \nu_{x_j}|^2 \le \rho^2$.*

Proof. Let $x', x'' \in \mathrm{bd}(K)$ and let $\nu' = \nu_{x'}$, $\nu'' = \nu_{x''}$. Note that, by the convexity of K, we must have $\langle \nu', x' - x''\rangle \ge 0$, $\langle \nu'', x'' - x'\rangle \ge 0$. Hence, we always have

$$|x' + \nu' - x'' - \nu''|^2 = |x' - x''|^2 + |\nu' - \nu''|^2 + 2(\langle \nu', x' - x''\rangle + \langle \nu'', x'' - x'\rangle) \ge |x' - x''|^2 + |\nu' - \nu''|^2,$$

and the second conclusion of the lemma follows immediately from the definition of the Bronshtein-Ivanov net $\{x_j : 1 \le j \le N\}$.

Now assume that $s', s'' \ge 0$. Write

$$|x' + \nu' + s'\nu' - x'' - \nu'' - s''\nu''|^2 = |x' + \nu' - x'' - \nu''|^2 + |s'\nu' - s''\nu''|^2 + 2s'\langle \nu', x' - x''\rangle + 2s''\langle \nu'', x'' - x'\rangle + 2(s' + s'')(1 - \langle \nu', \nu''\rangle) \ge |x' + \nu' - x'' - \nu''|^2.$$

Thus, if the balls of radius $\frac{\varrho}{2}$ centered at $x'+\nu'$ and $x''+\nu''$ are disjoint, so are the balls of radius $\frac{\varrho}{2}$ centered at $x'+(1+s')\nu'$ and $x''+(1+s'')\nu''$. From here we conclude that the balls of radius $\frac{\varrho}{2}$ centered at the points $x_j+(1+k\rho)\nu_{x_j}$, $0 \leq k \leq \frac{1}{\rho}$ are all disjoint (see Figure 2.3) and contained in $(d^2+3)\mathbf{B}^d$.

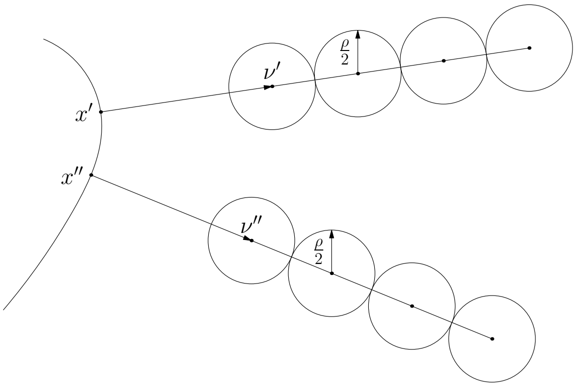

Figure 2.3: The disjoint balls

The total number of these balls is at least $\frac{N}{\rho}$ (for every point x_j in the net, there is a chain of at least $\frac{1}{\rho}$ balls corresponding to different values of k), whence $\frac{N}{\rho} \leq \left(\frac{d^2+3}{\frac{\varrho}{2}}\right)^d$ and the desired bound for N follows. □

2.7 The bound for cap diameters

The following lemma shows that ε-caps of convex bodies $K \in \mathcal{K}_R$ have small diameters.

Lemma 2.5. *Let $\varepsilon \in \left(0, \frac{1}{2}\right)$. Assume that $K \in \mathcal{K}_R$, $x \in \operatorname{bd}(K)$, and ν is the outer normal to $\operatorname{bd}(K)$ at x. If $y \in S(x, \varepsilon)$, i.e., $y \in K$ and $\langle y, \nu\rangle \geq (1-\varepsilon)\langle x, \nu\rangle$, then $|y-x| \leq \sqrt{2R}\, d\sqrt{\varepsilon}$.*

Proof. Let Q be the ball of radius R containing K whose boundary sphere touches K at x. Then $y \in Q$ and ν is the outer unit normal to Q at x, so Q is centered at $x - R\nu$. Note also that, since $0 \in K \subset d^2\mathbf{B}^d$, we have $0 \leq \langle x, \nu\rangle \leq d^2$. Now we have

$$R^2 \geq |y-x+R\nu|^2 = |y-x|^2 + 2R\langle y-x, \nu\rangle + R^2,$$

so

$$|y-x|^2 \leq 2R\langle x-y, \nu\rangle \leq 2R\varepsilon\langle x, \nu\rangle \leq 2Rd^2\varepsilon,$$

as required. □

2.8 Discretization

Lemma 2.6. *Let* $\varepsilon, \rho \in \left(0, \frac{1}{2}\right)$. *Let* $K \in \mathcal{K}_R$. *Let* $x, x', y \in \operatorname{bd}(K)$ *and let* ν *and* ν' *be the outer unit normals to* $\operatorname{bd}(K)$ *at* x *and* x' *respectively. Assume that* $|x-x'|^2 + |\nu-\nu'|^2 \le \rho^2$ *and* $\langle y, \nu\rangle \ge \left(1-\frac{\varepsilon}{2}\right)\langle x, \nu\rangle$. *Then*

$$\langle y, \nu'\rangle \ge \Big(1 - \frac{\varepsilon}{2} - 2\rho(\rho + \varepsilon d^2 + |y-x|)\Big)\langle x', \nu'\rangle.$$

Proof. We have

$$\begin{aligned}\langle y, \nu'\rangle = \langle x, \nu'\rangle + \langle y-x, \nu'\rangle = \\ \langle x', \nu'\rangle + \langle x-x', \nu'\rangle + \langle y-x, \nu\rangle + \langle y-x, \nu'-\nu\rangle \ge \\ \langle x', \nu'\rangle + \langle x-x', \nu'-\nu\rangle + \langle y-x, \nu\rangle + \langle y-x, \nu'-\nu\rangle \ge \\ \langle x', \nu'\rangle - \rho^2 - \frac{\varepsilon}{2}\langle x, \nu\rangle - \rho|y-x|.\end{aligned}$$

Here, when passing from the second line to the third one, we used the inequality $\langle x - x', \nu\rangle \ge 0$.

Note now that, since $\frac{1}{2}\mathbf{B}^d \subset K \subset d^2\mathbf{B}^d$, we have

$$\langle x, \nu\rangle = \langle x, \nu'\rangle + \langle x, \nu-\nu'\rangle \le \langle x', \nu'\rangle + \rho d^2$$

and $\langle x', \nu'\rangle \ge \frac{1}{2}$. So

$$\begin{aligned}\langle y, \nu'\rangle \ge \Big(1-\frac{\varepsilon}{2}\Big)\langle x', \nu'\rangle - \rho\Big(\rho + \frac{\varepsilon d^2}{2} + |y-x|\Big) \ge \\ \Big(1 - \frac{\varepsilon}{2} - 2\rho(\rho + \varepsilon d^2 + |y-x|)\Big)\langle x', \nu'\rangle.\end{aligned}$$

□

Recall that our task is to find a finite set of points $Y \subset \operatorname{bd}(K)$ such that $(1-\varepsilon)K \subset \operatorname{conv} Y$. This requirement is equivalent to the statement that for every $x \in \operatorname{bd}(K)$, there exists $y \in Y$ such that $\langle y, \nu\rangle \ge (1-\varepsilon)\langle x, \nu\rangle$, where ν is the outer unit normal to $\operatorname{bd}(K)$ at x.

Lemma 2.6 implies that it would suffice to show the existence of $y \in Y$ satisfying a slightly stronger inequality $\langle y, \nu\rangle \ge \left(1-\frac{\varepsilon}{2}\right)\langle x, \nu\rangle$ for every point x in the Bronshtein–Ivanov net only, provided that we can ensure that $2\rho(\rho + \varepsilon d^2 + |y-x|) \le \frac{\varepsilon}{2}$.

To this end, we apply Lemma 2.5, which shows that the inequality $\langle y, \nu\rangle \ge \left(1-\frac{\varepsilon}{2}\right)\langle x, \nu\rangle$ automatically implies the distance bound $|y-x| \le \sqrt{2R}d\sqrt{\frac{\varepsilon}{2}} = d\sqrt{R}\sqrt{\varepsilon}$. Thus, if we choose $\rho = \frac{1}{4(d^2+1+d\sqrt{R})}\sqrt{\varepsilon}$, we will be in good shape.

By Lemma 2.4, the size N of the corresponding Bronshtein-Ivanov net is at most $8^d(d^2+3)^d(d^2+1+d\sqrt{R})^d\varepsilon^{-\frac{d-1}{2}} = C(d)\varepsilon^{-\frac{d-1}{2}}$, which has the correct power of ε already. However, $C(d)$ is superexponential in d, which prevents us from just using the full Bronshtein–Ivanov net for Y and forces us to work a bit harder.

2.9 Rogers' trick

We now remind the reader of a simple abstract construction which may be traced back to Rogers [Rog57].

Lemma 2.7. *Let $\mathcal{S} = \{S_1, \ldots, S_N\}$ be a family of measurable subsets of a probability space (U, μ) such that for some $\vartheta > 0$, we have $\mu(S_i) \geq \vartheta$ for all $i = 1, \ldots, N$. Then there exists a set Y of cardinality at most $\lceil \vartheta^{-1}\log(N\vartheta)\rceil + \vartheta^{-1}$ that intersects each S_i.*

Here $\lceil z \rceil$ stands for the least non-negative integer greater than or equal to z.

Proof. First we choose M points randomly and independently according to μ and obtain a random set Y_0. For every fixed $i \in \{1, \ldots, N\}$, we have

$$\mathbb{P}\{Y_0 \cap S_i = \varnothing\} \leq (1-\vartheta)^M \leq e^{-\vartheta M}.$$

Hence, the expected number of sets $S_i \in \mathcal{S}$ disjoint from Y_0 is at most $Ne^{-\vartheta M}$. Choosing one additional point in each such set, we shall get a set Y of cardinality $Ne^{-\vartheta M} + M$ intersecting all S_i. Putting $M = \lceil \vartheta^{-1}\log(N\vartheta)\rceil$, we get the desired bound. □

Now, let $K \in \mathcal{K}_R$. Suppose that we can construct a probability measure μ on $\mathrm{bd}\,(K)$ such that for every $x \in \mathrm{bd}\,(K)$ and every $\varepsilon > 0$, we have $\mu(S(x,\varepsilon)) \geq p\varepsilon^{\frac{d-1}{2}}$ with some $p > 0$.

We take the Bronshtein–Ivanov net X of K constructed in Section 2.6. Its cardinality N does not exceed $C(d)\varepsilon^{-\frac{d-1}{2}}$, where $C(d)$ is of order $e^{O(d\log d)}$. Consider the caps $S(x, \frac{\varepsilon}{2}), x \in X$. By Lemma 2.7, there exists a set $Y \subset \mathrm{bd}\,(K)$ of cardinality at most $\lceil 2^{\frac{d-1}{2}}p^{-1}\varepsilon^{-\frac{d-1}{2}}\log(C(d)2^{-\frac{d-1}{2}}p)\rceil + 2^{\frac{d-1}{2}}p^{-1}\varepsilon^{-\frac{d-1}{2}}$ that intersects each of those caps. If $p = e^{O(d)}$, then the cardinality of Y is of order $e^{O(d)}\varepsilon^{-\frac{d-1}{2}}$.

2.10 The construction of the measure

Let n be a positive integer (we shall need both $n = d$ and $n = d - 1$). Recall that for a convex body $K \subset \mathbb{R}^n$ containing the origin in its interior, its polar body $K^\circ \subset \mathbb{R}^n$ is defined by

$$K^\circ = \{y \in \mathbb{R}^n \ : \ \langle x, y\rangle \leq 1 \quad \text{for all} \quad x \in K\}.$$

We shall need the following well-known (but, in part, highly non-trivial) facts about the polar bodies:
Fact 1. If K has a smooth boundary and is strictly convex, that is, K contains no line segment on its boundary, then the relation $\langle x, x^*\rangle = 1$, $x \in \mathrm{bd}\,(K)$, $x^* \in \mathrm{bd}\,(K^\circ)$, defines a continuous one to one mapping * from $\mathrm{bd}\,(K)$ to $\mathrm{bd}\,(K)^\circ$. The vector x^* is just $\frac{\nu}{\langle x,\nu\rangle}$, where ν is the outer unit normal to $\mathrm{bd}\,(K)$ at x (see [Sch14], Corollary 1.7.3, page 40).
Fact 2. For any convex body $K \subset \mathbb{R}^n$ containing the origin in its interior, we have $\mathrm{vol}_n(K)\mathrm{vol}_n(K^\circ) \geq e^{O(n)}n^{-n}$ (see [BM87], [GPV14], [Kup08], [Naz12]).
Fact 3. If K is a convex body with the center of mass at the origin, then

$$\mathrm{vol}_n(K)\,\mathrm{vol}_n(K^\circ) \leq e^{O(n)}n^{-n}$$

(see [MP90]).

Lemma 2.8. *Let $K \subset \mathbb{R}^d$ be a strictly convex body with smooth boundary. Assume that K contains the origin in its interior and satisfies the Santaló bound $\mathrm{vol}_d(K)\,\mathrm{vol}_d(K^\circ) \leq e^{O(d)}d^{-d}$. For any Borel set $S \subset \mathrm{bd}\,(K)$, define $S^* = \{x^* \in \mathrm{bd}\,(K^\circ) \ : \ x \in S\}$. Consider the "cones" $C(S) = \{rx \ : \ x \in S, 0 \leq r \leq 1\}$ and $C(S^*) = \{ry \ : \ y \in S^*, 0 \leq r \leq 1\}$ and put*

$$\mu(S) = \frac{1}{2}\Big(\frac{\mathrm{vol}_d(C(S))}{\mathrm{vol}_d(K)} + \frac{\mathrm{vol}_d(C(S^*))}{\mathrm{vol}_d(K^\circ)}\Big).$$

Then μ is a probability measure on $\mathrm{bd}\,(K)$ invariant under linear automorphisms of $\mathbb{R}^d$ and $\mu(S(x,\varepsilon)) \geq e^{O(d)}\varepsilon^{\frac{d-1}{2}}$ for all $x \in \mathrm{bd}\,(K)$ and all $\varepsilon \in (0, \frac{1}{2})$.

Proof. The invariance of μ under linear automorphisms of $\mathbb{R}^d$ follows immediately from the general properties of the volume with respect to linear transformations and the relation $(TK)^\circ = (T^{-1})^*K^\circ$.

Fix $x \in \mathrm{bd}\,(K)$. Apply an appropriate linear transformation to put the body K in such a position that $x = x^* = e = (0, \ldots, 0, 1) \in \mathbb{R}^d$. Then $S = S(x, \varepsilon)$ is given by $\langle x, e\rangle \geq 1 - \varepsilon$. Let $Q \subset e^\perp \cong \mathbb{R}^{d-1}$ be the convex body such that $(1 - \varepsilon)e + Q$ is the cross-section of K by the hyperplane $\{x \ : \ \langle x, e\rangle = 1 - \varepsilon\}$. Let $\widetilde{K} = K \cap \{x \ : \ \langle x, e\rangle \leq 1 - \varepsilon\}$.

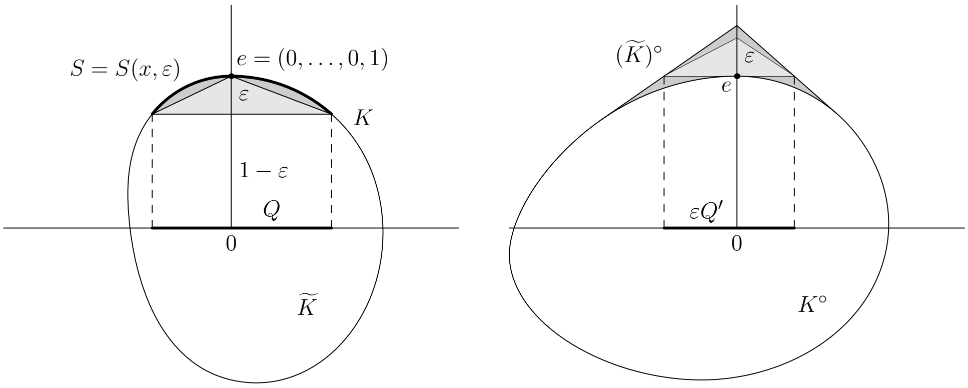

Figure 2.4: The regions $K \setminus \widetilde{K}$ and $(\widetilde{K})^\circ \setminus K^\circ$

Our first goal will be to show that

$$\mathrm{vol}_d(K \setminus \widetilde{K})\, \mathrm{vol}_d((\widetilde{K})^\circ \setminus K^\circ) \geq \frac{1}{d^2}\varepsilon^{d+1}\, \mathrm{vol}_{d-1}(Q)\, \mathrm{vol}_{d-1}(Q'),$$

where $Q' \subset e^\perp$ is the polar body to Q in $\mathbb{R}^{d-1}$.

To this end, note that $K \setminus \widetilde{K}$ contains the interior of the pyramid $\mathrm{conv}(\{e\} \cup (1-\varepsilon)e + Q)$ of height ε with the base $(1-\varepsilon)e + Q$, so

$$\mathrm{vol}_d(K \setminus \widetilde{K}) \geq \frac{1}{d}\varepsilon\, \mathrm{vol}_{d-1}(Q).$$

We claim now that the interior of the pyramid $\Pi = \mathrm{conv}\{(1+\varepsilon)e, e+\varepsilon Q'\}$ is contained in $(\widetilde{K})^\circ \setminus K^\circ$ (see Figure 2.4). Since $K^\circ \subset \{y \,:\, \langle y, e\rangle \leq 1\}$, and $\mathrm{int}\, \Pi \subset \{y \,:\, \langle y, e\rangle > 1\}$, it suffices to show that $\Pi \subset (\widetilde{K})^\circ$.

To this end, take $x \in \widetilde{K}$, and let $\langle x, e\rangle = 1 - t\varepsilon$, $t \geq 1$, so $x = (1-t\varepsilon)e + x'$, where $x' \in e^\perp$.

Since $e \in K$, by the convexity of K, $x' \in tQ$ (see Figure 2.5). Now, $\langle x, (1+\varepsilon)e\rangle = (1-t\varepsilon)(1+\varepsilon) \leq 1$, hence, $(1+\varepsilon)e \in (\widetilde{K})^\circ$. Let $y = e + \varepsilon y'$ with $y' \in Q'$. Then $\langle x, y\rangle = 1 - t\varepsilon + \varepsilon\langle x', y'\rangle \leq 1 - t\varepsilon + t\varepsilon = 1$. Thus, $e + \varepsilon Q' \subset (\widetilde{K})^\circ$. It follows by the convexity of $(\widetilde{K})^\circ$ that $\Pi \subset (\widetilde{K})^\circ$, and, therefore,

$$\mathrm{vol}_d((\widetilde{K})^\circ \setminus K^\circ) \geq \mathrm{vol}_d(\Pi) = \frac{1}{d}\varepsilon^d\, \mathrm{vol}_{d-1}(Q').$$

Multiplying these two estimates, we get the desired inequality.

On the other hand, we have $\mathrm{int}(K \setminus \widetilde{K}) \subset C(S) \setminus (1-\varepsilon)C(S)$, and

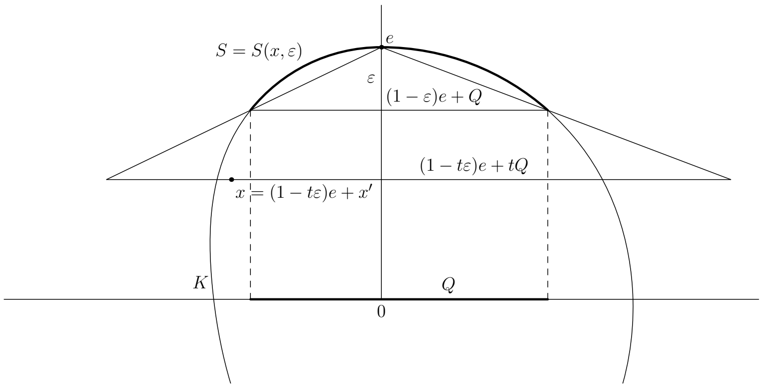

Figure 2.5: The cross-section of K by the hyperplane $\{x : \langle x, e\rangle = 1 - t\varepsilon\}$ is contained in tQ

$\mathrm{int}((\widetilde{K})^\circ \setminus K^\circ) \subset (1-\varepsilon)^{-1}C(S^*) \setminus C(S^*)$. Hence,

$$\mathrm{vol}_d(K \setminus \widetilde{K})\,\mathrm{vol}_d((\widetilde{K})^\circ \setminus K^\circ) \leq \\ (1-(1-\varepsilon)^d)((1-\varepsilon)^{-d} - 1)\,\mathrm{vol}_d(C(S))\,\mathrm{vol}_d(C(S^*)) \leq \\ e^{O(d)}\varepsilon^2\,\mathrm{vol}_d(C(S))\,\mathrm{vol}_d(C(S^*)).$$

Combining it with the previous estimate and using Fact 2, we get

$$\mathrm{vol}_d(C(S))\,\mathrm{vol}_d(C(S^*)) \geq e^{O(d)}\varepsilon^{d-1}\,\mathrm{vol}_{d-1}(Q)\,\mathrm{vol}_{d-1}(Q') \geq \\ e^{O(d)}\varepsilon^{d-1}(d-1)^{-(d-1)}.$$

Finally, since $\mathrm{vol}_d(K)\mathrm{vol}_d(K^\circ) \leq e^{O(d)}d^{-d}$, we get

$$\mu(S) \geq \frac{1}{2}\Big(\frac{\mathrm{vol}_d(C(S))}{\mathrm{vol}_d(K)} + \frac{\mathrm{vol}_d(C(S^*))}{\mathrm{vol}_d(K^\circ)}\Big) \geq \\ \sqrt{\frac{\mathrm{vol}_d(C(S))\,\mathrm{vol}_d(C(S^*))}{\mathrm{vol}_d(K)\mathrm{vol}_d(K^\circ)}} \geq e^{O(d)}\varepsilon^{\frac{d-1}{2}},$$

as required. □

This lemma, together with the discussion in Section 2.9, completes the proof of the theorem.

Part II

Covering

Chapter 3

Covering: some positive results

3.1 Introduction

This chapter is based on [Nas16a]. Given two sets K and L in $\mathbb{R}^d$ (resp. $\mathbb{S}^d$), and our goal is to cover K by as few translates (resp. rotated copies) of L as possible. Upper bounds for these kind of covering problems are often obtained by probabilistic methods, that is, by taking randomly chosen copies of L. We present a method that relies on an algorithmic result of Lovász [Lov75], D. Johnson [Joh74] and Stein [Ste74] independently, and yields proofs that are simple, non-probabilistic and quite uniform through different geometric settings.

For two Borel measurable sets K and L in $\mathbb{R}^d$, let $N(K, L)$ denote the *translative covering number* of K by L, ie. the minimum number of translates of L that cover K.

Definition 3.1. Let K and L be bounded Borel measurable sets in $\mathbb{R}^d$. A *fractional covering* of K by translates of L is a Borel measure μ on $\mathbb{R}^d$ with $\mu(x-L) \geq 1$ for all $x \in K$. The *fractional covering number* of K by translates of L is

$$N^*(K, L) =$$
$$\inf \left\{\mu(\mathbb{R}^d) \ : \ \mu \text{ is a fractional covering of } K \text{ by translates of } L\right\}.$$

Clearly, in Definition 3.8 we may assume that a fractional cover μ is supported on $\operatorname{cl}(K - L)$. According to Theorem 1.7 of [AAS15]), we have

$$\max\left\{\frac{\operatorname{vol}(K)}{\operatorname{vol}(L)}, 1\right\} \leq N^*(K, L) \leq \frac{\operatorname{vol}(K - L)}{\operatorname{vol}(L)}. \tag{3.1}$$

Here, the upper bound is easy to see, as the Lebesgue measure restricted to $K - L$ with the following scaling $\mu = \operatorname{vol}(\cdot) / \operatorname{vol}(L)$ is a fractional cover of K by translates of L.

For two sets $K, T \subset \mathbb{R}^d$, we denote their *Minkowski difference* by $K \sim T = \{x \in \mathbb{R}^d \, : \, T + x \subseteq K\}$.

Theorem 3.2. *Let K, L and T be bounded Borel measurable sets in $\mathbb{R}^d$ and let $\Lambda \subset \mathbb{R}^d$ be a finite set with $K \subseteq \Lambda + T$. Then*

$$N(K, L) \leq \tag{3.2}$$
$$(1 + \ln(\max_{x \in K-L} \operatorname{card}((x + (L \sim T)) \cap \Lambda))) \cdot N^*(K - T, L \sim T).$$

If $\Lambda \subset K$, then we have

$$N(K, L) \leq \tag{3.3}$$
$$(1 + \ln(\max_{x \in K-L} \operatorname{card}((x + (L \sim T)) \cap \Lambda))) \cdot N^*(K, L \sim T).$$

For a set $K \subset \mathbb{R}^d$ and $\delta > 0$, we denote the δ*-inner parallel body* of K by $K_{-\delta} := K \sim \mathbf{B}^d(o, \delta) = \{x \in K \, : \, \mathbf{B}^d(x, \delta) \subseteq K\}$, where $\mathbf{B}^d(x, \delta)$ denotes the Euclidean ball of radius δ centered at x. As an application of Theorem 3.2, we will obtain

Theorem 3.3. *Let $K \subseteq \mathbb{R}^d$ be a bounded measurable set. Then there is a covering of $\mathbb{R}^d$ by translated copies of K of density at most*

$$\inf_{\delta > 0} \left[\frac{\operatorname{vol}(K)}{\operatorname{vol}(K_{-\delta})} \left(1 + \ln \frac{\operatorname{vol}\left(K_{-\delta/2}\right)}{\operatorname{vol}\left(\mathbf{B}^d\left(o, \frac{\delta}{2}\right)\right)} \right) \right].$$

The δ-inner parallel body could be defined with respect to a norm that is distinct from the Euclidean. As is easily seen from the proof, the theorem would still hold.

Now, we turn to coverings on the sphere. We denote the Haar probability measure on $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ by σ, the closed spherical cap of spherical radius φ centered at $u \in \mathbb{S}^d$ by $C(u, \varphi)$, and its measure by $\Omega(\varphi) = \sigma(C(u, \varphi))$. For a set $K \subset \mathbb{S}^d$ and $\delta > 0$, we denote the δ*–inner parallel body* of K by $K_{-\delta} = \{u \in K \, : \, C(u, \delta) \subseteq K\}$.

A set $K \subset \mathbb{S}^d$ is called *spherically convex*, if it is contained in an open hemisphere and for any two of its points, it contains the shorter great circular arc connecting them.

The *spherical circumradius* of a subset of an open hemisphere of $\mathbb{S}^d$ is the spherical radius of the smallest spherical cap (the *circum-cap*) that contains the set.

Theorem 3.4. *Let $K \subseteq \mathbb{S}^d$ be a measurable set. Then there is a covering of $\mathbb{S}^d$ by rotated copies of K of density at most*

$$\inf_{\delta > 0} \left[\frac{\sigma(K)}{\sigma(K_{-\delta})} \left(1 + \ln \frac{\sigma\left(K_{-\delta/2}\right)}{\Omega\left(\frac{\delta}{2}\right)} \right) \right].$$

Corollary 3.5. *Let $K \subseteq \mathbb{S}^d$ be a spherically convex set of spherical circumradius ρ. Then there is a covering of $\mathbb{S}^d$ by rotated copies of K of density at most*

$$\inf_{\kappa>0\,:\,K_{-(\kappa\rho)}\neq\emptyset}\left[\frac{\sigma(K)}{\sigma(K)-\Omega(\rho)\left(1-(1-\kappa)^d\right)}\left(2d+d\ln\frac{1}{\kappa\rho}\right)\right].$$

We prove the Euclidean results in Section 3.4, and the spherical results in Section 3.5.

Main ideas

Given a set family $\mathcal{F}$ on a base set X, a *covering* of X is a subfamily of $\mathcal{F}$ whose union is X. Finding the minimum size of a covering is an integer programming problem (IP), which is generally difficult. Its linear programming (LP) relaxation, that is, when the integrality assumption on the variables is dropped is easier.

Our key tool is Lemma 3.9, a combinatorial result, which states that if in a set family, each set is of cardinality at most N, then the 'integrality gap', that is, the gap between the solution of the IP and the solution of the LP is roughly a $\ln N$ multiplicative factor. Moreover, it states that the solution to the IP can be found by a deterministic algorithm, up to this $\ln N$ multiplicative error factor.

In our geometric setting, X is not an abstract set, but a metric space (a convex body in $\mathbb{R}^d$, or the sphere, etc.). We first consider a metric net Λ of X, that is, we discretize the problem, and then apply this combinatorial result for Λ as the base set, and the intersection of members of $\mathcal{F}$ with Λ as a set family. Essentially, a volume argument will yield upper bounds on N, the maximum cardinality of sets thus obtained, and allow us to apply the combinatorial result described above.

3.2 History

An important point in the theory of coverings in geometry is the following theorem of Rogers [Rog57]. For a definition of the covering density, cf. [Rog64].

Theorem 3.6 (Rogers, [Rog57])**.** *Let K be a bounded convex set in $\mathbb{R}^d$ with non-empty interior. Then the covering density of K is at most*

$$\theta(K) \leq d\ln d + d\ln\ln d + 5d. \tag{3.4}$$

We will obtain Theorem 3.6 as a corollary to our more general Theorem 3.3.

Another classical example of a geometric covering problem is the following. Estimate the minimum number of spherical caps of radius φ needed to cover the sphere $\mathbb{S}^d$ in $\mathbb{R}^{d+1}$.

Theorem 3.7 (Böröczky Jr. and Wintsche, [BW03])**.** *Let* $0 < \varphi < \frac{\pi}{2}$*. Then there is a covering of* $\mathbb{S}^d$ *by spherical caps of radius* φ *with density at most* $d \ln d + d \ln \ln d + 5d$*.*

This estimate was proved in [BW03] improving an earlier result of Rogers [Rog63]. The current best bound is better when $\varphi < \frac{\pi}{3}$: Dumer [Dum07] gave a covering in this case of density at most $\frac{d \ln d}{2}$.

We will obtain Theorem 3.7 as a corollary to our more general Theorem 3.4.

The fractional version of $N(K, \operatorname{int}(K))$ (see Definition 3.8) first appeared in [Nas09] and in general for $N(K, L)$ in [AAR11] and [AAS15].

3.3 Preliminaries

We start with introducing some combinatorial notions.

Definition 3.8. Let Y be a set, $\mathcal{F}$ a family of subsets of Y and $X \subseteq Y$. A *covering* of X by $\mathcal{F}$ is a subset of $\mathcal{F}$ whose union contains X. The *covering number* $\tau(X, \mathcal{F})$ of X by $\mathcal{F}$ is the minimum cardinality of its coverings by $\mathcal{F}$.

A *fractional covering* of X by $\mathcal{F}$ is a measure μ on $\mathcal{F}$ with

$$\mu(\{F \in \mathcal{F} \;:\; x \in F\}) \geq 1 \quad \text{for all } x \in X.$$

The *fractional covering number* of $\mathcal{F}$ is

$$\tau^*(X, \mathcal{F}) = \inf \{\mu(\mathcal{F}) \;:\; \mu \text{ is a fractional covering of } X \text{ by } \mathcal{F}\}.$$

When a group G acts on Y and $\mathcal{F}$ is the set $\{g(A) \;:\; g \in G\}$ for some fixed subset A of Y, we will identify $F \in \mathcal{F}$ with $\{g \in G \;:\; g(A) = F\} \subseteq G$ and thus, we will call a measure μ on G a fractional covering of X by G if

$$\mu(\{g \in G \;:\; x \in g(A)\}) \geq 1 \quad \text{for all } x \in X.$$

For more on (fractional) coverings, cf. [Für88] in the abstract (combinatorial) setting and [Mat02] in the geometric setting.

The gap between τ and τ^* is bounded in the case of finite set families (hypergraphs) by the following result, independently obtain by Lovász [Lov75], Stein [Ste74] and D. Johnson [Joh74].

Lemma 3.9 (D. Johnson, Lovász, Stein)**.** *For any finite* Λ *and* $\mathcal{H} \subseteq 2^\Lambda$ *we have*

$$\tau(\Lambda, \mathcal{H}) < (1 + \ln(\max_{H\in\mathcal{H}} \operatorname{card}(H)))\tau^*(\Lambda, \mathcal{H}). \tag{3.5}$$

Furthermore, the greedy algorithm (always picking the set that covers the highest number of uncovered points) yields a covering of cardinality less than the right hand side in (3.5)*.*

The following straightforward corollary to Lemma 3.9 is a key element of our proofs.

Observation 3.10. Let Y be a set, $\mathcal{F}$ a family of subsets of Y, and $X \subseteq Y$. Let Λ be a finite subset of Y and $\Lambda \subseteq U \subseteq Y$. Assume that for another family $\mathcal{F}'$ of subsets of Y we have $\tau(X, \mathcal{F}) \leq \tau(\Lambda, \mathcal{F}')$. Then

$$\tau(X, \mathcal{F}) \leq \tau(\Lambda, \mathcal{F}') \leq (1 + \ln(\max_{F'\in\mathcal{F}'} \operatorname{card}(\Lambda \cap F'))) \cdot \tau^*(U, \mathcal{F}'). \tag{3.6}$$

We will rely on the following estimates of $\Omega(\varphi)$, the probability measure of a cap of spherical radius φ, by Böröczky and Wintsche [BW03].

Lemma 3.11 (Böröczky – Wintsche [BW03])**.** *Let* $0 < \varphi < \pi/2$*.*

$$\Omega(\varphi) > \frac{\sin^d \varphi}{\sqrt{2\pi(d+1)}}, \tag{3.7}$$

$$\Omega(\varphi) < \frac{\sin^d \varphi}{\sqrt{2\pi n}\cos\varphi}, \quad \text{if } \varphi \leq \arccos\frac{1}{\sqrt{d+1}}, \tag{3.8}$$

$$\Omega(t\varphi) < t^d\Omega(\varphi), \quad \text{if } 1 < t < \frac{\pi}{2\varphi}. \tag{3.9}$$

The following is known as Jordan's inequality:

$$\frac{2x}{\pi} \leq \sin x \quad \text{for} \quad x \in [0, \pi/2] \tag{3.10}$$

3.4 Proof of the covering results in $\mathbb{R}^d$

We present these proofs in the order of their difficulty. In this way, ideas and technicalities are –perhaps– easier to separate.

Proof of Theorem 3.2. The proof is simply a substitution into (3.6). We take $Y = \mathbb{R}^d$, $X = K$, $\mathcal{F} = \{L + x \,:\, x \in K - L\}$, $\mathcal{F}' = \{L \sim T + x \,:\, x \in K - L\}$. One can take $U = K - T$ as any member of Λ not in $K - T$ could be dropped from Λ and Λ would still have the property that $\Lambda + T \supseteq K$. That proves (3.2). To prove (3.3), we notice that in the case when $\Lambda \subset K$, one can take $U = K$. □

Proof of Theorem 3.3. Let C denote the cube $C = [-a, a]^d$, where $a > 0$ is large. Our goal is to cover C by translates of K economically.

Fix $\delta > 0$, and let $\Lambda \subset \mathbb{R}^d$ be a finite set such that $\Lambda + \mathbf{B}^d(o, \delta/2)$ is a saturated (ie. maximal) packing of $\mathbf{B}^d(o, \delta/2)$ in $C + \mathbf{B}^d(o, \delta/2)$. Thus, by the maximality, we have that Λ is a δ-net of C with respect to the Euclidean distance, ie. $\Lambda + \mathbf{B}^d(o, \delta) \supseteq C$.

By considering volume, for any $x \in \mathbb{R}^d$ we have

$$\operatorname{card}(\Lambda \cap (x + K_{-\delta})) \leq \frac{\operatorname{vol}\left(K_{-\delta} + \mathbf{B}^d(o, \delta/2)\right)}{\operatorname{vol}(\mathbf{B}^d(o, \delta/2))} \leq \frac{\operatorname{vol}\left(K_{-\delta/2}\right)}{\operatorname{vol}(\mathbf{B}^d(o, \delta/2))}. \tag{3.11}$$

Let $\varepsilon > 0$ be fixed. Clearly, if a is sufficiently large then

$$N^*(C + \mathbf{B}^d(o, \delta/2), K_{-\delta}) \leq \frac{\operatorname{vol}\left(C + \mathbf{B}^d(o, \delta/2) - K_{-\delta}\right)}{\operatorname{vol}(K_{-\delta})} \leq (1+\varepsilon)\frac{\operatorname{vol}(C)}{\operatorname{vol}(K_{-\delta})}. \tag{3.12}$$

By (3.2), (3.11) and (3.12) we have

$$N(C, K) \leq (1+\varepsilon)\left(1 + \ln \frac{\operatorname{vol}\left(K_{-\delta/2}\right)}{\operatorname{vol}(\mathbf{B}^d(o, \delta/2))}\right)\frac{\operatorname{vol}(C)}{\operatorname{vol}(K_{-\delta})}.$$

Finally,

$$\theta(K) \leq N(C, K)\operatorname{vol}(K) / \operatorname{vol}(C)$$

yields the promised bound. □

Proof of Theorem 3.6. Let C denote the cube $C = [-a, a]^d$, where $a > 0$ is large. Our goal is to cover C by translates of K economically. First, consider the case when $K = -K$.

Let $\delta > 0$ be fixed (to be chosen later) and let $\Lambda \subset \mathbb{R}^d$ be a finite set such that $\Lambda + \frac{\delta}{2}K$ is a saturated (ie. maximal) packing of $\frac{\delta}{2}K$ in $C - \frac{\delta}{2}K$. Thus, by the maximality, we have that Λ is a δ-net of C with respect to K, ie. $\Lambda + \delta K \supseteq C$. By considering volume, for any $x \in \mathbb{R}^d$ we have

$$\operatorname{card}(\Lambda \cap (x + (1-\delta)K)) \leq \frac{\operatorname{vol}\left((1-\delta)K + \frac{\delta}{2}K\right)}{\operatorname{vol}\left(\frac{\delta}{2}K\right)} \leq \left(\frac{2}{\delta}\right)^d. \tag{3.13}$$

Let $\varepsilon > 0$ be fixed. Clearly, if a is sufficiently large then

$$N^*(C - \delta K, (1-\delta)K) \leq (1+\varepsilon)\frac{\operatorname{vol}(C)}{(1-\delta)^d \operatorname{vol}(K)}. \tag{3.14}$$

By (3.2), (3.13) and (3.14) we have

$$N(C, K) \leq \frac{1+\varepsilon}{(1-\delta)^d}\left(1 + d\ln\left(\frac{2}{\delta}\right)\right)\frac{\operatorname{vol}(C)}{\operatorname{vol}(K)}.$$

On the other hand,

$$\theta(K) \leq N(C,K)\,\mathrm{vol}\,(K)\,/\,\mathrm{vol}\,(C) \leq \frac{1+\varepsilon}{(1-\delta)^d}\left(1+d\ln\left(\frac{2}{\delta}\right)\right). \tag{3.15}$$

We choose $\delta = \frac{1}{2d\ln d}$, and the following standard computation

$$(1+\varepsilon)^{-1}\theta(K) \leq (1+d\ln(4d\ln d))\exp(1/\ln d) \tag{3.16}$$
$$\leq (1+d\ln(4d\ln d))\,(1+2/\ln d) \leq (d\ln d + d\ln\ln d + 5d)\,,$$

yields the desired bound (as ε can be taken arbitrarily close to 0).

Next, consider the general case, that is when K is not necessarily symmetric about the origin. We need to make the following modifications. Milman and Pajor (cf. Corollary 3 of [MP00]) showed that, if the centroid (that is, the center of mass) of K is the origin, then $\mathrm{vol}\,(K\cap -K) \geq \frac{\mathrm{vol}(K)}{2^d}$. (Note that the existence of a translate of K for which this inequality holds was proved by Stein [Ste56] using a probabilistic argument.) We define Λ as a saturated packing of translates of $\frac{\delta}{2}(K\cap -K)$ in $C - \frac{\delta}{2}(K\cap -K)$. Thus, we have $C \subseteq \Lambda + \delta(K\cap -K) \subseteq \Lambda + \delta K$. Instead of (3.13), we now have

$$\mathrm{card}\,(\Lambda\cap(x+(1-\delta)K)) \leq \left(\frac{4}{\delta}\right)^d.$$

for any $x\in\mathbb{R}^d$. Rolling this change through the proof, at the end in place of (3.15), we obtain

$$\theta(K) \leq \frac{1+\varepsilon}{(1-\delta)^d}\left(1+d\ln\left(\frac{4}{\delta}\right)\right),$$

which, however, is still less than $(1+\varepsilon)\,(d\ln d + d\ln\ln d + 5d)$ with the same choice of $\delta = \frac{1}{2d\ln d}$. □

3.5 Proof of the spherical results

Proof of Theorem 3.4. Let Λ be the set of centers of a saturated (ie. maximal) packing of caps of radius $\delta/2$. Clearly, Λ is a δ-net of $\mathbb{S}^d$, and thus, if we cover Λ by rotated copies of radius $K_{-\delta}$, then the same rotations yield a covering of $\mathbb{S}^d$ by copies of K.

Let $\bar\sigma$ denote the probability Haar measure on $SO(d+1)$. Let $H\subset\mathbb{S}^d$ be a measurable set, and denote the family of rotated copies of H by $\mathcal{F}(H) = \{AH\ :\ A\in SO(d+1)\}$. Recall that for any fixed $u\in\mathbb{S}^d$ we have

$$\bar\sigma(\{A\in SO(d+1)\ :\ u\in AH\}) =$$
$$\bar\sigma(\{A\in SO(d+1)\ :\ u\in A^{-1}H\}) =$$
$$\bar\sigma(\{A\in SO(d+1)\ :\ Au\in H\}) = \sigma(H).$$

It follows that the measure $\frac{\bar{\sigma}}{\sigma(H)}$ on $SO(d+1)$ is a fractional cover of $\mathbb{S}^d$ by $\mathcal{F}(H)$ and thus, $\tau^*(\mathbb{S}^d, \mathcal{F}(H)) \le \frac{1}{\sigma(H)}$.

Thus by (3.6), we obtain the following for the density of a covering by rotated images of K:

$$\begin{aligned}\text{density } &\le \sigma(K)\tau(\mathbb{S}^d, \mathcal{F}(K)) \le \sigma(K)\tau(\Lambda, \mathcal{F}(K_{-\delta}))\\ &\le (1+\ln(\max_{A\in SO(d+1)} \text{card}\,(\Lambda\cap AK_{-\delta}))) \cdot \frac{\sigma(K)}{\sigma(K_{-\delta})}\\ &\le \frac{\sigma(K)}{\sigma(K_{-\delta})}\left(1+\ln\frac{\sigma\left(K_{-\delta/2}\right)}{\Omega\left(\frac{\delta}{2}\right)}\right).\end{aligned}$$

Since it holds for any $\delta > 0$, the theorem follows. □

Proof of Theorem 3.7. We will apply Theorem 3.4 with K being a cap of spherical radius φ. We set $\delta = \eta\varphi$, where η will be specified later. By Theorem 3.4 and (3.9), we obtain for the density of a covering of $\mathbb{S}^d$ by caps of radius φ:

$$\text{density } \le \left(1 + d\ln\left(\frac{2}{\eta}\right)\right)\cdot\left(\frac{1}{1-\eta}\right)^d.$$

We choose $\eta = \frac{1}{2d\ln d}$, and the same computation as in (3.16) yields the desired bound. □

Proof of Corollary 3.5. We set $\delta = \kappa\rho$. First, observe that the measure of the belt-like region $K \setminus K_{-\delta}$ at the boundary of K is at most as large as the measure of the belt-like region $C(v,\rho)\setminus C(c,\rho-\delta)$ at the boundary of the circum-cap $C(v,\rho)$ of K.

Next, combine $\ln\frac{\sigma\left(K_{-\delta/2}\right)}{\Omega\left(\frac{\delta}{2}\right)} \le \ln\frac{1}{\Omega\left(\frac{\delta}{2}\right)}$ with (3.9) and (3.10) to obtain the statement. □

Chapter 4

Covering: a negative result

4.1 Introduction

This chapter is based on [Nas16c]. Let K be a convex body in $\mathbb{R}^d$. Following Hadwiger [Had60], we say that a point $p \in \mathbb{R}^d \setminus K$ *illuminates* a boundary point $b \in \mathrm{bd}\,(K)$, if the ray $\{p + \lambda(b-p) \ : \ \lambda > 0\}$ emanating from p and passing through b intersects the interior of K. The main result of the chapter is the following.

Theorem 4.1. *Let $1 < D < 1.116$ be given. Then for any sufficiently large dimension d, there is an o-symmetric convex body K in $\mathbb{R}^d$, with illumination number*

$$\mathrm{i}(K) = N(K, \mathrm{int}\,(K)) \geq .05D^d, \tag{4.1}$$

for which

$$\frac{1}{D}\mathbf{B}^d \subset K \subset \mathbf{B}^d. \tag{4.2}$$

In all previous chapters, the goal was to show that a certain type of approximation or covering *is possible using few objects* (points, translates, etc.). Theorem 4.1 follows the opposite direction: it states that there is a convex body which *needs many* directions to illuminate, that is, many translates of its interior to cover it.

We will use a probabilistic construction to find K. We are not aware of any lower bound for the Illumination Problem that was obtained by a probabilistic argument.

For a point $u \in \mathbb{S}^{d-1}$, and $0 < \varphi < \pi/2$, let $C(u, \varphi) = \{v \in \mathbb{S}^{d-1} \ : \ \sphericalangle(u, v) \leq \varphi\}$ denote the spherical cap centered at u of angular radius φ. We denote the normalized Lebesgue measure (that is, the Haar probability measure on $\mathbb{S}^{d-1}$) of $C(u, \varphi)$ by $\Omega_{d-1}(\varphi)$.

In Theorem 4.2, we give an upper bound for the illumination number for bodies close to the Euclidean ball. It follows from [BK09] but, for the sake of completeness, we will sketch a proof.

Theorem 4.2. *Let K be a convex body in $\mathbb{R}^d$ such that $\frac{1}{D}\mathbf{B}^d \subset K \subset \mathbf{B}^d$ for some $D > 1$. Then the illumination number of K is at most*

$$\mathrm{i}(K) \leq \frac{d \ln d + d \ln \ln d + 5d}{\Omega_{d-1}(\alpha)}, \tag{4.3}$$

where $\alpha = \arcsin(1/D)$.

By combining Theorem 4.2 with the estimate (3.8) on Ω_{d-1}, one can see that (4.1) is asymptotically sharp, that is, the base D cannot be improved.

As mentioned in the Introduction of this dissertation, as an application of Theorem 4.1, we obtain that the illumination parameter $\mathrm{ill}(K)$ and the vertex index $\mathrm{vein}(K)$ are very far from each other for the convex body K that we constructed.

Main ideas

We pick N (to be determined later) points, $X_1, \ldots, X_N$ independently and uniformly on the Euclidean unit sphere, and set

$$K = \mathrm{conv}\left(\{\pm X_i \ : \ i \in [N]\} \cup \frac{1}{D}\mathbf{B}^{d+1}\right).$$

On the one hand, if N is not too large, then for any fixed $j \in [N]$, with high probability, the set of directions illuminating K at X_j is the same as the set of directions illuminating $\mathrm{conv}\left(\{X_j\} \cup \frac{1}{D}\mathbf{B}^{d+1}\right)$, which is a cap. In other words, with high probability, the other random points do not 'interfere' with X_j.

On the other hand, if N is not too small, then with high probability, these caps are difficult to hit, that is, there is no small cardinality set of directions such that for all $j \in [N]$, the cap of good directions corresponding to X_j is intersected by the set. It will follow from the fact that, with high probability, no direction belongs to too many caps of good directions.

We will show that for any d and D (given as in Theorem 4.1), there is an N, which is neither too large, nor too small.

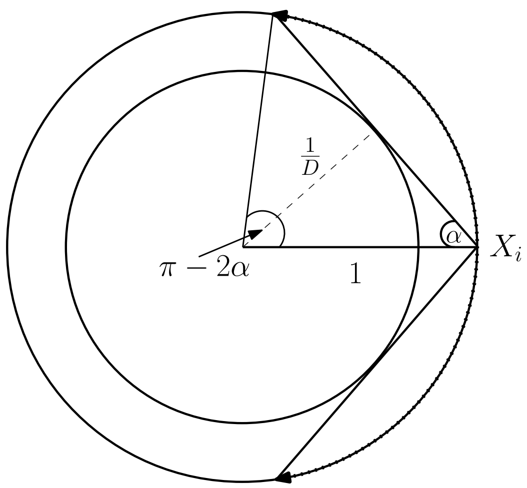

Figure 4.1: Event E_1: when X_j falls on the dotted cap (the arc with arrows at its endpoints) or on its reflection about the origin.

4.2 Construction of a Spiky Ball

We work in $\mathbb{R}^{d+1}$ instead of $\mathbb{R}^d$ to obtain slightly simpler formulas. We describe a probabilistic construction of $K \subset \mathbb{R}^{d+1}$ which is close to the Euclidean ball and has a large illumination number.

Let N be a fixed positive integer, to be given later. We pick N points, $X_1, \ldots, X_N$ independently and uniformly on the Euclidean unit sphere $\mathbb{S}^d$ of $\mathbb{R}^{d+1}$. Let

$$K = \operatorname{conv}\left(\{\pm X_i \ : \ i \in [N]\} \cup \frac{1}{D}\mathbf{B}^{d+1}\right). \tag{4.4}$$

Clearly, K is o-symmetric and $\frac{1}{D}\mathbf{B}^{d+1} \subset K \subset \mathbf{B}^{d+1}$. We need to bound the illumination number of K from below. Let $\frac{\pi}{4} < \alpha < \frac{\pi}{2}$ be such that $\sin\alpha = 1/D$.

We define two "bad" events, E_1 and E_2. Let E_1 be the event that there are $i \neq j \in [N]$ with $\sphericalangle(X_i, X_j) < \pi - 2\alpha$ or $\sphericalangle(-X_i, X_j) < \pi - 2\alpha$. We observe that if E_1 does not occur, then for all $i \in [N]$ we have

(4.5) *the set of directions (a subset of $\mathbb{S}^d$) that illuminate K at X_i is the spherical cap centered at $-X_i$ of spherical radius α.*

We want to prove that, with non-zero probability, no point of $\mathbb{S}^d$ belongs to too many of these caps. Thus, to illuminate K at each X_i, we will need many directions.

Let $T \in \mathbb{Z}^+$ be fixed, to be specified later. Let E_2 be the event that there is a direction $u \in \mathbb{S}^d$ with $\operatorname{card}\left(C(u,\alpha) \cap \{\pm X_i \ : \ i \in [N]\}\right) > T$.

Observe that if neither E_1 nor E_2 occur, then $\mathrm{i}(K) \geq 2N/T$. However, it is difficult to bound the probability of E_2. Thus, we will replace E_2 by a "more finite" condition E_2' as follows.

We fix a $\delta > 0$. We call a set $\Lambda \subset \mathbb{S}^d$ a *metric δ-net*, if $\cup_{v\in\Lambda} C(v,\delta) = \mathbb{S}^d$, that is, if the caps of radius δ centered at the points of Λ cover the sphere. By (3.7), the measure of a cap of radius δ is larger than $\frac{\sin^d(\delta)}{3\sqrt{d}}$. Thus, Theorem 1 of [Rog63] yields that, there is a covering of the sphere by at most $d^2/\sin^d(\delta)$ caps of radius δ. That is, there is a metric δ-net Λ of size at most $\mathrm{card}\,(\Lambda) \leq d^2/\sin^d(\delta)$.

Let $p = 2\Omega_d(\alpha+\delta)$. Let $\Theta > 1$ be fixed, and set $T = N\Theta p$. We define the event E_2' as follows: there is a direction $v \in \Lambda$ with $\mathrm{card}\,(C(v,\alpha+\delta) \cap \{\pm X_i \,:\, i \in [N]\}) > N\Theta p$. Clearly, if E_2 occurs, then so does E_2'. Thus, we have

$$(\mathrm{not}(E_1) \text{ and } \mathrm{not}(E_2')) \text{ implies } \mathrm{i}(K) \geq 2/(\Theta p). \tag{4.6}$$

Now, we need to set our parameters such that the event (*not*(E_1) *and* *not*(E_2')) is of positive probability and $2/(\Theta p)$ is exponentially large in the dimension.

Clearly,

$$\mathbb{P}(E_1) \leq N^2\Omega_d(\pi - 2\alpha). \tag{4.7}$$

Consider a fixed $v \in \Lambda$. When X_i is picked randomly, the probability that v is contained in $C(X_i, \alpha+\delta)$ or in $C(-X_i, \alpha+\delta)$ is p (recall that $p = 2\Omega_d(\alpha+\delta)$). Thus, the probability that v is contained in more than $N\Theta p$ caps of the form $C(\pm X_i, \alpha+\delta)$ is $\mathbb{P}(\xi > N\Theta p)$, where ξ is a binomial random variable of distribution $\mathtt{Binom}(N,p)$. Thus,

$$\mathbb{P}(E_2') \leq \frac{d^2}{\sin^d(\delta)}\mathbb{P}(\xi > N\Theta p) \quad \text{with } \xi \sim \mathtt{Binom}(N,p). \tag{4.8}$$

By a Chernoff-type inequality, (cf. p. 64 of [MU05]),

$$\mathbb{P}(\xi > N\Theta p) < 2^{-N\Theta p}, \quad \text{for any} \quad \Theta \geq 6. \tag{4.9}$$

Consider the following three inequalities.

$$N \leq \left(\frac{1}{4\Omega_d(\pi - 2\alpha)}\right)^{1/2}, \tag{4.10}$$

$$\frac{d^2}{\sin^d\delta}2^{-\Theta N p} \leq \frac{1}{4}, \tag{4.11}$$

$$6 \leq \Theta. \tag{4.12}$$

Combining (4.6), (4.7), (4.8) and (4.9), we obtain the following. If there are $N \in \mathbb{Z}^+, \delta > 0$ and $\Theta \geq 0$ (all depending on d) such that the three inequalities (4.10),(4.11) and (4.12) hold, then there is a $K \subset \mathbb{R}^{d+1}$ o-symmetric convex body with $\mathrm{i}(K) \geq 2/(\Theta p)$, where $p = 2\Omega_d(\alpha+\delta)$. In fact, in this case, our construction yields such a K with probability at least 1/2.

Now, (4.11) holds if $\Theta N p > 2d \log_2 \frac{1}{\sin\delta}$. Thus, an integer N satisfying (4.10) and (4.11) exists if

$$4d \log_2 \frac{1}{\sin\delta} \leq \Theta p \left(\frac{1}{4\Omega_d(\pi - 2\alpha)} \right)^{1/2},$$

which we rewrite as

$$\frac{1}{\Theta p} \leq \frac{1}{8d(\Omega_d(\pi-2\alpha))^{1/2} \log_2 \frac{1}{\sin\delta}}.$$

By (3.10), we can replace it by the following stronger inequality:

$$\frac{1}{\Theta p} \leq \frac{1}{24d(\Omega_d(\pi-2\alpha))^{1/2} \log_2 (1/\delta)}. \tag{4.13}$$

On the other hand, by substituting the value of p, we see that (4.12) is equivalent to

$$\frac{1}{\Theta p} \leq \frac{1}{12\Omega_d(\alpha+\delta)}. \tag{4.14}$$

Finally, let $\delta = \frac{\alpha}{d}$.

Since $1 < D = \frac{1}{\sin\alpha} < 1.116$, we have that $1.11 < \alpha < \pi/2$, and thus, $\sin^2(\alpha+\delta) > \sin(\pi-2\alpha)$. Now, by Lemma 3.11, (4.14) is a stronger inequality than (4.13). Thus, so far we have that if we can satisfy (4.14), then the proof is complete.

By (3.9), we have that (4.14) holds, if

$$\frac{1}{\Theta p} \leq \frac{1}{36\Omega_d(\alpha)}. \tag{4.15}$$

By (3.8), it holds for $\frac{1}{\Theta p} = \frac{1}{36} D^d$. Since $\mathrm{i}(K) \geq 2/(\Theta p)$, this finishes the proof of Theorem 4.1.

Remark 4.3. The body K is not a polytope. However, the construction can easily be modified to obtain a polytope. One simply replaces the ball of radius $1/D$ by a sufficiently dense finite subset A of this ball in the definition of K as follows: $K = \mathrm{conv}\left(\{\pm X_i \,:\, i \in [N]\} \cup A\right)$.

Proof of Theorem 4.2. Since $\frac{1}{D}\mathbf{B}^d \subset K \subset \mathbf{B}^d$, it follows that for any boundary point b of K, the set of directions (as a subset of $\mathbb{S}^{d-1}$) that illuminate K at b contains an open spherical cap of radius $\alpha = \arcsin(1/D)$. Thus, any subset A of $\mathbb{S}^{d-1}$ that pierces each such cap illuminates K. However, finding such A is equivalent to finding a covering of $\mathbb{S}^{d-1}$ by caps of radius α. Such a covering of the desired size exists by Theorem 3.4, cf. [Rog63], [BW03]. □

Part III

Quantitative Helly-type questions

Chapter 5

Solution of the Bárány–Katchalski–Pach Conjecture

5.1 Introduction and Preliminaries

This chapter is based on [Nas16b]. The main result of the present chapter is the confirmation of a conjecture of Bárány, Katchalski and Pach [BKP82] described in the Introduction of this dissertation.

Theorem 5.1. *Let $\mathcal{F}$ be a family of convex sets in $\mathbb{R}^d$ such that the volume of its intersection is* $\mathrm{vol}\,(\cap\mathcal{F}) > 0$. *Then there is a subfamily $\mathcal{G}$ of $\mathcal{F}$ with* $\mathrm{card}\,(\mathcal{G}) \leq 2d$ *and* $\mathrm{vol}\,(\cap\mathcal{G}) \leq e^{d+1}d^{2d+\frac{1}{2}}\,\mathrm{vol}\,(\cap\mathcal{F})$.

The order of magnitude d^{cd} in the Theorem (and in the conjecture in [BKP82]) is sharp as we show in Section 5.3.

Recently, other quantitative Helly type results have been obtained by De Loera, La Haye, Rolnick and Soberón [DLLHRS17].

We introduce notations and tools that we will use in the proof. The tensor product $u \otimes u$ is the rank one linear operator that maps any $x \in \mathbb{R}^d$ to the vector $(u \otimes u)x = \langle u, x\rangle\, u \in \mathbb{R}^d$. For a set $A \subset \mathbb{R}^d$, we denote its polar by $A^\circ = \{y \in \mathbb{R}^d : \langle x, y\rangle \leq 1, \text{ for all } x \in A\}$.

Definition 5.2. We say that a set of vectors $w_1, \ldots, w_m \in \mathbb{R}^d$ with weights $c_1, \ldots, c_m > 0$ form a *John's decomposition of the identity*, if

$$\sum_{i=1}^{m} c_i w_i = o \quad \text{and} \quad \sum_{i=1}^{m} c_i w_i \otimes w_i = I, \tag{5.1}$$

where I is the identity operator on $\mathbb{R}^d$.

We recall John's theorem [Joh48] (see also [Bal97]).

Lemma 5.3 (John's theorem). *For any convex body K in $\mathbb{R}^d$, there is a unique ellipsoid of maximal volume in K. Furthermore, this ellipsoid is $\mathbf{B}^d$ if, and only if, there are points $w_1, \dots, w_m \in \operatorname{bd}\left(\mathbf{B}^d\right) \cap \operatorname{bd}(K)$ (called* contact points*) and corresponding weights $c_1, \dots, c_m > 0$ that form a John's decomposition of the identity.*

It is not difficult to see that if $w_1, \dots, w_m \in \operatorname{bd}\left(\mathbf{B}^d\right)$ and corresponding weights $c_1, \dots, c_m > 0$ form a John's decomposition of the identity, then $\{w_1, \dots, w_m\}^\circ \subset d\mathbf{B}^d$, cf. [Bal97] or Theorem 5.1 in [GLMP04]. By polarity, we also obtain that $\frac{1}{d}\mathbf{B}^d \subset \operatorname{conv}(w_1, \dots, w_m)$.

One can verify that if Δ is a regular simplex in $\mathbb{R}^d$ such that the ball $\mathbf{B}^d$ is the largest volume ellipsoid in Δ, then

$$\operatorname{vol}(\Delta) = \frac{d^{d/2}(d+1)^{(d+1)/2}}{d!}. \tag{5.2}$$

We will use the following form of the Dvoretzky-Rogers lemma [DR50].

Lemma 5.4 (Dvoretzky-Rogers lemma). *Assume that $w_1, \dots, w_m \in \operatorname{bd}\left(\mathbf{B}^d\right)$ and $c_1, \dots, c_m > 0$ form a John's decomposition of the identity. Then there is an orthonormal basis $z_1, \dots, z_d$ of $\mathbb{R}^d$, and a subset $\{v_1, \dots, v_d\}$ of $\{w_1, \dots, w_m\}$ such that*

$$v_i \in \operatorname{span}\{z_1, \dots, z_i\}, \quad \textit{and} \quad \sqrt{\frac{d-i+1}{d}} \leq \langle v_i, z_i \rangle \leq 1, \quad \textit{for } i = 1, \dots, d. \tag{5.3}$$

This lemma is usually stated in the setting of John's theorem, that is, when the vectors are contact points of a convex body K with its maximal volume ellipsoid, which is $\mathbf{B}^d$. And often, it is assumed in the statement that K is symmetric about the origin, see for example [BGVV14]. Since we make no such assumption (in fact, we make no reference to K in the statement of Lemma 5.4), we give a proof in Section 5.4.

It may be challenging to give a geometric interpretation of John's decomposition, or of the Dvoretzky-Rogers lemma. In our application, it would be the following. If the unit ball is the largest volume ellipsoid in a convex body, then there are contact points that are not too unevenly distributed on the unit sphere. In fact, one can select d of them whose convex hull together with the origin forms a not too flat simplex. As we will see later, we can bound from below the volume of this not too flat simplex using the Dvoretzky-Rogers lemma.

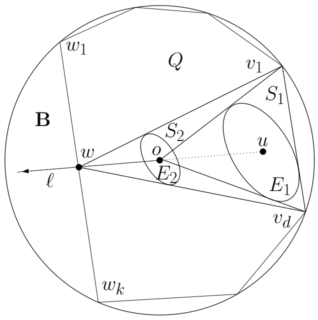

Figure 5.1:

Main ideas

Using John's theorem, we notice that the volume of a convex body, and its 'ellipsoid volume', that is, the volume of the largest ellipsoid that it contains are the same up to a multiplicative d^d, which is not a big loss in our setting. Thus, our plan is the following. By the affine invariance of the problem, we may assume that $\mathbf{B}^d$ is the largest volume ellipsoid of $\cap\mathcal{F}$. We may also assume that all members of $\mathcal{F}$ are supporting half-spaces of $\mathbf{B}^d$.

Clearly, it is sufficient to find $2d$ of these half-spaces whose intersection is of not too large volume.

By John's theorem, the contact points of these half-spaces form a John's decomposition of the identity (with appropriate weights). We find d of these contact points $v_1, \dots, v_d$ using the Dvoretzky-Reogers lemma that span a not too small volume simplex whose $(d+1)$-st vertex is the origin. The John ellipsoid of this simplex is not too small. By some additional geometric considerations, we can shrink this ellipsoid with a proper center of homothety to obtain a new ellipsoid whose volume is still not too small, and which is centered at the origin. Moreover, this new ellipsoid will be contained in the convex hull of $v_1, \dots, v_{2d}$, where $v_{d+1}, \dots, v_{2d}$ are properly chosen 'opposite' points for $v_1, \dots, v_d$. Finally, the polar of this not too small, origin centered ellipsoid is a not too large origin centered ellipsoid, which contains the intersection of the $2d$ half-spaces whose contact points are $v_1, \dots, v_{2d}$, just as needed.

5.2 Proof of Theorem 5.1

Without loss of generality, we may assume that $\mathcal{F}$ consists of closed half-spaces, and also that $\operatorname{vol}(\cap\mathcal{F}) < \infty$, that is, $\cap\mathcal{F}$ is a convex body in $\mathbb{R}^d$. As shown in [BKP84], by continuity, we may also assume that $\mathcal{F}$ is a finite family, that is $P = \cap\mathcal{F}$ is a d-dimensional polyhedron.

The problem is clearly affine invariant, so we may assume that $\mathbf{B}^d \subset P$ is the ellipsoid of maximal volume in P.

By Lemma 5.3, there are contact points $w_1, \dots, w_m \in \mathrm{bd}\left(\mathbf{B}^d\right) \cap \mathrm{bd}\,(P)$ (and weights $c_1, \dots, c_m > 0$) that form a John's decomposition of the identity. We denote their convex hull by $Q = \mathrm{conv}\,(w_1 \dots, w_m)$. Lemma 5.4 yields that there is an orthonormal basis $z_1, \dots, z_d$ of $\mathbb{R}^d$, and a subset $\{v_1, \dots, v_d\}$ of the contact points $\{w_1, \dots, w_m\}$ such that (5.3) holds.

Let $S_1 = \mathrm{conv}\,(o, v_1, v_2, \dots, v_d)$ be the simplex spanned by these contact points, and let E_1 be the largest volume ellipsoid contained in S_1. We denote the center of E_1 by u. Let ℓ be the ray emanating from the origin in the direction of the vector $-u$. Clearly, the origin is in the interior of Q. In fact, by the remark following Lemma 5.3, $\frac{1}{d}\mathbf{B}^d \subset Q$. Let w be the point of intersection of the ray ℓ with $\mathrm{bd}\,(Q)$. Then $|w| \geq 1/d$. Let S_2 denote the simplex $S_2 = \mathrm{conv}\,(w, v_1, v_2, \dots, v_d)$. See Figure 5.1.

We apply a contraction with center w and ratio $\lambda = \frac{|w|}{|w-u|}$ on E_1 to obtain the ellipsoid E_2. Clearly, E_2 is centered at the origin and is contained in S_2. Furthermore,

$$\lambda = \frac{|w|}{|u| + |w|} \geq \frac{|w|}{1 + |w|} \geq \frac{1}{d+1}. \tag{5.4}$$

Since w is on $\mathrm{bd}\,(Q)$, by Caratheodory's theorem, w is in the convex hull of some set of at most d vertices of Q. By re-indexing the vertices, we may assume that $w \in \mathrm{conv}\,(w_1, \dots, w_k)$ with $k \leq d$. Now,

$$E_2 \subset S_2 \subset \mathrm{conv}\,(w_1, \dots, w_k, v_1, \dots, v_d)\,. \tag{5.5}$$

Let $X = \{w_1, \dots, w_k, v_1, \dots, v_d\}$ be the set of these unit vectors, and let $\mathcal{G}$ denote the family of those half-space which support $\mathbf{B}^d$ at the points of X. Clearly, $|\mathcal{G}| \leq 2d$. Since the points of X are contact points of P and $\mathbf{B}^d$, we have that $\mathcal{G} \subseteq \mathcal{F}$. By (5.5),

$$\cap\mathcal{G} = X^\circ \subset E_2^\circ. \tag{5.6}$$

By (5.3),

$$\mathrm{vol}\,(S_1) \geq \frac{1}{d!} \cdot \frac{\sqrt{d!}}{d^{d/2}} = \frac{1}{\sqrt{d!}d^{d/2}}. \tag{5.7}$$

Since $\mathbf{B}^d \subset \cap\mathcal{F}$, by (5.6) and (5.4), (5.2), (5.7) we have

$$\frac{\mathrm{vol}\,(\cap\mathcal{G})}{\mathrm{vol}\,(\cap\mathcal{F})} \leq \frac{\mathrm{vol}\,(E_2^\circ)}{\mathrm{vol}\,(\mathbf{B}^d)} = \frac{\mathrm{vol}\left(\mathbf{B}^d\right)}{\mathrm{vol}\,(E_2)} \leq (d+1)^d \frac{\mathrm{vol}\left(\mathbf{B}^d\right)}{\mathrm{vol}\,(E_1)} = (d+1)^d \frac{\mathrm{vol}\,(\Delta)}{\mathrm{vol}\,(S_1)} \tag{5.8}$$

$$= \frac{d^{d/2}(d+1)^{(3d+1)/2}}{d!\,\mathrm{vol}\,(S_1)} = \frac{d^d d^{3d/2} e^{3/2}(d+1)^{1/2}}{(d!)^{1/2}} \leq e^{d+1} d^{2d+\frac{1}{2}},$$

where Δ is as defined above (5.2). This completes the proof of Theorem 5.1.

Remark 5.5. In the proof, in place of the Dvoretzky-Rogers lemma, we could select the d vectors $v_1, \dots, v_d$ from the contact points randomly: picking w_i with probability c_i/d for $i = 1, \dots, m$, and repeating this picking independently d times. Pivovarov proved (cf. Lemma 3 in [Piv10]) that the expected volume of the random simplex S_1 obtained this way is the same as the right hand side in (5.7).

5.3 A simple lower bound for $v(d)$

We outline a simple proof that one cannot hope a better bound in Theorem 5.1 than $d^{d/2}$ in place of $d^{2d+1/2}$. Indeed, consider the Euclidean ball $\mathbf{B}^d$, and a family $\mathcal{F}$ of (a large number of) supporting closed half space of $\mathbf{B}^d$ whose intersection is very close to $\mathbf{B}^d$. Suppose that $\mathcal{G}$ is a subfamily of $\mathcal{F}$ of $2d$ members. Denote by σ the Haar probability measure on the sphere $R \cdot \mathbb{S}^{d-1}$ of radius R, where $R = (d/(2\ln d))^{\frac{1}{2}}$. Let $H \in \mathcal{G}$ be one of the half spaces. Then,

$$\sigma(R \cdot \mathbb{S}^{d-1} \setminus H) \leq \exp\left(\frac{-d}{2R^2}\right) \leq 1/(4d).$$

It follows that

$$\mathrm{vol}\,(\cap\mathcal{G}) \geq R^d\,\mathrm{vol}\left(\mathbf{B}^d\right)\sigma(R \cdot \mathbb{S}^{d-1} \setminus (\cup\mathcal{G})) \geq \frac{1}{2} R^d\,\mathrm{vol}\left(\mathbf{B}^d\right) \geq d^{\frac{d}{2}-\varepsilon}\,\mathrm{vol}\,(\cap\mathcal{F})$$

for any $\varepsilon > 0$ if d is large enough.

5.4 Proof of Lemma 5.4

We follow the proof in [BGVV14].

Claim 5.6. *Assume that $w_1, \dots, w_m \in \mathrm{bd}\left(\mathbf{B}^d\right)$ and $c_1, \dots, c_m > 0$ form a John's decomposition of the identity. Then for any linear map $T : \mathbb{R}^d \to \mathbb{R}^d$ there is an $\ell \in \{1, \dots, m\}$ such that*

$$\langle w_\ell, T w_\ell \rangle \geq \frac{\mathrm{trace}\,(T)}{d}, \tag{5.9}$$

where trace (T) *denotes the trace of* T.

For matrices $A, B \in \mathbb{R}^{d\times d}$ we use $\langle A, B\rangle = \operatorname{trace}\left(AB^T\right)$ to denote their Frobenius product.

To prove the claim, we observe that

$$\frac{\operatorname{trace}(T)}{d} = \frac{1}{d}\langle T, I\rangle = \frac{1}{d}\sum_{i=1}^{m} c_i \langle T, w_i \otimes w_i\rangle = \frac{1}{d}\sum_{i=1}^{m} c_i \langle Tw_i, w_i\rangle .$$

Since $\sum_{i=1}^{m} c_i = d$, the right hand side is a weighted average of the values $\langle Tw_i, w_i\rangle$. Clearly, some value is at least the average, yielding Claim 5.6.

We define z_i and v_i inductively. First, let $z_1 = v_1 = w_1$. Assume that, for some $k < d$, we have found z_i and v_i, for all $i = 1, \dots, k$. Let $F = \operatorname{span}\{z_1, \dots, z_k\}$, and let T be the orthogonal projection onto the orthogonal complement $F^\perp$ of F. Clearly, $\operatorname{trace}(T) = \dim F^\perp = d - k$. By Claim 5.6, for some $\ell \in \{1, \dots, m\}$ we have

$$|Tw_\ell|^2 = \langle Tw_\ell, w_\ell\rangle \geq \frac{d-k}{d}.$$

Let $v_{k+1} = w_\ell$ and $z_{k+1} = \frac{Tw_\ell}{|Tw_\ell|}$. Clearly, $v_{k+1} \in \operatorname{span}\{z_1, \dots, z_{k+1}\}$. Moreover,

$$\langle v_{k+1}, z_{k+1}\rangle = \frac{\langle Tw_\ell, w_\ell\rangle}{|Tw_\ell|} = \frac{|Tw_\ell|^2}{|Tw_\ell|} = |Tw_\ell| \geq \sqrt{\frac{d-k}{d}},$$

finishing the proof of Lemma 5.4.

Note that in this proof, we did not use the fact that, in a John's decomposition of the identity, the vectors are balanced, that is $\sum_{i=1}^{m} c_i w_i = o$.

Chapter 6

A Colorful Quantitative Volume Theorem

6.1 Introduction

In this chapter, which is based on joint work with Damásdi and Földvári [DFN20], we consider a generalization of Helly's Theorem, known as the *Colorful Helly Theorem*, proved by Lovász, and later by Bárány [Bár82], see Theorem 6.7, and we prove quantitative variants of this colorful result.

6.1.1 Ellipsoids and volume

A well known consequence of John's Theorem (Lemma 5.3) that we used in the previous chapter, is that

(6.1) *if $\mathcal{E}$ is the unique maximal volume ellipsoid contained in the convex body K in $\mathbb{R}^d$, then $\mathcal{E}$ enlarged around its center by a factor d contains K.*

It follows that the volume of the largest ellipsoid contained in K is of volume at least $d^{-d}\,\mathrm{vol}\,(K)$. More precise bounds for this volume ratio are known (cf. [Bal97]), but we will not need them.

This observation yields that bounding the volume of intersections and bounding the volume of ellipsoids contained in the intersections are essentially equivalent problems: the only difference is a multiplicative factor d^d which is of no consequence, unless one wants to find the best constants in the exponent. Thus, from this point on, we phrase our results in terms of the volume of ellipsoids contained in intersections. Its benefit is that this is how in the proofs we actually "find volume": we find ellipsoids of large volume.

In short, we will use the following straight-forward corollary of Theorem 5.1.

Corollary 6.1 (Quantitative Volume Theorem with Ellipsoids). *Let $C_1, \dots, C_n$ be convex sets in $\mathbb{R}^d$. Assume that the intersection of any $2d$ of them contains an ellipsoid of volume at least 1. Then $\bigcap_{i=1}^{n} C_i$ contains an ellipsoid of volume at least $c^d d^{-3d}$ with an absolute constant $c > 0$.*

We could use the improved version of Theorem 5.1 by Brazitikos [Bra17] to obtain the bound $c^d d^{-5d/2}$ however, this constant in the exponent is of no consequence.

We note that quantitative Helly-type results are sometimes stated only for half-spaces and not convex sets in general, as is the case for example in [Bra17]. It yields no loss of generality, as any convex set can be approximated (in any meaningful metric) by the intersection of finitely many half-spaces.

6.1.2 Main result: few color classes

The main result of the present chapter is the following.

Theorem 6.2 (Colorful Quantitative Volume Theorem with Ellipsoids – Few Color Classes). *Let $\mathcal{C}_1, \dots, \mathcal{C}_{3d}$ be finite families of convex bodies in $\mathbb{R}^d$. Assume that for any colorful selection of $2d$ sets, $C_{i_k} \in \mathcal{C}_{i_k}$ for each $1 \le k \le 2d$ with $1 \le i_1 < \dots < i_{2d} \le 3d$, the intersection $\bigcap_{k=1}^{2d} C_{i_k}$ contains an ellipsoid of volume at least 1. Then, there exists an $1 \le i \le 3d$ such that $\bigcap_{C \in \mathcal{C}_i} C$ contains an ellipsoid of volume at least $c^{d^2} d^{-3d^2}$ with an absolute constant $c \ge 0$.*

We rephrase this theorem in terms of the volume of intersections, as this form may be more easily applicable.

Theorem 6.3 (Colorful Quantitative Volume Theorem – Few Color Classes). *Let $\mathcal{C}_1, \dots, \mathcal{C}_{3d}$ be finite families of convex bodies in $\mathbb{R}^d$. Assume that for any colorful selection of $2d$ sets, $C_{i_k} \in \mathcal{C}_{i_k}$ for each $1 \le k \le 2d$ with $1 \le i_1 < \dots < i_{2d} \le 3d$, the intersection $\bigcap_{k=1}^{2d} C_{i_k}$ is of volume at least 1.*

Then, there exists an $1 \le i \le 3d$ such that $\operatorname{vol}\left(\bigcap_{C \in \mathcal{C}_i} C\right) \ge c^{d^2} d^{-7d^2/2}$ with an absolute constant $c \ge 0$.

Observe that the smaller the number of color classes in a colorful Helly-type theorem, the stronger the theorem is. For example, the Colorful Helly Theorem (see p. 9) is stated with $d+1$ color classes, but it is easy to see that it implies the same result with $\ell \geq d+2$ color classes, as the last $\ell-(d+1)$ color classes make the assumption of the theorem stronger and the conclusion weaker. We note also that the Colorful Helly Theorem does not hold with less than $d+1$ color classes, as the number $d+1$ cannot be replaced by any smaller number in Helly's Theorem.

The novelty of the proof of Theorem 6.2 is the following. As we will see later, similar looking statements can be obtained by taking the Quantitative Volume Theorem as a "basic" Helly-type theorem, and combining it with John's Theorem and a combinatorial argument. This approach yields results with $d(d+3)/2$ color classes, but does not seem to yield results with fewer color classes. In order to achieve that, first, we introduce an ordering on the set of ellipsoids, and second, we give a finer geometric examination of the situation by comparing the maximum volume ellipsoid of a convex body K to other ellipsoids contained in K.

We find it an intriguing question whether one can decrease the number of color classes to $2d$ (possibly with an even weaker bound on the volume of the ellipsoid obtained), and whether an order d^{-cd} lower bound on the volume of the ellipsoid can be shown.

6.1.3 Earlier results and simple observations

In 1937, Behrend [Beh37] (see also Section 6.17 of the survey [DGK63] by Danzer, Grünbaum and Klee) proved a planar quantitative Helly-type result: *If the intersection of any 5 members of a finite family of convex sets in* $\mathbb{R}^2$ *contains an ellipse of area 1, then the intersection of all members of the family contains an ellipse of area 1.* We note that, since every convex set in $\mathbb{R}^2$ is the intersection of the half-planes containing it, the result is equivalent to the formally weaker statement where the family consists of half-planes only. This is the form in which it is stated in [DGK63].

In [DGK63, Section 6.17], it is mentioned that John's Theorem (Lemma 5.3) should be applicable to extend Behrend's result to higher dimensions. We spell out this argument, and present a straightforward proof of the following.

Proposition 6.4 (Helly-type Theorem with Ellipsoids). *Let $\mathcal{C}$ be a finite family of at least $d(d+3)/2$ convex sets in $\mathbb{R}^d$, and assume that for any selection $C_1, \dots, C_{d(d+3)/2} \in \mathcal{C}$, the intersection $\bigcap_{i=1}^{d(d+3)/2} C_i$ contains an ellipsoid* *of volume 1. Then* $\bigcap_{C\in\mathcal{C}} C$ *also contains an ellipsoid of volume 1.*

We prove a colorful version of Proposition 6.4.

Proposition 6.5 (Colorful Quantitative Volume Theorem with Ellipsoids – Many Color Classes)**.** *Let* $\mathcal{C}_1,\dots,\mathcal{C}_{d(d+3)/2}$ *be finite families of convex bodies in* $\mathbb{R}^d$*, and assume that for any colorful selection* $C_1\in\mathcal{C}_1,\dots,C_{d(d+3)/2}\in\mathcal{C}_{d(d+3)/2}$*, the intersection* $\bigcap_{i=1}^{d(d+3)/2} C_i$ *contains an ellipsoid of volume 1. Then for some* j*, the intersection* $\bigcap_{C\in\mathcal{C}_j} C$ *contains an ellipsoid of volume 1.*

The proof of Proposition 6.5 consists of two parts. First, as our contribution, in the *geometric part*, we introduce an ordering on the set of ellipsoids contained in a convex set, and study properties of this ordering, see Section 6.2.2. Second, a *combinatorial part* shows that this ordering yields the statement. This second part is essentially identical to the argument given by Lovász and Bárány [Bár82] in their proof of the Colorful Helly Theorem, and it was presented in an abstract setting in [DLLHORP17, Theorem 5.3] by De Loera et al.

Sarkar, Xue and Soberón [SXS21, Corollary 1.0.5], using matroids, recently obtained a result involving $d(d+3)/2$ color classes, but with the number of selected sets being $2d$.

Proposition 6.6 (Sarkar, Xue and Soberón [SXS21])**.** *Let* $\mathcal{C}_1,\dots,\mathcal{C}_{d(d+3)/2}$ *be finite families of convex bodies in* $\mathbb{R}^d$*. Assume that for any colorful selection of* $2d$ *sets,* $C_{i_k}\in\mathcal{C}_{i_k}$ *for each* $1\le k\le 2d$ *with* $1\le i_1<\dots<i_{2d}\le d(d+3)/2$*, the intersection* $\bigcap_{k=1}^{2d} C_{i_k}$ *contains an ellipsoid of volume at least 1. Then, there exists an* $1\le i\le d(d+3)/2$ *such that* $\bigcap_{C\in\mathcal{C}_i} C$ *has volume at least* $d^{-O(d)}$*.*

For completeness, in Section 6.3.3, we sketch a brief argument showing that Proposition 6.6 immediately follows from our Proposition 6.5 and the Quantitative Volume Theorem.

The structure of the chapter is the following. In Section 6.2, we introduce some preliminary facts and definitions, notably, an ordering on the family of ellipsoids of volume at least 1 that are contained in a convex body. Section 6.3 contains the proofs of our results.

6.2 Preliminaries

6.2.1 Colorful Helly Theorem

We recall the Colorful Helly Theorem, as one of its straightforward corollaries will be used.

Theorem 6.7 (Colorful Helly Theorem, Lovász, Bárány [Bár82]). *Let $\mathcal{C}_1, \dots, \mathcal{C}_{d+1}$ be finite families of convex bodies in $\mathbb{R}^d$, and assume that for any colorful selection $C_1 \in \mathcal{C}_1, \dots, C_{d+1} \in \mathcal{C}_{d+1}$, the intersection $\bigcap\limits_{i=1}^{d+1} C_i$ is non-empty. Then for some j, the intersection $\bigcap\limits_{C \in \mathcal{C}_j} C$ is also non-empty.*

Corollary 6.8. *Let $\mathcal{C}_1, \dots, \mathcal{C}_{d+1}$ be finite families of convex bodies, and L a convex body in $\mathbb{R}^d$. Assume that for any colorful selection $C_1 \in \mathcal{C}_1, \dots, C_{d+1} \in \mathcal{C}_{d+1}$, the intersection $\bigcap\limits_{i=1}^{d+1} C_i$ contains a translate of L. Then for some j, the intersection $\bigcap\limits_{C \in \mathcal{C}_j} C$ contains a translate of L.*

Proof of Corollary 6.8. We use the following operation, the Minkowski difference of two convex sets A and B:

$$A \sim B := \bigcap_{b \in B} (A - b).$$

It is easy to see that $A \sim B$ is the set of those vectors t such that $B + t \subseteq A$.

By the assumption, for any colorful selection $C_1 \in \mathcal{C}_1, \dots, C_{d+1} \in \mathcal{C}_{d+1}$, we have $\bigcap\limits_{i=1}^{d+1} (C_i \sim L) \neq \emptyset$. By Theorem 6.7, for some j, we have $\bigcap\limits_{C \in \mathcal{C}_j} (C \sim L) \neq \emptyset$, and thus, $\bigcap\limits_{C \in \mathcal{C}_j} C$ contains a translate of L. □

6.2.2 The lowest ellipsoid

We will follow Lovász' idea of the proof of the Colorful Helly Theorem. The first step is to fix an ordering of the objects of study. This time, we are looking for an ellipsoid and not a point in the intersection, therefore we need an *ordering on the ellipsoids.*

For an ellipsoid $\mathcal{E}$, we define its *height* as the largest value of the orthogonal projection of $\mathcal{E}$ on the last coordinate axis, that is, $\max\{x^T e_d \, : \, x \in \mathcal{E}\}$, where $e_d = (0, 0, \dots, 0, 1)^T$.

Lemma 6.9. *Let C be a convex body that contains an ellipsoid of volume $\omega_d := \mathrm{vol}\left(\mathbf{B}^d\right)$. Then there is a unique ellipsoid of volume ω_d such that every other ellipsoid of volume ω_d in C has larger height. Furthermore, if $\tau \in \mathbb{R}$ denotes the height of this ellipsoid, then the largest volume ellipsoid of the convex body $H_\tau \cap C$ is this ellipsoid, where H_τ denotes the closed half-space $H_\tau = \{x \in \mathbb{R}^d \,:\, x^T e_d \leq \tau\}$.*

We call this ellipsoid the *lowest ellipsoid* in C.

Proof of Lemma 6.9. It is not difficult to see that $H_\tau \cap C$ does not contain any ellipsoid of volume larger than ω_d. Indeed, otherwise for a sufficiently small $\varepsilon > 0$, the set $H_{\tau-\varepsilon} \cap C$ would contain an ellipsoid of volume equal to ω_d, where $H_{\tau-\varepsilon}$ denotes the closed half-space $H_{\tau-\varepsilon} = \{x \in \mathbb{R}^d \,:\, x^T e_d \leq \tau - \varepsilon\}$.

Thus, by Lemma 5.3, $\mathbf{B}^d$ is the unique largest volume ellipsoid of $H_\tau \cap C$. It follows that $\mathbf{B}^d$ is the unique lowest ellipsoid of C. □

6.3 Proofs

6.3.1 Proof of Proposition 6.4

We will prove the following statement, which is clearly equivalent to Proposition 6.4.

Assume that the largest volume ellipsoid contained in $\bigcap\limits_{C \in \mathcal{C}} C$ is of volume $\omega_d := \mathrm{vol}\left(\mathbf{B}^d\right)$. Then there are $d(d+3)/2$ sets in $\mathcal{C}$ such that the largest volume ellipsoid in their intersection is of volume ω_d.

The problem is clearly affine invariant, and thus, we may assume that the largest volume ellipsoid in $\bigcap\limits_{C \in \mathcal{C}} C$ is the unit ball $\mathbf{B}^d$.

By one direction of Lemma 5.3, there are contact points $u_1, \ldots, u_m \in bd(\bigcap\limits_{C \in \mathcal{C}} C) \cap bd(\mathbf{B}^d)$ and positive numbers $\lambda_1, \ldots, \lambda_m$ with $d+1 \leq m \leq \frac{d(d+3)}{2}$ satisfying the equations in Lemma 5.3. We can choose $C_1, \ldots C_m \in \mathcal{C}$ such that $u_i \in bd(C_i)$ for $i = 1, \ldots, m$.

By the other direction of Lemma 5.3, $\mathbf{B}^d$ is the largest volume ellipsoid of $\bigcap\limits_{i=1}^{m} C_i$, completing the proof of Proposition 6.4.

6.3.2 Proof of Proposition 6.5

Lemma 6.10. *Let $C_1, \ldots, C_{d(d+3)/2}$ be convex bodies in $\mathbb{R}^d$. Assume that $K := \bigcap\limits_{i=1}^{d(d+3)/2} C_i$ contains an ellipsoid of volume $\omega_d := \mathrm{vol}\left(\mathbf{B}^d\right)$. Set $K_j :=$*

$\bigcap_{i=1, i\neq j}^{d(d+3)/2} C_i$, *and let* $\mathcal{E}$ *denote the lowest ellipsoid in* K. *Then there exists a* j *such that* $\mathcal{E}$ *is also the lowest ellipsoid of* K_j.

Proof of Lemma 6.10. Let τ denote the height of $\mathcal{E}$. By Lemma 6.9, $\mathcal{E}$ is the largest volume ellipsoid of $K \cap H_\tau$, where H_τ is the half-space defined in Lemma 6.9.

Suppose that $\mathcal{E}$ is not the lowest ellipsoid in K_j for every $j \in \{1, \ldots, d(d+3)/2\}$. Since $\mathcal{E} \subset K \subset K_j$, this means that each K_j contains a lower ellipsoid than $\mathcal{E}$ of volume ω_d. Therefore we can choose a small $\varepsilon > 0$ such that $K_j \cap H_{\tau-\varepsilon}$ contains an ellipsoid of volume ω_d for each j, where $H_{\tau-\varepsilon}$ denotes the closed half-space $H_{\tau-\varepsilon} = \{x \in \mathbb{R}^d \ : \ x^T e_d \leq \tau - \varepsilon\}$.

Let us consider now the following $\frac{d(d+3)}{2} + 1$ sets: $K_1, K_2, \ldots, K_{d(d+3)/2}, H_{\tau-\varepsilon}$. If we take the intersection of $\frac{d(d+3)}{2}$ of these sets, we obtain either K, or $K_j \cap H_{\tau-\varepsilon}$ for some j. By our assumption, K contains an ellipsoid of volume ω_d. By the choice of ε, we have that $K_j \cap H_{\tau-\varepsilon}$ also contains an ellipsoid of volume ω_d. Hence, we can apply Proposition 6.4, which yields that $C_1 \cap \cdots \cap C_{d(d+3)/2} \cap H_{\tau-\varepsilon} = K \cap H_{\tau-\varepsilon}$ also contains an ellipsoid of volume ω_d. This contradicts the fact that $\mathcal{E}$ is the lowest ellipsoid in K, and thus, Lemma 6.10 follows. □

We will prove the following statement, which is clearly equivalent to Proposition 6.5.

Assume that for every colorful selection $C_1 \in \mathcal{C}_1, \ldots, C_{d(d+3)/2} \in \mathcal{C}_{d(d+3)/2}$, *the intersection* $\bigcap_{i=1}^{d(d+3)/2} C_i$ *contains an ellipsoid of volume* ω_d. *We will show that for some* j, *the intersection* $\bigcap_{C \in \mathcal{C}_j} C$ *contains an ellipsoid of volume* ω_d.

By Lemma 6.9, we can choose the lowest ellipsoid in each of these intersections. Let us denote the set of these ellipsoids as $\mathcal{B}$. Since we have finitely many intersections, there is a highest one among these ellipsoids. Let us denote this ellipsoid by $\mathcal{E}_{max}$.

$\mathcal{E}_{max}$ is defined by some $C_1 \in \mathcal{C}_1, \ldots, C_{d(d+3)/2} \in \mathcal{C}_{d(d+3)/2}$. Once again let $K_j = \bigcap_{i=1, i\neq j}^{d(d+3)/2} C_i$ and $K = \bigcap_{i=1}^{d(d+3)/2} C_i$. By Lemma 6.10, there is a j such that $\mathcal{E}_{max}$ is the lowest ellipsoid in K_j. We will show that $\mathcal{E}_{max}$ lies in every element of $\mathcal{C}_j$ for this j.

Fix a member C_0 of $\mathcal{C}_j$. Suppose that $\mathcal{E}_{max} \not\subset C_0$. Then $\mathcal{E}_{max} \not\subset C_0 \cap K_j$. By the assumption of Proposition 6.5, $C_0 \cap K_j$ contains an ellipsoid of volume ω_d, since it is the intersection of a colorful selection of sets. Since $C_0 \cap K_j \subset K_j$, the lowest ellipsoid of $C_0 \cap K_j$ is at least as high as the lowest ellipsoid of

K_j. But the unique lowest ellipsoid of K_j is $\mathcal{E}_{max}$, and $\mathcal{E}_{max} \not\subset C_0 \cap K_j$. So the lowest ellipsoid of $C_0 \cap K_j$ lies higher than $\mathcal{E}_{max}$. This contradicts that $\mathcal{E}_{max}$ was chosen to be the highest among the ellipsoids in $\mathcal{B}$. So $\mathcal{E}_{max} \subset C_0$. Since $C_0 \in \mathcal{C}_j$ was chosen arbitrarily, we obtain that $\mathcal{E}_{max} \subset \bigcap_{C \in \mathcal{C}_j} C$, completing the proof of Proposition 6.5.

6.3.3 Proof of Proposition 6.6

Consider an arbitrary colorful selection of $d(d+3)/2$ convex bodies. By Corollary 6.1, their intersection contains an ellipsoid of volume at least $c^d d^{-3d}$. It follows immediately from Proposition 6.5, that the intersection of one of the color classes contains an ellipsoid of volume at least $c^d d^{-3d}$, completing the proof of Proposition 6.6.

6.3.4 Proof of Theorem 6.2

We will prove the following statement, which is clearly equivalent to Theorem 6.2.

Assume that the intersection of all colorful selections of $2d$ *sets contains an ellipsoid of volume at least* $\omega_d := \mathrm{vol}\left(\mathbf{B}^d\right)$. *Then, there is an* $1 \leq i \leq 3d$ *such that* $\bigcap_{C \in \mathcal{C}_i} C$ *contains an ellipsoid of volume at least* $c^{d^2} d^{-3d^2} \omega_d$ *with an absolute constant* $c \geq 0$.

Lemma 6.11. *Assume that* $\mathbf{B}^d$ *is the largest volume ellipsoid contained in the convex set* C *in* $\mathbb{R}^d$. *Let* $\mathcal{E}$ *be another ellipsoid in* C *of volume at least* $\delta\omega_d$ *with* $0 < \delta < 1$, *where* $\omega_d = \mathrm{vol}\left(\mathbf{B}^d\right)$. *Then there is a translate of* $\frac{\delta}{d^{d-1}}\mathbf{B}^d$ *which is contained in* $\mathcal{E}$.

Proof of Lemma 6.11. If the length of all d semi-axes $a_1, \ldots, a_d$ of $\mathcal{E}$ are at least λ for some $\lambda > 0$, then clearly, $\lambda\mathbf{B}^d + c \subset \mathcal{E}$, where c denotes the center of $\mathcal{E}$. We will show that all the semi-axes are long enough.

By (6.1), $\mathcal{E} \subset C \subset d\mathbf{B}^d$. Therefore, $a_i \leq d$ for every $i = 1, \ldots, d$. Since the volume of $\mathcal{E}$ is $a_1 \cdots a_d \omega_d \geq \delta\omega_d$, we have $a_i \geq \frac{\delta}{d^{d-1}}$ for every $i = 1, \ldots, d$, completing the proof of Lemma 6.11. □

Consider the lowest ellipsoid in the intersection of all colorful selections of $2d-1$ sets. We may assume that the highest one of these ellipsoids is $\mathbf{B}^d$. By possibly changing the indices of the families, we may assume that the selection is $C_1 \in \mathcal{C}_1, \ldots, C_{2d-1} \in \mathcal{C}_{2d-1}$. We call $\mathcal{C}_{2d}, \mathcal{C}_{2d+1}, \ldots, \mathcal{C}_{3d}$ the *remaining families*.

Consider the half-space $H_1 = \{x \in \mathbb{R}^d \ : \ x^T e_d \leq 1\} \supset \mathbf{B}^d$. By Lemma 6.9, $\mathbf{B}^d$ is the largest volume ellipsoid contained in $M := C_1 \cap \cdots \cap C_{2d-1} \cap H_1$.

Next, take an arbitrary colorful selection $C_{2d} \in \mathcal{C}_{2d}, C_{2d+1} \in \mathcal{C}_{2d+1}, \ldots, C_{3d} \in \mathcal{C}_{3d}$ of the remaining $d+1$ families. We claim that the intersection of any $2d$ sets of

$$C_1, \ldots, C_{2d-1}, H_1, C_{2d}, \ldots, C_{3d}$$

contains an ellipsoid of volume at least ω_d. Indeed, if H_1 is not among those $2d$ sets, then our assumption ensures this. If H_1 is among them, then by the choice of H_1, the claim holds.

Therefore, by Theorem 6.1, the intersection

$$\bigcap_{i=1}^{3d} C_i \cap H_1$$

contains an ellipsoid $\mathcal{E}$ of volume at least $\delta\omega_d$, where $\delta := c^d d^{-3d}$. Clearly, $\mathcal{E} \subset M$.

Since $\mathbf{B}^d$ is the maximum volume ellipsoid contained in M, by Lemma 6.11, we have that there is a translate of $\frac{\delta}{d^{d-1}}\mathbf{B}^d$ which is contained in $\mathcal{E}$ and thus in $\bigcap_{i=2d}^{3d} C_i$.

Thus, we have shown that any colorful selection $C_{2d} \in \mathcal{C}_{2d}, C_{2d+1} \in \mathcal{C}_{2d+1}, \ldots, C_{3d} \in \mathcal{C}_{3d}$ of the remaining $d+1$ families, $\bigcap_{i=2d}^{3d} C_i$ contains a translate of the same convex body $c^d d^{-3d}\mathbf{B}^d$. It follows from Corollary 6.8 that there is an index $2d \leq i \leq 3d$ such that $\bigcap_{C \in \mathcal{C}_i} C$ contains a translate of $c^d d^{-3d}\mathbf{B}^d$, which is an ellipsoid of volume $c^{d^2} d^{-3d^2}\omega_d$, finishing the proof of Theorem 6.2.

6.3.5 Proof of Theorem 6.3

By (6.1), the volume of the largest ellipsoid in a convex body is at least d^{-d} times the volume of the body. Theorem 6.3 now follows immediately from Theorem 6.2.

Chapter 7

A Quantitative Steinitz-type Theorem

7.1 Introduction

The goal of this chapter, which is based on joint work with G. Ivanov [IN24], is to establish a quantitative version of the following classical result of E. Steinitz [Ste13].

Proposition 7.1 (Steinitz theorem). *Let the origin belong to the interior of the convex hull of a set $S \subset \mathbb{R}^d$. Then there is a subset of at most $2d$ points of S whose convex hull contains the origin in the interior.*

The first quantitative version of this result was obtained in [BKP82], where the following statement was proven.

Proposition 7.2 (Quantitative Steinitz theorem). *There exists a constant $r = r(d) > 0$ such that for any subset Q of $\mathbb{R}^d$ whose convex hull contains the Euclidean unit ball $\mathbf{B}^d$, there exists a subset F of Q of size at most $2d$ whose convex hull contains the ball $r\mathbf{B}^d$.*

It was also shown that $r(d) > d^{-2d}$.

The main result of this chapter is a polynomial bound on $r(d)$.

Theorem 7.3 (Q.S.T. with polynomial bound). *Let Q be a subset of $\mathbb{R}^d$ whose convex hull contains the Euclidean unit ball $\mathbf{B}^d$. Then there exist at most $2d$ points of Q whose convex hull Q' satisfies*

$$\frac{1}{6d^2}\mathbf{B}^d \subset Q'.$$

We conjecture the following.

Conjecture 7.4. *There is a constant $c > 0$ such that in any subset Q of $\mathbb{R}^d$ whose convex hull contains the Euclidean unit ball $\mathbf{B}^d$, there are at most $2d$ points whose convex hull Q' satisfies*

$$\frac{c}{\sqrt{d}}\mathbf{B}^d \subset Q'.$$

We provide an upper bound on $r(d)$ as well.

Theorem 7.5. *Let $u_1, \ldots, u_n$ be unit vectors in $\mathbb{R}^d$. Then their absolute convex hull, that is the convex hull of $\pm u_1, \ldots, \pm u_n$, does not contain the ball $\left(\frac{\sqrt{n}}{d} + \varepsilon\right)\mathbf{B}^d$ for any positive ε.*

It follows that if $u_1, \ldots, u_m$ form a sufficiently dense subset of the unit sphere (with a large m), then their convex hull is almost the unit ball, while for any n of them with $n \leq 2d$, we have that their convex hull does not contain the ball $\frac{2}{\sqrt{d}}\mathbf{B}^d$, which shows that the order of magnitude of $r(d)$ in Conjecture 7.4 is sharp if the conjecture holds.

We mention the following conjecture which is closely related to Theorem 7.5. It can be found in a different formulation in [BJB$^+$04, p.194], for a history of the conjecture, originally posed by Zong, see [SZ13] and [Zon05].

Conjecture 7.6. *Let $\{u_1, \ldots, u_{2d}\}$ be unit vectors in $\mathbb{R}^d$. Then there is a point in the set*

$$\bigcap_{i=1}^{2d}\{x \in \mathbb{R}^d \ : \ \langle u_i, x\rangle \leq 1\}$$

with norm $\sqrt{d}$.

7.2 The main steps in the proof of Theorem 7.3

Since $r(1) = 1$, we will assume that $d \geq 2$ throughout the chapter.

First, we reduce the problem for the polytopal case. By the classical Carathéodory theorem [Car11, p.200], any point of a convex hull of a subset Q of $\mathbb{R}^d$ can be represented as a convex combination of at most $d+1$ points of Q. Thus, taking a sufficiently dense subset of the unit sphere, we observe that for any $\varepsilon \in (0,1)$ and any set $Q \subset \mathbb{R}^d$ whose convex hull contains $\mathbf{B}^d$, there is a finite subset Q_f of Q whose convex hull contains the ball $(1-\varepsilon)\mathbf{B}^d$. Hence, Theorem 7.3 follows from the following polytopal version.

Theorem 7.7. *Let Q be a convex polytope in $\mathbb{R}^d$ containing the Euclidean unit ball $\mathbf{B}^d$. Then there are at most $2d$ vertices of Q whose convex hull Q' satisfies*

$$\frac{1}{5d^2}\mathbf{B}^d \subset Q'.$$

Proposition 7.2 was used in [BKP82] to prove certain quantitative versions of the Helly theorem. The connection between the quantitative Steinitz result and the quantitative Helly-type result is via polar duality. Recently, Ivanov and Naszódi [IN22a] proposed a new approach to quantitative Helly-type results via sparse approximation of polytopes. The connection between sparse approximation of polytopes and quantitative Helly-type results is via polar duality again. We state a refined version of the result on sparse approximation of polytopes obtained by Almendra–Hernández, Ambrus, and Kendall in [AHAK22, Theorem 1].

Proposition 7.8 (Almendra–Hernández et. al.)**.** *Let $\lambda > 0$, and $L \subset \mathbb{R}^d$ be a convex polytope such that $L \subset -\lambda L$. Then there exist at most $2d$ vertices of L whose convex hull L' satisfies*

$$L \subset -(\lambda + 2)d \cdot L'.$$

Choosing the origin smartly, one can achieve $\lambda = d$. For instance, the following statement holds.

Proposition 7.9. *Let K be a convex body in $\mathbb{R}^d$. Then the inclusion $(K - c) \subset -d(K - c)$ holds for some point c in the interior of K, for example, if c is the centroid of K or of a maximal volume simplex within K.*

Our idea of the proof of Theorem 7.7 is to use duality twice: We will start with translating the assertion of the theorem in terms of the polar polytope Q° of Q. Then we will choose a point c "deep" in Q° and consider $(Q^\circ - c)^\circ$. Roughly speaking, by changing the center of polarity, we obtain a more well-structured convex polytope. Next, we use Proposition 7.8 to obtain a sufficiently reasonable bound on $r(d)$, which is not destroyed on the way back to Q° and then to Q.

We use $[n]$ to denote the sets $\{1, \ldots, n\}$. The convex hull of a set S is denoted by $\operatorname{conv}(\,)S$. For a non-zero vector $v \in \mathbb{R}^d$, H_v denotes the half-space

$$H_v = \left\{x \in \mathbb{R}^d \;:\; \langle x, v\rangle \leq 1\right\}.$$

We use $\operatorname{vert} P$ to denote the vertex set of a polytope P.

For the sake of completeness, we provide a shortened original proof of Proposition 7.8.

Proof of Proposition 7.8. The condition $L \subseteq -\lambda L$ ensures that the origin belongs to the interior of L. Among all simplices with d vertices from the set of vertices of L and one vertex at the origin, consider a simplex $S = \mathrm{conv}\,(\{)\, 0, v_1, \ldots, v_d\}$ with maximal volume. The simplex S can be represented as

$$S = \left\{ x \in \mathbb{R}^d \ : \ x = \alpha_1 v_1 + \ldots + \alpha_d v_d \quad \text{for } \ \alpha_i \geq 0 \text{ and } \sum_{i=1}^{d} \alpha_i \leq 1 \right\}. \tag{7.1}$$

Define $P = \sum_{i \in [d]} [-v_i, v_i]$. It is easy to see that P is a parallelotope that can be represented as

$$P = \{x \in \mathbb{R}^d \ : \ x = \beta_1 v_1 + \ldots + \beta_d v_d \quad \text{for } \beta_i \in [-1, 1]\}. \tag{7.2}$$

Since S is chosen maximally, equation (7.2) shows that for any vertex v of L, $v \in P$. By convexity,

$$L \subset P. \tag{7.3}$$

Let $S' = -2dS + (v_1 + \ldots + v_d)$. By (7.1),

$$S' = \left\{ x \in \mathbb{R}^d \ : \ x = \gamma_1 v_1 + \ldots + \gamma_d v_d \quad \text{for } \gamma_i \leq 1 \text{ and } \sum_{i=1}^{d} \gamma_i \geq -d \right\},$$

which, together with (7.2), yields

$$P \subseteq S'. \tag{7.4}$$

Let y be the intersection of the ray emanating from 0 in the direction $-(v_1 + \cdots + v_d)$ and the boundary of L. By Carathéodory's theorem, we can choose $k \leq d$ vertices $\{v_1', \ldots, v_k'\}$ of L such that $y \in \mathrm{conv}\,(\{)\, v_1', \ldots, v_k'\}$. Set $L' = \mathrm{conv}\,(\{)\, v_1, \ldots, v_d, v_1', \ldots, v_k'\}$. Clearly, $\frac{v_1 + \cdots + v_d}{d} \in S \subset L$. Thus, $0 \in L'$, and consequently,

$$S \subseteq L'. \tag{7.5}$$

Since $L \subset -\lambda L$, we also have that

$$\frac{v_1 + \cdots + v_d}{d} \in -\lambda[y, 0] \subset -\lambda L'.$$

Combining it with (7.3), (7.4), (7.5), we obtain

$$L \subset P \subset S' = -2dS + (v_1 + \cdots + v_d) \subset -2d\,L' - \lambda d\,L' = -(\lambda + 2)d\,L', \tag{7.6}$$

Completing the proof of Proposition 7.8. □

7.3 Proof of Theorem 7.3

As was explained in the previous section, it suffices to prove Theorem 7.7, which we proceed to work with.

Set $K = Q^\circ$. Since $Q \supset \mathbf{B}^d$, $K \subset \mathbf{B}^d$. Also, it is easy to see that K is a convex polytope of the form

$$K = \bigcap_{v \in \operatorname{vert} Q} H_v, \tag{7.7}$$

containing the origin in its interior. By duality, it suffices to show that there are at most $2d$ half-spaces H_v with $v \in \operatorname{vert} Q$, whose intersection is contained in the ball $5d^2\mathbf{B}^d$.

Let c be a point in the interior of K such that the inclusion

$$K - c \subset -d(K - c)$$

holds. The existence of c follows from Proposition 7.9. Set $L = (K - c)^\circ$. Clearly,

$$L \subset -dL.$$

Now, we use Proposition 7.8 with $\lambda = d$. We obtain that there are $w_1, \ldots, w_m \in \operatorname{vert} L$ for some integer m satisfying $m \leq 2d$ such that

$$L \subset -(d+2)d \cdot \operatorname{conv}(w_i \ : \ i \in [m]).$$

Since $c \in K \subset \mathbf{B}^d$, one has that $K - c \subset 2\mathbf{B}^d$. Consequently, $L \supset \frac{1}{2}\mathbf{B}^d$. So,

$$\frac{1}{2}\mathbf{B}^d \subset L \subset -(d+2)d \cdot \operatorname{conv}(w_i \ : \ i \in [m]).$$

Considering the polar sets, we get

$$(\operatorname{conv}(w_i \ : \ i \in [m]))^\circ \subset 2(d+2)d\mathbf{B}^d.$$

Recall that c is an interior point of the polytope K. By (7.7), one has that for any $w \in \operatorname{vert} L$, $H_w = H_v - c$ for some $v \in \operatorname{vert} Q$. It means that

$$(\operatorname{conv}(w_i \ : \ i \in [m]))^\circ = \bigcap_{v_i \in [m]} (H_{v_i} - c)$$

for corresponding $v_i \in \operatorname{vert} Q$. Thus,

$$\bigcap_{v_i \in [m]} H_{v_i} = \bigcap_{v_i \in [m]} (H_{v_i} - c) + c \subset 2(d+2)d\mathbf{B}^d + c \subset (2(d+2)d + 1)\,\mathbf{B}^d.$$

Since $d \geq 2$, the desired bound for $Q' = \operatorname{conv}(v_i \ : \ i \in [m])$ follows. The proof of Theorem 7.7 is complete, which implies Theorem 7.3 as was discussed earlier.

7.4 Proof of Theorem 7.5

In this section, we prove Theorem 7.5, which is a dual version of [IN22a, Theorem 1.4] and immediately follows from it. For the sake of completeness, we prove Theorem 7.5 here. We first state the main ingredient of the proof obtained by K. Ball and M. Prodromou.

Proposition 7.10 ([BP09], Theorem 1.4). *Let vectors* $\{v_1, \dots, v_n\} \subset \mathbb{R}^d$ *satisfy* $\sum\limits_1^n v_i \otimes v_i = \mathrm{Id}$. *Then for any positive semi-definite operator* $T \colon \mathbb{R}^d \to \mathbb{R}^d$, *there is a point* p *in the intersection of the strips* $\{x \in \mathbb{R}^d \ : \ |\langle x, v_i \rangle| \leq 1\}$ *satisfying* $\langle p, Tp \rangle \geq \operatorname{trace}(T)$.

Proof of Theorem 7.5. There is nothing to prove if the absolute convex hull $\operatorname{conv}(\pm u_i \ : \ i \in [n])$ does not contain the origin in its interior. So, assume that $\operatorname{conv}(\pm u_i \ : \ i \in [n])$ contains the origin in its interior. Set $K = (\operatorname{conv}(\pm u_i \ : \ i \in [n]))^{\circ}$. By duality, it suffices to show that K contains a point of Euclidean norm $\frac{d}{\sqrt{n}}$.

Clearly, $\{u_i \ : \ i \in [n]\}$ spans $\mathbb{R}^d$. Consider $A = \sum\limits_{i \in [n]} u_i \otimes u_i$. Since the vectors span the space, A is positive definite. Using Proposition 7.10 with $v_i = A^{-1/2} u_i, i \in [n]$, and $T = A^{-1}$, we find a point p in

$$\bigcap_{i \in [n]} \{x \ : \ |\langle v_i, x \rangle| \leq 1\}.$$

such that

$$\left\langle p, A^{-1} p \right\rangle \geq \operatorname{trace}\left(A^{-1}\right).$$

Denote $q = A^{-1/2} p$. Then, by the choice of p,

$$1 \geq |\left\langle p, A^{-1/2} u_i \right\rangle| = |\left\langle A^{-1/2} p, u_i \right\rangle| = |\langle q, u_i \rangle|.$$

That is, $q \in K$. On the other hand,

$$|q|^2 = \left\langle A^{-1/2} p, A^{-1/2} p \right\rangle = \left\langle p, A^{-1} p \right\rangle \geq \operatorname{trace}\left(A^{-1}\right).$$

Finally, since $\operatorname{trace}(A) = n$ and by the Cauchy–Schwarz inequality, one sees that $\operatorname{trace}(A)^{-1}$ is at least $\frac{d^2}{n}$. Thus, $|q| \geq \frac{d}{\sqrt{n}}$. This completes the proof of Theorem 7.5. □

Chapter 8

The functional John ellipsoid

8.1 Main results and the structure of the chapter

In this chapter, which is joint work with G. Ivanov [IN22b], we extend the notion of the largest volume ellipsoid contained in a convex body in $\mathbb{R}^d$ to the setting of logarithmically concave functions, and present applications, most notably, a quantitative Helly-type result for the integral of the pointwise minimum of a family of logarithmically concave functions. The chapter is organized as follows.

Section 8.2 contains the *definition* of our main objects of study, the *John s-ellipsoid* (an ellipsoid in $\mathbb{R}^{d+1}$) and the *John s-ellipsoid function* (a function on $\mathbb{R}^d$) of a log-concave function f on $\mathbb{R}^d$. The rough idea is the following. Consider the graph of the function $f^{1/s}$, which is a set in $\mathbb{R}^{d+1}$, and turn it into a not necessarily convex body in $\mathbb{R}^{d+1}$, which we call the *s-lifting* of f. We define also a measure-like quantity, the *s-volume* of sets in $\mathbb{R}^{d+1}$. Then we look for the ellipsoid in $\mathbb{R}^{d+1}$ which is contained in the s-lifting of f and is of maximal s-volume. We call this ellipsoid in $\mathbb{R}^{d+1}$ the John s-ellipsoid of f. This ellipsoid defines a function on $\mathbb{R}^d$, which is the John s-ellipsoid function of f. This function is pointwise less than or equal to f.

In Section 8.3, we prove some *basic inequalities* about the quantities introduced before. As an immediate application of these inequalities, we obtain a compactness result that will (in the next Section) yield that the John s-ellipsoid exists, that is, the maximum in its definition is indeed attained.

Section 8.4 contains one of our main tools, *interpolation between ellipsoids.* In the classical theory of the John ellipsoid, the uniqueness of the largest volume ellipsoid contained in a convex body K in $\mathbb{R}^d$ may be proved in the following way. Assume that $E_1 = A_1\mathbf{B}^d + a_1$ and $E_2 = A_2\mathbf{B}^d + a_2$ are ellipsoids of the same volume contained in K, where $\mathbf{B}^d$ denotes the Euclidean unit ball, A_1, A_2 are matrices, and $a_1, a_2 \in \mathbb{R}^d$. Then the ellipsoid $\frac{A_1+A_2}{2}\mathbf{B}^d + \frac{a_1+a_2}{2}$ is also contained in K and its volume is larger than that of E_1 and E_2.

We are not be able to copy this argument in our setting in a straightforward manner, as the set we consider is not convex. However, we show that if two ellipsoids in $\mathbb{R}^{d+1}$ of the same s-volume are contained in the s-lifting of a log-concave function f, then one can define a third ellipsoid "between" the two ellipsoids which is of larger s-volume. This intermediate ellipsoid is obtained as a non-linear combination of the parameters determining the two ellipsoids.

As an immediate application, we obtain that the John s-ellipsoid is unique, see Theorem 8.11.

In Section 8.5, we state and prove a *necessary and sufficient condition* for the $(d+1)$-dimensional Euclidean unit ball $\mathbf{B}^{d+1}$ to be the John s-ellipsoid of a log-concave function f on $\mathbb{R}^d$, see Theorem 8.14. Here, we phrase a simple version of it.

Theorem 8.1. *Let $\overline{K} = \{(x,\xi) \in \mathbb{R}^{d+1} \ : \ |\xi| \leq f(x)/2\} \subseteq \mathbb{R}^{d+1}$ denote the symmetrized subgraph of an upper semi-continuous log-concave function f on $\mathbb{R}^d$ of positive non-zero integral. Assume that the $(d+1)$-dimensional Euclidean unit ball $\mathbf{B}^{d+1}$ is contained in $\overline{K}$. Then the following are equivalent.*

1. *The ball $\mathbf{B}^{d+1}$ is the unique maximum volume ellipsoid contained in $\overline{K}$.*
2. *There are contact points $\overline{u}_1, \ldots, \overline{u}_k \in \operatorname{bd}\left(\mathbf{B}^{d+1}\right) \cap \operatorname{bd}\left(\overline{K}\right)$, and weights $c_1, \ldots, c_k > 0$ such that*

$$\sum_{i=1}^{k} c_i \overline{u}_i \otimes \overline{u}_i = \overline{I} \quad \textit{and} \quad \sum_{i=1}^{k} c_i u_i = 0,$$

where u_i is the orthogonal projection of $\overline{u}_i$ to $\mathbb{R}^d$ and $\overline{I}$ is the $(d+1) \times (d+1)$ identity matrix.

The implication from (1) to (2) is proved in more or less the same way as John's fundamental theorem about convex bodies, there are hardly any additional difficulties. The converse however, is not straightforward, since $\overline{K}$ is not a convex body in general. That part of the proof relies heavily on the technique of interpolation between ellipsoids.

Section 8.6 contains the proof of our *quantitative Helly-type result.* This is a non-trivial application of the results of the previous sections. We describe it in detail here.

For a positive integer n, we denote by $[n]$ the set $[n] = \{1, 2, \dots, n\}$. For $m \leq n$, the family of subsets of $[n]$ of cardinality at most m is denoted by $\binom{[n]}{\leq m}$.

Observe that the pointwise minimum of a family of log-concave functions is again log-concave. Our quantitative Helly-type result is the following.

Theorem 8.2. *Let $f_1, \dots, f_n$ be upper semi-continuous log-concave functions on $\mathbb{R}^d$. For every $\sigma \subseteq [n]$, let f_σ denote the pointwise minimum:*

$$f_\sigma(x) = \min\{f_i(x) \ : \ i \in \sigma\}.$$

Then there is a set $\sigma \in \binom{[n]}{\leq 3d+2}$ of at most $3d+2$ indices such that, with the notation $f = f_{[n]}$, we have

$$\int_{\mathbb{R}^d} f_\sigma \leq 100^d d^{2d} \int_{\mathbb{R}^d} f. \tag{8.1}$$

We note that at the expense of obtaining a much worse bound in place of the multiplicative constant d^{2d}, we can show a similar result with Helly number $2d+1$ instead of $3d+2$. This is a joint unpublished result of G. Ivanov and Naszódi. However, in Subsection 8.6.6, we show that the number $2d+1$ cannot be decreased to $2d$.

In Section 8.7, we describe the *relationship between the approach of Alonso-Gutiérrez, Merino, Jiménez and Villa* [AGMJV18] *and our approach.*

In [AGMJV18], an ellipsoid in $\mathbb{R}^d$ is associated to any log-concave function f of finite positive integral on $\mathbb{R}^d$, which we call *the AMJV ellipsoid*. It is defined as follows.

For every $\beta \geq 0$, consider the superlevel set $\{x \in \mathbb{R}^d \ : \ f(x) \geq \beta\}$ of f. This is a bounded convex set with non-empty interior, we take its largest volume ellipsoid, and multiply the volume of this ellipsoid by β. As shown in [AGMJV18], there is a unique “height” $\beta_0 \in [0, \|f\|]$ such that this product is maximal, where $\|f\|$ denotes the L_∞ norm of f. The AMJV ellipsoid is the ellipsoid E in $\mathbb{R}^d$ obtained for this β_0.

In Theorem 8.28, we show that $\beta_0\chi_E$ is *the limit* (in a rather strong sense) of our John s-ellipsoid functions *as s tends to* 0.

This result is based on the comparison of the s-volumes of John s-ellipsoids for distinct values of s. We compare also these s-volumes and the integral of f.

Finally, in Section 8.8, we study our John s-ellipsoid functions *as s tends to infinity.* We show that the limit may only be a Gaussian distribution, see Theorem 8.38 . What is perhaps surprising is that the limit may be one Gaussian for a certain sequence $s_1, s_2, \dots \to \infty$, while it may be a different Gaussian for another sequence. We show however, that in this case, the two Gaussians are translates of each other, see Theorem 8.33.

8.1.1 Notation, Basic Terminology

We denote the Euclidean unit ball in $\mathbb{R}^n$ by $\mathbf{B}^n$, and we write $|\cdot|$ for the Euclidean norm.

We identify the hyperplane in $\mathbb{R}^{d+1}$ spanned by the first d standard basis vectors with $\mathbb{R}^d$. A set $C \subset \mathbb{R}^{d+1}$ is *d-symmetric*, if C is symmetric about $\mathbb{R}^d$ that is, if $(2P - I)C = C$, where $P : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ is the orthogonal projection onto $\mathbb{R}^d$.

For square matrices $A_1 \in \mathbb{R}^{d_1 \times d_1}$ and $A_2 \in \mathbb{R}^{d_2 \times d_2}$, we denote by $A_1 \oplus A_2$ the $(d_1 + d_2) \times (d_1 + d_2)$ matrix

$$A_1 \oplus A_2 = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}.$$

In particular, if $A \in \mathbb{R}^{d \times d}$ and $\alpha \in \mathbb{R}$ then we consider the scalar α as a 1×1 matrix, and write $A \oplus \alpha$, which is a $(d+1) \times (d+1)$ matrix.

For a function $f : \mathbb{R}^d \to \mathbb{R}$ and a scalar $\alpha \in \mathbb{R}$, we denote the *superlevel set* of f by $[f \geq \alpha] = \{x \in \mathbb{R}^d \ : \ f(x) \geq \alpha\}$. The *epigraph* of f is the set $\operatorname{epi}(f) = \{(x, \xi) \in \mathbb{R}^{d+1} \ : \ \xi \geq f(x)\}$ in $\mathbb{R}^{d+1}$. The L_∞ norm of a function f is denoted as $\|f\|$.

For two functions f_1, f_2, if f_1 is *pointwise* less than or equal to f_2, then we write $f_1 \leq f_2$.

A function $\psi : \mathbb{R}^d \to \mathbb{R} \cup \infty$ is called *convex*, if $\psi((1-\lambda)x + \lambda y) \leq (1-\lambda)\psi(x) + \lambda\psi(y)$ for every $x, y \in \mathbb{R}^d$ and $\lambda \in [0, 1]$. A function f on $\mathbb{R}^d$ is *logarithmically concave* (or, *log-concave* for short) if $f = e^{-\psi}$ for a convex function ψ on $\mathbb{R}^d$. We say that a log-concave function f on $\mathbb{R}^d$ is a *proper log-concave function*, if f is upper-semicontinuous and has finite positive integral.

We will use $\prec$ to denote the standard partial ordering on the cone of positive semi-definite matrices, that is, we will write $A \prec B$ if $B - A$ is positive definite. We recall the additive and the multiplicative form of *Minkowski's determinant inequality*. Let A and B be positive definite matrices of order d. Then, for any $\lambda \in (0, 1)$,

$$\left(\det\left(\lambda A + (1-\lambda)B\right)\right)^{1/d} \geq \lambda \left(\det A\right)^{1/d} + (1-\lambda)\left(\det B\right)^{1/d}, \tag{8.2}$$

with equality if and only if, $A = cB$ for some $c > 0$; and

$$\det\left(\lambda A + (1-\lambda)B\right) \geq \left(\det A\right)^{\lambda} \cdot \left(\det B\right)^{1-\lambda}, \tag{8.3}$$

with equality if and only if, $A = B$.

8.2 The s-volume, the s-lifting and the s-ellipsoids

8.2.1 Motivation for the definitions

One way to obtain a log-concave function f on $\mathbb{R}^d$ is to take a convex body $\overline{K}$ in $\mathbb{R}^{d+s}$ for some positive integer s, take the uniform measure on $\overline{K}$, and take the density of its marginal on $\mathbb{R}^d$. Furthermore, it is well known that any log-concave function is a limit of functions obtained this way.

If f is obtained this way, then it is natural to think of the marginal on $\mathbb{R}^d$ of the largest volume ($(d+s)$-dimensional) ellipsoid contained in $\overline{K}$ as the John ellipsoid of f.

Clearly, for a given f, the convex body $\overline{K}$ in $\mathbb{R}^{d+s}$ described above is not unique if it exists. One may take the symmetrization of any such $\overline{K}$ about $\mathbb{R}^d$ (see [BZ88, Section 9.2.1.I]) to obtain a new convex body in $\mathbb{R}^{d+s}$ which is now symmetric about $\mathbb{R}^d$ and still has the property that the density of the marginal on $\mathbb{R}^d$ of the uniform measure on it is f.

At this point, we realize that, because of the symmetry about $\mathbb{R}^d$, there is no need to consider a body in $\mathbb{R}^{d+s}$. Instead, we may consider the section of it by the linear subspace spanned by $\mathbb{R}^d$ and any vector, say e_{d+1} which is not in $\mathbb{R}^d$. We just need to remember that the last coordinate in $\mathbb{R}^{d+1}$ represents s coordinates when it comes to computing the marginal of the uniform distribution of a convex body in $\mathbb{R}^{d+1}$.

In what follows, we formalize this reasoning without referring to any $(d+s)$-dimensional convex body. An advantage of the formalism that follows is that it works for non-integer s as well.

8.2.2 The s-volume and its s-marginal

Fix a positive real s. For every $x \in \mathbb{R}^d$, we denote the line in $\mathbb{R}^{d+1}$ perpendicular to $\mathbb{R}^d$ at x by ℓ_x.

Let $\overline{C} \subset \mathbb{R}^{d+1}$ be a d-symmetric Borel set. The *s-volume* of $\overline{C}$ is defined by

$$^{(s)}\mu(\overline{C}) = \int_{\mathbb{R}^d} \left[\frac{1}{2}\text{length}\left(\overline{C} \cap \ell_x\right)\right]^s \mathrm{d}x.$$

Note that $^{(s)}\mu(\cdot)$ is not a measure on $\mathbb{R}^{d+1}$. However, clearly, for any d-symmetric Borel set $\overline{C}$ in $\mathbb{R}^{d+1}$, the *s-marginal* of $\overline{C}$ on $\mathbb{R}^d$ defined for any Borel set B in $\mathbb{R}^d$ as

$$^{(s)}\text{marginal}\left(\overline{C}\right)(B) = \int_{B} \left[\frac{1}{2}\text{length}\left(\overline{C} \cap \ell_x\right)\right]^s \mathrm{d}x \tag{8.4}$$

is a measure on $\mathbb{R}^d$.

We note that for any matrix $\overline{A} = A \oplus \alpha$, where $A \in \mathbb{R}^{d\times d}$ and $\alpha \in \mathbb{R}$, any d-symmetric set $\overline{C}$ in $\mathbb{R}^{d+1}$ and any Borel set B in $\mathbb{R}^d$, we have

$$\begin{cases} {}^{(s)}\mathrm{marginal}\left(\overline{A}\,\overline{C}\right)(AB) & = \ |\det A| \cdot |\alpha|^s \cdot {}^{(s)}\mathrm{marginal}\left(\overline{C}\right)(B), \text{ and} \\ {}^{(s)}\mu\left(\overline{A}\,\overline{C}\right) & = \ |\det A| \cdot |\alpha|^s \cdot {}^{(s)}\mu\left(\overline{C}\right). \end{cases} \tag{8.5}$$

8.2.3 The s-lifting of a function

Let $f : \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ be a function, and $s > 0$. The *s-lifting* of f is a d-symmetric set in $\mathbb{R}^{d+1}$ defined by

$$^{(s)}\overline{f} = \left\{(x, \xi) \in \mathbb{R}^{d+1} \ : \ |\xi| \leq (f(x))^{1/s}\right\},$$

Note the following *scaling property* of s-lifting: for any $\gamma > 0$,

$$^{(s)}\overline{(\gamma f)} = \left(I \oplus \gamma^{1/s}\right) {}^{(s)}\overline{f}. \tag{8.6}$$

Clearly, for any Borel set B in $\mathbb{R}^d$,

$$\int_B f \mathrm{d}x = {}^{(s)}\mu\left({}^{(s)}\overline{f} \cap (B \times \mathbb{R})\right),$$

that is, ${}^{(s)}\mathrm{marginal}\left({}^{(s)}\overline{f}\right)$ is the measure on $\mathbb{R}^d$ with density f.

We recall that for a $q > 0$, a function $f\colon \mathbb{R}^d \to [0, +\infty)$ is said to be *q-concave*, if f^q is a concave function on its convex support. The lifting introduced above allows us to represent q-concave functions as convex sets in $\mathbb{R}^{d+1}$: For any $f\colon \mathbb{R}^d \to [0, +\infty)$ upper semi-continuous function and $s > 0$, we have

$${}^{(s)}\overline{f} \text{ is convex if and only if } f \text{ is } 1/s\text{-concave.}$$

8.2.4 Ellipsoids

Let A be a positive definite matrix in $\mathbb{R}^{d\times d}$, and $a \in \mathbb{R}^d$. They determine an *ellipsoid* with center a associated to A defined by

$$A\left(\mathbf{B}^d\right) + a. \tag{8.7}$$

Note that $A\left(\mathbf{B}^d\right) + a = \{x \in \mathbb{R}^d \ : \ \langle A^{-1}x, A^{-1}x\rangle \leq 1\} + a$.

In $\mathbb{R}^{d+1}$, we will consider d-symmetric ellipsoids. To describe them, we introduce the vector space

$$\mathcal{M} = \left\{(\overline{A}, a) \ : \ \overline{A} \in \mathbb{R}^{(d+1)\times(d+1)},\ \overline{A}^{\mathsf{T}} = \overline{A},\ a \in \mathbb{R}^d\right\}, \tag{8.8}$$

and the cone

$$\mathcal{E} = \left\{(A \oplus \alpha, a) \in \mathcal{M},\ A \in \mathbb{R}^{d\times d} \text{ positive definite}, \alpha > 0\right\}. \tag{8.9}$$

Clearly, any d-symmetric ellipsoid in $\mathbb{R}^{d+1}$ is represented as

$$(A \oplus \alpha)\mathbf{B}^{d+1} + a,$$

in a unique way. Thus, from this point on, we identify $\mathcal{E}$ with the set of all d-symmetric ellipsoids in $\mathbb{R}^{d+1}$. We note that

$$\dim \mathcal{M} = \frac{(d+1)(d+2)}{2} + d. \tag{8.10}$$

8.2.5 Definition of the John s-ellipsoid of a function

Fix $s > 0$ and let $z(f, s)$ denote the supremum of the s-volumes of all d-symmetric ellipsoids $\overline{E}$ in $\mathbb{R}^{d+1}$ with $\overline{E} \subseteq {}^{(s)}\overline{f}$. Lemma 8.6 and a standard compactness argument yield that this supremum is attained. We will see (Theorem 8.11) that it is attained on a unique ellipsoid. We call this ellipsoid in $\mathbb{R}^{d+1}$ the *John s-ellipsoid* of f and denote it by $\overline{E}(f, s)$. We call the s-marginal of $\overline{E}(f, s)$ the *John s-ellipsoid function* of f, and denote its density by

$${}^{(s)}J_f = \text{ the density of } {}^{(s)}\text{marginal}\left(\overline{E}(f, s)\right).$$

Note the *scaling property* of s-ellipsoids, a consequence of (8.6): for any $s, \gamma > 0$,

$$\overline{E} \text{ is the John } s\text{-ellipsoid of } f \text{ if and only if } \left(I \oplus (\gamma^{1/s})\right)\overline{E} \tag{8.11}$$

$$\text{is the John } s\text{-ellipsoid of } \gamma f,$$

or, equivalently, ${}^{(s)}J_f$ is the John s-ellipsoid function of f if and only if, $\gamma \cdot {}^{(s)}J_f$ is the John s-ellipsoid function of γf.

Similarly, for any affine map $\mathcal{A} : \mathbb{R}^d \to \mathbb{R}^d$, ${}^{(s)}J_f$ is the John s-ellipsoid function of f if and only if, ${}^{(s)}J_f \circ \mathcal{A}$ is the John s-ellipsoid function of $f \circ \mathcal{A}$.

8.2.6 How the definitions described above implement the idea described in 8.2.1

As a closing note of this section, we return to the case when s is a positive integer. Assume that the log-concave function f is the density of the marginal on $\mathbb{R}^d$ of the uniform measure on a convex body $\overline{K}$ in $\mathbb{R}^{d+s}$. Consider the symmetrization $\overline{K}'$ about $\mathbb{R}^d$ of $\overline{K}$. Then $\overline{K}'$ is again a convex body in $\mathbb{R}^{d+s}$ which is symmetric about $\mathbb{R}^d$. Hence, its maximum volume ellipsoid, denote it by $\overline{\overline{E}}$, is symmetric about $\mathbb{R}^d$. It follows from our definitions that the section $\overline{E}$ of $\overline{\overline{E}}$ by $\mathbb{R}^{d+1}$ is the John s-ellipsoid of f, and the s-measure of $\overline{E}$ is equal to the $(d+s)$-dimensional volume of $\overline{\overline{E}}$.

8.3 Some basic inequalities

8.3.1 The s-volume of ellipsoids

We denote the s-volume of the ball $\mathbf{B}^{d+1}$ of unit radius centered at the origin in $\mathbb{R}^{d+1}$ by ${}^{(s)}\kappa_{d+1}$, and compute it using spherical coordinates.

$$ {}^{(s)}\kappa_{d+1} = {}^{(s)}\mu\big(\mathbf{B}^{d+1}\big) = \int\limits_{\mathbf{B}^d} \left(\sqrt{1-|x|^2}\right)^s \mathrm{d}x = \tag{8.12} $$

$$ \mathrm{vol}\,(d-1)\,S \int\limits_0^1 r^{d-1}(\sqrt{1-r^2})^s \mathrm{d}r = \frac{\mathrm{vol}\,(d-1)\,S}{2} \int\limits_0^1 t^{(d-2)/2}(1-t)^{s/2}\mathrm{d}t = $$

$$ \frac{d\,\mathrm{vol}\,(d)\,(\mathbf{B}^d)}{2} \frac{\Gamma(s/2+1)\,\Gamma(d/2)}{\Gamma(s/2+d/2+1)} = \pi^{d/2}\frac{\Gamma(s/2+1)}{\Gamma(s/2+d/2+1)}, $$

where $S = \mathrm{bd}\,\big(\mathbf{B}^d\big)$ denotes the unit sphere in $\mathbb{R}^d$, and $\Gamma(\cdot)$ is Euler's Gamma function.

Note that

$$ \lim_{s\to 0^+} {}^{(s)}\kappa_{d+1} = \mathrm{vol}\,(d)\,(\mathbf{B}^d). \tag{8.13} $$

Thus, ${}^{(s)}\kappa_{d+1}$, as a function of s on $[0,\infty)$ with ${}^{(0)}\kappa_{d+1} = \mathrm{vol}\,(d)\,(\mathbf{B}^d)$, is a strictly decreasing continuous function.

By (8.5), the s-volume of a d-symmetric ellipsoid can be expressed as

$$ {}^{(s)}\mu\big((A\oplus\alpha)\mathbf{B}^{d+1}+a\big) = {}^{(s)}\kappa_{d+1}\alpha^s \det A, \quad \text{for any } (A\oplus\alpha, a)\in\mathcal{E}. \tag{8.14} $$

8.3.2 The height function of an ellipsoid

For any $(A \oplus \alpha, a) \in \mathcal{E}$, we will say that α is the *height* of the ellipsoid $\overline{E} = (A \oplus \alpha)\mathbf{B}^{d+1} + a$. We define the *height function* of $\overline{E}$ as

$$h_{\overline{E}}(x) = \begin{cases} \alpha\sqrt{1 - \langle A^{-1}(x-a), A^{-1}(x-a)\rangle}, & \text{if } x \in A\mathbf{B}^d + a \\ 0, & \text{otherwise.} \end{cases}$$

Note that the height function of an ellipsoid is a proper log-concave function. Clearly, the inclusion $\overline{E} \subset {}^{(s)}\overline{f}$ holds if and only if

$$h_{\overline{E}}(x+a) \le f^{1/s}(x+a) \text{ for all } x \in A\mathbf{B}^d. \tag{8.15}$$

8.3.3 Bounds on f based on local behaviour

Lemma 8.3. *Let ψ_1 and ψ_2 be convex functions on $\mathbb{R}^d$ and $f_1 = e^{-\psi_1}$ and $f_2 = e^{-\psi_2}$. Let $f_2 \le f_1$ (pointwise) and $f_1(x_0) = f_2(x_0) > 0$ at some point x_0 in the interior of the domain of ψ_2. Assume that ψ_2 is differentiable at x_0. Then f_1 and f_2 are differentiable at x_0, $\nabla f_1(x_0) = \nabla f_2(x_0)$ and the following holds*

$$f_1(x) \le f_2(x_0)e^{-\langle \nabla\psi_2(x_0), x-x_0\rangle}$$

for all $x \in \mathbb{R}^d$.

Proof. Since $f_2 \le f_1$, the epigraph of ψ_1 contains the epigraph of ψ_2. Hence, $f_1(x_0) = f_2(x_0)$ implies that the subdifferential $\partial\psi_1$ at x_0 is contained in the subdifferential $\partial\psi_2$ at x_0. Since the latter consists of one vector $\nabla\psi_2(x_0)$, we conclude that ψ_1 is differentiable at x_0 and $\nabla\psi_1(x_0) = \nabla\psi_2(x_0)$. By the convexity of ψ_1, we have

$$\psi_1(x) \ge \psi_1(x_0) + \langle \nabla\psi_1(x_0), x - x_0\rangle = \psi_2(x_0) + \langle \nabla\psi_2(x_0), x - x_0\rangle$$

for all $x \in \mathbb{R}^d$. The result follows. □

Corollary 8.4. *Let f be a log-concave function on $\mathbb{R}^d$, and $s > 0$. Assume that $\mathbf{B}^{d+1} \subseteq {}^{(s)}\overline{f}$ and $\overline{u} \in \mathbb{R}^{d+1} \setminus \mathbb{R}^d$ is a* contact point *of $\mathbf{B}^{d+1}$ and ${}^{(s)}\overline{f}$, that is, $\overline{u} \in \operatorname{bd}\left(\mathbf{B}^{d+1}\right) \cap \operatorname{bd}\left({}^{(s)}\overline{f}\right) \setminus \mathbb{R}^d$. Then*

$$f(x) \le w^s e^{-\frac{s}{w^2}\langle u, x-u\rangle} \quad \textit{for all } x \in \mathbb{R}^d, \tag{8.16}$$

where u is the orthogonal projection of $\overline{u}$ to $\mathbb{R}^d$ and $w = \sqrt{1-|u|^2}$.

Note that since $u \notin \mathbb{R}^d$, we have $w > 0$.

Proof of Corollary 8.4. Applying Lemma 8.3 to the functions $f_1 = f^{1/s}$ and $f_2 = h_{\mathbf{B}^{d+1}}$ at $x_0 = u$, we obtain

$$f^{1/s}(x) \leq we^{-\left\langle \nabla\left[-\log h_{\mathbf{B}^{d+1}}\right](u), x-u\right\rangle} \quad \text{for all } x \in \mathbb{R}^d.$$

Since for any $y \in \operatorname{int}\left(\mathbf{B}^d\right)$, we have

$$\nabla\left[-\log h_{\mathbf{B}^{d+1}}\right](y) = -\frac{1}{2}\nabla\left[\log(1-|y|^2)\right] = \frac{y}{1-|y|^2},$$

inequality (8.16) follows. □

We will make use of the following well known bound (see [Lemma 2.2.1, Brazitikos book]).

Lemma 8.5. *For any proper log-concave function f on $\mathbb{R}^d$, there are $\Theta, \vartheta > 0$ such that*

$$f(x) \leq \Theta e^{-\vartheta|x|}, \text{ for all } x \in \mathbb{R}^d. \tag{8.17}$$

This bound implies that the integral of a proper log-concave function over any affine hyperplane in $\mathbb{R}^d$ is bounded by some constant. Hence, the integral over any slab is bounded from above by a linear function of its width, that is, for a proper log-concave function f, there exists Θ_f such that

$$\int\limits_{\{x\in\mathbb{R}^d \,:\, |\langle x-a,y\rangle|\leq\lambda\}} f\,\mathrm{d}x \leq \Theta_f\lambda \tag{8.18}$$

for arbitraty $a, y \in \mathbb{R}^d$ with $|y| = 1$.

8.3.4 Compactness

We show that ellipsoids of large s-volume contained in ${}^{(s)}\overline{f}$ are contained in a bounded region of $\mathbb{R}^{d+1}$. We state a bit more, allowing s to vary in an interval $(0, s_0]$.

Lemma 8.6 (Compactness)**.** *Let $f : \mathbb{R}^d \to [0, \infty)$ be a proper log-concave function, and $\delta, s_0 > 0$. Then there is a radius $\rho > 0$ such that for any $0 < s \leq s_0$, if $\overline{E} = (A \oplus \alpha)\mathbf{B}^{d+1} + a$, where $(A \oplus \alpha, a) \in \mathcal{E}$, is a d-symmetric ellipsoid in $\mathbb{R}^{d+1}$ with $\overline{E} \subseteq {}^{(s)}\overline{f}$ and ${}^{(s)}\mu\left(\overline{E}\right) \geq \delta$, then $A\mathbf{B}^d + a \subseteq \rho\mathbf{B}^d$ and $\alpha^s \leq \rho$.*

Proof of Lemma 8.6. By (8.13), ${}^{(s)}\kappa_{d+1}$ is bounded on the interval $(0, s_0]$, and its supremum is ${}^{(0)}\kappa_{d+1}$.

Let $\Theta, \vartheta > 0$ be such that (8.17) holds for f.

Denote the largest eigenvalue of A (that is, the length of the longest semi-axis of $A\mathbf{B}^d$) by β. By (8.14), we have

$$ {}^{(s)}\mu\big(\overline{E}\big) \leq {}^{(s)}\kappa_{d+1}\beta^d\alpha^s \leq {}^{(0)}\kappa_{d+1}\beta^d\alpha^s. \tag{8.19} $$

Let $m \in \mathbb{R}^d$ be the midpoint of a semi-axis of $\overline{E}$ of length β such that $\sphericalangle(0am) \geq \pi/2$ (if the center $a = 0$, then this angle condition can be ignored).

Since $\overline{E} \subseteq {}^{(s)}\overline{f}$, we have $h_{\overline{E}}(m) = \frac{\sqrt{3}\alpha}{2} \leq (f(m))^{1/s}$. Using the angle condition for m, we have $|m| \geq \beta/2$, and hence, (8.17) yields that $\frac{\sqrt{3}\alpha}{2} \leq \left(\Theta e^{-\vartheta\beta/2}\right)^{1/s}$. Thus, by (8.19),

$$ \delta \leq {}^{(s)}\mu\big(\overline{E}\big) \leq {}^{(0)}\kappa_{d+1}\beta^d\left(2/\sqrt{3}\right)^s\Theta e^{-\vartheta\beta/2} \leq {}^{(0)}\kappa_{d+1}\beta^d 2^{s_0}\Theta e^{-\vartheta\beta/2}. $$

The latter converges to 0 as $\beta \to \infty$, thus $\beta \leq \rho_1$ for some $\rho_1 > 0$ depending on f.

Similarly, we have $h_{\overline{E}}(a) = \alpha \leq (f(a))^{1/s} \leq \left(\Theta e^{-\vartheta|a|}\right)^{1/s}$. Thus, by (8.19),

$$ \delta \leq {}^{(s)}\mu\big(\overline{E}\big) \leq {}^{(0)}\kappa_{d+1}\Theta e^{-\vartheta|a|}\rho_1^d, $$

and hence, $|a| \leq \rho_2$ for some $\rho_2 > 0$ depending on f.

Since $\alpha^s \leq \|f\|$, the existence of a $\rho > 0$ follows immediately. This completes the proof of Lemma 8.6. $\square$

8.4 Interpolation between ellipsoids

In this section, we show that if two ellipsoids are contained in ${}^{(s)}\overline{f}$, then we can define a third ellipsoid that is also contained in ${}^{(s)}\overline{f}$, and we give a lower bound on its s-volume. The latter is a Brunn–Minkowski type inequality for the s-volume of ellipsoids.

After preliminaries, we present the main results of this section in Subsection 8.4.2, which is followed by immediate applications, one of which is the proof of Theorem 8.11.

8.4.1 Operations on functions: Asplund sum, epi-product

Following Section 9.5 of [Sch14], we define the *Asplund sum* (or *sup-convolution*) of two log-concave functions f_1 and f_2 on $\mathbb{R}^d$ as

$$(f_1 \star f_2)(x) = \sup_{x_1+x_2=x} f_1(x_1) f_2(x_2),$$

and the *epi-product* of a log-concave function f on $\mathbb{R}^d$ with a scalar $\lambda > 0$ as

$$(\lambda * f)(x) = f\left(\frac{x}{\lambda}\right)^{\lambda}.$$

Clearly,

$$\|f\| = \|f_1\| \, \|f_2\| \,, \text{ where } f = f_1 \star f_2.$$

The Asplund sum is closely related to the infimal convolution (See Section 1.6 of [Sch14]), and can be interpreted as follows. Let $f_1 = e^{-\psi_1}$ and $f_2 = e^{-\psi_2}$, where ψ_1, ψ_2 are convex functions, be proper log-concave functions. Then

$$f_1 \star f_2 = e^{-\psi},$$

where ψ is the function defined by taking the Minkowski sum of the epigraphs, that is,

$$\operatorname{epi} \psi = \operatorname{epi} \psi_1 + \operatorname{epi} \psi_2.$$

Thus, for any proper log-concave function f and $\lambda \in [0, 1]$, we have

$$(\lambda * f) \star ((1 - \lambda) * f) = f. \tag{8.20}$$

8.4.2 A non-linear combination of two ellipsoids

The following two lemmas are our key tools. They allow us to interpolate between two ellipsoids.

Lemma 8.7 (Containment of the interpolated ellipsoid)**.** *Fix $s_1, s_2, \beta_1, \beta_2 > 0$ with $\beta_1 + \beta_2 = 1$. Let f_1 and f_2 be two proper log-concave functions on $\mathbb{R}^d$, and $\overline{E_1}, \overline{E_2}$ be two d-symmetric ellipsoids represented by $(A_1 \oplus \alpha_1, a_1) \in \mathcal{E}$ and $(A_2 \oplus \alpha_2, a_2) \in \mathcal{E}$, respectively, such that*

$$\overline{E}_1 \subset {}^{(s_1)}\overline{f_1} \quad \text{and} \quad \overline{E_2} \subset {}^{(s_2)}\overline{f_2}. \tag{8.21}$$

Define

$$f = (\beta_1 * f_1) \star (\beta_2 * f_2) \quad \text{and} \quad s = \beta_1 s_1 + \beta_2 s_2.$$

Set

$$(A \oplus \alpha, a) = \left((\beta_1 A_1 + \beta_2 A_2) \oplus (\alpha_1^{\beta_1 s_1} \alpha_2^{\beta_2 s_2})^{1/s}, \beta_1 a_1 + \beta_2 a_2\right)$$

and

$$\overline{E} = (A \oplus \alpha)\mathbf{B}^{d+1} + a.$$

Then,

$$\overline{E} \subset {}^{(s)}\overline{f}. \tag{8.22}$$

Proof. Fix $x \in A\mathbf{B}^d$ and define

$$x_1 = A_1 A^{-1} x, \quad x_2 = A_2 A^{-1} x.$$

Clearly, $x_1 \in A_1 \mathbf{B}^d$ and $x_2 \in A_2 \mathbf{B}^d$. Thus, by (8.21) and (8.15), we have

$$f_1^{1/s_1}(x_1 + a_1) \geq h_{\overline{E}_1}(x_1 + a_1) \quad \text{and} \quad f_2^{1/s_2}(x_2 + a_2) \geq h_{\overline{E}_2}(x_2 + a_2). \tag{8.23}$$

By our definitions, we have that $\beta_1(x_1 + a_1) + \beta_2(x_2 + a_2) = x + a$. Therefore, by the definition of the Asplund sum, we have that

$$f(x + a) \geq f_1^{\beta_1}(x_1 + a_1) f_2^{\beta_2}(x_2 + a_2),$$

which, by (8.23) yields

$$f(x + a) \geq \left(h_{\overline{E}_1}(x_1 + a_1)\right)^{\beta_1 s_1} \left(h_{\overline{E}_2}(x_2 + a_2)\right)^{\beta_2 s_2}.$$

By the definition of the height function, and since $A^{-1}x = A_1^{-1}x_1 = A_2^{-1}x_2$, we have

$$\left(h_{\overline{E}_1}(x_1 + a_1)\right)^{\beta_1 s_1} \left(h_{\overline{E}_2}(x_2 + a_2)\right)^{\beta_2 s_2} =$$
$$\left(\alpha_1 \sqrt{1 - \langle A_1^{-1}x_1, A_1^{-1}x_1 \rangle}\right)^{\beta_1 s_1} \left(\alpha_2 \sqrt{1 - \langle A_2^{-1}x_2, A_2^{-1}x_2 \rangle}\right)^{\beta_2 s_2} =$$
$$\alpha_1^{\beta_1 s_1} \alpha_2^{\beta_2 s_2} \left(\sqrt{1 - \langle A^{-1}x, A^{-1}x \rangle}\right)^{\beta_1 s_1 + \beta_2 s_2} = \left(\alpha \sqrt{1 - \langle A^{-1}x, A^{-1}x \rangle}\right)^s =$$
$$\left(h_{\overline{E}}(x + a)\right)^s.$$

Combining this with the previous inequality, we obtain inequality (8.15). This completes the proof. □

Lemma 8.8 (Volume of the interpolated ellipsoid)**.** *Under the conditions of Lemma 8.7 with $s_1 = s_2$, the following inequality holds.*

$${}^{(s)}\mu(\overline{E}) \geq \left({}^{(s)}\mu(\overline{E}_1)\right)^{\beta_1} \left({}^{(s)}\mu(\overline{E}_2)\right)^{\beta_2}, \tag{8.24}$$

with equality if and only if, $A_1 = A_2$.

Proof. We set $s = s_1 = s_2$, and observe that by (8.14), inequality (8.24) is equivalent to

$$ {}^{(s)}\kappa_{d+1}(\alpha_1^{\beta_1}\alpha_2^{\beta_2})^s \cdot \det(\beta_1 A_1 + \beta_2 A_2) \geq {}^{(s)}\kappa_{d+1}(\alpha_1^{\beta_1}\alpha_2^{\beta_2})^s \cdot (\det A_1)^{\beta_1} (\det A_2)^{\beta_2}, $$

which holds if and only if,

$$ \det(\beta_1 A_1 + \beta_2 A_2) \geq (\det A_1)^{\beta_1} (\det A_2)^{\beta_2}. $$

Finally, (8.24) and the equality condition follow from Minkowski's determinant inequality (8.3) and the equality condition therein. □

8.4.3 Uniqueness of the John s-ellipsoids

We start with a simple but useful observation.

Lemma 8.9 (Interpolation between translated ellipsoids)**.** *Let f be a proper log-concave function on $\mathbb{R}^d$, and $s > 0$. Assume that the two d-symmetric ellipsoids $\overline{E}_1$ and $\overline{E}_2$ contained in ${}^{(s)}\overline{f}$ are translates of each other by a vector in $\mathbb{R}^d$. More specifically, assume that they are represented as $(A \oplus \alpha, a_1)$ and $(A \oplus \alpha, a_2)$ with $a_1 = -a_2 = \delta A e_1$, where e_1 is the first standard basis vector in $\mathbb{R}^d$. Then the origin centered ellipsoid*

$$ \overline{E}_0 = (A \oplus \alpha)\overline{M}\mathbf{B}^{d+1}, \quad \textit{where } \overline{M} = \operatorname{diag}(1+\delta, 1, \ldots, 1), $$

is contained in ${}^{(s)}\overline{f}$.

Proof. Since all super-level sets of $f^{1/s}$ are convex sets in $\mathbb{R}^d$, it is easy to see that for any convex set $\overline{H}$ and vector $v \in \mathbb{R}^d$, if $\overline{H} \subseteq {}^{(s)}\overline{f}$ and $\overline{H} + v \subseteq {}^{(s)}\overline{f}$, then $\operatorname{conv}\left(\overline{H} \cup (\overline{H} + v)\right) = \overline{H} + [0, v] \subseteq {}^{(s)}\overline{f}$.

Thus, ${}^{(s)}\overline{f}$ contains the 'sausage-like' body

$$ \overline{W} = \operatorname{conv}\left(\overline{E}_1 \cup \overline{E}_2\right) = (A \oplus \alpha)\left(\mathbf{B}^{d+1} + [A^{-1}a_2, A^{-1}a_1]\right) = $$

$$ (A \oplus \alpha)\left(\mathbf{B}^{d+1} + [-\delta e_1, \delta e_1]\right). $$

On the other hand, clearly, $\overline{E}_0 \subseteq \overline{W}$ completing the proof of Lemma 8.9. □

As a first application of Lemmas 8.7 and 8.8, we show that in the set of d-symmetric ellipsoids in ${}^{(s)}\overline{f}$ of height ρ, $0 < \rho < \|f\|$, there is a unique largest s-volume d-symmetric ellipsoid.

Lemma 8.10 (Uniqueness for a fixed height)**.** *Let f be a proper log-concave function on $\mathbb{R}^d$, and $s > 0$. Then, among all d-symmetric ellipsoids of height $\alpha, 0 < \alpha < \|f\|^{1/s}$, in ${}^{(s)}\overline{f}$, there is a unique one of maximal s-volume. Additionally, if there is a d-symmetric ellipsoid in ${}^{(s)}\overline{f}$ of height $\|f\|^{1/s}$, then among all d-symmetric ellipsoids of height $\alpha = \|f\|^{1/s}$ in ${}^{(s)}\overline{f}$, there is a unique one of maximal s-volume.*

Proof. Clearly, the maximum s-volume among d-symmetric ellipsoids of height α contained in ${}^{(s)}\overline{f}$ is positive, and by Lemma 8.6 and a standard compactness argument, it is attained.

We show that such an ellipsoid is unique. Assume that $\overline{E}_1 \subset {}^{(s)}\overline{f}$ and $\overline{E}_2 \subset {}^{(s)}\overline{f}$, represented by $(A_1 \oplus \alpha, a_1) \in \mathcal{E}$ and $(A_2 \oplus \alpha, a_2) \in \mathcal{E}$, are two d-symmetric ellipsoids of the maximal s-volume.

Define a new d-symmetric ellipsoid $\overline{E}$ represented by

$$\left(\frac{A_1 + A_2}{2} \oplus \alpha, \frac{a_1 + a_2}{2}\right) \in \mathcal{E}.$$

Applying (8.20) with $\lambda = 1/2$ and Lemma 8.7, we have $\overline{E} \subset {}^{(s)}\overline{f}$. Next, by the choice of the ellipsoids, we have that

$${}^{(s)}\mu\big(\overline{E}\big) \leq {}^{(s)}\mu\big(\overline{E}_1\big) = \sqrt{{}^{(s)}\mu\big(\overline{E}_1\big)\,{}^{(s)}\mu\big(\overline{E}_2\big)} = {}^{(s)}\mu\big(\overline{E}_2\big)\,.$$

By Lemma 4.2, we have that ${}^{(s)}\mu\big(\overline{E}\big) \geq \sqrt{{}^{(s)}\mu\big(\overline{E}_1\big)\,{}^{(s)}\mu\big(\overline{E}_2\big)}$, therefore equality holds. Thus, by the equality condition in Lemma 8.8, we conclude that $A_1 = A_2$.

To complete the proof, we need to show that $a_1 = a_2$. Assume the contrary: $a_1 \neq a_2$. By translating the origin and rotating the space $\mathbb{R}^d$, we may assume that $a_1 = -a_2 \neq 0$ and that $A_1^{-1} a_1 = \delta e_1$ for some $\delta > 0$.

By Lemma 8.9, the ellipsoid $\overline{E}_0 = (A_1 \oplus \alpha)\overline{M}\mathbf{B}^{d+1}$ is contained in ${}^{(s)}\overline{f}$, where $\overline{M} = \operatorname{diag}(1 + \delta, 1, \ldots, 1)$. However, ${}^{(s)}\mu\big(\overline{E}_0\big) > {}^{(s)}\mu\big(\mathbf{B}^{d+1}\big)$, which contradicts the choice of $\overline{E}_1$ and $\overline{E}_2$, completing the proof of Lemma 8.10. □

Theorem 8.11 (Existence and uniqueness of the John s-ellipsoid)**.** *Let $s > 0$ and f be a proper log-concave function on $\mathbb{R}^d$. Then, there exists a unique John s-ellipsoid of f.*

Proof of Theorem 8.11. As in the proof of Lemma 8.10, the existence of an s-ellipsoid follows from Lemma 8.6 and a standard compactness argument.

Assume that $\overline{E}_1 \subset {}^{(s)}\overline{f}$ and $\overline{E}_2 \subset {}^{(s)}\overline{f}$ are two d-symmetric ellipsoids of maximal s-volume, represented by $(A_1 \oplus \alpha_1, a_1) \in \mathcal{E}$ and $(A_2 \oplus \alpha_2, a_2) \in \mathcal{E}$, respectively. We define a new d-symmetric ellipsoid $\overline{E}$ represented by

$$\left(\frac{A_1 + A_2}{2} \oplus \sqrt{\alpha_1 \alpha_2}, \frac{a_1 + a_2}{2}\right) \in \mathcal{E}.$$

Applying (8.20) with $\lambda = 1/2$ and Lemma 8.7, we have $\overline{E} \subset {}^{(s)}\overline{f}$. Next, by the choice of the ellipsoids, we also have

$${}^{(s)}\mu\big(\overline{E}\big) \leq {}^{(s)}\mu\big(\overline{E}_1\big) = \sqrt{{}^{(s)}\mu\big(\overline{E}_1\big)\, {}^{(s)}\mu\big(\overline{E}_2\big)} = {}^{(s)}\mu\big(\overline{E}_2\big)\,,$$

which, combined with Lemma 8.8, yields ${}^{(s)}\mu\big(\overline{E}\big) = {}^{(s)}\mu\big(\overline{E}_1\big) = {}^{(s)}\mu\big(\overline{E}_2\big)$ and $A_1 = A_2$. This implies that $\alpha_1 = \alpha_2$, since the s-volume of $\overline{E}_1$ and $\overline{E}_2$ are equal. Therefore, by Lemma 8.10, the ellipsoids $\overline{E}_1$ and $\overline{E}_2$ coincide, completing the proof of Theorem 8.11. □

8.4.4 Bound on the height

Recall from Section 8.2.5 that ${}^{(s)}J_f$ denotes the density of the John s-ellipsoid function of f, that is, the density of the s-marginal of the John s-ellipsoid of f. The following result is an extension of the analogous result on the "height" of the AMJV ellipsoid [AGMJV18, Theorem 1.1] to the John s-ellipsoid with a similar proof.

Lemma 8.12. *Let f be a proper log-concave function on $\mathbb{R}^d$ and $s > 0$. Then,*

$$\left\| {}^{(s)}J_f \right\| \geq e^{-d} \left\| f \right\|. \tag{8.25}$$

We note that if the John s-ellipsoid of f is represented as $(A_0 \otimes \alpha_0, a_0)$ (that is, its height is α_0), then $\left\| {}^{(s)}J_f \right\| = \alpha_0^s$.

Proof. We define a function $\Psi : (0, \|f\|^{1/s}) \to \mathbb{R}^+$ as follows. By Lemma 8.10, for any $\alpha \in (0, \|f\|^{1/s})$, there is a unique d-symmetric ellipsoid of maximal s-volume among d-symmetric ellipsoids of height α in ${}^{(s)}\overline{f}$. Let this ellipsoid be represented by $(A_\alpha \otimes \alpha, a_\alpha) \in \mathcal{E}$. We set $\Psi(\alpha) = \det A_\alpha$.

Claim 8.13. *For any $\alpha_1, \alpha_2 \in (0, \|f\|^{1/s})$ and $\lambda \in [0, 1]$, we have*

$$\Psi\big(\alpha_1^{\lambda} \alpha_2^{1-\lambda}\big)^{1/d} \geq \lambda \Psi(\alpha_1)^{1/d} + (1 - \lambda)\Psi(\alpha_2)^{1/d}. \tag{8.26}$$

Proof. Let $(A_1 \otimes \alpha_1, a_1)$ and $(A_2 \otimes \alpha_2, a_2)$ represent the d-symmetric ellipsoids of maximum s-volume contained in ${}^{(s)}\overline{f}$ with the corresponding heights. By Lemma 8.7 and (8.14), we have that

$$\Psi\left(\alpha_1^{\lambda}\alpha_2^{1-\lambda}\right) \geq \det\left(\lambda A_1 + (1-\lambda)A_2\right).$$

Now, (8.26) follows immediately from Minkowski's determinant inequality (8.2). □

Set $\Phi(t) = \Psi(e^t)^{1/d}$ for all $t \in \left(-\infty, \frac{\log\|f\|}{s}\right)$. Inequality (8.26) implies that Φ is a concave function on its domain.

Let α_0 be the height of the John s-ellipsoid of f. Then, by (8.14), for any α in the domain of Ψ, we have that

$$\Psi(\alpha)\alpha^s \leq \Psi(\alpha_0)\alpha_0^s.$$

Setting $t_0 = \log\alpha_0$ and taking root of order d, we obtain

$$\Phi(t) \leq \Phi(t_0)e^{\frac{s}{d}(t_0-t)}$$

for any t in the domain of Φ. The expression on the right-hand side is a convex function of t, while Φ is a concave function. Since these functions take the same value at $t = t_0$, we conclude that the graph of Φ lies below the tangent line to graph of $\Phi(t_0)e^{\frac{s}{d}(t_0-t)}$ at point t_0. That is,

$$\Phi(t) \leq \Phi(t_0)\left(1 - \frac{s}{d}(t-t_0)\right).$$

Passing to the limit as $t \to \frac{\log\|f\|}{s}$ and since the values of Φ are positive, we get

$$0 \leq 1 - \frac{\log\|f\|}{d} + \frac{s}{d}t_0.$$

Or, equivalently, $t_0 \geq -\frac{d}{s} + \frac{\log\|f\|}{s}$. Therefore, $\alpha_0 \geq e^{-d/s}\|f\|^{1/s}$. Clearly, $\left\|{}^{(s)}J_f\right\| = \alpha_0^s \geq e^{-d}\|f\|$. This completes the proof of Lemma 8.12. □

8.5 John's condition — Proof of Theorem 8.1

Theorem 8.1 is an immediate consequence of the following theorem.

Theorem 8.14. *Let $\overline{K} \subseteq \mathbb{R}^{d+1}$ be a closed set which is symmetric about $\mathbb{R}^d$, and let $s > 0$. Assume that $\mathbf{B}^{d+1} \subseteq \overline{K}$. Then the following hold.*

1. *Assume that* $\mathbf{B}^{d+1}$ *is a* locally *maximal s-volume ellipsoid contained in* $\overline{K}$*, that is, in some neighborhood of* $\mathbf{B}^{d+1}$*, no ellipsoid contained in* $\overline{K}$ *is of larger s-volume.*

   Then there are contact points $\overline{u}_1, \dots, \overline{u}_k \in \operatorname{bd}\left(\mathbf{B}^{d+1}\right) \cap \operatorname{bd}\left(\overline{K}\right)$*, and weights* $c_1, \dots, c_k > 0$ *such that*

$$\sum_{i=1}^{k} c_i \overline{u}_i \otimes \overline{u}_i = \overline{S} \quad \text{and} \quad \sum_{i=1}^{k} c_i u_i = 0, \tag{8.27}$$

   where u_i is the orthogonal projection of $\overline{u}_i$ to $\mathbb{R}^d$ and $\overline{S} = \operatorname{diag}(1, \dots, 1, s) = I \oplus s$*. Moreover, such contact points and weights exist with some* $d+1 \le k \le \frac{(d+1)(d+2)}{2} + d + 1$*.*

2. *Assume that* $\overline{K} = {}^{(s)}\overline{f}$ *for a proper log-concave function f, and that there are contact points and weights satisfying* (8.27)*.*

   Then $\mathbf{B}^{d+1}$ *is the unique ellipsoid of (*globally*) maximum s-volume among ellipsoids contained in* $\overline{K}$*.*

In this section, we prove Theorem 8.14.

We equip $\mathcal{M}$ (for the definition, see (8.8)) with an inner product (that comes from the Frobenius product on the space of matrices and the standard inner product on $\mathbb{R}^d$) defined by

$$\left\langle (\overline{A}, a), (\overline{B}, b) \right\rangle = \operatorname{trace}\left(\overline{A}\,\overline{B}\right) + \langle a, b \rangle .$$

Thus, we may use the topology of $\mathcal{M}$ on the set $\mathcal{E}$ of ellipsoids in $\mathbb{R}^{d+1}$.

Denote the set of contact points by $C = \operatorname{bd}\left(\mathbf{B}^{d+1}\right) \cap \operatorname{bd}\left(\overline{K}\right)$, and consider

$$\widehat{C} = \{(\overline{u} \otimes \overline{u}, u) \ : \ \overline{u} \in C\} \subset \mathcal{M},$$

where u denotes the orthogonal projection of $\overline{u}$ to $\mathbb{R}^d$.

The proof of Part (1) of Theorem 8.14 is an adaptation of the argument given in [Bal97] and [Gru07] (see also [GLMP04, GPT01, BR02, Lew79] and [TJ89, Theorem 14.5]) to the s-volume. The idea is that, if there are no contact points and weights satisfying (8.27), then there is a *line segment* in the space $\mathcal{E}$ of ellipsoids starting from $\mathbf{B}^{d+1}$ such that the s-volume increases along the path and the path stays in the family of ellipsoids contained in $\overline{K}$.

Part (2) on the other hand, needs a finer argument as ${}^{(s)}\overline{f}$ is not necessarily convex. The idea is that, if $\mathbf{B}^{d+1}$ is not the global maximizer of the s-volume, then we will find a path in $\mathcal{E}$ starting from $\mathbf{B}^{d+1}$ such that the s-volume increases along the path, and the path stays in the family of ellipsoids contained in $\overline{K}$. This path *is not a line segment* since ${}^{(s)}\overline{f}$ is not necessarily convex. We will, however, be able to differentiate the s-volume along this path, and by doing so, we will show that $(\overline{S},0)$ is separated by a hyperplane from the points $\widehat{C}$ in $\mathcal{M}$, which in turn will yield that there are no contact points and weights satisfying (8.27).

First, as a standard observation, we state the relationship between (8.27) and separation by a hyperplane of the point $(\overline{S},0)$ from the set $\widehat{C}$ in the space $\mathcal{M}$.

Claim 8.15. *The following assertions are equivalent.*

1. *There are contact points and weights satisfying* (8.27)*.*
2. *There are contact points and weights satisfying a modified version of* (8.27)*, where in the second equation u_i is replaced by $\overline{u}_i$.*
3. $(\overline{S},0) \in \operatorname{pos}(\widehat{C})$.
4. $\frac{1}{d+s}(\overline{S},0) \in \operatorname{conv}\left(\widehat{C}\right)$.
5. *There is no $(\overline{H},h) \in \mathcal{M}$ with*
$$\left\langle (\overline{H},h),(\overline{S},0)\right\rangle > 0, \text{ and } \left\langle (\overline{H},h),(\overline{u}\otimes\overline{u},u)\right\rangle < 0 \text{ for all } \overline{u}\in C. \tag{8.28}$$
6. *There is no $(\overline{H},h) \in \mathcal{M}$ with*
$$\left\langle (\overline{H},h),(\overline{S},0)\right\rangle > 0, \text{ and } \left\langle (\overline{H},h),(\overline{u}\otimes\overline{u},u)\right\rangle \le 0 \text{ for all } \overline{u}\in C. \tag{8.29}$$

Proof. We leave it to the reader to verify the equivalence of (1) and (2) and (3), as well as that of (5) and (6).

To see that (1) is equivalent to (4), take trace in (8.27), and notice that $\operatorname{trace}(\overline{u}\otimes\overline{u}) = \operatorname{trace}\left(\frac{1}{d+s}\overline{S}\right) = 1$, which shows that $\sum_{i=1}^{k} c_i = d+s$.

Finally, observe that the convex cone $\operatorname{pos}(\widehat{C})$ in $\mathcal{M}$ does not contain the point $\left(\overline{S},0\right) \in \mathcal{M}$ if and only if, it is separated from the point by a hyperplane through the origin. This is what (8.28) expresses, showing that (3) is equivalent to (5), and hence, completing the proof of Claim 8.15. □

Claim 8.16. *If contact points and weights satisfying* (8.27) *exist for some k, then they exist for some $d+1 \le k \le \frac{(d+1)(d+2)}{2} + d + 1$.*

Proof. Since $\overline{u}\otimes\overline{u}$ is of rank 1, the lower bound on k is obvious. The upper bound follows from (4) in Claim 8.15 and Carathéodory's theorem applied in the vector space $\mathcal{M}$. □

Next, we show that if $(\overline{S},0)$ and $\widehat{C}$ are separated by a hyperplane in $\mathcal{M}$, then the normal vector of that hyperplane can be chosen to be of a special form.

Claim 8.17. *There is $(\overline{H},h)\in\mathcal{M}$ satisfying (8.28) if and only if, there is $(\overline{H}_0,h)$ satisfying (8.28), where $\overline{H}_0=H_0\oplus\gamma$ for some $H_0\in\mathbb{R}^{d\times d}$.*

Proof. For any $\overline{u}\in\mathbb{R}^{d+1}$, let $\overline{u}'$ denote the reflection of $\overline{u}$ about $\mathbb{R}^d$, that is, $\overline{u}'$ differs from $\overline{u}$ in the last coordinate only, which is the opposite of the last coordinate of $\overline{u}$. Since both $\overline{K}$ and $\mathbf{B}^{d+1}$ are symmetric about $\mathbb{R}^d$, if $\overline{u}$ is in C then so is $\overline{u}'$.

Let $\overline{H}_0$ denote the matrix obtained from $\overline{H}$ by setting the first d entries of the last row to zero, and the first d entries of the last column to zero. Thus, $\overline{H}_0$ is of the required form. We show that $(\overline{H}_0,h)$ satisfies (8.28). Clearly, $\operatorname{trace}\left(\overline{H}_0\overline{S}\right)=\operatorname{trace}\left(\overline{H}\ \overline{S}\right)$, and thus, the first inequality in (8.28) holds.

For the other inequality in (8.28), consider an arbitrary vector $\overline{u}\in C$. Then inequality hold $0>\left\langle(\overline{H},h),(\overline{u}\otimes\overline{u},u)\right\rangle$ and $0>\left\langle(\overline{H},h),(\overline{u}'\otimes\overline{u}',u)\right\rangle$. Note that in the $(d+1)\times(d+1)$ matrix $(\overline{u}'\otimes\overline{u}'+\overline{u}\otimes\overline{u})$, the first d entries of the last row as well as of the last column are 0. Thus,

$$0>\left\langle(\overline{H},h),\big((\overline{u}'\otimes\overline{u}'+\overline{u}\otimes\overline{u})/2,u\big)\right\rangle=$$

$$\left\langle(\overline{H}_0,h),\big((\overline{u}'\otimes\overline{u}'+\overline{u}\otimes\overline{u})/2,u\big)\right\rangle=\left\langle(\overline{H}_0,h),(\overline{u}\otimes\overline{u},u)\right\rangle,$$

completing the proof of Claim 8.17. □

In both parts of the proof of Theorem 8.14, we will consider a path in $\mathcal{E}$, and will need to compute the derivative of the s-volume at the start of this path. It is done in the following.

Claim 8.18. *Let $\varepsilon_0>0$ and let $\gamma:[0,\varepsilon_0]\to\mathbb{R}$ be a continuous function whose right derivative at 0 exists. Let $H\in\mathbb{R}^{d\times d}$ be an arbitrary matrix and $h\in\mathbb{R}^d$. Consider the path*

$$\overline{E}:[0,\varepsilon_0]\to\mathcal{E};\quad t\mapsto\Big(\overline{I}+t\big(H\oplus\gamma(t)\big)\Big)\mathbf{B}^{d+1}+th \tag{8.30}$$

in $\mathcal{E}$. Then the right derivative of the s-volume is

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0^+}\frac{{}^{(s)}\mu\big(\overline{E}(t)\big)}{{}^{(s)}\kappa_{d+1}}=\left\langle(H\oplus\gamma(0),h),(\overline{S},0)\right\rangle. \tag{8.31}$$

Proof. We apply (8.14),

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0^+}\frac{{}^{(s)}\mu\big(\overline{E}(t)\big)}{{}^{(s)}\kappa_{d+1}}=\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0^+}\Big[(1+t\gamma(t))^s\det(I+tH)\Big]=$$

$$(1+0\cdot\gamma(0))^s \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0^+}\Big[\det(I+tH)\Big]+\det(I+0\cdot H)\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0^+}\Big[(1+t\gamma(t))^s\Big]=$$

$$\operatorname{trace}(H)+s\gamma(0),$$

which is equal to the right hand side of (8.31) completing the proof of Claim 8.18. □

Claim 8.19. *If there is* $(H\oplus\gamma,h)\in\mathcal{M}$ *satisfying* (8.28)*, then* $\mathbf{B}^{d+1}$ *is not a locally maximal* s*-volume ellipsoid contained in* $\overline{K}$*.*

Proof. Let $\gamma(t)=\gamma$ be the constant function for $t\geq 0$, and consider the path (8.30) in $\mathcal{E}$. By Claim 8.18 and (8.28), the s-volume has positive derivative at the start of this path. Clearly, ${}^{(s)}\mu\big(\overline{E}(t)\big)$ is differentiable on some interval $[0,\varepsilon_0]$, and hence, there is an $\varepsilon_1>0$ such that for every $0<t<\varepsilon_1$, we have

$$ {}^{(s)}\mu\big(\overline{E}(t)\big)>{}^{(s)}\mu\big(\mathbf{B}^{d+1}\big)\,. \tag{8.32}$$

Now, it is sufficient to establish that there is an $\varepsilon_2>0$ such that for every $0<t<\varepsilon_2$, we have

$$\overline{E}(t)\subseteq\overline{K}. \tag{8.33}$$

Set $\overline{H}=H\oplus\gamma$. First, we fix an arbitrary contact point $\overline{u}\in C$. We claim that there is an $\varepsilon(\overline{u})>0$ such that for every $0<t<\varepsilon(\overline{u})$, we have $(\overline{I}+t\overline{H})\overline{u}+th\in\operatorname{int}\big(\mathbf{B}^{d+1}\big)$. Indeed,

$$\big\langle(\overline{I}+t\overline{H})\overline{u}+th,(\overline{I}+t\overline{H})\overline{u}+th\big\rangle=1+2t\big(\big\langle\overline{H}\overline{u},\overline{u}\big\rangle+\langle h,u\rangle\big)+o(t)=$$

$$1+2t\big\langle(\overline{H},h),(\overline{u}\otimes\overline{u},u)\big\rangle+o(t).$$

By (8.28), the latter is less than 1, if $t>0$ is sufficiently small. Next, the compactness of C yields that there is an $\varepsilon_3>0$ such that $(\overline{I}+\varepsilon_3\overline{H})C+\varepsilon_3 h\subseteq\operatorname{int}\big(\mathbf{B}^{d+1}\big)\subseteq\overline{K}$.

By the continuity of the map $x\mapsto(\overline{I}+\varepsilon_3\overline{H})x+\varepsilon_3 h$, there is an open neighborhood $\mathcal{W}$ of C in $\mathbf{B}^{d+1}$ such that $(\overline{I}+\varepsilon_3\overline{H})\mathcal{W}+\varepsilon_3 h\subseteq\operatorname{int}\big(\mathbf{B}^{d+1}\big)\subseteq\overline{K}$. The latter combined with $\mathcal{W}\subset\operatorname{int}\big(\mathbf{B}^{d+1}\big)$ and with the convexity of $\mathbf{B}^{d+1}$ yield that for every $0<t<\varepsilon_3$, we have $(\overline{I}+t\overline{H})\mathcal{W}+th\subseteq\operatorname{int}\big(\mathbf{B}^{d+1}\big)\subseteq\overline{K}$.

On the other hand, the compact set $\mathbf{B}^{d+1}\setminus\mathcal{W}$ is a subset of $\operatorname{int}\big(\overline{K}\big)$, and hence, there is an $\varepsilon_4>0$ such that for every $0<t<\varepsilon_4$, we have $(\overline{I}+t\overline{H})(\mathbf{B}^{d+1}\setminus\mathcal{W})+th\subseteq\operatorname{int}\big(\overline{K}\big)$. Thus, if $0<t<\varepsilon_2$ is less than $\min\{\varepsilon_3,\varepsilon_4\}$, then $(\overline{I}+t\overline{H})(\mathcal{W})+th\subseteq\operatorname{int}\big(\overline{K}\big)$ and $(\overline{I}+t\overline{H})(\mathbf{B}^{d+1}\setminus\mathcal{W})+th\subseteq\operatorname{int}\big(\overline{K}\big)$. Thus, (8.33) holds concluding the proof of Claim 8.19. □

8.5.1 Proof of part (1) of Theorem 8.14

Assume that there are no contact points and weights satisfying (8.27). By Claims 8.15 and 8.17, there is $(H \oplus \gamma, h) \in \mathcal{M}$ satisfying (8.28). Claim 8.19 yields that $\mathbf{B}^{d+1}$ is not a locally maximal s-volume ellipsoid contained in $\overline{K}$.

The bound on k follows from Claim 8.16, completing the proof of part (1) of Theorem 8.14.

8.5.2 Proof of part (2) of Theorem 8.14

Assume that there is an ellipsoid $\overline{E} = \overline{A}\mathbf{B}^{d+1} + a$ contained in int $\left({}^{(s)}\overline{f}\right)$ with ${}^{(s)}\mu\big(\overline{E}\big) > {}^{(s)}\mu\big(\mathbf{B}^{d+1}\big)$, where $\overline{A} = A \oplus \alpha \in \mathcal{E}$ and $a \in \mathbb{R}^d$.

Set $G = A - I \in \mathbb{R}^{d\times d}$, and define the function $\gamma(t) = \frac{\alpha^t - 1}{t}$ for $t \in (0, 1]$, which, with $\gamma(0) = \ln \alpha$ is a continuous function on $[0, 1]$ whose right derivative at 0 exists. Consider the path

$$\overline{E} : [0, 1] \to \mathcal{E};\;\; t \mapsto \left(\overline{I} + t\big(G \oplus \gamma(t)\big)\right)\mathbf{B}^{d+1} + ta$$

in $\mathcal{E}$ starting at $\overline{E}(0) = \mathbf{B}^{d+1}$ and ending at $\overline{E}(1) = \overline{E}$.

Claim 8.20.

$$0 \le \left\langle (G \oplus \gamma(0), a), (\overline{S}, 0)\right\rangle. \tag{8.34}$$

Proof. By Lemma 8.8, for every $t \in [0, 1]$, we have

$$\frac{{}^{(s)}\mu\big(\overline{E}(t)\big)}{{}^{(s)}\kappa_{d+1}} \ge 1,$$

and hence, for the right derivative, we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0^+} \frac{{}^{(s)}\mu\big(\overline{E}(t)\big)}{{}^{(s)}\kappa_{d+1}} \ge 0.$$

Claim 8.18 now yields the assertion of Claim 8.20. □

We want to have strict inequality in (8.34), thus we modify G a bit. Let

$$H = G + \delta I.$$

By Claim 8.20, we have

$$0 < \left\langle (H \oplus \gamma(0), a), (\overline{S}, 0)\right\rangle. \tag{8.35}$$

Moreover, since $\overline{A}\mathbf{B}^{d+1} + a \subset$ int $\left({}^{(s)}\overline{f}\right)$, we can fix $\delta > 0$ sufficiently small such that we also have that

$$(I + H) \oplus (1 + \gamma(1))\mathbf{B}^{d+1} + a \subset \text{int}\left({}^{(s)}\overline{f}\right). \tag{8.36}$$

Claim 8.21. *Set $\overline{H}_0 = H \oplus \gamma(0)$. Then*

$$\left\langle (\overline{H}_0, a), (\overline{u} \otimes \overline{u}, u) \right\rangle \leq 0 \tag{8.37}$$

for every contact point $\overline{u} \in C$.

Proof. Fix an $\overline{u} \in C$ and consider the curve $\xi : [0,1] \to \mathbb{R}^{d+1}; t \mapsto \overline{u} + t\big(H \oplus \gamma(t)\big)\overline{u} + ta$ in $\mathbb{R}^{d+1}$. By Lemma 8.7 and (8.36), the ellipsoid represented as $(\overline{I}, 0) + t(H \oplus \gamma(t), a)$ is contained in ${}^{(s)}\overline{f}$ for every $t \in [0,1]$, and in particular, the curve ξ is contained in ${}^{(s)}\overline{f}$. By convexity and (8.36), we have that the projection of ξ onto $\mathbb{R}^d$ is a subset of the closure of the support of f. Further, ξ is a smooth curve and its tangent vector $\xi'(0)$ is given by

$$\xi'(0) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0^+} \big(\overline{u} + t\big(H \oplus \gamma(t)\big)\overline{u} + ta\big) =$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0^+} \big((tH \oplus (\alpha^t - 1))\overline{u} + ta\big) = (H \oplus \ln \alpha)\overline{u} + a.$$

We consider two cases as to whether $\overline{u} \in \mathbb{R}^d$ or not.

First, if $\overline{u} \in \mathbb{R}^d$, then $\overline{u}$ belongs to the boundary of the support of f. Since the support of a log-concave function is a convex set, we conclude that $\overline{u}$ is the outer normal vector to the support of f at $\overline{u}$. Thus, $\langle \xi'(0), \overline{u} \rangle \leq 0$.

Second, if $\overline{u} \notin \mathbb{R}^d$, then Lemma 8.3 implies that $\operatorname{bd}\left({}^{(s)}\overline{f}\right)$ is a smooth hypersurface in $\mathbb{R}^{d+1}$ at $\overline{u}$, whose outer unit normal vector at $\overline{u}$ is $\overline{u}$ itself. Thus, the angle between the tangent vector vector $\xi'(0)$ of the curve ξ and the outer normal vector of the hypersurface $\operatorname{bd}\left({}^{(s)}\overline{f}\right)$ at $\overline{u}$ is not acute. That is, $\langle \xi'(0), \overline{u} \rangle \leq 0$.

Hence, in both cases, we have

$$0 \geq \langle \xi'(0), \overline{u} \rangle = \langle (H \oplus \ln \alpha)\overline{u} + a), \overline{u} \rangle,$$

which is (8.37) completing the proof of Claim 8.21. □

In summary, (8.35) and Claim 8.21 show that when $\big(\overline{H}_0, a\big)$ is substituted in the place of $(\overline{H}, h)$, then (8.29) holds. Hence, by Claim 8.15, the proof of part (2) of Theorem 8.14 is complete.

8.6 The Helly-type result — Proof of Theorem 8.2

In this section, we prove Theorem 8.2.

8.6.1 Assumption: the functions are supported on $\mathbb{R}^d$

We claim that we may assume that the support of each f_i is $\mathbb{R}^d$. Indeed, any log-concave function can be approximated in the L_1-norm by log-concave functions whose support is $\mathbb{R}^d$. We may approximate each function so that the f_σ (recall that f_σ is the pointwise minimum of functions $\{f_i\}_{i\in\sigma}$) are also all well approximated. One way to achieve this is to take the Asplund sum $f_i \star (e^{-\delta|x|^2})$ for a sufficiently large $\delta > 0$ (see Section 8.4.1).

8.6.2 Assumption: John position

Consider the s-lifting of our functions with $s = 1$. Clearly, the s-lifting of a pointwise minimum of a family of functions is the intersection of the s-liftings of the functions.

From our assumption in Subsection 8.6.1, it follows that $\int_{\mathbb{R}^d} f > 0$. By applying a linear transformation on $\mathbb{R}^d$, we may assume that, with $s = 1$, the largest s-volume ellipsoid in the s-lifting ${}^{(1)}\overline{f}$ of f is $\mathbf{B}^{d+1} \subset {}^{(1)}\overline{f}$.

By Theorem 8.14, there are contact points $\overline{u}_1, \ldots, \overline{u}_k \in \operatorname{bd}\left(\mathbf{B}^{d+1}\right) \cap \operatorname{bd}\left({}^{(1)}\overline{f}\right)$, and weights $c_1, \ldots, c_k > 0$ satisfying (8.27) with $s = 1$. For each $j \in [k]$, we denote by u_j the orthogonal projection of the contact point $\overline{u}_j$ to $\mathbb{R}^d$ and by $w_j = \sqrt{1 - |u_j|^2}$.

8.6.3 Reduction of the problem to finding P and η

Claim 8.22. *With the assumptions in Subsections 8.6.1 and 8.6.2, we can find a set of indices* $\eta \in \binom{[k]}{\leq 2d+1}$ *and an origin-symmetric convex body* P *in* $\mathbb{R}^d$ *with the following two properties.*

$$\operatorname{vol}(d)\, P \leq 50^d d^{3d/2} \operatorname{vol}(d)\,(\mathbf{B}^d) \tag{8.38}$$

and

$$\|x\|_P \leq \max\left\{\, \langle x, u_j\rangle \;:\; j \in \eta \right\} \quad \text{for every } x \in \mathbb{R}^d, \tag{8.39}$$

where $\|\cdot\|_P$ *is the gauge function of* P*, that is,* $\|x\|_P = \inf\{\lambda > 0 \;:\; x \in \lambda P\}$.

We will prove Claim 8.22 in Subsection 8.6.5.

In the present subsection, we show that Claim 8.22 yields the existence of the desired index set $\sigma \in \binom{[n]}{\leq 2d+1}$ *that satisfies* (8.1).

Let K be a set in $\mathbb{R}^n$, then its *polar* is defined by $K^\circ = \{p \in \mathbb{R}^n \;:\; \langle y, p\rangle \leq 1 \quad \forall y \in K\}$. Set $T = \{u_j \;:\; j \in \eta\}$. It is easy to see that (8.39) is equivalent to

$$T^\circ \subseteq P. \tag{8.40}$$

Notice that

$$\text{for all } x \in \mathbb{R}^d \setminus T^\circ \text{ there is } j \in \eta \text{ such that } \langle u_j, x - u_j \rangle \geq 0. \tag{8.41}$$

We will split the integral in (8.1) into two parts: the integral on $\mathbb{R}^d \setminus T^\circ$ and the integral on T°.

First, we find a set σ_1 of indices in $[n]$ that will help us bound the integral in (8.1) on $\mathbb{R}^d \setminus T^\circ$.

Fix a $j \in \eta$. Since $\overline{u}_j \in \operatorname{bd}\left({}^{(1)}\overline{f}\right)$, there is an index $i(j) \in [n]$ such that $\overline{u}_j \in \operatorname{bd}\left({}^{(1)}\overline{f}_{i(j)}\right)$. Let σ_1 be the set of these indices, that is, $\sigma_1 = \{i(j) : j \in \eta\}$.

By (8.16), for each $j \in \eta$, we have

$$f_{i(j)}(x) \leq w_j e^{-\frac{1}{w_j^2}\langle u_j, x-u_j \rangle} \leq e^{-\frac{1}{w_j^2}\langle u_j, x-u_j \rangle} \tag{8.42}$$

for all $x \in \mathbb{R}^d$.

Next, we find a set σ_2 of indices in $[n]$ that will help us bound the integral in (8.1) on T°.

It is easy to see that there is a $\sigma_2 \in \binom{[n]}{\leq d+1}$ such that $\|f\| = \|f_{\sigma_2}\|$. Indeed, for any $i \in [n]$, consider the following convex set in $\mathbb{R}^d$: $[f_i > \|f\|]$. By the definition of f, the intersection of these n convex sets in $\mathbb{R}^d$ is empty. Helly's theorem yields the existence of σ_2.

We combine the two index sets: let $\sigma = \sigma_1 \cup \sigma_2$. Clearly, σ is of cardinality at most $3d + 2$. We need to show that σ satisfies (8.1). Indeed, we have

$$\int_{\mathbb{R}^d} f_\sigma \leq \int_{T^\circ} \|f_\sigma\| + \int_{\mathbb{R}^d \setminus T^\circ} f_\sigma \overset{(8.25)}{\leq} \int_{T^\circ} e^d + \int_{\mathbb{R}^d \setminus T^\circ} f_\sigma \overset{(8.40)}{\leq}$$

$$e^d \operatorname{vol}(d)(P) + \int_{\mathbb{R}^d \setminus T^\circ} f_{\sigma_1} \overset{(8.42)}{\leq} e^d \operatorname{vol}(d)(P)$$

$$+ \int_{\mathbb{R}^d \setminus T^\circ} \exp\left(-\max\left\{\frac{1}{w_j^2}\langle u_j, x - u_j \rangle \ : \ j \in \eta\right\}\right) \overset{(8.41)}{\leq}$$

$$e^d \operatorname{vol}(d)(P) + \int_{\mathbb{R}^d \setminus T^\circ} \exp\left(-\max\left\{\langle u_j, x - u_j \rangle \ : \ j \in \eta\right\}\right) \leq$$

$$e^d \operatorname{vol}(d)(P) + e \int_{\mathbb{R}^d \setminus T^\circ} \exp\left(-\max\left\{\langle u_j, x \rangle \ : \ j \in \eta\right\}\right) \overset{(8.39)}{\leq}$$

$$e^d \operatorname{vol}(d)(P) + e \int_{\mathbb{R}^d \setminus T^\circ} \exp\left(-\|x\|_P\right) \leq$$

$$e^d \operatorname{vol}(d)(P) + e \int_{\mathbb{R}^d} \exp(-\|x\|_P) \le (e^d + e \cdot d!) \operatorname{vol}(d)(P) \le d^{d/2} \operatorname{vol}(d)(P).$$

By (8.38), and the fact that $\operatorname{vol}(d)(\mathbf{B}^d) \le d \operatorname{vol}(d+1)(\mathbf{B}^{d+1}) \le d \int_{\mathbb{R}^d} f$, we obtain (8.1).

8.6.4 The Dvoretzky-Rogers lemma

One key tool in proving Claim 8.22 is the Dvoretzky-Rogers lemma [DR50], which we recall here from Chapter 5 (see Lemma 5.4) with a slightly modified notation.

Lemma 8.23 (Dvoretzky-Rogers lemma)**.** *Assume that the points* $\overline{u}_1, \dots, \overline{u}_k \in \operatorname{bd}(\mathbf{B}^{d+1})$*, satisfy* (8.27) *for* $s = 1$ *with some weights* $c_1, \dots, c_k > 0$*. Then there is a sequence* $j_1, \dots, j_{d+1}$ *of* $d+1$ *distinct indices in* $[k]$ *such that*

$$\operatorname{dist}\left(\overline{u}_{j_t}, \operatorname{span}\{\overline{u}_{j_1}, \dots, \overline{u}_{j_{t-1}}\}\right) \ge \sqrt{\frac{d-t+2}{d+1}} \quad \text{for all } t = 2, \dots, d+1,$$

where dist *denotes the shortest Euclidean distance of a vector from a subspace.*

It follows immediately, that the determinant of the $(d+1) \times (d+1)$ matrix with columns $\overline{u}_{j_1}, \dots \overline{u}_{j_{d+1}}$ is at least

$$\left|\det\left[\overline{u}_{j_1}, \dots \overline{u}_{j_{d+1}}\right]\right| \ge \frac{\sqrt{(d+1)!}}{(d+1)^{(d+1)/2}}. \tag{8.43}$$

8.6.5 Finding P and η

In this subsection, we prove Claim 8.22, that is, we show that with the assumptions in Subsections 8.6.1 and 8.6.2, there is an origin symmetric convex body P and a set of indices $\eta \in \binom{[k]}{\le 2d+1}$ satisfying (8.38) and (8.39). Once it is shown, by Subsection 8.6.3, the proof of Theorem 8.2 is complete.

The proof in this section follows very closely the proof of the main result in [Nas16b] as refined by Brazitikos in [Bra17].

Let $\eta_1 \in \binom{[k]}{d+1}$ be the set of $d+1$ indices in $[k]$ given by Lemma 8.23, and let $\overline{\Delta}$ be the simplex $\overline{\Delta} = \operatorname{conv}(\{\overline{u}_j : j \in \eta_1\} \cup \{0\})$ in $\mathbb{R}^{d+1}$. Let $\overline{z} = \frac{\sum_{j \in \eta_1} \overline{u}_j}{d+1}$ denote the centroid of $\overline{\Delta}$, and $\overline{P}_1$ denote the intersection of $\overline{\Delta}$ and its reflection about $\overline{z}$, that is, $\overline{P}_1 = \overline{\Delta} \cap (2\overline{z} - \overline{\Delta})$, a polytope which is centrally symmetric about $\overline{z}$. It is well known [MP00, Corollary 3] (see also

[AS17, Section 4.3.5]), that $\mathrm{vol}\,(d+1)\,\overline{P}_1 \geq 2^{-(d+1)}\,\mathrm{vol}\,(d+1)\,\overline{\Delta}$, and hence, by (8.43), we have

$$\mathrm{vol}\,(d+1)\,\overline{P}_1 \geq 2^{-(d+1)}\frac{|\det[\overline{u}_j \ : \ j \in \eta_1]|}{(d+1)!} \geq \frac{1}{2^{d+1}\sqrt{(d+1)!}(d+1)^{(d+1)/2}}$$

Let P_1 denote the orthogonal projection of $\overline{P}_1$ to $\mathbb{R}^d$. Since $\overline{P}_1 \subset P_1 \times [-1,1]$, we have that

$$\mathrm{vol}\,(d)\,(P_1) \geq \frac{1}{2^{d+2}\sqrt{(d+1)!}(d+1)^{(d+1)/2}}. \tag{8.44}$$

Moreover, P_1 is symmetric about the orthogonal projection z of $\overline{z}$ to $\mathbb{R}^d$.

Let $\overline{Q}$ denote the convex hull of the contact points, $\overline{Q} = \mathrm{conv}\left(\mathrm{bd}\left({}^{(s)}\overline{f}\right) \cap \mathrm{bd}\left(\mathbf{B}^{d+1}\right)\right)$, and Q denote the orthogonal projection of $\overline{Q}$ to $\mathbb{R}^d$. As a well known consequence of (8.27) for $s=1$ [Bal97], we have $\frac{1}{d+1}\mathbf{B}^{d+1} \subset \overline{Q}$, and hence, $\frac{1}{d+1}\mathbf{B}^d \subset Q$.

Let ℓ be the ray in $\mathbb{R}^d$ emanating from the origin in the direction of the vector $-z$, and let y be the point of intersection of ℓ with the boundary (in $\mathbb{R}^d$) of Q, that is, $\{y\} = \ell \cap \mathrm{bd}\,(Q)$. Now, $\frac{1}{d+1}\mathbf{B}^d \subset Q$ yields that $|y| \geq 1/(d+1)$.

We apply a contraction with center y and ratio $\lambda = \frac{|y|}{|y-z|}$ on P_1 to obtain the polytope P_2. Clearly, P_2 is a convex polytope in $\mathbb{R}^d$ which is symmetric about the origin. Furthermore,

$$\lambda = \frac{|y|}{|y-z|} \geq \frac{|y|}{1+|y|} \geq \frac{1}{d+2}. \tag{8.45}$$

Let P be the polar $P = P_2^\circ$ of P_2 taken in $\mathbb{R}^d$. By the Blaschke–Santaló inequality [Gru07, Theorem 9.5], and using $\mathrm{vol}\,(d)\,(\mathbf{B}^d) \leq 10^d d^{-d/2}$, we obtain

$$\mathrm{vol}\,(d)\,(P) \leq \frac{\mathrm{vol}\,(d)\,(\mathbf{B}^d)^2}{\mathrm{vol}\,(d)\,(P_2)} = \frac{\mathrm{vol}\,(d)\,(\mathbf{B}^d)^2}{\lambda^d\,\mathrm{vol}\,(d)\,(P_1)} \leq \frac{10^d d^{-d/2}\,\mathrm{vol}\,(d)\,(\mathbf{B}^d)}{\lambda^d\,\mathrm{vol}\,(d)\,(P_1)},$$

which, by (8.44) and (8.45), yields that P satisfies (8.38).

To complete the proof, we need to find $\eta \in \binom{[k]}{\leq 2d+1}$ such that P and η satisfy (8.39).

Since y is on $\mathrm{bd}\,(Q)$, by Carathéodory's theorem, y is in the convex hull of some subset of at most d vertices of Q. Let this subset be $\{u_j \ : \ j \in \eta_2\}$, where $\eta_2 \in \binom{[k]}{\leq d}$.

We set $\eta = \eta_1 \cup \eta_2$, and claim that P and η satisfy (8.39).

Indeed, since $P_2 \subseteq \operatorname{conv}(\{u_j : j \in \eta_1\} \cup \{y\})$ and $y \in \operatorname{conv}(\{u_j : j \in \eta_2\})$, we have

$$P_2 \subseteq \operatorname{conv}(\{u_j : j \in \eta\}).$$

Taking the polar of both sides in $\mathbb{R}^d$, we obtain $P \supseteq \{u_j : j \in \eta\}^\circ$, which is equivalent to (8.39).

Thus, P and η satisfy (8.38) and (8.39), and hence, the proof of Theorem 8.2 is complete.

8.6.6 Lower bound on the Helly number

The number of functions selected in Theorem 8.2 is $3d+2$. In this subsection, we show that it cannot be decreased to $2d$. In fact, for any dimension d and any $\Delta > 0$, we give an example of $2d+1$ log-concave functions $f_1, \ldots, f_{2d+1}$ such that $\int f_{[n]} = 2^d$, but for any $I \in \binom{[2d+1]}{\leq 2d}$, the integral is $\int f_I > \Delta$. Our example is a simple extension of the standard one (the $2d$ supporting half-spaces of a cube) for convex sets.

Set

$$\varphi(t) = \begin{cases} 0, & \text{if } t < 0 \\ e^{\Delta}, & \text{otherwise.} \end{cases}$$

Clearly, φ is upper semi-continuous. Let $e_1, \ldots, e_d$ denote the standard basis in $\mathbb{R}^d$, and for each $i \in [d]$, define the functions $f_i(x) = \varphi(\langle e_i, x + e_i\rangle)$ and $f_{d+i} = \varphi(-\langle e_i, x - e_i\rangle)$, and let $f_{2d+1} = 1$. These functions are proper log-concave functions. The bounds on the integrals are easy.

8.7 Further inequalities and the limit as s tends to 0

8.7.1 Comparison of the s-volumes of John s-ellipsoids for distinct values of s

Lemma 8.24. *Let f be a proper log-concave function on $\mathbb{R}^d$, and $0 < s_1 < s_2$. Then,*

$$\sqrt{\left(\frac{s_2}{d+s_2}\right)^{s_2}\left(\frac{d}{d+s_2}\right)^{d}} \cdot \frac{{}^{(s_1)}\kappa_{d+1}}{{}^{(s_2)}\kappa_{d+1}} \leq \frac{{}^{(s_1)}\mu\big(\overline{E}(f,s_1)\big)}{{}^{(s_2)}\mu\big(\overline{E}(f,s_2)\big)} \leq \frac{{}^{(s_1)}\kappa_{d+1}}{{}^{(s_2)}\kappa_{d+1}}.$$

Proof. We start with the second inequality. We may assume that $\overline{E}(f,s_1) = \mathbf{B}^{d+1}$, and hence, its height function is $h_{\overline{E}(f,s_1)}(x) = \sqrt{1-|x|^2}$ for $x \in \mathbf{B}^d$. Since $s_1 < s_2$ and $h_{\overline{E}(f,s_1)}(x) \leq 1$, we have

$$\left(h_{\overline{E}(f,s_1)}(x)\right)^{s_2} \leq \left(h_{\overline{E}(f,s_1)}(x)\right)^{s_1} \leq f(x) \quad \text{for all } x \in \mathbf{B}^d.$$

That is, by (8.15), $\mathbf{B}^{d+1} \subset {}^{(s_2)}\overline{f}$, which yields ${}^{(s_2)}\mu\big(\mathbf{B}^{d+1}\big) \leq {}^{(s_2)}\mu\big(\overline{E}(f,s_2)\big)$. Hence,

$$\frac{{}^{(s_1)}\mu\big(\overline{E}(f,s_1)\big)}{{}^{(s_2)}\mu\big(\overline{E}(f,s_2)\big)} \leq \frac{{}^{(s_1)}\mu\big(\mathbf{B}^{d+1}\big)}{{}^{(s_2)}\mu\big(\mathbf{B}^{d+1}\big)} = \frac{{}^{(s_1)}\kappa_{d+1}}{{}^{(s_2)}\kappa_{d+1}}.$$

Next, we prove the first inequality of the assertion of the lemma. Now, we assume that $\overline{E}(f,s_2) = \mathbf{B}^{d+1}$. Therefore, for any $\rho \in (0,1)$, we have that ${}^{(s_2)}\overline{f}$ contains the cylinder $\rho\mathbf{B}^d \times [0, \sqrt{1-\rho^2}]$. Hence, ${}^{(s_1)}\overline{f}$ contains the ellipsoid $\overline{E}$, represented by $\left(\rho I \oplus \left(\sqrt{1-\rho^2}\right)^{s_2/s_1}, 0\right)$, whose s_1-volume by (8.14) is ${}^{(s_1)}\kappa_{d+1} \cdot \rho^d \cdot (1-\rho^2)^{s_2/2}$. Choosing $\rho = \sqrt{\frac{d}{d+s_2}}$, we obtain

$$\sqrt{\left(\frac{s_2}{d+s_2}\right)^{s_2}\left(\frac{d}{d+s_2}\right)^{d}} \cdot \frac{{}^{(s_1)}\kappa_{d+1}}{{}^{(s_2)}\kappa_{d+1}} = \frac{{}^{(s_1)}\mu\big(\overline{E}\big)}{{}^{(s_2)}\mu\big(\mathbf{B}^{d+1}\big)} \leq \frac{{}^{(s_1)}\mu\big(\overline{E}(f,s_1)\big)}{{}^{(s_2)}\mu\big(\overline{E}(f,s_2)\big)}.$$

□

8.7.2 Stability of the John s-ellipsoid

Lemma 8.25. *Fix the dimension d and a positive constant $C > 0$. Then there exist constants $\varepsilon_C > 0$ and $k_C > 0$ with the following property. Let $s \in (0,\infty)$, $\varepsilon \in [0, \varepsilon_C]$ and f be a proper log-concave function on $\mathbb{R}^d$, whose John s-ellipsoid $\overline{E}(f,s)$ is represented by $(A_1 \oplus \alpha_1, a_1)$, and let $\overline{E}_2$ denote another ellipsoid, represented by $(A_2 \oplus \alpha_2, a_2)$, with $\overline{E}_2 \subset {}^{(s)}\overline{f}$. Assume that*

$${}^{(s)}\mu\big(\overline{E}(f,s)\big) \geq C - \varepsilon \quad \text{and} \quad {}^{(s)}\mu\big(\overline{E}(f,s)\big) \geq {}^{(s)}\mu\big(\overline{E}_2\big) \geq {}^{(s)}\mu\big(\overline{E}(f,s)\big) - \varepsilon. \tag{8.46}$$

Then

$$\left\|\frac{A_1}{\|A_1\|} - \frac{A_2}{\|A_2\|}\right\| + \frac{|\alpha_1^s - \alpha_2^s|}{\|f\|} + \frac{|a_1 - a_2|}{\|A_1\| \cdot \|f\|} \leq k_C\sqrt{\varepsilon}. \tag{8.47}$$

In this subsection, we prove Lemma 8.25.

Let $\overline{E}$ denote the ellipsoid represented by

$$\left(\frac{A_1 + A_2}{2} \oplus \sqrt{\alpha_1\alpha_2}, \frac{a_1 + a_2}{2}\right).$$

Claim 8.26. *There are constants $\varepsilon_0 > 0$ and $k_0 > 0$ such that if the ellipsoids $\overline{E}(f,s)$ and $\overline{E_2}$ satisfy* (8.46) *for $\varepsilon \in [0, \varepsilon_0]$, then*

$$(1 - k_0\sqrt{\varepsilon})A_1 \prec A_2 \prec (1 + k_0\sqrt{\varepsilon})A_1, \tag{8.48}$$

and

$$\left\| \frac{A_1}{\|A_1\|} - \frac{A_2}{\|A_2\|} \right\| \le k_0\sqrt{\varepsilon} \quad \textit{and} \quad 1 - k_0\sqrt{\varepsilon} \le \frac{\det A_1}{\det A_2} \le 1 + k_0\sqrt{\varepsilon}.$$

Proof. By (8.14), we have

$$\frac{{}^{(s)}\mu\big(\overline{E}\big)}{\sqrt{{}^{(s)}\mu\big(\overline{E}(f,s)\big)\,{}^{(s)}\mu\big(\overline{E}_2\big)}} = \frac{1}{2^d}\frac{\det(A_1 + A_2)}{\sqrt{\det A_1 \det A_2}}$$

Since ${}^{(s)}\mu\big(\overline{E}(f,s)\big) \ge {}^{(s)}\mu\big(\overline{E}\big)$ and by (8.46), there exist $\varepsilon_1 > 0$ and $k_1 > 0$ such that the left-hand side of this is at most $1 + k_1 \cdot \varepsilon$ for all $\varepsilon \in [0, \varepsilon_1]$. Therefore, we have that

$$1 + k_1 \cdot \varepsilon \ge \frac{1}{2^d}\frac{\det(A_1 + A_2)}{\sqrt{\det A_1 \det A_2}}. \tag{8.49}$$

Let R be the square root of A_1, and U be the orthogonal transformation that diagonalizes $R^{-1}A_2R^{-1}$, that is, the matrix $D = UR^{-1}A_2R^{-1}U^T$ is diagonal. Let $D = \operatorname{diag}(\beta_1, \ldots, \beta_d)$. Then for $S = UR^{-1}$, we have $SA_1S^T = I, SA_2S^T = D$. By the multiplicativity of the determinant, inequality (8.49) is equivalent to

$$1 + k_1 \cdot \varepsilon \ge \prod_1^d \frac{1 + \beta_i}{2\sqrt{\beta_i}}.$$

Since $1 + \beta \ge 2\sqrt{\beta}$ for any $\beta > 0$, this implies that

$$1 + k_1 \cdot \varepsilon \ge \frac{1 + \beta_i}{2\sqrt{\beta_i}}$$

for every $i \in [d]$. Using the degree 2 Taylor expansion of the function $\beta \mapsto \sqrt{\beta}$ at $\beta = 1$, we obtain that there exist positive constants k_2 and ε_2 such that the inequality

$$1 - k_2\sqrt{\varepsilon} \le \beta_i \le 1 + k_2\sqrt{\varepsilon} \tag{8.50}$$

holds for all $\varepsilon \in [0, \varepsilon_2]$.

Clearly, $\frac{\det A_1}{\det A_2} = 1/\prod_{i=1}^d \beta_i$ and hence, the estimate on $\frac{\det A_1}{\det A_2}$ follows from (8.50).

On the other hand, (8.50) yields also that

$$(1-k_2\sqrt{\varepsilon})I \prec SA_2S^T \prec (1+k_2\sqrt{\varepsilon})I.$$

Thus, (8.48) follows. Hence, there exist positive constants k_3 and ε_3 such that the inequality

$$\left|\frac{\|A_2\|}{\|A_1\|}-1\right| \leq k_3\sqrt{\varepsilon}$$

holds for all $\varepsilon \in [0,\varepsilon_3]$. This and (8.48) yield that

$$\left\|\frac{A_1}{\|A_1\|}-\frac{A_2}{\|A_2\|}\right\| \leq \left\|\frac{A_1}{\|A_1\|}-\frac{A_2}{\|A_1\|}\right\| + \left\|\frac{A_2}{\|A_1\|}-\frac{A_2}{\|A_2\|}\right\| \leq (k_2+k_3)\sqrt{\varepsilon}$$

for all $\varepsilon \in [0,\min\{\varepsilon_2,\varepsilon_3\}]$. This completes the proof of Claim 8.26. □

Claim 8.27. *There are constants $\varepsilon_0 > 0$ and $k_0 > 0$ such that if the ellipsoids $\overline{E}(f,s)$ and $\overline{E_2}$ satisfy* (8.46) *for $\varepsilon \in [0,\varepsilon_0]$, then*

$$|\alpha_1^s - \alpha_2^s| \leq \|f\|\, k_0\sqrt{\varepsilon}.$$

Proof. By identity (8.14) and the inequalities in (8.46), we have that

$$1 \geq \frac{{}^{(s)}\mu\big(\overline{E}_2\big)}{{}^{(s)}\mu\big(\overline{E}(f,s)\big)} = \frac{\det A_2 \cdot \alpha_2^s}{\det A_1 \cdot \alpha_1^s} \geq 1 - \frac{\varepsilon}{{}^{(s)}\mu\big(\overline{E}(f,s)\big)}.$$

By this and by Claim 8.26, we get the following multiplicative inequality

$$\big(1+k_1\sqrt{\varepsilon}\big)\,\alpha_1^s \geq \alpha_2^s \geq \big(1-k_1\sqrt{\varepsilon}\big)\,\alpha_1^s$$

for all $\varepsilon \in [0,\varepsilon_1]$, where k_1 and ε_1 are some positive constants. Equivalently, we have that

$$k_1\sqrt{\varepsilon}\cdot\frac{\alpha_1^s}{\|f\|} \geq \frac{\alpha_2^s-\alpha_1^s}{\|f\|} \geq -k_1\sqrt{\varepsilon}\cdot\frac{\alpha_1^s}{\|f\|}.$$

The claim follows since $\frac{\alpha_1^s}{\|f\|} \leq 1$. □

To complete the proof of Lemma 8.25, we need to show that a_1 and a_2 are close. By translating the origin and rotating the space $\mathbb{R}^d$, we may assume that $a_1 = -a_2 \neq 0$ and that $A_1^{-1}a_1 = \delta e_1$ for some $\delta > 0$. Consider the ellipsoid

$$\overline{E}_0 = (A_1 \oplus \alpha_1)\overline{M}\mathbf{B}^{d+1}, \quad \text{where } \overline{M} = \operatorname{diag}(1+\delta, 1, \ldots, 1).$$

Clearly, ${}^{(s)}\mu\big(\overline{E}_0\big) = {}^{(s)}\mu\big(\overline{E}(f,s)\big)\left(1+\frac{|A_1^{-1}(a_1-a_2)|}{2}\right) \geq {}^{(s)}\mu\big(\overline{E}(f,s)\big)\left(1+\frac{|(a_1-a_2)|}{2\|A_1\|}\right)$.

By (8.48) and Claim 8.27, we have

$$\big((1-k_0\sqrt{\varepsilon})A_1 \oplus (1-k_0\|f\|\sqrt{\varepsilon})^{1/s}\alpha_1\big)\,\mathbf{B}^{d+1}+a_2 \subseteq {}^{(s)}\overline{f}.$$

On the other hand, clearly,

$$\big((1-k_0\sqrt{\varepsilon})A_1 \oplus (1-k_0\|f\|\sqrt{\varepsilon})^{1/s}\alpha_1\big)\,\mathbf{B}^{d+1}+a_1 \subseteq$$

$$(A_1\oplus\alpha_1)\,\mathbf{B}^{d+1}+a_1 \subseteq {}^{(s)}\overline{f}.$$

Thus, by Lemma 8.9,

$$\big((1-k_0\sqrt{\varepsilon})A_1 \oplus (1-k_0\|f\|\sqrt{\varepsilon})^{1/s}\alpha_1\big)\,\overline{M}\mathbf{B}^{d+1}$$

is contained in ${}^{(s)}\overline{f}$.

By (8.14), the s-volume of this ellipsoid is

$${}^{(s)}\mu\big(\overline{E}(f,s)\big)\left(1-k_0\|f\|\sqrt{\varepsilon}\right)\left(1-k_0\sqrt{\varepsilon}\right)^d\left(1+\frac{|(a_1-a_2)|}{2\|A_1\|}\right) \leq$$

$${}^{(s)}\mu\big(\overline{E}(f,s)\big)\,.$$

Thus, there exist constants $\varepsilon_1, k_1 > 0$ such that $\frac{|a_1-a_2|}{\|A_1\|} \leq k_1\|f\|\sqrt{\varepsilon}$ for any $\varepsilon \in [0,\varepsilon_1]$. From this and Claims 8.26 and 8.27, Lemma 8.25 follows.

8.7.3 The limit as $s \to 0$

We recall from Section 8.1 the approach of Alonso-Gutiérrez, Merino, Jiménez and Villa [AGMJV18].

Let f be a proper log-concave function on $\mathbb{R}^d$. For every $\beta \geq 0$, consider the superlevel set $[f \geq \beta]$ of f. This is a bounded convex set with non-empty interior in $\mathbb{R}^d$, we take its largest volume ellipsoid, and multiply the volume of this ellipsoid by β. As shown in [AGMJV18], there is a unique $\beta_0 \in [0, \|f\|]$ such that this product is maximal, where $\|f\|$ denotes the L_∞ norm of f. Furthermore, $\beta_0 \geq e^{-d}\|f\|$. We call the ellipsoid E in $\mathbb{R}^d$ obtained for this β_0 *the AMJV ellipsoid*.

We refer to a function of the form $\beta\chi_E$, where $E \subset \mathbb{R}^d$ is an ellipsoid in $\mathbb{R}^d$ and $\beta > 0$, as a *flat-ellipsoid function*. We will say that $(A\oplus\alpha, a) \in \mathcal{E}$ represents the flat-ellipsoid function $\alpha\chi_{A\mathbf{B}^d+a}$. Clearly, any flat-ellipsoid function is represented by a unique element of $\mathcal{E}$ and the AMJV ellipsoid is the maximal integral flat-ellipsoid function among all flat-ellipsoid functions that are pointwise less than f.

Theorem 8.28 (The AMJV ellipsoid is the 0-John ellipsoid)**.** *Let f be a proper log-concave function. Then there exists $(A \oplus \alpha, a) \in \mathcal{E}$ such that*

1. *The function $\alpha\chi_{A\mathbf{B}^d+a}$ is pointwise less than f.*
2. *The functions ${}^{(s)}J_f$ uniformly converge on the complement of any open neighborhood of the boundary in $\mathbb{R}^d$ of $A\mathbf{B}^d + a$ as s tends to 0 to $\alpha\chi_{A\mathbf{B}^d+a}$.*
3. *The function $\alpha\chi_{A\mathbf{B}^d+a}$ is a unique flat-ellipsoid function of maximal integral among all flat-ellipsoid functions that are pointwise less than f.*

In this subsection, we prove Theorem 8.28.

We start with the existence of the limit flat-ellipsoid function in (2). Let $\overline{E}(f,s)$ be represented by $(A_s \oplus \alpha_s, a_s)$ for every $s \in (0,1]$.

Claim 8.29. *The following limits exist.*

$$\lim_{s\to 0^+} {}^{(s)}\mu\big(\overline{E}(f,s)\big) = \mu > 0, \ \lim_{s\to 0^+} A_s = A, \ \lim_{s\to 0^+} \alpha_s^s = \alpha > 0 \tag{8.51}$$

and

$$\lim_{s\to 0^+} a_s = a,$$

where A is positive definite.

Proof. Since the John 1-ellipsoid exists, by (8.13) and Lemma 8.24, we have that

$$\limsup_{s\to 0^+} {}^{(s)}\mu\big(\overline{E}(f,s)\big) > 0.$$

By Lemma 8.6, the ellipsoids $A_s\mathbf{B}^d$ are uniformly bounded for all $s \in (0,1]$. Hence, the norms $\|A_s\|$ are uniformly bounded. By Lemma 8.12, $\alpha_s^s \in [e^{-d}\|f\|, \|f\|]$. Thus, there exists a sequence of positive reals $\{s_i\}_1^\infty$ with $\lim_{i\to\infty} s_i = 0$ such that

$${}^{(s_i)}\mu\big(\overline{E}(f,s_i)\big) \to \limsup_{s\to 0^+} {}^{(s)}\mu\big(\overline{E}(f,s)\big), \ A_{s_i} \to A, \ \alpha_{s_i}^{s_i} \to \alpha \quad \text{and} \quad a_{s_i} \to a$$

for some positive semidefinite matrix $A \in \mathbb{R}^{d\times d}$, an $\alpha > 0$ and $a \in \mathbb{R}^d$, as i tends to ∞. Inequality (8.18) implies that A is positive definite.

We use J_f to denote the flat-ellipsoid function represented by $(A \oplus \alpha, a)$. Clearly, J_f is pointwise less than f. Consider the ellipsoids $\overline{E}_s$ represented by $(A \oplus \alpha^{1/s}, a)$ for all $s \in (0,1]$. Then, $\overline{E}_s \subset {}^{(s)}\overline{f}$ for every $s \in (0,1]$. By (8.14) and (8.13), we have

$$^{(s)}\mu\big(\overline{E}_s\big) = \frac{^{(s)}\kappa_{d+1}}{\operatorname{vol}(d)\,(\mathbf{B}^d)} \int_{\mathbb{R}^d} J_f \mathrm{d}x \to \int_{\mathbb{R}^d} J_f \mathrm{d}x \text{ as } s \to 0^+.$$

That is, $\lim_{s\to 0^+} {}^{(s)}\mu\big(\overline{E}(f,s)\big) = \int_{\mathbb{R}^d} J_f \mathrm{d}x$. As an immediate consequence, Lemma 8.25 implies (8.51). □

Claim 8.30. *J_f, as defined in the proof of Claim 8.29, is the unique flat-ellipsoid function that is of maximal integral among those that are pointwise less than f.*

Proof. Assume that there is a flat-ellipsoid function J_E, represented by $(A_0 \oplus \alpha_0, a_0)$, such that $\int_{\mathbb{R}^d} J_E \mathrm{d}x \geq \int_{\mathbb{R}^d} J_f \mathrm{d}x$. Consider the ellipsoids $\overline{E}'_s$ represented by $(A_0 \oplus \alpha_0^{1/s}, a_0)$ for all $s \in (0,1]$. Clearly, $\overline{E}'_s \subset {}^{(s)}\overline{f}$ for every $s \in (0,1]$. By (8.14), $^{(s)}\mu\Big(\overline{E}'_s\Big) = \frac{^{(s)}\kappa_{d+1}}{\operatorname{vol}(d)(\mathbf{B}^d)} \int_{\mathbb{R}^d} J_E \mathrm{d}x$. By (8.13) and by the definition of the John s-ellipsoid, we have that

$$\int_{\mathbb{R}^d} J_E \mathrm{d}x = \lim_{s\to 0^+} {}^{(s)}\mu\Big(\overline{E}'_s\Big) \leq \lim_{s\to 0^+} {}^{(s)}\mu\big(\overline{E}(f,s)\big) = \int_{\mathbb{R}^d} J_f \mathrm{d}x.$$

Thus, for every positive integer i there is $s_i > 0$ such that

$$^{(s)}\mu\Big(\overline{E}'_s\Big) \geq {}^{(s)}\mu\big(\overline{E}(f,s)\big) - \frac{1}{i} \geq \int_{\mathbb{R}^d} J_f \mathrm{d}x - \frac{2}{i}$$

for all $s \in (0, s_i]$. Finally, by Lemma 8.25, we have that $\lim_{s_i\to 0^+} A_0 = A$, $\lim_{s_i\to 0^+} \alpha_0 = \alpha$ and $\lim_{s_i\to 0^+} a_0 = a$. That is, J_f and J_E coincide. □

Theorem 8.28 is an immediate consequence of Claims 8.29 and 8.30.

8.7.4 Integral ratio

For any $s \in [0,\infty)$, it is reasonable to define the *s-integral ratio* of f by

$$^{(s)}\mathrm{I.rat}(f) = \left(\frac{\int_{\mathbb{R}^d} f \mathrm{d}x}{\int_{\mathbb{R}^d} {}^{(s)}J_f \mathrm{d}x}\right)^{1/d}.$$

Corollary 1.3 of [AGMJV18] states that there exists $\Theta > 0$ such that

$$^{(0)}\mathrm{I.rat}(f) \leq \Theta\sqrt{d}$$

for any proper log-concave function f.

Using Lemma 8.24 and Theorem 8.28, we obtain the following.

Corollary 8.31. *Let f be a proper log-concave function. Then, for any $s > 0$, there exists Θ_s such that*

$$^{(s)}\mathrm{I.\,rat}(f) \leq \mathrm{B}(s/2+1, d/2)^{-\frac{1}{d}} \cdot {}^{(0)}\mathrm{I.\,rat}(f) \leq \Theta_s \sqrt{d},$$

where $\mathrm{B}(\cdot,\cdot)$ *denotes Euler's Beta function.*

8.8 Large s behavior

We will say that a function f_1 on $\mathbb{R}^d$ is *pointwise less than* a function f_2, if $f_1(x) \leq f_2(x)$ for all $x \in \mathbb{R}^d$.

We will say that a Gaussian density on $\mathbb{R}^d$ defined by $x \mapsto \alpha e^{-\left\langle A^{-1}(x-a), A^{-1}(x-a)\right\rangle}$ is *represented* by $(A \oplus \alpha, a) \in \mathcal{E}$. Clearly, any Gaussian density is represented by a unique element of $\mathcal{E}$. We will denote the Gaussian density represented by $(A \oplus \alpha, a)$ as $G[(A \oplus \alpha, a)]$. If a Gaussian density is represented by $(A \oplus \alpha, a) \in \mathcal{E}$, we will call α its *height*. We have that

$$\int_{\mathbb{R}^d} G[(A \oplus \alpha, a)] \mathrm{d}x = \alpha \pi^{d/2} \det A. \tag{8.52}$$

We will need the following property of Euler's Gamma function (see [AS48, 6.1.46])

$$\lim_{s \to \infty} \frac{\Gamma(s+t_1)}{\Gamma(s+t_2)} s^{t_2 - t_1} = 1. \tag{8.53}$$

8.8.1 Asymptotic bound on the operator

Lemma 8.32. *Let $f : \mathbb{R}^d \to [0, \infty)$ be a proper log-concave function, and $\delta, s_0 > 0$. Then there exist $\rho_1, \rho_2 > 0$ such that for any $s \geq s_0$, if $\overline{E} = (A \oplus \alpha)\mathbf{B}^{d+1} + a$, where $(A \oplus \alpha, a) \in \mathcal{E}$, is a d-symmetric ellipsoid in $\mathbb{R}^{d+1}$ with $\overline{E} \subseteq {}^{(s)}\overline{f}$ and ${}^{(s)}\mu\left(\overline{E}\right) \geq \delta$, then we have*

$$\rho_1 I \prec \frac{A}{\sqrt{s}} \prec \rho_2 I. \tag{8.54}$$

Proof. The existence of ρ_1 and ρ_2 on a finite interval $[s_0, s_1]$ follows from Lemma 8.6 and Lemma 8.24. Thus, it suffices to prove (8.54) for a sufficiently large s.

Let β be the smallest eigenvalue of A and w be the corresponding unit eigenvector.

We have the following chain of inequalities

$$\delta < {}^{(s)}\mu(\overline{E}) = \alpha^s \int\limits_{x\in A(\mathbf{B}^d)} \left(1 - \left\langle A^{-1}x, A^{-1}x\right\rangle\right)^{s/2} \mathrm{d}x \overset{(E)}{=}$$

$$\alpha^s \int\limits_{y\in[-\beta w,\beta w]} \int\limits_{z\in w^\perp\,:\,z+y\in A(\mathbf{B}^d)} \left(1 - \left\langle A^{-1}y, A^{-1}y\right\rangle - \left\langle A^{-1}z, A^{-1}z\right\rangle\right)^{s/2} \mathrm{d}y\mathrm{d}z =$$

$$\alpha^s \int\limits_{y\in[-\beta w,\beta w]} \left(1 - \left\langle A^{-1}y, A^{-1}y\right\rangle\right)^{s/2} \cdot$$

$$\int\limits_{z\in w^\perp\,:\,z+y\in A(\mathbf{B}^d)} \left(1 - \frac{\langle A^{-1}z, A^{-1}z\rangle}{1-\langle A^{-1}y, A^{-1}y\rangle}\right)^{s/2} \mathrm{d}z\mathrm{d}y \overset{(S)}{=}$$

$$\alpha^s \int\limits_{y\in[-\beta w,\beta w]} \left(1 - \left\langle A^{-1}y, A^{-1}y\right\rangle\right)^{s/2+(d-1)/2} \cdot$$

$$\int\limits_{t\in w^\perp\cap A(\mathbf{B}^d)} \left(1 - \left\langle A^{-1}t, A^{-1}t\right\rangle\right)^{s/2} \mathrm{d}t\mathrm{d}y \overset{(C)}{\leq}$$

$$\alpha^s \int\limits_{y\in[-\beta w,\beta w]} \left(1 - \left\langle A^{-1}y, A^{-1}y\right\rangle\right)^{s/2+(d-1)/2} \mathrm{d}y$$

$$\int\limits_{t\in(a+w^\perp)} f\mathrm{d}t \overset{(H)}{\leq}$$

$$C_f\alpha^s \int\limits_{y\in[-\beta,\beta]} \left(1 - \frac{y^2}{\beta^2}\right)^{s/2+(d-1)/2} \mathrm{d}y =$$

$$\sqrt{\pi}C_f\alpha^s \cdot \beta\frac{\Gamma(s/2+(d-1)/2+1)}{\Gamma(s/2+(d-1)/2+3/2)}$$

$$\leq \sqrt{\pi}C_f\left\|f\right\| \cdot \beta\frac{\Gamma(s/2+(d-1)/2+1)}{\Gamma(s/2+(d-1)/2+3/2)},$$

where in (E) we use that w is an eigenvector of A and A^{-1}; in (S) we make the substitution $t = \frac{z}{\sqrt{1-\langle A^{-1}y, A^{-1}y\rangle}}$; in (C) we use the inclusion $\overline{E}(f,s) \subset {}^{(s)}\overline{f}$; and in (H), we bound the integral over an affine hyperplane using (8.17).

Therefore, using this and (8.53), we conclude that $\frac{\beta}{\sqrt{s}}$ is bounded from below by some positive constant in a sufficiently small neighborhood of ∞. The existence of ρ_1 follows.

Since $\int\limits_{[f\geq\beta]} f\mathrm{d}x$ tends to $\int\limits_{\mathbb{R}^d} f\mathrm{d}x$ as β tends to 0, we conclude that $\alpha^s > c(\delta)$ for some positive constant $c(\delta)$. Thus, we get

$$\int\limits_{\mathbb{R}^d} f\mathrm{d}x \geq {}^{(s)}\mu(\overline{E}) = {}^{(s)}\kappa_{d+1}\alpha^s \det A \geq \pi^{d/2}\frac{\Gamma(s/2+1)}{\Gamma(s/2+d/2+1)} \cdot c(\delta)\cdot \beta^{d-1}\left\|A\right\|.$$

By (8.53), the ratio of Gamma functions here tends to $s^{-d/2}$ as s tends to ∞, and, since $\frac{\beta}{\sqrt{s}} > \rho_1$, we see that there exists $\rho_2 > 0$ such that $\rho_2 > \frac{\|A\|}{\sqrt{s}}$. This completes the proof of Lemma 8.32. □

8.8.2 Existence of a maximal Gaussian

Theorem 8.33. *Let f be a proper log-concave function. If there is a Gaussian density pointwise less than f, then there exists a Gaussian density pointwise less than f of maximal integral. All Gaussian densities of maximal integral pointwise less than f are translates of each other.*

Proof of Theorem 8.33. The proof mostly repeats the argument in Section 8.4.

Lemma 8.34 (Compactness for Gaussians)**.** *Let $f : \mathbb{R}^d \to [0,\infty)$ be a proper log-concave function. Then for any $\delta > 0$, the set S_δ of all $(A\oplus\alpha, a) \in \mathcal{E}$ such that $G[(A\oplus\alpha,a)]$ is pointwise less than f and $\int_{\mathbb{R}^d} G[(A\oplus\alpha,a)]\mathrm{d}x > \delta$ is bounded (and possibly empty).*

Proof. Assume that S_δ is not empty. Let $(A\oplus\alpha, a) \in S_\delta$. Since $\int\limits_{[f\geq\beta]} f dx$ tends to $\int_{\mathbb{R}^d} f\mathrm{d}x$ as β tends to 0, we conclude that $\alpha > c(\delta)$ for some positive constant $c(\delta)$. Obviously, $\alpha \leq \|f\|$. Since f is a proper log-concave function, the boundedness of α implies that a is bounded.

Next, we show that A is bounded as well. By (8.52), we have that $\det A$ is bounded. Hence, it is enough to show that the minimal eigenvalue of A is bounded from below. Let λ be the smallest eigenvalue of A and y be the corresponding unit eigenvector. Then, by the properties of Gaussian densities, we have that

$$\int\limits_{\{x\in\mathbb{R}^d\,:\,|\langle x-a,y\rangle|\leq\lambda\}} f\mathrm{d}x \geq$$

$$\int_{\{x\in\mathbb{R}^d\,:\,|\langle x-a,y\rangle|\leq\lambda\}} G[(A\oplus\alpha,a)](x)\mathrm{d}x > \frac{1}{2}\int_{\mathbb{R}^d} G[(A\oplus\alpha,a)]\mathrm{d}x > \delta/2.$$

Using (8.18), we see that the leftmost expression tends to zero as λ tends to zero. Lemma 8.34 follows. □

Lemma 8.34 implies that if there exists a Gaussian density pointwise less than f, then there is a Gaussian density of maximal integral among those that are pointwise less than f. .

Next, we show that this Gaussian density of maximal integral is unique up to translation.

First, we need the following extension of Lemmas 8.7 and 8.8.

Lemma 8.35 (Interpolation between Gaussians). *Fix $\beta_1, \beta_2 > 0$ with $\beta_1 + \beta_2 = 1$. Let f_1 and f_2 be two proper log-concave functions on $\mathbb{R}^d$, and G_1, G_2 be two Gaussian densities represented by $(A_1\oplus\alpha_1, a_1) \in \mathcal{E}$ and $(A_2\oplus\alpha_2, a_2) \in \mathcal{E}$, respectively, such that G_1 is pointwise less than f_1 and G_2 is pointwise less than f_2. Define*

$$f = (\beta_1 * f_1) \star (\beta_2 * f_2).$$

Set

$$(A\oplus\alpha, a) = (\beta_1 A_1 + \beta_2 A_2 \oplus \alpha_1^{\beta_1}\alpha_2^{\beta_2}, \beta_1 a_1 + \beta_2 a_2).$$

Then, $G[(A\oplus\alpha,a)] \leq f$ and the following inequality holds

$$\int_{\mathbb{R}^d} G[(A\oplus\alpha,a)]\mathrm{d}x \geq \left(\int_{\mathbb{R}^d} G_1\mathrm{d}x\right)^{\beta_1}\left(\int_{\mathbb{R}^d} G_2\mathrm{d}x\right)^{\beta_2}, \tag{8.55}$$

with equality if and only if $A_1 = A_2$.

Proof. Fix $x \in \mathbb{R}^d$ and define x_1, x_2 by

$$x_1 - a_1 = A_1A^{-1}(x-a), \quad x_2 - a_2 = A_2A^{-1}(x-a). \tag{8.56}$$

Since $G_1 \leq f_1, G_2 \leq f_2$, we have

$$f_1(x_1) \geq \alpha_1 e^{-\left\langle A_1^{-1}(x_1-a_1), A_1^{-1}(x_1-a_1)\right\rangle} \tag{8.57}$$

and

$$f_2(x_2) \geq \alpha_2 e^{-\left\langle A_2^{-1}(x_2-a_2), A_2^{-1}(x_2-a_2)\right\rangle}.$$

Since $\beta_1 x_1 + \beta_2 x_2 = x$ and by the definition of the Asplund sum, we have that

$$f(x) \geq f_1^{\beta_1}(x_1) f_2^{\beta_2}(x_2).$$

Combining this with inequalities (8.57) and (8.56), we obtain

$$f(x) \geq \alpha_1^{\beta_1}\alpha_2^{\beta_2} e^{-\beta_1\left\langle A_1^{-1}(x_1-a_1), A_1^{-1}(x_1-a_1)\right\rangle} e^{-\beta_2\left\langle A_2^{-1}(x_2-a_2), A_2^{-1}(x_2-a_2)\right\rangle} =$$
$$\alpha_1^{\beta_1}\alpha_2^{\beta_2} e^{-(\beta_1+\beta_2)\left\langle A^{-1}(x-a), A^{-1}(x-a)\right\rangle} = \alpha_1^{\beta_1}\alpha_2^{\beta_2} e^{-\left\langle A^{-1}(x-a), A^{-1}(x-a)\right\rangle}.$$

Thus, G is pointwise less than f.

We proceed with showing (8.55). Substituting (8.52), inequality (8.55) takes the form

$$\pi^{d/2}\alpha_1^{\beta_1}\alpha_2^{\beta_2} \cdot \det\left(\beta_1 A_1 + \beta_2 A_2\right) \geq \pi^{d/2}\alpha_1^{\beta_1}\alpha_2^{\beta_2} \cdot \left(\det A_1\right)^{\beta_1}\left(\det A_2\right)^{\beta_2},$$

or, equivalently,

$$\det\left(\beta_1 A_1 + \beta_2 A_2\right) \geq \left(\det A_1\right)^{\beta_1}\left(\det A_2\right)^{\beta_2}.$$

Thus, inequality (8.55) and the equality condition follow from Minkowski's determinant inequality (8.3) and the equality condition therein, completing the proof of Lemma 8.35. □

Let G_1, represented by $(A_1 \oplus \alpha_1, a_1)$, be a maximal integral Gaussian density that is pointwise less than f. Assume that there is another Gaussian density G_2 represented by $(A_2 \oplus \alpha_2, a_2)$ with the same integral as G_1 and pointwise less than f. Consider the Gaussian density G represented by

$$\left(\frac{A_1 + A_2}{2} \oplus \sqrt{\alpha_1\alpha_2}, \frac{a_1 + a_2}{2}\right) \in \mathcal{E}.$$

By (8.20) and Lemma 8.35, we have that G is pointwise less than f. Next, by the choice of the Gaussian densities, we also have

$$\int_{\mathbb{R}^d} G \mathrm{d}x \leq \int_{\mathbb{R}^d} G_1 \mathrm{d}x = \sqrt{\int_{\mathbb{R}^d} G_1 \mathrm{d}x \int_{\mathbb{R}^d} G_2 \mathrm{d}x} = \int_{\mathbb{R}^d} G_2 \mathrm{d}x,$$

which, combined with Lemma 8.35, yields

$$\int_{\mathbb{R}^d} G \mathrm{d}x = \int_{\mathbb{R}^d} G_1 \mathrm{d}x = \int_{\mathbb{R}^d} G_2 \mathrm{d}x, \text{ and } A_1 = A_2.$$

This, combining with (8.52), implies identity $\alpha_1 = \alpha_2$. This completes the proof of Theorem 8.33. □

8.8.3 Uniqueness does not hold for $s = \infty$

In this subsection, first, we show that it is possible that two Gaussians $G[(A\oplus\alpha, a_1)]$ and $G[(A\oplus\alpha, a_2)]$ with $a_1 \neq a_2$ are of maximal integral. Next, we show that uniqueness holds for a certain important class of log-concave functions.

Example 8.36. Consider

$$f = G[(A \oplus \alpha, a)] \star \chi_K,$$

where $(A \oplus \alpha, a) \in \mathcal{E}$ and K is a compact convex set in $\mathbb{R}^d$. It is not hard to show that the set

$$\{G[(A \oplus \alpha, a_m)] \ : \ a_m \in a + K\}$$

is the set of the maximal integral Gaussian densities that are pointwise less than f.

Uniqueness of the maximal Gaussian density pointwise less than f holds for an important class of log-concave functions.

Example 8.37. Let $K \subset \mathbb{R}^d$ be a compact convex set with non-empty interior, and let $\|\cdot\|_K$ denote the gauge function of K, that is, $\|x\|_K = \inf\{\lambda > 0 \ : \ x \in \lambda K\}$. Let $A\left(\mathbf{B}^d\right)$ be the John ellipsoid of K, where A is a positive definite matrix. Then the Gaussian density represented by $(A \oplus 1, 0)$ is the unique maximal integral Gaussian density pointwise less than the log-concave function $e^{-\|x\|_K^2}$.

8.8.4 Approximation by John s-ellipsoids

Theorem 8.38. *Let f be a proper log-concave function. Then the following hold.*

1. *There is a Gaussian density pointwise less than f if, and only if, $\limsup\limits_{s\to\infty} {}^{(s)}\mu\big(\overline{E}(f,s)\big) > 0$.*
2. *If $\limsup\limits_{s\to\infty} {}^{(s)}\mu\big(\overline{E}(f,s)\big) > 0$, then $\lim\limits_{s\to\infty} {}^{(s)}\mu\big(\overline{E}(f,s)\big) = \limsup\limits_{s\to\infty} {}^{(s)}\mu\big(\overline{E}(f,s)\big)$ and there exists a sequence $\{s_i\}_1^\infty$ with $\lim\limits_{i\to\infty} s_i = \infty$ such that the John s-ellipsoid functions ${}^{(s_i)}J_f$ uniformly converge on any bounded set $S \subset \mathbb{R}^d$ to a Gaussian density which is of maximal integral among those Gaussian densities that are pointwise less than f.*

Proof of Theorem 8.38. We start by showing that every Gaussian density G is the limit of ${}^{(s)}J_G$ as $s \to \infty$. We state a bit more, since we will need to characterize the convergence of a sequence of the John s-ellipsoid functions of a proper log-concave function.

Claim 8.39. *Let $\{s_i\}_1^\infty$ be a sequence of positive scalars such that $\lim\limits_{i\to\infty} s_i = \infty$, let $\{A_i\}_1^\infty$ be a sequence of positive definite operators such that $\lim\limits_{i\to\infty} \frac{A_i}{\|A_i\|} = I$ and the ellipsoids $\overline{E}_i$, represented by $(A_i \oplus 1, 0)$, satisfy $\lim\limits_{i\to\infty} {}^{(s_i)}\mu\big(\overline{E}_i\big) = \pi^{d/2}$. Then the height functions $h_{\overline{E}_i}^{s_i}$ uniformly converge to the standard Gaussian $G[(I \oplus 1, 0)]$ on any bounded set $S \subset \mathbb{R}^d$.*

Proof. Convergence of $\frac{A_i}{\|A_i\|} \to I$ as $i \to \infty$ yields two properties

$$\lim_{i\to\infty} \frac{\det A_i}{\|A_i\|^d} = 1, \tag{8.58}$$

and

$$\lim_{i\to\infty} \|A_i\| A_i^{-1} = I. \tag{8.59}$$

By (8.53), (8.12) and (8.14), we obtain

$$\pi^{d/2} = \lim_{i\to\infty} {}^{(s_i)}\mu\big(\overline{E}_i\big) = \lim_{i\to\infty} {}^{(s_i)}\kappa_{d+1} \det A_i =$$
$$\pi^{d/2} \lim_{i\to\infty} \frac{\Gamma(s_i/2+1)}{\Gamma(d/2+s_i/2+1)} \det A_i =$$
$$\pi^{d/2} \lim_{i\to\infty} \left(\frac{s_i}{2}\right)^{-d/2} \det A_i.$$

Thus, combining this with (8.58), we get

$$\lim_{i\to\infty} \frac{s_i}{2} \frac{1}{\|A_i\|^2} = 1. \tag{8.60}$$

Fix $\rho > 0$. By (8.60) and (8.59), the following identity holds for all $x \in \rho\mathbf{B}^d$ and a sufficiently large i:

$$\big(h_{\overline{E}_i}(x)\big)^{s_i} = \big(1 - \big\langle A_i^{-1}x, A_i^{-1}x\big\rangle\big)^{s_i/2} =$$
$$\left(1 - \frac{\big\langle \|A_i\|\, A_i^{-1}x, \|A_i\|\, A_i^{-1}x\big\rangle}{\|A_i\|^2}\right)^{\|A_i\|^2 \cdot \frac{s_i}{2} \frac{1}{\|A_i\|^2}}.$$

Again, by (8.60) and (8.59), for any $1 > \varepsilon > 0$, there exists i_ε such that the inequality

$$\left(1 - \frac{(1+\varepsilon)|x|^2}{\|A_i\|^2}\right)^{\|A_i\|^2(1+\varepsilon)} \le \big(h_{\overline{E}_i}(x)\big)^{s_i} \le \left(1 - \frac{(1-\varepsilon)|x|^2}{\|A_i\|^2}\right)^{\|A_i\|^2(1-\varepsilon)}$$

holds for all $x \in \rho\mathbf{B}^d$ and for all $i > i_\varepsilon$. This implies that the sequence of functions $\left\{\big(h_{\overline{E}_i}(x)\big)^{s_i}\right\}$ uniformly converge to $e^{-\langle x,x\rangle}$ on $\rho\mathbf{B}^d$. This completes the proof of Claim 8.39. □

Claim 8.40. *If* $\limsup_{s\to\infty} {}^{(s)}\mu\big(\overline{E}(f,s)\big) > 0$, *then there exists a sequence* $\{s_i\}_1^\infty$ *of positive reals with* $\lim_{i\to\infty} s_i = \infty$ *such that the John s-ellipsoid functions* ${}^{(s_i)}J_f$ *uniformly converge on any bounded set* $S \subset \mathbb{R}^d$ *to a Gaussian density that is pointwise less than* f *and of maximal integral.*

Proof. Let $(A_s \oplus \alpha_s, a_s)$ represent $\overline{E}(f,s)$. By Lemma 8.12, we have that $\left\|{}^{(s)}J_f\right\|$ belongs to the interval $[e^{-d}\|f\|, \|f\|]$. Hence, the set $\{a_s\}_{s>0}$ is bounded. Thus, there exists a sequence $\{s_i\}_1^\infty$ with $\lim_{i\to\infty} s_i = \infty$ such that

$$ {}^{(s_i)}\mu\big(\overline{E}(f,s_i)\big) \to \limsup_{s\to\infty} {}^{(s)}\mu\big(\overline{E}(f,s)\big), \ \frac{A_{s_i}}{\|A_{s_i}\|} \to A, $$

$$ \left\|{}^{(s_i)}J_f\right\| \to \alpha > 0 \quad \text{and} \quad a_{s_i} \to a $$

for some positive semidefinite matrix $A \in \mathbb{R}^{d\times d}$, an $\alpha > 0$ and $a \in \mathbb{R}^d$, as i tends to ∞.

Lemma 8.32 implies that A is positive definite. Hence, using (8.8.4) in Claim 8.39, we obtain that ${}^{(s_i)}J_f$ uniformly converge to the Gaussian $G[(A_\infty \oplus \alpha, a)]$ on any bounded set $S \subset \mathbb{R}^d$, where

$$ A_\infty = \frac{1}{\sqrt{\pi}} \left(\frac{\limsup_{s\to\infty} {}^{(s)}\mu\big(\overline{E}(f,s)\big)}{\det A} \right)^{1/d} A. $$

Clearly, $G[(A_\infty \oplus \alpha, a)]$ is pointwise less than f and ${}^{(s_i)}\mu\big(\overline{E}(f,s_i)\big) = \int_{\mathbb{R}^d} G[(A_\infty \oplus \alpha, a)]\mathrm{d}x$. The latter implies that there is no Gaussian density pointwise less than f with the integral strictly bigger than the integral of $G[(A_\infty \oplus \alpha, a)]$, since any Gaussian G is the limit of ${}^{(s)}J_G$ as $s \to \infty$. ☐

To complete the proof of Theorem 8.38, we need to show that the limit $\lim_{s\to\infty} {}^{(s)}\mu\big(\overline{E}(f,s)\big)$ exists. The latter follows from the following line

$$ {}^{(s)}\mu\big(\overline{E}(f,s)\big) \geq {}^{(s)}\mu\Big(\overline{E}\big({}^{(\infty)}J_f, s\big)\Big) \xrightarrow{s\to\infty} \int_{\mathbb{R}^d} {}^{(\infty)}J_f \mathrm{d}x = $$

$$ \limsup_{s\to\infty} {}^{(s)}\mu\big(\overline{E}(f,s)\big), $$

where ${}^{(\infty)}J_f$ is the Gaussian density of maximal integral constructed in Claim 8.40. ☐

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