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Combinatorics, geometry, and order

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Contents

Acknowledgments

First of all, I would like to thank those teachers who never spared their time or energy to explain the essence of mathematical thinking. Those who always encouraged me to think further, to discover the beauty and simplicity in what at first seemed unknown and incomprehensible. The list is not short, but unfortunately, it is not complete either.

First, I would like to thank my father, who, through his style and behavior, taught me to be humble, hardworking, and persistent.

During my university years, through László Lovász's lectures on combinatorics and geometry, as well as through personal conversations, I absorbed the idea that difficult mathematics can become beautiful and understandable—if viewed from the right angle. At his combinatorics seminar, it became natural to me that what I hear I must also truly understand, and the best way to do so is by asking good questions. Since I have my own seminar now, my goal has been to carry on this spirit.

It was Vilmos Totik who taught me that sets are not as simple as they might seem. Later, as my "boss," he showed me that doing science also requires good working conditions, something he always strove to ensure as head of our department—both professionally and personally.

Since my first year at the university of Szeged, György Turán has guided me toward the theoretical problems of algorithms and computers. Not only as a teacher but also as a research partner and friend, he has supported and shaped my work.

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After my time in Budapest, I went to the United States, thanks to László

Babai. Laci not only took me to a world-class research environment but also showed me how he managed to stay at the forefront of nearly everything. Unfortunately, I can only imitate his hard working and enthusiasm — I cannot match them. After the years in Chicago, I spent a year with Zoltán Füredi at M.I.T. During that very productive period after my PhD, I learned from Zoltán that every thought must be shared and explained to others, because in doing so, we ourselves understand it better. His explanations were always inspiring, leading to new questions, answers, and common results.

Back home in Szeged, I have tried to pass these lessons on to my students. They must be included among those I thank. Their receptiveness is evident from their successes. These successes have reassured me that I am on the right path — and, incidentally, I have learned a great deal from them as well.

Last but not least, I thank my family for their patience and understanding. Living with a mathematician is not easy. Their ability to sense when to leave me alone and when to encourage and support me was essential for this dissertation to come into being.

Thank you all.

Chapter 1

Introduction

Combinatorics is a young branch of mathematics with fast growing impact on classical fields and on its applications, especially on computer science.

The basic notions of combinatorics are simple, and powerful at the same time, for example hypergraph, i.e. certain subsets of a finite vertex set, is capable of grasping essential information on various structures. Extremal, Ramsey-type questions emerged in the first half of the 20th century. Since then, combinatorics became a deep field with many strong connections to algebra, geometry, analysis, and topology. These connections are two-sided. On one hand, the classical fields provide powerful methods, and on the other hand, they use combinatorial ideas and notions.

Our dissertation focuses on the interaction of various fields of mathematics and combinatorics. It concentrates especially on problems that have geometrical motivations and/or connections.

We are going to use the usual notations describing orders of magnitude. $g(n) = \mathcal{O}(f(n))$ means that f is non-negative, and there is a positive constant α such that $|g(n)| \le \alpha f(n)$ for all n. $f(n) = \Omega(g(n))$ means that f, g are non-negative, and there is a positive constant β such that $f(n) \ge \beta g(n)$ for all sufficiently large n. If $f = \mathcal{O}(g)$ and $f = \Omega(g)$, then we denote this by $f = \Theta(g)$. We also use $0 < F \ll G$, iff we have $F \le \alpha G$ for certain positive constant α, and we do not want to specify the constant.

1.1. Independent sets in geometrical hypergraphs

Often in a geometrical problem we translate the geometrical background to a combinatorial property, 'throw away' the geometry, and solve the remaining combinatorial problem. Very often, geometrical configuration determines a special hypergraph, and the question is equivalent to find a large independent set (a subset of vertices without any edge in it). Chapters 2, and 3 cover my results in that direction.

In these chapters we apply the so-called semi-random method (see Chapter 2) to improve previous results.

In Chapter 2, we consider a question asked by P. Erdős, and much later, independently by Gowers [65]. Given a planar point set $\mathcal{P}$, what is the minimal size of $\mathcal{P}$ that guarantees that one can find n points on a line or n independent points (no three on a line) in it? Gowers noted that the grid shows that $\frac{1}{4}n^2$ many points are necessary, and in the case of $2n^3$ many points without n points on a line a simple greedy algorithm finds n independent points. Payne and Wood [109] improved the upper bound to $\mathcal{O}(n^2 \log n)$. They also considered an arbitrary point set with many fewer points than n^3 and without n points on a line, but instead of the greedy algorithm they used Spencer's lemma, which is based on a simple probabilistic sparsification. They also used the Szemerédi–Trotter theorem to bound the number of edges of their hypergraph.

We improve the previous upper bound methods. First, we define the 3-uniform hypergraph of collinear triplets from the given point set. Next we apply a random sparsification. After some additional preparation (to get rid of 2-cycles) we can use the semi-random method (see [42]) to find a large independent set.

Theorem 1. *Let $\mathcal{P}$ be an arbitrary planar point set of size $\alpha\frac{n^2\log n}{\log\log n}$ for certain $\alpha > 0$ constant. Then there are n points in $\mathcal{P}$, that are incident to a line or independent.*

In Chapter 3 our application os the semi-random method is closely related to Heilbronn's triangle problem [115], [122], [116], [117], [118], [119], [90]. Take a "nice" unit area domain D (usually a square, disc, or a regular triangle). Place n points into D and find the smallest area among the triangles determined by the chosen points. Let $H_\triangle(n)$ denote the maximum of this area over all possible choices of n points.

Instead of triangles, we can take k-tuples of our point set and consider the area of the convex hull of the k chosen points. We denote the corresponding value by $H_k(n)$ (so $H_3(n) = H_\triangle(n)$). The best lower bound on $H_\triangle(n)$ [91], and some trivial observations are summarized in the next line:

$$\Omega\left(\frac{\sqrt{\log n}}{n^2}\right) = H_\triangle(n) \le H_4(n) \le H_5(n) \le \ldots = \mathcal{O}\left(\frac{1}{n}\right).$$

Our interest is in the lower bound on $H_4(n)$.

Theorem 2. *There exists a point set of size n in the unit square that does not contain four points with a convex hull of area smaller than $\beta n^{-3/2}(\log n)^{1/2}$, for certain β positive constant.*

This result improves Schmidt's theorem [122] by a logarithmic factor. Schmidt's original proof is based on a greedy method.

1.2. 0-1 matrices in combinatorics: Ramsey-type questions

When "distillating the combinatorial essence of a geometrical configuration" then very often we come up with a classical combinatorial object over an ordered vertex set.

The method is illustrated by a single example, which concerns a fundamental question posed by Erdős. The problem on the border of combinatorics and geometry. It is the following: How many unit distances can be determined by an n element point set on the plane? The special case when the points are in a convex position (they form a vertex set of a convex polygon) obtained special attention.

The best lower bound $2n-7$ (P. Hajnal and H. Edelsbrunner ([44], (1990)) was very far from the general upper bound: $\mathcal{O}(n^{4/3})$. Z. Füredi ([60], (1990)) improved this bound to $\mathcal{O}(n \log n)$ for convex point sets. He recognized the importance that in the convex case the points have a circular order, determined by convexity. The main lemma of his proof is an extremal theorem: If we have a 0-1 matrix of size $n \times n$, that does not contain four 1's forming the configuration $\begin{smallmatrix} 1 & & 1 \\ 1 & 1 & \end{smallmatrix}$, then the number of 1's in the matrix is $\mathcal{O}(n \log n)$.

The next two Chapters of the dissertation consider problems related to 0-1 matrices.

In Chapter 4 we consider a Ramsey-type problem of Gyárfás. The motivation is geometry and the essence is a problem on 0-1 matrices.

It is well-known (an early remark of Erdős and Rado, nowadays a standard introductory exercise in graph theory courses) that in any red/blue edge coloring of the complete graph a monochromatic spanning tree is guaranteed. It was in 1998 when Károlyi, Pach and Tóth [81] proved the geometric version of this fact: Take n independent points on the plane, connect all the pairs with a line segment, and color them arbitrarily with red and blue colors. Then a non-crossing spanning tree is guaranteed.

The bipartite case of the original graph theoretical question is easy: If we red/blue color the edges of $K_{n,n}$ (the balanced complete bipartite graph) then the largest monochromatic tree has at least n vertices if n is even and $n+1$ if n is odd. Furthermore, this bound is optimal, certain coloring does not contain a larger monochromatic tree.

The beautiful theorem from [81] led Gyárfás to consider the geometric problem when the underlying graph is a complete bipartite graph: Take any $2n$ points in convex position on the plane. Assume that a line separates them into two groups of n points. Connect all pairs, separated be the line and color them arbitrarily with red/blue colors. What is the size of the largest non-crossing monochromatic tree can be guaranteed?

The above/graph version of the basic question can be easily transformed to a matrix version as it was done in [29].

Let M be a 0-1 matrix. A 0-staircase is a sequence $\{s_i\}_{i=1}^{\ell}$ of zeroes in M such that s_{i+1} is either to the right of s_i in the same row, or it is below s_i in the same column ($i = 1, 2, \dots, \ell-1$). (We emphasize that s_i and s_{i+1} do not have to be neighboring elements in M.) A 1-staircase is defined similarly on ones of M. M is a (homogeneous) staircase in M iff it is either a 0- or a 1-staircase. The length/size of a staircase S is the number of elements in it, we denote it as $|S|$.

Conjecture (Gyárfás' Conjecture). *Any* 0-1 *matrix of size $n \times n$ contains a staircase of size $n-1$.*

Let $st(M)$ be the maximum among the lengths of the homogeneous staircases of M. Let $st_0(M)$ (resp. $st_1(M)$) be the maximum among the lengths of the homogeneous 0-staircases (resp. 1-staircases) of M. Hence $st(M) = \max\{st_0(M), st_1(M)\}$. The corresponding Ramsey-type thresholds:

$$st(n) = \min\{st(M) : M \in \{0,1\}^{n\times n}\}, st(n,N) = \min\{st(M) : M \in \{0,1\}^{n\times N}\}.$$

In the asymmetric case it is obvious that $st(n,N) = st(N,n)$. We always assume that $N \geq n$.

Instead of $st(M) = \max\{st_0(M), st_1(M)\}$ one can investigate

$$st_\Sigma(M) = st_0(M) + st_1(M).$$

It is natural to introduce

$$st_\Sigma(n) = \min\{st_\Sigma(M) : M \in \{0,1\}^{n\times n}\}, st_\Sigma(n,N) = \min\{st_\Sigma(M) : M \in \{0,1\}^{n\times N}\}.$$

Cai, Grindstaff, Gyárfás, and Shull stated a conjecture concerning the function $st_\Sigma(n)$ (see [28]), that turned out to be false (see the final remark in [29], the journal version of [28]). In section 2 we determine the exact value of $st_\Sigma(n, N)$.

Theorem 3. *For any* $n \leq N$,

$$st_\Sigma(n, N) = \left\lceil \frac{n}{2} \right\rceil + N - 1.$$

In section 3 we deal with $st(n, N)$. We present two constructions (hence we give upper bounds on $st(n, N)$).

Theorem 4. *For all* $N \geq \lfloor \frac{5}{2} n \rfloor - 1$,

$$st(n, N) = \left\lceil \frac{\lceil n/2 \rceil + N - 1}{2} \right\rceil.$$

Furthermore for all $n < N < \lfloor \frac{5}{2} n \rfloor - 1$,

$$st(n, N) \leq \left\lceil \frac{2n + N - 2}{3} \right\rceil.$$

We conjecture that our matrices give the right value of $st(n, N)$.

Conjecture 5. *For all* $n < N < \lfloor \frac{5}{2} n \rfloor - 1$,

$$st(n, N) = \left\lceil \frac{2n + N - 2}{3} \right\rceil.$$

Our conjecture is consistent with Gyárfás' conjecture. Unfortunately the case of square and "near-square" matrices are still unsolved. In the final section we give a significant improvement of the $\frac{4}{5} n$ bound in [66] (as opposed to the one given in [29]). We are going to prove the following theorem.

Theorem 6. *For any* $M \in \{0, 1\}^{n \times n}$

$$st(M) \geq \frac{5}{6} n - \frac{7}{12}.$$

1.3. 0-1 matrices in combinatorics: Enumerative questions

Chapter 5 investigates 0-1 matrices from a different point of view: enumeration.

One of the major open problems of this field was the so-called Stanley—Wilf conjecture: If there is any fixed forbidden non-trivial permutation pattern (τ) then the number of $\pi \in S_n$, not including τ is exponential in n (bounded from above by α^n for certain constant α). Note that $|S_n| = n!$, a super-exponential function. This problem is very central (see [21]), many special cases were treated, and this line of research led to powerful enumeration techniques, and resulted in several PhD theses.

Z. Füredi and P. Hajnal considered extremal problems for 0-1 matrices with forbidden configurations of 1's ([61], (1992)). Their contribution were for special forbidden matrices, and they also collected a long list of open questions that were decisive for further developments (see [84], [133], [110], [134]). One of those conjectures claimed that if the forbidden submatrix is a permutation matrix, then the extremal threshold for the maximal number of 1's for $n \times n$ matrices is linear in n.

M. Klazar ([88], (2000)) made an important observation: The extremal theory is closely related to enumeration of permutations with forbidden patterns: The Füredi—Hajnal conjecture implies the Stanley—Wilf conjecture. Extremal/Erdős-type combinatorics and enumerative combinatorics are two major branches of combinatorics with relatively few interactions. This result of Klazar was a breakthrough. A. Marcus and G. Tardos solved ([105], (2004)) the Füredi—Hajnal conjecture, so the Stanley—Wilf conjecture is settled affirmatively.

The Marcus—Tardos result gave more than Stanley and Wilf conjectured. They did not just give an upper bound on the number of permutations with a forbidden τ pattern, they gave an upper bound on the number of 0-1 matrices with a forbidden configuration of 1's. Counting 0-1 matrices with forbidden patterns became a new topic. Only sporadic results were know before. 0-1 matrices that are uniquely reconstructible from their row and column sum vectors are called *lonesum* matrices. We denote the set of lonesum matrices of size $n \times k$ as $\mathcal{L}_n^{(k)}$. Note that we allow $n = 0$ (and $k = 0$ too), in which case the empty matrix is counted as lonesum. These matrices can be characterized by two excluded submatrices by a classical result of D. Gale and H.J. Ryser ([63], [120], (1957)). Brewbaker determined the number of lonesum matrices of size $n \times k$.

Theorem (Brewbaker, [17], (2008))**.** *The number of* 01 *lonesum matrices of size $n \times k$ is given by a simple combinatorial formula:*

$$|\mathcal{L}_n^k| = \sum_{m=0}^{\min(n,k)} (m!)^2 \left\{ {n+1 \atop m+1} \right\} \left\{ {k+1 \atop m+1} \right\}.$$

$\left\{ {s \atop t} \right\}$ denotes the number of partitions of a set of size s into t classes, i.e. the Stirling number of the second kind. The right hand side of the equation is denoted by $B_n^{(-k)}$, an called as the poly-Bernoulli numbers. For their history see Chapter 5.

Table 1.1: The poly-Bernoulli numbers $B_n^{(-k)}$ for $n, k = 0, 1, 2, 3, 4, 5$

$n \backslash k$	0	1	2	3	4	5
0	1	1	1	1	1	1
1	1	2	4	8	16	32
2	1	4	14	46	146	454
3	1	8	46	230	1066	4718
4	1	16	146	1066	6906	41506
5	1	32	454	4718	41506	329462

The symmetry of the array in n and $-k$ is immediately conspicuous.

The original definition of poly-Bernoulli numbers was based on their double exponential generating function: $\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} B_n^{(-k)} \frac{x^n}{n!}\frac{y^k}{k!} = \frac{e^{x+y}}{e^x+e^y-e^{x+y}}$. From this, the formula $B_n^{(-k)} = \sum_{m=0}^{\min(n,k)} m!\left\{ {n+1 \atop m+1} \right\} m!\left\{ {k+1 \atop m+1} \right\}$ can be deduced, making obvious that the $B_n^{(-k)}$ numbers $(n, k \in \mathbb{N})$ are indeed natural numbers.

Brewbaker proved his theorem, above, and immediately realized that he can rephrase it as $|\mathcal{L}_n^k| = B_n^{(-k)}$. Alternatively (as we did above), one can define poly-Bernoulli numbers combinatorially: $B_n^{(-k)}$ is just the number of lonesum matrices of size $n \times k$.

From the combinatorial point of view we are interested only in the poly-Bernoulli numbers with negative k indices. From now on, we mean poly-Bernoulli numbers always with negative indices even if we do not emphasize it explicitly. For the sake of convenience, we denote in the rest of the dissertation $B_n^{(-k)}$ as $B_{n,k}$.

Definition 7. *The poly-Bernoulli numbers* $(B_{n,k})_{n,k=1}^{\infty}$ *are defined as*

$$B_{n,k} = |\mathcal{L}_{n,k}|,$$

the number of lonesum 0-1 *matrices of size* $n \times k$.

The two dimensional array can be found in OEIS [112] as A099594.

The realization of the fact that these numbers can be defined combinatorially made it possible to give combinatorial proofs of many known identities and recognize new properties. The fact that the poly-Bernoulli numbers are natural numbers with symmetry is obvious by this alternative definition. Also many alternative definitions can be given and their equivalence to the counting lonesum matrices can be proven bijectively. A summary of these results can be found in [11].

Now we consider lonesum matrices with further restrictions on the occurrence of all-0 columns resp. all-0 rows. More precisely let $\mathcal{L}_n^k(c|)$ denote the set of lonesum matrices with the property that each column contains at least one 1 entry and $\mathcal{L}_n^k(c|r|)$ the set of lonesum matrices with the property that each column and each row contains at least one 1.

Definition 8. $C_{n,k}$ *denotes* $|\mathcal{L}_n^k(c|)|$, *i.e. the number of lonesum matrices of size* $n \times k$ *without all-*0 *columns.*

$D_{n,k}$ *denotes* $|\mathcal{L}_n^k(c|r|)|$, *i.e. the number of lonesum matrices of size* $n \times k$ *without all-*0 *columns and rows.*

It turns out that $C_{n,k} = C_n^{(-k)}$. The connection to poly-Bernoulli numbers is obvious, and introducing $D_{n,k}$ is natural.

In Chapter 5 we give alternative combinatorial descriptions of these numbers and prove combinatorially old and new identities involving them. The combinatorial proofs are more natural than the analytical ones, they give more insights. We also mentioned that the definition and naming of poly-Bernoulli numbers are quite "young". But these numbers are already in L. Lovász' classical book Combinatorial Problems and Exercises, published in 1979 [102]. In chapter 4, exercise 36 implicitely deals with the $D_{n,k}$ numbers.

Theorem 9.

$$C_{n,k} = C_{k+1,n-1}.$$

Theorem 10.

$$B_{n,k} = C_{n,k} + C_{k,n} = C_{n,k} + C_{n+1,k-1}.$$

The proofs are combinatorial/bijective. They are simpler than the ones in the rest of the dissertation, but the topic exhibits the central role in enumerative combinatorics.

1.4. Saturated geometrical structures

In Chapter 6 and Chapter 7 we deal with graph drawings. First, we formalize the basic notions.

Let $G = (V, E)$ be a simple graph ($E \subset \binom{V}{2}$). Let $\mathcal{C}$ be a set of nice non-self-intersecting curves with two boundary points of the plane. For example we can consider open Jordan curves or simple polygonal curves. A δ drawing of G is a map defined on $V \cup E$ with the following property: $\delta|_V : V \to \mathbb{R}^2$ is a one-to-one map, i.e. δ assigns different points of the plane to the vertices of our graph. We refer to the elements of $\{\delta(v) : v \in V\}$ as vertex-points. $\delta|_E : E \to \mathcal{C}$, i.e. δ assigns nice curves to edges. We refer to the elements of $\{\delta(e) : e \in E\}$ as edge-curves. We assume that

(1) for any $e = xy \in E$ edge $\delta(e)$ is a curve connecting $\delta(x)$ and $\delta(y)$ and it does not go through any other vertex-point,

(2) any two different edge-curves meet at finitely many points and any meeting point — that is not a common endvertex — is a cross over for the two curves.

A drawing is nice if two different edge-curves meet only at a common endpoint. A drawing is simple iff two edge-curves have at most one common point. (Hence nice drawings are necessarily simple, but the reverse is not true.) A drawing is k-simple iff two edge-curves have at most k common points.

(G, δ), a graph with a drawing is called a topological graph. It is a simple topological graph iff the drawing is simple (our graphs are always simple). (G, δ) is a geometric graph iff all the edge-curves are segments. A geometric graph is a simple topological graph. Hence any graph G has a simple topological drawing, for example take vertex-points in general positions and consider the corresponding geometric drawing. (As opposed to nice drawing: The existence of nice drawing is a very restrictive assumption for a graph. It is the planarity property of graphs and for example implies sparsity.)

We are dealing with only topological graphs. Edges and edge-curves, vertices and vertex-points are usually not distinguished.

Saturated topological graphs/drawings are also very useful. Saturated nice drawings are triangulations. If we have n vertices (assume $n \geq 3$) the saturated nice drawings have the same, $3n - 6$ edges. [99] started to investigate saturated simple, and saturated k-simple graphs. A simple drawing of a complete graphs certainly saturated. One of the main results of [99] is that there are very sparse saturated simple, and k-simple topological graphs. Their upper bound on the fewest possible edges is a linear function of n/the number of vertices (17.5 in the simple, and $16n$ in the 2-simple case). We improve their bound for simple, and 2-simple topological graphs.

Theorem 11. *For any positive integer n there exists a saturated simple topological graph with at most $7n$ edges.*

[99] also gave a lower bound on the number of edges in a saturated simple topological graph based on the following observation: Let G be a simple topological graph. Let A be a vertex of degree of at most 2, and assume that A is not connected to all other vertices. Then G has a simple extension by an edge incident to A. I.e. in every saturated simple topological graph with at least four vertices, every vertex has degree at least 3.

We also considered this type of "local saturation" properties.

Theorem 12. *Let k be any positive integer. There is a saturated simple topological graph on $10k$ vertices that contains k vertices of degree 5.*

In Chapter 7 the basic construcion with few edges is extended to 2-simple drawings (Chapter 7).

Theorem 13. *For any positive integer n there exists a saturated 2-simple topological graph with at most $14.5n$ edges.*

1.5. Geometrical transversals

In this chapter we discuss a property of planar convex polygons, namely

Theorem 14. *If $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ are convex polygons in the plane with opposite orientation, then there exists a line that intersects each of the line segments $A_1B_1, \ldots, A_nB_n$.*

Thus, the claim is that the segments A_jB_j have a common transversal, see Figure 1.1 for illustration.

There is an alternative formulation of Theorem 14:

Theorem 15. *Let γ_1 and γ_2 be convex curves in the plane. Suppose that on each curve a car moves around, one of them in the clockwise, the other in the counterclockwise direction, returning to the starting point at the same time. Then there exists a line such that the two cars are always on opposite sides of that line.*

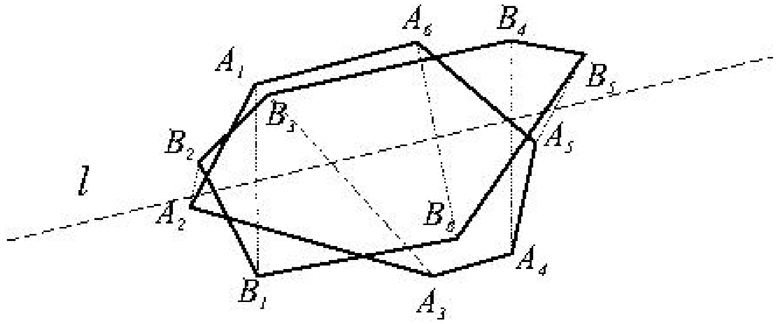

Figure 1.1: The convex polygons and a common transversal of A_iB_i

The line itself is considered to belong to both of its sides.

The cars may stop in their movement, but they cannot make retreats, see Figure 1.2.

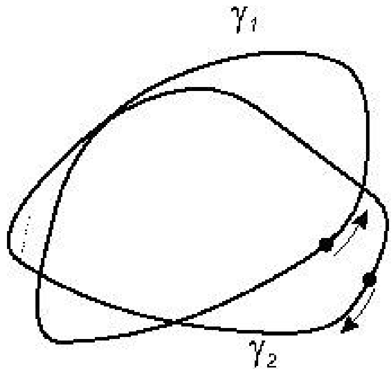

Figure 1.2: Two convex curves and one-one car on each

Clearly, Theorem 14 follows from Theorem 15. Indeed, given $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$, let us assume that the two cars start at A_1 and B_1, and each cover one side of the polygon (for example with uniform speed) in 1 minute. Then, after $i-1$ minutes, the first car is at A_i and the second car is at B_i ($i = 1, \ldots, n$). Let l be the line found in Theorem 2. Since, for each i, the points A_i and B_i are on the opposite sides of l, the line l intersects the segment A_iB_i.

A simple approximation argument (take as the vertices of the two polygons the positions of the two cars at times $t = (k/n)T$, $k = 1, \ldots, n$, where T is the time needed for the cars to make a full round) shows that conversely, Theorem 14 implies Theorem 15, so these two statements are equivalent.

We also consider higher dimensional analogues.

Proposition 16. *If $A_1 \cdots A_n$ and $B_1 \cdots B_n$ are oppositely oriented isometric polytopes, then there is a plane that intersects every segment A_iB_i, $i = 1, 2, \ldots, n$.*

Proposition 17. *Let $A_1 \cdots A_n$ be a polytope and $B_1 \cdots B_n$ an affine image of it. If these polytopes are oppositely oriented (in the sense that corresponding faces in them are oppositely oriented), then there is a plane that intersects every segment A_iB_i, $i = 1, 2, \ldots, m$.*

The original, planar result was extended to a different direction by A. F. Holmsen, J. Kincses and E. Roldán-Pensado in [74]. They showed that for any two convex curves C_1 and C_2 in $\mathbb{R}^d$ parametrized by $[0, 1]$ with opposite orientations, there exists a hyperplane H with the following property: For any $t \in [0, 1]$ the points $C_1(t)$ and $C_2(t)$ are never in the same open half space bounded by H.

Chapter 2

On a geometric Ramsey problem of Gowers

2.1. The semi-random method

The semi-random method was introduced for graphs in [4]. Later, it was extended to 3-uniform hypergraphs in [91]. The method was further extended in [5] and [42].

A hypergraph $\mathcal{H}$ on the vertex set V is a subset of $\mathcal{P}(V)$, the power set of V. I.e. $\mathcal{H}$ is a collection of certain subsets of V, called edges. If the edges have a common size, say k, then we say that $\mathcal{H}$ is k-uniform. In a hypergraph $\mathcal{H}$ a vertex set $I \subset V$ is called an independent set iff it does not contain any edge as a subset. The maximum size of the independent sets of $\mathcal{H}$ is denoted by $\alpha(\mathcal{H})$. There are several results concerning independent sets in 3-uniform uncrowded hypergraphs. From hypergraph theory, we recall that the degree of a vertex x ($deg(x)$) is the number of edges, containing x. Also a k-cycle ($k \geq 2$) in $\mathcal{H}$ is a sequence of k different vertices: $x_1, \ldots, x_{k-1}, x_k = x_0$ and a sequence of k different edges: $E_1, \ldots, E_k$ such that $x_{i-1}, x_i \in E_i$ for $i = 1, 2 \ldots, k$. The cycle, above is called a simple cycle iff $E_i \cap \left(\cup_{j:j\neq i} E_j\right) = \{x_{i-1}, x_i\}$ for $i = 1, 2 \ldots, k$.

We quote the earliest result on hypergraphs using the semi-random method.

Theorem ([91], Lemma 1)**.** *Let $\mathcal{H}$ be a 3-uniform hypergraph on v vertices. Let $\overline{d}$ denote the average degree of $\mathcal{H}$. Assume that $\overline{d} \leq t^2$ and $1 \ll t \ll v^{1/10}$.*

If $\mathcal{H}$ does not contain simple cycles of length at most 4, then

$$\alpha(\mathcal{H}) = \Omega\left(\frac{v}{t}\sqrt{\log t}\right).$$

We mention that $\alpha(\mathcal{H}) = \Omega(v/t)$ is an obvious application of the probabilistic method, observed by J. Spencer ([125], (1972)). This simplified bound can be considered as an extension of the classical graph theoretical Turán's theorem. In our applications, we might have many simple cycles of length 3 and 4. We need the following strengthening of the basic bound:

Theorem ([42], Theorem 2)**.** *Let $\mathcal{H}$ be a k-uniform hypergraph on v vertices. Let Δ be the maximum degree of $\mathcal{H}$. Assume that $\Delta \le t^{k-1}$ and $1 \ll t$. If $\mathcal{H}$ does not contain a 2-cycle (two edges with at least two common vertices), then*

$$\alpha(\mathcal{H}) = \Omega\left(\frac{v}{t}(\log t)^{\frac{1}{k-1}}\right).$$

Now we are ready to prove Theorem 1, stated in the Introduction.

2.2. The proof of Theorem 1

Let $\mathcal{P}$ be a planar point set of size $N = \frac{n^2 \log n}{\log\log n}$, not containing n points on a line. We must show that it contains a large independent set. The collinear triplets of $\mathcal{P}$ form a 3-uniform hypergraph $\mathcal{H}_3$. Independent subsets of the point set are the independent sets of the hypergraph. The collinear quadruples of $\mathcal{P}$ form a 4-uniform $\mathcal{H}_4$ hypergraph. Edges correspond to $K_4^{(3)}$'s (four vertices with all four triples as edges) in $\mathcal{H}_3$.

First we consider the size of $\mathcal{H}_3$, and $\mathcal{H}_4$. A line with i incident points determines $\binom{i}{3}$ many edges of $\mathcal{H}_3$ and $\binom{i}{4}$ many edges of $\mathcal{H}_4$. Let t_i denote the number of lines that contain exactly i points of $\mathcal{P}$. Our assumption gives that $0 = t_n = t_{n+1} = \ldots$. Similarly let $t_{\ge i}$ denote the number of lines that contain at least i points of $\mathcal{P}$ ($t_{\ge i} = t_i + t_{i+1} + \ldots + t_{n-1}$). Then

$$|\mathcal{H}_3| = \sum_{i=2}^{n-1}\binom{i}{3}t_i \le \sum_{i=2}^{n-1}\left(i^2\sum_{j=i}^{n-1}t_j\right) = \sum_{i=2}^{n-1} i^2 t_{\ge i}.$$

The Szemerédi–Trotter theorem [131] says that $t_{\ge i} = \Omega(|\mathcal{P}|^2/i^3 + |\mathcal{P}|/i) = \Omega(N^2/i^3)$ (in our case $N^2/i^3 \gg N/i$). For a suitable constant see [77] (Theorem 18.6 and Theorem 18.7): $t_{\ge i} \le 1000N^2/i^3$. Thus,

$$|\mathcal{H}_3| \le \sum_{i=2}^{n-1} i^2 t_{\ge i} \le \sum_{i=2}^{n-1} i^2 1000\frac{N^2}{i^3} = 1000N^2\sum_{i=2}^{n-1}\frac{1}{i} \le 2000N^2\log n = 2000\frac{n^4(\log n)^3}{(\log\log n)^2}.$$

Similarly

$$|\mathcal{H}_4| \leq \sum_{i=2}^{n-1} i^3 t_{\geq i} \leq \sum_{i=2}^{n-1} i^3 1000\frac{N^2}{i^3} = 1000N^2\sum_{i=2}^{n-1} 1 \leq 1000N^2 n = 1000\frac{n^5(\log n)^2}{(\log\log n)^2}.$$

Consider a random subset of $\mathcal{P}$, that we obtain by keeping each point with probability p, and throwing away with probability $1-p$ (and doing this independently).

Let

$$p = \frac{1}{100}\left(\frac{1}{n^{1/3}N^{1/3}}\right) = \frac{(\log\log n)^{1/3}}{100n(\log n)^{1/3}}.$$

After the random sparsification let $\boldsymbol{\mathcal{P}}$, $\boldsymbol{\mathcal{H}}_3$, and $\boldsymbol{\mathcal{H}}_4$ be the set of the surviving points, collinear triples, and collinear quadruples (these are random sets, throughout the paper we use bold face to denote random variables). It is obvious that

$$\mathbb{E}\left(|\boldsymbol{\mathcal{P}}|\right) = pN = \frac{n(\log n)^{2/3}}{100(\log\log n)^{2/3}},$$

$$\mathbb{E}\left(|\boldsymbol{\mathcal{H}}_3|\right) = p^3|\mathcal{H}_3| \leq \frac{2n(\log n)^2}{1000\log\log n},$$

$$\mathbb{E}\left(|\boldsymbol{\mathcal{H}}_4|\right) = p^4|\mathcal{H}_4| \leq \frac{n(\log n)^{2/3}}{100000(\log\log n)^{2/3}}.$$

We have chosen the probability so that the number of the surviving edges of $\mathcal{H}_4$ will be negligible compared to the number of surviving vertices.

Using elementary probability theory the following events will hold at the same time with high probability:

$$\frac{n(\log n)^{2/3}}{1000(\log\log n)^{2/3}} < |\boldsymbol{\mathcal{P}}| < \frac{n(\log n)^{2/3}}{10(\log\log n)^{2/3}},$$

$$|\boldsymbol{\mathcal{H}}_4| \leq \frac{n(\log n)^{2/3}}{10000(\log\log n)^{2/3}},$$

$$|\boldsymbol{\mathcal{H}}_3| < \frac{n(\log n)^2}{100\log\log n}.$$

Hence $\overline{d}(\boldsymbol{\mathcal{H}}_3) < \frac{30(\log n)^{4/3}}{(\log\log n)^{1/3}}$, where $\overline{d}$ denotes the average degree.

Now choose one of the good outcomes of the above probabilistic process such that all the above events hold. Let $\mathcal{H}_3^{(0)}$, and $\mathcal{H}_4^{(0)}$ be the corresponding 3- and 4-uniform hypergraphs.

Consider $\mathcal{H}_3^{(0)}$, and throw away all points of surviving quadruples of $\mathcal{H}_4^{(0)}$, and throw away each point that has degree higher than $\frac{100(\log n)^{4/3}}{(\log\log n)^{1/3}}$. Let $\mathcal{L}$ denote the "leftover" points with the "leftover" triples.

We are still left with at least one third of the points. Hence the leftover hypergraph has at least $\frac{n(\log n)^{2/3}}{3000(\log\log n)^{2/3}}$ many vertices. By throwing away the high degree vertices the maximal degree of $\mathcal{L}$ is at most $\frac{100(\log n)^{4/3}}{(\log\log n)^{1/3}}$. Furthermore $\mathcal{L}$ is very "uncrowded": we cannot have 2-cycles in $\mathcal{L}$. Indeed, two edges along the same pair of vertices would give a quadruple that is an edges in $\mathcal{H}_4$. Our process eliminated all of them, hence there are no 2-cycles in $\mathcal{L}$. The conditions in the quoted theorem about the semi-random method [[42], Theorem 2] are satisfied with $v = \Theta(\frac{n(\log n)^{2/3}}{(\log\log n)^{2/3}})$, $t = \frac{10(\log n)^{2/3}}{(\log\log n)^{1/6}}$, and $\log t = \Theta(\log\log n)$.

The rest of the proof is the application of semi-random method and simple arithmetic:

$$\alpha(\mathcal{H}_3) \geq \alpha(\mathcal{L}) \geq c\frac{v}{t}\sqrt{\log t} \geq c'n,$$

where c is the hidden constant in the quoted theorem [[42], Theorem 2] and c' can be determined from c based on our previuos calculation.

Theorem 1 can be easily deduced from this: Let $\nu = \min\{n, c'n\}$. Our point set contains $\Theta(\frac{\nu^2\log\nu}{\log\log\nu})$ points and our argument guarantees that it contains ν points on a line or ν independent points. The proof is complete.

Chapter 3

On Heilbronn quadrangle problem

Schmidt [122] proved that $H_4(n) = \Omega(n^{-3/2})$. The proof is a construction of a point set by a simple greedy algorithm. In [15] the authors provide a new proof, and extensions of this result. They also proposed an open question, which they have not yet been able to resolve: that is whether Schmidt's bound can be improved by a logarithmic factor. With the help of the semi-random method we are able to improve Schmidt's bound and settle the problem of [15].

3.1. The proof of Theorem 2

Let $S := \{(x,y) \in \mathbb{R}^2 : |x|, |y| \leq 1/2\}$ be a unit square on the plane. Choose N (a parameter that will be chosen later) random points (independently with uniform distribution) from $(1/2)S = \{(x/2, y/2); (x,y) \in S\}$. Let $\boldsymbol{\mathcal{P}}$ be the point set $\{P_1, P_2, \ldots, P_N\}$ we obtain this way. $\boldsymbol{\mathcal{P}}$ is a random point set. The reason we place our points into $(1/2)S$ is technical. This way we know that any connecting line of two points from $\mathcal{P}$ has an intersection with S of length $\Theta(1)$, furthermore any distance determined by points of $\boldsymbol{\mathcal{P}}$ is smaller than 0.9.

Consider the following 4-uniform hypergraph $\boldsymbol{\mathcal{Q}}$ on the vertex set $\boldsymbol{\mathcal{P}}$: A point set of size 4, $\{P, Q, R, S\}$ forms an edge iff $Area(PQRS) < \tau$, where $Area(PQRS)$ is the area of the convex hull of $\{P, Q, R, S\}$, and τ is a threshold, to be determined later. (Similarly $Area(PQR)$ is the area of the PQR

triangle.) $\mathcal{Q}$ is a random 4-uniform hypergraph.

The major part of the proof is bounding the expected values of the combinatorial parameters in $\mathcal{Q}$.

Let $A, B \in \mathcal{P}$ be two different points and

$$\mathbf{deg}(A,B) = |\{\{C,D\} : \{A,B,C,D\} \in \mathcal{Q}\}| \leq |\{(C,D) : \{A,B,C,D\} \in \mathbf{Q}\}|,$$

where the left hand side denotes the number of edges of $\mathcal{Q}$, that contains both A and B. The upper bound counts ordered pairs (hence it is an overcounting by a factor of 2). Our goal is to give an upper bound for this parameter. We will count how many ordered pairs of points C, D are considered when $deg(A,B)$ is given.

Let $strip(AB,w)$ denote the set of points from S, that are in the strip of width w with midline AB (see Figure 3.1). I.e. $strip(AB,w)$ contains those points of S that have distance at most $w/2$ from the line AB.

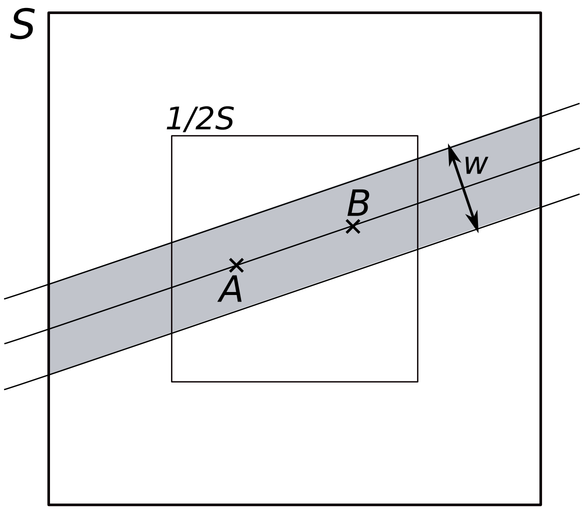

Figure 3.1: The shaded region is $strip(AB,w)$. Its area is $\Theta(w)$.

Fix A and B, and let $d = dist(A,B)(< 1)$. $deg(A,B)$ counts certain pairs of points, C, D, see (1). We distinguish cases according to the position of C, an arbitrary point from $\mathcal{P} - \{A,B\}$ and we bound the possible positions of the D's that contributes to $deg(A,B)$ with the current C.

Case 1: $C \notin strip(AB, 4\tau/d)$.

In this case the area of $ABC\triangle$ is at least τ, hence this C does not contributes to $deg(A,B)$.

Case 2: $C \in strip(AB, 4\tau/\sqrt{d})$. Note that $strip(AB, 4\tau/\sqrt{d})$ has area $\Theta(\tau/\sqrt{d})$. We distinguish two subcases:
Case 2a: $D \notin strip(AB, 4\tau/d)$. Similarly to Case 1 no D contributes to $deg(A, B)$.
Case 2b: $D \in strip(AB, 4\tau/d)$. Note that this strip has area $\Theta(\tau/d)$.
Case 3: $C \in strip(AB, 4\tau/d) \setminus strip(AB, 4\tau/\sqrt{d})$ (note that $d < \sqrt{d} < 1$). $strip(AB, 4\tau/d) \setminus strip(AB, 4\tau/\sqrt{d})$ has area $\Theta(\tau/d)$.

The contributing D's must come from $strip(AB, 4\tau/d) \cap strip(AC, 4\tau/dist(A, C))$.

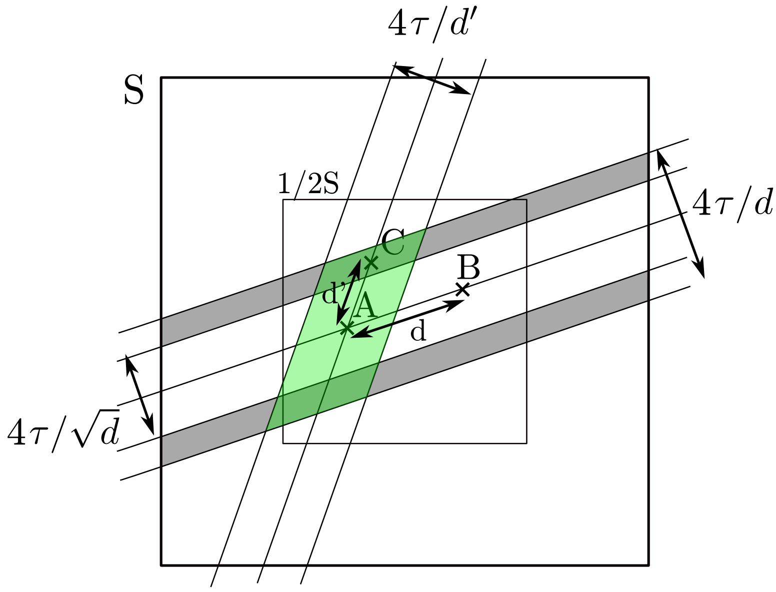

Figure 3.2: The shaded region is the space for those C's where Case 3 applies. The green region contains those D's, that can form an edge of $\boldsymbol{\mathcal{Q}}$ with A, B and C.

Elementary geometry gives that the above region has area $\Theta(\tau^2/Area(ABC\triangle))$ (see the green parallelogram on Figure 3.2), bounding the possible positions of contributing D's. Since we are in Case 3 we have $Area(ABC\triangle) = \Omega(d \cdot \tau/\sqrt{d}) = \Omega(\tau\sqrt{d})$, hence the parallelogram has area $\mathcal{O}(\tau/\sqrt{d})$.

The expected value of $\mathbf{deg}(A, B)$ can be bounded easily. The contributing C,D's are covered by Case 2b and Case 3. In both cases the contributing C's and D's are coming from a restricted domain with known area. Since the choice of C and D are independent the number of contributing (C, D)'s in expectation is a product of the two expectations, that we can bound. In each of the three cases this product is $\Omega(\tau^2 d^{-3/2} N^2)$. Hence

$$\mathbb{E}\ (\mathbf{deg}(A,B)) = \Omega(\tau^2 (dist(A,B))^{-\frac{3}{2}} N^2).$$

With a similar argument we obtain bound on the number of 2-cycles through $A, B \in \mathcal{P}$. In a 4-uniform hypergraph there are two types of 2-cycles: (I): $\{A,B,C,D\}$, $\{A,B,C',D'\}$ and (II): $\{A,B,C,D\}$, $\{A,B,C,D'\}$ (now different symbols denote different points). Let $\mathcal{C}_I(A,B)$, resp. $\mathcal{C}_{II}(A,B)$ denote the number of 2-cyles of type (I), resp. type (II) for given points, A and B. Bounding the expected value of $\mathcal{C}_I(A,B)$ is easy, based on the previous calculation

$$\mathbb{E}\ (\mathcal{C}_I(A,B)) = \Omega(\tau^4 (dist(A,B))^{-3} N^4).$$

Bounding the expected value of $\mathcal{C}_{II}(A,B)$ is a little bit more technical. We distinguish the contribution of C's that satisfy Case 2 and those that satisfy Case 3:

$$\mathbb{E}\ (\mathcal{C}_{II}(A,B)) = \Omega((N\tau/\sqrt{d})(N\tau/d)^2 + (N\tau/d)(N\tau/\sqrt{d})^2) = \Omega(\tau^3 N^3 d^{-2.5}).$$

Now we sparsify our point set a little bit in order to have a lower bound on the minimal distance determined by our points.

Let $\delta = \frac{1}{100} N^{-1/2}$. We count the number of pairs in $\mathcal{P}$ that are closer than δ. Let $C(\mathcal{P})$ be the set of these pairs (this is a random set). Let $C_A(\mathcal{P}) = \mathcal{P} \cap Disc(A;\delta)$, where $Disc(A;\delta)$ denote the disc of radius δ centered at A. It is clear that $|C(\mathcal{P})| = 1/2 \sum_{A \in \mathcal{P}} |C_A(\mathcal{P})|$ and $\mathbb{E}\ (|C_A(\mathcal{P})|) \le (N-1) Area(Disc(A;\delta)) = 1/2\pi \cdot \delta^2 N < 1/1000$. Hence

$$\mathbb{E}\ (|C(\mathcal{P})|) \le N/1000.$$

With high probability $|C(\mathcal{P})| \le N/4$. After deleting these pairs we obtain $\mathcal{P}_0$, our new point set. $\mathcal{P}_0$ has size at least $N/2$ with high probability, and the distance of any two points of it is at least δ.

Let $\mathcal{Q}_0$ be the restriction of $\mathcal{Q}$ to $\mathcal{P}_0$. From now on we will work with $\mathcal{Q}_0$.

Lemma 18. *Let $\mathcal{M}$ be a set of M points from S so that the minimal distance among them is at least δ. Let $P \in S$. Let $Ann_i(P,\delta)$ be the annulus*

$$Ann_i(P;\delta) = \{X \in \mathbb{R}^2 : (i-1)\delta < dist(P,X) \le i\delta\}.$$

$Ann_1(P;\delta)$, $Ann_2(P;\delta)$, ..., $Ann_{\mathcal{O}(\delta^{-1})}(P;\delta)$ are disjoint and cover S (hence they cover our point set). Furthermore at most $\mathcal{O}(i)$ of our M points can be covered by $Ann_i(P,\delta)$.

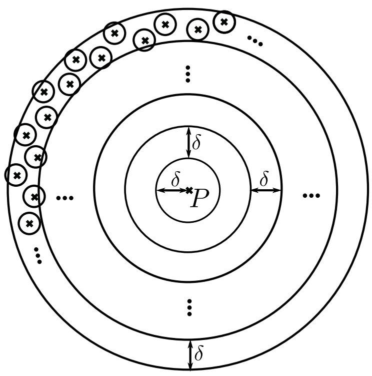

Figure 3.3: The annuli around P, and the elementary volume argument in the proof.

Proof. The covering property is obvious. The bound on the number of points in the annulus is a simple volume argument: Draw $Disc(A, \delta/3)$ for all points $A \in \mathcal{M} \cap Ann_i(P; \delta)$. These discs are disjoint subsets of $\{X \in \mathbb{R}^2 : (i-4/3)\delta < dist(P, X) \le (i+1/3)\delta\}$. The claim follows immediately. □

Using Lemma 5 and the earlier estimation for $deg(A, B)$'s it is easy to bound the expected number of edges, in $\boldsymbol{\mathcal{Q}}_0$:

$$\begin{aligned}
\mathbb{E}\ (deg_{\boldsymbol{\mathcal{Q}}_0}(A)) &\le \sum_{B \in \boldsymbol{\mathcal{P}}_0} \mathbb{E}\ (\mathbf{deg}(A,B)) = \sum_{i=1}^{\Omega(N^{1/2})} \sum_{B \in Ann_i(A,\delta) \cap \boldsymbol{\mathcal{P}}_0} \mathbb{E}\ (\mathbf{deg}(A,B)) \\
&\le \sum_{i=1}^{\Omega(N^{1/2})} \sum_{B \in Ann_i(A,\delta) \cap \boldsymbol{\mathcal{P}}_0} \Omega(\tau^2 N^2 (i/\sqrt{N})^{-3/2}) \\
&\le \sum_{i=1}^{\Omega(N^{1/2})} i \cdot \Omega(\tau^2 N^{2.75} i^{-3/2}) = \Omega(\tau^2 N^{2.75}) \sum_{i=1}^{\Omega(N^{1/2})} i \cdot i^{-3/2} \\
&= \Omega(\tau^2 N^{2.75}) \Omega(N^{0.25}) = \Omega(\tau^2 N^3).
\end{aligned}$$

Hence

$$\mathbb{E}\ (|\boldsymbol{\mathcal{Q}}_0|) = \Omega(\tau^2 N^4).$$

The bound of the expected number of 2-cycles ($\mathcal{C} = \mathcal{C}_I + \mathcal{C}_{II}$) is similar:

$$\begin{aligned}\mathbb{E}\,(\mathcal{C}_I) \le \sum_{A,B\in\mathcal{P}_0} \mathbb{E}\,(\mathcal{C}_I(A,B)) &= \sum_{A\in\mathcal{P}_0}\sum_{i=1}^{\Omega(N^{1/2})}\sum_{B\in Ann_i(A,\delta)\cap\mathcal{P}_0} \mathbb{E}\,(\mathcal{C}_I(A,B)) \\ &= \sum_{A\in\mathcal{P}_0}\sum_{i=1}^{\Omega(N^{1/2})}\sum_{B\in Ann_i(A,\delta)\cap\mathcal{P}_0} \Omega(\tau^4\cdot i^{-3}N^{1.5}\cdot N^4) \\ &= \sum_{A\in\mathcal{P}_0}\sum_{i=1}^{\Omega(N^{1/2})} \Omega(\tau^4\cdot i^{-2}\cdot N^{5.5}) = \sum_{A\in\mathcal{P}_0}\Omega(\tau^4 N^{5.5})\sum_{i=1}^{\Omega(N^{1/2})} i^{-2} = \\ &= \Omega(\tau^4 N^{6.5}).\end{aligned}$$

$$\begin{aligned}\mathbb{E}\,(\mathcal{C}_{II}) \le \sum_{A,B\in\mathcal{P}_0} \mathbb{E}\,(\mathcal{C}_{II}(A,B)) &= \sum_{A\in\mathcal{P}_0}\sum_{i=1}^{\Omega(N^{1/2})}\sum_{B\in Ann_i(A,\delta)\cap\mathcal{P}_0} \mathbb{E}\,(\mathcal{C}_{II}(A,B)) \\ &= \sum_{A\in\mathcal{P}_0}\sum_{i=1}^{\Omega(N^{1/2})}\sum_{B\in Ann_i(A,\delta)\cap\mathcal{P}_0} \Omega(\tau^3\cdot i^{-2.5}N^{1.25}\cdot N^3) \\ &= \sum_{A\in\mathcal{P}_0}\sum_{i=1}^{\Omega(N^{1/2})} \Omega(\tau^3\cdot i^{-1.5}\cdot N^{4.25}) = \sum_{A\in\mathcal{P}_0}\Omega(\tau^3 N^{4.25})\sum_{i=1}^{\Omega(N^{1/2})} i^{-1.5} = \\ &= \Omega(\tau^3 N^{5.25}).\end{aligned}$$

$$\mathbb{E}\,(\mathcal{C}) = \mathbb{E}\,(\mathcal{C}_I + \mathcal{C}_{II}) = \Omega(\tau^4 N^{6.5}) + \Omega(\tau^3 N^{5.25}).$$

Now choose one of the good outcomes of the above probabilistic process so that $\mathcal{P}_0$ and $\mathcal{Q}_0$ satisfy the following properties: the number of points is $N/2$, the number of quadruples is $\Omega(\tau^2 N^4)$, the number of 2-cycles is $\Omega(\tau^4 N^{6.5}) + \Omega(\tau^3 N^{5.25})$ (the two terms correspond to the two types of 2-cycles: the first term to cycles on six points, the second term to cycles on five points). Let $\mathcal{Q}_1$ be the 4-uniform hypergraph we obtained this way.

In order to get rid of the 2-cycles we need a random sparsification (as in the previous section): with probability p keep a point and with probability $1-p$ throw it away, and do this independently for all points. Let $\mathcal{Q}_1$ be the random 4-uniform hypergraph we obtain this way. Its parameters can easily be bounded:

$$\mathbb{E}\,(|V(\mathcal{Q}_1)|) = \Theta(pN),$$

$$\mathbb{E}\ (|\boldsymbol{\mathcal{Q}}_1|) = \Omega(p^4\tau^2N^4),$$
$$\mathbb{E}\ (\boldsymbol{\mathcal{C}}) = \Omega(p^6\tau^4N^{6.5}) + \Omega(p^5\tau^3N^{5.25}).$$

The end of the proof is straightforward: We choose p so that

$$\boldsymbol{\mathcal{C}} \ll |V(\boldsymbol{\mathcal{Q}}_1)|. \tag{3.1}$$

Choose one of the good outcomes of the above probabilistic process such that we obtain a 4-uniform hypergraph with the property that after deleting the points of the 2-cycles we obtain a leftover hypergraph with $\Theta(pN)$ points, and $\Omega(p^4\tau^2N^4)$ edges, and without 2-cycles. Let $\overline{d}$ denote the average degree. Throw away the points with degree at least $10\overline{d}$. The leftover hypergraph (without 2-cycles) is denoted by $\mathcal{L}$ and its parameters are:

$$|V(\mathcal{L})| = \Theta(pN),$$
$$|\mathcal{L}| = \Omega(p^4\tau^2N^4),$$
$$\Delta(\mathcal{L}) = \Omega(p^3\tau^2N^3).$$

Now we can apply Theorem 2.1. We choose N, τ such that $\alpha(\mathcal{L}) \geq n$ will hold. The n points forming an independent set will prove Theorem 2.

Set the parameters as follows:

$$p := n^{-0.001}, \quad N := n^{1.01}, \quad \tau := n^{-3/2}\sqrt{\log n}.$$

We check that with this choice of parameters (2) is satisfied:

$$\begin{aligned}\mathbb{E}\ (\boldsymbol{\mathcal{C}}) &= \Omega(p^6\tau^4N^{6.5}) + \Omega(p^5\tau^3N^{5.25})\\ &= \Omega(n^{0.006}(n^{-6}\log^2 n)\cdot n^{6.565}) + \Omega(n^{0.005}(n^{-4.5}\log n)n^{5.3025}) = o(n),\end{aligned}$$

and at the same time

$$\mathbb{E}\ (|V(\boldsymbol{\mathcal{Q}}_1)|) = \Theta(pN) = \Theta(n^{1.009}).$$

Hence getting rid of 2-cycles is easy.

In order to apply the semi-random method (see Chapter 1 [[42], Theorem 2]) we introduce a parameter t, such that $\Delta(\mathcal{L}) \leq t^3$. Based on our previous estimate $\Delta(\mathcal{L}) = \Omega(p^3\tau^2N^3)$, the right choice for t is

$$t = \Theta(p\tau^{2/3}N) = \Theta(n^{0.001}(n^{-1}\log^{1/3} n)n^{1.01}) = \Theta(n^{0.009}\log^{1/3} n).$$

Hence the semi-random method is applicable and it provides the following bound:

$$\alpha(\mathcal{L}) \geq \frac{\Omega(pN)}{t}\log^{1/3} t = \Omega(n).$$

Thus, $\alpha(\mathcal{L}) \geq cn$ for certain constant c, and the theorem is proven by the same scaling argument as Theorem 1.

Chapter 4

The staircases of Gyárfás

Let M be a 0-1 matrix. Recall, a 0-staircase is a sequence $\{s_i\}_{i=1}^{\ell}$ of zeroes in M such that s_{i+1} is either to the right of s_i in the same row, or it is below s_i in the same column ($i = 1, 2, \ldots, \ell - 1$). (We emphasize that s_i and s_{i+1} do not have to be neighboring elements in M.) A 1-staircase is defined similarly on ones of M. M is a homogeneous staircase in M iff it is either a 0- or a 1-staircase. The length of a homogeneous staircase S is the number of elements in it, we denote it as $|S|$. S can be viewed as $|S| - 1$ steps, where the steps can be right steps or down steps, formed by the consecutive elements of S. An element of S is a turning point if it is involved in two steps with different directions. $\mathrm{turn}(S)$ denotes the number of turns in S. For example $\mathrm{turn}(S) = 0$ iff S includes identical elements from a row, or from a column. $\mathrm{turn}(S) = 1$ iff the steps of S are some horizontal (staying in the same row) steps followed by some vertical steps or vice versa. In the first case, we say that S is a $\neg$-staircase. The first elements forming the horizontal steps will be referred to as the horizontal hand of the $\neg$-staircase. The last block of the elements that form the vertical steps will be referred to as the vertical hand of the $\neg$-staircase. In the second case, we say that S is a $\llcorner$-staircase. We use the notation of horizontal/vertical hands in this case, as well.

Let $st(M)$ be the maximum among the lengths of the homogeneous staircases of M. Let $st_0(M)$ (resp. $st_1(M)$) be the maximum among the lengths of the homogeneous 0-staircases (resp. 1-staircases) of M. Hence $st(M) = \max\{st_0(M), st_1(M)\}$.

We define a symmetric and an asymmetric version of the original Ramsey-type question.

$$st(n) = \min\{st(M) : M \in \{0,1\}^{n \times n}\},$$

$$st(n,N) = \min\{st(M) : M \in \{0,1\}^{n\times N}\}.$$

In the asymmetric case it is obvious that $st(n,N) = st(N,n)$. We always assume that $N \geq n$.

Gyárfás' conjecture (see the Introduction) states that $st(n) \geq n-1$. Since [29] contains an easy example that shows $st(n) \leq n-1$ (assuming $n > 1$) we can state Gyárfás' conjecture as $st(n) = n-1$ if $n > 1$.

Instead of $st(M) = \max\{st_0(M), st_1(M)\}$ one can investigate

$$st_\Sigma(M) = st_0(M) + st_1(M).$$

It is natural to introduce

$$st_\Sigma(n) = \min\{st_\Sigma(M) : M \in \{0,1\}^{n\times n}\},$$

$$st_\Sigma(n,N) = \min\{st_\Sigma(M) : M \in \{0,1\}^{n\times N}\}.$$

4.1. The exact value of $st_\Sigma(n,N)$

We start with an easy construction.

Construction 19. *Let $P^{(n,N)} \in \{0,1\}^{n\times N}$ be the matrix, where we have $P^{(n,N)}_{i,j} = 0$ iff $i+j \leq \lfloor \frac{n}{2} \rfloor + 1$ or $i+j \geq \lfloor \frac{n}{2} \rfloor + N + 1$.*

The following two examples help to understand the formalism:

$$P^{(6,8)} = \begin{pmatrix} 0&0&0&1&1&1&1&1\\ 0&0&1&1&1&1&1&1\\ 0&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&0\\ 1&1&1&1&1&1&0&0\\ 1&1&1&1&1&0&0&0 \end{pmatrix}, \qquad P^{(7,8)} = \begin{pmatrix} 0&0&0&1&1&1&1&1\\ 0&0&1&1&1&1&1&1\\ 0&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&0\\ 1&1&1&1&1&1&0&0\\ 1&1&1&1&1&0&0&0\\ 1&1&1&1&0&0&0&0 \end{pmatrix}$$

Note that the 0's form two triangular regions in the matrix (in the upper left and the lower right corners). Because of our convention ($n \leq N$) there is no 0-staircase that contains 0's from both triangular regions. Hence the following fact is easy:

Observation 20. *For any $n \le N$,*

$$st_0(P^{(n,N)}) = \left\lceil \frac{n}{2} \right\rceil, \qquad st_1(P^{(n,N)}) = N - 1,$$

and so

$$st_\Sigma(P^{(n,N)}) = \left\lceil \frac{n}{2} \right\rceil + N - 1.$$

We are going to prove a matching lower bound on $st_\Sigma(M)$ for any $M \in \{0,1\}^{n\times N}$ (see Theorem 3 in The Introduction).

Theorem 21. *For any $n \le N$,*

$$st_\Sigma(n, N) = \left\lceil \frac{n}{2} \right\rceil + N - 1.$$

Proof. Let $M \in \{0,1\}^{n\times N}$ arbitrary. Take the elements of the first column, that give the majority of this column. We can assume that they have the common value 1. Let a be the position of the lowest 1 in the first column. The number of 1's at and above a is at least $\lceil \frac{n}{2} \rceil$. Let S_1 the $\llcorner$-staircase centered at a. Let L_2 the staircase formed by the 0's in the row of a. It is easy to see that

$$|L_1| + |L_2| \ge \left\lceil \frac{n}{2} \right\rceil + N - 1.$$

This observation proves the theorem. □

4.2. An upper bound for asymmetric matrices

In this section we give an upper bound on $st(n, N)$. We will easily see that our bound is the truth for "wide" enough matrices. (The sharpness of the bound has been validated by computer for a few additional dimensions, too, but only for small n and N values.) We exhibit two constructions. They correspond to two different ranges of shapes.

Construction 22. *For $N \ge \lfloor \frac{5}{2} n \rfloor - 1$, we define the matrix $Q^{(n,N)}$ as follows: Let $Q^{(n,N)}_{i,j} = 1$ iff one of the following three possibilities holds:*

(1) $i \le \lceil \frac{n}{2} \rceil$ *and* $i + j \le \lceil \frac{n}{2} \rceil + 1$,

(2) $i \le \lceil \frac{n}{2} \rceil$ *and* $\lceil \frac{n}{2} \rceil + \left\lfloor \frac{\lceil n/2 \rceil + N - 1}{2} \right\rfloor + 1 < i + j$,

(3) $i > \lceil \frac{n}{2} \rceil$ *and* $\lceil \frac{n}{2} \rceil + N - \left\lfloor \frac{\lceil n/2 \rceil + N - 1}{2} \right\rfloor < i + j \le \lceil \frac{n}{2} \rceil + N$.

The reader is advised to check the example below to understand the structure of matrix $Q^{(n,N)}$: The first $\lceil n/2 \rceil$ rows of $Q^{(n,N)}$ forms an "isosceles right triangle" of 1's in the upper left corner (the length of the legs is $\lceil n/2 \rceil$), followed by a "parallelogram" of 0's with horizontal side of length $\left\lfloor \frac{\lceil n/2 \rceil + N - 1}{2} \right\rfloor$, and the rest of this upper region is filled with 1's. The last $\lfloor n/2 \rfloor$ rows are defined analogously: the triangle in the lower right corner has leg length $\lfloor n/2 \rfloor$, and the parallelogram of 1's has the same width as the parallelogram of 0's above.

$$Q^{(6,19)} = \begin{array}{c}
\overbrace{\hphantom{1\ 1\ 1}}^{\lceil \frac{n}{2} \rceil}\ \overbrace{\hphantom{0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0}}^{\left\lfloor \frac{\lceil n/2 \rceil + N - 1}{2} \right\rfloor}\hphantom{\ 1\ 1\ 1\ 1\ 1\ 1} \\
\begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0
\end{pmatrix} \\
\hphantom{1\ 1\ 1\ 0\ 0\ 0}\underbrace{\hphantom{1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1}}_{\left\lfloor \frac{\lceil n/2 \rceil + N - 1}{2} \right\rfloor}\ \underbrace{\hphantom{0\ 0\ 0}}_{\lfloor \frac{n}{2} \rfloor}
\end{array}
\begin{array}{l}
\left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} \lceil \frac{n}{2} \rceil \\
\left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} \lfloor \frac{n}{2} \rfloor
\end{array}$$

Claim 23. *For all* $N \ge \lfloor \frac{5}{2} n \rfloor - 1$,

$$st(Q^{(n,N)}) = \left\lceil \frac{\lceil n/2 \rceil + N - 1}{2} \right\rceil.$$

Proof. The 1's in the first row and last column form a 1-staircase of length $\left\lceil \frac{\lceil n/2 \rceil + N - 1}{2} \right\rceil$. We show that there are no longer homogeneous staircases in $Q^{(n,N)}$. We prove this for 1-staircases only, the case of 0-staircases is analogous.

The $\lceil n/2 \rceil$-th column, i.e. the column of the rightmost 1 in the upper left corner, precedes the column of the first 1 of the last row. (This can be seen by verifying that

$$\left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{\lceil n/2 \rceil + N - 1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor \le N$$

holds for $N \ge \lfloor \frac{5}{2} n \rfloor - 1$.) This means that every 1-staircase that starts from the upper left triangle, can only leave this triangle with a right step; which implies that if we translate this triangle of 1's to the right without overlapping any other 1's, the maximal length of 1-staircases does not decrease. We can translate this triangle to the right by $\left\lfloor \frac{\lceil n/2\rceil + N - 1}{2} \right\rfloor$ positions (the width of the upper parallelogram of 0's) without overlapping. It is easy to check that in the obtained matrix $\widetilde{Q}$, all the 1's lies in the region

$$\left\lceil \frac{n}{2} \right\rceil + N - \left\lfloor \frac{\lceil n/2\rceil + N - 1}{2} \right\rfloor + 1 \le i + j \le \left\lceil \frac{n}{2} \right\rceil + N.$$

Since in a staircase each step increases $i + j$, the sum of the "coordinates" of the actual position, it is obvious that there is no 1-staircase in $\widetilde{Q}$ with length greater than $\left\lceil \frac{\lceil n/2\rceil + N - 1}{2} \right\rceil$, as required. □

Theorem 21 says that $st_0(M) + st_1(M) \ge \lceil \frac{n}{2} \rceil + N - 1$, and so $st(M) \ge \left\lceil \frac{\lceil n/2\rceil + N - 1}{2} \right\rceil$ holds for every $M \in \{0,1\}^{n \times N}$. Hence we always have $st(n, N) \ge \left\lceil \frac{\lceil n/2\rceil + N - 1}{2} \right\rceil$. Claim 23 says that $st(n, N) \le \left\lceil \frac{\lceil n/2\rceil + N - 1}{2} \right\rceil$, for all $N \ge \lfloor \frac{5}{2} n \rfloor - 1$, so we obtained the following:

Theorem 24. *For all* $N \ge \lfloor \frac{5}{2} n \rfloor - 1$,

$$st(n, N) = \left\lceil \frac{\lceil n/2\rceil + N - 1}{2} \right\rceil.$$

Construction 25. *For* $n < N < \lfloor \frac{5}{2} n \rfloor - 1$, *we define the matrix* $R^{(n,N)}$ *as follows: Let* $R^{(n,N)}_{i,j} = 1$ *iff one of the following three possibilities holds:*

(1) $i \le \lceil \frac{n}{2} \rceil$ *and* $i + j \le \lfloor \frac{N-n+2}{3} \rfloor + 1$,

(2) $i \le \lceil \frac{n}{2} \rceil$ *and* $\lfloor \frac{N-n+2}{3} \rfloor + \lceil \frac{2n+N-2}{3} \rceil + 1 < i + j$,

(3) $i > \lceil \frac{n}{2} \rceil$ *and* $n + \lfloor \frac{N-n+2}{3} \rfloor < i + j \le n + N - \lceil \frac{N-n-1}{3} \rceil$.

$$
R^{(10,18)} = \begin{array}{c}
\overbrace{\hphantom{1\ 1\ 1}}^{\left\lfloor\frac{N-n+2}{3}\right\rfloor}\ \overbrace{\hphantom{0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0}}^{\left\lceil\frac{2n+N-2}{3}\right\rceil}\ \overbrace{\hphantom{1\ 1\ 1}}^{\left\lceil\frac{N-n-1}{3}\right\rceil} \\
\begin{pmatrix}
1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1\\
1&1&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1\\
1&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&0\\
0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&0&0\\
0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0
\end{pmatrix} \\
\underbrace{\hphantom{1\ 1\ 1}}_{\left\lfloor\frac{N-n+2}{3}\right\rfloor}\ \underbrace{\hphantom{0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0}}_{\left\lceil\frac{2n+N-2}{3}\right\rceil}\ \underbrace{\hphantom{1\ 1\ 1}}_{\left\lceil\frac{N-n-1}{3}\right\rceil}
\end{array}
\begin{array}{l}
\left.\vphantom{\begin{matrix}1\\1\\1\\1\\1\end{matrix}}\right\}\left\lceil\frac{n}{2}\right\rceil \\
\left.\vphantom{\begin{matrix}1\\1\\1\\1\\1\end{matrix}}\right\}\left\lfloor\frac{n}{2}\right\rfloor
\end{array}
$$

The reader should check that $\left\lfloor\frac{N-n+2}{3}\right\rfloor + \left\lceil\frac{2n+N-2}{3}\right\rceil + \left\lceil\frac{N-n-1}{3}\right\rceil$ is always equal to N, and the conditions imply that $\left\lfloor\frac{N-n+2}{3}\right\rfloor \le \left\lceil\frac{n}{2}\right\rceil$ and $\left\lceil\frac{N-n-1}{3}\right\rceil \le \left\lfloor\frac{n}{2}\right\rfloor$, as the above example suggests. After doing this and reading the proof of Claim 23, it should be straightforward to verify the following:

Claim 26. *For all $n < N < \lfloor\frac{5}{2}n\rfloor - 1$,*

$$st(R^{(n,N)}) = \left\lceil\frac{2n+N-2}{3}\right\rceil.$$

Corollary 27. *For all $n < N < \lfloor\frac{5}{2}n\rfloor - 1$,*

$$st(n,N) \le \left\lceil\frac{2n+N-2}{3}\right\rceil.$$

4.3. Improved lower bound

We prove Theorem 6, stated in the Introduction.

Let M be an arbitrary 0-1 matrix of size $n \times n$. Let a_1 be the last element of the first row. Without loss of generality we can assume that $a_1 = 1$. Let a_2 be the last 0 in the first row. Let a_3 be the top 0 (the 0 with the least first/row index) in the last column. Let a_4 denote the element/position in the intersection of the column of a_2 and the row of a_3.

It is possible that a_2 is not well defined (the first row does not contain any 0). Since in this case Gyárfás' conjecture is obviously true we assume that a_2 (and a_3 as well) is well defined.

$$\begin{pmatrix} & \overbrace{\cdots}^{x_1 \text{ many 1's}} & 0_{a_2} & 1 & \cdots & 1_{a_1} \\ & & & & & 1 \\ \vdots & & \vdots & & & \vdots \\ & & & & & 1 \\ & & ?_{a_4} & & & 0_{a_3} \\ \vdots & & \vdots & & & \vdots \end{pmatrix} \left.\right\} w_1 \text{ many 1's}$$

The summary of our notations, with some references to future parameters.

Observation 28. *If* $a_4 = 0$ *then Gyárfás' conjecture is true for our matrix.*

Proof. We consider two staircases: (1) The one that is formed by the 1's in the first row and the 1's in the last column. (2) The one that is formed by the 0's of the first row, a_4, the 0's above and on its right side of a_4, and the 0's of the last column. The staircase described in (1) is a 1-staircase, and (2) defines a 0-staircase.

$$\begin{pmatrix} & \cdots & 0 & 1 & \cdots & 1 \\ & & & & & 1 \\ \vdots & & \vdots & & & \vdots \\ & & & & & 1 \\ & & 0 & & & 0 \\ \vdots & & \vdots & & & \vdots \end{pmatrix} \qquad \begin{pmatrix} & \cdots & 0 & 1 & \cdots & 1 \\ & & & & & 1 \\ \vdots & & \vdots & & & \vdots \\ & & & & & 1 \\ & & 0 & & & 0 \\ \vdots & & \vdots & & & \vdots \end{pmatrix}$$

The highlighted elements just show the shape of the staircase. For the staircase we must consider only the elements with the same value as the turning points have.

It is easy to check that the sum of the length of the above two staircases is at least $2n$. Hence the longer one proves the conjecture. □

In the rest of the proof we assume that $a_4 = 1$. Let S_i denote the ¬-staircase centered at a_i ($i = 1, 2, 3, 4$). Let $s_i = |S_i|$.

$$S_1 = \begin{pmatrix} & \cdots & 0 & 1 & \cdots & 1 \\ & & & & & 1 \\ \vdots & & \vdots & & & \vdots \\ & & & & & 1 \\ & & 1 & & & 0 \\ \vdots & & \vdots & & & \vdots \end{pmatrix} \qquad S_2 = \begin{pmatrix} & \cdots & 0 & 1 & \cdots & 1 \\ & & & & & 1 \\ \vdots & & \vdots & & & \vdots \\ & & & & & 1 \\ & & 1 & & & 0 \\ \vdots & & \vdots & & & \vdots \end{pmatrix}$$

$$S_3 = \begin{pmatrix} & \cdots & 0 & 1 & \cdots & 1 \\ & & & & & 1 \\ \vdots & & \vdots & & & \vdots \\ & & & & & 1 \\ & & 1 & & & 0 \\ \vdots & & \vdots & & & \vdots \end{pmatrix} \qquad S_4 = \begin{pmatrix} & \cdots & 0 & 1 & \cdots & 1 \\ & & & & & 1 \\ \vdots & & \vdots & & & \vdots \\ & & & & & 1 \\ & & 1 & & & 0 \\ \vdots & & \vdots & & & \vdots \end{pmatrix}$$

Observation 29.

$$2s_1 + s_2 + s_3 + s_4 = 4n - 3 + x_1 + y_0 + z_0 + w_1.$$

where x_1 denotes the number of 1's in the first row before a_2, y_0 denotes the number of 0's in the column of a_2 between a_2 and a_4, z_0 denotes the number of 0's in the row of a_3 between a_4 and a_3, and w_1 denotes the number of 1's in the last column under a_3.

Proof. s_1, the number of 0's in the horizontal hands of S_2, and the number of 0's in the vertical hands of S_3 add up to $2n - 1$.

s_4, the number the number of 0's before a_4, and the number of 0's under a_4 (these 0's are counted in s_2 and s_3) add up to the number of the shaded positions in the figure of S_4. The second s_1 term counts the 1's in the last block of 1's of the first row and the 1's in the top block of 1's of the last column. The considered contribution of the sum of length give us $2n - 2$.

It is easy to see that the not counted contribution of the staircases gives us the last four terms on the right hand side. □

We can repeat the same argument with exchanging the role of rows and columns. Then the role of a_1 will be played by the last element of the first column (i.e. the first element of the last row). Let a be the value of this element. $\overline{a}$ denotes $1 - a$.

$$
w'_a \text{ many } a\text{'s } \left\{ \begin{pmatrix}
\vdots & & & \vdots & & \vdots \\
\overline{a}_{a_3{}'} & \cdots & & a_{a_4{}'} & \cdots & \\
a & & & & & \\
\vdots & & & \vdots & & \vdots \\
a & & & & & \\
a_{a_1{}'} & \cdots & a & \overline{a}_{a_2{}'} & \underbrace{\cdots \quad}_{x'_a \text{ many } a\text{'s}} &
\end{pmatrix} \right.
$$

All previous notations can be introduced in this setting (we use the same letters and a ' to distinguish from the originals). Our previous observation leads to

$$2{s_1}' + {s_2}' + {s_3}' + {s_4}' = 4n - 3 + {x_a}' + {y_{\overline{a}}}' + {z_{\overline{a}}}' + {w_a}'.$$

The order of magnitude of Gyárfás' original bound is already proven by our observations. We see some additional terms too, but the usage of them requires a case analysis.

Case 1: $s_1 + s'_a \geq 2n - 2$. In this case the longer of S_1 and ${S_1}'$ proves Gyárfás' conjecture and we are done.

Case 2: $s_1 + {s_a}' < 2n - 2$.

S_1 is a $\neg$-staircase, s_1 is its size. Let s_1^{-} denote the size of the horizontal hand of S_1, and let $s_1^{|}$ denote the size of its vertical hand. We introduce similar notation for its symmetric pair, the staircase, broken at the left bottom element of M (its size is denoted by ${s_a}'$).

Since $s_1 = s_1^{-} + s_1^{|} - 1$ and ${s_a}' = {s_a^{-}}' + {s_a^{|}}' - 1$, we have

$$2n - 2 > s_1 + {s_a}' = (s_1^{-} + s_1^{|} - 1) + ({s_a^{-}}' + {s_a^{|}}' - 1) = (s_1^{-} + {s_a^{-}}') + (s_1^{|} + {s_a^{|}}') - 2.$$

Without loss of generality we can assume, that $s_1^{-} + {s_a^{-}}' < n$. It is obvious that we are guaranteed to have a column in the matrix that has a 0 in the first/top position, a_5 and has an $\overline{a}$ at the last/bottom position, a_6. In the next picture we shaded this column and extended it to a staircase shape with further shaded positions (we are talking about staircase SHAPE, not about staircase!):

$$
\begin{pmatrix}
 & & & & \cdots & 0_{a_5} & \cdots & 0_{a_2} & 1 & \cdots & 1 \\
\vdots & & & & & \vdots & & & & & \vdots \\
a & \cdots & a & \overline{a}_{a_3{}'} & \cdots & \overline{a}_{a_6} & \cdots & & & &
\end{pmatrix}
$$

We define two staircases among the shaded elements. We distinguish two subcases:

Subcase 1: $0 = \overline{a}$. Let S_5 be the sequence of shaded 0's and let S_6 be the 1's of the shaded column.

Subcase 2: $0 \neq \overline{a}$. Let S_5 be a $\neg$-staircase centered at a_5, and Let S_6 be a $\llcorner$-staircase centered at a_6.

In both cases $|S_5|+|S_6|$ counts all shaded elements (that is $2n-1$ elements) except the shaded 1's in the first row and the shaded a's in the last row. We can upper bound their number with $x_1 + x_a{}'$. We obtained that

$$|S_5| + |S_6| \geq 2n - 1 - x_1 - x_a{}'.$$

Hence

$$2s_1+s_2+s_3+s_4+2s_1{}'+s_2{}'+s_3{}'+s_4{}'+|S_5|+|S_6| \geq 10n-7+y_0+z_0+w_1+y_{\overline{a}}{}'+z_{\overline{a}}{}'+w_a{}'.$$

We investigated 10 staircases. A weighted average of their length is at least $1/12(10n-7) = 5n/6 - 7/12$. So the longest staircase in our proof proves the theorem.

Remark 30. *At the beginning of the proof we excluded $a_4 = 0$ (and later we also excluded $a_4{}' = \overline{a}$). There, we used staircases S such that turn$(S) = 3$. All the other staircases in our proof have turning points at most 2. So the theorem can be stated in a stronger form: Any $M \in \{0,1\}^{n\times n}$ contains a staircase that has length at least $5n/6 - 7/12$, and it has at most 3 turning points.*

Gyárfás exhibited an example that shows that his $4n/5$ cannot be improved if we use only staircases with at most 1 turning point.

We do not know any bound for the case when we are allowed to use only staircases with at most 2 turning points.

Chapter 5

Poly-Bernoulli numbers, their relatives, and their combinatorics

5.1. From Bernoulli numbers to poly-Bernoulli numbers

In the Introduction we introduced poly-bernoulli numers combinatorially. The original definition of them came from a different direction.

The Dirichlet series $\zeta(k) = \sum_{m>0} \frac{1}{m^k}$ (convergent for $k > 1$), and its complex analytical continuation, the Riemann zeta function is at the center of mathematics. Its history started in the 17^{th} century when the so-called Basel problem was stated. It asked for evaluating $\zeta(2)$. The problem attracted a lot of attention and is strongly related to the Bernoulli numbers.

Definition. *The sequence $(B_n)_{n=0}^{\infty}$ of Bernoulli numbers defined by its exponential generating function*

$$\frac{xe^x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}.$$

The story of the Bernoulli numbers starts with investigating the sum of the m^{th} powers of the first n positive integers. The sums are polynomials in n. J. Bernoulli recognized the scheme in the coefficients of these polynomials. L. Euler made significant contributions, and named the sequence B_n after Bernoulli. Euler also solved the Basel problem. He proved that $\zeta(2) = \frac{\pi^2}{6}$. More generally, $\zeta(2k) = (-1)^{k-1}\frac{(2\pi)^{2k}}{2(2k)!}B_{2k}$, where $k = 1, 2, 3, \ldots$. For further results consult [8], [73], [121].

One extension of the ζ function is especially important for us. The multiple zeta value is a real number associated to each index set $(k_1, k_2, \ldots, k_n)$ of positive integers with $k_1 \geq 2$, defined by the convergent series

$$\zeta(k_1, k_2, \ldots, k_n) = \sum_{m_1 > m_2 > \ldots > m_n > 0} \frac{1}{m_1^{k_1} m_2^{k_2} \ldots m_n^{k_n}}.$$

This is a straightforward generalization of the Riemann zeta, whose study was initiated by Euler [51] in the case of $n = 2$. At the end of the 20$^{\text{th}}$ century connections to quantum field theory, knot theory, mixed Tate motive, or quantum groups were found ([25], [108], [64], [73]), the multiple zeta functions become a topic of intensive study.

Poly-Bernoulli numbers were introduced by M. Kaneko [78] in 1997 as a generalization of the classical Bernoulli numbers, during his investigations of multiple zeta values. As the values of the classical ζ function "contains" the Bernoulli numbers, we can "find" relevant numbers during investigating the values of multiple zeta functions. The original definition of poly-Bernoulli numbers is based on generating functions.

Definition (M. Kaneko, [79])**.** *The $B_n^{(k)}$, where n is a positive integer and k is an integer, poly–Bernoulli numbers are defined by the following exponential generating function*

$$\sum_{n=0}^{\infty} B_n^{(k)} \frac{x^n}{n!} = \frac{Li_k(1 - e^{-x})}{1 - e^{-x}}, \tag{5.1}$$

where

$$Li_k(z) = \sum_{i=1}^{\infty} \frac{z^i}{i^k},$$

i.e. $Li_k(z)$ is the kth poly-logarithm when $k > 0$ and a rational function when $k \leq 0$.

$B_n^{(1)}$ are just the ordinary Bernoulli numbers. After their introduction, poly-Bernoulli numbers were intensively studied. Several interesting properties were deduced using manipulating the generating functions. It was particularly interesting to notice that for negative indices poly-Bernoulli numbers were natural numbers. Brewbaker's theorem explained this phenomena and the combinatorial connection was "forgotten".

In [107] Y. Ohno considered the multiple "zeta star" functions

$$\zeta^*(k_1, \ldots, k_n) = \sum_{m_1 \geq m_2 \geq \ldots \geq m_n \geq 1} \frac{1}{m_1^{k_1} \ldots m_n^{k_n}}.$$

Kaneko continued the investigation [80], and introduced the numbers $C_n^{(k)}$ behind the ζ^* functions. He deduced simple recursions, formulas, and properties by formal methods. He also recognized the similarities and close connections to the poly-Bernoulli numbers.

5.2. Poly-Bernoulli numbers and their relatives

We consider lonesum matrices with further restrictions on the occurrence of all-0 columns resp. all-0 rows. More precisely let $\mathcal{L}_n^k(c|)$ denote the set of lonesum matrices with the property that each column contains at least one 1 entry and $\mathcal{L}_n^k(c|r|)$ the set of lonesum matrices with the property that each column and each row contains at least one 1.

Definition 31. *$C_{n,k}$ denotes $|\mathcal{L}_n^k(c|)|$, i.e. the number of lonesum matrices of $n \times k$ without all-0 columns.*

$D_{n,k}$ denotes $|\mathcal{L}_n^k(c|r|)|$, i.e. the number of lonesum matrices of $n \times k$ without all-0 columns and rows.

Let us see the first few values of these new numbers:

Table 5.1: PB-Relatives 1 $C_{n,k}$ and PB-Relatives 2 $D_{n,k}$

$n \backslash k$	0	1	2	3	4
1	1	1	1	1	1
2	1	3	7	15	31
3	1	7	31	115	391
4	1	15	115	675	3451
5	1	31	391	3451	25231

$n \backslash k$	1	2	3	4	5
1	1	1	1	1	1
2	1	5	13	29	61
3	1	13	73	301	1081
4	1	29	301	2069	11581
5	1	61	1081	11581	95401

First we give combiantorial formulas for the two new numbers (recall that $\left\{ {s \atop t} \right\}$ denote the number of partition of an s element set into t classes, the Stirling numbers of second kind):

Theorem 32. *(i)*

$$C_{n,k} = |\mathcal{L}_n^k(c|)| = \sum_{m=0}^{\min(n,k)} (m!)^2 \left\{ {n+1 \atop m+1} \right\} \left\{ {k \atop m} \right\} \quad n \geq 1 \text{ and } k \geq 0.$$

(ii)

$$D_{n,k} = |\mathcal{L}_n^k(c|r|)| = \sum_{m=0}^{\min(n,k)} (m!)^2 \left\{ {n \atop m} \right\} \left\{ {k \atop m} \right\} \quad n \geq 1 \text{ and } k \geq 1.$$

Proof. (Sketch) (i): We have the information that there is no all-0 column. Hence we do not need the extra column. The extra row ensures that the extended matrix has the class of all-0 rows. m denotes the number of classes of (k many non-0) columns. $m+1$ will be the number of classes of the $n+1$ rows. The rest is a straight forward repeat of the original argumnet.

(ii) is immediate by the same logic. □

From the combinatorial definition it is obvious that the series $B_{n,k}$ and $D_{n,k}$ are symmetric in n and k. The symmetry of the $C_{n,k}$ numbers ($C_{n,k} = C_{k+1,n-1}$) is also transparent from our table. But its proof is not straight forward. We will give it in a latter section.

The combinatorial definitions make it clear that the sequence $C_{n,k}$ is the binomial transform of $D_{n,k}$ and the sequence $B_{n,k}$ is the binomial transform of $C_{n,k}$. Precisely:

Observation 33. *(i)*

$$B_{n,k} = 1 + \sum_{i=1}^{k} \binom{k}{i} C_{n,i}, \qquad (k \geq 0, n \geq 0)$$

(ii)

$$C_{n,k} = \sum_{i=1}^{n} \binom{n}{i} D_{i,k}, \qquad (k \geq 0, n \geq 1).$$

(iii)

$$B_{n,k} = 1 + \sum_{i=1}^{n} \sum_{j=1}^{k} \binom{n}{i} \binom{k}{j} D_{i,j}, \qquad (k \geq 0, n \geq 0)$$

Proof. (i): To describe an arbitrary non-0 lonesum matrix we need to identify all its columns with at least one 1 (their number is denoted by $i(>0)$) and their entries in these i columns (that is describing a lonesum matrix of size $n \times i$ that contains at least one 1 in each column). This simple fact proves the formula of (i).

We obtain (ii) following the same argument on rows. (iii) summarizes (i) and (ii). □

There are other connections, recursions, combinatorial properties of the poly-Bernoulli numbers and its two relatives. They are not obvious, we discuss them when the appropiate combinatorial interpretations appear bellow.

These numbers seem to be just minor modifications of the original poly-Bernoulli numbers. In spite of the first impression, it turns out that these numbers appeared in a natural way in earlier papers. Now we summarize the analytical properties of these numbers (obtained by others). The rest of the paper is combinatorial.

5.3. Analytical results in the literature

Arakawa and Kaneko [9] introduced a function that are referred in the literature as the Arakawa-Kaneko function.

$$\xi_k(s) := \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} Li_k(1 - e^{-t}) dt.$$

The values of this function at non-positive integers are given by

$$\xi_k(-m) = (-1)^m C_m^{(k)},$$

where the generating function of the numbers $\{C_n^{(k)}\}$ (for arbitrary integers k) is given by

$$\sum_{n=0}^{\infty} C_n^{(k)} \frac{x^n}{n!} = \frac{Li_k(1 - e^{-x})}{e^x - 1}.$$

They computed the double exponential generating function of the $C_n^{(-k)}$ numbers. We know that the exponential functions of two number sequences differ only by an e^x (resp. e^y) factor when one sequence is the binomial transform of the other. From this observation we can conclude the binomial trasformation relation between poly-Bernoulli numbers and $\{C_n^{(-k)}\}$. It is immediate that $C_{n,k} = C_n^{(-k)}$. Furthermore we can obtain the generating function of $D_{n,k}$ numbers.

Theorem 34. *(i)*

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} C_{n,k}\frac{x^n}{n!}\frac{y^k}{k!} = \frac{e^x}{e^x+e^y-e^{x+y}},$$

(ii)

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} D_{n,k}\frac{x^n}{n!}\frac{y^k}{k!} = \frac{1}{e^x+e^y-e^{x+y}}.$$

Kaneko realized the importance of the C-relative and in a recent paper [80] summarized formulas and properties of the $B_n^{(k)}$ and $C_n^{(k)}$ parallel. Kaneko showed also a simple arithmetic connection between the two series:

$$B_{n,k} = C_{n,k} + C_{n+1,k-1}.$$

In our investigations we show this relation combinatorially using the variations of the so called Callan permutations. Moreover we prove a similar relation between the series $D_{n,k}$ and $C_{n,k}$.

5.4. 01 matrices with excluded submatrices

The study of matrices that are characterized by excluded submatrices is an active research area with many important results and applications. Given two matrices A and B we say that A avoids B whenever A does not contain B as a submatrix. (Given a matrix M a submatrix is a matrix that can be obtained from M by deletion of rows and columns.)

Generally we can set the following problem: Let $S = \{M_1,\ldots,M_r\}$ be a set of binary matrices. $\mathcal{M}_n^k(S)$ denote the $n\times k$ binary matrices that do not contain any matrix of the set S, $\mathcal{M}_n^k(S;c|)$ with the extra condition of containing in any column at least one 1, and $\mathcal{M}_n^k(S;r|c|)$ with the extra condition of containing in any column and any row at least one 1.

Lonesum matrices can be characterized also with the terminology of forbidden submatrices [120]. Lonesum matrices are matrices that avoid the following set of submatrices:

$$L = \left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right\}$$

I.e. $\mathcal{L}_n^k = \mathcal{M}_n^k(L)$

Interestingly beyond the set L there are sets that lead to the poly-Bernoulli numbers.

5.5. Γ-free matrices and recursions

In [11] the authors investigated the so called Γ-free matrices, matrices with the forbidden set:

$$\Gamma = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$$

and showed bijectively that the number of $n \times k$ Γ-free matrices (their set is denoted by $\mathcal{G}_n^k$) are the $B_{n,k}$ poly-Bernoulli numbers. Since the forbiddence of all 0 rows/resp. columns has the same effect in this case as in the case of the lonesum matrices we have:

Theorem 35. *(i)*

$$|\mathcal{G}_n^k(c|)| = C_{n,k},$$

(ii)

$$|\mathcal{G}_n^k(r|c|)| = D_{n,k}.$$

The structure of these matrices gives a transparent explanation of the recursive formula of poly-Bernoulli numbers that was first proven by Kaneko [9]. In the same spirit we can establish the recursions concerning the poly-Bernoulli relatives.

Theorem 36. *(i)*

$$C_{n,k+1} = \sum_{m=1}^{n} \binom{n}{m} C_{n-m+1,k},$$

(ii)

$$D_{n,k+1} = \sum_{m=1}^{n} \binom{n}{m} (D_{n-m,k} + D_{n-m+1,k}).$$

Proof. (i): $C_{n,k+1}$ counts the Γ-free matrices of size $n \times (k+1)$ without all-0 column. We partition these matrices according their rows Each row of a Γ-free matrix

A. starts with a 0 or

B. starts with a 1 followed only by 0s or

C. starts with a 1 and contains at least one more 1.

Let m denote the number of rows that starts with a 1. $m \geq 1$, since all columns contain at least one 1. We can choose these m rows $\binom{n}{m}$ ways. The first $m-1$ rows has to be of type B since a Γ would appear. The further $(n-m+1) \times k$ elements can be filled with an arbitrary Γ-free matrix that contain in any column at least one 1.

(ii): The proof above has to be slightly modified. Since now every row has to contain at least one 1 entry we have to handle separately two cases: the last row of the m rows that contain a 1 in the first column

a. contains at least one more 1 in another column. Then there are $(n-m+1) \times k$ elements to be filled arbitrary with a Γ-free matrix in $\mathcal{G}_n^k(r|c|)$.

b. has only 0-entries in the other columns. Then this row can be deleted and the remaining $(n-m) \times k$ matrix is an arbitrary Γ-free matrix in $\mathcal{G}_n^k(r|c|)$.

□

5.6. Permutation tableaux of size $n \times k$

Permutation tableaux were introduced by Postnikov [113] during his investigations of totally Grassmannian cells. The tableaux received this name because they are in bijection with permutations and are good tools for enumerating several statistics on permutations. Furthermore they are strongly connected with the PASEP model in statistical mechanics [36].

Permutation tableaux are usually defined as a 01 fillings of a Ferrers diagram with the next two conditions:

[(column)] each column contains at least one 1. [(1–hinge)] each cell with a 1 above in the same column and to its left in the same row must contain a 1.

In the special case when the Ferrers diagram is a $n \times k$ array the definition gives actually the set $\mathcal{M}_n^k(P; c|)$, where

$$P = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

Theorem 37. *(i)*

$$|\mathcal{M}_n^k(P)| = B_{n,k},$$

(ii)

$$|\mathcal{M}_n^k(P;c|)| = C_{n,k},$$

(iii)

$$|\mathcal{M}_n^k(P;r|c|)| = D_{n,k}.$$

Proof. We can make analogous considerations as were made for the Γ-free matrices in order to derive the recursion formula of the poly-Bernoulli numbers, resp. their relatives. The details are left to the reader. □

Remark 38. *Theorem 37 (ii) is contained in [87] without the recognition of the relation to the poly–Bernoulli numbers. In [138] in Lemma 4.3.5 the author proves that the number of $n \times k$ patterns of permutation diagrams is the poly-Bernoulli numbers $B_{n,k}$. For details see [138].*

5.7. A further excluded submatrix set

Computations of Brewbaker [18] suggested the following theorem that we prove by showing the recursion for the poly-Bernoulli numbers.

Theorem 39. *Let Q be the set*

$$Q = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\}.$$

Then we have

(i)

$$|\mathcal{M}_n^k(Q)| = B_{n,k},$$

(ii)

$$|\mathcal{M}_n^k(Q;c|)| = C_{n,k},$$

(iii)

$$|\mathcal{M}_n^k(Q;r|c|)| = D_{n,k}.$$

Proof. Let M be a matrix in $\mathcal{M}_n^k(Q)$ and $q(n,k) = |\mathcal{M}_n^k(Q)|$. Let $j_1 > j_2 > \cdots > j_m$ be the indices of the rows of the 1 entries in the first column ($m \geq 0$). When $m = 0$ or $m = 1$ the first row does not restrict the remainder $n \times (k-1)$ entries hence it can be filled with an arbitrary matrix in $\mathcal{M}_n^{k-1}(Q)$. When

$m \geq 2$ the rows $j_1, j_2, \ldots j_m$ coincide otherwise one of the submatrix in Q would appear. We have:

$$q(n,k) = (n+1)q(n,k-1) + \sum_{m=2}^{n} \binom{n}{m} q(n-m+1,k-1).$$

This recursion is equivalent to the recursion of the poly-Bernoulli numbers, hence the claim is proven. □

5.8. Permutations

In this section we consider classes of permutations that are enumerated by the poly-Bernoulli numbers resp. their relatives. As usual let $\{1,\ldots,n\} = [n]$ and S_{n+k} denote the set of permutations of $[n+k]$.

5.8.1. Vesztergombi permutations

The permutations we consider in this section are permutations that are restricted by constraints on the distance between their elements and their images. The enumeration of such permutation classes is a special case of a more general problem setting that were investigated by many authors. Given n subsets $A_1, A_2, \ldots A_n$ of $[n]$, determine the number of permutations π such that $\pi(i) \in A_i$ for all $i \in [n]$. The problem can be formulated as enumeration of the 1–factors of a bipartite graph or as the determination of the permanent of a 01 matrix or as the number of rook-placements on a given board. In general these formulations do not make the problem easier.

We want to use the results of Lovász and Vesztergombi ([136],[104], [102]) for derivation of the analogous formulas for $C_{n,k}$ and $D_{n,k}$ to the sieve formula for poly-Bernoulli numbers (see [17]). We recall some definitions and ideas for the sake of understanding. Detailed combinatorial proofs and analytical derivations can be found in the cited articles.

Let $f(r,n,k)$ denote the number of permutations $\pi \in S_{n+k}$ satisfying

$$-(k+r) < \pi(i) - i < n+r.$$

The main result is as follows:

Theorem 40. *[136]*

$$f(r,n,k) = \sum_{m=0}^{n} (-1)^{n+m} (m+r)!(m+r)^k \left\{ {n+1 \atop m+1} \right\}.$$

The original proof is analytic and depends on the solution of a certain differential equation for a generating function based on the $f(r,n,k)$ numbers. The differential equations capture the recursions that follow from the expanding rules of the corresponding permanent.

Launois [100] realized the connection of this formula to the poly-Bernoulli numbers, namely that $f(2,n,k) = B_{n,k}$.

Theorem 41. *[100] Let denote $\mathcal{V}_n^k$ the set of permutations π of $[n+k]$ such that*

$$-k \leq \pi(i) - i \leq n \quad \textit{for all} \quad i \in [n+k].$$

Then

$$|\mathcal{V}_n^k| = B_{n,k}$$

.

Beyond the analytical derivation of the formula there are combinatorial proofs of the theorem in the literature. In [86] the authors define an explicit bijection between Vesztergombi permutations and lonesum matrices. In [102] we find a combinatorial proof for a general case that includes the theorem. For the sake of completeness we present here the direct combinatorial proof from [14].

Proof. (Theorem 9.) $|\mathcal{V}_n^{(k)}|$ is the permanent of the $(n+k)\times(n+k)$ matrix $A = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1 & \text{if} \quad -k \leq i-j \leq n, \quad i = 1,\ldots,n+k \\ 0 & \text{otherwise.} \end{cases}$$

The permanent of A (denoted by per$A$) counts the number of expansion terms of the matrix A which do not contain a 0 term.

The matrix A is built up of 4 blocks:

$$A = \begin{bmatrix} J_{n,k} & B_n \\ B^k & J_{k,n} \end{bmatrix},$$

where $J_{n,k}(i,j) = 1$ for all $1 \leq i \leq n$ and $1 \leq j \leq k$, $J_{k,n}(i,j)$ analogously, $B_n(i,j) = 1$ iff $i \geq j$ and $B^k_{ij} = 1$ iff $i \leq j$.

For a term in the expansion of the permanent we have to select exactly one 1 from each row and each column. The number of ways of selecting 1s from the triangular matrices is given by the Stirling number of the second kind. (See proof for instance in [102].) So if a term contains m 1's from the upper left block $J_{k,n}$ ($m!$ ways), then it contains $n-m$ 1's from B_n ($\left\{ {n+1 \atop m+1} \right\}$ ways); m 1's from the lower right block ($m!$ ways) and finally $k-m$ 1's from B^k ($\left\{ {k+1 \atop m+1} \right\}$ ways). □

Suitable modifications of the definition of Vesztergombi permutations lead to the poly-Bernoulli relatives. Let $\mathcal{V}_n^{k*}$ the set of permutations π of $[n+k]$ such that

$$-k \leq \pi(i) - i < n \quad \text{for all} \quad i \in [n+k]$$

and $\mathcal{V}_n^{k**}$ the set of permutations π of $[n+k]$ such that

$$-k < \pi(i) - i < n \quad \text{for all} \quad i \in [n+k].$$

Theorem 42. *([136])*

(i)

$$|\mathcal{V}_n^{k*}| = C_{n,k},$$

(ii)

$$|\mathcal{V}_n^{k**}| = D_{n,k}.$$

Proof. In these cases the blocks B_n, B^k are slightly changed. The modifications are straightforward and hence the details are omitted. □

Corollary 43. *(i)*

$$C_{n,k} = \sum_{m=0}^{n} (-1)^{n+m} m!(m+1)^k \left\{ {n+1 \atop m+1} \right\},$$

(ii)

$$D_{n,k} = \sum_{m=0}^{n} (-1)^{n+m} m! m^k \left\{ {n+1 \atop m+1} \right\}.$$

Proof. Clearly $|\mathcal{V}_n^{k*}| = f(1,n,k-1)$ and $|\mathcal{V}_n^{k**}| = f(0,n,k)$. □

In [104] Theorem 1 describe the asymptotic behavior of $D_{n,n}$:.

Theorem 44. *[104]*

$$D_{n,n} \sim \sqrt{\frac{1}{2\pi(1-\ln 2)}} (n!)^2 \frac{1}{(\ln 2)^{2n}}.$$

5.8.2. Permutations with extremal excedance set

Permutations that have special restrictions on their excedance set are enumerated by the poly-Bernoulli numbers resp. their relatives. We note that the connection of this class of permutations to poly-Bernoulli numbers is not mentioned directly in the literature.

We call an index i an *excedance* (resp. *weak excedance*) of the permutation π when $\pi(i) > i$ (resp. $\pi(i) \geq i$). According that we define the set of excedances (resp. the set of weak excedances) of a permutation π as $E(\pi) := \{i|\pi(i) > i\}$ and $WE(\pi) := \{i|\pi(i) \geq i\}$. Further let $\mathcal{E}_n^k$ denote the set of permutations of $[n+k]$ with the condition $WE(\pi) = [k]$, $\mathcal{E}_n^{k*}$ the set of permutations with $E(\pi) = [k]$ and $\mathcal{E}_n^{k**}$ the set of permutations with the conditions (i.) $E(\pi) = [k]$ and (ii.) $\pi(i) \neq i$ for all $1 \leq i \leq n+k$. The main result in this line of research is summarized in the next theorem.

Theorem 45. *The following three statements hold:*

(i)
$$|\mathcal{E}_n^k| = B_{n,k},$$

(ii)
$$|\mathcal{E}_n^{k*}| = C_{n,k},$$

(iii)
$$|\mathcal{E}_n^{k**}| = D_{n,k}.$$

Proof. There are trivial bijections between these permutations and Vesztergombi permutations. The matrix A of that we need to count its permanent is essentially the same. Let consider this time the permutations of the set $\mathcal{E}_n^{k**}$ in details. The matrix is built up again of 4 blocks:

$$A = \begin{bmatrix} B_k & J_{k,n} \\ J_{n,k} & B^n \end{bmatrix},$$

where $J_{n,k}(i,j) = 1$ for all $1 \leq i \leq n$ and $1 \leq j \leq k$, $J_{k,n}$ analogously defined and $B_k(i,j) = 1$ iff $1 \leq j < i \leq k$, resp. $B_{ij}^n = 1$ iff $1 \leq i < j \leq j$.

□

We recall a simply bijection between permutation tableaux and permutations [26] that has the property that the excedances are given by the column labels and fixed points are given by empty rows. This bijection gives an another way how these permutation are connected to the poly-Bernoulli family.

Consider a $n \times k$ binary matrix that avoids the submatrices in the set P and contain in any column a 1. Label the rows from left to right by $[k]$, the columns from bottom to top by $[n]$. We define the zig-zag path by bouncing right or down every time we hit a 1. For i we find $\pi(i)$ by starting at the top of the column i (the left of the row i) following the zig-zag path until to boundary where we hit the row or column labeled by j. $\pi(i) = j$.

In [7] the authors determined the asymptotic of $C_{n,n}$ investigated as the number of the extremal excedance set statistic:

$$C_{n,n} \sim \left(\frac{1}{2\log 2\sqrt{(1-\log 2)}} + o(1) \right) \left(\frac{1}{2\log 2} \right)^{2n} (2n)!.$$

5.8.3. Callan permutations

Callan gave an alternative description of poly-Bernoulli numbers in a note in OEIS [30]. We repeat it, skecth the proof of his claim. We do this because Callan permutations play an important role in proving combinatorial properties of poly-Bernoulli relatives.

Definition 46. *Callan permutations are the permutations of* $[n+k]$ *in which each substring whose support belongs to* $N = \{1,2,\dots,n\}$ *or* $K = \{n+1, n+2, \dots, n+k\}$ *is increasing.*

We call the elements in N *left value elements* and that of K *right- value elements* and for the sake of convenience we rewrite $K \equiv \{\mathbf{1},\mathbf{2},\dots,\mathbf{k}\}$ ($N = \{1,2,\dots,n\}$). Actually we need just the distinction between the elements of the sets N and K and an order in N and K. So one can work with $N = \{0,2,3,\dots,n\}$ and talk about Callan permutations.

Let $\mathcal{C}_n^k$ denote the set of Callan permutations.

Theorem 47.

$$|\mathcal{C}_n^k| = \sum_{m=0}^{\min(n,k)} (m!)^2 \left\{ {n+1 \atop m+1} \right\} \left\{ {k+1 \atop m+1} \right\} = B_{n,k}.$$

Proof. (Sketch) Let $\pi \in \mathcal{C}_n^{(k)}$. Let $\widetilde{\pi} = 0\pi(\mathbf{k}+\mathbf{1})$, where 0 is a new left value and $\mathbf{k}+\mathbf{1}$ is a new right value. Divide $\widetilde{\pi}$ into maximal blocks of consequtive elements in such a way that each block is a subset of $\{0\} \cup N$ (left blocks) or a subset of $K \cup \{\mathbf{k}+\mathbf{1}\}$ (right blocks). The partition starts with a left block (the block of 0) and ends with a right block (the block of $\mathbf{k}+\mathbf{1}$). So the left and right blocks alternate, their number is the same, say $m+1$. To describe a Callan permutation is equivalent to specifying m, a partition $\Pi_{\widehat{N}}$ of $\widehat{N} = \{0\} \cup N$ into $m+1$ classes (one class is the class of 0, the other m ones are called ordinary classes), a partition $\Pi_{\widehat{K}}$ of $\widehat{K} = K \dot{\cup} \{\mathbf{k+1}\}$ into $m+1$ classes (m many of them not containing $\mathbf{k+1}$, the ordinary classes), and two orderings of the ordinary classes. This proves Callan's claim. □

The role of 0 and $\mathbf{k+1}$ were important. With the help of them we had the information how the left and right blocks follow each other.

Let $\mathcal{C}_n^k(*, l)$ be the set of Callan permutations of $N \dot{\cup} K$ that end with a left-value element (and hence with a left block). The star is to remind the reader that there is no assumption on the leading block of our permutation. Similarly let $\mathcal{C}_n^k(l, *)$ be the set of Callan permutations of $N \dot{\cup} K$ that start with a left-value element. Let $\mathcal{C}_n^k(l, r)$ be the set of Callan permutations of $N \dot{\cup} K$ that start with a left-value element and ends with a right element. The reader can easily define the sets $\mathcal{C}_n^k(r, *)$, $\mathcal{C}_n^k(*, r)$, $\mathcal{C}_n^k(r, l)$, $\mathcal{C}_n^k(l, l)$, and $\mathcal{C}_n^k(r, r)$.

If we take a Callan permutation, reverse the order of its blocks (leaving the order within each block) we obtain again a Callan permutation. This simple observation proves the following equalities:

$$|\mathcal{C}_n^k(*, l)| = |\mathcal{C}_n^k(l, *)|,$$

$$|\mathcal{C}_n^k(r, l)| = |\mathcal{C}_n^k(l, r)|.$$

Now we state a theorem that gives a new interpretation of poly-Bernoulli relatives with the help of Callan permutations.

Theorem 48. *(i)*

$$C_{n,k} = |\mathcal{C}_n^k(*, l)| = \sum_{m=0}^{\min(n,k)} (m!)^2 \left\{ {n+1 \atop m+1} \right\} \left\{ {k \atop m} \right\} \quad n \geq 1 \text{ and } k \geq 0.$$

(ii)

$$D_{n,k} = |\mathcal{C}_n^k(l, r)| = \sum_{m=0}^{\min(n,k)} (m!)^2 \left\{ {n \atop m} \right\} \left\{ {k \atop m} \right\} \quad n \geq 1 \text{ and } k \geq 1.$$

Proof. (i): Take a $\pi \in \mathcal{C}_n^k(*, l)$ and extend it with a starting 0 (an extra left value): $\widehat{\pi} = 0\pi$. One extra element is enough to control the structure of blocks: the block decomposition starts and ends with a left block. Let $m+1$ the number of left blocks, m is the number of right blocks (and the number of ordinary left blocks, i.e. blocks not containing 0). The rest of the proof is a straightforward modification of the previous one.

(ii): Without any extra element we control the starting and ending block. m denotes the common number of left and right blocks. The details are left to the reader. □

The next lemma is implicit in [11]. Since it is central for us, we present it here.

Lemma 49. *There is a bijection*

$$\varphi : \mathcal{C}_n^k(*, l) \to \mathcal{C}_{n-1}^{k+1}(*, r).$$

Proof. Take any $\pi \in \mathcal{C}_n^k(*, l)$. Find n (the largest left value) in it. It is the last element of one of the left blocks (possibly the very last element of π).

Assume that n is not the last element of π. Then it is followed by a right block R and by at least one left block. Exchange n to $\mathbf{k+1}$ and move R to the end of π. The permutation that we obtain this way will be $\varphi(\pi)$.

If n is the last element of π, then exchange it to $\mathbf{k+1}$.

In both case the described image is obviously in $C_{n-1}^{k+1}(*, r)$. In order to see that φ is a bijection we need to construct its inverse. This can be done easily based on $\mathbf{k+1}$. The details are left ot the reader. □

The lemma is obviously gives us a $\psi : \mathcal{C}_n^k(*, r) \to \mathcal{C}_{n+1}^{k-1}(*, l)$ bijection as well.

In [11] this lemma was used to prove that $\sum_{k,\ell:k+\ell=n}(-1)^k B_{k,\ell} = 0$. We use the lemma for a different purpose. First we combinatorially prove the symmetricity of the $C_{n,k}$ numbers.

Corollary 50.

$$C_{n,k} = C_{k+1,n-1}.$$

Proof. Change the role of left and right values. The two orderings remain, hence we obtain a Callan permutation (the blocks remain the same). This leads to a bijection between $\mathcal{C}_n^k(*, l)$ and $\mathcal{C}_k^n(*, r)$. Using the previuos lemma we obtain that

$$C_{n,k} = |\mathcal{C}_n^k(*, l)| = |\mathcal{C}_k^n(*, r)| = |\mathcal{C}_{k+1}^{n-1}(*, l)| = C_{k+1,n-1}.$$

□

The next application of the lemma will be a simple connection between poly-Bernoulli numbers and their C-relative. It was proved in [80] with analytical methods. Here we present a combinatorial method.

Theorem 51. *[80]*

$$B_{n,k} = C_{n,k} + C_{k,n} = C_{n,k} + C_{n+1,k-1}.$$

Proof. We know that $B_{n,k} = |\mathcal{C}_n^k|$, furthermore $\mathcal{C}_n^k = \mathcal{C}_n^k(*,l)\dot\cup\mathcal{C}_n^k(*,r)$. We have a bijection between $\mathcal{C}_n^k(*,r)$ and $\mathcal{C}_{n+1}^{k-1}(*,l)$. Hence

$$B_{n,k} = |\mathcal{C}_n^k| = |\mathcal{C}_n^k(*,l)| + |\mathcal{C}_n^k(*,r)| = C_{n,k} + |\mathcal{C}_{n+1}^{k-1}(*,l)| = C_{n,k} + C_{n+1,k-1}.$$

□

A similar connection is true between $C_{n,k}$ and $D_{n,k}$.

Theorem 52.

$$C_{n,k} = D_{n,k} + D_{n-1,k} + D_{n-1,k+1}.$$

Proof. We know that $C_{n,k} = |\mathcal{C}_n^k(*,l)|$, furthermore $\mathcal{C}_n^k(*,l) = \mathcal{C}_n^k(r,l)\dot\cup\mathcal{C}_n^k(l,l)$.

$$C_{n,k} = |\mathcal{C}_n^k(*,l)| = |\mathcal{C}_n^k(r,l)| + |\mathcal{C}_n^k(l,l)| = D_{n,k} + |\mathcal{C}_n^k(l,l)|.$$

The second term can be handled as we handled $\mathcal{C}_n^k(*,l)$ on our lemma: we present a bijection $\varphi : \mathcal{C}_n^k(l,l) \to \mathcal{C}_{n-1}^{k+1}(l,r)\dot\cup\mathcal{C}_{n-1}^k(r,l)$.

Let $\pi \in \mathcal{C}_n^k(l,l)$. Find the position of n (the largest left value) in π. It is the last element of one of the left blocks. If it is in the last block then simply rewrite it to $\mathbf{k+1}$. If it is not in the last block then there is a following right block R and at least one more left block. Then rewrite it again to $\mathbf{k+1}$ and at the same time move R to the end of π. The resulting permutation is $\varphi(\pi)$.

So far we did the same as we did in the proof of lemma. The only problem is that the image is not necessarily in $\mathcal{C}_{n-1}^{k+1}(l,r)$. It is possible that the block of n is the first block of π and it consists of only one element. Then the lemmas idea leads to $\varphi(\pi)$ where the leading element is $\mathbf{k+1}$. We do not want that. In this very special case (n is the first elmeent of π) we just erase n from π in order to obtain $\varphi(\pi)$.

Now it is clear that we defined a map with $\mathcal{C}_{n-1}^{k+1}(l,r)\dot\cup\mathcal{C}_{n-1}^k(r,l)$ as codomain. To see that it is a bijection we construct its inverse: If we have a permutation from $C_{n-1}^k(r,l)$, then the inverse puts a starting n in front of it. If we have a permutation from $C_{n-1}^{k+1}(l,r)$, then the inverse works as in our lemma.

The bijection leads to a fast end to our proof:

$$C_{n,k} = D_{n,k}+|\mathcal{C}_n^k(l,l)| = D_{n,k}+|\mathcal{C}_{n-1}^k(r,l)|+|\mathcal{C}_{n-1}^{k+1}(l,r)| = D_{n,k}+D_{n-1,k}+D_{n-1,k+1}.$$

□

5.9. Acyclic orientations of bipartite complete graphs

The connection of poly-Bernoulli numbers to acyclic orientations of the bipartite complete graph was discovered independently in two lines of research. (*Acyclic orientation* of a graph is an assignment of direction to each edge of the graph such that there are no directed cycles.)

Cameron, Glass and Schumacher [31] investigated the problem of maximizing the number of acyclic orientations of graphs with v vertices and e edges. Their conjecture is that if $v = 2n$ and $e = n^2$ then $K_{n,n}$ is the extremal graph. Along their research they counted the acyclic orientations of $K_{n,k}$, and established a bijection between these orientations and lonesum matrices of size $n \times k$.

In [45] the authors realized the connection of the permutations with extremal excedance sets (see subsection 5.8.2) and acyclic orientations with a unique sink. Without refering to the C-relatives of poly-Bernoulli numbers they gave an interpretation of the $C_{n,k}$ numbers in terms of acyclic orientations of complete bipartite graphs. Their proof is a specialization of general statements, we reprove the version, we need, by elementary means.

We extend the results in [45] with an interpretation for $D_{n,k}$ and summarize this line of research in the next theorem. We need some notation. Let $N = \{u_1, u_2, \dots, u_n\}$, $\widehat{N} = N \cup \{u\}$, $M = \{v_1, v_2, \dots, v_k\}$, $\widehat{M} = M \cup \{v\}$ be vertex sets. Let $K_{A,B}$ denotes the complete bipartite graphs on $A \dot\cup B$. Let $\mathcal{D}_n^k$ denote the set of acyclic orientations of $K_{N,M}$. Let $\mathcal{D}_n^{k\prime}$ denote the set of acyclic orientations of $K_{N,\widehat{M}}$, where v is the only sink (vertex without outgoing edge). Let $\mathcal{D}_n^{k\prime\prime}$ denote the set of acyclic orientations of $K_{\widehat{N},\widehat{M}}$, where u is the only source (vertex without ingoing edge) and v is the only sink.

Theorem 53. *(i) [31]*

$$|\mathcal{D}_n^k| = B_{n,k},$$

(ii) [45]

$$|\mathcal{D}_n^{k\prime}| = C_{n,k},$$

(iii)

$$|\mathcal{D}_n^{k\prime\prime}| = D_{n,k}.$$

Proof. (i)[31]: An acyclic orientation of $K_{N,K}$ can be coded by a binary matrix B of size $n \times k$ the following way: $b_{i,j} = 0$ whenever the edge $u_i v_j$ is oriented from u_i to v_j, and $b_{i,j} = 1$ whenever the edge $u_i v_j$ is oriented from v_j to u_i. It is east to check that the orientation is acyclic iff the corresponding matrix B does not contain any of the submatrix of the set L, hence B is lonesome. This establishes a bijection between $\mathcal{D}_n^k$ and $\mathcal{L}_n^k$. The claim follows from our previous results.

(ii): Take a binary matrix coding an orientation of a complete bipartite graph $K_{A,B}$. An all-0 column (the column of vertex $w \in B$) corresponds to the information that w is a sink. Hence if we take an arbitrary orientation of $K_{N,\widehat{M}}$ from $\mathcal{D}_n^{k\prime}$, then its restriction to $K_{N,M}$ will be an acyclic orientation. Its coding binary matrix cannot contain an all-0 column (an all-0 column would correspond to a second sink, that cannot exist in an orientation from $\mathcal{D}_n^{k\prime}$). Note that there is no restriction on rows. The elements of N cannot be sinks, since the edges connecting them to v are outgoing edges.

The above argument gave us a bijection between $\mathcal{D}_n^{k\prime}$ and $\mathcal{L}_n^k(c|)$, hence it proves our claim.

(iii): Straightforward extension of the previous proof. $\square$

Using classical results the connection to acyclic orientation of complete bipartite graphs immedietly leads to connections to the chromatic polynomials of complete bipartite graphs. The chromatic polynomial of a graph G is the polynomial $\mathrm{chr}_G(q)$, such that for any natural number k $\mathrm{chr}_G(k)$ gives the number of good k-colorings of G.

The famous result of Stanley [126] is that the number of acyclic orientations of a graph is equal to the absolute value of the chromatic polynomial of the graph evaluated at -1. Green and Zaslavsky [68] showed (up to sign) the coefficient of the linear term of the chromatic polynomial (see [67] for elementary proofs). Also in [68] it is proven, that the number of acyclic orientations of a graph G with a specified uv edge, such that u is the unique source and v is the unique sink is the derivative of the chromatic polynomial evaluated at 1 (the necessary signing is taken), the so called "Crapo's beta invariant". [67] presents an elementary discussion of this result. By putting together the information quoted above we obtain the following theorem. (We use the notation $[q^k]p(q)$ for the coefficient of q^k in the polynomial p.)

Theorem 54. *(i) [31]*

$$B_{n,k} = (-1)^{n+k} chr_{K_{n,k}}(-1),$$

(ii) [45]

$$C_{n,k} = (-1)^{n+k}[q] chr_{K_{n,k+1}}(q),$$

(iii)

$$D_{n,k} = (-1)^{n+k} \left(\frac{d}{dq} chr_{K_{n+1,k+1}} \right)(1).$$

The chromatic polynomials of complete bipartite graphs are well understood. We list a few results on this topic.

The exponential generating function of the chromatic polynomial of $K_{n,k}$ [127] Ex. 5.6:

$$\sum_{n\geq 0}\sum_{k\geq 0} \mathrm{chr}_{K_{n,k}}(q) \cdot \frac{x^n}{n!}\frac{y^k}{k!} = (e^x + e^y - 1)^q$$

There are several formulas for the chromatic polynomials of complete bipartite graphs (for example [128], [45], [75]):

$$\mathrm{chr}_{K_{n,k}}(q) = \sum_{i=0}^{n}\sum_{j=0}^{k} \left\{ {n \atop i} \right\} \left\{ {k \atop j} \right\} (q)_{i+j},$$

where $(q)_\ell = q(q-1)(q-2)\dots(q-\ell+1)$, the "falling factorial".

$$\mathrm{chr}_{K_{n,k}}(q) = \sum_{m\geq 0} \left(\sum_{i=0}^{n}\sum_{j=0}^{k} s(i+j,m) \left\{ {n \atop i} \right\} \left\{ {k \atop j} \right\} \right) q^m,$$

where $s(n,k)$ is the (signed) Stirling number of the first kind.

Simple arithmetic leads to the following theorem:

Theorem 55. *(i)*

$$B_{n,k} = (-1)^{n+k} \sum_{m=0}^{n+k}\sum_{i=0}^{n}\sum_{j=0}^{k} (-1)^m s(i+j,l) \left\{ {n \atop i} \right\} \left\{ {k \atop j} \right\},$$

(ii)

$$C_{n,k} = (-1)^{n+k} \sum_{i=0}^{n}\sum_{j=0}^{k} s(i+j,1) \left\{ {n \atop i} \right\} \left\{ {k+1 \atop j} \right\},$$

(iii)

$$D_{n,k} = (-1)^{n+k} \sum_{l=0}^{n+k+2} \sum_{i=0}^{n} \sum_{j=0}^{k} ls(i+j,l) \left\{ {n+1 \atop i} \right\} \left\{ {k+1 \atop j} \right\}.$$

We mention that the formula in (ii) is imprlicite in [45], without mentioning the poly-Bernoulli connection.

5.10. Algorithms for generating the series

In this section we recall simple algorithms that computes the arrays $B_{n,k}$, $C_{n,k}$ and $D_{n,k}$ by similar simple rules as Pascal's triangle the binomial coefficients. We will see that in this context the relatives $D_{n,k}$ arise naturally.

This line of research was initiated by the Akiyama–Tanigawa algorithms, that generates the Bernoulli numbers. Let define the array $a_{n,i}$ recursively (based on $\{a_{0,i}\}$) by the rule:

$$a_{n+1,i} = (i+1)(a_{n,i} - a_{n,i+1}).$$

Akiyama–Tanigawa proved that if the initial sequence is $a_{0,i} = \frac{1}{i}$ then $a_{n,0}$ are the n-th Bernoulli numbers. Legy denote by AT the trasformation $\{a_{0,i}\} \to \{a_{n,0}\}$. Akiyama–Tanigawa's theorem says that $AT(\{1/i\}_i) = \{B_i\}_i$.

Kaneko showed [79] that for any initial sequence $a_{0,i}$ it holds

$$a_{n,0} = \sum_{i=0}^{n} (-1)^i i! \left\{ {n+1 \atop i+1} \right\} a_{0,i}.$$

Based on the sieve formulas and simple arithmetic we obtain the following theorem.

Theorem 56. *(i)*

$$AT(\{(i+1)^k\}_i) = \{(-1)^i C_{i,k}\}_i,$$

(ii)

$$AT(\{i^k\}_i) = \{(-1)^i D_{i,k}\}_i$$

The poly–Bernoulli numbers itself can be generated by such simple rules, though the recursive rules has to be modified for that. Chen [32] presents the variant of the algorithm with these changes:

$$b_{n+1,i} = ib_{n,m} - (i+1)b_{n,i+1},$$

and shows that it holds

$$b_{n,0} = \sum_{i=0}^{n} (-1)^n m! \left\{ {n \atop m} \right\} b_{0,m}.$$

Therefore for the initial sequence $b_{0,i} = (i+1)^k$ we obtain $b_{n,0} = (-1)^n B_{n,k}$.

5.11. Diagonal sum of poly-Bernoulli numbers

The diagonal sum of poly-Bernoulli numbers resp. their relatives arise in analytical, number theoretical and combinatorial investigations [112], [80], [7]. However a nice formula is still missing. The diagonal sum of the poly-Bernoulli numbers

$$\sum_{n+k=N} B_{n,k}$$

are referred in OEIS [112] A098830: $1, 2, 4, 10, 32, 126, 588, 3170, \ldots$. The diagonal sum of the poly-Bernoulli relatives I are also referred in OEIS [112] A136127: $1, 2, 5, 16, 63, 294, \ldots$. The simple relation between $B_{n,k}$ and $C_{n,k}$ in Theorem 51 implies that (except the first entry) the A098830 is exactly the double of A136127.

From the combinatorial point of view the diagonal sum enumerates of course sets of the combinatorial objects we listed in this paper before. However there are combinatorial objects in that this sum appears naturally. One of this are the permutations with the ascending-to-max property [72], cycles without stretching pairs [7] and non-ambiguous trees [10]. The ascending-to-max property is one of the characteristic property of permutations that are suffix arrays of binary words. Suffix arrays play an important role in efficient searching algorithms of given patterns in a text. Stretching pairs in cycles arise in a result of Sharovsky in discrete dynamical systems. The introduction of the combinatorial non-ambiguous trees, that are compact embeddings of binary trees in a grid, was motivated by enumeration of parallelogram polynomios.

From the analytical results we recall here an interesting connection to the central binomial sum:

$$CB(k) = \sum_{n \geq 1} \frac{n^k}{\binom{2n}{n}}$$

Borwein and Girgensohn [24] showed that

$$3CB(N) = P_N + \pi\sqrt{3}Q_N$$

where P_N are integers and Q_N rationals. In [112] we find a conjecture of Stephan:

Conjecture 57.
$$\sum_{n+k=N} B_{n,k} = P_N.$$

Borwein and Girgensohn [24] determines P_N explicitly. Based on their formula, we can reformulate the Stephan's conjecture:

$$\sum_{n+k=N} B_{n,k} = P_N = (-1)^{N+1}\frac{1}{2}\sum_{j=1}^{N+1}\left(\frac{-1}{3}\right)^j j!\left\{ j \atop N+1 \right\}\binom{2j}{j}\sum_{i=0}^{j-1}\frac{3^i}{(2i+1)\binom{2i}{i}}.$$

It would be interesting to prove the conjecture or/and to find a simple expression for the diagonal sum.

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Chapter 6

Saturated simple topological graphs

6.1. Saturation technique in graph theory

Saturation is a very useful technique in combinatorics. In many cases we consider combinatorial structures with a certain property, say $\mathcal{T}$. A graph G is $\mathcal{T}$-saturated if G has property $\mathcal{T}$, but the addition of any edge joining two non-adjacent vertices of G ends property $\mathcal{T}$. Often structures with property $\mathcal{T}$ are quite hard to grasp, but $\mathcal{T}$-saturated structures might have useful character in some cases. See [47], [103], [83] for some applications of the saturation technique. A thorough survey [57] discusses the case when property $\mathcal{T}$ is 'not having F as a subhypergraph'.

Triangulations of a point set where the vertices of the triangulating triangles come from our points can be considered as a system of segments, that connects a pair of our points and saturated for the 'non-crossing' property.

In the rest of the Chapter we give a construction of a sparse simple topological graph. We prove that it is saturated. Finally we consider 'locally sparse' constructions.

6.2. The construction: our simple topological graph

The graph has $n = 6k$ vertices and $9k - 6$ edges. It consists of three vertex-disjoint *blue paths* $A_1^L A_2^L \ldots A_k^L$, $B_1^L B_2^L \ldots B_k^L$, $C_1^L C_2^L \ldots C_k^L$, three vertex-disjoint *red paths* $A_1^R A_2^R \ldots A_k^R$, $B_1^R B_2^R \ldots B_k^R$, $C_1^R C_2^R \ldots C_k^R$, and k disjoint *green triangles* $A_i^L B_i^L C_i^L$ which joint the blue paths together.

The drawing is best visualized on the surface of a long circular cylinder. Figure 6.1 shows an unrolling of the cylinder into the plane. The cylinder is obtained by cutting the drawing along the two dotted lines and gluing the top and the bottom together.

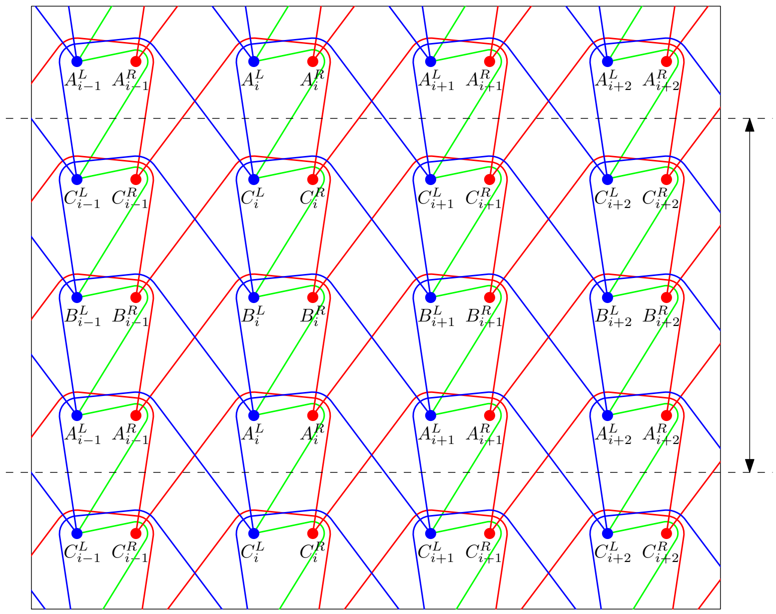

Figure 6.1: The graph G on an unrolled cylinder

The vertices are drawn together in pairs, with each vertex X_i^L on the *left* and the corresponding vertex X_i^R on the *right* ($X = A, B, C$).

The cylinder can be homeomorphically mapped into the plane, as shown in Figure 6.2 for the red and blue edges only. The horizontal directions turn into radial directions. But the resulting drawings suffer from large distortions, and the left-right symmetry is lost. We therefore prefer the cylindrical drawings, and we extend the cylinder surface periodically beyond the dotted lines (using the plane as a universal cover of the cylinder), because we found that these drawings are clearest and easiest to work with. One should however be aware that vertices (and edges) that appear as distinct in the figure may denote the same vertex, as indicated by the vertex labels.

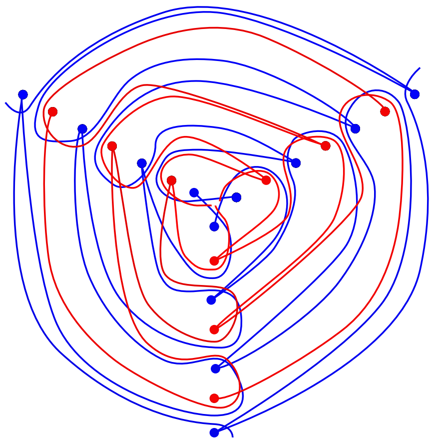

Figure 6.2: The graph G on an unrolled cylinder

We will first omit the green edges, because the resulting graph is more symmetric: with the exception of the vertices $X_1^{L/R}$ and $X_k^{L/R}$ near the boundary, all vertices look identical. Apart from these boundary effects, the drawing has a rotational symmetry, cyclically shifting the labels $A \to B \to C \to A$, a translational symmetry, shifting indices i up or down, and a mirror symmetry, exchanging left with right and blue with red. The green edges destroy the mirror symmetry: there are then two classes of vertices, the blue vertices X_i^L and the red vertices X_i^R.

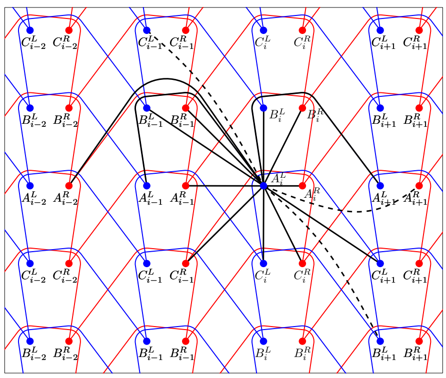

Figure 6.3: The 16 potential neigbors of a vertex

Let G_{RB} denote the topological graph without the green edges. We will show that the maximum degree in any saturated drawing which extends G_{RB} is 16. The 16 potential neighbors of a typical vertex A_i^L are shown in Figure 62.

This establishes that there are saturated drawings with n vertices and less than $8n$ edges. When the green edges are included, the three dashed edges in Figure 62 become impossible. Thus, each blue vertex has 13 potential neighbors. The red vertex A_{i+1}^R, which can be taken as a representative of a typical red vertex, loses A_l^L as a potential neighbor. Thus, each red vertex has at most 15 potential neighbors. This improves the upper bound for the smallest number of edges in a saturated drawings with n vertices to $7n$.

Theorem 58. *Any saturation of the topological graph G has at most $7n$ edges.*

The next section is devoted to proving the above theorem.

6.3. Proof of Theorem 58

Lemma 59. *The 16 potential neighbors of a typical vertex A_i^L in G_{RB} are all 11 vertices of levels $i-1$ and i ($A_{i-1}^L, B_{i-1}^L, C_{i-1}^L$; $A_{i-1}^R, B_{i-1}^R, C_{i-1}^R$; B_i^L, C_i^L; A_i^R, B_i^R, C_i^R) plus the 5 vertices A_{i-2}^R; $A_{i+1}^L, B_{i+1}^L, C_{i+1}^L$; A_{i+1}^R.*

When any of the neighbors listed above does not exist because $i \leq 2$ or $i = k$, the lemma still holds in the sense that the remaining vertices form the set of potential neighbors. In the proofs, when we exclude an edge between levels i and j, the arguments will not use edges outside this range.

In the following proofs, we will look at the given drawing of G or G_{RB} and argue about the additional edges that can be drawn. The implicit assumption is that these edges cannot cross any given edge more than once. Usually, we will regard a new edge as a directed edge, starting at some vertex and trying to reach another vertex.

6.3.1. Belts

A *belt* is formed by the 12 vertices of two successive layers with their 6 edges between them, see Figure 6.4. This subgraph separates a large face on the left from a large face on the right. More precisely, the belt is defined as the part of the plane (or the cylinder) which lies between these two large faces. (shaded area)

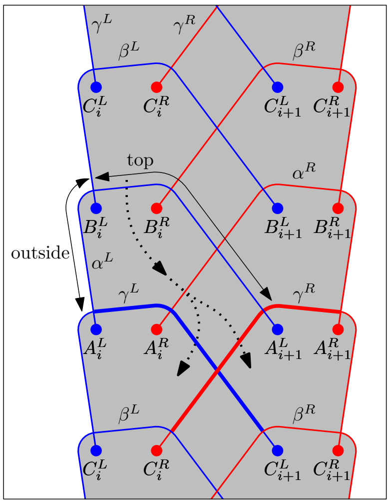

Figure 6.4: Escape from a belt is difficult (Observation 60 and Lemma 61).

We denote the six edges by $\alpha^L, \beta^L, \gamma^L, \alpha^R, \beta^R, \gamma^R$, as in Figure 6.4. Each edge is cut into six sections by the intersections with the other edges: Two sections are little “stumps” at the end vertices. One section belongs to the boundary between the belt and the *outside*. The remaining three sections form the *top part* of the edge. We say that a new (directed) edge crosses a belt edge *from outside* or *from the top* if it crosses the boundary part or the top part in the appropriate direction.

The following observation is obvious.

Observation 60. *If an edge which crosses α_L from outside or from the top, and it doesn't terminate at B_i^L or at B_i^R, then it must later cross γ^L or γ^R from the top.* □

The observation holds symmetrically for α_R instead of α_L, and cyclically for the other four belt edges. This implies that such an edge must cross another edge from the top, and then yet another, and so on, and hence it can never leave the belt. The following lemma summarizes this conclusion.

Lemma 61. *1. If an edge crosses a belt edge from the top or from outside, it must terminate in the belt.*

2. No edge can cross a belt. □

6.3.2. The Graph G_{RB}

After these preparations, we are ready to prove Lemma 59.

Let us first look at the potential neighbors on the left side. A connection from A_i^L to levels $j \le i-3$ is impossible, because it would have to cross a belt. Figure 6.5 investigates which vertices at level $i-2$ can be reached from A_i^L.

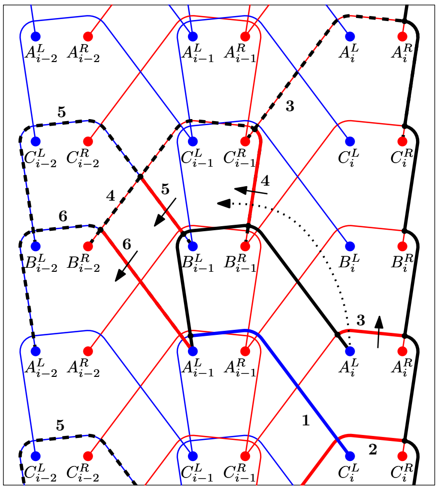

Figure 6.5: The potential neighbors of A_i^L in level $i-2$. We cannot cross the right boundary of the belt formed by levels $i-1$ and i, because then we would have to cross the whole belt to reach level $i-2$. If we cross edge 1 or 2 from the top, then, by Lemma 61, we are restricted to this belt. Thus we can regard edge 1 and 2 as closed from the top. (These edges can later be crossed from the bottom.) We successively conclude that the new edge must cross the red parts of edges 3, 4, 5, and 6. The endpoints $B_{i-2}^R, B_{i-2}^L, A_{i-2}^L$ of edges 4, 5, and 6 cannot be taken. C_{i-2}^L and C_{i-2}^R are enclosed in a small face delimited by edges 4, 5, and 6, and cannot be reached. A_{i-2}^R is thus the only reachable vertex of level $i-2$.

Let us turn to the potential neighbors on the right side. A connection from A_i^L to levels $j \ge i+3$ is impossible, because it would have to cross a belt. Vertices at level $i+2$ cannot be reached either, see Figure 6.6, and B_{i+1}^R and C_{i+1}^R are also excluded (Figure 6.7).

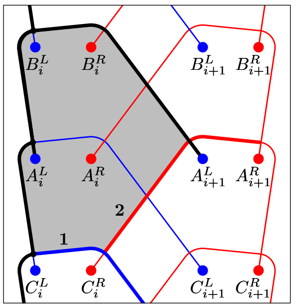

Figure 6.6: Level $i+2$ cannot be reached from A_i^L. We cannot cross the left boundary of the belt, because then we would have to cross the whole belt to reach level $i+2$. If we cross edge 1 or 2 from the top, then, by Lemma 61, we are restricted to this belt. This restricts the reachable region to the shaded area.

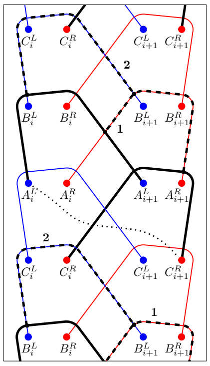

Figure 6.7: Exclusion of C_{i+1}^R as a potential neighbor of A_i^L. (The situation between A_i^L and B_{i+1}^R is symmetric.) The heavy edges are incident to A_i^L and C_{i+1}^R and therefore closed. We can successively close edge 1 and edge 2 by Observation (i). Then the connection between A_i^L and C_{i+1}^R is blocked.

This concludes the proof of Lemma 59.

6.3.3. The Graph G

Now we turn to G. The additional green edges exclude some of the possible edges from 59. Unfortunately the new lemma for our complet drawing must distinguish left and right vertices.

Lemma 62. *1. The* 13 *potential neighbors of a typical vertex* A_i^L *in* G *are all* 5 *vertices of level* i $(B_i^L, C_i^L;\ A_i^R, B_i^R, C_i^R)$, *all, but one vertex of level* $i-1$ $(A_{i-1}^L, B_{i-1}^L;\ A_{i-1}^R, B_{i-1}^R, C_{i-1}^R$; *plus the* 3 *vertices* A_{i-2}^R; A_{i+1}^L, C_{i+1}^L.

2. The 15 *potential neighbors of a typical vertex* A_i^R *in* G *are all* 11 *vertices of levels* i *and* $i+1$ $(B_i^L, C_i^L;\ A_i^R, B_i^R, C_i^R;\ A_{i+1}^L, B_{i+1}^L, C_{i+1}^L;\ A_{i=1}^R, B_{i+1}^R, C_{i+1}^R\ )$, *plus the* 4 *vertices* $A_{i-1}^R, B_{i-1}^R, C_{i-1}^R;\ A_{i+2}^L$.

The claim immediatley follows from the next two lemmas.

Lemma 63. *In a simple extension of* G, A_{i+1}^R *cannot be a neighbor of* A_i^L.

Proof. See Figure refstraight-right.

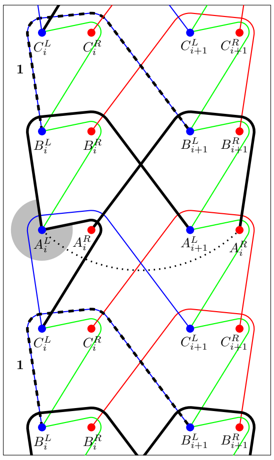

Figure 6.8:

We apply Observation (i) and identify edge $1 = B_i^L B_{i+1}^L$ as a closed edge. This cuts off the way to A_{i+1}^R. $\square$

Lemma 64. *In a simple extension of* G, B_{i+1}^L *cannot be a neighbor of* A_i^L.

By symmetry, C_{i-1}^L and A_i^L cannot be neighbors, and this concludes the proof of 62.

Proof. In Figure 6.9, some blocked edges incident to the end vertices are marked as heavy black edges.

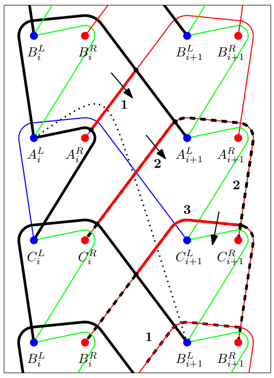

Figure 6.9:

The edge must leave A_i^L in the direction of the upper-right quadrant, because the other way is already blocked. After that, it must cross the red parts of edges $1 = A_i^R A_{i+1}^R$ and $2 = C_i^R C_{i+1}^R$. These edges are then blocked, and hence B_{i+1}^L cannot be reached. □

Lemma 62 imply that the avarege degree in a satutated extension of G is at most 14, hence Theorem 58 isproved.

We can count more carefully: If $k \geq 3$, then

1. the degrees of A_1^L, B_1^L, C_1^L are at most 7,
2. the degrees of A_1^R, B_1^R, C_1^R are at most 12,
3. the degrees of A_2^L, B_2^L, C_2^L are at most 12,
4. the degrees of A_i^R, B_i^R, C_i^R are at most 15, when $1 < i < k-1$,
5. the degrees of A_i^L, B_i^L, C_i^L are at most 13, when $2 < i < k$,
6. the degrees of $A_{k-1}^R, B_{k-1}^R, C_{k-1}^R$ are at most 14,
7. the degrees of A_k^L, B_k^L, C_k^L are at most 11,
8. the degrees of A_k^R, B_k^R, C_k^R are at most 8.

We immeditely obtain that if $k \geq 3$, then any saturated extension of G has at most $7n - 30$ edges. Hence we obtain that for any vertex set V if $6||V| > 12$, then there exists a saturated simple topological graph with at most $7|V| - 30$ edges.

We can extend our argument to any vertex size by a cloning method. Take a saturated simple topological graph and any vertex-point P of it. Add next to P s many new vertex-points/clones (of P). Connect the neighbors of P to each clone by an edge curve that are non-intersecting perturbations of the edge-curves leading to P. This way we obtain a simple drawing. Easy to see that a saturation of this drawing can include additional edges only edge-curves among the clones and P.

If we are given any $n \geq 18$, a vertex size, then we can write n as $6k + \rho$ where $\rho \in \{0, 1, 2, 3, 4, 5\}$. If $\rho = 0$ then we are done. If $\rho \geq 1$, then take a construction for a saturated simple topological graph with $6k$ vertices and its lowest degree vertex P. Add $\rho - 1$ clones of P and saturate. The simple topological graph, we obtain this way proves the Main Theorem (i). If $n \leq 15$ then Main Theorem (i) obviosly true. The cases $n = 16, 17$ can be handled easily, the rest is left to the reader.

6.4. Local saturation

The lower bound in [99] on the number of edges in a saturated simple topological graph is based on the following lemma.

Lemma 65. *[99] Let G be a simple topological graph. Let A be a vertex of degree of at most 2, and assume that A is not connected to all other vertices. Then G has a simple extension by an edge incident to A.*

Let G be a simple topological graph. We say that a vertex S is *locally saturated* if it cannot be connected to a non-neighbor vertex keeping simplicity.

A vertex F that is connected to all other vertices is locally saturated. In this case we say that F is *full*. The lemma says that a non-full saturated vertex must have degree at least three. Can we say more?

Observation 66. *For any positive integer n there is a simple topological graph on $5 + n$ vertices with a saturated vertex of degree 4.*

The observation is proved by Figure 6.10.

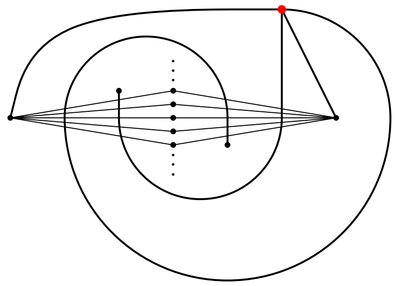

Figure 6.10: The red vertex/R is locally saturated

The local saturation property of R is obvious (the $n = 1$ case was discussed in [99], Figure 2).

Our crucial observation is summarized on Figure 6.11.

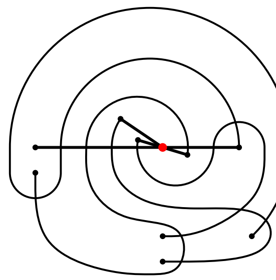

Figure 6.11: In the simple topological graph above the red vertex can not be connected by an edge-curve to any point in the outside/unbounded region with keeping simplicity

The explanation of the promised property can be easily verified, the details are left to the reader.

Theorem 67. *Let k be any positive integer. There is saturated simple topological graph on $10k$ vertices such that it contains k many vertices of degree 5.*

Proof. Take k copies of the drawing on Figure 6.11, and place them on the plane next to each other such a way that any of the copies is in the outside region of any other ones. Take any saturation. The k copies of the red vertex will remain degree 5 vertex no matter how the saturation was done. □

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Chapter 7

Saturated 2-simple topological graphs

We consider *2-simple saturated* planar drawings of graphs where *2-simple* means that any two edges of the drawing may have no more than 2 common points (including vertices) and *saturated* means that no more edges can be added to the drawing without violating 2-simplicity.

We follow the structure of the previous Chapter. First we describe the graph, state their properties, and prove our claims.

In what follows every graph is drawn on a horizontal strip representing an infinite cylinder – the top and bottom sides of the strip are identified.

We then use the standard mapping of a cylinder onto a plane without one point to get a drawing on a plane.

Clearly, this mapping preserves the property of the drawing to be 2-simple and saturated.

7.1. (3,2)-block

Before we proceed with the main results we revisit the construction of *grid-block* introduced by Kynčl, Pach, Radoičić and Tóth in [99].

In this section for simplicity we suppose that the cylinder is bounded both from left and right.

We call a *(3,2) grid-block* the arangements of lines homeomorphic to the one presented on the Fig. 7.1.

The block consists of 3 black and 3 red lines. 3 black lines are parallel translations of each other. They start on the leftmost side of the block and bend 2 times around the cylinder. 3 red lines are horizontal straight lines with right ends on the rightmost side of the cyliner. Each black line crosses each red line 2 times. We label every face of the drawing with 2 numbers, first is the distance of this face from the right side of the cylinder, second is the distance of this face from the bottom side of the cylinder.

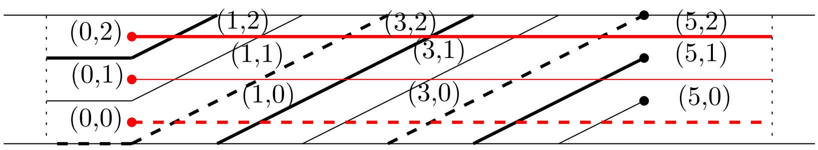


Figure 7.1: A (3,2) grid block with some labeled faces.

Observation 10 from [99] states that

Lemma 68. *Any path that crosses a (3,2)-grid block from left to right crosses lines participating in the block at least 5 times in total.*

We will need a more precise version of this statement:

Lemma 69. *Consider any path γ crossing the (3,2)-grid block that never visits faces $(0,i)$ and $(5,j)$ apart at its endpoints.*

Then this path can be modified to a path $\widetilde{\gamma}$ going through the (3,2)-grid block with the same endpoints, so that $\widetilde{\gamma}$

1. *first walks between the faces $(0,i)$, $0 \le i \le 2$,*
2. *then crosses some black lines to the right, going $(k,i) \to (k+1,i)$,*
3. *then crosses some red lines upward, going $(k,i) \to (k+1,i+1)$.*

Remark 70. *The arrangement of lines in the grid block forms a graph of the arrangement, G, and without loss of generality we view the path through the block as a walk on a dual graph G^*.*

Proof. Every path through the grid consists of edges of four different types: $\to$,$\leftarrow$,$\nearrow$ or $\swarrow$ (depending on which line of the grid it crosses and in which direction), see Fig. 7.2.

We execute the path simplification through a series of local path modifications on pairs of two consecutive steps: annihilation of two consecutive steps

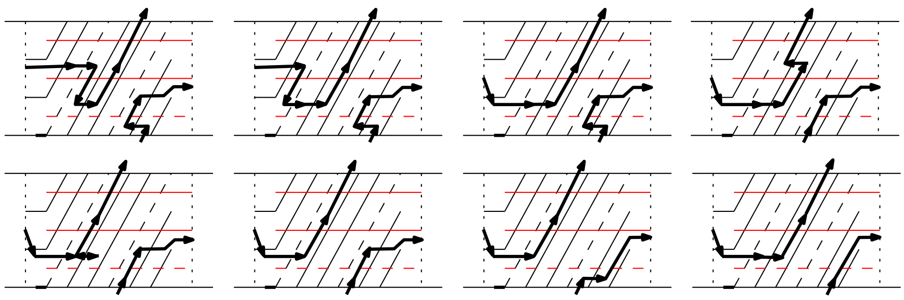


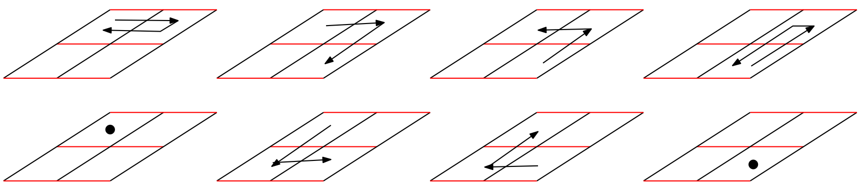
Figure 7.2: Process of simplification of a path going through a (3,2)-grid block.

in opposite directions and changing places of two consecutive steps that are not yet in a desired order.

The simplification goes in two stages. On the first stage (first 6 pictures on the Fig. 7.2) we get rid of "backward steps" $\swarrow$ and $\leftarrow$, while possibly increasing the number of steps the path $\widetilde{\gamma}$ walks between faces $(0,i)$, $0 \leq i \leq 2$. On the second step we reorder steps $\nearrow$ and $\rightarrow$ so that no $\nearrow$ precedes any $\rightarrow$

Stage 1.

We follow the path till we meet the first $\leftarrow$ or $\swarrow$ step. Together with the step before they form one of the 4 configurations from the Fig. 7.3. In cases (a) and (d) we simply annihilate two steps in different directions. In cases (b) and (c) the local modification is to reorder the two steps.



(a) (b) (c) (d)

Figure 7.3: 4 possible 2-step configurations involving "backward steps" as a second step before (first row) and after (second row) the appropriate local modification.

The reordering can be safely performed unless it forces the path to leave the (3,2)-grid. Clearly, this may happen only when the "backward" step starts from one faces labeled (2,i). Since this backward step is the first backward step of the path, this may happen only in four cases that are depicted in the Fig. 7.4 together with the accurate handling.

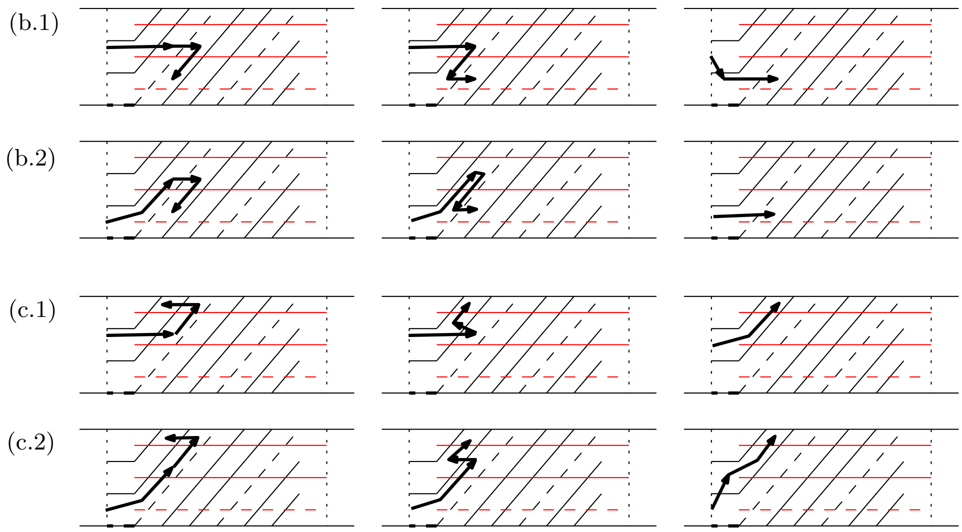


Figure 7.4: Handling of the 4 possible cases (each row for one case) when the local modification (b) or (c) forces path out of the grid.

We finish the proof of stage 1 using double induction on the number of “backward” edges and within it on the distance from the beginning of the path to the first bad edge.

Stage 2.

On stage 2 the path $\widetilde{\gamma}$ through the grid has only $\rightarrow$ and $\nearrow$ edges. These two type of edges can be reordered without changing the number of times the path crosses any line of the grid, and this reordering never puts the path out of the grid. $\square$

7.2. 3-block

The building blocks of the following constructions are black and red blocks, see Fig. 7.5. Any two lines forming the red block, as well as any two forming the black block, crosses exactly 2 times.

We combine 3 blocks in a way shown in a Fig. 7.6 to get a drawing that we call a *3-block*. Let us mention that red block is the symmetric image of the black block, so 3 block built form consecutive black-red-black blocks is a mirror image of the 3 block built from consecutive red-black-red blocks.

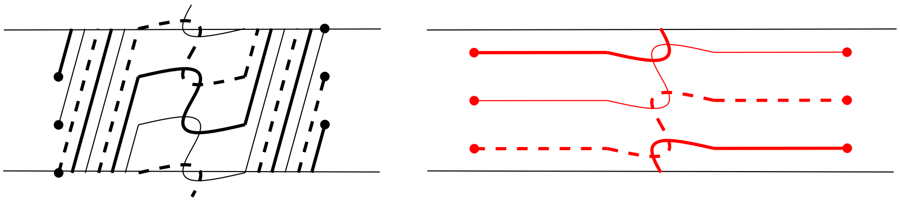


Figure 7.5: A black (left) and a red (right) blocks

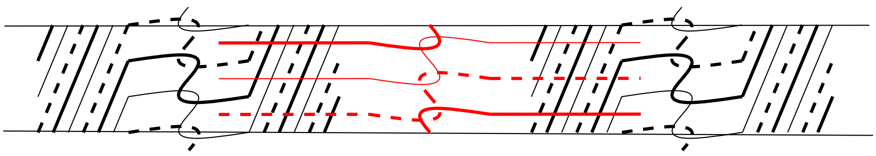


Figure 7.6: 3-block, formed by consecutive black, red and again black blocks

It is easy to see that every two lines forming the 3-block either do not cross or cross exactly 2 times.

The following theorem is our main observation.

Theorem 71. *Any path through the 3-block crosses at least one of the lines forming the 3-block at least 3 times.*

Proof. We view the path as a walk on a graph, dual to the initial arrangement. We label some of the faces of the arrangement as they are labeled on the Fig. 7.7.

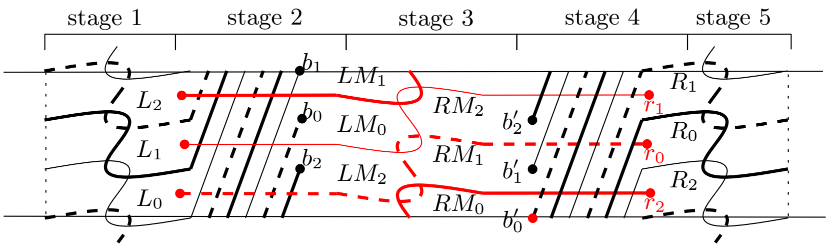


Figure 7.7: A 3-block with some distinguished faces (capital letters) and forming lines labeled. Lines forming the block are labeled b_i and r_i. The stages on which we devide the path are labeled above the strip.

To facilitate the analysis we divide the path onto the following stages (stages are edge disjoint, follow one another, and stage i's endpoint acts as a stage $(i+1)$'s start point):

1. the initial point $\to$ the last moment path is in the zone L,
2. last moment in $L \to$ first moment in LM,
3. first moment in $LM \to$ last moment in RM,
4. last moment in $RM \to$ first moment in R,
5. first moment in $R \to$ the endpoint.

Before we proceed we check that these stages are well defined (that the events defining the stages are linearly ordered in time).

For stages 1,3 and 5 it is trivial, while to check it for stages 2 (and, symmetric, 4), we prove that the last moment in L precedes the first moment in LM:

Lemma 72. *No path can visit zones $L \to LM \to L \to LM$ in this order.*

Proof. Zones L and LM are separated by the (3,2)-grid block. Passing through it requires at least 5 crossings of its forming lines. The path visiting $L \to LM \to L \to LM$ would cross the (3,2)-grid block at least 3 times, what results in $3 \times 5 = 15$ crossing of lines forming the grid block. But there are only 6 lines involved, so at least one of them will be crossed no less than 3 times. $\square$

Now we analyse the path by stages.

Stage 1.

Lemma 73. *Any path crossing the 3-block left-to-right crosses the line b_{i+1} at least once or one of the lines b_i and b_{i+2} at least twice on its first stage, where i is the subscript of L_i - the last point of the first stage of the path.*

Proof. This observation is evident. $\square$

Stage 2. During this stage the path crosses a (3,2)-grid block.

Lemma 69 defines the process of *path simplification* and shows that the intuition that the path should go through the (3,2)-grid block following the shortest route “right and up” is indeed correct.

The following lemma summarizes the behavior of the path on the first two stages:

Lemma 74. *Any path γ crossing the G_3 can be modified within its stage 2 to a path $\widetilde{\gamma}$ so that the path $\widetilde{\gamma}$:*

- *crosses red lines r_j, r_{j+1} before it first visits the zone LM at the point LM_j.*

Proof. We modify the path γ on the stage 2 following the simplification procedure described in lemma 69 to get the path $\widetilde{\gamma}$.

Lemma 69 implies that the stage 2 of $\widetilde{\gamma}$ consists of exactly 5 edges: first, $0 \le h \le 5$ edges crossing black lines $\rightarrow$ to the right, followed by $v = 5 - h$ edges crossing red lines $\nearrow$ upward.

As always, we denote the first point of the stage 2 of $\widetilde{\gamma}$ with L_i. Then h horizontal steps of this stage will cross black lines b_{i+1}, b_i, b_{i-1}, $\ldots$, $b_{i+1-(v-1)}$. But lemma 73 guarantees that already on the stage 1 the path $\widetilde{\gamma}$ crossed either b_{i+1} once or one of b_i or b_{i+2} twice. Since the path can not cross any of the black lines more than twice, $v \le 3$.

So, $h \ge 2$, what implies that the path $\widetilde{\gamma}$ crosses the lines r_{j-1}, r_j before it reaches the last point of its second stage, LM_j.

To finish the proof we recall that the path γ crosses every line of the 3-block at least as many times as $\widetilde{\gamma}$. □

Since the structure of the stage 4 is exactly the same as of the stage 2, the lemma 74 can be applied to the stage 4 as well and we summarize it into:

Lemma 75. *Any path γ crossing the G_3 can be modified within its stages 2 and 4 to a path $\widetilde{\gamma}$ so that the path $\widetilde{\gamma}$:*

- *crosses red lines r_j, r_{j+1} before it first visits the zone LM at the point LM_j.*

- *crosses red lines r_{k-1}, r_k after it last visits the zone RM at the point RM_k.*

We can now finish the proof of the theorem 71.

We prove by the contradiction. We consider any path γ crossing the G_3 and use lemma 75 to construct its modification $\widetilde{\gamma}$.

In what follows we use the notation from lemma 75. Lemma 75 describes the behaviour of the path on the stages 1, 2, 4, and 5. A short case distinction (k going through j, $j+1$ and $j+2$) shows that $\widetilde{\gamma}$ can not connect points LM_j and RM_k on the remaining stage 3 without crossing at least one of the red lines 3 times see Fig. 7.8. Fig. 7.8 depicts all the possible ways to continue the path $\widetilde{\gamma}$ beyond the point LM_j. Each of the possible continuations crosses

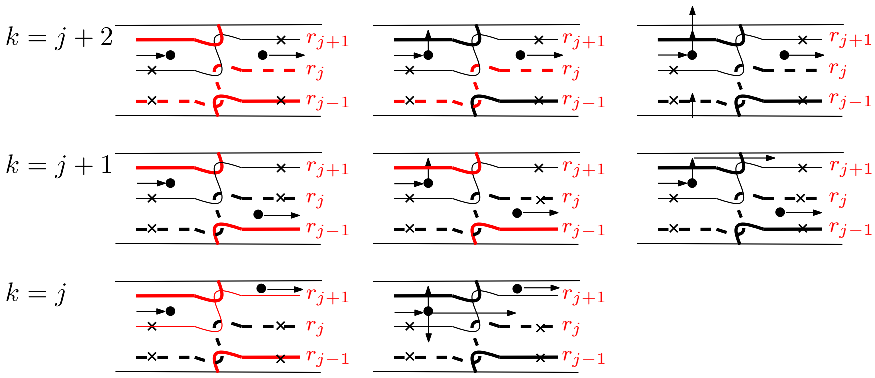


Figure 7.8: 3 rows depict 3 cases, $k = j+2$, $j+1$ and j. Black dots inside faces mark the faces LM_j (left) and RM_k (right). Black crosses on red lines mark the lines that are, following lemma 75, crossed by the path outside the stage 3. We colour red lines black as soon as they are crossed by the path 2 times, so no more crossings is possible. In the case $k = j$ the path can be continued in 3 different direction, in each of them the path is blocked after one step.

some of the red lines r_j, r_{j+1}, r_{j-1} twice and is blocked within one of the faces before it reaches the point RM_k.

□

7.3. Result

We denote with $s_2(n)$ the minimum number of edges that a 2-simple saturated drawing with n vertices may have.

Our goal is to prove Theorem 13 of the Introduction.

Theorem 76. *For n large enough*

$$s_2(n) \leq 14.5n.$$

Proof. We consider the drawing shown on the Fig. 7.9.

The horizontal strip denotes the cylinder. The drawing is formed by k consecutive black and red blocks, see Fig. 7.5. Each block contains 6 vertices, so the total number of vertices is $k \times 6$. Clearly, the drawing is 2-simple.

Now we add as many edges as it is possible without violating 2-simplicity, so that the drawing becomes saturated (this padding procedure is definitely

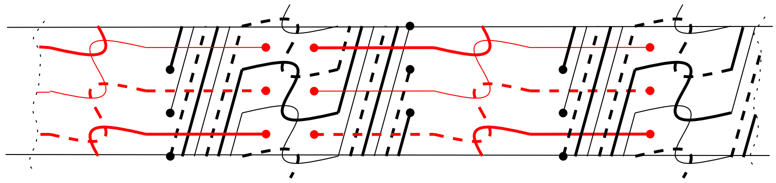


Figure 7.9:

non unique). The Theorem 71 implies that without violating 2-simplicity any vertex can be connected by an edge only to 29 other vertices, see Fig. 7.10 for internal vertices and Fig. 7.11 for vertices close to the left(right) boundary of the cylinder. That implies that the maximal number of edges in the resulting saturated 2-simple drawing is less or equal than $14.5n$.

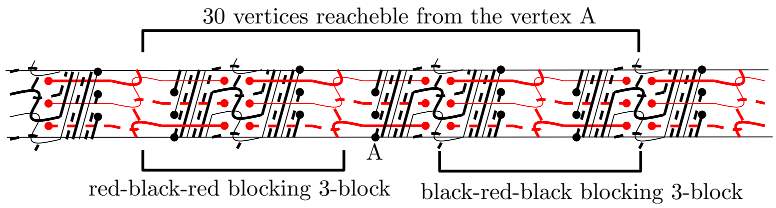


Figure 7.10:

For n not divisible by 6 we build the construction above with $k = \lfloor n/6 \rfloor$. We split the remaining $l = n - 6\lfloor n/6 \rfloor$ no more than 5 vertices onto two groups of no more than 3 vertices each, $l = l_1 + l_2$ and put l_1 vertices to the left and l_2 vertices to the right of the resulting arrangement.

The possible connections with the newly introduced vertices are illustrated on the Fig. 7.11. Since $l_1, l_2 \leq 3$, no vertex has degree greater than 29.

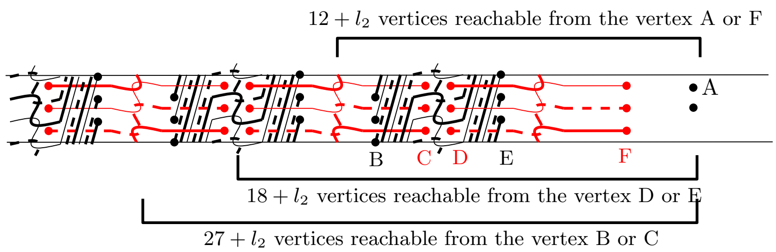


Figure 7.11:

□

Chapter 8

Geometric transversals

8.1. Polygons in the plane

Proof. Theorem 15. For definiteness assume that Car #1, the car on γ_1, moves in the positive, that is counterclockwise direction. It is also convenient to think of the cars not stopping after making a full round, but continuing their movement periodically forever.

If there is a line that separates the two curves (meaning that the two curves lie on opposite sides of the line including the line itself), then this line clearly satisfies the claim in the theorem. Furthermore, if the intersection of the interiors of two convex curves on the plane is empty, then the two curves can be separated by a line, so in what follows we may assume that there is a common interior point of the two regions enclosed by the curves.

First assume that both curves are strictly convex, i.e. neither of them contains a line segment and that the cars do not make stops.

We shall need to speak of directed lines. A direction α on the plane is given by a unit vector, say by a vector pointing from the origin to a point on the unit circle. We can parametrize such a direction by the angle α that the vector forms with the positive real half-line. So this parameter α lies in $[0, 2\pi)$, but it is convenient to extend the parametrization periodically to the whole real line. A directed line is just a line on a which a direction (parallel with the given line) is given. If we move on the line in the given direction, then we can speak of the left- resp. right-hand side of the line.

We claim that, for every direction α, there is a unique directed line $l = l(\alpha)$ with direction α such that Car #1 spends, within one period, exactly as much time on the right side of l as Car #2 spends on the left-side of l; note that it is NOT claimed that this happens during the same time interval. (This claim is somewhat similar to the Ham-Sandwich theorem, see [16].) To prove this claim, let us choose a line l_0 with direction α so that both curves lie on the right side of l_0 and move continuously this line, in one unit of time, using translations, to a position l_1, where both curves lie on the left-side of l_1. Let $T_1(t)$ be the time that Car #1 spends (in one period) on the right side of l_t (the line at time t) and $T_2(t)$ be the time that Car #2 spends on the left-side of l_t. The function $f(t) = T_1(t) - T_2(t)$, $f\colon[0,1] \to R$, is continuous, strictly monotone decreasing, $f(0) = 1$ and $f(1) = -1$ (continuity is due to the fact that we assumed strict convexity of the curves). Therefore, there is a unique point $t_0 \in [0,1]$ such that $f(t_0) = 0$. The line l_{t_0} is the one we are looking for.

Now we claim that the line $l(\alpha)$ intersects both curves in exactly 2 points. Indeed, if this line intersects one of the curves in no or one point, then this curve is completely on one side of $l(\alpha)$, so the other curve must be completely on the other side of $l(\alpha)$. But that would mean that $l(\alpha)$ separates the two curves, which we assumed not to be the case.

Let $B_j(\alpha)$ denote the point where the line $l(\alpha)$ enters the region enclosed by γ_j, and $K_j(\alpha)$ where it leaves that region for $j = 1,2$. Clearly, $B_j(\alpha)$ and $K_j(\alpha)$ are 2π-periodic functions from the real line into the complex plane. A simple argument shows that they are continuous. It is heuristically clear that, as α moves from 0 to 2π, the point $B_j(\alpha)$ goes around the curve γ_j once in the positive direction. Unfortunately, the points $B_j(\alpha)$ do not necessarily move always in the counterclockwise direction, so this is a subtle point, to which we shall return at the end of the proof.

Let $\phi : \gamma_2 \to \gamma_1$ be the following bijection between the two curves: Car #2 is at the point P on γ_2 exactly when Car #1 is at $\phi(P)$ on γ_1 (there is such a bijection since we assumed no stopping of the cars). Since the two cars move in opposite directions, the point $\phi(B_2(\alpha))$ goes around γ_1 once in the negative (clockwise) direction as α moves from 0 to 2π. It follows that there exists α_0 such that $B_1(\alpha_0) = \phi(B_2(\alpha_0))$ (see more explanation at the end of the proof). Note that both $B_1(\alpha_0)$ and $B_2(\alpha_0)$ lie on $l(\alpha_0)$, and the equality $B_1(\alpha_0) = \phi(B_2(\alpha_0))$ means that the two cars are in these two points of $l(\alpha_0)$ precisely at the same moment.

We claim that the line $l(\alpha_0)$ satisfies the condition of the theorem. Indeed, when Car #1 is at the point $B_1(\alpha_0)$, then Car #2 is at the point $B_2(\alpha_0)$. From that position Car #1 moves to the right side of $l(\alpha_0)$ (since Car #1 moves in the counterclockwise direction), while Car #2 moves to the left-side of $l(\alpha_0)$. Since Car #1 spends as much time on the right side of $l(\alpha_0)$ as Car #2 spends on the left-side of $l(\alpha_0)$, we must also have that Car #1 is at the point $K_1(\alpha_0)$ exactly when Car #2 is at the point $K_2(\alpha_0)$. From that position Car #1 moves to the left-side of $l(\alpha_0)$ and Car #2 moves to the right side of $l(\alpha_0)$, and then they hit the line $l(\alpha_0)$ again at the points $B_1(\alpha_0)$ and $B_2(\alpha_0)$. It follows that the two cars are always on opposite sides of the line $l(\alpha_0)$, and we are done.

In the preceding argument we extensively used continuity, which was due to the fact that the curves γ_j, $j = 1, 2$, are strictly convex. In the general case when this is not necessarily so, we can use approximation and take limit as follows. Still assuming that neither of the cars make a stop, let O_1 be a point inside γ_1 and let $\gamma_1^{(m)}$ be obtained from γ_1 by shrinking it from O_1 by the factor $1 - 1/m$, $m = 2, 3, \ldots$. Select a strictly convex curve $\gamma_{1,m}$ in between γ_1 and $\gamma_1^{(1)}$. At each moment project Car #1 onto $\gamma_{1,m}$ from O_1 — this way we get Car $\#1_m$ moving along $\gamma_{1,m}$ in the positive direction. We construct $\gamma_{2,m}$ and Car $\#2_m$ in the same way for the second curve. Now $\gamma_{1,m}$ and $\gamma_{2,m}$ are strictly convex, therefore, by the first part of the proof, there is a line l_m that separates Car $\#1_m$ and Car $\#2_m$ at every moment. We can select a subsequence $\{m_k\}$ of the natural numbers so that the lines l_{m_k} converge to some line l as $m_k \to \infty$, and it is simple to check that then l separates the original two cars at every moment.

The case when the cars make stops can be handled similarly: we approximate such movement by movements without stops, and take limit. We leave the details to the reader.

It remains to prove the heuristic claim that the points $B_j(\alpha)$ traverse (not necessarily monotonically) the curves γ_j once in the positive direction as α moves from 0 to 2π, and, due to that, there is an α_0 for which $B_1(\alpha_0) = \phi(B_2(\alpha_0))$. Consider γ_1, and all directed lines in the direction α which hit γ_1. There are two of them that are supporting lines to γ_1, one touching γ_1 on the right side of $l(\alpha)$ at a point $P_{1,\alpha}$ and one touching γ_1 on the left-side of $l(\alpha)$ at a point $Q_{1,\alpha}$. All other lines enter the interior of γ_1 somewhere on the arc $I_{1,\alpha} := Q_{1,\alpha}P_{1,\alpha}$ that goes from $Q_{1,\alpha}$ to $P_{1,\alpha}$ in the counterclockwise direction. In particular, $B_1(\alpha)$ is an interior point of that arc, see Figure 8.1. Note that $P_{1,\alpha}$ and $Q_{1,\alpha}$ move (as α increases) monotonically in the counterclockwise direction, and we express this fact by saying that the arc $I_{1,\alpha}$ moves in the counterclockwise direction.

Let us denote the analogue of $P_{1,\alpha}, Q_{1,\alpha}, I_{1,\alpha}$ on the curve γ_2 by $P_{2,\alpha}, Q_{2,\alpha}, I_{2,\alpha}$, and consider their image under the mapping ϕ. Because the cars move in opposite direction, $\phi(I_{2,\alpha})$ is a proper subarc of γ_1 with endpoints $\phi(P_{2,\alpha})$ and $\phi(Q_{2,\alpha})$ in the counterclockwise direction, and the arc $\phi(I_{2,\alpha})$ moves in

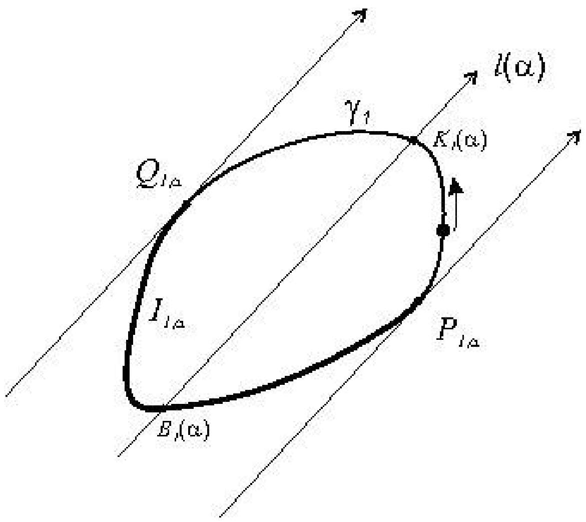


Figure 8.1: The curve γ_1, the line $l(\alpha)$, the two supporting lines parallel with it together with the corresponding touching points $P_{1,\alpha}$ and $Q_{1,\alpha}$, and the arc $I_{1,\alpha} = Q_{1,\alpha}P_{1,\alpha}$

the clockwise direction (again in the sense that its endpoints do so monotonically). Thus, on γ_1 the proper arc $I_{1,\alpha}$ moves continuously in the counterclockwise direction and the point $B_1(\alpha)$ is always on that arc and moves continuously, while the proper arc $\phi(I_{2,\alpha})$ moves in the clockwise direction and the point $\phi(B_2(\alpha))$ is always on that second arc and moves continuously, see Figure 8.2. The claim we are dealing with is that then the points $B_1(\alpha)$ and $\phi(B_2(\alpha))$ must meet, which is heuristically clear, since the arcs $I_{1,\alpha}$ and $\phi(I_{2,\alpha})$ "sweep through each other" (actually twice within one full round). A formal proof is as follows.

Let O be a fixed point in the interior of γ_1, which we may assume to be the origin. If $S(\alpha)$, $\alpha \in \mathbb{R}$, is a continuously and periodically moving point avoiding O (formally $S : \mathbb{R} \to \mathbb{R}^2 \setminus \{O\}$ is a continuous 2π-periodic mapping), then its winding number (relative to O) tells us how many times $S(\alpha)$ goes around O within one period (i.e. while α runs through $[0, 2\pi]$), where counterclockwise motions are counted as positive and clockwise motions are counted as negative (see [62, Chapter 3.a] for a precise definition). Now a basic fact is that if $S(\alpha)$ and $\tilde{S}(\alpha)$ are two such points continuously moving along γ_1 which do not meet, then their winding numbers must be the same. Indeed, if $S(\alpha) \neq \tilde{S}(\alpha)$ for any α, then the line segments joining

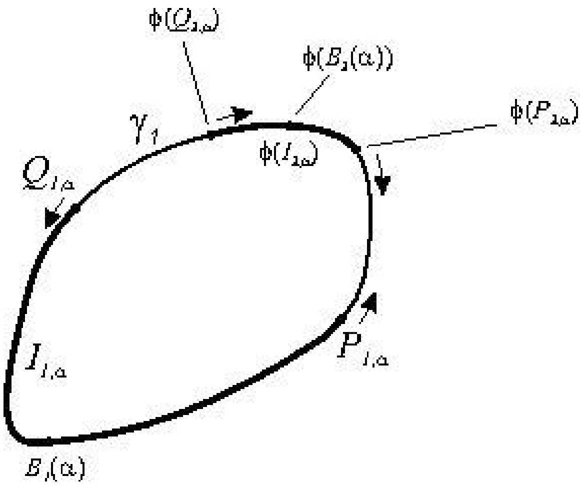


Figure 8.2: The oppositely moving arcs $I_{1,\alpha}$ and $\phi(I_{2,\alpha})$

$S(\alpha)$ and $-\tilde{S}(\alpha)$ never hit O, therefore, by the "Dog-on-a-Leash" theorem [62, Theorem 3.11], $S(\alpha)$ and $-\tilde{S}(\alpha)$ have the same winding numbers, which is the same that $S(\alpha)$ and $\tilde{S}(\alpha)$ have the same winding numbers.

After this, let us return to the points $B_1(\alpha)$ and $\phi(B_2(\alpha))$. Since $B_1(\alpha)$ and $P_{1,\alpha}$ do not meet, and the latter has winding number 1, we conclude that $B_1(\alpha)$ has winding number 1. In a similar fashion, $\phi(B_2(\alpha))$ has winding number -1, since it never meets $\phi(P_{2,\alpha})$ and this latter has winding number -1. Thus, $B_1(\alpha)$ and $\phi(B_2(\alpha))$ have different winding numbers, so they must meet. $\square$

8.2. Polytopes in 3-space

In this section, we briefly discuss if Theorem 14 is a purely 2-dimensional result or if it has an analogue in 3-space. As we shall see, there are difficulties in giving the full analogue of Theorem 1 in higher dimensions.

The analogue of a convex polygon in 3-space is a convex polytope $A_1, \ldots, A_n$, and if someone would like to speak of oppositely oriented polytopes, that is not as simple as in the plane. For example, it would be difficult to compare the orientation of a cube with that of a pyramid with a 7-gon base, even though both of them have 8 vertices. However, there is no problem with tetrahedrons: a tetrahedron $ABCD$ is said to be positively (negatively) oriented if the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$ are right-oriented (left-oriented), i.e. they follow each other as the thumb, forefinger and middle fingers on our right hand (left hand). And for tetrahedrons we have

Proposition 77. *If $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$ are oppositely oriented tetrahedrons, then there is a plane that intersects every segment A_iB_i, $i = 1,2,3,4$.*

Proof. . For $t \in [0,1]$ let a point $P_i(t)$ move continuously on the segment A_iB_i from A_i to B_i. Thus, $P_1(0)P_2(0)P_3(0)P_4(0)$ is the tetrahedron $A_1A_2A_3A_4$, while $P_1(1)P_2(1)P_3(1)P_4(1)$ is the tetrahedron $B_1B_2B_3B_4$. The vector triple $\overrightarrow{P_1(t)P_2(t)}$, $\overrightarrow{P_1(t)P_3(t)}$, $\overrightarrow{P_1(t)P_4(t)}$ changes continuously, and at $t = 0$ and at $t = 1$ its orientation is different. So somewhere this orientation must change, and that is possible only if at some $t_0 \in (0,1)$ these vectors lie in the same plane. Clearly, that plane then intersects each of the segments A_iB_i.

For the pedantic readers let us state here the precise meaning of "somewhere this orientation must change": if t_0 is the supremum of all t for which the vector triplet $\overrightarrow{P_1(t)P_2(t)}$, $\overrightarrow{P_1(t)P_3(t)}$, $\overrightarrow{P_1(t)P_4(t)}$ is right-oriented, then $t_0 \in (0,1)$, and for $t = t_0$ all these vectors must line in a plane.

Those familiar with signed volumes (which, for $P_1(t)P_2(t)P_3(t)P_4(t)$ is 1/6-times the mixed product of the vectors $\overrightarrow{P_1(t)P_2(t)}$, $\overrightarrow{P_1(t)P_3(t)}$, $\overrightarrow{P_1(t)P_4(t)}$) will recognize that this proof amounts to the same as with $P_1(t)P_2(t)P_3(t)P_4(t)$ we continuously move from a positive volume to a negative one, so at some point $P_1(t_0)P_2(t_0)P_3(t_0)P_4(t_0)$ must have zero volume, i.e. it must be degenerate: all four points $P_1(t_0)$, $P_2(t_0)$, $P_3(t_0)$, $P_4(t_0)$ lie on the same plane. $\square$

Next, consider more general polytopes in 3-space, but to avoid the above mentioned problem concerning a cube and a pyramid, let us suppose that $P: A_1 \cdots A_n$ and $Q: B_1 \cdots B_n$ are two polytopes with the property that whenever $A_{i_1} \cdots A_{i_k}$ form a face of P then $B_{i_1} \cdots B_{i_k}$ form a face of Q. We are not trying to define in general "opposite orientation" for such polytopes, but any meaningful definition should imply at least that all such corresponding faces $A_{i_1} \cdots A_{i_k}$ and $B_{i_1} \cdots B_{i_k}$ are oppositely oriented (by looking at them, say, from the outside). The simplest case when we can have such a correspondence in between the vertices of P and Q is when P and Q are isometric, and in this case we can state the following proposition, in which "opposite orientation" means that the corresponding faces are oppositely oriented.

Proposition 78. *If $A_1 \cdots A_n$ and $B_1 \cdots B_n$ are oppositely oriented isometric polytopes, then there is a plane that intersects every segment A_iB_i, $i =$*

$1, 2, \ldots, n$.

Of course, in this formulation we assume that the isometry in between the two polytopes moves A_i into B_i.

Proof. . It is known (see, e.g., [37, Chapter 7]) that in 3-space the isometries are the following:

a) screw (a rotation about some axis followed by a translation parallel to that axis),

b) glide reflection (reflection onto a plane followed by a translation with a vector that is parallel with the plane),

c) rotatory reflection (axial rotation followed by a reflection onto a plane that is perpendicular to the axis of rotation).

The first type preserve orientation, but the latter two types reverse it. Now, if the isometry moving $A_1 \cdots A_n$ into $B_1 \cdots B_n$ is of type **b)** or **c)**, then the plane of reflection in **b)** or **c)** intersects each segment A_iB_i. □

Next, we show that a slight distortion of the polytopes in Proposition 78 may result in "almost isometric" polytopes for which the proposition is no longer true. Let $A_1 \cdots A_5 A_6'$ and $B_1^* \cdots B_5^* B_6'$ be two isometric regular octahedrons of opposite orientation, see Figure 8.3.

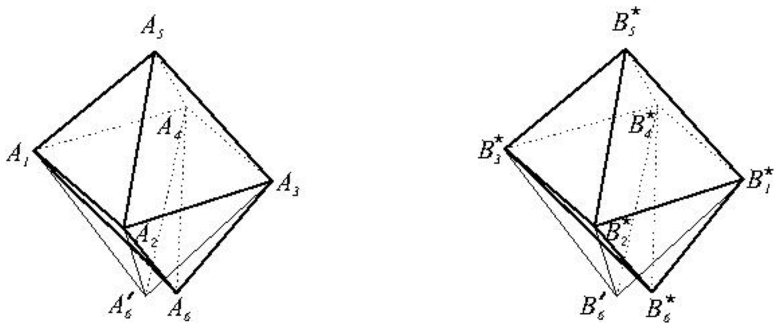


Figure 8.3: The two regular octahedrons and their modifications

Now move A_6' off the $A_5A_2A_4$ plane towards A_3 to get the octahedron $A_1 \cdots A_6$, and move B_6' off the $B_5^*B_2^*B_4^*$ plane towards B_1^* to get the octahedron $B_1^* \cdots B_6^*$. Finally move this last octahedron $B_1^* \cdots B_6^*$ so that B_2^* aligns with A_2; B_4^* aligns with A_4 and B_5^* aligns with A_5. Now if $B_1 \cdots B_6$ is the octahedron that we obtain after this last motion, then $A_2 = B_2$, $A_4 = B_4$ and $A_5 = B_5$, so a plane intersecting all segments A_iB_i would have to be the $A_2A_4A_5$ plane. But, by construction, A_6 and B_6 lie on the same side of that plane, so in this arrangement the segments A_iB_i do not have a common plane transversal.

Finally, we consider a situation that includes both the case of tetrahedrons and the case of isometric polytopes that have been discussed so far. This will also show the strength of algebraic methods in geometry.

An affine mapping is a mapping of $\mathbb{R}^3$ that preserves parallelism. Alternatively, if we use coordinates, then affine transformations can be defined as mappings $T(x_1, x_2, x_3) = (y_1, y_2, y_3)$, where

$$\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + b_1, \\ y_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + b_2, \\ y_3 &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + b_3 \end{aligned}$$

with some numbers a_{ij} and b_i. Such a T preserves orientation (say of faces) if the determinant $|a_{ij}|_{i,j=1}^3$ is positive, and it reverses orientation if it is negative. If $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$ are two tetrahedra, then there is a unique affine map taking $A_1A_2A_3A_4$ into $B_1B_2B_3B_4$. Isometries are affine maps in which the matrix $(a_{ij})_{i,j=1}^3$ is orthogonal, i.e. $a_{i1}a_{j1} + a_{i2}a_{j2} + a_{i3}a_{j3}$ is 0 if $i \neq j$ and is 1 if $i = j$, $1 \leq i, j \leq 3$. So the following statement contains both the tetrahedron and the isometric polytope cases discussed before.

Proposition 79. *Let $A_1 \cdots A_n$ be a polytope and $B_1 \cdots B_n$ an affine image of it. If these polytopes are oppositely oriented (in the sense that corresponding faces in them are oppositely oriented), then there is a plane that intersects every segment A_iB_i, $i = 1, 2, \ldots, m$.*

Proof. . Let T be the affine mapping in between $A_1 \cdots A_n$ and $B_1 \cdots B_n$ with matrix $\mathcal{A} = (a_{ij})_{i,j=1}^3$, so that if

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

then with some vector $\mathbf{b}$ we have $T\mathbf{x} = \mathcal{A}\mathbf{x} + \mathbf{b}$. Let

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

be the identity matrix. For $t \in [0,1]$ set $\mathcal{A}_t = (1-t)I + t\mathcal{A}$ and consider the transformation $T_t\mathbf{x} = \mathcal{A}_t\mathbf{x} + t\mathbf{b}$, so that T_0 is the identity and T_1 is the mapping T. By assumption, the latter reverses orientation, so $\mathcal{A}_1$ has negative determinant, while the determinant of $\mathcal{A}_0$ is 1. Therefore, there must be a $t_0 \in (0,1)$ such that the determinant of $\mathcal{A}_{t_0}$ is 0, which means that T_{t_0} is singular, i.e., it maps the whole space into a plane S. This S intersects every segment A_iB_i, the intersection points being $T_{t_0}(A_i)$. □

Note that these 3-D results are rather limited, e.g. even though in Proposition 79 convexity is not needed, the proposition itself is a quite restricted analogue of Theorem 14: in it the existence of a common transversal plane is due to the fact that there is a plane that intersects every segment that connects a point with its affine image (under the affine transformation considered). However, the example in Figure 8.3 shows that, in some sense, this is necessary: a slight distortion of the affine images may lead in 3-space to situations when there are no common transversal planes.

Similar results hold in higher dimensions (in $\mathbb{R}^d$) for simplices and affine polytopes.

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