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Inequalities in Discrete and Convex Geometry

D.Sc. dissertation

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Ajánlás

Ezt a disszertációt páromnak: Rozsnyai Kornéliának, és gyerekeimnek: Máténak, Lucának, Johannának és Blankának ajánlom. Az ő támogatásuk nélkül ez a dolgozat nem jött volna létre.

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Contents

List of symbols

$\langle \cdot, \cdot \rangle$	inner product of $\mathbb{R}^n$
$\|\|\cdot\|\|$	Euclidean norm
$\|\|\cdot\|\|_M$	norm in the normed space with unit ball M
$A + B$	Minkowski sum of A, B
$a(S)$	Borsuk number of S
$a_{frac}(S)$	fractional Borsuk number of S
$a_k(S)$	k-fold Borsuk number of S
$A[k]$	k-element Minkowski sum of A with itself
$\operatorname{aff}(X)$	affine hull of X
$\operatorname{area}(X)$	area of X
$\mathbf{B}^d$	closed unit ball of $\mathbb{R}^d$ centered at o
$\operatorname{bd} X$	boundary of X
$c(K)$	center of mass of K
$c_{tr}(K)$	largest (relative) volume of a translation body of K
$c_{tr}^{Bus}(K)$	largest Busemann volume of a translation body of K
$c_{tr}^{HT}(K)$	largest Holmes-Thompson volume of a translation body of K
$c_{tr}^{m}(K)$	largest Gromov's mass of a translation body of K
$c_{tr}^{m*}(K)$	largest Gromov's mass* of a translation body of K
$\operatorname{card} X$	cardinality of X
$\operatorname{conv}(X)$	convex hull of X
$\operatorname{cr}(A)$	circumradius of A
$\mathcal{D}_k$	symmetry group of a regular k-gon
$\operatorname{diam}(A)$	diameter of A
$d_H(A, B)$	Hausdorff distance of A, B
$\dim(X)$	dimension of X
$\operatorname{dist}(x, A)$	distance of $x \in \mathbb{R}^d$ from $A \subseteq \mathbb{R}^d$
$d_u(K)$	length of a longest chord of K in the direction of u
δ_K	Euclidean distance function of $\operatorname{bd} K$ from o
$F(K)$	flatness of K
$g(C)$	girth of C
$\mathcal{G}_{d,j}$	Grassmannian of j-dimensional linear subspaces of $\mathbb{R}^d$
h_K	support function of K
Id	identity operator on $\mathbb{R}^d$
$\operatorname{int} X$	interior of X

$\mathrm{ir}(A)$	inradius of A
K_n	complete graph on n vertices
κ_d	volume of $\mathbf{B}^d$
$\mathcal{K}^d$	family of convex bodies in $\mathbb{R}^d$
$\mathcal{K}_o^d$	family of o-symmetric convex bodies in $\mathbb{R}^d$
$\mathrm{mwidth}(K)$	mean width of K
$\mathcal{M}$	normed space with unit ball M
$M_p(\omega_n)$	ℓ_p-polarization of ω_n
$M_n^p(\mathbb{S}^{d-1})$	ℓ_p-polarization/Chebyshev constant of $\mathbb{S}^{d-1}$
$\mu(G)$	Mycielskian of graph G
$\mu_p(G)$	p-Mycielskian of graph G
o	origin in $\mathbb{R}^d$
$\mathcal{O}(d)$	the orthogonal group in dimension d
$[p, q]$	closed segment with endpoints p, q
$\mathrm{perim}(A)$	perimeter of A
$\mathbb{R}^d$	a d-dimensional real vector space
ρ_K	radial function of K
$\mathrm{relint}\, X$	relative interior of X
$\mathrm{relbd}\, X$	relative boundary of X
$\rho_1(\mathcal{M})$	edge density of $\mathcal{M}$
$\mathbb{S}^{d-1}$	unit sphere in $\mathbb{R}^d$ centered at o
$\mathrm{surf}(A)$	surface area of A
$\mathrm{Sym}(X)$	symmetry group of X
$(S, U)_c$	family of nondegenerate convex bodies in $\mathbb{R}^3$ with S stable, U unstable points and with positive Gaussians at equilibrium points
$(S, U)_p$	family of nondegenerate convex polyhedra with S stable, U unstable points
$\mathrm{skel}_1(P)$	1-skeleton of P
$\sigma_m^k(x_1, x_2, \ldots, x_m)$	kth elementary symmetric polynomial on $x_1, x_2, \ldots, x_m$
$\mathcal{T}_4$	symmetry group of a regular tetrahedron
$T(K)$	thinness of K
$v_1 \wedge v_2 \wedge \ldots \wedge v_k$	wedge product of $v_1, v_2, \ldots, v_k$
$V_i(K)$	ith intrinsic volume of K
$\mathrm{vol}(\cdot)/\mathrm{vol}_d(\cdot)$	d-dimensional volume
$\mathrm{vol}_M(\cdot)$	volume in the normed space with unit ball M
$\mathrm{vol}_M^{Bus}(\cdot)$	Busemann volume in the normed space with unit ball M
$\mathrm{vol}_M^{HT}(\cdot)$	Holmes-Thompson volume in the normed space with unit ball M
$\mathrm{vol}_M^{m}(\cdot)$	Gromov's mass in the normed space with unit ball M
$\mathrm{vol}_M^{m*}(\cdot)$	Gromov's mass* in the normed space with unit ball M
$W_i(K)$	ith quermassintegral of K
$\mathrm{width}(K)$	(minimal) width of K
$\mathrm{width}_u(K)$	width of K in the direction of u
$x + K$	translate of K by x
$X\|S$	orthogonal projection of X onto the linear subspace S
$X \Delta Y$	symmetric difference of X, Y
$\mathcal{Z}_{d,n}$	family of zonotopes in $\mathbb{R}^d$ generated by at most n segments

$\mathcal{Z}_p$	family of parallelotopes in $\mathbb{R}^d$
$\mathcal{Z}_{rd}$	family of rhombic dodecahedra in $\mathbb{R}^d$

Chapter 1

Summary

This work is included in my application for the *Doctor of the Hungarian Academy of Sciences* title (DSc., in Hungarian: MTA doktora). The topics discussed here belong to the areas of discrete geometry and convex geometry. All problems are related to *optimization problems*; i.e. to finding the extremal values of a geometric quantity in a family of geometric objects. The results presented in the dissertation cover seven of my papers, four of which are written by co-authors. These papers were published in refereed international journals, and are listed here in alphabetical order: Bezdek and Lángi [24], Fradelizi, Lángi and Zvavitch [75], Hujter and Lángi [114], Joós and Lángi [119], Lángi [131], Lángi [132], and Lángi [133].

The paper [24] investigates a conjecture of Goodman and Goodman [91] in 1945, stating that for any non-separable family of positive homothetic copies of a convex body K in Euclidean d-space, with homothety ratios $\tau_1, \tau_2, \ldots, \tau_k$, there is a homothetic copy of K with homothety ratio $\sum_{i=1}^k \tau_i$ that covers every element of the family.

The paper [75] deals with the behavior of the Minkowski average of compact sets. It is well-known that for any compact set A in Euclidean space, the Minkowski average $\frac{1}{k}A[k] = \frac{\overbrace{A + A + \ldots + A}^{k}}{k}$ approaches, and is contained in, the convex hull $\mathrm{conv}(A)$ of A with respect to Hausdorff distance as k tends to infinity. A conjecture of Bobkov, Madiman and Wang [31], made in 2011, states that the volume of the 'discrepancy' $\mathrm{conv}(A) \setminus \frac{1}{k}A[k]$ is a non-increasing sequence. The aim of this paper is to investigate whether this conjecture is true or false for certain classes of compact sets.

The aim of the paper [114] is to present a variant of a problem of Borsuk by defining the k-fold and fractional Borsuk numbers of sets, and examining their properties. In [119], isoperimetric problems are studied for zonotopes, involving their circumradii, inradii and intrinsic volumes. The paper [131] asks a normed variant of a problem of Rogers and Shephard [165], namely it proposes optimization problems on the volume of the convex hull of two intersecting translates of a convex body in a normed space, using different concepts of volume: Busemann volume, Holmes-Thompson volume, Gromov's mass, and Gromov's mass*. In [132], an isoperimetric problem is solved for 3-dimensional parallelohedra, comparing their volumes and mean widths. Finally, the paper [133] answers two questions of Conway and Guy [89], proposed in 1969, about monostable polyhedra; one of these questions was independently asked by Shephard in 1968 [173].

The structure of the dissertation is the following.

In Chapter 2 we give a brief historical background on the problems discussed here (Section 2.1), introduce the general setting and collect some of the tools used in the dissertation (Section 2.2). Chapter 3 contains the results about the Borsuk problem. Chapter 4 presents the results about the conjecture of Goodman and Goodman. The aim of Chapter 5 is to discuss the properties of equilibrium points of convex bodies, and to prove questions of Conway and Guy on monostable polyhedra. In Chapter 6 we investigate the monotonicity of the volume of the Minkowski average of a compact set. Chapter 7 presents a description of 3-dimensional parallelohedra and the solution of an isoperimetric problem regarding them. In Chapter 8 we discuss isoperimetric problems regarding zonotopes. Finally, in Chapter 9 we collect the properties of the volume of two intersecting translates of a convex body in a normed space.

Thesis 1: *We characterize the k-fold Borsuk numbers of all planar sets for all values of k, as well as those of a centrally symmetric or a smooth body in Euclidean d-space for any value of k. We give estimates for the k-fold Borsuk numbers of certain subclasses of finite point sets in Euclidean 3-space. These subclasses are defined by putting an extra condition on the point set regarding the girth of its diameter graph or its size. In addition, we verify some properties of the diameter graphs of finite point sets in Euclidean 3-space with certain symmetry groups.*

The first part of the thesis is supported by Theorems 3.2 and 3.3, and also by Theorem 3.4. The second part is supported by Theorems 3.5 and 3.6. The last part is supported by Theorems 3.7, 3.8, and 3.9.

Thesis 2: *We disprove the conjecture of Goodman and Goodman [91] in the general case, and prove it for the special case that the convex body K is centrally symmetric, and also for the so-called* k-impassable families *of positive homothets of a d-dimensional convex body for any $0 \leq k \leq d-2$. We give upper bounds for the smallest positive number $\lambda(K)$ with the property that any non-separable family of positive homothetic copies of the convex body K, with ratios $\tau_1, \tau_2, \ldots, \tau_k$ can be covered by a homothetic copy of K of ratio $\lambda(K) \sum_{i=1}^k \tau_i$.*

This thesis is supported by Example 4.1 and Remark 4.1 for the planar case $d = 2$. We note that by Remark 4.2, the above counterexamples can be extended to any dimension d. The theorems proving the conjecture for centrally symmetric bodies and k-impassable families are Theorems 4.4 and 4.5, respectively. We remark that for k-impassable families of positive homothets of a d-dimensional strictly convex body K, with $0 \leq k \leq d-2$, a stronger form of the conjecture is proved in Theorem 4.6. The last part of the thesis is supported by Theorems 4.2 and 4.3.

Thesis 3: *For any integer $k \geq 3$, we construct a monostable polyhedron that has k-fold rotational symmetry. Furthermore, we show that the infimum of the set of the diameter to girth ratios of monostable polyhedra is $\frac{1}{\pi}$.*

We note that even though the two parts of the thesis answer two separate questions of Conway and Guy in [89], in the proof of Theorem 5.1 we provide a construction answering these questions *simultaneously*. More specifically, we show in this theorem that for any fixed integer $k \geq 3$, a Euclidean ball can be approximated arbitrarily well by a monostable polyhedron P with k-fold rotational symmetry. This statement clearly implies the first half of the thesis, and, combining it with Corollary 5.1, also its second part.

Thesis 4: *We verify that for any $d \geq 2$ and star-shaped set S in the d-dimensional Euclidean space $\mathbb{R}^d$, the sequence of the volumes of the sets $\frac{1}{k}S[k]$, where $k \geq \max\{2, (d-1)(d-2)\}$, is non-decreasing. In addition, we prove a a similar statement for a set S with $k \geq 2$, where S is a connected planar set satisfying some additional properties.*

The thesis is supported by Theorems 6.1 and 6.2.

Thesis 5: *We prove that among unit volume 3-dimensional parallelohedra, regular truncated octahedra have minimal mean width.*

The thesis is supported by Theorem 7.1.

Thesis 6: *We generalize a decomposition theorem of Shephard [174], and prove various isoperimetric inequalities for rhombic dodecahedra. Furthemore, for any $0 < k < l \leq d$, we prove asymptotic upper bounds for the maximal lth intrinsic volume of a d-dimensional zonotope Z generated by n segments with a given kth intrinsic volume, as $n \to \infty$. We also prove similar estimates for the cases when one or both intrinsic volumes are replaced by the inradius or circumradius of Z.*

The first part of the thesis is supported by Theorem 8.1. Theorems 8.3, 8.4, 8.5, 8.6 and 8.7 present various isoperimetric inequalities for rhombic dodecahedra. The statement about zonotopes generated by n segments, where $n \to \infty$, is supported by Theorem 8.9, 8.10, 8.11 and 8.12.

Thesis 7: *For any Lebesgue measurable set S in the normed space $\mathcal{M}$, let vol_M^{Bus}, vol_M^{HT}, vol_M^{m}, and vol_M^{m*} denote the Busemann, Holmes-Thompson volume, Gromov's mass and Gromov's mass* of S. Furthermore, for any convex body K and $\tau \in \{Bus, HT, m, m*\}$, let $c_{tr}^{\tau}(K)$ denote the maximal value of $\mathrm{vol}_M^{\tau}(S)$, where $\mathcal{M}$ is generated by the relative norm of K, and S is the convex hull of a set $K \cup (x + K)$ with $K \cap (x + K) \neq \emptyset$. With this notation, for any $\tau \in \{Bus, HT, m, m*\}$, we find the extremal values of $c_{tr}^{\tau}(K)$ over the family of plane convex bodies, and also over the family of centrally symmetric plane convex bodies.*

The thesis is supported by Theorems 9.2 and 9.3. We note that by Remark 9.2, we also determine the maxima of $c_{tr}^{Bus}(K)$ over the families of d-dimensional convex bodies and also of centrally symmetric d-dimensional convex bodies, as well as the minimum of this quantity over the family of centrally symmetric d-dimensional convex bodies, for any $d \geq 2$.

Chapter 2

Introduction

2.1 Historical background

Geometric inequalities have played an important part in the history of mathematics. One of the first documented such inequality, the Isoperimetric Inequality, states that among regions in the plane with a given circumference, circles have maximal area. This statement is due to Zenodorus, who gave a proof of it around 150 BC [9], which was rigorous according to the mathematical standards of that time. Despite its ancient origin, the isoperimetric inequality has a far-reaching influence even today, with many variants and applications frequently appearing in the literature with a few of which included also in this dissertation.

The goal of my thesis is to present the solution of some such extremum problems in geometry, and investigate questions related to them. The problems examined here belong to the areas of discrete and convex geometry. These partly overlapping areas, despite their roots in the mathematics of ancient Greeks, are relatively young branches of mathematics, appearing in the late 19th and early 20th centuries, respectively. They cover topics related to properties of finite or discrete families of geometric objects in the first case, and of convex sets in the second case. Besides solving some centuries old problems (see e.g. [103]), they have also produced many new ones, some of which led to unexpected results.

One of these problems, Borsuk's problem, is discussed in Chapter 3. This problem dates back to a conjecture of the Polish mathematician Karol Borsuk, who in [37] made the conjecture that every set of diameter $D > 0$ in d-dimensional Euclidean space is the union of $d+1$ sets of diameters strictly less than D. From the 1930s, this conjecture has attracted a wide interest among geometers. The frequent attempts to prove it led to results in a number of special cases: for sets in $\mathbb{R}^2$ and $\mathbb{R}^3$ [37, 66, 96, 108], for smooth bodies or sets with certain symmetries [161, 162, 100, 101], etc., but the conjecture in general remained open till 1993, when it was disproved by Kahn and Kalai [121]. Their result did not mean that research on this problem stopped: the investigation of the so-called *Borsuk number* of a bounded set; that is, the minimum number of pieces of smaller diameters that it can be partitioned into, is still one of the fundamental problems of discrete geometry.

Since 1933, a large number of generalizations of Borsuk's problem has been introduced. Without completeness, we list only a few. The *generalized Borsuk problem* asks to find, for a fixed value of $0 < r < 1$, the minimum number m such that any set of diameter one in $\mathbb{R}^n$ can be partitioned into m pieces of diameters at most r. The *cylindrical Borsuk problem* makes restrictions on the method of partition [110]. Clearly, the original problem is meaningful for sets in any metric space, for instance, for finite dimensional normed spaces [189] or for binary codes equipped with Hamming distance. The latter one is called the $(0, 1)$-*Borsuk problem*, and is investigated, for example, in [158]. For more information on this problem and its generalizations, the reader is referred to the surveys [159, 122]. The problem of finding the k*-fold Borsuk numbers* of sets in Euclidean space was proposed in [114], and was generalized for normed spaces in [134]. This variant, for Euclidean space, is the object of Chapter 3.

It is well known that any two nonoverlapping convex bodies in Euclidean space can be separated by a hyperplane. Independently of each other, the property of separability, and non-separability, was formulated also for families of convex sets. In particular, a family of nonoverlapping convex bodies is called *totally separable*, if any two members of this family can be separated by a hyperplane that does not intersect the interior of any member of the family. This property was first defined by G. Fejes Tóth and L. Fejes Tóth [69] in 1973, and later studied e.g. in [125, 28, 26]. For variants of this concept, see also [16, 25].

As a 'counterpoint' to this definition, we say that a finite family of convex bodies in Euclidean space is *non-separable*, if any hyperplane that intersects the convex hull of the bodies intersects at least one of the bodies. This definition is due to Erdős, who conjectured that any non-separable family of circles can be covered by a circle whose radius is the sum of the radii of the elements of the family. The conjecture of Erdős was verified by Goodman and Goodman [91] in 1945, who made a more general conjecture, stating that any non-separable family of positive homothets of a convex body K, with homothety ratios $\tau_1, \tau_2, \ldots, \tau_n$, can be covered by a homothet of K of ratio $\sum_{i=1}^{n} \tau_i$. The conjecture of Goodman and Goodman was finally disproved by Bezdek and Lángi [24] in 2016, who, in addition, proved that it holds under some special conditions on K or the family. Recently, a spherical version of the result of Goodman and Goodman was proved by Polyanskii [157] in 2021, formulated as a strengthening of the famous zone conjecture of L. Fejes Tóth, proved by Jiang and Polyanskii [118]. Some inequalities from [24] regarding the Goodman-Goodman Conjecture were improved in [2]. Chapter 4 presents results related to this conjecture.

The study of static equilibrium points of convex bodies started with the work of Archimedes [106], and has been continued throughout the history of science in various disciplines: from geophysics and geology [63, 180] leading to examination of the possible existence of water on Mars [181], to robotics and manufacturing [185, 32] to biology and medicine [1, 57, 64].

In modern times, the mathematical aspects of this concept was started by a problem of Conway and Guy [45] in 1966 who conjectured that there is no homogeneous tetrahedron which can stand in rest only on one of its faces when placed on a horizontal plane, but there is a homogeneous convex polyhedron with the same property. These two questions were answered by Goldberg and Guy in [89] in 1969, respectively (for a more detailed proof of the first problem, see [52]), who called the convex polyhedra satisfying this property *monostable* or *unistable*. In addition, in [89] Guy presented some problems regarding monostable polyhedra, stating that three of them are due to Conway (for similar statements in the literature, see e.g. [47, 53, 65]). These three questions appear also in the problem collection of Croft, Falconer and Guy [47] as Problem B12. It is worth noting that, according to Guy [89], Conway showed that no body of revolution can be monostable, and also that the polyhedron constructed in [89] has a 2-fold rotational symmetry. The problems appear also in the problem collection of of Klamkin [126], and one of them (stated as Problem 5.1 in Chapter 5) and some other problems for monostable polyhedra appear in a 1968 collection of geometry problems of Shephard [173], who described these objects as 'a remarkable class of convex polyhedra' whose properties 'it would probably be very rewarding and interesting to make a study of'.

Some other problems proposed in [89] include the one of finding the minimal dimension d in which a d-simplex can be monostable. This problem has been investigated by Dawson et al. [52, 55, 54, 53], who proved that there is no monostable d-simplex if $d \leq 8$ and there is a monostable 11-simplex. With regard to Problem XVI in [173], asking about the minimum number of faces of a monostable polyhedron in $\mathbb{R}^3$, the original construction of Guy [89] with 19 faces (attributed also to Conway) was modified by Bezdek [17] to obtain a monostable polyhedron with 18 faces, while a computer-aided search by Reshetov [160] yields a monostable polyhedron with 14 faces. In [109], Heppes constructed a homogeneous tetrahedron in $\mathbb{R}^3$ with the property that putting it on a horizontal plane with a suitable face, it rolls twice before finding a stable position. Another interesting convex body is found by Dumitrescu and Tóth [65], who constructed a convex polyhedron P with the property that after placing it on a horizontal plane with a suitable face, it covers an arbitrarily large distance while rolling until it finds a stable position. Finally, we remark that a systematic study of the equilibrium properties of convex polyhedra was started in [59].

The main result discussed in Chapter 5 is the solution of two of the problems proposed by Conway and Guy in [89]. This result can be found in [133]. The main tool of the proof is a general theorem on approximation of smooth convex bodies by convex polytopes. It is worth noting that this tool led to a strengthening of the result described here, by characterizing the symmetry groups of all mono-monostatic (i.e. Gömböc-class [186]) convex bodies by Domokos, Lángi and Várkonyi [62].

For completeness, we also recall the remarkable result of Zamfirescu [193] stating that a typical convex body (in Baire category sense) has infinitely many equilibrium points, and note that critical points of the distance function from another point are examined in Riemannian manifolds, e.g. in [10, 81, 115].

The Minkowski sum of two sets K, L in Euclidean d-space is defined as $K + L = \{x + y : x \in K, y \in L\}$, where, for brevity, we set $A[k] = \sum_{i=1}^{k} A$, for any $k \in \mathbb{N}$ and any compact set A in this space. Since Minkowski sum preserves the convexity of the summands and the set $\frac{1}{k}A[k]$ consists of some particular convex combinations of elements of A, the containment $\frac{1}{k}A[k] \subseteq \operatorname{conv} A$, and, for the special case of convex sets, the equality $\frac{1}{k}A[k] = \operatorname{conv}(A)$ trivially holds; here $\operatorname{conv}(A)$ denotes the convex hull of A. We note that this observation also implies $\operatorname{vol}\left(\frac{1}{k}A[k]\right) \leq \operatorname{vol}(\operatorname{conv}(A))$, where $\operatorname{vol}(\cdot)$ denotes Lebesgue measure (volume). These observations suggest that for any compact set A, the set $\frac{1}{k}A[k]$ looks 'more convex' for larger values of k. This intuition was formalized by Starr [176, 177], crediting also Shapley and Folkman, and independently by Emerson and Greenleaf [67], by proving that the set $\frac{1}{k}A[k]$ approaches $\operatorname{conv}(A)$ in Hausdorff distance as k approaches infinity and by giving bounds on the speed of this convergence (see [77] for more discussion of this fact).

A further step in the investigation of the sequence $\left\{\frac{1}{k}A[k]\right\}$ is to examine the monotonicity of this convergence. Whereas this sequence is clearly not monotonous in terms of containment, a conjecture of Bobkov, Madiman, Wang [31] in 2011 stated that for any compact set A in Euclidean d-space, the sequence $\left\{\operatorname{vol}\left(\operatorname{conv}(A) \setminus \frac{1}{k}A[k]\right)\right\}_{k\geq 1}$ is non-increasing, or equivalently, the sequence $\left\{\operatorname{vol}\left(\frac{1}{k}A[k]\right)\right\}_{k\geq 1}$ is non-decreasing. This conjecture was partially resolved in [76, 77] by Fradelizi, Madiman, Marsiglietti and Zvavitch, who, following the approach of [99], proved it for any 1-dimensional compact set A, but constructed counterexamples in $\mathbb{R}^d$ for any $d \geq 12$, where it is worth noting that the constructed counterexample is a star-shaped set. In addition, the authors of the above two papers showed that with $k \geq 2$ fixed, for any sufficiently large d there is a compact, star-shaped set A in Euclidean d-space for which the inequality $\operatorname{vol}\left(\frac{1}{k}A[k]\right) > \operatorname{vol}\left(\frac{1}{k+1}A[k+1]\right)$ holds. Fradelizi, Lángi and Zvavitch [75] proved a fairly unexpected complement of this statement by showing that for any d fixed and for any compact, star-shaped set in Euclidean d-space, the sequence $\left\{\operatorname{vol}\left(\frac{1}{k}A[k]\right)\right\}_{k\geq 1}$ is non-decreasing for sufficiently large values of k. This result, and a similar result in [75] for compact, connected planar sets are discussed in Chapter 6.

Our next topic involves 3-dimensional parallelohedra, which are among the convex polyhedra in Euclidean 3-space $\mathbb{R}^3$ most known both within and outside mathematics. We recall that a *d-dimensional parallelohedron* is a convex polytope whose translates tile the Euclidean d-space. All parallelohedra in 3-space can be defined as a Minkowski sum of at most six segments with prescribed linear dependencies between the generating segments, and thus, they are a subclass of *zonotopes*. Parallelohedra are also related to the Voronoi cells of lattices via Voronoi's conjecture, stating (and proved in $\mathbb{R}^d$ up to $d \leq 5$, see [49, 72, 83]) that any parallelohedron is an affine image of such a cell. Well-known examples of parallelohedra are the cube, the regular rhombic dodecahedron, and the regular truncated octahedron, which are the Voronoi cells generated by the integer, the face-centered cubic and the body-centered cubic lattice, respectively. Despite their obvious importance, and the existence of a large number of isoperimetric type inequalities for zonotopes, apart from the celebrated proof of Kepler's Conjecture by Hales [103], which, as a byproduct, implies that among parallelohedra with a given inradius, the one with minimum volume is the regular rhombic dodecahedron, there has been only one known isoperimetric type result for 3-dimensional parallelohedra in the author's knowledge. Namely, a recent paper of Lángi [132] introduced a new representation of this class of convex polyhedra, which he used to find the unit volume 3-dimensional parallelohedra with minimal mean width. This result is discussed in Chapter 7.

The topic of Chapter 8 is isoperimetric inequalities for *zonotopes*. These objects have been in the focus of research since the middle of the 20th century, earning a separate chapter in the famous book [46] of Coxeter in 1943 and a place in the problem collection [172] of Shephard. They are connected to various branches of mathematics, as an example, we may mention the fact that their face lattices correspond to the combinatorial classes of central hyperplane arrangements in $\mathbb{R}^{d+1}$ [97, 191]. They play a central role in the theory of projections of convex polytopes, both being the projections of affine cubes (see e.g. [147]), and, by Cauchy's projection formula [84], being the projection bodies of convex polytopes in $\mathbb{R}^d$ (see also [132]). They are closely related to parallelohedra [68, 146], and appear in lattice geometry (see e.g. [137, 112, 11] or the survey [42]). Zonotopes are also investigated and often applied outside pure mathematics [3, 12, 13, 98].

There is a rich literature about isoperimetric type problems for zonotopes. From amongst these papers, we mention [19] of Bezdek comparing the inradius of a rhombic dodecahedron to its intrinsic volumes, [73] of Filliman comparing the volume of a zonotope to the total squared lengths of its generating vectors, and [50] of Deza, Pournin and Sukegawa giving a sharp asymptotic estimate on the maximum diameter of a primitive zonotope. Recently, Fradelizi et al. [78] investigated various volume inequalities for the Minkowski sums of zonoids. For open problems regarding zonotopes, the interested reader is referred also to [116].

An elegant, simple formula exists for the volume of a zonotope in terms of its generating segments; this formula is usually attributed to McMullen [145] or Shephard [174] and follows e.g. from a decomposition theorem for zonotopes in [174]. In [92], this formula was extended to zonotopes whose dimension is strictly less than that of the ambient space. Using an integral geometric approach, in a recent paper Brazitikos and McIntyre [43] found a generalization of this formula for all intrinsic volumes of a zonotope. Joós and Lángi [119] generalized Shephard's decomposition theorem reproving the formulas for the intrinsic volumes of a zonotope in [43], which they use to prove various isoperimetric inequalities for two types of zonotopes in Euclidean d-space: zonotopes generated by $d+1$ segments, and those generated by $n \gg d$ segments. These results can be applied to other problems of mathematics. As examples, we mention the so-called *ℓ_p-polarization problem* discussed e.g. in [8] (see also [153]), and a generalization of the well-known Maclaurin inequality [29] by Brazitikos and McIntyre [43], comparing the values of elementary symmetric functions evaluated at positive real numbers.

The volume of the convex hull of convex bodies in the Euclidean d-space $\mathbb{R}^d$ has been in the focus of research since the 1950s. Some of the first results in this area appeared in three papers of Rogers and Shephard [163, 164, 165], who, besides other cases, investigated this volume for two intersecting translates of the same convex body. They, for a d-dimensional convex body K, defined the *translation body* of K as the convex hull of $K \cup (x+K)$ for some $x \in \mathbb{R}^d$ satisfying $K \cap (x+K) \neq \emptyset$, and determined the extremal values of the quantity

$$c_{tr}(K) = \frac{1}{\mathrm{vol}(K)} \max\{\mathrm{vol}(\mathrm{conv}(K \cup (x+K)) : (x+K) \cap K \neq \emptyset, x \in \mathbb{R}^d\} \tag{2.1}$$

over the family of d-dimensional convex bodies. Their conjecture about the convex bodies minimizing $c_{tr}(K)$ remained open for almost fifty years. A proof of this conjecture, using measures in normed spaces, and another one based on more conventional tools, can be found in [142] and [85], respectively.

One way to find an analogue of this question in normed spaces is to measure the volume of a maximal volume translation body in a norm 'induced' by the body. This was done by Lángi [131] in 2016, who raised the problem of finding the extremal values of the Busemann, Holmes-Thompson volume, the Gromov's mass and Gromov's mass* of a maximal volume translation body, measured in the relative norm of the body. This problem is discussed in the last chapter, Chapter 9.

2.2 Preliminary concepts

2.2.1 Euclidean space

The results presented in the dissertation, apart from those in Chapter 9, deal with problems in the d-dimensional Euclidean space $\mathbb{R}^d$. We regard this space as a d-dimensional real vector space, equipped with the inner product $\langle \cdot, \cdot \rangle : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$. We denote the zero vector of this space by o, and call it the *origin*. For any point $x \in \mathbb{R}^d$, the *Euclidean norm* of x is the quantity $||x|| = \sqrt{\langle x, x \rangle}$. Unless specifically stated otherwise, by the *norm* of $x \in \mathbb{R}^d$ we always mean its Euclidean norm. The *Euclidean distance* or shortly *distance* of the points $p, q \in \mathbb{R}^d$ is $||q-p||$. Euclidean distance induces a topology on $\mathbb{R}^d$; for any set $X \subset \mathbb{R}^d$, we denote the *interior*, *boundary*, *relative interior*, *relative boundary* and *closure* of X by $\mathrm{int}\, X, \mathrm{bd}\, X, \mathrm{relint}\, X, \mathrm{relbd}\, X$ and $\mathrm{cl}\, X$, respectively, with respect to this topology. Furthermore, for any set $X \subset \mathbb{R}^d$, we denote by $\mathrm{card}\, X$ the *cardinality* of X.

Convex sets 'at large'

In this work, we mostly deal with *convex sets* in $\mathbb{R}^d$. To define them, for any points $p, q \in \mathbb{R}^d$, the *closed segment with endpoints* p, q is defined as the set $[p, q] = \{\lambda p + (1-\lambda) q : 0 \leq \lambda \leq 1]$. With this definition, a set $A \subseteq \mathbb{R}^d$ is called *convex*, if the closed segment connecting any two points of the set is contained in the set. The *convex hull of a set* $X \subseteq \mathbb{R}^d$ is the intersection of all convex sets containing X, this set, which we denote by $\mathrm{conv}(X)$, is also convex. An important description of the convex hull of a set is done by Carathéodory's theorem, which states that any point of the convex hull of a nonempty set $X \subseteq \mathbb{R}^d$ lies in the convex hull of at most $d+1$ points of X.

A convex set is called a *convex body* if it is compact, and it has nonempty interior; if $d = 2$, we may call such sets *convex disks* or simply *disks*. An important convex body is the closed unit ball centered at o, defined as the set of points with norm at most one. We denote this body by $\mathbf{B}^d$, and its boundary, consisting of all unit vectors in $\mathbb{R}^d$, as $\mathbb{S}^{d-1}$. We denote the family of convex bodies in $\mathbb{R}^d$ by $\mathcal{K}^d$.

One of the most elementary operations on the subsets of $\mathbb{R}^d$ is their *Minkowski sum*, which is defined for any nonempty sets $A, B \subseteq \mathbb{R}^d$ as

$$A + B = \{a + b : a \in A, b \in B\}. \tag{2.2}$$

Here, if one of the sets, say A, is a singleton $\{a\}$, we may simplify the notation $\{a\} + B$ to $a + B$, and call this set the *translate* of B by a. Similarly, we define the *dilation* of the set $A \subset \mathbb{R}^n$ by the factor $\lambda \in \mathbb{R}$ as

$$\lambda A = \{\lambda a : a \in A\}.$$

A set of the form $x+\lambda A$ with $x \in \mathbb{R}^d$, $\lambda \in \mathbb{R}$ and $A \subseteq \mathbb{R}^d$, is called a *homothetic copy* or *homothet* of A, of homothety ratio λ. Depending on the sign of λ, we may talk about *positive* or *negative homothets.* More generally, the composition of a nondegenerate linear transformation and a translation is called an *affine transformation*, and the image of a set $A \subset \mathbb{R}^d$ under an affine transformation is called an *affine image* of A. If $\sigma : \mathbb{R}^d \to \mathbb{R}^d$ is an isometry, $\sigma(A)$ is a *congruent copy* of $A \subset \mathbb{R}^d$. The family of isometries σ satisfying $\sigma(A) = A$ forms a group with respect to composition, which we call the *symmetry group* of A and denote by $\mathrm{Sym}(A)$. In particular, if $A \subseteq \mathbb{R}^n$ satisfies $2p - A = A$ for some $p \in \mathbb{R}^d$, we say that A is *p-symmetric*, or *centrally symmetric*. In the dissertation, we denote the family of o-symmetric convex bodies in $\mathbb{R}^d$ by $\mathcal{K}_o^d$.

The 'dual' operation of Minkowski addition is the Minkowski subtraction. More specifically, the *Minkowski difference* of the nonempty sets $A, B \subset \mathbb{R}^d$ is defined as the set

$$A \div B = \bigcap_{b \in B} (A - b)\,. \tag{2.3}$$

We note that that the equality $(A \div B) + B = A$ is *not* satisfied in general. It is satisfied if and only if B is an intersection of translates of B [169].

We can use Minkowski sum to define a metric on $\mathcal{K}^d$. More specifically, for any two compact sets $A, B \subset \mathbb{R}^d$, we define the *Hausdorff distance* of A, B as

$$d_H(A, B) = \inf\{\lambda \geq 0 : B \subseteq A + \lambda \mathbf{B}^d \text{ and } A \subseteq B + \lambda \mathbf{B}^d\}. \tag{2.4}$$

This function defines a metric on $\mathcal{K}^d$. Furthermore, it makes the family of compact, convex sets in $\mathbb{R}^d$ a *complete metric space.*

Translates of linear subspaces of dimension k are called *k-dimensional affine subspaces* or *k-flats.* In particular, 1-, 2- and $(d-1)$-flats are called *lines*, *planes* and *hyperplanes*, respectively. The intersection of affine subspaces containing a nonempty set $X \subseteq \mathbb{R}^d$, which is also an affine subspace, is the *affine hull* of X, denoted by $\mathrm{aff}(X)$. By the *dimension* of a nonempty set $X \subset \mathbb{R}^d$ we mean the dimension of $\mathrm{aff}(X)$, and denote it by $\dim(X)$.

A hyperplane H decomposes the space into two open, convex components, called *open half spaces* bounded by H. The union of H with one of these hyperplanes is called a *closed half space.* For any closed, convex set K, a closed half space containing K whose boundary intersects K is called a *supporting half space* of K, and its boundary a *supporting hyperplane* of K. Every closed, convex set coincides with the intersection of its supporting half spaces, which is equivalent to the fact that every $p \in \mathrm{bd}\, K$ belongs to a supporting hyperplane of K. In particular, these properties are satisfied for every convex body. A *face* of a closed, convex set is its intersection with one of its supporting hyperplanes. The $(d-1)$-dimensional faces of a d-dimensional convex polytope are also called *facets*.

The convex hull P of a finite set $X \subset \mathbb{R}^d$ is called a *convex polytope*; if $\dim(P) = 2$ or $\dim(P) = 3$, we may call P a *convex polygon* or *convex polyhedron*, respectively. Is is well known that the faces of a convex polytope P, together with $\emptyset$ and P, form an algebraic lattice, called the *face lattice* of P. Two convex polytopes are *combinatorially equivalent* if their face lattices are isomorphic.

Let H be a hyperplane. If the nonempty sets $X, Y \subset \mathbb{R}^d$ are contained in different closed half spaces bounded by H, we say that H *separates* X and Y. If, in addition, $X \cap H = Y \cap H = \emptyset$ is also satisfied, we say that H *strictly separates* X and Y. From amongst the numerous theorems describing the separability and strict separability of convex sets here we just state the simplest one: If $K, L \subset \mathbb{R}^d$ are disjoint, convex sets, then there is a hyperplane in $\mathbb{R}^d$ separating them.

Depending on its boundary structure, we distinguish two types of convex bodies. In particular, if the boundary of a convex body K does not contain a nondegenerate segment, we say that K is *strictly convex*. If for every $p \in \operatorname{bd} K$ there is a unique supporting hyperplane of K containing p, we say that K is *smooth*. More generally, any $p \in \operatorname{bd} K$ with a unique supporting hyperplane of K containing it is called a *smooth boundary point* of K.

For any compact set $A \subseteq \mathbb{R}^d$, the quantity $\sup\{||q-p||\colon p, q \in A\}$ is called the *diameter* of A, denoted by $\operatorname{diam}(A)$. The *inradius* $\operatorname{ir}(A)$ of A is the largest value of $r \geq 0$ such that $x + r\mathbf{B}^d \subseteq A$ for some $x \in \mathbb{R}^d$. Similarly, the *circumradius* $\operatorname{cr}(A)$ of A is the smallest value $R > 0$ such that $A \subseteq x + R\mathbf{B}^d$ for some $x \in \mathbb{R}^d$. In the dissertation, we denote the d-dimensional Lebesgue measure of A by $\operatorname{vol}(A)$ or (if it is necessary to emphasize the dimension) $\operatorname{vol}_d(A)$, and call it the d-dimensional *volume* of A. The $(d-1)$-dimensional Hausdorff measure of $\operatorname{bd} A$, if it exists, is called the *surface area* of A, denoted by $\operatorname{surf}(A)$. If $d = 2$, we call the volume and the surface area of A as the *area* and the *perimeter* of A, denoted by $\operatorname{area}(A)$ and $\operatorname{perim}(A)$, respectively. In particular, we set $\kappa_d = \operatorname{vol}(\mathbf{B}^d)$ for any $d \geq 2$. An elementary computation shows that $\kappa_d = \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)}$, where Γ denotes Euler's gamma function.

One of the most important inequalities related to volume is the Brunn-Minkowski Inequality, stating that if $K, L \subset \mathbb{R}^d$ are nonempty compact sets, then

$$(\operatorname{vol}(K+L))^{1/d} \geq (\operatorname{vol}(K))^{1/d} + (\operatorname{vol}(L))^{1/d}. \tag{2.5}$$

Here, if K, L are convex bodies in $\mathbb{R}^d$, then equality is attained in (2.5) if and only if K and L are positive homothetic copies of each other.

The volume of a convex body is also related to the notion of center of mass. In particular, if $K \in \mathcal{K}^d$ is a convex body, then the point

$$c(K) = \frac{1}{\operatorname{vol}(K)} \int_{x \in K} x \, dx, \tag{2.6}$$

where the integration is with respect to d-dimensional Lebesgue measure, is called the *center of mass* of K.

An important operation on nonempty sets $X \subseteq \mathbb{R}^d$ is the polarity operation. The *polar* of X is defined as

$$X^\circ = \left\{ y \in \mathbb{R}^d : \langle x, y \rangle \leq 1 \text{ for every } x \in X \right\}. \tag{2.7}$$

It is easy to see that the polar of every nonempty set is a closed, convex set containing o, and that exactly these sets satisfy the property $(X^\circ)^\circ = X$. If $K \in \mathcal{K}_o^d$, the quantity

$$\operatorname{vol}(K)\operatorname{vol}(K^\circ)$$

is called the *volume product* or *Mahler volume* of K. The Blaschke-Santaló Inequality states that this quantity attains its maximum on $\mathcal{K}_o^d$ at ellipsoids. We do not know the minimum of this quantity on $\mathcal{K}_o^d$. According to a famous unsolved conjecture of convex geometry, the so-called *Mahler Conjecture*, this quantity is minimal e.g. for cubes centered at o.

In the dissertation, we use two functions to characterize a convex body K with $o \in \operatorname{int} K$. The first one is called the *radial function* $\rho_K : \mathbb{S}^{d-1} \to \mathbb{R}$ of K, defined as

$$\rho_K(u) = \sup\{\lambda : \lambda u \in K\}. \tag{2.8}$$

The second one is called the *support function* $h_K : \mathbb{S}^{d-1} \to \mathbb{R}$ of K, defined as

$$h_K(u) = \sup\{\langle x, u\rangle : x \in K\}. \tag{2.9}$$

These two functions are connected by polarity via the identities $h_{K^\circ} = \frac{1}{\rho_K}$ and $\rho_{K^\circ} = \frac{1}{h_K}$ for every $K \in \mathcal{K}^d$ with $o \in \operatorname{int} K$ (see e.g. [87]). We note that the C^1-differentiability of the radial function is equivalent to the property that K is a smooth body, but the same is not true for the support function [169].

We note that the definition of the support function h_K in (2.9) can be naturally extended to any compact set K in $\mathbb{R}^d$ in the same way. Using this extension, for any $u \in \mathbb{S}^{d-1}$, we can define the *width of K in the direction of u* as

$$\operatorname{width}_u(K) = h_K(u) + h_K(-u). \tag{2.10}$$

Geometrically, this quantity is the distance between the two supporting hyperplanes of K with normal vector u. The support function is also related to the Hausdorff distance of the convex bodies K, L, as $d_H(K, L) = \sup\{|h_K(u) - h_L(u)|: u \in \mathbb{S}^{d-1}\}$, and satisfies the identity $h_{A+B} = h_A + h_B$ for any two compact, convex sets $A, B \subset \mathbb{R}^d$. It is easy to see that the maximum of $\operatorname{width}_u(K)$ over $u \in \mathbb{S}^{d-1}$ is the diameter of K.

The quantity

$$\operatorname{width}(K) = \min\{\operatorname{width}_u(K) : u \in \mathbb{S}^{d-1}\} \tag{2.11}$$

is called of *width* of K, or *minimal width* of K. If $\operatorname{width}(K) = \operatorname{diam}(K) = D$, or in other words, if $\operatorname{width}_K(u) = D$ for any $u \in \mathbb{S}^{d-1}$, we say that K is a *body of constant width* D in $\mathbb{R}^d$. The width function is also related to a certain symmetrization process on convex bodies. More specifically, for any convex body $K \in \mathcal{K}^d$, the o-symmetric convex body $\frac{1}{2}(K - K)$ is called the *central symmetral* of K. This body satisfies the property that for any $u \in \mathbb{S}^d$, $\operatorname{width}_u(K) = \operatorname{width}_u\left(\frac{1}{2}(K-K)\right)$.

Intrinsic volumes

The volume of the Minkowski sum of a compact, convex set K and a Euclidean ball $\lambda\mathbf{B}^d$ can be computed by Steiner's formula [169] as

$$\operatorname{vol}(K + \lambda\mathbf{B}^d) = \sum_{k=0}^{d} V_i(K)\kappa_{d-i}\lambda^{d-i}, \tag{2.12}$$

where the coefficients $V_i(K)$ are nonnegative, and are determined by K. The quantity $V_i(K)$ is called the *ith intrinsic volume* of K. It is well known that $V_d(K) = \operatorname{vol}(K)$, $V_0(K) = 1$. Furthermore, if K is a j-dimensional compact, convex set, then $V_j(K)$ is equal to $\operatorname{vol}_j(K)$. It is also well known that the jth intrinsic volume of K is also related to the average j-dimensonal volume of a projection of K onto a j-dimensional linear subspace of $\mathbb{R}^d$. More specifically, if $\mathcal{G}_{d,j}$ denotes the Grasmannian manifold of the j-dimensional linear subspaces of $\mathbb{R}^d$, and $X|S$ denotes the orthogonal projection of $X \subseteq \mathbb{R}^d$ onto the linear subspace S, then

$$V_j(K) = \binom{d}{j} \frac{\kappa_d}{\kappa_j \kappa_{d-j}} \int_{G_{d,j}} \mathrm{vol}_j(K|S)\, dS, \tag{2.13}$$

where the integration is with respect to the unique uniform probability measure on $\mathcal{G}_{d,j}$ [84].

In particular, the average length of the projection of the compact, convex set K onto a line, called the *mean width* of K and denoted by $\mathrm{mwidth}(K)$, is equal to

$$\mathrm{mwidth}(K) = \frac{2\kappa_{d-1}}{d\kappa_d} V_1(K). \tag{2.14}$$

An equivalent reformulation of the formula in (2.12) is that

$$\mathrm{vol}(K + \lambda \mathbf{B}^d) = \sum_{k=0}^{d} \binom{d}{i} W_i(K) \lambda^i, \tag{2.15}$$

where the quantity $W_i(K)$ is called the *ith quermassintegral* of K. Quermassintegrals and intrinsic volumes are connected via the identity $W_i(K) = \frac{\kappa_i}{\binom{d}{i}} V_{d-i}(K)$.

An important consequence of the Alexandrov-Fenchel inequality [84] regarding the intrinsic volumes of a convex body $K \in \mathcal{K}$ states that for any $0 < i < j \leq d$,

$$\left(\frac{V_i(K)}{V_i(\mathbf{B}^d)} \right)^{\frac{1}{i}} \geq \left(\frac{V_j(K)}{V_j(\mathbf{B}^d)} \right)^{\frac{1}{j}}. \tag{2.16}$$

Tilings, zonotopes and parallelohedra

In Chapter 8, we investigate properties of *zonotopes*, defined as the Minkowski sums of finitely many segments. A d-dimensional zonotope Z satisfies the property that for any $(d-2)$-dimensional face G of Z the facets of Z containing G form a *zone*, i.e. they can be arranged in a sequence $F_1, F_2, \ldots, F_k, F_{k+1} = F_1$ of facets of Z such that for all values of i, $F_i \cap F_{i+1}$ is a translate of G. A convex polytope is a zonotope if and only if its every 2-dimensional face is centrally symmetric.

Let $Z = \sum_{i=1}^{n} [o, p_i]$ be a d-dimensional zonotope. A well-known theorem of McMullen [147] gives an elegant analytic formula for the volume of Z, which can also be obtained from a decomposition theorem of Shephard in [174]. Namely, the volume of this zonotope is

$$\mathrm{vol}(Z) = \sum_{1 \leq i_1 < i_2 < \ldots < i_d \leq n} \left| \det\left([p_{i_1}, p_{i_2}, \ldots, p_{i_d}] \right) \right|, \tag{2.17}$$

where $[p_{i_1}, p_{i_2}, \ldots, p_{i_d}]$ is the matrix with $p_{i_1}, p_{i_2}, \ldots, p_{i_d}$ as columns.

A close relative of d-dimensional zonotopes is the family of d-dimensional parallelohedra, arisen in the theory of tilings. A *tiling* of $\mathbb{R}^d$ is a countable collection of compact sets $\mathcal{F} = \{C_i : i \in I\}$ with pairwise nonempty interiors, and satisfying $\bigcup_{i \in I} C_i = \mathbb{R}^d$. The elements of the tiling are called *cells*. A tiling is *convex* if every cell is convex. It is well known that in this case the cells are convex polytopes [171]. A convex tiling is called *face-to-face* (if $d = 2$, *edge-to-edge*), if for every facet F of every cell there is exactly one more cell containing F as a facet. A tiling is *translative* if every cell is a translate of a given compact set. A d-dimensional convex polytope P is a *d-dimensional parallelohedron* if there is a translative, convex tiling whose cells are translates of P. By a result of Dolbilin [51], for every d-dimensional parallelohedron there is a convex, face-to-face, translative tiling whose cells are translates of P.

Minkowski [151] and Venkov [188] proved that P is a d-dimensional parallelohedron if and only if the following holds:

(i) P and all its facets are centrally symmetric, and

(ii) the projection of P along any of its $(d-2)$-dimensional faces is a parallelogram or a centrally symmetric hexagon.

These properties show, in particular, that every 3-dimensional parallelohedron is a zonotope.

Let $X \subset \mathbb{R}^d$ be nonempty. A countable collection of compact sets $\mathcal{F} = \{C_i : i \in I\}$ satisfying $X \subseteq \bigcup_{i \in I} C_i$ is called a *covering* of X; in particular, any tiling of $\mathbb{R}^d$ is also a covering of $\mathbb{R}^d$. More generally, for any positive integer k, if every point of X belongs to at least k elements of $\mathcal{F}$, we say that $\mathcal{F}$ is a *k-fold covering* of X.

2.2.2 Normed spaces

Consider the d-dimensional real vector space $\mathbb{R}^d$, and let $f : \mathbb{R}^d \to \mathbb{R}$ be a function satisfying the following properties:

(i) $f(x) \geq 0$ for every $x \in \mathbb{R}^d$, and $f(x) = 0$ implies that $x = o$.

(ii) For any point $x \in \mathbb{R}^d$ and $\lambda \in \mathbb{R}$, we have $f(\lambda x) = |\lambda| f(x)$.

(iii) It satisfies the triangle inequality, i.e. $f(x+y) \leq f(x) + f(y)$ for any $x, y \in \mathbb{R}^d$.

Then $f(\cdot)$ is called a *norm* on $\mathbb{R}^d$. It is well known that the unit ball $\{x \in \mathbb{R}^d : f(x) \leq 1\}$ of the norm is an o-symmetric convex body in $\mathbb{R}^d$, and vice versa, any o-symmetric convex body $M \in \mathcal{K}_o^d$ induces a norm, defined as $f(x) = \inf\{\lambda \geq 0 : x \in \lambda M\}$. This observation establishes a natural one-to-one correspondence between norms and o-symmetric convex bodies in $\mathbb{R}^d$. We note that, like in a Euclidean space, norms can also be defined via a generalization of inner products, by the so-called *semi-inner products* (see e.g. [129, 87].

In the dissertation, for any $M \in \mathcal{K}_o^d$, we denote the norm induced by M as $||\cdot||_M$, and the space $\mathbb{R}^d$, equipped with this norm as $\mathbb{M}$. By its definition, every normed space is a metric space, where the distance of any two points $x, y \in \mathbb{M}$ is defined as $||x-y||_M$. This metric defines a topology on $\mathbb{R}^d$. It is easy to see that the Euclidean ball $\mathbf{B}^d$ defines the Euclidean norm $||\cdot||$ on $\mathbb{R}^d$, and the topology induced by any $M \in \mathcal{K}_o^d$ coincides with the one induced by $\mathbf{B}^d$. This implies that any statement in Euclidean space, involving only convexity or topological properties of sets, can be generalized for any normed space, whereas the same is not necessarily true if metric properties of sets are also involved.

The distance of two parallel hyperplanes H_1, H_2 in $\mathbb{M}$ is the minimum normed distance of any pair of points $p_1 \in H_1, p_2 \in H_2$. A convex body K is called a *body of constant width D in* $\mathbb{M}$ if the distance of any two parallel supporting hyperplanes of K have distance D. This property is equivalent to the property that the central symmetral of K is M, i.e. $M = \frac{1}{2}(K-K)$. For any such body K, the norm $||\cdot||_M$ is called the *relative norm* of K.

In this dissertation we investigate the volume of convex bodies in normed spaces. To do it, we recall that any finite dimensional real normed space can be equipped with a Haar measure, and that this measure is unique up to multiplication of the standard Lebesgue measure by a scalar. Depending on the choice of this scalar, one may define more than one version of normed volume. There are four variants that are regularly used in the literature. The *Busemann* and *Holmes-Thompson volume* of a set S in a d-dimensional normed space with unit ball M is defined as

$$\mathrm{vol}_M^{Bus}(S) = \frac{\kappa_d}{\mathrm{vol}(M)}\,\mathrm{vol}(S) \quad \text{and} \quad \mathrm{vol}_M^{HT}(S) = \frac{\mathrm{vol}(M^\circ)}{\kappa_d}\,\mathrm{vol}(S), \tag{2.18}$$

respectively, where, as usual, vol denotes d-dimensional Lebesgue measure. Note that the Busemann volume of the unit ball is equal to that of a Euclidean unit ball. For *Gromov's mass*, the scalar is chosen in such a way that the volume of a maximal volume cross-polytope, inscribed in the unit ball M is equal to $\frac{2^d}{d!}$, and for *Gromov's mass** (or *Benson's definition of volume*), the volume of a smallest volume parallelotope, circumscribed about M, is equal to 2^d (cf. [6]). We denote the two latter quantities by $\mathrm{vol}_M^m(S)$ and $\mathrm{vol}_M^{m*}(S)$, respectively.

Chapter 3

Multiple Borsuk numbers of sets

Recall from Chapter 2 the following conjecture of Borsuk [37] in 1933.

Conjecture 3.1 (Borsuk, 1933). *Every set of diameter $D > 0$ in the Euclidean d-space $\mathbb{R}^d$ is the union of $d+1$ sets of diameters less than D.*

Investigation of this conjecture led to the definition of Borsuk number of a set as follows.

Definition 3.1. *For any set $S \subset \mathbb{R}^d$ of diameter $D > 0$, the smallest positive integer m such that there is a cover of S with m sets of diameters strictly less than D, is called the* Borsuk number *of S, denoted by $a(S)$.*

Our aim is to generalize this concept. Our main definition is the following.

Definition 3.2. *Let $S \subset \mathbb{R}^d$ be a set of diameter $D > 0$. The smallest positive integer m such that there is a k-fold cover of S, with m sets of diameters strictly less than D, is called the* k-fold Borsuk number *of S. We denote this number by $a_k(S)$.*

Clearly, $a_1(S) = a(S)$.

Note that Definition 3.2 can be naturally adapted to almost any variant of the original Borsuk problem, and thus, raises many open questions that are not examined in the dissertation. Our goal is to investigate the properties of the k-fold Borsuk numbers of sets in $\mathbb{R}^d$.

We start with three observations. Then in Section 3.1 we characterize the k-fold Borsuk numbers of planar sets. In Section 3.2 we give estimates on the k-fold Borsuk numbers of smooth bodies, centrally symmetric sets, and convex bodies of constant width, and determine them for Euclidean balls. In Sections 3.3 and 3.4 we examine the k-fold Borsuk numbers of finite point sets in $\mathbb{R}^3$. In particular, in Section 3.3 we examine the sets with large k-fold Borsuk numbers, and in Section 3.4 we focus on sets with a nontrivial symmetry group. Finally, in Section 3.5 we make an additional remark and raise a related open question.

Our first observations are as follows.

Remark 3.1. *The sequence $a_k(S)$ is subadditive for every S. More precisely, for any positive integers k, l, we have $a_{k+l}(S) \leq a_k(S) + a_l(S)$.*

Remark 3.2. *For every set $S \subset \mathbb{R}^d$ of diameter $D > 0$ and for every $k \geq 1$, we have $a_k(S) \geq 2k$. Furthermore, for every value of k, if $a(S) = 2$, then $a_k(S) = 2k$, and if $a(S) > 2$, then $a_k(S) > 2k$.*

Proof. Without loss of generality, we may assume that S is compact. Let $[p,q]$ be a diameter of S. Since no set of diameter less than D contains both p and q, any k-fold cover of S with sets of smaller diameters has at least $2k$ elements. Furthermore, if $a(S)=2$, then by Remark 3.1, $a_k(S) \leq 2k$ for every value of k.

Now assume that $a_k(S)=2k$ for some value of k. Let $A_1, A_2, \ldots, A_{2k}$ be compact sets of diameters less than D that cover S k-fold. For every k-element subset J of $I=\{1,2,\ldots,2k\}$, let $\bar{J} = I \setminus J$, and let S_J be the (compact) set of points in S that are covered by A_i for every $i \in J$. Clearly, the union of the sets S_J is S, when J runs over the k-element subsets of I. We define two sets A and B in the following way: for any pair J and $\bar{J}$, we choose either S_J or $S_{\bar{J}}$ to add to A, and we add the other one to B. Then $S \subseteq A \cup B$.

Note that for any pair of points $p,q \in A$, there is an index i such that $p,q \in A_i$, and thus, $||p-q|| \leq \operatorname{diam}(A_i) < D$. This yields that $\operatorname{diam}(A) < D$. We may obtain similarly that $\operatorname{diam}(B) < D$, which implies that $a(S)=2$. □

Remark 3.3. *Let $S \subset \mathbb{R}^d$ be a set of positive diameter. Then for every value of k, $a_k(S) = a_k(\operatorname{bd} S)$.*

Proof. Without loss of generality, we may assume that S is compact and that $\operatorname{diam}(S)=1$. Then, clearly, $a_k(S) \geq a_k(\operatorname{bd} S)$ for every k.

On the other hand, assume that some sets $Q_1, Q_2, \ldots, Q_m$ form a k-fold cover of $\operatorname{bd} S$ where each Q_i is of diameter less than one. Without loss of generality, we may assume that $Q_i \subset S$ for every i. Let $\varepsilon > 0$ be chosen in such a way that $\operatorname{diam}(Q_i) < 1 - 2\varepsilon$ for all values of i. Then the sets $\bar{Q}_i = (Q_i + \varepsilon \mathbf{B}^d) \cap S$ form a k-fold cover of $(\operatorname{bd} S + \varepsilon \mathbf{B}^d) \cap S$ such that $\operatorname{diam}(\bar{Q}_i) < 1$. Let T denote the set $S \setminus (\operatorname{bd} S + \varepsilon \mathbf{B}^d)$. Observe that for any point $p \in T$ and $q \in S$, we have $||p-q|| \leq 1 - \varepsilon$. Thus, setting $Q'_i = \bar{Q}_i \cup T$ for every i, we have $\operatorname{diam}(Q'_i) \leq \max\{\operatorname{diam}(\bar{Q}_i), 1-\varepsilon\} < 1$, and the sets $Q'_1, Q'_2 \ldots, Q'_m$ form a k-fold cover of S. □

3.1 Sets in the Euclidean plane

To formulate our main results, we first recall the well-known fact that for any $S \subset \mathbb{R}^d$ of diameter D, there is a convex body $K \subset \mathbb{R}^d$ of constant width D such that $S \subseteq K$. The following characterization of the Borsuk numbers of plane sets was given by Boltyanskii (cf. [33], or alternatively [34]).

Theorem 3.1 (Boltyanskii, 1960). *Let $S \subset \mathbb{R}^2$ be of diameter $D > 0$. The Borsuk number of S is three if and only if there is a unique convex body of constant width D, containing S.*

Now we prove the following.

Theorem 3.2. *Let $S \subset \mathbb{R}^2$ be a set of diameter $D > 0$ with $a(S) = 3$, and let C be the unique plane convex body of constant width D that contains S. Then for every value of k, we have $a_k(S) = a_k(C)$.*

Our proof is based on the following lemma, used by Boltyanskii (cf. for example, Lemma 8, p. 29, [34])

Lemma 3.1 (Boltyanskii, 1971).

For any point $u \in (\operatorname{bd} C) \setminus S$, there is an open circle arc of radius D in $\operatorname{bd} C$, that contains u, such that the center p of the circle is contained in $\operatorname{bd} C$.

Proof of Theorem 3.2. Without loss of generality, let S be compact, and $D = 1$. Clearly, $a_k(C) \geq a_k(S)$. Hence, by Remark 3.3, it suffices to show that $a_k(\operatorname{bd} C) \leq a_k(S)$.

Assume that $Q_1, Q_2, \ldots, Q_m \subset S$ are sets of diameters less than one that form a k-fold cover of S. Without loss of generality, we may assume that $Q_i \subset C$ for every value of i.

Let $\delta > 0$ be chosen in a way that $\operatorname{diam}(Q_i) + 2\delta < 1$ for every value of i. Let A_j, where $j = 1, 2, \ldots, t$ denote the connected components of $(\operatorname{bd} C) \setminus S$ longer than 2δ, and let q_j and r_j be the two endpoints of A_j. Clearly, there are finitely many such arcs. First, we note that the sets $\bar{Q}_i = (Q_i + \delta \mathbf{B}^2) \cap C$ form a k-fold cover of $(\operatorname{bd} C) \setminus \left(\sum_{j=1}^{t} A_j \right)$, and that the diameter of any of these sets is less than one. We extend the sets $\bar{Q}_1, \bar{Q}_2, \ldots, \bar{Q}_m$ to cover k-fold all the A_js.

Using Lemma 3.1, for every value of j, A_j is an open unit circle arc with its center $p_j \in \operatorname{bd} C$. Since C is contained in the intersection of the two unit disks $q_j + \mathbf{B}^2$ and $r_j + \mathbf{B}^2$, we obtain that p_j is not a smooth point of $\operatorname{bd} C$, which yields, by the same lemma, that $p_j \in S$.

Let A_j^q be the set of the points of A_j that are not farther from q_j than from r_j. We define A_j^r analogously. Note that for any $u \in A_j$, the only point of C at distance one from u is p_j. Hence, for any index i such that $q_j \in \bar{Q}_i$, p_j has a neighborhood disjoint from $\bar{Q}_i$. This yields that $\operatorname{diam}(\bar{Q}_i \cup A_j^q) < 1$. We may obtain similarly that if $r_j \in \bar{Q}_i$, then $\operatorname{diam}(\bar{Q}_i \cup A_j^r) < 1$. Thus, we set

$$C_i = \operatorname{conv} \left(\bar{Q}_i \cup \bigcup_{q_j \in \bar{Q}_i} A_j^q \cup \bigcup_{r_j \in \bar{Q}_i} A_j^r \right),$$

and observe that, by induction on j, $\operatorname{diam}(C_i) < 1$ for every value of i, and that the sets C_i form a k-fold cover of $\operatorname{bd} C$. □

Now let us recall the notion of *Reuleaux polygons*. These polygons are constant width plane convex bodies bounded by finitely many circle arcs of the same diameter (of radius equal to the width of the body), called the *sides* of the polygon. It is a well-known fact that any such polygon has an odd number of sides (cf. [35] or [74]).

Theorem 3.3. *Let C be a constant width convex body in $\mathbb{R}^2$, and k be a positive integer. If C is a Reuleaux polygon with $2s+1$ sides, then $a_k(C) = 2k + \left\lceil \frac{k}{s} \right\rceil$, and otherwise $a_k(C) = 2k+1$.*

Proof. For simplicity, assume that $\operatorname{diam}(C) = 1$.

First, consider the case that C is a Reuleaux polygon with $2s+1$ sides. Let us call the common point of two consecutive sides of C a *vertex* of C. Let these vertices be $p_1, p_2, \ldots, p_{2s+1} = p_0$ in counterclockwise order in $\operatorname{bd} C$. Consider the diameter graph of the vertex set of C: The vertices of this graph are the vertices of C, and two vertices are connected with an edge if and only if they are endpoints of a diameter. Clearly, this graph is the cycle C_{2s+1}, of length $(2s+1)$. By [175] (see also [178]), the k-fold chromatic number of C_{2s+1} is $m = 2k + \left\lceil \frac{k}{s} \right\rceil$. This implies that $a_k(C) \geq m$.

Now we show the existence of some sets $Q_1, Q_2, \ldots, Q_m$, with diameters less than one, that form a k-fold cover of $\operatorname{bd} C$. This, by Remark 3.3, yields the assertion for Reuleaux polygons. Let $A_1, A_2, \ldots, A_m$ be sets of diameters less than one that form a k-fold cover of the vertices of C. Let $\widehat{p_i p_{i+1}}$ denote the side of C connecting p_i and p_{i+1}, and let G_i be the union of the points of the two sides $\widehat{p_i p_{i-1}}$ and $\widehat{p_i p_{i+1}}$ that are not farther from p_i than from p_{i-1} and p_{i+1}, respectively. Observe that the sets $Q_j = \bigcup_{p_i \in A_j} G_j$ form a k-fold cover of $\operatorname{bd} C$, and that their diameters are strictly less than one, which readily implies the assertion.

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In the remaining part we assume that C is *not* a Reuleaux polygon. By Remark 3.2, we have that $a_k(C) \geq 2k+1$. Hence, it suffices to construct a family of $2k+1$ sets of diameters less than one that form a k-fold cover of C or, by Remark 3.3, $\mathrm{bd}\, C$.

First, we carry out this construction for $C = \frac{1}{2}\mathbf{B}^2$. Consider $2k+1$ distinct diameters of C. Let these be $[p_1, q_1], [p_2, q_2], \ldots, [p_{2k+1}, q_{2k+1}]$, where the notation is chosen in such a way that the points $p_1, p_2, \ldots, p_{2k+1} = p_0$ and $q_1, q_2, \ldots, q_{2k+1} = q_0$ are in counterclockwise order in $\mathrm{bd}\, C$, and for every i, the shorter arc connecting p_i and p_{i+1} contains exactly one of the q_js (by exclusion, this point is q_{i+k}, see Figure 3.1). Observe that the points p_i have the property that the diameter containing any one of them divides $\mathrm{bd}\, C$ into two open half circles, each of which contains exactly k of the remaining $2k$ points. Let A_i denote the shorter arc in $\mathrm{bd}\, C$ connecting p_i and p_{i+k}. Observe that these sets form a k-fold cover of $\mathrm{bd}\, C$, and that their diameters are less than one.

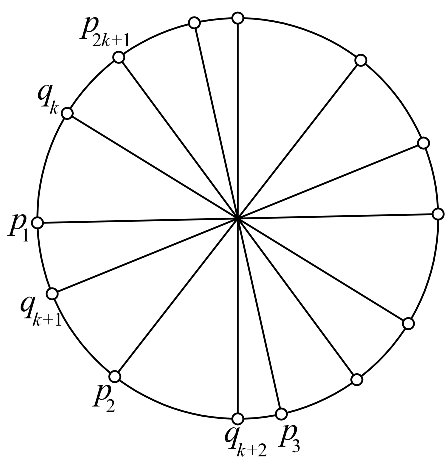

Figure 3.1: Covering a Euclidean disk

In the last step, we show that a similar family can be constructed for any C that is not a Reuleaux polygon. Before we do that, we recall the following simple property of plane convex bodies of constant width:

- No two diameters of a plane convex body of constant width are disjoint.

For any $p \in \mathrm{bd}\, C$, let $G(p) \subset \mathbb{S}^1$ be the *Gaussian image* of p; that is, the set of the external unit normal vectors of the lines supporting C at p. Observe that if $G(p)$ is not a singleton, then it is a closed arc in $\mathbb{S}^1$. Furthermore, in this case p is not a smooth point of $\mathrm{bd}\, C$, and thus, by Lemma 3.1 in the previous proof, the locus of the other endpoints of the diameters starting at p is a closed unit circle arc in $\mathrm{bd}\, C$. Apart from the endpoints, the points of this arc are smooth points of $\mathrm{bd}\, C$, and thus, their Gaussian images are singletons. This yields that if $G(p)$ is not a singleton, then, apart from its endpoints, the points of $-G(p)$ are decomposed into singleton Gaussian images.

Since C is not a Reuleaux polygon, we may choose $2k+1$ diameters of $\mathbf{B}^2$ such that the Gaussian image of any point of $\mathrm{bd}\, C$ intersects at most one of them, say $[p_1, q_1], [p_2, q_2], \ldots, [p_{2k+1}, q_{2k+1}]$. Indeed, as $\mathbb{S}^1$ is not covered by the union of finitely many Gaussian images and their antipodal arcs, at least one of the following holds:

Case 1, There are at least $2k+1$ Gaussian images in $\mathbb{S}^1$ that are not singletons: then we may choose $2k+1$ such arcs, and pick one point from each, different from the endpoints of the arc, as an endpoint of one of the chosen diameters.

Case 2, There are less than $2k+1$ Gaussian images that are not singletons. In this case there is an open arc $I \subset \mathbb{S}^1$ such that both I and $-I$ are decomposed into singleton Gaussian images. Thus, we may choose $2k+1$ pairwise distinct diameters with all their endpoints in $I \cup (-I)$.

Let us label the endpoints of these diameters as in the case that C is a Euclidean disk. That is, assume that the points $p_1, p_2, \ldots, p_{2k+1} = p_0$ are in counterclockwise order in $\operatorname{bd} C$, and for every i, the (counterclockwise) directed arc connecting p_i and p_{i+1} contains exactly q_{i+k} from amongst the q_js. Let $G^{-1}(u)$ denote the (unique) point v of $\operatorname{bd} C$ with the property that $u \in G(v)$. For every i, let F_i denote the closed arc of $\operatorname{bd} C$, with endpoints $G^{-1}(p_i)$ and $G^{-1}(p_{i+k})$, and containing $G^{-1}(p_{i+j})$ for $j = 1, 2, \ldots, k-1$. Observe that since any two diameters of C have a nonempty intersection, no arc F_i contains the endpoints of a diameter, and thus, $\operatorname{diam}(F_i) < 1$. On the other hand, these arcs form a k-fold cover of $\operatorname{bd} C$, which implies that $a_k(C) = 2k+1$. □

3.2 Centrally symmetric sets and smooth bodies

Two of the special cases for which Borsuk's original conjecture is proven, are when the set is centrally symmetric, or is a smooth convex body (cf. [161], [100] and [101]). The proofs in both cases are based on reducing the problem to the Euclidean d-ball $\mathbf{B}^d$, and then to finding $a(\mathbf{B}^d)$. In this section, we investigate the k-fold Borsuk numbers of these sets in the same way.

For preciseness, we first remark that we call a set $S \subset \mathbb{R}^d$ a *smooth body*, if S is homeomorphic to $\mathbf{B}^d$, and its boundary is a C^1-class submanifold of $\mathbb{R}^d$.

Theorem 3.4. *Let $S \subset \mathbb{R}^d$ be a set of diameter $D > 0$.*

(1) If S is a smooth body or centrally symmetric, then for every k, we have $a_k(S) \leq a_k(\mathbf{B}^d)$.

(2) If S is a convex body of constant width, then for every k, we have $a_k(S) \geq a_k(\mathbf{B}^d)$.

(3) For every k, we have $a_k(\mathbf{B}^d) = 2k + d - 1$.

Clearly, this theorem implies that if S is a smooth convex body of constant width, then for every k, $a_k(S) = 2k + d - 1$.

Proof. Let $D = 1$.

First we examine the case that S is a smooth body. For every $p \in \operatorname{bd} S$, let $G(p)$ denote the Gaussian image of p; that is, the unique external unit normal vector of S at p. Then $G : \operatorname{bd} S \to \mathbb{S}^{d-1}$ is a continuous mapping. Observe that if $[p, q]$ is a diameter of S, then $p, q \in \operatorname{bd} S$, and $G(p) = -G(q) = p - q$. Thus, any k-fold cover of $\mathbb{S}^{d-1}$ by m sets of smaller diameters induces a k-fold cover of $\operatorname{bd} S$, and thus S, by m sets of smaller diameters. This shows that $a_k(S) \leq a_k(\mathbf{B}^d)$.

Now, assume that S is a (not necessarily smooth) convex body of constant width. Then every point of $\operatorname{bd} S$ is an endpoint of some diameter, and thus, a k-fold cover of $\operatorname{bd} S$ induces a k-fold cover of $\mathbb{S}^{d-1}$ like in the previous paragraph. This yields $a_k(S) \geq a_k(\mathbf{B}^d)$ for every k.

Next, let S be symmetric to the origin. Observe that $S \subseteq \frac{1}{2}\mathbf{B}^d$. Then the set $S_D = S \cap \left(\frac{1}{2}\mathbb{S}^{d-1}\right)$ contains all the points of S that are endpoints of some diameter, and thus, the inequality $a_k(S) \leq a_k(\mathbf{B}^n)$ follows by an argument similar to the one used for smooth bodies.

We are left to show that $a_k(\mathbf{B}^d) = 2k+d-1$, or equivalently, that $a_k(\mathbb{S}^{d-1}) = 2k+d-1$. To show that $a_k(\mathbb{S}^{d-1}) \geq 2k+d-1$, we follow the idea of the proof for the usual Borsuk number of $\mathbb{S}^{d-1}$.

Consider a k-fold cover $\mathcal{F} = \{Q_1, Q_2, \ldots, Q_m\}$ of $\mathbb{S}^{d-1}$, with closed sets, such that no element of $\mathcal{F}$ contains a pair of antipodal points. Let us define the function $f : \mathbb{S}^{d-1} \to \mathbb{R}^{d-1}$ as

$$f(x) = (\operatorname{dist}(x, Q_1), \operatorname{dist}(x, Q_1), \ldots, \operatorname{dist}(x, Q_{d-1})),$$

where we set $\operatorname{dist}(y, A) = \inf\{||z-y||\colon z \in A\}$ for any $y \in \mathbb{R}^d$ and nonempty set $A \subseteq \mathbb{R}^d$. This function is clearly continuous, and hence, by the Borsuk-Ulam Theorem, there is a point $p \in \mathbb{S}^{d-1}$ such that $f(p) = f(-p)$. If some coordinate of $f(p)$ is zero, then both p and $-p$ are elements of one of the sets $A_1, A_2, \ldots, A_{d-1}$; a contradiction. Thus, $f(p)$ has no coordinate equal to zero, which means that neither p nor $-p$ belongs to $A_1 \cup \ldots A_{n-1}$. Since p and $-p$ are antipodal points, there are at least $2k$ elements of $\mathcal{F}$ that contain one of them, which yields that $m \geq 2k+n-1$. On the other hand, Gale proved (cf. Theorem II' in [82]) the existence of a family of $2k+d-1$ open hemispheres of $\mathbb{S}^{d-1}$ that form a k-fold cover of $\mathbb{S}^{d-1}$. Since contracting these open hemispheres one by one yields a k-fold cover of $\mathbb{S}^{d-1}$ with $2k+d-1$ closed spherical caps of radii strictly less than $\frac{\pi}{2}$, we obtain that $a_k(\mathbf{B}^d) = 2k+d-1$. □

3.3 Multiple Borsuk numbers of finite point sets in Euclidean 3-space

The fact that the (usual) Borsuk numbers of finite sets in 3-space are at most four was first shown by Heppes and Révész in [111], and it also follows from the proof of Vázsonyi's conjecture (cf. [107], [95], [127] or [179]), that stated that in any set $S \subset \mathbb{R}^3$ of cardinality m, $\operatorname{diam}(S)$ is attained between at most $2m-2$ pairs of points; or in other words, that the diameter graph of any set of m points in $\mathbb{R}^3$ has at most $2m-2$ edges. Later we use some of the ideas of these proofs.

A complete characterization of the Borsuk numbers of finite sets in $\mathbb{R}^3$, even of those with $a(S) = 4$ looks hopeless: indeed, by (2) of Theorem 3.4, if S is the vertex set of a Reuleaux polytope in $\mathbb{R}^3$, then $a(S) = 4$, and a result of Sallee [167] yields that the family of Reuleaux polytopes in $\mathbb{R}^3$ is an everywhere dense subfamily of the family of convex bodies of constant width in $\mathbb{R}^3$. Thus, unlike in Section 3.1, in this and the next sections we restrict our investigation to point sets with some special properties.

The main goal of this section is to find the finite sets $S \subset \mathbb{R}^3$ with $a_k(S) = 4k$ for every value of k. We observe that for a finite set $S \subset \mathbb{R}^3$, Remark 3.1 readily implies that $a_k(S) \leq 4k$ for every value of k, where we have equality, for example, for regular tetrahedra.

Using the next, naturally arising concept, we may rephrase our question in a different form.

Definition 3.3. *Let $S \subset \mathbb{R}^d$ be of diameter $D > 0$. Then the quantity*

$$a_{frac}(S) = \inf\left\{\frac{a_k(S)}{k} : k = 1, 2, 3\ldots\right\}$$

is called the fractional Borsuk number *of S.*

Clearly, $a_{frac}(S) \leq a(S)$ for every set S.

Problem 3.1. *Prove or disprove that if $S \subset \mathbb{R}^3$ is a finite point set with $a_{frac}(S) = 4$, then its diameter graph contains K_4 as a subgraph.*

We give only a partial answer to this problem. During the investigation, we denote the diameter graph of S by G_S, and recall that the *girth* of a graph G is the length of a shortest cycle in G. We denote this quantity by $g(G)$, and note that for any finite set $S \subset \mathbb{R}^3$, we have $a_k(S) = \chi_k(G_S)$, where $\chi_k(G_S)$ denotes the k-fold chromatic number of G_S.

Our main result is the following.

Theorem 3.5. *Let $S \subset \mathbb{R}^3$ be a finite set with $g(G_S) > 3$. Then $a_k(S) < 4k$ for some value of k.*

This theorem may be rephrased in the following form: For any finite set $S \subset \mathbb{R}^3$ with $a_{frac}(S) = 4$, G_S contains K_3 as a subgraph. The proof is based on Lemma 3.2.

Lemma 3.2. *There is an m-fold $(2m+1)$-coloring of the $(2m+1)$-cycle C with the property that any two nonconsecutive vertices have a common color.*

Proof of Lemma 3.2. Let the vertices of C be $v_1, v_2, \ldots, v_{2m+1} = v_0$ in counterclockwise order. Let the colors be $1, 2, \ldots, 2m+1$. We define a 3-coloring of C as follows with the colors $t, t+1, 2m+1$, where $t \in \{1, 3, \ldots, 2m-1\}$: only v_t is colored with $2m+1$, and the vertices $v_{t+1}, v_{t+2}, \ldots$ are colored with t and $t+1$, alternately. It is easy to see that the union of these m 3-colorings is an m-fold $(2m+1)$-coloring of C.

Consider any two vertices v_i and v_j with $|i-j| \geq 2$. Since C is an odd cycle, exactly one of the two connected components of $C \setminus \{v_i, v_j\}$ contains an even number of vertices. If this component does not contain a vertex colored with the color $2m+1$, then v_i and v_j are v_{2m-1} and v_1, both of which are colored with $2m+1$. If the even component contains the vertex a_t colored with $2m+1$, then both v_i and v_j are colored either with t or with $t+1$. □

Proof of Theorem 3.5. Let C be a shortest odd cycle in G_S, of length $2m+1 \geq 5$. We show that $\chi_m(G_S) \leq 4m-1$. Note that if G_S contains no odd cycle, then it is bipartite, and thus, the statement follows from $\chi(G_S) = 2$.

By [56], any odd cycle of G_S intersects C, or in other words, $G_S \setminus C$ is a bipartite graph. Let the two parts of $V(G_S \setminus C)$ in this partition be V_1 and V_2. Clearly, since G_S contains no triangle, no vertex of V_1 is connected to two consecutive vertices of C. Furthermore, no vertex of V_1 is connected to more than two vertices of C. Indeed, if a vertex v is connected to the distinct vertices $v_1, v_2, v_3 \in C$, then there is a path in C, of odd length at most $2m-3$, that connects two of v_1, v_2 and v_3. This yields that C is not a shortest odd cycle of G_S; a contradiction (for this argument, see also [56]).

Now we define an m-fold $(4m-1)$-coloring of G_S. We color each vertex of V_2 with the colors $3k, 3k+1, \ldots, 4k-1$, and use only the remaining colors for $C \cup V_1$. We color C in the way described in Lemma 3.2, using only the colors $1, 2, \ldots, 2m+1$. We color the vertices of V_1, using $1, 2, \ldots, 3m-1$, in the following way. Consider a vertex $v \in V_1$. Then v is connected to at most two vertices of C, which are not consecutive. Hence, by Lemma 3.2, there are at most $2m-1$ colors used for coloring them. Thus, there are at least $(3m-1) - (2m-1) = m$ colors, from amongst $1, 2, \ldots, 3m-1$, that do not color any neighbor of v in C. We color v with m such colors. □

In the remaining part we show that the statement of Problem 3.1 holds for any set S with $\operatorname{card} S \leq 7$. We start with finding the 4-critical subsets of diameter graphs of sets in $\mathbb{R}^3$ of at most seven points. Recall that a graph G is m-critical if $\chi(G) = m$, and for any proper subgraph H of G, $\chi(H) < m$.

Lemma 3.3. *If* $S \subset \mathbb{R}^3$ *with* $\operatorname{card} S \leq 7$ *and* $a(S) = 4$*, and* H *is a* 4*-critical subgraph of* G_S*, then* H *is either* K_4*, or the wheel graph* W_6*, or the Mycielskian* $\mu(C_3)$ *of the* 3*-cycle* C_3 *(cf. Figure 3.2).*

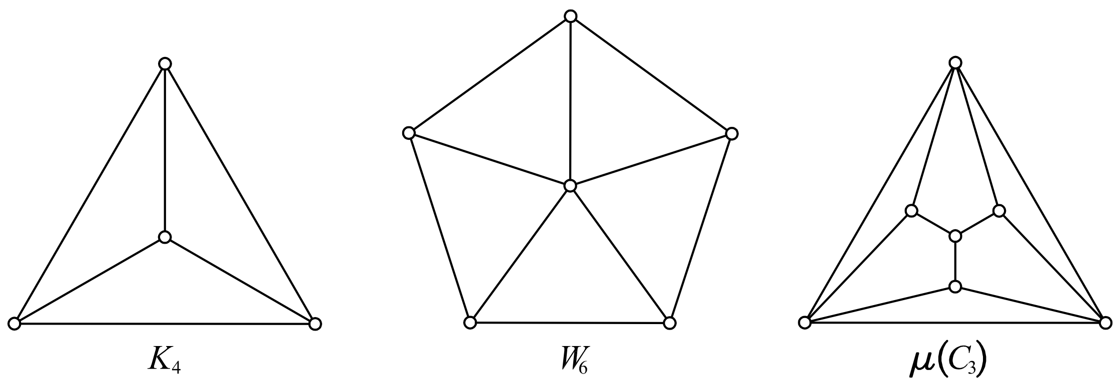

Figure 3.2: 4-critical subgraphs of diameter graphs

Proof. By the proof of Vázsonyi's conjecture, G_S has at most $2 \operatorname{card} S - 2$ edges, and by [56], any two odd cycles of G_S intersect. Clearly, these properties hold also for all the subgraphs of G_S. Thus, it suffices to prove the following, slightly more general statement: If G is a 4-critical graph with at most $m \leq 7$ vertices and at most $2m - 2$ edges such that any two odd cycles of G intersect, then G is either K_4, or W_6 or $\mu(C_3)$.

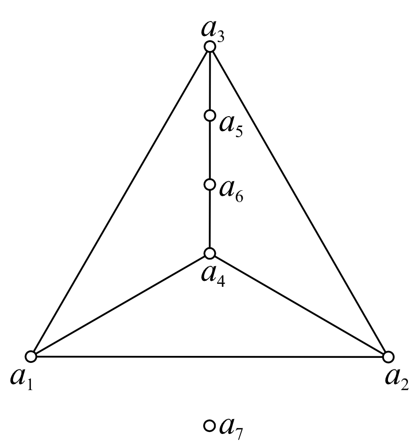

Figure 3.3: An illustration for the proof of Lemma 3.3

Answering a question of Toft [184] it was proven in [117] that if a 4-critical graph has at least one vertex of degree 3, then the graph contains a *fully odd subdivision* of K_4 as a subgraph, where a fully odd subdivision of a graph H is a graph, obtained from H in a way that the edges of H are replaced by paths with odd numbers of edges.

Since our graph G has at most $2m-2$ edges and, being 4-critical, the degree of any vertex is at least 3, G has at least four vertices of degree 3, and hence, by [117], it contains a fully odd subdivision of K_4. If G is not K_4, then, as G is 4-critical, this subdivision does not coincide with K_4, and thus, $m \geq 6$.

We leave it to the reader to show that the only 4-critical graph with six vertices and satisfying our conditions is W_6. We deal only with the case $m = 7$. For the proof we use the notations in Figure 3.3. Since G has at most 12 edges, there are at most four edges not shown in Figure 3.3. Note that as G contains no disjoint triangles and the degree of every vertex is at least 3, a_7 is connected to exactly one of a_5 or a_6. By symmetry, we may assume that a_5a_7 is an edge, and a_6a_7 is not. This implies also that the degrees of a_5, a_6 and a_7 are 3, and that G has exactly 12 edges.

By a similar argument, we may obtain that exactly one of a_1a_6 and a_2a_6 is an edge, say a_1a_6. Then the two additional edges of G connect a_7 to two of a_1, a_2, a_3 and a_4. It is an elementary exercise to check that if these edges are not a_2a_7 and a_4a_7, then G is 3-colorable or contains disjoint triangles. But if they are a_2a_7 and a_4a_7, then $G = \mu(C_3)$, which finishes the proof. □

Theorem 3.6. *If* $S \subset \mathbb{R}^3$ *with* $\operatorname{card} S \leq 7$, *then either* $a_k(S) < 4k$ *for every* $k > 1$, *or* G_S *contains* K_4 *as a subgraph.*

Proof. If $a_1(S) \leq 3$ or $a_2(S) \leq 7$, then the assertion readily follows from Remark 3.1. Thus, we may assume that $\operatorname{card} S \leq 7$ and that $a_2(S) = 2a_1(S) = 8$, or equivalently, that $2\chi(G_S) = \chi_2(G_S) = 8$. As a consequence of Lemma 3.3, G_S contains K_4, W_6 or $\mu(C_3)$ as a subgraph.

If G_S contains K_4, we are done. If G_S contains $\mu(C_3)$ as a subgraph, then $G_S = \mu(C_3)$, since $\mu(C_3)$ has 7 vertices and 12 edges, and by Vázsonyi's problem G_S has no more than 12 edges. It is easy to see that $\chi_2(\mu(C_3)) = 7$, which immediately implies the assertion.

Now we deal with the case that G_S contains W_6 as a subgraph. If $\operatorname{card} S = 6$, then by Vázsonyi's problem $G_S = W_6$, and thus, $a_2(S) = \chi_2(W_6) = 7$. Assume that $\operatorname{card} S = 7$. Then there are at most two edges of G_S not contained in the subgraph W_6. We may assume that W_6 is an induced subgraph of G_S, since if an additional edge of G_S connects two vertices of W_6, then G_S contains K_4 as a subgraph. Thus, G_S is obtained from W_6 by adding an additional vertex, and connecting it to at most two vertices. Depending on the choice of these vertices, it is an elementary exercise to find a 2-fold 7-coloring of G_S in each case. □

3.4 The Borsuk numbers of symmetric finite point sets in $\mathbb{R}^3$

In this section, our aim is to examine the Borsuk numbers of finite point sets in $\mathbb{R}^3$ with their Borsuk numbers equal to four, and with a nontrivial symmetry group. Our project is motivated by a result of Rogers [162], who proved Borsuk's conjecture for d-dimensional sets with symmetry groups containing that of a regular d-dimensional simplex.

In our investigation we denote the symmetry group of a regular tetrahedron by $\mathcal{T}_4$, and that of a regular k-gon by $\mathcal{D}_k$.

Theorem 3.7. *Let* $S \subset \mathbb{R}^3$ *be a finite set with* $\mathcal{T}_4 \subseteq \operatorname{Sym}(S)$. *If* $g(G_S) = 3$, *then* G_S *contains* K_4 *as a subgraph.*

Proof. Without loss of generality, let $\operatorname{diam}(S) = 1$, and let the regular tetrahedron, with symmetry group $\mathcal{T}_4$ and with unit edge length, be T.

Let $a, b, c \subseteq S$ be the vertices of a regular triangle of unit edge length. Let M be any plane reflection contained in $\mathrm{Sym}(S)$. Then the points $M(a), M(b), M(c)$ are contained in S. By [56] or [179], any two odd cycles of G_S intersect, and thus the sets $\{a, b, c\}$ and $\{M(a), M(b), M(c)\}$ are not disjoint. If a point is the reflecion of another one, say $b = M(a)$, then, clearly, $a = M(b)$, and then c is on the reflection plane; that is, $M(c) = c$. If a point is its own reflection, then the point is on the reflection plane. Thus, we have shown that each reflection plane of $\mathrm{Sym}(S)$ contains at least one vertex from any 3-cycle in G_S.

We leave it to the reader to show that since $\mathrm{diam}(S) = 1$, $\{a, b, c\}$ does not contain the center of T. Thus, there is an axis of rotation in T_4 that is disjoint from $\{a, b, c\}$. Let R be a rotation with angle $\frac{2\pi}{3}$ around this axis. Then the triples $\{a, b, c\}$ and $\{R(a), R(b), R(c)\}$ have a point in common; say, $b = R(a)$ (note that no point is the rotated copy of itself). In this case $\{a, b, R(b)\}$ is a 3-cycle in G_S which is invariant under R. As any 3-cycle has a point on each reflection plane, it implies that the vertices of this cycle are on the other three axes of rotation. Applying the symmetries of $\mathcal{T}_4$ to these vertices we obtain the vertices of a regular tetrahedron of unit edge length, which readily implies the assertion. $\square$

Remark 3.4. *Combining Theorems 3.7 and 3.5, we have that if for some finite set $S \subset \mathbb{R}^3$ we have $\mathcal{T}_4 \subseteq \mathrm{Sym}(S)$, and $a_k(S) = 4k$ for every k, then G_S contains K_4 as a subgraph.*

We note that by [179], for every finite set $S \subset \mathbb{R}^3$, G_S can be embedded in the projective plane. On the other hand, an example in [179] shows that not all these graphs are planar.

Remark 3.5. *It is known that the chromatic number of every triangle-free planar graph is at most three. Thus, Theorem 3.7 yields that if G_S is planar and $\mathcal{T}_4 \subseteq \mathrm{Sym}(S)$, then G_S contains K_4 as a subgraph.*

Problem 3.2. *Prove or disprove that if $S \subset \mathbb{R}^3$ is a finite set with $\mathcal{T}_4 \subseteq \mathrm{Sym}(S)$ and with $a(S) = 4$, then G_S contains K_4 as a subgraph.*

Our next aim is to examine sets S, with $a(S) = 4$ and with $\mathcal{D}_{2k+1} \subseteq \mathrm{Sym}(S)$ for some integer $k \geq 1$. We construct a family of sets satisfying these conditions.

In the construction we use the notion of the *p-Mycielskian* of a graph G (cf. [183]), denoted by $\mu_p(G)$. We regard the *wheel graph* W_{2k+2} as the 0th Mycielskian of the odd cycle C_{2k+1}.

Theorem 3.8. *For any $p \geq 0$, and $k > 0$, $\mu_p(C_{2k+1})$ is the diameter graph of a finite set $S \subset \mathbb{R}^3$.*

Proof. Let $p_1, p_2, \ldots, p_{2k+1} = p_0$ be the vertices of a regular $(2k+1)$-gon in the (x, y)-plane, centered at the origin. Assume that the diameter of the point set is $r \leq 1$. Consider the points $q = (0, 0, \sqrt{1 - r^2})$ and $r = (0, 0, \sqrt{1 - r^2} - 1)$. Note that for every i, $||p_i - q|| = 1$ and $||p_i - r|| < 1$.

Consider the pyramid $\mathrm{conv}\{p_1, \ldots, p_{2k+1}, q\}$, and let v_i denote the inner unit normal vector of its supporting plane of passing through the points $p_{i \pm k}$ and q. An elementary computation shows that $\langle v_i, p_j - q \rangle \geq 0$ for every i and j, and if $j \neq i, i + 1$, then we have strict inequality. Thus, for every i, we may choose a point q_i such that the points q_i are the vertices of a regular $(2k + 1)$-gon (and have equal z-coordinates), $||p_j - q_i|| \leq 1$ with equality if and only if $j = i \pm k$. Furthermore, if the points q_i are sufficiently close to q, then for the point r' on the negative half of the z-axis that satisfies $||q_i - r|| = 1$, we have $||p_i - r'|| < 1$.

Now, to obtain the required p-Mycielskian, we start with the wheel graph W_{2k+2}. This can be realized as the vertex set V_1 of a pyramid, with a regular $(2k + 1)$-gon of diameter one as its base, and with the property that the distance of its apex from any other vertex is one. To obtain a p-Mycielskian, we may apply the procedure described in the first two paragraphs $(p - 1)$ times. $\square$

Remark 3.6. *Let S be a point set with $G_S = \mu_p(C_{2k+1})$. Then $a(S) = 4$ for every p and k. Hence, since for $k \geq 2$ and $p > 1$ $\mu_p(C_{2k+1})$ is triangle-free, it is not a planar graph. On the other hand, it is easy to see that the number of edges in G_S is equal to $2 \operatorname{card} S - 2$. Thus, these sets form an infinite family of nonplanar Vázsonyi-critical graphs.*

Remark 3.7. *Clearly, if $p = 0$, then we have $a_k(S) = k + \chi_k(C_{2m+1}) = 3k + \lceil \frac{k}{m} \rceil$. By [138], for $p = 1$, we have*

$$a_k(S) = \begin{cases} 4 & \text{if } k = 1, \\ \frac{5k}{2} + 1 & \text{if } k \text{ is even}, \\ 2k + \frac{k+3}{2}, & \text{if } k \text{ is odd and } k \leq m \leq \frac{3k+3}{2}, \text{ and} \\ 2k + \frac{k+5}{2}, & \text{if } k \text{ is odd and } m \geq \frac{3k+5}{2}. \end{cases}$$

Theorem 3.9. *Let $S \subset \mathbb{R}^3$ be a finite set with $\mathcal{D}_{2m+1} \subseteq \operatorname{Sym}(S)$ for some $m \geq 2$. If $a(S) = 4$ and $g(G_S) = 3$, then G_S contains a topological wheel graph W_{2m+2} as a subgraph.*

In the proof, we use the following lemma.

Lemma 3.4. *If $S \subset \mathbb{R}^3$ is a finite set such that $\operatorname{Sym}(S)$ contains a reflection about the plane H, then every odd cycle of G_S has a vertex on H.*

Proof. Swanepoel [179, Theorem 2] showed that for any $S \subset \mathbb{R}^3$, G_S has a bipartite double cover, with a centrally symmetric drawing on $\mathbb{S}^2$: in this drawing, any point p is represented by a pair of antipodal points p_b and $p_r = -p_b$ that are colored differently, and a diameter of S, connecting p and q, corresponds to the two edges $p_b q_r$ and $p_r q_b$. In his construction, the point p_r representing p is an arbitrary relative interior point of the conic hull of the diameters of S starting at p. Using the geometric properties of the conic hulls of the diameters, he concluded that any two odd cycles of G_S, which are represented by centrally symmetric closed curves on $\mathbb{S}^2$, have a common vertex.

Now consider the plane H', parallel to H and containing o. We apply the construction of Swanepoel with a special choice of points. For any $p \in S$, let CH_p denote the conic hull of the diameters of S, starting at p. Then we choose $p_r \in \mathbb{S}^2$ as the projection of the center of mass of $\mathbf{B}^3 \cap CH_p$ on $\mathbb{S}^2$ from o. Clearly, p_r is on H' if, and only if p is on H.

Consider an odd cycle C in G_S. If its vertex set is symmetric about H, then H contains one of the vertices. Assume that C is not symmetric about H, and let C' denote its reflected copy about H. Clearly, the curves representing C and C' on $\mathbb{S}^2$ are symmetric about H', and thus, they intersect on H'. By Lemmas 1 and 2 of [179], these common points belong to common vertices of C and C', which yields that both cycles have a vertex on H. □

Proof of Theorem 3.9. Assume that $\operatorname{diam}(S) = 1$.

By Lemma 3.4, any odd cycle, and in particular any triangle, of G_S contains a point on each plane of symmetry in $\operatorname{Sym}(S)$. Since $\operatorname{Sym}(S)$ contains at least $2m + 1 \geq 5$ symmetry planes, any triangle T of G_S has a vertex on the axis L of the rotations of $\mathcal{D}_{2m+1}$. Clearly, this triangle T has at most two vertices on L. If T has exactly two vertices on L, then the diameter of the union of the rotated copies of T is stricly greater than one; a contradiction. Thus, we have that T has exactly one vertex on L, which we denote by a. Let the remanining two vertices of T be b and c. Let $b = b_1, b_2, \ldots, b_{2m+1}$, and $c = c_1, c_2, \ldots, c_{2m+1}$ denote the rotated copies of b and c, respectively, about L.

First, consider the case that the points b_i and c_j are pairwise distinct. Let C be a shortest odd cycle that does not contain a. Such a cycle exists, as otherwise $G \setminus \{a\}$ contains no odd cycle, and

$\chi(G) = 3$. Since any two odd cycles intersect, C contains at least one point from each pair $\{b_i, c_i\}$. Thus, the required subgraph is defined as the union of C, a, and for each i an edge connecting a to either b_i or c_i on C.

Finally, assume that from amongst the b_is and the c_js there are coinciding vertices. Note that since they are not on L, we have that $b_i = c_j$ for some i and j. But then $\{b_1, b_2, \ldots, b_{2m+1}\} = \{c_1, c_2, \ldots, c_{2m+1}\}$, and these vertices, and a, are the vertices of a subgraph W_{2m+2}. □

Corollary 3.1. *Let $S \subset \mathbb{R}^3$ be a finite set with $\mathcal{D}_{2m+1} \subseteq \mathrm{Sym}(S)$ for some $m \geq 2$. If $a(S) = 4$ and G_S is a plane graph, then G_S contains a topological wheel graph W_{2m+2} as a subgraph.*

Problem 3.3. *Is it true that if $S \subset \mathbb{R}^3$ is a finite set with $\mathcal{D}_{2m+1} \subseteq \mathrm{Sym}(S)$ for some $m \geq 2$, and with $a(S) = 4$, then G_S contains $\mu_p(C_{2t+1})$ as a subgraph, for some p and t satisfying $(2m+1)|(2t+1)$? If the answer is negative, is it true for Vázsonyi-critical graphs?*

3.5 An additional remark

Let $a_k(d)$ denote the maximum of the Borsuk numbers of d-dimensional sets of positive diameter, and let $a(d) = a_1(d)$. One of the fundamental questions regarding Borsuk's problem is to determine the asymptotic behavior of $a(d)$.

Presently, the best known asymptotic lower bound for $a(d)$ is due to Kahn and Kalai [121], who proved that for sufficiently large values of d, $a(d) \geq 1.2^{\sqrt{d}}$: they constructed a finite d-dimensional set S with the property that the independence number of its diameter graph is not greater than $\frac{\mathrm{card}\, S}{1.2^{\sqrt{d}}}$, if d is sufficiently large. Clearly, this property implies not only that $a(S) \geq 1.2^{\sqrt{d}}$ for large values of n, but also that $a_{frac}(S) \geq 1.2^{\sqrt{d}}$. Thus, we have the following.

Remark 3.8. *If d is sufficiently large, then for every value of k, we have $a_k(d) \geq k 1.2^{\sqrt{d}}$.*

We state the next problem from [114].

Problem 3.4. *Is it true that for every value of k and d, we have $a_k(d) = k a(d)$? If not, do the two sides have the same magnitude?*

Chapter 4

Non-separable families of positive homothets

In 1945 A.W. Goodman and R.E. Goodman [91] proved a conjecture of Erdős on non-separable families of circular disks in $\mathbb{R}^2$. In order to state that result we need the following definition.

Definition 4.1. *Let K be a convex body in $\mathbb{R}^d$ and let $\mathcal{G} = \{x_i + \tau_i K \ : \ x_i \in \mathbb{R}^d, \tau_i > 0, i = 1, 2, \ldots, n\}$, where $d \geq 2$ and $n \geq 2$. Assume that $\mathcal{G}$ is a* non-separable family *in short, an* NS-family*, meaning that every hyperplane intersecting* $\operatorname{conv}(\bigcup \mathcal{G})$ *intersects a member of $\mathcal{G}$ in $\mathbb{R}^d$, i.e., there is no hyperplane disjoint from $\bigcup \mathcal{G}$ that strictly separates some elements of $\mathcal{G}$ from all the other elements of $\mathcal{G}$ in $\mathbb{R}^d$. Then, let $\lambda(\mathcal{G}) > 0$ denote the smallest positive value λ such that a translate of $\lambda \left(\sum_{i=1}^{n} \tau_i \right) K$ covers $\bigcup \mathcal{G}$.*

If $F : \mathbb{R}^d \to \mathbb{R}^d$ is an arbitrary invertible affine map, then it is straightforward to see that $F(\mathcal{G}) = \{F(x_i + \tau_i K) \ : \ x_i \in \mathbb{R}^d, \tau_i > 0, i = 1, 2, \ldots, n\}$ is an NS-family of convex bodies homothetic to $F(K)$ with homothety ratios $\tau_1, \tau_2, \ldots, \tau_n$ such that $\lambda(\mathcal{G}) = \lambda(F(\mathcal{G}))$. Based on this property we can restate the main result of [91] as follows.

Theorem 4.1 (Goodman-Goodman, 1945)**.** *If $\mathcal{C}$ is an arbitrary NS-family of finitely many homothetic ellipses in $\mathbb{R}^2$, then $\lambda(\mathcal{C}) \leq 1$.*

For a completely different proof of Theorem 4.1 we refer the interested reader to Theorem 6.1 and its proof in [27]. On the other hand, on page 498 of [91] A.W. Goodman and R E. Goodman put forward the following conjecture.

Conjecture 4.1 (Goodman-Goodman, 1945)**.** *For every convex body K in $\mathbb{R}^d$ and every NS-family $\mathcal{G} = \{x_i + \tau_i K \ : \ x_i \in \mathbb{R}^d, \tau_i > 0, i = 1, 2, \ldots, n\}$ the inequality $\lambda(\mathcal{G}) \leq 1$ holds for all $d \geq 2$ and $n \geq 2$.*

We note that it is straightforward to prove Conjecture 4.1 for $n = \operatorname{card} \mathcal{G} = 2$ and $d \geq 2$. However, in what follows we give a counterexample to Conjecture 4.1 for all $\operatorname{card} \mathcal{G} \geq 3$ and $d \geq 2$. Furthermore, we give a sharp upper bound on $\lambda(\mathcal{G})$ for all NS-families $\mathcal{G}$ with $\operatorname{card} \mathcal{G} = 3$ in $\mathbb{R}^2$. Next we show that $\lambda(\mathcal{G}) \leq d$ holds for all $d \geq 2$. Then generalizing Theorem 4.1 we prove Conjecture 4.1 for all centrally symmetric convex bodies K in $\mathbb{R}^d$, $d \geq 2$. Moreover, we prove Conjecture 4.1 for natural subfamilies of the family of NS-families namely, for *k-impassable families* in short, *k-IP-families* of positive homothetic convex bodies in $\mathbb{R}^d$ whenever $0 \leq k \leq d-2$.

We conclude this section by inviting the interested reader to further investigate the basic problem of Goodman-Goodman rephrased as follows.

Problem 4.1. *Find $\sup_{\mathcal{G}} \lambda(\mathcal{G})$ for any given $d \geq 2$, where $\mathcal{G}$ runs over the NS-families of finitely many positive homothetic copies of an arbitrary convex body K in $\mathbb{R}^d$. In particular, is there an absolute constant $c > 0$ such that $\sup_{\mathcal{G}} \lambda(\mathcal{G}) \leq c$ holds for all $d \geq 2$?*

The results opresented in this chapter imply that $\frac{2}{3} + \frac{2}{3\sqrt{3}} = 1.0515\ldots \leq \sup_{\mathcal{G}} \lambda(\mathcal{K}) \leq d$ for all $d \geq 2$.

4.1 Counterexample to Conjecture 4.1 for $\operatorname{card} \mathcal{G} \geq 3$ in $\mathbb{R}^d$, $d \geq 2$

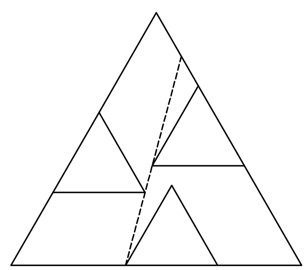

Figure 4.1: A counterexample in the plane for three triangles

Example 4.1. *Place three regular triangles $\mathcal{T} = \{T_1, T_2, T_3\}$ of unit side lengths into a regular triangle T of side length $2 + \frac{2}{\sqrt{3}} = 3.154700\ldots > 3$ such that*

- *each side of T contains a side of T_i, for $i = 1, 2, 3$, respectively (cf. Figure 4.1),*
- *for $i = 1, 2, 3$, the vertices of T_i contained in a side of T divide this side into three segments of lengths $\frac{2}{3} + \frac{1}{\sqrt{3}}$, 1, and $\frac{1}{3} + \frac{1}{\sqrt{3}}$, in counter-clockwise order.*

A simple computation yields that the convex hull of any two of the small triangles touches the third triangle, and hence, no two of them can be strictly separated from the third one. Thus, $\lambda(\mathcal{T}) = \frac{2}{3} + \frac{2}{3\sqrt{3}} = 1.0515\ldots > 1$ and $\mathcal{T}$ is a counterexample to Conjecture 4.1 for $n = 3$ in $\mathbb{R}^2$.

Remark 4.1. *Note that for any value $n \geq 4$, placing $n-3$ sufficiently small triangles inside T in Example 4.1 yields a counterexample to Conjecture 4.1 for n positive homothetic copies of a triangle in $\mathbb{R}^2$. Furthermore, we can extend Example 4.1 using $n-3$ translates of the small triangles of Example 4.1, as in Figure 4.2. As the small triangles have unit side length, the smallest* *triangle containing them has side length* $n - 1 + \frac{2}{\sqrt{3}} > n$ *and so, we get another counterexample to Conjecture 4.1 using* n *translates of a triangle in* $\mathbb{R}^2$.

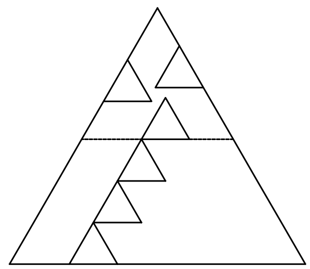

Figure 4.2: A counterexample in the plane for n triangles

Remark 4.2. *Let* $d \geq 2$*, and let* λ_s^d *denote the supremum of* $\lambda(\mathcal{G})$*, where* $\mathcal{G}$ *runs over the NS-families of finitely many positive homothetic* d*-simplices in* $\mathbb{R}^d$*. Then* λ_s^d *is a non-decreasing sequence of* d*.*

Proof. Let $\mathcal{G}$ be an NS-family of finitely many $(d-1)$-simplices in a hyperplane of $\mathbb{R}^d$. Clearly, we can extend the elements of $\mathcal{G}$ to homothetic d-simplices such that each element is a facet of its extension. Then, denoting this extended family by $\mathcal{G}'$, we have $\lambda(\mathcal{G}) = \lambda(\mathcal{G}')$, and $\mathcal{G}'$ is an NS-family. □

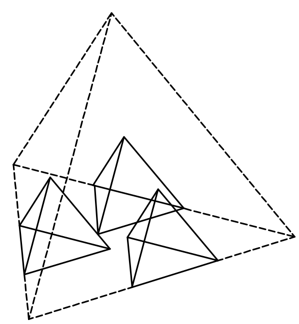

Figure 4.3: A counterexample in $\mathbb{R}^3$ for three tetrahedra

Figure 4.3 shows how to extend the configuration in Example 4.1 to $\mathbb{R}^3$, implying that $\lambda_s^d \geq \lambda_s^2 \geq \frac{2}{3} + \frac{2}{3\sqrt{3}} = 1.0515\ldots > 1$ for all $d \geq 3$.

Remark 4.3. *In fact,* $\lambda_s^d = \sup_{\mathcal{G}} \lambda(\mathcal{G})$ *for all* $d \geq 2$*, where* $\mathcal{G}$ *runs over the NS-families of finitely many positive homothetic copies of an arbitrary convex body* K *in* $\mathbb{R}^d$*.*

Proof. Clearly, $\lambda_s^d \leq \sup_{\mathcal{G}} \lambda(\mathcal{G})$. So, it is sufficient to show that for every convex body K in $\mathbb{R}^d$ and every NS-family $\mathcal{G} = \{x_i + \tau_i K \;:\; x_i \in \mathbb{R}^d, \tau_i > 0, i = 1, 2, \ldots, n\}$ a translate of $\lambda_s^d \left(\sum_{i=1}^{n} \tau_i \right) K$

covers $\bigcup \mathcal{G}$. Now, according to Lutwak's containment theorem [141] if K_1 and K_2 are convex bodies in $\mathbb{R}^d$ such that every circumscribed simplex of K_2 has a translate that covers K_1, then K_2 has a translate that covers K_1. (Here a circumscribed simplex of K_2 means a d-simplex of $\mathbb{R}^d$ that contains K_2 such that each facet of the d-simplex meets K_2.) Thus, if $\Delta(K)$ is a circumscribed simplex of K, then $\lambda_s^d \left(\sum_{i=1}^{n} \tau_i \right) \Delta(K)$ is a circumscribed simplex of $\lambda_s^d \left(\sum_{i=1}^{n} \tau_i \right) K$ and $x_i + \tau_i \Delta(K)$ is a circumscribed simplex of $x_i + \tau_i K$ for all $i = 1, 2, \ldots, n$. Furthermore, $\{x_i + \tau_i \Delta(K) \ : \ x_i \in \mathbb{R}^d, \tau_i > 0, i = 1, 2, \ldots, n\}$ is an NS-family and therefore $\lambda_s^d \left(\sum_{i=1}^{n} \tau_i \right) \Delta(K)$ has a translate that covers $\bigcup \{x_i + \tau_i \Delta(K) \ : \ x_i \in \mathbb{R}^d, \tau_i > 0, i = 1, 2, \ldots, n\} \supseteq \bigcup \mathcal{G}$, which completes our proof via Lutwak's containment theorem. □

4.2 Upper bounding $\lambda(\mathcal{G})$ for $\operatorname{card} \mathcal{G} = 3$ in $\mathbb{R}^2$

Theorem 4.2. *Let K be a convex body in $\mathbb{R}^2$. If $\mathcal{G}$ is an NS-family of three positive homothetic copies of K, with homothety ratios τ_1, τ_2, τ_3, respectively, then there is a translate of $\left(\frac{2}{3} + \frac{2}{3\sqrt{3}} \right) \left(\sum_{i=1}^{3} \tau_i \right) K$ containing $\mathcal{G}$. Furthermore, the constant $\frac{2}{3} + \frac{2}{3\sqrt{3}} = 1.0515\ldots$ is the smallest real number with this property.*

Proof. For simplicity, we set $\mu = \frac{2}{3} + \frac{2}{3\sqrt{3}}$.

First, note that by Example 4.1, there is a suitable 3-element NS-family $\mathcal{G}$ satisfying $\lambda(\mathcal{G}) = \mu$. Thus, we need only to show that for any 3-element NS-family, we have $\lambda(\mathcal{G}) \leq \mu$. On the other hand, using Lutwak's containment theorem in the same way as in Remark 4.3 we get that it is sufficient to prove Theorem 4.2 when K is a triangle T. Furthermore, as homothety ratios do not change under affine transformations, we may assume that T is a regular triangle. So, let $\mathcal{G} = \{T_1, T_2, T_3\}$ such that for $i = 1, 2, 3$, the homothety ratio of T_i is τ_i. Let $\overline{T} = \mu' (\sum_{i=1}^{3} \tau_i) T$ be the smallest positive homothetic copy of T that contains $\mathcal{G}$. Without loss of generality, we may assume that $\mu' > 1$.

Since $\mu' > 1$, it is easy to see that no two of T_1, T_2, and T_3 intersect. For $i = 1, 2, 3$, let p_i and q_i denote, respectively, the vertices of T_i on $\operatorname{bd} \overline{T}$, in counter-clockwise order, and let r_i denote the vertex of T_i in the interior of $\overline{T}$ (cf. Figure 4.4). Without loss of generality, we may assume that the closed line segment $[p_3, r_1]$ intersects T_2. Since $\mu' > 1$, from this it follows that $[p_1, r_2]$ intersects T_3, and that $[p_2, r_3]$ intersects T_1. Clearly, if each intersection contains more than one point, then, moving the triangles a little apart we obtain an NS-family of three homothetic copies of T that are contained only in a larger homothetic copy of T. We show that the same holds if at least one intersection contains more than one point. Assume, for example, that r_3 is contained in the interior of $\operatorname{conv}(T_1 \cup T_2)$. Then, slightly moving T_3 or T_1 as well as T_2 such that $\overline{T}$ is fixed, we obtain a configuration where each intersection contains more than one point. Thus, in the following we assume that each of the triplets $\{p_1, r_3, r_2\}$, $\{p_2, r_1, r_3\}$, and $\{p_3, r_2, r_1\}$ are collinear.

For $i = 1, 2, 3$ let p_i and q_i divide the corresponding side of $\overline{T}$ into segments of length x_i, t_i, and y_i, in counter-clockwise order, and assume that $x_i + t_i + y_i = 1$. Then the collinearity of the three triplets yields that $t_3 x_2 + x_1 y_3 - x_1 x_2 - y_2 y_3 = t_1 x_3 + x_2 y_1 - x_2 x_3 - y_1 y_3 = t_2 x_1 + x_3 y_2 - x_1 x_3 - y_1 y_2 = 0$. Now, for the nine variables x_i, t_i, y_i, $i = 1, 2, 3$ we have six conditions, and need to determine the minimum of the value of $f(t_1, t_2, t_3) = t_1 + t_2 + t_3$ on the interval $(0, 1)$. Eliminating some of the variables, we may apply Lagrange's method with the remaining conditions to obtain the critical

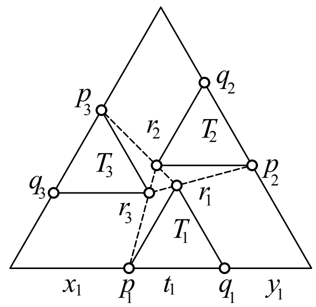

Figure 4.4: An illustration for the proof of Theorem 4.2

points of the expression $t_1+t_2+t_3$. We carried out this computation with a Maple 18.0 mathematical program, which provided five solutions. (The command list we used can be found in Section 4.6). A more careful analysis showed that two of the solutions were geometrically invalid (i.e., some of the parameters were outside the permitted range $(0,1)$). A third solution corresponded to the value $t_1+t_2+t_3=1$, which yields $\mu'=1$. The other two solutions were the following: $x_i = \frac{1}{4} + \frac{1}{4\sqrt{3}}$ and $t_i = \frac{3-\sqrt{3}}{4}$ for $i=1,2,3$, and $x_i = \frac{1}{4} - \frac{1}{4\sqrt{3}}$ and $t_i = \frac{3+\sqrt{3}}{4}$ for $i=1,2,3$. The values of $\mu' = \frac{1}{t_1+t_2+t_3}$ at these points are $\frac{2}{3} + \frac{2}{3\sqrt{3}}$ and $\frac{2}{3} - \frac{2}{3\sqrt{3}}$, respectively, from which the assertion readily follows. $\square$

Now, it is natural to ask the following.

Problem 4.2. *Let K be a convex body in $\mathbb{R}^2$. If $\mathcal{G}$ is an NS-family of k positive homothetic copies of K, with homothety ratios $\tau_1, \tau_2, \ldots, \tau_k$, respectively, and with $k \geq 4$, then prove or disprove that there is a translate of $\left(\frac{2}{3} + \frac{2}{3\sqrt{3}}\right)\left(\sum_{i=1}^{k} \tau_i\right) K$ containing $\mathcal{G}$.*

We note that an NS-family of (a large number of) convex bodies does not need to have NS-subfamilies as shown by the following example, which indicates the rather intricate nature of Problem 4.2.

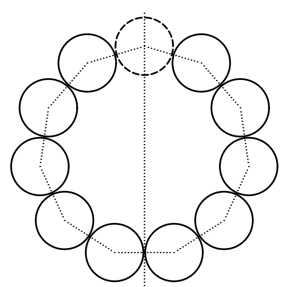

Figure 4.5: An illustration for Example 4.2: removing the dashed circle, the dotted line strictly separates the remaining circles

Example 4.2. *Let $k \geq 1$ and consider a regular $(2k+1)$-gon in $\mathbb{R}^2$. Place congruent closed circular disks at the vertices of the polygon as centers, and choose their radius such that they are disjoint, but consecutive circular disks 'almost' touch (cf. Figure 4.5). Then they form an NS-family of circular disks. On the other hand, removing an arbitrary subfamily of at most $2k-1$ circular disks, the remaining (at least two) circular disks can be strictly separated by a line.*

4.3 Upper bounding $\lambda(\mathcal{G})$ in $\mathbb{R}^d$, $d \geq 2$

Theorem 4.3. *If $\mathcal{G} = \{x_i + \tau_i K : x_i \in \mathbb{R}^d, \tau_i > 0, i = 1, 2, \ldots, n\}$ is an arbitrary NS-family of positive homothetic copies of the convex body K in $\mathbb{R}^d$, then*

$$\lambda(\mathcal{G}) \leq d$$

holds for all $n \geq 2$ and $d \geq 2$.

For the proof of Theorem 4.3 we need the following simple observations. For the first statement recall from Chapter 2 that if $u \in \mathbb{S}^{d-1}$ and Q is a convex body in $\mathbb{R}^d$, then the width of Q in the direction u is labelled by $\mathrm{width}_{\mathbf{u}}(Q)$ and it is equal to $\max\{\langle x, u\rangle : x \in Q\} - \min\{\langle x, u\rangle : x \in Q\}$.

Lemma 4.1. *Let $\mathcal{G} = \{x_i + \tau_i K : x_i \in \mathbb{R}^d, \tau_i > 0, i = 1, 2, \ldots, n\}$ be an arbitrary NS-family of positive homothetic copies of the convex body K in $\mathbb{R}^d, d \geq 2$. Then for any $u \in \mathbb{S}^{d-1}$ we have $\mathrm{width}_u \left(\mathrm{conv}\left(\bigcup \mathcal{G}\right)\right) \leq \left(\sum_{i=1}^n \tau_i\right) \mathrm{width}_u(K)$.*

Proof. As $\mathcal{G}$ is an NS-family, the orthogonal projection of $\mathrm{conv}(\bigcup \mathcal{G})$ onto a line L parallel to u is a line segment of length $\mathrm{width}_u\left(\mathrm{conv}\left(\bigcup \mathcal{G}\right)\right)$ which must be covered by the orthogonal projections of the convex bodies $x_i + \tau_i K, i = 1, 2, \ldots, n$ onto L having lengths equal to $\mathrm{width}_u(x_i + \tau_i K) = \tau_i \,\mathrm{width}_u(K), i = 1, 2, \ldots, n$. This yields the assertion. □

The following statement is well known. For the convenience of the reader we include its simple proof.

Lemma 4.2. *For any convex body Q in $\mathbb{R}^d$, there exist $x, y \in \mathbb{R}^d$ such that $x + \frac{2}{d+1}Q_0 \subseteq Q \subseteq y + \frac{2d}{d+1}Q_0$, where $Q_0 = \frac{1}{2}(Q - Q)$ is the central symmetral of Q with respect to the origin o in $\mathbb{R}^d$.*

Proof. Without loss of generality, let o be the center of a largest volume simplex inscribed in Q. Then, clearly, $Q \subseteq -dQ$. From this it follows that $(d+1)Q \subseteq d(Q - Q) = 2dQ_0$. Thus, $Q \subseteq \frac{2d}{d+1}Q_0$. On the other hand, as $-Q \subseteq dQ$ therefore $2Q_0 = Q - Q \subseteq (d+1)Q$. Hence, $\frac{2}{d+1}Q_0 \subseteq Q$. □

Proof of Theorem 4.3. Let $M = \mathrm{conv}\left(\bigcup \mathcal{G}\right)$. By Lemma 4.1 we get that the inequality $\mathrm{width}_u(M) \leq \left(\sum_{i=1}^n \tau_i\right) \mathrm{width}_u(K)$ holds for every $u \in \mathbb{S}^{d-1}$. As central symmetrization does not change the width of a convex body in any direction, $M_0 \subseteq \left(\sum_{i=1}^n \tau_i\right) K_0$, where M_0 and K_0 denotes the central symmetrals of M and K, respectively. Finally, by Lemma 4.2 there exist $m, k \in \mathbb{R}^d$ such that

$$M \subseteq m + \frac{2d}{d+1}M_0 \subseteq m + \frac{2d}{d+1}\left(\sum_{i=1}^n \tau_i\right) K_0 \subseteq m + \frac{2d}{d+1}\left(\sum_{i=1}^n \tau_i\right)\left(\frac{d+1}{2}(k + K)\right)$$

$$= \left(m + d \left(\sum_{i=1}^{n} \tau_i \right) k \right) + d \left(\sum_{i=1}^{n} \tau_i \right) K.$$

□

Remark 4.4. *From the above proof of Theorem 4.3, it follows that if K is an o-symmetric convex body in $\mathbb{R}^d$, i.e., $K = K_0$, then $\lambda(\mathcal{G}) \leq \frac{2d}{d+1}$. In the next section, using a completely different approach, we give an optimal improvement on this inequality.*

4.4 A proof of Conjecture 4.1 for centrally symmetric convex bodies in $\mathbb{R}^d$, $d \geq 2$

In this section we extend Theorem 4.1 to finite dimensional normed spaces by proving Conjecture 4.1 for all centrally symmetric convex bodies K in $\mathbb{R}^d$, $d \geq 2$.

Theorem 4.4. *For every o-symmetric convex body K_0 and every NS-family $\mathcal{G} = \{x_i + \tau_i K_0 \ : \ x_i \in \mathbb{R}^d, \tau_i > 0, i = 1, 2, \ldots, n\}$ the inequality $\lambda(\mathcal{G}) \leq 1$ holds for all $d \geq 2$ and $n \geq 2$.*

Our proof is a close variant of the proof of Theorem 1 by Goodman-Goodman in [91] and it is based on the following lemma, which is a strengthened version of Lemma 2 of [91].

Lemma 4.3. *Let $\mathcal{G} = \{[x_i - \tau_i, x_i + \tau_i] : \tau_i > 0, i = 1, 2, \ldots, n\}$ be a family of closed intervals in $\mathbb{R}$ such that $\bigcup \mathcal{G}$ is a single closed interval in $\mathbb{R}$. Let $x = \frac{\sum_{i=1}^{n} \tau_i x_i}{\sum_{i=1}^{n} \tau_i}$. Then the interval*

$$\left[x - \sum_{i=1}^{n} \tau_i, x + \sum_{i=1}^{n} \tau_i \right]$$

covers $\bigcup \mathcal{G}$.

Proof. We prove that the left endpoint of $\bigcup \mathcal{G}$ is covered by the interval $[x - \sum_{i=1}^{n} \tau_i, x + \sum_{i=1}^{n} \tau_i]$; to prove that the right endpoint is covered, we may apply a similar argument.

Without loss of generality, we may assume that the left endpoint of the interval $\bigcup_{i=1}^{n} [x_i - \tau_i, x_i + \tau_i]$ is the origin o, moreover, that the numbering of the intervals $[x_i - \tau_i, x_i + \tau_i]$, $i = 1, 2, \ldots, n$ is in the order in which the left endpoints of these intervals occur on $\mathbb{R}$ moving from left to right (in increasing order). Based on this it is clear that

$$x_i \leq 2\tau_1 + \ldots + 2\tau_{i-1} + \tau_i,$$

where, for convenience we set $\tau_0 = 0$, and therefore

$$x = \frac{\sum_{i=1}^{n} x_i \tau_i}{\sum_{i=1}^{n} \tau_i} \leq \frac{\sum_{i=1}^{n} (2\tau_1 + \ldots + 2\tau_{i-1} + \tau_i)\tau_i}{\sum_{i=1}^{n} \tau_i} = \frac{(\sum_{i=1}^{n} \tau_i)^2}{\sum_{i=1}^{n} \tau_i} = \sum_{i=1}^{n} \tau_i,$$

showing that o is indeed covered by the interval $[x - \sum_{i=1}^{n} \tau_i, x + \sum_{i=1}^{n} \tau_i]$. □

Proof of Theorem 4.4. Let $x = \frac{\sum_{i=1}^{n} \tau_i x_i}{\sum_{i=1}^{n} \tau_i}$, and set $K' = x + \left(\sum_{i=1}^{n} \tau_i\right) K_0$. We prove that K' covers $\bigcup \mathcal{G}$.

For any line L through the origin o, let $\mathrm{proj}_L : \mathbb{R}^d \to L$ denote the orthogonal projection onto L, and let $h_{\mathcal{G}} : \mathbb{S}^{d-1} \to \mathbb{R}$ and $h_{K'} : \mathbb{S}^{d-1} \to \mathbb{R}$ denote the support functions of $\mathrm{conv}\left(\bigcup \mathcal{G}\right)$ and K', respectively. Then $\mathrm{proj}_L\left(\bigcup \mathcal{G}\right)$ is a single interval, which, by Lemma 4.3, is covered by $\mathrm{proj}_L(K')$. Thus, for any $u \in \mathbb{S}^{d-1}$, we have that $h_{\mathcal{G}}(u) \leq h_{K'}(u)$, which readily implies that $\bigcup \mathcal{G} \subseteq K'$. □

Remark 4.5. *If the positive homothetic convex bodies of* $\mathcal{G} = \{x_i + \tau_i K_0 \ : \ x_i \in \mathbb{R}^d, \tau_i > 0, i = 1, 2, \ldots, n\}$ *have pairwise disjoint interiors with their centers* $\{x_i \ : \ i = 1, 2, \ldots, n\}$ *lying on a line* L *in* $\mathbb{R}^d$ *such that the consecutive elements of* $\mathcal{G}$ *along* L *touch each other, then* $\mathcal{G}$ *is an NS-family with* $\lambda(\mathcal{G}) = 1$.

4.5 A proof of Conjecture 4.1 for k-impassable families in $\mathbb{R}^d$ whenever $0 \leq k \leq d-2$

In this section we prove Conjecture 4.1 for the following natural subfamilies of the family of NS-families, which following G. Fejes Tóth and W. Kuperberg [70] we call *k-impassable families*, or in short, *k-IP-families* of (positive homothetic) convex bodies in $\mathbb{R}^d$, where $0 \leq k \leq d-2$.

Definition 4.2. *Let* $\mathcal{G} = \{x_i + \tau_i K : x_i \in \mathbb{R}^d, \tau_i > 0, i = 1, 2, \ldots, n\}$ *be a family of positive homothetic copies of the convex body* K *in* $\mathbb{R}^d$ *and let* $0 \leq k \leq d-1$. *We say that* $\mathcal{G}$ *is a* k-impassable arrangement, *in short, a* k-IP-family *if every k-dimensional affine subspace of* $\mathbb{R}^d$ *intersecting* $\mathrm{conv}\left(\bigcup \mathcal{G}\right)$ *intersects an element of* $\mathcal{G}$. *Let* $\lambda_k(\mathcal{G}) > 0$ *denote the smallest positive value* λ *such that some translate of* $\lambda \left(\sum_{i=1}^{n} \tau_i\right) K$ *covers* $\bigcup \mathcal{G}$, *where* $\mathcal{G}$ *is k-IP-family. A* $(d-1)$-IP-family *is simply called an* NS-family *and in that case* $\lambda_{d-1}(\mathcal{G}) = \lambda(\mathcal{G})$.

In order to state our theorem on k-IP-families of (positive homothetic) convex bodies in $\mathbb{R}^d$ we need the following definitions and statement from Section 3.2 of [169].

Definition 4.3. *Let* K_1' *and* K_2' *be convex bodies in* $\mathbb{R}^d, d \geq 2$. *We say that* K_2' *is a* summand *of* K_1' *if there exists a convex body* K' *in* $\mathbb{R}^d$ *such that* $K_2' + K' = K_1'$.

Definition 4.4. *Let* K_1' *and* K_2' *be convex bodies in* $\mathbb{R}^d, d \geq 2$. *We say that* K_2' slides freely *inside* K_1' *if to each boundary point* x *of* K_1' *there exists a translation vector* $y \in \mathbb{R}^d$ *such that* $x \in y + K_2' \subseteq K_1'$.

Lemma 4.4. *Let* K_1' *and* K_2' *be convex bodies in* $\mathbb{R}^d, d \geq 2$. *Then* K_2' *is a summand of* K_1' *if and only if* K_2' *slides freely inside* K_1'.

We prove Conjecture 4.1 for k-IP-families of (positive homothetic) convex bodies in $\mathbb{R}^d$ for all $0 \leq k \leq d-2$ in the following strong sense.

Theorem 4.5. *Let* K *be a d-dimensional convex body and* $\mathcal{G} = \{K_i = x_i + \tau_i K : x_i \in \mathbb{R}^d, \tau_i > 0, i = 1, 2, \ldots, n\}$ *be a k-IP family of positive homothetic copies of* K *in* $\mathbb{R}^d$, *where* $0 \leq k \leq d-2$. *Then* $\mathrm{conv}\left(\bigcup \mathcal{G}\right)$ *slides freely in* $\left(\sum_{i=1}^{n} \tau_i\right) K$ *(i.e.,* $\mathrm{conv}\left(\bigcup \mathcal{G}\right)$ *is a summand of* $\left(\sum_{i=1}^{n} \tau_i\right) K$*) and therefore* $\lambda_k(\mathcal{G}) \leq 1$.

Proof. Set $\overline{K} = \left(\sum_{i=1}^{n} \tau_i\right) K$. First we prove the assertion for $d = 2$. Then $k = 0$ and thus, $\bigcup \mathcal{G}$ is convex.

Assume that K is a polygon. Then any side of $\bigcup \mathcal{G}$ has the same external normal vector as some side S of K. Furthermore, the length of this side is at most $\left(\sum_{i=1}^{n} \tau_i\right) |S|$, where $|S|$ denotes the length of S. Hence, we may apply a theorem of Alexandrov [4], that states that under this condition there is a translate of $\bigcup \mathcal{G}$ contained in $\overline{K}$. In addition, as is remarked in [18], a translation that translates a side of $\bigcup \mathcal{G}$ into the corresponding side of $\overline{K}$ translates also $\bigcup \mathcal{G}$ into $\overline{K}$. This yields that $\bigcup \mathcal{G}$ slides freely in $\overline{K}$, or, in particular, that $\lambda_0(\mathcal{G}) \leq 1$.

Now we consider a general plane convex body K. First, we show that for any $i \neq j$, the intersection $(K_i \cap K_j) \cap (\operatorname{bd} \bigcup \mathcal{G})$ consists of at most two segments. Indeed, let x be a point of $K_i \cap K_j$ in $\operatorname{bd} \bigcup \mathcal{G}$, and let L_x be a supporting line of $\bigcup \mathcal{G}$ at x. If K_i and K_j are translates, then the vector translating K_i into K_j is parallel to L_x, and, clearly, in this case every other point of $K_i \cap K_j$ and $\operatorname{bd} \bigcup \mathcal{G}$ is contained in either L_x, or the supporting line of $\bigcup \mathcal{G}$ parallel to L_x. If K_i and K_j are homothetic copies, then the center of the homothety transforming K_i into K_j lies on L_x. Thus, the claim follows from the fact that there is no point of $\mathbb{R}^2$ lying on three distinct supporting lines of $\bigcup \mathcal{G}$.

Let x be an arbitrary boundary point of $\overline{K}$. We show that there is vector $y \in \mathbb{R}^2$ such that $x \in y + \bigcup \mathcal{G} \subseteq \overline{K}$. Without loss of generality, we may assume that x is the origin of $\mathbb{R}^2$, implying, in particular, that $x \in \operatorname{bd} K$.

Let L_t, $t = 1, 2, \ldots, m$ be supporting lines of $\bigcup \mathcal{G}$, in counterclockwise order in $\operatorname{bd} \bigcup \mathcal{G}$ that contain the points of $\operatorname{bd} \bigcup \mathcal{G}$ belonging to more than one member of $\mathcal{G}$. Let Γ_t denote the arc of $\operatorname{bd} \bigcup \mathcal{G}$ between L_t and L_{t+1} for every value of t. Since every member of $\mathcal{G}$ intersects $\operatorname{bd} \bigcup \mathcal{G}$ in a closed set, there is a value i_t of i such that Γ_t belongs to K_{i_t}.

For any positive integer s, let Q^s be a convex polygon obtained as the intersection of some (closed) supporting halfplanes of $\bigcup \mathcal{G}$, including those bounded by the L_t's. Similarly, let P^s be the convex polygon, circumscribed about K, bounded by those translates of the supporting halfplanes defining Q^s that support K. Since we can choose arbitrarily many supporting halfplanes of $\bigcup \mathcal{G}$, we may assume that $P^s \subseteq (1 + 1/s)K$ and that a sideline L of P^s contains x. Note that then L is a supporting line of $\overline{K}$ as well. Finally, we set $P_i^s = x_i + \tau_i P^s$. Then $Q^s = \bigcup_{i=1}^n P_i^s$. Indeed, any point of $Q^s \setminus \bigcup \mathcal{G}$ is contained in a region bounded by two sidelines of Q^s and an arc Γ_t for some value of t, which is covered by $P_{i_t}^s$. Thus, we may apply our result for polygons, which yields that any translation transferring the side of Q^s associated to L into the side of $(\sum_{i=1}^n \tau_i) P^s$ contained in L, transfers also Q^s into $(\sum_{i=1}^n \tau_i) P^s \subseteq (1 + 1/s)\overline{K}$. Let y_s be an associated translation vector such that $x \in y_s + \bigcup \mathcal{G} \subseteq y_s + Q^s$. Choosing the limit y of a convergent subsequence of these vectors we have that $x \in y + \bigcup \mathcal{G} \subseteq \overline{K}$, implying that $\bigcup \mathcal{G}$ slides freely in $\overline{K}$ and therefore $\lambda_0(\mathcal{G}) \leq 1$.

Now we prove Theorem 4.5 for $d > 2$. For any 2-dimensional linear subspace H of $\mathbb{R}^d$ let $\operatorname{proj}_H : \mathbb{R}^d \to H$ denote the orthogonal projection onto H. As $\operatorname{proj}_H (\operatorname{conv}(\bigcup \mathcal{G})) = \operatorname{conv}(\bigcup_{i=1}^n \operatorname{proj}_H(K_i))$ therefore $\{\operatorname{proj}_H K_i : i = 1, 2, \ldots, n\}$ is a 0-IP family in H. Thus, the above proof of Theorem 4.5 for $d = 2$ and Lemma 4.4 imply that $\bigcup_{i=1}^n \operatorname{proj}_H(K_i) = \operatorname{conv}(\bigcup_{i=1}^n \operatorname{proj}_H(K_i)) = \operatorname{proj}_H (\operatorname{conv}(\bigcup \mathcal{G}))$ is a summand of $\operatorname{proj}_H \overline{K}$. Now we apply Lemma 3.2.6 from [169], which states that if K_1' and K_2' are two convex bodies in $\mathbb{R}^d$, and $\operatorname{proj}_H K_1'$ is a summand of $\operatorname{proj}_H K_2'$ for all 2-dimensional linear subspaces H in $\mathbb{R}^d$, then K_1' is a summand of K_2'. Hence, $\operatorname{conv}(\bigcup \mathcal{G})$ is a summand of $\overline{K}$ and therefore Lemma 4.4 implies that $\operatorname{conv}(\bigcup \mathcal{G})$ slides freely in $\overline{K}$, or in particular, that $\lambda_k(\mathcal{G}) \leq 1$. □

Remark 4.6. *We note that for* 0*-impassable families, Theorem 4.5 can be proved in a simple way. Consider two intersecting homothetic copies* $K_1 = x_1 + \tau_1 K$ *and* $K_2 = x_2 + \tau_2 K$ *of* K. *Let* K' *be the smallest positive homothetic copy of* K *that covers* $K_1 \cup K_2$. *Then there is a pair of parallel supporting hyperplanes* H_1 *and* H_2 *of* K' *such that for* $i = 1, 2$, $H_i \cap K_i \neq \emptyset$. *Let* $u \in \mathbb{S}^{d-1}$ *be a normal vector of* H_1 *and* H_2. *Since* $\operatorname{width}_u(K') \leq \{\operatorname{width}_u(K_1) + \operatorname{width}_u(K_2)$, *it follows that the homothety ratio of* K' *is at most* $\tau_1 + \tau_2$. *Thus, if* $\mathcal{G} = \{K_i = x_i + \tau_i K : x_i \in \mathbb{R}^2, \tau_i > 0, i = 1, 2, \ldots, n\}$ *is* 0*-impassable,* *applying this observation yields the assertion. In particular, this proves Theorem 4.5 for* $d = 2$*. On the other hand, to be able to apply the argument in the last paragraph of the proof of Theorem 4.5, we need to prove more for* $d = 2$*, namely that* $\bigcup \mathcal{G}$ *is a summand of* $\left(\sum_{i=1}^{n} \tau_i\right) K$*.*

For strictly convex bodies one can do more.

Theorem 4.6. *Let* K *be strictly convex body in* $\mathbb{R}^d, d \geq 2$ *and* $0 \leq k \leq d-2$*. If* $\mathcal{G} = \{K_i = x_i + \tau_i K : x_i \in \mathbb{R}^d, \tau_i > 0, i = 1, 2, \ldots, n\}$ *is a* k*-IP-family of positive homothetic copies of* K *in* $\mathbb{R}^d$*, then* $\bigcup \mathcal{G} = x_{i^*} + \tau_{i^*} K = K_{i^*}$ *for* $\tau_{i^*} = \max\{\tau_i \ : \ i = 1, 2, \ldots, n\}$*.*

Proof. Let A be an arbitrary k-dimensional linear subspace A of $\mathbb{R}^d$ and let $A^\perp$ denote the orthogonal complement of A in $\mathbb{R}^d$. Moreover, let $\mathrm{proj}_{A^\perp} : \mathbb{R}^d \to A^\perp$ denote the orthogonal projection onto $A^\perp$. Note that for every A, we have $\mathrm{conv}\left(\bigcup_{i=1}^{n} \mathrm{proj}_{A^\perp}(K_i)\right) = \bigcup_{i=1}^{n} \mathrm{proj}_{A^\perp}(K_i)$. Furthermore, if K is strictly convex, then so is $\mathrm{proj}_{A^\perp}(K)$.

We show that there is an $i^* = i^*(A)$ such that $\mathrm{conv}\left(\bigcup_{i=1}^{n} \mathrm{proj}_{A^\perp}(K_i)\right) = \mathrm{proj}_{A^\perp}(K_{i^*})$. Let $K_i^p = \mathrm{proj}_{A^\perp}(K_i)$, and assume, for contradiction, that $\mathrm{bd}(\bigcup_{i=1}^{n} K_i^p)$ is not covered by any of the (K_i^p)'s. Then there is a point $p \in \mathrm{bd}(\bigcup_{i=1}^{n} K_i^p)$, contained in $K_i^p \cap K_j^p$ for some $i \neq j$ such that $K_i^p \not\subseteq K_j^p \not\subseteq K_i^p$. Let h be the (positive) homothety in $A^\perp$ that transforms K_i^p into K_j^p, and let $q = h(p)$. Then, if H is a closed supporting halfspace of $\bigcup_{i=1}^{n} K_i^p$ in $A^\perp$, then H and $h(H)$ supports K_j^p at p and at q, respectively. As H and $h(H)$ have the same outer unit normals, they coincide. Thus, if $q \neq p$, then the segment $[p, q]$ is contained in K_j^p, contradicting our assumption that K is strictly convex. Thus, $q = p$ is the center of h, which implies that $K_i^p \subseteq K_j^p$, or that $K_j^p \subseteq K_i^p$, a contradiction.

Hence, there exists an $i^* = i^*(A)$ such that $\mathrm{conv}\left(\bigcup_{i=1}^{n} \mathrm{proj}_{A^\perp}(K_i)\right) = \mathrm{proj}_{A^\perp}(K_{i^*})$. We note that $\mathrm{conv}\left(\bigcup_{i=1}^{n} \mathrm{proj}_{\overline{A}^\perp}(K_i)\right)$ contains $\mathrm{proj}_{\overline{A}^\perp}(K_{i^*})$ for all k-dimensional linear subspaces $\overline{A}$ of $\mathbb{R}^d$, that is, a translate of $\lambda_{i^*} \mathrm{proj}_{\overline{A}^\perp}(K)$, and thus, i^* is independent of A. This yields that $\mathrm{conv}\left(\bigcup \mathcal{G}\right) = K_{i^*}$. □

4.6 The Maple program used in the proof of Theorem 4.2

For the convenience of the reader we include here the Maple code used in the the proof of Theorem 4.2.

```
>eq1 := t3*x2-x1*x2+x1*y3-y2*y3; eq2 := t1*x3-x2*x3+x2*y1-y1*y3;
>eq3 := t2*x1-x1*x3+x3*y2-y1*y2;
>Eq1 := simplify(subs(y1 = 1-x1-t1, subs(y2 = 1-x2-t2, subs(y3 = 1-x3-t3, eq1))));
>Eq2 := simplify(subs(y1 = 1-x1-t1, subs(y2 = 1-x2-t2, subs(y3 = 1-x3-t3, eq2))));
>Eq3 := simplify(subs(y1 = 1-x1-t1, subs(y2 = 1-x2-t2, subs(y3 = 1-x3-t3, eq3))));
>T := solve([Eq1, Eq2], [t2, t3], explicit = true);
>F := simplify(t1+subs(T[1][1], t2)+subs(T[1][2], t3)+a*subs(T[1][1], subs(T[1][2], Eq3)));
>F1 := factor(simplify(diff(F, t1))); F2 := factor(simplify(diff(F, x1)));
>F3 := factor(simplify(diff(F, x2))); F4 := factor(simplify(diff(F, x3)));
>F5 := factor(simplify(diff(F, a)));
>sols := solve([F1, F2, F3, F4, F5], [t1, x1, x2, x3, a], explicit = true,
allsolutions = true);
```

Chapter 5

Monostable polyhedra

The aim of this chapter is to investigate three problems proposed by Conway and Guy in [89] in 1968. These three questions appear also in the problem collection of Croft, Falconer and Guy [47] as Problem B12.

Before stating these problems, we recall that, informally, a monostable polyhedon is a convex polhyhedron of uniform density, which can be balanced on a horizontal plane only on one of its faces. Later, in Section 5.1, we give a more precise definition of this concept.

Problem 5.1. *Can a monostable polyhedron in the Euclidean 3-space $\mathbb{R}^3$ have an n-fold axis of symmetry for $n > 2$?*

Before the next problem, recall that the *girth* of a convex body in $\mathbb{R}^3$ is the minimum perimeter of an orthogonal projection of the body onto a plane [65].

Problem 5.2. *What is the smallest possible ratio of diameter to girth for a monostable polyhedron?*

Problem 5.3. *What is the set of convex bodies uniformly approximable by monostable polyhedra, and does this contain the sphere?*

We recall from Chapter 2 that, according to Guy [89], Conway showed that no body of revolution can be monostable, and also that the polyhedron constructed in [89] has a 2-fold rotational symmetry.

Our main result is the following.

Theorem 5.1. *For any $n \geq 3$, $n \in \mathbb{Z}$ and $\varepsilon > 0$ there is a monostable polyhedron P such that P has an n-fold rotational symmetry and $d_H(P, \mathbf{B}^3) < \varepsilon$.*

Theorem 5.1 answers Problem 5.1, and also the case of a sphere in Problem 5.3. In addition, from it we may deduce Corollary 5.1 (for the second part, see also [65]). This solves Problem 5.2. Here, for any convex body $K \subset \mathbb{R}^3$, we denote by $g(K)$ the girth of K.

Corollary 5.1. *For any $\varepsilon > 0$, there is a monostable polyhedron P with $\frac{\operatorname{diam}(P)}{g(P)} < \frac{1}{\pi} + \varepsilon$. Furthermore, we have $\frac{\operatorname{diam}(K)}{g(K)} \geq \frac{1}{\pi}$ for any convex body $K \subset \mathbb{R}^3$.*

The proof of Theorem 5.1 is based on a general theorem on approximation of smooth convex bodies by convex polytopes. Before stating it, we briefly introduce some elementary concepts regarding their equilibrium properties. Let $K \subset \mathbb{R}^3$ be a smooth *centered* convex body; i.e. assume that K is a convex body whose center of mass is o. Let $\delta_K : \operatorname{bd} K \to \mathbb{R}$ be the Euclidean distance function measured from o. The critical points of δ_K are called *equilibrium points* of K. To avoid degeneracy, it is usually assumed that δ_K is a Morse function; i.e. it has finitely many critical points, $\operatorname{bd} K$ is twice continuously differentiable at least in a neighborhood of each critical point, and at each such point the Hessian of δ_K is nondegenerate [194]. Depending on the number of negative eigenvalues of the Hessian, we distinguish between *stable*, *unstable* and *saddle-type* equilibrium points, corresponding to the local minima, maxima and saddle points of δ_K, respectively. The Poincaré-Hopf Theorem implies that under these conditions, the numbers S, U and H of the stable, unstable and saddle points of K, respectively, satisfy the equation $S - H + U = 2$.

Answering a conjecture of Arnold, Domokos and Várkonyi [186] proved that there is a homogeneous convex body with only one stable and one unstable point. They called the body they contructed 'Gömböc' (for more information, see [90]). In addition to the existence of Gömböc, in their paper [186] Domokos and Várkonyi proved the existence of a convex body with S stable and U unstable equilibrium points for any $S, U \geq 1$. This investigation was extended in [61] to the combinatorial equivalence classes defined by the Morse-Smale complexes of ρ_K, and in [58] for transitions between these classes. Based on these results, for any $S, U \geq 1$ we define the set $(S,U)_c$ as the family of smooth convex bodies K having S stable and U unstable equilibrium points, where K has no degenerate equilibrium point, and at each such point $\operatorname{bd} K$ has a positive Gaussian curvature. We define the class $(S,U)_p$ analogously for convex polyhedra, where stable and unstable points of a convex polyhedron are defined formally in Section 5.1. Our theorem is the following, where we call a convex body *centered* if its center of mass is the origin o.

Theorem 5.2. *Let $\varepsilon > 0$, $S, U \geq 1$ be arbitrary, and let G be any subgroup of the orthogonal group $\mathcal{O}(3)$. Then for any centered, G-invariant convex body $K \in (S,U)_c$, there is a centered G-invariant convex polyhedron $P \in (S,U)_p$ such that $d_H(K,P) < \varepsilon$.*

Here we note that the fact that any nondegenerate convex polyhedron can be approximated arbitrarily well by a smooth convex body with the same number of equilibrium points is regarded as 'folklore' (we use a simple argument to show it in Section 5.1). On the other hand, it is shown in [60] that any sufficiently fine approximation of a smooth convex body K by a convex polyhedron P, using an equidistant partition of the parameter range of the boundary of K, has strictly more stable, unstable and saddle points in general than the corresponding quantities for K.

Even though the convex body constructed in [186] is not C^2-class at its two equilibrium points, in [61] it is shown that class $(1,1)_c$ is not empty. Thus, Theorem 5.2 readily implies the existence of a polyhedron in class $(1,1)_p$.

Corollary 5.2. *There is a convex polyhedron with a unique stable and a unique unstable point.*

Furthermore, we remark that the elegant construction in the paper [65] of Dumitrescu and Tóth yields an *inhomogeneous* monostable convex polyhedron arbitrarily close to a sphere. Nevertheless, we must add that dropping the requirement of uniform density may significantly change the equilibrium properties of a convex body. To show it we recall the construction of Conway (see [53]) of an inhomogeneous monostable tetrahedron in $\mathbb{R}^3$, and observe that spheres with inhomogeneous density, as also roly-poly toys, yield trivial solutions to Arnold's conjecture.

In Section 5.1, we introduce our notation and collect the necessary tools for proving Theorems 5.1 and 5.2, including more precise definitions for some of the concepts defined here. In Section 5.2 we prove Theorem 5.1 and Corollary 5.1. In Section 5.3 we prove Theorem 5.2. Finally, in Section 5.4 we collect some additional remarks and ask some open questions.

5.1 Preliminaries

Let $K \subset \mathbb{R}^3$ be a convex body. The *center of mass* $c(K)$ of K is defined by the fraction $c(K) = \frac{1}{\mathrm{vol}(K)} \int_{x \in K} x \, dv$, where v denotes 3-dimensional Lebesgue measure (see also Chapter 2). We remark that the integral in this definition is called the *first moment* of K, and note that we clearly have $c(K) \in \mathrm{int}\, K$ for any convex body K. If $q \in \mathrm{bd}\, K$ satisfies the property that the plane through q and orthogonal to the vector $q - c(K)$ supports K, then we say that q is an *equilibrium point* of K. Here, if K is smooth, then the equilibrium points of K coincide with the critical points of the Euclidean distance function measured from $c(K)$ and restricted to $\mathrm{bd}\, K$. We recall that a convex body $K \subset \mathbb{R}^3$ is called *smooth* if for any boundary point x of K there is a unique supporting plane of K at q; this property coincides with the property that $\mathrm{bd}\, K$ is a C^1-class submanifold of $\mathbb{R}^3$ (cf. [169]).

We define nondegenerate equilibrium points only in two special cases. If K is smooth, $q \in \mathrm{bd}\, K$ is an equilibrium point of K with a C^2-class neighborhood in $\mathrm{bd}\, K$, and the Hessian of the Euclidean distance function on $\mathrm{bd}\, K$, measured from $c(K)$, is nondegenerate, we say that q is *nondegenerate*. In this case q is called a *stable, saddle-type* or *unstable* point of K if the number of the negative eigenvalues of the Hessian at q is $0, 1$ or 2, respectively [61]. Consider now the case that K is a convex polyhedron in $\mathbb{R}^3$, and $q \in \mathrm{bd}\, K$ is an equilibrium point of K. Then there is a unique vertex, edge or face of K that contains q in its relative interior. Let F denote this face, and let H be the supporting plane of K through q that is perpendicular to $q - c(K)$. Observe that $F \subset K \cap H$. We say that q is *nondegenerate* if $F = K \cap H$. In this case we call q a *stable, saddle-type* or *unstable* point of K if the dimension of F is $2, 1$ or 0, respectively [59]. In both the smooth and the polyhedral cases K is called nondegenerate if it has only finitely many equilibrium points, and each such point is nondegenerate. We note that in the above definitions, we may replace the center of mass of K by any fixed reference point $c \in K$. In this case we write about equilibrium points *relative to c*. We emphasize that in this chapter, unless it is stated otherwise, if the reference point is not specified, then it is meant to be the center of mass of the body.

Let $K \subset \mathbb{R}^3$ be a nondegenerate smooth convex body with S stable, H saddle-type and U unstable equilibrium points. Using a standard convolution technique, we may assume that K has a C^∞-class boundary, and hence, by the Poincaré-Hopf Theorem, we have $S - H + U = 2$ [61]. We show that the same holds if K is a nondegenerate convex polyhedron. Indeed, let $\tau > 0$ be sufficiently small, and set $K(\tau) = (K \div (\tau \mathbf{B}^3)) + (\tau \mathbf{B}^3)$, where $\div$ denotes Minkowski difference and $+$ denotes Minkowski addition (see Chapter 2 or [169]). Then for any $\tau > 0$, $K(\tau)$ is a smooth nondegenerate convex body having the same numbers of stable, saddle-type and unstable points relative to $c(K)$; hence, we may apply the Poincaré-Hopf Theorem for $K(\tau)$ (here we note that by Lemma 5.2 the same property holds relative to $c(K(\tau))$ as well). Thus, for any nondegenerate convex body, the numbers of stable and unstable points determine the number of saddle-type points. We define class $(S, U)_c$ as the family of nondegenerate, smooth convex bodies $K \subseteq \mathbb{R}^3$ with S stable, U unstable points with the additional assumption that at each equilibrium point of K, the principal curvatures of $\mathrm{bd}\, K$ are strictly positive. Similarly, by $(S, U)_p$ we mean the family of nondegenerate convex polyhedra with S stable and U unstable points. Observe that if K is nondegenerate, the point of $\mathrm{bd}\, K$ closest to or farthest from $c(K)$ is necessarily a stable or unstable point, respectively, implying that the numbers S, U in the above symbol are necessarily positive.

For the following remark, see Lemma 7 from [61].

Remark 5.1. *Let $K \in (S, U)_c^E$ and for any equilibrium point q of K, let V_q be an arbitrary compact neighborhood of q containing no other equilibrium point of K. Then $c(K)$ has an open neighborhood U*

such that for any $x \in U$, K has S stable and U unstable points relative to x, and for any equilibrium point q of K relative to $c(K)$, V_q contains exactly one equilibrium point of K relative to x, and the type of this point is the same as the type of q.

For Remark 5.2, see the paragraph in [60] after Definition 2.

Remark 5.2. *Let q be an equilibrium point of a centered convex body K in $(S,U)_c$ for some $S,U \geq 1$. Let $||q||= \rho$, and let k_1, k_2 denote the principal curvatures of $\operatorname{bd} K$ at q. Then $k_1, k_2 \neq \frac{1}{\rho}$. Furthermore, $0 \leq k_1, k_2 < \frac{1}{\rho}$ if and only if q is a stable point, $k_1, k_2 > \frac{1}{\rho}$ if and only if q is an unstable point, and $0 < \min\{k_1,k_2\} < \frac{1}{\rho} < \max\{k_1,k_2\}$ if and only if q is a saddle-type equilibrium point.*

Lemma 5.1. *The symmetry group of any nondegenerate convex body K is finite.*

Proof. Let K be a nondegenerate convex body with symmetry group G. Without loss of generality, assume that K is centered, i.e. $c(K) = o$. Since $c(K)$ is clearly a fixed point of any symmetry in G, we have that G is a subgroup of the orthogonal group $\mathcal{O}(3)$. Clearly, G is closed in $\mathcal{O}(3)$, and thus, it is a Lie group embedded in $\mathcal{O}(3)$ by Cartan's Closed Subgroup Theorem. On the other hand, the Lie subgroups of $\mathcal{O}(3)$ are well known, and in particular we have that if G is infinite, then it contains, up to conjugacy, $\mathcal{SO}(2)$ as a subgroup. In other words, K is rotationally symmetric. Thus, by nondegeneracy, K has exactly one stable and one unstable equilibrium point. But this property contradicts Conway's result mentioned in the beginning of this chapter that no rotationally symmetric convex body is monostable. □

We finish Section 5.1 with two lemmas and two remarks, where $X \triangle Y$ denotes the symmetric difference of the sets X, Y.

Lemma 5.2. *Let $K(\tau) \subset \mathbb{R}^3$ be a 1-parameter family of convex bodies, where $\tau \in [0, \tau_0]$ for some $\tau_0 > 0$. For any $\tau \in [0, \tau_0]$, let $c(\tau)$ denote the center of mass of $K(\tau)$, and let $K = K(0)$ and $c = c(0)$. Assume that for some $C > 0$ and $m > 0$, $\operatorname{vol}(K(\tau) \triangle K) \leq C\tau^m$ holds for any sufficiently small value of τ. Then there is some $C' > 0$ such that $||c(\tau) - c|| \leq C'\tau^m$ holds for any sufficiently small value of τ.*

Proof. Without loss of generality, we may assume that $K(\tau) \subseteq r\mathbf{B}^3$ for some $r > 0$ if τ is sufficiently small. By definition, $c(\tau) = \frac{\int_{x\in K(\tau)} x \, dv}{\operatorname{vol}(K(\tau))}$. On the other hand, by the conditions, we have $|\operatorname{vol}(K(\tau)) - \operatorname{vol}(K)| \leq C\tau^m$, and $|\int_{x \in K(\tau)} x \, d\lambda - \int_{x\in K} x \, dv| \leq rC\tau^m$ for all sufficiently small values of τ. From these inequalities and the fact that $\operatorname{vol}(K) > 0$, the assertion readily follows. □

Lemma 5.3. *Let $p \in \operatorname{int} \mathbf{B}^2 \subset \mathbb{R}^2$ and $q \in \mathbb{S}^1$ such that p, q and o are not collinear, and let L be a line through p such that L does not separate o and q. Furthermore, if A denotes the convex angular region with $q \in A$ and bounded by a half line of L starting at p, and the half line starting at p and containing o, then assume that the angle of A is obtuse. Then there is a convex polygon $Q \subset \mathbf{B}^2$ with vertices $o, x_0 = q, x_1, \ldots, x_k = p$ in cyclic order in $\operatorname{bd} Q$ such that $x_{i-1}x_i o\angle > \frac{\pi}{2}$ for all values of i, and L supports Q.*

We remark that the conditions in Lemma 5.3 imply that the Euclidean distance function $x \mapsto ||x||$, $x \in \mathbb{R}^2$ strictly decreases along the curve $\bigcup_{i=1}^{k} [x_{i-1}, x_i]$ from q to p.

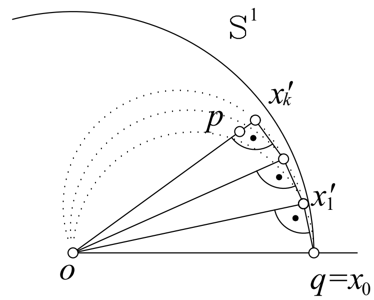

Figure 5.1: The construction of the points x_i' in the proof of Lemma 5.3. The dotted curves indicate arcs in the Thales circles of the segments $[o, x_i']$.

Proof. Without loss of generality, we may assume that $q = (1,0)$ and the y-coordinate of p is positive. Set $poq\angle = \beta \in (0,\pi)$, and choose an arbitrary positive integer k. For any $i = 0,1,\ldots,k$, define the point $x_i' = \left(r_i \cos\frac{i\beta}{k}, r_i \sin\frac{i\beta}{k} \right)$, where $r_i = \cos^i \frac{\beta}{k}$. Then $x_0' = q$, and x_i' is on the Thales circle of the segment $[0, x_{i-1}']$, and thus, $x_{i-1}' x_i' o\angle = \frac{\pi}{2}$ (cf. Figure 5.1) for all $i = 1,2,\ldots,k$. Using elementary calculus, we obtain that $\lim_{k\to\infty} \cos^k \frac{\beta}{k} = 1$, which yields that there is some value of k such that $||x_k'|| > ||p||$. Since x_k' and p are on the same half line, we may decrease the values of r_i for $i = 1,2,\ldots,k$ slightly such that for the points x_i obtained in this way the convex polygon $Q = \operatorname{conv}(\{o, x_0, x_1, \ldots, x_k\})$ satisfies the required conditions apart from the one for L. Now, if L supports Q, we are done. On the other hand, if L does not support Q, then we may take the polygon obtained as the intersection of Q and the closed half plane bounded by L and containing o in its interior. $\square$

Remark 5.3. *Let $a,b>0$, where $a \neq b$, and let $E \subset \mathbb{R}^2$ be the ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$. Then, for any $\delta > 0$ there is some $\varepsilon > 0$ such that if $K \subset \mathbb{R}^2$ is a plane convex body satisfying $E \subseteq K \subseteq (1+\delta)E$, and the vector w is perpendicular to a supporting line of K through $w \in \operatorname{bd} K$, then the angle between w and the x-axis or the y-axis is at most δ.*

Remark 5.4. *Let f,g be two real functions defined in a neighborhood of $a \in \mathbb{R}$. If f,g are both locally strictly increasing (resp., decreasing) at a, then so are $\min\{f,g\}$ and $\max\{f,g\}$.*

Finally, we remark that in the proof of Theorem 5.2, we use ideas also from [61, 58, 65].

5.2 Proofs of Theorem 5.1 and Corollary 5.1

First, we show how Theorem 5.2 implies Theorem 5.1.

Let $n \geq 3$ be a positive integer and let $\varepsilon > 0$ be a sufficiently small fixed value. By Theorem 5.2, it is sufficient to construct a smooth convex body $K \in (1,m)_c$ for some value of m with n-fold rotational symmetry and satisfying $d_H(K, \mathbf{B}^3) \leq \varepsilon$. Let P be a regular n-gon inscribed in a fixed circle C on $\mathbf{B}^3$ parallel to, but not contained in the (x,y)-plane. Let the vertices of P be p_i, $i = 1,2,\ldots,n$. Let $Q(\varepsilon) = \operatorname{conv}\left(\mathbf{B}^3 \cup \{(1+\varepsilon)p_1, \ldots, (1+\varepsilon)p_n\}\right)$. Then $Q(\varepsilon)$ is the union of $\mathbf{B}^3$ and n cones C_i,

$i = 1, 2, \ldots, n$, with spherical circles centered at the points p_i as directrixes. By symmetry, the center of mass c of $Q(\varepsilon)$ is on the z-axis, and by the Thales Theorem and Lemma 5.2, its distance from o is of magnitude $O(\varepsilon^2)$. Thus, the points $(1+\varepsilon)p_i$ are equilibrium points of $Q(\varepsilon)$ if ε is sufficiently small. Furthermore, we have $c \neq o$. On one hand, from this we have that there are exactly two equilibrium points of $Q(\varepsilon)$ on $\mathbb{S}^2$, namely the points $(0,0,1)$ and $(0,0,-1)$, and exactly one of these points is stable, and the other one is unstable. On the other hand, this also implies that $Q(\varepsilon)$ has exactly one equilibrium point on each cone C_i apart from its vertex; this point is a saddle point in the relative interior of a generating segment of C_i (cf. Figure 5.2).

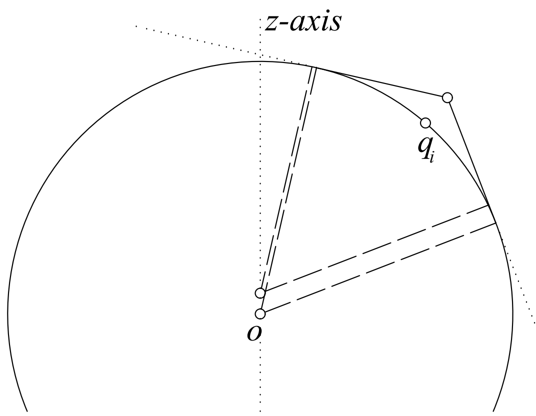

Figure 5.2: An illustration for the proof of Theorem 5.1.

Now, we set $Q'(\varepsilon) = (Q(\varepsilon) \div (\tau \mathbf{B}^3)) + (\tau \mathbf{B}^3)$, where $\tau > 0$ is negligible compared to ε. Then $Q'(\varepsilon)$ is a smooth convex body which has 1 stable, n saddle-type and $(n+1)$ unstable points by Lemma 5.2. To guarantee that the body has positive principal curvatures at each equilibrium point, we may replace the generating segments of the cones by circular arcs of radius $R > 0$, where $\frac{1}{R}$ is negligible compared to τ. The obtained convex body $K(\varepsilon) \in (1, n+1)_c$ satisfies the required conditions.

Finally, we prove Corollary 5.1. Clearly, $\operatorname{diam}(\mathbf{B}^3) = 2$ and $g(\mathbf{B}^3) = 2\pi$, and hence, the first statement follows from the continuity of diameter and girth with respect to Hausdorff distance. On the other hand, let $K \subset \mathbb{R}^3$ be a convex body, and recall mwidth$(\cdot)$ and perim$(\cdot)$ denote mean width and perimeter, respectively. Then, for any projection M of K, we have $\leq \operatorname{mwidth}(M) \leq \operatorname{diam}(M) \leq \operatorname{diam}(K)$. On the other hand, it is well known that $\operatorname{mwidth}(M) = \frac{\operatorname{perim}(M)}{\pi}$, implying that $\frac{\operatorname{diam}(K)}{\operatorname{perim}(M)} \geq \frac{1}{\pi}$. From this, we readily obtain $\frac{\operatorname{diam}(K)}{g(K)} \geq \frac{1}{\pi}$.

5.3 Proof of Theorem 5.2

First, observe that by Lemma 5.1, G is finite.

We construct P by truncating K with finitely many suitably chosen planes; or more precisely by taking its intersection with finitely many suitably chosen closed half spaces. We carry out the construction of P in three steps.

In Step 1, we replace some small regions of $\operatorname{bd} K$ by polyhedral regions disjoint from all equilibrium points of K. These polyhedral regions will serve as 'controlling regions'; that is, after constructing a polyhedron with S stable and U unstable points relative to o, we modify these regions to move back the center of mass of the constructed polyhedron to o. In Step 2, we truncate a neighborhood of each equilibrium point to replace it by a polyhedral surface in such a way that each polyhedral surface contains exactly one equilibrium point relative to o, and the type of this point is the same as the type of the corresponding equilibrium point of K. Finally, in Step 3 we truncate the remaining part of $\operatorname{bd} K$ such that no new equilibrium point is created.

In the proof, we denote by $\mathcal{E}$ the set of the equilibrium points of K, and for any point $q \in \operatorname{bd} K$, we denote by H_q the unique supporting plane of K at q. Observe that by the definition of $(S,U)_c$, $H_q \cap K = \{q\}$ for any $q \in \mathcal{E}$, and set $X = (\operatorname{bd} K) \setminus \mathcal{E}$. Finally, by F we denote the set of the fixed points of G, and note that F is a linear subspace of $\mathbb{R}^3$ that contains the center of mass of any G-invariant convex body.

Step 1.

We distinguish two cases depending on $\dim(F)$.

Case 1, if $F = \mathbb{R}^3$. By Carathéodory's theorem, there are points $z_1, z_2, z_3, z_4 \in X$ such that $o \in \operatorname{conv}(\{z_1, z_2, z_3, z_4\})$. Since X is open in $\operatorname{bd} K$, we may choose these points to satisfy $o \in \operatorname{int}\operatorname{conv}(\{z_1, z_2, z_3, z_4\})$. By the definition of $(S,U)_c$, we have that H_{z_i} is disjoint from H_q for any $1 \leq i \leq 4$ and $q \in \mathcal{E}$. We show that the z_is can be chosen such that the planes H_{z_i} are pairwise distinct. Suppose for contradiction that, say, three of these planes coincide. Without loss of generality, assume that $H_{z_1} = H_{z_2} = H_{z_3}$, and denote this common plane by H. Then there are points $z_1', z_2', z_3' \in \operatorname{relbd}(K \cap H)$ such that $\operatorname{conv}(\{z_1, z_2, z_3\}) \subseteq \operatorname{conv}(\{z_1', z_2', z_3'\})$. Now we may replace z_2' and z_3' by two points $z_2'', z_3'' \notin H$ such that z_i'' is sufficiently close to z_i' for $i = 2, 3$. Then we have $o \in \operatorname{int}\operatorname{conv}(\{z_1', z_2'', z_3'', z_4\})$, where no supporting plane of K contains three of the points. If a supporting plane of K contains two of these points, we may repeat the above procedure, and finally obtain some points $w_1, \ldots, w_4 \in X$ such that $o \in \operatorname{int}\operatorname{conv}(\{w_1, \ldots, w_4\})$, and the sets $H_{w_i} \cap K$ are pairwise disjoint.

Let $\delta > 0$, and let us truncate K by planes $H_1, \ldots, H_4$ such that for all is H_i is parallel to H_{z_i} and it is at the distance δ from it in the direction of o. We denote the truncated convex body by K' and its center of mass by c'. By Remark 5.1, if δ is sufficiently small, then K has S stable and U unstable equilibrium points relative to o', and each such equilibrium point is contained in $(\operatorname{bd} K') \setminus (\bigcup_{i=1}^4 H_i)$. Furthermore, if δ is sufficiently small, then for any point $q \in H_i \cap (\operatorname{bd} K)$ and any plane H supporting K' at q, q is not perpendicular to H. Finally, since $o \in \operatorname{int}\operatorname{conv}(\{w_1, \ldots, w_4\})$, we may choose points $w_i' \in \operatorname{relint}(H_i \cap K')$ such that $c' \operatorname{int}\operatorname{conv}(\{w_1', \ldots, w_4'\})$. For any w_i', choose some convex n_i-gon $P_i \subset \operatorname{relint}(H_i \cap K')$ such that the center of mass of P_i is w_i'. Now we obtain the body K'' by truncating K' by n_i planes almost parallel to H_i such that for each i, every side of P_i is contained in one of the truncating planes, and we have $P_i = H_i \cap K''$. We choose the truncating planes such that the center of mass c'' of K'' satisfies $c'' \in \operatorname{int}\operatorname{conv}(\{w_1', \ldots, w_4'\})$, and K'' has S stable and U unstable points on the smooth part of its boundary, and no equilibrium point on the non-smooth part. Now, we set $K_1 = K'' - c''$, and $q_i' = q_i - c''$, $P_i' = P_i - c''$ for all is, and for some sufficiently small $\bar{\tau} > 0$ we define four 1-parameter families of $C_i(\tau_i) = \operatorname{conv}(P_i' \cup \{(1 + \tau_i) q_i'\})$, $\tau_i \in [0, \bar{\tau}]$, $i = 1, 2, 3, 4$. Furthermore, for later use, we set $K_0 = K - c''$, and call the set $X_1 = K_1 \cap \operatorname{bd} K_0$ the *non-truncated part of* $\operatorname{bd} K_1$.

If $\bar{\tau}$ is sufficiently small, then $K_1 \cup \bigcup_{i=1}^4 C_i(\tau_i)$ is convex for all values of the parameters τ_i. Furthermore, the first moment of $\bigcup i = 1^4 C_i(\tau_i)$ is $\sum_{i=1}^4 \alpha_i \tau_i q_i'$ for some suitable constants $\alpha_i > 0$, which implies that it is surjective in a neighborhood of o. Thus, since K_1 is centered, after we replace the non-truncated part of $\operatorname{bd} K_1$ by a polyhedral surface in Steps 2 and 3, we may choose values of the τ_is in such a way that the sum of the first moment of $\bigcup_{i=1}^4 C_i(\tau_i)$ and of the first moment of the polyhedron P obtained after Step 3 is equal o. This makes the polyhedron $P \cup \bigcup i = 1^4 C_i(\tau_i)$ centered. Finally, we observe that by choosing sufficiently small values of δ and $\bar{\tau}$, for all values of the parameters, no point of $C_i(\tau_i)$ is an equilibrium point of K_1 relative to o.

Case 2, if $F \neq \mathbb{R}^3$. In this case F is a plane or a line through o, or $F = \{o\}$. Consider the case that F is a plane. Then, by the properties of isometries, the orbit of any point p under G consists of p and its reflection about F. Let $K_F = F \cap K$, and observe that since K is symmetric about F, for any $q \in \operatorname{bd} K$ H_q is either disjoint from K_F or $q \in \operatorname{relbd}(K_F)$. Thus, we may apply the argument in Case 1 for K_F, and obtain some points $z_1, z_2, z_3 \in X \cap K_F$ such that $o \in \operatorname{relint} \operatorname{conv}(\{z_1, z_2, z_3\})$ and the planes H_{z_i} are pairwise disjoint. But then there are some points z_3' and z_3'', sufficiently close to H_{z_3} such that z_3'' is the reflected copy of z_3' about L, $o \in \int \operatorname{conv}(\{z_1, z_2, z_3', z_3''\})$, and the supporting planes at these points are pairwise disjoint. Clearly, the set $\{z_1, z_2, z_3', z_3''\}$ is G-invariant. From now on, we may apply the argument in Case 1. If F is a line, we may repeat the argument in the previous paragraph. Finally, if $F = \{o\}$, then any G-invariant convex body (and in particular the convex polyhedron constructed in Steps 2 and 3) is centered. Thus, in this case we may skip Step 1.

Based on the existence of the families $C_i(\tau_i)$, in Steps 2 and 3 all equilibrium points are meant to be *relative to o*. We denote by $\mathcal{E}_1$ the set of the equilibrium points of K_1.

Step 2.

In this step we take all points $q \in \mathcal{E}_1$, and truncate neighborhoods of them in $\operatorname{bd} K_1$ simultaneously for all points in the orbit of q. Here we observe that the orbit of an equilibrium point consists of equilibrium points. We carry out the truncations in such a way that the regions truncated in Step 1 or Step 2 are pairwise disjoint. We denote the convex body obtained in this step by K_2, and set $X_2 = (\operatorname{bd} K_1) \cap K_2$. We construct K_2 in such a way that for any point $p \in X_2$ there is no supporting plane H of K_2 through p which contains an equilibrium point of K_2.

Consider some $q \in \mathcal{E}_1$. Without loss of generality, we may assume that $q = (0, 0, \rho)$ for some $\rho > 0$, and denote by e_x, e_y, and e_z the vectors of the standard orthonormal basis. With a little abuse of notation, for any $p \in \operatorname{bd} K_0$, we denote by H_p the unique supporting plane of K_0 at p.

Case 1, the stabilizer of q in G is the identity; i.e. q not fixed under any element of G other than the identity.

Subcase 1.1, q is a stable point of K_1. In this case we truncate K_1 by a plane H_q' parallel to, and sufficiently close to H_q. Then we truncate K_1 by finitely many additional planes such that any point of $H_q' \cap (\operatorname{bd} K_1)$ is truncated by at least one of them, and for any point p of the non-truncated part X_2 of $\operatorname{bd} K_1$ there is no supporting plane H of K_2 through p which contains an equilibrium point of K_2 relative to o.

Subcase 1.2, q is a saddle-type equilibrium point. Note that by Remark 5.2, q is not an umbilic point of $\operatorname{bd} K_1$, and its principal curvatures $k_1 < k_2$ satisfy the inequalities $0 < k_1 < \frac{1}{\rho} < k_2$.

Without loss of generality, we may assume that the sectional curvature of $\operatorname{bd} K_1$ in the (x, z)-plane is k_1, and in the (y, z)-plane it is k_2. For any $\tau > 0$, let $K_1(\tau)$ denote the set of points of K_1 with z-coordinates at least $\rho - \tau$, and observe that by the fact that $k_2 > k_1 > 0$, for any $\varepsilon > 0$ there is some $\tau > 0$ such that $K(\tau)$ is contained in the neighborhood of q of radius ε. For any $\{i, j\} \subset \{x, y, z\}$, let H_{ij} denote the (i, j) coordinate plane, and proj_{ij} denote the orthogonal projection of $\mathbb{R}^3$ onto H_{ij}.

For any $\eta > 0$, let $C(\eta)$ be the set of the points of the circular disk $y^2 + (z - \rho + \eta)^2 \leq \eta^2$ in H_{yz} whose z-coordinates are at least $\rho - \tau$. Then, since $\operatorname{bd} K_1$ is C^2-class in a neighborhood of q, we have that for any $\eta_1, \eta_2 > 0$ satisfying $\frac{1}{\rho} < \frac{1}{\eta_1} < k_2 < \frac{1}{\eta_2}$, if τ is sufficiently small, then

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$C(\eta_2) \subseteq \text{proj}_{yz}(K(\tau)) \subseteq C(\eta_1)$ holds. Since $\text{proj}_{yz}(K_1)$ is convex, $\text{relbd}(\text{proj}_{yz}(K_1))$ has exactly two points with their z-coordinates equal to $\rho - \tau$. Let these points be $q^- = (0, \sigma^-, \rho - \tau)$ and $q^+ = (0, \sigma^+, \rho - \tau)$ such that $\sigma^- < 0 < \sigma^+$. Then there are some supporting lines L_-, L_+ of $\text{proj}_{yz}(K)$ passing through q^- and q^+, respectively. Clearly, for $i \in \{-,+\}$, L^i is the orthogonal projection of some supporting plane H^i of K_1 onto H_{yz}. Let r^i be a point of L^i, on the open half line starting at q^i such that the line through $[o, q^i]$ do not separate q and r_i. Then, for any fixed values of η_1 and η_2 and sufficiently small value of τ, the angles $oq^ir^i\angle$ are obtuse. Now we choose some sufficiently small value of $\zeta > 0$, and define $q^{i'} = \zeta e_z + q^i$, $r^{i'} = \zeta e_z + r^i$, $L^{i'} = \zeta e_z + L^i$, $H^{i'} = \zeta e_z + H^i$ and $q' = -\zeta e_z + q$. Then we may assume that $||q^{i'}|| < ||q'||$, the angles $oq^{i'}r^{i'}\angle$ are obtuse, and the planes $H^{i'}$ are disjoint from K_1.

Thus, by Lemma 5.3, for $i \in \{-,+\}$, there is a polygonal curve Γ_i in H_{yz}, connecting q' to $q^{i'}$ such that the Euclidean distance measured from the points of Γ_i to o is strictly decreasing as we move from q' to $q^{i'}$ (see the remark after Lemma 5.3), Γ_i is contained in $\text{relbd}\,\text{conv}(\Gamma_i \cup \{o\})$, and the latter set is supported by ${L_i}'$ in H_{yz}. Consider the closed, convex set $C_H \subset H_{yz}$ bounded by $\Gamma_- \cup \Gamma_+$, the half line of $L^{+'}$ starting at $q^{+'}$ and not containing $r^{+'}$, and the half line of $L^{-'}$ starting at $q^{-'}$ and not containing $r^{-'}$, and set $C = \text{proj}_{yz}^{-1}(C_H) \subset \mathbb{R}^3$. By the previous consideration, C is an infinite convex cylinder with the properties that $o \in \text{int}\, C$, $K_1 \setminus C \subseteq K_1(\tau)$, and the equilibrium points of C relative to o are q and two stable points on ${L_+}'$ and ${L_-}'$, respectively. To construct K_2, we truncate K_1 by C, and show that, apart from the saddle point q', no new equilibrium point is created by this truncation.

Observe that by our construction, any new equilibrium point is a point of $(\text{bd}\, K_1) \cap (\text{bd}\, C)$. Suppose that there is some equilibrium point $q \in (\text{bd}\, K_1) \cap (\text{bd}\, C)$ of $K_1 \cap C$. To reach a contradiction, we identify H_{xy} with $\mathbb{R}^2$, and parametrize $\text{bd}\, K_1$ in a neighborhood of q as the graph of a function $f : \mathbb{R}^2 \to \mathbb{R}$ and $\text{bd}\, C$ in a neighborhood of q' as the graph of a function $g : \mathbb{R}^2 \to \mathbb{R}$. Note that by the nondegeneracy of q, for some value of $\phi > 0$, o has a neighborhood $U \subset \mathbb{R}^2$ such that for any $w = (x_0, y_0) \in U$ whose angle with the x-axis is at most ϕ, $||(x, y, f(x, y)||$ is locally strictly increasing at w as a function of x if $x_0 > 0$ and locally strictly decreasing if $x_0 < 0$; furthermore, if the angle of w with the y-axis is at most ϕ, then $||(x, y, f(x, y)||$ is locally strictly decreasing as a function of y if $y_0 > 0$ and locally strictly increasing if $y_0 < 0$. Note that by Remark 5.4, the same property holds for $\min\{||(x, y, f(x, y))||, ||(x, y, g(x, y))||\}$ as well. Observe that since q is an equilibrium point of $K_1 \cap C$, it is an equilibrium point of the section of $K_1 \cap C$ with the plane through q parallel to H_{xy}. Thus, by Remark 5.3, if $\tau > 0$ is chosen small enough, then the angle of $\text{proj}_{xy}(q)$ with the x-axis or the y-axis is at most ϕ. But this contradicts our previous observation that at such a point $\min\{||(x, y, f(x, y))||, ||(x, y, g(x, y))||\}$ is locally strictly increasing or decreasing parallel to the x- or the y-axis.

Finally, to exclude the possibility that a support plane of $K_1 \cap C$ through a point in $(\text{bd}\, K_1) \cap (\text{bd}\, C_1)$ contains q', we truncate all points of $(\text{bd}\, K_1) \cap (\text{bd}\, C)$ by planes, not containing q', whose intersections with $K_1 \cap C$ do not contain equilibrium point.

Subcase 1.3, q is an unstable point. In this case both principal curvatures k_1, k_2 of $\text{bd}\, K_1$ at q satisfy $k_1, k_2 > \frac{1}{\rho} > 0$, and thus, there is some constant $\max\left\{\frac{1}{k_1}, \frac{1}{k_2}\right\} < \eta < \rho$ such that the ball $\frac{\rho - \eta}{\rho} q + \eta \mathbf{B}^3$ contains a neighborhood of q in $\text{bd}\, K_1$. We parametrize $\text{bd}\, K_1$ in a neighborhood of q as the graph $\{z = f(x, y)\}$ of a function $(x, y) \mapsto f(x, y)$, and note that by nondegeneracy, the function $||(x, y, f(x, y)||$ is strictly decreasing in a neighborhood of $(0, 0)$ as a function of $\sqrt{x^2 + y^2}$.

For any $\tau > 0$, let $K_1(\tau)$ denote the set of points of K_1 with z-coordinates at least $\rho - \tau$, let H_τ denote the plane with equation $\{z = \rho - \tau\}$. Let $\tau > 0$ be sufficiently small. Then there is a

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circle C_0 centered at $(0,0,\rho-\tau)$ which is contained in $H_\tau \cap \operatorname{int}(\eta \mathbf{B}^3)$, and is disjoint from K_1. Let H be a plane supporting K_1 at a point of H_τ such that its angle α with H_τ is minimal among these supporting planes. Let H' be the translate of H touching C_0 such that H strictly separates o and H', and let the intersection point of H' and the z-axis be r. Consider the infinite cone C with apex r and base C_0, and observe that it contains $K_1 \setminus K_1(\tau)$ in its interior. Now, let $q' = q - \zeta e_z$ for some suitably small $\zeta > 0$, and let Γ be a polygonal curve connecting q' to a point $p \in C_0$ such that the plane of o, p, q' contains Γ, $\Gamma \subset \operatorname{relbd}(\operatorname{conv}(\Gamma \cup \{o\}))$, and the Euclidean distance function is strictly decreasing along Γ from q' to p. Let L_p denote the closed half line in the line of $[r,p]$ starting at p and not containing r, and let $\Gamma' = \Gamma \cup L_p$. Let $m \geq 3$ be arbitrary, and for any $i = 0, 1, \ldots, m-1$, let Γ'_i denote the rotated copy of Γ' around the z-axis, with angle $\frac{2\pi i}{m}$.Let $P' = \operatorname{conv}\left(\bigcup_{i=0}^{m-1} \Gamma'_i\right)$. Then P' is a convex polyhedral domain such that $K_1 \setminus P' \subseteq K_1(\tau)$, and if m is sufficiently large, then at any boundary point of P' with z-coordinate greater than $\rho - \tau$, $||(x,y,g(x,y))||$ is strictly locally increasing in a neighborhood of $(0,0)$ as a function of $\sqrt{x^2+y^2}$, where $\operatorname{bd} P'$ is given as the graph of the function $z = g(x,y)$. Thus, by Remark 5.4 and following the idea at the end of Subcase 1.2 in Step 2, we may truncate a neighborhood of q in $\operatorname{bd} K_1$ by a convex polyhedral region P' such that the only equilibrium point of the truncated body on $\operatorname{bd} P'$ is the unstable point q', and the truncated body has no non-truncated boundary point where some supporting plane contains an equilibrium point.

The procedure discussed in Subcases 1.1-1.3 for q are applied for any equilibrium point in the orbit of q in an analogous way.

Case 2, if the stabilizer of q in G is not the identity. In this case the procedure described in Case 1 is carried out in such a way that the truncating polyhedral domain is invariant under any element of G fixing q.

Summing up, to construct K_2 in Step 2 we truncated a neighborhood of each equilibrium point of K_1 by a polyhedral region in such a way that each region contains exactly one equilibrium point relative to o, and no plane supporting K_2 at any point of $X_2 = (\operatorname{bd} K_1) \cap K_2$ contains an equilibrium point of K_2 relative to o. In addition, K_2 is G-invariant.

Step 3.

In this step let $Y = X_1 \cap X_2$. Furthermore, for any plane H let o_H denote the orthogonal projection of o onto H, and let $\mathcal{H}$ denote the family of planes H with the property that $o_H \in K_2$. Note that $\mathcal{H}$ consists of all planes through o, and for any $p \in K_2 \setminus \{o\}$ the (unique) plane passing through p and perpendicular to $[o,p]$.

Observe that Y is compact, and by our construction, for any plane H supporting K_2 at some point $p \in Y$, we have $o_H \notin H \cap K_2$; or equivalently, for any $H \in \mathcal{H}$, $H \cap Y = \emptyset$. Thus, by compactness, there is some $\delta > 0$ such that for any $H \in \mathcal{H}$ and $p \in Y$, the distance of H and p is at least $\delta > 0$. Now, for any $p \in Y$, let H_p denote the unique closed half space whose boundary supports K_0 at p and which satisfies $(\operatorname{int} H_p) \cap K_1 = \emptyset$. Let u_p denote the outer unit normal vector of K at p, and for any $\zeta > 0$, set $H_p(\zeta) = H_p - \zeta u_p$, and $Y(\zeta) = K_2 \cap \left(\bigcup_{p \in Y} H_p(\zeta)\right)$. Clearly, $Y(\zeta)$ tends to Y with respect to Hausdorff distance as $\zeta \to 0^+$. Thus, there is some sufficiently small $\zeta_0 > 0$ such that for any $H \in \mathcal{H}$, H is disjoint from $Y(\zeta_0)$.

Now, for any $p \in Y$, set $U(p) = Y \cap \operatorname{int} H_p(\zeta_0)$. Then $U(p)$ is an open neighborhood of p in Y. Thus, by the compactness of Y, there are finitely many points $p_1, \ldots, p_m$ such that $\bigcup_{i=1}^m U(p_i) = Y$. Then, clearly $P = K_2 \cap \left(\bigcap_{i=1}^m \left(\mathbb{R}^3 \setminus \operatorname{int} H_{p_i}(\zeta_0)\right)\right)$ is a convex polytope contained in K_2. Furthermore, since $\zeta_0 > 0$ can be arbitrarily small, P can be arbitrarily close to K_2.

We show that no point $q \in \operatorname{bd} P$, contained in some $\operatorname{bd} H_{p_i}(\zeta_0)$ is an equilibrium point of P.

Indeed, if q was such a point, then the plane H through q and perpendicular to $[o,q]$ is contained in $\mathcal{H}$. On the other hand, $q \in (\operatorname{bd} P) \cap (\operatorname{bd} H_{p_i}(\zeta_0)) \subset Y(\zeta_0)$, which is impossible by our choice of ζ_0.

Finally, we may choose the points $p_1, p_2, \ldots, p_m$ in such a way that the set $\{p_1, \ldots, p_m\}$ is invariant under the act of any element of G.

5.4 Remarks and open questions

First, we remark that by using truncations instead of conic extensions in the proof of Theorem 5.1, we readily obtain Theorem 5.3. Here, a *mono-unstable* convex body is meant to be a nondegenerate convex body with a unique unstable point.

Theorem 5.3. *For any $n \geq 3$, $n \in \mathbb{Z}$ and $\varepsilon > 0$ there is a homogeneous mono-unstable polyhedron P such that P has an n-fold rotational symmetry and $d_H(P, \mathbf{B}^3) < \varepsilon$.*

It is also worth noting that Domokos, Lángi and Várkonyi recently strengthened Theorems 5.1 and 5.3 [62], by proving the following.

Theorem 5.4 (Domokos, Lángi and Várkonyi, 2023)**.** *For any $n \geq 3$, $n \in \mathbb{Z}$ and $\varepsilon > 0$ there is some convex polyhedron $P \in (1,1)_p$ such that P has an n-fold rotational symmetry and $d_H(P, \mathbf{B}^3) < \varepsilon$.*

In light of the words of Shephard in [173] from Chapter 2, we remark that a consequence of Theorem 5.2 is that to study the metric properties of nondegenerate polyhedra, in particular monostable polyhedra, it is sufficient to study the metric properties of their smooth counterparts, which seems to be much more tractable.

Next, we conjecture that Theorem 5.2 remains true if we omit the condition that the principal curvatures of $\operatorname{bd} K$ at every equilibrium point of K are strictly positive.

Finally, to propose a conjecture for the first part of Problem 5.3, we recall the following concept from [186], where the function $\rho_K : \mathbb{S}^2 \to \mathbb{R}$, $\rho_K(x) = \max\{\lambda : \lambda x \in K\}$ is called the *radial function* of the convex body K (see Chapter 2).

Definition 5.1. *Let $K \in \mathbb{R}^3$ be a centered convex body, and for any simple, closed curve $\Gamma \subset \mathbb{S}^2$, let Γ^+ and Γ^- denote the two compact, connected domains in $\mathbb{S}^2$ bounded by Γ. Then the quantities*

$$F(K) = \sup_{\Gamma} \sup_{p_1 \in \Gamma^+, p_2 \in \Gamma^-} \frac{\min\{\rho_K(s) : s \in \Gamma\}}{\max\{\rho_K(p_1), \rho_K(p_2)\}}$$

and

$$T(K) = \sup_{\Gamma} \sup_{p_1 \in \Gamma^+, p_2 \in \Gamma^-} \frac{\min\{\rho_K(p_1), \rho_K(p_2)\}}{\max\{\rho_K(s) : s \in \Gamma\}}$$

are called the flatness *and the* thinness *of K, respectively.*

Domokos and Várkonyi in [186] proved that for any nondegenerate, centered smooth convex body K, $F(K) = 1$ if and only if K is monostable, and $T(K) = 1$ if and only if K is mono-unstable.

Recall that a nondegenerate convex body is *mono-monostatic* if it has a unique stable and a unique unstable point [186]. We conjecture the following.

Conjecture 5.1. *For any centered convex body $K \subset \mathbb{R}^3$, K can be uniformly approximated by monostable convex polyhedra if and only if $F(K) = 1$, by mono-unstable convex polyhedra if and only if $T(K) = 1$, and by mono-monostatic convex polyhedra if and only if $F(K) = T(K) = 1$.*

Chapter 6

Monotonicity of the volume of the Minkowski average of compact sets

Recall from Chapter 2 that the Minkowski sum of two sets $K, L \subset \mathbb{R}^d$ is defined as $K + L = \{x + y : x \in K, y \in L\}$, where, for brevity, we set $A[k] = \sum_{i=1}^{k} A$, for any $k \in \mathbb{N}$ and any compact set $A \subset \mathbb{R}^d$. Since Minkowski sum preserves the convexity of the summands and the set $\frac{1}{k}A[k]$ consists of some particular convex combinations of elements of A, the containment $\frac{1}{k}A[k] \subseteq \operatorname{conv}(A)$, and, for the special case of convex sets, the equality $\frac{1}{k}A[k] = \operatorname{conv}(A)$ trivially holds. These observations suggest that for any compact set A, the set $\frac{1}{k}A[k]$ looks 'more convex' for larger values of k.

The aim of this chapter is the following conjecture of Bobkov, Madiman, Wang [31], relating the volumes of the elements of the sequence.

Conjecture 6.1 (Bobkov-Madiman-Wang, 2011). *Let A be a compact set in $\mathbb{R}^d$ for some $d \in \mathbb{N}$. Then the sequence*

$$\left\{ \operatorname{vol}\left(\operatorname{conv}(A) \setminus \left(\frac{1}{k} A[k] \right) \right) \right\}_{k \geq 1}$$

is non-increasing in k, or equivalently, the sequence

$$\left\{ \operatorname{vol}\left(\frac{1}{k} A[k] \right) \right\}_{k \geq 1}$$

is non-decreasing in k.

Thus, this conjecture asks whether for any integer $k \geq 1$ and compact set $A \subset \mathbb{R}^d$, the following inequality holds

$$\operatorname{vol}\left(\frac{1}{k} A[k] \right) \leq \operatorname{vol}\left(\frac{1}{k+1} A[k+1] \right). \tag{6.1}$$

This inequality trivially holds for any compact set A if $k = 1$ since $A \subseteq \frac{1}{2}A[2]$. In the same way, it is easy to find monotone subsequences of the sequence $\{\operatorname{vol}(\frac{1}{k}A[k])\}_{k \geq 1}$ by the same argument; one such example is $\{\operatorname{vol}(\frac{1}{2^m}A[2^m])\}_{m \geq 0}$. On the other hand, even the first nontrivial case; that is, the inequality $\operatorname{vol}\left(\frac{1}{2}A[2]\right) \leq \operatorname{vol}\left(\frac{1}{3}A[3]\right)$ seems to require new methods to approach. Conjecture 6.1 was partially resolved in [76, 77], where, following the approach of [99], the authors proved it for any 1-dimensional compact set A, but constructed counterexamples in $\mathbb{R}^d$ for any $d \geq 12$. More precisely, they showed that for every $k \geq 2$, there is $d_k \in \mathbb{N}$ such that for every $d \geq d_k$ there is a compact set $A \subset \mathbb{R}^d$ such that $\operatorname{vol}\left(\frac{1}{k}A[k]\right) > \operatorname{vol}\left(\frac{1}{k+1}A[k+1]\right)$. In particular, one has $d_2 = 12$, whence Conjecture 6.1 fails for $\mathbb{R}^d$ if $d \geq 12$.

Our goal is to find additional conditions on A and k under which the statement in Conjecture 6.1, or more precisely when the inequality (6.1) is satisfied.

In this chapter, for any set $A \subset \mathbb{R}^d$ we denote by $\dim A$ the dimension of the smallest affine subspace containing A. To state our main result, let us recall the following well-known concept.

Definition 6.1. *A nonempty set $S \subset \mathbb{R}^d$ is called* star-shaped *with respect to a point p if for any $q \in S$, we have $[p, q] \subseteq S$.*

Our main result is the following.

Theorem 6.1. *Let $d \geq 2$ and $k \geq \max\{2, (d-1)(d-2)\}$ be integers. Then for any compact, star-shaped set $S \subset \mathbb{R}^d$ we have*

$$\operatorname{vol}\left(\frac{1}{k+1}S[k+1]\right) \geq \operatorname{vol}\left(\frac{1}{k}S[k]\right),$$

with equality if only if $\dim(S) < d$ or $\frac{1}{k}S[k] = \operatorname{conv}(S)$.

We notice that Theorem 6.1 establishes Conjecture 6.1 for star-shaped compact sets in dimensions 2 and 3. It is worth remarking that the compact sets A constructed in [77] as counterexamples to Conjecture 6.1 are star-shaped, which makes Theorem 6.1 fairly unexpected, as we already observed in Chapter 2.

We prove Theorem 6.1 in Section 6.1. In Section 6.2 we adapt our techniques to investigate connected sets. Our main result in this section is summarized in Theorem 6.2. Finally, in Section 6.3 we collect some additional remarks and questions, and, in particular, we construct low dimensional counterexamples to a generalization of Conjecture 6.1, which also appeared in [31].

6.1 Conjecture 6.1 for star-shaped sets: the proof of Theorem 6.1

We start this section with a couple of Lemmas which are needed for the proof. Throughout this section, we denote $X_d(t) = \{(x_1, \ldots, x_d) \in \mathbb{N}^d : x_1 + \ldots + x_d = t\}$ and $N_d(t) = \operatorname{card} X_d(t)$ to be the number of elements of $X_d(t)$. Here and in the rest of the chapter we will denote by $\mathbb{N}$ the set of non-negative integers.

Lemma 6.1. *For any integer $t \geq 1$, and $d \geq 2$, we have $N_d(t) = \binom{t+d-1}{d-1}$.*

Proof. If $d = 2$, then, clearly, $N_2(t) = t + 1 = \binom{t+2-1}{1}$. On the other hand, by induction, we have

$$N_d(t) = \sum_{s=0}^{t} N_{d-1}(s) = \sum_{s=0}^{t} \binom{s+d-2}{d-2} = \binom{t+d-1}{d-1}.$$

□

Lemma 6.2. *Let $d \geq 2$ and o be the origin of $\mathbb{R}^d$, $(p_1, \ldots, p_d)$ be a basis of $\mathbb{R}^d$, and, let $B = \bigcup_{i=1}^d [o, p_i]$. Consider a compact set $M \subset \mathbb{R}^d$ such that $B[k] \subseteq M \subseteq k \operatorname{conv}(B)$ for some $k \geq \max\{2, (d-1)(d-2)\}$, then*

$$\operatorname{vol}\left(\frac{1}{k+1}(M+B)\right) \geq \operatorname{vol}\left(\frac{1}{k}M\right), \tag{6.2}$$

where, equality holds if and only if $M = k \operatorname{conv}(B)$. Furthermore, if $\operatorname{vol}\left(\frac{1}{k}M\right) \geq \operatorname{vol}\left(\frac{1}{k+1}(M+B)\right) - \delta$ for some $\delta \geq 0$, then $\operatorname{vol}(M) \geq \operatorname{vol}(k \operatorname{conv}(B)) - C(d,k)\delta$, where the constant $C(d,k) = k^d(1 - \frac{k^d}{(k-d+2)(k+1)^{d-1}})^{-1}$ depends only on d and k.

Proof. Since the inequality (6.2) is independent of a non-degenerate linear transformation applied to B and M simultaneously, we may assume that $(p_1, \ldots, p_d)$ is the canonical basis of $\mathbb{R}^d$. Let

$$V(t) = \operatorname{vol}\{(x_1, \ldots, x_d) \in [0,1]^d : x_1 + \ldots + x_d \leq t\}.$$

Let $C_i = i + [0,1]^d$, $i \in \mathbb{Z}^d$ be the unit cube cells of the lattice $\mathbb{Z}^d$, and set $\mu_i = \operatorname{vol}(C_i \cap M)$, and $\lambda_i = \operatorname{vol}(C_i \cap (M+B))$.

Note that for any $i \in X_d(t)$, $\operatorname{vol}(C_i \cap k \operatorname{conv}(B))$ is independent of i, namely it is equal to 1, if $t \leq k-d$, and to $V(k-t)$ if $t = k-d+1, \ldots, k-1$. A similar statement holds for $\operatorname{vol}(C_i \cap (k+1) \operatorname{conv}(B))$. The number of unit cells contained in $k \operatorname{conv}(B)$ is equal to the number of the solutions of the inequality $x_1 + x_2 + \ldots + x_d \leq k$, where each variable is a positive integer, and thus, it is $\binom{k}{d}$. Hence, if $Y_d(k)$ denotes the union of these cells, then we have that

$$\operatorname{vol}(Y_d(k)) = k^{\underline{d}} V, \tag{6.3}$$

where $k^{\underline{d}} = k(k-1)\ldots(k-d+1)$, and $V = \operatorname{vol}(\operatorname{conv} B) = \frac{1}{d!}$. Thus,

$$\operatorname{vol}(M) = k^{\underline{d}} V + \sum_{t=k-d+1}^{k-1} \sum_{i \in X_d(t)} \mu_i, \tag{6.4}$$

and

$$\operatorname{vol}(M+B) = (k+1)^{\underline{d}} V + \sum_{t=k-d+2}^{k} \sum_{i \in X_d(t)} \lambda_i.$$

In the following step, we give a lower bound on the λ_is depending on the values of the μ_is. We say that $i \in X_d(t)$ and $i' \in X_d(t+1)$ are *adjacent* if the corresponding cells C_i and $C_{i'}$ have a common facet, or in other words, if $i' - i$ coincides with one of the standard basis vectors p_j. In this case we write $ii' \in I$. Let $i \in X_d(t)$, and let $S = M \cap C_i$. Then, for every $j = 1, 2, \ldots, d$, $S + p_j \subset (M+B) \cap C_{i'}$ with $i' = i + p_j$. Thus, for any $i \in X_d(t+1)$,

$$\lambda_i \geq \max\{\mu_{i'} : i' \in X_d(t) \text{ is adjacent to } i\}. \tag{6.5}$$

Note that the right-hand side of this inequality is not less than any convex combination of the corresponding $\mu_{i'}$s. Using a suitable convex combination for each $i \in X_d(t+1)$, we show that this inequality implies that

$$\sum_{i \in X_d(t+1)} \lambda_i \geq \frac{t+d}{t+1} \sum_{i \in X_d(t)} \mu_i. \tag{6.6}$$

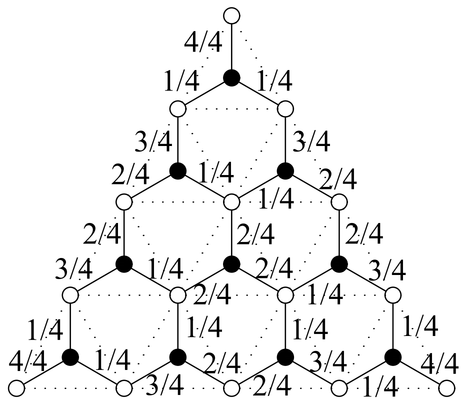

Figure 6.1: Illustration on choosing the weights if $d = 3$ and $t = 3$. The black and empty dots represent the elements of the set $X_3(3)$ and $X_3(4)$, respectively. Dots illustrating adjacent indices are connected by a segment. The weight assigned to the segment connecting the dots representing i and i' is equal to $\alpha_{ii'}$.

Consider some $i = (i_1, i_2, \ldots, i_d) \in X_d(t+1)$. Then the indices in $X_d(t)$ adjacent to i are all of the form $i - p_j$ for some $j = 1, 2, \ldots, d$. Furthermore, $i - p_j$ is adjacent to i iff $i_j \geq 1$, or in other words, iff $i_j \neq 0$. Now, for any $i' \in X_d(t)$ adjacent to i we set $\alpha_{ii'} = \frac{i_j}{t+1}$, where $i - i' = p_j$ (cf. Figure 6.1). Then, since $i \in X_d(t+1)$, we clearly have $1 = \sum_{j=1}^{d} \frac{i_j}{t+1} = \sum_{i' \in X_d(t), ii' \in I} \alpha_{ii'}$. Thus, by (6.5), we have

$$\lambda_i \geq \sum_{i' \in X_d(t), ii' \in I} \alpha_{ii'} \mu_{i'} \tag{6.7}$$

for all $i \in X_d(t+1)$. Now, let $i' \in X_d(t)$, and $i' = (i'_1, i'_2, \ldots, i'_d)$. Then the indices in $X_d(t+1)$ adjacent to i' are exactly those of the form $i' + p_j$ for some $i = 1, 2, \ldots, d$. Hence,

$$\sum_{i \in X_d(t+1), ii' \in I} \alpha_{ii'} = \sum_{j=1}^{d} \frac{i'_j + 1}{t+1} = \frac{t+d}{t+1}. \tag{6.8}$$

Finally, by (6.7) and (6.8)

$$\sum_{i \in X_d(t+1)} \lambda_i \geq \sum_{i \in X_d(t+1)} \sum_{i' \in X_d(t), ii' \in I} \alpha_{ii'} \mu_{i'} = \sum_{i' \in X_d(t)} \left(\sum_{i \in X_d(t+1), ii' \in I} \alpha_{ii'} \right) \mu_{i'}$$
$$= \frac{t+d}{t+1} \sum_{i' \in X_d(t)} \mu_{i'}.$$

Using this inequality and the assumption that $B[k] \subseteq M \subseteq k\,\mathrm{conv}(B)$, we obtain

$$\mathrm{vol}(M+B) \geq (k+1)^{\underline{d}} V + \sum_{t=k-d+1}^{k-1} \frac{t+d}{t+1} \sum_{i \in X_d(t)} \mu_i.$$

Note that the sequence $\left\{\frac{t+d}{t+1}\right\}$, where $t = 0, 1, 2, \ldots$, is strictly decreasing. Hence, using the fact that if $i \in X_d(t)$, then $\mu_i \leq V(k-t)$, one has, for $k-d+1 \leq t \leq k-1$,

$$\begin{aligned}\frac{t+d}{t+1} \sum_{i \in X_d(t)} \mu_i &\geq \frac{k+1}{k-d+2} \sum_{i \in X_d(t)} \mu_i + \left(\frac{t+d}{t+1} - \frac{k+1}{k-d+2} \right) V(k-t) N_d(t) \\ &\geq \frac{k+1}{k-d+2} \left(\sum_{i \in X_d(t)} \mu_i - V(k-t) N_d(t) \right) + \frac{t+d}{t+1} V(k-t) N_d(t).\end{aligned}$$

Hence

$$\begin{aligned}\operatorname{vol}(M+B) \geq (k+1)^{\underline{d}} V + \frac{k+1}{k-d+2} \sum_{t=k-d+1}^{k-1} \left(\sum_{i \in X_d(t)} \mu_i - V(k-t) N_d(t) \right) \\ + \sum_{t=k-d+1}^{k-1} \frac{t+d}{t+1} V(k-t) N_d(t). \quad (6.9)\end{aligned}$$

Observe that $\sum_{t=k-d+1}^{k-1} V(k-t) N_d(t) = (k^d - k^{\underline{d}}) V$, since it is the volume of the part of $k \operatorname{conv}(B)$ belonging to the cells that are not contained in $k \operatorname{conv}(B)$, and the equality follows by (6.3). Similarly, since

$$\frac{t+d}{t+1} N_d(t) = \frac{t+d}{t+1} \binom{t+d-1}{d-1} = \binom{t+d}{d-1} = N_d(t+1)$$

we deduce that

$$\sum_{t=k-d+1}^{k-1} \frac{t+d}{t+1} V(k-t) N_d(t) = \sum_{t=k-d+1}^{k-1} V(k-t) N_d(t+1) = \sum_{t'=k-d+2}^{k} V(k+1-t') N_d(t') = ((k+1)^d - (k+1)^{\underline{d}}) V,$$

since it is the volume of the part of $(k+1) \operatorname{conv}(B)$ belonging to cells that are not contained in $(k+1) \operatorname{conv}(B)$. Substituting these into (6.9) and using (6.4), we obtain

$$\begin{aligned}\operatorname{vol}(M+B) &\geq (k+1)^{\underline{d} V + \frac{k+1}{k-d+2} \left(\operatorname{vol}(M) - k^d V \right) + ((k+1)^d - (k+1)^{\underline{d}}) V} \\ &\geq \frac{k+1}{k-d+2} \operatorname{vol}(M) + \left((k+1)^d - \frac{k+1}{k-d+2} k^d \right) V.\end{aligned}$$

Thus,

$$\begin{aligned}\operatorname{vol}\left(\frac{1}{k+1}(M+B) \right) \geq \frac{k^d}{(k-d+2)(k+1)^{d-1}} \operatorname{vol}\left(\frac{1}{k} M \right) \\ + \left(1 - \frac{k^d}{(k-d+2)(k+1)^{d-1}} \right) V. \quad (6.10)\end{aligned}$$

Since $\operatorname{vol}\left(\frac{1}{k} M\right) \leq V$, to prove the first inequality of the lemma, it is sufficient to show that the right-hand side of (6.10) is a convex combination of the volumes, namely that the second coefficient is nonnegative. This is clear if $d = 2$, while for $d \geq 3$ using the Binomial Theorem, one has

$$\begin{aligned}(k-d+2)(k+1)^{d-1} - k^d &> (k-d+2)\left(k^{d-1} + (d-1)k^{d-2}\right) - k^d \\ &= k^{d-1} - (d-1)(d-2)k^{d-2},\end{aligned}$$

which is nonnegative for $k \geq (d-1)(d-2)$.

Now we prove the equality case. By (6.10), equality in the lemma implies that $\mathrm{vol}\left(\frac{1}{k}M\right) = V$, or equivalently, $\mathrm{vol}(k\,\mathrm{conv}(B) \setminus M) = 0$. Note that since

$$\mathrm{vol}(k\,\mathrm{conv}(B)) > 0,$$

its interior is not empty. Thus, $k\,\mathrm{conv}(B)$ is equal to the closure of its interior. On the other hand, $\mathrm{vol}(k\,\mathrm{conv}(B) \setminus M) = 0$ implies that $\mathrm{int}(k\,\mathrm{conv}(B)) \subset M$, but as M is compact, $M = k\,\mathrm{conv}(B)$ follows.

Finally, if $\mathrm{vol}\left(\frac{1}{k+1}(M+B)\right) - \delta \leq \mathrm{vol}\left(\frac{1}{k}M\right)$, then in the same way (6.10) yields the inequality $\mathrm{vol}(M) \geq \mathrm{vol}(k\,\mathrm{conv}(B)) - C(d,k)\delta$, with

$$C(d,k) = \frac{k^d}{1 - \frac{k^d}{(k-d+2)(k+1)^{d-1}}}. \tag{6.11}$$

□

Proof of Theorem 6.1. Without loss of generality, we may assume that S is star-shaped with respect to the origin. Let $\varepsilon > 0$ be an arbitrary positive number. By Carathéodory's theorem, we may choose a finite point set $A_0 \subset S$ such that $\mathrm{vol}(\mathrm{conv}(S)) - \varepsilon \leq \mathrm{vol}(\mathrm{conv}(A_0))$, and without loss of generality, we may assume that the points of A_0 are in convex position. Clearly, the star-shaped set $A = \bigcup_{a \in A_0} [o, a]$ is a subset of S, satisfying $\mathrm{vol}(\mathrm{conv}(S)) - \varepsilon \leq \mathrm{vol}(\mathrm{conv}(A))$. Consider a simplicial decomposition $\mathcal{F}$ of the boundary of $\mathrm{conv}(A)$ such that all vertices of $\mathcal{F}$ are vertices of $\mathrm{conv}(A)$. Let the $(d-1)$-dimensional faces of $\mathcal{F}$ be $F_1, F_2, \ldots, F_m$, and for $j = 1, 2, \ldots, m$, let $B_j = \bigcup_{t=1}^{d} [o, p_t^j]$, where $p_1^j, p_2^j, \ldots, p_d^j$ are the vertices of F_j. Then $B_j \subseteq S$ for all values of j, the sets $\mathrm{conv}(B_j)$ are mutually non-overlapping, and $\mathrm{conv}(A) = \bigcup_{j=1}^{m} \mathrm{conv}(B_j)$. Finally, let $M_j = S[k] \cap (k\,\mathrm{conv}(B_j))$. Then, since $B_j \subseteq S$, we have $B_j[k] \subseteq M_j \subseteq (k\,\mathrm{conv}(B_j))$. Thus, Lemma 6.2 implies that $\mathrm{vol}\left(\frac{1}{k+1}(M_j + B_j)\right) \geq \mathrm{vol}\left(\frac{1}{k}M_j\right)$. Thus, we have

$$\begin{aligned}\mathrm{vol}\left(\frac{S[k]}{k} \cap \mathrm{conv}(A)\right) &= \sum_{j=1}^{m} \mathrm{vol}\left(\frac{S[k]}{k} \cap \mathrm{conv}(B_j)\right) = \sum_{j=1}^{m} \mathrm{vol}\left(\frac{M_j}{k}\right) \\ &\leq \sum_{j=1}^{m} \mathrm{vol}\left(\frac{M_j + B_j}{k+1}\right) \leq \mathrm{vol}\left(\frac{S[k+1]}{k+1}\right).\end{aligned}$$

On the other hand, since $0 \leq \mathrm{vol}(\mathrm{conv}(S)) - \mathrm{vol}(\mathrm{conv}(A)) \leq \varepsilon$, we have

$$0 < \mathrm{vol}\left(\frac{S[k]}{k}\right) - \mathrm{vol}\left(\mathrm{conv}(A)\right) \leq \varepsilon,$$

implying that

$$\mathrm{vol}\left(\frac{S[k]}{k}\right) - \varepsilon \leq \sum_{j=1}^{m} \mathrm{vol}\left(\frac{M_j}{k}\right) \leq \sum_{j=1}^{m} \mathrm{vol}\left(\frac{M_j + B_j}{k+1}\right) \leq \mathrm{vol}\left(\frac{S[k+1]}{k+1}\right). \tag{6.12}$$

This inequality is satisfied for all positive ε, and thus, the inequality part of Theorem 6.1 holds.

Now, assume that

$$\mathrm{vol}\left(\frac{S[k]}{k}\right) = \mathrm{vol}\left(\frac{S[k+1]}{k+1}\right),$$

and that $\dim(S) = d$. Then, from inequality (6.12) we deduce that

$$\sum_{j=1}^{m}\left(\mathrm{vol}\left(\frac{M_j + B_j}{k+1}\right) - \mathrm{vol}\left(\frac{M_j}{k}\right)\right) \leq \varepsilon.$$

For $j = 1, 2, \ldots, m$, set $\delta_j = \mathrm{vol}\left(\frac{1}{k+1}(M_j + B_j)\right) - \mathrm{vol}\left(\frac{1}{k}M_j\right)$. Then, clearly $\sum \delta_j \leq \varepsilon$. On the other hand, by Lemma 6.2, for every $j = 1, 2, \ldots, m$, we have $\mathrm{vol}(k\,\mathrm{conv}(B_j)) - \mathrm{vol}(M_j) \leq C(k,d)\delta_j$, where $C(k,d)$ is defined in (6.11). Thus, summing on j, it follows that

$$\varepsilon C(k,d) \geq \mathrm{vol}(\mathrm{conv}(kA)) - \mathrm{vol}(S[k] \cap \mathrm{conv}(kA)),$$

implying that $\varepsilon\left(k^d + C(k,d)\right) \geq \mathrm{vol}(\mathrm{conv}(kS)) - \mathrm{vol}(S[k])$. This inequality holds for any value $\varepsilon > 0$, and hence, $\mathrm{vol}(\mathrm{conv}(S)) = \mathrm{vol}\left(\frac{1}{k}S[k]\right)$, or equivalently, $\mathrm{vol}\left(\mathrm{conv}(S) \setminus \frac{1}{k}S[k]\right) = 0$. Since $\mathrm{conv}(S)$ is a compact, convex set with nonempty interior, and $\frac{1}{k}S[k]$ is compact, to show the equality $\mathrm{conv}(S) = \frac{1}{k}S[k]$, we may apply the argument at the end of the proof of Lemma 6.2. □

6.2 Conjecture 6.1 for connected sets

In the first few lemmas we collect some elementary properties of the Minkowski sum of connected sets. Throughout this section, e_1, e_2 denotes the elements of the standard orthonormal basis of $\mathbb{R}^2$.

Lemma 6.3. *Let $A \subset \mathbb{R}^d$ be a compact set with a connected boundary and let $\mathrm{bd}\,A \subseteq B \subseteq A$. Then $B + B = A + A$.*

Proof. We have $\mathrm{bd}\,A + \mathrm{bd}\,A \subseteq B + B \subseteq A + A$. Thus it is sufficient to prove that $\mathrm{bd}\,A + \mathrm{bd}\,A = A + A$. Clearly, $A + A \supseteq \mathrm{bd}\,A + \mathrm{bd}\,A$. We show that $\frac{A+A}{2} \subseteq \frac{\mathrm{bd}\,A + \mathrm{bd}\,A}{2}$, which then yields the assertion. Consider a point $p \in \frac{A+A}{2}$. Then p is the midpoint of a segment whose endpoints are points of A. Let $\chi_p : \mathbb{R}^d \to \mathbb{R}^d$ be the reflection about p defined by $\chi_p(x) = 2p - x$, for $x \in \mathbb{R}^d$. To prove that $p \in \frac{\mathrm{bd}\,A + \mathrm{bd}\,A}{2}$ we need to show that for some $q \in \mathrm{bd}\,A$, we have $\chi_p(q) \in \mathrm{bd}\,A$. To do this, let us define $f_p(x)$ ($x \in \mathbb{R}^d$) as the signed distance of $\chi_p(x)$ from the boundary of A, where the sign is positive if $\chi_p(x) \notin A$, and not positive if $\chi_p(x) \in A$. Here we remark that since A is compact, $\mathrm{bd}\,A$ is compact as well. Let x_1 be a point of $\mathrm{bd}\,A$ farthest from p. If $\chi_p(x_1) \in A$ then $\chi_p(x_1) \in \mathrm{bd}\,A$, and we are done. Thus, assume that $\chi_p(x_1) \notin A$, implying that $f_p(x_1) > 0$. Now, since $p \in \frac{A+A}{2}$, we have some $y \in A$ such that $\chi_p(y) \in A$. Let L be the line through y, p and $\chi_p(y)$. Let y' and y'' be points of $L \cap \mathrm{bd}\,A$ closest to y and $\chi_p(y)$, respectively. Then the segments $[y, y']$ and $[\chi_p(y), y'']$ are included in A. If $0 < ||y' - y|| \leq ||y'' - \chi_p(y)||$, then $y' \in \mathrm{bd}\,A$ and $\chi_p(y') \in [\chi_p(y), y''] \subset A$. If $0 < ||y'' - \chi_p(y)|| \leq ||y' - y||$, then the same holds for y'' in place of y'. Thus, it follows that for some point $x_2 \in \mathrm{bd}\,A$, $\chi_p(x_2) \in A$. If $\chi_p(x_2) \in \mathrm{bd}\,A$, then we are done, and so we may assume that $\chi_p(x_2) \in \mathrm{int}\,A$, which yields that $f_p(x_2) < 0$.

We have shown that $f_p : \mathrm{bd}\,A \to \mathbb{R}$ attains both a positive and a negative value on its domain. On the other hand, since f is continuous and $\mathrm{bd}\,A$ is connected, $f_p(q) = 0$ for some $q \in \mathrm{bd}\,A$, from which the assertion readily follows. □

Remark 6.1. *Lemma 6.3 holds also for the boundary of the external connected component of* $\mathbb{R}^d \setminus A$ *in place of* $\operatorname{bd} A$*.*

Remark 6.2. *We note that the equality* $A_1 + A_2 = \operatorname{bd} A_1 + \operatorname{bd} A_2$ *does not hold in general for different compact sets* A_1, A_2 *with connected boundaries. To show it, one may consider the sets* $A_1 = \mathbf{B}^2$ *and* $A_2 = \varepsilon \mathbf{B}^2$ *for some sufficiently small value of* ε*.*

Remark 6.3. *Lemma 6.3 does not hold if we omit the condition that* $\operatorname{bd} A$ *is connected. To show it, we may choose* A *as the union of* $\mathbf{B}^2$ *and a singleton* $\{p\}$ *with* $||p||$ *being sufficiently large.*

Corollary 6.1. *If* A *is a compact set with a connected boundary then* $A+A = A+\operatorname{bd} A = \operatorname{bd} A+\operatorname{bd} A$*. Thus, for any positive integer* $k \geq 2$*, we have* $A[k] = \operatorname{bd} A[k]$*.*

Corollary 6.2. *Let* $d \geq 2$ *and* $k \geq \max\{2, (d-1)(d-2)\}$*. Let* A *be a compact set such that* $\partial S \subseteq A \subseteq S$ *for some compact, star-shaped set* $S \subset \mathbb{R}^d$*. Then we have*

$$\operatorname{vol}\left(\frac{1}{k} A[k]\right) \leq \operatorname{vol}\left(\frac{1}{k+1} A[k+1]\right).$$

Proof. Without loss of generality, we may assume that S is star-shaped with respect to the origin. Set $S' = S + \varepsilon \mathbf{B}^d$ for some small value $\varepsilon > 0$.

First, we show that $\partial S'$ is path-connected. Let L be a ray starting at o. Since $o \in \operatorname{int} S'$, $L \cap \operatorname{bd} S' \neq \emptyset$. Let $p \in L \cap \operatorname{bd} S'$. Then there is a point $q \in S$ such that $||q-p|| = \varepsilon$. Now, if x is any relative interior point of $[o,q]$, then the line through x and parallel to $[p,q]$ intersects $[o,q]$ at a point at distance less than ε from x. Since $[o,q] \subseteq S$, from this it follows that $x \in S + \varepsilon \operatorname{int} \mathbf{B}^d \subseteq \operatorname{int} S'$. In other words, for any $p \in \operatorname{bd} S'$, all points of $[o,p]$ but p lie in $\operatorname{int} S'$. Thus, $L \cap \operatorname{bd} S'$ is a singleton for any ray L starting at o.

Let $0 < r < R$ such that $\operatorname{bd} S' \subset H = R\mathbf{B}^d \setminus (r \operatorname{int} \mathbf{B}^d)$. Let $P : H \to \mathbb{S}^{d-1}$ be the central projection to $\mathbb{S}^{d-1}$. Note that P is Lipschitz, and thus continuous on H, and its restriction $P|_{\operatorname{bd} S'}$ to $\operatorname{bd} S'$ is bijective. On the other hand, since $\operatorname{bd} S'$ (as also S') are compact, this implies that the inverse of $P|_{\operatorname{bd} S'}$ is continuous, that is, $\operatorname{bd} S'$ and $\mathbb{S}^{d-1}$ are homeomorphic. Thus, $\operatorname{bd} S'$ is path-connected.

On the other hand, $\operatorname{bd} S \subseteq A \subseteq S$ implies that $A' = A + \varepsilon \mathbf{B}^d \subseteq S'$, and $\operatorname{bd} S' \subseteq \operatorname{bd} S + \varepsilon \mathbb{S}^{d-1} \subseteq \operatorname{bd} S + \varepsilon \mathbf{B}^d \subseteq A'$. Now, we may apply Lemma 6.3 and Corollary 6.1, and obtain that for any value of $k \geq 2$, $A'[k] = S'[k]$. Thus, by Theorem 6.1 it follows that

$$\operatorname{vol}\left(\frac{A[k]}{k} + \varepsilon \mathbf{B}^d\right) = \operatorname{vol}\left(\frac{A'[k]}{k}\right) \leq \operatorname{vol}\left(\frac{A'[k+1]}{k+1}\right) = \operatorname{vol}\left(\frac{A[k+1]}{k+1} + \varepsilon \mathbf{B}^d\right).$$

On the other hand, for any compact set C the function $t \mapsto \operatorname{vol}(C + t\mathbf{B}^d)$ is continuous on $[0, +\infty)$, see for example [79], hence $\lim_{\varepsilon \to 0^+} \operatorname{vol}\left(\frac{1}{m} A[m] + \varepsilon \mathbf{B}^d\right) = \operatorname{vol}\left(\frac{1}{m} A[m]\right)$, for any integer m which implies the corollary. □

Let us denote the closure of a set $A \subset \mathbb{R}^d$ by $\operatorname{cl}(A)$.

Proposition 6.1. *Let* $\gamma \subset \mathbb{R}^2$ *be a simple continuous curve connecting* o *and* e_1 *such that its intersection with the* x*-axis is* $\{o, e_1\}$*. Let* D *be the interior of the closed Jordan curve* $\gamma \cup [o, e_1]$*. For* $i = 0, 1$*, let* $\gamma_i = \frac{i}{2} e_1 + \frac{1}{2}\gamma$*, and* $D_i = \frac{i}{2} e_1 + \frac{1}{2} D$*. Then* $\operatorname{cl}(D \setminus (D_0 \Delta D_1)) \subseteq \frac{1}{2}\gamma[2]$*, where* Δ *denotes symmetric difference.*

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Proof. For convenience, we assume that γ lies in the half plane $\{y \leq 0\}$. As in the proof of Lemma 6.3, let $\chi_p : \mathbb{R}^2 \to \mathbb{R}^2$ denote the reflection about $p \in \mathbb{R}^2$ defined by $\chi_p(x) = 2p - x$, and note that $p \in \frac{1}{2}\gamma[2]$ if and only if there is some point $q \in \gamma$ such that $\chi_p(q) \in \gamma$, or in other words, if $\gamma \cap \chi_p(\gamma) \neq \emptyset$. Let L denote the x-axis, $L_p = \chi_p(L)$, and let S be the infinite strip between L and L_p (cf. Figure 6.2).

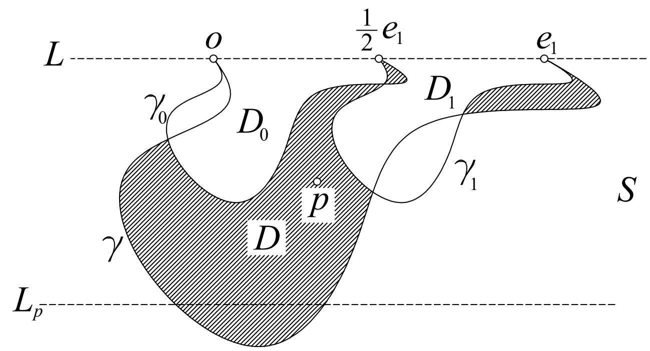

Figure 6.2: An illustration for Proposition 6.1. The dashed region belongs to $\frac{1}{2}\gamma[2]$.

First, observe that $o, e_1 \in \gamma$ yields that $\gamma_0 \cup \gamma_1 \subset \frac{1}{2}\gamma[2]$, and $\gamma \subset \frac{1}{2}\gamma[2]$ trivially holds. Thus, we need to show that if for some point p we have $p \in D \setminus \mathrm{cl}(D_0 \cup D_1)$ or $p \in D_0 \cap D_1 \cap D$, then $p \in \frac{1}{2}\gamma[2]$. We do it only for the case $p \in D \setminus \mathrm{cl}(D_0 \cup D_1)$ since for the second case a similar argument can be applied.

Consider some point $p \in D \setminus (D_0 \cup D_1)$. Then $p \notin \mathrm{cl}(D_0 \cup D_1)$ yields that $\chi_p(o) = 2p \notin \mathrm{cl}\, D$, and the relation $\chi_p(e_1) \notin \mathrm{cl}\, D$ follows similarly.

Case 1: $\gamma \subset S$. Note that in this case $\chi_p(\gamma) \subset S$. Since $p \in D$ and $\chi_p(o) \notin \mathrm{cl}\, D$, $\mathrm{bd}\, D = \gamma \cup [o, e_1]$ and $[\chi_p(o), p] \cap [o, e_1] = \emptyset$, it follows by the continuity of γ that $\gamma \cap [\chi_p(o), p] \neq \emptyset$. Hence, by the compactness of γ, there is a point $x \in \gamma \cap [\chi_p(o), p]$ closest to p. By its choice, $\chi_p(x) \in D \cup \gamma$. If $\chi_p(x) \in \gamma$, we are done, and thus, we assume that $\chi_p(x) \in D$. This implies that $\chi_p(\gamma)$ contains both interior and exterior points of D. On the other hand, since $\chi_p(\gamma) \subset S$, this implies that $\chi_p(\gamma) \cap \gamma \neq \emptyset$.

Case 2: $\gamma \not\subset S$. Let $\gamma_p = \gamma \cap S$, and let $\bar{\gamma}_0$ and $\bar{\gamma}_1$ denote the connected components of γ_p containing o and e_1, respectively. For $i = 0, 1$, we denote the endpoint of $\bar{\gamma}_i$ on L_p by x_i. Clearly, since γ is simple and continuous, x_0 is on the left-hand side of x_1, and the curve $\bar{\gamma}_0 \cup [x_0, x_1] \cup \bar{\gamma}_1 \cup [o, e_1]$ is a Jordan curve. We denote the interior of this curve by D_p.

Consider the case where $p \notin D_p$. Then p is an exterior point of D_p, and there is a connected component γ^* of γ_p, with endpoints on L_p, that separates p from L. Since the reflections of the endpoints of γ^* about p lie on L, we may apply the argument in Case 1, and obtain that $\emptyset \neq \gamma^* \cap \chi_p(\gamma^*) \subseteq \gamma \cap \chi_p(\gamma)$. Thus, we may assume that $p \in D_p$.

If $\chi_p(x_0) \in [o, e_1]$, then the continuity of $\bar{\gamma}_0$ and $\chi_p(o) \notin \mathrm{cl}\, D$ implies that $\emptyset \neq \gamma \cap \chi_p(\bar{\gamma}_0) \subseteq \gamma \cap \chi_p(\gamma)$. If $\chi_p(x_1) \in [o, e_1]$, then we may apply a similar argument, and thus we may assume that $\chi_p(x_0), \chi_p(x_1) \notin [o, e_1]$. This implies that either $[o, p_1] \subset [\chi_p(x_0), \chi_p(x_1)]$, or $[\chi_p(x_0), \chi_p(x_1)]$ and $[o, p_1]$ are disjoint.

The relation $[o, p_1] \subset [\chi_p(x_0), \chi_p(x_1)]$ yields $[\chi_p(o), \chi_p(p_1)] \subset [x_0, x_1]$, and, by the previous argument, we have $\emptyset \neq \chi_p(\bar{\gamma}_0) \cap \bar{\gamma}_1 \subseteq \gamma \cap \chi_p(\gamma)$. Thus, we are left with the case where $[\chi_p(x_1), \chi_p(x_2)]$ and $[o, p_1]$ are disjoint; without loss of generality we may assume that $\chi_p(x_1)$, $\chi_p(x_0)$, o and e_1 are in this consecutive order on L. Let U be the closure of the connected component of $S \setminus \bar{\gamma}_0$ containing $\bar{\gamma}_1$. Then $\chi_p(p) = p \in \operatorname{int} U \cap \chi_p(U)$, implying that $\emptyset \neq \gamma_1 \cap \chi_p(\gamma_1) \subseteq \gamma \cap \chi_p(\gamma)$. $\square$

The proof of Lemma 6.4 is based on the idea of the proof of Proposition 6.1, with some necessary modifications.

Lemma 6.4. *Let $k \geq 2$, and let $\gamma \subset \mathbb{R}^2$ be a convex, continuous curve connecting o and e_1 such that its intersection with the x-axis is $\{o, e_1\}$. Let D be the interior of the closed Jordan curve $\gamma \cup [o, e_1]$. For $i = 0, 1, \ldots, k-1$, let $\gamma_i = \frac{i}{k}e_1 + \frac{1}{k}\gamma$, and $D_i = \frac{i}{k}e_1 + \frac{1}{k}D$. Then $\operatorname{cl}\left(D \setminus (\bigcup_{i=1}^k D_i)\right) \subseteq \frac{1}{k}\gamma[k]$, and for any $i \neq j$, $D_i \cap D_j \subseteq \frac{1}{k}\gamma[k]$.*

Proof. First observe that D is convex, hence D_i is contained in D for all values of i. Let us denote the x-axis by L and, for any $p \in \mathbb{R}^2$, let $\chi_p^k : \mathbb{R}^2 \to \mathbb{R}^2$ be the homothety with center p and ratio $-\frac{1}{k-1}$ defined by $\chi_p^k(x) = \frac{k}{k-1}p - \frac{x}{k-1}$, for $x \in \mathbb{R}^2$. Furthermore, we set $L_p^k = \chi_p^k(L)$, and denote the infinite strip between L and L_p^k by S. The assertion for $k = 2$ is a special case of Proposition 6.1. To prove it for $k \geq 3$, we apply induction on k, and assume that the lemma holds for $\gamma[k-1]$.

Let $p \in \operatorname{cl}\left(D \setminus (\bigcup_{i=1}^k D_i)\right)$. Clearly, since $(\operatorname{bd} D) \setminus (\bigcup_{i=1}^k D_i) = \gamma \subseteq \gamma[k]$, we may assume that $p \in D$. By the induction hypothesis for $\frac{k-1}{k}\gamma$, if $p \in X_1 = \frac{k-1}{k} \operatorname{cl} D$, then $p \in \frac{k-1}{k} \cdot \frac{1}{k-1}\gamma[k-1] = \frac{1}{k}\gamma[k-1] \subseteq \frac{1}{k}\gamma[k]$. Similarly, if $p \in X_2 = \frac{1}{k}e_1 + \frac{k-1}{k} \operatorname{cl} D$, then $p \in \frac{1}{k}e_1 + \frac{1}{k}\gamma[k-1] \subseteq \frac{1}{k}\gamma[k]$. Thus, assume that $p \notin X_1 \cup X_2$, which yields that $\chi_p^k(o)$ and $\chi_p^k(e_1)$ are in the exterior of D. Let the (unique) intersection point of $[p, \chi_p^k(o)]$ and γ be q_1 and the (unique) intersection point of $[p, \chi_p^k(e_1)]$ and γ be q_2. As $\chi_p^k(q_1) \in [o, p]$, the convexity of D implies that $\chi_p^k(q_1) \in D$, and the containment $\chi_p^k(q_2) \in D$ follows similarly.

Similarly like in Proposition 6.1, if $\gamma \subset S$, then by continuity, $\gamma \cap \chi_p^k(\gamma) \neq \emptyset$, which implies the containment $p \in \frac{1}{k}\gamma[k]$. Assume that $\gamma \not\subset S$. Then $S \cap \gamma$ has two connected components γ_1, γ_2, where we choose the indices such that $o \in \gamma_1$, and $e_1 \in \gamma_2$. Clearly, we have either $q_1 \in \gamma_2$, $q_2 \in \gamma_1$, or both. If $q_1 \in \gamma_2$, then the containment relations $\chi(q_1) \in D$, $\chi(e_1) \notin \operatorname{cl} D$, and $\chi_p^k(\gamma_2) \subset S$ yield that $\emptyset \neq \gamma_1 \cap \chi_p^k(\gamma_2) \subset \gamma \cap \chi_p^k(\gamma)$. If $q_2 \in \gamma_1$, then the assertion follows by a similar argument.

Finally, we consider the case that $p \in D_i \cap D_j$ for some $i < j$. In this case the convexity of D implies that $p \in D_s$ for any $i \leq s \leq j$. This yields that there are some distinct values $i, j \leq k-1$ or $i, j \geq 2$ such that $p \in D_i \cap D_j$. Thus, the assertion readily follows from the induction hypothesis. $\square$

Lemma 6.5 is a variant of Lemma 6.2 for some path-connected sets in $\mathbb{R}^2$.

Lemma 6.5. *Let $k \geq 2$ and γ be a bounded convex curve in $\mathbb{R}^2$, and let $\gamma[k] \subseteq M \subseteq k \operatorname{conv}(\gamma)$. Then*

$$\operatorname{area}\left(\frac{1}{k}M\right) \leq \operatorname{area}\left(\frac{1}{k+1}(M + \gamma)\right).$$

Proof. If γ is closed, then Lemma 6.3 yields that $\frac{1}{k}\gamma[k] = \operatorname{conv}(\gamma)$ for all $k \geq 2$, which clearly implies the statement. Assume that γ is not closed. Since the inequalities in Lemma 6.5 do not change under affine transformations, we may assume that the endpoints of γ are o and e_1, and the x-axis is a supporting line of $\operatorname{conv}(\gamma)$.
Let us define

$$D = \operatorname{conv}(\gamma), \alpha = \operatorname{area}(D \cap (e_1 + D)), \text{ and } \beta = \operatorname{area}\left(D \cap ((e_1 + D) \cup (-e_1 + D))\right).$$

Note that $0 \leq \alpha \leq \beta \leq 2\alpha$. Let $D_i = ie_1 + D$ for $i = 0, 1, \ldots, k$. For $0 \leq i \leq k-1$, let μ_i be the area of the region of M in D_i that do not belong to any D_j, $j \neq i$, where we note that since $k \geq 2$, by Lemma 6.4 we have that all other points of D_i belong to M. Similarly, for $0 \leq i \leq k$, let λ_i be the area of the region of $M + \gamma$ in D_i that do not belong to any D_j, $j \neq i$. An elementary computation shows that

$$\begin{aligned}\operatorname{area}(M) =& k^2 \operatorname{area}(D) - 2(\operatorname{area}(D) - \alpha) - (k-2)(\operatorname{area}(D) - \beta) + \sum_{i=0}^{k-1} \mu_i \\ =& (k^2 - k) \operatorname{area}(D) + 2\alpha + (k-2)\beta + \sum_{i=0}^{k-1} \mu_i,\end{aligned} \tag{6.13}$$

and similarly,

$$\operatorname{area}(M + \gamma) = (k^2 + k) \operatorname{area}(D) + 2\alpha + (k-1)\beta + \sum_{i=0}^{k} \lambda_i. \tag{6.14}$$

Since $o, e_1 \in \gamma$, we have $M, e_1 + M \subseteq M + \gamma$. Thus, $\lambda_0 \geq \mu_0$, $\lambda_k \geq \mu_{k-1}$, $\lambda_1 \geq \max\{\mu_0 - (\beta - \alpha), \mu_1\}$, $\lambda_{k-1} \geq \max\{\mu_{k-2}, \mu_{k-1} - (\beta - \alpha)\}$, and for $2 \leq i \leq k-2$, $\lambda_i \geq \max\{\mu_{i-1}, \mu_i\}$. Since $\lambda_i \geq \frac{i}{k}\mu_{i-1} + \frac{k-i}{k}\mu_i$ if $2 \leq i \leq k-2$, and $\lambda_i \geq \frac{i}{k}\mu_{i-1} + \frac{k-i}{k}\mu_i - \frac{1}{k}(\beta - \alpha)$ if $i = 1$ or $i = k-1$, it follows that

$$\sum_{i=0}^{k} \lambda_i \geq \frac{k+1}{k} \sum_{i=1}^{k-1} \mu_i - \frac{2}{k}(\beta - \alpha).$$

Thus, by (6.13),

$$\sum_{i=0}^{k} \lambda_i \geq \frac{k+1}{k} \left(\operatorname{area}(M) - (k^2 - k) \operatorname{area}(D) - 2\alpha - (k-2)\beta \right) - \frac{2}{k}(\beta - \alpha).$$

After substituting this into (6.14) and simplifying, we obtain

$$\operatorname{area}(M + \gamma) \geq \frac{k+1}{k} \operatorname{area}(M) + (k+1) \operatorname{area}(D),$$

which yields

$$\operatorname{area}\left(\frac{1}{k+1}(M + \gamma)\right) \geq \frac{k}{k+1} \operatorname{area}\left(\frac{1}{k}M\right) + \frac{1}{k+1} \operatorname{area}(D).$$

Thus, the inequality $\operatorname{area}\left(\frac{1}{k}M\right) \leq \operatorname{area}(D)$ yields the assertion. □

In Theorem 6.2, by an open topological disc we mean the bounded connected component defined by a Jordan curve, and recall that a convex body is a compact, convex set with nonempty interior.

Theorem 6.2. *Let $k \geq 2$. Let K be a plane convex body, and let $\mathcal{F} = \{F_i : i \in I\}$ be a family of pairwise disjoint open topological discs such that if $F_i \cap \operatorname{bd} K \neq \emptyset$ then $F_i \cap \operatorname{bd} K$ is a connected curve and F_i is convex. Let $X = K \setminus \left(\bigcup_{i \in I} F_i\right)$. Then*

$$\operatorname{area}\left(\frac{1}{k}X[k]\right) \leq \operatorname{area}\left(\frac{1}{k+1}X[k+1]\right).$$

Proof. Clearly, we may assume that each F_i intersects K, and also for each F_i, $(\operatorname{bd} K) \setminus F_i$ is infinite, since removing the first type discs does not change X, and if there is some F_i such that $(\operatorname{bd} K) \setminus F_i$ is finite, then X is either $\emptyset$ or a singleton, and in both cases the statement is trivial. Thus, we have that if F_i intersects $\operatorname{bd} K$, then the boundary of the convex set $F_i \cap K$ consists of the two connected, convex curves $F_i \cap \operatorname{bd} K$ and $K \cap \operatorname{bd} F_i$.

First, note that since each member of $\mathcal{F}$ has positive area, it has countably many elements; indeed, for any $\delta > 0$ there are only finitely many elements F_i of $\mathcal{F}$ for which $\operatorname{area}(F_i \cap K) \geq \delta$, and thus, we may list the elements according to area. Furthermore, since X is compact, $\operatorname{area}(X)$ exists.

By Lemma 6.3, we may assume that every member of $\mathcal{F}$ intersects $\operatorname{bd} K$; indeed, if some F_i does not intersect $\operatorname{bd} K$, then $\operatorname{bd} F_i$ is a compact, connected set in X, implying that $F_i \subseteq \frac{1}{k}(\operatorname{bd} F_i)[k] \subseteq \frac{1}{k}X[k]$ for all $k \geq 2$. For any $i \in I$, let γ_i denote the part of $\operatorname{bd} F_i$ in K. Clearly, γ_i is a convex curve, and the segment connecting its endpoints lie in K by convexity. As the two endpoints of γ_i are in $\operatorname{bd} K$, the line through them supports $K \setminus F_i$. Choose some finite subfamily $I_\varepsilon \subseteq I$ such that $\operatorname{area}(X_\varepsilon \setminus X) \leq \varepsilon$, where $X_\varepsilon = K \setminus \left(\bigcup_{i \in I_\varepsilon} F_i\right)$. This is possible, since for any ordering of the elements, $\sum_{i \in I} \operatorname{area}(K \cap F_i)$ is a bounded series with positive elements, and hence, it is absolute convergent, and convex sets with small area and bounded diameter are contained in a small neighborhood of their boundary.

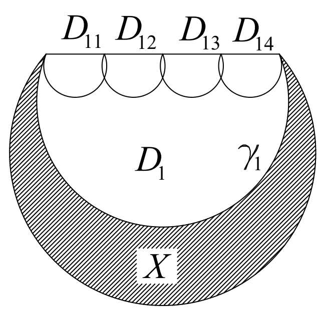

Figure 6.3: An illustration for the proof of Theorem 6.2.

For any $i \in I_\varepsilon$, we set $D_i = F_i \cap K$, and observe that D_i is a convex set separated from X_ε by the convex curve γ_i. Let the endpoints of γ_i be q_i^1 and q_i^2, and let D_{i1} be the homothetic copy of D_i with ratio $\frac{1}{k}$ and center $\_i^1$. Furthermore, for $j = 2, 3, \ldots, k$, let $D_{ij} = \frac{j-1}{k}\left(q_i^2 - q_i^1\right) + D_{i1}$ (cf. Figure 6.3). Then, by Lemma 6.4, $\frac{1}{k}\gamma_i[k] \subseteq \frac{1}{k}X_\varepsilon[k]$ contains all points of D_i belonging to none of the D_{ij}s or to at least two of them. Let $M_i = (X[k] \cap (kD_i))$. Then $M_i \subseteq \operatorname{conv}(kD_i)$, and thus, Lemma 6.5 yields that

$$\operatorname{area}\left(\frac{1}{k}M_i\right) \leq \operatorname{area}\left(\frac{1}{k+1}(M_i + \gamma_i)\right).$$

On the other hand, with the notation $D_\varepsilon = \bigcup_{i \in I_\varepsilon} D_i$, we have

$$\operatorname{area}\left(\frac{1}{k}X[k] \cap D_\varepsilon\right) = \sum_{i \in I_\varepsilon} \operatorname{area}\left(\frac{1}{k}M_i\right),$$

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and

$$\operatorname{area}\left(\frac{1}{k+1}X[k+1]\cap D_\varepsilon\right) \geq \sum_{i\in I_\varepsilon} \operatorname{area}\left(\frac{1}{k+1}(M_i+\gamma_i)\right),$$

and thus, we have $\operatorname{area}\left(\frac{1}{k}X[k]\cap D_\varepsilon\right) \leq \operatorname{area}\left(\frac{1}{k+1}X[k+1]\cap D_\varepsilon\right)$. On the other hand, since $\operatorname{area}(X_\varepsilon \setminus X) < \varepsilon$, $X_\varepsilon \cup D_\varepsilon = \operatorname{conv}(X)$, and $X \subseteq X_\varepsilon$, we have that $\operatorname{area}(\frac{1}{m}X[m]\setminus D_\varepsilon) \leq \varepsilon$ for all $m \geq 1$. This implies that

$$\operatorname{area}\left(\frac{1}{k}X[k]\right) \leq \operatorname{area}\left(\frac{1}{k+1}X[k+1]\right) - \varepsilon.$$

This holds for all $\varepsilon > 0$, which yields the assertion. □

6.3 Additional remarks and questions

Remark 6.4. *One can ask if the statement of Theorem 6.1 holds for arbitrary measure instead of volume. The answer to this question is negative. Indeed, consider the measure* $\mu(K) = \operatorname{vol}(K\cap C)$, *where* $C = \left[-\frac{1}{d}, \frac{1}{d}\right]^d$ *and* $S = \bigcup_{i=1}^d [o, e_i]$, *where* $e_1, e_2, \ldots, e_d$ *are the vectors of the standard orthonormal basis. Then, clearly, we have*

$$\mu\left(\frac{1}{2k}S[2k]\right) = \frac{1}{2^d}\operatorname{vol}(C) > \mu\left(\frac{1}{2k+1}S[2k+1]\right).$$

Remark 6.5. *The statement of Theorem 6.1 does not hold for arbitrary measures even for rotationally invariant measures in the plane: for any value of* k *there is a compact, star-shaped set* $S \subset \mathbb{R}^2$ *such that* $\operatorname{vol}\left(\frac{1}{k}S[k]\cap \mathbf{B}^2\right) > \operatorname{vol}\left(\frac{1}{k+1}S[k+1]\cap\mathbf{B}^2\right)$. *To prove this, set* $S = [o, e_1]\cup[o, e_2]$, *and let* E *denote the ellipse centered at* o *and containing the points* $(1-1/k, 0)$ *and* $(1-2/k, 1/k)$. *It is an elementary computation to check that in this case* $\operatorname{vol}\left(\frac{1}{k}S[k]\cap E\right) = \frac{1}{4}\operatorname{vol}(E)$. *On the other hand, the boundary point* $(1-2/(k+1), 1/(k+1))$ *of* $\frac{1}{k+1}S[k+1]$ *lies in* $\operatorname{int} E$, *which implies that* $\operatorname{vol}\left(\frac{1}{k+1}S[k+1]\cap E\right) < \frac{1}{4}\operatorname{vol}(E)$. *Now, if* $f:\mathbb{R}^2\to\mathbb{R}^2$ *is defined as the linear transformation mapping* E *into* $\mathbf{B}^2$, *then* $f(S)$ *satisfies the required conditions.*

One can use star-shaped sets together with ideas from [77] to give a negative answer to a more general version of Conjecture 6.1, also from [31].

Conjecture 6.2 (Bobkov, Madiman, Wang, 2011)**.** *For any* $k \geq 2$ *and compact sets* $A_1, A_2, \ldots, A_{k+1}$ *in* $\mathbb{R}^d$, *we have*

$$\operatorname{vol}\left(\sum_{i=1}^{k+1} A_i\right)^{1/d} \geq \frac{1}{k}\sum_{i=1}^{k+1}\operatorname{vol}\left(\sum_{j\neq i} A_j\right)^{1/d}.$$

In particular, for $k = 2$,

$$\begin{aligned}\operatorname{vol}(A_1+A_2+A_3)^{1/d}\\ \geq \frac{1}{2}\left(\operatorname{vol}(A_1+A_2)^{1/d} + \operatorname{vol}(A_1+A_3)^{1/d} + \operatorname{vol}(A_2+A_3)^{1/d}\right).\end{aligned} \tag{6.15}$$

The above conjecture is trivial for convex sets. Moreover, (6.15) is true when $A_1 = A_2$ and A_1 is convex. Indeed, in this case (6.15) is equivalent to

$$\operatorname{vol}(A_1 + A_1 + A_3)^{1/d} \geq \operatorname{vol}(A_1)^{1/d} + \operatorname{vol}(A_1 + A_3)^{1/d},$$

which follows from the Brunn-Minkowski inequality [169].

It was proved in [77] that Conjecture 6.2 is true in $\mathbb{R}$. Since an affirmative answer to Conjecture 6.2 implies also Conjecture 6.1, the former is also false for $d \geq 12$ by [76, 77]. Here we show that Conjecture 6.2 is false in $\mathbb{R}^d$ for $d \geq 7$.

Proposition 6.2. *For any $d \geq 7$, there are compact, star-shaped sets $A_1, A_2, A_3 \subset \mathbb{R}^d$ satisfying*

$$\operatorname{vol}(A_1 + A_2 + A_3)^{1/d} < \frac{1}{2}\left(\operatorname{vol}(A_1 + A_2)^{1/d} + \operatorname{vol}(A_1 + A_3)^{1/d} + \operatorname{vol}(A_2 + A_3)^{1/d}\right).$$

Proof. We give the proof for $d = 7$ and the result follows for $d > 7$ by taking direct products with a cube. Consider the sets

$$A_1 = [0,1]^4 \times \{0\}^3; A_2 = \{0\}^4 \times [0,1]^3 \text{ and } A_3 = ([0,a]^4 \times \{0\}^3) \cup (\{0\}^4 \times [0,b]^3),$$

where we select $a, b > 0$ later. Since these sets are lower dimensional, one has $\operatorname{vol}(A_1) = \operatorname{vol}(A_2) = \operatorname{vol}(A_3) = 0$. An elementary consideration shows that

$$\operatorname{vol}(A_1 + A_3) = b^3, \operatorname{vol}(A_2 + A_3) = a^4 \text{ and } \operatorname{vol}(A_1 + A_2) = 1,$$

and

$$\operatorname{vol}(A_1 + A_2 + A_3) = (a+1)^4 + (b+1)^3 - 1.$$

The last step is to show that, with $a = 3$ and $b = 6$, the quantity

$$((a+1)^4 + (b+1)^3 - 1)^{1/7} - \frac{1}{2}\left(a^{4/7} + b^{3/7} + 1\right)$$

is negative, which gives a counterexample to (6.15). □

Chapter 7

An isoperimetric inequality for 3-dimensional parallelohedra

In this chapter we deal with 3-dimensional parallelohedra (see Section 2.2). Our first goal is to present a new representation of 3-dimensional parallelohedra that might be useful to investigate their geometric properties. Next, we use this representation to prove Theorem 7.1.

Theorem 7.1. *Among unit volume* 3*-dimensional parallelohedra, regular truncated octahedra have minimal mean width.*

The structure of Chapter 7 is as follows. In Section 7.1 we introduce and parametrize 3-dimensional parallelohedra, and describe the tools and the notation necessary to prove our result. In Section 7.2 we prove Theorem 7.1. Finally, in Section 7.3, we collect additional remarks, and recall some old, and raise some new problems.

7.1 Preliminaries

7.1.1 Properties of three-dimensional parallelohedra

The combinatorial classes of 3-dimensional parallelohedra are well known (cf. Figure 7.1). More specifically, any 3-dimensional parallelohedron is combinatorially isomorphic to one of the following:

(1) a cube,

(2) a hexagonal prism,

(3) Kepler's rhombic dodecahedron (which we call a regular rhombic dodecahedron),

(4) an elongated rhombic dodecahedron,

(5) a regular truncated octahedron.

We call a parallelohedron combinatorially isomorphic to the polyhedron in (i) a *type (i) parallelohedron*. Recall from Chapter 2 that every parallelohedron P in $\mathbb{R}^3$ is a zonotope. More specifically, a type (1)-(5) parallelohedron can be attained as the Minkowski sum of $3, 4, 4, 5, 6$ segments, respectively. A typical parallelohedron is of type (5), as every other parallelohedron can be obtained by removing some of the generating vectors of a type (5) parallelohedron.

Since every 3-dimensional parallelohedron $P \subset \mathbb{R}^3$ is a zonotope, every edge of P is a translate of one of the segments generating P. Furthermore, for any generating segment S, the faces of P that contain a translate of S form a *zone*, i.e. they can be arranged in a sequence $F_1, F_2, \ldots, F_k, F_{k+1} = F_1$ of faces of P such that for all values of i, $F_i \cap F_{i+1}$ is a translate of S. From property (ii) in the list of the properties characterizing parallelohedra in Subsection 2.2.1, this sequence contains 4 or 6 faces. In these cases we say that S generates a 4-belt or a 6-belt in P, respectively. The numbers of 4- and 6-belts of a type (i) parallelohedron are 3 and 0, 3 and 1, 0 and 4, 1 and 4, and 0 and 6 for $i = 1, 2, 3, 4, 5$, respectively.

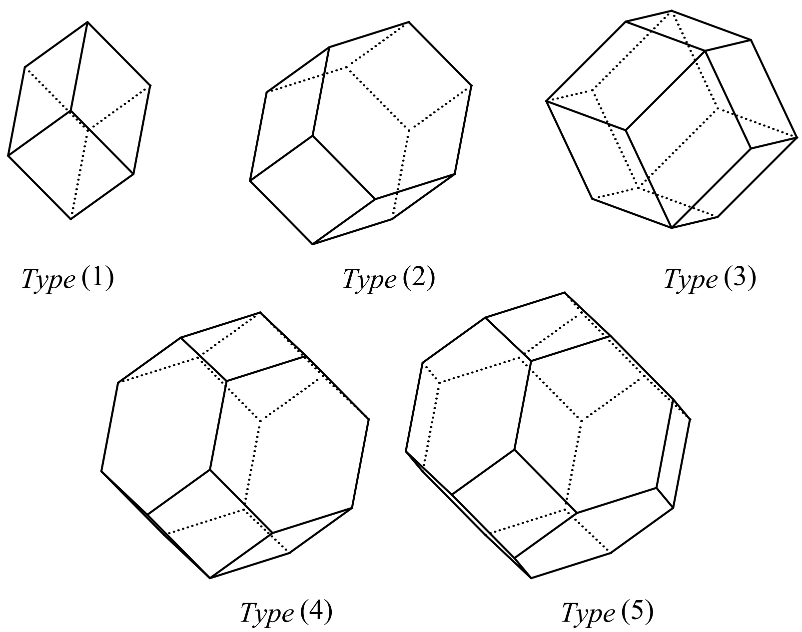

Figure 7.1: The five combinatorial types of 3-dimensional parallelohedra. The type (5) parallelohedron in the picture is the regular truncated octahedron generated by the six segments connecting the midpoints of opposite edges of a cube. The type (3) polyhedron is the regular rhombic dodecahedron generated by the four diagonals of the same cube. The rest of the polyhedra in the picture are obtained by removing some generating segments from the type (5) polyhedron.

Remark 7.1. *Let $P = \sum_{i=1}^{m} [o, v_i] \subset \mathbb{R}^3$ be a parallelohedron, and set $I = \{1, 2, \ldots, m\}$. Then the volume, the surface area and the mean width of P is*

$$\mathrm{vol}(P) = \sum_{\{i,j,k\} \subset I} |V_{ijk}|, \tag{7.1}$$

$$\mathrm{surf}(P) = 2 \sum_{\{i,j\} \subset I} ||v_i \times v_j||, \text{ and} \tag{7.2}$$

$$\mathrm{mwidth}(P) = \frac{1}{2} \sum_{i=1}^{m} ||v_i||, \tag{7.3}$$

respectively, where V_{ijk} is the determinant of the matrix with column vectors v_i, v_j, v_k.

Proof. The formula in (7.1) is the well-known volume formula for zonotopes (cf. e.g. [174]). The one in (7.2) can be proved directly for type (5) parallelohedra, which implies its validity for all parallelohedra. Finally, Steiner's formula [169] yields that the second quermassintegral $W_2(P)$ of P is $\frac{1}{6} \sum_{ij} (\pi - \alpha_{ij}) l_{ij}$, where α_{ij} and l_{ij} is the dihedral angle and the length of the edge between the ith and jth faces of P. Thus, from the zone property it follows that $W_2(P) = \frac{\pi}{3} \sum_{i=1}^{m} ||v_i||$. On the other hand, for any convex body $K \subset \mathbb{R}^3$ we have $\mathrm{mwidth}(K) = \frac{3}{2\pi} W_2(K)$, which implies (7.3). $\square$

If we choose the generating segments in the form $[o, p_i]$ for some vectors $p_i \in \mathbb{R}^3$, then in case of a type (1) or type (4) parallelohedron, the vectors v_i are in general position, i.e. any three of them are linearly independent, whereas in case of the other types there are triples of vectors that are restricted to be co-planar. In particular, if P is a type (5) parallelohedron, then each of the four pairs of hexagon faces defines a linear dependence relation on the generating segments. More specifically, if v_1, v_2, v_3, v_4 are normal vectors of the four pairs of hexagon faces of the parallelohedron, then the directions of the six generating segments are the ones defined by the six cross-products $v_i \times v_j$, $i \neq j$. Note that all triples of the v_is are linearly independent, as otherwise some of the generating segments of P are collinear. Thus, we have

$$P = \sum_{1 \leq i < j \leq 4} [o, \beta_{ij}(v_i \times v_j)] \tag{7.4}$$

for some real numbers β_{ij}, which, without loss of generality, we may assume to be non-negative. Every 3-dimensional parallelohedron can be obtained from a type (5) parallelohedron by removing some of the generating segments, or equivalently, by setting some of the β_{ij}s to be zero. In particular, P is of type (5), or (4), or (3), or (2), or (1) if and only if

- $\beta_{ij} > 0$ for all $i \neq j$, or
- exactly one of the β_{ij}s is zero, or
- exactly two of the β_{ij}s is zero: $\beta_{i_1 j_1} = \beta_{i_2, j_2} = 0$, and they satisfy $\{i_1, j_1\} \cap \{i_2, j_2\} = \emptyset$, or
- exactly two of the β_{ij}s is zero: $\beta_{i_1 j_1} = \beta_{i_2, j_2} = 0$, and they satisfy $\{i_1, j_1\} \cap \{i_2, j_2\} \neq \emptyset$, or
- exactly three of the β_{ij}s are zero. and there is no $s \in \{1, 2, 3, 4\}$ such that the indices of all nonzero β_{ij}s contain s, respectively.

It is easy to check that in the remaining cases P is planar.

Thus, we can use this representation for all parallelohedra appearing in the paper with the assumption that $\beta_{ij} \geq 0$ for all $1 \leq i < j \leq 4$.

Since no three of the vectors v_i are linearly independent, but clearly all four of them are, up to multiplying by a constant they have a unique nontrivial linear combination $\lambda_1 v_i + \lambda_2 v_2 + \lambda_3 v_3 + \lambda_4 v_4 = o$. Since in the representation of P we considered only the directions of the v_is, we may clearly assume that $v_1+v_2+v_3+v_4=o$. This implies that the tetrahedron $\operatorname{conv}(\{v_1,v_2,v_3,v_4\})$ is *centered*, that is, its center of mass is o. From the equation $v_1+v_2+v_3+v_4=o$ it follows that the absolute values of the determinants of any three of the v_is are equal. We choose this common value 1, which yields that the volume of the tetrahedron is $\frac{2}{3}$. Throughout the paper for any $i,j,k \in \{1,2,3,4\}$, we denote by V_{ijk} the value of the determinant with v_i,v_j,v_k as column vectors. We choose the indices in such a way that $V_{123}=1$. Since $v_1+v_2+v_3+v_4=o$, we have that for any $\{i,j,s,t\}=\{1,2,3,4\}$, the plane containing o,v_i,v_j strictly separates v_s and v_t, which implies that $V_{ijs}=-V_{ijt}$.

In the proof we often use the function $f:\mathbb{R}^6\to\mathbb{R}$,

$$f(\tau_{12},\tau_{13},\tau_{14},\tau_{23},\tau_{24},\tau_{34})=\tau_{12}\tau_{13}\tau_{23}+\tau_{12}\tau_{14}\tau_{24}+\tau_{13}\tau_{14}\tau_{34}+\tau_{23}\tau_{24}\tau_{34}+ \tag{7.5}$$

$$+(\tau_{12}+\tau_{34})(\tau_{13}\tau_{24}+\tau_{14}\tau_{23})+(\tau_{13}+\tau_{24})(\tau_{12}\tau_{34}+\tau_{14}\tau_{23})+(\tau_{14}+\tau_{23})(\tau_{12}\tau_{34}+\tau_{13}\tau_{24}).$$

We remark that f is *not* the third elementary symmetric function on six variables, since after expansion it has only $16<20=\binom{6}{3}$ members.

An elementary computation yields that for any $i,j,k,l,s,t\in\{1,2,3,4\}$, we have

$$|v_i\times v_j, v_k\times v_l, v_s\times v_t| = V_{ijl}V_{stk}-V_{ijk}V_{stl}.$$

Using this formula and the properties of cross-products, we may express the volume, surface area and mean width of P in our representation.

Remark 7.2. *For the parallelohedron $P=\sum_{i\neq j}\beta_{ij}[o,v_i\times v_j]$ satisfying the conditions above, we have*

$$\operatorname{vol}(P)=f(\beta_{12},\beta_{13},\beta_{14},\beta_{23},\beta_{24},\beta_{34}), \tag{7.6}$$

$$\operatorname{surf}(P)=2((\beta_{12}\beta_{13}+\beta_{12}\beta_{14}+\beta_{13}\beta_{14})||v_1||+\ldots(\beta_{14}\beta_{24}+\beta_{14}\beta_{34}+\beta_{24}\beta_{34})||v_4||+ \tag{7.7}$$

$$+\beta_{12}\beta_{34}(||v_1+v_2||)+\ldots+\beta_{14}\beta_{23}||v_1+v_4||),$$

$$\operatorname{mwidth}(P)=\frac{1}{2}\sum_{1\le i<j\le 4}\beta_{ij}||v_i\times v_j||, \tag{7.8}$$

respectively.

7.1.2 Preliminary lemmas

The following lemma is used more than once in the paper.

Lemma 7.1. *Let $T=\operatorname{conv}(\{p_1,p_2,p_3,p_4\})$ be an arbitrary centered tetrahedron with volume $V>0$ where the vertices are labelled in such a way that the determinant with columns p_1,p_2,p_3 is positive. For any $\{i,j,s,t\}=\{1,2,3,4\}$, let $\gamma_{ij}=-\langle p_s,p_t\rangle$ and $\zeta_{ij}=\gamma_{ij}||p_i\times p_j||^2$. Then*

1. *$f(\gamma_{12},\gamma_{13},\gamma_{14},\gamma_{23},\gamma_{24},\gamma_{34})=\frac{9}{4}V^2$,*

2. *$\sum_{1\le i<j\le 4}\zeta_{ij}=\frac{27}{4}V^2$,*

where f is the function defined in (7.5).

Proof. Let $\chi_{ij} = \langle p_i, p_j \rangle$ for all i, j. Consider the Gram matrix G defined by p_1, p_2, p_3, and observe that as T is centered, the volume of the parallelepiped spanned by them is $\frac{3}{2}V$. Since the determinant of the Gram matrix of n linearly independent vectors in $\mathbb{R}^n$ is the square of the volume of the parallelotope spanned by the vectors, it follows that $\det(G) = \frac{9}{4}V^2$. Furthermore, since T is centered, we have $\chi_{14} = -\chi_{11} - \chi_{12} - \chi_{13}$, and we may obtain similar formulas for χ_{24} and χ_{34}. Now, substituting these formulas into $f(\gamma_{12}, \ldots, \gamma_{34}) = f(-\chi_{34}, \ldots, -\chi_{12})$, we obtain that

$$f(\gamma_{12}, \ldots, \gamma_{34}) = \chi_{11}\chi_{22}\chi_{33} + 2\chi_{12}\chi_{13}\chi_{23} - \chi_{11}\chi_2^2 - \chi_{22}\chi_{13}^2 - \chi_{33}\chi_{12}^2 = \det(G),$$

which implies the first identity.

To obtain the second identity, observe that for any $\{i, j, s, t\} = \{1, 2, 3, 4\}$, we have $\zeta_{ij} = -\langle p_s, p_t \rangle ||p_i \times p_j||^2 = -\chi_{st}\left(\chi_{ii}\chi_{jj} - \chi_{ij}^2\right)$. This observation and an argument similar to the one in the previous paragraph yields that $\sum_{1 \leq i < j \leq 4} \zeta_{ij} = 3\det(G) = \frac{27}{4}V^2$. □

Lemma 7.2. *Under the condition that* $\sum_{1 \leq i < j \leq 4} \tau_{ij} = C > 0$ *and* $\tau_{ij} \geq 0$ *for all* $1 \leq i < j \leq 4$, *we have*

$$f(\tau_{12}, \tau_{13}, \tau_{14}, \tau_{23}, \tau_{24}, \tau_{34}) \leq \frac{2C^3}{27},$$

with equality if and only if $\tau_{ij} = \frac{C}{6}$ *for all* $1 \leq i < j \leq 4$. *Furthermore, if, in addition,* $\tau_{34} = 0$, *then*

$$f(\tau_{12}, \tau_{13}, \tau_{14}, \tau_{23}, \tau_{24}, 0) \leq \frac{16C^3}{243},$$

with equality if and only if $2\tau_{12} = \tau_{13} = \ldots = \tau_{24} = \frac{2C}{9}$.

Proof. Without loss of generality, we assume that $C = 1$. Since the set of points in $\mathbb{R}^6$ satisfying the conditions of Lemma 7.2 is compact, f attains its maximum on it. Assume that f is maximal at some point $\tau = (\tau_{12}, \ldots, \tau_{34})$.

Case 1, τ has only positive coordinates. By the Lagrange multiplier method, the gradient of f at τ is is parallel to the gradient of the function $\sum_{i \neq j} \tau_{ij} - 1$ defining the condition. In other words, all partial derivatives of f are equal. Let us assume that this common value is some $t \in \mathbb{R}$. Since f is a linear function of any of its variables, $\partial_{\tau_{ij}} f$ is equal to the coefficient of τ_{ij} in f. On the other hand, any member of the sum in f contributes to partial derivatives with respect to three of the variables. Thus, we have that $t = \sum_{i \neq j} t\tau_{ij} = \sum_{i \neq j} (\partial_{\tau_{ij}} f)(\tau)\tau_{ij} = 3f(\tau)$.

Computing the partial derivatives, we have that

$$(\partial_{\tau_{12}} f)(\tau) = \tau_{13}\tau_{23} + \tau_{14}\tau_{24} + \tau_{13}\tau_{24} + \tau_{14}\tau_{23} + (\tau_{13} + \tau_{24} + \tau_{14} + \tau_{23})\tau_{34},$$

and we obtain similar expressions for all partial derivatives. Solving the system of equations $(\partial_{\tau_{1i}} f)(\tau) = 3f(\tau)$, $i = 2, 3, 4$ for $\tau_{12}, \tau_{13}, \tau_{14}$, and substituting back the solutions into the definition of f, we obtain that $f(\tau) = \frac{4}{27}(\tau_{23} + \tau_{24} + \tau_{34})$. By a similar argument, $f(\tau) = \frac{4}{27}(\tau_{12} + \tau_{13} + \tau_{23}) = \frac{4}{27}(\tau_{12} + \tau_{14} + \tau_{24}) = \frac{4}{27}(\tau_{13} + \tau_{14} + \tau_{34})$ also follows. Adding up, we obtain that in this case $4f(\tau) = \frac{8}{27}\sum_{i \neq j} \tau_{ij} = \frac{8}{27}$, implying that $f(\tau) = \frac{2}{27}$.

We show that in case of equality all coordinates of τ are equal. The equations in the previous paragraph yield that $\tau_{12} + \tau_{13} + \tau_{23} = \ldots = \tau_{23} + \tau_{24} + \tau_{34} = \frac{1}{2}$, and also that $\tau_{12} + \tau_{13} + \tau_{14} = \ldots = \tau_{14} + \tau_{24} + \tau_{34} = \frac{1}{2}$. Comparing suitable equations it readily follows that $\tau_{12} = \tau_{34}$, $\tau_{13} = \tau_{24}$ and $\tau_{14} = \tau_{23}$. Replacing τ_{34}, τ_{24} and τ_{23} by τ_{12}, τ_{13} and τ_{14}, respectively, in the equations $(\partial_{\tau_{12}} f)(\tau) = (\partial_{\tau_{13}} f)(\tau) = (\partial_{\tau_{14}} f)(\tau)$, we obtain $\tau_{12}^2 = \tau_{13}^2 = \tau_{14}^2$, which characterizes equality in this case.

Case 2, if exactly one coordinate of τ is zero. Without loss of generality, we may assume that $\tau_{34} = 0$, and let $\tau' \in \mathbb{R}^5$ be the vector obtained by removing the last coordinate of τ. Let us define the function $g(\tau_{12}, \ldots, \tau_{24}) = f(\tau_{12}, \ldots, \tau_{24}, 0)$. Similarly like in Case 1, if $f(\tau)$ is a local maximum, we have that all five partial derivatives of g are equal at τ'. Again, from this it follows that this common value is equal to $3g(\tau')$. Factorizing the left-hand sides of the equations $\partial_{\tau_{13}} - \partial_{\tau_{14}} = 0$ and $(\partial_{\tau_{23}} g)(\tau') - (\partial_{\tau_{24}} g)(\tau') = 0$, we obtain that $\tau_{13} = \tau_{14}$, $\tau_{23} = \tau_{24}$, respectively. After substituting these equalities into the equation $(\partial_{\tau_{13}} g)(\tau') - (\partial_{\tau_{23}} g)(\tau') = 0$ and factorizing the left-hand side, we obtain that $\tau_{13} = \tau_{23}$. Again, substituting back into $(\partial_{\tau_{12}} g)(\tau') - (\partial_{\tau_{13}} g)(\tau') = 0$ and factorizing its left-hand side yields $\tau_{13} = 2\tau_{12}$. From this, we have $\tau_{12} = \frac{1}{9}$ and $\tau_{ij} = \frac{2}{9}$ for any $i \in \{1, 2\}$ and $j \in \{3, 4\}$, which implies that $g(\tau') = \frac{16}{243}$.

Case 3, if at least two coordinates of τ are zero. Then, using an argument similar to that in the previous cases, we obtain that the maximum of $f(\tau)$ is $\frac{1}{16}$, $\frac{1}{27}$ and 0 if exactly two, exactly three or more than three coordinates of τ are zero, respectively. □

7.2 Proof of Theorem 7.1

Recall from Subsection 7.1.1 that we represent P in the form

$$P = \sum_{1 \leq i < j \leq 4} [o, \beta_{ij} v_i \times v_j]$$

for some $v_1, v_2, v_3, v_4 \in \mathbb{R}^3$ satisfying $\sum_{i=1}^{4} v_i = o$, and for some $\beta_{ij} \geq 0$. By our assumptions, $|V_{ijk}| = 1$ for any $\{i, j, k\} \subset \{1, 2, 3, 4\}$, where V_{ijk} denotes the determinant with v_i, v_j, v_k as columns, and we assumed that $V_{123} = 1$, which implies, in particular, that $V_{124} = -1$. Then $T = \operatorname{conv}(\{v_1, v_2, v_3, v_4\})$ is a centered tetrahedron, with volume $\operatorname{vol}(T) = \frac{2}{3}$.

By Remark 7.2, we need to find the minimum value of $\operatorname{mwidth}(P) = \frac{1}{2} \sum_{1 \leq i < j \leq 4} \beta_{ij} ||v_i \times v_j||$ under the conditions that $T = \operatorname{conv}(\{v_1, v_2, v_3, v_4\})$ is a centered tetrahedron with volume $\operatorname{vol}(T) = \frac{2}{3}$, and the value of $f(\beta_{12}, \ldots, \beta_{34})$ being fixed. Equivalently, we need to find the maximum value of $f(\beta_{12}, \ldots, \beta_{34})$ under the condition that $\operatorname{mwidth}(P) = \frac{1}{2} \sum_{1 \leq i < j \leq 4} \beta_{ij} ||v_i \times v_j||$ is fixed, where $T = \operatorname{conv}(\{v_1, v_2, v_3, v_4\})$ is an arbitrarily chosen centered tetrahedron with volume $\operatorname{vol}(T) = \frac{2}{3}$.

Our main goal is to reduce this optimization problem to the special case where T is a regular tetrahedron, and then apply Lemma 7.2. We do it in two steps. In the first step we find an upper bound for f depending only on T (and thus, we eliminate all β_{ij}s from the conditions). In the second step we show that our upper bound from the first step can be bounded from above by a value of f under the additional condition that T is regular (or in other words, we eliminate the tetrahedron T from the conditions at the price of bringing back the parameters β_{ij}).

Step 1.

Our main tool in this step is Lemma 7.3, proved by Petty [155] and later rediscovered by Giannopoulos and Papadimitrakis [88] for arbitrary convex bodies (see (3.4) in [88]). Before stating it, we remark that any affine image of a parallelohedron is also a parallelohedron.

Lemma 7.3. *Let $P \subset \mathbb{R}^d$ be a convex polytope with outer unit facet normals $u_1, \ldots, u_k$. Let F_i denote the $(d-1)$-dimensional volume of the ith facet of P. Then, up to congruence, there is a unique volume preserving affine transformation L such that* $\operatorname{surf}(L(P))$ *is minimal in the affine class of P. Furthermore, P satisfies this property if and only if its surface area measure is isotropic, that*

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is, if

$$\sum_{i=1}^{k} \frac{dF_i}{\mathrm{surf}(P)} u_i \otimes u_i = \mathrm{Id} \tag{7.9}$$

where Id *denotes the identity matrix.*

Any convex polytope satisfying the conditions in (7.9) is said to be in *surface isotropic position*. We note that the volume of the projection body of any convex polyhedron is invariant under volume preserving linear transformations (cf. [156]). On the other hand, from Cauchy's projection formula and the additivity of the support function (see. [84]) it follows that the projection body of the polytope in Lemma 7.3 is the zonotope $\sum_{i_1}^{k}[o, F_i u_i]$ (for more information, see Remark 8.2). Note that the solution to Minkowski's problem [169] states that some unit vectors $u_1, \ldots, u_k \in \mathbb{R}^d$ and positive numbers $F_1, \ldots, F_k$ are the outer unit normals and volumes of the facets of a convex polytope if and only if the u_is span $\mathbb{R}^d$, and $\sum_{i=1}^{k} F_i u_i = o$. On the other hand, a translate of the parallelohedron P in our investigation can be written in the form $\frac{1}{2}\sum_{1 \leq i < j \leq 4}[o, \pm \beta_{ij} v_i \times v_j]$, which, by the previous observation, is the projection body of a centrally symmetric polytope with outer unit facet normals $\pm \frac{v_i \times v_j}{||v_i \times v_j||}$, and volumes of the corresponding facets $\frac{\beta_{ij}}{2}||v_i \times v_j||$, where $1 \leq i < j \leq 4$. Since $u \otimes u = (-u) \otimes (-u)$ for any $u \in \mathbb{R}^d$, any solution P to our optimization problem satisfies the conditions in (7.9) with the vectors $\frac{v_i \times v_j}{||v_i \times v_j||}$ in place of the u_is, and the quantities $\beta_{ij}||v_i \times v_j||$ in place of the F_is. Thus, applying also the formula in Remark 7.2 for mwidth(P), it follows that

$$\sum_{1 \leq i < j \leq 4} \frac{3\beta_{ij}}{2\,\mathrm{mwidth}(P)||v_i \times v_j||}(v_i \times v_j) \otimes (v_i \times v_j) = \mathrm{Id}, \tag{7.10}$$

which in the following we assume to hold for P.

Recall that $x \otimes y = xy^T$ and $\langle x, y \rangle = x^T y$ for any column vectors $x, y \in \mathbb{R}^n$. Hence, multiplying both sides in (7.10) by v_3^T from the left and v_4 from the right, it follows that

$$\langle v_3, v_4 \rangle = \frac{3\beta_{12}}{2\,\mathrm{mwidth}(P)||v_1 \times v_2||} V_{123} V_{124}.$$

Since $V_{123} = -V_{124} = 1$, we can express β_{12} as $\beta_{12} = -\langle v_3, v_4 \rangle ||v_1 \times v_2|| \frac{2\,\mathrm{mwidth}(P)}{3}$. By symmetry, for any $\{i,j,s,t\} = \{1,2,3,4\}$ it follows that $\beta_{ij} = -\langle v_s, v_t \rangle ||v_i \times v_j|| \frac{2\,\mathrm{mwidth}(P)}{3}$, where it may be worth noting that as $\beta_{ij} \geq 0$ holds for any value of i, j, if P satisfies (7.10), then $\langle v_s, v_t \rangle \leq 0$ for all s, t. Set $\zeta_{ij} = -\langle v_s, v_t \rangle ||v_i \times v_j|| = \frac{3\beta_{ij}}{2\,\mathrm{mwidth}(P)}$ for all i, j. Substituting back these quantities into the formulas for vol(P) and mwidth(P) and simplifying, we may rewrite our optimization problem in the following form: find the maximum of $\frac{27\,\mathrm{vol}(P)}{(2\,\mathrm{mwidth}(P))^3} = f(\zeta_{12}, \ldots, \zeta_{34})$, where ζ_{ij} is defined as above, in the family of all centered tetrahedra T with vol$(T) = \frac{2}{3}$, under the condition that $\sum_{1 \leq i < j \leq 4} \zeta_{ij}||v_i \times v_j|| = 3$. Here, it is worth noting that the last condition is satisfied for any centered tetrahedron T with volume $\frac{2}{3}$ by Lemma 7.1, and thus, it is redundant.

Step 2.

Consider the function $f(\zeta_{12}, \ldots, \zeta_{34})$, where $\zeta_{ij} = -\langle v_s, v_t \rangle ||v_i \times v_j||$ for $\{i,j,s,t\} = \{1,2,3,4\}$, and v_1, v_2, v_3, v_4 are the vertices of a centered tetrahedron T with volume $\frac{2}{3}$. Set $\gamma_{ij} = -\langle v_s, v_t \rangle$ and $\tau_{ij} = \gamma_{ij}||v_i \times v_j||^2$ for all $\{i,j,s,t\} = \{1,2,3,4\}$. To give an upper bound on the value of f, we apply the Cauchy-Schwartz Inequality, which states that for any nonnegative real numbers x_i, y_i,

$i = 1, 2, \ldots, k$, we have $\sum_{i=1}^{k} x_i y_i \leq \sqrt{\sum_{i=1}^{k} x_i^2} \sqrt{\sum_{i=1}^{k} y_i^2}$, with equality if and only if the x_is and the y_is are proportional. To do this, we write each member of f as the product $\zeta_{ij}\zeta_{kl}\zeta_{mn} = \sqrt{\gamma_{ij}\gamma_{kl}\gamma_{mn}}\sqrt{\tau_{ij}\tau_{kl}\tau_{mn}}$. Thus, we obtain that

$$f(\zeta_{12}, \ldots, \zeta_{34}) \leq \sqrt{f(\gamma_{12}, \ldots, \gamma_{34})}\sqrt{f(\tau_{12}, \ldots, \tau_{34})} = \sqrt{f(\tau_{12}, \ldots, \tau_{34})},$$

where we used the fact that by Lemma 7.1, $f(\gamma_{12}, \ldots, \gamma_{34}) = 1$ for all centered tetrahedra with volume $\frac{2}{3}$. Furthermore, observe that by Lemma 7.1 we have $\sum_{1 \leq i \leq j \leq 4} \tau_{ij} = 3$ for all such tetrahedra. Hence, by Lemma 7.2, we have $f(\zeta_{12}, \ldots, \zeta_{34}) \leq \sqrt{2}$.

Now, assume that $f(\zeta_{12}, \ldots, \zeta_{34}) = \sqrt{2}$. Then, by Lemma 7.2, $\tau_{ij} = \frac{1}{2}$ for all values $i \neq j$. This, by the Cauchy-Schwartz Inequality, implies that for some $t \in \mathbb{R}$, $\gamma_{ij} = t$ for all values $i \neq j$. Since T is centered, this implies that $\gamma_{ii} = 3t$ for all is. In other words, the Gram matrix of the vectors $v_1, \ldots, v_4$ is a scalar multiple of the matrix $4\,\mathrm{Id} - E$, where E is the matrix with all entries equal to 1. Since the Gram matrix of a vector system determines the vectors up to orthogonal transformations, and it is easy to check that $4\,\mathrm{Id} - E$ is the Gram matrix of the vertex set of a centered regular tetrahedron of circumradius $\sqrt{3}$, the equality part in Theorem 7.1 follows.

7.3 Remarks and open problems

First, in the next table we collect the results of our investigation for the minimal values of the mean widths of the different types of parallelohedra.

Recall (cf. Section 7.1) that the different types of parallelohedra correspond to the following polyhedra.

- Type (1) parallelohedra: parallelopipeds.
- Type (2) parallelohedra: hexagonal prisms.
- Type (3) parallelohedra: rhombic dodecahedra.
- Type (4) parallelohedra: elongated rhombic dodecahedra.
- Type (5) parallelohedra: truncated octahedra.

Type	Minimum of mwidth(P)	Optimal parallelohedra
(1)	$\frac{3}{2}$	cube
(2)	$\frac{3^{7/6}}{2^{4/3}} \approx 1.43$	regular hexagon based right prism, with base and lateral edges of lengths $\frac{2^{2/3}}{3^{5/6}}$ and $\frac{3^{1/6}}{2^{1/3}}$, respectively
(3)	$\frac{3^{1/2}}{2^{1/3}} \approx 1.37$	regular rhombic dodecahedron with edge length $\frac{\sqrt{3}}{2^{4/3}}$
(4)	$\geq \frac{3^{4/3}}{2^{5/3}} \approx 1.36$	not known
(5)	$\frac{3}{2^{7/6}} \approx 1.34$	regular truncated octahedron of edge length $\frac{1}{2^{7/6}}$

Table 7.1: Minima of the mean widths of different types of unit volume parallelohedra

In the remaining part of this section we collect some old conjectures about 3-dimensional parallelohedra, and propose some new ones.

The origin of the Honeycomb Conjecture, stating that in a decomposition of the Euclidean plane into regions of equal area, the regular hexagonal grid has the least perimeter, can be traced back to ancient times [187]. This problem has been in the focus of research throughout the second half of the 20th century [71, 152], and was finally settled by Hales [102].

The most famous analogous conjecture for mosaics in 3-dimensional Euclidean space is due to Lord Kelvin [124], who in 1887 conjectured that in a tiling of space with cells of unit volume, the mosaic with minimal surface area is composed of slightly modified truncated octahedra. Even though this conjecture was disproved by Weaire and Phelan in 1994 [190], who discovered a tiling of space with two slighly curved 'polyhedra' of equal volume and with less total surface area than in Lord Kelvin's mosaic, the original problem of finding the mosaics with equal volume cells and minimal surface area has been extensively studied (see, e.g. [128, 80, 154]). On the other hand, in the author's knowledge, there is no subfamily of mosaics for which Kelvin's problem is solved. This is the motivation for the following conjecture, where we note that a 3-dimensional parallelohedron of unit volume has minimal surface area if and only if the same holds for the translative, convex mosaic generated by it. We note that Conjecture 7.1 was first stated by Bezdek [21], and restated in [132] in 2022.

Conjecture 7.1 (Bezdek, 2006)**.** *Among 3-dimensional, unit volume parallelohedra P, $\operatorname{surf}(P)$ is minimal if and only if P is a regular truncated octahedron.*

The following conjecture, called Rhombic Dodecahedral Conjecture, can be found in [22] (see also [19]). We note that this conjecture is the lattice variant of the so-called Strong Dodecahedral Conjecture [20] proved by Hales in [104]. For other variants of the Dodecahedral Conjecture, see also [23] or [105].

Conjecture 7.2 (Bezdek, 2000)**.** *The surface area of any parallelohedron in $\mathbb{R}^3$ with unit inradius is at least as large as $12\sqrt{2} \approx 16.97$, which is the surface area of the regular rhombic dodecahedron of unit inradius.*

As it was mentioned in the introduction, Voronoi cells of lattice packings of congruent balls is an important subfamily of 3-dimensional parallelohedra. Regarding this subclass, Daróczy [48] gave an example of a packing whose Voronoi cells have a smaller mean width than that of the regular rhombic dodecahedron of the same inradius. This is the motivation behind our last question. Here we note that the mean width of a cube, a regular truncated octahedron and a regular rhombic dodecahedron of unit inradius is $3 > \sqrt{6} = \sqrt{6}$, respectively, which, in our opinion, makes the question indeed intriguing.

Problem 7.1. *Find the minimum of the mean widths of 3-dimensional parallelohedra of unit inradius.*

As a final result, we mention that in a recent manuscript of Kadlicskó and Lángi [120], the method of the proof of Theorem 7.1 is generalized to prove Theorem 7.2 and Corollary 7.1 (see Theorem 1 and Corollary 1 of [120], respectively). In their formulations, by the *edge density* of a convex tiling $\mathcal{M}$ we mean the quantity

$$\rho_1(\mathcal{M}) = \lim \frac{L(\operatorname{skel}_1(\mathcal{M}) \cap (R\mathbf{B}^3))}{\operatorname{vol}(R\mathbf{B}^3)},$$

where $L(\cdot)$ denotes arclength, and $\text{skel}_1(\mathcal{M})$ denotes the 1-skeleton of $\mathcal{M}$. We note that for a convex polyhedron $P \subset \mathbb{R}^3$, $L(\text{skel}_1(P))$ coincides with the sum of the edge lengths of P.

Theorem 7.2 (Kadlicskó, Lángi, 2023). *For any translative, convex tiling $\mathcal{M}$ with unit volume cells, we have*

$$\rho_1(\mathcal{M}) \geq 3,$$

with equality if and only if $\mathcal{M}$ is a face-to-face mosaic with unit cubes as cells.

Corollary 7.1 (Kadlicskó, Lángi, 2023). *For any 3-dimensional parallelohedron P of unit volume, we have*

$$L(\text{skel}_1(P)) \geq 12,$$

with equality if and only if P is a unit cube.

We note that Corollary 7.1 solves a still open conjecture of Melzak, stated in 1965 [148], for the special case of 3-dimensional parallelohedra; this conjecture states that the total edge length of a unit volume convex polyhedron is at least $2^{2/3}3^{11/6} \approx 11.8962$, with equality attained by a certain regular-triangle-based right prism.

Chapter 8

Isoperimetric problems for zonotopes

In this chapter we investigate properties of zonotopes.

An elegant, simple formula exists for the volume of a zonotope in terms of its generating segments; this formula is usually attributed to McMullen [145] or Shephard [174] and follows e.g. from a decomposition theorem for zonotopes in [174]. In [92], this formula was extended to zonotopes whose dimension is strictly less than that of the ambient space. Using an integral geometric approach, in a very recent paper Brazitikos and McIntyre [43] found a generalization of this formula for all intrinsic volumes of a zonotope.

As a preliminary result, we prove a generalization of Shephard's decomposition theorem [174] reproving the formulas for the intrinsic volumes of a zonotope in [43]. Our proof is presented in Section 8.1, where we also collect all the tools required in our investigation. In the main part of the chapter we prove isoperimetric inequalities for zonotopes. We focus on two types of zonotopes. In particular, we investigate zonotopes in $\mathbb{R}^d$ generated by d or $d+1$ segments, and also zonotopes in $\mathbb{R}^d$ generated by $n \gg d$ segments. We collect our results for these two types of zonotopes in Sections 8.2 and 8.3, respectively.

We also point out applications of our results to other problems in mathematics. We focus on two problems, one of which is the so-called *ℓ_p-polarization problem* discussed e.g. in [8] (see also [153]), and the other one is a generalization of the well known Maclaurin inequality [29] by Brazitikos and McIntyre [43], comparing the values of elementary symmetric functions evaluated at positive real numbers. We introduce these problems and their relations to zonotopes in detail in Section 8.1, and describe our related results at their appropriate places in Sections 8.2 and 8.3.

Finally, in Section 8.4 we collect some remarks and open questions.

8.1 Preliminaries

For any $v_1, v_2, \ldots, v_d \in \mathbb{R}^d$, we denote the $d \times d$ matrix with the v_i as column vectors by $[v_1, v_2, \ldots, v_n]$. We remark that since every zonotope Z is centrally symmetric, the quantities $2\,\mathrm{cr}(Z)$ and $2\,\mathrm{ir}(Z)$ are equal to the diameter and the minimum width of Z. We distinguish zonotopes satisfying some additional property. More specifically, if the lengths of all generating vectors of a zonotope Z are equal (resp., equal to 1), we say that Z is an *equilateral* (resp., *unit edge length*) zonotope. Note that up to translation, any zonotope Z in $\mathbb{R}^d$ can be defined as $Z = \sum_{i=1}^n [o, p_i]$ for some $p_1, p_2, \ldots, p_n \in \mathbb{R}^d$. In this case we say that $Z = \sum_{i=1}^n [o, p_i]$ is given in a *canonical form*. Clearly, the zonotope $Z = \sum_{i=1}^{n} [o, p_i]$ is symmetric to the origin o if and only if $\sum_{i=1}^{n} p_i = o$; in this case we say that Z has a *centered canonical form*. We denote the family of zonotopes in $\mathbb{R}^d$ generated by at most n segments by $\mathcal{Z}_{d,n}$.

We note that the zonotope $Z = \sum_{i=1}^{n} [o, p_i]$, $p_i \in \mathbb{R}^d$ is the translate of the zonotope $Z' = \frac{1}{2} \sum_{i=1}^{n} [-p_i, p_i]$. Thus, since Z' is o-symmetric and every vertex of Z' is of the form $\sum_{i=1}^{n} \varepsilon_i p_i$ with some $\varepsilon_i \in \{-1, 1\}$, we have that

$$\operatorname{cr}(Z) = \frac{1}{2} \max \left\{ \left\| \sum_{i=1}^{n} \varepsilon_i p_i \right\| : \varepsilon_i \in \{-1, 1\}, i = 1, 2, \ldots, n \right\}. \tag{8.1}$$

Recall that the support function of a convex body $K \subset \mathbb{R}^d$ is defined as $h_K : \mathbb{S}^{d-1} \to \mathbb{R}$, $h_K(u) = \sup\{\langle x, u\rangle : x \in K\}$. Furthermore, recall that for any convex bodies $K, L \subset \mathbb{R}^d$, we have $h_{K+L} = h_K + h_L$, and $h_{[-q,q]}(u) = |\langle q, u\rangle|$ for any $q \in \mathbb{R}^n$. As $\operatorname{cr}(K) = \max\{h_K(u) : u \in \mathbb{S}^{d-1}\}$ and $\operatorname{ir}(K) = \min\{h_K(u) : u \in \mathbb{S}^{d-1}\}$, this yields that for $Z = \sum_{i=1}^{n} [o, p_i]$, $p_i \in \mathbb{R}^d$,

$$\operatorname{cr}(Z) = \frac{1}{2} \max \left\{ \sum_{i=1}^{n} |\langle u, p_i\rangle| : u \in \mathbb{S}^{d-1} \right\}, \text{ and } \operatorname{ir}(Z) = \frac{1}{2} \min \left\{ \sum_{i=1}^{n} |\langle u, p_i\rangle| : u \in \mathbb{S}^{d-1} \right\}. \tag{8.2}$$

Our first goal is to prove a generalization of the decomposition theorem in [174], which yields an elementary proof of the formula in [43]. To state this result, we need some preparation.

Let $d \geq 2$, and consider a zonotope $Z = \sum_{i=1}^{n} [o, p_i]$ with $p_i \in \mathbb{R}^d$ for all values of i. We set $\mathcal{P}^Z = \{p_1, \ldots, p_n\}$, and for any $0 \leq k \leq d$, by $\mathcal{P}_k^Z$ the family of k-element subsets of $\mathcal{P}^Z$ containing linearly independent vectors. Furthermore, for any $I \in \mathcal{P}_k^Z$ with $k \geq 1$, we set $P(I) = \sum_{i \in I} [o, p_i]$, and for later use, we extend this notation for the case $k = 0$ by setting $P(\emptyset) = \{o\}$, which we regard as a 0-dimensional parallelotope. Note that for any $I = \{i_1, i_2, \ldots, i_k\} \in \mathcal{P}_k^Z$, the k-dimensional volume of $P(I)$ is $V_k(P(I))$, which is equal to the length of the wedge product of the vectors p_{i_j} [43], i.e. $V_k(P(I)) = |p_{i_1} \wedge p_{i_2} \wedge \ldots \wedge p_{i_k}|$. In addition, for any parallelotope $P(I)$ with $I \in \mathcal{P}_k^Z$, we set $B^\perp(I) = \mathbf{B}^d \cap (\operatorname{aff}(P(I)))^\perp$ and $\mathcal{S}^\perp(I) = \mathbb{S}^{d-1} \cap (\operatorname{aff}(P(I)))^\perp$, where $(\operatorname{aff}(P(I)))^\perp$ denotes the orthogonal complement of $\operatorname{aff}(P(I))$. Finally, by a decomposition of a convex body K we mean is a finite family of mutually non-overlapping convex bodies whose union is K.

Theorem 8.1. *Using the notation in the previous paragraph, for any $t \geq 0$, the set $Z + t\mathbf{B}^d$ can be decomposed into a family $\mathcal{F}_Z$ of mutually non-overlapping convex bodies of the form $X + tB_X$ such that*

1. *for any $X + tB_X \in \mathcal{F}_Z$, X is a translate of some parallelotope $P(I)$ with $I \in \mathcal{P}_k^Z$ for some $0 \leq k \leq d$, and $B_X \subseteq B^\perp(I)$ is the convex hull of o and a spherically convex, compact subset of $S^\perp(I)$;*
2. *if for any $0 \leq k \leq d$ and $I \in \mathcal{P}_k^Z$, $\mathcal{F}_Z(I)$ denotes the subfamily of the elements $X + tB_X$ of $\mathcal{F}_Z$, where X is a translate of $P(I)$, then $\{B_X : X + tB_X \in \mathcal{F}_Z(I)\}$ is a decomposition of $B^\perp(I)$.*

We note that the case $t = 0$ of Theorem 8.1 coincides with the decomposition theorem in [174].

Proof. Recall [174] that a zonotope Z is called *cubical* if every facet of Z is an affine cube, or equivalently, if any at most d of its generating vectors in a canonical form are linearly independent. We prove the statement only for cubical zonotopes. In particular, we show that in this case Theorem 8.1

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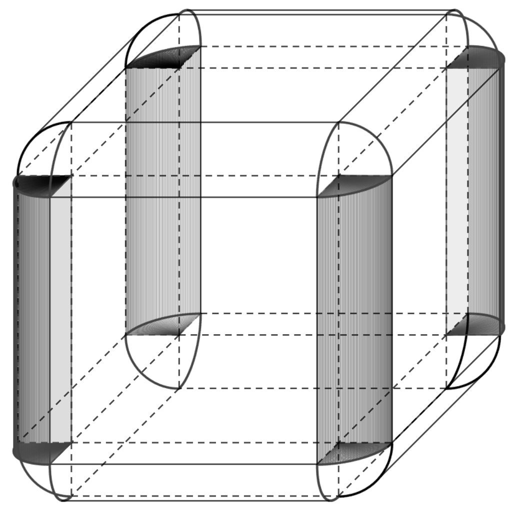

Figure 8.1: The body $Z + t\mathbf{B}^d$ if Z is a cube generated by 3 mutually orthogonal segments. There are 4 translates of every generating segment appearing as edges of Z. The solid bodies in the picture correspond to the sets $X + tB_X$, where X is a translate of a fixed generating segment.

holds with the additional assumption that if $X + tB_X \in \mathcal{F}_Z$ with $\dim X < d$, then X is a face of Z and B_X is the set of the outer normal vectors of X of length at most one, and observe that the general case follows from this by a standard limit argument.

We prove the theorem by induction on the dimension d. Observe that for $d = 1$ the statement is trivial. Assume that it holds for any zonotope of dimension strictly less than d, and let $Z = \sum_{i=1}^n [o, p_i]$, $p_i \in \mathbb{R}^n$ be a zonotope. Let F be a k-face of Z. Then there is a supporting hyperplane H of Z with $F = H \cap Z$. By the definition of a zonotope, F is a translate of the generating segments $[o, p_i]$ with the property that p_i is parallel to H. Thus, since Z is cubical, any k-face of Z is a translate of some $P(I)$ with $I \in \mathcal{P}_k^Z$. Let $\mathrm{proj} : \mathbb{R}^d \to (\mathrm{aff}(P(I)))^\perp$ be the orthogonal projection onto $(\mathrm{aff}(P(I)))^\perp$. Then $\mathrm{proj}(Z)$ is a zonotope generated by the projections of the generating segments of Z, and since Z is cubical, the vertices of $\mathrm{proj}(Z)$ coincide with the projections of the k-faces of Z that are translates of $P(I)$. Since it is well known that the sets of the outer unit normal vectors of a convex polytope in $\mathbb{R}^{d-k}$ is a decomposition of $\mathbb{S}^{d-k-1}$ into spherically convex sets (see, e.g. [169]), we proved that the closure of $(Z + t\mathbf{B}^d) \setminus Z$ can be decomposed with sets $X + tB_X$, where X is a proper face of Z, B_X is the set of outer normal vectors of X of length at most one, and for any $I \in \mathcal{P}_k^Z$, $0 \le k < d$, $\{B_X : X + tB_X \in \mathcal{F}_Z(I)\}$ is a decomposition of $B^\perp(I)$.

We are left to show that Z can be decomposed into translates of $P(I)$ with $I \in \mathcal{P}_d^Z$ such that for any $I \in \mathcal{P}_d^Z$, there is exactly one translate of $P(I)$ in the decomposition. We prove it by induction on n. If $n = d$, the statement is obvious. Assume that it holds for cubical zonotopes generated by at most $n-1$ segments. Let $W = \sum_{i=1}^{n-1} [o, p_i]$. Then W can be decomposed into translates of $P(I)$ with $I \in \mathcal{P}_d^W$ according to the requirements. Furthermore, by the previous paragraph, for any $J \in \mathcal{P}_{d-1}^W$ there are exactly two facets of W and Z that are translates of $P(J)$. By the definition of a zonotope, if F_1 and F_2 are the above two facets of W, then either F_1 and $p_n + F_2$, or F_2 and

$p_n + F_1$ are facets of Z. Thus, the closure of $Z \setminus W$ can be decomposed into translates of $P(J \cup \{n\})$, where $J \in \mathcal{P}_{d-1}^W$, such that for each $J \in \mathcal{P}_{d-1}^W$, there is exactly one translate of $P(J \cup \{n\})$ in the decomposition. This proves Theorem 8.1. □

For Corollary 8.1, see also [43].

Corollary 8.1. *For any zonotope $Z = \sum_{i=1}^n [o, p_i]$ in $\mathbb{R}^d$, and any $0 \leq k \leq d$, the kth intrinsic volume of Z is*

$$V_k(Z) = \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} |p_{i_1} \wedge p_{i_2} \wedge \ldots \wedge p_{i_k}|. \tag{8.3}$$

Proof. Theorem 8.1 yields that for any $t > 0$

$$V_d(Z + t\mathbf{B}^d) = \kappa_d t^d + \sum_{k=1}^{d} \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} |p_{i_1} \wedge p_{i_2} \wedge \ldots \wedge p_{i_k}| \kappa_{d-k} t^{d-k}.$$

Corollary 8.1 readily follows from comparing this formula with Steiner's formula in (2.12). □

Remark 8.1. *Recall that the mean width* $\operatorname{mwidth}(K)$ *of a convex body $K \subset \mathbb{R}^d$ is* $\operatorname{mwidth}(K) = \frac{2\kappa_{d-1}}{d\kappa_d} V_1(K)$ *(see [84]). Thus, the mean width of a zonotope $Z = \sum_{i=1}^n [o, p_i]$ in $\mathbb{R}^d$ is* $\operatorname{mwidth}(Z) = \frac{2\kappa_{d-1}}{d\kappa_d} \sum_{i=1}^n ||p_i||$.

Remark 8.2. *Let P be a convex polytope in $\mathbb{R}^d$ with m facets such that the outer unit normal vector of the ith facet of P is u_i and its $(d-1)$-dimensional volume is F_i. Recall that for a convex body $K \in \mathbb{R}^n$, the function $h : \mathbb{S}^{d-1} \to \mathbb{R}$, $h(u) = V_{d-1}(K|x^{\perp})$, where $K|x^{\perp}$ is the orthogonal projection of K onto the hyperplane through o and with unit normal vector x, is the support function of a convex body; this body, denoted by ΠK, is called the* projection body *of K [84]. Cauchy's projection formula [84] states that $V_{d-1}(K|x^{\perp}) = \int_{\mathbb{S}^{d-1}} |\langle x, u \rangle| \, dS(K, u)$, where $S(K, \cdot)$ is the surface area measure of K. Applying this for P, we obtain that $h_{\Pi K}(x) = \sum_{i=1}^m F_i |\langle x, u_i \rangle|$ for any $x \in \mathbb{S}^{d-1}$. Thus, the additivity of the support function with respect to Minkowski sums of convex bodies yields that*

$$\Pi P = \sum_{i=1}^{m} [o, F_i u_i]$$

In particular, by Corollary 8.1 and since both mean width and surface area are continuous functions with respect to Hausdorff distance, for any convex body $K \subset \mathbb{R}^d$, we have $\operatorname{mwidth}(\Pi K) = \frac{2\kappa_{d-1}}{d\kappa_d} \operatorname{surf}(K)$.

In the remaining part of Section 8.1, we introduce two problems that are related to isoperimetric problems for zonotopes.

For $p > 0$ and a multiset $\omega_n = \{x_1, x_2, \ldots, x_n\}$ in $\mathbb{S}^{d-1}$, the *ℓ_p-polarization of ω_n* is defined as

$$M_p(\omega_n) = \max \left\{ \sum_{i=1}^{n} |\langle x_i, u \rangle|^p : u \in \mathbb{S}^{d-1} \right\},$$

and the quantity

$$M_n^p(\mathbb{S}^{d-1}) = \min \left\{ M_p(\omega_n) : \omega_n \subset \mathbb{S}^{d-1} \right\}$$

is called the *ℓ_p-polarization (or Chebyshev) constant of* $\mathbb{S}^{d-1}$. The ℓ_p-polarization problem on the sphere asks for determining the value of $M_n^p(\mathbb{S}^{d-1})$ for all values of p and d.

Note that by (8.1) and (8.2), if $Z = \sum_{i=1}^n [o, x_i]$ is a zonotope in $\mathbb{R}^d$, then

$$\mathrm{cr}(Z) = \frac{1}{2} \max \left\{ \left\| \sum_{i=1}^n \varepsilon_i x_i \right\| : \varepsilon_i \in \{-1, 1\}, i = 1, 2, \ldots, n \right\}, \tag{8.4}$$

or equivalently (using the definition of support function),

$$\mathrm{cr}(Z) = \frac{1}{2} \max \left\{ \sum_{i=1}^n |\langle u, x_i \rangle| : u \in \mathbb{S}^{d-1} \right\}. \tag{8.5}$$

Thus, the ℓ_1-polarization constant of $\mathbb{S}^{d-1}$ is equal to twice the minimal circumradius of a zonotope generated by n segments of unit length. By Remark 8.1, the mean width of a zonotope is a scalar multiple of the total length of its generating vectors. Thus, this problem can be restated as finding the minimum circumradius of a zonotope generated by n segments of unit length. In particular, for the special case of unit vectors, the equality of the expressions in (8.4) and (8.5) is proved as Proposition 3 of [8] by the Lagrange multiplier method.

The function $\sigma_m^k(x_1, \ldots, x_m) = \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq m} x_{i_1} x_{i_2} \ldots x_{i_k}$ is called the kth elementary symmetric function on the m variables $x_1, \ldots, x_m$. The following statement, called Maclaurin's inequality, can be found e.g. in [29].

Lemma 8.1 (Maclaurin, 1729). *Let $1 \leq k < m$ be integers, and $x_1, \ldots, x_m > 0$ be positive real numbers. Then*

$$\left(\frac{\sigma_m^k(x_1, x_2, \ldots, x_m)}{\binom{m}{k}} \right)^{\frac{1}{k}} \geq \left(\frac{\sigma_m^{k+1}(x_1, x_2, \ldots, x_m)}{\binom{m}{k+1}} \right)^{\frac{1}{k+1}},$$

with equality if and only if all x_i are equal.

A generalization of this inequality for a set of vectors in $\mathbb{R}^d$ was conjectured in [43] as follows.

Conjecture 8.1 (Brazitikos, McIntyre, 2022). *Let $x_1, x_2, \ldots, x_n \in \mathbb{R}^d$ be given with $1 \leq d \leq n$. Then for any $p \in [0, \infty]$ and $2 \leq k \leq d$, we have*

$$\left(\frac{\sum_{1 \leq i_1 < \ldots < i_k \leq n} |x_{i_1} \wedge x_{i_2} \wedge \ldots \wedge x_{i_k}|^p}{\binom{n}{k}} \right)^{\frac{1}{pk}} \leq \left(\frac{\sum_{1 \leq i_1 < \ldots < i_{k-1} \leq n} |x_{i_1} \wedge x_{i_2} \wedge \ldots \wedge x_{i_{k-1}}|^p}{\binom{n}{k-1}} \right)^{\frac{1}{p(k-1)}}, \tag{8.6}$$

with equality if and only if $n = d$ and the vectors form an orthonormal basis.

The authors of [43] proved this conjecture for $p = 0$ and $p = \infty$, for $p = 2$ and $n = d$, and for $p = 1$, $n = d$ and $k = 2, 3, d$. They also pointed out (see also Corollary 8.1) that if $p = 1$, the numerators in (8.6) correspond to the kth and $(k-1)$st intrinsic volumes of the zonotope $\sum_{i=1}^n [o, x_i]$, showing that this case of Conjecture 8.1 directly leads to isoperimetric problems for zonotopes.

8.2 Isoperimetric problems for zonotopes generated by a small number of segments

Zonotopes in $\mathcal{Z}_{d,d}$ are called parallelotopes, and those in $\mathcal{Z}_{d,d+1}$ are called rhombic dodecahedra [19]. The goal of this section is to prove isoperimetric inequalities for them. We note that if $Z = \sum_{i=1}^{d+1} [o, p_i]$ is a rhombic dodecahedron where the points p_i are the vertices of a regular simplex centered at o, then Z is called a *regular rhombic dodecahedron*. A regular rhombic dodecahedron in $\mathbb{R}^3$ is the Voronoi cell of a face-centered cubic lattice. Thus, by the seminal result of Hales [103] proving Kepler's conjecture, among rhombic dodecahedra in $\mathbb{R}^3$ with unit inradius, the ones with minimal volume are the regular ones. This consequence of the result of Hales was strengthened in [19], which we quote for completeness.

Theorem 8.2 (Bezdek, 2000)**.** *Let $1 \leq k \leq d$ be arbitrary. Among rhombic dodecahedra in $\mathbb{R}^d$ of unit inradius, the ones with minimal kth intrinsic volumes are the regular ones.*

Intuitively, regular rhombic dodecahedra appear as natural candidates for solutions in isoperimetric problems for rhombic dodecahedra. Nevertheless, as we will see in Subsection 8.2.2, this does not always hold.

In Subsection 8.2.1 we find the minimal intrinsic volumes and circumradii of the elements of $\mathcal{Z}_{d,d}$ and $\mathcal{Z}_{d,d+1}$ with a given volume. In Subsections 8.2.2 and 8.2.3 we investigate the problems of finding the minimal circumradius and the second intrinsic volume $V_2(\cdot)$, respectively, of a parallelotope or a rhombic dodecahedron of a given mean width, respectively.

In Subsection 8.2.4 our investigation is motivated by a result of Tanner [182], who defined the *total squared k-content of a simplex $S \subset \mathbb{R}^d$* as the sum of the squares of the k-volumes of its k-faces, and proved that among simplices with a given squared 1-content, regular simplices have maximal squared k-content for all $1 \leq k \leq d$. In this subsection we define the total squared k-volume of a zonotope with $1 \leq k \leq d$, and find its minimum among the parallelotopes and rhombic dodecahedra with a given squared l-volume for all $k < l \leq d$.

In our investigation, we set $\mathcal{Z}_{\mathrm{p}} = \mathcal{Z}_{d,d}$ and $\mathcal{Z}_{\mathrm{rd}} = \mathcal{Z}_{d,d+1}$, and we denote by $Z_{\mathrm{p}}^{\mathrm{reg}} \in \mathcal{Z}_{d,d}$ a cube, and by $Z_{\mathrm{rd}}^{\mathrm{reg}} \in \mathcal{Z}_{d,d+1}$ a regular rhombic dodecahedron.

8.2.1 The intrinsic volumes and circumradius of a zonotope with a given volume

Our main result in Subsection 8.2.1 is Theorem 8.3.

Theorem 8.3. *Let $1 \leq k \leq d-1$. Then, for any $Z_i \in \mathcal{Z}_i$ with $V_d(Z_i) = V_d(Z_i^{\mathrm{reg}})$, where $i \in \{\mathrm{p}, \mathrm{rd}\}$, we have*

$$V_k(Z_i) \geq V_k(Z_i^{\mathrm{reg}})$$

with equality if and only if Z_i is congruent to Z_i^{reg}. Furthermore, we have

$$\mathrm{cr}(Z_i) \geq \mathrm{cr}(Z_i^{\mathrm{reg}}).$$

We note that our next corollary readily follows from Theorems 8.2 and 8.3.

Corollary 8.2. *For any $Z_i \in \mathcal{Z}_i$ with $\mathrm{ir}(Z_i) = \mathrm{ir}(Z_i^{\mathrm{reg}})$, where $i \in \{\mathrm{p}, \mathrm{rd}\}$, we have*

$$\mathrm{cr}(Z_i) \geq \mathrm{cr}(Z_i^{\mathrm{reg}})$$

with equality if and only if Z_i is congruent to Z_i^{reg}.

We prove Theorem 8.3 only for $i = \mathrm{rd}$, as the same argument can be used to prove it for $i = \mathrm{p}$. We note that the case $i = \mathrm{p}$ and $k = d-1$ of our theorem was also proved in Theorem 5 of [43]. Our proof is based on an application of shadow systems [168] (also called linear parameter systems [165]) and Steiner symmetrization. We start with two lemmas needed for the proof.

Lemma 8.2. *Let $k \geq 2$, $p_1, \ldots, p_k, v \in \mathbb{R}^d$, and $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$. For $i = 1, 2, \ldots, k$ and all $t \in \mathbb{R}$, set $p_i(t) = p_i + \lambda_i t v$ and $S(t) = \operatorname{conv}(\{p_1(t), \ldots, p_k(t)\})$. Then the function $f : \mathbb{R} \to \mathbb{R}$, $f(t) = \operatorname{vol}_{k-1}(S(t))$ is a convex function of t, and if the points $p_1, \ldots, p_k, v$ are affinely independent, then f is strictly convex.*

Proof. Since $\bigcup_{t \in \mathbb{R}} S(t)$ is contained in a k-dimensional affine subspace of $\mathbb{R}^d$, without loss of generality, we may assume that $k = d$. Furthermore, since volume is invariant under translations, $V_d(S(t)) = V_d(S(t) - p_d(t))$, and thus, we may also assume that $p_d = o$ and $\lambda_d = 0$. Define the linear functional $L_t : \mathbb{R}^d \to \mathbb{R}$, $L_t(x) = \det([p_1(t), \ldots, p_{d-1}(t), x])$. This functional can be written in the form $L_t(x) = \langle u(t), x \rangle$, where $u(t)$ is the vector whose entries are the (signed) minors of size $(d-1) \times (d-1)$ of the $d \times (d-1)$ matrix $[p_1(t), \ldots, p_{d-1}(t)]$. Note that by the properties of determinants, $u(t)$ is a linear function of t, and hence, it can be written as $u(t) = u + tw$ for some $u, w \in \mathbb{R}^n$. Furthermore, $f(t) = \operatorname{vol}_{d-1}(S(t)) = \frac{1}{(d-1)!} ||u(t)||$. Thus,

$$f''(t) = \frac{\langle w, w \rangle \langle u(t), u(t) \rangle - \langle w, u(t) \rangle^2}{(d-1)! \, ||u(t)||^3} \geq 0$$

by the Cauchy-Schwartz Inequality. This implies the convexity of f.

Now, assume that $p_1, \ldots, p_k, v$ are affinely independent. Then, as $p_d = o$, it follows that $p_1, \ldots, p_{d-1}, v$ are linearly independent. Note that $u(t) \neq o$ for any $t \in \mathbb{R}$. Indeed, if $u(t) = o$ for some value of t, then, by the definition of $L_t(x)$, $p_1(t), \ldots, p_{d-1}(t)$ are linearly dependent for some value of t, which yields the existence of a nontrivial linear combination of $p_1, \ldots, p_{d-1}, v$ equal to o. Thus, we have $u(t) \neq o$ for any $t \in \mathbb{R}$, implying that u is not a scalar multiple of w. We show that $w \neq o$. Indeed, suppose for contradiction that $w = o$. Then the value of $L_t(x)$ is independent of t for any value of x. In particular, the kernel $\ker 1(L_t)$ of L_t is independent of t, implying that $p_1, \ldots, p_{d-1}, v \in \ker 1(L_t)$. Thus, combined with the fact that L_t is not identically zero, this yields that $p_1, \ldots, p_{d-1}, v$ are linearly dependent, contradicting our assumptions. Now, we have shown that $w \neq o$, from which we also have that $u(t)$ and w are not scalar multiples of each other for any value of t. From this, the Cauchy-Schwartz Inequality implies that $f''(t) > 0$ for any value of t, and hence f is strictly convex. □

Lemma 8.3. *For any simplex $S \subset \mathbb{R}^d$ and $1 \leq k \leq d-1$, let $\mathcal{F}_k(S)$ denote the family of the $(k-1)$-faces of S, and set $f_k(S) = \sum_{F \in \mathcal{F}_k(S)} V_k(\{o\} \cup F)$. Furthermore, if $S = \operatorname{conv}(\{p_1, p_2, \ldots, p_{d+1}\})$, set*

$$g(S) = \max \left\{ \left\| \sum_{i=1}^{d+1} \varepsilon_i p_i \right\| : \varepsilon_i \in \{-1, 1\}, i = 1, 2, \ldots, d+1 \right\}.$$

Let S_{reg} be a regular simplex centered at o. Then, for any $1 \leq k \leq d-1$ and any simplex $S \subset \mathbb{R}^d$ with $V_d(S) = V_d(S_{\mathrm{reg}})$, we have

$$f_k(S) \geq f_k(S_{\mathrm{reg}})$$

with equality if and only if S is a regular simplex centered at o. Furthermore, we also have

$$g(S) \geq g(S_{\mathrm{reg}}).$$

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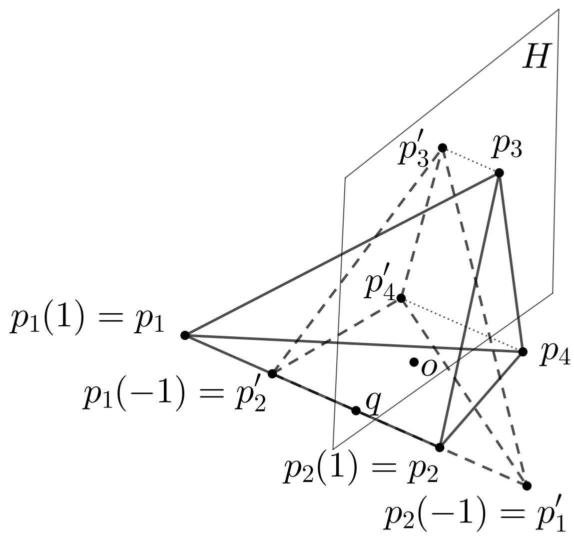

Figure 8.2: Lemma 2 in case $d = 3$.

Proof. Without loss of generality, let $V_d(S_{\text{reg}}) = 1$. Clearly, the function $f_k(\cdot)$ attains its minimum value in the family of simplices with unit volume. Thus, it is sufficient to show that if f_k is minimal at a simplex S with $V_d(S) = 1$, then it is a regular simplex centered at o.

Consider a simplex S with $V_d(S) = 1$, which is not a regular simplex centered at o. Let $S = \operatorname{conv}(\{p_1, p_2, \ldots, p_{d+1}\})$, and let H be the unique hyperplane perpendicular to the line through the edge $E = [p_1, p_2]$ and passing through o. Without loss of generality, assume that S is not symmetric to H, and let S' denote the reflected copy of S about H. Let v be a unit normal vector of H and let δ_i denote the signed distance of p_i from H, where the sign is chosen in such a way that distances are positive in the open half space bounded by H and containing v. Note that for any value of i, $p_i - \delta_i v$ is the orthogonal projection of p_i onto H, which we denote by q_i, and set $q = q_1 = q_2$. For any $i \geq 3$, let $p_i(t) = q_i + \delta_i t v$, and let $p_1(t) = q + \frac{\delta_1 - \delta_2}{2} v + \frac{\delta_1 + \delta_2}{2} t v$ and $p_2(t) = q + \frac{\delta_2 - \delta_1}{2} v + \frac{\delta_1 + \delta_2}{2} t v$. Finally, let $S(t) = \operatorname{conv}(\{p_1(t), \ldots, p_{d+1}(t)\})$. Clearly, $S(1) = S$, $S(-1) = S'$, and $S(0)$ is the Steiner symmetral of S with respect to H, and hence, it is symmetric to H. By Lemma 8.2, $t \mapsto f_k(S(t))$ is a strictly convex even function on $[-1, 1]$, and hence, it attains a unique minimum at $t = 0$. This yields the assertion for $f_k(S)$.

Finally, we consider the function g. Let $\mathcal{F}_{\min}$ denote the subfamily of the family of unit volume simplices that minimize g in this family. Recall that $f_1(S) = \sum_{i=1}^{d+1} ||p_i||$. Note that f_1 attains its minimal value in $\mathcal{F}_{\min}$. Thus, to prove the assertion it is sufficient to show that if S is not a unit volume regular simplex centered at o, then there is a unit volume simplex S' such that $g(S') < g(S)$, or $g(S') = g(S)$ and $f_1(S') < f_1(S)$. To do it we use the notation of the previous paragraph. Note that the function $t \mapsto ||\sum_{i=1}^{d+1} \varepsilon_i p_i(t)||$ is convex for any $(\varepsilon_1, \ldots, \varepsilon_{d+1}) \in \{-1, 1, \}^{d+1}$. Since the maximum of convex functions is convex, $g(S(t))$ is an even convex function on $[-1, 1]$, implying that

$g(S(0)) \leq g(S(1))$. Clearly, since S is not symmetric to H, we also have that $f_1(S(0)) < f_1(S(1))$, which yields the statement. □

Now we prove Theorem 8.3.

Proof of Theorem 8.3. Let $Z_{\text{rd}}^{\text{reg}} = \sum_{i=1}^{d+1}[o, q_i]$, and set $S_{\text{reg}} = \operatorname{conv}(\{q_1, q_2, \ldots, q_{d+1}\})$. Then S_{reg} is a regular simplex centered at o, and by Corollary 8.1, $V_d(S_{\text{reg}}) = \frac{1}{d!} V_d(Z_{\text{rd}}^{\text{reg}})$. Now, consider any rhombic dodecahedron Z with $V_d(Z) = V_d(Z_{\text{rd}}^{\text{reg}})$. Since the statement in Theorem 8.3 is invariant under translating Z, we may assume that $Z = \sum_{i=1}^{d+1}[o, p_i]$, where $o \in S = \operatorname{conv}(\{p_1, p_2, \ldots, p_{d+1}\})$. Clearly, this implies that S is a nondegenerate simplex with volume $V_d(S_{\text{reg}})$. Thus, applying Lemma 8.3 for S and S_{reg} readily implies Theorem 8.3. □

By Theorem 8.3 and the convexity of the function $x \mapsto x^p$, where $p > 1$, we readily obtain the following.

Remark 8.3. *Let $x_1, x_2, \ldots, x_n \in \mathbb{R}^d$ with $d \leq n \leq d+1$. Then for all $1 \leq k \leq d$,*

$$\left(\frac{\sum_{1 \leq i_1 < \ldots < i_k \leq n} |x_{i_1} \wedge x_{i_2} \wedge \ldots \wedge x_{i_k}|}{\binom{n}{k}} \right)^{\frac{1}{k}} \geq \left(\frac{\sum_{1 \leq i_1 < \ldots < i_d \leq n} |x_{i_1} \wedge x_{i_2} \wedge \ldots \wedge x_{i_d}|}{\binom{n}{d}} \right)^{\frac{1}{d}},$$

with equality if and only if $n = d$ and the vectors are pairwise orthogonal and are of equal length, or if the linear hull of the x_i is of dimension at most $k-1$. Furthermore, for any $p > 1$, we have

$$\left(\frac{\sum_{1 \leq i_1 < \ldots < i_k \leq d} |x_{i_1} \wedge x_{i_2} \wedge \ldots \wedge x_{i_k}|^p}{\binom{n}{k}} \right)^{\frac{1}{pk}} \geq |x_1 \wedge x_2 \wedge \ldots \wedge x_d|^{\frac{1}{d}},$$

with equality if and only if the vectors are pairwise orthogonal and are of equal length, or if the linear hull of the x_i is of dimension at most $k-1$.

For any simplex $S \subset \mathbb{R}^d$, integer $1 \leq k \leq d-1$, and real number $m \in \mathbb{R}$, let $g_k^m(S)$ denote the sum of the mth powers of the volumes of the k-faces of S. Tanner [182] showed that if $2 \leq k \leq d-1$ and $m \in (0, 2]$, among simplices S in $\mathbb{R}^n$ with $g_1^2(S)$ fixed, $g_k^m(S)$ is maximal if and only if S is regular. Our last result in this subsection is Proposition 8.1, which can be regarded as a complement of the above result of Tanner. This result follows by a natural modification of the proof of Lemma 8.3, and the observation that the composition of convex functions is convex.

Proposition 8.1. *Let $1 \leq k \leq d-1$ be an integer, and $m \geq 1$ be a real number. Then, for any simplex $S \subset \mathbb{R}^d$ with unit volume, $g_k^m(S)$ is maximal if and only if S is regular.*

8.2.2 The circumradius of a zonotope with a given mean width

Our aim in this section is to find the parallelotopes or rhombic dodecahedra Z of a given mean width $\operatorname{mwidth}(Z)$ that have minimal circumradius. For parallelotopes, we have the following theorem.

Theorem 8.4. *If $Z \in \mathcal{Z}_{\text{p}}$ satisfies $\operatorname{mwidth}(Z) = \operatorname{mwidth}(Z_{\text{p}}^{\text{reg}})$, then $\operatorname{cr}(Z) \geq \operatorname{cr}(Z_{\text{p}}^{\text{reg}})$, with equality if and only if Z is a cube.*

Proof. By (8.1),

$$4 \cdot 2^d \cdot (\mathrm{cr}(Z))^2 \geq \sum_{(\varepsilon_1,\ldots,\varepsilon_d)\in\{-1,1\}^d} \left\| \sum_{i=1}^d \varepsilon_i p_i \right\|^2 = 2^d \sum_{i=1}^d ||p_i||^2,$$

with equality if and only if the vectors p_i are pairwise orthogonal. On the other hand,

$$\sqrt{\frac{\sum_{i=1}^d ||p_i||^2}{d}} \geq \frac{\sum_{i=1}^d ||p_i||}{d},$$

with equality if and only if all p_i are of equal length. By Remark 8.1, this yields Theorem 8.4. □

Next, we examine $\mathcal{Z}_{\mathrm{rd}}$. We prove that in the family of rhombic dodecahedra having a centered canonical form, the ones with a given mean width and minimal circumradius are regular, whereas in the family of all rhombic dodecahedra this statement is not true. Recall that a zonotope $\sum_{i=1}^n [o, p_i]$ is given in a centered canonical form if $\sum_{i=1}^n p_i = o$.

We start with a simple observation.

Remark 8.4. *Let* $Z_{\mathrm{rd}}^{\mathrm{reg}} = \sum_{i=1}^{d+1} [o, q_i]$ *with* $q_i \in \mathbb{S}^{d-1}$ *for all values of* i. *Then*

$$\mathrm{cr}(Z_{\mathrm{rd}}^{\mathrm{reg}}) = \begin{cases} \frac{\sqrt{d+2}}{2} & \text{if } d \text{ is even,} \\ \frac{d+1}{2\sqrt{d}} & \text{if } d \text{ is odd.} \end{cases}$$

Furthermore, for some $\varepsilon_i \in \{-1, 1\}$, $i = 1, 2, \ldots, d+1$ *we have* $\frac{1}{2}||\sum_{i=1} \varepsilon_i q_i|| = \mathrm{cr}(Z_{\mathrm{rd}}^{\mathrm{reg}})$ *if and only if the number of the* ε_i *equal to* -1 *is* $\lfloor \frac{d+1}{2} \rfloor$ *or* $\lceil \frac{d+1}{2} \rceil$.

Proof. Clearly, the value of $||\sum_{i=1}^{d+1} \varepsilon_i p_i||$ depends only on the numbers of the positive and negative coefficients. Thus, we may assume that there is some $0 < k \leq d+1$ such that $\varepsilon_1 = \ldots = \varepsilon_k = 1$ and $\varepsilon_{k+1} = \ldots = \varepsilon_{d+1} = -1$. Furthermore, observe that $\sum_{i=1}^{d+1} p_i = o$ implies that $\frac{1}{2}||\sum_{i=1}^k p_i - \sum_{i=k+1}^{d+1} p_i|| = ||\sum_{i=1}^k p_i||$. By the same fact and since the value of $\langle p_i, p_j \rangle$ is independent of i, j for all $i \neq j$, we also have $\langle p_i, p_j \rangle = -\frac{1}{d}$ for any $i \neq j$. Thus, an elementary computation yields that

$$||p_1 + \ldots + p_k||^2 = \langle p_1 + \ldots + p_k, p_1 + \ldots + p_k \rangle = k + 2\binom{k}{2}\left(-\frac{1}{d}\right) = \frac{k(d+1-k)}{d}.$$

As k is an integer, from this we have that the maximum of $||p_1 + \ldots + p_k||$ is attained if and only if $k = \lfloor \frac{d+1}{2} \rfloor$ or $k = \lceil \frac{d+1}{2} \rceil$. □

We note that Theorem 8.5 in the special case that d is even and all p_i are unit vectors is proved in [7, Theorem 6] based on a different idea.

Theorem 8.5. *Let* $d \geq 2$ *and let* $Z_{\mathrm{rd}}^{\mathrm{reg}} = \sum_{i=1}^{d+1} [o, q_i]$. *Then for any* $Z = \sum_{i=1}^{d+1} [o, p_i] \in \mathcal{Z}_{\mathrm{rd}}$ *having a centered canonical form with* $\mathrm{mwidth}(Z) = \mathrm{mwidth}(Z_{\mathrm{rd}}^{\mathrm{reg}})$, *we have*

$$\mathrm{cr}(Z) \geq \mathrm{cr}(Z_{\mathrm{rd}}^{\mathrm{reg}}), \tag{8.7}$$

with equality if and only if Z *is congruent to* $Z_{\mathrm{rd}}^{\mathrm{reg}}$. *Furthermore, if* d *is odd, then there is a rhombic dodecahedron* $Z' = \sum_{i=1}^{d+1} [o, p'_i]$ *with* $\mathrm{mwidth}(Z') = \mathrm{mwidth}(Z_{\mathrm{rd}}^{\mathrm{reg}})$ *and* $\mathrm{cr}(Z') < \mathrm{cr}(Z_{\mathrm{rd}}^{\mathrm{reg}})$.

Proof. Let $Z=\sum_{i=1}^{d+1}[o,p_i]$ with $\mathrm{mwidth}(Z)=\mathrm{mwidth}(Z_{\mathrm{rd}}^{\mathrm{reg}})$. Without loss of generality, we assume that $\sum_{i=1}^{d+1}||q_i||= d+1$, which by Remark 8.1 implies that $\sum_{i=1}^{d+1}||p_i||= d+1$. First, we prove the inequality $\mathrm{cr}(Z)\geq \mathrm{cr}(Z_{\mathrm{rd}}^{\mathrm{reg}})$ for any $Z\in\mathcal{Z}_{\mathrm{rd}}$ with $\mathrm{mwidth}(Z)=\mathrm{mwidth}(Z_{\mathrm{rd}}^{\mathrm{reg}})$ and having a centered canonical form. To do it, we prove an inequality valid for any rhombic dodecahedron with $\mathrm{mwidth}(Z)=\mathrm{mwidth}(Z_{\mathrm{rd}}^{\mathrm{reg}})$. On the other hand, for simplicity we derive this formula only in the special case that d is odd; if d is even, the same argument can be applied with a slightly different computation. In the proof, we denote the coordinates of any $\varepsilon\in\{-1,1\}^{d+1}$ by $\varepsilon=(\varepsilon_1,\ldots,\varepsilon_{d+1})$, and set $\mathcal{E}=\left\{\varepsilon\in\{-1,1\}^{d+1}:\sum_{i=1}^{d+1}\varepsilon_i=0\right\}$.

Assume that $d=2m-1$. Then we clearly have

$$2\,\mathrm{cr}(Z)\geq\max\left\{\left\|\sum_{i=1}^{d+1}\varepsilon_i p_i\right\|:\varepsilon\in\mathcal{E}\right\}\geq\sqrt{\frac{1}{\binom{d+1}{m}}\sum_{\varepsilon\in\mathcal{E}}\left\|\sum_{i=1}^{d+1}\varepsilon_i p_i\right\|^2}. \tag{8.8}$$

Furthermore,

$$\sum_{\varepsilon\in\mathcal{E}}\left\|\sum_{i=1}^{d+1}\varepsilon_i p_i\right\|^2=\sum_{\varepsilon\in\mathcal{E}}\left\langle\sum_{i=1}^{d+1}\varepsilon_i p_i,\sum_{i=1}^{d+1}\varepsilon_i p_i\right\rangle=\binom{d+1}{m}\sum_{i=1}^{d+1}||p_i||^2+2\sum_{\varepsilon\in\mathcal{E}}\sum_{1\leq i<j\leq d+1}\varepsilon_i\varepsilon_j\langle p_i,p_j\rangle. \tag{8.9}$$

An elementary computation shows that for any $i<j$, the number of the elements $\varepsilon\in\mathcal{E}$ with the property that $\varepsilon_i=\varepsilon_j$ is $2\binom{2m-2}{m-2}$ and the number of elements with $\varepsilon_i=-\varepsilon_j$ is $2\binom{2m-2}{m-1}$. As $2\binom{2m-2}{m-2}-2\binom{2m-2}{m-1}=-\frac{1}{2m-1}\binom{2m}{m}=-\frac{1}{d}\binom{d+1}{m}$, this implies that

$$\sum_{\varepsilon\in\mathcal{E}}\sum_{1\leq i<j\leq d+1}\varepsilon_i\varepsilon_j\langle p_i,p_j\rangle=-\frac{1}{d}\binom{d+1}{m}\sum_{1\leq i<j\leq d+1}\langle p_i,p_j\rangle. \tag{8.10}$$

On the other hand,

$$\sum_{1\leq i<j\leq d+1}\langle p_i,p_j\rangle=\frac{1}{2}\left(\left\langle\sum_{i=1}^{d+1}p_i,\sum_{j=1}^{d+1}p_j\right\rangle-\sum_{i=1}^{d+1}||p_i||^2\right)=\frac{1}{2}\left(||p||^2-\sum_{i=1}^{d+1}||p_i||^2\right),$$

where $p=\sum_{i=1}^{d+1}p_i$. Substituting it back into (8.10) and applying (8.9) and (8.8), we obtain that

$$2\,\mathrm{cr}(Z)\geq\sqrt{\frac{d+1}{d}\sum_{i=1}^{d+1}||p_i||^2-\frac{1}{d}||p||^2}.$$

Now, if Z is given in a centered canonical form, then $p=o$, which implies, together with the inequality between the arithmetic and the quadratic means, that

$$\mathrm{cr}(Z)\geq\frac{1}{2}\sqrt{\frac{d+1}{d}\sum_{i=1}^{d+1}||p_i||^2}\geq\frac{d+1}{2\sqrt{d}}\cdot\frac{\sum_{i=1}^{d+1}||p_i||}{d+1}\geq\frac{d+1}{2\sqrt{d}}.$$

This, combined with Remark 8.4, yields the inequality in (8.4). Assume that we have equality in (8.4). Then, by the inequality between the arithmetic and the quadratic means, $||p_i||=1$ for all values of i. Furthermore, by (8.10), for any $\varepsilon \in \mathcal{E}$ we have $\left\|\sum_{i=1}^{d+1} \varepsilon_i p_i\right\| = \frac{d+1}{\sqrt{d}}$, or equivalently, for any subset $J \subset \{1,2,\ldots,d+1\}$ consisting of m indices, we have that $\left\|\sum_{j\in J} p_j\right\| = \frac{d+1}{2\sqrt{d}}$.

Consider an arbitrary $J \subset \{1,2,\ldots,d+1\}$ such that $\operatorname{card} J = m-1$ and $1,2,\notin J$. Then, by the previous observation, $\left\|p_1 + \sum_{j\in J} p_j\right\|^2 = \left\|p_2 + \sum_{j\in J} p_j\right\|^2$. Computing both sides and using the fact that $||p_1|| = ||p_2|| = 1$, we obtain that $\left\langle p_1, \sum_{j\in J} p_j\right\rangle = \left\langle p_2, \sum_{j\in J} p_j\right\rangle$, which yields also that $\left\|p_1 - \sum_{j\in J} p_j\right\| = \left\|p_2 - \sum_{j\in J} p_j\right\|$. Thus, $\sum_{j\in J} p_j$ lies in the bisector H_{12} of the segment $[p_1,p_2]$. Note that $o \in H_{12}$ follows from the fact that $\|p_1\| = \|p_2\|$. Observe that our conditions imply that for any $i,j \geq 3$, we have $p_i - p_j \in H_{12}$. Thus, if some p_j with $j \geq 3$, say p_3 does not lie in H_{12}, then all p_j lie in the hyperplane $p_3 + H_{12}$. This yields that for any J satisfying the conditions, $\sum_{j\in J} p_j \in (m-1)p_3 + H_{12} \neq H_{12}$, a contradiction. Hence, we have that $p_j \in H_{12}$ for all $j \geq 3$, implying that $\operatorname{conv}(\{p_1,p_2,\ldots,p_{d+1}\})$ is symmetric to the hyperplane H_{12}. Repeating the same argument for any pair of indices in place of 1 and 2, we have that $S = \operatorname{conv}(\{p_1,p_2,\ldots,p_{d+1}\})$ is symmetric to any hyperplane through o and a $(d-2)$-face of S. This yields that S is a regular simplex centered at o, or in other words, Z is a regular rhombic dodecahedron.

Now we show that for d odd, if we drop the condition that Z is centered, then, among rhombic dodecahedra with a given mean width, non-regular ones have minimal circumradius. Consider the regular rhombic dodecahedron $Z_{\mathrm{rd}}^{\mathrm{reg}} = \sum_{i=1}^{d+1}[o,q_i]$, and let $v \neq o$ be an arbitrary vector. Let $\bar{Z} = \sum_{i=1}^{d+1}[o,q_i+v]$. By Remark 8.4, if $||v||$ is sufficiently small, then

$$\operatorname{cr}(\bar{Z}) = \left\|\sum_{i=1}^{d+1} \varepsilon_i (q_i + v)\right\|, \varepsilon \in \{-1,1\}^{d+1}$$

implies that $\sum_{i=1}^{d+1} \varepsilon_i = 0$. As d is odd, from this we have that $\sum_{i=1}^{d+1} \varepsilon_i(q_i+v) = \sum_{i=1}^{d+1} \varepsilon_i q_i$, which yields that $\operatorname{cr}(\bar{Z}) = \operatorname{cr}(Z_{\mathrm{reg}})$.

We show that $\operatorname{mwidth}(\bar{Z}) > \operatorname{mwidth}(Z_{\mathrm{rd}}^{\mathrm{reg}})$. To do it it is sufficient to show that the unique point $x \in S_{\mathrm{reg}} = \operatorname{conv}(\{q_1,q_2,\ldots,q_{d+1}\})$ for which the function $\sum_{i=1}^{d+1}||x-q_i||$ is minimal is o. To prove it, let h denote the length of the altitudes of S_{reg}, and for $i = 1,2,\ldots,d+1$, let h_i denote the distance of x from the facet of S_{reg} opposite of q_i. It is well known that for any $x \in S_{\mathrm{reg}}$, $\sum_{i=1}^{d+1} h_i = h$. Thus, for any $x \in S_{\mathrm{reg}}$, we have

$$\sum_{i=1}^{d+1}||x-q_i|| \geq \sum_{i=1}^{d+1}(h-h_i) = dh,$$

with equality if and only if x lies on all altitudes of S_{reg}, or in other words, if $x = o$.

We have shown that $\operatorname{cr}(\bar{Z}) = \operatorname{cr}(Z_{\mathrm{rd}}^{\mathrm{reg}})$ and $\operatorname{mwidth}(\bar{Z}) > \operatorname{mwidth}(Z_{\mathrm{reg}})$. Thus, if we set $t = \frac{\operatorname{mwidth}(Z_{\mathrm{rd}}^{\mathrm{reg}})}{\operatorname{mwidth}(\bar{Z})}$, then for $Z' = tZ$ we have $\operatorname{mwidth}(Z') = \operatorname{mwidth}(Z_{\mathrm{rd}}^{\mathrm{reg}})$ and $\operatorname{cr}(Z') < \operatorname{cr}(Z_{\mathrm{rd}}^{\mathrm{reg}})$, which proves the last statement of Theorem 8.5. □

8.2.3 The second intrinsic volume of a zonotope with a given mean width

We start with a simple observation for parallelotopes.

Proposition 8.2. *For any $Z \in \mathcal{Z}_{\mathrm{p}}$ with $\operatorname{mwidth}(Z) = \operatorname{mwidth}(Z_{\mathrm{p}}^{\mathrm{reg}})$, we have $V_2(Z) \leq V_2(Z_{\mathrm{p}}^{\mathrm{reg}})$, with equality if and only if Z is a cube.*

Proof. Let $Z = \sum_{i=1}^{d} [o, p_i]$, and set $\lambda_i = ||p_i||$ where, without loss of generality, we may assume that all the λ_i are positive.. Then, for any $i \neq j$, we have $|p_i \wedge p_j| \leq \lambda_i \lambda_j$, with equality if and only p_i and p_j are orthogonal. Thus, we need to show that if $\sum_{i=1}^{d} \lambda_i$ is fixed, then $A = \sum_{1 \leq i < j \leq d} \lambda_i \lambda_j$ is maximal if and only if all λ_i are equal. But this follows from the Maclaurin inequality (see Lemma 8.1). □

In the remaining part of Subsection 8.2.3, we find the equilateral rhombic dodecahedra with a given mean width and maximal second intrinsic volume. Our proof is a modification of a proof of [7, Theorem 3]. We remark that the 'dual' problem, that is, minimizing the ith intrinsic volume of a zonotope with a fixed mean width was considered in [113] for the special class of zonotopes with an isotropic generating measure.

Theorem 8.6. *Let $Z_{\mathrm{rd}}^{\mathrm{reg}} = \sum_{i=1}^{d+1} [o, q_i]$, where $q_i \in \mathbb{S}^{d-1}$ for all values of i. Then, if $Z = \sum_{i=1}^{d+1} [o, p_i]$ is a rhombic dodecahedron with $p_i \in \mathbb{S}^{d-1}$ for all values of i, then*

$$V_2(Z) \geq V_2(Z_{\mathrm{rd}}^{\mathrm{reg}}),$$

with equality if and only if Z is regular.

Proof. It is well known that for any vectors $x_1, x_2, \ldots, x_k$ in $\mathbb{R}^d$ with $k \leq d$, the square of the k-dimensional volume of the parallelotope spanned by $x_1, x_2, \ldots, x_k$ is equal to the determinant of the Gram matrix of the vectors (see e.g. [92]). In particular, for any $x_1, x_2 \in \mathbb{R}^d$, the square of the area of the parallelogram spanned by x_1, x_2 is

$$|x_1 \wedge x_2|^2 = \det([x_1, x_2]^T [x_1, x_2]) = \langle x_1, x_1 \rangle \langle x_2, x_2 \rangle - \langle x_1, x_2 \rangle^2.$$

This and Corollary 8.1 imply that

$$V_2(Z) = \sum_{1 \leq i < j \leq d+1} \sqrt{\langle p_i, p_i \rangle \langle p_j, p_j \rangle - \langle p_i, p_j \rangle^2}.$$

We note that this yields, in particular, that $V_2(Z_{\mathrm{reg}}) = \binom{d+1}{2} \cdot \sqrt{\frac{d^2-1}{d^2}} = \frac{1}{2}(d+1)\sqrt{d^2-1}$. By applying the fact that Z is of unit edge length and also the inequality between the arithmetic and the quadratic means we obtain that

$$(V_2(Z))^2 \leq \binom{d+1}{2} \left(\binom{d+1}{2} - \sum_{1 \leq i < j \leq d+1} \langle p_i, p_j \rangle^2 \right), \tag{8.11}$$

with equality if and only if Z is regular. In the following part we determine the minimum value of $\sum_{1 \leq i < j \leq d+1} \langle p_i, p_j \rangle^2$.

Since the vectors $p_1, p_2, \ldots, p_{d+1}$ are linearly dependent, there are constants $\lambda_i \in \mathbb{R}$, where $i = 1, 2, \ldots, d+1$, such that $\sum_{i=1}^{d+1} \lambda_i p_i = o$ and $\sum_{i=1}^{d+1} \lambda_i^2 > 0$. Without loss of generality, we may assume that $\sum_{i=1}^{d+1} \lambda_i^2 = 1$. Multiplying both sides of this vector equation by itself and using the equalities $\langle p_i, p_i \rangle = 1$, we obtain that

$$0 = \sum_{i=1}^{d+1} \lambda_i^2 + 2 \sum_{1 \leq i < j \leq d+1} \lambda_i \lambda_j \langle p_i, p_j \rangle = 1 + 2 \sum_{1 \leq i < j \leq d+1} \lambda_i \lambda_j \langle p_i, p_j \rangle.$$

Thus, by the Cauchy-Schwarz Inequality, we have

$$1=-2\sum_{1\leq i<j\leq d+1}\lambda_i\lambda_j\langle p_i,p_j\rangle \leq 2\sqrt{\sum_{1\leq i<j\leq d+1}\lambda_i^2\lambda_j^2}\sqrt{\sum_{1\leq i<j\leq d+1}\langle p_i,p_j\rangle^2},$$

implying that

$$\sum_{1\leq i<j\leq d+1}\langle p_i,p_j\rangle^2 \geq \frac{1}{4\sum_{1\leq i<j\leq d+1}\lambda_i^2\lambda_j^2}, \tag{8.12}$$

which yields by Lemma 8.1 that

$$\sum_{1\leq i<j\leq d+1}\langle p_i,p_j\rangle^2 \geq \frac{2d}{d+1}.$$

Combining this inequality with (8.11), we obtain that

$$V_2(Z)\leq \sqrt{\binom{d+1}{2}\left(\binom{d+1}{2}-\frac{d+1}{2d}\right)}=\frac{1}{2}(d+1)\sqrt{d^2-1}=V_2(Z_{\text{reg}}),$$

with equality if and only if Z is regular. □

8.2.4 The squared k-volumes of a zonotope

Recall that for any simplex $S\subset\mathbb{R}^d$, integer $1\leq k\leq d-1$, and real number $m\in\mathbb{R}$, $g_k^m(S)$ denotes the sum of the mth powers of the volumes of the k-faces of S [182]. Similarly, Filliman [73] investigated the problem of finding the zonotopes of maximal volume with a fixed value of the squared lengths of its generating vectors. These results, Corollary 8.1, and the results of Brazitikos and McIntyre in [43] is the motivation behind our next definition.

Definition 8.1. *Let $Z=\sum_{i=1}^n[o,p_i]$ be a zonotope in $\mathbb{R}^d$ and let $\alpha>0$. We call the quantity*

$$V_{k,\alpha}(Z)=\sum_{1\leq i_1<i_2<\ldots<i_k\leq n}V_k^{\alpha}(P(i_1,\ldots,i_k))$$

the total k-volume of power α *of Z.*

Our next result is an Alexandrov-Fenchel type inequality for the total squared k-volumes of rhombic dodecahedra. We remark that an analogous statement for parallelotopes can be found in [43].

Theorem 8.7. *Let Z be a rhombic dodecahedron in $\mathbb{R}^d$. Then, for any $1\leq k<m\leq d$, the quantity*

$$\frac{(V_{k,2}(Z))^m}{(V_{m,2}(Z))^k}$$

is minimal if and only if Z is regular.

First, we prove a slight generalization of Lemma 8.1.

Lemma 8.4. *Let $1 \leq k < m$ be integers, and $x_1, \ldots, x_m \geq 0$ be nonnegative real numbers. Then*

$$\left(\frac{\sigma_m^k(x_1, x_2, \ldots, x_m)}{\binom{m}{k}} \right)^{\frac{1}{k}} \geq \left(\frac{\sigma_m^{k+1}(x_1, x_2, \ldots, x_m)}{\binom{m}{k+1}} \right)^{\frac{1}{k+1}}, \tag{8.13}$$

with equality if and only if all x_i are equal, or at least $m-k+1$ of them is zero.

Proof. Since both sides in (8.13) are continuous functions of their variables, the inequality in (8.13) clearly holds. Similarly, the equality case in Lemma 8.4 holds if all x_i are positive. Note that for any values of $m, k, x_1, x_2, \ldots, x_m$, $\sigma_m^k(x_1, \ldots, x_m) = \sigma_{m+1}^k(x_1, \ldots, x_m, 0)$. Thus, to prove Lemma 8.4 for all nonnegative x_i, it is sufficient to show that if $\sqrt[k]{\frac{A}{\binom{m}{k}}} \geq \sqrt[k+1]{\frac{B}{\binom{m}{k+1}}}$ for some reals $A, B > 0$ and integers $1 \leq k < m$, then $\sqrt[k]{\frac{A}{\binom{m+1}{k}}} > \sqrt[k+1]{\frac{B}{\binom{m+1}{k+1}}}$. In other words, we need to show that

$$\left(\frac{\binom{m}{k}}{\binom{m+1}{k}} \right)^{\frac{1}{k}} > \left(\frac{\binom{m}{k+1}}{\binom{m+1}{k+1}} \right)^{\frac{1}{k+1}} \tag{8.14}$$

for all $1 \leq k < m$. An elementary computation shows that the inequality in (8.14) is equivalent to

$$\left(1 - \frac{k}{m+1} \right)^{\frac{1}{k}} > \left(1 - \frac{k+1}{m+1} \right)^{\frac{1}{k+1}},$$

and thus, the assertion follows from the fact that for any $0 < a < 1$, the function $x \mapsto (1-ax)^{\frac{1}{x}}$ is strictly decreasing on the interval $\left(0, \frac{1}{a}\right]$. □

Proof of Theorem 8.7. Consider the zonotope $Z = \sum_{i=1}^{d+1} [o, p_i]$. Let G be the Gram matrix of the vectors p_i; i.e. $G = [\langle p_i, p_j \rangle]$. Then G is a $(d+1) \times (d+1)$ symmetric, positive semidefinite matrix. Let $p(\lambda) = \det(G - \lambda E)$ denote the characteristic polynomial of G. Clearly, the coefficient of λ^{d+1-k} is equal to the signed sum of the principal minors of size $k \times k$ [149], where a principal minor of size $k \times k$ of G is the determinant of a $k \times k$ submatrix of G symmetric to the main diagonal of G. On the other hand, it is also well known that this coefficient is equal to the kth elementary symmetric function of the eigenvalues of G. Note that the rank of G is equal to the dimension of the linear hull of the vectors v_i, and thus, 0 is an eigenvalue of G with multiplicity at least one. Thus, the inequality in Theorem 8.7, together with the equality part, follows from Lemma 8.4 applied for d eigenvalues of G, containing all nonzero ones. □

We also note that a straightforward modification of the proof of Theorem 8.7 verifies Conjecture 8.1 for $p = 2$ in the following form.

Theorem 8.8. *Let $n \geq d$ and $Z = \sum_{i=1}^{n} [o, x_i]$ be a zonotope in $\mathbb{R}^d$. Then, for any $1 \leq k < d$, the quantity*

$$\left(\frac{V_{k,2}(Z)}{\binom{n}{k}} \right)^{\frac{1}{k}} \geq \left(\frac{V_{k+1,2}(Z)}{\binom{n}{k+1}} \right)^{\frac{1}{k+1}},$$

with equality if and only if $n = d$ and Z is a cube, or if the dimension of Z is at most $k-1$.

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8.3 Isoperimetric problems for zonotopes generated by a large number of segments

Recall that for any $d \geq 2$ and $n \geq d$, $\mathcal{Z}_{d,n}$ denotes the family of d-dimensional zonotopes generated by n segments. In this section we investigate isoperimetric problems for zonotopes generated by sufficiently many segments. It is worth noting that as the Euclidean ball can be approached arbitrarily well by zonotopes, and in Euclidean space a ball is the solution of most isoperimetric problems among all convex bodies, the problems discussed in this section are closely related to the problem of how well a Euclidean ball can be approached with a zonotope generated by a given number of segments. The latter problem was extensively studied in the literature, motivated by the problem of determining the minimum number of directions we need to estimate the surface area of a convex body up to error ε from the areas of the projections in the chosen directions. The order of magnitude of the number of directions was established in most dimensions in a series of papers [15, 140, 38, 40, 144] by Linhart, Bourgain, Lindestrauss, Milman and Matoušek. Namely, it is known that for $d \geq 2$ there are constants $c_1 = c_1(d)$ and $c_2 = c_2(d)$ depending only on d with the following property: if for any $\varepsilon > 0$, $N(\varepsilon)$ denotes the minimum number n such that there is a zonotope $Z \in \mathcal{Z}_{d,n}$ with $\mathbf{B}^d \subseteq Z \subseteq (1+\varepsilon)\mathbf{B}^d$, then

$$c_1 \varepsilon^{\frac{-2(d-1)}{d+2}} \leq N(\varepsilon) \leq \begin{cases} c_2 \varepsilon^{\frac{-2(d-1)}{d+2}}, & \text{if } d = 2 \text{ or } d \geq 5, \\ c_2 \left(\varepsilon^{-2} \log|\varepsilon|\right)^{\frac{(d-1)}{d+2}}, & \text{otherwise.} \end{cases} \tag{8.15}$$

Furthermore, it was proved in [39] that the weaker bound on the right-hand side of (8.15) can be attained with an equilateral zonotope, i.e. there is a zonotope Z generated by $n \leq c_2 \left(\varepsilon^{-2} \log|\varepsilon|\right)^{\frac{(d-1)}{d+2}}$ segments of equal length such that $\mathbf{B}^d \subseteq Z \subseteq (1+\varepsilon)\mathbf{B}^d$.

Similarly like in the previous problem, in our investigation we regard the dimension d as a fixed parameter, and the number n of generating segments as a variable. Furthermore, we set

$$U_d(n) = \begin{cases} \frac{\sqrt{\log n}}{n^{\frac{d+2}{2d-2}}}, & \text{if } d = 3 \text{ or } d = 4, \\ \frac{1}{n^{\frac{d+2}{2d-2}}}, & \text{if } d = 2 \text{ or } d \geq 5. \end{cases}$$

Our first result is an immediate consequence of (8.15), and we omit its proof.

Theorem 8.9. *Let $d \geq 2$ be fixed. Then there are positive constants $c = c(d)$ and $C = C(d)$ depending only on the dimension such that for any $n \geq d+1$,*

$$\frac{c}{n^{\frac{d+2}{2d-2}}} \leq \min\left\{ \frac{\mathrm{cr}(Z)}{\mathrm{ir}(Z)} - 1 : Z \in \mathcal{Z}_{d,n} \right\} \leq C U_d(n). \tag{8.16}$$

In the remaining part of this section we investigate similar problems. More specifically, in Subsections 8.3.1 and 8.3.2 we estimate the intrinsic volumes of a zonotope $Z \in \mathcal{Z}_{d,n}$ with a given inradius or circumradius, respectively. Finally, in Subsection 8.3.3 we compare two intrinsic volumes of a zonotope in $\mathcal{Z}_{d,n}$.

8.3.1 Intrinsic volumes of a zonotope with a given inradius

Our main result in this subsection is Theorem 8.10.

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Theorem 8.10. *Let $1 \leq i \leq d$. Then there is a positive constant $C = C(d)$ depending only on d such that for any sufficiently large value of n, we have*

$$\frac{4i}{5dn^2} \leq \min\left\{ \frac{V_i(Z)}{V_i(\mathbf{B}^d)} - 1 : Z \in \mathcal{Z}_{d,n}, \operatorname{ir}(Z) = 1 \right\} \leq CU_d(n).$$

To prove it we need the following lemma.

Lemma 8.5. *Let $2 \leq d \leq n$. Then we have*

$$\min\{V_d(Z) : Z \in \mathcal{Z}_{d,n}, \operatorname{ir}(Z) \geq 1\} \geq \kappa_d \left(1 + \frac{\pi^2}{12n^2}\right).$$

Proof. First, let $d = 2$. Consider any $Z \in \mathcal{Z}_{2,n}$ with $\operatorname{ir}(Z) \geq 1$. Without loss of generality, assume that $\mathbf{B}^2 \subset Z$. Then, we clearly have that the minimum of $V_2(Z)$ is the area of the regular $(2n)$-gon circumscribed about $\mathbf{B}^2$. Thus, an elementary computation shows that $V_2(Z) \geq 2n \tan\frac{\pi}{2n} = \pi \cdot \frac{\tan\frac{\pi}{2n}}{\frac{\pi}{2n}}$. On the other hand, from the third order Taylor polynomial of the tangent function with the Lagrange remainder form, and using the fact that $(\tan x)^{(4)} > 0$ for $x \in \left(0, \frac{\pi}{2}\right)$, it follows that $\tan x > x + \frac{1}{3}x^3$. Hence, we obtain that

$$V_2(Z) > \kappa_2 \left(1 + \frac{\pi^2}{12n^2}\right).$$

In the remaining part of the proof we apply an induction on d. Let $Z \in \mathcal{Z}_{d,n}$ with $\operatorname{ir}(Z) \geq 1$. Assume that $\mathbf{B}^d \subset Z$. For any unit vector $u \in \mathbb{S}^{d-1}$, let $Z|u^\perp$ denote the orthogonal projection of Z onto the hyperplane through o and with normal vector u. Note that for any $u \in \mathbb{S}^{d-1}$, $Z|u^\perp$ is a zonotope in $\mathcal{Z}_{d-1,n}$ satisfying $\operatorname{ir}(Z) \geq 1$. Thus, by the induction hypothesis, $V_{d-1}(Z|u^\perp) \geq \kappa_{d-1}\left(1 + \frac{\pi^2}{12n^2}\right)$. Recall that by Cauchy's surface area formula [84], we have

$$\operatorname{surf}(Z) = \frac{1}{\kappa_{d-1}} \int_{\mathbb{S}^{d-1}} V_{d-1}(Z|u^\perp)\, du,$$

where $\operatorname{surf}(Z)$ denotes the surface area of Z. Then, applying the previous estimate and the fact that the surface area of $\mathbb{S}^{d-1}$ is $d\kappa_d$, after simplification we obtain that

$$\operatorname{surf}(Z) \geq d\kappa_d \left(1 + \frac{\pi^2}{12n^2}\right). \tag{8.17}$$

Now, let F be any facet of Z. Then, by the convexity of Z and since $\mathbf{B}^d \subset Z$, the distance of $\operatorname{aff}(F)$ from o is at least 1. Thus, $V_d(\operatorname{conv}(F \cup \{o\})) \geq \frac{1}{d} V_{d-1}(F)$, implying that $\operatorname{surf}(Z) \leq dV_d(Z)$. This, combined with (8.17), yields the assertion. □

Proof of Theorem 8.10. Let Z be an arbitrary zonotope in $\mathcal{Z}_{d,n}$ satisfying $\operatorname{ir}(Z) = 1$. By the Alexandrov-Fenchel inequality for intrinsic volumes, we have that

$$\left(\frac{V_d(Z)}{V_d(\mathbf{B}^d)}\right)^{\frac{1}{d}} \leq \left(\frac{V_{d-1}(Z)}{V_{d-1}(\mathbf{B}^d)}\right)^{\frac{1}{d-1}} \leq \ldots \leq \frac{V_1(Z)}{V_1(\mathbf{B}^d)}. \tag{8.18}$$

Thus,

$$\left(\frac{V_d(Z)}{V_d(\mathbf{B}^d)}\right)^{\frac{i}{d}} - 1 \leq \frac{V_i(Z)}{V_i(\mathbf{B}^d)} - 1 \leq \left(\frac{V_1(Z)}{V_1(\mathbf{B}^d)}\right)^{i} - 1.$$

By Lemma 8.5 and using that $\lim_{x\to 0} \frac{(1+x)^{\alpha}-1}{x} = \alpha$ and that $\frac{\pi^2}{12} > \frac{4}{5}$, for sufficiently large values of n, we have

$$\frac{4i}{5dn^2} \leq \left(\frac{V_d(Z)}{V_d(\mathbf{B}^d)}\right)^{\frac{i}{d}} - 1 \leq \frac{V_i(Z)}{V_i(\mathbf{B}^d)} - 1.$$

This yields the left-hand side inequality in Theorem 8.10. To prove the right-hand side inequality, observe that if Z is a zonotope in $\mathcal{Z}_{d,n}$ with $\mathbf{B}^d \subseteq Z \subseteq (1+\varepsilon)\mathbf{B}^d$, then $2 \leq \operatorname{mwidth}(Z) \leq 2(1+\varepsilon)$. This, combined with Theorem 8.9, yields that if n is sufficiently large, then

$$\left(\frac{V_1(Z)}{V_1(\mathbf{B}^d)}\right)^{i} - 1 \leq CU_d(n)$$

for some suitably chosen $Z \in \mathcal{Z}_{d,n}$ with $\operatorname{ir}(Z) = 1$. □

8.3.2 Intrinsic volumes of a zonotope with a given circumradius

We prove Theorem 8.11.

Theorem 8.11. *Let $1 \leq i \leq d$. Then there is a positive constant $C = C(d)$ such that for any sufficiently large value of n,*

$$\frac{2i}{5n^2} \leq \min\left\{1 - \frac{V_i(Z)}{V_i(\mathbf{B}^d)} : Z \in \mathcal{Z}_{d,n}, \operatorname{cr}(Z) = 1\right\} \leq CU_d(n).$$

Lemma 8.6. *Let $d \geq 2$ and $n \geq d$. Then, for any $Z \in \mathcal{Z}_{n,d}$ with $\operatorname{cr}(Z) \geq 1$, we have*

$$1 - \frac{V_1(Z)}{V_1(\mathbf{B}^d)} \geq \frac{\pi^2}{24n^2} - \frac{\pi^4}{1920n^4}.$$

Proof. First, we prove the inequality for $d = 2$. Let $Z \in \mathcal{Z}_{n,2}$ with $\operatorname{cr}(Z) \geq 1$, and assume that $Z \subset \mathbf{B}^2$. By Dowker's theorem, we have that the perimeter of Z is at most equal to the perimeter of the regular $(2n)$-gon inscribed in $\mathbf{B}^2$. Thus, we have $\operatorname{perim}(Z) \leq 4n \sin \frac{\pi}{2n}$. Note that $V_1(Z) = \frac{1}{2}\operatorname{perim}(Z)$ (cf. [169, p.210]). Thus, applying the Taylor-expansion of the sine function, we have that

$$V_1(Z) \leq \pi \left(1 - \frac{\pi^2}{24n^2} + \frac{\pi^4}{1920n^4}\right).$$

Since $V_1(\mathbf{B}^d) = \frac{d\kappa_d}{\kappa_{d-1}}$, and in particular $V_1(\mathbf{B}^2) = \pi$, this implies the assertion for $d = 2$. To prove it for $d > 2$, we recall Kubota's integral recursion formula from [84, A.46]. To state it, recall from Chapter 2 the notation $\mathcal{G}_{d,k}$ for the Grassmannian manifold of the k-dimensional linear subspaces of $\mathbb{R}^d$, $K|S$ for the orthogonal projection of a compact, convex set $K \subset \mathbb{R}^d$ to some $S \in \lceil, \|$. Then this formula states that if K is any compact, convex set in $\mathbb{R}^d$, and $1 \leq i \leq k \leq d-1$, then

$$V_i(K) = \frac{\binom{d}{i}\kappa_{k-i}\kappa_d}{\binom{k}{i}\kappa_{d-i}\kappa_k} \int_{\mathcal{G}_{d,k}} V_i(K|S)\, dS, \tag{8.19}$$

where the integration is with respect to the unique Haar probability measure of $\mathcal{G}_{d,k}$. Let $Z \in \mathcal{Z}_{d,n}$ be a zonotope with $\operatorname{cr}(Z) \leq 1$. Then, for any $S \in \mathcal{G}_{d,2}$, $Z|S$ is a zonotope in $Z_{2,n}$ with $\operatorname{cr}(Z|S) \leq 1$.

Thus, $V_1(Z|S) \leq \pi\left(1 - \frac{\pi^2}{24n^2} + \frac{\pi^4}{1920n^4}\right)$ for any $S \in \mathcal{G}_{d,2}$. Now, applying (8.19) for Z with $i = 1$ and $k = 2$, we obtain that

$$V_1(Z) \geq \frac{d\kappa_d}{\kappa_{d-1}}\left(1 - \frac{\pi^2}{24n^2} + \frac{\pi^4}{1920n^4}\right),$$

from which the assertion readily follows. □

Proof of Theorem 8.11. By the Alexandrov-Fenchel inequality for any zonotope $Z \in \mathcal{Z}_{d,n}$ with $\operatorname{cr}(Z) = 1$, we have

$$1 - \left(\frac{V_d(Z)}{V_d(\mathbf{B}^d)}\right)^{\frac{i}{d}} \geq 1 - \frac{V_i(Z)}{V_i(\mathbf{B}^d)} \geq 1 - \left(\frac{V_1(Z)}{V_1(\mathbf{B}^d)}\right)^i.$$

Thus, the inequalities in Theorem 8.11 follow by an argument analogous to that in the proof of Theorem 8.10. □

Remark 8.5. *It was shown in [8, Theorem 1] (and remarked also in [7]) that*

$$M_n^1(\mathbb{S}^{d-1}) = n\mu_{d,1} + o\left(\frac{n}{\sqrt{d}}\right)$$

if $n, d \to \infty$ *and* $n = \omega\left(d^2 \log d\right)$, *where* $\mu_{d,1} = \frac{\Gamma\left(\frac{d}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{d+1}{2}\right)}$. *Theorem 8.11 yields another estimate for this quantity. Namely, for any fixed* $d \geq 2$, *there is a constant* $C = C(d)$ *such that*

$$\left|M_n^1(\mathbb{S}^{d-1}) - n\mu_{d,1}\right| \leq Cn^{\frac{d-4}{2d-2}}\sqrt{\log n}.$$

Proof. By Theorem 8.11, there is a zonotope $Z \in \mathcal{Z}_{d,n}$ with $\operatorname{cr}(Z) = 1$ satisfying

$$1 - \frac{V_1(Z)}{V_1(\mathbf{B}^d)} \leq C\frac{\sqrt{\log n}}{n^{\frac{d+2}{2d-2}}}$$

where $C > 0$ may depend on d. Here, by [39], we may assume that Z is an equilateral zonotope, i.e. that all generating segments of Z are of length t for some $t > 0$. The definition of intrinsic volume and an elementary computation shows that $V_1(\mathbf{B}^d) = \frac{d\kappa_d}{\kappa_{d-1}} = \frac{2\sqrt{\pi}\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} = \frac{2}{\mu_{d,1}}$. Let $Z' = \sum_{i=1}^n [o, p_i] = \frac{1}{t}Z$. Then $V_1(Z') = n$, and $\operatorname{cr}(Z') = \frac{1}{t} = \frac{1}{2}\max_{v \in \mathbb{S}^{d-1}} \sum_{i=1}^n |\langle v, p_i\rangle|$, which implies that

$$1 - \frac{n\mu_{d,1}}{2\operatorname{cr}(Z')} \leq C\frac{\sqrt{\log n}}{n^{\frac{d+2}{2d-2}}}.$$

From this inequality and the inequality $\frac{1}{1-x} < 1 + 2x$ holding for any sufficiently small $x > 0$, we obtain that

$$2\operatorname{cr}(Z') \leq n\mu_{d,1}\left(1 + 2C\frac{\sqrt{\log n}}{n^{\frac{d+2}{2d-2}}}\right),$$

which yields the assertion. □

8.3.3 Comparing two intrinsic volumes of a zonotope

Our main result here is Theorem 8.12.

Theorem 8.12. *Let $1 \leq i < k \leq d$. Then there are positive constants c, C depending only on d such that for any sufficiently large value of n,*

$$\frac{c}{n^{\frac{(d+2)(d+3)}{4d-4}}} \leq \min\left\{ \frac{(V_i(Z))^{\frac{1}{i}}}{(V_k(Z))^{\frac{1}{k}}} - \frac{\left(V_i(\mathbf{B}^d)\right)^{\frac{1}{i}}}{(V_k(\mathbf{B}^d))^{\frac{1}{k}}} : Z \in \mathcal{Z}_{d,n} \right\} \leq \frac{C}{n}. \tag{8.20}$$

Furthermore, there is a constant $\bar{c} > 0$ depending on d such that

$$\frac{\bar{c}}{n^2} \leq \min\left\{ \frac{(V_{d-1}(Z))^{\frac{1}{d-1}}}{(V_d(Z))^{\frac{1}{d}}} - \frac{\left(V_{d-1}(\mathbf{B}^d)\right)^{\frac{1}{d-1}}}{(V_d(\mathbf{B}^d))^{\frac{1}{d}}} : Z \in \mathcal{Z}_{d,n} \right\}. \tag{8.21}$$

To prove this theorem, we need some preparation. In the following, let ω_d denote the surface area of the Euclidean sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$.

Remark 8.6. *For any $d \geq 2$, we have*

$$\frac{V_d(\mathbf{B}^d)}{\left(V_1(\mathbf{B}^d)\right)^d} = \frac{2\omega_{d+1}^{d-1}}{\omega_d^d d!}.$$

Proof. Note that $\omega_d = d\kappa_d$, and $\kappa_d = \frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}+1\right)}$, implying also the identity $\kappa_{d+1} = \frac{2\pi}{d+1}\kappa_{d-1}$. Furthermore, it is well known that $V_1(\mathbf{B}^d) = \frac{d\kappa_d}{\kappa_{d-1}}$. Applying these identities, we have

$$\frac{2\omega_{d+1}^{d-1}}{\omega_d^d d!} \cdot \frac{\left(V_1(\mathbf{B}^d)\right)^d}{V_d(\mathbf{B}^d)} = \frac{2(d+1)^{d-1}\kappa_{d+1}^{d-1}}{d!\,\kappa_{d-1}^d \kappa_d} = \frac{2^d \pi^{d-1}}{d!\,\kappa_d \kappa_{d-1}} = \frac{2^d \Gamma\left(\frac{d+2}{2}\right)\Gamma\left(\frac{d+1}{2}\right)}{d!\sqrt{\pi}} = 1,$$

where in the last step we used the Legendre duplication formula $\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\Gamma(2z)$ for $z \in \mathbb{C}$, and the identity $\Gamma(k) = (k-1)!$ for all positive integers k. □

Our next lemma is a special case of [170, Theorem 8.2.3], and hence, we omit its proof.

Lemma 8.7. *Let $p_1, \ldots, p_d \in \mathbb{S}^{d-1}$ be unit vectors chosen independently according to the uniform probability distribution on $\mathbb{S}^{d-1}$. Then the expected value of $|p_1 \wedge p_2 \wedge \ldots \wedge p_d|$ is $\frac{2\omega_{d+1}^{d-1}}{\omega_d^d}$.*

Proof of Theorem 8.12. First, we prove the lower bound in (8.20) for $i = 1$ and $k = 2$. Let $Z \in \mathcal{Z}_{d,n}$. Since the statement is invariant under rescaling Z, we can assume that $V_1(Z) = V_1(\mathbf{B}^d)$ and that Z is centered at the origin. Then the Steiner ball of Z, defined as the ball centered at the Steiner point of Z and having mean width equal to that of Z [169], coincides with $\mathbf{B}^d$. Our proof in this case is based on a stability version of the Alexandrov-Fenchel inequality, stated in (7.123) of [169] as a consequence of [169, Theorem 7.6.6] and implying that $\frac{1}{\kappa_d}\left(W_{d-1}(Z)\right)^2 - W_{d-2}(Z) \geq \frac{d+1}{d(d-1)}\left(\delta_2(Z, \mathbf{B}^d)\right)^2$, where $W_i(K)$ denotes the ith quermassintegral of the convex body K, and $\delta_2(K, L) = \sqrt{\int_{u \in \mathbb{S}^{d-1}} |h_K(u) - h_L(u)|^2\, du}$, with h_K and h_L denoting the support functions of the convex bodies of K and L, respectively. Applying the well known relation $W_{d-i}(Z) = \frac{\kappa_{d-i}}{\binom{d}{i}} V_i(Z)$ for $i = 1, 2$, and using $W_i(\mathbf{B}^d) = \kappa_d$ for all values of i, we obtain that

$$\frac{(V_1(Z))^2}{(V_1(\mathbf{B}^d))^2} - \frac{V_2(Z)}{V_2(\mathbf{B}^d)} = 1 - \frac{V_2(Z)}{V_2(\mathbf{B}^d)} \geq \frac{d+1}{\kappa_d d(d-1)} \left(\delta_2(K, \mathbf{B}^d)\right)^2 .$$

We note that by Lemma 7.6.5 of [169] (see also Lemmas 1 and 2 in [93]), we have

$$\left(\delta_2(Z, \mathbf{B}^d)\right)^2 \geq c' \left(\delta(Z, \mathbf{B}^d)\right)^{\frac{d+3}{2}} ,$$

where $\delta(\cdot, \cdot)$ denotes the Hausdorff distance, and $c' > 0$ is a constant depending only on the dimension d. If for some $\varepsilon > 0$, $\delta(Z, \mathbf{B}^d) \geq \varepsilon$ for all $Z \in \mathcal{Z}_{d,n}$ with Steiner ball $\mathbf{B}^d$ and for all sufficiently large values of n, we are done. Thus, we may assume that $\delta(Z, \mathbf{B}^d) \geq \frac{1}{2}$, implying that $\frac{1}{2} \leq \mathrm{ir}(Z)$. Observe that since the Steiner ball of Z is $\mathbf{B}^d$, we have $\delta(Z, \mathbf{B}^d) \geq \frac{1}{2} \left(\mathrm{cr}(Z) - \mathrm{ir}(Z)\right) \geq \frac{1}{4} \left(\frac{\mathrm{cr}(Z)}{\mathrm{ir}(Z)} - 1 \right)$. Hence, from Theorem 8.9 it follows that

$$1 - \frac{V_2(Z)}{V_2(\mathbf{B}^d)} \geq c'' \left(\delta(Z, \mathbf{B}^d)\right)^{\frac{d+3}{2}} \geq \frac{c'''}{n^{\frac{(d+2)(d+3)}{4d-4}}}$$

for some constant $c''' > 0$ depending on the dimension. After some elementary calculations, this implies the required bound.

Now we prove the lower bound in (8.20) for any value of i with $k = i + 1$, and observe that this clearly yields the lower bound in (8.20) for all values of i and k. By the Alexandrov-Fenchel inequality (see e.g. [14, Theorem 10 (ii)], for any $2 \leq j \leq d-1$ and any $Z \in \mathcal{Z}_{d,n}$, we have $\left(\frac{V_j(Z)}{V_j(\mathbf{B}^d)} \right)^2 \geq \frac{V_{j-1}(Z)}{V_{j-1}(\mathbf{B}^d)} \frac{V_{j+1}(Z)}{V_{j+1}(\mathbf{B}^d)}$, implying $\frac{\left(V_j(Z)/V_j(\mathbf{B}^d)\right)^{1/j}}{\left(V_{j+1}(Z)/V_{j+1}(\mathbf{B}^d)\right)^{1/(j+1)}} \geq \left(\frac{\left(V_{j-1}(Z)/V_{j-1}(\mathbf{B}^d)\right)^{1/(j-1)}}{\left(V_j(Z)/V_j(\mathbf{B}^d)\right)^{1/j}} \right)^{\frac{j-1}{j+1}}$. From this, it follows that

$$\frac{\left(V_i(Z)/V_i(\mathbf{B}^d)\right)^{1/i}}{\left(V_{i+1}(Z)/V_{i+1}(\mathbf{B}^d)\right)^{1/(i+1)}} \geq \left(\frac{V_1(Z)/V_1(\mathbf{B}^d)}{\sqrt{V_2(Z)/V_2(\mathbf{B}^d)}} \right)^{\frac{2}{i(i+1)}} .$$

We investigate the left-hand side. Similarly like in the previous paragraph, we may assume that $\frac{V_1(Z)/V_1(\mathbf{B}^d)}{\sqrt{V_2(Z)/V_2(\mathbf{B}^d)}} \leq 2$. Under this assumption, we obtain that

$$\left(\frac{V_1(Z)/V_1(\mathbf{B}^d)}{\sqrt{V_2(Z)/V_2(\mathbf{B}^d)}} \right)^{\frac{2}{i(i+1)}} = \left(1 + \frac{V_1(Z)/\sqrt{V_2(Z)}}{V_1(\mathbf{B}^d)/\sqrt{V_2(\mathbf{B}^d)}} - 1 \right)^{\frac{2}{i(i+1)}} \geq$$

$$\geq 1 + \frac{\sqrt{V_2(\mathbf{B}^d)}}{V_1(\mathbf{B}^d)} \cdot \frac{\frac{V_1(Z)}{\sqrt{V_2(Z)}} - \frac{V_1(\mathbf{B}^d)}{\sqrt{V_2(\mathbf{B}^d)}}}{2^{(i^2+i)/2} - 1} \geq 1 + \frac{\hat{c}}{n^{\frac{(d+2)(d+3)}{4d-4}}}$$

for some $\hat{c} > 0$ depending on d. From this a similar computation yields the lower bound in (8.20) for $k = i + 1$.

Now we prove the inequality in (8.21). Let Q be the convex polytope with the same outer unit normals as Z and circumscribed about $\mathbf{B}^d$. Then, by Lindelöf's theorem [94], $\frac{(V_{d-1}(Z))^{\frac{1}{d-1}}}{(V_d(Z))^{\frac{1}{d}}} \geq$

$\frac{(V_{d-1}(Q))^{\frac{1}{d-1}}}{(V_d(Q))^{\frac{1}{d}}}$. On the other hand, applying the volume formula of a cone for the convex hull of each facet of Q with $\{o\}$, we obtain that $\mathrm{surf}(Q) = dV_d(Q)$, implying $V_{d-1}(Q) = \frac{d}{2}V_d(Q)$. Thus, applying an asymptotic estimate on the minimal volume of a convex polytope with a given number of facets and circumscribed about $\mathbf{B}^d$ [94, 14], we obtain that

$$\frac{(V_{d-1}(Z))^{\frac{1}{d-1}}}{(V_d(Z))^{\frac{1}{d}}} \geq \left(\frac{d}{2}\right)^{\frac{1}{d-1}} \cdot (V_d(Q))^{\frac{1}{d(d-1)}} \geq \left(\frac{d}{2}\right)^{\frac{1}{d-1}} \kappa_d^{\frac{1}{d(d-1)}} \cdot \left(1 + \frac{\gamma}{\left(\binom{n}{d-1}\right)^{\frac{2}{d-1}}}\right)^{\frac{1}{d(d-1)}}. \tag{8.22}$$

for any zonotope $Z \in \mathcal{Z}_{d,n}$. Since $\left(\frac{d}{2}\right)^{\frac{1}{d-1}} \kappa_d^{\frac{1}{d(d-1)}} = \frac{\left(V_{d-1}(\mathbf{B}^d)\right)^{\frac{1}{d-1}}}{(V_d(\mathbf{B}^d))^{\frac{1}{d}}}$, and $\binom{n}{d-1}^{\frac{2}{d-1}} \leq \frac{n^2}{((d-1)!)^{\frac{2}{d-1}}}$, this implies the bound in (8.21).

Finally, to prove the upper bound in (8.20), first, observe that by (8.18), for any $1 \leq i \leq k \leq d$, and zonotope Z, we have

$$\frac{(V_i(Z))^{\frac{1}{i}} / \left((V_i(\mathbf{B}^d)\right)^{\frac{1}{i}}}{(V_k(Z))^{\frac{1}{k}} / ((V_k(\mathbf{B}^d))^{\frac{1}{k}}} \leq \frac{V_1(Z)/V_1(\mathbf{B}^d)}{(V_d(Z))^{\frac{1}{d}} / (V_d(\mathbf{B}^d))^{\frac{1}{d}}}. \tag{8.23}$$

We give an upper bound on $\min\left\{\frac{V_1(Z)}{(V_d(Z))^{\frac{1}{d}}} : Z \in \mathcal{Z}_{d,n}\right\}$. To do it we choose n unit vectors $p_i \in \mathbb{S}^{d-1}$ independently and using uniform distribution, and set $Z = \sum_{i=1}^n [o, p_i]$. Then, by the linearity of expectation and Lemma 8.7, we have that the expected value of $V_d(Z)$ is $\binom{n}{d} \cdot \frac{2\omega_{d+1}^{d-1}}{\omega_d^d}$. Since $V_1(Z) = n$ for any such zonotope and by Remark 8.6, there is some $Z \in \mathcal{Z}_{d,n}$ such that

$$\frac{V_1(Z)}{(V_d(Z))^{\frac{1}{d}}} \leq n \cdot \left(\frac{\omega_d^d}{2\binom{n}{d}\omega_{d+1}^{d-1}}\right)^{\frac{1}{d}} = \frac{n}{\sqrt[d]{(n(n-1)\dots(n-d+1))}} \frac{V_1(\mathbf{B}^d)}{(V_d(\mathbf{B}^d))^{\frac{1}{d}}} \leq$$

$$\leq \left(1 + \frac{d-1}{n-d+1}\right) \frac{V_1(\mathbf{B}^d)}{(V_d(\mathbf{B}^d))^{\frac{1}{d}}} \leq \left(1 + \frac{2d}{n}\right) \frac{V_1(\mathbf{B}^d)}{(V_d(\mathbf{B}^d))^{\frac{1}{d}}},$$

for any $n \geq 2d$. Combining it with (8.23), we obtain that

$$\frac{(V_i(Z))^{\frac{1}{i}}}{(V_k(Z))^{\frac{1}{k}}} \leq \frac{\left((V_i(\mathbf{B}^d)\right)^{\frac{1}{i}}}{((V_k(\mathbf{B}^d))^{\frac{1}{k}}} \cdot \left(1 + \frac{2d}{n}\right),$$

from which the upper bound in Theorem 8.12 readily follows. □

Remark 8.7 investigates the difference of somewhat rearranged form of the fractions from the previous theorem.

Remark 8.7. *We remark that if* $Z_0 \in \mathcal{Z}_{d,n}$ *satisfies the condition that* $\mathrm{ir}(Z_0) = 1$ *and*

$$V_i(Z_0) = \min\{V_i(Z) : Z \in \mathcal{Z}_{d,n}, \mathrm{ir}(Z) = 1\},$$

then by (8.18) and Theorem 8.10, for any $1 \leq i < k \leq d$ *we have*

$$0 \leq \left(\frac{V_i(Z_0)}{V_d(\mathbf{B}^d)}\right)^{\frac{1}{i}} - \left(\frac{V_k(Z_0)}{V_k(\mathbf{B}^d)}\right)^{\frac{1}{k}} \leq \left(\frac{V_i(Z_0)}{V_d(\mathbf{B}^d)}\right)^{\frac{1}{i}} - 1 \leq CU_d(n).$$

for some constant $C > 0$ *and sufficiently large value of* n.

8.4 Remarks and open problems

In Theorem 8.3, we have shown that a regular rhombic dodecahedron has minimal circumradius among the regular dodecahedra of the same volume.

Problem 8.1. *Is it true that the circumradius of a unit volume rhombic dodecahedron* Z *is minimal only if* Z *is regular?*

Problem 8.2. *Let* $1 \leq k < d$. *Find the elements* $Z \in \mathcal{Z}_{d,d+1}$ *with* $\mathrm{cr}(Z) = 1$ *and maximal* k*th intrinsic volume.*

We note that in Problem 8.2, by Theorem 8.3, Z is a regular rhombic dodecahedron if $k = d$, and by Theorem 8.5 Z is not regular if $k = 1$ and d is odd.

Problem 8.3. *Prove or disprove that for any* $1 < k < d$, *the elements* $Z \in \mathcal{Z}_{d,d+1}$ *with* $\mathrm{mwidth}(Z) = 1$ *and maximal value of* $V_k(Z)$ *are regular rhombic dodecahedra.*

Problem 8.4. *Find exact orders of magnitudes in the problems discussed in Section 8.3.*

Chapter 9

The volume of translation bodies in normed spaces

In this chapter we deal with the normed volume of the translation body of a convex body $K \in \mathcal{K}^d$. To do it, recall from Chapter 2 that for any $K \in \mathcal{K}^d$, the *translation body* of K is defined as the convex hull of K and a translate of K intersecting K. For such a body, Rogers and Shephard [165] defined the quantity

$$c_{tr}(K) = \frac{1}{\operatorname{vol}(K)} \max\{\operatorname{vol}(\operatorname{conv}(K \cup (x+K)) : (x+K) \cap K \neq \emptyset, x \in \mathbb{R}^d\}, \tag{9.1}$$

and determined its extremal values over $\mathcal{K}^d$.

Since, as we stated in the introduction, by Haar's result, any 'meaningful' definition of volume in a d-dimensional normed space is a scalar multiple of d-dimensional Lebesgue measure, the result of Rogers and Shephard immediately implies the following theorem.

Theorem 9.1. *Let $\mathcal{M}$ be a normed space with volume vol_M. Then, for any convex body $K \in \mathcal{K}^d$, we have*

$$1 + \frac{2\kappa_{d-1}}{\kappa_d} \leq \frac{\max\{\operatorname{vol}_M(\operatorname{conv}(K \cup (x+K))) : (x+K) \cap K \neq \emptyset, x \in \mathbb{R}^d\}}{\operatorname{vol}_M(K)} \leq 1 + d.$$

We observe that there is equality on the left if, and only if K is an ellipsoid (cf. [85]), and on the right if, and only if K is a *pseudo-double-pyramid* (cf. [165]).

In the remaining part we use a different approach. Recall from Chapter 2 that for any $K \in \mathcal{K}^d$, we say that the *relative norm* of K is the norm with the central symmetral $\frac{1}{2}(K - K)$ of K as its unit ball (cf. [135] or [130]). Observe that, up to multiplication by a scalar, the relative norm of K is the unique norm in which K is a body of constant width. We introduce the following quantities.

Definition 9.1. *Let $K \in \mathcal{K}^d$ and $\mathcal{M}$ be the space with its relative norm. For $\tau \in \{Bus, HT, m, m*\}$, let*

$$c_{tr}^{\tau}(K) = \max\{\operatorname{vol}_M^{\tau}(\operatorname{conv}(K \cup (x+K))) : (x+K) \cap K \neq \emptyset, x \in \mathbb{R}^d\}. \tag{9.2}$$

Note that the quantities in Definition 9.1 do not change under affine transformations. Our aim is to characterize the extremal values of these quantities in the planar case. To formulate our main result we need to define the following plane convex body.

Consider the square S_0 with vertices $\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$ in a Cartesian coordinate system. Replace the two horizontal edges of S_0 by the corresponding arcs of the ellipse with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where $a = 1.61803\ldots$, and $b = \frac{a}{\sqrt{2a^2-1}}$. Note that the vertices of S_0 are points of this ellipse. Replace the vertical edges of S_0 by rotated copies of these elliptic arcs by $\frac{\pi}{2}$. We denote the plane convex body, obtained in this way and bounded by four congruent elliptic arcs, by M_0. We remark that the value of a is obtained as a root of a transcendent equation, and has the property that the value of $\text{area}(M_0^\circ)\,(\text{area}(M_0)+4)$ is maximal for all possible values of $a > 1$.

Our main result is the following.

Theorem 9.2. *Let $K \in \mathcal{K}^2$. Then*

9.2.1. we have $2\pi \leq c_{tr}^{Bus}(K) \leq 3\pi$, with equality on the left if, and only if K is a triangle, and on the right if, and only if K is a parallelogram.

9.2.2. We have $\frac{18}{\pi} \leq c_{tr}^{HT}(K) \leq 7.81111\ldots$, with equality on the left if , and only if K is a triangle, and on the right if K is an affine image of M_0.

9.2.3. We have $6 \leq c_{tr}^{m}(K) \leq \pi+4$, with equality on the left if, and only if K is a (possibly degenerate) convex quadrilateral, and on the right if, and only if K is an ellipse.

9.2.4. We have $6 \leq c_{tr}^{m^*}(K) \leq 12$, with equality on the left if, and only if K is a triangle, and on the right if, and only if K is a parallelogram.

It is a natural question to ask for the extremal values of these four quantities over the family of centrally symmetric plane convex bodies. This question is answered in the next theorem.

Theorem 9.3. *Let $M \in \mathcal{K}_o^2$. Then*

9.2.1. we have $\pi + 4 \leq c_{tr}^{Bus}(M) \leq 3$, with equality on the left if, and only if M is an ellipse, and on the right if, and only if M is a parallelogram.

9.2.2. We have $\frac{21}{\pi} \leq c_{tr}^{HT}(M) \leq 7.81111\ldots$, with equality on the left if, and only if M is an affine-regular hexagon, and on the right if M is an affine image of M_0.

9.2.3. We have $6 \leq c_{tr}^{m}(M) \leq \pi+4$, with equality on the left if, and only if M is a parallelogram, and on the right if, and only if M is an ellipse.

9.2.4. We have $7 \leq c_{tr}^{m^*}(M) \leq 12$, with equality on the left if, and only if M is an affine-regular hexagon, and on the right if, and only if M is a parallelogram.

The proof of Theorem 9.3 is a straightforward modification of the proof of Theorem 9.2, and thus, we omit it.

In Section 9.1, we prove the left-hand side inequality about Holmes-Thompson area. In Section 9.2 we deal with the right-hand side inequality regarding it. In Section 9.3 we examine Busemann area, Gromov's mass and its dual. Finally, in Section 9.4, we collect our remarks, and propose some open questions.

9.1 The proof of the left-hand side inequality in 9.2.2

Let $K \in \mathcal{K}^2$ and $M = \frac{1}{2}(K - K)$. From (2.18) and (9.2), one can deduce that

$$c_{tr}^{HT}(K) = \frac{\text{area}(M^\circ)}{\pi} \left(\text{area}(K) + \max\{d_u(K)\,\text{width}_{u^\perp}(K) : u \in \mathbb{S}^1\}\right), \tag{9.3}$$

where $d_u(K)$ is the length of a longest chord of K in the direction of u, and $\text{width}_{u^\perp}(K)$ is the width of K in the direction perpendicular to u (cf. also the proof of Theorem 1 in [85]).

Observe that for any direction u, we have $d_u(K) = d_u(M)$ and $\text{width}_{u^\perp}(K) = \text{width}_{u^\perp}(M)$, which yields that minimizing $c_{tr}^{Bus}(K)$, over the class of convex disks with a given central symmetral, is equivalent to minimizing $\text{area}(K)$ within this class. For the special case that M is a Euclidean unit ball, this problem is solved by a theorem of Blaschke [30] and Lebesgue [136], which states that the smallest area convex disks of constant width two are the Reuleaux triangles of width two. This result was generalized by Chakerian [44] for normed planes in the following way.

Let $M \subset \mathcal{K}^2$ be an o-symmetric convex disk. Then, for every $x \in \text{bd}\, M$, there is an affine-regular hexagon, inscribed in M, with x as a vertex. Let y be a consecutive vertex of this hexagon. By joining the points o, x and y with the corresponding arcs in $\text{bd}\, M$ we obtain a 'triangle' T with three arcs from $\text{bd}\, M$ as its 'sides' (cf. Figure 9.1). These 'triangles', and their homothetic copies, are called the *Reuleaux triangles in the norm of* M. Chakerian proved that, given a normed plane $\mathcal{M}$, the area of any convex disk K of constant width two in the norm of $\mathcal{M}$ is minimal for some Reuleaux triangle in the norm. It is not too difficult to see, and was also proven by Chakerian, that the area of such a triangle is equal to $\text{area}(K) = 2\,\text{area}(M) - \frac{4}{3}\,\text{area}(H)$, where H is a largest area affine-regular hexagon inscribed in the unit disk M.

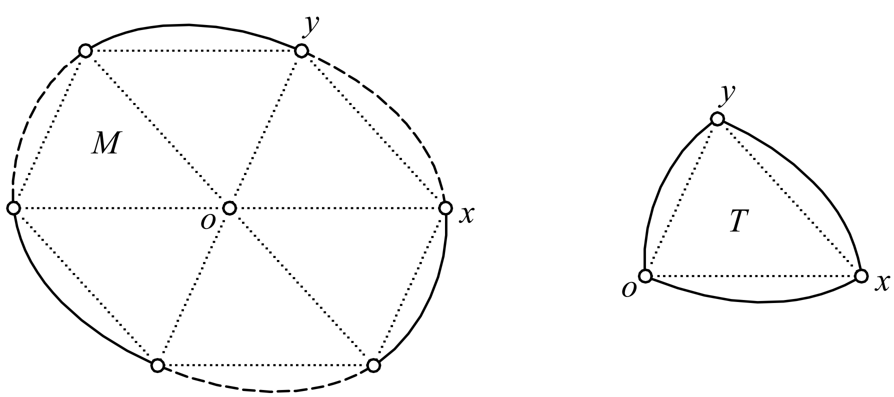

Figure 9.1: The construction of Reuleaux triangles in a normed plane

Now, assume that $K \in \mathcal{K}^2$ is a minimizer of $c_{tr}^{HT}(K)$ over $\mathcal{K}^2$; by compactness arguments, such a minimizer exists. Then, from Chakerian's result, we obtain that K is a Reuleaux triangle in its relative norm, and that its area is $\text{area}(K) = 2\,\text{area}(M) - \frac{4}{3}\,\text{area}(H)$, where H is a largest area affine-regular hexagon inscribed in M. Now let P be a largest area parallelogram inscribed in M. Then, by (9.3) and the equality

$$\max\{d_u(K)\,\text{width}_{u^\perp}(K) : u \in \mathbb{S}^1\} = 2\,\text{area}(P),$$

we have

$$c_{tr}^{HT}(K) = \frac{\text{area}(M^\circ)}{\pi}\left(2\,\text{area}(M) - \frac{4}{3}\,\text{area}(H) + 2\,\text{area}(P)\right). \tag{9.4}$$

It is easy to see that if K is a triangle, then M is an affine-regular hexagon, and vice versa, if M is an affine-regular hexagon, then the smallest area Reuleaux triangles in its norm are (Euclidean) triangles. Thus, we only need to show that the quantity in (9.4) is minimal if, and only if $M = H$. Observe that $\text{area}(H) \leq \text{area}(M)$, and hence, it suffices to prove that

$$f(M) = \frac{\text{area}(M^\circ)\left(\frac{2}{3}\,\text{area}(M) + 2\,\text{area}(P)\right)}{\pi} \tag{9.5}$$

is minimal if, and only if M is an affine-regular hexagon.

Now we show that if $f(M)$ is minimal for M, then its norm is a *Radon norm* (cf. [143] or [5]). Recall that a norm is Radon if, for some affine image C of its unit disk, the polar C° is a rotated copy of C by $\frac{\pi}{2}$; in this case the boundary of the unit disk is called a *Radon curve*.

Since $f(M)$ is an affine invariant quantity, we may assume that P is a square, with vertices $(\pm 1, 0)$ and $(0, \pm 1)$ in a Cartesian coordinate system. Note that as P is a largest area inscribed parallelogram, the lines $x = \pm 1$ and $y = \pm 1$ support M. Thus, the arc of $\operatorname{bd} M$ in the first quadrant determines the corresponding part of $\operatorname{bd} M^\circ$. On the other hand, the maximality of the area of P yields that for any point $p \in \operatorname{bd} M$, the two lines, parallel to the segment $[0,p]$ and at the distance $\frac{1}{||p||}$ from the origin, are either disjoint from M or support it. Thus, the rotated copy of M° by $\frac{\pi}{2}$ contains M, and the two bodies coincide if, and only if $\operatorname{bd} M$ is a Radon curve.

Let Q_1 and Q_2 denote the parts of M in the first and the second quadrant, respectively. We define Q_1° and Q_2° similarly for M°. Then $\text{area}(Q_2^\circ) = \text{area}(Q_1) + x_1$ and $\text{area}(Q_1^\circ) = \text{area}(Q_2) + x_2$ for some $0 \leq x_1, x_2 \leq \frac{1}{2}$. Using this notation, we have $f(M) = \frac{1}{\pi}\left(\text{area}(M) + 2x_1 + 2x_2\right)\left(\frac{2}{3}\,\text{area}(M) + 4\right)$. Let M_1 denote the convex disk obtained by replacing the part of $\operatorname{bd} M$ in the second and fourth quadrants by the rotated copy of the arc of $\operatorname{bd} M^\circ$ in the first quadrant (cf. Figure 9.2). Similarly, let M_2 be the disk obtained by replacing the part of $\operatorname{bd} M$ in the other two quadrants by the rotated copy of the arc of $\operatorname{bd} M^\circ$ in the second quadrant.

By our previous observations, we have that M_1 and M_2 are unit disks of Radon norms, and $M \subset M_1$ and $M \subset M_2$. On the other hand, the area of a largest area parallelogram inscribed in M_1 or M_2 is equal to $\text{area}(P) = 2$. Now an elementary computation shows that

$$f(M_i) = \frac{1}{\pi}\left(\text{area}(M) + 2x_{i+1}\right)\left(\frac{2}{3}(\text{area}(M) + 2x_{i+1}) + 4\right) \quad \text{for } i = 1, 2,$$

which, since $0 \leq x_1, x_2 \leq \frac{1}{2}$, yields that

$$2f(M) - f(M_1) - f(M_2) = \frac{1}{\pi}\left(8x_1 + 8x_2 - \frac{8}{3}x_1^2 - \frac{8}{3}x_2^2\right) \geq 0,$$

with equality if, and only if $x_1 = x_2 = 0$. From this, it follows that $f(M) \geq \min\{f(M_1), f(M_2)\}$, with equality if, and only if $x_1 = x_2 = 0$ and $M_1 = M_2 = M$. This readily implies that if $f(M)$ is minimal for M, then M is the unit disk of a Radon norm.

In the following, we assume that the norm of M is Radon. Observe that, under our assumption about P, we have $\text{area}(M) = \text{area}(M^\circ)$, since M° is a rotated copy of M. On the other hand,

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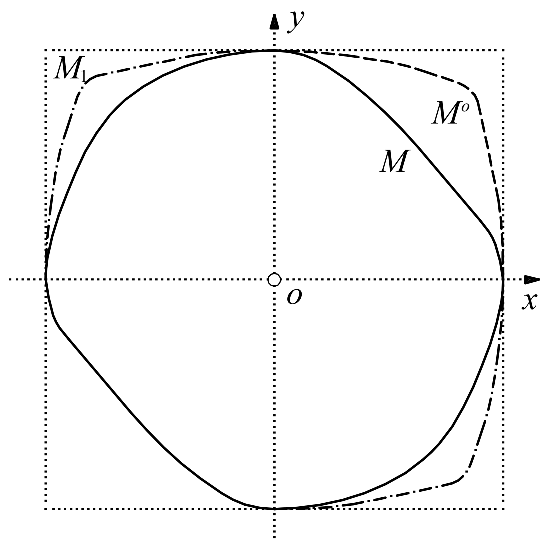

Figure 9.2: The extension of M to the unit disk of a Radon norm

since the *volume product* $\text{area}(M)\,\text{area}(M^\circ)$ of M (cf. e.g. [41]) does not change under affine transformations, the definition of Radon norm implies that, in general,

$$\text{area}(M^\circ) = \frac{4\,\text{area}(M)}{(\text{area}(P))^2}.$$

Since $\text{vol}^m_M(M) = \frac{2}{\text{area}(P)}\,\text{area}(M)$ (cf. the definition in Chapter 2, or [6]), this yields that

$$f(M) = \frac{4\,\text{area}(M)}{(\pi\,\text{area}(P))^2}\left(\frac{2}{3}\,\text{area}(M) + 2\,\text{area}(P)\right) = \frac{2}{3\pi}\,(\text{vol}^m_M(M))^2 + \frac{2}{\pi}\,\text{vol}^m_M(M).$$

Hence, we need to find the minimum of $\text{vol}^m_M(M)$ under the condition that M defines a Radon norm. This problem was examined in [5], where the authors proved that for any Radon norm with unit disk M, $\text{vol}^m_M(M)$ is at least 3, with equality if, and only if M is an affine-regular hexagon. Thus, the left-hand side of 9.2.2 immediately follows.

9.2 The proof of the right-hand side inequality in 9.2.2

Assume that $c^{HT}_{tr}(K)$ is maximal for some $K \in \mathcal{K}^2$ and let $M = \frac{1}{2}(K - K)$. Note that by the Brunn-Minkowski Inequality, we have $\text{area}(K) \leq \text{area}(M)$, with equality if, and only if K is centrally symmetric. Thus, (9.3) implies that K is centrally symmetric and, without loss of generality, we may assume that $K = M$.

Let P be a largest area parallelogram inscribed in M. Since $c^{HT}_{tr}(M)$ is affine invariant, we may assume that P is the square with vertices $(\pm 1, 0)$ and $(0, \pm 1)$ in a Cartesian coordinate system. Then the lines $x = \pm 1$ and $y = \pm 1$ support M. Let σ be a Steiner symmetrization with a symmetry axis of P as its axis, and let $M^* = \sigma(M)$. Then, clearly, $\text{area}(M^*) = \text{area}(M)$. Observe that P is inscribed in M^* as well, which yields that if P^* is a maximal area parallelogram inscribed in M^*, then $\operatorname{area}(P^*) \geq \operatorname{area}(P)$. For the Euclidean version of the problem, we have

$$c_{tr}(M) = 1 + \frac{2\operatorname{area}(P)}{\operatorname{area}(M)}. \tag{9.6}$$

Then, Theorem 1 of [165] yields that $c_{tr}(M)$ does not increase under Steiner symmetrization, which implies that $\operatorname{area}(P^*) \leq \operatorname{area}(P)$. Thus, we have $\operatorname{area}(P^*) = \operatorname{area}(P)$.

Now we apply a result of Meyer and Pajor [150] about the Blascke-Santaló Inequality, who proved that volume product does not decrease under Steiner symmetrizations, which yields that $\operatorname{area}((M^*)^\circ) \geq \operatorname{area}(M^\circ)$. Thus, since M maximizes $c_{tr}^{HT}(M)$, (9.3) implies that $\operatorname{area}((M^*)^\circ) = \operatorname{area}(M^\circ)$. Unfortunately, no geometric condition is known that characterizes the equality case for Steiner symmetrization. Nevertheless, we may apply another method, used by Saint-Raymond in [166], which he used to characterize the equality case of the Blaschke-Santaló Inequality. This method, described also in [192], is as follows.

Let C be an o-symmetric convex body in $\mathbb{R}^d$, and let H be the hyperplane with the equation $x_d = 0$. For any $t \in \mathbb{R}$, let C_t be the section of C with the hyperplane $\{x_d = t\}$. Define $\bar{C}$ as the union of the $(d-1)$-dimensional convex bodies $te_d + \frac{1}{2}(C_t - C_t)$, where e_d is the dth coordinate unit vector. Then we have the following (cf. Lemma 5.3.1 and the proof of Theorem 5.3.2 of [192]).

- $\bar{C}$ is an o-symmetric convex body.
- $\operatorname{vol}(\bar{C}) \geq \operatorname{vol}(C)$, with equality if, and only if every t-section C_t has a center of symmetry.
- $\operatorname{vol}(\bar{C}^\circ) \geq \operatorname{vol}(C^\circ)$.
- If $\operatorname{vol}(\bar{C}^\circ)\operatorname{vol}(\bar{C}) = \operatorname{vol}(C^\circ)\operatorname{vol}(C)$, then the centers of symmetry of the sets C_t lie on a straight line segment.

We note that this symmetrization procedure in the plane coincides with the Steiner symmetrization with respect to the second coordinate axis.

Let L be the axis of σ. Then, since in our case $\operatorname{area}(M^*) = \operatorname{area}(M)$ and $\operatorname{area}((M^*)^\circ) = \operatorname{area}(M^\circ)$, it follows from the theorem of Saint-Raymond that the midpoints of the chords of M, perpendicular to L, lie on a straight line segment. On the other hand, as $\sigma(P) = P$, we have that this segment is contained in L. Thus, M is symmetric to L. Since L was an arbitrary symmetry axis of P, we obtain that the symmetry group of M contains that of P, and, in particular, M has a 4-fold rotational symmetry.

Observe that in this case $M \subseteq \mathbf{B}^2$. Indeed, if for some $p \in M$ we have $||p|| > 1$, then, by the 4-fold rotational symmetry of M, it follows that M contains a square of area greater than $\operatorname{area}(P) = 2$, which contradicts our assumption that P is a largest area parallelogram inscribed in M. Since it is easy to check that $c_{tr}^{HT}(M)$ is not maximal if $M = \mathbf{B}^2$, this implies, in particular, that $\operatorname{area}(M) < \pi$. Note that in our case the area of the part of M in each quadrant is equal.

In the next step, we use the following Proposition from [36].

Proposition 9.1 (Böröczky Jr., Makai Jr., 2014)**.** *Let $Q = \operatorname{conv}\{o, a, c, b\}$ be a convex deltoid symmetric about the line containing the diagonal $[o, c]$. Assume that $a, b \in \mathbb{S}^1$ and that the lines containing $[a, c]$ and $[b, c]$ support $\mathbf{B}^2$. Let C be any o-symmetric convex disk such that $a, b \in \operatorname{bd} C$ and the lines containing $[a, c]$ and $[b, c]$ support C, and set $K = C \cap Q$ and $K^\circ = C^\circ \cap Q$. Let $\operatorname{area}(K) = \alpha \leq \operatorname{area}(Q \cap \mathbf{B}^2)$ be fixed. Then $\operatorname{area}(K^\circ)$ is maximal, e.g., if C is an o-symmetric ellipse E satisfying $\operatorname{area}(E \cap Q) = \alpha$.*

Applying this theorem for the part of M, say, in the first quadrant, we have that, under our assumption about P, M is a convex body bounded by four congruent elliptic arcs, having centers at o. Then it is a matter of computation to verify that $f(M)$ is maximal for a rotated copy of the body M_0 described in the introduction.

9.3 The proofs of 9.2.1, 9.2.3 and 9.2.4

First, we prove 9.2.1. Observe that for any $K \in \mathcal{K}^2$,

$$c_{tr}^{Bus}(K) = \frac{\pi}{\operatorname{area}(M)} \left(\operatorname{area}(K) + 2\operatorname{area}(P)\right),$$

where $M = \frac{1}{2}(K - K)$, and P is a largest area parallelogram inscribed in M. By the result of Chakerian [44] described in Section 9.1, we have that if K minimizes $c_{tr}^{Bus}(K)$ over $\mathcal{K}^2$, then K is a minimal area Reuleaux triangle in the norm of M, and its area is

$$\operatorname{area}(K) = 2\operatorname{area}(M) - \frac{4}{3}\operatorname{area}(H), \tag{9.7}$$

where H is a largest area affine-regular hexagon inscribed in M. Thus, we may assume, without loss of generality, that

$$c_{tr}^{Bus}(K) = \frac{\pi}{\operatorname{area}(M)} \left(2\operatorname{area}(M) + 2\operatorname{area}(P) - \frac{4}{3}\operatorname{area}(H)\right). \tag{9.8}$$

Note that in this case K is a (Euclidean) triangle if, and only if $M = H$.

From (9.8), it readily follows that

$$c_{tr}^{Bus}(K) = 2\pi + 2\pi\frac{3\operatorname{area}(P) - 2\operatorname{area}(H)}{3\operatorname{area}(M)}.$$

Observe that H contains a parallelogram of area $\operatorname{area}(\bar{P}) = \frac{2}{3}\operatorname{area}(H)$. Since $H \subseteq M$, this yields that $\operatorname{area}(P) \geq \frac{2}{3}\operatorname{area}(H)$, with equality if, and only if $M = H$. This means that $c_{tr}^{Bus}(K) \geq 2\pi$, with equality if, and only if $M = H$, which proves the left-hand side inequality about Busemann area.

Now we prove the right-hand side inequality. The formula in (9.7) and the Brunn-Minkowski Inequality shows, like in Section 9.2, that if $c_{tr}^{Bus}(K)$ is maximal over $\mathcal{K}^2$, then K is centrally symmetric. Thus we may apply Theorem 3 of [165] about the maximum of $c_{tr}(K)$, which yields the assertion.

Next, we prove 9.2.3. Let P be a largest area parallelogram inscribed in $M = \frac{1}{2}(K - K)$. Then we have

$$c_{tr}^{m}(K) = \frac{2\left(\operatorname{area}(K) + 2\operatorname{area}(P)\right)}{\operatorname{area}(P)} = 4 + \frac{2\operatorname{area}(K)}{\operatorname{area}(P)}. \tag{9.9}$$

Observe that for any $K \in \mathcal{K}^2$, we have

$$c_{tr}(K) = \frac{\operatorname{area}(K) + 2\operatorname{area}(P)}{\operatorname{area}(K)} = 1 + \frac{2\operatorname{area}(P)}{\operatorname{area}(K)}.$$

By Theorem 3 of [165], the latter expression is maximal if, and only if K is a convex quadrilateral, and by Theorem 1 of [85], it is minimal if, and only if, K is an ellipse. Thus the assertion readily follows.

Our next case is the left-hand side inequality of 9.2.4. Observe that

$$c_{tr}^{m^*}(K) = \frac{4\,(\operatorname{area}(K) + 2\operatorname{area}(P))}{\operatorname{area}(P')}, \tag{9.10}$$

where P is a largest area inscribed, and P' is a smallest area circumscribed parallelogram in $M = \frac{1}{2}(K - K)$.

As in the previous sections, if $c_{tr}^{m^*}(K)$ is minimal for some $K \in \mathcal{K}^2$, then, by [44], we may assume that K is a Reuleaux triangle in its relative norm, and its area is $\operatorname{area}(K) = 2\operatorname{area}(M) - \frac{4}{3}\operatorname{area}(H)$, where H is a largest area affine-regular hexagon inscribed in M. Thus, $\operatorname{area}(M) \geq \operatorname{area}(H)$ implies that

$$c_{tr}^{m^*}(K) \geq \frac{8\,(\operatorname{area}(M) + 3\operatorname{area}(P))}{3\operatorname{area}(P')}. \tag{9.11}$$

On the other hand, we clearly have $\operatorname{area}(P) \geq \frac{1}{2}\operatorname{area}(P')$, where we have equality, for example, if M is an affine-regular hexagon. Furthermore, Corollary 5.1 of [5] states that Gromov's mass* of any o-symmetric convex disk is at least three, with equality if, and only if M is an affine-regular hexagon. This implies that $\operatorname{area}(M) \geq \frac{3}{4}\operatorname{area}(P')$, and thus, we obtain $c_{tr}^{m^*}(K) \geq 6$. Here, we have equality if, and only if M is an affine-regular hexagon, which immediately implies that K is a triangle.

Finally, we prove the right-hand side of 9.2.4. Similarly like in the previous sections, we may assume that $K = M$. But then, clearly, $\operatorname{area}(M) \leq \operatorname{area}(P')$, $\operatorname{area}(P) \leq \operatorname{area}(P')$ and (9.10) yields that $c_{tr}^{m^*}(K) \leq 12$. Since in both inequalities equality is possible only if M is a parallelogram, the assertion follows.

9.4 Concluding remarks and open problems

Our first question is to find the plane convex bodies K for which the quantity $c_{tr}^{HT}(K)$ is maximal.

Problem 9.1. *Prove or disprove such that if $c_{tr}^{HT}(K)$ is maximal for some $K \in \mathcal{K}^2$, then K is an affine image of the body M_0 described in the beginning of this chapter.*

Remark 9.1. *For any $K \in \mathcal{K}^d$ and direction $u \in \mathbb{S}^{d-1}$, let $d_u(K)$ denote the length of a maximal chord of K in the direction u, and let $K|u^\perp$ be the orthogonal projection of K onto the hyperplane, through o, that is perpendicular to u. Then the maximal volume of the convex hull of two intersecting translates of K (that is, the numerator in the definition of $c_{tr}(K)$), is*

$$\operatorname{vol}_d(K) + \max\{d_u(K)\operatorname{vol}_{d-1}(K|u^\perp) : u \in \mathbb{S}^{d-1}\}. \tag{9.12}$$

This observation can also be found in the proof of Theorem 1 of [85]. Note that for any $u \in \mathbb{S}^{d-1}$, the central symmetral of $K|u^\perp$ is $\left(\frac{1}{2}(K-K)\right)|u^\perp$. Thus, by the Brunn-Minkowski Inequality, the expression in (9.12) does not decrease under central symmetrization, with equality if, and only if K is centrally symmetric. This yields that if $c_{tr}^{\tau}(K)$ is maximal for some $K \in \mathcal{K}^d$ for any $\tau \in \{Bus, HT, m, m*\}$, then K is centrally symmetric.

Remark 9.2. *By Remark 9.1, to find the maximal value of $c_{tr}^{Bus}(K)$, it suffices to find the maximum of $c_{tr}(K)$ over the family of d-dimensional centrally symmetric convex bodies. Thus, from Theorem 3 of [165] it follows that*

$$c_{tr}^{Bus}(K) \leq d+1,$$

with equality if, and only if K is a centrally symmetric pseudo-double-pyramid in the sense of [165]. Similarly, by [165] and [85], over the family of d-dimensional o-symmetric convex bodies, we have

$$c_{tr}^{Bus}(K) \geq 1 + \frac{2\kappa_{d-1}}{\kappa_d},$$

with equality if, and only if K is an ellipsoid.

Problem 9.2. *For $d \geq 3$ and $\tau \in \{HT, m, m*\}$, find the maximal values of $c_{tr}^{\tau}(K)$ over $\mathcal{K}^d$.*

Problem 9.3. *For $d \geq 3$ and $\tau \in \{Bus, HT, m, m*\}$, find the minimal values of $c_{tr}^{\tau}(K)$ over $\mathcal{K}^d$.*

When finding the minimal value of $c_{tr}^{Bus}(K)$ over $K \in \mathcal{K}^2$, we had to examine the smallest area convex disks of constant width two in a fixed normed plane. Nevertheless, in $\mathbb{R}^3$, even for the Euclidean norm, this question has been open for a long while (cf. [123]).

Other problems arise if, instead of two translates of a convex body, we consider other families related to the body. This was done also by Rogers and Shephard, who, among other objects, studied the extrema of the volumes of *differences bodies* or *reflection bodies*. We remark that a more general treatment of this type of questions can be found in [85] (see also [86]).

Our problem applied to the case of difference bodies has already appeared in the literature in a different setting. The Busemann volume of the difference body of K is 2^d for any $K \in \mathcal{K}^d$. For Holmes-Thompson volume, its value is a constant multiple of the volume product of the central symmetral of K, and thus, its maximum is attained for ellipsoids, and the problem of finding its minimum leads to the famous Mahler Conjecture. For Gromov's mass, we have

$$\frac{4^d}{d!} \leq \mathrm{vol}_M^m(K-K) \leq 2^d \kappa_d$$

for every $K \in \mathcal{K}^d$ (cf. [6]), and these inequalities are sharp. For Gromov's mass*, we have

$$\mathrm{vol}_M^{m*}(K-K) \leq 4^d$$

and finding its minimum is also connected to the Mahler Conjecture (cf. [6]).

Another possibility is to examine the reflection bodies of K, which are defined as the convex hull of K with one of its reflections about some point $x \in K$.

Definition 9.2. *Let $K \in \mathcal{K}^d$ and $M = \frac{1}{2}(K-K)$. For $\tau \in \{Bus, HT, m, m*\}$, set*

$$c_p^{\tau}(K) = \max\{\mathrm{vol}_M^{\tau}(\mathrm{conv}(K \cup 2x - K)) : x \in K\}. \tag{9.13}$$

Problem 9.4. *For $d \geq 2$ and $\tau \in \{Bus, HT, m, m*\}$, find the minimal and the maximal values of $c_p^{\tau}(K)$ over $\mathcal{K}^d$.*

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