Kvantum Grupoidok
Doktori Értekezés
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Előszó

A 'kvantum csoport' kifejezést több rokon értelemben is használják. A népszerű *WikipediA* „Quantum group" szócikke a következőt írja [77]. *„A matematikában és az elméleti fizikában a 'kvantum csoport' kifejezés különféle további struktúrákkal ellátott nem kommutatív algebrákat jelent. Általában a 'kvantum csoport' valamiféle Hopf-algebra. Nincs egyetlen, az összes változatot magába foglaló definíció, hanem lényegében hasonló objektumok egy családjára kell gondolni. [...]"* Az elnevezés nyilvánvalóan onnan ered, hogy bármely csoport elemei által tetszőleges test fölött kifeszített vektortér természetesen ellátható egy 'kvantum csoport' azaz Hopf-algebra struktúrával; s a motiváló példákat ezen 'klasszikus' esetek deformálásával vagyis 'kvantálásával' konstruálták.

Hasonlóképpen, a dolgozatban szereplő 'kvantum grupoidok' Hopf-algebrák különféle általánosításai nem feltétlenül kommutatív (de mindig asszociatív és egységelemes) alap gyűrűk esetére. Akárcsak a kvantum csoportok esetén, az elnevezés eredete a motivációul szolgáló példa: egy (véges sok objektummal rendelkező) grupoid elemei által kifeszített vektortér. Noha ezen példában az identitás morfizmusok által generált alapgyűrű kommutatív, megfigyelhetjük azt a nem kommutatív vonást, hogy jobb és bal hatása a grupoidon különböző.

Az irodalomban előforduló hasonló fogalmak közül kettővel foglalkozunk részletesen. A Szlachányi Kornéllal közösen bevezetett *Hopf-algebroidokkal* és az ezek speciális esetét jelentő, de történetileg előbb, szintén Szlachányival együttműködésben született *gyenge Hopf-algebrákkal.* Mindkettőt általánosítja a Schauenburg által javasolt '$\times_R$-Hopf-algebra' [66], másfelől a gyenge Hopf-algebra általánosabb, mint a Hayashi féle ' lap algebra' (face algebra) [42], a Yamanouchi féle általánosított Kac-algebra [73] és az Ocneanu féle 'para csoport' [60]. Hopf-algebroidok (de nem feltétlenül gyenge Hopf-algebrák) az algebrai topológiában szintén Hopf-algebroidnak nevezett kogrupoid objektumok a kommutatív algebrák kategóriájában [64, Appendix I].

Noha a Hopf-algebrák vizsgálata rendkívül gazdag és sikeres tudományterület immár több, mint ötven éve, az 1990-es években egyre több és több kérdés motiválta egy általánosabb struktúra bevezetését. A Poisson geometriában a dinamikai Yang-Baxter egyenlet megoldásai ún. dinamikai kvantum csoportokkal kapcsolatosak, melyek nem Hopf-algebrák. A topológiában bizonyos újabb invariánsok nem származtathatók Hopf-algebrákból. A Connes féle nem kommutatív geometriában Hopf-algebrák nem kommutatív algebrákkal való kiterjesztései jelentek meg szimmetriaként. Faktorok véges mélységű, de reducibilis kiterjesztései (az irreducibilis esettel szemben) nem írhatók Hopf-algebrákkal vett kereszt szorzatként. Az alacsony dimenziós kvantumtérelméletekben a nem egész értékű kvantum dimenziók fellépése kizárja, hogy a szuperszelekciós szimmetriát Hopf-algebra írja le. A klasszikus Galois-elmélet Hopf-algebrai általánosításának gyenge pontja, hogy egy Hopf–Galois-kiterjesztés nem jellemezhető a szimmetriát leíró Hopf-algebrára való explicit hivatkozás nélkül, tisztán a kiterjesztés tulajdonságainak megfogalmazásával. Mint az utóbbi évtizedben kiderült, mindezen kérdésekben sikeresen alkalmazhatók a Hopf-algebroidok.

A dolgozat eredményei három fő téma köré csoportosíthatók. Az első a Hopf-algebroidok tisztán algebrai axiomatikus tárgyalása. A második alkalmazásuk a nem kommutatív Galois-elméletben. Harmadikként olyan kategóriaelméleti eredményeket ismertetünk, melyek gyenge Hopf-algebrákhoz kapcsolódó konstrukciókat helyeznek el egy tágabb fogalomkörben illetve általánosítják azokat. Keletkezési idejüket tekintve a kiválasztott publikációk a szerző munkásságának bő tíz évét ölelik fel, a PhD fokozat megszerzésétől a legutóbbi időkig.

Köszönetnyilvánítás. Köszönetet szeretnék mondani minden társszerzőmnek, akár a dolgozatban szereplő témákon, akár más problémákon dolgoztunk együtt. Legelőször is Szlachányi Kornélnak, egykori doktori témavezetőmnek és első társszerzőmnek, akivel mindenkinél hosszabb ideig dolgoztam együtt és akitől mindenkinél többet tanultam. A többiek, akikkel mind öröm volt együtt dolgozni, Alessandro Ardizzoni, Bajnok Zoltán, Tomasz Brzeziński, Steve Lack, Claudia Menini, Dragoş Ştefan, Ross Street, Takács Gábor, Joost Vercruysse és Robert Wisbauer. Köszönettel tartozom még számos kollégának, akikkel ugyan (talán még) nem publikáltunk közösen, de akikkel való beszélgetésekből nagyon sokat tanulhattam. Ők Stef Caenepeel, José Gómez Torrecillas, Piotr M. Hajac, Lars Kadison, Laiachi El Kaoutit, Zoran Škoda és Vecsernyés Péter. Nagy-nagy köszönettel tartozom családomnak, akik megértése és szerető támogatása nélkül munkám nem lenne lehetséges. Férjemnek, aki mindig természetesen és szívesen vállalt egyenlő szerepet az otthoni feladatokban, szüleimnek, akik rengeteg terhet vettek és vesznek le a vállamról, valamint három gyerekemnek, akik megértik, bár derűsen mosolyognak a mama furcsa szenvedélyén a matematika iránt...

Tartalomjegyzék

1. fejezet

Bevezetés

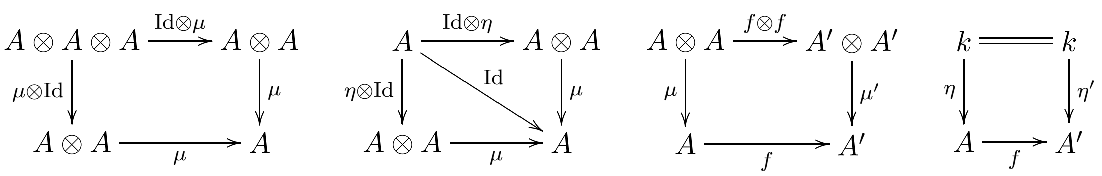
Ezen bevezető fejezet kettős céllal íródott. Egyfelől, hogy tömör áttekintést adjon mindazon fogalmakról és előismeretekről, melyekre a dolgozat későbbi fejezetei (a kiválasztott publikációk) épülnek. Másfelől, összegyűjtöttük itt a későbbi fejezetben bemutatásra kerülő legfontosabb eredményeket. Ezen tételek teljes bizonyítása megtalálható a 2-7. fejezetben mellékelt dolgozatokban. E bevezetés bizonyításokat nem tartalmaz. Ehelyett igyekeztünk itt felvázolni a munkánkat motiváló kérdéseket, a bizonyításokhoz felhasznált fő gondolatokat és elhelyezni ezeket az eredményeket a témakör irodalmában.

A fejezetben a következő általános érvényű jelöléseket használjuk. Mindvégig, k egy kommutatív, asszociatív egység elemes gyűrű. A díszítetlen $\otimes$ szimbólum k-modulusok tenzor szorzatát jelöli. Minden előforduló gyűrű asszociatív és egység elemes. Egy R gyűrű jobb- illetve bal modulusainak kategóriáját $\mathcal{M}_R$-rel illetve ${}_R\mathcal{M}$-rel jelöljük, $P \to Q$ homomorfizmusaik halmazát $\mathrm{Hom}_R(P,Q)$-val illetve ${}_R\mathrm{Hom}(P,Q)$-val. A bimodulus kategória jelölése ${}_R\mathcal{M}_R$, $P \to Q$ homomorfizmusainak halmazáé ${}_R\mathrm{Hom}_R(P,Q)$. Az R-modulus tenzor szorzatot $\otimes_R$ jelöli.

1.1. Néhány szó a Hopf-algebrák klasszikus elméletéről

Ebben a fejezetben röviden áttekintjük a Hopf-algebrák elméletének azon kérdéseit, melyek általánosításai szerepelnek a későbbiekben. Bővebb ismeretek forrásaként javasoljuk [70] és [54] monográfiákat.

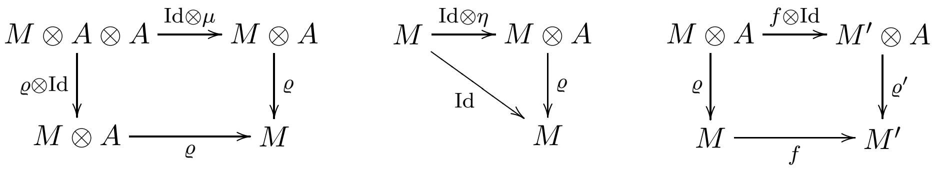
1.1. Definíció. Egy *k-algebra* (a továbbiakban csak *algebra*) egy k-modulus A, ellátva $\eta : k \to A$ (egység) és $\mu : A \otimes A \to A$ (szorzás) k-modulus leképezésekkel, melyekre az első két diagrammal ábrázolt asszociativitási és egység feltételek teljesülnek.



Egy $f : A \to A'$ k-modulus leképezést algebra homomorfizmusnak mondunk, ha a fenti második két diagram kommutatív.

Az a és b elemek szorzatát ab-vel jelöljük, míg k egység elemének η általi képét 1-gyel. Egy algebra *ellentettje* ugyanazon k-modulus A a fordított sorrendű $a \otimes b \mapsto ba$ szorzással.

1.2. Definíció. Egy A algebra *jobb modulusai* M k-modulusok, ellátva egy $\varrho : M \otimes A \to M$ k-modulus leképezéssel, melyre az első két diagrammal ábrázolt asszociativitási és egység feltételek teljesülnek.



Egy $f : M \to M'$ k-modulus leképezést A*-modulus homomorfizmusnak* mondunk, ha a fenti harmadik diagram kommutatív.

Tetszőleges $m \in M$, $a \in A$ elemeken a $\varrho(m \otimes a) = m.a$ jelölést használjuk. Szimmetrikusan értelmezzük a bal modulusokat és homomorfizmusaikat.

A koalgebra az algebra fogalom duálisa:

1.3. Definíció. Egy k*-koalgebra* (a továbbiakban csak *koalgebra*) egy k-modulus C, ellátva $\varepsilon : C \to k$ (koegység) és $\delta : C \to C \otimes C$ (koszorzás) k-modulus leképezésekkel. Az ezekre vonatkozó koasszociativitási és koegység feltételeket úgy kapjuk, hogy az algebra axiómák diagramjaiban minden nyilat megfordítunk. A *koalgebra homomorfizmusok* a koszorzással és koegységgel kompatibilis lineáris leképezések.

Egy c elem koszorzatára a Sweedler-Heynemann index jelölest használjuk; $\delta(c) = c_1 \otimes c_2$, ami alatt véges sok tag összege értendő $C \otimes C$-ben. Az összegzést nem jelöljük expliciten, arra csak az indexek jelenléte utal. A megkövetelt koasszociativitás miatt az egyenlő $c_{11} \otimes c_{12} \otimes c_2$ és $c_1 \otimes c_{21} \otimes c_{22}$ kifejezések helyett írhatunk egyszerűen $c_1 \otimes c_2 \otimes c_3$-at. Egy koalgebra *ellentettje* ugyanazon k-modulus C a fordított sorrendű $c \mapsto c_2 \otimes c_1$ koszorzással. Bármely C-koalgebra $\mathrm{Hom}(C, k)$ duálisa algebra a transzponált struktúrával. Ha egy algebra A végesen generált projektív k-modulus (így $\mathrm{Hom}(A \otimes A, k) \cong \mathrm{Hom}(A, k) \otimes \mathrm{Hom}(A, k)$), akkor $\mathrm{Hom}(A, k)$ koalgebra a transzponált struktúrával.

1.4. Definíció. Egy C koalgebra *jobb komodulusai* M k-modulusok, ellátva egy $\varrho : M \to M \otimes C$ k-modulus leképezéssel, az ún. *kohatással*, melyre vonatkozó koasszociativitási és koegység feltételeket úgy kapjuk, hogy a megfelelő algebrai axiómák diagramjaiban megfordítjuk a nyilakat. Egy $f : M \to M'$ k-modulus leképezést C*-komodulus homomorfizmusnak* mondunk, ha a modulus homomorfizmus definíciójának megfelelő diagram a nyilak megfordításával kommutatív.

A jobb C komodulusok és homomorfizmusaik alkotta kategóriát $\mathcal{M}^C$-vel jelöljük, a $P \to Q$ homomorfizmusok halmazát $\mathrm{Hom}^C(P, Q)$-val. Szimmetrikusan értelmezzük és jelöljük a bal komodulusok ${}^C\mathcal{M}$ kategóriáját. Hasonlóan a koszorzáshoz, a (jobb) kohatásra is implicit összegzést jelentő index jelölést használunk: $\varrho(m) = m_0 \otimes m_1$. A megkövetelt koasszociativitás miatt az egyenlő $m_{00} \otimes m_{01} \otimes m_1$ és $m_0 \otimes m_{11} \otimes m_{12}$ kifejezések helyett írhatunk egyszerűen $m_0 \otimes m_1 \otimes m_2$-t.

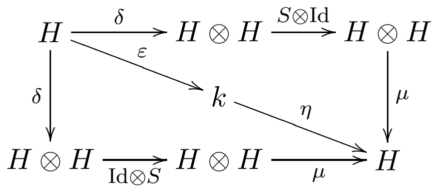
1.5. Definíció. Egy k*-bialgebra* (a továbbiakban csak *bialgebra*) egy k-modulus B, ellátva egy (η, μ) algebra struktúrával és egy (ε, δ) koalgebra struktúrával úgy, hogy ε és δ algebra homomorfizmusok (vagy ami evvel ekvivalens, η és μ koalgebra homomorfizmusok). A *bialgebra homomorfizmusok* egyszerre algebra- és koalgebra homomorfizmusok.

Tetszőleges $a, b \in B$ elemeken kiírva, a bialgebra axiómák a következők:

$$1_1 \otimes 1_2 = 1 \otimes 1, \qquad (ab)_1 \otimes (ab)_2 = a_1 b_1 \otimes a_2 b_2, \qquad \varepsilon(1) = 1, \qquad \varepsilon(ab) = \varepsilon(a)\varepsilon(b).$$

Egy bialgebrában akár a szorzást, akár a koszorzást az ellentettjére cserélve ismét bialgebrát kapunk. Ha egy bialgebra B végesen generált projektív k-modulus, akkor a lineáris duális $\mathrm{Hom}(B, k)$ is bialgebra. Az algebra struktúra a koalgebra struktúra transzponáltjaként, a koalgebra struktúra pedig az algebra struktúra transzponáltjaként adódik.

1.6. Definíció. Egy *k-Hopf-algebra* (a továbbiakban csak *Hopf-algebra*) egy bialgebra H ellátva egy *antipódnak* nevezett $S : H \to H$ k-modulus leképezéssel, melyre az alábbi feltételek teljesülnek.



A *Hopf-algebra homomorfizmusok* az $f : H \to H'$ bialgebra homomorfizmusok. Ezek szükségképpen őrzik az antipódokat, azaz $fS = S'f$.

Tetszőleges $a \in H$ elemen kiírva, a Hopf-algebra axiómák a következők:

$$a_1 S(a_2) = \varepsilon(a)1 = S(a_1)a_2.$$

A $H \to H$ k-modulus homomorfizmusok halmaza algebrává tehető az $f \otimes g \mapsto \mu(f \otimes g)\delta$ ún. konvolúció szorzással Az antipód axiómák olvashatók úgy, hogy ebben az algebrában S inverze az identitás leképezésnek.

Ha az antipód létezik, akkor egyértelmű, továbbá (ko)algebra homomorfizmus H-ból az ellentett (ko)algebrába. Ha H végesen generált projektív k-modulus, akkor igazolható, hogy az antipód bijektív és a lineáris duális $\mathrm{Hom}(H, k)$ is Hopf-algebra. Az duális antipód H antipódjának transzponáltjaként adódik.

1.7. Példa. Legyen G egy tetszőleges csoport. A G elemei által generált szabad k-modulus, kG ellátható Hopf-algebra struktúrával a következőképpen. A szorzást a csoport szorzás lineáris kiterjesztéseként definiáljuk, a koszorzást pedig a csoport elemeken diagonális $g \mapsto g \otimes g$ leképezés lineáris kiterjesztéseként. Az egység elem tehát a csoport egysége lesz, a koegység a csoport elemeken azonosan 1 függvény lineáris kiterjesztése, míg az antipód a csoport inverz műveletének lineáris kiterjesztése. Ha G véges csoport, akkor a lineáris duális $k(G)$ – azaz a $G \to k$ függvények által generált k-modulus – is Hopf-algebra a transzponált struktúrával.

1.1.1. Bialgebrák (ko)modulusai

A k-bialgebrák jellemezhetők, mint pontosan azok a (ko)algebrák, melyek (ko)modulusainak kategóriája monoidális, méghozzá úgy, hogy a felejtő funktor a (ko)modulus kategóriából a k-modulusok kategóriájába szigorúan monoidális.

Konkrétan, mivel a koegység algebra homomorfizmus, bármely B bialgebrának jobb (vagy szimmetrikusan bal) modulusa maga az alapgyűrű:

$$\kappa.h = \kappa\varepsilon(h), \qquad \kappa \in k, \; h \in B.$$

A koegység kiolvasható B hatásából k-n. Mivel a koszorzás algebra homomorfizmus, bármely M és N jobb B-modulusok esetén $M \otimes N$ is jobb B-modulus az ún. *diagonális hatás* révén:

$$(m \otimes n).h = m.h_1 \otimes n.h_2, \qquad m \in M,\ n \in N,\ h \in B.$$

A koszorzás kiolvasható B hatásából $B \otimes B$-n. A k-modulusok kategóriájának $(M \otimes N) \otimes P \cong M \otimes (N \otimes P)$ asszociátor izomorfizmusai és $M \otimes k \cong M \cong k \otimes M$ egység izomorfizmusai B-modulus homomorfizmusok, a koalgebra axiómáknak köszönhetően.

Ugyanígy, mivel az egység koalgebra homomorfizmus, bármely B bialgebrának jobb (vagy szimmetrikusan bal) komodulusa maga az alapgyűrű:

$$k \to k \otimes B \cong B \qquad \kappa \mapsto \kappa 1, \qquad \kappa \in k.$$

Az egység kiolvasható B kohatásából k-n. Mivel a szorzás koalgebra homomorfizmus, bármely M és N jobb B-komodulusok esetén $M \otimes N$ is jobb B-komodulus az ún. *diagonális kohatás* révén:

$$M \otimes N \to M \otimes N \otimes B \quad m \otimes n \mapsto (m \otimes n)_0 \otimes (m \otimes n)_1 := m_0 \otimes n_0 \otimes m_1 n_1, \quad m \in M,\ n \in N.$$

A szorzás kiolvasható B kohatásából $B \otimes B$-n. A k-modulusok kategóriájának $(M \otimes N) \otimes P \cong M \otimes (N \otimes P)$ asszociátor izomorfizmusai és $M \otimes k \cong M \cong k \otimes M$ egység izomorfizmusai B-komodulus homomorfizmusok, az algebra axiómáknak köszönhetően.

1.1.2. Hopf-algebrák integrálelmélete

Az integrálok a Hopf-algebrák kitüntetett elemei, melyeknek vizsgálatával fontos információk nyerhetők az algebrai szerkezetről.

1.8. Definíció. Legyen B egy bialgebra és M egy bal B-modulus. *M invariánsainak* hívjuk az

$$M^{inv} := \{n \in M | \forall a \in B,\ a.n = \varepsilon(a)n\} \cong {}_B\mathrm{Hom}(k, M)$$

k-részmodulus elemeit. Szimmetrikusan értelmezzük a jobb modulusok invariánsait. A *bal integrálok* alatt B-ben a bal reguláris ábrázolás invariánsait értjük, azaz a

$$B^{inv} := \{l \in B | \forall a \in B,\ al = \varepsilon(a)l\}$$

jobb ideál elemeit. Szimmetrikusan, a jobb integrálok a jobb reguláris ábrázolás invariánsai.

1.9. Példa. Egy véges G csoport elemei által generált kG Hopf-algebrában az integrálok a $\sum_{g \in G} g$ Haar elem számszorosai.

Maschke klasszikus, csoport algebrák félig egyszerűségére vonatkozó tételét általánosítva, a következő igazolható.

1.10. Tétel (Sweedler [70], Caenepeel–Militaru [27]). *Egy H Hopf-algebrára vonatkozó következő állítások ekvivalensek.*

- *H szeparábilis k-algebra;*
- *H félig egyszerű k fölött (azaz egy bal vagy jobb H-modulus epimorfizmus pontosan akkor felhasadó, ha mint k-modulus epimorfizmus felhasadó);*
- *A koegység ε felhasadó (jobb vagy bal) H-modulus epimorfizmus;*

- *létezik H-ban egy (jobb vagy bal) integrál l, ami normált az $\varepsilon(l) = 1$ értelemben.*

Egy másik fontos vonás, melyet az integrálok segítségével vizsgálhatunk, a (kvázi) Frobenius-tulajdonság. (Egy algebra A rendelkezik a Frobenius-tulajdonsággal, ha végesen generált projektív k-modulus és izomorf $\mathrm{Hom}(A, k)$-val mint (ekvivalens módon jobb vagy bal) A-modulus. Jobb illetve bal kvázi Frobenius algebráról beszélünk, ha A végesen generált projektív k-modulus és kvázi-izomorf $\mathrm{Hom}(A, k)$-val mint jobb illetve bal A-modulus [55].) Mint Pareigis igazolta, egy Hopf-algebra pontosan akkor jobb kvázi-Frobenius- algebra ha bal kvázi-Frobenius-algebra, ami pontosan akkor teljesül, ha végesen generált projektív k-modulus [61]. A Frobenius-tulajdonságról a következő tudható.

1.11. Tétel (Larson–Sweedler [49], Pareigis [61])**.** *Egy H Hopf-algebrára vonatkozó következő állítások ekvivalensek.*

- *H Frobenius-k-algebra;*
- $\mathrm{Hom}(H, k)$ *Frobenius-k-algebra;*
- *létezik H-ban egy (jobb vagy bal) integrál l, amely nem degenerált abban az értelemben, hogy a következő leképezések bijektívek.*

$$\mathrm{Hom}(H, k) \to H \qquad \phi \mapsto (\mathrm{Id} \otimes \phi)\delta(l) \equiv l_1\phi(l_2)$$
$$\mathrm{Hom}(H, k) \to H \qquad \phi \mapsto (\phi \otimes \mathrm{Id})\delta(l) \equiv \phi(l_1)l_2$$

Ha k Picard csoportja triviális (például k egy test), akkor a fenti tulajdonságok pontosan akkor teljesülnek, ha H végesen generált projektív (véges dimenziós).

1.1.3. Hopf–Galois-elmélet

Nem kommutatív algebrák Hopf-algebrákkal való kiterjesztéseit – mint a kommutatív gyűrűk csoporttal való kiterjesztésének általánosításait – először Chase és Sweedler tanulmányozta [30] munkájában illetve Kreimer és Takeuchi [47]-ben. A Hopf–Galois-elmélet egységes leírását adja korábban függetlenül vizsgált struktúráknak (pl. a csoport gradált algebráknak). Algebrai vonatkozásaik mellett a Hopf–Galois-kiterjesztések a nem kommutatív geometria nézőpontjából is érdekesek, ahol a principális nyalábok nem kommutatív megfelelőiként jelennek meg.

Az 1.1.1 fejezetben láttuk, hogy egy H bialgebra jobb komodulusainak $\mathcal{M}^H$ kategóriája monoidális. A monoidokat ebben a kategóriában *H-komodulus algebráinak* hívjuk. Expliciten, egy H-komodulus algebra egy algebra A, aminek szorzása és egysége H-komodulus homomorfizmus. Azaz

$$1_0 \otimes 1_1 = 1 \otimes 1; \qquad (a'a)_0 \otimes (a'a)_1 = a'_0 a_0 \otimes a'_1 a_1, \qquad a', a \in A.$$

A komodulus invariánsai (röviden *koinvariánsai*) alatt a

$$A^{coH} := \{b \in A | b_0 \otimes b_1 = b \otimes 1\}$$

részalgebra elemeit értjük. Dolgozatunkban mindvégig jobb komodulus algebrákkal dolgozunk. Ez nem jelenti a szimmetria megsértését, hiszen B bal komodulusai jobb komodulusai annak a bialgebrának melyben (az algebra struktúrát megfordítva vagy változatlanul hagyva) B koalgebra struktúráját az ellentettjére cseréljük.

1.12. Definíció. Legyen H egy bialgebra. Egy $B \subseteq A$ algebra kiterjesztést *jobb H-Galois-kiterjesztésnek* mondunk, ha A jobb H-komodulus algebra, $B = A^{coH}$ és az alábbi kanonikus leképezés izomorfizmus.

$$\text{can} : A \otimes_B A \to A \otimes H \qquad a' \otimes_B a \mapsto a'a_0 \otimes a_1. \tag{1.1}$$

1.13. Példa. A klasszikus Galois test bővítések véges csoportokkal Hopf–Galois-kiterjesztések, l. [33, Example 6.4.3]. Legyen G egy véges csoport és F egy test amin G automorfizmusokkal hat, azaz létezik egy csoport homomorfizmus $G \to \text{Aut}(F)$, $g \mapsto \alpha_g$. Bármely $k \subseteq F$ testre F $k(G)$-komodulus algebra az $a \mapsto \sum_{g\in G} \alpha_g(a) \otimes \delta_g$ kohatás révén, ahol $\delta_g \in k(G)$ értéke g-n 1, az összes többi csoport elemen 0. A koinvariáns részalgebra $F^{co\, k(G)}$ elemei éppen a G-hatás invariánsai és $F^{co\, k(G)} \subseteq F$ egy $k(G)$-Galois-kiterjesztés.

1.14. Példa. Ez a példa a Hopf–Galois-kiterjesztések geometriai interpretációjával kapcsolatos. Hasson egy véges G csoport balról egy véges X halmazon. Az X-en értelmezett függvények $k(X)$ algebrája (a pontonkénti szorzással) $k(G)$-komodulus algebra az $f \mapsto \sum_{g\in G} f(g.-)\otimes\delta_g$ kohatás révén. A $k(X)^{co\, k(G)}$ koinvariáns részalgebra elemei pontosan a G-pályákon konstans függvények. $k(X)^{co\,(G)} \subseteq k(X)$ pontosan akkor $k(G)$-Galois-kiterjesztés, ha a G-hatás X-en szabad és tranzitív, azaz X bármely x, y elemeire létezik pontosan egy g elem G-ben, amire $y = x.g$. (Ez esetben persze $k(X)^{co\, k(G)} \cong k$.)

1.15. Példa. Bármely B bialgebra maga is B-komodulus algebra a koszorzás mint kohatás révén. A B^{coB} koinvariáns részalgebra elemei pontosan az egység elem számszorosai. A $k \subseteq B$ kiterjesztés akkor és csakis akkor B-Galois ha B Hopf-algebra.

Számos olyan tétel ismert, amely a kanonikus leképezés bijektivitására vonatkozó elégséges feltételeket fogalmaz meg. Példaként álljon itt egy klasszikus.

1.16. Tétel (Kreimer–Takeuchi [47]). *Legyen H egy véges dimenziós Hopf-algebra tetszőleges test fölött, és legyen A egy jobb H-komodulus algebra. Ezen feltevések mellett az* (1.1) *kanonikus leképezés pontosan akkor bijektív ha szürjektív.*

Tekintsünk egy H bialgebrát és egy A jobb H-komodulus algebrát. Definíció szerint, A egy monoid $\mathcal{M}^H$-ban így tekinthetjük a jobb A-modulusok kategóriáját $\mathcal{M}^H$-ban. Ezen modulusokat *relatív Hopf-modulusoknak* hívjuk. Expliciten, egy (A, H) relatív Hopf-modulus egy jobb A-modulus és jobb H-komodulus M úgy, hogy az A-hatás H-komodulus homomorfizmus, azaz

$$(m.a)_0 \otimes (m.a)_1 = m_0.a_0 \otimes m_1a_1, \qquad m \in M,\ a \in A.$$

Az (A, H) relatív Hopf-modulusok homomorfizmusai jobb A-modulus és jobb H-komodulus homomorfizmusok. A relatív Hopf-modulusok és homomorfizmusaik kategóriáját $\mathcal{M}_A^H$-val jelöljük. A koinvariánsainak részalgebráját B-vel jelölve, az $(-) \otimes_B A : \mathcal{M}_B \to \mathcal{M}_A$ bal adjungált funktor rendelkezik egy $(-) \otimes_B A : \mathcal{M}_B \to \mathcal{M}_A^H$ felhúzással (azaz bármely P jobb B-modulus esetén $P \otimes_B A$ relatív Hopf-modulus A nyilvánvaló relatív Hopf-modulus struktúrája révén). A felhúzott funktor jobb adjungáltja $(-)^{coH} : \mathcal{M}_A^H \to \mathcal{M}_B$. A Hopf–Galois-elmélet központi kérdése, hogy ez a funktor mikor teli és hű (gyenge struktúra-tételek) illetve ekvivalencia (erős struktúra tételek). Történetileg az egyik első ilyen tétel a következő volt.

1.17. Tétel (A Hopf-modulusok alaptétele [70]). *Bármely H Hopf-algebrára $(-) \otimes H$: $\mathcal{M}_k \to \mathcal{M}_H^H$ ekvivalencia, azaz a reguláris komodulus algebrára az erős struktúra-tétel teljesül.*

Ha H végesen generált projektív k-modulus, akkor bármely A jobb H-komodulus algebra esetén található egy alkalmas algebra, melynek modulus kategóriája izomorf a relatív (A, H)-Hopf-modulusok kategóriájával. Ez a keresett algebra a duális, $H^* := \mathrm{Hom}(H, k)$ és A féldirekt szorzataként írható. Mint k-modulus, megegyezik $A \otimes H^*$-val. A szorzás

$$(a \otimes \phi)(b \otimes \psi) = a_0 b \otimes (a_1.\psi)\phi,$$

ahol $h.\psi = \phi(-h)$ H bal hatása H^*-on. Ebben az esetben tehát az erős és gyenge struktúratételek modulus kategóriák összehasonlítására vezetnek, így a Morita-elmélet alkalmazásával bizonyíthatóak.

1.1.4. Doi–Hopf-modulusok

Az 1.1.3 fejezetben egy B bialgebra (jobb vagy bal) komodulus algebráit úgy definiáltuk, mint monoidokat a (jobb vagy bal) B-komodulusok monoidális kategóriájában. Szimmetrikusan definiálhatunk (jobb vagy bal) B-*modulus koalgebrákat* mint komonoidokat a (jobb vagy bal) B-modulusok monoidális kategóriájában.

1.18. Definíció (Doi [36], Koppinen [46])**.** (Jobb-jobb) *Doi–Hopf-adatok* alatt olyan (A, B, C) hármasokat értünk, ahol B egy bialgebra, A egy jobb B-komodulus algebra és C egy jobb B-modulus koalgebra.

Elegendő jobb-jobb Doi–Hopf-adatokat bevezetnünk, hiszen a többi lehetőséget megkapjuk, ha B (ko)algebra struktúráját az ellentettjére cseréljük, és az így nyert bialgebra jobb-jobb Doi–Hopf-adatait tekintjük.

Bármely (A, B, C) Doi–Hopf-adat esetén tekinthetjük az alábbi k-modulus leképezést:

$$\psi : C \otimes A \to A \otimes C \qquad c \otimes a \mapsto a_0 \otimes c.a_1,$$

mely kommutatívvá teszi az alábbi diagramokat.

$$\begin{array}{ccc} C \otimes A \otimes A & \xrightarrow{\mathrm{Id}\otimes\mu} & C \otimes A \\ \downarrow{\scriptstyle \psi\otimes\mathrm{Id}} & & \\ A \otimes C \otimes A & & \downarrow{\scriptstyle \psi} \\ \downarrow{\scriptstyle \mathrm{Id}\otimes\psi} & & \\ A \otimes A \otimes C & \xrightarrow[\mu\otimes\mathrm{Id}]{} & A \otimes C \end{array} \quad \begin{array}{ccc} C & \xrightarrow{\mathrm{Id}\otimes\eta} & C \otimes A \\ \| & & \downarrow{\scriptstyle \psi} \\ C & \xrightarrow[\eta\otimes\mathrm{Id}]{} & A \otimes C \end{array} \quad \begin{array}{ccc} C \otimes A & \xrightarrow{\delta\otimes\mathrm{Id}} & C \otimes C \otimes A \\ & & \downarrow{\scriptstyle \mathrm{Id}\otimes\psi} \\ \downarrow{\scriptstyle \psi} & & C \otimes A \otimes C \\ & & \downarrow{\scriptstyle \psi\otimes\mathrm{Id}} \\ A \otimes C & \xrightarrow[\mathrm{Id}\otimes\delta]{} & A \otimes C \otimes C \end{array} \quad \begin{array}{ccc} C \otimes A & \xrightarrow{\varepsilon\otimes\mathrm{Id}} & A \\ \downarrow{\scriptstyle \psi} & & \| \\ A \otimes C & \xrightarrow[\mathrm{Id}\otimes\varepsilon]{} & A \end{array}$$

A kategóriaelméletben szokásos terminológiával ez azt jelenti, hogy ψ egy *vegyes* (monádot és komonádot összekulcsoló) *disztributív szabály* [7]. Ugyanerre a Hopf-algebrai szakirodalom az *összekulcsoló struktúra* (entwining structure) elnevezést használja [24]. [7] szerint minden $C \otimes A \to A \otimes C$ vegyes disztributív szabály indukál egy $(-) \otimes A$ monádot a C-komodulusok kategóriáján (az $(\mathcal{M}_k, (-)\otimes A)$ monád felhúzását) és egy $(-)\otimes C$ komonádot az A-modulusok kategóriáján (a $(\mathcal{M}_k, (-) \otimes C)$ komonád felhúzását). Ezen monád és komonád Eilenberg–Moore-kategóriái izomorfak és szokásos jelölésük $\mathcal{M}_A^C$. Expliciten, objektumai $\varrho_A : M \otimes A \to M$ jobb A-modulusok és $\varrho^C : M \to M \otimes C$ jobb C-komodulusok, amikre az alábbi diagram kommutatív.

$$\begin{array}{ccccc} M \otimes A & & \xrightarrow{\qquad\quad \varrho_A \quad\qquad} & & M \\ \downarrow{\scriptstyle \varrho^C\otimes\mathrm{Id}} & & & & \downarrow{\scriptstyle \varrho^C} \\ M \otimes C \otimes A & \xrightarrow[\mathrm{Id}\otimes\psi]{} & M \otimes A \otimes C & \xrightarrow[\varrho_A\otimes\mathrm{Id}]{} & M \otimes C \end{array}$$

A morfizmusok A-modulus homomorfizmusok és C-komodulus homomorfizmusok. Ha A a triviális B-komodulus algebra k, akkor $\mathcal{M}_A^C$ megegyezik a C-komodulusok kategóriájával. Ha C a triviális B-modulus koalgebra k, akkor $\mathcal{M}_A^C$ megegyezik az A-modulusok kategóriájával. Ha C a reguláris B-modulus koalgebra B, akkor $\mathcal{M}_A^C$ megegyezik az (A,B)-relatív Hopf-modulusok kategóriájával. Számos érdekes eset megkapható még A és C alkalmas választásával, például az ún. Long dimodulusok vagy a Yetter–Drinfel'd-modulusok kategóriája. Ha C végesen generált projektív k-modulus, akkor konstruálható egy alkalmas algebra, melynek modulus kategóriája izomorf $\mathcal{M}_A^C$-val. Ez a keresett algebra a duális, $C^* := \mathrm{Hom}(C,k)$ és A féldirekt szorzataként írható.

Az Eilenberg–Moore-kategóriák általános tulajdonságai szerint az $\mathcal{M}_A^C \to \mathcal{M}^C$ felejtő funktor bal adjungáltja $(-)\otimes A : \mathcal{M}^C \to \mathcal{M}_A^C$, ahol az A-hatás $M\otimes A$-n $M\otimes\mu : M\otimes A\otimes A \to M\otimes A$ és a C-kohatás

$$M\otimes A \xrightarrow{\varrho^C\otimes \mathrm{Id}} M\otimes C\otimes A \xrightarrow{\mathrm{Id}\otimes\psi} M\otimes A\otimes C\,.$$

Szimmetrikusan, az $\mathcal{M}_A^C \to \mathcal{M}_A$ felejtő funktor jobb adjungáltja $(-)\otimes C : \mathcal{M}_A \to \mathcal{M}_A^C$, ahol a C-kohatás $M\otimes C$-n $M\otimes\delta : M\otimes C \to M\otimes C\otimes C$ és az A-hatás

$$M\otimes C\otimes A \xrightarrow{\mathrm{Id}\otimes\psi} M\otimes A\otimes C \xrightarrow{\varrho_A\otimes \mathrm{Id}} M\otimes C\,.$$

1.1.5. Hopf-algebrai konstrukciók és a monádok formális elmélete

Mint arra néhol utaltunk is, az 1.1.1 és az 1.1.4 fejezetekben szereplő állítások közül sok következik a monádok jóval általánosabb ún. *formális elméletéből* [69], [48].

Bármely $\mathcal{K}$ bikategóriához hozzárendelhetjük a $\mathcal{K}$-beli monádok $\mathrm{EM}(\mathcal{K})$ bikategóriáját [48]. Ennek objektumai (vagy 0-cellái) a $\mathcal{K}$-beli monádok, azaz $t : A \to A$ 1-cellák ellátva $\eta : \mathrm{Id}_A \to t$ (egység) és $\mu : tt \to t$ (szorzás) 2-cellákkal, melyekre a szokásos asszociativitási és egység feltételek teljesülnek:

$$\begin{array}{ccc} ttt & \xrightarrow{\mathrm{Id}\,\mu} & tt \\ {\scriptstyle \mu\,\mathrm{Id}}\downarrow & & \downarrow{\scriptstyle \mu} \\ tt & \xrightarrow[\mu]{} & t \end{array} \qquad\qquad \begin{array}{ccc} t & \xrightarrow{\mathrm{Id}\,\eta} & tt \\ {\scriptstyle \eta\,\mathrm{Id}}\downarrow & \searrow^{\mathrm{Id}} & \downarrow{\scriptstyle \mu} \\ tt & \xrightarrow[\mu]{} & t \end{array} \tag{1.2}$$

Az $(A,t) \to (A',t')$ 1-cellák $\mathrm{EM}(\mathcal{K})$-ban $\mathcal{K}$-beli $x : A \to A'$ 1-cellák ellátva egy $\psi : t'x \to xt$ 2-cellával, melyre az alábbi diagramok kommutatívak.

$$\begin{array}{ccccc} t't'x & \xrightarrow{\mathrm{Id}\,\psi} & t'xt & \xrightarrow{\psi\,\mathrm{Id}} & xtt \\ {\scriptstyle \mu'\,\mathrm{Id}}\downarrow & & & & \downarrow{\scriptstyle \mathrm{Id}\,\mu} \\ t'x & & \xrightarrow[\psi]{} & & xt \end{array} \qquad\qquad \begin{array}{ccc} x & = & x \\ {\scriptstyle \eta'\,\mathrm{Id}}\downarrow & & \downarrow{\scriptstyle \mathrm{Id}\,\eta} \\ t'x & \xrightarrow[\psi]{} & xt \end{array} \tag{1.3}$$

Az $(x,\psi) \to (y,\phi)$ 2-cellák $\mathrm{EM}(\mathcal{K})$-ban $\mathcal{K}$-beli $\varrho : x \to yt$ 2-cellák, melyekre az alábbi feltétel teljesül.

$$\begin{array}{ccccc} t'x & \xrightarrow{\mathrm{Id}\,\varrho} & t'xt & \xrightarrow{\psi\,\mathrm{Id}} & xtt \\ {\scriptstyle \psi}\downarrow & & & & \downarrow{\scriptstyle \mathrm{Id}\,\mu} \\ xt & \xrightarrow[\varrho\,\mathrm{Id}]{} & xtt & \xrightarrow[\mathrm{Id}\,\mu]{} & xt \end{array} \tag{1.4}$$

EM($\mathcal{K}$) monádjait *koszorúknak* (wreath) hívják. A mondottak szerint a koszorú egy $(s,\psi):(A,t)\to(A,t)$ 1-cella EM($\mathcal{K}$)-ban, ellátva szorzással, mely $\nu: ss\to st$ 2-cella; és egységgel, ami $\vartheta:\mathrm{Id}\to st$ 2-cella $\mathcal{K}$-ban. Ha $((A,t),(s,\psi))$ koszorú, akkor (A,st) monád $\mathcal{K}$-ban, az (A,t) monád és az s 1-cella ún. *koszorú szorzata* [48].

1.19. Példa. Legyen (A,B,C) egy jobb-jobb Doi–Hopf-adat. Ez meghatároz egy $((k,A),(C^*,\psi))$ koszorút a Gyűrűk; Bimodulusok; Bimodulus homomorfizmusok Bim-mel jelölt bikategóriájában, ahol

$$\psi: A\otimes C^*\to C^*\otimes A, \qquad a\otimes\phi\mapsto\phi(-a_1)\otimes a_0;$$

a koszorú szorzása és egysége pedig

$$\nu: C^*\otimes C^*\to C^*\otimes A,\quad \phi\otimes\phi'\mapsto\phi'\phi\otimes 1 \quad\text{és}\quad \vartheta: k\to C^*\otimes A,\quad 1\mapsto\varepsilon\otimes 1,$$

ahol C^* szorzása $\phi'\otimes\phi\mapsto\phi'\phi:=(\phi'\otimes\phi)\delta$. Az adódó koszorú szorzat izomorf C^* és A féldirekt szorzatával (l. 1.1.3, 1.1.4 fejezet).

Legyenek (x,ψ) és (y,ϕ) párhuzamos 1-cellák EM($\mathcal{K}$)-ban. Feltehetjük azt a kérdést, hogy mely $\mathcal{K}$-beli $x\stackrel{\omega}{\to}y$ 2-cellák esetén lesz $x\stackrel{\omega}{\to}y\stackrel{\mathrm{Id}\,\eta}{\to}yt$ 2-cella EM($\mathcal{K}$)-ban (x,ψ)-ből (y,ϕ)-be. A válasz az, hogy pontosan akkor, ha a következő diagram kommutatív.

$$\begin{array}{ccc} t'x & \xrightarrow{\mathrm{Id}\,\omega} & t'y \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\phi} \\ xt & \xrightarrow[\omega\,\mathrm{Id}]{} & yt \end{array} \tag{1.5}$$

A $\mathcal{K}$-beli monádok, mint 0-cellák; az (1.3)-ban definiált 1-cellák; és az (1.5)-tel jellemzett $\mathcal{K}$-beli 2-cellák, mint 2-cellák; bikategóriát alkotnak, melynek szokásos jelölése Mnd($\mathcal{K}$). Ezen bikategória fontos szerepet játszik az ún. *felhúzás* problémában.

1.20. Definíció. Bármely $\mathcal{K}$ bikategória esetén tekintsük a „diagonális" $\mathcal{K}\to\mathrm{Mnd}(\mathcal{K})$ (szigorú) bifunktort, mely egy A objektumot az (A,Id) monádba, egy $x:A\to B$ 1-cellát az $(x,\mathrm{Id}):(A,\mathrm{Id})\to(B,\mathrm{Id})$ 1-cellába és egy $\omega:x\to y$ 2-cellát önmagába képez. Ha ennek van jobb adjungáltja, akkor azt mondjuk, hogy léteznek $\mathcal{K}$-ban az *Eilenberg–Moore objektumok.* Expliciten, A^t objektum $\mathcal{K}$-ban az (A,t) monád Eilenberg–Moore-objektuma, ha létezik egy

$$\mathcal{K}(B,A^t)\cong\mathrm{Mnd}(\mathcal{K})((B,\mathrm{Id}),(A,t))$$

B-ben természetes izomorfizmus.

1.21. Példa. Az Eilenberg–Moore-objektum definícióját a Cat ((kis) kategóriák; funktorok; természetes transzformációk 2-kategóriája) példa motiválja. Ha $t:A\to A$ monád Cat-ban, akkor A^t létezik és az ún. Eilenberg–Moore-algebrák kategóriája. Expliciten, objektumai (a,α) párok, ahol a objektum A kategóriában $\alpha: t(a)\to a$ pedig morfizmus A-ban melyre a szokásos asszociativitási és egység feltételek teljesülnek.

1.22. Példa. A gyűrűk; bimodulusok; bimodulus homomorfizmusok Bim-mel jelölt bikategóriájában is minden monádnak létezik Eilenberg–Moore objektuma. Ebben az esetben egy (A,t) monád leírható egy $\eta: A\to t$ gyűrű homomorfizmussal (l. bővebben az 1.2 fejezetben). Az Eilenberg–Moore objektum maga a t gyűrű.

1.23. Tétel (Street [69]). *Ha egy tetszőleges $\mathcal{K}$ bikategóriában az (A, t) monádnak létezik A^t Eilenberg–Moore-objektuma, akkor van $\mathcal{K}$-ban egy $f \dashv v : A^t \to A$ adjunkció, melyre $t = vf$; melynek egysége a monád η egysége; és melynek ϵ koegysége segítségével a monád szorzása $\mu = v\epsilon f$ alakban írható.*

1.24. Példa. Cat-ban $v : A^t \to A$ a felejtő funktor. Bal adjungáltja $f : A \to A^t$ egy a objektumot $(t(a), \mu(a))$-ba, egy h morfizmust $t(h)$-ba visz.

Bim-ben v-t és f-et úgy kapjuk, hogy t-t mint A-t bimodulust, illetve t-A bimodulust tekintjük.

A fenti Eilenberg–Moore-objektumok segítségével a *felhúzás* probléma a következőképpen fogalmazható meg. Legyen adott egy $\mathcal{K}$ bikategóriában egy (A, t) és egy (B, s) monád, melyeknek létezik A^t illetve B^s Eilenberg–Moore-objektuma. Egy $x : A \to B$ 1-cella felhúzásai alatt olyan $\overline{x} : A^t \to B^s$ 1-cellákat értünk, amikre a bal oldali (1-cellákra vonatkozó) diagram kommutatív.

$$\begin{array}{ccc} A^t & \xrightarrow{\overline{x}} & B^s \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle v} \\ A & \xrightarrow[x]{} & B \end{array} \qquad\qquad \begin{array}{ccc} v\overline{x} & = & xv \\ {\scriptstyle \mathrm{Id}\,\overline{\omega}}\downarrow & & \downarrow{\scriptstyle \omega\,\mathrm{Id}} \\ v\overline{y} & = & yv \end{array}$$

Hasonlóan, egy $\omega : x \to y$ 2-cella felhúzásai alatt olyan $\overline{\omega} : \overline{x} \to \overline{y}$ 2-cellákat értünk, amikre a jobb oldali (2-cellákra vonatkozó) diagram kommutatív.

1.25. Tétel (Street [69]). *Egy adott $x : A \to B$ 1-cella felhúzásai bijektív kapcsolatban vannak* Mnd($\mathcal{K}$) *(x, ψ) alakú 1-celláival. Egy $\omega : x \to y$ 2-cellának pontosan akkor van $\overline{\omega} : \overline{x} \to \overline{y}$ felhúzása, ha ω 2-cella* Mnd($\mathcal{K}$)*-ban a megfelelő (x, ψ) és (y, ϕ) 1-cellák között; ekkor a felhúzás egyértelmű.*

Street tételéből azonnal származtatható néhány klasszikus Hopf-algebrai eredmény.

1.26. Példa. Bármely adott B algebra esetén bijektív kapcsolat van az alábbi struktúrák között.

- B lehetséges bialgebra struktúrái;
- $\mathcal{M}_k$ monoidális struktúrájának lehetséges felhúzásai $\mathcal{M}_B$-re.

1.27. Példa. Bármely A algebra és C koalgebra esetén bijektív kapcsolat van az alábbi struktúrák között.

- Az $(\mathcal{M}_k, (-) \otimes A)$ monád lehetséges felhúzásai $\mathcal{M}^C$-re;
- az $(\mathcal{M}_k, (-) \otimes C)$ komonád lehetséges felhúzásai $\mathcal{M}_A$-ra;
- az A-t és C-t összekulcsoló vegyes disztributív szabályok (azaz $((k, A), (C, \psi))$ komonádok Mnd(Bim)-ben).

1.2. Hopf-algebroidok

A bialgebra fogalom általánosítása nem kommutatív alapgyűrűk esetére Takeuchi nevéhez fűződik és több, mint harminc éve fellelhető [72]. Takeuchi eredetileg a $\times_R$-bialgebra elnevezést használta de (elsősorban J.H. Lu [51] munkája nyomán) napjainkban a *bialgebroid* elnevezés elfogadottabb. A bialgebrák számos ekvivalens leírása közül sok szépen általánosítható bialgebroidokra. Például, egy R gyűrű feletti bialgebroidok jellemezhetők modulus kategóriájuk monoidalitásával és egy, az R bimodulusok kategóriájába menő felejtő funktor szigorú monoidalitásával.

Az irodalomban több alternatív javaslat is található arra, milyen további tulajdonságot követeljünk meg egy *Hopf-algebroidtól*. A Hopf-algebra antipódja a $H \to H$ identitás leképezés inverze az $(f * g)(h) := f(h_1)g(h_2)$ konvolúció szorzásra nézve. Egy R gyűrű feletti B bialgebroid esetén azonban az identitás leképezés nem eleme (az alap R-kogyűrű és az alap R-gyűrű közötti bimodulus leképezések alkotta) konvolúció algebrának. Ezen probléma áthidalására J.H. Lu az antipód axiómák megfogalmazásához a $B \otimes B \twoheadrightarrow B \otimes_R B$ epimorfizmus egy szelését használta [51], ami azonban sok szempontból nem természetes.

Szlachányi Kornéllal közös vállalkozásunk **célja** egy olyan Hopf-algebroid fogalom megalkotása volt, amely kategóriaelméleti szempontból is természetes, amely alkalmas a szimmetria leírására (minél több) olyan helyzetben, amikor a Hopf-algebrák már nem használhatóak, s amelyre a Hopf-algebrákra vonatkozó eredmények minél szélesebb köre kiterjeszthető. Az általunk javasolt definíció szerint egy Hopf-algebroidban egy adott algebrán nem egy, hanem két kompatibilis bialgebroid struktúra van jelen R illetve R^{op} fölött. Ennek megfelelően van két konvolúció algebra, melyeket egy Morita-összefüggés köt össze. Az identitás leképezés a Morita-összefüggés egyik bimodulusának eleme, az antipód a másiknak – s ezek egymás inverzei a Morita-összefüggés szorzására nézve [14]. A Hopf-algebroidokról írt első, [19] cikkben arra az esetre szorítkoztunk, amikor az így definiált antipód ráadásul bijektív, így az egyik bialgebroid struktúra redundáns információt jelent: megkapható a másikból az antipód és annak inverze által.

Mint Day és Street [34]-ben tárgyalta, ezek az axiómák annak felelnek meg, hogy a szigorúan monoidális felejtő funktor a B-modulusok kategóriájából az R-bimodulusok kategóriájába ráadásul őrzi a $*$-autonóm struktúrát is. Schauenburg [66]-ban egy általánosabb Hopf-algebroid fogalmat javasolt, melyben a felejtő funktor a zártságot (vagyis a belső homokat) őrzi. A Schauenburg féle definícióból nem következik azonban egy $H \to H$ antipód létezése.

1.2.1. Az építőkövek: R-gyűrűk; R-kogyűrűk; bialgebroidok

Egy k kommutatív gyűrű fölötti (ko)algebrák (ko)monoidok a k-modulusok monoidális kategóriájában. Egy tetszőleges, nem (feltétlenül) kommutatív R (k fölötti) algebra esetén a (jobb vagy bal) R-modulusok kategóriája nem monoidális. Tekinthetjük azonban az R-bimodulusok kategóriáját, mely monoidális az R-modulus tenzor szorzás révén; a monoidális egység a reguláris bimodulus R. (Hangsúlyozzuk azonban, hogy ez a keret módosítását jelenti a „triviális” $R = k$ esetben is – a k-bimodulusok kategóriája teljes részkategóriaként tartalmazza a k-modulusok kategóriáját.)

1.28. Definíció. A monoidokat az R-bimodulusok kategóriájában *R-gyűrűknek* hívjuk. Expliciten, egy R-gyűrű egy R-bimodulus A, ellátva $\mu : A \otimes_R A \to A$ (szorzás) és $\eta : R \to A$ (egység) R-bimodulus leképezésekkel, melyek a szokásos asszociativitási és egység feltételeknek tesznek eleget. R-gyűrűk *homomorfizmusai* a szorzással és egységgel kompatibilis R-bimodulus homomorfizmusok.

Vegyük észre, hogy egy R-gyűrű leírható egyetlen $R \to A$ algebra homomorfizmussal. Csakugyan, ha A egy R-gyűrű, akkor ellátható egy algebra struktúrával. Az egységet az R algebra $k \to R$ egységét az A R-gyűrű $R \to A$ egységével komponálva kapjuk, a szorzást pedig az $A \otimes A \twoheadrightarrow A \otimes_R A$ epimorfizmust az R-gyűrű $A \otimes_R A \to A$ szorzásával komponálva. Felhasználva, hogy mind η mind μ (jobb) R-modulus homomorfizmus és η (jobb) egység, könnyen látható, hogy η algebra homomorfizmus. Fordítva, egy $\eta : R \to A$ algebra homomorfizmus R-bimodulus struktúrát indukál A-n. A szorzása kiegyensúlyozott R-ben, így felírható, mint az $A \otimes A \twoheadrightarrow A \otimes_R A$ epimorfizmus és egy egyértelmű $\mu : A \otimes_R A \to A$ leképezés kompozíciója. A μ szorzás és az η egység révén A egy R-gyűrű.

1.29. Definíció. Egy A R-gyűrű *jobb modulusa* egy jobb R-modulus M, ellátva egy $M \otimes_R A \to M$ jobb R-modulus leképezéssel, mely a szokásos asszociativitási és egység feltételeknek tesz eleget. Az előző bekezdés gondolatmenetéhez hasonlóan könnyű belátni, hogy ez éppen ugyanaz, mint az A algebra jobb modulusa. Egy A R-gyűrű jobb modulusainak *homomorfizmusai* jobb R-modulus homomorfizmusok, melyek kompatibilisek a hatásokkal. Ez megegyezik az A algebra jobb modulus homomorfizmusaival.

Szimmetrikusan definiáljuk R-gyűrűk bal modulusait és azok homomorfizmusait.

1.30. Példa. Bármely R algebra esetén az $R \to R$ k-modulus homomorfizmusok egy $\mathrm{End}(R)$-rel jelölt R-gyűrűt alkotnak. Az algebra struktúrát a leképezések kompozíciója adja és

$$R \to \mathrm{End}(R) \qquad r \mapsto r(-)$$

algebra homomorfizmus.

A jobb- és bal hatások szerepének felcserélése bijektív kapcsolatot teremt az R-bimodulusok és az R^{op}-bimodulusok között. Az indukált ${}_R\mathcal{M}_R \to {}_{R^{op}}\mathcal{M}_{R^{op}}$ izomorfizmus anti-monoidális. Egy R-gyűrű ellentettjén azt az R^{op}-gyűrűt értjük, melybe ezen anti-monoidális izomorfizmus viszi őt. Egy $\eta : R \to A$ algebra homomorfizmussal jellemzett R-gyűrű esetén ez az $\eta : R^{op} \to A^{op}$ algebra homomorfizmussal jellemzett R^{op}-gyűrűt jelenti.

A következőkben különösen fontosak lesznek számunkra az $R \otimes R^{op}$-gyűrűk. A fentiek szerint, egy $R \otimes R^{op}$-gyűrű leírható egy $\eta : R \otimes R^{op} \to A$ algebra homomorfizmussal. Ekvivalens módon, η helyett tekinthetjük az $s := \eta(- \otimes 1) : R \to A$ és a $t := \eta(1 \otimes -) : R^{op} \to A$ algebra homomorfizmusokat, melyek értékkészletei kommutálnak A-ban. Egy A $R \otimes R^{op}$-gyűrűn négy kommutáló R-hatás van jelen:

- Az ${}_RA$-val jelölt bal R-moduluson a hatás $R \otimes A \to A$, $r \otimes h \mapsto s(r)h$;
- Az A_R-val jelölt jobb R-moduluson a hatás $A \otimes R \to A$, $h \otimes r \mapsto t(r)h$;
- Az ${}^RA$-val jelölt bal R-moduluson a hatás $R \otimes A \to A$, $r \otimes h \mapsto ht(r)$;
- Az A^R-val jelölt jobb R-moduluson a hatás $A \otimes R \to A$, $h \otimes r \mapsto hs(r)$.

A fenti hatások kombinációjával négy R-bimodulus struktúrát definiálhatunk A-n. Az ${}^RA_R \otimes_R {}_RA^R$ R-bimodulus centrumát $A\,{}_R\!\times A$-val jelöljük, és A önmagával vett *Takeuchi szorzatának* nevezzük. $A\,{}_R\!\times A$ maga is $R \otimes R^{op}$-gyűrű. Algebra struktúráját a (jól definiált!) faktoronkénti szorzás adja és

$$R \otimes R^{op} \to A\,{}_R\!\times A \qquad r \otimes r' \mapsto s(r) \otimes_R t(r')$$

algebra homomorfizmus. Szimmetrikusan, a fentiek helyett tekinthetjük az ${}_RA^R \otimes_R {}^RA_R$ R-bimodulus centrumát is, melyre jelölésünk legyen $A \times_R A$.

1.31. Definíció. A komonoidokat az R-bimodulusok kategóriájában *R-kogyűrűknek* hívjuk. Expliciten, egy R-kogyűrű egy R-bimodulus C, ellátva $\delta : C \to C \otimes_R C$ (koszorzás) és $\varepsilon : C \to R$ (koegység) R-bimodulus leképezésekkel, melyek a szokásos koasszociativitási és koegység feltételeknek tesznek eleget. R-kogyűrűk *homomorfizmusai* a koszorzással és koegységgel kompatibilis R-bimodulus homomorfizmusok.

Egy R-kogyűrű ellentettjén azt az R^{op}-kogyűrűt értjük, amibe őt az ${}_R\mathcal{M}_R \to {}_{R^{op}}\mathcal{M}_{R^{op}}$ anti-monoidális izomorfizmus viszi. Expliciten, egy (C, δ, ε) kogyűrű esetén C-t mint R^{op}-bimodulust tekintjük, a koszorzás $\delta : C \to C \otimes_{R^{op}} C$, $c \mapsto c_2 \otimes_{R^{op}} c_1$, a koegység $\varepsilon : C \to R^{op}$.

1.32. Definíció. Egy C R-kogyűrű *jobb komodulusa* egy jobb R-modulus M, ellátva egy $M \to M \otimes_R C$ jobb R-modulus leképezéssel, mely a szokásos koasszociativitási és koegység feltételeknek tesz eleget. Egy C R-kogyűrű jobb komodulusainak *homomorfizmusai* jobb R-modulus homomorfizmusok melyek kompatibilisek a kohatásokkal.

Szimmetrikusan definiáljuk R-kogyűrűk bal komodulusait és azok homomorfizmusait.

1.33. Definíció. Egy *bal R-bialgebroid* B áll egy $(s : R \to B, t : R^{op} \to B)$ $R \otimes R^{op}$-gyűrűből és egy $({}_R B_R, \delta, \varepsilon)$ R-kogyűrűből, melyekre az alábbi axiómák teljesülnek:

- δ felírható, mint egy $B \to B\,{}_R\!\times B$ algebra homomorfizmus és a $B\,{}_R\!\times B \hookrightarrow B \otimes_R B$ inklúzió kompozíciója;
- A $B \to \mathrm{End}(R)$, $b \mapsto \varepsilon(bs(-))$ leképezés algebra homomorfizmus.

Egy bal bialgebroid koszorzására a szokásos $\delta(h) = h_1 \otimes_R h_2$ Sweeler-Heynemann index jelölést használjuk. B tetszőleges h, h' elemein és R tetszőleges r elemén kiírva a bialgebroid axiómák a következő alakot öltik.

$$\begin{array}{ll} h_1 t(r) \otimes_R h_2 = h_1 \otimes h_2 s(r); & \\ 1_1 \otimes_R 1_2 = 1 \otimes_R 1; & (h'h)_1 \otimes_R (h'h)_2 = h'_1 h_1 \otimes_R h'_2 h_2; \\ \varepsilon(1) = 1; & \varepsilon(h'h) = \varepsilon(h' s\varepsilon(h)). \end{array}$$

A második sorban szereplő második axióma jobb oldalán álló kifejezés az első sorban szereplő axióma miatt jól definiált.

Tekintsünk egy B bal R bialgebroidot. Az $s : R \to B$ algebra homomorfizmus révén a ${}_R B^R$ R-bimodulus rendelkezik egy R-gyűrű struktúrával. Mivel definíció szerint a ${}_R B_R$ R-bimodulus rendelkezik egy R-kogyűrű struktúrával, a bimodulus leképezések $\mathrm{Hom}({}_R B_R, {}_R B^R)$ halmaza algebra a

$$(f * g)(h) := f(h_1) g(h_2), \qquad f, g \in \mathrm{Hom}({}_R B_R, {}_R B^R),\ h \in B,$$

konvolúció szorzás révén; $s\varepsilon$ egységgel.

Ha $(s : R \to B, t : R^{op} \to B)$ egy $R \otimes R^{op}$-gyűrű, akkor minden bal B-modulus természetesen R-bimodulus is az $r.m.r' := s(r)t(r').m$ hatás révén. A B-modulus homomorfizmusok R bimodulus homomorfizmusok is. Más szóval, létezik egy ${}_B\mathcal{M} \to {}_R\mathcal{M}_R$ felejtő funktor.

1.34. Tétel (Schauenburg [67])**.** *Bármely $(s : R \to B, t : R^{op} \to B)$ $R \otimes R^{op}$-gyűrű esetén bijektív kapcsolat van az alábbi struktúrák között.*

- *Bal bialgebroid struktúrák az adott $R \otimes R^{op}$-gyűrűn;*

- *monoidális struktúrák a bal B-modulusok kategóriáján úgy, hogy a ${}_B\mathcal{M} \to {}_R\mathcal{M}_R$ felejtő funktor szigorúan monoidális.*

Ha B egy bal R bialgebroid, akkor $(t : R \to B^{op}, s : R^{op} \to B^{op})$ is $R \otimes R^{op}$-gyűrű. A bal bialgebroid axiómák helyett azonban más, hasonló alakú axiómáknak tesz eleget:

1.35. Definíció. Egy *jobb R-bialgebroid* B áll egy $(s : R \to B, t : R^{op} \to B)$ $R \otimes R^{op}$-gyűrűből és egy $({}^R B^R, \delta, \varepsilon)$ R-kogyűrűből, melyekre az alábbi axiómák teljesülnek:

- δ felírható, mint egy $B \to B \times_R B$ algebra homomorfizmus és a $B \times_R B \hookrightarrow B \otimes_R B$ inklúzió kompozíciója;
- A $B \to \mathrm{End}(R)^{op}$, $b \mapsto \varepsilon(s(-)b)$ leképezés algebra homomorfizmus.

Egy jobb bialgebroid koszorzására a szokásos $\delta(h) = h^1 \otimes_R h^2$ Sweeler-Heynemann jelölést használjuk – ezúttal alsó helyett felső indexekkel. A jobb bialgebroidok pontosan azok az $R \otimes R^{op}$-gyűrűk, melyek jobb modulus kategóriája monoidális és a felejtő funktor az R-bimodulusok kategóriájába szigorúan monoidális.

1.36. Példa. Bármely B bialgebra bal és jobb bialgebroid is $R := k$ fölött. Az $R \otimes R^{op}$-gyűrűt leíró $s : k \to B$ és $t : k \to B$ algebra homomorfizmusok mindegyike megegyezik B egységével. Az R-kogyűrű struktúrát B koalgebra struktúrája adja.

Ha egy B bal R-bialgebroidban ${}_R B$ végesen generált projektív bal R-modulus, akkor a ${}_R\mathrm{Hom}(B, R)$ bal R-duális ellátható egy jobb R-bialgebroid struktúrával. Ha B_R végesen generált projektív jobb R-modulus, akkor a $\mathrm{Hom}_R(B, R)$ jobb R-duális látható el egy jobb R-bialgebroid struktúrával. Szimmetrikusan, egy megfelelő oldalon végesen generált projektív jobb bialgebroid bal- illetve jobb duálisa természetes módon bal bialgebroid, l. [45].

1.2.2. A Hopf-algebroid axiómák és következményeik

A Hopf-algebroid axiómáinak alábbi megfogalmazása [14]-ből való.

Legyenek L és R tetszőleges algebrák. Tekintsünk egy H algebrát, mely rendelkezik egy bal L-bialgebroid struktúrával – $s_L : L \to H$ és $t_L : L^{op} \to H$ kommutáló értékkészletű algebra homomorfizmusokkal; $\delta_L : {}_L H_L \to {}_L H_L \otimes_L {}_L H_L$ koszorzással; illetve $\varepsilon_L : {}_L H_L \to L$ koegységgel – valamint egy jobb R-bialgebroid struktúrával – $s_R : R \to H$ és $t_R : R^{op} \to H$ kommutáló értékkészletű algebra homomorfizmusokkal; $\delta_R : {}^R H^R \to {}^R H^R \otimes_R {}^R H^R$ koszorzással; illetve $\varepsilon : {}^R H^R \to R$ koegységgel. A két bialgebroid között követeljük meg az alábbi kompatibilitási axiómákat:

$$s_L \varepsilon_L t_R = t_R; \qquad t_L \varepsilon_L s_R = s_R; \qquad s_R \varepsilon_R t_L = t_L; \qquad t_R \varepsilon_R s_L = s_L. \tag{1.6}$$

Könnyen látható, hogy ezen axiómák következtében $\varepsilon_R s_L : L \to R^{op}$ algebra izomorfizmus, melynek inverze $\varepsilon_L t_R$. Ugyanígy, $\varepsilon_L s_R : R \to L^{op}$ algebra izomorfizmus, melynek inverze $\varepsilon_R t_L$. (1.6) teljesülése esetén értelmes megkövetelni a következő axiómákat.

$$(\delta_R \otimes_L \mathrm{Id})\delta_L = (\mathrm{Id} \otimes_R \delta_L)\delta_R; \qquad (\delta_L \otimes_R \mathrm{Id})\delta_R = (\mathrm{Id} \otimes_L \delta_R)\delta_L, \tag{1.7}$$

mint $H \to {}^R H^R \otimes_R {}^R H_L \otimes_L {}_L H_L$ illetve $H \to {}_L H_L \otimes_L {}_L H^R \otimes_R {}^R H^R$ leképezések. Két, a fenti kapcsolatban lévő bialgebroid meghatároz egy Morita-összefüggést a $\mathrm{Hom}({}_L H_L, {}_L H^L)$ és $\mathrm{Hom}({}^R H^R, {}_R H^R)$ konvolúció algebrák között. A két szereplő bimodulus $\mathrm{Hom}({}_L H^R, {}_L H^R)$

és $\mathrm{Hom}({}^R H_L, {}_R H^L)$. A Morita-összefüggés valamennyi művelete a megfelelő konvolúció szorzással adott; azaz H bármely h elemére,

$$\begin{aligned}
(\phi * \phi')(h) &= \phi(h_1)\phi'(h_2), && \phi, \phi' \in \mathrm{Hom}({}_L H_L, {}_L H^L); \\
(\psi * \psi')(h) &= \psi(h^1)\psi'(h^2), && \psi, \psi' \in \mathrm{Hom}({}^R H^R, {}_R H^R); \\
(\phi * \alpha)(h) &= \phi(h_1)\alpha(h_2), && \phi \in \mathrm{Hom}({}_L H_L, {}_L H^L),\ \alpha \in \mathrm{Hom}({}_L H^R, {}_L H^R); \\
(\alpha * \psi)(h) &= \alpha(h^1)\psi(h^2), && \psi \in \mathrm{Hom}({}^R H^R, {}_R H^R),\ \alpha \in \mathrm{Hom}({}_L H^R, {}_L H^R); \\
(\psi * \beta)(h) &= \psi(h^1)\beta(h^2), && \psi, \in \mathrm{Hom}({}^R H^R, {}_R H^R),\ \beta \in \mathrm{Hom}({}^R H_L, {}_R H^L); \\
(\beta * \phi)(h) &= \beta(h_1)\phi(h_2), && \phi \in \mathrm{Hom}({}_L H_L, {}_L H^L),\ \beta \in \mathrm{Hom}({}^R H_L, {}_R H^L); \\
(\alpha * \beta)(h) &= \alpha(h^1)\beta(h^2), && \alpha \in \mathrm{Hom}({}_L H^R, {}_L H^R),\ \beta \in \mathrm{Hom}({}^R H_L, {}_R H^L); \\
(\beta * \alpha)(h) &= \beta(h_1)\alpha(h_2), && \alpha \in \mathrm{Hom}({}_L H^R, {}_L H^R),\ \beta \in \mathrm{Hom}({}^R H_L, {}_R H^L),
\end{aligned} \tag{1.8}$$

ahol a korábban bevezetett $\delta_L(h) = h_1 \otimes_L h_2$ és $\delta_R(h) = h^1 \otimes_R h^2$ jelöléseket használtuk. Nyilvánvalóan, a $H \to H$ identitás leképezés a $\mathrm{Hom}({}_L H^R, {}_L H^R)$ bimodulus eleme.

1.37. Definíció. Egy *Hopf-algebroid* alatt a következő struktúrát értjük. Egy H algebrát, ellátva egy bal L-bialgebroid és egy jobb R-bialgebroid struktúrával, melyekre (1.6) és (1.7) axiómák teljesülnek, továbbá az (1.8) Morita-összefüggésben $\mathrm{Id} \in \mathrm{Hom}({}_L H^R, {}_L H^R)$ invertálható. Expliciten, az utóbbi követelmény azt jelenti, hogy létezik $S \in \mathrm{Hom}({}^R H_L, {}_R H^L)$, melyre

$$h^1 S(h^2) = s_L \varepsilon_L(h); \qquad S(h_1)h_2 = s_R \varepsilon_R(h), \qquad \forall h \in H. \tag{1.9}$$

1.38. Tétel ([10] Proposition 2.3)**.** *Egy H Hopf-algebroid S antipódjára a következő állítások teljesülnek.*

- *S homomorfizmus az $(s_R : R \to H, t_R : R^{op} \to H)$ $R \otimes R^{op}$-gyűrűből az $(s_R = t_L \varepsilon_L s_R : R \to H, s_L \varepsilon_L s_R : R^{op} \to H)$ $R \otimes R^{op}$-gyűrű ellentettjébe;*
- *S homomorfizmus az $(s_L : L \to H, t_L : L^{op} \to H)$ $L \otimes L^{op}$-gyűrűből az $(s_L = t_R \varepsilon_R s_L : L \to H, s_R \varepsilon_R s_L : L^{op} \to H)$ $L \otimes L^{op}$-gyűrű ellentettjébe;*
- *S homomorfizmus a $({}_L H_L, \delta_L, \varepsilon_L)$ L-kogyűrűből abba a kogyűrűbe, melybe az $\varepsilon_L s_R : R^{op} \to L$ algebra izomorfizmus által indukált ${}_{R^{op}}\mathcal{M}_{R^{op}} \cong {}_L\mathcal{M}_L$ monoidális izomorfizmus viszi $({}^R H^R, \delta_R, \varepsilon_R)$ R-kogyűrű ellentettjét;*
- *S homomorfizmus a $({}^R H^R, \delta_R, \varepsilon_R)$ R-kogyűrűből abba a kogyűrűbe, melybe az $\varepsilon_R s_L : L^{op} \to R$ algebra izomorfizmus által indukált ${}_{L^{op}}\mathcal{M}_{L^{op}} \cong {}_R\mathcal{M}_R$ monoidális izomorfizmus viszi $({}_L H_L, \delta_L, \varepsilon_L)$ L-kogyűrű ellentettjét.*

Az 1.38 Tételből nyilvánvaló, hogy abban az esetben, ha egy Hopf-algebroid antipódja bijektív, a jobb bialgebroidot definiáló valamennyi leképezés kifejezhető a bal bialgebroid struktúrát leíró leképezésekkel, az antipóddal illetve annak inverzével. A Hopf-algebroidok axiómái ezen adatokkal megfogalmazva [19] 4. fejezetében találhatók.

1.39. Megjegyzés. Nagyon fontos hangsúlyoznunk, hogy az (1.7) axiómák $H \to {}^R H^R \otimes_R {}^R H_L \otimes_L {}_L H_L$ illetve $H \to {}_L H_L \otimes_L {}_L H^R \otimes_R {}^R H^R$ leképezésekre vonatkoznak. Tudjuk ugyan, hogy az első egyenlőség bal oldalán szereplő leképezés faktorizálódik $(H \times_R H) \times_L H$-n keresztül, a jobb oldalon álló pedig $H \times_R (H \times_L H)$-n keresztül. Ezek azonban nem részhalmazai ${}^R H^R \otimes_R {}^R H_L \otimes_L {}_L H_L$-nak, melyeknek lehetne a metszetét venni. Noha a nyilvánvaló

$(H \times_R H) \times_L H \to (H \times_R H) \otimes_L H$ leképezés injektív, $(H \times_R H) \otimes_L H \to (H \otimes_R H) \otimes_L H$ nem az, legalábbis további feltevések nélkül nem. Így az olyan φ leképezések esetén, melyek mondjuk $(H \times_R H) \times_L H$-en vannak értelmezve, de nem terjeszthetők ki $H \otimes_R H \otimes_L H$-ra (pl. $H \otimes_R \varepsilon_R \otimes_L H : (H \times_R H) \times_L H \to H \times_L H)$, a $\varphi(\delta_R \otimes_L \mathrm{Id})\delta_L$ kifejezés átalakítására nem alkalmazható az (1.7) axióma. Sweedler indexeket használva, noha ${h_1}^1 \otimes_R {h_1}^2 \otimes_L h_2 = h^1 \otimes {h^2}_1 \otimes_L {h^2}_2$, mint ${}^R H^R \otimes_R {}^R H_L \otimes_L {}_L H_L$ elemei, a $\varphi({h_1}^1 \otimes_R {h_1}^2 \otimes_L h_2) = \varphi(h^1 \otimes {h^2}_1 \otimes_L {h^2}_2)$ egyenlőség hibás, mivel a jobb oldal rosszul definiált. Sajnos ilyen típusú – saját magam által, vagy hivatkozott cikkben elkövetett – hiba miatt több cikkem javításra szorult.

A Hopf-algebroidok a gyenge Hopf-algebrák (így a Hopf-algebrák) általánosításai a következő értelemben.

1.40. Tétel ([19] Example 4.8). *Legyen B egy (gyenge) bialgebra $B \to B \otimes B$, $b \mapsto b_1 \otimes b_2$ koszorzással és $\varepsilon : B \to k$ koegységgel (l. 1.56 Definíció). Ezen adatok meghatároznak egy jobb bialgebroid struktúrát B-n az $\{1_1 \varepsilon(b 1_2)\}_{b \in B}$ részalgebra fölött és egy bal bialgebroid struktúrát az $\{\varepsilon(1_1 b) 1_2\}_{b \in B}$ részalgebra fölött. Ha továbbá B (gyenge) Hopf-algebra S antipóddal (l. 1.59 Definíció), akkor a fenti bialgebroidok és S Hopf-algebroid struktúrát definiálnak B-n.*

Hopf-algebroidok fontos alkalmazása, hogy leírják algebrák kettes mélységű Frobenius-kiterjesztéseinek szimmetriáját. A Frobenius-tulajdonság azt jelenti, hogy M végesen generált projektív (jobb vagy bal) N-modulus és a (jobb vagy bal) modulus homomorfizmusokból álló $\mathrm{Hom}_N(M, N)$ izomorf M-mel mint (N-M vagy M-N) bimodulus. A kettes mélység feltétel azt jelenti, hogy $M \otimes_N M$ direkt összeadandó M véges sok példányának direkt összegében – mint N-M bimodulus és M-N bimodulus.

1.41. Tétel ([19] Section 3). *Tekintsünk egy $N \subseteq M$ kettes mélységű és Frobenius-tulajdonságú algebra kiterjesztést. Ezen feltevések mellett mind M N-bimodulus endomorfizmusainak algebrája, mind az $M \otimes_N M$ M-bimodulus centruma (melynek algebra struktúrája a faktoronkénti szorzással ill. ellentett szorzással adott) Hopf-algebroidok bijektív antipóddal.*

Az 1.41 Tétel általánosításaként, bármely k-lineáris bikategória Frobenius-tulajdonságú és kettes mélységű 1-cellája meghatároz két (megfelelő értelemben duális) Hopf-algebroidot, l. [20].

További példák Hopf-algebroidokra [19] 4.3 fejezetében találhatók.

1.42. Tétel ([19] Proposition 4.2). *Egy L és R algebrák fölötti Hopf-algebroidban a*

$$H^L \otimes_L {}_L H \to H^R \otimes_R {}^R H \qquad h' \otimes_L h \mapsto h' h^1 \otimes_R h^2$$

leképezés bijektív (azaz $s_L : L \to H$ Galois-kiterjesztés az alap jobb bialgebroiddal, l. 1.4 fejezet). Vagyis [66] szóhasználatával, minden Hopf-algebroid $\times_R$-Hopf-algebra.

Ezt a megfigyelést kombinálva Schauenburg ([66] Theorem and Definition 3.5) eredményével, azt látjuk, hogy az $\mathcal{M}_H \to {}_R\mathcal{M}_R$ felejtő funktor őrzi a jobbról zárt struktúrát. Ebből következik, hogy azon jobb H-modulusoknak, melyek végesen generált projektív bal R-modulusok (így van bal duálisuk ${}_R\mathcal{M}_R$-ben), van bal duálisuk $\mathcal{M}_H$-ban is, melyet az $\mathcal{M}_H \to {}_R\mathcal{M}_R$ felejtő funktor őriz.

Day és Street [34]-ben (a problémát általánosabban, monoidális bikategóriákban megfogalmazva) azt a kérdést vizsgálták, hogy az $\mathcal{M}_H \to {}_R\mathcal{M}_R$ felejtő funktor mely H jobb R-bialgebroidokra őrzi a jobbról zártságnál erősebb $*$-autonóm struktúrát. Az ehhez szükséges további struktúrát H-n *$*$-autonóm* struktúrának nevezték [34, Section 9].

1.43. Tétel ([19] Theorem 4.7)**.** *Legyen B egy jobb bialgebroid. Egy erős *-autonóm struktúra (a [34]-ben tárgyalt értelemben) B-n ekvivalens egy bijektív antipóddal mely B-t Hopf-algebroiddá teszi.*

1.3. Hopf-algebroidok integrálelmélete

Ahogyan a Hopf-algebrák esetében, ugyanúgy a Hopf-algebroidok vizsgálatában is fontos információk nyerhetők az algebrai szerkezetről az integrálok tanulmányozásával. E fejezet – és az alapjául szolgáló [10] munka – **célja** ennek a kérdéskörnek a vizsgálata; annak a megfelelő integrál fogalomnak a megtalálása, mely lehetőséget teremt olyan algebrai tulajdonságok leírására, mint a félig (ko)egyszerűség, a (ko)szeparabilitás vagy a (kvázi-) Frobenius-tulajdonság. A megfelelő integrál fogalom segítségével a Hopf-algebroidokra a Hopf-algebrákkal megnyugtató módon analóg tételek igazolhatóak. A lényeges különbség, hogy míg a Hopf-algebrák esetén egyetlen kommutatív gyűrű feletti algebra (ill. koalgebra) tulajdonságairól van szó, egy Hopf-algebroidban négy (páronként izomorf) gyűrűt (ill. két kogyűrűt) kell vizsgálnunk, nem kommutatív algebrák fölött.

1.3.1. Integrálok

Schauenburg 1.34 Tételben idézett észrevétele szerint, ha H egy bal L-bialgebroid, akkor L bal H-modulus a $h.l := \varepsilon(hs(l))$ hatás révén. Az alábbi definíció ezt a tényt használja fel.

1.44. Definíció. Legyen B egy bal L-bialgebroid. Egy M bal B-modulus *invariánsainak* nevezzük a

$$_B\mathrm{Hom}(L, M) \cong \{n \in M | \forall h \in B,\ h.n = s\varepsilon(h).n\}$$

k-részmodulus elemeit. A *bal integrálok B-ben* a bal reguláris B-modulus invariánsai.

Szimmetrikusan definiáljuk egy jobb bialgebroid jobb modulusainak invariánsait és az integrálokat mint a jobb reguláris modulus invariánsait. Egy Hopf-algebroidban egyszerre van jelen egy jobb és egy bal bialgebroid struktúra, így ez esetben mind bal- mind jobb integrálokat értelmezhetünk. Egy bal (vagy jobb) integrál antipód általi képe jobb (vagy bal) integrál.

Egy bialgebroid komodulusain az alap kogyűrű komodulusait értjük. Bármely B bal L-bialgebroid esetén L jobb B-komodulus az $L \to L \otimes_L B \cong B$, $l \mapsto s(l)$ kohatás révén és bal B-komodulus is az $L \to B \otimes_L L \cong B$, $l \mapsto t(l)$ kohatás révén. Így a *B-beli* integrálok duálisaként bevezethetjük az alábbi fogalmakat is.

1.45. Definíció. Legyen B egy bal L-bialgebroid. Egy *s-integrál B-n* egy $B \to L$ bal B-komodulus homomorfizmus, azaz egy $\varrho : B \to L$ leképezés, amire

$$\varrho(s(l)h) = l\varrho(h); \qquad t\varrho(h_2)h_1 = s\varrho(h), \qquad\qquad l \in L,\ h \in H.$$

Egy *t-integrál B-n* egy $B \to L$ jobb B-komodulus homomorfizmus, azaz egy $\varrho : B \to L$ leképezés, amire

$$\varrho(t(l)h) = \varrho(h)l; \qquad s\varrho(h_1)h_2 = t\varrho(h), \qquad\qquad l \in L,\ h \in H.$$

Szimmetrikusan definiálunk integrálokat egy B jobb R-bialgebroidon, mint bal- illetve jobb komodulus homomorfizmusokat $B \to R$.

Ha egy B bal L-bialgebroidban ${}_LB$ végesen generált projektív bal L-modulus, így ${}_L\mathrm{Hom}(B,L)$ jobb L bialgebroid, akkor egy s-integrál B-n pontosan ugyanaz mint egy jobb integrál ${}_L\mathrm{Hom}(B,L)$-ban. Ha B_L végesen generált projektív jobb L-modulus, így $\mathrm{Hom}_L(B,L)$ jobb L bialgebroid, akkor egy t-integrál B-n pontosan ugyanaz mint egy jobb integrál $\mathrm{Hom}_L(B,L)$-ban.

Egy Hopf-algebroidban egyszerre van jelen egy jobb és egy bal bialgebroid struktúra, így egy Hopf-algebroidon négyféle integrál értelmezhető: s- és t-integrálok az alap bal- és jobb bialgebroidokon. Ha ϱ s-integrál a bal bialgebroidon akkor ϱS t-integrál ugyanezen a bal bialgebroidon.

1.3.2. Maschke-típusú tételek

Legyen H egy Hopf-algebroid L és R (szükségképpen anti-izomorf) algebrák fölött. A megfelelő értelemben (l. 1.46 Tétel) normált integrálok létezésének H-ban a négy $s_L : L \to H$, $t_L : L^{op} \to H$, $s_R : R \to H$ illetve $t_R : R^{op} \to H$ algebra kiterjesztés félig egyszerűségéhez illetve szeparabilitásához van köze. (Egy $R \to H$ algebra kiterjesztést *szeparábilisnak* mondunk, ha a $H \otimes_R H \to H$ szorzás felhasadó H-bimodulus epimorfizmus. Az $R \to H$ algebra kiterjesztés jobbról (illetve balról) *félig egyszerű*, ha minden olyan jobb (illetve bal) H-modulus homomorfizmus felhasad, amely felhasadó jobb (illetve bal) R-modulus epimorfizmus.)

1.46. Tétel ([10] Theorem 3.1). *Legyen H egy Hopf-algebroid L és R algebrák fölött. A következő állítások ekvivalensek.*

- *Az $s_R : R \to H$ kiterjesztés szeparábilis;*
- *A $t_R : R^{op} \to H$ kiterjesztés szeparábilis;*
- *Az $s_L : L \to H$ kiterjesztés szeparábilis;*
- *Az $t_L : L^{op} \to H$ kiterjesztés szeparábilis;*
- *Az $s_R : R \to H$ kiterjesztés jobbról félig egyszerű;*
- *A $t_R : R^{op} \to H$ kiterjesztés jobbról félig egyszerű;*
- *Az $s_L : L \to H$ kiterjesztés balról félig egyszerű;*
- *A $t_L : L^{op} \to H$ kiterjesztés balról félig egyszerű;*
- *Létezik egy bal integrál ℓ az alap bal L-bialgebroidban, ami normált az $\varepsilon_L(\ell) = 1$ értelemben;*
- *Létezik egy jobb integrál ℓ az alap jobb R-bialgebroidban, ami normált az $\varepsilon_R(\ell) = 1$ értelemben;*
- *$\varepsilon_L : H \to L$ felhasadó epimorfizmus a bal H-modulusok kategóriájában;*
- *$\varepsilon_R : H \to R$ felhasadó epimorfizmus a jobb H-modulusok kategóriájában.*

Hasonlóképpen, megfelelő értelemben (l. 1.47 Tétel) normált integrálok létezésének H-n a két, L illetve R fölötti kogyűrű félig koegyszerűségéhez illetve koszeparabilitásához van köze. (Egy R fölötti H kogyűrűt *koszeparábilisnak* mondunk, ha a $H \to H \otimes_R H$ koszorzás felhasadó

H-bikomodulus monomorfizmus. A H R-kogyűrű jobbról (illetve balról) *félig koegyszerű*, ha minden olyan jobb (illetve bal) H-komodulus homomorfizmus felhasad, amely felhasadó jobb (illetve bal) R-modulus monomorfizmus.)

1.47. Tétel ([10] Theorem 3.2). *Tekintsünk egy H egy Hopf-algebroidot L és R algebrák fölött. A következő állítások ekvivalensek.*

- *H mint R-kogyűrű koszeparábilis;*
- *H mint L-kogyűrű koszeparábilis;*
- *H mint R-kogyűrű jobbról félig koegyszerű;*
- *H mint R-kogyűrű balról félig koegyszerű;*
- *H mint L-kogyűrű jobbról félig koegyszerű;*
- *H mint L-kogyűrű balról félig koegyszerű;*
- *Létezik egy s-integrál λ az alap R-bialgebroidon, ami normált a $\lambda(1) = 1$ értelemben;*
- *Létezik egy t-integrál λ az alap R-bialgebroidon, ami normált a $\lambda(1) = 1$ értelemben;*
- *Létezik egy s-integrál λ az alap L-bialgebroidon, ami normált a $\lambda(1) = 1$ értelemben;*
- *Létezik egy t-integrál λ az alap L-bialgebroidon, ami normált a $\lambda(1) = 1$ értelemben;*
- *$s_R : R \to H$ felhasadó monomorfizmus az alap jobb R-bialgebroid jobb komodulusainak kategóriájában;*
- *$t_R : R \to H$ felhasadó monomorfizmus az alap jobb R-bialgebroid bal komodulusainak kategóriájában;*
- *$s_L : L \to H$ felhasadó monomorfizmus az alap bal L-bialgebroid bal komodulusainak kategóriájában;*
- *$t_L : L \to H$ felhasadó monomorfizmus az alap bal L-bialgebroid jobb komodulusainak kategóriájában.*

1.3.3. A (kvázi-)Frobenius-tulajdonságról

Legyen H egy Hopf-algebroid L és R algebrák fölött. A megfelelő értelemben (l. 1.48 Tétel) nem degenerált integrálok létezése a négy $s_L : L \to H$, $t_L : L^{op} \to H$, $s_R : R \to H$ illetve $t_R : R^{op} \to H$ algebra kiterjesztés Frobenius-tulajdonságával (l. 1.41 Tételt megelőző bekezdés) kapcsolatos.

1.48. Tétel ([10] *Erratum*, Corollary 3). *Tekintsünk egy H egy Hopf-algebroidot L és R algebrák fölött. A következő állítások ekvivalensek.*

- *Az $s_R : R \to H$ és a $t_R : R^{op} \to H$ kiterjesztések mindegyike Frobenius-kiterjesztés;*
- *Az $s_L : L \to H$ és a $t_L : L^{op} \to H$ kiterjesztések mindegyike Frobenius-kiterjesztés;*
- *H^R végesen generált projektív jobb R-modulus és létezik egy s-integrál λ az alap jobb R-bialgebroidon, amire a $H \to \mathrm{Hom}_R(H, R)$, $h \mapsto \lambda(h-)$ leképezés bijektív;*

- *Az S antipód bijektív, ${}^{R}H$ végesen generált projektív bal R-modulus és létezik egy t-integrál λ az alap jobb R-bialgebroidon, amire a $H \to {}_R\mathrm{Hom}(H,R)$, $h \mapsto \lambda(h-)$ leképezés bijektív;*
- *${}_LH$ végesen generált projektív bal L-modulus és létezik egy s-integrál λ az alap bal L-bialgebroidon, amire a $H \to {}_L\mathrm{Hom}(H,L)$, $h \mapsto \lambda(-h)$ leképezés bijektív;*
- *Az S antipód bijektív, ${}_LH$ végesen generált projektív jobb L-modulus és létezik egy t-integrál λ az alap bal L-bialgebroidon, amire a $H \to \mathrm{Hom}_L(H,L)$, $h \mapsto \lambda(-h)$ leképezés bijektív;*
- *Létezik egy bal integrál ℓ az alap bal L-bialgebroidban, ami nem degenerált abban az értelemben, hogy mindkét*
$$\mathrm{Hom}_R(H,R) \to H,\ \varphi \mapsto \ell^1 s_R \varphi(\ell^2) \quad \textit{és} \quad {}_R\mathrm{Hom}(H,R) \to H,\ \psi \mapsto \ell^2 t_R \psi(\ell^1)$$
leképezés bijektív;
- *Létezik egy jobb integrál ℓ az alap jobb R-bialgebroidban, ami nem degenerált abban az értelemben, hogy mindkét*
$${}_L\mathrm{Hom}(H,L) \to H,\ \varphi \mapsto s_L \varphi(\ell_1)\ell_2 \quad \textit{és} \quad \mathrm{Hom}_L(H,L) \to H,\ \psi \mapsto t_L \psi(\ell_2)\ell_1$$
leképezés bijektív.

A fenti ekvivalens tulajdonságokkal rendelkező Hopf-algebroidot Frobenius- Hopf-algebroidnak *mondjuk.*

A fenti tétel ([10] Theorem 4.7)-ben megjelent formája nem helyes – egy, a 1.39 Megjegyzésben ismertetett hiba miatt.

Az integrálok vizsgálatával szükséges és elégséges feltételek adhatók meg a négy, $s_L : L \to H$, $t_L : L^{op} \to H$, $s_R : R \to H$ illetve $t_R : R^{op} \to H$ algebra kiterjesztés kvázi-Frobenius-tulajdonságára is, l. ([10], Theorem 5.2).

1.3.4. (Frobenius-) Hopf-algebroidok dualitása

Mint azt az 1.2.1 fejezetben tárgyaltuk, ha egy (bal vagy jobb) R-bialgebroidon valamelyik R-modulus végesen generált projektív, akkor a megfelelő duális rendelkezik egy (jobb vagy bal) bialgebroid struktúrával. Ha tehát egy Hopf-algebroid végesen generált projektív valamelyik értelemben, akkor a megfelelő duális rendelkezik egy bialgebroid struktúrával. Nem ismert azonban az általánosságnak ezen a szintjén, hogy a duális Hopf-algebroid-e. A természetes jelölt, a Hopf-algebroid antipódjával való kompozíció ugyanis nem definiál antipódot egyik duálison sem, mert erre a műveletre nézve egyikük sem zárt. Az antipóddal való kompozíció az egyik duálisból egy másikba való leképezés. Hopf-algebroidok egy szűkebb osztályáról tudható csak, hogy zárt a dualitásra nézve. Tekintsünk egy H Frobenius-Hopf-algebroidot L és R algebrák fölött. Ez esetben H összes L- illetve R- modulus struktúrája végesen generált projektív, tehát mind a négy duális rendelkezik (bal vagy jobb) bialgebroid struktúrával. Sőt, az 1.48 Tételben látott izomorfizmusok ezen duálisok ill. ellentettjeik közötti bialgebroid izomorfizmusokká kombinálhatók (l. [19] Theorem 5.16). Ezekkel komponálva az eredeti (bijektív) antipód transzponáltját, bármely duális ellátható egy bijektív antipóddal. Mi több, az alábbi tétel áll fenn.

1.49. Tétel ([19] Theorem 5.17 and Proposition 5.19)**.** *Egy Frobenius-Hopf-algebroid négy duálisa (anti-)izomorf Frobenius-Hopf-algebroid.*

1.4. Galois-kiterjesztések Hopf-algebroid szimmetriával

Mint azt az 1.1.3 fejezetben láttuk, a Hopf–Galois-elmélet alkalmazása igen széles – a csoport gradált algebrák elegáns tárgyalásától a nem kommutatív differenciálgeometria principális nyalábjainak megfogalmazásáig. Jelen fejezet **célja** az első lépések bemutatása a Galois-elmélet Hopf-algebroidokra való általánosítása felé. Az adódó elmélet magában foglalja a grupoid gradált algebrák és a grupoid nyalábok nem kommutatív megfelelőinek leírását.

Egy klasszikus Galois-testbővítés jellemezhető a Galois-csoportra való hivatkozás nélkül: mint véges szeparábilis normális bővítés. Evvel szemben nem ismerünk olyan inherens tulajdonságokat, amelyekkel a Hopf-algebrára való explicit hivatkozás nélkül jellemezhetnénk Hopf–Galois-kiterjesztéseket. Nem ismert továbbá a kapcsolat egy adott Hopf–Galois-kiterjesztés szimmetriáját leíró lehetséges Hopf-algebrák között. Mint Bálint és Szlachányi [6] kiváló munkájában megmutatta, mindezen kérdések megválaszolhatók a (Frobenius-) Hopf-algebroidokkal vett Galois-kiterjesztések tágabb körében.

Mint azt az 1.1.3 fejezetben áttekintettük, egy Hopf-algebrával való Galois-kiterjesztés alatt tulajdonképpen az alap bialgebrával való kiterjesztést értjük. Mégis, a Hopf-algebrák további tulajdonságait kihasználva, Hopf-algebrákkal való kiterjesztésekre erősebb állítások igazolhatók, mint bialgebrákkal való kiterjesztésekre. A bialgebroidokkal illetve Hopf-algebroidokkal való kiterjesztések között sokkal alapvetőbb elvi különbségek vannak, abból adódóan, hogy egy Hopf-algebroidban egyszerre két bialgebroid (egy bal és egy jobb) van jelen. Tekinthetünk Galois-kiterjesztéseket ezek bármelyikével; az első kérdés ezek viszonyának tisztázása. Fontos látnunk továbbá, hogy a Hopf–Galois-kiterjesztésekre vonatkozó tételek közül melyek terjeszthetők ki Hopf-algebroidokra. A kanonikus leképezés bijektivitására (azaz a Galois-tulajdonságra) vonatkozó elégséges feltételeket megfogalmazó klasszikus tételek közül Kreimer és Takeuchi 1.16 Tételét [11]-ben, Schneider híres I. tételét [5]-ben (Ardizzonival és Meninivel együttműködve) általánosítottam Hopf-algebroidokra. Erős és gyenge struktúra-tételeket (a Morita-elmélet segítségével) szintén [11]-ben igazoltam.

1.4.1. Hopf-algebroidok komodulusai

Értelemszerűen, egy B (mondjuk jobb) R-bialgebroid (bal vagy jobb) komodulusai alatt az alap R-kogyűrű komodulusait értjük. Az 1.32 Definíció szerint tehát egy (mondjuk) jobb B-komodulus egy jobb R-modulus M, ellátva egy $\varrho : M \to M \otimes_R {}^R B^R$, $m \mapsto m^0 \otimes_R m^1$ jobb R-modulus leképezéssel (ahol implicit összegzés értendő), amelyre a szokásos koasszociativitási és koegység feltételek teljesülnek. Noha egy jobb B-komodulus definíció szerint csak jobb R-modulus struktúrával rendelkezik, ([63] Lemma 1.4.1) szerint R-bimodulussá tehető az $r.m := m^0.\varepsilon(s(r)m^1)$ bal R-hatás bevezetésével. Erre a hatásra nézve minden komodulus homomorfizmus R-bimodulus homomorfizmus is. Mi több, a jobb B-komodulusok $\mathcal{M}^B$ kategóriája monoidális és a fent konstruált $\mathcal{M}^B \to {}_R\mathcal{M}_R$ funktor szigorúan monoidális. Hasonlóan, B bal komodulusainak ${}^B\mathcal{M}$ kategóriája is monoidális és van egy szigorúan monoidális funktor ${}^B\mathcal{M} \to {}_{R^{op}}\mathcal{M}_{R^{op}}$. Egy B bal L-bialgebroid bal vagy jobb komodulusainak kategóriája is monoidális és el van látva ${}^B\mathcal{M} \to {}_{L^{op}}\mathcal{M}_{L^{op}}$ illetve $\mathcal{M}^B \to {}_L\mathcal{M}_L$ szigorúan monoidális funktorral.

Egy B jobb bialgebroid jobb *komodulus algebráit* tehát definiálhatjuk, mint monoidokat a jobb B-komodulusok monoidális kategóriájában. Egy tetszőleges M B-komodulus *koinvariánsai* alatt az $M^{coB} := \{n \in M | n^0 \otimes_R n^1 = n \otimes_R 1\}$ halmaz elemeit értjük. Egy A jobb B-komodulus algebra koinvariánsai A^{coB} részalgebrát alkotnak. Így tekinthetjük a

$$\mathrm{can} : A \otimes_{A^{coB}} A \to A \otimes_R B \quad a' \otimes_{A^{coB}} a \mapsto a'a^0 \otimes_R a^1$$

kanonikus leképezést. Ha ez bijektív, akkor az $A^{coB} \to A$ algebra kiterjesztést B-*Galois-kiterjesztésnek* mondjuk.

Egy H Hopf-algebroidban egyszerre van jelen egy jobb R-bialgebroid és egy bal L-bialgebroid. Egyikük komodulusait sincs jobb okunk a Hopf-algebroid komodulusainak hívni. Ehelyett tekinthetjük az alábbi szimmetrikus fogalmat.

1.50. Definíció ([11] Definition 3.2)**.** Egy L és R algebrák fölötti H Hopf-algebroid *jobb komodulusa* alatt egy M k-modulust értünk, amely egyszerre komodulusa az alap jobb R-bialgebroidnak – valamely jobb R-hatás és egy $\varrho^R : M \to M \otimes_R {}^R M^R$ kohatás révén – és komodulusa az alap bal L-bialgebroidnak – valamely jobb L-hatás és egy $\varrho^L : M \to M \otimes_L {}_L M_R$ kohatás révén – továbbá ϱ^R jobb L-modulus homomorfizmus, ϱ^L jobb R-modulus homomorfizmus, és a két kohatásra az alábbi egyenlőségek teljesülnek.

$$(\mathrm{Id} \otimes_R \delta_L)\varrho^R = (\varrho^R \otimes_L \mathrm{Id})\varrho^L \qquad \text{és} \qquad (\mathrm{Id} \otimes_L \delta_R)\varrho^L = (\varrho^L \otimes_R \mathrm{Id})\varrho^R.$$

Azaz ϱ^R komodulus homomorfizmusa az alap bal L-bialgebroidnak és ϱ^L komodulus homomorfizmusa az alap jobb R-bialgebroidnak. H-komodulusok *homomorfizmusai* komodulus homomorfizmusai mind az alap bal L-bialgebroidnak mind az alap jobb R-bialgebroidnak.

A jobb H-komodulusok és homomorfizmusaik kategóriáját $\mathcal{M}^H$-val jelöljük. Megkülönböztetésül, az alap bal L-bialgebroid komodulusainak kategóriáját $\mathcal{M}^{H_L}$-val, míg az alap jobb R-bialgebroid komodulusainak kategóriáját $\mathcal{M}^{H_R}$-val jelöljük. Korábbi konvenciónkhoz hasonlóan, az alap jobb R-bialgebroid kohatására az $m \mapsto m^0 \otimes_R m^1$ index jelölést alkalmazzuk, míg az alap bal L-bialgebroid kohatására $m \mapsto m_0 \otimes_L m_1$-et, ahol mindkét esetben implicit összegzés értendő. Szimmetrikusan értelmezzük egy Hopf-algebroid bal komodulusait.

Ha M egy L és R algebrák fölötti H Hopf-algebroid jobb komodulusa, akkor tekinthetjük M koinvariánsait az alap jobb R-bialgebroid kohatására – ezek M azon m elemei, melyekre $m^0 \otimes_R m^1 = m \otimes_R 1$ – illetve M koinvariánsait az alap bal L-bialgebroid kohatására – ezek M azon m elemei, melyekre $m_0 \otimes_L m_1 = m \otimes_L 1$. A ([14] *Corrigendum*, Proposition 3) alapján, az előbbi értelemben vett koinvariánsok koinvariánsok az utóbbi értelemben is; és a két koinvariáns fogalom egybeesik mindazon esetekben, amikor az antipód bijektív.

1.51. Tétel ([14] *Corrigendum*, Theorem 6)**.** *Bármely, L és R algebrák fölötti H Hopf-algebroid esetén a H-komodulusok $\mathcal{M}^H$ kategóriája monoidális; az alábbi diagram kommutatív; és a benne szereplő (felejtő) funktorok szigorúan monoidálisak.*

$$\begin{array}{ccccc}
\mathcal{M}^H & & \xrightarrow{\quad G_R \quad} & & \mathcal{M}^{H_R} \\
{\scriptstyle G_L}\downarrow & & & & \downarrow \\
\mathcal{M}^{H_L} & \longrightarrow & {}_{L^{op}}\mathcal{M}_{L^{op}} & \xrightarrow{\cong} & {}_R\mathcal{M}_R
\end{array}$$

Hangsúlyozzuk, hogy – bár ellenpéldát nem ismerünk – tetszőleges Hopf-algebroidok esetén nem bizonyított, hogy a fenti diagramban szereplő G_L és G_R funktorok izomorfizmusok. Az ezt állító ([11] Theorem 3.1) bizonyításában az 1.39 Megjegyzésben leírt hiba található. A G_L és G_R funktorok izomorfizmusok bizonyos további feltevések mellett, például, ha H lapos bal L-modulus és lapos bal R-modulus.

Az 1.51 Tételre alapozva megfogalmazható a következő.

1.52. Definíció. Egy H Hopf-algebroid (jobb vagy bal) *komodulus algebrái* monoidok a (jobb vagy bal) H-komodulusok kategóriájában.

Expliciten, ha H Hopf-algebroid L és R algebrák fölött, akkor egy jobb H-komodulus algebra egy R-gyűrű, melynek $R \to H$ egysége és $H \otimes_R H \to H$ szorzása H-komodulus homomorfizmusok. Egy H-komodulus algebra komodulus algebrája mindkét alap bialgebroidnak.

Ha A egy L és R algebrák fölötti H Hopf-algebroid jobb komodulus algebrája, akkor – a két alap bialgebroid kohatásának megfelelően – két kanonikus Galois-leképezést tekinthetünk:

$$\begin{aligned} \mathrm{can}_R &: A \otimes_{B_R} A \to A \otimes_R {}^R H^R, \quad a' \otimes_{B_R} a \mapsto a'a^0 \otimes_R a^1 \quad \text{és} \\ \mathrm{can}_L &: A \otimes_{B_L} A \to A \otimes_L {}_L H_L, \quad a' \otimes_{B_L} a \mapsto a'_0 a \otimes_L a'_1, \end{aligned} \tag{1.10}$$

ahol B_R az A koinvariánsainak részalgebráját jelöli az alap jobb R-bialgebroid kohatására, míg B_L az A koinvariánsainak részalgebráját jelöli az alap bal L-bialgebroid kohatására.

Ha H antipódja bijektív, akkor $B_R = B_L$; és can_R pontosan akkor bijektív, ha can_L bijektív. Azaz ebben az esetben egy jobb H-komodulus algebra A pontosan akkor Galois-kiterjesztése $B_R = B_L$-nek az alap jobb R-bialgebroiddal ha Galois-kiterjesztése az alap bal L-bialgebroiddal.

1.4.2. Egy Kreimer–Takeuchi-típusú tétel

Kreimer és Takeuchi klasszikus 1.16 Tétele a következőképpen általánosítható Hopf-algebroidokra.

1.53. Tétel ([11] Lemma 3.3 and Corollary 4.3). *Tekintsünk egy H Hopf-algebroidot L és R k-algebrák fölött, amiben a H^R, ${}^R H$, ${}_L H$ és H_L modulusok mindegyike végesen generált projektív és aminek az antipódja bijektív. Legyen A egy jobb H-komodulus algebra (ami jelen esetben ugyanaz, mint bármelyik alap bialgebroid komodulus algebrája) és legyen B a koinvariáns részalgebra (a jobb- vagy ekvivalens módon a bal alap bialgebroid kohatására nézve). Ha $(A \otimes A)^{coH} \cong A \otimes B$ (pl. mert A lapos k-modulus), akkor a következő állítások ekvivalensek.*

- *$B \to A$ Galois-kiterjesztés az alap jobb R-bialgebroiddal, azaz az* (1.10)*-ben definiált* can_R *kanonikus leképezés bijektív;*
- *$B \to A$ Galois-kiterjesztés az alap bal L-bialgebroiddal azaz az* (1.10)*-ben definiált* can_L *kanonikus leképezés bijektív;*
- can_R *szürjektív;*
- can_L *szürjektív.*

A [11] munkában a fenti tétel egy sokkal általánosabb tételből (([68] Theorem 3.1) általánosításából) következik. Az 1.1.4 fejezetben látottakhoz hasonlóan, egy B jobb R-bialgebroid A jobb komodulus algebrája meghatároz egy

$$\psi : B \otimes_R A \to A \otimes_R B \qquad b \otimes_R a \mapsto a^0 \otimes_R ba^1 \tag{1.11}$$

vegyes disztributív szabályt az A R-gyűrű és a B R-kogyűrű között (melyben 1 csoportszerű elem, azaz $1^1 \otimes_R 1^2 = 1 \otimes_R 1$ és $\varepsilon(1) = 1$). Ha B az alap jobb bialgebroid egy Hopf-algebroidban, melynek antipódja bijektív, akkor a fenti ψ bijektív. Ha ráadásul ebben a Hopf-algebroidban az összes fellépő R-modulus végesen generált projektív, akkor ezek a modulusok laposak is; továbbá B projektív mint akár bal, akár jobb B-komodulus (abban az értelemben, hogy a $\mathrm{Hom}^B(B, -)$ funktor a B-komodulusok kategóriájából a k-modulusok kategóriájába őrzi az epimorfizmusokat), azaz ([11], Theorem 4.2) összes feltevése teljesül.

1.4.3. Erős és gyenge struktúra-tételek a Morita-elméletből

Hasonlóan az 1.1.3 fejezetben látottakhoz, ahogy azt [11] 5. fejezete tárgyalja, bármely H jobb R-bialgebroid A jobb komodulus algebrája esetén tekinthetjük az ún. *relatív Hopf-modulusok* $\mathcal{M}_A^H$-val jelölt kategóriáját. Ezt úgy definiáljuk, mint a jobb A modulusok kategóriáját a jobb H-komodulusok $\mathcal{M}^H$ kategóriájában (hiszen definíció szerint, A egy monoid $\mathcal{M}^H$-ban). Felhasználva azt az észrevételt, hogy A meghatároz egy vegyes disztributív szabályt (l. (1.11)), $\mathcal{M}_A^H$-ra tekinthetünk úgy is, mint egy $A \otimes_R H$ A-kogyűrű komodulusainak kategóriájára. Az A koinvariánsainak részalgebráját B-vel jelölve, tekinthetjük az

$$\mathcal{M}_B \xrightarrow{(-)\otimes_B A} \mathcal{M}_A^H \quad \dashv \quad \mathcal{M}_A^H \xrightarrow{(-)^{coH}} \mathcal{M}_B \tag{1.12}$$

adjungált funktor párt. Ebben a szövegkörnyezetben az erős és gyenge struktúra tételek a jobb adjungált hű teliségére; illetve ekvivalencia voltára vonatkozó feltételeket fogalmaznak meg.

Ha H végesen generált projektív bal R-modulus, akkor $\mathcal{M}_A^H$ izomorf az $A \otimes_R H$ A-kogyűrű duálisaként adódó A-gyűrű modulusainak kategóriájával. Ugyanúgy, mint a bialgebrák esetében, ez a duális gyűrű most is A-nak és H bal R-duálisának, ${}^*H := {}_R\mathrm{Hom}(H, R)$-nak ${}^*H \ltimes A$ féldirekt szorzataként írható (mely most is egy alkalmas koszorú szorzat). Ebben az esetben tehát az erős és gyenge struktúra-tételek visszavezethetők a következő Morita-összefüggés vizsgálatára. A két szereplő algebra B illetve ${}^*H \ltimes A$; a fellépő bimodulusok pedig A-n illetve $({}^*H \ltimes A)^{coH}$-n vannak definiálva. Ez a Morita-összefüggés speciális esete azon Morita-összefüggéseknek, melyet Caenepeel és társai [28]-ban csoportszerű elemmel rendelkező A-kogyűrűkhöz rendeltek, azon feltevés mellett, hogy a kogyűrű végesen generált projektív bal A-modulus. Szimmetrikusan kezelhetjük egy bal bialgebroid jobb komodulus algebráit. A következő *gyenge struktúra-tételt* úgy kapjuk, hogy a Morita-elmélet eredményeit kombináljuk azzal az előző fejezetben látott ténnyel, hogy az alap jobb- illetve bal bialgebroiddal vett Galois-kiterjesztések egybeesnek azon Hopf-algebroidok esetén, melyeknek antipódja bijektív s amelyekben az összes releváns modulus struktúra végesen generált projektív.

1.54. Tétel. *([11] Theorem 5.4) Tekintsünk egy H Hopf-algebroidot L és R algebrák fölött, amiben a H^R, ${}^RH$, ${}_LH$ és H_L modulusok mindegyike végesen generált projektív és aminek az antipódja bijektív. Legyen A egy jobb H-komodulus algebra (ami jelen esetben ugyanaz, mint bármelyik alap bialgebroid komodulus algebrája) és legyen B a koinvariáns részalgebra (a jobb- vagy ekvivalens módon a bal alap bialgebroid kohatására nézve). Ezen feltevések mellett a következő állítások ekvivalensek.*

- *$B \to A$ Galois-kiterjesztés az alap jobb bialgebroiddal;*
- *A projektív bal B-modulus és a kanonikus ${}^*H \ltimes A \to {}_B\mathrm{End}(A)$ algebra anti-homomorfizmus izomorfizmus;*
- *A generátor a jobb ${}^*H \ltimes A$-modulusok kategóriájában;*
- *$(-)^{coH} : \mathcal{M}_A^H \to \mathcal{M}_B$ hű és teli;*
- *A $({}^*H \ltimes A)^{coH} \otimes_B A \to {}^*H \ltimes A$ Morita-leképezés szürjekció (így bijekció);*
- *$B \to A$ Galois-kiterjesztés az alap bal bialgebroiddal;*

- *A projektív jobb B-modulus és a kanonikus ${}_*H \ltimes A \to \mathrm{End}_B(A)$ algebra anti-homomorfizmus izomorfizmus (ahol ${}_*H = {}_L\mathrm{Hom}(H, L)$ az alap bal bialgebroid bal duálisa);*

- *A generátor a bal ${}_*H \ltimes A$-modulusok kategóriájában;*

- *$(-)^{coH} : {}_A\mathcal{M}^H \to {}_B\mathcal{M}$ hű és teli (ahol ${}_A\mathcal{M}^H$ A bal modulusainak kategóriája a jobb H-komodulusok kategóriájában);*

- *Az $A \otimes_B ({}_*H \ltimes A)^{coH} \to {}_*H \ltimes A$ Morita-leképezés szürjekció (így bijekció).*

Egy C R-kogyűrű valamely M jobb komodulusát *relatíve injektívnek* mondjuk, ha mindazon $f : P \to Q$ jobb komodulus homomorfizmusokra, amelyek felhasadó jobb R-modulus monomorfizmusok, $\mathrm{Hom}^C(f, M) : \mathrm{Hom}^C(Q, M) \to \mathrm{Hom}^C(P, M)$ szürjekció. Egyfelől ([13], Proposition 4.1); másfelől ([5] Proposition 3.4) szerint, az 1.54 Tételben tárgyalt Galois-kiterjesztésekben A pontosan akkor relatíve injektív jobb/bal komodulusa az alap jobb (vagy bal) bialgebroidnak, ha a $B \to A$ beágyazás felhasadó jobb/bal B-modulus monomorfizmus, vagy ami evvel ekvivalens, A hűen lapos jobb/bal B-modulus. ([5] Theorem 4.1) szerint, az antipód bijektivitásának köszönhetően, A pontosan akkor relatíve injektív jobb komodulus ha relatíve injektív bal komodulus. Mindezen észrevételeket ötvözve ([23] Theorem 5.6)-tal illetve a Morita-elmélet eredményeivel, a következő *erős struktúra-tételre* jutunk.

1.55. Tétel ([11] Proposition 5.5, [5] Proposition 4.2)**.** *Tekintsünk egy H Hopf-algebroidot L és R algebrák fölött, amiben a H^R, ${}^RH$, ${}_LH$ és H_L modulusok mindegyike végesen generált projektív és aminek az antipódja bijektív. Legyen A egy jobb H-komodulus algebra (ami jelen esetben ugyanaz, mint bármelyik alap bialgebroid komodulus algebrája) és legyen B a koinvariáns részalgebra (a jobb- vagy ekvivalens módon a bal alap bialgebroid kohatására nézve). Ezen feltevések mellett a következő állítások ekvivalensek.*

- *$B \to A$ Galois-kiterjesztés az alap jobb (vagy bal) bialgebroiddal és A hűen lapos bal B-modulus;*

- *A projektív generátor a bal B-modulusok kategóriájában és a kanonikus ${}^*H \ltimes A \to {}_B\mathrm{End}(A)$ algebra anti-homomorfizmus izomorfizmus;*

- *$(-)^{coH} : \mathcal{M}_A^H \to \mathcal{M}_B$ ekvivalencia;*

- *A B és ${}^*H \ltimes A$ közötti Morita-összefüggés szigorú;*

- *$B \to A$ Galois-kiterjesztés az alap jobb (vagy bal) bialgebroiddal és A hűen lapos jobb B-modulus;*

- *A projektív generátor a jobb B-modulusok kategóriájában és egy és a kanonikus ${}_*H \ltimes A \to \mathrm{End}_B(A)$ algebra anti-homomorfizmus izomorfizmus;*

- *$(-)^{coH} : {}_A\mathcal{M}^H \to {}_B\mathcal{M}$ ekvivalencia;*

- *A B és ${}_*H \ltimes A$ közötti Morita-összefüggés szigorú.*

1.5. Gyenge Hopf-algebrák Doi–Hopf-modulusai

Ha R tetszőleges algebra egy k kommutatív gyűrű felett, akkor – mint láttuk – minden R-gyűrű k-algebra is. Ugyanakkor egy R-kogyűrű nem feltétlenül k-koalgebra, hacsak R nem rendelkezik további tulajdonságokkal.

Mint az 1.1.2 fejezetben felidéztük, R-et Frobenius-k-algebrának mondjuk, ha végesen generált és projektív k-modulus, továbbá R és $\mathrm{Hom}(R,k)$ izomorfak mint (jobb vagy bal) R-modulusok. Nem nehéz látni, hogy ez ekvivalens egy $\psi \in \mathrm{Hom}(R,k)$ ún. *Frobenius-funkcionál* és egy $\sum_i e_i \otimes f_i \in R \otimes_k R$ ún. *duális bázis* létezésével, melyek a $\sum_i \psi(re_i)f_i = r = e_i\psi(f_i r)$ egyenlőségeknek tesznek eleget, R minden r elemére. Ebből következik, hogy $\sum_i re_i \otimes f_i = \sum_i e_i \otimes f_i r$, R minden r elemére. Egy R Frobenius-algebra *indexe* a $\sum_i e_i f_i$ elem R centrumában. Ha ez egyenlő R egység elemével, akkor bármely M és N R-bimodulusok esetén az $M \otimes_R N \to M \otimes_k N$, $m \otimes_R n \mapsto \sum_i m.e_i \otimes_k f_i.n$ R-bimodulus homomorfizmus az $M \otimes_k N \twoheadrightarrow M \otimes_R N$ kanonikus epimorfizmus szelése. Eszerint R szeparábilis, hiszen az $R \to R \otimes R$, $r \mapsto \sum_i re_i \otimes f_i = \sum_i e_i \otimes f_i r$ R-bimodulus homomorfizmus a szorzás egy szelése.

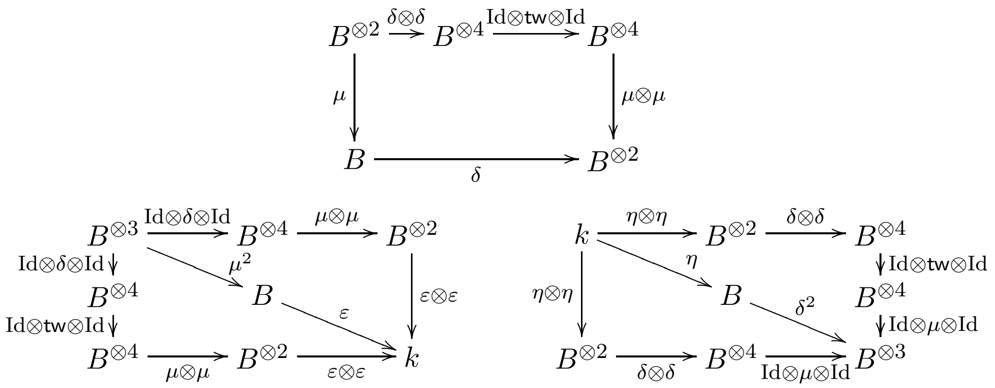
Ha B egy R-bialgebroid, ahol R 1 indexű Frobenius-algebra k felett, akkor B k-algebra és k-koalgebra is. A bialgebroid $B \to B \otimes_R B$ koszorzását komponálva a $B \otimes_R B \rightarrowtail B \otimes_k B$ szeléssel nyerjük a koalgebrai koszorzást. A koegység szerepét a Frobenius-funkcionál játssza. Az így nyert algebra és koalgebra struktúrák kielégítik a gyenge bialgebra axiómákat sőt, minden gyenge bialgebra előáll ilyen módon egy 1 indexű Frobenius-algebra fölötti bialgebroidból. A gyenge Hopf-algebrák pontosan azok a H, L és R anti-izomorf 1 indexű Frobenius-algebrák fölötti Hopf-algebroidok, melyekben a két alap (L- illetve R-) kogyűrű ugyanazon koalgebrából adódik a két $H \otimes_k H \twoheadrightarrow H \otimes_R H$ illetve $H \otimes_k H \twoheadrightarrow H \otimes_L H$ epimorfizmus széléseinek segítségével.

Ilyetén módon a gyenge bialgebrákra illetve gyenge Hopf-algebrákra vonatkozó eredmények egy része megkapható bialgebroidokra illetve Hopf-algebroidokra vonatkozó állítások speciális eseteként. Történetileg azonban számos ilyen tétel korábban, függetlenül született meg. Vannak továbbá olyan, erősebb tételek, melyek általánosítása tetszőleges Hopf-algebroidokra nem lehetséges – vagy más esetekben legalábbis nem ismert.

1.5.1. Gyenge Hopf-algebrák

A gyenge Hopf-algebrák axiómáinak első változata [18]-ban jelent meg, a gyenge bialgebrákat pedig [59] említi először. A struktúra részletes vizsgálata [17]-ben található. A *gyenge* jelző arra utal, hogy a bialgebra axiómák közül a koszorzás és az egység kompatibilitását megkövetelő $\delta(1) = 1 \otimes 1$; valamint a szorzás és a koegység kompatibilitását megkövetelő $\varepsilon(ab) = \varepsilon(a)\varepsilon(b)$ axiómák gyengíttetnek.

1.56. Definíció ([17] Definition 2.1)**.** Egy *gyenge bialgebra* (valamely k kommutatív gyűrű felett) egy k-modulus B, ellátva egy (η, μ) algebra struktúrával és egy (δ, ε) koalgebra struktúrával, melyekre az alábbi diagramokkal megfogalmazott kompatibilitási feltételek teljesülnek (ahol $\mathsf{tw} : V \otimes W \to W \otimes V$, $v \otimes w \mapsto w \otimes v$ a felcserélést, azaz a k-modulusok kategóriájának szimmetria operációját jelöli).



Tetszőleges B-beli a, b és c elemeken kiírva a gyenge bialgebra axiómák az alábbi alakot öltik.

$$(ab)_1 \otimes (ab)_2 = a_1 b_1 \otimes a_2 b_2$$
$$\varepsilon(ab_1)\varepsilon(b_2 c) = \varepsilon(abc) = \varepsilon(ab_2)\varepsilon(b_1 c) \quad 1_1 \otimes 1_2 1_{1'} \otimes 1_{2'} = 1_1 \otimes 1_2 \otimes 1_3 = 1_1 \otimes 1_{1'} 1_2 \otimes 1_{2'},$$

ahol $1_1 \otimes 1_2 = 1_{1'} \otimes 1_{2'}$ a $\delta(1)$ $B \otimes B$-beli elem példányait jelöli.

Vegyük észre, hogy a gyenge bialgebra axiómák önduálisak, azaz az őket reprezentáló diagramok összessége invariáns a nyilak megfordítására. Következésképpen, ha egy gyenge bialgebra B végesen generált projektív k-modulus (azaz létezik duálisa a k-modulusok kategóriájában) akkor a $\mathrm{Hom}(B, k)$ duális is gyenge bialgebra a transzponált struktúrával.

Egy gyenge bialgebrában a (ko)szorzást az ellentettjére cserélve ismét gyenge bialgebrát kapunk.

1.57. Állítás. *([17] Proposition 2.4 and Proposition 2.11) Bármely B gyenge bialgebrában a*

$$\sqcap^L : B \to B \quad b \mapsto \varepsilon(1_1 b)1_2 \qquad \text{és} \qquad \sqcap^R : B \to B \quad b \mapsto 1_1\varepsilon(b1_2)$$

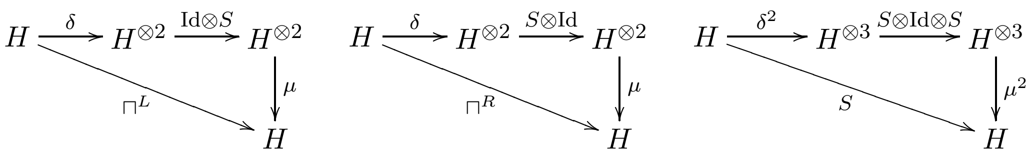
leképezések idempotensek; továbbá $B^L := \sqcap^L(B)$ és $B^R := \sqcap^R(B)$ (az ún. bal- illetve jobb részalgebrák*) egymással kommutáló, anti-izomorf 1-indexű (így szeparábilis) Frobenius-algebrák, melyekre $\delta(1) \in B^R \otimes B^L$.*

1.58. Lemma ([17] page 391.)**.** *Bármely B gyenge bialgebrára a következő állítások ekvivalensek:*

- *B bialgebra;*
- *$\delta(1) = 1 \otimes 1$;*
- *$\varepsilon(ab) = \varepsilon(a)\varepsilon(b)$, $\forall a, b \in B$;*
- *$\sqcap^R(a) = \varepsilon(a)1$, $\forall a \in B$;*
- *$\sqcap^L(a) = \varepsilon(a)1$, $\forall a \in B$.*

A 1.40 Tétel szerint minden B gyenge bialgebra meghatároz egy jobb bialgebroidot B^R fölött és egy bal bialgebroidot B^L fölött. A jobb bialgebroid $B^R \otimes (B^R)^{op}$-gyűrű struktúrája a $B^R \hookrightarrow B$ és a $(B^R)^{op} \to B$, $r \mapsto \varepsilon(r1_1)1_2$ algebra homomorfizmusokkal adott. A koszorzást a gyenge bialgebra $B \to B \otimes_k B$ koszorzását a $B \otimes_k B \twoheadrightarrow B^R \otimes_R {}^R B$ epimorfizmussal komponálva nyerjük a koegység pedig $\sqcap^R$. Szimmetrikusan, a bal bialgebroid $B^L \otimes (B^L)^{op}$-gyűrű struktúrája a $B^L \hookrightarrow B$ és a $(B^L)^{op} \to B$, $l \mapsto 1_1\varepsilon(1_2 l)$ algebra homomorfizmusokkal adott. A koszorzást a gyenge bialgebra $B \to B \otimes_k B$ koszorzását a $B \otimes_k B \twoheadrightarrow B_L \otimes_L {}_L B$ epimorfizmussal komponálva nyerjük a koegység pedig $\sqcap^L$. Ebből a megfigyelésből azonnal következik, hogy B jobb illetve bal modulusainak kategóriája monoidális a B^R illetve B^L fölötti modulus tenzor szorzás révén. (Mivel B^L és B^R 1 indexű Frobenius-algebrák, a B^R illetve B^L fölötti modulus tenzor szorzat izomorf a k-modulus tenzor szorzat megfelelő retraktumával). Igazolható, hogy egy gyenge bialgebra (azaz alap koalgebrája) komodulusainak kategóriája izomorf a megfelelő (akár bal akár jobb) bialgebroid komodulusainak kategóriájával, így az is monoidális a megfelelő modulus tenzor szorzás révén (ami izomorf a k-modulus tenzor szorzat megfelelő retraktumával).

1.59. Definíció ([17] Definition 2.1)**.** Egy *gyenge Hopf-algebra* egy gyenge bialgebra H, ellátva egy $H \to H$ *antipódnak* nevezett lineáris leképezéssel, mely az alábbi diagramokkal megfogalmazott axiómáknak tesz eleget.



Tetszőleges $h \in H$ elemen kiírva, a gyenge Hopf-algebra axiómák a következőek.

$$h_1 S(h_2) = \varepsilon(1_1 h)1_2 \qquad S(h_1)h_2 = 1_1\varepsilon(h 1_2) \qquad S(h_1)h_2 S(h_3) = S(h).$$

A gyenge Hopf-algebra axiómák önduálisak, így a végesen generált gyenge Hopf-algebrák duálisai is gyenge Hopf-algebrák a transzponált struktúrával.

Egy gyenge Hopf-algebrában mind a szorzást mind a koszorzást az ellentettjére cserélve ismét gyenge Hopf-algebrát kapunk az eredeti antipóddal. Ha az antipód izomorfizmus, akkor vagy a szorzást vagy a koszorzást az ellentettjére cserélve is gyenge Hopf-algebrát kapunk, ahol az antipód az eredeti antipód inverze.

1.60. Állítás ([17] Lemma 2.8 and Theorem 2.10)**.** *Egy gyenge Hopf-algebra antipódja egyértelmű (adott gyenge bialgebra esetén). Az antipód algebra anti-homomorfizmus és koalgebra anti-homomorfizmus. Egy test fölötti véges dimenziós gyenge Hopf-algebra antipódja bijektív.*

Bármely gyenge Hopf-algebra meghatároz egy Hopf-algebroidot a fent leírt bialgebroid struktúrákkal a bal- illetve jobb részalgebrák fölött, a gyenge Hopf-algebrai antipóddal.

1.61. Példa. Nyilvánvalóan, minden bialgebra gyenge bialgebra és minden Hopf-algebra gyenge Hopf-algebra. Gyenge bialgebra (gyenge Hopf-algebra) továbbá bialgebrák (Hopf-algebrák) véges direkt összege. A bal- és a jobb részalgebrát a direkt összeadandók egység elemei feszítik ki.

1.62. Példa. Bármely B gyenge bialgebrában a B^R és B^L részalgebrák által generált részalgebra gyenge Hopf-algebra. A koalgebra struktúra B koalgebra struktúrájának megszorításával adódik, az antipód $S(rl) := \varepsilon(r1_1)1_2\varepsilon(1_3 l)$, ha $r \in B^R$, $l \in B^L$. Az 1.60 Állítás szerint ezen S megszorításai $B^L \to (B^R)^{op}$ illetve $B^R \to (B^L)^{op}$ algebra izomorfizmusok.

1.63. Példa. Legyen $\mathcal{C}$ egy kis kategória véges sok objektummal és legyen k egy kommutatív gyűrű. A $\mathcal{C}$ morfizmusai által generált szabad k-modulus ellátható egy gyenge bialgebra struktúrával. Két morfizmus szorzatát a kompozíciójukként definiáljuk, ha az utóbbi létezik, egyébként 0-ként, majd ezt a szorzás műveletet k-lineárisan kiterjesztjük. Az egység elem az identitás morfizmusok összege $\mathcal{C}$ összes objektumára. A koszorzást a morfizmusokon az $f \mapsto f \otimes f$ diagonális leképezésként definiáljuk, majd ezt a műveletet k-lineárisan kiterjesztjük. A koegység értéke a morfizmusokon 1, majd ezt a műveletet k-lineárisan kiterjesztjük. A bal- és a jobb részalgebrát az identitás morfizmusok feszítik ki. Ha $\mathcal{C}$ grupoid (azaz minden morfizmus izomorfizmus), akkor a fenti gyenge bialgebra gyenge Hopf-algebra. Az antipódot a morfizmusokon az inverzként definiáljuk, majd ezt a műveletet k-lineárisan kiterjesztjük.

A fenti módon bármely n pozitív egész szám esetén az $n \times n$-es k-beli elemű mátrixok algebrája ellátható gyenge Hopf-algebra struktúrával. A mátrix egységeken a koszorzás a diagonális leképezés; az antipód a mátrix transzponálás. A bal- és a jobb részalgebrát a diagonális mátrixok alkotják.

1.64. Példa ([17] Appendix)**.** Bármely R 1 indexű Frobenius-algebra esetén a $B \otimes B^{op}$ algebra gyenge Hopf-algebra. A koszorzás az $\sum_i e_i \otimes f_i \in R \otimes R$ duális bázis segítségével definiálható $x \otimes y \mapsto \sum_i (x \otimes e_i) \otimes (f_i \otimes y)$ módon. A koegység a ψ Frobenius-funkcionált használva $x \otimes y \mapsto \psi(xy)$ alakú. Az antipód $x \otimes y \mapsto \sum_i y \otimes e_i \psi(x f_i)$.

1.5.2. Gyenge Doi–Hopf-modulusok

Mint az 1.1.4 fejezetben láttuk, egy bialgebra különböző Hopf- típusú modulusainak egységes leírása adható a Doi illetve Koppinen által bevezetett ún. Doi–Hopf-modulusok segítségével. Ezen fejezet – és az alapjául szolgáló [9] publikáció – **célja** egy analóg, a gyenge bialgebrák Hopf-típusú modulusait egységesítő fogalom kidolgozása és vizsgálata. Az itt tárgyalt gyenge Doi–Hopf-modulusok inspirálták a vegyes disztributív szabályok (vagy összekulcsoló struktúrák) Caenepeel és De Groot nevéhez fűződő „gyenge" általánosítását, aminek absztrakt kategóriaelméleti megalapozása [12]-ben található.

Egy gyenge bialgebra jobb komodulus algebrája monoid a jobb komodulusok monoidális kategóriájában. Expliciten, ez a következőt jelenti.

1.65. Definíció ([9] Definition 2.1)**.** Egy B gyenge bialgebra *jobb komodulus algebrája* egy algebra A ellátva egy $a \mapsto a_0 \otimes a_1$ jobb B komodulus struktúrával úgy, hogy minden $a, a' \in A$ elemre

$$(aa')_0 \otimes (aa')_1 = a_0 a'_0 \otimes a_1 a'_1 \qquad 1_0 a \otimes 1_1 = a_0 \otimes \sqcap^L(a_1).$$

Egy gyenge bialgebra jobb modulus koalgebrája komonoid a jobb modulusok monoidális kategóriájában. Expliciten, ez a következőt jelenti.

1.66. Definíció ([9] Definition 2.1)**.** Egy B gyenge bialgebra *jobb modulus koalgebrája* egy koalgebra C ellátva egy jobb B modulus struktúrával úgy, hogy minden $c \in C$, $b \in B$ elemre

$$(c.b)_1 \otimes (c.b)_2 = c_1.b_1 \otimes c_2.b_2 \qquad c. \sqcap^L (b) = \varepsilon(c_1.b)c_2.$$

1.67. Definíció ([9] Definition 2.2)**.** *(Jobb-jobb) gyenge Doi–Hopf-adatok* alatt olyan (A, B, C) hármasokat értünk, ahol B gyenge bialgebra, A jobb B-komodulus algebra és C jobb B-modulus koalgebra.

Elegendő jobb-jobb Doi–Hopf-adatokat bevezetnünk, hiszen a többi lehetőséget megkapjuk, ha a gyenge bialgebra (ko)szorzását az ellentettjére cseréljük és az így adódó gyenge bialgebra jobb-jobb Doi–Hopf-adatait tekintjük.

Gyenge Doi–Hopf-adatokhoz hozzárendelhetjük a következő modulus fogalmat.

1.68. Definíció ([9] Definition 3.1). Tekintsünk egy (A, B, C) gyenge Doi–Hopf-adatot. Ezen adatok *gyenge Doi–Hopf-modulusain* olyan M k-modulusokat értünk, melyek egyszerre jobb A modulusok és jobb C-komodulusok és minden $m \in M$, $a \in A$ elemre az $(m.a)_0 \otimes (m.a)_1 = m_0.a_0 \otimes m_1.a_1$ egyenlőség teljesül. Gyenge Doi–Hopf-modulusok morfizmusai jobb A-modulus jobb C-komodulus homomorfizmusok. A gyenge Doi–Hopf-modulusok és morfizmusaik kategóriáját $\mathcal{M}_A^C$-val jelöljük.

1.69. Tétel ([9] Section 3, Examples). *Legyen B egy gyenge bialgebra.*

- *Ha $A = B$ a reguláris B-komodulus algebra és $C = B^R$ a triviális B-modulus koalgebra ($r \mapsto r1_1 \otimes S(1_2) \equiv 1_1 \otimes S(1_2)r$ koszorzással, B koegységének megszorításával és az $r.b := \sqcap^R(rb)$ jobb B-hatással) akkor $\mathcal{M}_A^C$ izomorf a jobb B-modulusok kategóriájával;*
- *Ha $C = B$ a reguláris B-modulus koalgebra és $A = B^R$ a triviális B-komodulus algebra (az $r \mapsto 1_1 \otimes r1_2$ kohatással), akkor $\mathcal{M}_A^C$ izomorf a jobb B-komodulusok kategóriájával;*
- *Ha $C = B$ a reguláris B-modulus koalgebra és $A = B$ a reguláris B-komodulus algebra, akkor $\mathcal{M}_A^C$ izomorf B Hopf-modulusainak kategóriájával;*
- *Ha H egy gyenge Hopf-algebra, $B = H^{op} \otimes H$, $C = H$ és $A = H$ a*

  $$c.(h \otimes h') := hch' \qquad \text{és} \qquad a \mapsto a_2 \otimes (S(a_1) \otimes a_3)$$

  hatással illetve kohatással, akkor $\mathcal{M}_A^C$ izomorf H Yetter–Drinfel'd-modulusainak [56], [29] kategóriájával.

A ([9] Proposition 3.3) szerint az $\mathcal{M}_A^C \to \mathcal{M}_A$ felejtő funktor jobb adjungált, az $\mathcal{M}_A^C \to \mathcal{M}^C$ felejtő funktor pedig bal adjungált.

1.70. Tétel ([9] Proposition 4.2). *Ha egy (A, B, C) gyenge Doi–Hopf-adatban C végesen generált projektív k-modulus, akkor található egy alkalmas algebra amelynek modulus kategóriája izomorf $\mathcal{M}_A^C$-val.*

Ez a *gyenge féldirekt szorzatnak* nevezett algebra a következőképpen konstruálható meg. C koalgebra struktúrájának transzponálásával $C^* := \mathrm{Hom}(C, k)$ algebra; és a $C \otimes B \to C$ hatás transzponálásával C^* bal B-modulus. A szokásos $(a \otimes \phi)(b \otimes \psi) := a_0 b \otimes (a_1.\psi)\phi$ féldirekt szorzás formula $A \otimes C^*$-on asszociatív szorzást definiál. Ennek a szorzásnak azonban nincs egység eleme, az $1 \otimes \varepsilon$ elem centrális idempotens. Az általa generált ideál a keresett egység elemes asszociatív algebra, C^* és A ún. gyenge féldirekt szorzata. Ez $A \otimes C^*$ k-modulus retraktuma és elemei $1_0 a \otimes 1_1.\phi$ alakúak, ahol $a \in A$, $\phi \in C^*$.

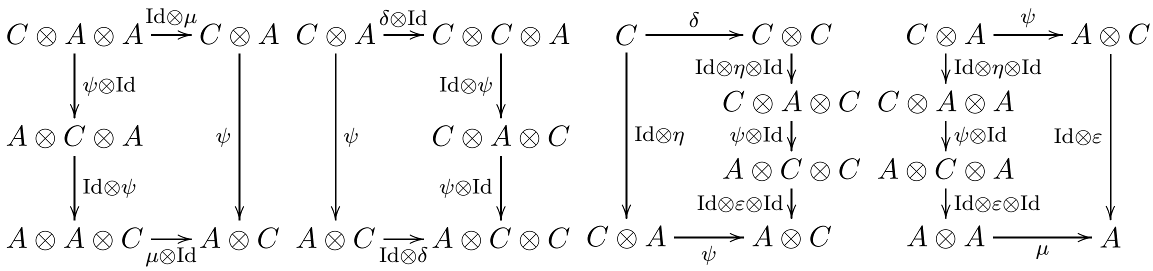
1.71. Tétel ([9] Section 4 Examples and Appendix). *Legyen B végesen generált projektív gyenge bialgebra.*

- *Ha $A = B$ a reguláris B-komodulus algebra és $C = B^R$ a triviális B-modulus koalgebra (l. 1.69 Tétel), akkor C^* és A gyenge féldirekt szorzata izomorf B-vel;*
- *Ha $C = B$ a reguláris B-modulus koalgebra és $A = B^R$ a triviális B-komodulus algebra (l. 1.69 Tétel), akkor C^* és A gyenge féldirekt szorzata izomorf B-duális algebrájával;*
- *Ha $C = B$ a reguláris B-komodulus algebra és $A = B$ a reguláris B-komodulus algebra, akkor C^* és A gyenge féldirekt szorzata izomorf B Heisenberg-duplumával;*

- *Ha H egy végesen generált projektív gyenge Hopf-algebra, $B = H^{op} \otimes H$, $C = H$ és $A = H$ az 1.69 Tételben látott hatással illetve kohatással, akkor C^* és A gyenge féldirekt szorzata izomorf H Drinfel'd-duplumával.*

Mint azt az 1.1.4 fejezetben láttuk, a bialgebrákra vonatkozó hasonló konstrukciók illetve állítások szépen illeszkednek a vegyes disztributív szabályok általánosabb keretébe. Természetesen merül fel a kérdés, beleilleszthetők-e a fenti általánosítások valamilyen gyenge vegyes disztributív szabályok, avagy gyenge összekulcsoló struktúrák elméletébe. Ezen motivációk alapján a következő definíciót Caenepeel és De Groot javasolták [25]-ben.

1.72. Definíció (Caenepeel and De Groot [25]). Egy *gyenge vegyes disztributív szabály* avagy *gyenge összekulcsoló struktúra* áll egy A algebrából, egy C koalgebrából és egy $\psi : C \otimes A \to A \otimes C$ k-modulus homomorfizmusból, melyre az alábbi diagramok kommutálnak.



Az első két diagram, mely ψ (ko)szorzással való kompatibilitását fejezi ki, ugyanaz, mint a Beck féle vegyes disztributív szabály esetén. Csak a második két diagram gyengíttetett, mely ψ (ko)egységgel való kompatibilitását fejezi ki.

Minden (A, B, C) gyenge Doi–Hopf-adat meghatároz egy (A, C, ψ) gyenge összekapcsoló struktúrát, ahol $\psi(c \otimes a) = a_0 \otimes c.a_1$.

A gyenge Doi–Hopf-modulusok kategóriáját általánosítva, gyenge összekulcsoló struktúrákhoz hozzárendelhető *összekulcsolt modulusaik* $\mathcal{M}_A^C$ kategóriája. Igazolható, hogy a $\mathcal{M}_A^C \to \mathcal{M}_A$ felejtő funktor jobb adjungált, a $\mathcal{M}_A^C \to \mathcal{M}^C$ felejtő funktor pedig bal adjungált. Ha egy gyenge összekulcsoló struktúrában C végesen generált projektív k-modulus, akkor $\mathcal{M}_A^C$ izomorf egy, az A és $C^* := \mathrm{Hom}(C, k)$ algebrák gyenge féldirekt szorzataként adódó algebra modulus kategóriájával. Mindennek az 1.1.4 fejezetben látotthoz hasonló, absztrakt kategóriaelméleti leírása csak mintegy tíz évvel később, [12]-ben történt meg, l. 1.6 fejezet.

1.6. Gyenge Hopf-algebrákra épülő konstrukciók és a monádok gyenge elmélete

E fejezet – és az alapjául szolgáló [12] munka – **célja**, hogy gyenge bialgebrákra épülő konstrukcióknak – így az 1.5 fejezet definícióinak – kategóriaelméleti megalapozását adja. Kidolgozzuk a monádok 1.1.5 fejezetben megismert „formális elméletének” alkalmas általánosítását, amely olyanformán képes leírni ezeket a konstrukciókat, ahogy a klasszikus elmélet leírta bialgebrai megfelelőiket, l. 1.1.5 fejezet.

1.6.1. $\mathrm{EM}^w(\mathcal{K})$ és a gyenge koszorú szorzat

Az 1.1.5 fejezetben bialgebrák modulus algebráinak és modulus koalgebrái duálisának féldirekt szorzatát mint koszorú szorzatot írtuk le – azaz megmutattuk, hogy egy EM(Bim)-

mel jelölt bikategória monádjából származtatható. A következőkben a féldirekt szorzat 1.5 fejezetben szereplő gyenge általánosításának hasonló szellemű leírását adjuk.

Legyen $\mathcal{K}$ egy tetszőleges bikategória. Rendeljük hozzá a következő, $\mathrm{EM}^w(\mathcal{K})$-val jelölt bikategóriát. A 0-cellák legyenek a monádok $\mathcal{K}$-ban, l. (1.2). Az $(A,t) \to (A',t')$ 1-cellák legyenek $x : A \to A'$ 1-cellák $\mathcal{K}$-ban, ellátva egy $\psi : t'x \to xt$ $\mathcal{K}$-beli 2-cellával, melyre (1.3) első diagramja kommutatív. A második diagramot egyszerűen vessük el. Hogy mégis bikategóriát kapjunk, ezt a 2-cellákra kirótt további feltétellel kompenzáljuk. Az $(x,\psi) \to (y,\phi)$ 2-cellák $\mathcal{K}$-beli $\varrho : x \to yt$ 2-cellák, melyekre az (1.4) diagram és

$$\begin{array}{ccccc} x & & \xrightarrow{\varrho} & & yt \\ \big\downarrow{\scriptstyle\varrho} & & & & \big\uparrow{\scriptstyle \mathrm{Id}\mu} \\ yt & \xrightarrow[\eta'\mathrm{Id}]{} & t'yt & \xrightarrow[\phi\mathrm{Id}]{} & ytt \end{array}$$

kommutatív. Az $\mathrm{EM}^w(\mathcal{K})$ bikategória lokálisan teli rész-bikategóriaként tartalmazza $\mathrm{EM}(\mathcal{K})$-t. Vegyük észre, hogy a $\mathcal{K}$-beli $\mathrm{Id}\eta : x \to xt$ 2-cella nem 2-cella $\mathrm{EM}^w(\mathcal{K})$-ban. Az $(x,\psi) \to (x,\psi)$ identitás 2-cella $\psi.\eta'\mathrm{Id} : x \to xt$. Az (1.3) második diagramjának elvetése miatt

$$xt \xrightarrow{\eta'\mathrm{Id}} t'xt \xrightarrow{\psi\mathrm{Id}} xtt \xrightarrow{\mathrm{Id}\mu} xt$$

nem identitás 2-cella $\mathcal{K}$-ban, de idempotens.

Az $\mathrm{EM}^w(\mathcal{K})$ bikategória monádjait hívhatjuk akár *gyenge koszorúknak* is.

A monád fogalom általánosításaként tekintsük az alábbi definíciót.

1.73. Definíció ([12] Definition 2.1)**.** Egy *pre-monád* egy $\mathcal{K}$ bikategóriában egy $t : A \to A$ 1-cella, ellátva $\eta : \mathrm{Id}_A \to t$ (pre-egység) és $\mu : tt \to t$ (szorzás) 2-cellákkal, melyekre az alábbi diagramok kommutatívak.

$$\begin{array}{ccc} ttt & \xrightarrow{\mu\mathrm{Id}} & tt \\ {\scriptstyle\mathrm{Id}\mu}\big\downarrow & & \big\downarrow{\scriptstyle\mu} \\ tt & \xrightarrow[\mu]{} & t \end{array} \qquad \begin{array}{ccc} t & \xrightarrow{\eta\mathrm{Id}} & tt \\ {\scriptstyle\mathrm{Id}\eta}\big\downarrow & & \big\downarrow{\scriptstyle\mu} \\ tt & \xrightarrow[\mu]{} & t \end{array} \qquad \begin{array}{ccc} 1_A & \xrightarrow{\eta\eta} & tt \\ & {\scriptstyle\eta}\searrow & \big\downarrow{\scriptstyle\mu} \\ & & t \end{array} \qquad \begin{array}{ccc} tt & \xrightarrow{\eta\mathrm{Id}} & ttt \\ & {\scriptstyle\mu}\searrow & \big\downarrow{\scriptstyle\mu^2} \\ & & t \end{array}$$

Bármely pre-monádra a $\mu.t\eta : t \to t$ 2-cella idempotens. Tegyük fel, hogy ez az idempotens 2-cella felhasad, azaz létezik egy $\widehat{t} : A \to A$ 1-cella valamint $i : \widehat{t} \to t$ és $p : t \to \widehat{t}$ 2-cellák, melyekre $p.i = \mathrm{Id}$ és $i.p$ egyenlő a kérdéses idempotens 2-cellával. Ekkor $\widehat{t}$ monád $\mathcal{K}$-ban,

$$\mathrm{Id}_A \xrightarrow{\eta} t \xrightarrow{p} \widehat{t} \quad \text{és} \quad \widehat{t}\,\widehat{t} \xrightarrow{ii} tt \xrightarrow{\mu} t \xrightarrow{p} \widehat{t}$$

egységgel illetve szorzással.

1.74. Tétel ([12] Theorem 2.3)**.** *Tekintsünk egy $\mathcal{K}$ bikategóriát, egy (A,t) monádot és egy $s : A \to A$ 1-cellát $\mathcal{K}$-ban. Bijektív kapcsolat van az alábbi struktúrák között.*

- $((A,t),(s,\psi))$ *alakú monádok* $\mathrm{EM}^w(\mathcal{K})$*-ban;*
- (A,st) *alakú pre-monádok $\mathcal{K}$-ban, melyek $\Theta : stst \to st$ szorzására $\Theta.sts\mu = s\mu.\Theta t$.*

Ha a fenti tételben az (A,st) pre-monád idempotens $st \to st$ 2-cellája felhasad, akkor az ezáltal meghatározott $(A,\widehat{st})$ retraktum monádot $\mathcal{K}$-ban hívhatjuk t és s *gyenge koszorú szorzatának.*

1.75. Példa. Legyen (A, B, C) egy jobb-jobb gyenge Doi–Hopf-adat. Ehhez hozzárendelhetünk egy $((k, A), (C^*, \psi))$ monádot $\mathrm{EM}^w(\mathrm{Bim})$-ben, ahol

$$\psi : A \otimes C^* \to C^* \otimes A \qquad a \otimes \phi \mapsto \phi(-a_1) \otimes a_0.$$

A hozzátartozó pre-monád $C^* \otimes A$ a $(\phi \otimes a)(\phi' \otimes a') = \phi'(-a_1)\phi \otimes a_0 a'$ szorzással és az $\varepsilon(-1_1) \otimes 1_0$ pre-egység elemmel. A

$$C^* \otimes A \to C^* \otimes A \qquad \phi \otimes a \mapsto \phi(-1_1) \otimes 1_0 a$$

idempotens leképezés felhasad, így képe egység elemes és asszociatív algebra, mely (a tenzor faktorok felcserélése révén) izomorf C^* és A gyenge féldirekt szorzatával, l. 1.5 fejezet.

További példaként a gyenge Hopf-algebrákkal [65]-ben vett kereszt szorzatok is gyenge koszorú szorzatok. A gyenge koszorú szorzat fenti konstrukciója egybeesik a [40]-ben (bármiféle kategória elméleti megfontolásra való hivatkozás nélkül) javasolt gyenge kereszt szorzattal.

Tekintsünk egy $\mathcal{K}$ bikategóriát, amelyben minden idempotens 2-cella felhasad, és amelyben léteznek a monádok Eilenberg–Moore-objektumai (azaz a $\mathcal{K} \to \mathrm{Mnd}(\mathcal{K})$ diagonális bifunktornak van egy R jobb adjungáltja, l. 1.1.5 fejezet). Ezen feltevések mellett konstruálható egy $J^w : \mathrm{EM}^w(\mathcal{K}) \to \mathcal{K}$ pszeudo-funktor, l. ([12] Theorem 3.5). („Pszeudo" volta annyit tesz, hogy a horizontális kompozíciót nem szigorúan, csak koherens izomorfizmusok erejéig őrzi). A 0-cellákon (monádokon) legyen $J^w(A, t) := R(A, t) = A^t$ a szokásos Eilenberg–Moore-objektum. Egy $(x, \psi) : (A, t) \to (A', t')$ 1-cella esetén, az 1.23 Tételben bevezetett $f \dashv v : A^t \to A$ adjunkció és annak ϵ koegysége segítségével tekintsük az

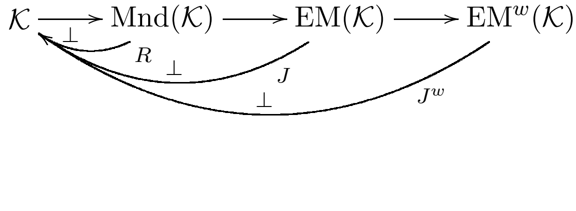
$$xv \xrightarrow{\eta' \mathrm{Id}} t'xv \xrightarrow{\psi} xtv \xrightarrow{\mathrm{Id}\epsilon} xv \tag{1.13}$$

idempotens 2-cellát $\mathcal{K}$-ban. Feltevésünk szerint ez felhasad, azaz létezik egy $p : xv \to \widehat{x}$ epi 2-cella $i : \widehat{x} \to xv$ szeléssel úgy, hogy $i.p$ egyenlő az (1.13) idempotens 2-cellával. Nem nehéz látni, hogy $(\widehat{x}, \widehat{\psi} := p.\mathrm{Id}\epsilon.\psi\mathrm{Id}.\mathrm{Id}i) : (A^t, \mathrm{Id}) \to (A', t')$ 1-cella $\mathrm{Mnd}(\mathcal{K})$-ban. Így értelmezhető $J^w(x, \psi) := R(\widehat{x}, \widehat{\psi})$. Hasonlóan, egy $\varrho : (x, \psi) \to (y, \phi)$ 2-cellára

$$\widehat{\varrho} := (\ \widehat{x} \xrightarrow{i} xv \xrightarrow{\varrho \mathrm{Id}} ytv \xrightarrow{\mathrm{Id}\epsilon} yv \xrightarrow{p} \widehat{y}\ ) : (\widehat{x}, \widehat{\psi}) \to (\widehat{y}, \widehat{\phi})$$

2-cella $\mathrm{Mnd}(\mathcal{K})$-ban, így értelmezhető $J^w(\varrho) := R(\widehat{\varrho})$. Jegyezzük meg, hogy J^w konstrukciójához idempotens 2-cellák felhasadását használtuk. A felhasadás azonban csak izomorfizmus erejéig egyértelmű, így J^w csak pszeudo-természetes izomorfizmus erejéig egyértelmű. Ugyancsak a felhasadás nem egyértelműségéből következik, hogy általában J^w csak pszeudo-funktor (nem bifunktor).

Az alábbi diagram balról jobbra mutató nyilai rendre a diagonális bifunktort; a 0- és 1-cellákon identitásként ható és egy ω 2-cellát $\mathrm{Id}\eta.\omega$-ba vivő bifunktort; illetve a nyilvánvaló beágyazást jelölik. A J bifunktor J^w megszorítása. Az R, J és J^w bifunktorok mindegyike jobb biadjungált, ahogy az ábra mutatja.



1.6.2. Gyenge felhúzás

Mivel $(x,\psi):(A,t)\to(A',t')$ $\mathrm{EM}^w(\mathcal{K})$-beli 1-cellák esetén $\mathrm{Id}\eta : x\to xt$ nem 2-cella $\mathrm{EM}^w(\mathcal{K})$-ban, nem természetes kérdés felvetés, hogy mely ω $\mathcal{K}$-beli 2-cellákra lesz $\mathrm{Id}\eta.\omega$ 2-cella $\mathrm{EM}^w(\mathcal{K})$-ban. Ehelyett megkérdezhetjük, hogy mely ω esetén lesz

$$x \xrightarrow{\eta'\mathrm{Id}} t'x \xrightarrow{\psi} xt \xrightarrow{\omega\mathrm{Id}} yt \qquad \text{illetve} \qquad x \xrightarrow{\omega} y \xrightarrow{\eta'\mathrm{Id}} t'y \xrightarrow{\phi} yt$$

$(x,\psi)\to(y,\phi)$ 2-cella $\mathrm{EM}^w(\mathcal{K})$-ban. A válasz az, hogy pontosan akkor, ha a

$$\begin{array}{ccccc} t'x & \xrightarrow{\psi} & xt & \xrightarrow{\omega\mathrm{Id}} & yt \\ {\scriptstyle \mathrm{Id}\eta'\mathrm{Id}}\downarrow & & & & \uparrow{\scriptstyle \mathrm{Id}\mu} \\ t't'x & & & & \\ {\scriptstyle \mathrm{Id}\psi}\downarrow & & & & \\ t'xt & \xrightarrow[\mathrm{Id}\omega\mathrm{Id}]{} & t'yt & \xrightarrow[\phi\mathrm{Id}]{} & ytt \end{array} \qquad \text{illetve} \qquad \begin{array}{ccccc} t'x & \xrightarrow{\mathrm{Id}\omega} & t'y & \xrightarrow{\phi} & yt \\ {\scriptstyle \psi}\downarrow & & & & \uparrow{\scriptstyle \mathrm{Id}\mu} \\ xt & & & & \\ {\scriptstyle \omega\mathrm{Id}}\downarrow & & & & \\ yt & \xrightarrow[\eta'\mathrm{Id}]{} & t'yt & \xrightarrow[\phi\mathrm{Id}]{} & ytt \end{array} \tag{1.14}$$

diagram kommutatív. Ha (x,ψ) 1-cella $\mathrm{EM}(\mathcal{K})$-ban (azaz kompatibilis a monádok egységével (1.3) második diagramjának értelmében), akkor az (1.14)-beli első diagram redukálódik (1.5)-té, ha (y,ϕ) 1-cella $\mathrm{EM}(\mathcal{K})$-ban akkor az (1.14)-beli második diagram redukálódik (1.5)-té. Tetszőleges (x,ψ) és (y,ϕ) $\mathrm{EM}^w(\mathcal{K})$-beli 1-cellák esetén a két (1.14)-beli diagram szimultán kommutativitása ekvivalens (1.5)-tel.

Vezessük be a következő két, $\mathrm{Mnd}^i(\mathcal{K})$-val illetve $\mathrm{Mnd}^p(\mathcal{K})$-val jelölt bikategóriát. A 0-cellák és 1-cellák mindkettőben legyenek ugyanazok, mint $\mathrm{EM}^w(\mathcal{K})$-ban. Az $(x,\psi)\to(y,\phi)$ 2-cellák $\mathrm{Mnd}^i(\mathcal{K})$-ban illetve $\mathrm{Mnd}^p(\mathcal{K})$-ban legyenek $\omega : x\to y$ $\mathcal{K}$-beli 2-cellák, amikre (1.14) első illetve második diagramja kommutatív. Ellenőrizhető, hogy ezek bikategóriák és vannak $G^i : \mathrm{Mnd}^i(\mathcal{K}) \to \mathrm{EM}^w(\mathcal{K}) \leftarrow \mathrm{Mnd}^p(\mathcal{K}) : G^p$ bifunktorok, melyek identitás leképezésként hatnak a 0-cellákon és az 1-cellákon, egy ω 2-cellát pedig $\omega\mathrm{Id}.\psi.\eta'\mathrm{Id}$-be illetve $\phi.\eta'\mathrm{Id}.\omega$-ba visznek. Ezek a bikategóriák fontos szerepet játszanak az alábbi *gyenge felhúzás* problémában.

1.76. Definíció ([12] Definition 4.1)**.** Legyen $\mathcal{K}$ egy bikategória, melyben léteznek a monádok Eilenberg–Moore-objektumai. Legyen adott $\mathcal{K}$-ban egy (A,t) és egy (B,s) monád. Egy $\overline{x} : A^t \to B^s$ 1-cellát az $x : A\to B$ 1-cella *gyenge felhúzásának* mondjuk, ha léteznek

$$\begin{array}{ccc} A^t & \xrightarrow{\overline{x}} & B^s \\ {\scriptstyle v}\downarrow & \Downarrow_i & \downarrow{\scriptstyle v} \\ A & \xrightarrow[x]{} & B \end{array} \qquad \text{és} \qquad \begin{array}{ccc} A^t & \xrightarrow{\overline{x}} & B^s \\ {\scriptstyle v}\downarrow & \Uparrow_p & \downarrow{\scriptstyle v} \\ A & \xrightarrow[x]{} & B \end{array}$$

2-cellák, melyekre $p.i = \mathrm{Id}$.

Egy 2-cella gyenge felhúzásait kétféle értelemben definiálhatjuk.

1.77. Definíció ([12] Definition 4.2)**.** Legyen $\mathcal{K}$ egy bikategória, melyben léteznek a monádok Eilenberg–Moore-objektumai. Legyen adott $\mathcal{K}$-ban egy (A,t) és egy (B,s) monád továbbá $x,y : A\to B$ 1-cellák $\overline{x},\overline{y} : A^t \to B^s$ gyenge felhúzásokkal. Egy $\overline{\omega} : \overline{x}\to\overline{y}$ 2-cellát az $\omega : x\to y$ 2-cella *i-felhúzásának* mondunk ha az alábbi első ($\mathcal{K}$-beli 2-cellákra vonatkozó) diagram kommutatív. $\overline{\omega}$ az ω *p-felhúzása* ha az alábbi második diagram kommutatív.

$$\begin{array}{ccc} v\overline{x} & \xrightarrow{i} & xv \\ {\scriptstyle \mathrm{Id}\,\overline{\omega}}\downarrow & & \downarrow{\scriptstyle \omega\,\mathrm{Id}} \\ v\overline{y} & \xrightarrow[i]{} & yv \end{array} \qquad\qquad \begin{array}{ccc} v\overline{x} & \xleftarrow{p} & xv \\ {\scriptstyle \mathrm{Id}\,\overline{\omega}}\downarrow & & \downarrow{\scriptstyle \omega\,\mathrm{Id}} \\ v\overline{y} & \xleftarrow[p]{} & yv \end{array}$$

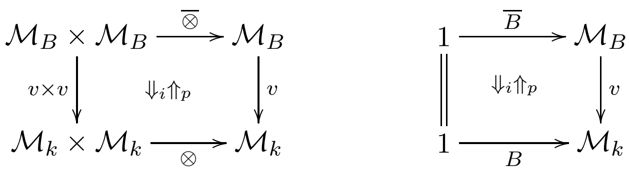
Ha egy $\mathcal{K}$ bikategóriában léteznek a monádok Eilenberg–Moore objektumai, akkor adott (A,t) és (B,s) $\mathcal{K}$-beli monádokra tekinthetjük az alábbi $\mathrm{Lift}^i((A,t),(B,s))$-vel illetve $\mathrm{Lift}^p((A,t),(B,s))$-vel jelölt kategóriákat. Mindkét kategória objektumai legyenek $(x:A\to B, \overline{x}:A^t\to B^s, i:v\overline{x}\to xv, p:xv\to v\overline{x})$ négyesek, ahol $p.i=\mathrm{Id}$ (így $\overline{x}$ az x gyenge felhúzása). Az $(x,\overline{x},i,p)\to(x',\overline{x}',i',p')$ morfizmusok legyenek $(\omega:X\to x', \overline{\omega}:\overline{x}\to\overline{x}')$ párok, ahol $\overline{\omega}$ az ω i-felhúzása illetve p-felhúzása.

1.78. Tétel ([12] Theorem 4.4)**.** *Legyen $\mathcal{K}$ egy bikategória, melyben léteznek a monádok Eilenberg–Moore-objektumai és melyben minden idempotens 2-cella felhasad. Bármely (A,t) és (B,s) monádokra $\mathcal{K}$-ban az alábbi kategóriák ekvivalensek.*

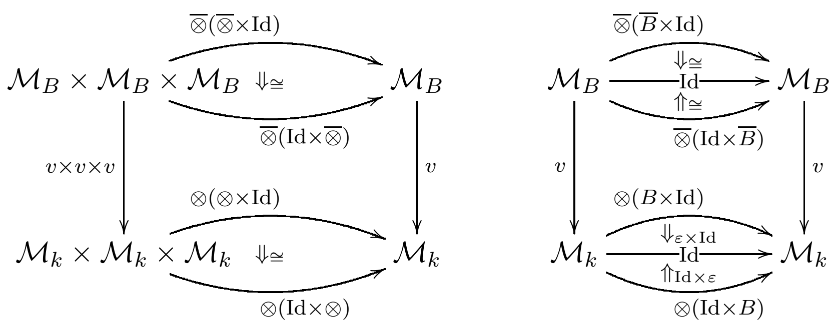
- $\mathrm{Lift}^i((A,t),(B,s))$ *és* $\mathrm{Mnd}^i(\mathcal{K})((A,t),(B,s))$*;*
- $\mathrm{Lift}^p((A,t),(B,s))$ *és* $\mathrm{Mnd}^p(\mathcal{K})((A,t),(B,s))$.

Ha $\omega:(x,\psi)\to(x',\psi')$ morfizmus (mondjuk) $\mathrm{Mnd}^i(\mathcal{K})((A,t),(B,s))$-ben, akkor a fenti ekvivalencia általi képe az $(\omega, J^wG^i(\omega)):(x,J^wG^i(x,\psi),i,p)\to(x',J^wG^i(x',\psi'),i',p')$ morfizmus $\mathrm{Lift}^i((A,t),(B,s))$-ben, ahol i és p (i' és p') a J^w konstrukciójához használt 2-cellák $\mathcal{K}$-ban. A fenti tételben látható ekvivalenciák általában nem izomorfizmusok (szemben az 1.25 Tétellel), mivel idempotens morfizmusok felhasadása csak izomorfizmus erejéig egyértelmű.

1.79. Példa. Bármely B gyenge bialgebrára $\mathcal{M}_B$ monoidális struktúrája gyenge felhúzással adódik. Míg a monoidális szorzás $\mathcal{M}_k$ monoidális szorzásának (mint $\otimes:\mathcal{M}_k\times\mathcal{M}_k\to\mathcal{M}_k$ funktornak) a gyenge felhúzása, a monoidális egység nem $\mathcal{M}_k$ monoidális egységének, hanem a $B:1\to\mathcal{M}_k$ funktornak a gyenge felhúzása, ahol 1 a terminális kategóriát jelöli, melynek egyetlen objektuma van és egyetlen morfizmusa ennek identitása. Rajzban tehát, léteznek



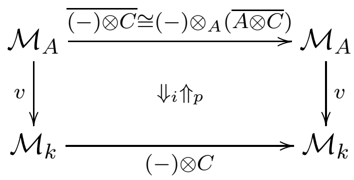
természetes transzformációk, melyek segítségével az $\mathcal{M}_B$ monoidális kategória koherencia természetes izomorfizmusai a rajz szerinti i-felhúzással adódnak.



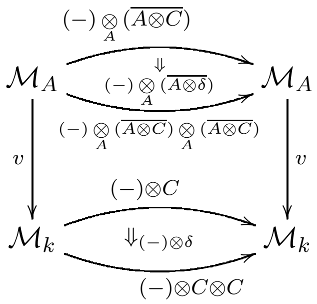
A fenti ábrákon $v:\mathcal{M}_B\to\mathcal{M}_k$ a felejtő funktort jelöli, l. 1.24 Példa.

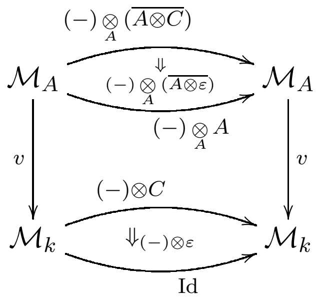
Az állítás megfordításához, azaz ahhoz a következtetéshez, hogy valamely adott B algebra rendelkezik gyenge bialgebra struktúrával, nem elég feltenni, hogy $\mathcal{M}_B$ monoidális valamely gyenge felhúzással adódó monoidális struktúrával – ráadásul a gyenge felhúzásért felelős $\mathrm{Mnd}^i(\mathrm{Cat})$-beli 1-cellák alakjára is feltevést kell tenni, l. 1.84 és 1.85 Tétel.

1.80. Példa. Tekintsünk egy A algebrát és egy C koalgebrát valamely k kommutatív gyűrű fölött, és közöttük egy $\psi : C \otimes A \to A \otimes C$ gyenge vegyes disztributív szabályt Bim-ben. Ez ugyanaz, mint egy $(-)\otimes\psi$ gyenge vegyes disztributív szabály Cat-ban a $(\mathcal{M}_k, (-)\otimes A)$ monád és a $(\mathcal{M}_k, (-) \otimes C)$ komonád között. Ezen feltevések mellett a $(\mathcal{M}_k, (-) \otimes C)$ komonádnak létezik i-felhúzása $\mathcal{M}_A$-ra. Rajzban, léteznek



természetes transzformációk (vagy ami ugyanaz, $A \otimes C \twoheadrightarrow \overline{A \otimes C}$ és $\overline{A \otimes C} \rightarrowtail A \otimes C$ bal A-modulus homomorfizmusok), melyek segítségével a gyengén felhúzott $(\mathcal{M}_A, (-) \otimes_A \overline{(A \otimes C)})$ komonád koszorzása és koegysége a rajz szerinti i-felhúzással adódik.





Több is igaz, ([12] Proposition 5.7) szerint egy A algebra, egy C koalgebra és egy ψ : $C \otimes A \to A \otimes C$ lineáris leképezés pontosan akkor alkotnak gyenge vegyes disztributív szabályt Bim-ben, ha ψ az $(\mathcal{M}_k, (-) \otimes C)$ komonád i-felhúzását indukálja $\mathcal{M}_A$-ra a fenti értelemben – azaz $((k, A), (C, \psi))$ komonád $\mathrm{Mnd}^i(\mathrm{Bim})$-ben – és az $(\mathcal{M}_k, (-) \otimes A)$ monád p-felhúzását indukálja $\mathcal{M}^C$-re a duális értelemben – azaz $((k, C), (A, \psi))$ monád $\mathrm{Mnd}^i(\mathrm{Bim}_*)_*$-ben, ahol $(-)_*$ az ellentett vertikális kompozíciójú bikategóriát jelöli. Ebben az esetben a gyengén felhúzott monád és komonád Eilenberg–Moore-kategóriái izomorfak [16] szerint és visszaadják a Caenepeel és De Groot által bevezetett összekulcsolt modulusok kategóriáját, l. 1.5 fejezet.

Vegyes disztributív szabályok más általánosításait – a Caenepeel és Janssen által [26]-ban bevezetett parciális- illetve lax összekulcsoló struktúrákat – ([12] Section 5) vizsgálja a gyenge felhúzások nézőpontjából.

1.7. Gyenge bimonádok

Mint az 1.1.1 fejezetben felidéztük, a bialgebrák jellemzhetők, mint pontosan azok a B k-algebrák, melyek modulusainak kategóriája (azaz az $(\mathcal{M}_k, (-)\otimes B)$ monád Eilenberg–Moore-kategóriája) monoidális úgy, hogy a felejtő funktor a k-modulusok kategóriájába szigorúan monoidális Ezen a leíráson alapszik a bialgebrák Moerdijk által [53]-ban javasolt általánosítása, melyet ő eredetileg „Hopf-monádnak” hívott, de amelyre azóta inkább a kifejezőbb „bimonád” név használatos.

A monádok (e fejezetben konkrétan a Cat 2-kategóriában) az algebrák természetes általánosításai. Definíció szerint, egy monád (Cat-ban) egy hármas, mely áll egy $T : \mathcal{C} \to \mathcal{C}$

funktorból, egy $m : T^2 \to T$ és egy $u : \mathrm{Id} \to T$ természetes transzformációból, melyek a szokásos asszociativitás és egység feltételeknek tesznek eleget. Egy T monádhoz hozzárendelhetjük az ún. Eilenberg–Moore kategóriáját (l. 1.21 Példa).

Moerdijk definíciója szerint egy *bimonád* egy monád valamely monoidális kategórián, ellátva mindazon struktúrákkal, melyek ekvivalensek e monád Eilenberg–Moore-kategóriájának monoidalitásával úgy, hogy az alap kategóriába menő felejtő funktor szigorún monoidális. (Eszerint egy B jobb R-bialgebroid is meghatároz egy $(-)\otimes_{R^{op}\otimes R} B$ bimonádot az R-bimodulusok kategóriáján.) Mint McCrudden [52]-ben megmutatta, egy bimonád ugyanaz, mint egy monád a Monoidális Kategóriák; Opmonoidális Funktorok; Opmonoidális Természetes Transzformációk 2-kategóriájában.

Természetesen felvetődik a kérdés, mi a bialgebrák két különböző irányú általánosítását – a bimonádokat és a gyenge bialgebrákat – egyaránt általánosító fogalom. E kérdés megválaszolását tűzte ki **céljául** Steve Lackkal Ross Streettel közös [15] munkánk.

1.7.1. Az axiómák és gyenge bimonádok Eilenberg–Moore-kategóriái

Szlachányi [71] munkájából ismert, hogy a gyenge bialgebrák jellemzhetők, mint pontosan azok a k-algebrák, melyek modulusainak kategóriája monoidális, és a felejtő funktor a k-modulusok kategóriájába rendelkezik a következő, ún. szeparábilis Frobenius-szerkezettel.

1.81. Definíció (Szlachányi [71]). Tekintsünk egy F funktort az $(\mathcal{N}, \boxtimes, R)$ monoidális kategóriából az $(\mathcal{M}, \otimes, K)$ monoidális kategóriába. Azt mondjuk, hogy F *szeparábilis Frobenius* szerkezetű, ha el van látva egy $p_{X,Y} : FX \otimes FY \to F(X \boxtimes Y)$, $p_0 : K \to FR$ monoidális struktúrával és egy $i_{X,Y} : F(X \boxtimes Y) \to FX \otimes FY$, $i_0 : FR \to K$ opmonoidális struktúrával úgy, hogy $\mathcal{N}$ minden X, Y, Z objektumára a következő diagramok kommutatívak.

$$\begin{array}{ccc} FX\otimes F(Y\boxtimes Z) & \xrightarrow{\mathrm{Id}\otimes i_{Y,Z}} & FX\otimes FY\otimes FZ \\ {\scriptstyle p_{X,Y\boxtimes Z}}\downarrow & & \downarrow{\scriptstyle p_{X,Y}\otimes \mathrm{Id}} \\ F(X\boxtimes Y\boxtimes Z) & \xrightarrow[i_{X\boxtimes Y,Z}]{} & F(X\boxtimes Y)\otimes FZ \end{array} \qquad \begin{array}{ccc} F(X\boxtimes Y)\otimes FZ & \xrightarrow{i_{X,Y}\otimes \mathrm{Id}} & FX\otimes FY\otimes FZ \\ {\scriptstyle p_{X\boxtimes Y,Z}}\downarrow & & \downarrow{\scriptstyle \mathrm{Id}\otimes p_{Y,Z}} \\ F(X\boxtimes Y\boxtimes Z) & \xrightarrow[i_{X,Y\boxtimes Z}]{} & FX\otimes F(Y\boxtimes Z) \end{array}$$

$$\begin{array}{ccccc} & \overset{i_{X,Y}}{\nearrow} & FX\otimes FY & \overset{p_{X,Y}}{\searrow} & \\ F(X\boxtimes Y) & & = \!= \!= \!= & & F(X\boxtimes Y) \end{array}$$

Szigorúan monoidális funktorok nyilvánvalóan rendelkeznek szeparábilis Frobenius-szerkezettel. A következő definíció tehát általánosítja mind a bimonádokat, mind a gyenge Hopf-algebrákat.

1.82. Definíció ([15] Definition 1.3). Egy *gyenge bimonád* alatt egy T monádot értünk valamely $(\mathcal{M}, \otimes, K)$ monoidális kategórián, ellátva mindazon struktúrákkal, melyek ekvivalensek az $\mathcal{M}^T$ Eilenberg–Moore-kategória monoidalitásával úgy, hogy az $\mathcal{M}^T \to \mathcal{M}$ felejtő funktor rendelkezik szeparábilis Frobenius-szerkezettel.

A fenti világos jelentésű, ámde meglehetősen implicit definíció aprópénzre váltható, ha ráadásul az $\mathcal{M}$ monoidális kategória Cauchy-teljes, azaz idempotens morfizmusai felhasadnak. Ez azt jelenti, hogy minden olyan $\mathcal{M}$-beli $e : A \to A$ morfizmusra, melyre $e^2 = e$, létezik egy Q objektum valamint $p : A \to Q$ és $i : Q \to A$ morfizmusok $\mathcal{M}$-ben, melyekre $pi = \mathrm{Id}_Q$ és $ip = e$.

1.83. Tétel ([15] Theorem 1.5). *Tekintsünk egy (T, m, u) monádot egy $(\mathcal{M}, \otimes, K)$ Cauchy-teljes monoidális kategórián. Egy gyenge bimonád struktúra T-n pontosan ugyanaz, mint egy (T, τ, τ_0) opmonoidális struktúra, melyre az alábbi diagramok kommutatívak.*

$$
\begin{array}{ccccccc}
T^2(X\otimes TK) & \xrightarrow{T\tau_{X,TK}} & T(TX\otimes T^2K) & \xrightarrow{T(\mathrm{Id}\otimes m_K)} & T(TX\otimes TK) & \xrightarrow{T(\mathrm{Id}\otimes \tau_0)} & T^2X \\
{\scriptstyle Tu_{X\otimes TK}}\uparrow & & & & & & \downarrow{\scriptstyle m_X} \\
T(X\otimes TK) & \xrightarrow[\tau_{X,TK}]{} & TX\otimes T^2K & \xrightarrow[\mathrm{Id}\otimes m_K]{} & TX\otimes TK & \xrightarrow[\mathrm{Id}\otimes\tau_0]{} & TX
\end{array}
$$

$$
\begin{array}{ccccccc}
T^2(TK\otimes X) & \xrightarrow{T\tau_{TK,X}} & T(T^2K\otimes TX) & \xrightarrow{T(m_K\otimes \mathrm{Id})} & T(TK\otimes TX) & \xrightarrow{T(\tau_0\otimes \mathrm{Id})} & T^2X \\
{\scriptstyle Tu_{TK\otimes X}}\uparrow & & & & & & \downarrow{\scriptstyle m_X} \\
T(TK\otimes X) & \xrightarrow[\tau_{TK,X}]{} & T^2K\otimes TX & \xrightarrow[m_K\otimes\mathrm{Id}]{} & TK\otimes TX & \xrightarrow[\tau_0\otimes\mathrm{Id}]{} & TX
\end{array}
$$

$$
\begin{array}{ccccccc}
X\otimes T(Y\otimes Z) & \xrightarrow{\mathrm{Id}\otimes\tau_{Y,Z}} & X\otimes TY\otimes TZ & \xrightarrow{u_{X\otimes TY}\otimes\mathrm{Id}} & T(X\otimes TY)\otimes TZ & \xrightarrow{\tau_{X,TY}\otimes\mathrm{Id}} & TX\otimes T^2Y\otimes TZ \\
\uparrow{\scriptstyle \mathrm{Id}\otimes u_{Y\otimes Z}} & & & & & & {\scriptstyle \mathrm{Id}\otimes m_Y\otimes\mathrm{Id}}\downarrow \\
X\otimes Y\otimes Z & \xrightarrow[u_{X\otimes Y\otimes Z}]{} & T(X\otimes Y\otimes Z) & \xrightarrow[\tau_{X\otimes Y,Z}]{} & T(X\otimes Y)\otimes TZ & \xrightarrow[\tau_{X,Y}\otimes\mathrm{Id}]{} & TX\otimes TY\otimes TZ
\end{array}
$$

$$
\begin{array}{ccccccc}
T(X\otimes Y)\otimes Z & \xrightarrow{\tau_{X,Y}\otimes\mathrm{Id}} & TX\otimes TY\otimes Z & \xrightarrow{\mathrm{Id}\otimes u_{TY\otimes Z}} & TX\otimes T(TY\otimes Z) & \xrightarrow{\mathrm{Id}\otimes\tau_{TY,Z}} & TX\otimes T^2Y\otimes TZ \\
\uparrow{\scriptstyle u_{X\otimes Y}\otimes\mathrm{Id}} & & & & & & {\scriptstyle \mathrm{Id}\otimes m_Y\otimes\mathrm{Id}}\downarrow \\
X\otimes Y\otimes Z & \xrightarrow[u_{X\otimes Y\otimes Z}]{} & T(X\otimes Y\otimes Z) & \xrightarrow[\tau_{X\otimes Y,Z}]{} & T(X\otimes Y)\otimes TZ & \xrightarrow[\tau_{X,Y}\otimes\mathrm{Id}]{} & TX\otimes TY\otimes TZ
\end{array}
$$

$$
\begin{array}{ccccc}
T^2(X\otimes Y) & \xrightarrow{T\tau_{X,Y}} & T(TX\otimes TY) & \xrightarrow{\tau_{TX,TY}} & T^2X\otimes T^2Y \\
{\scriptstyle m_{X\otimes Y}}\downarrow & & & & \downarrow{\scriptstyle m_X\otimes m_Y} \\
T(X\otimes Y) & & \xrightarrow[\tau_{X,Y}]{\qquad\qquad\qquad} & & TX\otimes TY
\end{array}
$$

A fenti tételben fellépő opmonoidális struktúra a gyenge bialgebra koalgebra struktúráját általánosítja, az öt szereplő diagram pedig rendre megfeleltethető az 1.56 Definícióban látott öt diagramnak (vö. 1.88 Tétel).

Egy gyenge bimonád Eilenberg–Moore-kategóriájának, és alap kategóriájának monoidális struktúrái közötti kapcsolat megértése az alábbi tételen alapszik.

1.84. Tétel ([15] Theorem 1.10). *Legyen (T, m, u) egy monád valamely $(\mathcal{M}, \otimes, K)$ monoidális kategórián, melyben $(T, \tau, \tau_0) : (\mathcal{M}, \otimes, K) \to (\mathcal{M}, \otimes, K)$ egy opmonoidális funktor. Ezen feltevések mellett az 1.83 Tételben szereplő diagramok pontosan akkor kommutatívak, ha az alábbiak teljesülnek.*

- *A K, mint a terminális kategóriából $\mathcal{M}$-be menő funktor segítségével definiált $1 \stackrel{K}{\to} \mathcal{M} \stackrel{T}{\to} \mathcal{M}$ funktor; és a*

$$\sqcap := \big(\, TK \xrightarrow{u_{TK}} T^2K \xrightarrow{\tau_{K,TK}} TK\otimes T^2K \xrightarrow{\mathrm{Id}\otimes m_K} TK\otimes TK \xrightarrow{\mathrm{Id}\otimes\tau_0} TK \,\big)$$

  természetes transzformációból felépülő $T^2K \stackrel{m_K}{\to} TK \stackrel{\sqcap}{\to} TK$ természetes transzformáció $(1, \mathrm{Id}) \to (\mathcal{M}, T)$ 1-cellát alkotnak $\mathsf{Mnd}^i(\mathsf{Cat})$-ben;

- *Az* $\mathcal{M}\times\mathcal{M}\stackrel{\otimes}{\to}\mathcal{M}$ *funktor és a* $T(\bullet\otimes\bullet)\stackrel{\tau}{\to}T(\bullet)\otimes T(\bullet)$ *természetes transzformáció* $(\mathcal{M},T)\times(\mathcal{M},T)\to(\mathcal{M},T)$ *1-cellát alkotnak* $\mathsf{Mnd}^i(\mathsf{Cat})$*-ben;*

- *Az*

$$\begin{array}{ccc} \mathcal{M}\times\mathcal{M} & \xrightarrow{\mathrm{Id}\times T} & \mathcal{M}\times\mathcal{M} \\ {\scriptstyle \mathrm{Id}\times K}\uparrow & \Downarrow \mathrm{Id}\times\tau_0 & \downarrow{\scriptstyle\otimes} \\ \mathcal{M} & \xrightarrow[\mathrm{Id}]{} & \mathcal{M} \end{array} \qquad \begin{array}{ccc} \mathcal{M}\times\mathcal{M} & \xrightarrow{T\times\mathrm{Id}} & \mathcal{M}\times\mathcal{M} \\ {\scriptstyle K\times\mathrm{Id}}\uparrow & \Downarrow \tau_0\times\mathrm{Id} & \downarrow{\scriptstyle\otimes} \\ \mathcal{M} & \xrightarrow[\mathrm{Id}]{} & \mathcal{M} \end{array}$$

természetes transzformációk 2-cellák $\mathsf{Mnd}^i(\mathsf{Cat})$*-ben;*

- *Az alábbi formulákkal definiált* $E_{TX,TY}$ *és* $E^{(3)}_{TX,TY,TZ}$ *természetes transzformációk*

$$TX\otimes TY \xrightarrow{u_{TX\otimes TY}} T(TX\otimes TY) \xrightarrow{\tau_{TX,TY}} T^2X\otimes T^2Y \xrightarrow{m_X\otimes m_Y} TX\otimes TY \quad \textit{és}$$

$$TX\otimes TY\otimes TZ \xrightarrow{u_{TX\otimes TY\otimes TZ}} T(TX\otimes TY\otimes TZ) \xrightarrow{\tau^{(3)}_{TX,TY,TZ}} T^2X\otimes T^2Y\otimes T^2Z \xrightarrow{m_X\otimes m_Y\otimes m_Z} TX\otimes TY\otimes TZ\ ,$$

kommutatívvá teszik az alábbi diagramot,

$$\begin{array}{ccc} TX\otimes TY\otimes TZ & \xrightarrow{E_{TX,TY}\otimes\mathrm{Id}} & TX\otimes TY\otimes TZ \\ {\scriptstyle \mathrm{Id}\otimes E_{TY,TZ}}\downarrow & \searrow^{E^{(3)}_{TX,TY,TZ}} & \downarrow{\scriptstyle \mathrm{Id}\otimes E_{TY,TZ}} \\ TX\otimes TY\otimes TZ & \xrightarrow[E_{TX,TY}\otimes\mathrm{Id}]{} & TX\otimes TY\otimes TZ \end{array}$$

$\mathcal{M}$ *minden* X,Y,Z *objektumára.*

Ezt együtt alkalmazva az 1.78 Tétellel, a következő igazolható.

1.85. Tétel ([15] Proof of Proposition 1.11). *Egy* T *gyenge bimonádra valamely* $(\mathcal{M},\otimes,K)$ *Cauchy-teljes monoidális kategórián a következő állítások teljesülnek.*

- $\mathcal{M}^T$ *monoidális egysége az* $1\stackrel{K}{\to}\mathcal{M}\stackrel{T}{\to}\mathcal{M}$ *funktor gyenge felhúzása;*
- $\mathcal{M}^T$ *monoidális szorzása* $\mathcal{M}$ *monoidális szorzásának gyenge felhúzása;*
- $\mathcal{M}^T$ *asszociátor izomorfizmusa* $\mathcal{M}$ *asszociátor izomorfizmusának i-felhúzása;*
- $\mathcal{M}^T$ *monoidális egységgel kapcsolatos koherencia izomorfizmusai az 1.84 Tétel harmadik pontjában szereplő természetes transzformációk i-felhúzásai.*

1.7.2. Gyenge bimonádok mint bimonádok

Ahogy a gyenge bialgebrák speciális bialgebroidok, sejthető, hogy a gyenge bimonádok is megkaphatók, mint speciális bimonádok. Ebben a fejezetben ([15] Section 2) alapján áttekintjük, hogy csakugyan, a gyenge bimonádok kategóriája ekvivalens bimonádok egy alkalmas kategóriájával.

1.86. Definíció ([15] Definition 2.6)**.** Egy $\mathcal{M}$ monoidális kategórián értelmezett gyenge bimonádok közötti *morfizmusokat* úgy definiáljuk, mint opmonoidális monád morfizmusokat. Azaz mint $g: T \to T'$ természetes transzformációkat, melyek opmonoidálisak abban az értelemben, hogy a

$$\begin{array}{ccc} T(X\otimes Y) & \xrightarrow{g_{X\otimes Y}} & T'(X\otimes Y) \\ {\scriptstyle \tau_{X,Y}}\downarrow & & \downarrow{\scriptstyle \tau'_{X,Y}} \\ TX\otimes TY & \xrightarrow[g_X\otimes g_Y]{} & T'X\otimes T'Y \end{array} \qquad \begin{array}{ccc} TK & \xrightarrow{g_K} & T'K \\ {\scriptstyle \tau_0}\downarrow & & \downarrow{\scriptstyle \tau'_0} \\ K & = & K \end{array}$$

diagramok kommutatívak $\mathcal{M}$ minden X és Y objektumára, és amelyek monád morfizmusok a következő kommutatív diagramok értelmében, $\mathcal{M}$ minden X objektumára.

$$\begin{array}{ccccc} T^2X & \xrightarrow{Tg_X} & TT'X & \xrightarrow{g_{T'X}} & T'^2X \\ {\scriptstyle m_X}\downarrow & & & & \downarrow{\scriptstyle m'_X} \\ TX & & \xrightarrow[g_X]{} & & T'X \end{array} \qquad \begin{array}{ccc} X & = & X \\ {\scriptstyle u_X}\downarrow & & \downarrow{\scriptstyle u'_X} \\ TX & \xrightarrow[g_X]{} & T'X \end{array} .$$

A gyenge bimonádok $\mathcal{M}$-en, mint objektumok, és morfizmusaik, mint nyilak, egy $\mathsf{Wbm}(\mathcal{M})$-mel jelölt kategóriát alkotnak, amiben a bimonádok kategóriája teljes részkategória.

Azt mondjuk, hogy egy (R, μ, η) monoid valamely $(\mathcal{M}, \otimes, K)$ monoidális kategóriában rendelkezik a *Frobenius*-tulajdonsággal, ha a felejtő funktor az R-modulusok kategóriájából $\mathcal{M}$-be olyan, hogy jobb és bal adjungáltja megegyezik. Ez ekvivalens azzal, hogy létezik egy (R, δ, ε) komonoid is $\mathcal{M}$-ben és a következő diagram kommutatív.

$$\begin{array}{ccccc} R\otimes R & & \xrightarrow{\delta\otimes \mathrm{Id}} & & R\otimes R\otimes R \\ & \searrow^{\mu} & & & \\ {\scriptstyle \mathrm{Id}\otimes\delta}\downarrow & & R & & \downarrow{\scriptstyle \mathrm{Id}\otimes\mu} \\ & & & \searrow^{\delta} & \\ R\otimes R\otimes R & & \xrightarrow[\mu\otimes\mathrm{Id}]{} & & R\otimes R \end{array}$$

Szeparábilis Frobenius-monoidról beszélünk, ha továbbá δ a μ szorzás szelése azaz $\mu\delta = \mathrm{Id}_R$. (A látszólag átfedő terminológia magyarázata az, hogy egy R monoid pontosan akkor (szeparábilis) Frobenius-tulajdonságú, ha a felejtő funktor az R-bimodulusok kategóriájából $\mathcal{M}$-be (szeparábilis) Frobenius-szerkezetű.)

Tekintsük a következő, $\mathsf{Sfbm}(\mathcal{M})$-mel jelölt kategóriát. Objektumai legyenek $(R, \widetilde{T})$ párok, ahol R szeparábilis Frobenius-monoid $\mathcal{M}$-ben és $\widetilde{T}$ bimonád az R bimodulusok kategóriáján. Az $(R, \widetilde{T}) \to (R', \widetilde{T}')$ morfizmusok legyenek (γ, Γ) párok, ahol $\gamma: R \to R'$ monoid és komonoid izomorfizmus (így egy γ^* izomorfizmust indukál az R-bimodulusok és az R'-bimodulusok monoidális kategóriája között) Γ pedig $\widetilde{T} \to \gamma^*\widetilde{T}\gamma^{*-1}$ bimonád morfizmus (az 1.86 Definíció értelmében).

1.87. Tétel ([15] Theorem 2.11)**.** *Bármely $\mathcal{M}$ Cauchy-teljes monoidális kategória esetén a* $\mathsf{Wbm}(\mathcal{M})$ *és* $\mathsf{Sfbm}(\mathcal{M})$ *kategóriák ekvivalensek.*

1.7.3. Gyenge bimonádok és gyenge bialgebrák

A gyenge bialgebrák 1.56 Definíciója, illetve a gyenge Hopf-algebrák 1.59 Definíciója minden nehézség nélkül megismételhető a k-modulusok kategóriája helyett tetszőleges szimmetrikus, vagy akár fonott monoidális kategóriában, l. [62], [3].

1.88. Tétel ([15] Theorem 3.1)**.** *Tekintsünk egy (B, μ, η) monoidot valamely $(\mathcal{M}, \otimes, K, c)$ Cauchy-teljes fonott monoidális kategóriában. Bijektív kapcsolat áll fent a következő struktúrák között.*

- *$(B, \mu, \eta, \delta, \varepsilon)$ alakú gyenge bialgebrák $\mathcal{M}$-ben;*
- *$((-) \otimes B, (-) \otimes \mu, (-) \otimes \eta, \tau, \tau_0)$ alakú gyenge bimonádok $\mathcal{M}$-en, melyekre az alábbi diagram kommutatív, $\mathcal{M}$ bármely X, Y objektumára.*

$$\begin{array}{ccc} X \otimes Y \otimes B & \xrightarrow{\mathrm{Id}\otimes\mathrm{Id}\otimes\tau_{K,K}} & X \otimes Y \otimes B \otimes B \\ & \searrow^{\tau_{X,Y}} & \downarrow{\scriptstyle \mathrm{Id}\otimes c_{Y,B}\otimes\mathrm{Id}} \\ & & X \otimes B \otimes Y \otimes B \end{array} \tag{1.15}$$

Ha valamely B monoidra egy $(\mathcal{M}, \otimes, K, c)$ Cauchy-teljes fonott monoidális kategóriában a $(-)\otimes B$ funktor gyenge bimonád $\mathcal{M}$-en, akkor a természetesség miatt minden $f : K \to X$, $g : K \to Y$, $h : K \to B$ morfizmusra az alábbi diagram kommutatív.

$$\begin{array}{ccccc} K & \xrightarrow{f\otimes g\otimes h} & X \otimes Y \otimes B & \xrightarrow{\mathrm{Id}\otimes\mathrm{Id}\otimes\tau_{K,K}} & X \otimes Y \otimes B \otimes B \\ {\scriptstyle h}\downarrow & \searrow^{f\otimes g\otimes h} & & & \\ B & & X \otimes Y \otimes B & & \downarrow{\scriptstyle \mathrm{Id}\otimes c_{Y,B}\otimes\mathrm{Id}} \\ {\scriptstyle \tau_{K,K}}\downarrow & & & \searrow^{\tau_{X,Y}} & \\ B \otimes B & & \xrightarrow[f\otimes\mathrm{Id}\otimes g\otimes\mathrm{Id}]{} & & X \otimes B \otimes Y \otimes B \end{array}$$

Így (1.15) teljesül, ha a K monoidális egység „köbös generátor" a következő értelemben: Ha valamely $p, q : X \otimes Y \otimes Z \to W$ $\mathcal{M}$-beli morfizmusokra $p \circ (f \otimes g \otimes h) = q \circ (f \otimes g \otimes h)$ *minden* $f : K \to X$, $g : K \to Y$, $h : K \to Z$ morfizmus esetén, akkor $p = q$.

A monoidális egység „köbös generátor" például a k-modulusok szimmetrikus monoidális kategóriájában. Így tehát a fenti tétel magában foglalja a k fölötti gyenge bialgebrák, mint $\mathcal{M}_k$-n értelmezett gyenge bimonádok leírását (l. Szlachányi [71]).

1.7.4. Gyenge Hopf-monádok

Egy $(\mathcal{M}, \otimes, K)$ monoidális kategóriát *jobbról zártnak* mondunk, ha $\mathcal{M}$ minden X objektumára a $(-) \otimes X : \mathcal{M} \to \mathcal{M}$ funktor bal adjungált. Például, a k-modulusok kategóriája jobbról zárt. Jobbról zárt továbbá egy B k-bialgebra modulusainak kategóriája is, és Schauenburg [66] észrevétele szerint, az $\mathcal{M}_B \to \mathcal{M}_k$ felejtő funktor pontosan akkor kompatibilis a zárt struktúrákkal, ha B Hopf-algebra. Ezen az alapon kézenfekvő egy bimonádot egy jobbról zárt monoidális kategórián *jobb Hopf- monádnak* mondani, ha Eilenberg–Moore-kategóriája is jobbról zárt úgy, hogy a felejtő funktor kompatibilis a zárt struktúrákkal.

Lawvere (Schauenburg fenti észrevételénél általánosabb) tétele (l. [50]) szerint egy jobb adjungált funktor (pl. a felejtő funktor) pontosan akkor kompatibilis a zárt struktúrákkal, ha az ún. Frobenius-reciprocitás teljesül. Egy (T, m, u, τ, τ_0) bimonád esetén ez azt jelenti, hogy az $\mathcal{M}^T \to \mathcal{M}$ felejtő funktor pontosan akkor kompatibilis a jobbról zárt struktúrákkal, ha a

$$\mathsf{can}_{X,Y} := \big(\, T(TX \otimes Y) \xrightarrow{\tau_{TX,Y}} T^2X \otimes TY \xrightarrow{m_X \otimes \mathrm{Id}} TX \otimes TY \,\big) \tag{1.16}$$

kanonikus természetes transzformáció izomorfizmus. Bruguières és társai [22]-ben azt a definíciót javasolták, hogy egy bimonádot egy tetszőleges monoidális kategórián nevezzünk *jobb Hopf-monádnak*, ha (1.16) természetes izomorfizmus. Szimmetrikusan definiálhatóak *bal Hopf-monádok* a $T(X \otimes TY) \to TX \otimes TY$ kanonikus természetes transzformáció izomorfizmus voltával.

Az (1.16) természetes transzformációt tekinthetjük egy gyenge bimonád esetén is. Egy gyenge Hopf-algebra által indukált gyenge bimonádra azonban (1.16) nem természetes izomorfizmus, de természetes izomorfizmust indukál $T(TX \otimes Y)$ és $TX \otimes TY$ egy-egy alkalmas retraktuma között. A gyenge Hopf-monád definíciójához az 1.84 Tételben látott $E_{TX,TY}$: $TX \otimes TY \to TX \otimes TY$; és egy, T-hez szintén kanonikusan rendelt (l. [15] (4.3)) $F_{X,Y}$: $T(TX \otimes Y) \to T(TX \otimes Y)$ idempotens természetes transzformációt használunk.

1.89. Definíció ([15] Definition 4.1). Egy T gyenge bimonádot valamely $(\mathcal{M}, \otimes, K)$ monoidális kategórián *gyenge jobb Hopf-monádnak* mondunk, ha létezik egy $\chi_{X,Y} : TX \otimes TY \to T(TX \otimes Y)$ természetes transzformáció, melyre

$$\chi_{X,Y} E_{TX,TY} = \chi_{X,Y} = F_{X,Y}\chi_{X,Y}, \quad \chi_{X,Y}\mathsf{can}_{X,Y} = F_{X,Y}, \quad \mathsf{can}_{X,Y}\chi_{X,Y} = E_{TX,TY},$$

$\mathcal{M}$ bármely X, Y objektuma esetén. Szimmetrikusan definiáljuk a *gyenge bal Hopf-monádokat* a $T(X \otimes TY) \to TX \otimes TY$ kanonikus természetes transzformáció segítségével.

E definíció jogosságát igazolja a következő.

1.90. Tétel ([15] Theorem 4.2). *Tekintsünk egy T gyenge bimonádot valamely $(\mathcal{M}, \otimes, K)$ Cauchy-teljes monoidális kategórián, és az 1.87 Tétel szerint neki megfelelő $\widetilde{T}$ bimonádot (egy szeparábilis Frobenius-monoid bimodulus kategóriáján). A következő állítások ekvivalensek.*

- *$\widetilde{T}$ jobb (ill. bal) Hopf-monád;*
- *T gyenge jobb (ill. bal) Hopf-monád.*

Legyen T egy gyenge jobb Hopf-monád valamely $(\mathcal{M}, \otimes, K)$ Cauchy-teljes monoidális kategórián és legyen R az 1.87 Tétel szerint hozzárendelt szeparábilis Frobenius-monoid. Ha az R-bimodulusok kategóriája jobbról zárt, akkor a fenti tétel szerint $\mathcal{M}^T$ is jobbról zárt úgy, hogy a felejtő funktor az R-bimodulusok kategóriájába kompatibilis a zárt struktúrákkal. Ha $\mathcal{M}$ jobbról zárt, akkor az R-bimodulusok kategóriája is jobbról zárt (a megkívánt jobb adjungáltak alkalmas idempotens természetes transzformációk felhasadásával adódnak).

Végül, a remélt kapcsolat bizonyítható gyenge Hopf-monádok és gyenge Hopf-algebrák között.

1.91. Tétel ([22] Theorems 4.6 and 4.8). *Legyen B egy gyenge bialgebra valamely $(\mathcal{M}, \otimes, K, c)$ Cauchy-teljes fonott monoidális kategóriában. Az indukált $(-) \otimes B$ funktor pontosan akkor gyenge jobb Hopf-monád, ha B gyenge Hopf-algebra. Továbbá $(-)\otimes B$ pontosan akkor gyenge bal Hopf-monád <u>is</u>, ha a B gyenge Hopf-algebra antipódja invertálható.*

1.8. Kvantum grupoidok alkalmazásai

Értekezésem témájául munkásságom azon (részben társszerzőkkel írt) darabjait választottam ki, melyek a Hopf-algebrákat általánosító algebrai struktúrák – gyenge Hopf-algebrák, illetve a még általánosabb Hopf-algebroidok – definíciójának kimunkálásával, a struktúra vizsgálatával, kategóriaelméleti megalapozásával illetve néhány elemi alkalmazásával foglalkoznak.

Mondhatjuk, hogy mára e struktúrák (főként a speciálisabb gyenge Hopf-algebrák) alkalmazási köre viszonylag széles. Faktorok bővítésének leírására használták például Enock [37], David [35], Vallin [75], Nikshych és Vainerman [58], Das és Kodiyalam [32]. Frobenius-kiterjesztések szimmetriájaként használta Kadison és Nikshych [44]. Dupla grupoidokkal való kapcsolatukat mutatták meg Andruskiewitsch és Natale [4], dinamikai kvantum csoportokat írtak le velük Etingof és Nikshych [38], általánosított Kac-Moody algebrákkal összefüggésben merültek föl Wu munkájában [79], Cartan mátrixokhoz rendelt gyenge Hopf-algebrákat Yang [74], bizonyos kvantum csoportokhoz Aizawa és Isaac [1]. A (dinamikai) Yang-Baxter egyenlet megoldására alkalmazta Etingof és Nikshych [39], csomó invariánsok konstrukciójához használta Nikshych, Turaev és Vainerman [57].

A matematika mellett ezek a struktúrák felléptek fizikai alkalmazásokban is. Peremes konform térelméletek szimmetriájaként jelentek meg Coquereaux és társszerzői [31], Behrend, Pierce, Petkova és Zuber [8] cikkeiben. Rács modellekben bukkant fel Alekseev, Faddeev, Fröhlich és Schomerus [2] munkájában.

[18] és [17] munkánk általánosításaként (kommutatív gyűrűk modulus kategóriája helyett) fonott monoidális kategóriákban definiáltak és vizsgáltak gyenge Hopf-algebrákat Pastro és Street [62] illetve Alonso Álvárez és társai [3]. A [9]-beli gyenge Doi–Hopf-modulusok inspirálták Caenepeel és De Groot gyenge vegyes disztributív szabályát [25] és Wisbauer gyenge kogyűrűjét [78].

A Hopf-algebroidokat általánosító (megint egyszer kvantum grupoidnak nevezett) struktúrát vezetett be Day és Street [34]. Számos cikk tárgya a hatalmas Hopf-algebrai irodalom különböző erdményeinek Hopf-algebroidokra (elsősorban gyenge Hopf-algebrákra) való általánosítása, l. pl. [76]. [10] kvázi-Frobenius- kiterjeztésekre vonatkozó eredményeit alkalmazta [41]. A Hopf-algebroidok Galois-elméletének [11]-ben megkezdett tanulmányozását folytatta [6], [14].

A monádok gyenge elméletét kidolgozó [12] nyomán indult meg együttműködésem a Sydney-i kategóriaelméleti csoporttal, Steve Lackkal és Ross Streettel. Ennek eredménye eddig két elkészült [15], [16] és további két folyamatban lévő, részben [12]-re épülő munka.

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2. fejezet

Doi-Hopf modules over weak Hopf algebras

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Doi-Hopf Modules over Weak Hopf Algebras

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Abstract

The theory of Doi-Hopf modules [8, 11] is generalized to Weak Hopf Algebras [1, 14, 2].

1 Introduction

The category ${}^C\mathcal{M}(H)_A$ of Doi-Hopf Modules over the bialgebra H was introduced in [8] and independently in [11]. It is the category of the modules over the algebra A which are also comodules over the coalgebra C and satisfy certain compatibility condition involving H. The study of ${}^C\mathcal{M}(H)_A$ turned out to be very useful: It was shown in [8, 4] that many categories investigated independently before – such as the module and comodule categories over bialgebras, the Hopf modules category [17], and the Yetter-Drinfel'd category [18, 16] – are special cases of ${}^C\mathcal{M}(H)_A$. Using this observation many results known for module categories over bialgebras or Hopf algebras were generalized to this more general setting [5, 6, 7].

In this paper we generalize the definition of Doi-Hopf modules to the case when H is a Weak Bialgebra (WBA). Our definitions are supported by the fact that many results of [11, 5, 6, 7] remain valid in this case.

Weak Bialgebras (Weak Hopf Algebras – WHA's –) are generalizations of bialgebras (Hopf algebras) see [1, 2] and [14] (latter one using somewhat different terminology). In contrast to another direction of generalization, the quasi-Hopf algebras and weak quasi-Hopf algebras, WBA's are coassociative. Though their counit is not an algebra map, their structure is designed such a way that their (left or right) (co-)

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module category carries a monoidal structure [14, 3] (and some more in the WHA case [3]).

To be more concrete a WBA H is a unital associative algebra and a counital coassociative coalgebra over the field k with the structural maps $u : k \to H$ (*unit*), $m : H \otimes H \to H$ (*multiplication*), $\varepsilon : H \to k$ (*counit*) and $\Delta : H \to H \otimes H$ (*comultiplication*). The coproduct Δ is required to be a (typically non-unital) algebra map and the following compatibility conditions hold true:

$$\begin{aligned} \varepsilon(xyz) &= \varepsilon(xy_{(1)})\varepsilon(y_{(2)}z) \\ &= \varepsilon(xy_{(2)})\varepsilon(y_{(1)}z) \qquad (1.1)\\ \mathbb{1}_{(1)} \otimes \mathbb{1}_{(2)} \otimes \mathbb{1}_{(3)} &= \mathbb{1}_{(1)} \otimes \mathbb{1}_{(2)}\mathbb{1}_{(1)'} \otimes \mathbb{1}_{(2)'} \\ &= \mathbb{1}_{(1)} \otimes \mathbb{1}_{(1)'}\mathbb{1}_{(2)} \otimes \mathbb{1}_{(2)'} \qquad (1.2)\end{aligned}$$

for all $x, y, z \in H$. We adopted the conventions [17] that $u(1) = \mathbb{1}$, $m(x \otimes y) = xy$, $\Delta(x) = x_{(1)} \otimes x_{(2)}$ – summation implicitly understood – and for the multiple coproduct $(\Delta \otimes \mathrm{id}) \circ \Delta(x) = (\mathrm{id} \otimes \Delta) \circ \Delta(x)$ we write $x_{(1)} \otimes x_{(2)} \otimes x_{(3)}$ for $x, y \in H$.

H is a WHA if in addition there exists an antipode map $S : H \to H$ such that

$$\begin{aligned} x_{(1)}S(x_{(2)}) &= \varepsilon(\mathbb{1}_{(1)}x)\mathbb{1}_{(2)} \qquad (1.3)\\ S(x_{(1)})x_{(2)} &= \mathbb{1}_{(1)}\varepsilon(x\mathbb{1}_{(2)}) \qquad (1.4)\\ S(x_{(1)})x_{(2)}S(x_{(3)}) &= S(x) \qquad (1.5)\end{aligned}$$

holds for all $x \in H$.

In the study of WBA's (WHA's) the maps $\Pi^L(x) := \varepsilon(\mathbb{1}_{(1)}x)\mathbb{1}_{(2)}$ and $\Pi^R(x) := \mathbb{1}_{(1)}\varepsilon(x\mathbb{1}_{(2)})$ for $x \in H$ – standing on the right hand sides of axioms (1.3) and (1.4) – turn out to have important roles. Their images are subalgebras generalizing the one dimensional subalgebra $k\mathbb{1}$ of the scalars in many senses [1, 2, 14].

WHA's have relevance for example in describing depth 2 reducible Jones inclusions [15, 13].

As the bialgebra (Hopf algebra) also the WBA (WHA) is a self-dual structure: The dual space of a finite dimensional WBA (WHA) carries naturally a WBA (WHA) structure [1, 2].

The paper is organized as follows: we define and examine the structures such as the weak Doi-Hopf datum (generalizing the Doi-Hopf datum of [8]) the weak Doi-Hopf module (generalizing the Doi-Hopf module of [8]) the weak smash product (generalizing the analogous notion of [11]) and the weak Doi-Hopf integral (generalizing definitions of [6, 7, 9]). We illustrate these notions on the same four examples generalizing some classical examples of [8, 4].

2 The Weak Doi-Hopf Data

In this Section H is a Weak Bialgebra (WBA) in the sense of [2] over the field k.

Definition 2.1 *Let H be a WBA over the field k. The k-algebra A is a* left H-comodule algebra *if there exists a left weak coaction ρ of H on A which is also an algebra map. I.e. a map $\rho : A \to H \otimes A$ such that*

$$(\mathrm{id}_H \otimes \rho) \circ \rho = (\Delta \otimes \mathrm{id}_A) \circ \rho \tag{2.1a}$$
$$(\mathbb{1} \otimes a)\rho(1_A) = (\Pi^R \otimes \mathrm{id}_A) \circ \rho(a) \tag{2.1b}$$
$$\rho(ab) = \rho(a)\rho(b) \tag{2.1c}$$

for all $a, b \in A$. *We use the standard notation* $\rho(a) = a_{<-1>} \otimes a_{<0>}$ *and* $(\Delta \otimes \mathrm{id}_A) \circ \rho(a) = a_{<-2>} \otimes a_{<-1>} \otimes a_{<0>} = (\mathrm{id}_H \otimes \rho) \circ \rho(a)$.

Throughout the paper the left weak coaction ρ *is required to be* non-degenerate *i.e.* $(\varepsilon \otimes \mathrm{id}_A) \circ \rho = \mathrm{id}_A$ *or, equivalently,* $(\varepsilon \otimes \mathrm{id}_A) \circ \rho(1_A) = 1_A$. *For non-degenerate left weak coactions* ρ *(2.1b) has an equivalent form (compare with [15])* $(\Delta \otimes \mathrm{id}_A) \circ \rho(1_A) = (\mathbb{1} \otimes \rho(1_A))(\Delta(\mathbb{1}) \otimes 1_A)$.

Similarly, A *is a* right H-comodule algebra *if there exists a right weak coaction* ρ *of* H *on* A *which is also an algebra map. I.e. a map* $\rho : A \to A \otimes H$ *such that*

$$(\rho \otimes \mathrm{id}_H) \circ \rho = (\mathrm{id}_A \otimes \Delta) \circ \rho \tag{2.2a}$$
$$\rho(1_A)(a \otimes \mathbb{1}) = (\mathrm{id}_A \otimes \Pi^L) \circ \rho(a) \tag{2.2b}$$
$$\rho(ab) = \rho(a)\rho(b) \tag{2.2c}$$

for all $a, b \in A$. *We also denote* $\rho(a) = a_{<0>} \otimes a_{<1>}$.

The right weak coaction ρ *is required to be* non-degenerate *i.e.* $(\mathrm{id}_A \otimes \varepsilon) \circ \rho = \mathrm{id}_A$ *or, equivalently, if* $(\mathrm{id}_A \otimes \varepsilon) \circ \rho(1_A) = 1_A$. *For non-degenerate right weak coactions* ρ *(2.2b) has an equivalent form* $(\mathrm{id}_A \otimes \Delta) \circ \rho(1_A) = (1_A \otimes \Delta(\mathbb{1}))(\rho(1_A) \otimes \mathbb{1})$.

The dual notion to comodule algebra is the module coalgebra defined as follows: The k*-coalgebra* C *is a* right H-module coalgebra *if there exists a right weak action of* H *on* C *which is also a coalgebra map. I.e. a map* $\cdot : C \times H \to C$ *such that*

$$(c \cdot g) \cdot h = c \cdot (gh) \tag{2.3a}$$
$$c \cdot \Pi^L(h) = \varepsilon_C(c_{(1)} \cdot h)c_{(2)} \tag{2.3b}$$
$$\Delta_C(c \cdot h) = \Delta_C(c) \cdot \Delta(h) \tag{2.3c}$$

for all $c \in C$, $g, h \in H$.

The right weak action $\cdot$ *is required to be* non-degenerate *i.e.* $c \cdot \mathbb{1} = c$ *for all* $c \in C$ *or, equivalently, if* $\varepsilon_C(c \cdot \mathbb{1}) = \varepsilon_C(c)$ *for all* $c \in C$. *For non-degenerate right weak actions* $\cdot$ *(2.3b) has the equivalent reformulation as* $\varepsilon_C(c \cdot h) = \varepsilon_C(c \cdot \Pi^L(h))$.

Similarly, C *is a* left H-module coalgebra *if there exists a left weak action of* H *on* C *which is also a coalgebra map. I.e. a map* $\cdot : H \times C \to C$ *such that*

$$g \cdot (h \cdot c) = (gh) \cdot c \tag{2.4a}$$
$$\Pi^R(h) \cdot c = c_{(1)}\varepsilon_C(h \cdot c_{(2)}) \tag{2.4b}$$
$$\Delta_C(h \cdot c) = \Delta(h) \cdot \Delta_C(c) \tag{2.4c}$$

for all $c \in C$ $g, h \in H$.

The left weak action $\cdot$ *is required to be* non-degenerate *i.e.* $\mathbb{1} \cdot c = c$ *for all* $c \in C$ *or, equivalently, if* $\varepsilon_C(\mathbb{1} \cdot c) = \varepsilon_C(c)$ *for all* $c \in C$. *For non-degenerate left weak actions* $\cdot$ *(2.4b) has the equivalent reformulation* $\varepsilon_C(h \cdot c) = \varepsilon_C(\Pi^R(h) \cdot c)$.

Notice, that in contrast to the case when H is an ordinary bialgebra the unit preserving property of ρ and the counit preserving property of $\cdot$ are not required and the form of condition (b) in each group is somewhat different from the usual one.

Definition 2.2 *A* right Weak Doi-Hopf datum *is a triple* (H, A, C), *where* H *is a WBA over* k, A *a left* H*-comodule algebra and* C *a right* H*-module coalgebra.*

A left Weak Doi-Hopf datum *is a triple* (H, A, C) *where* H *is a WBA over* k, A *a right* H*-comodule algebra and* C *a left* H*-module coalgebra.*

Examples:

1 Let H be a WBA over k, $A := H$ as an algebra with the coaction $\rho := \Delta$, $C := H^L$ with the coalgebra structure

$$\begin{aligned}\Delta_{H^L}(a^L): &= \mathbb{1}_{(2)}a^L \otimes S(\mathbb{1}_{(1)}) \equiv \mathbb{1}_{(2)} \otimes a^L S(\mathbb{1}_{(1)}) \\ \varepsilon_{H^L}(a^L): &= \varepsilon(a^L)\end{aligned}$$

and the action $a^L \cdot h := \mathbb{1}_{(2)}\varepsilon(a^L h \mathbb{1}_{(1)})$ for all $a^L \in H^L, h \in H$. Then $(H, A = H, C = H^L)$ is a right Weak Doi-Hopf datum.

2 Let H be a WBA over k, $A := H^L$ as the subalgebra of H with the coaction $\rho := \Delta|_{H^L}$, $C := H$ as a coalgebra with the action $c \cdot h := ch$ for all $c, h \in H$. Then $(H, A = H^L, C = H)$ is a right Weak Doi-Hopf datum.

3 Let H be a WBA over k, $A := H$ as an algebra with the coaction $\rho := \Delta$, $C := H$ as a coalgebra with the action $c \cdot h = ch$ for all $c, h \in H$. Then $(H, A = H, C = H)$ is a right Weak Doi-Hopf datum.

4 Let K be a WHA over k, $H := K^{op} \otimes K$ as a bialgebra. (K^{op} is the bialgebra with the same coalgebra structure as K and the opposite algebra structure.) $A := K$ as an algebra with the coaction $\rho(a) := (S^{-1}(a_{(3)}) \otimes a_{(1)}) \otimes a_{(2)}$ for all $a \in K$, $C := K$ as a coalgebra with the action $c \cdot (a \otimes b) := acb$ for all $c \in K, (a \otimes b) \in H$. Then $(H = K^{op} \otimes K, A = K, C = K)$ is a right weak Doi-Hopf datum.

Let us call a (left or right) weak Doi-Hopf datum *finite dimensional* if all H, A and C are finite dimensional as k-spaces. There is a well defined notion of duality for finite dimensional weak Doi- Hopf data sending a left weak Doi-Hopf datum to a right one and vice versa:

Introduce the following notations: For any finite dimensional k- space M let $\hat{M}$ denote the dual k-space. If A is a finite dimensional algebra then by $\hat{A}$ we mean the dual space equipped with the dual coalgebra structure. Similarly, for a finite dimensional coalgebra C denote the dual algebra by $\hat{C}$ and finally for a finite dimensional bialgebra H denote the dual bialgebra by $\hat{H}$.

Proposition 2.3 *For a right weak Doi-Hopf datum* (H, A, C) *the triple* $(\hat{H}, \hat{C}, \hat{A})$ *is a left weak Doi-Hopf datum – called the dual of* (H, A, C) *– with*

$$\begin{aligned}\hat{\rho}(\hat{c}): &= b_i \triangleright \hat{c} \otimes \beta^i \\ \phi \cdot \hat{a}: &= (\phi \otimes \hat{a}) \circ \rho\end{aligned} \tag{2.5}$$

where $\hat{c} \in \hat{C}$, $\{b_i\}$ *is any basis in* H *and* $\{\beta^i\}$ *is the dual basis in* $\hat{H}$, $\phi \in \hat{H}$, $\hat{a} \in \hat{A}$ *and* $(h \triangleright \hat{c})(d) = \hat{c}(d \cdot h)$ *for* $\hat{c} \in \hat{C}, d \in C, h \in H$.

Similarly, for a left weak Doi-Hopf datum (H, A, C) *the triple* $(\hat{H}, \hat{C}, \hat{A})$ *is a right weak Doi-Hopf datum – called the dual of* (H, A, C) *– with*

$$\hat{\rho}(\hat{c}): = \beta^i \otimes \hat{c} \triangleleft b_i$$
$$\hat{a} \cdot \phi = (\hat{a} \otimes \phi) \circ \rho \qquad (2.6)$$

with the obvious notation. The above duality transformation is involutive.

Proof: We leave the proof to be easily done by the reader. ∎

3 Weak Doi-Hopf Modules

Definition 3.1 *The k-space M is a* right weak Doi-Hopf module *over the right weak Doi-Hopf datum (H, A, C) if it is a right A- module and a left C-comodule i.e. there exists an action $\cdot : M \times A \to M$ for which $m \cdot 1_A = m$ for all $m \in M$ and a coaction $\rho_M : M \to C \otimes M$ for which $(\varepsilon_C \otimes \mathrm{id}_M) \circ \rho_M = \rho_M$ such that the compatibility condition*

$$\rho_M(m \cdot a) = m_{<-1>} \cdot a_{<-1>} \otimes m_{<0>} \cdot a_{<0>} \qquad (3.1)$$

holds for $\rho_M(m) \equiv m_{<-1>} \otimes m_{<0>}$.

Similarly, M is a left weak Doi-Hopf module *over the left Doi-Hopf datum (H, A, C) if it is a left A-module (with A-action $\cdot$) and a right C-comodule (with C-coaction ρ_M) such that*

$$\rho_M(a \cdot m) = a_{<0>} \cdot m_{<0>} \otimes a_{<1>} \cdot m_{<1>}. \qquad (3.2)$$

The category ${}^C\mathcal{M}(H)_A$ has as objects the finite dimensional right weak Doi-Hopf modules M over the right weak Doi-Hopf datum (H, A, C) and arrows $T : M \to M'$ which intertwine both the A-actions and the C-coactions:

$$T(m \cdot a) = T(m) \cdot a \qquad \rho_{M'} \circ T = (\mathrm{id}_C \otimes T) \circ \rho_M \qquad (3.3)$$

for all $m \in M, a \in A$.

Similarly, ${}_A\mathcal{M}(H)^C$ is the category of finite dimensional left weak Doi-Hopf modules over the left Doi-Hopf datum (H, A, C).

Let us see what categories ${}^C\mathcal{M}(H)_A$ are in our earlier examples:

Examples:

1 ${}^C\mathcal{M}(H)_A$ is equivalent to $\mathcal{M}_A \equiv \mathcal{M}_H$, the category of right H-modules. The equivalence functor $F : {}^C\mathcal{M}(H)_A \to \mathcal{M}_A$ is the forgetful functor.

2 ${}^C\mathcal{M}(H)_A$ is equivalent to ${}^C\mathcal{M} \equiv {}^H\mathcal{M}$, the category of left H-comodules. The equivalence functor $\hat{F} : {}^C\mathcal{M}(H)_A \to {}^C\mathcal{M}$ is the forgetful functor.

3 ${}^C\mathcal{M}(H)_A$ is equivalent to ${}^H\mathcal{M}_H$, the category of weak Hopf modules [1, 2] over H.

4 ${}^C\mathcal{M}(H)_A$ is equivalent to $\mathcal{YD}(K)$, the category of Yetter-Drinfel'd modules over K. (For the definition of Yetter- Drinfel'd modules over WHA's see the Appendix).

Proposition 3.2 *Let (H, A, C) be a finite dimensional right weak Doi-Hopf datum and $(\hat{H}, \hat{C}, \hat{A})$ its dual. Then the categories ${}^C\mathcal{M}(H)_A$ and ${}_{\hat{C}}\mathcal{M}(\hat{H})^{\hat{A}}$ are equivalent.*

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Proof: Let us define the functor $D : {}^C\mathcal{M}(H)_A \to_{\hat{C}} \mathcal{M}(\hat{H})^{\hat{A}}$

$$\begin{aligned} D(M) &:= \hat{M} \text{ as a } k-\text{space} & \hat{c}\cdot\mu &:= (\hat{c}\otimes\mu)\circ\rho_M \\ & & \hat{\rho}_{\hat{M}}(\mu) &:= a_i \triangleright \mu \otimes \alpha^i \\ D(T) &:= T^t \end{aligned} \tag{3.4}$$

where M is an object and T an arrow in ${}^C\mathcal{M}(H)_A$, ^{t} means transposition of linear operators, $\hat{c} \in \hat{C}$, $\mu \in \hat{M}$, $(a \triangleright \mu)(m) = \mu(m \cdot a)$ for $a \in A, \mu \in \hat{M}, m \in M$, $\{a_i\}$ is a basis for A and $\{\alpha^i\}$ is the dual basis for $\hat{A}$. One checks by direct calculation that D defines an equivalence functor. ∎

Proposition 3.3 *Let (H, A, C) be a right weak Doi-Hopf datum. Then the forgetful functor $F : {}^C\mathcal{M}(H)_A \to \mathcal{M}_A$ has a left adjoint and $\hat{F} : {}^C\mathcal{M}(H)_A \to {}^C\mathcal{M}$ has a right adjoint.*

Proof: Define $G : \mathcal{M}_A \to {}^C\mathcal{M}(H)_A$ by

$$\begin{aligned} G(M) :=\ & C\cdot 1_{A<-1>} \otimes M \cdot 1_{A<0>} \quad \text{as a } k-\text{space} \\ & (c\otimes m)\cdot a := c\cdot a_{<-1>} \otimes m\cdot a_{<0>} \\ & \rho_{G(M)} := (\Delta_C \otimes \mathrm{id}_M)|_{G(M)} \\ G(T) :=\ & (id_C \otimes T) \end{aligned} \tag{3.5}$$

for M an object and T an arrow in ${}^C\mathcal{M}(H)_A$. $a \in A$, $(c\otimes m) \in G(M) \subset C \otimes M$.

The fact that G is a left adjoint of F is justified by the existence of unit and counit natural homomorphisms $\rho : \mathrm{id}_{{}^C\mathcal{M}(H)_A} \to G \circ F$ and $\delta : F \circ G \to \mathrm{id}_{\mathcal{M}_A}$ satisfying

$$\delta_{F(M)} \circ F(\rho_M) = \mathrm{id}_{F(M)} \tag{3.6}$$
$$G(\delta_M) \circ \rho_{G(M)} = \mathrm{id}_{G(M)}. \tag{3.7}$$

Define them as

$$\begin{aligned} \rho_M : M \to G(M) \qquad & \rho_M(m) := m_{<-1>} \otimes m_{<0>} \\ \delta_M : G(M) \to M \qquad & \delta_M := (\varepsilon_C \otimes \mathrm{id}_M)|_{G(M)}. \end{aligned} \tag{3.8}$$

It is straightforward to show that $\rho_M \in (M, G(M))_{{}^C\mathcal{M}(H)_A}$, and ρ is natural. The proof of $\delta_M \in (G(M), M)_{\mathcal{M}_A}$ lies on the following

Lemma 3.4 *Let (H, A, C) be a right weak Doi-Hopf datum. Then for any $c \in C$ and $a \in A$*

$$(i) \qquad \Delta_C(c\cdot 1_{A<-1>}) \otimes 1_{A<0>} = c_{(1)} \otimes c_{(2)} \cdot 1_{A<-1>} \otimes 1_{A<0>} \tag{3.9}$$
$$(ii) \qquad \Pi^L(a_{<-1>}) \otimes a_{<0>} = \Pi^L(1_{A<-1>}) \otimes 1_{A<0>}a. \tag{3.10}$$

Lemma 3.4 (ii) implies $\varepsilon_C(c\cdot a_{<-1>})a_{<0>} = \varepsilon_C(c\cdot 1_{A<-1>})1_{A<0>}a$ and hence $\delta_M \in (G(M), M)_{\mathcal{M}_A}$. Naturality of δ is obvious. The conditions (3.6) and (3.7) are checked by substitution.

One can proceed the same way in the case of $\hat{F}$ using now Lemma 3.4 (i). Define $\hat{G} :^C \mathcal{M} \to {}^C\mathcal{M}(H)_A$ as

$$\begin{aligned} G(M) &:= \{\varepsilon_C(m_{<-1>} \cdot a_{<-1>})m_{<0>} \otimes a_{<0>} | m \in M, a \in A\} \text{ as a } k-\text{space} \\ &\quad (m \otimes a) \cdot b := \varepsilon_C(m_{<-1>} \cdot a_{<-1>} b_{<-1>})m_{<0>} \otimes a_{<0>} b_{<0>} \\ &\quad \rho_{\hat{G}(M)}(m \otimes a) := m_{<-1>} \cdot a_{<-1>} \otimes m_{<0>} \otimes a_{<0>} \\ \hat{G}(T) &:= T \otimes \mathrm{id}_A \end{aligned} \tag{3.11}$$

for M an object and T an arrow in ${}^C\mathcal{M}(H)_A$, $(m \otimes a) \in \hat{G}(M) \subset M \otimes A, b \in A$.

The unit and counit natural homomorphisms $\hat{\rho} : \mathrm{id}_{{}^C\mathcal{M}} \to \hat{F} \circ \hat{G}$ and $\hat{\delta} : \hat{G} \circ \hat{F} \to \mathrm{id}_{{}^C\mathcal{M}(H)_A}$ can be given by

$$\begin{aligned} &\hat{\rho}_M : M \to \hat{G}(M) \qquad && \hat{\rho}_M(m) := \varepsilon_C(m_{<-1>} \cdot 1_{A<-1>})m_{<0>} \otimes 1_{A<0>} \\ &\hat{\delta}_M : \hat{G}(M) \to M \qquad && \hat{\delta}_M(m \otimes a) := m \cdot a \end{aligned} \tag{3.12}$$

∎

4 The Weak Smash Product

Definition 4.1 *For the right weak Doi-Hopf datum (H, A, C) define the weak smash product algebra $A\#\hat{C}$ as the k-space $1_{A<0>}A \otimes 1_{A<-1>} \triangleright \hat{C}$ equipped with the multiplication rule*

$$(a\#\hat{c})(b\#\hat{d}) := (a_{<0>}b\#\hat{c}(a_{<-1>} \triangleright \hat{d})) \tag{4.1}$$

for $(a\#\hat{c}), (b\#\hat{d}) \in A\#\hat{C}$.

One checks that (4.1) makes $A\#\hat{C}$ an associative algebra with unit element $1_{A<0>}\# 1_{A<-1>} \triangleright 1_{\hat{C}}$.

Let us see what algebras $A\#\hat{C}$ are in our earlier examples.

Examples:

1 $A\#\hat{C} \equiv H\#\hat{H}^R$ is isomorphic to H, the isomorphism being given by $\iota : A\#\hat{C} \to H$, $\iota := \mathrm{id}_H \otimes \varepsilon_{\hat{H}}$.

2 $A\#\hat{C} \equiv H^L\#\hat{H}$ is isomorphic to $\hat{H}$, the isomorphism being given by $\iota : A\#\hat{C} \to \hat{H}$, $\iota := \varepsilon \otimes \mathrm{id}_{\hat{H}}$.

3 $A\#\hat{C} \equiv H\#\hat{H}$ is isomorphic to the Weyl algebra or Heisenberg double $\hat{H}\rtimes H$ [1, 2], the isomorphism being given by $\iota : A\#\hat{C} \to \hat{H}\rtimes H$, $\iota(\mathbb{1}_{(2)}a\#(\mathbb{1}_{(1)} \rightharpoonup \phi)) := \phi a$. (In all of the examples $a \rightharpoonup .\ (. \leftharpoonup a) : \hat{H} \to \hat{H}$ for $a \in H$ denotes the Sweedler's arrow [17] which is the transpose of the right (left) multiplication in H with respect to the canonical pairing $\hat{H} \otimes H \to k$: $(a \rightharpoonup \phi)(b) := \phi(ba)$ $((\phi \leftharpoonup a)(b) := \phi(ab))$ for $\phi \in \hat{H}$ and $a, b \in H$.)

4 $A\#\hat{C} \equiv K\#\hat{K}$ is isomorphic to the Drinfel'd double $\mathcal{D}(K)$ (For the definition of the Drinfel'd double of WHA's see the Appendix). The equivalence is given by $\iota : A\#\hat{C} \to \mathcal{D}(K)$, $\iota(\mathbb{1}_{(2)}a\#(\mathbb{1}_{(1)} \rightharpoonup \phi \leftharpoonup S^{-1}(\mathbb{1}_{(3)}))) := \mathcal{D}(\phi)\mathcal{D}(a)$.

Proposition 4.2 *Let (H, A, C) be a right weak Doi-Hopf datum such that C is finite dimensional as a k-space. Then the categories ${}^C\mathcal{M}(H)_A$ and $\mathcal{M}_{A\#\hat{C}}$ are isomorphic.*

Proof: We have the functor $P : {}^C\mathcal{M}(H)_A \to \mathcal{M}_{A\#\hat{C}}$

$$P(M) := M \text{ as a } k-\text{space} \qquad m\cdot(a\#\hat{c}) := \hat{c}(m_{<-1>})m_{<0>}\cdot a$$
$$P(T) := T \tag{4.2}$$

for M an object and T an arrow in ${}^C\mathcal{M}(H)_A$. $(a\#\hat{c}) \in A\#\hat{C}, m \in M$.

If C is finite dimensional as a k-space then let $\{c_i\}$ be any basis for C and $\{\gamma^i\}$ the dual basis for $\hat{C}$ and construct the inverse functor $P' : \mathcal{M}_{A\#\hat{C}} \to {}^C\mathcal{M}(H)_A$ of P:

$$P'(M) := M \text{ as a } k-\text{space} \qquad m\cdot a := m\cdot(1_{A<0>}a\#1_{A<-1>} \triangleright 1_{\hat{C}})$$
$$\rho_M(m) := c_i \otimes m\cdot(1_{A<0>}\#1_{A<-1>} \triangleright \gamma^i)$$
$$P'(T) := T \tag{4.3}$$

for M an object and T an arrow of ${}^C\mathcal{M}(H)_A$, $a \in A, m \in M$.

■

5 Integrals for Weak Doi-Hopf Data

Let (H, A, C) be a right weak Doi-Hopf datum where H is a weak *Hopf* algebra with antipode S, $F : {}^C\mathcal{M}(H)_A \to \mathcal{M}_A$ the forgetful functor, G its left adjoint as in Proposition 3.3. Let V be the k-space of the natural homomorphisms $\nu : G \circ F \to \mathrm{id}_{{}^C\mathcal{M}(H)_A}$, called the *space of integrals* for the weak Doi-Hopf datum (H, A, C). We have a straightforward generalization of Theorem 2.3 of [7]:

Theorem 5.1 *The space V is isomorphic to the space $\tilde{V}$ (the weak generalization of V_4 of [7]):*

$$\begin{aligned}\tilde{V} := \{\gamma : C \to (C.A)_{\mathrm{Lin}} \quad &| \quad \forall c, d \in C \quad a \in A \\ \gamma(c)(d)a \quad &= \quad a_{<0>}\gamma(c\cdot a_{<-2>})(d\cdot a_{<-1>}) \\ c_{(1)} \otimes \gamma(c_{(2)})(d) \quad &= \quad d_{(2)}\cdot\gamma(c)(d_{(1)})_{<-1>} \otimes \gamma(c)(d_{(1)})_{<0>}\}.\end{aligned} \tag{5.1}$$

Furthermore the isomorphism $f : V \to \tilde{V}$ takes $\nu \in V$ to a normalized element *of $\tilde{V}$ i.e. to an element $\gamma \in \tilde{V}$ such that $\gamma(c_{(1)})(c_{(2)}) = \varepsilon_C(c\cdot 1_{A<-1>})1_{A<0>}$ if and only if ν is a splitting of the unit natural homomorphism $\rho : \mathrm{id}_{{}^C\mathcal{M}(H)_A} \to G \circ F$.*

As it is proven in [7] the message of Theorem 5.1 is that the sufficient and necessary condition for the forgetful functor $F : {}^C\mathcal{M}(H)_A \to \mathcal{M}_A$ to be separable is the existence of normalized elements in $\tilde{V}$.

Let us turn to the investigation of the space of integrals over the weak Doi-Hopf datum (H, A, C) in our earlier examples. In doing so we use the fact [19] that the WHA's H in the Examples **2** and **3** and K in **4** are *Frobenius* algebras. By means of their non-degenerate integrals [2] we identify the space of integrals for the weak Doi-Hopf datum (H, A, C) in these examples with a subspace of the smash product algebra $A\#\hat{C}$. Also the normalization condition is formulated in these cases as a relation in the algebra $A\#\hat{C}$.

The simple forms of our results – obtained by individual inspection of our Examples 1-4 and tedious calculations – suggest that the description of $\tilde{V}$ and its normalized elements can be translated to the language of $A\#\hat{C}$ but until now we were not able to establish this correspondence in general.

In all of the examples r be a non-degenerate right integral in H and ρ the dual right integral [2] in $\hat{H}$.

Examples:

1 The space of Doi-Hopf integrals over (H, A, C) is isomorphic to $V_0 := \mathrm{Center} H$. Construct the isomorphism $f : \tilde{V} \to V_0$ as

$$f(\gamma) := \gamma(\mathbb{1})(\mathbb{1}). \tag{5.2}$$

The unique normalized element of V_0 is the unit element $\mathbb{1}$ of H.

2 The space of the Doi-Hopf integrals is isomorphic to $V_0 := (\hat{H}^R)' \cap \hat{H}$, the commutant of the right subalgebra in $\hat{H}$. Let us construct the isomorphism $f : \tilde{V} \to V_0$ as

$$[f(\gamma)](h) := \varepsilon(\gamma(r)(h)) \tag{5.3}$$

for all $h \in H$.

An element $\xi \in V_0$ is normalized if

$$\hat{S}^{-1}(\rho_{(2)})\xi\rho_{(1)} = \hat{\mathbb{1}} \tag{5.4}$$

holds in $\hat{H}$.

The space V_0 is *not* isomorphic to the space $\mathcal{I}^L(\hat{H})$ of left integrals in $\hat{H}$ generalizing the situation in 3.3 of [7] –. It is its subspace $\hat{H}^L$ which is isomorphic to $\mathcal{I}^L(\hat{H})$ via the isomorphism $g : \mathcal{I}^L(\hat{H}) \to \hat{H}^L$, $g(\lambda) := \hat{S}(\lambda \leftharpoonup r)$. However, it is true that the existence of normalized elements in $\mathcal{I}^L(\hat{H})$ and V_0 are equivalent.

3 The space of the Doi-Hopf integrals is isomorphic to $V_0 := H' \cap (\hat{H} \rtimes H)$, the commutant of H in the Weyl algebra. The isomorphism $f : \tilde{V} \to V_0$ is given by

$$f(\gamma) := \beta^i \gamma(r)(b_i) \tag{5.5}$$

with the help of the basis $\{b_i\}$ of H and the dual basis $\{\beta^i\}$ of $\hat{H}$.

The element $w \in V_0$ is normalized if

$$\hat{S}^{-1}(\rho_{(2)})w\rho_{(1)} = 1_{\hat{H} \rtimes H} \tag{5.6}$$

holds in the Weyl algebra $\hat{H} \rtimes H$.

4 The space of the Doi-Hopf integrals is isomorphic to $V_0 := \{u \in \mathcal{D}(K) \mid u\mathcal{D}(b) = \mathcal{D}(b_{(1)})u\mathcal{D}(S^{-1}(r)S^{-2}(b_{(2)}) \rightharpoonup \rho)\}$. The isomorphism $f : \tilde{V} \to V_0$ is given by

$$f(\gamma) := \mathcal{D}(\beta^i)\mathcal{D}(\gamma(r)(b_i)) \tag{5.7}$$

with the help of the basis $\{b_i\}$ of K and the dual basis $\{\beta^i\}$ of $\hat{K}$.

$u \in V_0$ is normalized if

$$\hat{S}^{-1}(\rho_{(2)})u\rho_{(1)} = 1_{\mathcal{D}(K)} \tag{5.8}$$

holds in the double $\mathcal{D}(K)$.

6 Appendix: Yetter-Drinfel'd modules over WHA's and Drinfel'd doubles

For the convenience of the reader we give here the generalization of the double construction due to Drinfel'd [10] and of the corresponding theory of Yetter-Drinfel'd modules [18, 16] to WHA's. The Drinfel'd double of WHA's – also without the details – was discussed in [1].

Definition 6.1 *Let H be a finite dimensional WHA over the field k. Its* Drinfel'd double *$\mathcal{D}(H)$ is the WHA defined below:*

As a k-space $\mathcal{D}(H)$ is an amalgamated tensor product $H_{H^L\equiv\hat{H}^R}\otimes_{H^R\equiv\hat{H}^L}\hat{H}$ with the amalgamation relations $a^R\otimes\hat{\mathbb{1}}\equiv\mathbb{1}\otimes(\hat{\mathbb{1}}\leftharpoonup a^R)$; $(a^L\otimes\hat{\mathbb{1}})\equiv\mathbb{1}\otimes(a^L\rightharpoonup\hat{\mathbb{1}})$ for $a^L\in H^L, a^R\in H^R$. Denote by $\mathcal{D}(a)\mathcal{D}(\phi)$ the image of $H\otimes\hat{H}\ni a\otimes\phi$ under the amalgamation and $\mathcal{D}(a)\equiv\mathcal{D}(a)\mathcal{D}(\hat{\mathbb{1}}), \mathcal{D}(\phi)\equiv\mathcal{D}(\mathbb{1})\mathcal{D}(\phi)$.

The algebra structure is defined by

$$\begin{aligned}\mathcal{D}(a)\mathcal{D}(b) &= \mathcal{D}(ab)\\ \mathcal{D}(\phi)\mathcal{D}(\psi) &= \mathcal{D}(\phi\psi)\\ \mathcal{D}(\phi)\mathcal{D}(a) &= \mathcal{D}(a_{(2)})\mathcal{D}(\phi_{(2)})\langle\phi_{(1)}|a_{(3)}\rangle\langle\phi_{(3)}|S^{-1}(a_{(1)})\rangle.\end{aligned}\tag{6.1}$$

One checks that (6.1) is compatible with the amalgamation relations and makes $\mathcal{D}(H)$ an associative algebra with unit $\mathcal{D}(\mathbb{1})\equiv\mathcal{D}(\hat{\mathbb{1}})$.

The coalgebra structure is given by

$$\begin{aligned}\Delta_{\mathcal{D}}(\mathcal{D}(a)\mathcal{D}(\phi)) &= \mathcal{D}(a_{(1)})\mathcal{D}(\phi_{(2)})\otimes\mathcal{D}(a_{(2)})\mathcal{D}(\phi_{(1)})\\ \varepsilon_{\mathcal{D}}(\mathcal{D}(a)\mathcal{D}(\phi)) &= \varepsilon(a(\phi\rightharpoonup\mathbb{1}))\equiv\hat{\varepsilon}((\hat{\mathbb{1}}\leftharpoonup a)\phi).\end{aligned}\tag{6.2}$$

One checks that (6.2) makes $\mathcal{D}(H)$ a WBA. Finally the antipode is

$$S_{\mathcal{D}}(\mathcal{D}(a)\mathcal{D}(\phi))=\mathcal{D}(\hat{S}^{-1}(\phi))\mathcal{D}(S(a))\tag{6.3}$$

making $\mathcal{D}(H)$ a WHA.

Definition 6.2 *Let H be a WBA over the field k. The k-space M is a right* Yetter-Drinfel'd module *over H if it is a right H-module and a left H comodule s.t.*

$$\begin{aligned}m_{<-1>}a_{(1)}\otimes m_{<0>}\cdot a_{(2)} &= a_{(2)}(m\cdot a_{(1)})_{<-1>}\otimes(m\cdot a_{(1)})_{<0>}\\ m_{<-1>}\mathbb{1}_{(1)}\otimes m_{<0>}\cdot\mathbb{1}_{(2)} &= m_{<-1>}\otimes m_{<0>}\end{aligned}\tag{6.4}$$

for all $m\in M, a\in A$.

Notice that if H is also a WHA then (6.4) can be replaced by the single relation

$$(m\cdot a)_{<-1>}\otimes(m\cdot a)_{<0>}=S^{-1}(a_{(3)})m_{<-1>}a_{(1)}\otimes m_{<0>}\cdot a_{(2)}.\tag{6.5}$$

By the category $\mathcal{YD}(H)$ we mean the category with objects the finite dimensional right Yetter-Drinfel'd modules over H and arrows $T: M\to M'$ intertwining both the H-module and the H-comodule structures of M and M'.

If H is a finite dimensional WHA then by our Proposition 4.2 and Example 4. the category $\mathcal{YD}(H)$ is equivalent to the category of the right modules over the WHA $\mathcal{D}(H)$ hence carries (among others) a monoidal structure [1, 2].

We claim that also for a finite dimensional WBA H the category $\mathcal{YD}(H)$ of its right Yetter-Drinfel'd modules is a monoidal category. If M and N are Yetter-Drinfel'd modules then their monoidal product is defined within $\mathcal{M}_H$ the category of right H-modules. This k-space $M \times N$ can be equipped also with a left H-comodule structure as

$$\rho_{M\times N}(m \otimes n) := n_{<-1>} m_{<-1>} \mathbb{1}_{(1)} \otimes m_{<0>} \cdot \mathbb{1}_{(2)} \otimes n_{<0>} \cdot \mathbb{1}_{(3)}.$$

Also the monoidal unit H^L of $\mathcal{M}_H$ has a left H-comodule structure via

$$\rho_{H^L} := \Delta|_{H^L}.$$

The reader may check using some WBA calculus that all $M \times N$ and H^L are Yetter-Drinfel'd modules over H if M and N are.

Also the intertwiners $u^L_M \in (M, H^L \times M)_{\mathcal{M}_H}$, $u^R_M \in (M, M \times H^L)_{\mathcal{M}_H}$ are intertwiners in $\mathcal{YD}(H)$ too.

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Hopf algebroids with bijective antipodes: axioms, integrals, and duals

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Abstract

Motivated by the study of depth 2 Frobenius extensions, we introduce a new notion of Hopf algebroid. It is a 2-sided bialgebroid with a bijective antipode which connects the two, left and right handed, structures. While all the interesting examples of the Hopf algebroid of J.H. Lu turn out to be Hopf algebroids in the sense of this paper, there exist simple examples showing that our definition is not a special case of Lu's. Our Hopf algebroids, however, belong to the class of $\times_L$-Hopf algebras proposed by P. Schauenburg. After discussing the axioms and some examples, we study the theory of non-degenerate integrals in order to obtain duals of Hopf algebroids.

1. Introduction

There is a consensus in the literature that bialgebroids, invented by Takeuchi [24] as $\times_R$-bialgebras, are the proper generalizations of bialgebras to non-commutative base rings [5,13,17,20,21,25]. The situation of Hopf algebroids, i.e., bialgebroids with some sort of antipode, is less understood. The antipode proposed by J.H. Lu [13] is burdened by the need of a section for the canonical epimorphism $A \otimes A \to A \otimes_R A$ the precise role of which remained unclear. The $\times_R$-Hopf algebras proposed by P. Schauenburg in [18] have a clearcut categorical meaning. They are the bialgebroids A over R such that the

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forgetful functor ${}_A\mathcal{M} \to {}_R\mathcal{M}_R$ is not only strict monoidal, which is the condition for A to be a bialgebroid over R, but preserves the closed structure as well. In this very general quantum groupoid, however, antipode, as a map $A \to A$, does not exist.

Our proposal of an antipode, announced in [2], is based on the following simple observation. The antipode of a Hopf algebra H is a bialgebra map $S : H \to H^{\rm op}_{\rm cop}$. The opposite of a bialgebroid A, however, is not a bialgebroid in the same sense. In the terminology of [8] there are left bialgebroids and right bialgebroids; corresponding to whether ${}_A\mathcal{M}$ or $\mathcal{M}_A$ is given a monoidal structure. This suggests that the existence of antipode on a bialgebroid should be accompanied with a two-sided bialgebroid structure and the antipode should swap the left and right handed structures. More explicit guesses for what to take as a definition of the antipode can be obtained by studying depth 2 Frobenius extensions. In [8] it has been shown that for a depth 2 ring extension $N \subset M$ the endomorphism ring $A = \mathrm{End}_N\, M_N$ has a canonical left bialgebroid structure over the centralizer $R = C_M(N)$. If $N \subset M$ is also Frobenius and a Frobenius homomorphism $\psi : M \to N$ is given, then A has a right bialgebroid structure, too. There is a candidate for the antipode $S : A \to A$ as transposition w.r.t. the bilinear form $m, m' \in M \mapsto \psi(mm')$, see (3.6). In fact, this definition of S does not require the depth 2 property, so in this example antipode exists prior to comultiplication. Assuming the extension M/N is either H-separable or Hopf–Galois, L. Kadison has shown [9,10] that the bialgebroid A or its dual $B = (M \otimes_N M)^N$ has an antipode in the sense of [13].

In a recent paper [7] B. Day and R. Street give a new characterization of bialgebroids in the framework of symmetric monoidal autonomous bicategories. They also introduce a new notion called *Hopf bialgebroid*. It is more restrictive than Schauenburg's $\times_R$-Hopf algebra since it requires star autonomy [1] rather than to be closed. We will show in Section 4.2 that the categorical definition [7] of the Hopf bialgebroid is equivalent to our purely algebraic Definition 4.1 of *Hopf algebroid*, apart from the tiny difference that we allow S^2 to be nontrivial on the base ring. This freedom can be adjusted to the Nakayama automorphism of ψ in case of the Frobenius depth 2 extensions. It is a new feature of our Hopf algebroids, compared to (weak) Hopf algebras, that the antipode is not unique. The various antipodes on a given bialgebroid were shown to be in one-to-one correspondence with the generalized characters (called twists) in [2].

It is encouraging that one can find 'quantum groupoids' in the literature that satisfy our axioms. Such are the weak Hopf algebras with bijective antipode, the examples of Lu–Hopf algebroids in [5], and the extended Hopf algebras in [11]. In particular, the Connes–Moscivici algebra [6] is a Hopf algebroid it this sense. We also present an example which is a Hopf algebroid in the sense of this paper but does not satisfy the axioms of [13]. This proves that the two notions of Hopf algebroid—the one in the sense of this paper and the one in the sense of [13]—are not equivalent. Until now we could neither prove nor exclude by examples the possibility that the latter was a special case of the former.

The left and right *integrals* in Hopf algebra theory are introduced as the invariants of the left and right regular module, respectively. In this analogy one can define left integrals in a left bialgebroid and right integrals in a right bialgebroid. Since a Hopf algebroid has both left and right bialgebroid structures both left and right integrals can be defined.

The properties of the integrals in a (weak) Hopf algebra over a field k carry information about its algebraic structure. For example, the Maschke's theorem [3,12] states that it is a semi-simple algebra if and only if it has a normalized integral. The Larson–Sweedler theorem [12,26] implies that it is finite dimensional over k if and only if it has a non-degenerate left (hence also a right) integral. In this case the k-dual also has a (weak) Hopf algebra structure.

Therefore, in Section 5, we analyze the consequences of the existence of a non-degenerate integral in a Hopf algebroid. We show that if there exists a non-degenerate integral in a Hopf algebroid $\mathcal{A}$ over the base L then the ring extension $L \to A$ is a Frobenius extension hence also finitely generated projective. We do *not* investigate, however, the opposite implication, i.e., we do not study the question under what conditions on the Hopf algebroid the existence of a non-degenerate integral follows.

If some of the L-module structures of a Hopf algebroid is finitely generated projective then the corresponding dual can be equipped with a bialgebroid structure [8]. There is no obvious way, however, how to equip it with an antipode in general. We show that in the case of Hopf algebroids possessing a non-degenerate integral the dual bialgebroids are all (anti-) isomorphic and they combine into a Hopf algebroid—depending on the choice of the non-degenerate integral. Therefore, we do not associate a dual Hopf algebroid to a given Hopf algebroid rather a dual isomorphism class to an isomorphism class of Hopf algebroids. The well-known k-dual of a finite (weak) Hopf algebra H over the commutative ring k turns out to be the unique (distinguished) (weak) Hopf algebra in the dual isomorphism class of the isomorphism class of the Hopf algebroid H.

The paper is organized as follows. In Section 2 we introduce some technical conventions about bialgebroids that are used in this paper. Our motivating example, the Hopf algebroid corresponding to a depth 2 Frobenius extension of rings is discussed in Section 3. In Section 4 we give some equivalent definitions of Hopf algebroids. We prove that our definition is equivalent to the one in [7] hence gives a special case of the one in [18]. In the final subsection of Section 4 we present a collection of examples. In Section 5 we propose a theory of non-degenerate integrals as a tool for the definition of the dual Hopf algebroid.

2. Preliminaries on bialgebroids

In this technical section we summarize our notations and the basic definitions of bialgebroids that will be used later on. For more about bialgebroids we refer to the literature [5,8,18,19,21,22,24].

Definition 2.1. A *left bialgebroid* (or Takeuchi $\times_L$-bialgebra) $\mathcal{A}_L$ consists of the data $(A, L, s_L, t_L, \gamma_L, \pi_L)$. The A and L are associative unital rings, the total and base rings, respectively. The $s_L : L \to A$ and $t_L : L^{\mathrm{op}} \to A$ are ring homomorphisms such that the images of L in A commute making A an L-L-bimodule via

$$l \cdot a \cdot l' := s_L(l) t_L(l') a. \tag{2.1}$$

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The bimodule (2.1) is denoted by ${}_LA_L$. The triple $({}_LA_L, \gamma_L, \pi_L)$ is a comonoid in ${}_L\mathcal{M}_L$, the category of L-L-bimodules. Introducing the Sweedler's notation $\gamma_L(a) \equiv a_{(1)} \otimes a_{(2)} \in A \otimes_L A$ the identities

$$a_{(1)}t_L(l) \otimes a_{(2)} = a_{(1)} \otimes a_{(2)}s_L(l), \tag{2.2}$$

$$\gamma_L(1_A) = 1_A \otimes 1_A, \tag{2.3}$$

$$\gamma_L(ab) = \gamma_L(a)\gamma_L(b), \tag{2.4}$$

$$\pi_L(1_A) = 1_L, \tag{2.5}$$

$$\pi_L\big(as_L \circ \pi_L(b)\big) = \pi_L(ab) = \pi_L\big(at_L \circ \pi_L(b)\big) \tag{2.6}$$

are required for all $l \in L$ and $a, b \in A$. The requirement (2.4) makes sense in the view of (2.2).

The L actions of the bimodule ${}_LA_L$ in (2.1) are given by left multiplication. Using right multiplication there exists another L-L-bimodule structure on the total ring A of a left bialgebroid $\mathcal{A}_L$:

$$l \cdot a \cdot l' := at_L(l)s_L(l'). \tag{2.7}$$

This L-L-bimodule is called ${}^LA^L$. This way A carries four commuting actions of L.

If $\mathcal{A}_L = (A, L, s_L, t_L, \gamma_L, \pi_L)$ is a left bialgebroid then so is its co-opposite: $\mathcal{A}_{L\,\mathrm{cop}} = (A, L^{\mathrm{op}}, t_L, s_L, \gamma_L^{\mathrm{op}}, \pi_L)$. The opposite $\mathcal{A}_L^{\mathrm{op}} = (A^{\mathrm{op}}, L, t_L, s_L, \gamma_L, \pi_L)$ has a different structure that was introduced under the name *right bialgebroid* in [8].

Definition 2.2. A *right bialgebroid* $\mathcal{A}_R$ consists of the data $(A, R, s_R, t_R, \gamma_R, \pi_R)$. The A and R are associative unital rings, the total and base rings, respectively. The $s_R : R \to A$ and $t_R : R^{\mathrm{op}} \to A$ are ring homomorphisms such that the images of R in A commute making A an R-R-bimodule:

$$r \cdot a \cdot r' := as_R(r')t_R(r). \tag{2.8}$$

The bimodule (2.8) is denoted by ${}^RA^R$. The triple $({}^RA^R, \gamma_R, \pi_R)$ is a comonoid in ${}_R\mathcal{M}_R$. Introducing the Sweedler's notation $\gamma_R(a) \equiv a^{(1)} \otimes a^{(2)} \in A \otimes_R A$ the identities

$$\begin{aligned} s_R(r)a^{(1)} \otimes a^{(2)} &= a^{(1)} \otimes t_R(r)a^{(2)}, \\ \gamma_R(1_A) &= 1_A \otimes 1_A, \\ \gamma_R(ab) &= \gamma_R(a)\gamma_R(b), \\ \pi_R(1_A) &= 1_R, \end{aligned}$$

$$\pi_R\big(s_R \circ \pi_R(a)b\big) = \pi_R\,(ab) = \pi_R\big(t_R \circ \pi_R(a)b\big)$$

are required for all $r \in R$ and $a, b \in A$.

For the right bialgebroid $\mathcal{A}_R$ we introduce the R-R-bimodule ${}_RA_R$ via

$$r \cdot a \cdot r' := s_R(r)t_R(r')a. \tag{2.9}$$

This way A caries four commuting actions of R.

Left (right) bialgebroids can be characterized by the property that the forgetful functor ${}_A\mathcal{M} \to {}_L\mathcal{M}_L$ ($\mathcal{M}_A \to {}_R\mathcal{M}_R$) is strong monoidal [17,20].

It is natural to consider the homomorphisms of bialgebroids to be ring homomorphisms preserving the comonoid structure. We do not want to make difference however between bialgebroids over isomorphic base rings. This leads to the following.

Definition 2.3 [21]. A left bialgebroid homomorphism $\mathcal{A}_L \to \mathcal{A}'_L$ is a pair of ring homomorphisms ($\Phi : A \to A', \phi : L \to L'$) such that

$$s'_L \circ \phi = \Phi \circ s_L, \qquad t'_L \circ \phi = \Phi \circ t_L,$$
$$\pi'_L \circ \Phi = \phi \circ \pi_L, \qquad \gamma'_L \circ \Phi = (\Phi \otimes \Phi) \circ \gamma_L.$$

The last condition makes sense since by the first two conditions $\Phi \otimes \Phi$ is a well-defined map $A \otimes_L A \to A \otimes_{L'} A$.

The pair (Φ, ϕ) is an isomorphism of left bialgebroids if it is a bialgebroid homomorphism such that both Φ and ϕ are bijective.

A right bialgebroid homomorphism (isomorphism) $\mathcal{A}_R \to \mathcal{A}'_R$ is a left bialgebroid homomorphism (isomorphism) $(\mathcal{A}_R)^{\mathrm{op}} \to (\mathcal{A}'_R)^{\mathrm{op}}$.

Let $\mathcal{A}_L$ be a left bialgebroid. The equation (2.1) describes two L-modules A_L and ${}_LA$. Their L-duals are the additive groups of L-module maps

$$\mathcal{A}_* := \{\phi_* : A_L \to L_L\} \quad \text{and} \quad {}_*\mathcal{A} := \{{}_*\phi : {}_LA \to {}_LL\},$$

where ${}_LL$ stands for the left regular and L_L for the right regular L-module. Both $\mathcal{A}_*$ and ${}_*\mathcal{A}$ carry left A module structures via the transpose of the right regular action of A. For $\phi_* \in \mathcal{A}_*$, ${}_*\phi \in {}_*\mathcal{A}$, and $a, b \in A$ we have

$$(a \rightharpoonup \phi_*)(b) = \phi_*(ba) \quad \text{and} \quad (a \rightharpoondown {}_*\phi)(b) = {}_*\phi(ba).$$

Similarly, in the case of a right bialgebroid $\mathcal{A}_R$—denoting the left and right regular R-modules by ${}^RR$ and R^R, respectively—the two R-dual additive groups

$$\mathcal{A}^* := \{\phi^* : A^R \to R^R\} \quad \text{and} \quad {}^*\mathcal{A} := \{{}^*\phi : {}^RA \to {}^RR\}$$

carry right A-module structures:

$$(\phi^* \leftharpoonup a)(b) = \phi^*(ab) \quad \text{and} \quad ({}^*\phi \leftharpoondown a)(b) = {}^*\phi(ab).$$

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The comonoid structures can be transposed to give monoid (i.e., ring) structures to the duals. In the case of a left bialgebroid $\mathcal{A}_L$

$$(\phi_*\psi_*)(a) = \psi_*\big(s_L \circ \phi_*(a_{(1)})a_{(2)}\big) \quad \text{and}$$
$$({}_*\phi_*\psi)(a) = {}_*\psi\big(t_L \circ {}_*\phi(a_{(2)})a_{(1)}\big) \tag{2.10}$$

for ${}_*\phi, {}_*\psi \in {}_*\mathcal{A}$, $\phi_*, \psi_* \in \mathcal{A}_*$, and $a \in A$.

Similarly, in the case of a right bialgebroid $\mathcal{A}_R$,

$$(\phi^*\psi^*)(a) = \phi^*\big(a^{(2)}t_R \circ \psi^*\big(a^{(1)}\big)\big) \quad \text{and}$$
$$({}^*\phi^*\psi)(a) = {}^*\phi\big(a^{(1)}s_R \circ {}^*\psi\big(a^{(2)}\big)\big) \tag{2.11}$$

for $\phi^*, \psi^* \in \mathcal{A}^*$, ${}^*\phi, {}^*\psi \in {}^*\mathcal{A}$, and $a \in A$.

In the case of a left bialgebroid $\mathcal{A}_L$ also the ring A has right $\mathcal{A}_*$- and right ${}_*\mathcal{A}$-module structures:

$$a \leftharpoonup \phi_* = s_L \circ \phi_*(a_{(1)})a_{(2)} \quad \text{and} \quad a \leftharpoondown {}_*\phi = t_L \circ {}_*\phi(a_{(2)})a_{(1)} \tag{2.12}$$

for $\phi_* \in \mathcal{A}_*$, ${}_*\phi \in {}_*\mathcal{A}$ and $a \in A$.

Similarly, in the case of a right bialgebroid $\mathcal{A}_R$ the ring A has left $\mathcal{A}^*$- and left ${}^*\mathcal{A}$-structures:

$$\phi^* \rightharpoonup a = a^{(2)}t_R \circ \phi^*\big(a^{(1)}\big) \quad \text{and} \quad {}^*\phi \rightharpoondown a = a^{(1)}s_R \circ {}^*\phi\big(a^{(2)}\big) \tag{2.13}$$

for $\phi^* \in \mathcal{A}^*$, ${}^*\phi \in {}^*\mathcal{A}$, and $a \in A$.

In the case when the L (R) module structure on A is finitely generated projective then the corresponding dual has also a bialgebroid structure: if $\mathcal{A}_L$ is a left bialgebroid such that the L-module A_L is finitely generated projective then $\mathcal{A}_*$ is a right bialgebroid over the base L as follows:

$$\big(s_{*R}(l)\big)(a) = \pi_L\big(as_L(l)\big), \qquad \big(t_{*R}(l)\big)(a) = l\pi_L(a),$$
$$\gamma_{*R}(\phi_*) = b_i \rightharpoonup \phi_* \otimes \beta_*^i, \qquad \pi_{*R}(\phi_*) = \phi_*(1_A),$$

where $\{b_i\}$ is an L-basis in A_L and $\{\beta_*^i\}$ is the dual basis in $\mathcal{A}_*$.

Similarly, if $\mathcal{A}_L$ is a left bialgebroid such that the L-module ${}_LA$ is finitely generated projective then ${}_*\mathcal{A}$ is a right bialgebroid over the base L as follows:

$$\big({}_*s_R(l)\big)(a) = \pi_L(a)l, \qquad \big({}_*t_R(l)\big)(a) = \pi_L\big(at_L(l)\big),$$
$${}_*\gamma_R({}_*\phi) = {}_*\beta^i \otimes b_i \rightharpoondown {}_*\phi, \qquad {}_*\pi_R({}_*\phi) = {}_*\phi(1_A),$$

where $\{b_i\}$ is an L-basis in ${}_LA$ and $\{{}_*\beta^i\}$ is the dual basis in ${}_*\mathcal{A}$.

If $\mathcal{A}_R$ is a right bialgebroid such that the R-module A^R is finitely generated projective then $\mathcal{A}^*$ is a left bialgebroid over the base R as follows:

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$$\big(s_L^*(r)\big)(a) = r\pi_R(a), \quad \big(t_L^*(r)\big)(a) = \pi_R\big(s_R(r)a\big),$$
$$\gamma_L^*(\phi^*) = \phi^* \leftharpoonup b_i \otimes \beta^{*i}, \qquad \pi_L^*(\phi^*) = \phi^*(1_A),$$

where $\{b_i\}$ is an R-basis in A^R and $\{\beta^{*i}\}$ is the dual basis in $\mathcal{A}^*$.

If $\mathcal{A}_R$ is a right bialgebroid such that the R-module ${}^R A$ is finitely generated projective then ${}^*\!\mathcal{A}$ is a left bialgebroid over the base R as follows:

$$\big({}^*s_L(r)\big)(a) = \pi_R\big(t_R(r)a\big), \qquad \big({}^*t_L(r)\big)(a) = \pi_R(a)r,$$
$${}^*\gamma_L({}^*\phi) = {}^*\beta^i \otimes {}^*\phi \leftharpoondown b_i, \qquad {}^*\pi_L({}^*\phi) = {}^*\phi(1_A),$$

where $\{b_i\}$ is an R-basis in ${}^R A$ and $\{{}^*\beta^i\}$ is the dual basis in ${}^*\!\mathcal{A}$.

3. The motivating example: D2 Frobenius extensions

3.1. The forefather of antipodes

In this subsection $N \to M$ denotes a Frobenius extension of rings. This means the existence of N-N-bimodule maps $\psi : M \to N$ possessing quasibases. An element $\sum_i u_i \otimes v_i \in M \otimes_N M$ is called the quasibasis of ψ [27] if

$$\sum_i \psi(mu_i) \cdot v_i = m = \sum_i u_i \cdot \psi(v_i m), \quad m \in M. \tag{3.1}$$

As we shall see, already in this general situation there exist anti-automorphisms S on the ring $A := \mathrm{End}_N M_N$, one for each Frobenius homomorphism ψ. The S will become an antipode if the extension $N \subset M$ is also of depth 2, so A also has coproduct(s).

The idea of writing the antipode as the difference of two Fourier transforms goes back to Radford's paper [16] but plays important role in Szymanski's treatment of finite index subfactors [23], too. If l is a non-degenerate left integral in a finite dimensional Hopf algebra H and $\lambda \in H^*$ is the dual left integral, i.e., $\lambda \rightharpoonup l = 1$, then the antipode can be written as $S(h) = (\lambda \leftharpoonup h) \rightharpoonup l$. This formula generalizes also to weak Hopf algebras [3]. Here we will give the analogous formula for the two-step centralizer A of any Frobenius extension $N \subset M$.

Instead of a dual algebra of A we have the second two-step centralizer B in the Jones tower of $N \subset M$ which is the center of the N-N-bimodule $M \otimes_N M$

$$B := (M \otimes_N M)^N \equiv \{X \in M \otimes_N M \mid n \cdot X = X \cdot n \ \forall n \in N\}.$$

It is a ring with multiplication $(b^1 \otimes b^2)(b'^1 \otimes b'^2) = b'^1 b^1 \otimes b^2 b'^2$ and unit $1_B = 1_M \otimes 1_M$. Note that the ring structures of neither A nor B depend on the Frobenius structure. But if there is a Frobenius homomorphism ψ then Fourier transformation makes A and B isomorphic as additive groups.

Fixing a Frobenius homomorphism ψ with quasibasis $\sum_i u_i \otimes v_i$, we can introduce convolution products on both A and B as follows. From now on we omit the summation symbol for summing over the quasibasis

$$\alpha,\beta \in A \quad \mapsto \quad \alpha * \beta := \alpha(u_i)\beta(v_i\_) \in A,$$
$$a,b \in B \quad \mapsto \quad a * b := a^1\psi\bigl(a^2b^1\bigr) \otimes b^2 \in B.$$

The convolution product lends A and B new ring structures. The unit of A is ψ and the unit of B is $u_i \otimes v_i$. The Fourier transformation is to relate these new algebra structures to the old ones. There are two natural candidates for a Fourier transformation:

$$\mathcal{F}: A \to B, \quad \mathcal{F}(\alpha) := u_i \otimes \alpha(v_i), \qquad \dot{\mathcal{F}}: A \to B, \quad \dot{\mathcal{F}}(\alpha) := \alpha(u_i) \otimes v_i,$$
$$\mathcal{F}^{-1}: B \to A, \quad \mathcal{F}^{-1}(b) = \psi\bigl(\_b^1\bigr)b^2, \qquad \dot{\mathcal{F}}^{-1}: B \to A, \quad \dot{\mathcal{F}}^{-1}(b) = b^1\psi\bigl(b^2\_\bigr).$$

They relate the convolution and ordinary products or their opposites as follows:

$$\mathcal{F}(\alpha * \beta) = \mathcal{F}(\alpha)\mathcal{F}(\beta), \qquad \dot{\mathcal{F}}(\alpha * \beta) = \dot{\mathcal{F}}(\beta)\dot{\mathcal{F}}(\alpha),$$
$$\mathcal{F}(\alpha\beta) = \mathcal{F}(\beta) * \mathcal{F}(\alpha), \qquad \dot{\mathcal{F}}(\alpha\beta) = \dot{\mathcal{F}}(\alpha) * \dot{\mathcal{F}}(\beta).$$

The difference between $\mathcal{F}$ and $\dot{\mathcal{F}}$ is therefore an anti-automorphism on both A and B. This leads to the "antipodes"

$$S_A : A \to A^{\mathrm{op}}, \quad S_A := \dot{\mathcal{F}}^{-1} \circ \mathcal{F}, \quad S_A(\alpha) = u_i\psi\bigl(\alpha(v_i)\_\bigr), \tag{3.2}$$
$$S_B : B \to B^{\mathrm{op}}, \quad S_B := \dot{\mathcal{F}} \circ \mathcal{F}^{-1}, \quad S_B(b) = \psi\bigl(u_ib^1\bigr)b^2 \otimes v_i \tag{3.3}$$

with inverses

$$S_A^{-1}(\alpha) = \psi\bigl(\_\alpha(u_i)\bigr)v_i, \tag{3.4}$$
$$S_B^{-1}(b) = u_i \otimes b^1\psi\bigl(b^2v_i\bigr). \tag{3.5}$$

Notice that S_A is just transposition w.r.t. the bi-N-linear form $(m,m') = \psi(mm')$ since

$$\psi\bigl(mS_A(\alpha)(m')\bigr) = \psi\bigl(\alpha(m)m'\bigr). \tag{3.6}$$

These antipodes behave well also relative to the bimodule structures over the centralizer $C_M(N) := \{c \in M \mid cn = nc, \forall n \in N\}$. Let us consider S_A. The centralizer is embedded into A twice: via left multiplications and right multiplications,

$$L \xrightarrow{\lambda} A \xleftarrow{\rho} R,$$

where L stands for $C_M(N)$ and R for $C_M(N)^{\mathrm{op}}$. Clearly, $\lambda(L) \subset C_M(\rho(R))$. Introducing the Nakayama automorphism

$$\nu : C_M(N) \to C_M(N), \quad \nu(c) := \psi(u_i c) v_i$$

of ψ and using its basic identities

$$\psi(mc) = \psi\big(\nu(c)m\big), \quad m \in M,\ c \in C_M(N),$$
$$u_i c \otimes v_i = u_i \otimes \nu(c) v_i, \quad c \in C_M(N),$$

we obtain

$$S_A \circ \lambda = \rho \circ \nu^{-1}, \qquad S_A \circ \rho = \lambda$$

and therefore

$$S_A\big(\lambda(l)\rho(r)\alpha\big) = S_A(\alpha)\lambda(r)\rho\big(\nu^{-1}(l)\big), \quad l \in L,\ r \in R,\ \alpha \in A. \tag{3.7}$$

In order to interpret the latter relation as the statement that S_A is a bimodule map we define L-L- and R-R-bimodule structures on A by

$$l_1 \cdot \alpha \cdot l_2 := s_L(l_1) t_L(l_2) \alpha, \tag{3.8}$$
$$r_1 \cdot \alpha \cdot r_2 := \alpha t_R(r_1) s_R(r_2), \tag{3.9}$$

where we introduced the ring homomorphisms

$$s_L := L \xrightarrow{\lambda} A, \qquad s_R := R \xrightarrow{\rho} A,$$
$$t_L := L^{\mathrm{op}} \xrightarrow{\mathrm{id}} R \xrightarrow{\rho} A, \qquad t_R := R^{\mathrm{op}} \xrightarrow{\nu} L \xrightarrow{\lambda} A. \tag{3.10}$$

Also using the notation θ for the inverse of the Nakayama automorphism when considered as a map

$$\theta : L \xrightarrow{\nu^{-1}} R^{\mathrm{op}},$$

Eq. (3.7) can be read as

$$S_A(l_1 \cdot \alpha \cdot l_2) = \theta(l_2) \cdot S_A(\alpha) \cdot \theta(l_1). \tag{3.11}$$

Remark 3.1. The apparent asymmetry between t_L and t_R in (3.10) disappears if one repeats the above construction for the more general situation of a Frobenius N-M-bimodule X instead of the ${}_N M_M$ arising from a Frobenius extension of rings. As a matter of fact, denoting by $\overline{X}$ the (two-sided) dual of X and setting $A = \operatorname{End} X \otimes_M \overline{X}$, $L = \operatorname{End} X$, and $R = \operatorname{End} \overline{X}$, we find the obvious ring homomorphisms

$$s_L(l) = l \otimes \overline{X}, \qquad s_R(r) = X \otimes r;$$

but there is no distinguished map $L \to R^{\mathrm{op}}$ like the identity is in the case of $X = {}_N M_M$. Instead we have two distinguished maps given by the left and right dual functors (transpositions). It is easy to check that in case of $X = {}_N M_M$ they are the identity and the ν^{-1}, respectively, as we used in (3.10).

In order to restore the symmetry, let us introduce the counterpart of θ which is the identity as a homomorphism $\iota : R \xrightarrow{\mathrm{id}} L^{\mathrm{op}}$. Then, in addition to (3.11) the antipode satisfies also

$$S_A(r_1 \cdot \alpha \cdot r_2) = \iota(r_2) \cdot S_A(\alpha) \cdot \iota(r_1). \tag{3.12}$$

The most important consequence of (3.11) and (3.12) is the existence of a tensor square of S_A. In the case of Hopf algebras, one often uses expressions like $(S \otimes S) \circ \Sigma$, where Σ is the symmetry $A \otimes A \to A \otimes A$, $x \otimes y \mapsto y \otimes x$ in the category of k-modules. Now we have bimodule categories ${}_L\mathcal{M}_L$ and ${}_R\mathcal{M}_R$ without braiding, so Σ does not exist and neither do $S_A \otimes S_A$ nor $S_A^{-1} \otimes S_A^{-1}$ because S_A is not a bimodule map. Instead we have the *twisted bimodule properties* (3.11) and (3.12) which guarantee the existence of the 'composite of' $S_A \otimes S_A$ and Σ although individually they do not exist. More precisely, there exist twisted bimodule maps

$$S_{A \otimes_L A} : A \otimes_L A \to A \otimes_R A, \quad \alpha \otimes_L \beta \mapsto S_A(\beta) \otimes_R S_A(\alpha),$$
$$S_{A \otimes_R A} : A \otimes_R A \to A \otimes_L A, \quad \alpha \otimes_R \beta \mapsto S_A(\beta) \otimes_L S_A(\alpha).$$

For later convenience let us record some useful formulas following directly from (3.2) and (3.10):

$$S_A \circ s_L = t_L \circ \nu^{-1}, \qquad S_A \circ t_L = s_L, \qquad t_L \circ \iota = s_R,$$
$$S_A \circ s_R = t_R \circ \nu^{-1}, \qquad S_A \circ t_R = s_R, \qquad t_R \circ \theta = s_L. \tag{3.13}$$

Notice also that $s_L(L)$ and $t_R(R)$ are the same subrings of A and similarly $t_L(L) = s_R(R)$.

3.2. Two-sided bialgebroids

Recall from [8] that for depth 2 extensions $N \to M$ the A has a canonical left bialgebroid structure over L in which the coproduct $\gamma_L : A \to A \otimes_L A$ is an L-L-bimodule map with respect to the bimodule structure (3.8). If $N \to M$ is also Frobenius then there is another right bialgebroid structure on A, canonically associated to a choice of ψ, in which R is the base and A is an R-R-bimodule via (3.9). Moreover, these two structures are related by the antipode. This two-sided structure is our motivating example of a Hopf algebroid.

We start with a technical lemma on the left and right quasibases. A and B denotes the rings as before.

Lemma 3.2. *Let $N \to M$ be a Frobenius extension and ψ, $u_i \otimes v_i$ be a fixed Frobenius structure. Let n be a positive integer and β_i, $\gamma_i \in A$ and b_i, $c_i \in B$, for $i = 1, \dots, n$. Assume they are related via $b_i = \mathcal{F}(\gamma_i)$ and $c_i = \dot{\mathcal{F}}(\beta_i)$. Then the following conditions are equivalent* (*summation symbols over i suppressed*):

(i) $b_i^1 \otimes_N b_i^2 \beta_i(m) = m \otimes_N 1_M$, $m \in M$;

(ii) $\gamma_i(m) c_i^1 \otimes_N c_i^2 = 1_M \otimes_N m$, $m \in M$;

(iii) $\gamma_i(m)\beta_i(m') = \psi(mm')$, $m, m' \in M$;

(iv) $b_i^1 \otimes_N b_i^2 c_i^1 \otimes_N c_i^2 = u_k \otimes_N 1_M \otimes_N v_k$.

If such elements exist the extension is called D2, i.e., of depth 2. The first two conditions are meaningful also in the non-Frobenius case and therefore $\{b_i, \beta_i\}$ was called in [8] a left D2 quasibasis and $\{c_i, \gamma_i\}$ a right D2 quasibasis. The equivalence of conditions (i) and (ii) was shown in [8, Proposition 6.4]. The rest of the proof is left to the reader.

For D2 extensions the map $\alpha \otimes \beta \mapsto \{m \otimes m' \mapsto \alpha(m)\beta(m')\}$ is an isomorphism

$$A \otimes_L A \xrightarrow{\sim} \mathrm{Hom}_{N-N}(M \otimes_N M, M),$$

see [8, Proposition 3.11]. Then the coproduct $\gamma_L : A \to A \otimes_L A$ is the unique map $\alpha \mapsto \alpha_{(1)} \otimes \alpha_{(1)}$ which satisfies

$$\alpha(mm') = \alpha_{(1)}(m)\alpha_{(1)}(m').$$

We can dualize this construction for D2 Frobenius extensions. We have the isomorphism

$$A \otimes_R A \xrightarrow{\sim} \mathrm{Hom}_{N-N}(M, M \otimes_N M), \quad \alpha \otimes_R \beta \mapsto \alpha(\_u_i) \otimes_N \beta(v_i).$$

Then $\gamma_R : A \to A \otimes_R A$ is defined as the unique map $\alpha \mapsto \alpha^{(1)} \otimes \alpha^{(2)}$ for which

$$\alpha(m)u_i \otimes_N v_i = \alpha^{(1)}(mu_i) \otimes_N \alpha^{(2)}(v_i). \tag{3.14}$$

Explicit formulas for both coproducts, as well as their counits, are given in the corollary below. But even without these formulas we can find out how the two coproducts are related by the antipode.

Theorem 3.3. *For a D2 Frobenius extension $N \to M$ of rings the endomorphism ring $A = \mathrm{End}_N M_N$ is a left bialgebroid over $L = C_M(N)$ and a right bialgebroid over $R = L^{\mathrm{op}}$ such that the antipode defined in* (3.2) *gives rise to isomorphisms*

$$(A, R, s_R, t_R, \gamma_R, \pi_R)^{\mathrm{op}} \xrightarrow{(S_A,\iota)} (A, L, s_L, t_L, \gamma_L, \pi_L)_{\mathrm{cop}},$$
$$(A, L, s_L, t_L, \gamma_L, \pi_L) \xrightarrow{(S_A,\theta)} (A, R, s_R, t_R, \gamma_R, \pi_R)^{\mathrm{op}}_{\mathrm{cop}} \tag{3.15}$$

of left bialgebroids. That is to say,

$$S_A : A \to A^{\mathrm{op}} \textit{ is a ring isomorphism}, \tag{3.16}$$
$$S_A \circ s_R = s_L \circ \iota, \qquad S_A \circ t_R = t_L \circ \iota,$$
$$S_A \circ s_L = s_R \circ \theta, \qquad S_A \circ t_L = t_R \circ \theta, \tag{3.17}$$
$$\gamma_L \circ S_A = S_{A\otimes_R A} \circ \gamma_R, \qquad \gamma_R \circ S_A = S_{A\otimes_L A} \circ \gamma_L, \tag{3.18}$$
$$\pi_L \circ S_A = \iota \circ \pi_R, \quad \textit{and} \quad \pi_R \circ S_A = \theta \circ \pi_L. \tag{3.19}$$

Proof. The left bialgebroid structure of A has been constructed in [8, Theorem 4.1]. The right bialgebroid structure will follow automatically after establishing the four properties of the antipode. The first two have already been discussed before. In order to prove (3.18) recall the definition (3.14) of γ_R. Thus, (3.18) is equivalent to

$$S_A^{-1}\big(S_A(\alpha)_{(1)}\big)(mu_i) \otimes_N S_A^{-1}\big(S_A(\alpha)_{(1)}\big)(v_i) = \alpha(m)u_i \otimes_N v_i, \tag{3.20}$$
$$S_A\big(S_A^{-1}(\alpha)_{(1)}\big)(mu_i) \otimes_N S_A\big(S_A^{-1}(\alpha)_{(1)}\big)(v_i) = u_i \otimes_N v_i\alpha(m) \tag{3.21}$$

for all $m \in M$. Expanding the left-hand side of (3.20) then using (3.6), then (3.4), then the definition of γ_L, and finally (3.6) again we obtain

$$\begin{aligned}
&S_A^{-1}\big(S_A(\alpha)_{(1)}\big)(mu_i)\psi\big(S_A^{-1}\big(S_A(\alpha)_{(1)}\big)(v_i)u_k\big) \otimes_N v_k \\
&\quad = S_A^{-1}\big(S_A(\alpha)_{(1)}\big)(mu_i)\psi\big(v_i S_A(\alpha)_{(1)}(u_k)\big) \otimes_N v_k \\
&\quad = S_A^{-1}\big(S_A(\alpha)_{(1)}\big)\big(mS_A(\alpha)_{(1)}(u_k)\big) \otimes_N v_k \\
&\quad = \psi\big(mS_A(\alpha)_{(1)}(u_k)S_A(\alpha)_{(1)}(u_j)\big)v_j \otimes_N v_k \\
&\quad = \psi\big(mS_A(\alpha)(u_k u_j)\big)v_j \otimes_N v_k = \psi\big(\alpha(m)u_k u_j\big)v_j \otimes_N v_k \\
&\quad = \alpha(m)u_k \otimes_N v_k
\end{aligned}$$

and analogously (3.21). Now it is easy to see that both $\pi_L \circ S_A$ and $\theta \circ \pi_L \circ S_A^{-1}$ are counits for γ_R. Therefore, both are equal to *the* counit π_R. This finishes the proof of the isomorphisms (3.15). □

Corollary 3.4. *Explicit formulas for the left and right bialgebroid structures can be given using the quasibases of Lemma* 3.2 *as follows*:

$$\gamma_L(\alpha) = \gamma_i \otimes_L c_i^1 \alpha\big(c_i^2 \_\big) = \alpha\big(\_ b_i^1\big) b_i^2 \otimes_L \beta_i = \gamma_i \otimes_L \beta_i * \alpha = \alpha * \gamma_i \otimes_L \beta_i,$$
$$\pi_L(\alpha) = \alpha(1_M),$$
$$\gamma_R(\alpha) = \alpha\big(\_ c_i^1\big) c_i^2 \otimes_R \psi\big(\_ \gamma_i(u_k)\big) v_k = u_k \psi\big(\beta_i(v_k)\_\big) \otimes_R b_i^1 \alpha\big(b_i^2 \_\big)$$
$$= \alpha * S_A(\beta_i) \otimes_R S_A(\gamma_i) = S_A(\beta_i) \otimes_R S_A(\gamma_i) * \alpha,$$
$$\pi_R(\alpha) = u_i \psi \circ \alpha(v_i).$$

4. Hopf algebroids

4.1. The definition

The total ring of a Hopf algebroid carries eight canonical module structures over the base ring—modules of the kind (2.1), (2.7)–(2.9). In this situation the standard notation for the tensor product of modules, e.g., $A \otimes_R A$, would be ambiguous. In order to avoid any misunderstandings, we therefore put marks on both modules, as in $A^R \otimes {}^R A$ for example, that indicate the module structures taking part in the tensor product. Other module structures (commuting with those taking part in the tensor product) are usually unadorned and should be clear from the context.

For coproduts of left bialgebroids we use the Sweedler's notation in the form $\gamma_L(a) = a_{(1)} \otimes a_{(2)}$ and of right bialgebroids $\gamma_R(a) = a^{(1)} \otimes a^{(2)}$.

Definition 4.1. The Hopf algebroid is a pair $(\mathcal{A}_L, S)$ consisting of a left bialgebroid $\mathcal{A}_L = (A, L, s_L, t_L, \gamma_L, \pi_L)$ and an anti-automorphism S of the total ring A satisfying

$$\text{(i)} \quad S \circ t_L = s_L \quad \text{and} \tag{4.1}$$
$$\text{(ii)} \quad S^{-1}(a_{(2)})_{(1')} \otimes S^{-1}(a_{(2)})_{(2')} a_{(1)} = S^{-1}(a) \otimes 1_A, \tag{4.2}$$
$$S(a_{(1)})_{(1')} a_{(2)} \otimes S(a_{(1)})_{(2')} = 1_A \otimes S(a) \tag{4.3}$$

as elements of $A_L \otimes {}_L A$, for all $a \in A$.

The axiom (4.3) implies that

$$S(a_{(1)}) a_{(2)} = t_L \circ \pi_L \circ S(a) \tag{4.4}$$

for all $a \in A$. Introduce the map $\theta_L := \pi_L \circ S \circ s_L : L \to L$. Owing to (4.4), it satisfies

$$t_L \circ \theta_L(l) = t_L \circ \pi_L \circ S \circ s_L(l) = S \circ s_L(l),$$
$$\theta_L(l)\theta_L(l') = \pi_L \circ S \circ s_L(l) \pi_L \circ S \circ s_L(l') = \pi_L\big(t_L \circ \pi_L \circ S \circ s_L(l') S \circ s_L(l)\big)$$
$$= \pi_L\big(S \circ s_L(l') S \circ s_L(l)\big) = \theta_L(ll'). \tag{4.5}$$

In view of (4.5) S is a *twisted bimodule map* ${}^RA^R \to {}_LA_L$ where R is a ring isomorphic to L^{op} and the R-R-bimodule structure of A is given by fixing an isomorphism $\mu : L^{\mathrm{op}} \to R$:

$$r \cdot a \cdot r' := as_L \circ \theta_L^{-1} \circ \mu^{-1}(r)t_L \circ \mu^{-1}(r'). \tag{4.6}$$

The usage of the same notation ${}^RA^R$ as in (2.8) is not accidental. It will turn out from the next Proposition 4.2 that there exists a right bialgebroid structure on the total ring A over the base R for which the R-R-bimodule (2.8) is (4.6).

It makes sense to introduce the maps

$$\begin{aligned} &S_{A\otimes_L A} : A_L \otimes {}_LA \to A^R \otimes {}^RA, \quad a \otimes b \mapsto S(b) \otimes S(a) \quad \text{and} \\ &S_{A\otimes_R A} : A^R \otimes {}^RA \to A_L \otimes {}_LA, \quad a \otimes b \mapsto S(b) \otimes S(a). \end{aligned} \tag{4.7}$$

It is useful to give some alternative forms of the Definition 4.1.

Proposition 4.2. *The following are equivalent*:

(i) $(\mathcal{A}_L, S)$ *is a Hopf algebroid*;

(ii) $\mathcal{A}_L = (A, L, s_L, t_L, \gamma_L, \pi_L)$ *is a left bialgebroid and* S *is an anti-automorphism of the total ring* A *satisfying* (4.1), (4.4), *and*

$$S_{A\otimes_L A} \circ \gamma_L \circ S^{-1} = S^{-1}_{A\otimes_R A} \circ \gamma_L \circ S, \tag{4.8}$$

$$(\gamma_L \otimes \mathrm{id}_A) \circ \gamma_R = (\mathrm{id}_A \otimes \gamma_R) \circ \gamma_L, \qquad (\gamma_R \otimes \mathrm{id}_A) \circ \gamma_L = (\mathrm{id}_A \otimes \gamma_L) \circ \gamma_R, \tag{4.9}$$

where we introduced the ring R *and the* R-R-*bimodule* ${}^RA^R$ *as in* (4.6) *and the map*

$$\gamma_R := S_{A\otimes_L A} \circ \gamma_L \circ S^{-1} \equiv S^{-1}_{A\otimes_R A} \circ \gamma_L \circ S : A \to A^R \otimes {}^RA.$$

The equations in (4.9) *are equalities of maps* $A \to A_L \otimes {}_LA^R \otimes {}^RA$ *and* $A \to A^R \otimes {}^RA_L \otimes {}_LA$, *respectively.*

(iii) $\mathcal{A}_L = (A, L, s_L, t_L, \gamma_L, \pi_L)$ *is a left bialgebroid and* $\mathcal{A}_R = (A, R, s_R, t_R, \gamma_R, \pi_R)$ *is a right bialgebroid such that the base rings are related to each other via* $R \simeq L^{\mathrm{op}}$. S *is a bijection of additive groups and*

$$s_L(L) = t_R(R), \qquad t_L(L) = s_R(R) \quad \text{as subrings of } A, \tag{4.10}$$

$$(\gamma_L \otimes \mathrm{id}_A) \circ \gamma_R = (\mathrm{id}_A \otimes \gamma_R) \circ \gamma_L, \qquad (\gamma_R \otimes \mathrm{id}_A) \circ \gamma_L = (\mathrm{id}_A \otimes \gamma_L) \circ \gamma_R, \tag{4.11}$$

$$S\big(t_L(l)at_L(l')\big) = s_L(l')S(a)s_L(l), \qquad S\big(t_R(r')at_R(r)\big) = s_R(r)S(a)s_R(r'), \tag{4.12}$$

$$S(a_{(1)})a_{(2)} = s_R \circ \pi_R(a), \qquad a^{(1)}S\big(a^{(2)}\big) = s_L \circ \pi_L(a) \tag{4.13}$$

hold true for all $l, l' \in L$, $r, r' \in R$, *and* $a \in A$.

(iv) *$\mathcal{A}_L$ is a left bialgebroid over L and $\mathcal{A}_R$ is a right bialgebroid over R such that the base rings are related to each other via $R \simeq L^{\mathrm{op}}$ and Eqs.* (4.10) *and* (4.11) *hold true. Furthermore, the maps of additive groups*

$$\alpha : A^R \otimes_R A \to A_L \otimes_L A, \quad a \otimes b \mapsto a_{(1)} \otimes a_{(2)} b \quad \textit{and}$$
$$\beta : A_R \otimes {}^R A \to A_L \otimes_L A, \quad a \otimes b \mapsto b_{(1)} a \otimes b_{(2)} \tag{4.14}$$

are bijective. (*All modules appearing in* (4.14) *are the canonical modules introduced in Section* 2.)

Each characterization of Hopf algebroids in Proposition 4.2 will be relevant in what follows. The one in (ii) is as similar to [13] as possible which will be useful in Section 4.3 both in checking that concrete examples of Lu–Hopf algebroids satisfy the axioms in Definition 4.1 and also in constructing Hopf algebroids in the sense of Definition 4.1 which do not satisfy the Lu axioms.

As it will turn out from the following proof of Proposition 4.2, the Definition 4.1 implies the existence of a right bialgebroid structure on the total ring of the Hopf algebroid. The characterization in (iii) uses the left and right bialgebroids underlying a Hopf algebroid in a perfectly symmetric way. This characterization will be appropriate in developing the theory of integrals in Section 5.

The characterization in (iv) is formulated in the spirit of [18], that is the bijectivity of certain Galois maps is required. The relevance of this form of the definition is that it shows that the Hopf algebroid in the sense of Definition 4.1 is a special case of Schauenburg's $\times_L$-Hopf algebra.

Proof. (ii) ⇒ (iii). We construct a right bialgebroid $\mathcal{A}_R$ such that $(\mathcal{A}_L, \mathcal{A}_R, S)$ satisfies the requirements in (iii). Let R be a ring isomorphic to L^{op} and $\mu : L^{\mathrm{op}} \to R$ a fixed isomorphism. Set

$$\mathcal{A}_R = \big(A, R,\ s_R := t_L \circ \mu^{-1},\ t_R := S^{-1} \circ t_L \circ \mu^{-1},\ \gamma_R := S^{-1}_{A \otimes_R A} \circ \gamma_L \circ S,$$
$$\pi^R := \mu \circ \pi_L \circ S\big). \tag{4.15}$$

(iii) ⇒ (iv). We construct the inverses of the maps (4.14):

$$\alpha^{-1} : A_L \otimes_L A \to A^R \otimes_R A, \quad a \otimes b \mapsto a^{(1)} \otimes S\big(a^{(2)}\big) b \quad \text{and}$$
$$\beta^{-1} : A_L \otimes_L A \to A_R \otimes {}^R A, \quad a \otimes b \mapsto S^{-1}\big(b^{(1)}\big) a \otimes b^{(2)}. \tag{4.16}$$

(iv) ⇒ (i). Recall that the requirements in (iv) imply that both left bialgebroids $\mathcal{A}_L$ and $\mathcal{A}_{L\,\mathrm{cop}}$ are $\times_L$-Hopf algebras in the sense of [18]. In particular, [18, Proposition 3.7] holds true for both. That is, denoting $\alpha^{-1}(a \otimes 1_A) := a_+ \otimes a_-$ and $\beta^{-1}(1_A \otimes a) := a_{[-]} \otimes a_{[+]}$, we have

$$
\begin{aligned}
&\text{(i)} && a_{+(1)} \otimes a_{+(2)}a_- = a \otimes 1_A, \qquad a_{[+](1)}a_{[-]} \otimes a_{[+](2)} = 1_A \otimes a,\\
&\text{(ii)} && a_{(1)+} \otimes a_{(1)-}a_{(2)} = a \otimes 1_A, \qquad a_{(2)[-]}a_{(1)} \otimes a_{(2)[+]} = 1_A \otimes a,\\
&\text{(iii)} && (ab)_+ \otimes (ab)_- = a_+b_+ \otimes b_-a_-, \qquad (ab)_{[-]} \otimes (ab)_{[+]} = b_{[-]}a_{[-]} \otimes a_{[+]}b_{[+]},\\
&\text{(iv)} && (1_A)_+ \otimes (1_A)_- = 1_A \otimes 1_A, \qquad (1_A)_{[-]} \otimes (1_A)_{[+]} = 1_A \otimes 1_A,\\
&\text{(v)} && a_{+(1)} \otimes a_{+(2)} \otimes a_- = a_{(1)} \otimes a_{(2)+} \otimes a_{(2)-},\\
& && a_{[-]} \otimes a_{[+](1)} \otimes a_{[+](2)} = a_{(1)[-]} \otimes a_{(1)[+]} \otimes a_{(2)},\\
&\text{(vi)} && a_+ \otimes a_{-(1)} \otimes a_{-(2)} = a_{++} \otimes a_- \otimes a_{+-},\\
& && a_{[-](2)} \otimes a_{[-](1)} \otimes a_{[+]} = a_{[-]} \otimes a_{[+][-]} \otimes a_{[+][+]},\\
&\text{(vii)} && a = a_+t_L \circ \pi_L(a_-), \qquad a = a_{[+]}s_L \circ \pi_L(a_{[-]}),\\
&\text{(viii)} && a_+a_- = s_L \circ \pi_L(a), \qquad a_{[+]}a_{[-]} = t_L \circ \pi_L(a).
\end{aligned} \tag{4.17}
$$

We define the antipode as

$$S(a) := s_R \circ \pi_R(a_+)a_- \tag{4.18}$$

and what is going to be its inverse as

$$S'(a) := t_R \circ \pi_R(a_{[+]})a_{[-]}. \tag{4.19}$$

The maps (4.18) and (4.19) are well-defined due to the R-module map property of π_R.

Since $\alpha(1_A \otimes s_L(l)) = 1_A \otimes s_L(l) \equiv t_L(l) \otimes 1_A$, the requirement (4.1) holds true. By making use of (vi) and (i) of (4.17), one verifies

$$
\begin{aligned}
S(a_{(1)})_{(1)'}a_{(2)} \otimes S(a_{(1)})_{(2)'} &= a_{(1)-(1)'}a_{(2)} \otimes s_R \circ \pi_R(a_{(1)+})a_{(1)-(2)'}\\
&= a_{(1)-}a_{(2)} \otimes S(a_{(1)+}) = 1 \otimes S(a)
\end{aligned}
$$

and similarly

$$S'(a_{(2)})_{(1)'} \otimes S'(a_{(2)})_{(2)'}a_{(1)} = S'(a) \otimes 1_A,$$

which becomes the requirement (4.2) once we proved $S' = S^{-1}$. As a matter of fact by (vi) of (4.17)

$$S(a)_{(1)} \otimes S(a)_{(2)} = a_{-(1)} \otimes s_R \circ \pi_R(a_+)a_{-(2)} = a_- \otimes S(a_+)$$

hence, using (ii) of (4.17),

$$\beta\big(a_{(2)} \otimes S(a_{(1)})\big) = a_{(1)-}a_{(2)} \otimes S(a_{(1)+}) = 1_A \otimes S(a),$$

so, by (viii) of (4.17) and (4.10),

$$S' \circ S(a) = t_R \circ \pi_R\big(s_R \circ \pi_R(a_{(1)+})a_{(1)-}\big)a_{(2)} = t_R \circ \pi_R(a_{(1)+}a_{(1)-})a_{(2)}$$
$$= s_L \circ \pi_L(a_{(1)})a_{(2)} = a.$$

In a similar way one checks that $S \circ S' = \mathrm{id}_A$.

The anti-multiplicativity of S is proven as follows. We have $\alpha(t_R(r) \otimes 1_A) = t_R(r) \otimes 1_A$ and by $\beta(1_A \otimes s_R(r)) = 1_A \otimes s_R(r)$ and $S' = S^{-1}$ also $S(t_R(r)a) = S(a)s_R(r)$ hence,

$$S(ab) = s_R \circ \pi_R\big((ab)_+\big)(ab)_- = s_R \circ \pi_R(a_+b_+)b_-a_- = s_R \circ \pi_R\big(t_R \circ \pi_R(a_+)b_+\big)b_-a_-$$
$$= s_R \circ \pi_R\big(\big[t_R \circ \pi_R(a_+)b\big]_+\big)\big[t_R \circ \pi_R(a_+)b\big]_- a_- = S\big(t_R \circ \pi_R(a_+)b\big)a_-$$
$$= S(b)s_R \circ \pi_R(a_+)a_- = S(b)S(a).$$

(i) $\Rightarrow$ (ii). The requirements (4.1) and (4.4) hold obviously true. One easily checks that the maps α and β in (4.14) are bijective with inverses

$$\alpha^{-1}(a \otimes b) = S^{-1}\big(S(a)_{(2)}\big) \otimes S(a)_{(1)}b,$$
$$\beta^{-1}(a \otimes b) = S^{-1}(b)_{(2)}a \otimes S\big(S^{-1}(b)_{(1)}\big).$$

This implies that the [18, Proposition 3.7] holds true both in $\mathcal{A}_L$ and $\mathcal{A}_{L\,\mathrm{cop}}$. In particular, introducing the maps

$$\gamma_R : A \to A^R \otimes {}^R A, \quad a \mapsto \big(\mathrm{id}_A \otimes S^{-1}\big) \circ \alpha^{-1}(a \otimes 1_A),$$
$$\gamma'_R : A \to A^R \otimes {}^R A, \quad a \mapsto (S \otimes \mathrm{id}_A) \circ \beta^{-1}(1_A \otimes a)$$

the part (v) of (4.17) reads as

$$(\gamma_L \otimes \mathrm{id}_A) \circ \gamma_R = (\mathrm{id}_A \otimes \gamma_R) \circ \gamma_L, \tag{4.20}$$
$$(\mathrm{id}_A \otimes \gamma_L) \circ \gamma'_R = (\gamma'_R \otimes \mathrm{id}_A) \circ \gamma_L. \tag{4.21}$$

This means that both (4.8) and (4.9) follow provided $\gamma_R = \gamma'_R$. Using the Sweedler's notation $\gamma_R(a) = a^{(1)} \otimes a^{(2)}$ and $\gamma'_R(a) = a^{\langle 1\rangle} \otimes a^{\langle 2\rangle}$ by the repeated use of (4.20) and (4.21), we obtain

$$(\mathrm{id}_A \otimes \gamma_R) \circ \gamma'_R(a) = a^{\langle 1\rangle} \otimes s_L \circ \pi_L\big({a^{\langle 2\rangle}}_{(1)}\big){{a^{\langle 2\rangle}}_{(2)}}^{(1)} \otimes {{a^{\langle 2\rangle}}_{(2)}}^{(2)}$$
$$= {a_{(1)}}^{\langle 1\rangle} \otimes s_L \circ \pi_L\big({a_{(1)}}^{\langle 2\rangle}\big){a_{(2)}}^{(1)} \otimes {a_{(2)}}^{(2)}$$
$$= {{a^{(1)}}_{(1)}}^{\langle 1\rangle} \otimes s_L \circ \pi_L\big({{a^{(1)}}_{(1)}}^{\langle 2\rangle}\big){a^{(1)}}_{(2)} \otimes a^{(2)}$$
$$= a^{(1)\langle 1\rangle} \otimes s_L \circ \pi_L\big({a^{(1)\langle 2\rangle}}_{(1)}\big){a^{(1)\langle 2\rangle}}_{(2)} \otimes a^{(2)}$$
$$= \big(\mathrm{id}_A \otimes \gamma'_R\big) \circ \gamma_R(a).$$

Since by (viii) of (4.17) both $a^{(1)}S(a^{(2)}) = s_L \circ \pi_L(a)$ and $a^{\langle 1\rangle}S(a^{\langle 2\rangle}) = s_L \circ \pi_L(a)$, we have

$$\varepsilon(a) := S \circ s_L \circ \pi_L \circ S^{-1}(a) = S(a_{(1)})a_{(2)} = S^{-1} \circ s_L \circ \pi_L \circ S(a)$$

and

$$\begin{aligned}
&(m_A \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \varepsilon \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \gamma_R) \circ \gamma'_R(a) \\
&\quad = a^{\langle 1\rangle}\varepsilon\big(a^{\langle 2\rangle(1)}\big) \otimes a^{\langle 2\rangle(2)} = a^{\langle 1\rangle} \otimes a^{\langle 2\rangle(2)} S^{-2} \circ s_L \circ \pi_L \circ S\big(a^{\langle 2\rangle(1)}\big) \\
&\quad = a^{\langle 1\rangle} \otimes S^{-1}\big(S\big(a^{\langle 2\rangle}\big)_{(1)}\big) S^{-1} \circ t_L \circ \pi_L\big(S\big(a^{\langle 2\rangle}\big)_{(2)}\big) = \gamma'_R(a), \\
&(m_A \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \varepsilon \otimes \mathrm{id}_A) \circ \big(\gamma'_R \otimes \mathrm{id}_A\big) \circ \gamma_R(a) \\
&\quad = a^{(1)\langle 1\rangle}\varepsilon\big(a^{(1)\langle 2\rangle}\big) \otimes a^{(2)} = a^{(1)\langle 1\rangle} S \circ s_L \circ \pi_L \circ S^{-1}\big(a^{(1)\langle 2\rangle}\big) \otimes a^{(2)} \\
&\quad = S\big(S^{-1}\big(a^{(1)}\big)_{(2)}\big) S \circ s_L \circ \pi_L\big(S^{-1}\big(a^{(1)}\big)_{(1)}\big) \otimes a^{(2)} = \gamma_R(a)
\end{aligned}$$

hence by the equality of the left-hand sides $\gamma_R = \gamma'_R$. □

The following is a consequence of the proof of Proposition 4.2.

Proposition 4.3. *Let $(\mathcal{A}_L, S)$ be a Hopf algebroid and $\mathcal{A}_R$ a right bialgebroid such that $(\mathcal{A}_L, \mathcal{A}_R, S)$ satisfies the requirements of Proposition* 4.2(iii). *Then both $(S : A \to A^{\mathrm{op}}, \nu := \pi_R \circ s_L : L \to R^{\mathrm{op}})$ and $(S^{-1} : A \to A^{\mathrm{op}}, \mu := \pi_R \circ t_L : L \to R^{\mathrm{op}})$ are left bialgebroid isomorphisms $\mathcal{A}_L \to (\mathcal{A}_R)^{\mathrm{op}}_{\mathrm{cop}}$. In particular, $\mathcal{A}_R$ is unique up to an isomorphism of the form (id_A, ϕ).*

One easily checks that $\mu^{-1} \circ \nu = \theta_L$. For the sake of symmetry we introduce also $\theta_R := \nu \circ \mu^{-1}$ with the help of which the right analogue of (4.5) holds true:

$$S \circ s_R = t_R \circ \theta_R.$$

Proposition 4.3 has an interpretation in terms of the forgetful functors $\Phi_R : \mathcal{M}_A \to {}_R\mathcal{M}_R$ and $\Phi_L : {}_A\mathcal{M} \to {}_R\mathcal{M}_R$ as follows. The antipode map defines two functors $\mathcal{S}$ and $\mathcal{S}' : \mathcal{M}_A \to {}_A\mathcal{M}$. They have object maps $(M, \triangleleft) \mapsto (M, \triangleleft \circ S)$ and $(M, \triangleleft) \mapsto (M, \triangleleft \circ S^{-1})$, respectively, and the identity maps on the morphisms. It is clear that $\mathcal{S}$ and $\mathcal{S}'$ are strict antimonoidal equivalence functors. The ring automorphisms μ and ν define endo-functors $\underline{\mu}$ and $\underline{\nu}$ of ${}_R\mathcal{M}_R$. The object maps are $(M, \triangleright, \triangleleft) \mapsto (M, \triangleleft \circ \mu, \triangleright \circ \mu)$ and $(M, \triangleright, \triangleleft) \mapsto (M, \triangleleft \circ \nu, \triangleright \circ \nu)$, respectively, and the identity map on the morphisms. The $\underline{\mu}$ and $\underline{\nu}$ are also strict antimonoidal equivalence functors. We have then equalities of strong monoidal functors $\Phi_L \circ \mathcal{S} = \underline{\nu} \circ \Phi_R$ and $\Phi_L \circ \mathcal{S}' = \underline{\mu} \circ \Phi_R$.

Finally, we define the morphisms of Hopf algebroids.

Definition 4.4. A *Hopf algebroid homomorphism* (*isomorphism*) $(\mathcal{A}_L, S) \to (\mathcal{A}'_L, S')$ is a left bialgebroid homomorphism (isomorphism) $\mathcal{A}_L \to \mathcal{A}'_L$. A Hopf algebroid homomorphism (Φ, ϕ) is *strict* if $S' \circ \Phi = \Phi \circ S$.

The existence of non-strict isomorphisms of Hopf algebroids—that is the non-uniqueness of the antipode in a Hopf algebroid—is a new feature compared to (weak) Hopf algebras. The antipodes making a given left bialgebroid into a Hopf algebroid are characterized in [2].

In the following (in particular in Section 5) we are going to call a triple $(\mathcal{A}_L, \mathcal{A}_R, S)$ satisfying Proposition 4.2(iii) a *symmetrized form* of the Hopf algebroid $(\mathcal{A}_L, S)$. The $\mathcal{A}_R$ is called the right bialgebroid underlying $(\mathcal{A}_L, S)$. In the view of Proposition 4.3 the symmetrized form is unique up to the choice of the base ring R of $\mathcal{A}_R$ within the isomorphism class of L^{op}—the opposite of the base ring of $\mathcal{A}_L$—and the isomorphism $\mu : L^{\text{op}} \to R$ in (4.15).

Let us define the *homomorphisms* of symmetrized Hopf algebroids $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S) \to \mathcal{A}' = (\mathcal{A}'_L, \mathcal{A}'_R, S')$ as pairs of bialgebroid homomorphisms $(\Phi_L, \phi_L) : \mathcal{A}_L \to \mathcal{A}'_L$, $(\Phi_R, \phi_R) : \mathcal{A}_R \to \mathcal{A}'_R$. Then by Proposition 4.3 the Hopf algebroid homomorphisms $(\Phi, \phi) : (\mathcal{A}_L, S) \to (\mathcal{A}'_L, S')$ are injected into the homomorphisms of symmetrized Hopf algebroids $(\mathcal{A}_L, \mathcal{A}_R, S) \to (\mathcal{A}'_L, \mathcal{A}'_R, S')$ via $(\Phi, \phi) \mapsto ((\Phi, \phi), (S'^{-1} \circ \Phi \circ S, \mu' \circ \phi \circ \mu^{-1}))$—where $\mu = \pi_R \circ t_L$ and $\mu' = \pi'_R \circ t'_L$ are the ring isomorphisms introduced in Proposition 4.3. This implies that two symmetrized Hopf algebroids $(\mathcal{A}_L, \mathcal{A}_R, S)$ and $(\mathcal{A}'_L, \mathcal{A}'_R, S')$ are isomorphic if and only if the Hopf algebroids $(\mathcal{A}_L, S)$ and $(\mathcal{A}'_L, S')$ are isomorphic.

A homomorphism $((\Phi_L, \phi_L), (\Phi_R, \phi_R))$ of symmetrized Hopf algebroids $\mathcal{A} \to \mathcal{A}'$ is *strict* if $\Phi_L = \Phi_R$ as homomorphisms of rings $A \to A'$. We leave it to the reader to check that this is equivalent to the requirement that (Φ_L, ϕ_L) is a strict homomorphism of Hopf algebroids $(\mathcal{A}_L, S) \to (\mathcal{A}'_L, S')$, that is two symmetrized Hopf algebroids $(\mathcal{A}_L, \mathcal{A}_R, S)$ and $(\mathcal{A}'_L, \mathcal{A}'_R, S')$ are strictly isomorphic if and only if the Hopf algebroids $(\mathcal{A}_L, S)$ and $(\mathcal{A}'_L, S')$ are strictly isomorphic.

The usage of symmetrized Hopf algebroids allows for the definition of the opposite and co-opposite structures $\mathcal{A}^{\text{op}} = (\mathcal{A}_R^{\text{op}}, \mathcal{A}_L^{\text{op}}, S^{-1})$ and $\mathcal{A}_{\text{cop}} = (\mathcal{A}_{L\,\text{cop}}, \mathcal{A}_{R\,\text{cop}}, S^{-1})$, respectively.

4.2. Relation to the Hopf bialgebroid of Day and Street

In [22] left bialgebroids over the base L have been characterized as opmonoidal monads T on the category ${}_L\mathcal{M}_L$ such that their underlying functors T have right adjoints. The endofunctor T is given by tensoring over $L^e = L \otimes L^{\text{op}}$ with the L^e-L^e-bimodule A. The monad structure makes A into an L^e-ring via an algebra map $\eta : L^e \to A$, while the opmonoidal structure comprises the coproduct and counit of the bialgebroid A. In a recent preprint [7] Day and Street put this into the more general context of pseudomonoids in monoidal bicategories. Using the notion of $*$-autonomy [1] they propose a definition of *Hopf bialgebroid* as a $*$-autonomous structure on a bialgebroid. We are going to show in this section that their definition coincides with our Definition 4.1.

Working with k-algebras the base category $\mathcal{V}$ is the category $\mathcal{M}_k$ of k-modules. Then the monoidal bicategory in question is $\mathbf{Mod}(\mathcal{V})$ having as objects the k-algebras and as hom-categories $\mathbf{Mod}(\mathcal{V})(A, B)$ the category ${}_B\mathcal{M}_A$ of B-A-bimodules. The tensor product $\otimes = \otimes_k$ of k-modules makes $\mathbf{Mod}(\mathcal{V})$ monoidal. A pseudomonoid in $\mathbf{Mod}(\mathcal{V})$ is a triple $\langle A, M, J \rangle$ where A is an object, $M : A \otimes A \to A$ and $J : k \to A$ are 1-cells satisfying associativity and unitality up to invertible 2-cells that in turn satisfy the pentagon and the triangle constraints. If $\langle A, L, s, t, \gamma, \pi\rangle$ is a left bialgebroid in $\mathcal{V}$ then we have a strong monoidal morphism

$$\eta^* : \langle A, M, J\rangle \to \langle L^e, m, j\rangle \tag{4.22}$$

of pseudomonoids where

$$M = A \otimes_L A \quad \text{with actions } a \cdot (x \otimes_L y) \cdot (a_1 \otimes a_2) = a_{(1)} x a_1 \otimes_L a_{(2)} y a_2, \tag{4.23}$$

$$J = L \quad \text{with action } a \cdot x = \pi\big(as(x)\big), \tag{4.24}$$

$$m = L^e \otimes_L L^e \quad \text{with actions } (l \otimes l') \cdot \big((x \otimes x') \otimes_L (y \otimes y')\big) \cdot \big(\big(l_1 \otimes l_1'\big) \otimes \big(l_2 \otimes l_2'\big)\big)$$
$$= \big(l x l_1 \otimes l_1' x'\big) \otimes_L \big(y l_2 \otimes l_2' y' l'\big), \tag{4.25}$$

$$j = L \quad \text{with action } (l \otimes l') \cdot x = l x l' \tag{4.26}$$

and the bimodule $\eta^* = {}_{L^e}A_A$, induced by the algebra map η, has a left adjoint $\eta_* = {}_A A_{L^e}$ with the counit $\epsilon : \eta_* \circ \eta^* \to A$ induced by multiplication of A. The connection of this bimodule picture with module categories can be seen by applying the monoidal pseudofunctor $\mathbf{Mod}(\mathcal{V})(k, -) : \mathbf{Mod}(\mathcal{V}) \to \mathsf{Cat}$. Then pseudomonoids become monoidal categories, η^* becomes the strong monoidal forgetful functor $\eta^* \otimes_A - : {}_A\mathcal{M} \to {}_L\mathcal{M}_L$, and η_* its opmonoidal left adjoint. An important piece of the structure is the bimodule morphism

$$\psi : \eta_* \circ m \to M \circ (\eta_* \otimes \eta_*), \quad a \otimes (l \otimes l') \mapsto a_{(1)} s(l) \otimes a_{(2)} t(l'), \tag{4.27}$$

which makes η_* opmonoidal and encodes the comultiplication $\gamma : A \to A \otimes_L A$ of the bialgebroid.

In [7] a Hopf bialgebroid is defined to be a bialgebroid A over L together with a strong $*$-autonomous structure on the opmonoidal morphism $\eta_* : L^e \to A$. The latter means:

(1) A $*$-autonomous structure on the pseudomonoid A, i.e.,
   (a) a right $A \otimes A$-module σ defined on the k-module A in terms of an algebra isomorphism $\xi : A \to A^{\mathrm{op}}$ by

$$x \cdot (a \otimes b) = \xi^{-1}(b) x a, \quad x, a, b \in A, \quad \text{and} \tag{4.28}$$

   (b) an isomorphism

$$\Gamma : \sigma \circ (M \otimes A) \to \sigma \circ (A \otimes M).$$

(2) A “canonical” $*$-autonomous structure on the pseudomonoid L^e which consists of the following:

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(a) A right $L^e \otimes L^e$-module σ_0 defined on L^e using the isomorphism $\xi_0 : L^e \to (L^e)^{\text{op}}$, $l \otimes l' \mapsto \theta^{-1}(l') \otimes l$ for some algebra automorphism $\theta : L \to L$. Thus

$$(x \otimes x') \cdot \big(\big(l_1 \otimes l_1'\big) \otimes \big(l_2 \otimes l_2'\big)\big) = \theta^{-1}\big(l_2'\big) x l_1 \otimes l_1' x' l_2$$

for $x, x', l_1, l_1', l_2, l_2' \in L$ where juxtaposition is always multiplication in L and never in L^{op}. (Note that allowing a non-trivial θ in the definition of σ_0 is a slight deviation from [7] which is, however, well motivated by Section 3.)

(b) The isomorphism

$$\Gamma_0 : \sigma_0 \circ \big(m \otimes L^e\big) \to \sigma_0 \circ \big(L^e \otimes m\big).$$

(3) The arrow η_* is strongly $*$-autonomous in the following sense: There exists a 2-cell $\tau : \sigma_0 \circ (\eta_* \otimes \eta_*) \to \sigma$ such that

$$\begin{aligned} &\big[\Gamma_0 \circ (\eta_* \otimes \eta_* \otimes \eta_*)\big] \bullet \big[\sigma \circ (\psi \otimes \eta_*)\big] \bullet \big[\tau \circ \big(m \otimes L^e\big)\big] \\ &\quad = \big[\sigma \circ (\eta_* \otimes \psi)\big] \bullet \big[\tau \circ \big(L^e \otimes m\big)\big] \bullet \Gamma_0 \end{aligned} \tag{4.29}$$

and such that the map

$$\begin{aligned} &\tau^l : \sigma_0 \circ \big(\eta^* \otimes L^e\big) \to \sigma \circ (A \otimes \eta_*), \\ &\tau^l = \big[\sigma \circ (\epsilon \otimes \eta_*)\big] \bullet \big[\tau \circ \big(\eta^* \otimes L^e\big)\big] \end{aligned} \tag{4.30}$$

is an isomorphism. (We used $\circ$ and $\bullet$ to denote horizontal and vertical compositions in the bicategory $\mathbf{Mod}(\mathcal{V})$.)

Now we are going to translate this categorical definition into simple algebraic expressions.

Lemma 4.5. *$*$-autonomous structures on the pseudomonoid $\langle A, M, J\rangle$ are in one-to-one correspondence with the data $\langle \xi, \sum_k e_k \otimes f_k\rangle$ where ξ is an anti-automorphism of the ring A and $\sum_k e_k \otimes f_k \in A \otimes_L A$ is such that for all $a \in A$*

$$\sum_k \xi(a_{(1)})_{(1')} e_k a_{(2)} \otimes \xi(a_{(1)})_{(2')} f_k = \sum_k e_k \otimes f_k \xi(a) \tag{4.31}$$

and such that there exists $\sum_j g_j \otimes h_j \in A \otimes_L A$ satisfying, for all $a \in A$:

$$\sum_j \xi^{-1}(a_{(2)})_{(1')} g_j \otimes \xi^{-1}(a_{(2)})_{(2')} h_j a_{(1)} = \sum_j g_j \xi^{-1}(a) \otimes h_j, \tag{4.32}$$

$$\sum_{j,k} \xi(g_j)_{(1)} e_k h_j \otimes \xi(g_j)_{(2)} f_k = 1_A \otimes 1_A, \tag{4.33}$$

$$\sum_{j,k} \xi^{-1}(f_k)_{(1)} g_j \otimes \xi^{-1}(f_k)_{(2)} h_j e_k = 1_A \otimes 1_A \tag{4.34}$$

as elements of $A \otimes_L A$.

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Proof. In order to find explicit formulas for Γ and its inverse, we introduce the isomorphisms

$$\varphi_+ : \sigma \circ (M \otimes A) \to A \otimes_L A, \tag{4.35}$$

$$x \otimes_{A\otimes A} (a \otimes_L b \otimes c) \mapsto \big(\xi^{-1}(c)x\big)_{(1)} a \otimes_L \big(\xi^{-1}(c)x\big)_{(2)} b, \tag{4.36}$$

$$\varphi_- : \sigma \circ (A \otimes M) \to A \otimes_L A, \tag{4.37}$$

$$x \otimes_{A\otimes A} (a \otimes b \otimes_L c) \mapsto \xi(xa)_{(1)} b \otimes_L \xi(xa)_{(2)} c,$$

with inverses

$$\varphi_+^{-1}(a \otimes_L b) = 1 \otimes_{A\otimes A} (a \otimes_L b \otimes 1), \qquad \varphi_-^{-1}(b \otimes_L c) = 1 \otimes_{A\otimes A} (1 \otimes b \otimes_L c).$$

Then $\Gamma' := \varphi_- \circ \Gamma \circ \varphi_+^{-1}$ is a twisted $A^{\otimes 3}$-automorphism of $A \otimes_L A$ in the sense of

$$\Gamma'\big(\xi^{-1}(c) \cdot (x \otimes_L y) \cdot (a \otimes b)\big) = \xi(a) \cdot \Gamma'(x \otimes_L y) \cdot (b \otimes c). \tag{4.38}$$

Hence, Γ is uniquely determined by $\sum_i e_i \otimes f_i = \Gamma'(1 \otimes 1)$ as

$$\Gamma\big(x \otimes_{A\otimes A} (a \otimes_L b \otimes c)\big) = 1 \otimes_{A\otimes A} \left(x_{(1)} a \otimes \sum_k e_k x_{(2)} b \otimes_L f_k c \right) \tag{4.39}$$

and $\sum_k e_k \otimes f_k$ satisfies (4.31). Invertibility then implies that $\Gamma'^{-1}(1 \otimes 1) = \sum_j g_j \otimes h_j$ satisfies the remaining equations. □

We need also the expression for Γ_0. We leave it to the reader to check that

$$\begin{aligned}&\Gamma_0\big((x \otimes x') \otimes_{L^e \otimes L^e} \big(l_1 \otimes l_1'\big) \otimes_L \big(l_2 \otimes l_2'\big) \otimes \big(l_3 \otimes l_3'\big)\big) \\ &\quad = (1 \otimes 1) \otimes_{L^e \otimes L^e} \big(x l_1 \otimes l_1'\big) \otimes \big(l_2 \otimes l_2' x'\big) \otimes_L \big(l_3 \otimes l_3'\big)\end{aligned} \tag{4.40}$$

is well-defined and is invertible.

Lemma 4.6. *The $*$-autonomous property of η_* is equivalent to the existence of an element $i \in A$ satisfying*

$$i\eta\big(\theta^{-1}(l') \otimes l\big) = \xi^{-1}\big(\eta(l \otimes l')\big)i, \quad l, l' \in L, \tag{4.41}$$

$$\sum_k e_k \otimes f_k \xi(i) = \xi(i)_{(1)} \otimes \xi(i)_{(2)}, \tag{4.42}$$

while strong $*$-autonomy adds the requirement that i be invertible.

Proof. Since $1 \otimes 1$ is a cyclic vector in the $L^e \otimes L^e$-module σ_0, the module map τ is uniquely determined by $i := \tau(1 \otimes 1)$ as

$$\tau(x \otimes x') = i\eta(x \otimes x'). \tag{4.43}$$

Tensoring with $\eta_* = {}_A A_{L^e}$ from the right being the restriction via $\eta : L^e \to A$ the element i is subject to condition (4.41) due to (4.28). Using the expressions (4.43), (4.39), (4.40), and (4.27) the $*$-autonomy condition (4.29) becomes equation (4.42). The mate of τ in (4.30) can now be written as

$$\tau^l\big((x \otimes x') \otimes_{L^e \otimes L^e} \big(a \otimes (l \otimes l')\big)\big) = i\eta(x \otimes x')a \otimes_{A \otimes L^e} \big(1 \otimes (l \otimes l')\big).$$

Hence τ^l is invertible iff the map $a \mapsto ia$ is, i.e., iff i is invertible. □

Theorem 4.7. *A strong $*$-autonomous structure on the bialgebroid A over L in the sense of* [7] *is equivalent to a Hopf algebroid structure $(\mathcal{A}_L, S)$ in the sense of Definition* 4.1.

Proof. Using invertibility of $i \in A$ condition (4.41) has the equivalent form

$$\xi\big(i\eta(l' \otimes l)i^{-1}\big) = \eta\big(l \otimes \theta(l')\big) \tag{4.44}$$

and (4.42) can be used to express the element $\sum_k e_k \otimes f_k$ in terms of i,

$$e_i \otimes f_i = \xi(i)_{(1)} \otimes \xi(i)_{(2)}\xi(i)^{-1}.$$

The conditions (4.31)–(4.34) are then equivalent to

$$g_j \otimes h_j = i_{(1)}i^{-1} \otimes i_{(2)}, \tag{4.45}$$

$$\big[i^{-1}\xi^{-1}(a_{(2)})i\big]_{(1')} \otimes \big[i^{-1}\xi^{-1}(a_{(2)})i\big]_{(2')}a_{(1)} = i^{-1}\xi^{-1}(a)i \otimes 1_A, \tag{4.46}$$

$$\xi\big(ia_{(1)}i^{-1}\big)_{(1')}a_{(2)} \otimes \xi\big(ia_{(1)}i^{-1}\big)_{(2')} = 1_A \otimes \xi\big(iai^{-1}\big). \tag{4.47}$$

This means that introducing $S : A \to A^{\mathrm{op}}$, $a \mapsto \xi(iai^{-1})$, the conditions (4.44), (4.46), and (4.47) are identical to the axioms (4.1)–(4.3), respectively. □

4.3. Examples

In addition to our motivating example in Section 3, let us collect some more examples of Hopf algebroids.

Example 4.8. *Weak Hopf algebras with bijective antipode.* Let $(H, \Delta, \varepsilon, S)$ be a weak Hopf algebra [3,4,14] (WHA) over the commutative ring k with bijective antipode. This means that H is an associative unital k algebra, $\Delta : H \to H \otimes_k H$ is a coassociative coproduct.

It is an algebra map (i.e., multiplicative) but not unit preserving in general. In its stead we have *weak* comultiplicativity of the unit

$$1_{[1]} \otimes 1_{[2]}1_{[1']} \otimes 1_{[2']} = 1_{[1]} \otimes 1_{[2]} \otimes 1_{[3]} = 1_{[1]} \otimes 1_{[1']}1_{[2]} \otimes 1_{[2']},$$

where $1_{[1]} \otimes 1_{[2]} = \Delta(1)$. The map $\varepsilon : H \to k$ is the counit of the coproduct Δ. Instead of being multiplicative, it is *weakly* multiplicative:

$$\varepsilon\big(ab_{[1]}\big)\varepsilon\big(b_{[2]}c\big) = \varepsilon(abc) = \varepsilon\big(ab_{[2]}\big)\varepsilon\big(b_{[1]}c\big) \quad \text{for } a, b, c \in H.$$

The bijective map $S : H \to H$ is the antipode, subject to the axioms

$$h_{[1]}S\big(h_{[2]}\big) = \varepsilon\big(1_{[1]}h\big)1_{[2]}, \qquad S\big(h_{[1]}\big)h_{[2]} = 1_{[1]}\varepsilon\big(h1_{[2]}\big), \qquad S\big(h_{[1]}\big)h_{[2]}S\big(h_{[3]}\big) = S(h)$$

for all $h \in H$. (If H is finite over k then the assumption made about the bijectivity of S is redundant.) The WHA $(H, \Delta, \varepsilon, S)$ is a Hopf algebra if and only if Δ is unit preserving.

The algebra H contains two commuting subalgebras: R is the image of H under the projection $\sqcap^R : h \mapsto 1_{[1]}\varepsilon(h1_{[2]})$ and L under $\sqcap^L : h \mapsto \varepsilon(1_{[1]}h)1_{[2]}$—generalizing the subalgebra of the scalars in a Hopf algebra. Both maps S and S^{-1} restrict to algebra anti-isomorphisms $R \to L$. We have four commuting actions of L and R on H:

$$H^R\!: \quad h \cdot r := hr, \qquad {}^R H\!: \quad r \cdot h := hS^{-1}(r),$$
$$H_L\!: \quad h \cdot l := S^{-1}(l)h, \qquad {}_L H\!: \quad l \cdot h := lh.$$

Introduce the canonical projections $\mathrm{p}_R : H \otimes_k H \to H^R \otimes {}^R H$ and $\mathrm{p}_L : H \otimes_k H \to H_L \otimes {}_L H$. There exists a left and a right bialgebroid structure corresponding to the weak Hopf algebra:

$$\mathcal{H}_R := \big(H, R, \mathrm{id}_R, S^{-1}|_R, \mathrm{p}_R \circ \Delta, \sqcap^R\big),$$
$$\mathcal{H}_L := \big(H, L, \mathrm{id}_L, S^{-1}|_L, \mathrm{p}_L \circ \Delta, \sqcap^L\big).$$

We leave it as an exercise to the reader to check that $(\mathcal{H}_L, \mathcal{H}_R, S)$ satisfies the requirements of Proposition 4.2(iii).

Notice that the examples of the above class are not necessarily finite dimensional and not even finitely generated over R. To have a trivial counterexample, think of the group Hopf algebra kG of an infinite group.

Example 4.9. *An example that does not satisfy the Lu-axioms* [13]. Let k be a field the characteristic of which is different from 2. Consider the group bialgebra kZ_2 with presentation

$$kZ_2 = \text{bialg-}\big\langle t \;\big|\; t^2 = 1,\ \Delta(t) = t \otimes t,\ \varepsilon(t) = 1\big\rangle$$

as a left bialgebroid over the base k. That is to say, we set $\mathcal{A}_L = (kZ_2, k, \eta, \eta, \Delta, \varepsilon)$ where η is the unit map $k \to kZ_2$, $\lambda \mapsto \lambda 1$. Introduce the would-be-antipode $S : kZ_2 \to kZ_2$, $t \mapsto -t$.

Proposition 4.10. *The pair* $(\mathcal{A}_L, S)$ *in Example* 4.9 *satisfies the axioms in Definition* 4.1 *but not the Lu-axioms.*

Proof. One easily checks the conditions of Proposition 4.2(ii) on the single algebraic generator t, proving that $(\mathcal{A}_L, S)$ is a Hopf algebroid in the sense of Definition 4.1.

Now, the base ring L being k itself, the canonical projection $kZ_2 \otimes_k kZ_2 \to kZ_2 \otimes_L kZ_2$ is the identity map leaving us with the only section $kZ_2 \otimes_L kZ_2 \to kZ_2 \otimes_k kZ_2$, the identity map. Since $t_{(1)} S(t_{(2)}) = -1$ and $\eta \circ \varepsilon(t) = 1$, this contradicts to that $(\mathcal{A}_L, S)$ is a Lu–Hopf algebroid. □

In the Example 4.8 the left and right coproducts γ_L and γ_R are compositions of a coproduct $\Delta : A \to A \otimes_k A$ with the canonical projections p_L and p_R, respectively. Actually, many other examples can be found this way—by allowing for Δ not to be counital.

Proposition 4.11. *Let* $\mathcal{A}_L$ *be a left bialgebroid such that* A *and* L *are* k*-algebras over some commutative ring* k, S *an anti-automorphism of* A *such that the axioms* (4.1) *and* (4.4) *hold true. Suppose that* $\gamma_L = \mathrm{p}_L \circ \Delta$, *where* p_L *is the canonical projection* $A \otimes_k A \to A_L \otimes {}_L A$ *and* $\Delta : A \to A \otimes_k A$ *is a coassociative* (*possibly non-counital*) *coproduct satisfying*

$$\mathrm{p}_L \circ (S \otimes S) \circ \Delta^{\mathrm{op}} = \mathrm{p}_L \circ \Delta \circ S, \qquad \mathrm{p}_L \circ \left(S^{-1} \otimes S^{-1}\right) \circ \Delta^{\mathrm{op}} = \mathrm{p}_L \circ \Delta \circ S^{-1}. \quad (4.48)$$

Then $(\mathcal{A}_L, S)$ *is a Hopf algebroid in the sense of Definition* 4.1.

Proof. We leave it to the reader to check that all the requirements of Proposition 4.2(ii) are satisfied. □

Example 4.12. *The groupoid Hopf algebroid.* Let $\mathcal{G}$ be a groupoid that is a small category with all morphisms invertible. Denote the object set by $\mathcal{G}^0$ and the set of morphisms by $\mathcal{G}^1$. For a commutative ring k the groupoid algebra is the k-module spanned by the elements of $\mathcal{G}^1$ with the multiplication given by the composition of the morphisms if the latter makes sense and 0 otherwise. It is an associative algebra and if $\mathcal{G}^0$ is finite it has a unit $1 = \sum_{a \in \mathcal{G}^0} a$. The groupoid algebra admits a left bialgebroid structure over the base subalgebra $k\mathcal{G}^0$. The map $s_L = t_L$ is the canonical embedding, γ_L is the diagonal map $g \mapsto g \otimes_L g$, and $\pi_L(g) := \mathrm{target}(g)$. This left bialgebroid together with the antipode $S(g) := g^{-1}$ is a Hopf algebroid in the sense of Definition 4.1. Actually this example is of the kind described in Proposition 4.11 with $\Delta(g) := g \otimes_k g$.

Example 4.13. *The algebraic quantum torus.* Let k be a field and T_q the unital associative k algebra generated by two invertible elements U and V subject to the relation $UV = qVU$ where q is an invertible element in k. As it is explained in [11], the algebra T_q admits a

Lu–Hopf algebroid structure over the base subalgebra L generated by U: the map $s_L = t_L$ is the canonical embedding,

$$\gamma_L\big(U^n V^m\big) := U^n V^m \otimes_L V^m \equiv V^m \otimes_L U^n V^m, \qquad \pi_L\big(U^n V^m\big) := U^n,$$

and the antipode $S(U^n V^m) := V^{-m} U^n$. The section ξ of the canonical projection $\mathrm{p}_L : T_q \otimes_k T_q \to T_q \otimes_L T_q$ appearing in the Lu axioms is of the form

$$\xi\big(U^n V^m \otimes_L U^k V^l\big) := U^{(n+k)} V^m \otimes_k V^l.$$

The reader may check that these maps satisfy the Definition 4.1 as well. This example is also of the type considered in Proposition 4.11 with $\Delta(U^n V^m) := U^n V^m \otimes_k V^m$.

Example 4.14. *Examples by Brzezinski and Militaru* [5]. In the paper [5] a wide class of examples of Lu–Hopf algebroids is described. Some other examples [13,15] turn out to belong also to this class.

The examples of [5] are Lu–Hopf algebroids of the type considered in Proposition 4.11. Let $(H, \Delta_H, \varepsilon_H, \tau)$ be a Hopf algebra over the field k with bijective antipode τ and the triple $(L, \cdot, \rho)$ a braided commutative algebra in the category ${}_H\mathcal{D}^H$ of Yetter–Drinfel'd modules over H. Then the crossed product algebra $L \# H$ carries a left bialgebroid structure over the base algebra L:

$$\begin{gathered} s_L(l) = l \# 1_H, \qquad t_L(l) = \rho(l) \equiv l_{\langle 0\rangle} \# l_{\langle 1\rangle}, \\ \gamma_L(l \# h) = (l \# h_{(1)}) \otimes_L (1_L \# h_{(2)}), \qquad \pi_L(l \# h) = \varepsilon_H(h) l, \end{gathered} \tag{4.49}$$

where $h_{(1)} \otimes_k h_{(2)} \equiv \Delta_H(h)$. It is proven in [5] that the left bialgebroid (4.49) and the bijective antipode

$$S(l \# h) := \big(\tau(h_{(2)})\tau^2(l_{\langle 1\rangle})\big) \cdot l_{\langle 0\rangle} \# \tau(h_{(1)})\tau^2(l_{\langle 2\rangle}) \tag{4.50}$$

form a Lu–Hopf algebroid. It is obvious that γ_L is of the form $\mathrm{p}_L \circ \Delta$ with $\Delta(l \# h) := (l \# h_{(1)}) \otimes_k (1_L \# h_{(2)})$. The map Δ is well-defined since $L \# H$ is $L \otimes_k H$ as a k-space and Δ_H maps H into $H \otimes_k H$. We leave it to the reader to check that Δ satisfies (4.48) hence the left bialgebroid (4.49) and the antipode (4.50) form a Hopf algebroid in the sense of Definition 4.1.

The Example 4.9 is not of the type considered in Proposition 4.11. Although γ_L is of the form $\mathrm{p}_L \circ \Delta$, the Δ does not satisfy (4.48). In [11] data $(\mathcal{A}_L, S, \widetilde{S})$ satisfying compatibility conditions somewhat analogous to (4.48) were introduced under the name *extended Hopf algebra*. The next proposition states that extended Hopf algebras with S bijective (such as Example 4.9) provide examples of Hopf algebroids.

Proposition 4.15. *Let $(\mathcal{A}_L, S, \widetilde{S})$ be an extended Hopf algebra. This means that $\mathcal{A}_L$ is a left bialgebroid such that A and L are k-algebras over some commutative ring k. The maps S and $\widetilde{S}$ are anti-automorphisms of the algebra A, $\widetilde{S}^2 = \mathrm{id}_A$ and both pairs $(\mathcal{A}_L, S)$ and*

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$(\mathcal{A}_L, \widetilde{S})$ *satisfy* (4.1) *and* (4.4). *The map* γ_L *is a composition of a coassociative coproduct* $\Delta : A \to A \otimes_k A$ *and the canonical projection* $\mathrm{p}_L : A \otimes_k A \to A \otimes_L A$. *The compatibility relations*

$$\Delta \circ S = (S \otimes S) \circ \Delta^{\mathrm{op}} \quad \textit{and} \quad \Delta \circ \widetilde{S} = \big(S \otimes \widetilde{S}\big) \circ \Delta^{\mathrm{op}}$$

hold true. Then the pair $(\mathcal{A}_L, \widetilde{S})$ *is a Hopf algebroid in the sense of Definition* 4.1.

Proof. We leave to the reader to check that the condition (4.9)—hence all requirements of Proposition 4.2(ii)—hold true. □

5. Integral theory and the dual Hopf algebroid

In this section we generalize the notion of non-degenerate integrals in (weak) bialgebras to bialgebroids. We examine the consequences of the existence of a non-degenerate integral in a Hopf algebroid. We do *not* address the question, however, under what conditions on the Hopf algebroid does the existence of a non-degenerate integral follow. That is we do not give a generalization of the Larson–Sweedler theorem on bialgebroids and neither of the (weaker) [3, Theorem 3.16] stating that a weak Hopf algebra possesses a non-degenerate integral if and only if it is a Frobenius algebra. (About the implications in one direction see however [2, Theorem 6.3] and Theorem 5.5 below, respectively.)

Assuming the existence of a non-degenerate integral in a Hopf algebroid we show that the underlying bialgebroids are finite. The duals of finite bialgebroids w.r.t. the base rings were shown to have bialgebroid structures [8] but there is no obvious way how to transpose the antipode to (either of the four) duals. As the main result of this section we show that *if there exists a non-degenerate integral* in a Hopf algebroid then the four dual bialgebroids are all (anti-) isomorphic and they can be made Hopf algebroids. This dual Hopf algebroid structure is unique up to isomorphism (in the sense of Definition 4.4).

For the considerations of this section the "symmetric definition" of Hopf algebroids, i.e., the characterization of Proposition 4.2(iii) is the most appropriate. Throughout the section we use the symmetrized form of the Hopf algebroid introduced at the end of Section 4.1.

It is important to emphasize that although the Definitions 5.1 and 5.3 are formulated in terms of a particular symmetrized Hopf algebroid $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$, actually they depend only on the Hopf algebroid $(\mathcal{A}_L, S)$. That is to say, if ℓ is a (non-degenerate) left integral in a symmetrized Hopf algebroid then it is one in any other symmetrized form of the same Hopf algebroid. Therefore, ℓ can be called a (non-degenerate) left integral of the Hopf algebroid. Analogously, although the anti-automorphism ξ in Lemma 5.9 is defined in terms of a particular symmetrized Hopf algebroid, it is invariant under the change of the underlying right bialgebroid.

For a symmetrized Hopf algebroid $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ we use the notations of Section 2: the $\mathcal{A}_*$ and ${}_*\mathcal{A}$ are the L-duals of $\mathcal{A}_L$, the $\mathcal{A}^*$ and ${}^*\!\mathcal{A}$ the R-duals of $\mathcal{A}_R$. Also for the coproducts of $\mathcal{A}_L$ and $\mathcal{A}_R$ we write $\gamma_L(a) = a_{(1)} \otimes a_{(2)}$ and $\gamma_R(a) = a^{(1)} \otimes a^{(2)}$, respectively.

5.1. Non-degenerate integrals

Definition 5.1. The *left integrals* in a left bialgebroid $\mathcal{A}_L = (A, L, s_L, t_L, \gamma_L, \pi_L)$ are the invariants of the left regular A module

$$\mathcal{I}^L(\mathcal{A}) := \{\ell \in A \mid a\ell = s_L \circ \pi_L(a)\ell \ \forall a \in A\}.$$

The *right integrals* in a right bialgebroid $\mathcal{A}_R = (A, R, s_R, t_R, \gamma_R, \pi_R)$ are the invariants of the right regular A module

$$\mathcal{I}^R(\mathcal{A}) := \{\Upsilon \in A \mid \Upsilon a = \Upsilon s_R \circ \pi_R(a) \ \forall a \in A\}.$$

The left/right integrals in a symmetrized Hopf algebroid $(\mathcal{A}_L, \mathcal{A}_R, S)$ are the left/right integrals in $\mathcal{A}_L/\mathcal{A}_R$.

Lemma 5.2. *For a symmetrized Hopf algebroid $\mathcal{A}$ the following properties of the element ℓ of A are equivalent*:

(i) $\ell \in \mathcal{I}^L(\mathcal{A})$;
(ii) $a\ell = t_L \circ \pi_L(a)\ell$ *for all* $a \in A$;
(iii) $S(\ell) \in \mathcal{I}^R(\mathcal{A})$;
(iv) $S^{-1}(\ell) \in \mathcal{I}^R(\mathcal{A})$;
(v) $S(a)\ell^{(1)} \otimes \ell^{(2)} = \ell^{(1)} \otimes a\ell^{(2)}$ *as elements of* $A^R \otimes^R A$, *for all* $a \in A$.

Proof. Left to the reader. □

Definition 5.3. The left integral ℓ in the symmetrized Hopf algebroid $\mathcal{A}$ is *non-degenerate* if the maps

$$\ell_R : \mathcal{A}^* \to A, \quad \phi^* \mapsto \phi^* \rightharpoonup \ell, \quad \text{and}$$
$$_R\ell : {}^*\mathcal{A} \to A, \quad {}^*\phi \mapsto {}^*\phi \rightharpoondown \ell \tag{5.1}$$

are bijective. The right integral Υ in the symmetrized Hopf algebroid $\mathcal{A}$ is *non-degenerate* if $S(\Upsilon)$ is a non-degenerate left integral, i.e., if the maps

$$\Upsilon_L : \mathcal{A}_* \to A, \quad \phi_* \mapsto \Upsilon \leftharpoonup \phi_*, \quad \text{and}$$
$$_L\Upsilon : {}_*\mathcal{A} \to A, \quad {}_*\phi \mapsto \Upsilon \leftharpoondown {}_*\phi \tag{5.2}$$

are bijective.

Remark 5.4. If ℓ is a non-degenerate left integral in the symmetrized Hopf algebroid $\mathcal{A}$ then so is in $\mathcal{A}_{\text{cop}}$ and when replacing $\mathcal{A}$ with $\mathcal{A}_{\text{cop}}$ the roles of ℓ_R and $_R\ell$ become interchanged. Hence, any statement proven in a symmetrized Hopf algebroid possessing a non-degenerate left integral on ℓ_R implies that the co-opposite statement holds true on $_R\ell$.

Theorem 5.5. *Let $\mathcal{A}$ be a symmetrized Hopf algebroid possessing a non-degenerate left integral. Then the ring extensions $s_R : R \to A$, $t_R : R^{\mathrm{op}} \to A$, $s_L : L \to A$, and $t_L : L^{\mathrm{op}} \to A$ are all Frobenius extensions.*

Proof. Let ℓ be a non-degenerate left integral in $\mathcal{A}$. With its help we construct the Frobenius system for the extension $s_R : R \to A$. It consists of a Frobenius map

$$\lambda^* := \ell_R^{-1}(1_A) : {}_R A^R \to R \tag{5.3}$$

and a quasi-basis (in the sense of (3.1)) for it

$$\ell^{(1)} \otimes S\big(\ell^{(2)}\big) \in A^R \otimes {}_R A.$$

As a matter of fact the λ^* is a right R-module map $A^R \to R$ by construction. We claim that it is also a left R-module map ${}_R A \to R$. Since

$$\big(\lambda^* \leftharpoonup S(a)\big) \rightharpoonup \ell = \ell^{(2)} t_R \circ \lambda^*\big(S(a)\ell^{(1)}\big) = a(\lambda^* \rightharpoonup \ell) = a, \tag{5.4}$$

the inverse ℓ_R^{-1} maps $a \in A$ to $\lambda^* \leftharpoonup S(a)$. This implies that $\lambda^* \leftharpoonup s_R(r) = \ell_R^{-1} \circ t_R(r)$. Now for a given element $r \in R$ the map $\chi(r)^* : A^R \to R$, $a \mapsto r\lambda^*(a)$ is also equal to $\ell_R^{-1} \circ t_R(r)$ as

$$\chi(r)^* \rightharpoonup \ell = \ell^{(2)} t_R\big(r\lambda^*\big(\ell^{(1)}\big)\big) = (\lambda^* \rightharpoonup \ell) t_R(r) = t_R(r).$$

Applying the two equal maps $\lambda^* \leftharpoonup s_R(r)$ and $\chi(r)^*$ to an element $a \in A$, we obtain

$$\lambda^*\big(s_R(r)a\big) = r\lambda^*(a). \tag{5.5}$$

This proves that λ^* is an R-R-bimodule map ${}_R A^R \to A$. Also

$$s_R \circ \lambda^*\big(a\ell^{(1)}\big) S\big(\ell^{(2)}\big) = S(\lambda^* \rightharpoonup \ell) a = a \quad \text{and} \tag{5.6}$$

$$\ell^{(1)} s_R \circ \lambda^*\big(S\big(\ell^{(2)}\big)a\big) = a\ell^{(1)} s_R \circ \lambda^* \circ S\big(\ell^{(2)}\big). \tag{5.7}$$

Now we claim that $\ell^{(1)} s_R \circ \lambda^* \circ S(\ell^{(2)}) \equiv \lambda^* \circ S \rightharpoondown \ell$ is equal to 1_A, which proves the claim. (Recall that by (5.5) $\lambda^* \circ S$ is an element of ${}^*\!\mathcal{A}$.) Since $\ell_R^{-1}(a) = \lambda^* \leftharpoonup S(a)$,

$$\phi^*(a) = \big[\lambda^* \leftharpoonup S(\phi^* \rightharpoonup \ell)\big](a) = \lambda^*\big(s_R \circ \phi^*\big(\ell^{(1)}\big) S\big(\ell^{(2)}\big)a\big) \tag{5.8}$$

for all $\phi^* \in \mathcal{A}^*$ and $a \in A$.

By the bijectivity of ${}_R\ell$, we can introduce the element ${}^*\lambda := {}_R\ell^{-1}(1_A) \in {}^*\!\mathcal{A}$. Analogously to (5.5) and (5.4), we have

$${}^*\lambda\big(t_R(r)a\big) = {}^*\lambda(a)r \quad \text{and} \tag{5.9}$$

$${}_R\ell^{-1}(a) = {}^*\lambda \leftharpoondown S^{-1}(a). \tag{5.10}$$

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Since both ${}_R\ell$ and S^{-1} are bijective, so is the map $A \to {}^*\mathcal{A}$, $a \mapsto {}^*\lambda \leftharpoondown a$. Using the identities (5.9), (5.5), and (5.8), compute

$$\big({}^*\lambda \leftharpoondown S^{-1}(\lambda^* \circ S \rightharpoondown \ell)\big)(a) = {}^*\lambda\big(t_R \circ \lambda^* \circ S\big(\ell^{(2)}\big)S^{-1}\big(\ell^{(1)}\big)a\big)$$
$$= \lambda^*\big(s_R \circ {}^*\lambda \circ S^{-1}\big(\ell^{(1)}\big)S\big(\ell^{(2)}\big)S(a)\big) = {}^*\lambda(a)$$

for all $a \in A$. This is equivalent to $\lambda^* \circ S \rightharpoondown \ell = 1_A$ that is ${}^*\lambda = \lambda^* \circ S$ proving that $(\lambda^*, \ell^{(1)} \otimes S(\ell^{(2)}))$ is a Frobenius system for the extension $s_R : R \to A$.

By repeating the same proof in $\mathcal{A}_{\mathrm{cop}}$, we obtain the Frobenius system $({}^*\lambda, \ell^{(2)} \otimes S^{-1}(\ell^{(1)}))$ for the extension $t_R : R \to A$.

It is straightforward to check that $(\mu^{-1} \circ \lambda^*, \ell^{(1)} \otimes S(\ell^{(2)}))$ is a Frobenius system for the extension $t_L : L \to A$, and $(\nu^{-1} \circ {}^*\lambda, \ell^{(2)} \otimes S^{-1}(\ell^{(1)}))$ is a Frobenius system for the extension $s_L : L \to A$. □

From now on, let ℓ be a non-degenerate left integral in the symmetrized Hopf algebroid $\mathcal{A}$, set $\lambda^* = \ell_R^{-1}(1_A)$ and ${}^*\lambda = {}_R\ell^{-1}(1_A)$.

Theorem 5.5 implies that for a symmetrized Hopf algebroid $\mathcal{A}$ possessing a non-degenerate integral the modules A^R, ${}^RA$, A_L, and ${}_LA$ are finitely generated projective. Hence, by the result of [8], their duals $\mathcal{A}^*$ and ${}^*\mathcal{A}$ carry left bialgebroid structures over the base R, and $\mathcal{A}_*$ and ${}_*\mathcal{A}$ carry right bialgebroid structures over the base L

$$\begin{aligned}
&s_L^*(r)(a) = r\pi_R(a), && {}^*s_L(r)(a) = \pi_R\big(t_R(r)a\big),\\
&t_L^*(r)(a) = \pi_R\big(s_R(r)a\big), && {}^*t_L(r)(a) = \pi_R(a)r,\\
&\gamma_L^*(\phi^*) = \phi^* \leftharpoonup \ell^{(1)} \otimes \ell_R^{-1}\big(\ell^{(2)}\big), && {}^*\gamma_L({}^*\phi) = {}_R\ell^{-1}\big(\ell^{(1)}\big) \otimes {}^*\phi \leftharpoondown \ell^{(2)},\\
&\pi_L^*(\phi^*) = \phi^*(1_A), && {}^*\pi_L({}^*\phi) = {}^*\phi(1_A),\\
&s_{*R}(l)(a) = \pi_L\big(as_L(l)\big), && {}_*s_R(l)(a) = \pi_L(a)l,\\
&t_{*R}(l)(a) = l\pi_L(a), && {}_*t_R(l)(a) = \pi_L\big(at_L(l)\big),\\
&\gamma_{*R}(\phi_*) = \ell_{(1)} \rightharpoonup \phi_* \otimes \ell_L^{-1}(\ell_{(2)}), && {}_*\gamma_R({}_*\phi) = {}_L\ell^{-1}(\ell_{(1)}) \otimes \ell_{(2)} \rightharpoondown {}_*\phi,\\
&\pi_{*R}(\phi_*) = \phi_*(1_A), && {}_*\pi_R({}_*\phi) = {}_*\phi(1_A).
\end{aligned} \tag{5.11}$$

Lemma 5.6. *Let ℓ be a non-degenerate left integral in the symmetrized Hopf algebroid $\mathcal{A}$. Then for $\lambda^* = \ell_R^{-1}(1_A)$, ${}^*\lambda = {}_R\ell^{-1}(1_A)$, and any element $a \in A$, the identities*

$$\lambda^* \rightharpoonup a = s_R \circ \lambda^*(a), \tag{5.12}$$

$${}^*\lambda \rightharpoondown a = t_R \circ {}^*\lambda(a) \tag{5.13}$$

hold true.

Proof. One checks that

$$\phi^*\lambda^* = \ell_R^{-1}(\phi^* \rightharpoonup 1_A) = s_L^* \circ \phi^*(1_A)\lambda^*$$

for all $\phi^* \in \mathcal{A}^*$. This implies that $\phi^*(\lambda^* \rightharpoonup a) = \phi^*(s_R \circ \lambda^*(a))$ for all $\phi^* \in \mathcal{A}^*$. Since A^R is finitely generated projective by Theorem 5.5, this proves (5.12). The identity (5.13) follows by Remark 5.4. □

The left integrals in a Hopf algebroid were defined in Definition 5.1 as the left integrals in the underlying left bialgebroid. The non-degeneracy of the left integral was defined in Definition 5.3 using however the underlying right bialgebroid as well, that is it relies to the whole of the Hopf algebroid structure. Therefore, it is not obvious whether the non-strict isomorphisms of Hopf algebroids preserve non-degenerate integrals. In the rest of this subsection, we prove that this is the case:

Proposition 5.7. *Let both $(\mathcal{A}_L, S)$ and $(\mathcal{A}_L, S')$ be Hopf algebroids. Then their non-degenerate left integrals coincide.*

Proof. A left integral ℓ in $(\mathcal{A}_L, S)$ is a left integral in $(\mathcal{A}_L, S')$ by definition.

Let $\mathcal{A}_R = (A, R, s_R, t_R, \gamma_R, \pi_R)$ and $\mathcal{A}'_R = (A, R', s'_R, t'_R, \gamma'_R, \pi'_R)$ be the right bialgebroids underlying the Hopf algebroids $(\mathcal{A}_L, S)$ and $(\mathcal{A}_L, S')$, respectively. It follows from the uniqueness of the maps α^{-1} and β^{-1} in (4.16) that the coproducts $\gamma_R(a) = a^{(1)} \otimes a^{(2)}$ of $\mathcal{A}_R$ and $\gamma'_R(a) = a^{\{1\}} \otimes a^{\{2\}}$ of $\mathcal{A}'_R$ are related as

$$a^{(1)} \otimes S'^{-1} \circ S\big(a^{(2)}\big) = a^{\{1\}} \otimes a^{\{2\}} = S' \circ S^{-1}\big(a^{(1)}\big) \otimes a^{(2)}. \tag{5.14}$$

With the help of the maps $\mu = \pi_R \circ t_L$, $\mu' = \pi'_R \circ t_L$, $\nu = \pi_R \circ s_L$, and $\nu' = \pi'_R \circ s_L$, we can introduce the isomorphisms of additive groups:

$$\mathcal{A}^* \to \mathcal{A}'^*, \quad \phi^* \mapsto \mu' \circ \mu^{-1} \circ \phi^*,$$
$${}^*\mathcal{A} \to {}^*\mathcal{A}', \quad {}^*\phi \mapsto \nu' \circ \nu^{-1} \circ {}^*\phi.$$

Then the canonical actions (2.13) of $\mathcal{A}'^*$ and $\mathcal{A}^*$ and of ${}^*\mathcal{A}'$ and ${}^*\mathcal{A}$ on A are related as

$$\mu' \circ \mu^{-1} \circ \phi^* \overset{\prime}{\rightharpoonup} a = S'^{-1} \circ S(\phi^* \rightharpoonup a),$$
$$\nu' \circ \nu^{-1} \circ {}^*\phi \overset{\prime}{\rightharpoondown} a = S' \circ S^{-1}({}^*\phi \rightharpoondown a)$$

what implies the non-degeneracy of the left integral ℓ in $(\mathcal{A}_L, S')$ provided it is non-degenerate in $(\mathcal{A}_L, S)$. □

5.2. Two-sided non-degenerate integrals

Proposition 5.7 above proves that the structure of the non-degenerate left integrals is the same within an isomorphism class of Hopf algebroids. In this subsection we prove that for a non-degenerate left integral ℓ in the Hopf algebroid $(\mathcal{A}_L, S)$ there exists a distinguished representative $(\mathcal{A}_L, S'_\ell)$ in the isomorphism class of $(\mathcal{A}_L, S)$ with the property that ℓ is not only a non-degenerate left integral in $(\mathcal{A}_L, S'_\ell)$ but also a non-degenerate right integral.

The Hopf algebroids with two-sided non-degenerate integral are of particular interest. Both the Hopf algebroid structure constructed on the dual of a Hopf algebroid in Section 5.3 and the one associated to a depth 2 Frobenius extension in Section 3 belong to this class.

Lemma 5.8. *Let ℓ be a non-degenerate left integral in a symmetrized Hopf algebroid $\mathcal{A}$. Set $\lambda^* := \ell_R^{-1}(1_A)$ and ${}^*\lambda := {}_R\ell^{-1}(1_A)$. Then any (not necessarily non-degenerate) left integral $\ell' \in \mathcal{I}^L(\mathcal{A})$ satisfies*

$$\ell s_R \circ \lambda^*(\ell') = \ell' = \ell t_R \circ {}^*\lambda(\ell').$$

Proof. Observe that for ${}^*\phi \in {}^*\mathcal{A}$ and $\ell' \in \mathcal{I}^L(\mathcal{A})$ we have ${}^*\phi \leftharpoondown S^{-1}(\ell') = {}^*t_L \circ {}^*\phi \circ S^{-1}(\ell')$; hence

$$\ell' = \big({}^*\lambda \leftharpoondown S^{-1}(\ell')\big) \rightharpoondown \ell = {}^*t_L \circ {}^*\lambda \circ S^{-1}(\ell') \rightharpoondown \ell = \ell s_R \circ \lambda^*(\ell').$$

The identity $\ell' = \ell t_R \circ {}^*\lambda(\ell')$ follows by Remark 5.4. □

Lemma 5.9. *Let ℓ be a non-degenerate left integral in a symmetrized Hopf algebroid $\mathcal{A}$. Set $\lambda^* := \ell_R^{-1}(1_A)$. Then the map $\xi : A \to A$; $a \mapsto S((\lambda^* \leftharpoonup \ell) \rightharpoonup a)$ is a ring anti-automorphism.*

Proof. Using Lemma 5.8 one checks that

$$\begin{aligned}
\big[(\lambda^* \leftharpoonup \ell) \rightharpoonup a\big]\big[(\lambda^* \leftharpoonup \ell) \rightharpoonup b\big] &= a^{(2)} t_R \circ \lambda^*\big(\ell a^{(1)}\big) b^{(2)} t_R \circ \lambda^*\big(\ell b^{(1)}\big) \\
&= a^{(2)} b^{(2)} t_R \circ \lambda^*\big(\ell s_R \circ \lambda^*\big(\ell a^{(1)}\big) b^{(1)}\big) \\
&= a^{(2)} b^{(2)} t_R \circ \lambda^*\big(\ell a^{(1)} b^{(1)}\big) \\
&= (\lambda^* \leftharpoonup \ell) \rightharpoonup ab
\end{aligned}$$

for $a, b \in A$, hence the map ξ is anti-multiplicative. By analogous calculations the reader may check that it is bijective with inverse $\xi^{-1}(a) = S^{-1}(({}^*\lambda \leftharpoondown \ell) \rightharpoondown a)$, where ${}^*\lambda := {}_R\ell^{-1}(1_A) = \lambda^* \circ S$. □

Proposition 5.10. *Let ℓ be a non-degenerate left integral in a symmetrized Hopf algebroid $\mathcal{A}$. Then the following maps are bijective:*

$$\ell_L : \mathcal{A}_* \to A, \quad \phi_* \mapsto \ell \leftharpoonup \phi_*,$$
$$_L\ell : {}_*\mathcal{A} \to A, \quad {}_*\phi \mapsto \ell \leftharpoondown {}_*\phi.$$

Proof. We claim that with the help of the ring isomorphism ν (introduced in Proposition 4.3) we have $\ell \leftharpoondown {}_*\phi = \xi^{-1} \circ \ell_R(\nu \circ {}_*\phi \circ S^{-1})$, which implies the bijectivity of $_L\ell$. As a matter of fact,

$$\begin{aligned}
&\ell \leftharpoondown {}_*\big[\nu^{-1} \circ \ell_R^{-1}(a) \circ S\big] \\
&\quad = t_L \circ \nu^{-1} \circ \big(\lambda^* \leftharpoonup S(a)\big) \circ S(\ell_{(2)})\ell_{(1)} = S^{-1} \circ t_R \circ {}^*\lambda(\ell_{(2)}a)\ell_{(1)} \\
&\quad = S^{-1}\big({}^*\lambda \rightharpoondown \ell_{(2)}a\big)\ell_{(1)} = S^{-1}\big(S(\ell_{(1)})\ell_{(2)}{}^{(1)}a^{(1)}s_R \circ {}^*\lambda\big(\ell_{(2)}{}^{(2)}a^{(2)}\big)\big) \\
&\quad = S^{-1}\big(s_R \circ \pi_R\big(\ell^{(1)}\big)a^{(1)}s_R \circ {}^*\lambda\big(\ell^{(2)}a^{(2)}\big)\big) = S^{-1}\big(a^{(1)}s_R \circ {}^*\lambda\big(\ell a^{(2)}\big)\big) \\
&\quad = S^{-1}\big(({}^*\lambda \leftharpoondown \ell) \rightharpoondown a\big) = \xi^{-1}(a).
\end{aligned} \tag{5.15}$$

Similarly, by the application of (5.15) to $\mathcal{A}_{\mathrm{cop}}$,

$$\ell \leftharpoonup \phi_* = \xi \circ {}_R\ell(\mu \circ \phi_* \circ S), \tag{5.16}$$

hence ℓ_L is also bijective. □

Using (5.16) we have an equivalent form of the anti-automorphism ξ introduced in Lemma 5.9:

$$\xi(a) = \ell \leftharpoonup \big(a \rightharpoonup \ell_L^{-1}(1_A)\big). \tag{5.17}$$

Lemma 5.11. *Let ℓ be a non-degenerate left integral in a symmetrized Hopf algebroid $\mathcal{A}$. Then for all elements $a, b \in A$ we have the identities*

$$\ell_R^{-1}(b) \rightharpoonup a = {}_R\ell^{-1}(a) \rightharpoondown b, \tag{5.18}$$
$$a \leftharpoonup \ell_L^{-1}(b) = b \leftharpoondown {}_L\ell^{-1}(a), \tag{5.19}$$
$$\ell_R^{-1}(b) \rightharpoonup a = a \leftharpoonup \ell_L^{-1}(b), \tag{5.20}$$
$${}_R\ell^{-1}(b) \rightharpoondown a = a \leftharpoondown {}_L\ell^{-1}(b). \tag{5.21}$$

Proof. We illustrate the proof on (5.18). Use Lemma 5.6 to see that

$$\begin{aligned}
{}_R\ell^{-1}(a) \rightharpoondown b &= b^{(1)}\big(\lambda^* \rightharpoonup S\big(b^{(2)}\big)a\big) = s_L \circ \pi_L(b_{(1)})a^{(2)}t_R \circ \lambda^*\big(S(b_{(2)})a^{(1)}\big) \\
&= \ell_R^{-1}(b) \rightharpoonup a,
\end{aligned}$$

where $\lambda^* = \ell_R^{-1}(1_A)$. The rest of the proof is analogous. □

Lemma 5.12. *Let ℓ be a non-degenerate left integral in a symmetrized Hopf algebroid $\mathcal{A}$. Set $\lambda^* = \ell_R^{-1}(1_A)$. Then the map $\kappa : R \to R$, $r \mapsto \lambda^*(\ell t_R(r))$ is a ring automorphism.*

Proof. It follows from Lemma 5.8 that κ is multiplicative: for $r, r' \in R$

$$\kappa(r)\kappa(r') = \lambda^*\big(\ell t_R(r)\big)\lambda^*\big(\ell t_R(r')\big) = \lambda^*\big(\ell s_R \circ \lambda^*\big(\ell t_R(r')\big)t_R(r)\big) = \lambda^*\big(\ell t_R(r')t_R(r)\big)$$
$$= \kappa(rr').$$

In order to show that κ is bijective, we construct the inverse $\kappa^{-1} : r \mapsto {}^*\lambda(\ell s_R(r))$ where ${}^*\lambda = {}_R\ell^{-1}(1_A) = \lambda^* \circ S$. □

Proposition 5.13. *Let ℓ be a non-degenerate left integral in the Hopf algebroid $(\mathcal{A}_L, S)$. Then there exists a unique Hopf algebroid $(\mathcal{A}_L, S'_\ell)$ such that ℓ is a two-sided non-degenerate integral in $(\mathcal{A}_L, S'_\ell)$.*

Proof. *Uniqueness.* Suppose that $(\mathcal{A}_L, S'_\ell)$ is a Hopf algebroid of the required kind. Denote the underlying right bialgebroid by $\mathcal{A}'_R = (A, R', s'_R, t'_R, \gamma'_R, \pi'_R)$. Define $\lambda'^* \in \mathcal{A}'^*$ with the property that $\lambda'^* \overset{\prime}{\rightharpoonup} \ell = 1_A$ (where $\overset{\prime}{\rightharpoonup}$ denotes the canonical action (2.13) of $\mathcal{A}'^*$ on A). Introducing the notation $\gamma'_R(a) = a^{\{1\}} \otimes a^{\{2\}}$, one checks that

$$S'^{-1}(\ell) = \big(\lambda'^* \overset{\prime}{\leftharpoonup} \ell\big) \overset{\prime}{\rightharpoonup} \ell = \ell^{\{2\}} t'_R \circ \lambda'^*\big(\ell\ell^{\{1\}}\big) = \ell^{\{2\}} t'_R \circ \lambda'^*\big(\ell s'_R \circ \pi'_R\big(\ell^{\{1\}}\big)\big)$$
$$= \ell t'_R \circ \lambda'^*(\ell) = \ell.$$

With the help of the element $\pi_L \circ S^{-1} \circ S' \in \mathcal{A}_*$, we have

$$S\big(a \leftharpoonup \pi_L \circ S^{-1} \circ S'\big) = S \circ S'^{-1}\big(S'(a)^{\{1\}}\big)s_R \circ \pi_R\big(S'(a)^{\{2\}}\big) = S'(a)$$

for all $a \in A$, where in the last step the relation (5.14) has been used. Then the condition $S'(\ell) = \ell$ is equivalent to

$$S'(a) = S\big(a \leftharpoonup \ell_L^{-1} \circ S^{-1}(\ell)\big).$$

This proves the uniqueness of S'.

Existence. Let ξ be the anti-automorphism of A introduced in Lemma 5.9. We claim that $(\mathcal{A}_L, \xi)$ is a Hopf algebroid of the required kind. Introduce the right bialgebroid $\mathcal{A}'_R$ on the total ring A over the base R with structural maps

$$s'_R = s_R, \qquad t'_R = \xi^{-1} \circ s_R, \qquad \gamma'_R = \xi^{-1}_{A \otimes_L A} \circ S_{A \otimes_R A} \circ \gamma_R \circ S^{-1} \circ \xi,$$
$$\pi'_R = \pi_R \circ S^{-1} \circ \xi,$$

where $\mathcal{A}_R = (A, R, s_R, t_R, \gamma_R, \pi_R)$ is the right bialgebroid underlying $(\mathcal{A}_L, S)$. First, we check that the triple $(\mathcal{A}_L, \mathcal{A}'_R, \xi)$ satisfies Proposition 4.2(iii). Since

$$\xi^{-1} \circ S \circ s_R = \xi^{-1} \circ t_R \circ \theta_R = S^{-1} \circ t_R \circ \theta_R = s_R \quad \text{and} \quad \xi^{-1} \circ S \circ t_R = \xi^{-1} \circ s_R,$$

the $\mathcal{A}'_R$ is a right bialgebroid isomorphic to $\mathcal{A}_R$ via the isomorphism $(\xi^{-1}\circ S, \mathrm{id}_R)$.

The requirement $s'_R(R) \equiv s_R(R) = t_L(L)$ obviously holds true. Since

$$t'_R(r) = \xi^{-1}\circ s_R(r) = S^{-1}\circ s_R \circ {}^*\lambda\big(\ell s_R(r)\big) = t_R\circ\kappa^{-1}(r),$$

also $t'_R(R) \equiv t_R(R) = s_L(L)$. Since

$$\begin{aligned}
\gamma'_R(a) &\equiv a^{\{1\}}\otimes a^{\{2\}} = \xi^{-1}_{A\otimes_L A}\circ\gamma_L\circ\xi(a) = a^{(1)}\otimes\xi^{-1}\circ S\big(a^{(2)}\big)\\
&= a^{(1)}\otimes S^{-1}\big({}_R\ell^{-1}\circ S(\ell)\rightharpoondown S\big(a^{(2)}\big)\big) = a^{(1)}\otimes S^{-1}\big(S\big(a^{(2)}\big)\leftharpoondown {}_L\ell^{-1}\circ S(\ell)\big)\\
&= a^{(1)}s_R\circ\kappa\circ\mu\circ{}_L\ell^{-1}\circ S(\ell)\circ S\big(a^{(2)}\big)\otimes a^{(3)},
\end{aligned}$$

we have

$$\begin{aligned}
(\gamma_L\otimes\mathrm{id}_A)\circ\gamma'_R(a) &= a^{(1)}{}_{(1)}\otimes a^{(1)}{}_{(2)}\otimes\xi^{-1}\circ S\big(a^{(2)}\big) = a_{(1)}\otimes a_{(2)}{}^{(1)}\otimes\xi^{-1}\circ S\big(a_{(2)}{}^{(2)}\big)\\
&= \big(\mathrm{id}_A\otimes\gamma'_R\big)\circ\gamma_L(a),\\
(\mathrm{id}_A\otimes\gamma_L)\circ\gamma'_R(a) &= a^{(1)}s_R\circ\kappa\circ\mu\circ{}_L\ell^{-1}\circ S(\ell)\circ S\big(a^{(2)}\big)\otimes a^{(3)}{}_{(1)}\otimes a^{(3)}{}_{(2)}\\
&= a_{(1)}{}^{(1)}s_R\circ\kappa\circ\mu\circ{}_L\ell^{-1}\circ S(\ell)\circ S\big(a_{(1)}{}^{(2)}\big)\otimes a_{(1)}{}^{(3)}\otimes a_{(2)}\\
&= \big(\gamma'_R\otimes\mathrm{id}_A\big)\circ\gamma_L(a).
\end{aligned}$$

By Lemma 5.9, the ξ is an anti-automorphism of the ring A. The identity $\xi\circ t'_R = s'_R$ is obvious and also

$$\xi\circ t_L = \xi\circ s_R\circ\mu = S\circ s_R\circ\mu = s_L.$$

Finally,

$$\begin{aligned}
&\xi(a_{(1)})a_{(2)} = s_R\circ\pi_R\big((\lambda^*\leftharpoonup\ell)\rightharpoonup a\big) = s_R\circ\pi_R\circ S^{-1}\circ\xi(a) = s'_R\circ\pi'_R(a),\\
&a^{\{1\}}\xi\big(a^{\{2\}}\big) = a^{(1)}\xi\circ\xi^{-1}\circ S\big(a^{(2)}\big) = s_L\circ\pi_L(a).
\end{aligned}$$

This proves that $\mathcal{A}'_\ell = (\mathcal{A}_L, \mathcal{A}'_R, \xi)$ satisfies the conditions in Proposition 4.2(iii); hence $(\mathcal{A}_L, \xi)$ is a Hopf algebroid. Since

$$\xi(\ell) = S\big((\lambda^*\leftharpoonup\ell)\rightharpoonup\ell\big) = S\circ S^{-1}(\ell) = \ell,$$

the ℓ is a two-sided non-degenerate integral in $\mathcal{A}'_\ell$. □

5.3. Duality

It follows from Theorem 5.5 that for a symmetrized Hopf algebroid $\mathcal{A}$ possessing a non-degenerate left integral ℓ the dual rings (with respect to the base ring) carry bialgebroid structures. These bialgebroids (5.11) are independent of the particular choice of the non-degenerate integral. In this subsection we analyze these bialgebroids. We show that the four bialgebroids (5.11) are all (anti-) isomorphic and can be equipped with an ℓ-dependent Hopf algebroid structure. Because of the ℓ-dependence of this Hopf algebroid structure the duality of Hopf algebroids is sensibly defined on the isomorphism classes of Hopf algebroids.

Lemma 5.14. *Let ℓ be a non-degenerate left integral in the symmetrized Hopf algebroid $\mathcal{A}$. Then with the help of the anti-automorphism ξ of Lemma* 5.9 *we have the equalities*

$$\xi^{-1}\big(\ell^{(2)}\big)\otimes S^{-1}\big(\ell^{(1)}\big)=\ell_{(1)}\otimes\ell_{(2)}=S\big(\ell^{(2)}\big)\otimes\xi\big(\ell^{(1)}\big) \tag{5.22}$$

in $A_L\otimes{}_LA$.

Proof. The element $\xi^{-1}(\ell^{(2)})\otimes S^{-1}(\ell^{(1)})$ is in $A_L\otimes{}_LA$ since $S^{-1}\circ s_R=t_R=s_L\circ\nu^{-1}$ and $\xi^{-1}\circ t_R=S^{-1}\circ t_R=t_L\circ\nu^{-1}$. Using (5.15), in $A_L\otimes{}_LA$ we have

$$\xi^{-1}\big(\ell^{(2)}\big)\otimes S^{-1}\big(\ell^{(1)}\big)=\ell_{(1)}\otimes t_R\circ{}^*\lambda\big(\ell_{(2)}\ell^{(2)}\big)S^{-1}\big(\ell^{(1)}\big)=\ell_{(1)}\otimes\ell_{(2)}.$$

The other equality follows by repeating the proof in $\mathcal{A}_{\mathrm{cop}}$. □

Corollary 5.15. *For a non-degenerate left integral ℓ in the symmetrized Hopf algebroid $\mathcal{A}$ the maps ℓ_L and ${}_L\ell$ satisfy the identities*

$$\ell_L(a\rightharpoonup\phi_*)=\ell_L(\phi_*)\xi(a), \tag{5.23}$$

$${}_L\ell(a\rightharpoondown{}_*\phi)={}_L\ell({}_*\phi)\xi^{-1}(a), \tag{5.24}$$

where ξ is the anti-automorphism of A introduced in Lemma 5.9.

Theorem 5.16. *Let ℓ be a non-degenerate left integral in the symmetrized Hopf algebroid $\mathcal{A}=(\mathcal{A}_L,\mathcal{A}_R,S)$. Then the left bialgebroids $\mathcal{A}^*{}_L$, ${}^*\mathcal{A}_L$, $(\mathcal{A}_{*R})^{\mathrm{op}}_{\mathrm{cop}}$, and $({}_*\mathcal{A}_R)^{\mathrm{op}}_{\mathrm{cop}}$ in* (5.11) *are isomorphic via the isomorphisms*

$$\begin{array}{ccc}
(\mathcal{A}_{*R})^{\mathrm{op}}_{\mathrm{cop}} & \xrightarrow{\ ({}_L\ell^{-1}\circ\xi^{-1}\circ\ell_L,\,\mathrm{id}_R)\ } & ({}_*\mathcal{A}_R)^{\mathrm{op}}_{\mathrm{cop}} \\
{\scriptstyle(\ell_R^{-1}\circ\ell_L,\,\nu)}\big\downarrow & & \big\downarrow{\scriptstyle({}_R\ell^{-1}\circ{}_L\ell,\,\mu),} \\
\mathcal{A}^*{}_L & \xrightarrow{\ ({}_R\ell^{-1}\circ\xi^{-1}\circ\ell_R,\,\theta_R^{-1})\ } & {}^*\mathcal{A}_L
\end{array}$$

where ξ is the anti-automorphism of A introduced in Lemma 5.9 *and the maps μ, ν, and θ_R are the ring isomorphisms introduced in Proposition* 4.3.

Proof. By Proposition 4.3, the map ν is a ring isomorphism $L^{\mathrm{op}} \to R$. By Proposition 5.10, $\ell_R^{-1} \circ \ell_L$ is bijective. Its anti-multiplicativity follows from (5.20). The comultiplicativity follows by the successive use of the identity $\ell_R(\phi^* \leftharpoonup a) = S^{-1}(a)\ell_R(\phi^*)$, the integral property of ℓ, (5.22), and (5.23):

$$\begin{aligned}
\gamma_L^* \circ \ell_R^{-1} \circ \ell_L(\phi_*) &= \ell_R^{-1} \circ \ell_L(\phi_*) \leftharpoonup \ell^{(1)} \otimes \ell_R^{-1}\big(\ell^{(2)}\big) \\
&= \ell_R^{-1}\big(S^{-1}\big(\ell^{(1)}\big)\ell_L(\phi_*)\big) \otimes \ell_R^{-1}\big(\ell^{(2)}\big) \\
&= \ell_R^{-1}\big(S^{-1}\big(\ell^{(1)}\big)\big) \otimes \ell_R^{-1}\big(\ell_L(\phi_*)\ell^{(2)}\big) \\
&= \ell_R^{-1}(\ell_{(2)}) \otimes \ell_R^{-1}\big(\ell_L(\phi_*)\xi(\ell_{(1)})\big) \\
&= \ell_R^{-1} \circ \ell_L\big(\ell_L^{-1}(\ell_{(2)})\big) \otimes \ell_R^{-1} \circ \ell_L(\ell_{(1)} \rightharpoonup \phi_*) \\
&= \big(\ell_R^{-1} \circ \ell_L \otimes \ell_R^{-1} \circ \ell_L\big) \circ \gamma_{*R}{}^{\mathrm{op}}(\phi_*).
\end{aligned}$$

One checks also

$$\begin{aligned}
\big(\ell_R^{-1} \circ \ell_L \circ s_{*R}(l)\big)(a) &= {}^*\lambda\big(S^{-1}(a)s_L \circ \pi_L\big[\ell_{(1)}s_L(l)\big]\ell_{(2)}\big) = \nu(l)\pi_R(a) = \big(s_L^* \circ \nu(l)\big)(a), \\
\big(\ell_R^{-1} \circ \ell_L \circ t_{*R}(l)\big)(a) &= {}^*\lambda\big(S^{-1}(a)s_L\big[l\pi_L(\ell_{(1)})\big]\ell_{(2)}\big) = \pi_R\big(s_R \circ \nu(l)a\big) = \big(t_L^* \circ \nu(l)\big)(a), \\
\pi_L^* \circ \ell_R^{-1} \circ \ell_L(\phi_*) &= \nu \circ \phi_*\big(t_L \circ \nu^{-1} \circ {}^*\lambda(\ell_{(2)})\ell_{(1)}\big) = \nu \circ \phi_*(1_A) = \nu \circ \pi_{*R}(\phi_*).
\end{aligned}$$

This proves that $(\ell_R^{-1} \circ \ell_L, \nu)$ is a bialgebroid isomorphism $(\mathcal{A}_{*R})^{\mathrm{op}}_{\mathrm{cop}} \to \mathcal{A}^*{}_L$. By Remark 5.4, $({}_R\ell^{-1} \circ {}_L\ell, \mu)$ is a bialgebroid isomorphism $({}_*\mathcal{A}_R)^{\mathrm{op}}_{\mathrm{cop}} \to {}^*\mathcal{A}_L$.

By (5.16), ${}_R\ell^{-1} \circ \xi^{-1} \circ \ell_R = \mu \circ \ell_L^{-1} \circ \ell_R(\phi^*) \circ S$, hence we have to prove that $(\mu \circ - \circ S, \mu)$ is a bialgebroid isomorphism $(\mathcal{A}_{*R})^{\mathrm{op}}_{\mathrm{cop}} \to {}^*\mathcal{A}_L$.

The map μ is a ring isomorphism $L^{\mathrm{op}} \to R$ by Proposition 4.3. The map $\phi_* \mapsto \mu \circ \phi_* \circ S$ is bijective. Its anti-multiplicativity is obvious. The anti-comultiplicativity follows from (5.16) and (5.22):

$$\begin{aligned}
{}^*\gamma_L(\mu \circ \phi_* \circ S) &= {}_R\ell^{-1}\big(\ell^{(1)}\big) \otimes \mu \circ \phi_* \circ S \leftharpoonup \ell^{(2)} \\
&= \mu \circ \ell_L^{-1} \circ \xi\big(\ell^{(1)}\big) \circ S \otimes \mu \circ \big(S\big(\ell^{(2)}\big) \rightharpoonup \phi_*\big) \circ S \\
&= \mu \circ \phi_*^{(2)} \circ S \otimes \mu \circ \phi_*^{(1)} \circ S.
\end{aligned}$$

Finally, by Proposition 4.3

$$\begin{aligned}
\big(\mu \circ s_{*R}(l) \circ S\big)(a) &= \mu \circ \pi_L\big(S(a)s_L(l)\big) = \pi_R\big(s_R \circ \mu(l)a\big) = \big({}^*s_L \circ \mu(l)\big)(a), \\
\big(\mu \circ t_{*R}(l) \circ S\big)(a) &= \mu\big(l\pi_L \circ S(a)\big) = \pi_R(a)\mu(l) = \big({}^*t_L \circ \mu(l)\big)(a), \\
{}^*\pi_L(\mu \circ \phi_* \circ S) &= \mu \circ \phi_* \circ S(1_A) = \mu \circ \pi_{*R}(\phi_*).
\end{aligned}$$

This proves the theorem. □

Theorem 5.17. *Let ℓ be a non-degenerate left integral in the symmetrized Hopf algebroid $\mathcal{A}$. Then the left bialgebroid $\mathcal{A}^{\ell}_{*L} = (\mathcal{A}_*, R, s_{*L}, t_{*L}, \gamma_{*L}, \pi_{*L})$ where*

$$s_{*L}(r)(a) = \mu^{-1}(r)\pi_L(a), \qquad t_{*L}(r)(a) = \pi_L\big(at_R \circ \kappa^{-1}(r)\big),$$
$$\gamma_{*L}(\phi_*) = \xi^{-2}(\ell_{(2)}) \rightharpoonup \phi_* \otimes {\ell_L}^{-1}(\ell_{(1)}), \qquad \pi_{*L}(\phi_*) = \lambda^*(\ell \leftharpoonup \phi_*),$$

the right bialgebroid $\mathcal{A}_{*R}$ in (5.11), and the antipode $S^{\ell}_* := \ell_L^{-1} \circ \xi \circ \ell_L$ form a symmetrized Hopf algebroid denoted by $\mathcal{A}^{\ell}_*$.

Proof. We show that the triple $\mathcal{A}^{\ell}_* := (\mathcal{A}^{\ell}_{*L}, \mathcal{A}_{*R}, S^{\ell}_*)$ satisfies the conditions in Proposition 4.2(iii).

The $\mathcal{A}^{\ell}_{*L}$ is a left bialgebroid isomorphic to $({}_*\mathcal{A}_R)^{\mathrm{op}}_{\mathrm{cop}}$ via the isomorphism $({}_L\ell^{-1} \circ \ell_L, \mu)$. Also

$$s_{*L}(R) = t_{*R}(L) \quad \text{and} \quad t_{*L}(R) = s_{*R}(L)$$

hold obviously true. Making use of the identities (5.22) and (5.23), one checks that

$$\begin{aligned}
(\mathrm{id}_{\mathcal{A}_*} \otimes \gamma_{*R}) \circ \gamma_{*L}(\phi_*) &= \xi^{-2}(\ell_{(2)}) \rightharpoonup \phi_* \otimes \ell_{(1')} \rightharpoonup \ell_L^{-1}(\ell_{(1)}) \otimes \ell_L^{-1}(\ell_{(2')}) \\
&= \xi^{-2}(\ell_{(2)}) \rightharpoonup \phi_* \otimes \ell_L^{-1}\big(\ell_{(1)}\xi(\ell_{(1')})\big) \otimes \ell_L^{-1}(\ell_{(2')}) \\
&= \xi^{-2}\big(\ell_{(2)}\xi^2(\ell_{(1')})\big) \rightharpoonup \phi_* \otimes \ell_L^{-1}(\ell_{(1)}) \otimes \ell_L^{-1}(\ell_{(2')}) \\
&= (\gamma_{*L} \otimes \mathrm{id}_{\mathcal{A}_*}) \circ \gamma_{*R}(\phi_*), \\
(\gamma_{*R} \otimes \mathrm{id}_{\mathcal{A}_*}) \circ \gamma_{*L}(\phi_*) &= \ell_{(1')}\xi^{-2}(\ell_{(2)}) \rightharpoonup \phi_* \otimes \ell_L^{-1}(\ell_{(2')}) \otimes \ell_L^{-1}(\ell_{(1)}) \\
&= \ell_{(1')} \rightharpoonup \phi_* \otimes \ell_L^{-1}\big(\ell_{(2')}\xi^{-1}(\ell_{(2)})\big) \otimes \ell_L^{-1}(\ell_{(1)}) \\
&= \ell_{(1')} \rightharpoonup \phi_* \otimes \xi^{-2}(\ell_{(2)}) \rightharpoonup \ell_L^{-1}(\ell_{(2')}) \otimes \ell_L^{-1}(\ell_{(1)}) \\
&= (\mathrm{id}_{\mathcal{A}_*} \otimes \gamma_{*L}) \circ \gamma_{*R}(\phi_*).
\end{aligned}$$

The $S^{\ell}_* = (\ell_L^{-1} \circ \xi \circ {}_L\ell) \circ ({}_L\ell^{-1} \circ \ell_L)$ is a composition of ring isomorphisms ${}_L\ell^{-1} \circ \ell_L : (\mathcal{A}_*)^{\mathrm{op}} \to {}_*\mathcal{A}$ and $\ell_L^{-1} \circ \xi \circ {}_L\ell : {}_*\mathcal{A} \to \mathcal{A}_*$, hence it is an anti-automorphism of the ring $\mathcal{A}_*$. Also

$$\begin{aligned}
S^{\ell}_* \circ t_{*R}(l)(a) &= \mu^{-1} \circ {}_R\ell^{-1} \circ \ell_L \circ t_{*R}(l) \circ S^{-1}(a) \\
&= \mu^{-1} \circ {}^*\lambda\big(S^{-1}\big[s_L\big(l\pi_L(\ell_{(1)})\big)\ell_{(2)}\big]S^{-1}(a)\big) \\
&= \mu^{-1} \circ \pi_R \circ S^{-1}\big(as_L(l)\big) = \pi_L\big(as_L(l)\big) = s_{*R}(l)(a), \\
S^{\ell}_* \circ t_{*L}(r)(a) &= \mu^{-1} \circ {}_R\ell^{-1} \circ \ell_L \circ t_{*L}(r) \circ S^{-1}(a) \\
&= \mu^{-1} \circ \lambda^*\big(as_L \circ \pi_L\big(\ell_{(1)}t_R \circ \kappa^{-1}(r)\big)\ell_{(2)}\big) \\
&= \mu^{-1} \circ \lambda^*\big(t_L \circ \pi_L(a)\ell t_R \circ \kappa^{-1}(r)\big) = \mu^{-1}(r)\pi_L(a) = s_{*L}(r)(a).
\end{aligned}$$

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Since

$$\gamma_L \circ \xi^{-1}(a) = \xi^{-1}\big(a^{(2)}\big) \otimes S^{-1}\big(a^{(1)}\big) \quad \text{and} \quad \gamma_R \circ \xi^{-1}(a) = \xi^{-1}(a_{(2)}) \otimes S^{-1}(a_{(1)}),$$

we have $\gamma_L \circ \xi^{-2}(a) = \xi^{-1} \circ S^{-1}(a_{(1)}) \otimes S^{-1} \circ \xi^{-1}(a_{(2)})$. Then we can compute

$$\begin{aligned}
\big[S_*^{\ell-1}(\phi_{*(2)})\phi_{*(1)}\big](a) &= \big[\ell_L^{-1} \circ \xi^{-1}(\ell_{(1)})\big(\xi^{-2}(\ell_{(2)}) \rightharpoonup \phi_*\big)\big](a) \\
&= \phi_*\big(\big[a \leftharpoonup \ell_L^{-1} \circ \xi^{-1}(\ell_{(1)})\big]\xi^{-2}(\ell_{(2)})\big) \\
&= \phi_*\big(\big[\xi^{-1}(\ell_{(1)}) \leftharpoonup {}_L\ell^{-1}(a)\big]\xi^{-2}(\ell_{(2)})\big) \\
&= \phi_* \circ \xi^{-2}\big(\xi\big(\ell^{(2)}{}_{(1)}\ell^{(2)}{}_{(2)}\big)_L\ell^{-1}(a) \circ S^{-1}\big(\ell^{(1)}\big)\big) \\
&= \phi_*(1_A)\pi_L(\ell_{(1)})_L\ell^{-1}(a)(\ell_{(2)}) = \phi_*(1_A)\pi_L(a);
\end{aligned}$$

hence $S_*^\ell(\phi_{*(1)})\phi_{*(2)} = s_{*R} \circ \pi_{*R}(\phi_*)$ for all $\phi_* \in \mathcal{A}_*$. Also

$$\begin{aligned}
\big[\phi_*^{(1)} S_*^\ell\big(\phi_*^{(2)}\big)\big](a) &= \big[(\ell_{(1)} \rightharpoonup \phi_*)\ell_L^{-1} \circ \xi(\ell_{(2)})\big](a) \\
&= \mu^{-1} \circ {}_R\ell^{-1}(\ell_{(2)}) \circ S^{-1}\big(s_L \circ \phi_*(a_{(1)}\ell_{(1)})a_{(2)}\big) \\
&= \mu^{-1} \circ \lambda^*\big(s_L \circ \phi_*(a_{(1)}\ell_{(1)})a_{(2)}\ell_{(2)}\big) = \mu^{-1} \circ \lambda^*(a\ell \leftharpoonup \phi_*) \\
&= \mu^{-1} \circ \lambda^*\big(t_L \circ \pi_L(a)\ell \leftharpoonup \phi_*\big) = \mu^{-1} \circ \lambda^*(\ell \leftharpoonup \phi_*)\pi_L(a) \\
&= \big[s_{*L} \circ \pi_{*L}(\phi_*)\big](a).
\end{aligned}$$

This proves that $\mathcal{A}_*^\ell = (\mathcal{A}_{*L}^\ell, \mathcal{A}_{*R}, S_*^\ell)$ is a symmetrized Hopf algebroid. □

Obviously, the strong isomorphism class of the Hopf algebroid $(\mathcal{A}_{*L}^\ell, S_*^\ell)$ depends only on the Hopf algebroid $(\mathcal{A}_L, S)$ and the non-degenerate left integral ℓ of it. It is insensitive to the particular choice of the underlying right bialgebroid $\mathcal{A}_R$.

The antipode S_*^ℓ has a form analogous to (5.17):

$$S_*^\ell(\phi_*)(a) = \big[(\ell \leftharpoonup \phi_*) \rightharpoonup \ell_L^{-1}(1_A)\big](a). \tag{5.25}$$

Using the left bialgebroid isomorphisms of Theorem 5.16, also the dual left bialgebroids $({}_*\mathcal{A}_R)^{\mathrm{op}}_{\mathrm{cop}}$, $\mathcal{A}_L^*$, and ${}^*\!\mathcal{A}_L$ can be made Hopf algebroids, all strictly isomorphic to the above Hopf algebroid $(\mathcal{A}_{*L}^\ell, S_*^\ell)$. They have the antipodes

$${}_*S^\ell = {}_L\ell^{-1} \circ \xi \circ {}_L\ell, \tag{5.26}$$

$$S_\ell^* = \ell_R^{-1} \circ \xi \circ \ell_R, \tag{5.27}$$

$${}^*S_\ell = {}_R\ell^{-1} \circ \xi \circ {}_R\ell. \tag{5.28}$$

Let us turn to the interpretation of the role of the Hopf algebroid $(\mathcal{A}_{*L}^\ell, S_*^\ell)$. As $\mathcal{A}_{*L}^\ell$ is isomorphic to $({}_*\mathcal{A}_R)^{\mathrm{op}}_{\mathrm{cop}}$ and the right bialgebroid underlying $(\mathcal{A}_{*L}^\ell, S_*^\ell)$ is $\mathcal{A}_{*R}$, on the first sight it seems to be natural to consider it as some kind of a dual of $(\mathcal{A}_L, S)$. There are however two arguments against this interpretation. First, the Hopf algebroid $(\mathcal{A}^{\ell}_{*L}, S^{\ell}_{*})$ depends on ℓ. Second, as it is proven in Proposition 5.19, $(\mathcal{A}^{\ell}_{*L}, S^{\ell}_{*})$ belongs to a special kind of Hopf algebroids: it possesses a two-sided non-degenerate integral.

Lemma 5.18. *Let $\mathcal{A}$ be a symmetrized Hopf algebroid such that the R-module A^R in* (4.6) *is finitely generated projective. Then a left integral $\ell \in \mathcal{I}^L(\mathcal{A})$ is non-degenerate if and only if the map ℓ_R is bijective.*

Proof. The *only if part* is trivial.

In order to prove the *if part*, recall that by the proof of Lemma 5.6 for $\lambda^* := \ell_R^{-1}(1_A)$ and all $a \in A$ the identity $\lambda^* \rightharpoonup a = s_R \circ \lambda^*(a)$ holds true, λ^* is an R-R-bimodule map ${}_RA^R \to R$, and the inverse of ℓ_R reads as $\ell_R^{-1}(a) = \lambda^* \leftharpoonup S(a)$. Then

$$\phi^*(a) = \ell_R^{-1} \circ \ell_R(\phi^*)(a) = \lambda^*\big(s_R \circ \phi^*\big(\ell^{(1)}\big)S\big(\ell^{(2)}\big)a\big) = \phi^*\big(a\ell^{(1)}s_R \circ \lambda^* \circ S\big(\ell^{(2)}\big)\big)$$

for all $\phi^* \in \mathcal{A}^*$ and $a \in A$. Using the finitely generated projectivity of A^R, we have $\ell^{(1)}s_R \circ \lambda^* \circ S(\ell^{(2)}) = 1_A$. Since $\lambda^* \circ S \in {}^*\mathcal{A}$, the inverse ${}_R\ell^{-1}$ can be defined as ${}_R\ell^{-1}(a) = \lambda^* \circ S \leftharpoonup S^{-1}(a)$. $\square$

Proposition 5.19. *Let $\mathcal{A}$ be a symmetrized Hopf algebroid possessing a non-degenerate left integral ℓ. Then the element $\ell_L^{-1}(1_A)$ in $\mathcal{A}_*$ is a two-sided non-degenerate integral in the symmetrized Hopf algebroid $\mathcal{A}^{\ell}_{*}$ (constructed in Theorem* 5.17).

Proof. It follows from (5.16) that $\ell_L^{-1}(1_A) = \mu^{-1} \circ \lambda^*$ where $\lambda^* := \ell_R^{-1}(1_A)$ and μ is the ring isomorphism introduced in Proposition 4.3. For all $\phi_* \in \mathcal{A}_*$ and $a \in A$, we have

$$\big[\big(\mu^{-1} \circ \lambda^*\big)\phi_*\big](a) = \phi_* \circ S\big({}^*\lambda \rightharpoondown S^{-1}(a)\big) = \phi_*(1_A)\mu^{-1} \circ \lambda^*(a),$$

hence

$$\big[\big(\mu^{-1} \circ \lambda^*\big)t_{*R} \circ \pi_{*R}(\phi_*)\big](a) = t_{*R} \circ \pi_{*R}(\phi_*)(1_A)\mu^{-1} \circ \lambda^*(a) = \phi_*(1_A)\mu^{-1} \circ \lambda^*(a),$$

which proves that $\mu^{-1} \circ \lambda^*$ is a right integral. Using (5.17),

$$S^{\ell}_{*}\big(\mu^{-1} \circ \lambda^*\big) = \ell_L^{-1} \circ \xi \circ \ell_L\big(\mu^{-1} \circ \lambda^*\big) = \ell_L^{-1} \circ \xi^2(1_A) = \ell_L^{-1}(1_A) = \mu^{-1} \circ \lambda^*,$$

hence $\mu^{-1} \circ \lambda^*$ is also a left integral.

As it is proven in [8], since A_L is finitely generated projective, so is the left $L_* \equiv L$-module ${}^{L_*}(\mathcal{A}_*)$. The corresponding dual bialgebroid ${}^*(\mathcal{A}_*)_L$ is isomorphic to $\mathcal{A}_L$ via the isomorphism (ι, id_L) of left bialgebroids where

$$\iota : A \to {}^*(\mathcal{A}_*), \quad \iota(a)(\phi_*) := \phi_*(a). \tag{5.29}$$

Since

$$\begin{aligned}\big(\iota(a) \rightharpoondown \phi_*\big)(b) &= \pi_L\big(\big[b \leftharpoonup (\ell_{(1)} \rightharpoonup \phi_*)\big] s_L \circ \ell_L^{-1}(\ell_{(2)})(a)\big)\\ &= \phi_*\big(b s_R \circ \lambda^*\big(a\ell^{(1)}\big) S\big(\ell^{(2)}\big)\big) = (a \rightharpoonup \phi_*)(b),\end{aligned}$$

the map ${}_{L_*}(\mu^{-1} \circ \lambda^*) : {}^*(\mathcal{A}_*) \to \mathcal{A}_*$, $\iota(a) \mapsto \iota(a) \rightharpoondown \mu^{-1} \circ \lambda^* \equiv a \rightharpoonup \mu^{-1} \circ \lambda^*$ is bijective with inverse

$${}_{L_*}\big(\mu^{-1} \circ \lambda^*\big)^{-1} : \phi_* \mapsto \iota \circ \ell_L \circ S_*^{\ell -1} \equiv \iota \circ {}_R\ell(\mu \circ \phi_* \circ S).$$

The application of Lemma 5.18 finishes the proof. □

In the view of Proposition 5.7, the following definition makes sense.

Definition 5.20. The *dual of the isomorphism class* of a Hopf algebroid $(\mathcal{A}_L, S)$ possessing a non-degenerate left integral ℓ is the isomorphism class of the Hopf algebroid $(\mathcal{A}^\ell_{*L}, S^\ell_*)$ (constructed in the Theorem 5.17).

The next proposition shows that this notion of duality is involutive.

Proposition 5.21. *Let ℓ be a non-degenerate left integral in the Hopf algebroid $(\mathcal{A}_L, S)$. Then the Hopf algebroid*

$$\Big(\big(\mathcal{A}^\ell_*\big)^{\ell_L^{-1}(1_A)}_{*L}, \big(S^\ell_*\big)^{\ell_L^{-1}(1_A)}_*\Big) \tag{5.30}$$

is strictly isomorphic to $(\mathcal{A}_L, S'_\ell)$—the Hopf algebroid constructed in Proposition 5.13. *In particular, the Hopf algebroid* (5.30) *is isomorphic to $(\mathcal{A}_L, S)$.*

Proof. Since the Hopf algebroids

$$\Big(\big(\mathcal{A}^\ell_*\big)^{\ell_L^{-1}(1_A)}_{*L}, \big(S^\ell_*\big)^{\ell_L^{-1}(1_A)}_*\Big) \quad \text{and} \quad \Big({}^*(\mathcal{A}_*)_L, {}^*\big(S^\ell_*\big)_{\ell_L^{-1}(1_A)}\Big)$$

are strictly isomorphic, it suffices to show that the isomorphism (ι, id_L) of left bialgebroids $\mathcal{A}_L \to {}^*(\mathcal{A}_*)_L$ in (5.29) extends to a strict isomorphism of Hopf algebroids $(\mathcal{A}_L, S'_\ell) \to ({}^*(\mathcal{A}_*)_L, {}^*(S^\ell_*)_{\ell_L^{-1}(1_A)})$.

By (5.28) for a non-degenerate left integral ℓ in the Hopf algebroid $(\mathcal{A}_L, S)$, the antipode ${}^*S_\ell$ of the Hopf algebroid $({}^*\mathcal{A}_L, {}^*S_\ell)$ reads as

$${}^*S_\ell({}^*\phi) = {}^*\lambda \leftharpoondown \big({}^*\phi \rightharpoondown S^{-1}(\ell)\big).$$

Applying it to the non-degenerate integral $\ell_L^{-1}(1_A)$ in $(\mathcal{A}^\ell_{*L}, S^\ell_*)$, we obtain

$$\begin{aligned}{}^*\big(S^\ell_*\big)_{\ell_L^{-1}(1_A)}\big(\iota(a)\big) &= \iota(\ell) \leftharpoondown \big[\iota(a) \rightharpoondown S_*^{\ell -1} \circ \ell_L^{-1}(1_A)\big]\\ &= \iota\big(\ell \leftharpoonup \big(a \rightharpoonup \ell_L^{-1}(1_A)\big)\big) = \iota \circ \xi(a).\end{aligned}$$ □

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The duality of (weak) Hopf algebras is re-obtained from Definition 5.20 as follows. Let H be a finite weak Hopf algebra over a commutative ring k. Let $(\mathcal{H}_L, S)$ be the corresponding Hopf algebroid (introduced in the Example 4.8) and let ℓ be a non-degenerate left integral in H. Recall that, in order to reconstruct the weak Hopf algebra from the Hopf algebroid in Example 4.8, one needs a distinguished separability structure on the base ring. The dual weak Hopf algebra is the unique weak Hopf algebra in the isomorphism class of $(\mathcal{H}^{\ell}_{*L}, S^{\ell}_{*})$ corresponding to the same separability structure on L as H corresponds to.

If H is a Hopf algebra over k then—since the separability structure on k is unique—the dual Hopf algebra is the only Hopf algebra in the isomorphism class of $(\mathcal{H}^{\ell}_{*L}, S^{\ell}_{*})$.

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4. fejezet

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Integral Theory for Hopf Algebroids

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Abstract. The theory of integrals is used to analyze the structure of Hopf algebroids. We prove that the total algebra of a Hopf algebroid is a separable extension of the base algebra if and only if it is a semi-simple extension and if and only if the Hopf algebroid possesses a normalized integral. It is a Frobenius extension if and only if the Hopf algebroid possesses a nondegenerate integral. We give also a sufficient and necessary condition in terms of integrals, under which it is a quasi-Frobenius extension, and illustrate by an example that this condition does not hold true in general. Our results are generalizations of classical results on Hopf algebras.

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1. Introduction

The notion of *integrals* in Hopf algebras has been introduced by Sweedler [33]. The integrals in Hopf algebras over principal ideal domains were analyzed in [19, 32] where the following – by now classical – results have been proven:

– A free, finite-dimensional bialgebra over a principal ideal domain is a Hopf algebra if and only if it possesses a nondegenerate left integral. (Larson–Sweedler Theorem.)
– The antipode of a free, finite-dimensional Hopf algebra over a principal ideal domain is bijective.
– A Hopf algebra over a field is finite-dimensional if and only if it possesses a nonzero left integral.
– The left integrals in a finite-dimensional Hopf algebra over a field form a one dimensional subspace.
– A Hopf algebra over a field is semi-simple if and only if it possesses a normalized left integral. (Maschke's Theorem.)

There are numerous generalizations of these results in the literature. Historically the first is due to Pareigis [27] who proved the following statements on a finitely generated and projective Hopf algebra (H, Δ, ϵ, S) over a commutative ring k:

– H is a Frobenius extension of k if and only if there exists a Frobenius functional $\psi\colon H \to k$ satisfying $(H \otimes \psi) \circ \Delta = 1_H \psi(\_)$.
– The antipode, S, is bijective.
– The left integrals form a projective rank 1 direct summand of the k-module H.
– H is a quasi-Frobenius extension of k.
– A finitely generated and projective bialgebra over a commutative ring k, such that $\mathrm{pic}(k) = 0$, is a Hopf algebra if and only if it possesses a nondegenerate left integral.

The generalization of the Maschke theorem to Hopf algebras H over commutative rings k states that the existence of a normalized left integral in H is equivalent to the separability of H over k, which is further equivalent to its relative semi-simplicity in the sense of [15, 17] that any H-module is (H, k)-projective [12, 20]. This is equivalent to the true semi-simplicity of H (i.e. the true projectivity of any H-module [28]) if and only if k is a semi-simple ring [20].

As a nice review on these results we recommend Section 3.2 in [13].

Similar results are known also for the generalizations of Hopf algebras. Integrals for finite-dimensional quasi-Hopf algebras [14] over fields were studied in [16, 25, 26, 11] and for finite-dimensional weak Hopf algebras [4, 3] over fields in [3, 40].

The purpose of the present paper is to investigate which of the above results generalizes to *Hopf algebroids*.

Hopf algebroids with bijective antipode have been introduced in [1, 5]. It is important to emphasize that this notion of Hopf algebroid is not equivalent to the one introduced under the same name by Lu in [21]. Here we generalize the definition of [5, 1] by relaxing the requirement of the bijectivity of the antipode. A Hopf algebroid consists of a compatible pair of a left and a right bialgebroid structure [21, 34, 35, 38] on the common total algebra A. The antipode relates these two left- and right-handed structures. Left/right integrals *in* a Hopf algebroid are defined as the invariants of the left/right regular A-module in terms of the counit of the left/right bialgebroid. Integrals *on* a Hopf algebroid are the comodule maps from the total algebra to the base algebra (reproducing the integrals *in* the dual bialgebroids, provided the duals possess bialgebroid structures).

The total algebra of a bialgebroid can be looked at as an extension of the base algebra or its opposite via the source and target maps, respectively. This way there are four algebra extensions associated to a Hopf algebroid. The main results of the paper relate the properties of these extensions to the existence of integrals with special properties:

- A Maschke-type theorem, proving that the separability, and also the (in two cases left in two cases right) semi-simplicity of any of the four extensions is equivalent to the existence of a normalized integral *in* the Hopf algebroid (Theorem 3.1).

- Any of the four extensions is a Frobenius extension if and only if there exists a nondegenerate integral *in* the Hopf algebroid (Theorem 4.7).
- Any of the four extensions is (in two cases a left in two cases a right) quasi-Frobenius extension if and only if the total algebra is a finitely generated and projective module, and the (left or right) integrals *on* the Hopf algebroid form a flat module, over the base algebra (Theorem 5.2).

Our main tool in proving the latter two points is the Fundamental Theorem for Hopf modules over Hopf algebroids (Theorem 4.2).

The paper is organized as follows: We start Section 2 with reviewing some results on bialgebroids from [9, 18, 21, 30, 31, 34–36, 38], the knowledge of which is needed for the understanding of the paper. Then we give the definition of Hopf algebroids and discuss some of its immediate consequences. Integrals both *in* and *on* Hopf algebroids are introduced and some equivalent characterizations are given.

In Section 3 we prove two Maschke-type theorems. The first collects some equivalent properties (in particular the separability) of the inclusion of the base algebra in the total algebra of a Hopf algebroid. These equivalent properties are related to the existence of a normalized integral *in* the Hopf algebroid. The second collects some equivalent properties (in particular the coseparability) of the coring, underlying the Hopf algebroid. These equivalent properties are shown to be equivalent to the existence of a normalized integral *on* the Hopf algebroid.

In Section 4 we prove the Fundamental Theorem for Hopf modules over a Hopf algebroid. This theorem is somewhat stronger than the one that can be obtained by the application of [7, Theorem 5.6], to the present situation. The main result of the section is Theorem 4.7. In proving it we follow an analogous line of reasoning as in [19]. That is, assuming that one of the module structures of the total algebra over the base algebra is finitely generated and projective, we apply the Fundamental Theorem to the Hopf module, constructed on the dual of the Hopf algebroid (w.r.t. the base algebra). Similarly to the case of Hopf algebras, our result implies the existence of nonzero integrals *on* any finitely generated projective Hopf algebroid. Since the dual of a (finitely generated projective) Hopf algebroid is not known to be a Hopf algebroid in general, we have no dual result, that is, we do not know whether there exist nonzero integrals *in* any finitely generated projective Hopf algebroid. As a byproduct, also a sufficient and necessary condition on a finitely generated projective Hopf algebroid is obtained, under which the antipode is bijective. We do not know, however, whether this condition follows from the axioms.

In Section 5 we use the results of Section 4 to obtain conditions which are equivalent to the (either left or right) quasi-Frobenius property of any of the four extensions behind a Hopf algebroid. In order to show that these conditions do not hold true in general, we construct a counterexample.

Throughout the paper we work over a commutative ring k. That is, the total and base algebras of our Hopf algebroids are k-algebras. For an (always associative and unital) k-algebra $A \equiv (A, m_A, 1_A)$ we denote by ${}_A\mathcal{M}$, $\mathcal{M}_A$ and ${}_A\mathcal{M}_A$ the categories of left, right, and bimodules over A, respectively. For the k-module of morphisms in ${}_A\mathcal{M}$, $\mathcal{M}_A$ and ${}_A\mathcal{M}_A$ we write ${}_A\mathrm{Hom}(\ ,\ )$, $\mathrm{Hom}_A(\ ,\ )$ and ${}_A\mathrm{Hom}_A(\ ,\ )$, respectively.

2. Integrals for Hopf Algebroids

Hopf algebroids with bijective antipode have been introduced in [5], where several equivalent reformulations of the definition [5, Definition 4.1] have been given. The definition we give in this section generalizes the form in [5, Proposition 4.2(iii)] by allowing the antipode not to be bijective.

Integrals *in* Hopf algebroids have also been introduced in [5]. As we shall see, the definition [5, Definition 5.1] applies also in our more general setting. In this section we introduce integrals also *on* Hopf algebroids.

In order for the paper to be self-contained we recall some results on bialgebroids from [38, 21, 34, 35, 18]. For more on bialgebroids we refer to the papers [30, 9, 31, 36].

The notions of Takeuchi's $\times_R$-bialgebra [38], Lu's bialgebroid [21] and Xu's bialgebroid with anchor [41] have been shown to be equivalent in [9]. We are going to use the definition in the following form:

DEFINITION 2.1. A *left bialgebroid* $\mathcal{A}_L = (A, B, s, t, \gamma, \pi)$ consists of two algebras A and B over the commutative ring k, which are called the total and base algebras, respectively. A is a $B \underset{k}{\otimes} B^{\mathrm{op}}$-ring (i.e. a monoid in ${}_{B\otimes B^{\mathrm{op}}}\mathcal{M}_{B\otimes B^{\mathrm{op}}}$) via the algebra homomorphisms $s\colon B \to A$ and $t\colon B^{\mathrm{op}} \to A$, called the source and target maps, respectively. In terms of s and t one equips A with a B–B bimodule structure ${}_BA_B$ as

$$b \cdot a \cdot b' := s(b)t(b')a \quad \text{for } a \in A,\ b, b' \in B.$$

The triple $({}_BA_B, \gamma, \pi)$ is a B-coring, that is a comonoid in ${}_B\mathcal{M}_B$. Introducing Sweedler's convention $\gamma(a) = a_{(1)} \underset{B}{\otimes} a_{(2)}$ for $a \in A$, the axioms

$$a_{(1)}t(b) \underset{B}{\otimes} a_{(2)} = a_{(1)} \underset{B}{\otimes} a_{(2)}s(b), \tag{2.1}$$
$$\gamma(1_A) = 1_A \underset{B}{\otimes} 1_A, \tag{2.2}$$
$$\gamma(aa') = \gamma(a)\gamma(a'), \tag{2.3}$$
$$\pi(1_A) = 1_B, \tag{2.4}$$
$$\pi(a\ s \circ \pi(a')) = \pi(aa'), \tag{2.5}$$
$$\pi(a\ t \circ \pi(a')) = \pi(aa') \tag{2.6}$$

are required for all $b \in B$ and $a, a' \in A$.

Notice that – although $A \underset{B}{\otimes} A$ is not an algebra – axiom (2.3) makes sense in view of (2.1).

The homomorphisms of left bialgebroids $\mathcal{A}_L = (A, B, s, t, \gamma, \pi) \to \mathcal{A}'_L = (A', B', s', t', \gamma', \pi')$ are pairs of k-algebra homomorphisms $(\Phi\colon A \to A'$,

$\phi\colon B \to B'$) satisfying

$$s' \circ \phi = \Phi \circ s, \tag{2.7}$$

$$t' \circ \phi = \Phi \circ t, \tag{2.8}$$

$$\gamma' \circ \Phi = (\Phi \underset{B}{\otimes} \Phi) \circ \gamma, \tag{2.9}$$

$$\pi' \circ \Phi = \phi \circ \pi. \tag{2.10}$$

The bimodule ${}_BA_B$, appearing in Definition 2.1, is defined in terms of multiplication on the left. Hence – following the terminology of [18] – we use the name *left* bialgebroid for this structure. In terms of right multiplication one defines right bialgebroids analogously. For the details we refer to [18].

Once the map $\gamma\colon A \to A \underset{B}{\otimes} A$ is given we can define $\gamma^{\mathrm{op}}\colon A \to A \underset{B^{\mathrm{op}}}{\otimes} A$ via $a \mapsto a_{(2)} \otimes a_{(1)}$. It is straightforward to check that if $\mathcal{A}_L = (A, B, s, t, \gamma, \pi)$ is a left bialgebroid then $\mathcal{A}_{L\ \mathrm{cop}} = (A, B^{\mathrm{op}}, t, s, \gamma^{\mathrm{op}}, \pi)$ is also a left bialgebroid and $\mathcal{A}_L^{\mathrm{op}} = (A^{\mathrm{op}}, B, t, s, \gamma, \pi)$ is a right bialgebroid.

In the case of a left bialgebroid $\mathcal{A}_L = (A, B, s, t, \gamma, \pi)$ the category ${}_A\mathcal{M}$ of left A-modules is a monoidal category. As a matter of fact, any left A-module is a B–B bimodule via s and t. The monoidal product in ${}_A\mathcal{M}$ is defined as the B-module tensor product with A-module structure

$$a \cdot (m \underset{B}{\otimes} m') := a_{(1)} \cdot m \underset{B}{\otimes} a_{(2)} \cdot m' \quad \text{for } a \in A,\ m \underset{B}{\otimes} m' \in M \underset{B}{\otimes} M'.$$

Just the same way as axiom (2.3), also this definition makes sense in the view of (2.1). The monoidal unit is B with A-module structure

$$a \cdot b := \pi(as(b)) \quad \text{for } a \in A,\ b \in B.$$

Analogously, in the case of a right bialgebroid $\mathcal{A}_R$ the category $\mathcal{M}_A$ of right A-modules is a monoidal category.

The B-coring structure $({}_BA_B, \gamma, \pi)$, underlying the left bialgebroid $\mathcal{A}_L = (A, B, s, t, \gamma, \pi)$, gives rise to a k-algebra structure on any of the B-duals of ${}_BA_B$ [10, 17.8]. The multiplication on the k-module ${}_*\mathcal{A} := {}_B\mathrm{Hom}(A, B)$, for example, is given by

$$({}_*\phi\, {}_*\psi)(a) = {}_*\psi(t \circ {}_*\phi(a_{(2)})a_{(1)}) \quad \text{for } {}_*\phi, {}_*\psi \in {}_*\mathcal{A},\ a \in A \tag{2.11}$$

and the unit is π. ${}_*\mathcal{A}$ is a left A-module and A is a right ${}_*\mathcal{A}$-module via

$$a \rightharpoonup {}_*\phi := {}_*\phi(\_\, a) \quad \text{and} \quad a \leftharpoonup {}_*\phi := t \circ {}_*\phi(a_{(2)})\, a_{(1)} \tag{2.12}$$

for ${}_*\phi \in {}_*\mathcal{A}$, $a \in A$. As it is well known [39, 18], ${}_*\mathcal{A}$ is also a $B \underset{k}{\otimes} B^{\mathrm{op}}$-ring via the inclusions

$$ {}_*s\colon B \to {}_*\mathcal{A}, \qquad b \mapsto \pi(\_)b,$$

$$ {}_*t\colon B^{\mathrm{op}} \to {}_*\mathcal{A}, \qquad b \mapsto \pi(\_\, s(b)).$$

Both maps ${}_*s$ and ${}_*t$ are split injections of B-modules with common left inverse ${}_*\pi\colon {}_*\mathcal{A} \to B, {}_*\phi \mapsto {}_*\phi(1_A)$. What is more, if A is finitely generated and projective as a left B-module, then ${}_*\mathcal{A}$ has also a right bialgebroid structure (with source and target maps ${}_*s$ and ${}_*t$, respectively, and counit ${}_*\pi$).

Notice that the algebra ${}_*\mathcal{A}$ reduces to the opposite of the usual dual algebra if $({}_BA_B, \gamma, \pi)$ is a coalgebra over a commutative ring B. In the case when A is a finitely generated projective left B-module, also the coproduct specializes to the opposite of the usual one in the case when $\mathcal{A}$ is a bialgebra. This convention is responsible for duality to flip the notions of left and right bialgebroids.

Applying the above formulae to the left bialgebroid $(\mathcal{A}_L)_{\rm cop}$ we obtain a $B \underset{k}{\otimes} B^{\rm op}$-ring structure on $\mathcal{A}_* := {\rm Hom}_B(A, B)$. The inclusions $B \to \mathcal{A}_*$ and $B^{\rm op} \to \mathcal{A}_*$ will be denoted by s_* and t_*, respectively. In particular, $\mathcal{A}_*$ is a left A-module and A is a right $\mathcal{A}_*$-module via

$$a \rightharpoonup \phi_* := (\_\, a) \quad \text{and} \quad a \leftharpoonup \phi_* := s \circ \phi_*(a_{(1)})\, a_{(2)}. \tag{2.13}$$

If the module A is finitely generated and projective as a right B-module then $\mathcal{A}_*$ is also a right bialgebroid.

In the case of a right bialgebroid $\mathcal{A}_R = (A, B, s, t, \gamma, \pi)$ the application of the opposite of the multiplication formula (2.11) to $(\mathcal{A}_R)^{\rm op}_{\rm cop}$ and to $(\mathcal{A}_R)^{\rm op}$ results $B \underset{k}{\otimes} B^{\rm op}$-ring structures on $\mathcal{A}^* := {\rm Hom}_B(A, B)$ and ${}^*\mathcal{A} := {}_B{\rm Hom}(A, B)$, respectively. We have the inclusions $s^*\colon B \to \mathcal{A}^*$, $t^*\colon B^{\rm op} \to \mathcal{A}^*$, ${}^*s\colon B \to {}^*\mathcal{A}$ and ${}^*t\colon B^{\rm op} \to {}^*\mathcal{A}$.

In particular, $\mathcal{A}^*$ and ${}^*\mathcal{A}$ are right A-modules and A is a left $\mathcal{A}^*$-module and a left ${}^*\mathcal{A}$-module via the formulae

$$\phi^* \leftharpoonup a := \phi^*(a\, \_) \quad \text{and} \quad \phi^* \rightharpoonup a := a^{(2)}\, t \circ \phi^*(a^{(1)}), \tag{2.14}$$

$${}^*\phi \leftharpoondown a := {}^*\phi(a\, \_) \quad \text{and} \quad {}^*\phi \rightharpoondown a := a^{(1)}\, s \circ {}^*\phi(a^{(2)}) \tag{2.15}$$

for $\phi^* \in \mathcal{A}^*$, ${}^*\phi \in {}^*\mathcal{A}$ and $a \in A$. If A is finitely generated and projective as a right, or as a left B-module then the corresponding dual is also a left bialgebroid.

Before defining the structure that is going to be the subject of the paper let us stop here and introduce some notations. Analogous notations were already used in [5].

When dealing with a $B \underset{k}{\otimes} B^{\rm op}$-ring A, we have to face the situation that A carries different module structures over the base algebra B. In this situation the usual notation $A \underset{B}{\otimes} A$ would be ambiguous. Therefore we make the following notational convention. In terms of the maps $s\colon B \to A$ and $t\colon B^{\rm op} \to A$ we introduce four B-modules

$$\begin{aligned} {}_BA &: \quad b \cdot a := s(b)a, \\ A_B &: \quad a \cdot b := t(b)a, \\ A^B &: \quad a \cdot b = as(b), \\ {}^BA &: \quad b \cdot a = at(b). \end{aligned} \tag{2.16}$$

(Our notation can be memorized as left indices stand for left modules and right indices for right modules. Upper indices for modules defined in terms of right multiplication and lower indices for the ones defined in terms of left multiplication.)

In writing B-module tensor products we write out explicitly the module structures of the factors that are taking part in the tensor products, and do not put marks under the symbol $\otimes$. For example, we write $A_B \otimes {}_B A$. Normally we do not denote the module structures that are not taking part in the tensor product, this should be clear from the context. In writing elements of tensor product modules we do not distinguish between the various module tensor products. That is, we write both $a \otimes a' \in A_B \otimes {}_B A$ and $c \otimes c' \in A^B \otimes {}_B A$, for example.

A left B-module can be considered as a right B^{op}-module, and sometimes we want to take a module tensor product over B^{op}. In this case we use the name of the corresponding B-module and the fact that the tensor product is taken over B^{op} should be clear from the order of the factors. For example, ${}_B A \otimes A_B$ is the B^{op}-module tensor product of the right B^{op} module defined via multiplication by $s(b)$ on the left, and the left B^{op}-module defined via multiplication by $t(b)$ on the left.

In writing multiple tensor products we use different types of letters to denote which module structures take part in the same tensor product. For example, the B-module tensor product $A_B \otimes {}^B A$ can be given a right B module structure via multiplication by $t(b)$ on the left in the second factor. The tensor product of this right B-module with ${}_B A$ is denoted by $A_B \otimes {}^B A_{\mathbf{B}} \otimes {}_{\mathbf{B}} A$.

We are ready to introduce the structure that is going to be the subject of the paper:

DEFINITION 2.2. A *Hopf algebroid* $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ consists of a left bialgebroid $\mathcal{A}_L = (A, L, s_L, t_L, \gamma_L, \pi_L)$, a right bialgebroid $\mathcal{A}_R = (A, R, s_R, t_R, \gamma_R, \pi_R)$ and a k-module map $S\colon A \to A$, called the antipode, such that the following axioms hold true:

(i) $s_L \circ \pi_L \circ t_R = t_R, \qquad t_L \circ \pi_L \circ s_R = s_R$ and
$s_R \circ \pi_R \circ t_L = t_L, \qquad t_R \circ \pi_R \circ s_L = s_L,$ (2.17)

(ii) $(\gamma_L \otimes {}^R A) \circ \gamma_R = (A_L \otimes \gamma_R) \circ \gamma_L$
as maps $A \to A_L \otimes {}_L A^R \otimes {}^R A$ and
$(\gamma_R \otimes {}_L A) \circ \gamma_L = (A^R \otimes \gamma_L) \circ \gamma_R$
as maps $A \to A^R \otimes {}^R A_L \otimes {}_L A,$ (2.18)

(iii) S is both an L–L bimodule map ${}^L A_L \to {}_L A^L$
and an R–R bimodule map ${}^R A_R \to {}_R A^R,$ (2.19)

(iv) $m_A \circ (S \otimes {}_L A) \circ \gamma_L = s_R \circ \pi_R$ and
$m_A \circ (A^R \otimes S) \circ \gamma_R = s_L \circ \pi_L.$ (2.20)

If $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ is a Hopf algebroid then so is $\mathcal{A}^{\mathrm{op}}_{\mathrm{cop}} = ((\mathcal{A}_R)^{\mathrm{op}}_{\mathrm{cop}}, (\mathcal{A}_L)^{\mathrm{op}}_{\mathrm{cop}}, S)$ and if S is bijective then also $\mathcal{A}_{\mathrm{cop}} = ((\mathcal{A}_L)_{\mathrm{cop}}, (\mathcal{A}_R)_{\mathrm{cop}}, S^{-1})$ and $\mathcal{A}^{\mathrm{op}} = ((\mathcal{A}_R)^{\mathrm{op}}, (\mathcal{A}_L)^{\mathrm{op}}, S^{-1})$.

The following modification of Sweedler's convention will turn out to be useful. For a Hopf algebroid $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ we use the notation $\gamma_L(a) = a_{(1)} \otimes a_{(2)}$

with lower indices, and $\gamma_R(a) = a^{(1)} \otimes a^{(2)}$ with upper indices for $a \in A$ in the case of the coproducts of $\mathcal{A}_L$ and of $\mathcal{A}_R$, respectively. The axioms (2.18) read in this notation as

$$\begin{aligned} {a^{(1)}}_{(1)} \otimes {a^{(1)}}_{(2)} \otimes a^{(2)} &= a_{(1)} \otimes {a_{(2)}}^{(1)} \otimes {a_{(2)}}^{(2)}, \\ {a_{(1)}}^{(1)} \otimes {a_{(1)}}^{(2)} \otimes a_{(2)} &= a^{(1)} \otimes {a^{(2)}}_{(1)} \otimes {a^{(2)}}_{(2)} \end{aligned}$$

for $a \in A$.

Examples of Hopf algebroids (with bijective antipode) are collected in [5].

PROPOSITION 2.3. (1) *The base algebras L and R of the left and right bialgebroids in a Hopf algebroid are anti-isomorphic.*

(2) *For a Hopf algebroid $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ the pair $(S, \pi_L \circ s_R)$ is a left bialgebroid homomorphism $(\mathcal{A}_R)^{\mathrm{op}}_{\mathrm{cop}} \to \mathcal{A}_L$ and $(S, \pi_R \circ s_L)$ is a left bialgebroid homomorphism $\mathcal{A}_L \to (\mathcal{A}_R)^{\mathrm{op}}_{\mathrm{cop}}$.*

Proof. (1) Both $\pi_R \circ s_L$ and $\pi_R \circ t_L$ are anti-isomorphisms $L \to R$ with inverses $\pi_L \circ t_R$ and $\pi_L \circ s_R$, respectively.

(2) We have seen that the map $\pi_L \circ s_R \colon R^{\mathrm{op}} \to L$ is an algebra homomorphism. It follows from (2.19), (2.20) and some bialgebroid identities that $S\colon A^{\mathrm{op}} \to A$ is an algebra homomorphism, as for $a, b \in A$ we have

$$\begin{aligned} S(1_A) &= 1_A\, S(1_A) = s_L \circ \pi_L(1_A) = 1_A \quad \text{and} \\ S(ab) &= S[t_L \circ \pi_L(a_{(2)})\, a_{(1)}\, b] \\ &= S[a_{(1)}\, t_L \circ \pi_L(b_{(2)})\, b_{(1)}]\, {a_{(2)}}^{(1)} S({a_{(2)}}^{(2)}) \\ &= S[{a^{(1)}}_{(1)} {b^{(1)}}_{(1)}]\, {a^{(1)}}_{(2)} {b^{(1)}}_{(2)} S(b^{(2)}) S(a^{(2)}) \\ &= s_R \circ \pi_R(a^{(1)} b^{(1)})\, S(b^{(2)})\, S(a^{(2)}) \\ &= S[b^{(2)}\, t_R \circ \pi_R(t_R \circ \pi_R(a^{(1)})\, b^{(1)})] S(a^{(2)}) \\ &= S(b)\, s_R \circ \pi_R(a^{(1)})\, S(a^{(2)}) = S(b) S(a). \end{aligned}$$

The properties (2.7)–(2.8) follow from (2.19) and (2.17) as

$$\begin{aligned} s_L \circ \pi_L \circ s_R &= S \circ t_L \circ \pi_L \circ s_R = S \circ s_R, \\ t_L \circ \pi_L \circ s_R &= s_R = S \circ t_R. \end{aligned}$$

The properties (2.9)–(2.10) are checked on an element $a \in A$ as

$$\begin{aligned} \gamma_L \circ S(a) &= S(a_{(1)})_{(1)}\, s_L \circ \pi_L(a_{(2)}) \otimes S(a_{(1)})_{(2)} \\ &= S({a^{(1)}}_{(1)})_{(1)} {a^{(1)}}_{(2)} S(a^{(2)}) \otimes S({a^{(1)}}_{(1)})_{(2)} \\ &= S({a^{(1)}}_{(1)})_{(1)}\, t_L \circ \pi_L({a^{(1)}}_{(2)(2)})\, {a^{(1)}}_{(2)(1)}\, S(a^{(2)}) \otimes S({a^{(1)}}_{(1)})_{(2)} \\ &= S({a^{(1)(1)}}_{(1)})_{(1)} {a^{(1)(1)}}_{(2)(1)} S(a^{(2)}) \\ &\quad \otimes S({a^{(1)(1)}}_{(1)})_{(2)} {a^{(1)(1)}}_{(2)(2)} S(a^{(1)(2)}) \\ &= S(a^{(2)}) \otimes s_R \circ \pi_R(a^{(1)(1)})\, S(a^{(1)(2)}) = (S \otimes S) \circ \gamma_R^{\mathrm{op}}(a) \quad \text{and} \\ \pi_L \circ S(a) &= \pi_L[S(a_{(1)})\, s_L \circ \pi_L(a_{(2)})] = \pi_L \circ s_R \circ \pi_R(a). \end{aligned}$$

The proof is completed by the observation that in passing from the Hopf algebroid $\mathcal{A}$ to $\mathcal{A}^{\mathrm{op}}_{\mathrm{cop}}$ the roles of $(S, \pi_L \circ s_R)$ and $(S, \pi_R \circ s_L)$ become interchanged. □

PROPOSITION 2.4. *The left bialgebroid $\mathcal{A}_L$ in a Hopf algebroid $\mathcal{A}$ is a $\times_L$-Hopf algebra in the sense of* [30]. *That is, the map*

$$\alpha\colon\ {}^L A \otimes A_L \to A_L \otimes {}_L A, \quad a \otimes b \to a_{(1)} \otimes a_{(2)} b$$

is bijective.

Proof. The inverse of α is given by

$$\alpha^{-1}\colon\ A_L \otimes {}_L A \to {}^L A \otimes A_L, \quad a \otimes b \mapsto a^{(1)} \otimes S(a^{(2)}) b. \qquad \square$$

The relation between the left and the right bialgebroids in a Hopf algebroid $\mathcal{A}$ implies relations between the dual algebras $\mathcal{A}^* \equiv \mathrm{Hom}_R(A^R, R)$ and $\mathcal{A}_* \equiv \mathrm{Hom}_L(A_L, L)$ and also between ${}^*\mathcal{A} \equiv {}_R\mathrm{Hom}({}^R A, R)$ and ${}_*\mathcal{A} \equiv {}_L\mathrm{Hom}({}_L A, L)$:

LEMMA 2.5. *For a Hopf algebroid $\mathcal{A}$ there exist algebra anti-isomorphisms $\sigma\colon {}_*\mathcal{A} \to {}^*\mathcal{A}$ and $\chi\colon \mathcal{A}^* \to \mathcal{A}_*$ satisfying*

$$a \leftharpoondown {}_*\phi = \sigma({}_*\phi) \rightharpoondown a \tag{2.21}$$

and

$$\phi^* \rightharpoonup a = a \leftharpoonup \chi(\phi^*) \tag{2.22}$$

for all ${}_*\phi \in {}_*\mathcal{A}$, $\phi^* \in \mathcal{A}^*$ and $a \in A$.

Proof. We leave it to the reader to check that the maps

$$\sigma\colon\ {}_*\mathcal{A} \to {}^*\mathcal{A}, \quad {}_*\phi \mapsto \pi_R(\_ \leftharpoondown {}_*\phi)$$

and

$$\chi\colon\ \mathcal{A}^* \to \mathcal{A}_*, \quad \phi^* \mapsto \pi_L(\phi^* \rightharpoonup \_)$$

are algebra anti-homomorphisms satisfying (2.21)–(2.22). The inverses are given by

$$\sigma^{-1}\colon\ {}^*\mathcal{A} \to {}_*\mathcal{A}, \quad {}^*\phi \mapsto \pi_L({}^*\phi \rightharpoondown \_)$$

and

$$\chi^{-1}\colon\ \mathcal{A}_* \to \mathcal{A}^*, \quad \phi_* \mapsto \pi_R(\_ \leftharpoonup \phi_*). \qquad \square$$

LEMMA 2.6. *The following properties of a Hopf algebroid $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ are equivalent*:

(1.a) *The module A_L is finitely generated and projective.*
(1.b) *The module A^R is finitely generated and projective.*

The following are also equivalent:

(2.a) *The module* ${}_L A$ *is finitely generated and projective.*
(2.b) *The module* ${}^R A$ *is finitely generated and projective.*

If furthermore S is bijective then all the four properties (1.a)*,* (1.b)*,* (2.a) *and* (2.b) *are equivalent.*

Proof. (1.a) $\Rightarrow$ (1.b) In terms of the dual bases, $\{b_i\} \subset A$ and $\{\beta_*^i\} \subset \mathcal{A}_*$ for the module A_L, the dual bases, $\{k_j\} \subset A$ and $\{\kappa_j^*\} \subset \mathcal{A}^*$ for the module A^R, can be constructed by the requirement that

$$\sum_j k_j \otimes \kappa_j^* = \sum_i b_i^{(1)} \otimes [\chi^{-1}(\beta_*^i) \leftharpoonup s_R \circ \pi_R \circ s_L \circ \pi_L(b_i^{(2)})]$$

as elements of $A^R \otimes {}_R\mathcal{A}^*$, where χ is the isomorphism (2.22). The expression on the right-hand side is well defined since – though the map

$$A_L \otimes {}^L\mathcal{A}_* \to \mathcal{A}^*, \quad a \otimes \phi_* \mapsto \chi^{-1}(\phi_*) \leftharpoonup s_R \circ \pi_R \circ s_L \circ \pi_L(a)$$

is not a left R-module map ${}^R A_L \otimes {}^L\mathcal{A}_* \to {}_R\mathcal{A}^*$ – its restriction to the R-submodule $\{\sum_k a_k \otimes \phi_*^k \in A_L \otimes {}^L\mathcal{A}_* \mid \sum_k a_k t_L(l) \otimes \phi_*^k = \sum_k a_k \otimes \phi_*^k s_*(l)\ \forall l \in L\}$ is so.

(2.a) $\Rightarrow$ (2.b) Similarly, in terms of the dual bases, $\{b_i\} \subset A$ and $\{{}_*\beta^i\} \subset {}_*\mathcal{A}$ for the module ${}_L A$, the dual bases, $\{k_j\} \subset A$ and $\{{}^*\kappa_j\} \subset {}^*\mathcal{A}$ for the module ${}^R A$, can be constructed by the requirement that

$$\sum_j {}^*\kappa_j \otimes k_j = \sum_i [\sigma({}_*\beta^i) \leftharpoondown t_R \circ \pi_R \circ t_L \circ \pi_L(b_i^{(1)})] \otimes b_i^{(2)}$$

as elements of ${}^*\mathcal{A}_R \otimes {}^R A$, where σ is the isomorphism (2.21).

(1.b) $\Rightarrow$ (1.a) follows by applying (2.a) $\Rightarrow$ (2.b) to the Hopf algebroid $\mathcal{A}_{\mathrm{cop}}^{\mathrm{op}}$.

(2.b) $\Rightarrow$ (2.a) follows by applying (1.a) $\Rightarrow$ (1.b) to the Hopf algebroid $\mathcal{A}_{\mathrm{cop}}^{\mathrm{op}}$.

Now suppose that S is bijective.

(1.a) $\Rightarrow$ (2.b) In terms of the dual bases, $\{b_i\} \subset A$ and $\{\beta_*^i\} \subset \mathcal{A}_*$ for the module A_L, the dual bases, $\{k_j\} \subset A$ and $\{{}^*\kappa_j\} \subset {}^*\mathcal{A}$ for the module ${}^R A$, can be constructed by the requirement that

$$\sum_j {}^*\kappa_j \otimes k_j = \sum_i \pi_R \circ t_L \circ \beta_*^i \circ S \otimes S^{-1}(b_i) \quad \text{as elements of } {}^*\mathcal{A}_R \otimes {}^R A.$$

(2.b) $\Rightarrow$ (1.a) follows by applying (1.a) $\Rightarrow$ (2.b) to the Hopf algebroid $\mathcal{A}_{\mathrm{cop}}^{\mathrm{op}}$. $\square$

Now we turn to the study of the notion of integrals *in* Hopf algebroids. For a left bialgebroid $\mathcal{A}_L = (A, L, s_L, t_L, \gamma_L, \pi_L)$ and a left A-module M the *invariants* of M with respect to $\mathcal{A}_L$ are the elements of

$$\mathrm{Inv}(M) := \{n \in M \mid a \cdot n = s_L \circ \pi_L(a) \cdot n\ \forall a \in A\}.$$

Clearly, the invariants of M with respect to $(\mathcal{A}_L)_{\rm cop}$ coincide with its invariants with respect to $\mathcal{A}_L$. The invariants of a right A-module M with respect to a right bialgebroid $\mathcal{A}_R$ are defined as the invariants of M (viewed as a left $A^{\rm op}$-module) with respect to $(\mathcal{A}_R)^{\rm op}$.

DEFINITION 2.7. The *left integrals in a left bialgebroid* $\mathcal{A}_L$ are the invariants of the left regular A-module with respect to $\mathcal{A}_L$.

The *right integrals in a right bialgebroid* $\mathcal{A}_R$ are the invariants of the right regular A-module with respect to $\mathcal{A}_R$.

The *left/right integrals in a Hopf algebroid* $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ are the left/right integrals in $\mathcal{A}_L/\mathcal{A}_R$, that is the elements of

$$\mathcal{L}(\mathcal{A}) = \{\ell \in A \mid a\ell = s_L \circ \pi_L(a)\, \ell\ \forall a \in A\}$$

and

$$\mathcal{R}(\mathcal{A}) = \{\wp \in A \mid \wp a = \wp\, s_R \circ \pi_R(a)\ \forall a \in A\}.$$

For any Hopf algebroid $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ we have $\mathcal{L}(\mathcal{A}) = \mathcal{R}(\mathcal{A}^{\rm op}_{\rm cop})$ and if S is bijective then also $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}_{\rm cop}) = \mathcal{R}(\mathcal{A}^{\rm op})$. Since for $\ell \in \mathcal{L}(\mathcal{A})$ and $a \in A$,

$$S(\ell)a = S[t_L \circ \pi_L(a_{(1)})\, \ell]a_{(2)} = S(a_{(1)}\ell)a_{(2)} = S(\ell)\, s_R \circ \pi_R(a),$$

we have $S(\mathcal{L}(\mathcal{A})) \subseteq \mathcal{R}(\mathcal{A})$ and, similarly, $S(\mathcal{R}(\mathcal{A})) \subseteq \mathcal{L}(\mathcal{A})$.

SCHOLIUM 2.8. *The following properties of an element* $\ell \in A$ *are equivalent*:

(1.a) $\ell \in \mathcal{L}(\mathcal{A})$,
(1.b) $S(a)\ell^{(1)} \otimes \ell^{(2)} = \ell^{(1)} \otimes a\ell^{(2)} \quad \forall a \in A$,
(1.c) $a\ell^{(1)} \otimes S(\ell^{(2)}) = \ell^{(1)} \otimes S(\ell^{(2)})a \quad \forall a \in A$.

The following properties of the element $\wp \in A$ *are also equivalent*:

(2.a) $\wp \in \mathcal{R}(\mathcal{A})$,
(2.b) $\wp_{(1)} \otimes \wp_{(2)}S(a) = \wp_{(1)}a \otimes \wp_{(2)} \quad \forall a \in A$,
(2.c) $S(\wp_{(1)}) \otimes \wp_{(2)}a = aS(\wp_{(1)}) \otimes \wp_{(2)} \quad \forall a \in A$.

By comodules over a left bialgebroid $\mathcal{A}_L = (A, L, s_L, t_L, \gamma_L, \pi_L)$ we mean comodules over the L-coring $({}_LA_L, \gamma_L, \pi_L)$, and by comodules over a right bialgebroid $\mathcal{A}_R = (A, R, s_R, t_R, \gamma_R, \pi_R)$ comodules over the R-coring $({}^RA^R, \gamma_R, \pi_R)$. The pair $({}_LA, \gamma_L)$ is a left comodule, and (A_L, γ_L) is a right comodule over the left bialgebroid $\mathcal{A}_L$. Since the L-coring $({}_LA_L, \gamma_L, \pi_L)$ possesses a grouplike element 1_A, also (L, s_L) is a left comodule and (L, t_L) is a right comodule over $\mathcal{A}_L$ (see [10, 28.2]). Similarly, (A^R, γ_R) and (R, s_R) are right comodules, and $({}^RA, \gamma_R)$ and (R, t_R) are left comodules over $\mathcal{A}_R$.

DEFINITION 2.9. An *s-integral on a left bialgebroid* $\mathcal{A}_L = (A, L, s_L, t_L, \gamma_L, \pi_L)$ is a left $\mathcal{A}_L$-comodule map ${}_*\rho\colon ({}_LA, \gamma_L) \to (L, s_L)$. That is, an element of

$$\mathcal{R}({}_*\mathcal{A}) := \{{}_*\rho \in {}_*\mathcal{A} \mid (A_L \otimes {}_*\rho) \circ \gamma_L = s_L \circ {}_*\rho\}.$$

A *t-integral on* $\mathcal{A}_L$ is a right $\mathcal{A}_L$-comodule map $(A_L, \gamma_L) \to (L, t_L)$. That is, an element of

$$\mathcal{R}(\mathcal{A}_*) := \{\rho_* \in \mathcal{A}_* \mid (\rho_* \otimes {}_L A) \circ \gamma_L = t_L \circ \rho_*\}.$$

An *s-integral on a right bialgebroid* $\mathcal{A}_R = (A, R, s_R, t_R, \gamma_R, \pi_R)$ is a right $\mathcal{A}_R$-comodule map $(A^R, \gamma_R) \to (R, s_R)$. That is, an element of

$$\mathcal{L}(\mathcal{A}^*) := \{\lambda^* \in \mathcal{A}^* \mid (\lambda^* \otimes {}^R A) \circ \gamma_R = s_R \circ \lambda^*\}.$$

A *t-integral on* $\mathcal{A}_R$ is a left $\mathcal{A}_R$-comodule map $({}^R A, \gamma_R) \to (R, t_R)$. That is, an element of

$$\mathcal{L}({}^*\mathcal{A}) := \{{}^*\lambda \in {}^*\mathcal{A} \mid (A^R \otimes {}^*\lambda) \circ \gamma_R = t_R \circ {}^*\lambda\}.$$

The *right/left s- and t-integrals on a Hopf algebroid* $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ are the s- and t-integrals on $\mathcal{A}_L/\mathcal{A}_R$.

The integrals on a *left/right* bialgebroid are checked to be invariants of the appropriate *right/left* regular module – justifying our usage of the terms *'right'* and *'left'* integrals for them (cf. the remark in Section 2 about using the opposite–co-opposite of the convention, usual in the case of bialgebras, when defining the dual bialgebroids ${}_*\mathcal{A}$ and $\mathcal{A}^*$). As a matter of fact, for example, if ${}_*\rho \in \mathcal{R}({}_*\mathcal{A})$ then

$$\begin{aligned}[{}_*\rho\ {}_*\phi](a) &= {}_*\phi(a \leftharpoonup {}_*\rho) = {}_*\phi(s_L \circ {}_*\rho(a)) = {}_*\rho(a)\ {}_*\phi(1_A) \\ &= [{}_*\rho\ \ {}_*s \circ {}_*\pi({}_*\phi)](a) \end{aligned} \tag{2.23}$$

for all ${}_*\phi \in {}_*\mathcal{A}$ and $a \in A$. If the module ${}_L A$ is finitely generated and projective (hence ${}_*\mathcal{A}$ is a right bialgebroid) then also the converse is true, so in this case the s-integrals on $\mathcal{A}_L$ are the same as the right integrals in ${}_*\mathcal{A}$. Similar statements hold true on the elements of $\mathcal{R}(\mathcal{A}_*)$, $\mathcal{L}(\mathcal{A}^*)$ and $\mathcal{L}({}^*\mathcal{A})$.

The reader should be warned that integrals on Hopf algebras H over commutative rings k are defined in the literature sometimes as comodule maps $H \to k$ – similarly to our Definition 2.9 –, sometimes by the analogue of the weaker invariant condition (2.23).

For any Hopf algebroid $\mathcal{A}$ we have $\mathcal{R}({}_*\mathcal{A}) = \mathcal{L}((\mathcal{A}^{\mathrm{op}}_{\mathrm{cop}})^*)$ and $\mathcal{R}(\mathcal{A}_*) = \mathcal{L}({}^*(\mathcal{A}^{\mathrm{op}}_{\mathrm{cop}}))$. If the antipode is bijective then also $\mathcal{R}({}_*\mathcal{A}) = \mathcal{R}((\mathcal{A}_{\mathrm{cop}})_*) = \mathcal{L}({}^*(\mathcal{A}^{\mathrm{op}}))$.

SCHOLIUM 2.10. *Let* $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ *be a Hopf algebroid. The following properties of an element* ${}_*\rho \in {}_*\mathcal{A}$ *are equivalent*:

(1.a) ${}_*\rho \in \mathcal{R}({}_*\mathcal{A})$,
(1.b) $\pi_R \circ s_L \circ {}_*\rho \in \mathcal{L}({}^*\mathcal{A})$,
(1.c) $s_L \circ {}_*\rho(aS(b_{(1)}))\, b_{(2)} = t_L \circ {}_*\rho(a_{(2)}S(b))\, a_{(1)} \quad \forall a, b \in A.$

The following properties of an element $\rho_* \in \mathcal{A}_*$ are equivalent:

(2.a) $\rho_* \in \mathcal{R}(\mathcal{A}_*)$,
(2.b) $\pi_R \circ t_L \circ \rho_* \in \mathcal{L}(\mathcal{A}^*)$,
(2.c) $t_L \circ \rho_*(ab^{(1)})\, S(b^{(2)}) = s_L \circ \rho_*(a_{(1)}b)\, a_{(2)} \quad \forall a, b \in A$.

The following properties of an element $\lambda^* \in \mathcal{A}^*$ are equivalent:

(3.a) $\lambda^* \in \mathcal{L}(\mathcal{A}^*)$,
(3.b) $\pi_L \circ s_R \circ \lambda^* \in \mathcal{R}(\mathcal{A}_*)$,
(3.c) $a^{(1)}\, s_R \circ \lambda^*(S(a^{(2)})b) = b^{(2)}\, t_R \circ \lambda^*(S(a)b^{(1)}) \quad \forall a, b \in A$.

The following properties of an element ${}^*\lambda \in {}^*\mathcal{A}$ are equivalent:

(4.a) ${}^*\lambda \in \mathcal{L}({}^*\mathcal{A})$,
(4.b) $\pi_L \circ t_R \circ {}^*\lambda \in \mathcal{R}({}_*\mathcal{A})$,
(4.c) $S(a_{(1)})\, t_R \circ {}^*\lambda(a_{(2)}b) = b^{(1)}\, s_R \circ {}^*\lambda(ab^{(2)}) \quad \forall a, b \in A$.

In particular, for ${}_*\rho \in \mathcal{R}({}_*\mathcal{A})$ the element ${}_*\rho \circ S$ belongs to $\mathcal{R}(\mathcal{A}_*)$ and for $\lambda^* \in \mathcal{L}(\mathcal{A}^*)$ the element $\lambda^* \circ S$ belongs to $\mathcal{L}({}^*\mathcal{A})$.

3. Maschke Type Theorems

The most classical version of Maschke's theorem [22] considers group algebras over fields. It states that the group algebra of a finite group G over a field F is semi-simple if and only if the characteristic of F does not divide the order of G. This result has been generalized to finite-dimensional Hopf algebras H over fields F by Sweedler [32] proving that H is a separable F-algebra if and only if it is semi-simple and if and only if there exists a normalized left integral in H. The proof goes as follows. It is a classical result that a separable algebra over a field is semi-simple. If H is semi-simple then, in particular, the H-module on F, defined in terms of the counit, is projective. This means that the counit, as an H-module map $H \to F$, splits. Its right inverse maps the unit of F into a normalized integral. Finally, in terms of a normalized integral one can construct an H-bilinear right inverse for the multiplication map $H \underset{F}{\otimes} H \to H$.

The only difficulty in the generalization of Maschke's theorem to Hopf algebras over commutative rings comes from the fact that in the case of an algebra A over a commutative base ring k, separability does not imply the semi-simplicity of A in the sense [28] that every (left or right) A-module was projective. It implies [15, 17], however, that every A-module is (A, k)-projective, i.e. that every epimorphism of A-modules which is k-split, is also A-split. In order to avoid confusion, we will say that the k-algebra A is *semi-simple* [28] if it is an Artinian semi-simple ring, i.e. if any A-module is projective. By the terminology of [15] we call A a (left or right) *semi-simple extension* of k if any (left or right) A-module is (A, k)-projective.

Since the counit of a Hopf algebra H over a commutative ring k is a split epimorphism of k-modules, the Maschke theorem generalizes to this case in the following form [12, 20]. The extension $k \to H$ is separable if and only if it is (left and right) semi-simple and if and only if there exist normalized (left and right) integrals in H.

In this section we investigate the properties of the total algebra of a Hopf algebroid, as an extension of the base algebra, that are equivalent to the existence of normalized integrals *in* the Hopf algebroid. Dually, we investigate also the properties of the coring over the base algebra, underlying a Hopf algebroid, that are equivalent to the existence of normalized integrals *on* the Hopf algebroid (in any of the four possible senses).

A Maschke-type theorem on certain Hopf algebroids can be obtained also by the application of [37, Theorem 4.2]. Notice, however, that the Hopf algebroids occurring this way are only the Frobenius Hopf algebroids (discussed in Section 4 below), that is the Hopf algebroids possessing nondegenerate integrals (which are called Frobenius integrals in [37]).

The following Theorem 3.1 generalizes results from [12, Proposition 4.7] and [20, Theorem 3.3].

THEOREM 3.1 (Maschke Theorem for Hopf algebroids). *The following assertions on a Hopf algebroid $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ are equivalent*:

- (1.a) *The extension s_R: $R \to A$ is separable. That is, the multiplication map $A^R \otimes {}_R A \to A$ splits as an A–A bimodule map.*
- (1.b) *The extension t_R: $R^{\mathrm{op}} \to A$ is separable. That is, the multiplication map ${}^R A \otimes A_R \to A$ splits as an A–A bimodule map.*
- (1.c) *The extension s_L: $L \to A$ is separable. That is, the multiplication map $A^L \otimes {}_L A \to A$ splits as an A–A bimodule map.*
- (1.d) *The extension t_L: $L^{\mathrm{op}} \to A$ is separable. That is, the multiplication map ${}^L A \otimes A_L \to A$ splits as an A–A bimodule map.*
- (2.a) *The extension s_R: $R \to A$ is right semi-simple. That is, any right A-module is (A, R)-projective.*
- (2.b) *The extension t_R: $R^{\mathrm{op}} \to A$ is right semi-simple. That is, any right A-module is (A, R^{op})-projective.*
- (2.c) *The extension s_L: $L \to A$ is left semi-simple. That is, any left A-module is (A, L)-projective.*
- (2.d) *The extension t_L: $L^{\mathrm{op}} \to A$ is left semi-simple. That is, any left A-module is (A, L^{op})-projective.*
- (3.a) *There exists a normalized left integral in $\mathcal{A}$. That is, an element $\ell \in \mathcal{L}(\mathcal{A})$ such that $\pi_L(\ell) = 1_L$.*
- (3.b) *There exists a normalized right integral in $\mathcal{A}$. That is, an element $\wp \in \mathcal{R}(\mathcal{A})$ such that $\pi_R(\wp) = 1_R$.*
- (4.a) *The epimorphism π_R: $A \to R$ splits as a right A-module map.*
- (4.b) *The epimorphism π_L: $A \to L$ splits as a left A-module map.*

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Proof. (1.a) $\Rightarrow$ (2.a), (1.b) $\Rightarrow$ (2.b), (1.c) $\Rightarrow$ (2.c) and (1.d) $\Rightarrow$ (2.d) It is proven in [17, Proposition 2.6] that a separable extension is both left and right semi-simple.

(2.a) $\Rightarrow$ (4.a) ((2.b) $\Rightarrow$ (4.a)) The epimorphism π_R is split as a right (left) R-module map by s_R (by t_R), hence it is split as a right A-module map.

(4.a) $\Rightarrow$ (3.b) Let $\nu\colon\ R \to A$ be the right inverse of π_R in $\mathcal{M}_A$. Then $\wp := \nu(1_R)$ is a normalized right integral in $\mathcal{A}$.

(3.a) $\Leftrightarrow$ (3.b) By part (2) of Proposition 2.3 the antipode takes a normalized left/right integral to a normalized right/left integral.

(3.a) $\Rightarrow$ (1.a) and (3.b) $\Rightarrow$ (1.b) If ℓ is a normalized left integral in $\mathcal{A}$ then, by Scholium 2.8, the required right inverse of the multiplication map $A^R \otimes_R A \to A$ is given by the A–A bimodule map $a \mapsto a\ell^{(1)} \otimes S(\ell^{(2)}) \equiv \ell^{(1)} \otimes S(\ell^{(2)})a$. Similarly, if $\wp$ is a normalized right integral in $\mathcal{A}$ then the right inverse of the multiplication map ${}^R A \otimes A_R \to A$ is given by $a \mapsto aS(\wp_{(1)}) \otimes \wp_{(2)} \equiv S(\wp_{(1)}) \otimes \wp_{(2)} a$.

The proof is completed by applying the above arguments to the Hopf algebroid $\mathcal{A}^{\rm op}_{\rm cop}$. □

Let us make a comment on the semi-simplicity of the algebra A (cf. [17, Proposition 1.3]). If R is a semi-simple algebra and the equivalent conditions of Theorem 3.1 hold true, then A – being a semi-simple extension of a semi-simple algebra – is a semi-simple algebra. On the other hand, notice that condition (4.a) in Theorem 3.1 is equivalent to the projectivity of the right A-module R. Hence if A is a semi-simple k-algebra then the equivalent conditions of the theorem hold true. It is not true, however, that the semi-simplicity of the total algebra implies the semi-simplicity of the base algebra (which was shown by Lomp to be the case in Hopf algebras [20]). A counterexample can be constructed as follows: If B is a Frobenius algebra over a commutative ring k then $A := \mathrm{End}_k(B)$ has a Hopf algebroid structure over the base B [6]. If B is a Frobenius algebra over a field – which can be non-semi-simple! – then A is a Hopf algebroid with semi-simple total algebra.

The following Theorem 3.2 can be considered as a dual of Theorem 3.1 in the sense that it speaks about corings over the base algebras instead of algebra extensions. It is important to emphasize, however, that the two theorems are independent results. Even in the case of Hopf algebroids such that all module structures (2.16) are finitely generated and projective, the duals are not known to be Hopf algebroids.

Recall that the dual notion of that of a relative projective module is the relative injective comodule. Namely, a comodule M for an R-coring A is called (A, R)-*injective* [10, 18.18] if any monomorphism of A-comodules from M, which splits as an R-module map, splits also as an A-comodule map.

THEOREM 3.2 (Dual Maschke Theorem for Hopf algebroids). *The following assertions on a Hopf algebroid $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ are equivalent*:

(1.a) *The R-coring $({}^RA^R, \gamma_R, \pi_R)$ is coseparable. That is, the comultiplication $\gamma_R\colon A \to A^R \otimes {}^RA$ splits as an $\mathcal{A}_R$–$\mathcal{A}_R$ bicomodule map.*
(1.b) *The L-coring $({}_LA_L, \gamma_L, \pi_L)$ is coseparable. That is, the comultiplication $\gamma_L\colon A \to A_L \otimes {}_LA$ splits as an $\mathcal{A}_L$–$\mathcal{A}_L$ bicomodule map.*
(2.a) *Any right $\mathcal{A}_R$-comodule is $(\mathcal{A}_R, R)$-injective.*
(2.b) *Any left $\mathcal{A}_R$-comodule is $(\mathcal{A}_R, R)$-injective.*
(2.c) *Any left $\mathcal{A}_L$-comodule is $(\mathcal{A}_L, L)$-injective.*
(2.d) *Any right $\mathcal{A}_L$-comodule is $(\mathcal{A}_L, L)$-injective.*
(3.a) *There exists a normalized left s-integral on $\mathcal{A}$. That is, an element $\lambda^* \in \mathcal{L}(\mathcal{A}^*)$ such that $\lambda^*(1_A) = 1_R$.*
(3.b) *There exists a normalized left t-integral on $\mathcal{A}$. That is, an element ${}^*\lambda \in \mathcal{L}({}^*\mathcal{A})$ such that ${}^*\lambda(1_A) = 1_R$.*
(3.c) *There exists a normalized right s-integral on $\mathcal{A}$. That is, an element ${}_*\rho \in \mathcal{R}({}_*\mathcal{A})$ such that ${}_*\rho(1_A) = 1_L$.*
(3.d) *There exists a normalized right t-integral on $\mathcal{A}$. That is, an element $\rho_* \in \mathcal{R}(\mathcal{A}_*)$ such that $\rho_*(1_A) = 1_L$.*
(4.a) *The monomorphism $s_R\colon R \to A$ splits as a right $\mathcal{A}_R$-comodule map.*
(4.b) *The monomorphism $t_R\colon R \to A$ splits as a left $\mathcal{A}_R$-comodule map.*
(4.c) *The monomorphism $s_L\colon L \to A$ splits as a left $\mathcal{A}_L$-comodule map.*
(4.d) *The monomorphism $t_L\colon L \to A$ splits as a right $\mathcal{A}_L$-comodule map.*

Proof. (1.a) $\Rightarrow$ (2.a), (2.b) is proven in [10, 26.1].

(2.a) $\Rightarrow$ (4.a) ((2.b) $\Rightarrow$ (4.b)) The monomorphism s_R (t_R) is split as a right (left) R-module map by π_R hence it is split as a right (left) $\mathcal{A}_R$-comodule map.

(4.a) $\Rightarrow$ (3.a) and (4.b) $\Rightarrow$ (3.b) The left inverse λ^* of s_R in the category of right $\mathcal{A}_R$-comodules is a normalized s-integral on $\mathcal{A}_R$ by its very definition. Similarly, the left inverse ${}^*\lambda$ of t_R in the category of left $\mathcal{A}_R$-comodules is a normalized t-integral on $\mathcal{A}_R$.

(3.a) $\Rightarrow$ (3.b) If λ^* is a normalized s-integral on $\mathcal{A}_R$ then $\lambda^* \circ S$ is a normalized t-integral on $\mathcal{A}_R$ by Scholium 2.10.

(3.b) $\Rightarrow$ (1.a) In terms of the normalized t-integral ${}^*\lambda$ on $\mathcal{A}_R$ the required right inverse of the coproduct γ_R is constructed as the map

$$A^R \otimes {}^RA \to A, \quad a \otimes b \mapsto t_R \circ {}^*\lambda(aS(b_{(1)}))\, b_{(2)}.$$

It is checked to be an $\mathcal{A}_R$–$\mathcal{A}_R$ bicomodule map using that by Scholium 2.10, (4.b) and (1.c) we have $t_R \circ {}^*\lambda(aS(b_{(1)}))\, b_{(2)} = a^{(1)}\, s_R \circ \pi_R[t_R \circ {}^*\lambda(a^{(2)}S(b_{(1)}))\, b_{(2)}]$ for all a, b in A.

(3.a) $\Leftrightarrow$ (3.d) follows from Scholium 2.10, (2.b).
The remaining equivalences are proven by applying the above arguments to the Hopf algebroid $\mathcal{A}^{\mathrm{op}}_{\mathrm{cop}}$. □

The proofs of Theorem 3.1 and 3.2 can be unified if one formulates them as equivalent statements on the forgetful functors from the category of A-modules, and from the category of $\mathcal{A}_L$ or $\mathcal{A}_R$-comodules, respectively, to the category of L- or R-modules – as it is done in the case of Hopf algebras over commutative rings in [12]. We believe (together with the referee), however, that the above formulation in terms of algebra extensions and corings, respectively, is more appealing.

4. Frobenius Hopf Algebroids and Nondegenerate Integrals

A left or right integral ℓ in a Hopf algebra (H, Δ, ϵ, S) over a commutative ring k is called nondegenerate [19] if the maps

$$\mathrm{Hom}_k(H, k) \to H, \quad \phi \mapsto (\phi \otimes H) \circ \Delta(\ell) \quad \text{and}$$
$$\mathrm{Hom}_k(H, k) \to H, \quad \phi \mapsto (H \otimes \phi) \circ \Delta(\ell)$$

are bijective.

The notion of nondegenerate integrals is made relevant by the Larson–Sweedler Theorem [19] stating that a free and finite-dimensional bialgebra over a principal ideal domain is a Hopf algebra if and only if there exists a nondegenerate left integral in H.

The Larson–Sweedler Theorem has been extended by Pareigis [27] to Hopf algebras over commutative rings with trivial Picard group. He proved also that a bialgebra over an arbitrary commutative ring k, which is a Frobenius k-algebra, is a Hopf algebra if and only if there exists a Frobenius functional $\psi\colon H \to k$ satisfying

$$(H \otimes \psi) \circ \Delta = 1_H \psi(\_).$$

As a matter of fact, based on the results of [27] the following variant of [13, 3.2 Theorem 31] can be proven:

THEOREM 4.1. *The following properties of a Hopf algebra (H, Δ, ϵ, S) over a commutative ring k are equivalent*:

(1) *H is a Frobenius k-algebra.*
(2) *There exists a nondegenerate left integral in H.*
(3) *There exists a nondegenerate right integral in H.*
(4) *There exists a nondegenerate left integral on H. That is, a Frobenius functional $\psi\colon H \to k$ satisfying $(H \otimes \psi) \circ \Delta = 1_H \psi(\_)$.*
(5) *There exists a nondegenerate right integral on H. That is, a Frobenius functional $\psi\colon H \to k$ satisfying $(\psi \otimes H) \circ \Delta = 1_H \psi(\_)$.*

The main subject of the present section is the generalization of Theorem 4.1 to Hopf algebroids.

The most important tool in the proof of Theorem 4.1 is the Fundamental Theorem for Hopf modules [19]. A very general form of it has been proven by Brzeziński [7, Theorem 5.6], see also [10, 28.19] in the framework of corings. It can be applied in our setting as follows.

Hopf modules over bialgebroids are examples of Doi–Koppinen modules over algebras, studied in [8]. A left-left Hopf module over a left bialgebroid $\mathcal{A}_L = (A, L, s_L, t_L, \gamma_L, \pi_L)$ is a left comodule for the comonoid (A, γ_L, π_L) in the category of left A-modules. That is, a pair (M, τ) where M is a left A-module, hence a left L-module ${}_L M$ via s_L. The pair $({}_L M, \tau)$ is a left $\mathcal{A}_L$-comodule such that $\tau\colon M \to A_L \otimes {}_L M$ is a left A-module map to the module

$$a \cdot (b \otimes m) := a_{(1)} b \otimes a_{(2)} \cdot m \quad \text{for } a \in A,\ b \otimes m \in A_L \otimes {}_L M.$$

The right–right Hopf modules over a right bialgebroid $\mathcal{A}_R$ are the left–left Hopf modules over $(\mathcal{A}_R)^{\mathrm{op}}_{\mathrm{cop}}$.

It follows from [8, Proposition 4.1] that the left–left Hopf modules over $\mathcal{A}_L$ are the left comodules over the A-coring

$$\mathcal{W} := (A_L \otimes {}_L A, \gamma_L \otimes {}_L A, \pi_L \otimes {}_L A), \tag{4.1}$$

where the A–A bimodule structure is given by

$$a \cdot (b \otimes c) \cdot d := a_{(1)} b \otimes a_{(2)} cd \quad \text{for } a, d \in A,\ b \otimes c \in A_L \otimes {}_L A.$$

The coring (4.1) was studied in [2]. It was shown to possess a group-like element $1_A \otimes 1_A \in A_L \otimes {}_L A$ and corresponding coinvariant subalgebra $t_L(L)$ in A. The coring (4.1) is Galois (w.r.t. the group-like element $1_A \otimes 1_A$) if and only if $\mathcal{A}_L$ is a $\times_L$-Hopf algebra in the sense of [30]. Since in a Hopf algebroid $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ the left bialgebroid $\mathcal{A}_L$ is a $\times_L$-Hopf algebra by Proposition 2.4, the A-coring (4.1) is Galois in this case. Denote the category of left–left Hopf modules over $\mathcal{A}_L$ (i.e. of left comodules over the coring (4.1)) by ${}^{\mathcal{W}}\mathcal{M}$. The application of [7, Theorem 5.6] results that if $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ is a Hopf algebroid, such that the module ${}^L A$ is faithfully flat, then the functor

$$\begin{aligned} &G\colon {}^{\mathcal{W}}\mathcal{M} \to \mathcal{M}_L, \\ &(M, \tau) \mapsto \mathrm{Coinv}(M)_L := \{m \in M \mid \tau(m) = 1_A \otimes m \in A_L \otimes {}_L M\} \end{aligned} \tag{4.2}$$

(where the right L-module structure on $\mathrm{Coinv}(M)$ is given via t_L) and the induction functor

$$F\colon \mathcal{M}_L \to {}^{\mathcal{W}}\mathcal{M}, \quad N_L \mapsto ({}^L A \otimes N_L, \gamma_L \otimes N_L) \tag{4.3}$$

(where the left A-module structure on ${}^L A \otimes N_L$ is given by left multiplication in the first factor) are inverse equivalences.

In the case of Hopf algebras H over commutative rings k, these arguments lead to the Fundamental Theorem only for faithfully flat Hopf algebras. The proof of the Fundamental Theorem in [19], however, does not rely on any assumption on the k-module structure of H.

Since the Hopf algebroid structure is more restrictive than the $\times_L$-Hopf algebra structure, one hopes to prove the Fundamental Theorem for Hopf algebroids also under milder assumptions – using the whole strength of the Hopf algebroid structure.

THEOREM 4.2 (Fundamental Theorem for Hopf algebroids). *Let* $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ *be a Hopf algebroid and* $\mathcal{W}$ *the* A*-coring* (4.1). *The functors* $G\colon {}^{\mathcal{W}}\mathcal{M} \to \mathcal{M}_L$ *in* (4.2) *and* $F\colon \mathcal{M}_L \to {}^{\mathcal{W}}\mathcal{M}$ *in* (4.3) *are inverse equivalences.*

Proof. We construct the natural isomorphisms $\alpha\colon F\circ G \to {}^{\mathcal{W}}\mathcal{M}$ and $\beta\colon G\circ F \to \mathcal{M}_L$. The map

$$\alpha_M\colon {}^L A \otimes \mathrm{Coinv}(M)_L \to M, \quad a\otimes m \mapsto a\cdot m$$

is a left $\mathcal{W}$-comodule map and natural in M. The isomorphism property is proven by constructing the inverse

$$\alpha_M^{-1}\colon M \to {}^L A \otimes \mathrm{Coinv}(M)_L, \quad m \mapsto {m_{\langle -1\rangle}}^{(1)} \otimes S({m_{\langle -1\rangle}}^{(2)}) \cdot m_{\langle 0\rangle},$$

where we used the standard notation $\tau(m) = m_{\langle -1\rangle} \otimes m_{\langle 0\rangle}$. It requires some work to check that $\alpha_M^{-1}(m)$ belongs to ${}^L A \otimes \mathrm{Coinv}(M)_L$. Let us introduce the right L-submodule X of $A_L \otimes {}_L A_{\mathbf{L}} \otimes {}_{\mathbf{L}} M$ as

$$X := \left\{ \sum_i a_i \otimes b_i \otimes m_i \in A_L \otimes {}_L A_{\mathbf{L}} \otimes {}_{\mathbf{L}} M \;\middle|\; \sum_i a_i t_L(l) \otimes b_i \otimes m_i = \sum_i a_i \otimes b_i s_L(l) \otimes m_i \ \forall l \in L \right\}$$

with L-module structure $[\sum_i a_i \otimes b_i \otimes m_i]\cdot l := \sum_i a_i\, t_R\circ\pi_R\circ t_L(l)\otimes b_i \otimes m_i$, and the map

$$\omega\colon A_L \otimes {}_L A_{\mathbf{L}} \otimes {}_{\mathbf{L}} M \to M,$$
$$\sum_i a_i \otimes b_i \otimes m_i \mapsto \sum_i S[s_L\circ\pi_L(a_i)\, b_i]\cdot m_i.$$

Making M a right L-module via t_L, the restriction of ω becomes a right L-module map $X \to M_L$. The image of the map $\omega\circ(A_L\otimes\tau)\colon A_L \otimes {}_L M \to M$ lies in $\mathrm{Coinv}(M)$, since for any $a\otimes m \in A_L \otimes {}_L M$ we have

$$\begin{aligned}
&\tau\circ\omega\circ(A_L\otimes\tau)(a\otimes m)\\
&\quad = S({m_{\langle -1\rangle}}^{(2)}) m_{\langle 0\rangle\langle -1\rangle} \otimes S[s_L\circ\pi_L(a)\, {m_{\langle -1\rangle}}^{(1)}]\cdot m_{\langle 0\rangle\langle 0\rangle}\\
&\quad = s_R\circ\pi_R({m_{\langle -1\rangle}}^{(2)}) \otimes S[s_L\circ\pi_L(a)\, {m_{\langle -1\rangle}}^{(1)}]\cdot m_{\langle 0\rangle}\\
&\quad = 1_A \otimes \omega\circ(A_L\otimes\tau)(a\otimes m).
\end{aligned}$$

Since $\alpha_M^{-1} = [{}^L A \otimes \omega\circ(A_{\mathbf{L}}\otimes\tau)]\circ(\gamma_R\otimes_{\mathbf{L}} M)\circ\tau$, it follows that $\alpha_M^{-1}(m)$ belongs to ${}^L A\otimes\mathrm{Coinv}(M)_L$ for all $m\in M$, as stated.

The coinvariants of the left $\mathcal{W}$-comodule ${}^L A\otimes N_L$ are the elements of

$$\mathrm{Coinv}({}^L A\otimes N_L) = \left\{ \sum_i a_i\otimes n_i \in {}^L A\otimes N_L \;\middle|\; \sum_i a_i\otimes n_i = \sum_i s_R\circ\pi_R(a_i)\otimes n_i \right\},$$

hence the map

$$\beta_N\colon \operatorname{Coinv}({}^{L}A \otimes N_L) \to N,$$
$$\sum_i a_i \otimes n_i \mapsto \sum_i n_i \cdot \pi_L \circ S(a_i) \equiv \sum_i n_i \cdot \pi_L(a_i)$$

is a right L-module map and is natural in N. It is an isomorphism with inverse

$$\beta_N^{-1}\colon N \to \operatorname{Coinv}({}^{L}A \otimes N_L), \quad n \mapsto 1_A \otimes n. \qquad \square$$

An analogous result for right–right Hopf modules over $\mathcal{A}_R$ can be obtained by applying Theorem 4.2 to the Hopf algebroid $\mathcal{A}^{\mathrm{op}}_{\mathrm{cop}}$.

PROPOSITION 4.3. *Let $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ be a Hopf algebroid and (M, τ) a left–left Hopf module over $\mathcal{A}_L$. Then* Coinv(M) *is a k-direct summand of M.*

Proof. The canonical inclusion Coinv$(M) \to M$ is split by the k-module map

$$E_M\colon M \to \operatorname{Coinv}(M), \quad m \mapsto S(m_{\langle -1\rangle}) \cdot m_{\langle 0\rangle}. \tag{4.4}$$

$\square$

As the next step towards our goal, let us assume that $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ is a Hopf algebroid such that the module A^R – and hence by Lemma 2.6 also A_L – is finitely generated and projective. Under this assumption we are going to equip $\mathcal{A}^*$ with the structures of a left–left Hopf module over $\mathcal{A}_L$ and a right–right Hopf module over $\mathcal{A}_R$.

Let $\{b_i\} \subset A$ and $\{\beta_*^i\} \subset \mathcal{A}_*$ be dual bases for the module A_L. A left $\mathcal{A}_L$-comodule structure on $\mathcal{A}^*$ can be introduced via the L-module structure

$${}_L\mathcal{A}^*\colon l \cdot \phi^* := \phi^* \leftharpoonup S \circ s_L(l) \quad \text{for } l \in L,\ \phi^* \in \mathcal{A}^*$$

and the left coaction

$$\tau_L\colon \mathcal{A}^* \to A_L \otimes {}_L\mathcal{A}^*, \quad \phi^* \mapsto \sum_i b_i \otimes \chi^{-1}(\beta_*^i)\phi^*. \tag{4.5}$$

Similarly, a right $\mathcal{A}_R$-comodule structure on $\mathcal{A}^*$ can be introduced by the right R-module structure

$$\mathcal{A}^*_R\colon \phi^* \cdot r := \phi^* \leftharpoonup s_R(r) \quad \text{for } r \in R,\ \phi^* \in \mathcal{A}^*$$

and the right coaction

$$\tau_R\colon \mathcal{A}^* \to \mathcal{A}^*_R \otimes {}^{R}A, \quad \phi^* \mapsto \sum_i \chi^{-1}(\beta_*^i)\phi^* \otimes S(b_i), \tag{4.6}$$

where $\chi\colon \mathcal{A}^* \to \mathcal{A}_*$ is the algebra anti-isomorphism (2.22).

PROPOSITION 4.4. *Let $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ be a Hopf algebroid such that the module A^R is finitely generated and projective.*

(1) *Introduce the left A-module*

$${}_A\mathcal{A}^*\colon\ a\cdot\phi^* := \phi^* \leftharpoonup S(a) \quad \textit{for } a\in A,\ \phi^*\in\mathcal{A}^*.$$

Then $({}_A\mathcal{A}^*, \tau_L)$ *– where* τ_L *is the map* (4.5) *– is a left–left Hopf module over* $\mathcal{A}_L$.

(2) *Introduce the right A-module*

$$\mathcal{A}^*_A\colon\ \phi^*\cdot a := \phi^* \leftharpoonup a \quad \textit{for } a\in A,\ \phi^*\in\mathcal{A}^*.$$

Then $(\mathcal{A}^*_A, \tau_R)$ *– where* τ_R *is the map* (4.6) *– is a right–right Hopf module over* $\mathcal{A}_R$.

The coinvariants of both Hopf modules $({}_A\mathcal{A}^*, \tau_L)$ *and* $(\mathcal{A}^*_A, \tau_R)$ *are the elements of* $\mathcal{L}(\mathcal{A}^*)$.

Proof. (1) We have to show that τ_L is a left A-module map. That is, for all $a\in A$ and $\phi^*\in\mathcal{A}^*$,

$$\sum_i b_i\otimes\chi^{-1}(\beta^i_*)(\phi^*\leftharpoonup S(a)) = \sum_i a_{(1)}b_i\otimes(\chi^{-1}(\beta^i_*)\phi^*)\leftharpoonup S(a_{(2)}) \tag{4.7}$$

as elements of $A_L\otimes_L\mathcal{A}^*$. Since for any $\phi_*\in\mathcal{A}_*$ and $a\in A$,

$$\sum_i \chi^{-1}(\beta^i_*)\leftharpoonup S[s_L\circ\phi_*(a_{(1)}b_i)\,a_{(2)}] = \chi^{-1}(\phi_*)\leftharpoonup s_L\circ\pi_L(a),$$

the following identity holds true in $A_L\otimes_L\mathcal{A}^*$ for all $a\in A$:

$$\begin{aligned}
&\sum_i a_{(1)}b_i\otimes\chi^{-1}(\beta^i_*)\leftharpoonup S(a_{(2)})\\
&\quad=\sum_{i,j} t_L\circ\beta^j_*(a_{(1)}b_i)\,b_j\otimes\chi^{-1}(\beta^i_*)\leftharpoonup S(a_{(2)})\\
&\quad=\sum_{i,j} b_j\otimes\chi^{-1}(\beta^i_*)\leftharpoonup S[s_L\circ\beta^j_*(a_{(1)}b_i)\,a_{(2)}]\\
&\quad=\sum_j b_j\otimes\chi^{-1}(\beta^j_*)\leftharpoonup s_L\circ\pi_L(a).
\end{aligned}\tag{4.8}$$

Since for all $\phi^*,\psi^*\in\mathcal{A}^*$ and $a\in A$,

$$(\phi^*\psi^*)\leftharpoonup a = (\phi^*\leftharpoonup a^{(2)})(\psi^*\leftharpoonup a^{(1)}), \tag{4.9}$$

the identity (4.8) is equivalent to (4.7).

(2) We have to show that τ_R is a right A-module map. That is, for all $a\in A$ and $\phi^*\in\mathcal{A}^*$,

$$\sum_i \chi^{-1}(\beta^i_*)(\phi^*\leftharpoonup a)\otimes S(b_i) = \sum_i(\chi^{-1}(\beta^i_*)\phi^*)\leftharpoonup a^{(1)}\otimes S(b_i)a^{(2)} \tag{4.10}$$

as elements of $\mathcal{A}_R^* \otimes {}^R A$. Recall from the proof of Lemma 2.6 that the dual bases, $\{b_i\} \subset A$ and $\{\beta_*^i\} \subset \mathcal{A}_*$ for the module A_L, and the dual bases, $\{k_j\} \subset A$ and $\{\kappa_j^*\} \subset \mathcal{A}^*$ for A^R, are related to each other by

$$\sum_i b_i \otimes \beta_*^i = \sum_j k_{j(1)} \otimes \chi[s^* \circ \pi_R(k_{j(2)})\, \kappa_j^*] \quad \text{as elements of } A_L \otimes {}^L \mathcal{A}_*.$$

This implies that $\tau_R(\phi^*) = \sum_j s^* \circ \pi_R(k_{j(2)})\ \kappa_j^*\ \phi^* \otimes S(k_{j(1)})$. The following identity holds true in $\mathcal{A}_R^* \otimes {}^R A$ for all $a \in A$:

$$\begin{aligned}
&\sum_j [s^* \circ \pi_R(k_{j(2)})\, \kappa_j^*] \leftharpoonup a^{(1)} \otimes S(k_{j(1)}) a^{(2)} \\
&\quad = \sum_j s^* \circ \pi_R(a^{(1)}{}_{(2)} k_{j(2)})\, \kappa_j^* \otimes S(a^{(1)}{}_{(1)} k_{j(1)}) a^{(2)} \\
&\quad = \sum_j s^* \circ \pi_R[s_R \circ \pi_R(a_{(2)}{}^{(1)})\, k_{j(2)}] \kappa_j^* \otimes S(a_{(1)} k_{j(1)}) a_{(2)}{}^{(2)} \\
&\quad = \sum_j [s^* \circ \pi_R(k_{j(2)})\, \kappa_j^*] \leftharpoonup s_R \circ \pi_R(a_{(2)}{}^{(1)}) \otimes S(a_{(1)} k_{j(1)}) a_{(2)}{}^{(2)} \\
&\quad = \sum_j s^* \circ \pi_R(k_{j(2)})\, \kappa_j^* \otimes S(k_{j(1)})\, s_R \circ \pi_R(a) \\
&\quad = \sum_j [s^* \circ \pi_R(k_{j(2)})\, \kappa_j^*] \leftharpoonup t_R \circ \pi_R(a) \otimes S(k_{j(1)}). \qquad (4.11)
\end{aligned}$$

Here we used the identity $(s^*(r)\phi^*) \leftharpoonup a = s^*(r)\ (\phi^* \leftharpoonup a)$ for $r \in R$, $\phi^* \in \mathcal{A}^*$ and $a \in A$, the property of the dual bases $\sum_j k_j \otimes \kappa_j^* \leftharpoonup a = \sum_j a k^j \otimes \kappa_j^*$ for all $a \in A$ as elements of $A^R \otimes_R \mathcal{A}^*$, the right analogue of the bialgebroid axiom (2.6) and the Hopf algebroid axioms (2.19) and (2.20). In view of (4.9) the identity (4.11) is equivalent to (4.10).

In the cases of the Hopf modules $({}_A\mathcal{A}^*, \tau_L)$ and $(\mathcal{A}_A^*, \tau_R)$ a projection onto the coinvariants is given by the map (4.4) and its right–right version, respectively, both yielding

$$E_{\mathcal{A}^*} \colon \mathcal{A}^* \to \mathrm{Coinv}(\mathcal{A}^*), \quad \phi^* \mapsto \sum_i \chi^{-1}(\beta_*^i)\phi^* \leftharpoonup S^2(b_i). \qquad (4.12)$$

A left s-integral λ^* on $\mathcal{A}$ is a coinvariant, since it is an invariant of the left regular $\mathcal{A}^*$-module and so for all $a \in A$,

$$\begin{aligned}
E_{\mathcal{A}^*}(\lambda^*)(a) &= \sum_i \chi^{-1}(\beta_*^i)(1_A)\lambda^*(S^2(b_i)a) \\
&= \lambda^*[S^2(t_L \circ \beta_*^i(1_A)\, b_i)a] = \lambda^*(a).
\end{aligned}$$

On the other hand, for all $a \in A$,

$$\sum_i S(b_i)(a \leftharpoonup \beta_*^i) = S[t_L \circ \beta_*^i(a_{(1)})\, b_i] a_{(2)} = s_R \circ \pi_R(a), \qquad (4.13)$$

hence for all $\phi^* \in \mathcal{A}^*$,

$$\begin{aligned}
&E_{\mathcal{A}^*}(\phi^*) \rightharpoonup a \\
&\quad = \sum_i a^{(2)}\, t_R \circ \pi_R\{[\phi^* \rightharpoonup S^2(b_i)a^{(1)}] \leftharpoonup \beta_*^i\} \\
&\quad = \sum_i t_R \circ \pi_R \circ S^2(b_i^{(2)})\, a^{(2)}\, t_R \circ \pi_R\{[\phi^* \rightharpoonup S^2(b_i^{(1)})a^{(1)}] \leftharpoonup \beta_*^i\} \\
&\quad = \sum_i S(b_{i(2)})\, (S^2(b_{i(1)})a)^{(2)}\, t_R \circ \pi_R\{[\phi^* \rightharpoonup (S^2(b_{i(1)})a)^{(1)}] \leftharpoonup \beta_*^i\} \\
&\quad = \sum_{i,j} S(b_{i(2)})\{[\phi^* \rightharpoonup S^2(t_L \circ \beta_*^j(b_{i(1)})\, b_j)a] \leftharpoonup \beta_*^i\} \\
&\quad = \sum_{i,j} S(b_i \leftharpoonup \beta_*^j)\{[\phi^* \rightharpoonup S^2(b_j)a] \leftharpoonup \beta_*^i\} \\
&\quad = \sum_j s_R \circ \pi_R\{[\phi^* \rightharpoonup S^2(b_j)a] \leftharpoonup \beta_*^j\} = s_R \circ E_{\mathcal{A}^*}(\phi^*)(a).
\end{aligned}$$

That is, any coinvariant is an s-integral on $\mathcal{A}_R$. Here we used (4.12), the right analogue of (2.1), the identity $t_R \circ \pi_R \circ S^2 = S \circ s_R \circ \pi_R$, (2.20), the right analogue of (2.3), the identity $\gamma_R[(\phi^* \rightharpoonup a) \leftharpoonup \psi_*] = (\phi^* \rightharpoonup a^{(1)}) \leftharpoonup \psi_* \otimes a^{(2)}$, holding true for all $a \in A$, $\phi^* \in \mathcal{A}^*$ and $\psi_* \in \mathcal{A}_*$, the right L-linearity of the map $(\phi^* \rightharpoonup \_) \leftharpoonup \psi_*\colon A_L \to A_L$ and (4.13). □

The application of Theorem 4.2 to the Hopf modules of Proposition 4.4 results in isomorphisms

$$\alpha_L\colon {}^L A \otimes \mathcal{L}(\mathcal{A}^*)^L \to \mathcal{A}^*, \quad a \otimes \lambda^* \mapsto \lambda^* \leftharpoonup S(a) \quad \text{and} \tag{4.14}$$

$$\alpha_R\colon {}^R\mathcal{L}(\mathcal{A}^*) \otimes A_R \to \mathcal{A}^*, \quad \lambda^* \otimes a \mapsto \lambda^* \leftharpoonup a \tag{4.15}$$

of left–left Hopf modules over $\mathcal{A}_L$ and of right–right Hopf modules over $\mathcal{A}_R$, respectively. (The right L-module structure on $\mathcal{L}(\mathcal{A}^*)$ is given by $\lambda^* \cdot l := \lambda^* \leftharpoonup s_L(l)$ and the left R-module structure is given by $r \cdot \lambda^* := \lambda^* \leftharpoonup t_R(r)$ – see the explanation after (4.2).)

COROLLARY 4.5. *For a Hopf algebroid $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$, such that any of the modules A^R, ${}^RA$, ${}_LA$ and A_L is finitely generated and projective, there exist nonzero elements in all of $\mathcal{L}(\mathcal{A}^*)$, $\mathcal{L}({}^*\mathcal{A})$, $\mathcal{R}({}_*\mathcal{A})$ and $\mathcal{R}(\mathcal{A}_*)$.*

Proof. Suppose that the module A^R (equivalently, by Proposition 2.6 the module A_L) is finitely generated and projective. It follows from Proposition 4.4 and Theorem 4.2 that the map (4.14) is an isomorphism, hence there exist nonzero elements in $\mathcal{L}(\mathcal{A}^*)$.

For any element λ^* of $\mathcal{L}(\mathcal{A}^*)$, $\lambda^* \circ S$ is a (possibly zero) element of $\mathcal{L}({}^*\mathcal{A})$ by Scholium 2.10. Now we claim that it is excluded by the bijectivity of the map (4.14) that $\lambda^* \circ S = 0$ for all $\lambda^* \in \mathcal{L}(\mathcal{A}^*)$. For if so, then by the surjectivity of the map (4.14) we have $\phi^*(1_A) = 0$ for all $\phi^* \in \mathcal{A}^*$. But this is impossible, since $\pi_R(1_A) = 1_R$, by definition.

It follows from Scholium 2.10, (3.b) and (4.b) that also $\mathcal{R}({}_*\mathcal{A})$ and $\mathcal{R}(\mathcal{A}_*)$ must contain nonzero elements.

The case when the module ${}_L A$ (equivalently, by Proposition 2.6 the module ${}^R A$) is finitely generated and projective can be treated by applying the same arguments to the Hopf algebroid $\mathcal{A}^{\mathrm{op}}_{\mathrm{cop}}$. □

Since none of the duals of a Hopf algebroid is known to be a Hopf algebroid, it does not follow from Theorem 4.2, however, that for a Hopf algebroid, in which the total algebra is finitely generated and projective as a module over the base algebra, also $\mathcal{L}(A)$ and $\mathcal{R}(A)$ contain nonzero elements. At the moment we do not know under what necessary conditions the existence of nonzero integrals in a Hopf algebroid follows.

It is well known [27, Proposition 4] that the antipode of a finitely generated and projective Hopf algebra over a commutative ring is bijective. We do not know whether a result of the same strength holds true on Hopf algebroids. Our present understanding on this question is formulated in

PROPOSITION 4.6. *The following statements on a Hopf algebroid* $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ *are equivalent*:

(1) *The antipode* S *is bijective and any of the modules* ${}_L A$, A_L, A^R *and* ${}^R A$ *is finitely generated and projective.*
(2) *There exists an invariant* $\sum_k x_k \otimes \lambda^*_k$ *of the left* A*-module* ${}^R A \otimes \mathcal{L}(\mathcal{A}^*)^R$ *– defined via left multiplication in the first factor – with respect to* $\mathcal{A}_L$*, satisfying* $\sum_k \lambda^*_k(x_k) = 1_R$. *(The right* R*-module structure of* $\mathcal{L}(\mathcal{A}^*)$ *is defined by the restriction of the one of* $(\mathcal{A}^*)^R$*, i.e. as* $\lambda^* \cdot r := \lambda^*(\_\, t_R(r))$*.)*

Proof. For any invariant $\sum_k x_k \otimes \lambda^*_k$ of the left A-module ${}^R A \otimes \mathcal{L}(\mathcal{A}^*)^R$ and any element $a \in A$ the identities

$$\sum_k S(a) x_k^{(1)} \otimes x_k^{(2)} \otimes \lambda^*_k = \sum_k x_k^{(1)} \otimes a x_k^{(2)} \otimes \lambda^*_k \quad \text{and}$$
$$\sum_k a x_k^{(1)} \otimes S(x_k^{(2)}) \otimes \lambda^*_k = \sum_k x_k^{(1)} \otimes S(x_k^{(2)}) a \otimes \lambda^*_k$$

hold true as identities in ${}^R A^{\mathbf{R}} \otimes {}^{\mathbf{R}} A \otimes \mathcal{L}(\mathcal{A}^*)^R$ and in ${}^R A^{\mathbf{R}} \otimes {}_{\mathbf{R}} A \otimes \mathcal{L}(\mathcal{A}^*)^R$, respectively.

(2) $\Rightarrow$ (1) In terms of the invariant $\sum_k x_k \otimes \lambda^*_k$ the inverse of the antipode is constructed explicitly as

$$S^{-1}\colon A \to A, \quad a \mapsto \sum_k (\lambda^*_k \leftharpoondown a) \rightharpoonup x_k.$$

The dual bases $\{b_i\} \subset A$ and $\{{}^*\beta_i\} \subset {}^*\mathcal{A}$ for the module ${}^RA$ are introduced by the requirement that

$$\sum_i {}^*\beta_i \otimes b_i = \sum_k \lambda_k^*(S(\_)x_k^{(1)}) \otimes x_k^{(2)}$$

as elements of ${}^*\mathcal{A}_R \otimes {}^RA$. Together with Lemma 2.6 this proves the implication (2) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2) If S is bijective then in the case of the Hopf algebroid $\mathcal{A}_{\rm cop}$ the isomorphism (4.14) takes the form

$$\alpha_L^{\rm cop}\colon\ A^L \otimes {}^L\mathcal{L}({}^*\mathcal{A}) \to {}^*\mathcal{A}, \quad a \otimes {}^*\lambda \mapsto {}^*\lambda \leftharpoondown S^{-1}(a),$$

where the left L-module structure on $\mathcal{L}({}^*\mathcal{A})$ is defined by $l \cdot {}^*\lambda := {}^*\lambda \leftharpoondown t_L(l)$.

In terms of $\sum_k x_k \otimes {}^*\lambda_k := (\alpha_L^{\rm cop})^{-1}(\pi_R)$ the required invariant of ${}^RA \otimes \mathcal{L}(\mathcal{A}^*)^R$ is given by $\sum_k x_k \otimes {}^*\lambda_k \circ S^{-1}$. □

In any Hopf algebroid $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$, in which the module A_L is finitely generated and projective, the extensions $s_R\colon R \to A$ and $t_L\colon L^{\rm op} \to A$ satisfy the left depth two (or D2, for short) condition and the extensions $t_R\colon R^{\rm op} \to A$ and $s_L\colon L \to A$ satisfy the right D2 condition of [18]. If furthermore S is bijective then all the four extensions satisfy both the left and the right D2 conditions. This means [18, Lemma 3.7] in the case of $s_R\colon R \to A$, for example, the existence of finite sets (the so called D2 quasi-bases) $\{d_k\} \subset A^R \otimes {}_RA$, $\{\delta_k\} \subset {}_R\mathrm{End}_R({}_RA^R)$, $\{f_l\} \subset A^R \otimes {}_RA$ and $\{\phi_l\} \subset {}_R\mathrm{End}_R({}_RA^R)$ satisfying

$$\sum_k d_k \cdot m_A \circ (\delta_k \otimes {}_RA)(u) = u$$

and

$$\sum_l m_A \circ (A^R \otimes \phi_l)(u) \cdot f_l = u$$

for all elements u in $A^R \otimes {}_RA$, where the A–A bimodule structure on $A^R \otimes {}_RA$ is defined by left multiplication in the first factor and right multiplication in the second factor.

The D2 quasi-bases for the extension $s_R\colon R \to A$ can be constructed in terms of the invariants $\sum_i x_i \otimes \lambda_i^* := \alpha_L^{-1}(\pi_R)$ and $\sum_j x'_j \otimes {}^*\lambda'_j := (\alpha_L^{\rm cop})^{-1}(\pi_R)$ via the requirements that

$$\sum_k d_k \otimes \delta_k = \sum_i {x_i}_{(1)}{}^{(1)} \otimes S({x_i}_{(1)}{}^{(2)}) \otimes [\lambda_i^* \leftharpoonup S(x_{i(2)})] \rightharpoonup \_$$

and

$$\sum_l \phi_l \otimes f_l = \sum_j \_ \leftharpoonup [x'_{j(1)} \rightharpoonup \pi_L \circ s_R \circ {}^*\lambda'_j \circ S^{-1}] \otimes \otimes {x'_{j(2)}}^{(1)} \otimes S({x'_{j(2)}}^{(2)})$$

as elements of $A^R \otimes_R A^L \otimes_L [{}_R\mathrm{End}_R({}_RA^R)]$ and of $[{}_R\mathrm{End}_R({}_RA^R)]_L \otimes_L A^R \otimes_R A$, respectively. (The L–L bimodule structure on ${}_R\mathrm{End}_R({}_RA^R)$ is given by

$$l_1 \cdot \Psi \cdot l_2 = s_L(l_1)\Psi(\_)s_L(l_2) \quad \text{for } l_1, l_2 \in L,\ \Psi \in {}_R\mathrm{End}_R({}_RA^R).)$$

The D2 property of the extensions $t_R\colon\ R^{\mathrm{op}} \to A$, $s_L\colon\ L \to A$ and $t_L\colon\ L^{\mathrm{op}} \to A$ follows by applying these formulae to the Hopf algebroids $\mathcal{A}_{\mathrm{cop}}$, $\mathcal{A}^{\mathrm{op}}_{\mathrm{cop}}$ and $\mathcal{A}^{\mathrm{op}}$, respectively.

The following theorem, characterizing Frobenius Hopf algebroids $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ – that is, Hopf algebroids such that the extensions, given by the source and target maps of the bialgebroids $\mathcal{A}_L$ and $\mathcal{A}_R$, are Frobenius extensions – is the main result of this section.

Recall that for a homomorphism $s\colon\ R \to A$ of k-algebras the canonical R–A bimodule ${}_RA_A$ is a 1-cell in the additive bicategory of [k-algebras, bimodules, bimodule maps], possessing a right dual, the bimodule ${}_AA_R$. If A is finitely generated and projective as a left R-module, then ${}_RA_A$ possesses also a left dual, the bimodule ${}_A[{}_R\mathrm{Hom}(A, R)]_R$ defined as

$$a \cdot \phi \cdot r = \phi(\_\, a)r \quad \text{for } r \in R,\ a \in A,\ \phi \in {}_R\mathrm{Hom}(A, R).$$

A monomorphism of k-algebras $s\colon\ R \to A$ is called a *Frobenius extension* if the module ${}_RA$ is finitely generated and projective and the left and right duals

$${}_AA_R \quad \text{and} \quad {}_A[{}_R\mathrm{Hom}(A, R)]_R$$

of the bimodule ${}_RA_A$ are isomorphic. Equivalently, if A_R is finitely generated and projective and the left and right duals

$${}_RA_A \quad \text{and} \quad {}_R[\mathrm{Hom}_R(A, R)]_A$$

of the bimodule ${}_AA_R$ are isomorphic. This property holds if and only if there exists a *Frobenius system* $(\psi, \sum_i u_i \otimes v_i)$, where $\psi\colon\ A \to R$ is an R–R bimodule map and $\sum_i u_i \otimes v_i$ is an element of $A \underset{R}{\otimes} A$ such that

$$\sum_i s \circ \psi(au_i)\, v_i = a = \sum_i u_i\ s \circ \psi(v_i a) \quad \text{for all } a \in A.$$

THEOREM 4.7. *The following statements on a Hopf algebroid $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ are equivalent*:

(1.a) *The map $s_R\colon\ R \to A$ is a Frobenius extension of k-algebras.*
(1.b) *The map $t_R\colon\ R^{\mathrm{op}} \to A$ is a Frobenius extension of k-algebras.*
(1.c) *The map $s_L\colon\ L \to A$ is a Frobenius extension of k-algebras.*
(1.d) *The map $t_L\colon\ L^{\mathrm{op}} \to A$ is a Frobenius extension of k-algebras.*
(2.a) *The module A^R is finitely generated and projective and the module $\mathcal{L}(\mathcal{A}^*)^L$, defined by $\lambda^* \cdot l := \lambda^* \leftharpoonup s_L(l)$, is free of rank* 1.
(2.b) *S is bijective, the module ${}^RA$ is finitely generated and projective and the module ${}^L\mathcal{L}({}^*\mathcal{A})$, defined by $l \cdot {}^*\lambda := {}^*\lambda \leftharpoondown t_L(l)$, is free of rank* 1.

(2.c) *The module* $_LA$ *is finitely generated and projective and the module* $_R\mathcal{R}(_*\mathcal{A})$, *defined by* $r \cdot {}_*\rho := s_R(r) \rightharpoondown {}_*\rho$, *is free of rank* 1.

(2.d) *S is bijective, the module* A_L *is finitely generated and projective and the module* $\mathcal{R}(\mathcal{A}_*)_R$, *defined by* $\rho_* \cdot r := t_R(r) \rightharpoonup \rho_*$, *is free of rank* 1.

(3.a) *The module* A^R *is finitely generated and projective and there exists an element* $\lambda^* \in \mathcal{L}(\mathcal{A}^*)$ *such that the map*

$$\mathcal{F}\colon A \to \mathcal{A}^*, \quad a \mapsto \lambda^* \leftharpoonup a \tag{4.16}$$

is bijective.

(3.b) *S is bijective, the module* ^RA *is finitely generated and projective and there exists an element* $^*\lambda \in \mathcal{L}(^*\mathcal{A})$ *such that the map* $A \to {}^*\mathcal{A}$, $a \mapsto {}^*\lambda \leftharpoondown a$ *is bijective.*

(3.c) *The module* $_LA$ *is finitely generated and projective and there exists an element* $_*\rho \in \mathcal{R}(_*\mathcal{A})$ *such that the map* $A \to {}_*\mathcal{A}$, $a \mapsto a \rightharpoondown {}_*\rho$ *is bijective.*

(3.d) *S is bijective, the module* A_L *is finitely generated and projective and there exists an element* $\rho_* \in \mathcal{R}(\mathcal{A}_*)$ *such that the map* $A \to \mathcal{A}_*$, $a \mapsto a \rightharpoonup \rho_*$ *is bijective.*

(4.a) *There exists a left integral* $\ell \in \mathcal{L}(\mathcal{A})$ *such that the map*

$$\mathcal{F}^*\colon \mathcal{A}^* \to A, \quad \phi^* \mapsto \phi^* \rightharpoonup \ell \tag{4.17}$$

is bijective.

(4.b) *S is bijective and there exists a left integral* $\ell \in \mathcal{L}(\mathcal{A})$ *such that the map*

$$^*\mathcal{F}\colon {}^*\mathcal{A} \to A, \quad {}^*\phi \mapsto {}^*\phi \rightharpoondown \ell \tag{4.18}$$

is bijective.

(4.c) *There exists a right integral* $\wp \in \mathcal{R}(\mathcal{A})$ *such that the map* $_*\mathcal{A} \to A$, $_*\phi \mapsto \wp \leftharpoondown {}_*\phi$ *is bijective.*

(4.d) *S is bijective and there exists a right integral* $\wp \in \mathcal{R}(\mathcal{A})$ *such that the map* $\mathcal{A}_* \to A$, $\phi_* \mapsto \wp \leftharpoonup \phi_*$ *is bijective.*

In particular, the integrals λ^*, $^*\lambda$, $_*\rho$ *and* ρ_* *on* $\mathcal{A}$ *satisfying the condition in* (3.a), (3.b), (3.c) *and* (3.d), *respectively, are Frobenius functionals themselves for the extensions* $s_R\colon R \to A$, $t_R\colon R^{\mathrm{op}} \to A$, $s_L\colon L \to A$ *and* $t_L\colon L^{\mathrm{op}} \to A$, *respectively.*

What is more, under the equivalent conditions of the theorem the left integrals $\ell \in \mathcal{L}(\mathcal{A})$ *satisfying the conditions in* (4.a) *and* (4.b) *can be chosen to be equal, that is, to be a* nondegenerate *left integral in* $\mathcal{A}$. *Similarly, the right integrals* $\wp \in \mathcal{R}(\mathcal{A})$ *satisfying the conditions in* (4.c) *and* (4.d) *can be chosen to be equal, that is to be a* nondegenerate *right integral in* $\mathcal{A}$.

Proof. (4.a) $\Rightarrow$ (1.a) In terms of the left integral ℓ in (4.a) define $\lambda^* := \mathcal{F}^{*-1}(1_A) \in \mathcal{A}^*$. We claim that λ^* is a left s-integral on $\mathcal{A}$. The element $\ell \otimes \lambda^* \in {}^R\mathcal{L}(A) \otimes \mathcal{L}(\mathcal{A}^*)^R$ is an invariant of the left A-module $^RA \otimes \mathcal{L}(\mathcal{A}^*)^R$, hence by Proposition 4.6 the antipode is bijective and the modules A^R and ${}^RA$ are finitely generated and projective. Since for all $\phi^* \in \mathcal{A}^*$,

$$\phi^*\lambda^* = \mathcal{F}^{*-1}(\phi^* \rightharpoonup 1_A) = \mathcal{F}^{*-1}(s^* \circ \phi^*(1_A) \rightharpoonup 1_A) = s^* \circ \pi^*(\phi^*)\, \lambda^*,$$

λ^* is an s-integral on $\mathcal{A}_R$, so in particular an R–R bimodule map ${}_RA^R \to R$.

Since for all $a \in A$,

$$\ell^{(2)}\, t_R \circ \lambda^*(S(a)\ell^{(1)}) = a,$$

we have $\mathcal{F}^{*-1}(a) = \lambda^* \leftharpoonup S(a)$ hence $\ell^{(1)}\, s_R \circ \lambda^* \circ S(\ell^{(2)}) = 1_A$. A Frobenius system for the extension $s_R\colon R \to A$ is provided by $(\lambda^*, \ell^{(1)} \otimes S(\ell^{(2)}))$.

(1.a) $\Rightarrow$ (2.a) The module A^R is finitely generated and projective by assumption. In terms of a Frobenius system $(\psi, \sum_i u_i \otimes v_i)$ for the extension $s_R\colon R \to A$ one constructs an isomorphism of right L-modules as

$$\kappa\colon \mathcal{L}(\mathcal{A}^*) \to L, \quad \lambda^* \mapsto \pi_L\Big[\sum_i s_R \circ \lambda^*(u_i)v_i\Big] \tag{4.19}$$

with inverse

$$\kappa^{-1}\colon L \to \mathcal{L}(\mathcal{A}^*), \quad l \mapsto E_{\mathcal{A}^*}(\psi \leftharpoonup s_L(l)), \tag{4.20}$$

where $E_{\mathcal{A}^*}$ is the map (4.12). The right L-linearity of κ follows from the property of the Frobenius system $(\psi, \sum_i u_i \otimes v_i)$ that $\sum_i au_i \otimes v_i = \sum_i u_i \otimes v_i a$ for all $a \in A$, the bialgebroid axiom (2.5), and left R-linearity of the map $\lambda^*\colon {}_RA \to R$ and the right L-linearity of $\pi_L\colon {}_LA \to L$.

The maps κ and κ^{-1} are mutual inverses as

$$\begin{aligned}
&\kappa^{-1} \circ \kappa(\lambda^*)\\
&\quad = \sum_{i,j}[\chi^{-1}(\beta_*^j)\psi] \leftharpoonup s_L \circ \pi_L(s_R \circ \lambda^*(u_i)v_i)\; S^2(b_j)\\
&\quad = \sum_{i,j}[\chi^{-1}(\beta_*^j)\psi] \leftharpoonup S^2(b_j^{(2)})\; t_R \circ \pi_R[t_R \circ \pi_R \circ S(s_R \circ \lambda^*(u_i)v_i)\; S^2(b_j^{(1)})]\\
&\quad = \sum_{i,j}[\chi^{-1}(\beta_*^j)\psi] \leftharpoonup S^2(b_j^{(2)})\; s_L \circ \pi_L[S(b_j^{(1)})\; s_R \circ \lambda^*(u_i)v_i] = \lambda^*,
\end{aligned} \tag{4.21}$$

where in the first step we used (4.9), in the second step the fact that by Proposition 2.3 we have $s_L \circ \pi_L = t_R \circ \pi_R \circ S$, then the right analogue of (2.5) and finally in the last step the identity in ${}^R\mathcal{L}(\mathcal{A}^*) \otimes A_R$:

$$\begin{aligned}
&\sum_{i,j}[\chi^{-1}(\beta_*^j)\psi] \leftharpoonup S^2(b_j^{(2)}) \otimes S(b_j^{(1)})\; s_R \circ \lambda^*(u_i)v_i\\
&\quad = \alpha_R^{-1}\Big(\sum_i \psi \leftharpoonup s_R \circ \lambda^*(u_i)v_i\Big) = \lambda^* \otimes 1_A,
\end{aligned}$$

which follows from the explicit form of the inverse of the map (4.15). In a similar way, also

$$\begin{aligned}
\kappa\circ\kappa^{-1}(l) &= \sum_{i,j}\pi_L[s_R\circ(\chi^{-1}(\beta_*^j)\psi)(s_L(l)S^2(b_j)u_i)\ v_i]\\
&= \sum_{i,j}\pi_L[s_R\circ(\chi^{-1}(\beta_*^j)\ \psi)(s_L(l)u_i)\ v_i\ S^2(b_j)]\\
&= \sum_{i,j}\pi_L[s_R\circ(\chi^{-1}(\beta_*^j)\ \psi)(s_L(l)u_i)\ v_i\ t_L\circ\pi_L\circ S^2(b_j)]\\
&= \sum_{i,j}\pi_L[s_R\circ(\chi^{-1}(\beta_*^j)\ \psi)(s_L(l)\ t_L\circ\pi_L\circ S^2(b_j)\ u_i)\ v_i]\\
&= \sum_{i,j}\pi_L\{s_R\circ[(\chi^{-1}(\beta_*^j)\leftharpoondown t_L\circ\pi_L\circ S^2(b_j))\psi](s_L(l)u_i)\ v_i\}\\
&= l,
\end{aligned}$$

where in the last step we used that $\sum_j \chi^{-1}(\beta_*^j)\leftharpoondown t_L\circ\pi_L\circ S^2(b_j)=\chi^{-1}(\sum_j \beta_*^j\ t_*\circ\pi_L(b_j))=\pi_R$.

(2.a) $\Rightarrow$ (3.a) If $\kappa\colon\ \mathcal{L}(\mathcal{A}^*)^L\to L$ is an isomorphism of L-modules then $\pi_R\circ s_L\circ\kappa\colon\ {}^R\mathcal{L}(\mathcal{A}^*)\to R$ is an isomorphism of R-modules. Introduce the cyclic and separating generator $\lambda^*:=\kappa^{-1}(1_L)$ for the module $\mathcal{L}(\mathcal{A}^*)^L$. The map $\mathcal{F}$ in (4.16) is equal to $\alpha_R\circ(\kappa^{-1}\circ\pi_L\circ t_R\otimes A_R)$ – where α_R is the isomorphism (4.15) – hence bijective.

(3.a) $\Rightarrow$ (4.a), (4.b) A Frobenius system for the extension $s_R\colon\ R\to A$ is given in terms of the dual bases $\{b_i\}\subset A$ and $\{\beta_i^*\}\subset\mathcal{A}^*$ for the module A^R as $(\lambda^*,\sum_i b_i\otimes\mathcal{F}^{-1}(\beta_i^*))$.

The element $\ell:=\sum_i b_i\ t_L\circ\pi_L\circ\mathcal{F}^{-1}(\beta_i^*)$ is a left integral in $\mathcal{A}$. Using the identities

$$\begin{aligned}
\lambda^*\rightharpoonup\ell &= s_R\circ\lambda^*\left[\sum_i b_i\ t_L\circ\pi_L\circ\mathcal{F}^{-1}(\beta_i^*)\right]\\
&= t_L\circ\pi_L\left[\sum_i s_R\circ\lambda^*(b_i)\ \mathcal{F}^{-1}(\beta_i^*)\right]=1_A,\\
\ell^{(1)}\otimes S(\ell^{(2)}) &= \sum_i b_i\ s_R\circ\lambda^*[\mathcal{F}^{-1}(\beta_i^*)\ell^{(1)}]\otimes S(\ell^{(2)})\\
&= \sum_i b_i\otimes S[\ell^{(2)}\ t_R\circ\lambda^*(\ell^{(1)})]\mathcal{F}^{-1}(\beta_i^*)\\
&= \sum_i b_i\otimes\mathcal{F}^{-1}(\beta_i^*)
\end{aligned}$$

one checks that the inverse of the map $\mathcal{F}^*$ in (4.17) is given by $\mathcal{F}\circ S$. This implies, in particular, that S is bijective.

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The inverse of the map ${}^*\mathcal{F}$ in (4.18) – defined in terms of the same left integral ℓ – is the map

$$A \to {}^*\mathcal{A}, \quad a \mapsto \lambda^* \circ S \leftharpoondown S^{-1}(a).$$

(1.a) $\Leftrightarrow$ (1.d) The datum $(\psi, \sum_i u_i \otimes v_i)$ is a Frobenius system for the extension $s_R\colon R \to A$ if and only if $(\pi_L \circ s_R \circ \psi, \sum_i u_i \otimes v_i)$ is a Frobenius system for $t_L\colon L^{\mathrm{op}} \to A$, where $\pi_L \circ s_R\colon R \to L^{\mathrm{op}}$ was claimed to be an isomorphism of k-algebras in part (1) of Proposition 2.3.

(1.a) $\Rightarrow$ (1.c) We have already seen that (1.a) $\Rightarrow$ (3.a) $\Rightarrow$ S is bijective. If the datum $(\psi, \sum_i u_i \otimes v_i)$ is a Frobenius system for the extension $s_R\colon R \to A$ then $(\pi_L \circ s_R \circ \psi \circ S^{-1}, S(v_i) \otimes S(u_i))$ is a Frobenius system for $s_L\colon L \to A$.

(4.c) $\Rightarrow$ (1.c) $\Rightarrow$ (2.c) $\Rightarrow$ (3.c) $\Rightarrow$ (4.c), (1.c) $\Leftrightarrow$ (1.b) and (1.c) $\Rightarrow$ (1.a) follow by applying (4.a) $\Rightarrow$ (1.a) $\Rightarrow$ (2.a) $\Rightarrow$ (3.a) $\Rightarrow$ (4.a), (1.a) $\Leftrightarrow$ (1.d) and (1.a) $\Rightarrow$ (1.c) to the Hopf algebroid $\mathcal{A}^{\mathrm{op}}_{\mathrm{cop}}$.

(1.b) $\Rightarrow$ (2.b) $\Rightarrow$ (3.b) $\Rightarrow$ (4.b) $\Rightarrow$ (1.b) We have seen that (1.b) $\Leftrightarrow$ (1.c) $\Rightarrow$ S is bijective. Hence we can apply (1.a) $\Rightarrow$ (2.a) $\Rightarrow$ (3.a) $\Rightarrow$ (4.a) $\Rightarrow$ (1.a) to the Hopf algebroid $\mathcal{A}_{\mathrm{cop}}$.

(1.d) $\Rightarrow$ (2.d) $\Rightarrow$ (3.d) $\Rightarrow$ (4.d) $\Rightarrow$ (1.d) follows by applying (1.b) $\Rightarrow$ (2.b) $\Rightarrow$ (3.b) $\Rightarrow$ (4.b) $\Rightarrow$ (1.b) to the Hopf algebroid $\mathcal{A}^{\mathrm{op}}_{\mathrm{cop}}$. □

It is proven in [5, Theorem 5.17] that under the equivalent conditions of Theorem 4.7 the duals, $\mathcal{A}^*$, ${}^*\mathcal{A}$, ${}_*\mathcal{A}$ and $\mathcal{A}_*$ of the Hopf algebroid $\mathcal{A}$, possess (anti-) isomorphic Hopf algebroid structures.

The Hopf algebroids, satisfying the equivalent conditions of Theorem 4.7, provide examples of distributive Frobenius double algebras [37]. (Notice that the integrals, which we call nondegenerate, are called Frobenius integrals in [37].)

Our result naturally raises the question, under what conditions on the base algebra the equivalent conditions of Theorem 4.7 hold true. That is, what is the generalization of Pareigis' condition – the triviality of the Picard group of the commutative base ring of a Hopf algebra – to the noncommutative base algebra of a Hopf algebroid. We are going to return to this problem in a different publication.

5. The Quasi-Frobenius Property

It is known [27, Theorem added in proof], that any finitely generated projective Hopf algebra over a commutative ring k is (both a left and a right) quasi-Frobenius extension of k in the sense of [23]. In this section we examine in what Hopf algebroids the total algebra is (a left or a right) quasi-Frobenius extension of the base algebra.

The quasi-Frobenius property of an extension $s\colon R \to A$ of k-algebras has been introduced by Müller [23] as a weakening of the Frobenius property (see the paragraph preceding Theorem 4.7). The extension $s\colon R \to A$ is *left quasi-Frobenius* (or left QF, for short) if the module ${}_RA$ is finitely generated and projective (hence the bimodule ${}_R A_A$ possesses both a right dual ${}_A A_R$ and a left dual ${}_A[{}_R\mathrm{Hom}(A, R)]_R$) and the bimodule ${}_A A_R$ is a direct summand in a finite direct sum of copies of ${}_A[{}_R\mathrm{Hom}(A, R)]_R$.

The extension $s\colon R \to A$ is *right QF* if s, considered as a map $R^{\mathrm{op}} \to A^{\mathrm{op}}$, is a left QF extension. That is, if the module A_R is finitely generated and projective and the left dual bimodule ${}_R A_A$ is a direct summand in a finite direct sum of copies of the right dual bimodule ${}_R[\mathrm{Hom}_R(A, R)]_A$.

To our knowledge it is not known whether the notions of left and right QF extensions are equivalent (except in particular cases, such as central extensions, where the answer turns out to be affirmative [29]; and Frobenius extensions, which are also both left and right QF [23]).

A powerful characterization of a Frobenius extension $s\colon R \to A$ is the existence of a Frobenius system – see the paragraph preceding Theorem 4.7. In the following lemma a generalization to quasi-Frobenius extensions is introduced:

LEMMA 5.1.

(1) *An algebra extension $s\colon R \to A$ is left QF if and only if the module ${}_R A$ is finitely generated and projective and there exist finite sets $\{\psi_k\} \subset {}_R\mathrm{Hom}_R(A, R)$ and $\{\sum_i u_i^k \otimes v_i^k\} \subset A \underset{R}{\otimes} A$ satisfying*

$$\sum_{i,k} u_i^k\, s \circ \psi_k(v_i^k) = 1_A$$

and

$$\sum_{i,k} a u_i^k \otimes v_i^k = u_i^k \otimes v_i^k a \quad \textit{for all } a \in A.$$

The datum $\{\psi_k, \sum_i u_i^k \otimes v_i^k\}$ is called a left QF-system *for the extension $s\colon R \to A$.*

(2) *An algebra extension $s\colon R \to A$ is right QF if and only if the module A_R is finitely generated and projective and there exist finite sets $\{\psi_k\} \subset {}_R\mathrm{Hom}_R(A, R)$ and $\{\sum_i u_i^k \otimes v_i^k\} \subset A \underset{R}{\otimes} A$ satisfying*

$$\sum_{i,k} s \circ \psi_k(u_i^k)\, v_i^k = 1_A$$

and

$$\sum_{i,k} a u_i^k \otimes v_i^k = u_i^k \otimes v_i^k a \quad \textit{for all } a \in A.$$

The datum $\{\psi_k, \sum_i u_i^k \otimes v_i^k\}$ is called a right QF-system *for the extension $s\colon R \to A$.*

Proof. Let us spell out the proof in the case (1). Suppose that there exists a left QF system $\{\psi_k, \sum_i u_i^k \otimes v_i^k\}$ for the extension $s\colon R \to A$. The bimodule ${}_A A_R$

is a direct summand in a finite direct sum of copies of ${}_A[{}_R\mathrm{Hom}(A, R)]_R$ by the existence of A–R bimodule maps

$$\Phi_k\colon\ {}_R\mathrm{Hom}(A, R) \to A, \quad \phi \mapsto \sum_i u_i^k\, s \circ \phi(v_i^k) \quad \text{and}$$

$$\Phi'_k\colon\ A \to {}_R\mathrm{Hom}(A, R), \quad a \mapsto \psi_k(\,\_\, a)$$

satisfying $\sum_k \Phi_k \circ \Phi'_k = A$.

Conversely, in terms of the A–R bimodule maps $\{\Phi_k\colon\ {}_R\mathrm{Hom}(A, R) \to A\}$ and $\{\Phi'_k\colon\ A \to {}_R\mathrm{Hom}(A, R)\}$, satisfying $\sum_k \Phi_k \circ \Phi'_k = A$, and the dual bases, $\{b_j\} \subset A$ and $\{\beta_j\} \subset {}_R\mathrm{Hom}(A, R)$ for the module ${}_RA$, a left QF system can be constructed as

$$\psi_k := \Phi'_k(1_A) \in {}_R\mathrm{Hom}_R(A, R)$$

and

$$\sum_i u_i^k \otimes v_i^k := \sum_j \Phi_k(\beta_j) \otimes b_j \in A \underset{R}{\otimes} A. \qquad \square$$

Lemma 5.1 implies, in particular, that for a left/right QF extension $R \to A$, A is finitely generated and projective also as a right/left R-module.

THEOREM 5.2. *The following properties of a Hopf algebroid* $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ *are equivalent*:

(1.a) $s_R\colon\ R \to A$ *is a left QF extension.*
(1.b) $t_L\colon\ L^{\mathrm{op}} \to A$ *is a left QF extension.*
(1.c) *The modules* A^R *and* $\mathcal{L}(\mathcal{A}^*)^L$ *– defined by* $\lambda^* \cdot l := \lambda^* \leftharpoonup s_L(l)$ *– are finitely generated and projective.*
(1.d) *The module* A^R *is finitely generated and projective and the module* $\mathcal{L}(\mathcal{A}^*)^L$ *is flat.*
(1.e) *The module* A^R *is finitely generated and projective and the invariants of the left* A*-module* ${}^LA \otimes \mathcal{L}(\mathcal{A}^*)^L$ *– defined via left multiplication in the first factor – with respect to* $\mathcal{A}_L$ *are the elements of* ${}^L\mathcal{L}(\mathcal{A}) \otimes \mathcal{L}(\mathcal{A}^*)^L$.
(1.f) *There exist finite sets* $\{\ell_k\} \subset \mathcal{L}(\mathcal{A})$ *and* $\{\lambda_k^*\} \subset \mathcal{L}(\mathcal{A}^*)$ *satisfying* $\sum_k \lambda_k^* \circ S(\ell_k) = 1_R$.
(1.g) *The left* A*-module* ${}_A\mathcal{A}^*$ *– defined by* $a \cdot \phi^* := \phi^* \leftharpoonup S(a)$ *– is finitely generated and projective with generator set* $\{\lambda_k^*\} \subset \mathcal{L}(\mathcal{A}^*)$.

The following properties of $\mathcal{A}$ *are also equivalent*:

(2.a) $s_L\colon\ L \to A$ *is a right QF extension.*
(2.b) $t_R\colon\ R^{\mathrm{op}} \to A$ *is a right QF extension.*
(2.c) *The modules* ${}_LA$ *and* ${}_R\mathcal{R}({}_*\mathcal{A})$ *– defined by* $r \cdot {}_*\rho := s_R(r) \rightharpoondown {}_*\rho$ *– are finitely generated and projective.*
(2.d) *The module* ${}_LA$ *is finitely generated and projective and the module* ${}_R\mathcal{R}({}_*\mathcal{A})$ *is flat.*

(2.e) *The module* ${}_L A$ *is finitely generated and projective and the invariants of the right A-module* ${}_R\mathcal{R}({}_*\mathcal{A}) \otimes A_R$ *– defined via right multiplication in the second factor – with respect to* $\mathcal{A}_R$ *are the elements of* ${}_R\mathcal{R}({}_*\mathcal{A}) \otimes \mathcal{R}(\mathcal{A})_R$.

(2.f) *There exist finite sets* $\{\wp_k\} \subset \mathcal{R}(\mathcal{A})$ *and* $\{{}_*\rho_k\} \subset \mathcal{R}({}_*\mathcal{A})$ *satisfying* $\sum_k {}_*\rho_k \circ S(\wp_k) = 1_L$.

(2.g) *The right A-module* ${}_*\mathcal{A}_A$ *– defined by* ${}_*\phi \cdot a := S(a) \rightharpoondown {}_*\phi$ *– is finitely generated and projective with generator set* $\{{}_*\rho_k\} \subset \mathcal{R}({}_*\mathcal{A})$.

If furthermore the antipode is bijective, then the conditions (1.a)–(1.g) *and* (2.a)–(2.g) *are equivalent to each other and also to*

(1.h) *The left* ${}^*\mathcal{A}$*-module on A – defined by* ${}^*\phi \cdot a := {}^*\phi \rightharpoondown a$ *– is finitely generated and projective with generator set* $\{\ell_k\} \in \mathcal{L}(\mathcal{A})$.

(2.h) *The right* $\mathcal{A}_*$*-module on A – defined by* $a \cdot \phi_* := a \leftharpoonup \phi_*$ *– is finitely generated and projective with generator set* $\{\wp_k\} \in \mathcal{R}(\mathcal{A})$.

Proof. (1.a) $\Leftrightarrow$ (1.b) It follows from part (1) of Proposition 2.3 that the module A_L is finitely generated and projective if and only if ${}_R A$ is, and the datum $\{\psi_k, \sum_i u_i^k \otimes v_i^k\}$ is a left QF system for the extension $s_R\colon R \to A$ if and only if $\{\pi_L \circ s_R \circ \psi_k, \sum_i u_i^k \otimes v_i^k\}$ is a left QF system for $t_L\colon L^{\mathrm{op}} \to A$.

(1.a) $\Rightarrow$ (1.c) The module A^R is finitely generated and projective by Lemma 5.1. In terms of the left QF system, $\{\psi_k, \sum_i u_i^k \otimes v_i^k\}$ for the extension $s_R\colon R \to A$, the dual bases for the module $\mathcal{L}(\mathcal{A}^*)^L$ are given with the help of the map (4.12) as $\{E_{\mathcal{A}^*}(\psi_k)\} \subset \mathcal{L}(\mathcal{A}^*)$ and $\{\kappa_k := \pi_L[\sum_i s_R \circ \_ (u_i^k)\, v_i^k]\} \subset \mathrm{Hom}_L(\mathcal{L}(\mathcal{A}^*)^L, L)$.

The right L-linearity of the maps $\kappa_k\colon \mathcal{L}(\mathcal{A}^*) \to L$ is checked similarly to the right L-linearity of the map (4.19). Notice that for any R–R bimodule map $\psi\colon {}_R A^R \to R$ we have

$$\begin{aligned}
E_{\mathcal{A}^*}(\psi) \leftharpoonup s_L(l) &= \sum_j [\chi^{-1}(\beta_*^j)\psi] \leftharpoonup s_L(l) S^2(b_j) \\
&= \sum_j [\chi^{-1}(t_* \circ \pi_L \circ t_R \circ \pi_R \circ t_L(l)\ \beta_*^j)\psi] \leftharpoonup S^2(b_j) \\
&= \sum_j [\chi^{-1}(\beta_*^j)\ t^* \circ \pi_R \circ t_L(l)\ \psi] \leftharpoonup S^2(b_j) \\
&= \sum_j [\chi^{-1}(\beta_*^j)\ s^* \circ \pi_R \circ t_L(l)\ \psi] \leftharpoonup S^2(b_j) \\
&= \sum_j [\chi^{-1}(s_* \circ \pi_L \circ t_R \circ \pi_R \circ t_L(l)\ \beta_*^j)\ \psi] \leftharpoonup S^2(b_j) \\
&= E_{\mathcal{A}^*}(\psi \leftharpoonup s_L(l))
\end{aligned}$$

for all $l \in L$, where in the first step we used (4.12) and (4.9), in the second step the property of the dual bases $\{b_j\} \subset A$ and $\{\beta_*^j\} \subset \mathcal{A}_*$ that $\sum_j \beta_*^j \otimes s_L(l) b_j = \sum_j t_*(l)\beta_*^j \otimes b_j$ for all $l \in L$ as elements of ${}^L\mathcal{A}_* \otimes A_L$, in the third step the identity

$\chi^{-1} \circ t_* = t^* \circ \pi_R \circ s_L$, in the fourth step the fact that by the left R-linearity of ψ we have $t^*(r)\psi = s^*(r)\psi$ for all $r \in R$, in the fifth step $\chi^{-1} \circ s_* = s^* \circ \pi_R \circ s_L$, and finally $\sum_j \beta_*^j \otimes b_j s_L(l) = \sum_j s_*(l)\beta_*^j \otimes b_j$, holding true for all $l \in L$ as an identity in ${}^{L}\mathcal{A}_* \otimes A_L$.

The dual basis property of the sets $\{E_{\mathcal{A}^*}(\psi_k)\}$ and $\{\kappa_k\}$ is verified by the property that $\sum_{i,k} E_{\mathcal{A}^*}(\psi_k \leftharpoonup s_L \circ \kappa_k(\lambda^*)) = \lambda^*$ for all $\lambda^* \in \mathcal{L}(\mathcal{A}^*)$, which is checked similarly to (4.21).

(1.c) $\Rightarrow$ (1.d) is a standard result.

(1.d) $\Rightarrow$ (1.e) If the module A^R – equivalently, by Lemma 2.6 the module A_L – is finitely generated and projective then the invariants of any left A-module M with respect to $\mathcal{A}_L$ are the elements of the kernel of the map

$$\zeta_M \colon M \to {}^{L}\mathcal{A}_* \otimes M_L, \quad m \mapsto \left(\sum_i \beta_*^i \otimes b_i \cdot m \right) - \pi_L \otimes m,$$

where the right L module M_L is defined via t_L, and the sets $\{b_i\} \subset A$ and $\{\beta_*^i\} \subset \mathcal{A}_*$ are dual bases for the module A_L.

The map ζ_A, corresponding to the left regular A-module, is a left L-module map ${}^{L}A \to {}^{\mathbf{L}}\mathcal{A}^* \otimes {}^{L}A_{\mathbf{L}}$ and $\zeta_{{}^{L}A \otimes \mathcal{L}(\mathcal{A}^*)^L} = \zeta_A \otimes \mathcal{L}(\mathcal{A}^*)^L$. Since tensoring with $\mathcal{L}(\mathcal{A}^*)^L$ is an exact functor by assumption, it preserves the kernels, that is the invariants in this case.

(1.e) $\Rightarrow$ (1.f) With the help of the map (4.14) introduce

$$\sum_k \ell_k \otimes \lambda_k^* := \alpha_L^{-1}(\pi_R) \in \mathrm{Inv}({}^{L}A \otimes \mathcal{L}(\mathcal{A}^*)^L) \equiv {}^{L}\mathcal{L}(\mathcal{A}) \otimes \mathcal{L}(\mathcal{A}^*)^L.$$

It satisfies $\sum_k \lambda_k^* \circ S(\ell_k) = \alpha_L \circ \alpha_L^{-1}(\pi_R)(1_A) = 1_R$.

(1.f) $\Rightarrow$ (1.a) In terms of the sets $\{\ell_k\} \subset \mathcal{L}(\mathcal{A})$ and $\{\lambda_k^*\} \subset \mathcal{L}(\mathcal{A}^*)$ a left QF system for the extension $s_R \colon R \to A$ can be constructed as $\{\lambda_k^*, \ell_k^{(1)} \otimes S(\ell_k^{(2)})\}$.

The module A^R is finitely generated and projective since there exist dual bases $\{b_i\} \subset A$ and $\{\beta_i^*\} \subset \mathcal{A}^*$ defined by $\sum_i b_i \otimes \beta_i^* = \sum_k \ell_k^{(1)} \otimes \lambda_k^*[S(\ell_k^{(2)})\ \_\ ]$, as elements of $A^R \otimes {}_R\mathcal{A}^*$. The module A_L is finitely generated and projective by Lemma 2.6, hence so is ${}_RA$.

(1.f) $\Rightarrow$ (1.g) In terms of the sets $\{\ell_k\} \subset \mathcal{L}(\mathcal{A})$ and $\{\lambda_k^*\} \subset \mathcal{L}(\mathcal{A}^*)$ the dual bases for the module ${}_A\mathcal{A}^*$ are given by $\{\lambda_k^*\} \subset \mathcal{L}(\mathcal{A}^*)$ and $\{\ \_ \rightharpoonup \ell_k\} \subset {}_A\mathrm{Hom}({}_A\mathcal{A}^*, A)$.

(1.g) $\Rightarrow$ (1.f) In terms of the dual bases $\{\lambda_k^*\} \subset \mathcal{L}(\mathcal{A}^*)$ and $\{\Xi_k\} \subset {}_A\mathrm{Hom}({}_A\mathcal{A}^*, A)$ one defines the required left integrals $\ell_k := \Xi_k(\pi_R)$ in $\mathcal{A}$.

The equivalence of the conditions (2.a)–(2.g) follows by applying the above results to the Hopf algebroid $\mathcal{A}_{\mathrm{cop}}^{\mathrm{op}}$.

Now assume that S is bijective. Then

(1.f) $\Leftrightarrow$ (2.f) follows from Scholium 2.10.

(1.f) $\Rightarrow$ (1.h) Scholium 2.8, (1.b) and Scholium 2.10, (3.c) can be used to show that in terms of the sets $\{\ell_k\} \subset \mathcal{L}(\mathcal{A})$ and $\{\lambda_k^*\} \subset \mathcal{L}(\mathcal{A}^*)$ the dual bases for the

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left ${}^*\mathcal{A}$-module on A are given by $\{\ell_k\} \subset \mathcal{L}(\mathcal{A})$ and $\{\lambda_k^* \circ S \leftharpoonup S^{-1}(\_)\} \subset {}_{{}^*\mathcal{A}}\mathrm{Hom}(A, {}^*\mathcal{A})$.

(1.h) $\Rightarrow$ (1.f) Let $\{\ell_k\} \subset \mathcal{L}(A)$ and $\{\chi_k\} \subset {}_{{}^*\mathcal{A}}\mathrm{Hom}(A, {}^*\mathcal{A})$ be dual bases for the left ${}^*\mathcal{A}$-module A. Since for all $a \in A$ we have $\sum_k \ell_k^{(1)} s_R \circ \chi_k(a)(\ell^{(2)}) = a$, the module A^R, and hence by Proposition 2.6 also ${}^R A$, is finitely generated and projective. For any value of the index k the element $\chi_k(1_A)$ is an invariant of the left regular ${}^*\mathcal{A}$-module, hence a t-integral on $\mathcal{A}_R$. By Scholium 2.10 the elements $\lambda_k^* := \chi_k(1_A) \circ S^{-1}$ are s-integrals on $\mathcal{A}_R$, satisfying

$$\sum_k \lambda_k^* \circ S(\ell_k) = \pi_R\left[\sum_k \chi_k(1_A) \rightharpoonup \ell_k\right] = 1_R.$$

(2.f) $\Leftrightarrow$ (2.h) follows by applying (1.f) $\Leftrightarrow$ (1.h) to the Hopf algebroid $\mathcal{A}^{\mathrm{op}}_{\mathrm{cop}}$. □

If the antipode of a Hopf algebroid $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ is bijective then the application of Theorem 5.2 to the Hopf algebroid $\mathcal{A}^{\mathrm{op}}$ results in equivalent conditions under which the extensions $s_R\colon R \to A$ and $t_L\colon L^{\mathrm{op}} \to A$ are right QF, and $s_L\colon L \to A$ and $t_R\colon R^{\mathrm{op}} \to A$ are left QF.

In order to show that – in contrast to Hopf algebras over commutative rings – not any finitely generated projective Hopf algebroid is quasi-Frobenius, let us give here an example (with bijective antipode) such that the total algebra is finitely generated and projective as a module over the base algebra (in all the four senses listed in (2.16)) and the total algebra is neither a left nor a right QF extension of the base algebra.

The example is taken from [21, Example 3.1] where it is shown that for any algebra B over a commutative ring k the k-algebra $A := B \underset{k}{\otimes} B^{\mathrm{op}}$ has a left bialgebroid structure, $\mathcal{A}_L$, over the base B with structural maps

$$\begin{aligned}
&s_L\colon B \to A, && b \mapsto b \otimes 1_B,\\
&t_L\colon B^{\mathrm{op}} \to A, && b \mapsto 1_B \otimes b,\\
&\gamma_L\colon A \to A_B \otimes {}_B A, && b_1 \otimes b_2 \mapsto (b_1 \otimes 1_B) \otimes (1_B \otimes b_2),\\
&\pi_L\colon A \to B, && b_1 \otimes b_2 \mapsto b_1 b_2.
\end{aligned} \tag{5.1}$$

The bialgebroid $\mathcal{A}_L$ satisfies the Hopf algebroid axioms of [21] with the involutive antipode S, equal to the flip map

$$S\colon B \underset{k}{\otimes} B^{\mathrm{op}} \to B^{\mathrm{op}} \underset{k}{\otimes} B, \quad b_1 \otimes b_2 \mapsto b_2 \otimes b_1. \tag{5.2}$$

The reader may check that A has a Hopf algebroid structure also in the sense of this paper with left bialgebroid structure (5.1), antipode (5.2) and right bialgebroid structure $\mathcal{A}_R = (A, B^{\mathrm{op}}, S \circ s_L, S \circ t_L, (S \otimes S) \circ \gamma_L^{\mathrm{op}} \circ S, \pi_L \circ S)$.

If B is finitely generated and projective as a k-module then all modules $A^{B^{\mathrm{op}}}$, ${}^{B^{\mathrm{op}}}A$, A_B and ${}_B A$ are finitely generated and projective, and vice versa. What is more, we have

LEMMA 5.3. *Let B be an algebra over the commutative ring k with trivial center. The following statements are equivalent*:

(1) *The extension $k \to B$ is left QF.*
(2) *The extension $k \to B$ is right QF.*
(3) *The extension $B \to B \underset{k}{\otimes} B^{\mathrm{op}}$, $b \mapsto b \otimes 1_B$ is left QF.*
(4) *The extension $B \to B \underset{k}{\otimes} B^{\mathrm{op}}$, $b \mapsto b \otimes 1_B$ is right QF.*

The equivalence (1) $\Leftrightarrow$ (2) is proven in [29] and the rest can be proven using the techniques of quasi-Frobenius systems.

In view of Lemma 5.3 it is easy to construct a finitely generated projective Hopf algebroid which is not QF. Let us choose, for example, B to be the algebra of $n \times n$ upper triangle matrices with entries in the commutative ring k. Then B has trivial center and it is neither a left nor a right QF extension of k, hence $A = B \underset{k}{\otimes} B^{\mathrm{op}}$ is neither a left nor a right QF extension of B.

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Erratum to: "Integral Theory for Hopf Algebroids" [Algebra Represent. Theory (2005) 8:563–599]

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Abstract Because of similar errors in Lemma 2.6 and Theorem 4.2, we need to add stronger assumptions in several statements.

At several points in the paper we repeated essentially the same incorrect step, thanks to which the to-be-dual-bases in the proof of Lemma 2.6 (1.a)⇔(1.b) and (2.a)⇔(2.b), and also the map $[{}^L A \otimes \omega \circ (A_{\mathbf{L}} \otimes \tau)] \circ (\gamma_R \otimes_{\mathbf{L}} M) \circ \tau$ in the proof of Theorem 4.2, are ill-defined. As a consequence of these errors, the published proofs of Lemma 2.6, Theorem 4.2 and Proposition 4.4(2) are not correct. Since in this way also some unjustified claims were used to prove them, Corollary 4.5, Proposition 4.6, some considerations on page 587, Theorems 4.7 and 5.2 become valid only by adding some finitely generated projectivity assumptions. Although the obtained results are somewhat weaker than their (eventually incorrect) versions in the original paper, the most interesting cases are still covered.

Below we go through the necessary corrections.

In Lemma 2.6, only the equivalences (1.a)⇔(2.b) and (1.b)⇔(2.a) are justified, for a Hopf algebroid with a bijective antipode.

Theorem 4.2 is replaced by Theorem 1 below. In its formulation Sweedler's index notation $\tau(m) = m_{\langle -1 \rangle} \otimes m_{\langle 0 \rangle}$ (with implicit summation) is used, for the left coaction $\tau : M \to A_L \otimes_L M$ of the constituent left L-bialgebroid $\mathcal{A}_L$ in a Hopf algebroid $\mathcal{A}$, on a left $\mathcal{A}_L$-comodule M and $m \in M$.

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Theorem 1 *Let $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ be a Hopf algebroid and $\mathcal{W}$ be the A-coring* (4.1). *Assume that the kernel of the maps*

$$M \to A_L \otimes {}_L M, \qquad m \mapsto \left(m_{\langle -1\rangle} \otimes m_{\langle 0\rangle}\right) - (1_A \otimes m) \tag{1}$$

is preserved by the functor ${}^L A \otimes - : \mathcal{M}_L \to \mathcal{M}_L$, for any $M \in {}^{\mathcal{W}}\mathcal{M}$ (e.g. ${}^L A$ is a flat module). Then the functors (4.2) *and* (4.3) *are inverse equivalences.*

Proof We only need to check that $\alpha_M^{-1}(m) = m_{\langle -1\rangle}{}^{(1)} \otimes S(m_{\langle -1\rangle}{}^{(2)}) \cdot m_{\langle 0\rangle}$ belongs to ${}^L A \otimes \mathrm{Coinv}(M)_L$, the rest of the proof in the paper is valid. By the assumption that the kernel of Eq. 1 is preserved by the functor ${}^L A \otimes - : \mathcal{M}_L \to \mathcal{M}_L$, we need to show only that

$$\begin{aligned} &m_{\langle -1\rangle}{}^{(1)} \otimes \left(S\left(m_{\langle -1\rangle}{}^{(2)}\right) \cdot m_{\langle 0\rangle}\right)_{\langle -1\rangle} \otimes \left(S\left(m_{\langle -1\rangle}{}^{(2)}\right) \cdot m_{\langle 0\rangle}\right)_{\langle 0\rangle} \\ &\quad = m_{\langle -1\rangle}{}^{(1)} \otimes 1_A \otimes S\left(m_{\langle -1\rangle}{}^{(2)}\right) \cdot m_{\langle 0\rangle}, \end{aligned} \tag{2}$$

as elements of ${}^L A \otimes A_{\mathbf{L}} \otimes {}_{\mathbf{L}} M_L$, for all $m \in M$. Compose the well defined map

$$A^R \otimes {}^R A_L \otimes {}_L A \to A^R \otimes {}_R A, \qquad a \otimes b \otimes c \mapsto a \otimes S(b)c$$

with the equal maps $(\gamma_R \otimes {}_L A) \circ \gamma_L = (A^R \otimes \gamma_L) \circ \gamma_R : A \to A^R \otimes {}^R A_L \otimes {}_L A$ (cf. (2.18)) in order to conclude that, for any $a \in A$,

$$a_{(1)}{}^{(1)} \otimes S\left(a_{(1)}{}^{(2)}\right) a_{(2)} = a^{(1)} \otimes S\left(a^{(2)}{}_{(1)}\right) a^{(2)}{}_{(2)} = a^{(1)} \otimes s_R \circ \pi_R\left(a^{(2)}\right) = a \otimes 1_A. \tag{3}$$

In Eq. 3, in the second equality (2.20) was used and the last equality follows by the counitality of γ_R. Using the left A-linearity of the coaction $\tau : M \to \mathcal{W} \otimes_A M \cong A_L \otimes {}_L M$, anti-comultiplicativity of the antipode (cf. Proposition 2.3(2)), coassociativity of τ and γ_R and finally Eq. 3, the left hand side of Eq. 2 is computed to be equal to

$$\begin{aligned} &m_{\langle -2\rangle}{}^{(1)} \otimes S\left(m_{\langle -2\rangle}{}^{(2)}\right)_{(1)} m_{\langle -1\rangle} \otimes S\left(m_{\langle -2\rangle}{}^{(2)}\right)_{(2)} \cdot m_{\langle 0\rangle} \\ &\quad = m_{\langle -2\rangle}{}^{(1)} \otimes S\left(m_{\langle -2\rangle}{}^{(2)(2)}\right) m_{\langle -1\rangle} \otimes S\left(m_{\langle -2\rangle}{}^{(2)(1)}\right) \cdot m_{\langle 0\rangle} \\ &\quad = m_{\langle -1\rangle(1)}{}^{(1)(1)} \otimes S\left(m_{\langle -1\rangle(1)}{}^{(2)}\right) m_{\langle -1\rangle(2)} \otimes S\left(m_{\langle -1\rangle(1)}{}^{(1)(2)}\right) \cdot m_{\langle 0\rangle} \\ &\quad = m_{\langle -1\rangle}{}^{(1)} \otimes 1_A \otimes S\left(m_{\langle -1\rangle}{}^{(2)}\right) \cdot m_{\langle 0\rangle}. \end{aligned}$$

Thus it follows that $\alpha_M^{-1}(m)$ belongs to ${}^L A \otimes \mathrm{Coinv}(M)_L$ for all $m \in M$, as stated. □

In the arXiv version of [1] a more restrictive notion of a comodule of a Hopf algebroid is studied, cf. [1, arXiv version, Definition 2.20]. The total algebra A of any Hopf algebroid $\mathcal{A}$ can be regarded as a monoid in the monoidal category of $\mathcal{A}$-comodules in this more restrictive sense. In this setting, the category of A-modules in the category of $\mathcal{A}$-comodules, and the category of modules for the base algebra L of $\mathcal{A}$, were proven to be equivalent without any further (equalizer preserving) assumption, see [1, Theorem 3.27 and Remark 3.28]. That is, in the arXiv version of [1] a weaker statement is proven under weaker assumptions, compared to Theorem 1 above.

Since the original version of Lemma 2.6 turned out to be incorrect, so is the proof of Proposition 4.4(2) built on it. In order to replace Proposition 4.4(2) by a justified claim, take a Hopf algebroid $\mathcal{A}$, such that the module ${}^R A$ is finitely generated and projective, and let $\{k_j\} \subset A$ and $\{{}^*\kappa^j\} \subset {}^*\!\mathcal{A}$ be dual bases for it. Alternatively to (4.6), a right $\mathcal{A}_R$-comodule structure on $\mathcal{A}^*$ can be introduced by the right R-action

$$\mathcal{A}^*{}_R : \quad \phi^* \cdot r :\, = \phi^* \leftharpoonup s_R(r) \qquad \text{for } r \in R,\ \phi^* \in \mathcal{A}^*$$

and the right coaction

$$\tau_R : \mathcal{A}^* \to \mathcal{A}^*{}_R \otimes {}^R A \qquad \phi^* \mapsto \sum_i \chi^{-1}(\pi_L \circ t_R \circ {}^*\kappa^j \circ S)\phi^* \otimes k_j, \tag{4}$$

where $\chi : \mathcal{A}^* \to \mathcal{A}_*$ is the algebra anti-isomorphism (2.22). Note that if both modules A_L and ${}^R A$ are finitely generated and projective, then the dual bases $\{b_i\} \subset A$, $\{\beta_*^i\} \subset \mathcal{A}_*$ for A_L, and $\{k_j\} \subset A$, $\{{}^*\kappa^j\} \subset {}^*\!\mathcal{A}$ for ${}^R A$, are related via the identity

$$\sum_i \beta_*^i \otimes S(b_i) = \sum_j \pi_L \circ t_R \circ {}^*\kappa_j \circ S \otimes k_j$$

in $\mathcal{A}_*{}^R \otimes {}^R A$. Hence in this case the coaction (4) is equal to (4.6).

Proposition 4.4 is replaced by the following.

Proposition 2 *Let $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ be a Hopf algebroid.*

(1) *Introduce the left A-module*

$$_A\mathcal{A}^* : \quad a \cdot \phi^* :\, = \phi^* \leftharpoonup S(a) \qquad \textit{for } a \in A,\ \phi^* \in \mathcal{A}^*.$$

If the module A_L is finitely generated and projective, then $({}_A\mathcal{A}^*, \tau_L)$—where τ_L is the map (4.5)*—is a left-left Hopf module over $\mathcal{A}_L$.*

(2) *Introduce the right A-module*

$$\mathcal{A}^*{}_A : \quad \phi^* \cdot a :\, = \phi^* \leftharpoonup a \qquad \textit{for } a \in A,\ \phi^* \in \mathcal{A}^*.$$

If the module ${}^R A$ is finitely generated and projective, then $(\mathcal{A}^*{}_A, \tau_R)$—where τ_R is the map (4)*—is a right-right Hopf module over $\mathcal{A}_R$.*

The coinvariants of both Hopf modules $({}_A\mathcal{A}^*, \tau_L)$ and $(\mathcal{A}^*{}_A, \tau_R)$ are the elements of $\mathcal{L}(\mathcal{A}^*)$.

Proof Part (1) of Proposition 2 is proven in the paper, and part (2) follows by similar steps. □

Since Theorem 1 and Proposition 2 contain stronger assumptions than their original counterparts, we need stronger assumptions than those in the paper to conclude that the maps (4.14) and (4.15) are isomorphisms. Applying Theorem 1 to the Hopf modules in Proposition 2, the following are obtained. The map (4.14) is an isomorphism of left-left Hopf modules over $\mathcal{A}_L$ provided that the module A_L is finitely generated and projective, and the kernel of the map

$${}^L A \otimes \mathcal{A}^{*L} \to {}^{\mathbf{L}}A \otimes A_L \otimes {}_L\mathcal{A}^{*\mathbf{L}}, \qquad a \otimes \phi^* \mapsto a \otimes \tau_L(\phi^*) - a \otimes 1_A \otimes \phi^*$$

is equal to ${}^L A \otimes \mathcal{L}(\mathcal{A}^*)^L$. The map (4.14) is an isomorphism, in particular, if both modules A_L and ${}^L A$ are finitely generated and projective. The map (4.15) is an isomorphism of right-right Hopf modules over $\mathcal{A}_R$ provided that the module ${}^R A$ is finitely generated and projective, and the kernel of

$$ {}^R\mathcal{A}^* \otimes A_R \to {}^{\mathbf{R}}\mathcal{A}^*{}_R \otimes {}^R A \otimes A_{\mathbf{R}}, \qquad \phi^* \otimes a \mapsto \tau_R(\phi^*) \otimes a - \phi^* \otimes 1_A \otimes a $$

is equal to ${}^R\mathcal{L}(\mathcal{A}^*) \otimes A_R$. The map (4.15) is an isomorphism, in particular, if both modules ${}^R A$ and A_R are finitely generated and projective.

The proofs of Corollary 4.5, Proposition 4.6, considerations about depth 2 properties on page 587, Theorems 4.7 and 5.2 all resort to the bijectivity of (4.14) and/or (4.15). Hence, as it is explained above, they are valid only under more restrictive assumptions. In each case, it has to be added to the hypothes is that

$\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ *is a Hopf algebroid such that all of the modules* A^R, ${}^R A$, ${}_L A$ *and* A_L *are finitely generated and projective.*

Accordingly, all claims about finitely generated projectivity of these modules need to be removed in the statements (together with their verifications in the proofs).

Based on the corrected form of Theorem 4.7, we obtain the following generalization of Theorem 4.1.

Corollary 3 *For any Hopf algebroid* $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$, *the following assertions are equivalent.*

- (1.a) *Both maps* $s_R : R \to A$ *and* $t_R : R^{op} \to A$ *are Frobenius extensions of k-algebras.*
- (1.b) *Both maps* $s_L : L \to A$ *and* $t_L : L^{op} \to A$ *are Frobenius extensions of k-algebras.*
- (2.a) *The module* A^R *is finitely generated and projective and there exists an element* $\lambda^* \in \mathcal{L}(\mathcal{A}^*)$ *such that the map* $\mathcal{F} : A \to \mathcal{A}^*, \quad a \mapsto \lambda^* \leftharpoonup a$ *is bijective.*
- (2.b) *S is bijective, the module* ${}^R A$ *is finitely generated and projective and there exists an element* ${}^*\lambda \in \mathcal{L}({}^*\!\mathcal{A})$ *such that the map* $A \to {}^*\!\mathcal{A}, \quad a \mapsto {}^*\lambda \leftharpoondown a$ *is bijective.*
- (2.c) *The module* ${}_L A$ *is finitely generated and projective and there exists an element* ${}_*\rho \in \mathcal{R}({}_*\mathcal{A})$ *such that the map* $A \to {}_*\mathcal{A}, \quad a \mapsto a \rightharpoondown {}_*\rho$ *is bijective.*
- (2.d) *S is bijective, the module* A_L *is finitely generated and projective and there exists an element* $\rho_* \in \mathcal{R}(\mathcal{A}_*)$ *such that the map* $A \to \mathcal{A}_*, \quad a \mapsto a \rightharpoonup \rho_*$ *is bijective.*
- (3.a) *There exists a* non-degenerate *left integral, that is an element* $\ell \in \mathcal{L}(\mathcal{A})$ *such that both maps* $\mathcal{F}^* : \mathcal{A}^* \to A, \quad \phi^* \mapsto \phi^* \rightharpoonup \ell$ *and* ${}^*\mathcal{F} : {}^*\!\mathcal{A} \to A, \quad {}^*\phi \mapsto {}^*\phi \rightharpoondown \ell$ *are bijective.*
- (3.b) *There exists a* non-degenerate right integral, *that is, an element* $\wp \in \mathcal{R}(\mathcal{A})$ *such that both maps* ${}_*\mathcal{A} \to A, \quad {}_*\phi \mapsto \wp \leftharpoondown {}_*\phi$ *and* $\mathcal{A}_* \to A, \quad \phi_* \mapsto \wp \leftharpoonup \phi_*$ *are bijective.*

Proof (1.*a*) ⇔ (1.*b*): This follows by the same reasoning used to prove (1.a)⇔(1.d) and (1.b)⇔(1.c) in Theorem 4.7.

(1.*a*) ⇒ (2.*a*): Since $s_R : R \to A$ is a Frobenius extension by assumption, the modules A^R and ${}_R A$ (hence also A_L) are finitely generated and projective by definition. Similarly, since $t_R : R^{op} \to A$ is a Frobenius extension, the modules ${}^R A$ and ${}_L A$ are finitely generated and projective. Thus this implication follows by (the corrected form of) Theorem 4.7 (1.a)⇒(3.a).

(2.a) ⇒ (3.a) and S is bijective: This is proven by repeating the same steps used to prove (3.a)⇒(4.a) and (4.b) in Theorem 4.7.

(3.a) ⇒ (1.a) and S is bijective: Putting $\lambda^* := \mathcal{F}^{*-1}(1_A)$, the map $a \mapsto (\lambda^* \leftharpoonup a) \rightharpoonup \ell$ is checked to be the inverse of S.

For any $r \in R$, $\mathcal{F}^*(r\lambda^*(-)) = t_R(r) = \mathcal{F}^*(\lambda^* \leftharpoonup s_R(r))$. So by the bijectivity of $\mathcal{F}^*$, we conclude that λ^* is a left R-module map ${}_RA \to R$. Therefore the module ${}^RA$ is finitely generated and projective with dual basis $\lambda^*(S(-)\ell^{(1)}) \otimes \ell^{(2)} \in {}^*\!\mathcal{A}_R \otimes {}^RA$. Since the antipode is bijective, the module A_L is finitely generated and projective by Lemma 2.6 (2.b)⇒(1.a). Applying the same reasoning to the Hopf algebroid $\mathcal{A}_{cop}$, we conclude that by the bijectivity of ${}^*\mathcal{F}$ also the modules A^R and ${}_LA$ are finitely generated and projective. Hence the claim follows by (the corrected form of) Theorem 4.7, (4.a)⇒(1.a) and (1.b).

(1.b) ⇔ (2.c) ⇔ (3.b): This follows by applying (1.a)⇔(2.a)⇔(3.a) to the Hopf algebroid $\mathcal{A}_{cop}^{op}$.

(1.a) ⇔ (2.b): Since we proved that (1.a) implies the bijectivity of the antipode, we can apply (1.a)⇔(2.a) to the Hopf algebroid $\mathcal{A}_{cop}$.

(1.b) ⇔ (2.d): This follows by applying (1.a)⇔(2.b) to the Hopf algebroid $\mathcal{A}_{cop}^{op}$.

□

Finally, we would like to correct some regrettable typos in the paper.

In both parts of Lemma 5.1, instead of the condition $\sum_{i,k} au_i^k \otimes v_i^k = \sum_{i,k} u_i^k \otimes v_i^k a$, for all $a \in A$, the conditions $\sum_i au_i^k \otimes v_i^k = \sum_i u_i^k \otimes v_i^k a$ need to hold, for all possible values of k and all $a \in A$. (In the published version also the summation symbols are missing on the right hand sides.)

In the computation on page 595, the first and the last expressions are interchanged.

In the penultimate line on page 597, the phrase "and vice versa" has to be erased.

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5. fejezet

Galois theory for Hopf algebroids

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Galois theory for Hopf algebroids

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SUNTO – Un'estensione $B \subset A$ di algebre su un anello commutativo k è una $\mathcal{H}$-estensione per un L-bialgebroide $\mathcal{H}$ se A è una $\mathcal{H}$-comodulo algebra e B è la sottoalgebra dei suoi coinvarianti. Essa è $\mathcal{H}$-Galois se l'applicazione canonica $A \otimes_A B \to A \otimes_L \mathcal{H}$ è un isomorfismo o, equivalentemente, se il coanello canonico $(A \otimes_L \mathcal{H} : A)$ è un coanello di Galois.
Nel caso di un algebroide di Hopf $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$ si dimostra che ogni $\mathcal{H}_R$-estensione è una $\mathcal{H}_L$-estensione. Se l'antipode è biiettivo allora si dimostra che anche le nozioni di estensioni $\mathcal{H}_R$-Galois e $\mathcal{H}_L$-Galois coincidono.
I risultati per le strutture biiettive entwining sono estesi alle strutture entwining su algebre non commutative, al fine di dimostrare un teorema simile al Teorema dii Kreimer-Takeuchi per un Hopf algebroide $\mathcal{H}$ proiettivo finitamente generato con antipode biiettivo. Il teorema afferma che ogni estensione $\mathcal{H}$-Galois $B \subset A$ è proiettiva e se A è k-piatto allora la suriettività dell'applicazione canonica è sufficiente a garantire la proprietà di Galois.
La teoria di Morita, sviluppata per i coanelli da Caenepeel, Vercruysse e Wang, viene applicata per ottenere criteri equivalenti per la proprietà di Galois per estensioni di algebroidi di Hopf. Questo conduce a risultati analoghi, per algebroidi di Hopf, a quelli ottenuti da Doi per estensioni di algebre di Hopf e da Cohen Fishman e Montgomery nel caso degli algebroidi di Hopf Frobenius.

ABSTRACT – An extension $B \subset A$ of algebras over a commutative ring k is an $\mathcal{H}$-extension for an L-bialgebroid $\mathcal{H}$ if A is an $\mathcal{H}$-comodule algebra and B is the subalgebra of its coinvariants. It is $\mathcal{H}$-Galois if the canonical map $A \otimes_B A \to A \otimes_L \mathcal{H}$ is an isomorphism or, equivalently, if the canonical coring $(A \otimes_L \mathcal{H} : A)$ is a Galois coring.
In the case of a Hopf algebroid $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$ any $\mathcal{H}_R$-extension is shown to be also an $\mathcal{H}_L$-extension. If the antipode is bijective then also the notions of $\mathcal{H}_R$-Galois extensions and of $\mathcal{H}_L$-Galois extensions are proven to coincide.
Results about bijective entwining structures are extended to entwining structures over non-commutative algebras in order to prove a Kreimer-Takeuchi type theorem for a finitely generated projective Hopf algebroid $\mathcal{H}$ with bijective antipode. It states that any $\mathcal{H}$-Galois extension $B \subset A$ is projective, and if A is k-flat then already the surjectivity of the canonical map implies the Galois property.

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The Morita theory, developed for corings by Caenepeel, Vercruysse and Wang, is applied to obtain equivalent criteria for the Galois property of Hopf algebroid extensions. This leads to Hopf algebroid analogues of results for Hopf algebra extensions by Doi and, in the case of Frobenius Hopf algebroids, by Cohen, Fishman and Montgomery.

1. – Introduction.

An extension $B \subset A$ of algebras over a commutative ring k is an H-extension for a k-bialgebra H if A is a right H-comodule algebra and B is the subalgebra of its coinvariants i.e. of elements $b \in A$ such that $b_{\langle 0\rangle} \otimes b_{\langle 1\rangle} = b \otimes 1_H$ – where the map $A \to A \otimes_k H$, $a \mapsto a_{\langle 0\rangle} \otimes a_{\langle 1\rangle}$ is the coaction of H on A (summation understood). An H-extension $B \subset A$ is H-Galois if the canonical map $A \otimes_B A \to A \otimes_k H$, $a \otimes a' \mapsto aa'_{\langle 0\rangle} \otimes a'_{\langle 1\rangle}$ is an isomorphism of k-modules.

In many cases it is technically much easier to check the surjectivity of the canonical map than its injectivity. A powerful tool in the study of H-extensions is the Kreimer-Takeuchi theorem [25] stating that if H is a finitely generated projective Hopf algebra then the surjectivity of the canonical map implies its bijectivity and also the fact that A is projective both as a left and as a right B-module.

The proof of the Kreimer-Takeuchi theorem went through both simplification and generalization in the papers [30, 32, 9, 31]. In the present paper we adopt the method of Brzeziński [9] and of Schauenburg and Schneider [31], who used the following observation. A comodule algebra A for a bialgebra H determines a canonical entwining structure [11] consisting of the algebra A, the coalgebra underlying the bialgebra H, and the entwining map $H \otimes_k A \to A \otimes_k H$, $h \otimes a \mapsto a_{\langle 0\rangle} \otimes ha_{\langle 1\rangle}$. In the case when the bialgebra H possesses a skew antipode, this entwining map is a bijection. The proof of (a wide generalization of) the Kreimer-Takeuchi theorem both in [9] and in [31] is based on the study of bijective entwining structures, under slightly different assumptions. In Section 4 below we are going to show that these arguments can be repeated almost without modification by using entwining structures over non-commutative algebras [4].

In the paper [21] Doi constructed a Morita context for an H-extension $B \subset A$. If H is finitely generated and projective as a k-module then the surjectivity of one of the connecting maps is equivalent to the projectivity and the Galois property of the extension $B \subset A$, while the strictness of the Morita context is equivalent to faithful flatness and the Galois property. This observation made it possible to use all results of Morita theory for characterizing H-Galois extensions. In the case when H is a finite dimensional Hopf algebra over a field (or a Frobenius Hopf algebra over a commutative ring), the Morita context of Doi is equivalent to another Morita context, introduced by Cohen, Fishman and Montgomery [20].

One of the most beautiful applications of the theory of corings [14] is the observation [8] that the Galois property of an H-extension $B \subset A$ is equivalent to the Galois property of a canonical A-coring $A \otimes H$. In [18] the construction of the

Morita context by Doi has been extended to any A-coring C possessing a grouplike element (i.e. such that A is a C-comodule). In the case when C is a finitely generated projective A-module (or an A-progenerator, see [15]) the application of Morita theory results then several equivalent criteria for the Galois property of the coring C and the projectivity (or faithful flatness) of A as a module for the subalgebra of coinvariants in A. In the case when the A-dual algebra of the coring C is a Frobenius extension of A, also the Morita context in [20] has been generalized to the general setting of corings and the precise relation of the two Morita contexts has been explained.

The notion of bialgebra extensions has been generalized to bialgebroids by Kadison [23] as an extension $B \subset A$ of k-algebras such that A is a comodule algebra and B is the subalgebra of coinvariants. The Galois property of a bialgebroid extension can be formulated also as the Galois property of a canonical coring. This implies that the general theory, developed in [18], can be applied also to bialgebroid extensions.

In the present paper we study Hopf algebroid extensions. The notion of Hopf algebroids has been introduced in [7, 3] and studied further in [5]. It consists of two compatible (left and right) bialgebroid structures on the same algebra which are related by the antipode.

It is shown in Section 3 below that any comodule for the constituent right bialgebroid $\mathcal{H}_R$ of the Hopf algebroid possesses a unique comodule structure for the constituent left bialgebroid $\mathcal{H}_L$ in such a way that the $\mathcal{H}_R$-coaction is $\mathcal{H}_L$-colinear and the $\mathcal{H}_L$-coaction is $\mathcal{H}_R$-colinear. In particular, an $\mathcal{H}_R$-comodule algebra possesses a distinguished $\mathcal{H}_L$-comodule algebra structure and the $\mathcal{H}_R$- and the $\mathcal{H}_L$-coinvariants coincide. What is more, – just as in the case of Hopf algebras – if $\mathcal{H}$ is a Hopf algebroid with bijective antipode then the canonical entwining structure (over the non-commutative base algebra of $\mathcal{H}$), associated to an $\mathcal{H}$-extension, is bijective. This fact is used to prove a Kreimer-Takeuchi type theorem in Section 4.

In Section 3 we show that if the antipode of the Hopf algebroid $\mathcal{H}$ is bijective then an $\mathcal{H}$- extension is Galois for the left bialgebroid structure of $\mathcal{H}$ if and only if it is Galois for its right bialgebroid structure.

In Section 5 we apply the Morita theory for corings to a Hopf algebroid extension $B \subset A$, looked at as a right bialgebroid extension. In the finitely generated projective case this results equivalent criteria, under which it is a projective right bialgebroid Galois extension. Similarly, we can look at $B \subset A$ as a left bialgebroid extension and obtain equivalent conditions for its projective left bialgebroid Galois property. Making use of the results about Hopf algebroid extensions in Section 3, and the Kreimer-Takeuchi type theorem proven in Section 4, we conclude that if $\mathcal{H}$ is a finitely generated projective Hopf algebroid with bijective antipode then the two equivalent sets of conditions are equivalent also to each other. In the case of Frobenius Hopf algebroids [5] we obtain a direct generalization of ([20], Theorem 1.2).

Throughout the paper k is a commutative ring. By an algebra $R = (R, \mu, \eta)$ we mean an associative unital k-algebra. Instead of the unit homomorphism η we use sometimes the unit element $1_R := \eta(1_k)$. We denote by ${}_R\mathcal{M}$, $\mathcal{M}_R$ and ${}_R\mathcal{M}_R$ the categories of left, right, and bimodules for R, respectively. For the k-module of morphisms in ${}_R\mathcal{M}$, $\mathcal{M}_R$ and ${}_R\mathcal{M}_R$ we write ${}_R\mathrm{Hom}(\ ,\ )$, $\mathrm{Hom}_R(\ ,\ )$ and ${}_R\mathrm{Hom}_R(\ ,\ )$, respectively.

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2. – Preliminaries.

2.1 – *Bialgebroids and Hopf algebroids.*

L-bialgebroids [26, 37, 34, 28] or, what were shown in [12] to be equivalent to it, $\times_L$-bialgebras [36] are generalizations of bialgebras to the case of non-commutative base algebras. This means that instead of coalgebras and algebras over commutative rings one works with corings and rings over non-commutative base algebras. Recall that a coring over a k-algebra L is a comonoid in ${}_L\mathcal{M}_L$ while an L-ring is a monoid in ${}_L\mathcal{M}_L$. The notion of L-rings is equivalent to a pair, consisting of a k-algebra A and an algebra homomorphism $L \to A$.

Definition 2.1. A *left bialgebroid* is a 6-tuple $\mathcal{H} = (H, L, s, t, \gamma, \pi)$, where H and L are k-algebras. H is an $L \otimes_k L^{op}$-ring via the algebra homomorphisms $s : L \to H$ and $t : L^{op} \to H$, the images of which are required to commute in H. In terms of the maps s and t one equips H with an L-L bimodule structure as

$$l \cdot h \cdot l' := s(l)t(l')h \qquad \text{for } h \in H,\ l, l' \in L. \tag{2.1}$$

The triple (H, γ, π) is an L-coring. Introducing Sweedler's notation $\gamma(h) = h_{(1)} \otimes_L h_{(2)}$ for $h \in H$ the axioms

$$h_{(1)}t(l) \otimes_L h_{(2)} = h_{(1)} \otimes_L h_{(2)}s(l) \tag{2.2}$$

$$\gamma(1_H) = 1_H \otimes_L 1_H \tag{2.3}$$

$$\gamma(hh') = \gamma(h)\gamma(h') \tag{2.4}$$

$$\pi(1_H) = 1_L \tag{2.5}$$

$$\pi(h\ s \circ \pi(h')) = \pi(hh') = \pi(h\ t \circ \pi(h')) \tag{2.6}$$

are required for all $l \in L$ and $h, h' \in H$.

Notice that – although $H \underset{L}{\otimes} H$ is not an algebra – the axiom (2.4) makes sense in view of (2.2).

The bimodule (2.1) is defined in terms of multiplication by s and t on the left. The R-R bimodule structure in a *right bialgebroid* $\mathcal{H} = (H, R, s, t, \gamma, \pi)$ is defined in terms of multiplication on the right. For the details we refer to [24].

The *opposite* of a left bialgebroid $\mathcal{H} = (H, L, s, t, \gamma, \pi)$ is the right bialgebroid $\mathcal{H}^{op} = (H^{op}, L, t, s, \gamma, \pi)$ where H^{op} is the algebra opposite to H. Its *co-opposite* is the left bialgebroid $\mathcal{H}_{cop} = (H, L^{op}, t, s, \gamma^{op}, \pi)$ where $\gamma^{op} : H \to H \underset{L^{op}}{\otimes} H$ is the opposite coproduct $h \mapsto h_{(2)} \underset{L^{op}}{\otimes} h_{(1)}$.

It has been observed in [24] that for a left bialgebroid $\mathcal{H} = (H, L, s, t, \gamma, \pi)$ such that H is finitely generated and projective as a left or right L-module the corresponding L-*dual* possesses a right bialgebroid structure over the base algebra L. Analogously, the R-duals of a finitely generated projective right bialgebroid possess left bialgebroid structures. For the explicit forms of the dual bialgebroid structures consult [24].

Before defining the notion of Hopf algebroid let us introduce some notations. Analogous notations were used already in [7, 5].

When dealing with an $L \underset{k}{\otimes} L^{op}$-ring H we have to face the situation that H carries different module structures over the base algebra L. In this situation the usual notation $H \underset{L}{\otimes} H$ would be ambiguous. Therefore we make the following notational convention. In terms of the algebra homomorphisms $s : L \to H$ and $t : L^{op} \to H$ (with commuting images in H) we introduce four L-modules

$$
\begin{aligned}
{}_LH &: \quad l \cdot h := s(l)h \\
H_L &: \quad h \cdot l := t(l)h \\
H^L &: \quad h \cdot l = hs(l) \\
{}^LH &: \quad l \cdot h = ht(l).
\end{aligned}
\tag{2.7}
$$

This convention can be memorized as left indices stand for left modules and right indices for right modules. Upper indices for modules defined in terms of right multiplication and lower indices for the ones defined in terms of left multiplication.

In writing L-module tensor products we write out explicitly the module structures of the factors that are taking part in the tensor products, and do not put marks under the symbol $\otimes$. E.g. we write $H_L \otimes {}_LH$. In writing elements of tensor product modules we do not distinguish between the various module tensor products. That is, we write both $h \underset{L}{\otimes} h' \in H_L \otimes {}_LH$ and $g \underset{L}{\otimes} g' \in H^L \otimes {}_LH$, for example.

A left L-module can be considered as a right L^{op}-module, and sometimes we want to take a module tensor product over L^{op}. In this case we use the name of the corresponding L-module and the fact that the tensor product is taken over

L^{op} should be clear from the order of the factors.

In writing multiple tensor products we use different types of letters to denote which module structures take part in the same tensor product.

DEFINITION 2.2. A *Hopf algebroid* $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$ consists of a left bialgebroid $\mathcal{H}_L = (H, L, s_L, t_L, \gamma_L, \pi_L)$ and a right bialgebroid $\mathcal{H}_R = (H, R, s_R, t_R, \gamma_R, \pi_R)$, with common total algebra H, and a k-module map $S : H \to H$, called the antipode, such that the following axioms hold true:

i) $s_L \circ \pi_L \circ t_R = t_R$, $t_L \circ \pi_L \circ s_R = s_R$ and $s_R \circ \pi_R \circ t_L = t_L$, $t_R \circ \pi_R \circ s_L = s_L$

ii) $(\gamma_L \otimes {}^R H) \circ \gamma_R = (H_L \otimes \gamma_R) \circ \gamma_L$ as maps $H \to H_L \otimes {}_L H^R \otimes {}^R H$ and $(\gamma_R \otimes {}_L H) \circ \gamma_L = (H^R \otimes \gamma_L) \circ \gamma_R$ as maps $H \to H^R \otimes {}^R H_L \otimes {}_L H$

iii) S is both an $L-L$ bimodule map ${}^L H_L \to {}_L H^L$ and an $R-R$ bimodule map ${}^R H_R \to {}_R H^R$

iv) $\mu_H \circ (S \otimes {}_L H) \circ \gamma_L = s_R \circ \pi_R$ and $\mu_H \circ (H^R \otimes S) \circ \gamma_R = s_L \circ \pi_L$.

If $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$ is a Hopf algebroid then so is $\mathcal{H}^{op}_{cop} = ((\mathcal{H}_R)^{op}_{cop}, (\mathcal{H}_L)^{op}_{cop}, S)$ and if S is bijective then also $\mathcal{H}_{cop} = ((\mathcal{H}_L)_{cop}, (\mathcal{H}_R)_{cop}, S^{-1})$ and $\mathcal{H}^{op} = ((\mathcal{H}_R)^{op}, (\mathcal{H}_L)^{op}, S^{-1})$.

We are going to use the following variant of Sweedler's convention. For a Hopf algebroid $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$ we use the notation $\gamma_L(h) = h_{(1)} \otimes h_{(2)}$ with lower indices and $\gamma_R(h) = h^{(1)} \otimes h^{(2)}$ with upper indices for $h \in H$ in the case of the coproducts of $\mathcal{H}_L$ and $\mathcal{H}_R$, respectively. The axioms (Definition 2.2 (ii)) read in this notation as

$$h^{(1)}{}_{(1)} \otimes h^{(1)}{}_{(2)} \otimes h^{(2)} = h_{(1)} \otimes h_{(2)}{}^{(1)} \otimes h_{(2)}{}^{(2)}$$
$$h_{(1)}{}^{(1)} \otimes h_{(1)}{}^{(2)} \otimes h_{(2)} = h^{(1)} \otimes h^{(2)}{}_{(1)} \otimes h^{(2)}{}_{(2)}$$

for $h \in H$.

It is proven in ([5], Proposition 2.3) that the base algebras L and R of the left and right bialgebroids $\mathcal{H}_L$ and $\mathcal{H}_R$ in a Hopf algebroid $\mathcal{H}$ are anti-isomorphic via any of the isomorphisms $\pi_L \circ s_R$ and $\pi_L \circ t_R$. The antipode is a homomorphism of left bialgebroids $\mathcal{H}_L \to (\mathcal{H}_R)^{op}_{cop}$ and also $(\mathcal{H}_R)^{op}_{cop} \to \mathcal{H}_L$ in the sense that it is an anti-algebra endomorphism of H and the pair of maps $(S, \pi_L \circ s_R)$ is a coring homomorphism from the R^{op}-coring $(H, \gamma_R^{op}, \pi_R)$ to the L-coring (H, γ_L, π_L) and the pair of algebra homomorphisms $(S, \pi_R \circ s_L)$ is a coring homomorphism $(H, \gamma_L, \pi_L) \to (H, \gamma_R^{op}, \pi_R)$.

By ([5], Lemma 2.6) the module H_L is finitely generated and projective if and only if H^R is and the module ${}_L H$ is finitely generated and projective if and only if ${}^R H$ is. If the antipode is bijective then the finitely generated projectivity of all the four listed modules are equivalent properties. Therefore we term a Hopf algebroid $\mathcal{H}$, the antipode of which is bijective and one (and hence, a fortiori, all) of the modules H^R, ${}^RH$, H_L and ${}_LH$ is finitely generated and projective, as a *finitely generated projective* Hopf algebroid with bijective antipode.

The left *integrals* in a left bialgebroid $\mathcal{H}_L$ are defined ([7], Definition 5.1) as the invariants of the left regular H-module i.e. the elements of

$$\mathcal{L}(H) := \{ \ \ell \in H \mid h\ell = s_L \circ \pi_L(h)\ \ell \quad \forall h \in H \ \}.$$

By ([5], Scholium 2.8) an element ℓ of a Hopf algebroid $\mathcal{H}$ is a left integral if and only if $h\ell^{(1)} \underset{R}{\otimes} S(\ell^{(2)}) = \ell^{(1)} \underset{R}{\otimes} S(\ell^{(2)})h$ for all $h \in H$.

A left integral ℓ in a Hopf algebroid $\mathcal{H}$ is called *non-degenerate* ([7], Definition 5.3) if both maps

$$\ell_R : H^* := \mathrm{Hom}_R(H^R, R) \to H \qquad \phi^* \mapsto \phi^* \rightharpoonup \ell \equiv \ell^{(2)}\ t_R \circ \phi^*(\ell^{(1)}) \quad \text{and}$$

$${}_R\ell : {}^*H := {}_R\mathrm{Hom}({}^RH, R) \to H \qquad {}^*\phi \mapsto {}^*\phi \rightharpoonup \ell \equiv \ell^{(1)}\ s_R \circ {}^*\phi(\ell^{(2)})$$

are isomorphisms. By ([7], Proposition 5.10) for a non-degenerate left integral ℓ in a Hopf algebroid $\mathcal{H}$ also the maps

$$\ell_L : H_* := \mathrm{Hom}_L(H_L, L) \to H \qquad \phi_* \mapsto \ell \leftharpoonup \phi_* \equiv s_L \circ \phi_*(\ell_{(1)})\ \ell_{(2)} \quad \text{and}$$

$${}_L\ell : {}_*H := {}_L\mathrm{Hom}({}_LH, L) \to H \qquad {}_*\phi \mapsto \ell \leftharpoonup {}_*\phi \equiv t_L \circ {}_*\phi(\ell_{(2)})\ \ell_{(1)}$$

are isomorphisms. It is shown in ([5], Theorem 4.7) that the existence of a non-degenerate left integral in a Hopf algebroid $\mathcal{H}$ is equivalent to the Frobenius property of any of the four extensions $s_R : R \to H$, $t_R : R^{op} \to H$, $s_L : L \to H$ and $t_L : L^{op} \to H$ and it implies the bijectivity of the antipode. What is more, if the Hopf algebroid $\mathcal{H}$ possesses a non-degenerate left integral then also the four duals H^*, ${}^*H$, H_* and ${}_*H$ possess (anti-) isomorphic Hopf algebroid structures with non-degenerate integrals ([7], Theorem 5.17 and Proposition 5.19). Motivated by these results we term a Hopf algebroid possessing a non-degenerate left integral as a *Frobenius* Hopf algebroid. Recall from [35] that a Frobenius Hopf algebroid is equivalent to a distributive Frobenius double algebra.

2.2 – *Module and comodule algebras.*

The category ${}_H\mathcal{M}$ of left modules for the total algebra H of a left bialgebroid $(H, L, s, t, \gamma, \pi)$ is a monoidal category. As a matter of fact, any H-module is an L-L bimodule via s and t. The monoidal product in ${}_H\mathcal{M}$ is the L-module tensor product with H-module structure

$$h \cdot (m \underset{L}{\otimes} n) := h_{(1)} \cdot m \underset{L}{\otimes} h_{(2)} \cdot n \qquad \text{for } h \in H,\ m \underset{L}{\otimes} n \in M \underset{L}{\otimes} N$$

and the monoidal unit is L with H-module structure

$$h \cdot l := \pi(hs(l)) \qquad \text{for } h \in H, l \in L.$$

A left H-*module algebra* is defined as a monoid in the monoidal category ${}_H\mathcal{M}$.

A left H -module algebra A is in particular an L-ring via the homomorphism

$$L \to A \qquad l \mapsto l \cdot 1_A \equiv 1_A \cdot l.$$

The *invariants* of A are the elements of

$$A^H := \{ \, a \in A \mid h \cdot a = s \circ \pi(h) \cdot a \quad \forall h \in H \, \}.$$

Just the same way, the category $\mathcal{M}_H$ of right modules for the total algebra H of a right R-bialgebroid is a monoidal category with monoidal product the R-module tensor product and monoidal unit R. A right H-module algebra is a monoid in $\mathcal{M}_H$. A right module algebra is in particular an R-ring. The invariants are defined analogously to the left case in terms of the counit.

By a comodule for a left bialgebroid $\mathcal{H} = (H, L, s, t, \gamma, \pi)$ we mean a comodule for the L-coring (H, γ, π). Recall that the category of left $\mathcal{H}$-comodules is also a monoidal category in the following way. Any left $\mathcal{H}$-comodule (M, τ) can be equipped with a right L-module structure via

(2.8) $$m \cdot l := \pi(m_{\langle -1 \rangle} s(l)) \cdot m_{\langle 0 \rangle} \qquad \text{for } m \in M, \ l \in L$$

where $m_{\langle -1 \rangle} \otimes_L m_{\langle 0 \rangle}$ stands for $\tau(m)$ (summation understood). Indeed, (2.12) is the unique right action via which M becomes an L-L bimodule and τ becomes an L-L bimodule map from ${}_LM_L$ to the Takeuchi product $H \times_L M$. Recall from [36] that $H \times_L M$ is the L-L submodule of ${}_LH^L_{\mathrm{L}} \otimes {}_{\mathrm{L}}M$ the elements $\sum_i h_i \otimes_L m_i$ of which satisfy

(2.9) $$\sum_i h_i \otimes_L m_i \cdot l = \sum_i h_i t(l) \otimes_L m_i \qquad \text{for } \ l \in L.$$

This observation amounts to saying that our definition of left $\mathcal{H}$-comodules is equivalent to ([28], Definition 5.5). Hence, without loss of generality, from now on we can think of a comodule in this latter sense. On the basis of ([28], Definition 5.5) the category ${}^H\mathcal{M}$ of left $\mathcal{H}$ -comodules was shown in ([28], Proposition 5.6) to be monoidal. The monoidal product is the L-module tensor product with comodule structure

$$M \otimes_L N \to H \otimes_L M \otimes_L N \qquad m \otimes_L n \mapsto m_{\langle -1 \rangle} n_{\langle -1 \rangle} \otimes_L m_{\langle 0 \rangle} \otimes_L n_{\langle 0 \rangle}$$

and the monoidal unit is L with comodule structure

$$L \to L \otimes_L H \simeq H \qquad l \mapsto s(l).$$

Following ([28], Definition 5.7), a left *comodule algebra* for a left bialgebroid $\mathcal{H}$ is a monoid in the monoidal category ${}^H\mathcal{M}$. Notice that – in view of the equivalence of the two definitions of $\mathcal{H}$ -comodules – this definition of comodule algebras is equivalent to ([10], Definition 3.4).

By similar arguments also the category of right $\mathcal{H}$-comodules – that is of right comodules for the L-coring (H, γ, π) – is monoidal. The monoidal product is the L-module tensor product with coaction

$$M \otimes_L N \to M \otimes_L N \otimes_L H \qquad m \otimes_L n \mapsto m_{\langle 0\rangle} \otimes_L n_{\langle 0\rangle} \otimes_L n_{\langle 1\rangle} m_{\langle 1\rangle}$$

and the monoidal unit is L with comodule structure

$$L \to H \qquad l \mapsto t(l),$$

where the left L-module structure of a right (H, γ, π)-comodule M is defined as $l \cdot m := m_{\langle 0\rangle} \cdot \pi(m_{\langle 1\rangle} s(l))$. Similarly to the case of left comodule algebras we expect the coaction of a right comodule algebra A to be multiplicative – i.e. such that $(aa')_{\langle 0\rangle} \otimes_L (aa')_{\langle 1\rangle} = a_{\langle 0\rangle} a'_{\langle 0\rangle} \otimes_L a_{\langle 1\rangle} a'_{\langle 1\rangle}$ for $a, a' \in A$. Therefore we consider the monoidal category $(\mathcal{M}^H)^{op}$, the monoidal structure of which is the opposite of $\mathcal{M}^H$ (i.e. it comes from the monoidal structure of ${}_{L^{op}}\mathcal{M}_{L^{op}}$). A right $\mathcal{H}$-comodule algebra is defined as a monoid in the category $(\mathcal{M}^H)^{op}$.

Notice that a left $\mathcal{H}$-comodule algebra is in particular an L-ring while a right $\mathcal{H}$-comodule algebra is an L^{op}-ring.

The *coinvariants* of a left (right) $\mathcal{H}$-comodule algebra (A, τ) are the elements of the subalgebra

$$A^{coH} := \{\ a \in A \mid \tau(a) = 1_H \otimes_L a\ \} \quad (\ A^{coH} := \{\ a \in A \mid \tau(a) = a \otimes_L 1_H\ \}\ ).$$

Recall that for a left L-bialgebroid $\mathcal{H}$ the left and right L-duals ${}_*H$ and H_* are rings. The category $\mathcal{M}_{{}_*H}$ of right ${}_*H$-modules is a full subcategory in the category $\mathcal{M}^H$ of right $\mathcal{H}$-comodules and the two categories are equivalent if and only if the module ${}_LH$ is finitely generated and projective. The category $\mathcal{M}_{H_*}$ is a full subcategory in ${}^H\mathcal{M}$ and they are equivalent if and only if the module H_L is finitely generated and projective.

The left and right comodules, comodule algebras and their coinvariants for a right bialgebroid are defined analogously.

2.3 – *Entwining structures over non-commutative algebras.*

Entwining structures over non-commutative algebras were introduced in [4] as mixed distributive laws in the bicategory of [Algebras, Bimodules, Bimodule maps]. This definition is clearly equivalent to a monad in the bicategory of corings i.e. the bicategory of comonads [33] in the bicategory of [Algebras, Bimodules, Bimodule maps]. Explicitly, we have

DEFINITION 2.3. An *entwining structure* over an algebra R is a triple (A, C, ψ) where $A = (A, \mu, \eta)$ is an R-ring, $C = (C, \Delta, \varepsilon)$ is an R-coring and ψ is an R-R bimodule map $C \underset{R}{\otimes} A \to A \underset{R}{\otimes} C$ satisfying

$$\psi \circ (C \underset{R}{\otimes} \eta) = \eta \underset{R}{\otimes} C$$

$$(A \underset{R}{\otimes} \varepsilon) \circ \psi = \varepsilon \underset{R}{\otimes} A$$

$$(\mu \underset{R}{\otimes} C) \circ (A \underset{R}{\otimes} \psi) \circ (\psi \underset{R}{\otimes} A) = \psi \circ (C \underset{R}{\otimes} \mu)$$

$$(A \underset{R}{\otimes} \Delta) \circ \psi = (\psi \underset{R}{\otimes} C) \circ (C \underset{R}{\otimes} \psi) \circ (\Delta \underset{R}{\otimes} A).$$

An entwining structure is *bijective* if ψ is an isomorphism.

It is shown in ([4], Example 4.5) that an entwining structure (A, C, ψ) over the algebra R determines an A-coring structure on $A \underset{R}{\otimes} C$ with A-A bimodule structure

$$a_1 \cdot (a \underset{R}{\otimes} c) \cdot a_2 = a_1 a \psi(c \underset{R}{\otimes} a_2) \qquad \text{for } a_1, a_2 \in A,\ a \underset{R}{\otimes} c \in A \underset{R}{\otimes} C,$$

coproduct $A \underset{R}{\otimes} \Delta$ and counit $A \underset{R}{\otimes} \varepsilon$.

DEFINITION 2.4. A right *entwined module* over an R-entwining structure (A, C, ψ) is a right comodule over the corresponding A-coring $A \underset{R}{\otimes} C$. Explicitly, it is a triple (M, ρ, τ), where (M, ρ) is a right A-module, making M in particular a right R-module. The pair (M, τ) is a right C-comodule such that τ is a right A-module map i.e.

$$\tau \circ \rho = (\rho \underset{R}{\otimes} C) \circ (M \underset{R}{\otimes} \psi) \circ (\tau \underset{R}{\otimes} A).$$

A *morphism* of entwined modules is a morphism of comodules for the A-coring $A \underset{R}{\otimes} C$, that is, an A-linear and C-colinear map. The category of entwined modules will be denoted by $\mathcal{M}_A^C$.

By ([14], 18.13 (2)) the forgetful functor $\mathcal{M}_A^C \to \mathcal{M}_A$ possesses a right adjoint, the functor

$$(2.10) \qquad \_ \underset{R}{\otimes} C : \mathcal{M}_A \to \mathcal{M}_A^C \quad (M, \rho) \mapsto (M \underset{R}{\otimes} C\ ,\ (\rho \underset{R}{\otimes} C) \circ (M \underset{R}{\otimes} \psi)\ ,\ M \underset{R}{\otimes} \Delta),$$

where the right R-module structure of M comes from its A-module structure. What is more, by the self-duality of the notion of R-entwining structures, also ([14], 32.8 (3)) extends to entwining structures over non-commutative algebras. That is, also the forgetful functor $\mathcal{M}_A^C \to \mathcal{M}^C$ possesses a left adjoint, the functor

$$(2.11) \qquad \_ \underset{R}{\otimes} A : \mathcal{M}^C \to \mathcal{M}_A^C \quad (M, \tau) \mapsto (M \underset{R}{\otimes} A\ ,\ (M \underset{R}{\otimes} \mu)\ ,\ (M \underset{R}{\otimes} \psi) \circ (\tau \underset{R}{\otimes} A)).$$

This implies, in particular, that both $A \underset{R}{\otimes} C$ and $C \underset{R}{\otimes} A$ are entwined modules. The morphism ψ becomes a morphism of entwined modules.

If the R-coring C possesses a grouplike element, e, then also the A-coring $A \underset{R}{\otimes} C$ possesses a grouplike element, $1_A \underset{R}{\otimes} e$. Hence A is an entwined module via the right regular A-action and the C-coaction

$$A \to A \underset{R}{\otimes} C \qquad a \mapsto \psi(e \underset{R}{\otimes} a).$$

The *coinvariants* of an entwined module are defined as its coinvariants as a comodule for the A-coring $A \underset{R}{\otimes} C$, w.r.t. the grouplike element $1_A \underset{R}{\otimes} e$. This is the same as its C-coinvariants w.r.t. e.

In this case also the functors

$$(2.12) \qquad (\_)^{coC} : \mathcal{M}_A^C \to \mathcal{M}_{A^{coC}} \qquad \text{and} \qquad \_ \underset{A^{coC}}{\otimes} A : \mathcal{M}_{A^{coC}} \to \mathcal{M}_A^C$$

are adjoints ([14], 28.8). (The entwined module structure of $N \underset{A^{coC}}{\otimes} A$ for a right A^{coC}-module N is defined via the second tensor factor). The unit and the counit of the adjunction are

$$\eta_N : N \to (N \underset{A^{coC}}{\otimes} A)^{coC} \qquad n \mapsto n \underset{A^{coC}}{\otimes} 1_A \quad \text{and}$$
$$\mu_M : M^{coC} \underset{A^{coC}}{\otimes} A \to M \qquad m \underset{A^{coC}}{\otimes} a \mapsto m \cdot a$$

for any right A^{coC}-module N and entwined module M.

2.4 – *Morita theory for corings.*

In the paper [18] a Morita context $(A^{^*C}, {}^*C, A, {}^*C^{^*C}, \upsilon, \mu)$ has been associated to an A-coring C possessing a grouplike element e. Here the ring ${}^*C = {}_A\mathrm{Hom}(C, A)$ is the left A-dual of the A-coring C with multiplication $(fg)(c) = g(c_{(1)} \cdot f(c_{(2)}))$. The invariants of a right ${}^*C$-module M are defined with the help of the grouplike element e as the elements of

$$M^{^*C} := \{ \, m \in M \mid m \cdot f = m \cdot [\varepsilon(\_)f(e)] \quad \forall f \in {}^*C \, \}.$$

In terms of the grouplike element e the k-module A can be equipped with a right ${}^*C$-module structure as

$$a \cdot f := f(e \cdot a) \qquad \text{for } a \in A, f \in {}^*C.$$

The ring $A^{^*C}$ is the subring of ${}^*C$-invariants of A i.e.

$$A^{^*C} = \{ \, b \in A \mid f(e \cdot b) = bf(e) \quad \forall f \in {}^*C \, \}.$$

A is an $A^{^*C}$ - ${}^*C$ bimodule via

$$b \cdot a \cdot f := bf(e \cdot a) = f(e \cdot ba) \qquad \text{for } b \in A^{^*C}, a \in A, f \in {}^*C.$$

${}^{*}C^{{}^{*}C}$ is the k-module of ${}^{*}C$-invariants of the right regular ${}^{*}C$-module i.e.

$${}^{*}C^{{}^{*}C} = \{\ q \in {}^{*}C \mid f(c_{(1)} \cdot q(c_{(2)})) = f(q(c) \cdot e) \quad \forall f \in {}^{*}C,\ c \in C\ \}.$$

It is a ${}^{*}C$ - $A^{{}^{*}C}$ bimodule via

$$(f \cdot q \cdot b)(c) := q(c_{(1)} \cdot f(c_{(2)}))b \qquad \text{for } f \in {}^{*}C,\ q \in {}^{*}C^{{}^{*}C},\ b \in A^{{}^{*}C}, c \in C.$$

The connecting maps υ and μ are given as

$$\upsilon : A \underset{{}^{*}C}{\otimes} {}^{*}C^{{}^{*}C} \to A^{{}^{*}C} \qquad a \underset{{}^{*}C}{\otimes} q \mapsto q(e \cdot a) \quad \text{and}$$

$$\mu : {}^{*}C^{{}^{*}C} \underset{A^{{}^{*}C}}{\otimes} A \to {}^{*}C \qquad q \underset{A^{{}^{*}C}}{\otimes} a \mapsto (\ c \mapsto q(c)a\ ).$$

In [18] the following theorem has been proven.

THEOREM 2.5. *Let C be an A-coring possessing a grouplike element e, and let $(A^{{}^{*}C}, {}^{*}C, A, {}^{*}C^{{}^{*}C}, \upsilon, \mu)$ be the Morita context associated to it.*

(1) ([18], *Theorem* 3.5). *If C is finitely generated and projective as a left A-module (hence the categories $\mathcal{M}^{C}$ and $\mathcal{M}_{{}^{*}C}$ are isomorphic and the C-coinvariants coincide with the ${}^{*}C$-invariants) then the following assertions are equivalent.*

(a) *The map μ is surjective (and, a fortiori, bijective).*

(b) *The functor $(\ \_\ )^{coC} : \mathcal{M}^{C} \to \mathcal{M}_{A^{coC}}$ is fully faithful.*

(c) *A is a right ${}^{*}C$-generator.*

(d) *A is projective as a left A^{coC}-module and the map*

$${}^{*}C \to {}_{A^{coC}}\mathrm{End}\,(A) \qquad f \mapsto (\ a \mapsto f(e \cdot a)\ )$$

is an algebra anti-isomorphism.

(e) *A is projective as a left A^{coC}-module and the A-coring C with grouplike element e is a Galois coring.*

(2) ([18], *Theorem* 2.7). *If the algebra extension*

$$A \to {}^{*}C \qquad a \mapsto (\ c \mapsto \varepsilon(c)a\ )$$

is a Frobenius extension with Frobenius system $(\psi, u_i \underset{A}{\otimes} v_i)$ then the Morita context $(A^{{}^{*}C}, {}^{*}C, A,\ {}^{*}C^{{}^{*}C}, \upsilon, \mu)$ is equivalent to the Morita context $(A^{{}^{*}C}, {}^{*}C, A, A, \upsilon', \mu')$ via the isomorphism

$$A \to {}^{*}C^{{}^{*}C} \qquad a \mapsto (\ c \mapsto \sum_i v_i[c \cdot a u_i(e)]\ ).$$

3. – Hopf algebroid extensions.

An extension $B \subset A$ is an $\mathcal{H}_R$ -extension for a right bialgebroid $\mathcal{H}_R = (H, R, s, t, \gamma, \pi)$ if A is a right $\mathcal{H}_R$-comodule algebra and B is the subalgebra of $\mathcal{H}_R$-coinvariants of A. In this situation – denoting the R-coring (H, γ, π) by C – the triple $(A, \mathcal{H}_R, C)$ is a Doi-Koppinen datum over R in the sense of ([10], Definition 3.6). This implies that the R-R bimodule map

$$\psi : H \underset{R}{\otimes} A \to A \underset{R}{\otimes} H \qquad h \underset{R}{\otimes} a \mapsto a^{\langle 0 \rangle} \underset{R}{\otimes} ha^{\langle 1 \rangle} \tag{3.13}$$

gives rise to an entwining structure (A, C, ψ) over R. Hence the R-R bimodule $A \underset{R}{\otimes} H$ possesses an A-coring structure with A-A bimodule structure

$$a_1 \cdot (a \underset{R}{\otimes} h) \cdot a_2 = a_1 a a_2^{\langle 0 \rangle} \underset{R}{\otimes} h a_2^{\langle 1 \rangle} \qquad \text{for } a_1, a_2 \in A, \ a \underset{R}{\otimes} h \in A \underset{R}{\otimes} H,$$

coproduct $A \underset{R}{\otimes} \gamma$ and counit $A \underset{R}{\otimes} \pi$. This coring possesses a grouplike element $1_A \underset{R}{\otimes} 1_H$. The $\mathcal{H}_R$-extension $B \subset A$ was termed $\mathcal{H}_R$-Galois in [23] if the A-coring $A \underset{R}{\otimes} H$, associated to it above, is a Galois coring. This property is equivalent to the bijectivity of the canonical map

$$\mathrm{can}_R : A \underset{B}{\otimes} A \to A \underset{R}{\otimes} H \qquad a \underset{B}{\otimes} a' \mapsto aa'^{\langle 0 \rangle} \underset{R}{\otimes} a'^{\langle 1 \rangle}. \tag{3.14}$$

Analogously, in the case of a right comodule algebra A for the left bialgebroid $\mathcal{H}_L = (H, L, s, t, \gamma, \pi)$ the $\mathcal{H}_L$-Galois property of the extension $A^{co\mathcal{H}_L} \subset A$ means the bijectivity of the canonical map

$$\mathrm{can}_L : A \underset{A^{co\mathcal{H}_L}}{\otimes} A \to A \underset{L}{\otimes} H \qquad a \underset{B}{\otimes} a' \mapsto a_{\langle 0 \rangle} a' \underset{L}{\otimes} a_{\langle 1 \rangle}. \tag{3.15}$$

This is equivalent to the Galois property of the A-coring $A \underset{L}{\otimes} H$ with A-A bimodule structure

$$a_1 \cdot (a \underset{L}{\otimes} h) \cdot a_2 = a_{1\langle 0 \rangle} a a_2 \underset{L}{\otimes} a_{1\langle 1 \rangle} h \qquad \text{for } a_1, a_2 \in A, \ a \underset{L}{\otimes} h \in A \underset{L}{\otimes} H,$$

coproduct $A \underset{L}{\otimes} \gamma$, counit $A \underset{L}{\otimes} \pi$ and grouplike element $1_A \underset{L}{\otimes} 1_H$. (Recall from Subsection 2.2 that by our convention A is an L^{op}-ring in this case.)

A Hopf algebroid $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$ possesses two compatible bialgebroid structures, $\mathcal{H}_L$ and $\mathcal{H}_R$, on the same total algebra H. It turns out that the notions of comodules (comodule algebras) for the constituent left bialgebroid $\mathcal{H}_L$ and for the constituent right bialgebroid $\mathcal{H}_R$ are related as follows.

PROPOSITION 3.1. (1) *Let $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$ be a Hopf algebroid and (M, τ_R) a right $\mathcal{H}_R$-comodule. Making M an L-L bimodule using its R-R bimodule structure with*

$$l \cdot m \cdot l' = \pi_R \circ t_L(l') \cdot m \cdot \pi_R \circ t_L(l), \tag{3.16}$$

there exists a unique right $\mathcal{H}_L$-coaction τ_L on M such that τ_R is right $\mathcal{H}_L$-colinear and τ_L is right $\mathcal{H}_R$-colinear. That is, the identities

$$(\tau_R \otimes_L H) \circ \tau_L = (M \otimes_R \gamma_L) \circ \tau_R \tag{3.17}$$

$$(\tau_L \otimes_R H) \circ \tau_R = (M \otimes_L \gamma_R) \circ \tau_L \tag{3.18}$$

hold true. Furthermore, the $\mathcal{H}_R$-coinvariants of M coincide with its $\mathcal{H}_L$-coinvariants.

(2) *Let $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$ be a Hopf algebroid and (A, τ_R) a right $\mathcal{H}_R$-comodule algebra. Then A is a right $\mathcal{H}_L$-comodule algebra with the $\mathcal{H}_L$-coaction τ_L in* (1).

(3) *Let $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$ be a Hopf algebroid and (M, τ_R^M) and (N, τ_R^N) two right $\mathcal{H}_R$-comodules. Considering the right $\mathcal{H}_L$-comodule structures of M and N as in* (1), *any right $\mathcal{H}_R$-colinear map is also $\mathcal{H}_L$-colinear.*

PROOF. Denote $\tau_R(m) = m^{\langle 0\rangle} \otimes_R m^{\langle 1\rangle}$ for $m \in M$. Recall from Subsection 2.2 that M is an R-R bimodule with

$$r \cdot m \cdot r' = m^{\langle 0\rangle} \cdot \pi_R(s_R(r) m^{\langle 1\rangle} s_R(r')) \equiv m^{\langle 0\rangle} \cdot \pi_R(s_R(r) m^{\langle 1\rangle})\, r', \tag{3.19}$$

and an L-L bimodule with (3.16). The required right $\mathcal{H}_L$-coaction on M is given by the map

$$\tau_L : M \to M \otimes_L H \qquad m \mapsto m^{\langle 0\rangle} \cdot \pi_R({m^{\langle 1\rangle}}_{(1)}) \otimes_L {m^{\langle 1\rangle}}_{(2)}. \tag{3.20}$$

In order to see that the map (3.20) is well defined, we have to show that it makes sense to apply the map $(M \otimes_R \pi_R) \otimes_L H$ to the element $m^{\langle 0\rangle} \otimes_R {m^{\langle 1\rangle}}_{(1)} \otimes_L {m^{\langle 1\rangle}}_{(2)}$ of $M \otimes_R H \otimes_L H$. This can be seen, on one hand, by using the left R-linearity of γ_L (i.e. the identity $\gamma_L(h t_R(r)) = h_{(1)} t_R(r) \otimes_L h_{(2)}$) and the left R-linearity of π_R. On the other hand, the image of τ_R is in the Takeuchi product $M \times_R H$ (see Subsection 2.2), hence $m^{\langle 0\rangle} \otimes_R {m^{\langle 1\rangle}}_{(1)} \otimes_L {m^{\langle 1\rangle}}_{(2)}$ is an element of $(M \times_R H) \otimes_L H$. Since the restriction of the map $M \otimes_R \pi_R : M \otimes_R H \to M$ to $M \times_R H$ is right L-linear, the expression on the right hand side of (3.20) makes sense.

In order to see that M is a right $\mathcal{H}_L$-comodule we have to check the counitality and the coassociativity of the map (3.20). It goes as follows.

$$\begin{aligned}(M \otimes_L \pi_L) \circ \tau_L(m) =& (m^{\langle 0\rangle} \cdot \pi_R({m^{\langle 1\rangle}}_{(1)})) \cdot \pi_L({m^{\langle 1\rangle}}_{(2)})\\ =& {m^{\langle 0\rangle}}^{\langle 0\rangle} \cdot \pi_R[t_L \circ \pi_L({m^{\langle 1\rangle}}_{(2)}) {m^{\langle 0\rangle}}^{\langle 1\rangle} s_R \circ \pi_R({m^{\langle 1\rangle}}_{(1)})]\\ =& m^{\langle 0\rangle} \cdot \pi_R[t_L \circ \pi_L({m^{\langle 1\rangle}}_{(2)}) {{m^{\langle 1\rangle}}_{(1)}}^{(1)} s_R \circ \pi_R({{m^{\langle 1\rangle}}_{(1)}}^{(2)})] = m,\end{aligned}$$

where the second equality follows from (3.16) and (3.19), the third one from the coassociativity of τ_R and the Hopf algebroid axiom (Definition 2.2 (ii)) and the last one from the counitality of the right and left coproducts γ_R and γ_L together with the counitality of τ_R.

$$
\begin{aligned}
(M \underset{L}{\otimes} \gamma_L)\circ\tau_L(m) &= m^{\langle 0\rangle}\cdot\pi_R(m^{\langle 1\rangle}{}_{(1)}) \underset{L}{\otimes} m^{\langle 1\rangle}{}_{(2)(1)} \underset{L}{\otimes} m^{\langle 1\rangle}{}_{(2)(2)}\\
&= m^{\langle 0\rangle}\cdot\pi_R(m^{\langle 1\rangle}{}_{(1)}) \underset{L}{\otimes} m^{\langle 1\rangle}{}_{(2)(1)}{}^{(1)} s_R\circ\pi_R(m^{\langle 1\rangle}{}_{(2)(1)}{}^{(2)}) \underset{L}{\otimes} m^{\langle 1\rangle}{}_{(2)(2)}\\
&= m^{\langle 0\rangle\langle 0\rangle}\cdot\pi_R(m^{\langle 0\rangle\langle 1\rangle}{}_{(1)}) \underset{L}{\otimes} m^{\langle 0\rangle\langle 1\rangle}{}_{(2)} s_R\circ\pi_R(m^{\langle 1\rangle}{}_{(1)}) \underset{L}{\otimes} m^{\langle 1\rangle}{}_{(2)}\\
&= (\tau_L \underset{L}{\otimes} H)\circ\tau_L(m),
\end{aligned}
$$

where the second equality follows from the counitality of γ_R, the penultimate one from the Hopf algebroid axiom (Definition 2.2 (ii)) and the coassociativity of τ_R and the last one from the right R-linearity of τ_R and of γ_L (i.e. of τ_L).

The $\mathcal{H}_R$-colinearity of τ_L is checked as

$$
\begin{aligned}
(\tau_L \underset{R}{\otimes} H)\circ\tau_R(m) &= m^{\langle 0\rangle\langle 0\rangle}\cdot\pi_R(m^{\langle 0\rangle\langle 1\rangle}{}_{(1)}) \underset{L}{\otimes} m^{\langle 0\rangle\langle 1\rangle}{}_{(2)} \underset{R}{\otimes} m^{\langle 1\rangle}\\
&= m^{\langle 0\rangle}\cdot\pi_R(m^{\langle 1\rangle}{}_{(1)}) \underset{L}{\otimes} m^{\langle 1\rangle}{}_{(2)}{}^{(1)} \underset{R}{\otimes} m^{\langle 1\rangle}{}_{(2)}{}^{(2)}\\
&= (M \underset{L}{\otimes} \gamma_R)\circ\tau_L(m),
\end{aligned}
$$

where the second equality follows from the coassociativity of τ_R and the Hopf algebroid axiom (Definition 2.2 (ii)). Also τ_R is $\mathcal{H}_L$-colinear as

$$
\begin{aligned}
(\tau_R \underset{L}{\otimes} H)\circ\tau_L(m) &= m^{\langle 0\rangle\langle 0\rangle} \underset{R}{\otimes} m^{\langle 0\rangle\langle 1\rangle} s_R\circ\pi_R(m^{\langle 1\rangle}{}_{(1)}) \underset{L}{\otimes} m^{\langle 1\rangle}{}_{(2)}\\
&= m^{\langle 0\rangle} \underset{R}{\otimes} m^{\langle 1\rangle}{}_{(1)}{}^{(1)} s_R\circ\pi_R(m^{\langle 1\rangle}{}_{(1)}{}^{(2)}) \underset{L}{\otimes} m^{\langle 1\rangle}{}_{(2)}\\
&= (M \underset{R}{\otimes} \gamma_L)\circ\tau_R(m),
\end{aligned}
$$

where the first equality follows from the right R-linearity of τ_R, the second one from the coassociativity of τ_R and the Hopf algebroid axiom (Definition 2.2 (ii)) and the last one from the counitality of γ_R.

The $\mathcal{H}_L$-coaction (3.20) is unique since any right $\mathcal{H}_R$-colinear right $\mathcal{H}_L$-coaction $m \mapsto m_{\langle 0\rangle} \underset{L}{\otimes} m_{\langle 1\rangle}$ on M satisfies

$$
m_{\langle 0\rangle} \underset{L}{\otimes} m_{\langle 1\rangle} = m_{\langle 0\rangle}{}^{\langle 0\rangle}\cdot\pi_R(m_{\langle 0\rangle}{}^{\langle 1\rangle}) \underset{L}{\otimes} m_{\langle 1\rangle} = m^{\langle 0\rangle}\cdot\pi_R(m^{\langle 1\rangle}{}_{(1)}) \underset{L}{\otimes} m^{\langle 1\rangle}{}_{(2)} = \tau_L(m).
$$

It is clear from (3.20) that any $\mathcal{H}_R$-coinvariant in M is $\mathcal{H}_L$-coinvariant. Writing $\tau_L(m) = m_{\langle 0\rangle} \underset{L}{\otimes} m_{\langle 1\rangle}$ and using the right $\mathcal{H}_R$-colinearity of τ_L the relation (3.20) can be inverted to yield

$$
m^{\langle 0\rangle} \underset{R}{\otimes} m^{\langle 1\rangle} = m_{\langle 0\rangle}\cdot\pi_L(m_{\langle 1\rangle}{}^{(1)}) \underset{R}{\otimes} m_{\langle 1\rangle}{}^{(2)},
$$

which shows that any $\mathcal{H}_L$-coinvariant in M is also $\mathcal{H}_R$-coinvariant.

(2) Let (A, τ_R) be a right $\mathcal{H}_R$-comodule algebra. Then by (1) A is an $\mathcal{H}_L$-co-

module. It remains to check that the $\mathcal{H}_L$-coaction τ_L on A is multiplicative. Indeed, for $a, a' \in A$

$$\begin{aligned}
\tau_L(a)\tau_L(a') =& [a^{\langle 0\rangle} \cdot \pi_R(a^{\langle 1\rangle}{}_{(1)})][a'^{\langle 0\rangle} \cdot \pi_R(a'^{\langle 1\rangle}{}_{(1)})] \otimes_L a^{\langle 1\rangle}{}_{(2)} a'^{\langle 1\rangle}{}_{(2)} \\
=& a^{\langle 0\rangle}[\pi_R(a^{\langle 1\rangle}{}_{(1)}) \cdot a'^{\langle 0\rangle} \cdot \pi_R(a'^{\langle 1\rangle}{}_{(1)})] \otimes_L a^{\langle 1\rangle}{}_{(2)} a'^{\langle 1\rangle}{}_{(2)} \\
=& a^{\langle 0\rangle}[a'^{\langle 0\rangle} \cdot \pi_R\Big(t_R \circ \pi_R(a^{\langle 1\rangle}{}_{(1)}) a'^{\langle 1\rangle}{}_{(1)}\Big)] \otimes_L a^{\langle 1\rangle}{}_{(2)} a'^{\langle 1\rangle}{}_{(2)} \\
=& (a^{\langle 0\rangle} a'^{\langle 0\rangle}) \cdot \pi_R(a^{\langle 1\rangle}{}_{(1)} a'^{\langle 1\rangle}{}_{(1)}) \otimes_L a^{\langle 1\rangle}{}_{(2)} a'^{\langle 1\rangle}{}_{(2)} \\
=& \tau_L(aa'),
\end{aligned}$$

where the second equality follows from the fact that A is an R-ring, the third one from the fact that the image of τ_R is in the Takeuchi product $A \times_R H$, the fourth one from the right bialgebroid version of axiom (2.6) and the R-ring structure of A and the last one from the multiplicativity of τ_R and of γ_L.

(3) Suppose that $f : M \to N$ is a right $\mathcal{H}_R$-colinear map. Denote the $\mathcal{H}_L$-coactions on M and on N as in (1) by $\tau_L^M(m) = m_{\langle 0\rangle} \otimes_L m_{\langle 1\rangle}$ for $m \in M$ and $\tau_L^N(n) = n_{\langle 0\rangle} \otimes_L n_{\langle 1\rangle}$ for $n \in N$, respectively. Then

$$\begin{aligned}
f(m)_{\langle 0\rangle} \otimes_L f(m)_{\langle 1\rangle} =& f(m)_{\langle 0\rangle}{}^{\langle 0\rangle} \cdot \pi_R(f(m)_{\langle 0\rangle}{}^{\langle 1\rangle}) \otimes_L f(m)_{\langle 1\rangle} \\
=& f(m)^{\langle 0\rangle} \cdot \pi_R(f(m)^{\langle 1\rangle}{}_{(1)}) \otimes_L f(m)^{\langle 1\rangle}{}_{(2)} \\
=& f(m^{\langle 0\rangle}) \cdot \pi_R(m^{\langle 1\rangle}{}_{(1)}) \otimes_L m^{\langle 1\rangle}{}_{(2)} \\
=& f(m^{\langle 0\rangle} \cdot \pi_R(m^{\langle 1\rangle}{}_{(1)})) \otimes_L m^{\langle 1\rangle}{}_{(2)} \\
=& f(m_{\langle 0\rangle}{}^{\langle 0\rangle} \cdot \pi_R(m_{\langle 0\rangle}{}^{\langle 1\rangle})) \otimes_L m_{\langle 1\rangle} = f(m_{\langle 0\rangle}) \otimes_L m_{\langle 1\rangle},
\end{aligned}$$

where the first and the last equalities follow from the counitality of τ_R, the second and the penultimate ones from the right $\mathcal{H}_L$-colinearity of τ_R, the third one from the $\mathcal{H}_R$-colinearity of f and the fourth one from the right R-linearity of f. ■

In view of Proposition 3.1 the usage of both the $\mathcal{H}_L$- and the $\mathcal{H}_R$-comodule structures of a comodule for a Hopf algebroid $(\mathcal{H}_L, \mathcal{H}_R, S)$ is redundant. On the other hand, it turns out that a symmetric treatment makes the theory more transparent. Therefore we make the following

DEFINITION 3.2. A right *comodule for the Hopf algebroid* $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$ is a triple (M, τ_L, τ_R), where the pair (M, τ_R) is a right $\mathcal{H}_R$-comodule and (M, τ_L) is a right $\mathcal{H}_L$-comodule such that the R-R and the L-L bimodule structures of M are related via (3.16) and the compatibility relations (3.21-3.22) hold true.

The right $\mathcal{H}$-comodule (A, τ_L, τ_R) is a right *$\mathcal{H}$-comodule algebra* if (A, τ_R) is a right $\mathcal{H}_R$-comodule algebra. Equivalently, if (A, τ_L) is a right $\mathcal{H}_L$-comodule algebra.

An extension $B \subset A$ is an *$\mathcal{H}$-extension* if A is a right $\mathcal{H}$ -comodule algebra such that B is the coinciding subalgebra of the $\mathcal{H}_L$- and of the $\mathcal{H}_R$-coinvariants.

An $\mathcal{H}$-extension $B \subset A$ is *$\mathcal{H}$-Galois* if it is both $\mathcal{H}_L$-Galois and $\mathcal{H}_R$-Galois.

We follow the convention of using upper indices to denote the components of the coaction of a right bialgebroid and lower indices in the case of a left bialgebroid.

LEMMA 3.3. *Let $\mathcal{H}$ be a Hopf algebroid with bijective antipode and let $B \subset A$ be an $\mathcal{H}$-extension. The canonical map*

$$\mathrm{can}_R : A \underset{B}{\otimes} A \to A \underset{R}{\otimes} H \qquad a \underset{B}{\otimes} a' \mapsto aa'^{\langle 0\rangle} \underset{R}{\otimes} a'^{\langle 1\rangle}$$

is injective/surjective/bijective if and only if the canonical map

$$\mathrm{can}_L : A \underset{B}{\otimes} A \to A \underset{L}{\otimes} H \qquad a \underset{B}{\otimes} a' \mapsto a_{\langle 0\rangle} a' \underset{L}{\otimes} a_{\langle 1\rangle}$$

is injective/surjective/bijective.

PROOF. For any right $\mathcal{H}$-comodule (M, τ_L, τ_R) there exists an isomorphism

$$\Phi_M : M \underset{R}{\otimes} H \to M \underset{L}{\otimes} H \qquad m \underset{R}{\otimes} h \mapsto m_{\langle 0\rangle} \underset{L}{\otimes} m_{\langle 1\rangle} S(h)$$

with inverse

$$\Phi_M^{-1} : M \underset{L}{\otimes} H \to M \underset{R}{\otimes} H \qquad m \underset{L}{\otimes} h \mapsto m^{\langle 0\rangle} \underset{R}{\otimes} S^{-1}(h) m^{\langle 1\rangle}.$$

Using the $\mathcal{H}_R$-colinearity of τ_L, the Hopf algebroid axiom (Definition 2.2 (iv)), the fact that the image of τ_L is in the Takeuchi product $A \times_L H$, the L^{op}-ring structure of A and the counitality of τ_L one checks that for an $\mathcal{H}$-comodule algebra (A, τ_L, τ_R) the identity $\Phi_A \circ \mathrm{can}_R = \mathrm{can}_L$ holds true. ■

EXAMPLE 3.4. A k-bialgebra $(H, \mu, \eta, \Delta, \varepsilon)$ is an example both of a left- and of a right bialgebroid via the correspondence $\mathcal{B} := (H, k, \eta, \eta, \Delta, \varepsilon)$. A Hopf algebra $(H, \mu, \eta, \Delta, \varepsilon, S)$ is an example of a Hopf algebroid via $\mathcal{H} := (\mathcal{B}, \mathcal{B}, S)$.

A right H-comodule (algebra) (A, τ) is an example of a right $\mathcal{B}$-comodule (algebra) and gives rise to an $\mathcal{H}$-comodule (algebra) via (A, τ, τ). The H-coinvariants are obviously the same as the $\mathcal{B}$-coinvariants and the extension $A^{coH} \subset A$ is H-Galois if and only if it is $\mathcal{B}$-Galois.

EXAMPLE 3.5. A weak bialgebra ([27, 6]) has been shown to determine a left bialgebroid in ([22, 29, 34]). Weak comodule algebras ([2, 16, 17]) over a weak bialgebra have been shown in ([10], Proposition 3.9) to be equivalent to comodule algebras for the corresponding left bialgebroid.

Now just the same way as the weak bialgebra determines a left bialgebroid, it determines also a right bialgebroid, and a weak Hopf algebra determines a Hopf algebroid ([7], Example 4.8). A weak comodule algebra for the weak bialgebra is equivalent also to a comodule algebra for the corresponding right bialgebroid, and a weak comodule algebra for a weak Hopf algebra is equivalent to a comodule algebra for the corresponding Hopf algebroid.

The coinvariants of a weak comodule algebra for a weak bialgebra are the same as its coinvariants for the corresponding left or right bialgebroid.

A weak bialgebra extension is Galois in the sense of [17] if and only if the corresponding right bialgebroid extension is Galois. In the case of a weak Hopf algebra with bijective antipode this is equivalent also to the Galois property of the corresponding left bialgebroid extension.

EXAMPLE 3.6. The total algebra H of a right bialgebroid $\mathcal{H}_R = (H, R, s, t, \gamma, \pi)$ is a right $\mathcal{H}_R$-comodule algebra via γ. We have $H^{co\mathcal{H}_R} = t(R)$.

The total algebra H of a left bialgebroid $\mathcal{H}_L = (H, L, s, t, \gamma, \pi)$ is a right $\mathcal{H}_L$-comodule algebra via γ and $H^{co\mathcal{H}_L} = s(L)$.

For a Hopf algebroid $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$ the total algebra is then a right $\mathcal{H}$-comodule algebra and the canonical map

$$\mathrm{can}_R : {}^RH \otimes H_R \to H^R \otimes {}^RH \qquad h \underset{R^{op}}{\otimes} h' \mapsto hh'^{(1)} \underset{R}{\otimes} h'^{(2)}$$

is bijective with inverse

$$\mathrm{can}_R^{-1} : H^R \otimes {}^RH \to {}^RH \otimes H_R \qquad h \underset{R}{\otimes} h' \mapsto hS(h'_{(1)}) \underset{R^{op}}{\otimes} h'_{(2)}.$$

That is, the extension $t_R : R^{op} \to H$ is $\mathcal{H}_R$-Galois. If the antipode S is bijective then also the canonical map

$$\mathrm{can}_L : H^L \otimes {}_LH \to H_L \otimes {}_LH \qquad h \underset{L}{\otimes} h' \mapsto h_{(1)}h' \underset{L}{\otimes} h_{(2)}$$

is bijective with inverse

$$\mathrm{can}_L^{-1} : H_L \otimes {}_LH \to H^L \otimes {}_LH \qquad h \underset{L}{\otimes} h' \mapsto h'^{(2)} \underset{L}{\otimes} S^{-1}(h'^{(1)})h,$$

that is also the extension $s_L : L \to H$ is $\mathcal{H}_L$-Galois.

EXAMPLE 3.7. Let $\mathcal{H}$ be a Hopf algebroid and A a right $\mathcal{H}_R$-module algebra. The smash product algebra [24] $A \rtimes H$ is defined as the k-module $A^R \otimes {}^RH$ with multiplication

$$(a \rtimes h)(a' \rtimes h') := a'(a \cdot h'^{(1)}) \rtimes hh'^{(2)}.$$

With this definition $A \rtimes H$ is an R-ring via the homomorphism

$$R \to A \rtimes H \qquad r \mapsto 1_A \rtimes s_R(r)$$

or an L^{op}-ring via the anti-homomorphism

$$L \to A \rtimes H \qquad l \mapsto 1_A \rtimes t_L(l).$$

One can introduce right $\mathcal{H}_L$- and right $\mathcal{H}_R$-comodule structures on $A \rtimes H$ via $\tau_L := A \otimes_R \gamma_L$ and $\tau_R := A \otimes_R \gamma_R$, respectively. The triple $(A \rtimes H, \tau_L, \tau_R)$ is a right $\mathcal{H}$-comodule algebra. We have $(A \rtimes H)^{co\mathcal{H}_R} = \{a \rtimes 1_H\}_{a \in A}$ and the canonical map

$$\mathrm{can}_R : (A \rtimes H) \otimes_A (A \rtimes H) \simeq A^R \otimes {}^R H_{\mathbf{R}} \otimes {}^{\mathbf{R}} H \to (A \rtimes H) \otimes_R H \simeq A^R \otimes {}^R H^{\mathbf{R}} \otimes {}^{\mathbf{R}} H$$

$$a \otimes_R h \otimes_{\mathbf{R}} h' \mapsto a \otimes_R h' h^{(1)} \otimes_{\mathbf{R}} h^{(2)}$$

is bijective with inverse

$$\mathrm{can}_R^{-1} : A^R \otimes {}^R H^{\mathbf{R}} \otimes {}^{\mathbf{R}} H \to A^R \otimes {}^R H_{\mathbf{R}} \otimes {}^{\mathbf{R}} H \qquad a \otimes_R h \otimes_{\mathbf{R}} h' \mapsto a \otimes_R h'_{(2)} \otimes_{\mathbf{R}} h S(h'_{(1)}).$$

This means that the extension $A \subset A \rtimes H$ is $\mathcal{H}_R$-Galois. If the antipode of $\mathcal{H}$ is bijective then it is also $\mathcal{H}_L$-Galois.

EXAMPLE 3.8. In ([23], Theorem 5.1) Kadison has shown that a depth 2 extension $B \subset A$ of k-algebras – if it is balanced or faithfully flat – is a Galois extension for the right bialgebroid, constructed in [24] on the total algebra $(A \otimes_B A)^B$ (the centralizer of B in the canonical bimodule $A \otimes_B A$).

Recall that if the extension $B \subset A$ is in addition a Frobenius extension, $(A \otimes_B A)^B$ possesses a Frobenius Hopf algebroid structure [7]. Extending the result of [23] considerably, Bálint and Szlachányi have shown in ([1], Theorem 3.7) that an extension $B \subset A$ of k-algebras is $\mathcal{H}$-Galois for some Frobenius Hopf algebroid $\mathcal{H}$ if and only if it is a balanced depth 2 Frobenius extension.

4. – A Kreimer-Takeuchi type theorem for Hopf algebroids.

In this section we investigate $\mathcal{H}$-extensions for finitely generated projective Hopf algebroids $\mathcal{H}$ with bijective antipode. We show that for an $\mathcal{H}$-Galois extension $B \subset A$ the algebra A is projective both as a left and as a right B-module and – under the additional assumption that $(A \otimes_k A)^{co\mathcal{H}} \simeq A \otimes_k B$, see below – the surjectivity of the canonical map (3.18) implies its bijectivity. This is a generalization of the classical theorem for finitely generated projective Hopf algebras by Kreimer and Takeuchi ([25], Theorem 1.7).

Recently Brzeziński [9] and Schauenburg and Schneider [31] have used new ideas to prove the Kreimer-Takeuchi theorem and generalizations of it. Their arguments are formulated in terms of entwining structures [11] over a commutative ring. In what follows we claim that their line of reasoning can be applied almost without modification to entwining structures over non-commutative algebras so to prove a Kreimer-Takeuchi type theorem for Hopf algebroids.

As we have seen at the beginning of Section 3, a right comodule algebra A for a right R -bialgebroid $\mathcal{H}_R$ determines an entwining structure (3.17) over R.

LEMMA 4.1. *Let $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$ be a Hopf algebroid with bijective antipode and A a right $\mathcal{H}$-comodule algebra. The map ψ in* (3.17), *corresponding to the right $\mathcal{H}_R$-comodule algebra structure of A, is an isomorphism.*

PROOF. The inverse of ψ is constructed using the $\mathcal{H}_L$-comodule algebra structure of A as

$$\psi^{-1}: A \underset{R}{\otimes} H \to H \underset{R}{\otimes} A \qquad a \underset{R}{\otimes} h \mapsto hS^{-1}(a_{\langle 1 \rangle}) \underset{R}{\otimes} a_{\langle 0 \rangle}.$$ ■

Motivated by Lemma 4.1 we study bijective entwining structures over R. We are going to generalize ([31], Theorem 3.1) which is, indeed, a variant of ([9], Theorem 4.4). Recall that for any entwined module M and any k-module V the k-module $V \underset{k}{\otimes} M$ is an entwined module via the second tensor factor. The elements of $V \underset{k}{\otimes} M^{coC}$ form a subset of $(V \underset{k}{\otimes} M)^{coC}$. We have $V \underset{k}{\otimes} M^{coC} = (V \underset{k}{\otimes} M)^{coC}$ if, for example, the k-module V is flat.

PROPOSITION 4.2. *Let (A, C, ψ) be a bijective entwining structure over the algebra R, such that the R-coring C possesses a grouplike element e. Denote the corresponding right C-coaction on A by $a_{\langle 0 \rangle} \underset{R}{\otimes} a_{\langle 1 \rangle} := \psi(e \underset{R}{\otimes} a)$ and the subring of its coinvariants by B. Assume that C is flat as a left (right) R-module and projective as a right (left) C-comodule.*

(1) *Suppose that $(A \underset{k}{\otimes} A)^{coC} = A \underset{k}{\otimes} B$ and the canonical map*

$$\text{can}: A \underset{B}{\otimes} A \to A \underset{R}{\otimes} C \qquad a \underset{B}{\otimes} a' \mapsto aa'_{\langle 0 \rangle} \underset{R}{\otimes} a'_{\langle 1 \rangle} \tag{4.21}$$

is surjective. Under these assumptions the canonical map (4.25) *is bijective.*

(2) *If the canonical map* (4.25) *is bijective then A is projective as a right (left) B-module.*

PROOF. The proof is actually the same as the proof of ([31], Theorem 3.1), so we present only a sketchy proof here.

<u>(1)</u> Let us use the assumption that C is projective as a right C-comodule. By the bijectivity of ψ and the adjunction (2.15) for any entwined module M we have $\mathrm{Hom}_A^C(A \underset{R}{\otimes} C, M) \simeq \mathrm{Hom}^C(C, M)$. By the flatness of the left R-module C the

projectivity of the right C-comodule C implies the projectivity of the entwined module $A \otimes_R C$. Hence the surjective map

$$A \otimes_k A \to A \otimes_R C \qquad a \otimes_k a' \mapsto aa'_{\langle 0 \rangle} \otimes_R a'_{\langle 1 \rangle}$$

is a split epimorphism of entwined modules. This means that $A \otimes_R C$ is a direct summand of $A \otimes_k A$.

Notice that $(A \otimes_R C)^{coC} \simeq A$ via the isomorphism

$$a : A \to (A \otimes_R C)^{coC} \qquad a \mapsto a \otimes_R e,$$

hence the canonical map (4.25) is related to the unit of the adjunction (2.16) as $\mathrm{can} = \mu_{A \otimes_R C} \circ (a \otimes_B A)$. This means that can is bijective provided $\mu_{A \otimes_k A}$ is bijective. Tensoring a k-free resolution

$$\dots P_n \xrightarrow{\partial_n} P_{n-1} \longrightarrow \dots \longrightarrow P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} A \longrightarrow 0$$

of A with A over k we can write $A \otimes_k A$ as the cokernel of the morphism $\partial_1 \otimes_k A : P_1 \otimes_k A \to P_0 \otimes_k A$ between entwined modules that are direct sums of copies of A. The functoriality of μ implies that

$$\mu_{A \otimes_k A} \circ [(\partial_0 \otimes_k B) \otimes_B A] = (\partial_0 \otimes_k A) \circ (P_0 \otimes_k \mu_A).$$

Since μ_A is an isomorphism we can use the universality of the cokernel to define the inverse of $\mu_{A \otimes_k A}$ as the unique map $\phi : A \otimes_k A \to (A \otimes_k A)^{coC} \otimes_B A = (A \otimes_k B) \otimes_B A$ which satisfies

$$\phi \circ (\partial_0 \otimes_k A) = [(\partial_0 \otimes_k B) \otimes_B A] \circ (P_0 \otimes_k \mu_A^{-1}).$$

This proves the bijectivity of $\mu_{A \otimes_k A}$, hence of the canonical map (4.25).

If C is projective as a left C-comodule and flat as a right R-module then ψ has to be replaced with its inverse in the above line of reasoning.

<u>(2)</u> The projectivity of the right B-module A is proven from the projectivity of the right C-comodule C as follows. By the bijectivity of the canonical map (4.25) and the adjunction (2.16) for any entwined module M we have $\mathrm{Hom}^C(C, M) \simeq$ $\simeq \mathrm{Hom}_B(A, M^{coC})$, hence the functor $\mathrm{Hom}_B(A, (\ \_\ )^{coC}) : \mathcal{M}_A^C \to \mathcal{M}_k$ is exact.

Let $f : \oplus_I B \to A$ be an epimorphism in $\mathcal{M}_B$ for some index set I. Then $f \otimes_B A$ is an epimorphism in $\mathcal{M}_A^C$ which is mapped by the functor $\mathrm{Hom}_B(A, (\ \_\ )^{coC})$ to the epimorphism $\mathrm{Hom}_B(A, f)$ in $\mathcal{M}_k$.

In order to prove the projectivity of the left B-module A from the projectivity of the left C-comodule C replace ψ by its inverse in the above arguments. ■

As it has been explained to us by Tomasz Brzeziński, there exists an alternative, more general argument proving the bijectivity of the canonical map (4.25) from its split surjectivity in $\mathcal{M}_A^C$. Namely, the application of ([13], Theorem 2.1) to the A-coring $A \otimes_R C$, corresponding to the R-entwining structure (A, C, ψ) with grouplike element $e \in C$, and its comodule $M = A$, implies the claim. Indeed, in this case the condition b) of ([13], Theorem 2.1) reduces to $(A \otimes_k A)^{coC} \simeq A \otimes_k A^{coC}$.

We have formulated Proposition 4.2 (1) in terms of the assumption $(A \otimes_k A)^{coC} = A \otimes_k A^{coC}$. It has the advantage that in certain situations (e.g. if k is a field) it obviously holds true. We do not know, however, whether it is also a necessary condition for the claim of Proposition 4.2 (1). On the other hand, notice that if the canonical map (4.25) is an isomorphism in $\mathcal{M}_A^C$ then

$$(A \otimes_B A)^{coC} \simeq (A \otimes_R C)^{coC} = A, \tag{4.22}$$

where B stands for A^{coC}, as before. The application of ([38], Proposition 5.1) to the A-coring $A \otimes_R C$, corresponding to the R-entwining structure (A, C, ψ) with grouplike element $e \in C$, and its comodule $M = A$, implies that the bijectivity of the canonical map (4.25) follows from its split surjectivity also under the (sufficient and necessary) condition (4.26).

Let us turn to the application of Proposition 4.2 to Hopf algebroid extensions. Let $\mathcal{H}$ be a finitely generated projective Hopf algebroid with bijective antipode. It follows from the Fundamental Theorem for Hopf algebroids ([5], Theorem 4.2) and the existence of a Hopf module structure on $H^* = \mathrm{Hom}_R(H, R)$ with coinvariants $\mathcal{L}(H^*)$ ([5], Proposition 4.4) that for such a Hopf algebroid the map

$$a_L : {}^L H \otimes \mathcal{L}(H^*)^L \to H^* \qquad h \underset{L^{op}}{\otimes} \lambda^* \mapsto \lambda^* \leftharpoonup S(h) \equiv \lambda^*[S(h)\ \_\ ] \tag{4.23}$$

is an isomorphism of Hopf modules, hence in particular of left H-modules. (The left H-module structure on ${}^L H \otimes \mathcal{L}(H^*)^L$ is given by left multiplication in the first tensor factor, and on H^* by $h \cdot \phi^* := \phi^* \leftharpoonup S(h) \equiv \phi^*[S(h)\ \_\ ]$.) This implies that the element $a_L^{-1}(\pi_R)$ is an invariant of the left H-module ${}^L H \otimes \mathcal{L}(H^*)^L$. The elements $\{x_i\} \subset H$ and $\{\lambda_i^*\} \subset \mathcal{L}(H^*)$ satisfying $\sum_i x_i \underset{L^{op}}{\otimes} \lambda_i^* = a_L^{-1}(\pi_R)$ can be used to construct dual bases for the left ${}^*H$-module on H defined as ${}^*\phi \rightharpoonup h := h^{(1)}\, s_R \circ {}^*\phi(h^{(2)})$. As a matter of fact, for $\lambda^* \in \mathcal{L}(H^*)$ we have $\lambda^* \circ S \in \mathcal{L}({}^*H)$ ([5], Scholium 2.10), and ${}^*H$ is a right H module via ${}^*\phi \leftharpoonup h = {}^*\phi(h\ \_\ )$. We leave it to the reader to check that the sets $\{x_i\} \subset H$ and $\{\lambda_i^* \circ S \leftharpoonup S^{-1}(\ \_\ )\} \subset {}_{{}^*H}\mathrm{Hom}(H, {}^*H)$ are dual bases, showing that H is a finitely generated and projective ${}^*H$-module.

Since the antipode was assumed to be bijective, we can apply the same argument to the co-opposite Hopf algebroid $\mathcal{H}_{cop}$ to conclude on the finitely generated projectivity of the left H^*-module on H defined as $\phi^* \rightharpoonup h := = h^{(2)}\, t_R \circ \phi^*(h^{(1)})$.

Now the projectivity of H as a left ${}^*H$ module implies that it is projective as a right $\mathcal{H}_R$-comodule and its projectivity as a left H^*-module implies that it is projective as a left $\mathcal{H}_R$-comodule. These observations allow for the application of Proposition 4.2 to the entwining structure (3.17) – which is bijective by Lemma 4.1.

COROLLARY 4.3. *Let $\mathcal{H}$ be a finitely generated projective Hopf algebroid with bijective antipode and let $B \subset A$ be an $\mathcal{H}$-extension.*

(1) *Suppose that $(A \underset{k}{\otimes} A)^{co\mathcal{H}} = A \underset{k}{\otimes} B$ (e.g. A is k-flat). If the canonical map*

$$\mathrm{can}_R : A \underset{B}{\otimes} A \to A \underset{R}{\otimes} H \qquad a \underset{B}{\otimes} a' \mapsto aa'^{\langle 0 \rangle} \underset{R}{\otimes} a'^{\langle 1 \rangle}$$

is surjective then the extension $B \subset A$ is $\mathcal{H}_R$-Galois (equivalently, $\mathcal{H}_L$-Galois).

(2) *If the extension $B \subset A$ is $\mathcal{H}$-Galois then A is projective both as a left and as a right B-module.*

5. – Morita theory for Hopf algebroid extensions.

As we have seen at the beginning of Section 3, an $\mathcal{H}_R$-extension $B \subset A$ for a right R-bialgebroid $\mathcal{H}_R$ determines an A-coring structure on $A \underset{R}{\otimes} H$. One can apply the Morita theory for corings, developed in [18], to this coring. In particular, if A is finitely generated and projective as a left R-module, then Theorem 2.5 (1) can be used to obtain criteria which are equivalent to the $\mathcal{H}_R$-Galois property of the extension $B \subset A$ and the projectivity of the left B-module A. Analogously, one can apply Theorem 2.5 (1) to obtain criteria which are equivalent to the Galois property of an $\mathcal{H}_L$-extension $B \subset A$ for a finitely generated projective left bialgebroid $\mathcal{H}_L$ and the projectivity of the right B-module A.

Applying the results of the previous sections we prove that if $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$ is a finitely generated projective Hopf algebroid with bijective antipode and $B \subset A$ is an $\mathcal{H}$-extension then the equivalent conditions, derived from Theorem 2.5 (1) for $B \subset A$ as an $\mathcal{H}_R$-extension, on one hand, and as an $\mathcal{H}_L$-extension, on the other hand, are equivalent also to each other.

Let $\mathcal{H}_R = (H, R, s, t, \gamma, \pi)$ be a right bialgebroid such that H is finitely generated and projective as a left R-module and let A be a right $\mathcal{H}_R$-comodule algebra. As a first step, let us identify the Morita context $(A^{*C}, {}^*C, A, {}^*C^{*C}, \upsilon, \mu)$, associated to the A-coring $C = A \underset{R}{\otimes} H$ (as it is explained in Subsection 2.4).

Recall that A – being a right $\mathcal{H}_R$-comodule – is also a left ${}^*H$-module via ${}^*\phi \cdot a = a^{\langle 0 \rangle} \cdot {}^*\phi(a^{\langle 1 \rangle})$. Since the module ${}^R H$ is finitely generated and projective, ${}^*H$ possesses a left bialgebroid structure. Let us introduce the smash product algebra ${}^*H \ltimes A$ as the k-module ${}^*H \underset{R}{\otimes} A$ (where the right R-module structure of ${}^*H$ is given by $({}^*\phi \cdot r)(h) = {}^*\phi(h)r$) with multiplication

$$({}^*\phi \ltimes a)({}^*\psi \ltimes a') := {}^*\psi_{(1)}{}^*\phi \ltimes ({}^*\psi_{(2)} \cdot a)a'. \tag{5.24}$$

With this definition ${}^*H \ltimes A$ is an A-ring via the homomorphism

$$i_A : A \to {}^*H \ltimes A \qquad a \mapsto \pi \ltimes a.$$

Define a right-right *relative* $(A, \mathcal{H}_R)$ *-module* to be an entwined module for the R -entwining structure (3.17), i.e. a right comodule for the A-coring $A \otimes_R H$. This means a right A-module (and hence in particular a right R-module) and a right $\mathcal{H}_R$-comodule M such that the compatibility condition

$$(m \cdot a)^{\langle 0 \rangle} \otimes_R (m \cdot a)^{\langle 1 \rangle} = m^{\langle 0 \rangle} \cdot a^{\langle 0 \rangle} \otimes_R m^{\langle 1 \rangle} a^{\langle 1 \rangle}$$

holds true for any $m \in M$ and $a \in A$. Clearly, A is itself a relative $(A, \mathcal{H}_R)$-module. It is straightforward to check that the category $\mathcal{M}_A^H$ of relative $(A, \mathcal{H}_R)$-modules is isomorphic to the category $\mathcal{M}_{{}^*H \ltimes A}$ of right ${}^*H \ltimes A$-modules and the ${}^*H \ltimes A$-invariants are the same as the $\mathcal{H}_R$-coinvariants. This fact can be easily understood in view of

LEMMA 5.1. *Let $\mathcal{H}_R$ be a right R- bialgebroid such that H is finitely generated and projective as a left R-module and let A be a right $\mathcal{H}_R$-comodule algebra.*

The left A-dual algebra of the A-coring $A \otimes_R H$ is isomorphic to the smash product algebra ${}^*H \ltimes A$.

PROOF. The required algebra isomorphism is constructed as

$${}^*H \ltimes A \to {}_A\mathrm{Hom}(A \otimes_R H, A) \qquad {}^*\phi \ltimes a \mapsto (\, a' \otimes_R h \mapsto a'({}^*\phi(h) \cdot a)). \qquad \blacksquare$$

The Morita context, associated to the $\mathcal{H}_R$-extension $B \subset A$, is then $(B, {}^*H \ltimes A, A, ({}^*H \ltimes A)^{co\mathcal{H}_R}\ \upsilon_R, \mu_R)$ with connecting maps

$$\upsilon_R : A \otimes_{{}^*H \ltimes A} ({}^*H \ltimes A)^{co\mathcal{H}_R} \to B \qquad a' \otimes_{{}^*H \ltimes A} (\sum_i {}^*\phi_i \ltimes a_i) \mapsto \sum_i ({}^*\phi_i \cdot a')a_i \tag{5.25}$$

$$\mu_R : ({}^*H \ltimes A)^{co\mathcal{H}_R} \otimes_B A \to {}^*H \ltimes A \qquad (\sum_i {}^*\phi_i \ltimes a_i) \otimes_B a' \mapsto \sum_i {}^*\phi_i \ltimes a_i a'. \tag{5.26}$$

Since H is finitely generated and projective as a left R-module, so is the left A-module $A \otimes_R H$. Hence part (1) in Theorem 2.5 implies

PROPOSITION 5.2. *Let $\mathcal{H}_R$ be a right R- bialgebroid such that H is finitely generated and projective as a left R-module and let $B \subset A$ be an $\mathcal{H}_R$-extension. The following assertions are equivalent.*

(a) *The map μ_R in* (5.30) *is surjective (and, a fortiori, bijective).*

(b) *The functor $(\ \_\ )^{co\mathcal{H}_R} : \mathcal{M}_A^H \to \mathcal{M}_B$ is fully faithful.*

(c) *A is a right ${}^*H \ltimes A$-generator.*

(d) *A is projective as a left B-module and the map*

$${}^*H \ltimes A \to {}_B\mathrm{End}(A) \qquad {}^*\phi \ltimes a \mapsto (\, a' \mapsto ({}^*\phi \cdot a')a\ ) \tag{5.27}$$

is an algebra anti-isomorphism.

(e) *A is projective as a left B-module and the extension $B \subset A$ is $\mathcal{H}_R$-Galois.*

The arguments, leading to Theorem 5.2, can be repeated by replacing the right bialgebroid $\mathcal{H}_R$ with a left L-bialgebroid $\mathcal{H}_L$ such that H is finitely generated and projective as a left L-module. Indeed, in this case the left L-dual, ${}_*H$, possesses a right bialgebroid structure and A is a right ${}_*H$-module via $a \cdot {}_*\phi = a_{\langle 0\rangle} \cdot {}_*\phi(a_{\langle 1\rangle})$. The right A-dual of the A-coring $A \underset{L}{\otimes} H$ is isomorphic to the smash product algebra ${}_*H \ltimes A$, which is defined as the k-module ${}_*H \underset{L}{\otimes} A$, (where the right L-module structure on ${}_*H$ is given by $({}_*\phi \cdot l)(h) = {}_*\phi(h)l$), with multiplication

$$({}_*\phi \ltimes a)({}_*\psi \ltimes a') := {}_*\psi {}_*\phi^{(1)} \ltimes a(a' \cdot {}_*\phi^{(2)}).$$

Since A is an L^{op}-ring, a left A-module is in particular a right L-module and we have the isomorphism $(A \underset{L}{\otimes} H) \underset{A}{\otimes} M \simeq M \underset{L}{\otimes} H$ for any left A-module M. A left comodule for the A-coring $A \underset{L}{\otimes} H$ is then equivalent to a left A-module (hence in particular a right L-module) and a right $\mathcal{H}_L$-comodule M such that the compatibility condition

$$(a \cdot m)_{\langle 0\rangle} \underset{L}{\otimes} (a \cdot m)_{\langle 1\rangle} = a_{\langle 0\rangle} \cdot m_{\langle 0\rangle} \underset{L}{\otimes} a_{\langle 1\rangle} m_{\langle 1\rangle}$$

holds true for $m \in M$ and $a \in A$. Such modules are called left-right relative $(A, \mathcal{H}_L)$-modules and their category is denoted by ${}_A\mathcal{M}^H$. It follows that the category ${}_A\mathcal{M}^H$ is isomorphic also to ${}_{{}_*H \ltimes A}\mathcal{M}$, the category of left ${}_*H \ltimes A$-modules. The Morita context associated to the $\mathcal{H}_L$-extension $B \subset A$ is $(B, {}_*H \ltimes A, A, ({}_*H \ltimes A)^{co\mathcal{H}_L}, \upsilon_L, \mu_L)$ with connecting maps

$$\upsilon_L : ({}_*H \ltimes A)^{co\mathcal{H}_L} \underset{{}_*H\ltimes A}{\otimes} A \to B \qquad (\sum_i {}_*\phi_i \ltimes a_i) \underset{{}_*H\ltimes A}{\otimes} a' \mapsto \sum_i a_i(a' \cdot {}_*\phi_i) \tag{5.28}$$

$$\mu_L : A \underset{B}{\otimes} ({}_*H \ltimes A)^{co\mathcal{H}_L} \to {}_*H \ltimes A \qquad a' \underset{B}{\otimes} (\sum_i {}_*\phi_i \ltimes a_i) \mapsto \sum_i {}_*\phi_i \ltimes a' a_i\,. \tag{5.29}$$

Part (1) of Theorem 2.5 implies

PROPOSITION 5.3. *Let $\mathcal{H}_L$ be a left L- bialgebroid such that H is finitely generated and projective as a left L-module and let $B \subset A$ be an $\mathcal{H}_L$-extension. The following assertions are equivalent.*

(a) *The map μ_L in (5.33) is surjective (and, a fortiori, bijective).*

(b) *The functor $(\ \_\ )^{co\mathcal{H}_L} : {}_A\mathcal{M}^H \to {}_B\mathcal{M}$ is fully faithful.*

(c) *A is a left ${}_*H \ltimes A$-generator.*

(d) *A is projective as a right B-module and the map*

$${}_*H \ltimes A \to \mathrm{End}_B(A) \qquad {}_*\phi \ltimes a \mapsto (\ a' \mapsto a(a' \cdot {}_*\phi)\ ) \tag{5.30}$$

is an algebra isomorphism.

(e) *A is projective as a right B-module and the extension $B \subset A$ is $\mathcal{H}_L$-Galois.*

Combining Proposition 5.2, 5.3, Corollary 4.3 and Lemma 3.3 we can state our main result.

THEOREM 5.4. *Let $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$ be a finitely generated projective Hopf algebroid with bijective antipode and let $B \subset A$ be an $\mathcal{H}$-extension. The following assertions are equivalent.*

(a) *The extension $B \subset A$ is $\mathcal{H}_R$ -Galois.*

(b) *A is projective as a left B-module and the map* (5.31) *is an algebra anti-isomorphism.*

(c) *A is a right ${}^*H \ltimes A$-generator.*

(d) *The functor $(\_)^{co\mathcal{H}_R} : \mathcal{M}_A^H \to \mathcal{M}_B$ is fully faithful.*

(e) *The map μ_R in* (5.30) *is surjective (and, a fortiori, bijective).*

(a′) *The extension $B \subset A$ is $\mathcal{H}_L$ -Galois.*

(b′) *A is projective as a right B-module and the map* (5.34) *is an algebra isomorphism.*

(c′) *A is a left ${}_*H \ltimes A$-generator.*

(d′) *The functor $(\_)^{co\mathcal{H}_L} : {}_A\mathcal{M}^H \to {}_B\mathcal{M}$ is fully faithful.*

(e′) *The map μ_L in* (5.33) *is surjective (and, a fortiori, bijective).*

PROOF. It follows from part (2) of Corollary 4.3 that (a) is equivalent to any of the assertions

(5.31) A is projective as a left B module and the extension $B \subset A$ is $\mathcal{H}_R$−Galois.

(5.32) A is projective as a right B module and the extension $B \subset A$ is $\mathcal{H}_R$−Galois.

By Lemma 3.3 the assertions (a) and (a′) are equivalent and (5.36) is equivalent to

(5.33) A is projective as a right B module and the extension $B \subset A$ is $\mathcal{H}_L$−Galois.

The rest of the proof follows from Theorem 5.2 and 5.3. ∎

Let us mention that in ([15], Theorem 4.7) a stronger version of Theorem 2.5 (1) has been proven. Its application to bialgebroid extensions implies

PROPOSITION 5.5. *Let $\mathcal{H}_R$ be a right R- bialgebroid such that H is finitely generated and projective as a left R-module and let $B \subset A$ be an $\mathcal{H}_R$-extension. The following assertions are equivalent.*

(a) *The Morita context $(B, {}^*H \ltimes A, A, ({}^*H \ltimes A)^{co\mathcal{H}_R}, \upsilon_R, \mu_R)$ is strict.*

(b) *The functor $(\_)^{co\mathcal{H}_R} : \mathcal{M}_A^H \to \mathcal{M}_B$ is an equivalence with inverse $\_ \otimes_B A : \mathcal{M}_B \to \mathcal{M}_A^H$.*

(c) *The map* (5.31) *is an algebra anti-isomorphism and A is a left B-progenerator.*

(d) *A is faithfully flat as a left B-module and the extension $B \subset A$ is $\mathcal{H}_R$-Galois.*

Also the analogue of Proposition 5.5 for left bialgebroid extensions can be proven. We can not prove, however, that for an $\mathcal{H}$-extension, in the case of any finitely generated projective Hopf algebroid $\mathcal{H}$ with bijective antipode, the equivalent conditions in Proposition 5.5 and their counterparts on the left bialgebroid of $\mathcal{H}$ are equivalent also to each other (as it was seen to be the case with Proposition 5.2 and 5.3).

The Morita context $(A^{*C}, {}^*C, A, {}^*C^{*C}, \upsilon, \mu)$, associated in [18] to an A-coring C with a grouplike element, is the generalization of the Morita context, associated to a bialgebra extension in [21]. In the case of a finite dimensional Hopf algebra over a field (or a Frobenius Hopf algebra over a commutative ring) another Morita context has been associated to a Hopf algebra extension in [20, 19]. The relation of the two Morita contexts is of the type described in part (2) of Theorem 2.5. In order to see what is the analogue of the Morita context of Cohen, Fishman and Montgomery in the case of Hopf algebroids, in the rest of the section we assume that $\mathcal{H}$ is a *Frobenius* Hopf algebroid.

LEMMA 5.6. *Let $\mathcal{H}$ be a Frobenius Hopf algebroid and A a left $\mathcal{H}_L$-module algebra. Consider the smash product algebra $H \ltimes A$, which is the k-module $H \underset{L}{\otimes} A$ with multiplication*

$$(h \ltimes a)(g \ltimes a') := g_{(1)}h \ltimes (g_{(2)} \cdot a)a'.$$

The extension

$$i : A \to H \ltimes A \qquad a \mapsto 1_H \ltimes a$$

is a Frobenius extension.

PROOF. Recall (from Subsection 2.1) that a Frobenius Hopf algebroid possesses non-degenerate left integrals. Let us fix such an integral ℓ and denote by ρ_* the unique element in H_*, for which $\ell \leftharpoonup \rho_* \equiv s_L \circ \rho_*(\ell_{(1)})\ell_{(2)} = 1_H$. A Frobenius functional $\Phi : H \ltimes A \to A$ is given by $h \ltimes a \mapsto \rho_*(h) \cdot a$. A Hopf algebroid calculation shows that it is an A-A bimodule map and possesses a dual basis $(S(\ell^{(2)}) \ltimes 1_A) \underset{A}{\otimes} (\ell^{(1)} \ltimes 1_A)$. ∎

Recall that for a Frobenius Hopf algebroid $\mathcal{H}$ also the left bialgebroid ${}^*H$ possesses a Frobenius Hopf algebroid structure. Applying Lemma 5.6 together with part (2) of Theorem 2.5 we conclude that the Morita context $(A^{co\mathcal{H}}, {}^*H \ltimes A, A, ({}^*H \ltimes A)^{co\mathcal{H}}, \upsilon_R, \mu_R)$, associated to the right $\mathcal{H}_R$ -comodule algebra structure of a right $\mathcal{H}$-comodule algebra A for the Frobenius Hopf algebroid $\mathcal{H}$, is equivalent to the Morita context $(A^{co\mathcal{H}}, {}^*H \ltimes A, A, A, \upsilon', \mu')$ with connecting maps

$$\upsilon' : A \underset{{}^*H \ltimes A}{\otimes} A \to A^{co\mathcal{H}} \qquad a \underset{{}^*H \ltimes A}{\otimes} a' \mapsto {}^*\lambda \cdot (aa') \tag{5.34}$$

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$$(5.35)\qquad \mu' : A \underset{A^{co\mathcal{H}}}{\otimes} A \to {}^*H \ltimes A \qquad a \underset{A^{co\mathcal{H}}}{\otimes} a' \mapsto (\pi_R \ltimes a)({}^*\lambda \ltimes 1_A)(\pi_R \ltimes a'),$$

where ${}^*\lambda$ is a non-degenerate left integral in ${}^*H$.

COROLLARY 5.7. *If we add to the conditions of Theorem* 5.4 *the requirement that $\mathcal{H}$ be a Frobenius Hopf algebroid then we can add to the equivalent assertions* (a)-(e$'$) *also*

(f) *For any non-degenerate left integral ${}^*\lambda$ in the Frobenius Hopf algebroid ${}^*H$ the map*

$$A \underset{B}{\otimes} A \to {}^*H \ltimes A \qquad a \underset{B}{\otimes} a' \mapsto (\pi_R \ltimes a)({}^*\lambda \ltimes 1_A)(\pi_R \ltimes a')$$

is surjective (and, a fortiori, bijective).

By ([7], Lemma 5.14) for any non-degenerate left integral ℓ in a Hopf algebroid $\mathcal{H}$, for $\rho_* := \ell_L^{-1}(1_H) \in H_*$ and any element h of H we have $\ell_{(1)}h \underset{L}{\otimes} \ell_{(2)} = \ell_{(1)} \underset{L}{\otimes} \ell_{(2)} t_L \circ \rho_*(\ell h^{(1)}) S(h^{(2)})$. This implies that the image of the map (5.39) is an ideal in ${}^*H \ltimes A$. Hence – just as it has been proven for finite dimensional Hopf algebras in ([20], Corollary 1.3) – we see that it is true also for $\mathcal{H}$-extensions $B \subset A$ for a Frobenius Hopf algebroid $\mathcal{H}$ that if the k-algebra ${}^*H \ltimes A$ is simple then the extension $B \subset A$ is $\mathcal{H}$-Galois.

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6. fejezet

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The weak theory of monads

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Abstract

We construct a 'weak' version $\mathsf{EM}^w(\mathcal{K})$ of Lack and Street's 2-category of monads in a 2-category $\mathcal{K}$, by replacing their compatibility constraint of 1-cells with the units of monads by an additional condition on the 2-cells. A relation between monads in $\mathsf{EM}^w(\mathcal{K})$ and composite pre-monads in $\mathcal{K}$ is discussed. If $\mathcal{K}$ admits Eilenberg–Moore constructions for monads, we define two symmetrical notions of 'weak liftings' for monads in $\mathcal{K}$. If moreover idempotent 2-cells in $\mathcal{K}$ split, we describe both kinds of weak lifting via an appropriate pseudo-functor $\mathsf{EM}^w(\mathcal{K}) \to \mathcal{K}$. Weak entwining structures and partial entwining structures are shown to realize weak liftings of a comonad for a monad in these respective senses. Weak bialgebras are characterized as algebras and coalgebras, such that the corresponding monads weakly lift for the corresponding comonads and also the comonads weakly lift for the monads.

Keywords: 2-Category; Monad; Weak lifting

0. Introduction

Many constructions, developed independently in Hopf algebra theory, turn out to fit more general situations studied in category theory. For example, crossed products with a Hopf algebra in [21,2,12] are examples of a wreath product in [16]. As another example, the comonad induced by the underlying coalgebra in a Hopf algebra H, has a lifting to the category of modules over any H-comodule algebra. So-called Hopf modules are comodules (also called coalgebras) for the lifted comonad. Galois property of an algebra extension by a Hopf algebra turns out to correspond to comonadicity of an appropriate functor [14].

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Motivated by noncommutative differential geometry, these constructions were extended from Hopf algebras to coalgebras (over a commutative base ring) and to corings (over an arbitrary base ring), see e.g. the pioneering paper [8]. The resulting theory turns out to fit the same categorical framework, only the occurring (co)monads have slightly more complicated forms [14].

For a study of non-Tannakian monoidal categories (i.e. those that admit no strict monoidal fiber functor to the module category of some commutative ring), another direction of generalization was proposed in [6]. The essence of this approach is a weakening of the unitality of some maps and it leads to the replacing of a Hopf algebra by a 'weak' Hopf algebra. In the last decade many Hopf algebraic constructions were extended to the weak setting. Weak crossed products were studied e.g. in [9,13] and [18]. Weak Galois theory was developed, among other papers, in [9,8]. However, just because in these generalizations one deals with non-unital maps, they do not fit the categorical framework of (co)monads, their wreath products and liftings.

The aim of the current paper is to provide a categorical framework for 'weak' constructions. For this purpose, in Section 1 we construct, for any 2-category $\mathcal{K}$, a 2-category $\mathsf{EM}^w(\mathcal{K})$ which contains the 2-category $\mathsf{EM}(\mathcal{K})$ in [16] as a vertically full 2-subcategory. In $\mathsf{EM}^w(\mathcal{K})$ 0-cells are the same as in $\mathsf{EM}(\mathcal{K})$, i.e. monads in $\mathcal{K}$. Since we aim to describe constructions in terms of non-unital maps, in the definition of a 1-cell in $\mathsf{EM}^w(\mathcal{K})$ we impose the same compatibility condition with the multiplications of monads which is required in $\mathsf{EM}(\mathcal{K})$, but we relax the compatibility condition in $\mathsf{EM}(\mathcal{K})$ with the units of monads. Certainly, without compensating it with some other requirements, we would not obtain a 2-category. We show that imposing one further axiom on the 2-cells in addition to the axiom in $\mathsf{EM}(\mathcal{K})$, $\mathsf{EM}^w(\mathcal{K})$ becomes a 2-category, with the same horizontal and vertical composition laws used in $\mathsf{EM}(\mathcal{K})$.

It was observed in [9] that smash products (and more generally crossed products [18]) by weak bialgebras are not unital algebras. Motivated by the definition of a pre-unit in [9], we study pre-monads (defined in Section 2) in any 2-category $\mathcal{K}$. In Section 2 we interpret 'weak crossed products' in [13] as monads in $\mathsf{EM}^w(\mathcal{K})$. This leads to a bijection between monads $t \stackrel{s}{\to} t$ in $\mathsf{EM}^w(\mathcal{K})$ and pre-monad structures (in $\mathcal{K}$) on the composite 1-cell st with a 't-linear' multiplication.

Starting from Section 3, we restrict our studies to 2-categories $\mathcal{K}$ which admit Eilenberg–Moore constructions for monads (in the sense of [19]) and in which idempotent 2-cells split. These assumptions are motivated by applications to bimodules. The bicategory BIM of `[Algebras; Bimodules; Bimodule maps]`, over a commutative, associative and unital ring k, satisfies both assumptions. However, in order to avoid technical complications caused by non-strictness of the horizontal composition in a bicategory, we prefer to restrict to 2-categories. In the examples, instead of the bicategory BIM, we can work with its image in the 2-category $\mathsf{CAT} =$ `[Categories; Functors; Natural transformations]`, under the hom 2-functor $\mathsf{BIM}(k, -) : \mathsf{BIM} \to \mathsf{CAT}$, which image is a 2-category with the desired properties.

For a 2-category $\mathcal{K}$ which admits Eilenberg–Moore constructions for monads, the inclusion 2-functor $I : \mathcal{K} \to \mathsf{EM}(\mathcal{K})$ possesses a right 2-adjoint J, cf. [16]. In Section 3 we use the splitting property of idempotent 2-cells in $\mathcal{K}$ to construct a factorization of J through the inclusion 2-functor $\mathsf{EM}(\mathcal{K}) \hookrightarrow \mathsf{EM}^w(\mathcal{K})$ and an appropriate pseudo-functor $J^w : \mathsf{EM}^w(\mathcal{K}) \to \mathcal{K}$. For a monad $t \stackrel{s}{\to} t$ in $\mathsf{EM}^w(\mathcal{K})$ and any 0-cell k in $\mathcal{K}$, we prove that both monads $\mathcal{K}(k, J^w(s))$ and $\mathcal{K}(k, \widehat{st})$ in CAT possess isomorphic Eilenberg–Moore categories, where $\widehat{st}$ is a canonical retract monad of the pre-monad st.

In a 2-category $\mathcal{K}$ which admits Eilenberg–Moore constructions for monads, any monad $k \stackrel{t}{\to} k$ determines an adjunction $(k \stackrel{f}{\to} J(t), J(t) \stackrel{v}{\to} k)$ in $\mathcal{K}$, cf. [19]. A lifting of a 1-cell $k \stackrel{V}{\to} k'$

in $\mathcal{K}$ for monads $k \xrightarrow{t} k$ and $k' \xrightarrow{t'} k'$ is, by definition, a 1-cell $J(t) \xrightarrow{\overline{V}} J(t')$, such that $v'\overline{V} = Vv$, cf. [17]. In Section 4 we define a 'weak' lifting by replacing this equality with the existence of a 2-cell $v'\overline{V} \overset{\iota}{\Rightarrow} Vv$, possessing a retraction $Vv \overset{\pi}{\Rightarrow} v'\overline{V}$. This leads to two symmetrical notions of weak lifting of a 2-cell $V \overset{\omega}{\Rightarrow} W$ for the monads t and t'. A weak ι-lifting $\overline{V} \overset{\vec{\omega}}{\Rightarrow} \overline{W}$ is defined by the condition $\iota * v'\vec{\omega} = \omega v * \iota$ and a weak π-lifting $\overline{V} \overset{\overleftarrow{\omega}}{\Rightarrow} \overline{W}$ is defined by $v'\overleftarrow{\omega} * \pi = \pi * \omega v$. We show that any weak ι-lifting and any weak π-lifting of a 2-cell in $\mathcal{K}$, if it exists, is isomorphic to the image of an appropriate 2-cell in $\mathsf{EM}^w(\mathcal{K})$ under the pseudo-functor J^w. Both a weak ι-lifting and a weak π-lifting are proven to strictly preserve vertical composition and to preserve horizontal composition up to a coherent isomorphism. We also give sufficient and necessary conditions for the existence of weak ι- and π-liftings of a 2-cell in $\mathcal{K}$.

A powerful tool to treat algebra extensions by weak bialgebras is provided by 'weak entwining structures' in [9]. A weak entwining structure in a 2-category $\mathcal{K}$ consists of a monad t and a comonad c, together with a 2-cell $tc \Rightarrow ct$ relating both structures in a way which generalizes a mixed distributive law in [1]. It was observed in [8] that any weak entwining structure (in BIM) induces a comonad (called a 'coring' in the particular case of the bicategory BIM). In Section 5 we show that – in the same way as mixed distributive laws in a 2-category $\mathcal{K}$ provide examples of comonads in $\mathsf{EM}(\mathcal{K})$ – weak entwining structures provide examples of comonads in $\mathsf{EM}^w(\mathcal{K})$. Moreover, if the 2-category $\mathcal{K}$ satisfies the assumptions in Section 3, then the comonad in $\mathcal{K}$, induced by a weak entwining structure, is an example of a weak ι-lifting of a comonad for a monad.

Studying partial coactions of Hopf algebras, in [10] another generalization of a mixed distributive law, a so-called 'partial entwining structure' was introduced. Partial entwining structures (in BIM) were proven to induce comonads as well. We show that also partial entwining structures in a 2-category $\mathcal{K}$ provide examples of a comonad in $\mathsf{EM}^w(\mathcal{K})$. Moreover, if the 2-category $\mathcal{K}$ satisfies the assumptions in Section 3, then the comonad in $\mathcal{K}$, induced by a partial entwining structure, is an example of a weak π-lifting of a comonad for a monad.

As a final application, weak bialgebras are characterized via weak liftings. If a module H, over a commutative, associative and unital ring k, possesses both an algebra and a coalgebra structure, then it induces two monads $t_R = (-) \otimes_k H$ and $t_L = H \otimes_k (-)$, and two comonads $c_R = (-) \otimes_k H$ and $c_L = H \otimes_k (-)$ on the category of k-modules. We relate weak bialgebra structures of H to weak ι-liftings of c_R and c_L for t_R and t_L, respectively, and weak π-liftings t_R and t_L for c_R and c_L, respectively.

Notation. We assume that the reader is familiar with the theory of 2-categories. For a review of the occurring notions (such as a 2-category, a 2-functor and a 2-adjunction, monads, adjunctions and Eilenberg–Moore construction in a 2-category) we refer to the article [15].

In a 2-category $\mathcal{K}$, horizontal composition is denoted by juxtaposition and vertical composition is denoted by $*$, 1-cells are represented by an arrow $\rightarrow$ and 2-cells are represented by $\Rightarrow$.

For any 2-category $\mathcal{K}$, $\mathsf{Mnd}(\mathcal{K})$ denotes the 2-category of monads in $\mathcal{K}$ as in [19] and $\mathsf{Cmd}(\mathcal{K}) := \mathsf{Mnd}(\mathcal{K}_*)_*$ denotes the 2-category of comonads in $\mathcal{K}$, where $(-)_*$ refers to the vertical opposite of a 2-category. We denote by $\mathsf{EM}(\mathcal{K})$ the extended 2-category of monads in [16]. We use the reduced form of 2-cells in $\mathsf{EM}(\mathcal{K})$, see [16].

1. The 2-category $\mathsf{EM}^w(\mathcal{K})$

For any 2-category $\mathcal{K}$, the 2-category $\mathsf{EM}^w(\mathcal{K})$ introduced in this section extends the 2-category $\mathsf{EM}(\mathcal{K})$ in [16].

Theorem 1.1. *For any* 2*-category* $\mathcal{K}$*, the following data constitute a* 2*-category, to be denoted by* $\mathsf{EM}^w(\mathcal{K})$.

0-cells are monads $(k \xrightarrow{t} k, tt \overset{\mu}{\Rightarrow} t, k \overset{\eta}{\Rightarrow} t)$ *in* $\mathcal{K}$.
1-cells $(t,\mu,\eta) \to (t',\mu',\eta')$ *are pairs* (V,ψ)*, consisting of a* 1*-cell* $k \xrightarrow{V} k'$ *and a* 2*-cell* $t'V \overset{\psi}{\Rightarrow} Vt$ *in* $\mathcal{K}$*, such that*

$$V\mu * \psi t * t'\psi = \psi * \mu' V. \tag{1.1}$$

The identity 1*-cell is* $t \xrightarrow{(k,t)} t$.
2-cells $(V,\psi) \Rightarrow (W,\phi)$ *are* 2*-cells* $V \overset{\varrho}{\Rightarrow} Wt$ *in* $\mathcal{K}$*, such that*

$$W\mu * \varrho t * \psi = W\mu * \phi t * t'\varrho, \tag{1.2}$$

$$\varrho = W\mu * \phi t * \eta' Wt * \varrho. \tag{1.3}$$

The identity 2*-cell is* $(W,\phi) \xrightarrow{\phi * \eta' W} (W,\phi)$.

Horizontal composition of 2*-cells* $(V,\psi) \overset{\varrho}{\Rightarrow} (W,\phi)$, $(V',\psi') \overset{\varrho'}{\Rightarrow} (W',\phi')$ *(for* 1*-cells* $(V,\psi),(W,\phi): t \to t'$ *and* $(V',\psi'),(W',\phi'): t' \to t''$*) is given by*

$$\varrho' \circ \varrho := W'W\mu * W'\varrho t * W'\psi * \varrho' V. \tag{1.4}$$

Vertical composition of 2*-cells* $(V,\psi) \overset{\varrho}{\Rightarrow} (W,\phi) \overset{\tau}{\Rightarrow} (U,\theta)$ *(for* 1*-cells* $(V,\psi),(W,\phi)$ *and* $(U,\theta): t \to t'$*) is given by*

$$\tau \bullet \varrho := U\mu * \tau t * \varrho. \tag{1.5}$$

Proof. We verify only those axioms whose proof is different from the proof of the respective axiom for $\mathsf{EM}(\mathcal{K})$.

The vertical composite of 2-cells $(V,\psi) \overset{\varrho}{\Rightarrow} (W,\phi) \overset{\tau}{\Rightarrow} (U,\theta)$ in $\mathsf{EM}^w(\mathcal{K})$ is checked to satisfy (1.2) by the same computation used to verify that the vertical composite of 2-cells in $\mathsf{EM}(\mathcal{K})$ is a 2-cell in $\mathsf{EM}(\mathcal{K})$. In order to see that $\tau \bullet \varrho$ satisfies (1.3), use the interchange law in $\mathcal{K}$, associativity of the multiplication μ of the monad t and the fact that τ satisfies (1.3):

$$\begin{aligned} U\mu * \theta t * \eta' Ut * (\tau \bullet \varrho) &= U\mu * \theta t * \eta' Ut * U\mu * \tau t * \varrho \\ &= U\mu * U\mu t * \theta tt * \eta' Utt * \tau t * \varrho = U\mu * \tau t * \varrho = \tau \bullet \varrho. \end{aligned}$$

Associativity of the vertical composition follows by the same reasoning as in the case of $\mathsf{EM}(\mathcal{K})$. Using the constraint (1.1), it follows for any 1-cell $t \xrightarrow{(W,\phi)} t'$ in $\mathsf{EM}^w(\mathcal{K})$ that

$$W\mu * \phi t * t'\phi * t'\eta' W = \phi * \mu' W * t'\eta' W = \phi \quad \text{and}$$

$$W\mu * \phi t * \eta' Wt * \phi = W\mu * \phi t * t'\phi * \eta' t' W = \phi * \mu' W * \eta' t' W = \phi.$$

Thus the 2-cell $(W,\phi) \overset{\phi * \eta' W}{\Longrightarrow} (W,\phi)$ in $\mathcal{K}$ satisfies (1.2) and (1.3), proving that it is a 2-cell in $\mathsf{EM}^w(\mathcal{K})$. It is immediate by condition (1.3) that for any 2-cell $(V,\psi) \overset{\varrho}{\Rightarrow} (W,\phi)$ in $\mathsf{EM}^w(\mathcal{K})$,

$$\big(\phi * \eta' W\big) \bullet \varrho = W\mu * \phi t * \eta' W t * \varrho = \varrho.$$

Using (1.2), the interchange law in $\mathcal{K}$ and then (1.3), one checks that also

$$\varrho \bullet \big(\psi * \eta' V\big) = W\mu * \varrho t * \psi * \eta' V = W\mu * \phi t * t'\varrho * \eta' V = W\mu * \phi t * \eta' W t * \varrho = \varrho.$$

Hence there are identity 2-cells of the stated form.

On identity 2-cells $(V,\psi) \overset{\psi * \eta' V}{\Longrightarrow} (V,\psi)$ and $(V',\psi') \overset{\psi' * \eta'' V'}{\Longrightarrow} (V',\psi')$, the horizontal composite comes out as

$$\big(\psi' * \eta'' V'\big) \circ \big(\psi * \eta' V\big) = V'\psi * \psi' V * \eta'' V' V,$$

where we applied (1.1) for ψ and unitality of the monad t'. That is to say, the horizontal composite of the 1-cells $t \xrightarrow{(V,\psi)} t' \xrightarrow{(V',\psi')} t''$ is the 1-cell $(V'V, V'\psi * \psi' V)$. By the same computations used in the case of $\mathsf{EM}(\mathcal{K})$, the horizontal composite of 2-cells $(V,\psi) \overset{\varrho}{\Rightarrow} (W,\phi)$, $(V',\psi') \overset{\varrho'}{\Rightarrow} (W',\phi')$ is checked to satisfy condition (1.2). It satisfies also (1.3), as

$$\begin{aligned}
W'W\mu \,*\, & W'\phi t * \phi' W t * \eta'' W' W t * \big(\varrho' \circ \varrho\big) \\
&= W'W\mu * W'\phi t * \phi' W t * \eta'' W' W t * W' W \mu * W' \varrho t * W' \psi * \varrho' V \\
&= W'W\mu * W'W\mu t * W'\phi t t * W' t' \varrho t * \phi' V t * \eta'' W' V t * W' \psi * \varrho' V \\
&= W'W\mu * W'W\mu t * W'\varrho t t * W' \psi t * \phi' V t * \eta'' W' V t * W' \psi * \varrho' V \\
&= W'W\mu * W'\varrho t * W' V \mu * W' \psi t * W' t' \psi * \phi' t' V * \eta'' W' t' V * \varrho' V \\
&= W'W\mu * W'\varrho t * W'\psi * W' \mu' V * \phi' t' V * \eta'' W' t' V * \varrho' V \\
&= W'W\mu * W'\varrho t * W'\psi * \varrho' V = \varrho' \circ \varrho.
\end{aligned}$$

The first and last equalities follow by (1.4). In the second and fourth equalities we used the interchange law in $\mathcal{K}$ and associativity of the multiplication μ of the monad t. In the third equality we used that ϱ satisfies (1.2). The fifth equality is derived by using that ψ satisfies (1.1). The penultimate equality follows by using that ϱ' satisfies (1.3). This proves that the horizontal composite of 2-cells is a 2-cell. Associativity of the horizontal composition in $\mathsf{EM}^w(\mathcal{K})$ is checked in the same way as it is done in $\mathsf{EM}(\mathcal{K})$. Obviously, for any 2-cell $(V,\psi) \overset{\varrho}{\Rightarrow} (W,\phi)$, $\varrho \circ \eta = W\mu * W\eta t * \varrho = \varrho$. By (1.2) and (1.3), also $\eta' \circ \varrho = W\mu * \varrho t * \psi * \eta' V = \varrho$. Hence the identity 2-cell $(k,t) \overset{\eta}{\Rightarrow} (k,t)$ is a unit for the horizontal composition, proving the stated form (k,t) of the identity 1-cell $t \to t$.

The interchange law in $\mathsf{EM}^w(\mathcal{K})$ is checked in the same way as it is done in $\mathsf{EM}(\mathcal{K})$. □

Clearly, any 1-cell in $\mathsf{EM}(\mathcal{K})$ is a 1-cell also in $\mathsf{EM}^w(\mathcal{K})$. For a 1-cell $t \xrightarrow{(W,\phi)} t'$ in $\mathsf{EM}(\mathcal{K})$, any 2-cell ϱ in $\mathcal{K}$ of target Wt satisfies (1.3). Hence 2-cells in $\mathsf{EM}^w(\mathcal{K})$ between 1-cells of $\mathsf{EM}(\mathcal{K})$

are the same as the 2-cells in $\mathsf{EM}(\mathcal{K})$. Comparing the formulae of the horizontal and vertical compositions in $\mathsf{EM}(\mathcal{K})$ and $\mathsf{EM}^w(\mathcal{K})$, we conclude that $\mathsf{EM}(\mathcal{K})$ is a vertically full 2-subcategory of $\mathsf{EM}^w(\mathcal{K})$.

One may ask what 2-subcategory of $\mathsf{EM}^w(\mathcal{K})$ plays the role of the 2-subcategory, obtained as an image of $\mathsf{Mnd}(\mathcal{K})$ in $\mathsf{EM}(\mathcal{K})$. As the lemmata below show, there seems to be no unique answer to this question. This is because, for some 1-cells $t \xrightarrow{(V,\psi)} t'$ and $t \xrightarrow{(W,\phi)} t'$ in $\mathsf{EM}^w(\mathcal{K})$ and a 2-cell $V \overset{\omega}{\Rightarrow} W$ in $\mathcal{K}$, the 2-cells $\omega t * \psi * \eta' V$ and $\phi * \eta' W * \omega$ in $\mathcal{K}$ need not be equal. Still, there are two distinguished sets of 2-cells in $\mathsf{EM}^w(\mathcal{K})$ on equal footing, both closed under the horizontal and vertical compositions and both containing the identity 2-cells.

Lemma 1.2. *For any* 2*-category* $\mathcal{K}$*, let* $t \xrightarrow{(V,\psi)} t'$ *and* $t \xrightarrow{(W,\phi)} t'$ *be* 1*-cells in* $\mathsf{EM}^w(\mathcal{K})$ *and* $V \overset{\omega}{\Rightarrow} W$ *be a* 2*-cell in* $\mathcal{K}$*.*

(1) *The following assertions are equivalent:*
  (i) $\omega t * \psi * \eta' V : (V,\psi) \Rightarrow (W,\phi)$ *is a* 2*-cell in* $\mathsf{EM}^w(\mathcal{K})$*;*
  (ii) $\omega t * \psi = W\mu * \phi t * t'\omega t * t'\psi * t'\eta' V$*;*
  (iii) $W\mu * \phi t * \eta' W t * \omega t * \psi * \eta' V = \omega t * \psi * \eta' V$ *and* $W\mu * \phi t * \eta' W t * \omega t * \psi = W\mu * \phi t * t'\omega t * t'\psi * t'\eta' V$.
(2) *The following assertions are equivalent:*
  (i) $\phi * \eta' W * \omega : (V,\psi) \Rightarrow (W,\phi)$ *is a* 2*-cell in* $\mathsf{EM}^w(\mathcal{K})$*;*
  (ii) $\phi * t'\omega = W\mu * \phi t * \eta' W t * \omega t * \psi$*;*
  (iii) $W\mu * \phi t * \eta' W t * \omega t * \psi * \eta' V = \phi * \eta' W * \omega$ *and* $W\mu * \phi t * \eta' W t * \omega t * \psi = W\mu * \phi t * t'\omega t * t'\psi * t'\eta' V$.
(3) *The following assertions are equivalent:*
  (i) $\phi * \eta' W * \omega$ *and* $\omega t * \psi * \eta' V$ *are (necessarily equal)* 2*-cells* $(V,\psi) \Rightarrow (W,\phi)$ *in* $\mathsf{EM}^w(\mathcal{K})$*;*
  (ii) $\phi * t'\omega = \omega t * \psi$.

Proof. <u>(1)</u> <u>(i) ⇔ (ii)</u> Using that ψ satisfies (1.1) together with the unitality of the monad t', condition (1.2) for $\varrho := \omega t * \psi * \eta' V$ comes out as the equality in part (ii). Hence in order to prove the equivalence of assertions (i) and (ii), we need to show that (ii) implies that ϱ satisfies (1.3). Indeed, applying the equality in part (ii) in the second step, we obtain

$$W\mu * \phi t * \eta' W t * \omega t * \psi * \eta' V = W\mu * \phi t * t'\omega t * t'\psi * t'\eta' V * \eta' V = \omega t * \psi * \eta' V. \quad (1.6)$$

<u>(ii) ⇔ (iii)</u> We have seen in the proof of equivalence (i) ⇔ (ii) above, that assertion (ii) implies (1.6), i.e. the first condition in part (iii). The second condition is checked as follows:

$$\begin{aligned} W\mu * \phi t * \eta' W t * \omega t * \psi &= W\mu * \phi t * \eta' W t * \omega t * V\mu * \psi t * t'\psi * \eta' t' V \\ &= W\mu * W\mu t * \phi t t * \eta' W t t * \omega t t * \psi t * \eta' V t * \psi \\ &= W\mu * \omega t t * \psi t * \eta' V t * \psi \\ &= \omega t * \psi = W\mu * \phi t * t'\omega t * t'\psi * t'\eta' V. \end{aligned} \quad (1.7)$$

The first equality follows by (1.1) and unitality of the monad t'. In the second equality we used associativity of μ and the interchange law. The third equality is obtained by using (1.6). The penultimate equality follows by (1.1) and unitality of the monad t' again. The last equality follows by assertion (ii).

Conversely, assume that the identities in part (iii) hold. Then

$$\begin{aligned}\omega t * \psi &= \omega t * V\mu * \psi t * t'\psi * \eta' t' V = W\mu * W\mu t * \phi tt * \eta' Wtt * \omega tt * \psi t * \eta' Vt * \psi\\ &= W\mu * \phi t * \eta' Wt * \omega t * \psi = W\mu * \phi t * t'\omega t * t'\psi * t'\eta' V.\end{aligned}$$

The first equality follows by (1.1) and unitality of the monad t'. In the second equality we applied the first condition in part (iii). The third equality is obtained by using the associativity of μ, the interchange law and (1.1) again. The last equality follows by the second condition in part (iii).

(2) (i) $\Leftrightarrow$ (ii) Using that ϕ satisfies (1.1) together with the unitality of the monad t', condition (1.2) for $\varrho := \phi * \eta' W * \omega$ comes out as the equality in part (ii). Condition (1.3) holds for ϱ automatically, i.e. it follows by applying (1.1) for ϕ.

(ii) $\Leftrightarrow$ (iii) If (iii) holds, then

$$\begin{aligned}\phi * t'\omega &= W\mu * \phi t * t'\phi * t'\eta' W * t'\omega\\ &= W\mu * \phi t * t' W\mu * t'\phi t * t'\eta' Wt * t'\omega t * t'\psi * t'\eta' V\\ &= W\mu * \phi t * t'\omega t * t'\psi * t'\eta' V = W\mu * \phi t * \eta' Wt * \omega t * \psi.\end{aligned}$$

The first equality follows by applying (1.1) for ϕ, together with the unitality of the monad t'. The second equality is obtained by the first identity in part (iii). In the penultimate equality we applied the interchange law, associativity of μ, (1.1) on ϕ and unitality of t'. The last equality follows by the second condition in part (iii).

Conversely, if assertion (ii) holds, then the first condition in part (iii) is proven by composing both sides of the equality in part (ii) by $\eta' V$ on the right. The second condition, i.e. equality (1.7), is proven by the following computation:

$$\begin{aligned}W\mu * \phi t * t'\omega t * t'\psi * t'\eta' V &= W\mu * W\mu t * \phi tt * t'\phi t * t'\eta' Wt * t'\omega t * t'\psi * t'\eta' V\\ &= W\mu * \phi t * t'\phi * t'\eta' W * t'\omega\\ &= \phi * t'\omega = W\mu * \phi t * \eta' Wt * \omega t * \psi.\end{aligned}$$

The first equality follows by applying (1.1) for ϕ, together with the unitality of the monad t'. The second equality is obtained by using the associativity of μ, the interchange law and the first condition in part (iii). The third equality follows by applying (1.1) for ϕ, together with the unitality of the monad t' again. The last equality follows by part (ii).

(3) Assume first that assertion (3)(i) holds. Then by parts (1) and (2), also (1)(ii) and (2)(ii) hold, implying (1.7). Hence

$$\omega t * \psi = W\mu * \phi t * t'\omega t * t'\psi * t'\eta' V' = W\mu * \phi t * \eta' Wt * \omega t * \psi = \phi * t'\omega.$$

Conversely, if assertion (3)(ii) holds, then

$$\begin{aligned}&W\mu * \phi t * t'\omega t * t'\psi * t'\eta' V = W\mu * \omega tt * \psi t * t'\psi * t'\eta' V = \omega t * \psi \quad \text{and}\\ &W\mu * \phi t * \eta' Wt * \omega t * \psi = W\mu * \phi t * \eta' Wt * \phi * t'\omega = \phi * t'\omega,\end{aligned}$$

where in both computations the first equality follows by (3)(ii) and the second equality follows by (1.1) and the unitality of the monad t'. We conclude by parts (1) and (2) that both $\omega t * \psi * \eta' V$ and $\phi * \eta' W * \omega$ are 2-cells in $\mathsf{EM}^w(\mathcal{K})$. It follows by comparing the first identities in (1)(iii) and (2)(iii) that $\phi * \eta' W * \omega = \omega t * \psi * \eta' V$, as stated. □

Next we investigate the behaviour of the correspondences in Lemma 1.2 with respect to the horizontal and vertical compositions in $\mathcal{K}$ and $\mathsf{EM}^w(\mathcal{K})$.

Lemma 1.3. *For any 2-category $\mathcal{K}$, let (V, ψ), (W, ϕ) and (U, θ) be 1-cells $t \to t'$ and (V', ψ') and (W', ϕ') be 1-cells $t' \to t''$ in $\mathsf{EM}^w(\mathcal{K})$.*

(1) *If some 2-cells $V \overset{\omega}{\Rightarrow} W$ and $V' \overset{\omega'}{\Rightarrow} W'$ in $\mathcal{K}$ satisfy the equivalent conditions in Lemma 1.2(1), then $(\omega' t' * \psi' * \eta'' V') \circ (\omega t * \psi * \eta' V) = \omega' \omega t * V' \psi * \psi' V * \eta'' V' V$. Hence in particular also $\omega' \omega$ satisfies the equivalent conditions in Lemma 1.2(1).*
(2) *If some 2-cells $V \overset{\omega}{\Rightarrow} W$ and $V' \overset{\omega'}{\Rightarrow} W'$ in $\mathcal{K}$ satisfy the equivalent conditions in Lemma 1.2(2), then $(\phi' * \eta'' W' * \omega') \circ (\phi * \eta' W * \omega) = W' \phi * \phi' W * \eta'' W' W * \omega' \omega$. Hence in particular also $\omega' \omega$ satisfies the equivalent conditions in Lemma 1.2(2).*
(3) *If some 2-cells $V \overset{\omega}{\Rightarrow} W \overset{\kappa}{\Rightarrow} U$ in $\mathcal{K}$ satisfy the equivalent conditions in Lemma 1.2(1), then $(\kappa t * \phi * \eta' W) \bullet (\omega t * \psi * \eta' V) = \kappa t * \omega t * \psi * \eta' V$. Hence in particular also $\kappa * \omega$ satisfies the equivalent conditions in Lemma 1.2(1).*
(4) *If some 2-cells $V \overset{\omega}{\Rightarrow} W \overset{\kappa}{\Rightarrow} U$ in $\mathcal{K}$ satisfy the equivalent conditions in Lemma 1.2(2), then $(\theta * \eta' U * \kappa) \bullet (\phi * \eta' W * \omega) = \theta * \eta' U * \kappa * \omega$. Hence in particular also $\kappa * \omega$ satisfies the equivalent conditions in Lemma 1.2(2).*

Proof. (1) This compatibility with the horizontal composition follows by applying (1.1) for ψ, and unitality of the monad t'.

(2) This follows by using that ω obeys Lemma 1.2(2)(ii).

(3) This compatibility with the vertical composition follows using that, by Lemma 1.2(1), $\omega t * \psi * \eta' V$ satisfies (1.3).

(4) This assertion follows by using that κ satisfies Lemma 1.2(2)(ii). □

The message of Lemmas 1.2 and 1.3 can be summarized as follows.

Corollary 1.4. *Consider an arbitrary 2-category $\mathcal{K}$.*

(1) *There is a 2-category, to be denoted by $\mathsf{Mnd}^\iota(\mathcal{K})$, defined by the following data:*
 0-cells are monads t in $\mathcal{K}$;
 1-cells $t \xrightarrow{(V,\psi)} t'$ are the same as 1-cells in $\mathsf{EM}^w(\mathcal{K})$, cf. (1.1);
 2-cells $(V, \psi) \overset{\omega}{\Rightarrow} (W, \phi)$ are 2-cells $V \overset{\omega}{\Rightarrow} W$ in $\mathcal{K}$, satisfying the equivalent conditions in Lemma 1.2(1);
 horizontal and vertical compositions are the same as in $\mathcal{K}$.
 Furthermore, there is a 2-functor $G^\iota : \mathsf{Mnd}^\iota(\mathcal{K}) \to \mathsf{EM}^w(\mathcal{K})$, acting on the 0- and 1-cells as the identity map and taking a 2-cell $(V, \psi) \overset{\omega}{\Rightarrow} (W, \phi)$ to $\omega t * \psi * \eta' V$.
(2) *There is a 2-category, to be denoted by $\mathsf{Mnd}^\pi(\mathcal{K})$, defined by the following data:*
 0-cells are monads t in $\mathcal{K}$;

1-cells $t \xrightarrow{(V,\psi)} t'$ *are the same as* 1*-cells in* $\mathsf{EM}^w(\mathcal{K})$*, cf.* (1.1)*;*
2-cells $(V,\psi) \overset{\omega}{\Rightarrow} (W,\phi)$ *are* 2*-cells* $V \overset{\omega}{\Rightarrow} W$ *in* $\mathcal{K}$*, satisfying the equivalent conditions in Lemma* 1.2(2)*;*
horizontal and vertical compositions are the same as in $\mathcal{K}$*.*

Furthermore, there is a 2*-functor* $G^\pi : \mathsf{Mnd}^\pi(\mathcal{K}) \to \mathsf{EM}^w(\mathcal{K})$*, acting on the* 0*- and* 1*-cells as the identity map and taking a* 2*-cell* $(V,\psi) \overset{\omega}{\Rightarrow} (W,\phi)$ *to* $\phi * \eta' W * \omega$*.*

Clearly, both $\mathsf{Mnd}^\iota(\mathcal{K})$ and $\mathsf{Mnd}^\pi(\mathcal{K})$ contain $\mathsf{Mnd}(\mathcal{K})$ as a vertically full subcategory.

2. Monads in $\mathsf{EM}^w(\mathcal{K})$ and pre-monads in $\mathcal{K}$

It was observed in [16, p. 257] that monads in $\mathsf{EM}(\mathcal{K})$ induce monads in $\mathcal{K}$. The aim of this section is to give a similar interpretation of monads in $\mathsf{EM}^w(\mathcal{K})$.

A monad in $\mathsf{EM}^w(\mathcal{K})$ is given by a triple $((s,\psi),\nu,\vartheta)$, consisting of a 1-cell $(t,\mu,\eta) \xrightarrow{(s,\psi)} (t,\mu,\eta)$ and 2-cells $(s,\psi)\circ(s,\psi) \overset{\nu}{\Rightarrow} (s,\psi)$ and $(k,t) \overset{\vartheta}{\Rightarrow} (s,\psi)$ in $\mathsf{EM}^w(\mathcal{K})$, such that

$$\nu \bullet \big(\nu \circ (s,\psi)\big) = \nu \bullet \big((s,\psi)\circ \nu\big),$$
$$\nu \bullet \big(\vartheta \circ (s,\psi)\big) = (s,\psi) = \nu \bullet \big((s,\psi)\circ \vartheta\big).$$

In light of Theorem 1.1, this means a 1-cell $k \xrightarrow{s} k$, and 2-cells $ts \overset{\psi}{\Rightarrow} st$, $ss \overset{\nu}{\Rightarrow} st$ and $k \overset{\vartheta}{\Rightarrow} st$ in $\mathcal{K}$, subject to the following identities.

$$\psi * \mu s = s\mu * \psi t * t\psi, \tag{2.1}$$
$$s\mu * \psi t * t\nu = s\mu * \nu t * s\psi * \psi s, \tag{2.2}$$
$$s\mu * \psi t * \eta st * \nu = \nu, \tag{2.3}$$
$$s\mu * \psi t * t\vartheta = s\mu * \vartheta t, \tag{2.4}$$
$$s\mu * \nu t * s\nu = s\mu * \nu t * s\psi * \nu s, \tag{2.5}$$
$$s\mu * \nu t * s\psi * \vartheta s = \psi * \eta s, \tag{2.6}$$
$$s\mu * \nu t * s\vartheta = \psi * \eta s. \tag{2.7}$$

Condition (2.1) expresses the requirement that (s,ψ) is a 1-cell in $\mathsf{EM}^w(\mathcal{K})$, (2.2) and (2.3) together mean that ν is a 2-cell in $\mathsf{EM}^w(\mathcal{K})$, (2.4) means that ϑ is a 2-cell in $\mathsf{EM}^w(\mathcal{K})$ (condition (1.3) on ϑ follows by the interchange law in $\mathcal{K}$, (2.4) and unitality of the monad t). Conditions (2.5), (2.6) and (2.7) express associativity and unitality of the monad $((s,\psi),\nu,\vartheta)$, after being simplified using (2.1), (2.2), (2.3) and (2.4).

Note that a monad $(t \xrightarrow{(s,\psi)} t,\nu,\vartheta)$ in $\mathsf{EM}^w(\mathcal{K})$, for a monad $k \xrightarrow{t} k$ in $\mathcal{K}$, is identical to a 'crossed product system' (t,s,ψ,ν) in the monoidal category $\mathcal{K}(k,k)$, in the sense of [13, Definition 3.5], subject to the 'twisted' and 'cocycle' conditions in [13, Definitions 3.3 and 3.6], the normalization condition in [13, Proposition 3.7] and identities (11), (12) and (13) in [13, Theorem 3.11], for ϑ.

The following definition is inspired by [9, Section 3.1] (see also [13, Definition 2.3]).

Definition 2.1. A *pre-monad* in a 2-category $\mathcal{K}$ is a triple (t, μ, η), consisting of a 1-cell $k \xrightarrow{t} k$ and 2-cells $tt \overset{\mu}{\Rightarrow} t$ and $k \overset{\eta}{\Rightarrow} t$, such that the following conditions hold:

$$\mu * \mu t = \mu * t\mu, \tag{2.8}$$

$$\mu * \eta t = \mu * t\eta, \tag{2.9}$$

$$\mu * \eta\eta = \eta, \tag{2.10}$$

$$\mu * \mu t * \eta tt = \mu. \tag{2.11}$$

Note that if $k \xrightarrow{t} k$ is a 1-cell in a 2-category $\mathcal{K}$ and some 2-cells $tt \overset{\mu}{\Rightarrow} t$ and $k \overset{\eta}{\Rightarrow} t$ satisfy $\mu * \mu t = \mu * t\mu$ and $\mu * \eta t = \mu * t\eta = \mu * \mu t * \eta\eta t$ as in [9, Section 3.1], then $(t, \mu' := \mu * \mu t * \eta tt, \eta' := \mu * \eta\eta)$ is a pre-monad in the sense of Definition 2.1.

The motivation for a study of pre-monads stems from the following observation.

Lemma 2.2. *Consider a pre-monad (t, μ, η) in an arbitrary 2-category $\mathcal{K}$.*

(1) *The 2-cell $\mu * \eta t$ is idempotent.*
(2) *Assume that there exists a 1-cell $\hat{t}$ and 2-cells $t \overset{\pi}{\Rightarrow} \hat{t}$ and $\hat{t} \overset{\iota}{\Rightarrow} t$ in $\mathcal{K}$, such that $\mu * \eta t = \iota * \pi$ and $\hat{t} = \pi * \iota$. Then $(\hat{t}, \hat{\mu} := \pi * \mu * \iota\iota, \hat{\eta} := \pi * \eta)$ is a monad in $\mathcal{K}$.*

Proof. The proof is an easy computation using Definition 2.1 of a pre-monad and the properties ι and π obey, so it is left to the reader. □

Improving [13, Theorem 3.11], we obtain the following generalization of a correspondence between monads in $\mathsf{EM}(\mathcal{K})$ and in $\mathcal{K}$, observed by Lack and Street in [16].

Theorem 2.3. *For any monad $(k \xrightarrow{t} k, \mu, \eta)$ and any 1-cell $k \xrightarrow{s} k$ in an arbitrary 2-category $\mathcal{K}$, there is a bijective correspondence between the following structures:*

(i) *A monad $(t \xrightarrow{(s,\psi)} t, \nu, \vartheta)$ in $\mathsf{EM}^w(\mathcal{K})$;*
(ii) *A pre-monad (st, Θ, ϑ) in $\mathcal{K}$, such that*

$$\Theta * sts\mu = s\mu * \Theta t. \tag{2.12}$$

Proof. The proof is built on the same constructions as [13, Theorem 3.11].

Assume first that there exist 2-cells ψ and ν as in part (i). A multiplication Θ for the pre-monad in part (ii) is given by the same formula of a 'wreath product' in [16, p. 256]:

$$\Theta := s\mu * \nu t * ss\mu * s\psi t. \tag{2.13}$$

Its associativity is checked by the same computation as in the case of the wreath product in [16]. By (2.6) on one hand, and by (2.4) and (2.7) on the other,

$$\Theta * \vartheta st = s\mu * \psi t * \eta st = \Theta * st\vartheta, \tag{2.14}$$

proving (2.9). By applying (2.4) again, we conclude that $\Theta * \vartheta\vartheta = \vartheta$, i.e. also (2.10) holds true. Condition (2.11) is proven by the following computation:

$$\begin{aligned}\Theta * \Theta st * \vartheta stst &= s\mu * \nu t * ss\mu * s\psi t * s\mu st * \psi tst * \eta stst \\ &= s\mu * \nu t * ss\mu * ss\mu t * s\psi tt * st\psi t * \psi tst * \eta stst \\ &= s\mu * s\mu t * s\mu tt * \nu ttt * s\psi tt * \psi stt * ts\psi t * \eta stst \\ &= s\mu * s\mu t * s\mu tt * \psi ttt * t\nu tt * ts\psi t * \eta stst \\ &= s\mu * s\mu t * \nu tt * s\psi t = \Theta.\end{aligned}$$

The second equality follows by (2.1). The fourth and the fifth equalities follow by (2.2) and (2.3), respectively. Condition (2.12) follows by associativity of μ. This proves that the data in part (i) determine a pre-monad as in part (ii).

Conversely, assume that there is a 2-cell Θ as in part (ii). The 2-cells ψ and ν in part (i) are constructed as

$$\psi := \Theta * s\mu st * \vartheta tst * ts\eta, \qquad \nu := \Theta * s\eta s\eta. \tag{2.15}$$

By (2.12),

$$s\mu * \psi t = \Theta * s\mu st * \vartheta tst \quad \text{and} \quad s\mu * \nu t = \Theta * s\eta st. \tag{2.16}$$

Moreover, by (2.8), (2.12) and (2.9),

$$\Theta * st\psi = \Theta * \Theta st * sts\mu st * st\vartheta tst * stts\eta = \Theta * s\mu st * \Theta tst * \vartheta sttst * stts\eta. \tag{2.17}$$

With identities (2.16), (2.17), (2.12) and (2.10) at hand, (2.1) is verified as

$$\begin{aligned}s\mu * \psi t * t\psi &= \Theta * st\psi * s\mu ts * \vartheta tts = \Theta * s\mu st * \Theta tst * sts\mu tst * \vartheta\vartheta ttst * tts\eta \\ &= \Theta * s\mu st * s\mu tst * \vartheta ttst * tts\eta = \psi * \mu s.\end{aligned}$$

Use next (2.16), (2.17), (2.12) and (2.11) to compute

$$s\mu * \nu t * s\psi = \Theta * st\psi * s\eta ts = \Theta * \Theta st * \vartheta stst * sts\eta = \Theta * sts\eta. \tag{2.18}$$

In order to prove that (2.2) holds, apply (2.18), (2.8) and (2.16):

$$\begin{aligned}s\mu * \nu t * s\psi * \psi s &= \Theta * \psi st * tss\eta = \Theta * \Theta st * s\mu stst * \vartheta tstst * ts\eta s\eta \\ &= s\mu * \psi t * t\nu.\end{aligned} \tag{2.19}$$

Condition (2.3) is verified by comparing the last and third expressions in (2.19), and using (2.11):

$$s\mu * \psi t * \eta st * \nu = \Theta * \Theta st * \vartheta stst * s\eta s\eta = \Theta * s\eta s\eta = \nu.$$

Condition (2.4) is proven by using (2.16), (2.9), (2.12) and (2.10):

$$s\mu * \psi t * t\vartheta = \Theta * st\vartheta * s\mu * \vartheta t = \Theta * sts\mu * \vartheta\vartheta t = s\mu * \Theta t * \vartheta\vartheta t = s\mu * \vartheta t.$$

Condition (2.5) follows by (2.18), (2.8) and (2.16):

$$s\mu * \nu t * s\psi * \nu s = \Theta * \Theta st * s\eta s\eta s\eta = \Theta * s\eta st * s\Theta * ss\eta s\eta = s\mu * \nu t * s\nu.$$

Condition (2.6) is checked by applying (2.18):

$$s\mu * \nu t * s\psi * \vartheta s = \Theta * \vartheta st * s\eta = \Theta * s\mu st * \vartheta tst * \eta s\eta = \psi * \eta s. \tag{2.20}$$

Finally, (2.7) is proven by making use of (2.16), (2.9) and comparing the second and last expressions in (2.20):

$$s\mu * \nu t * s\vartheta = \Theta * st\vartheta * s\eta = \Theta * \vartheta st * s\eta = \psi * \eta s.$$

This proves that the data in part (ii) determine a monad as in part (i).

It remains to show that the above constructions are mutual inverses. Take 2-cells ν and ψ as in part (i). Use (2.13) to associate a 2-cell Θ as in part (ii) to them, and then use (2.15) to define 2-cells ν' and ψ' as in part (i) again. We obtain

$$\begin{aligned} \nu' &= s\mu * \nu t * s\psi * s\eta s = s\mu * s\mu t * \nu tt * s\nu t * ss\vartheta = s\mu * \nu t * ss\mu * s\psi t * st\vartheta * \nu \\ &= s\mu * s\mu t * \nu tt * s\vartheta t * \nu = s\mu * \psi t * \eta st * \nu = \nu. \end{aligned}$$

The first equality follows by unitality of μ and the second equality follows by (2.7) and associativity of μ. The third equality is obtained by (2.5) and the fourth equality follows by (2.4) and associativity of μ. The penultimate equality is a consequence of (2.7) and the last one follows by (2.3). Also,

$$\psi' = s\mu * \nu t * s\psi * s\mu s * \vartheta ts = s\mu * s\mu t * \nu tt * s\psi t * \vartheta st * \psi = s\mu * \psi t * t\psi * \eta ts = \psi.$$

The first equality follows by unitality of μ, the second equality follows by (2.1) and associativity of μ, and the third equality is obtained by (2.6). The last equality follows by (2.1) and by unitality of μ.

In the opposite order, start with a 2-cell Θ as in part (ii). Apply (2.15) to construct 2-cells ψ and ν as in part (i) and then apply (2.13) to obtain a 2-cell Θ' as in part (ii). It satisfies

$$\begin{aligned} \Theta' &= s\mu * \nu t * ss\mu * s\psi t = \Theta * s\eta st * ss\mu * s\psi t = s\mu * \Theta t * st\psi t * s\eta tst \\ &= \Theta * s\mu st * \Theta tst * \vartheta sttst * s\eta tst = \Theta * \Theta st * \vartheta stst = \Theta. \end{aligned}$$

The second equality follows by the second identity in (2.16). The third equality is obtained by (2.12). The fourth equality is a consequence of (2.17), (2.12) and unitality of μ. The penultimate equality follows by (2.12) and unitality of μ. The last equality is obtained by (2.11). □

Examples 2.4. Examples of a composite pre-monad as in Theorem 2.3(ii) are given, first of all, by wreath products in [16]. It is shown in [16, Example 3.3] that crossed products by Hopf algebras in [21,2,12] are examples of a wreath product. As it was observed by Ross Street [20] (see also [13, Example 3.18]), so are the crossed products with coalgebras in [7] and their generalizations to comonads in [22, Section 4.8].

Examples of a composite pre-monad, which are not monads, are provided by 'weak smash products' in [9, Section 3] (for a review see [13, Example 3.16]). This includes smash products with weak bialgebras [6]. Crossed products with weak bialgebras in [18] are also shown in [13, Section 4] to provide examples.

Note, however, that (weak) crossed products with bialgebroids in [5, Section 4 & Appendix] are not (pre-)monads of the kind in Theorem 2.3(ii). Let k be a commutative and associative unital ring. A k-algebra B, measured by a left bialgebroid H over a k-algebra L, determines two monads, $(-) \otimes_k B$ on the category of k-modules and $(-) \otimes_L B$ on the category of L-modules. In terms of the measuring $\cdot : H \otimes_k B \to B$ and the comultiplication $h \mapsto \sum h_1 \otimes_L h_2$ in H, consider the left L-module map

$$H \otimes_k B \to B \otimes_L H, \qquad h \otimes_k b \mapsto \sum h_1 \cdot b \otimes_L h_2.$$

It equips the left L-module (or L–k bimodule) H with the structure of a 1-cell $(-) \otimes_L B \xrightarrow{(-)\otimes_L H} (-) \otimes_k B$ in $\mathsf{EM}(\mathsf{CAT})$ (or in $\mathsf{EM}^w(\mathsf{CAT})$ if $\cdot$ is only a weak measuring). However, if L is a non-trivial k-algebra, this 1-cell has different source and target. Although in [5] the composite endofunctor $- \otimes_k B \otimes_L H$ on the category of k-modules is proven to carry a monad structure, it is not a composite of a monad with an endofunctor.

3. A pseudo-functor $\mathsf{EM}^w(\mathcal{K}) \to \mathcal{K}$

Throughout this section (and the next one), we make two basic assumptions on the 2-category $\mathcal{K}$ we deal with:

(i) Idempotent 2-cells in $\mathcal{K}$ split;
(ii) $\mathcal{K}$ admits Eilenberg–Moore constructions (EM constructions, for short) for monads.

In more details, assumption (i) means that for any 2-cell $V \overset{\varpi}{\Rightarrow} V$ in $\mathcal{K}$, such that $\varpi * \varpi = \varpi$, there exist a 1-cell $\hat{V}$ and 2-cells $\hat{V} \overset{\iota}{\Rightarrow} V$ and $V \overset{\pi}{\Rightarrow} \hat{V}$, such that $\pi * \iota = \hat{V}$ and $\iota * \pi = \varpi$. It is easy to see that the datum $(\hat{V}, \iota, \pi)$ is unique up to an isomorphism $\hat{V} \overset{\iota}{\Rightarrow} V \overset{\pi'}{\Rightarrow} \hat{V}'$.

Property (ii) of a 2-category was introduced by Street in [19, p. 151]. It means that the inclusion 2-functor $\mathcal{K} \to \mathsf{Mnd}(\mathcal{K})$ (with underlying maps $k \mapsto (k \xrightarrow{k} k, k, k)$, $(k \xrightarrow{V} k') \mapsto (k \xrightarrow{V} k', V \overset{V}{\Rightarrow} V)$, $(V \overset{\varrho}{\Rightarrow} W) \mapsto (V \overset{\varrho}{\Rightarrow} W)$ on the 0-, 1-, and 2-cells, respectively) possesses a right 2-adjoint. By [16, Section 1], property (ii) can be formulated equivalently by saying that the inclusion 2-functor $I : \mathcal{K} \to \mathsf{EM}(\mathcal{K})$ possesses a right 2-adjoint J. Important properties of 2-categories admitting EM constructions for monads are formulated in the following theorem. It is recalled from [19, Theorem 2] and [16, Section 1].

Theorem 3.1. *In a 2-category $\mathcal{K}$ which admits EM constructions for monads, any monad $(k \xrightarrow{t} k, \mu, \eta)$ determines an adjunction $(k \xrightarrow{f} J(t), J(t) \xrightarrow{v} k, k \overset{\eta}{\Rightarrow} vf, fv \overset{\epsilon}{\Rightarrow} k)$ in $\mathcal{K}$, such that*

$(t,\mu,\eta)=(vf, v\epsilon f,\eta)$. One can choose $JI=\mathcal{K}$, and the isomorphism corresponding to the 2-adjunction (I,J) is given by the mutually inverse functors

$$\mathcal{K}\big(l,J(t)\big)\to \mathsf{EM}(\mathcal{K})\big(I(l),t\big),\qquad (V\overset{\omega}{\Rightarrow}W)\mapsto \big((vV,v\epsilon V)\overset{v\omega}{\Longrightarrow}(vW,v\epsilon W)\big),$$

$$\mathsf{EM}(\mathcal{K})\big(I(l),t\big)\to \mathcal{K}\big(l,J(t)\big),\qquad \big((A,\alpha)\overset{\varrho}{\Rightarrow}(B,\beta)\big)\mapsto \big(J(A,\alpha)\overset{J(\varrho)}{\Longrightarrow}J(B,\beta)\big),\tag{3.1}$$

for any 0-cell l and monad t in $\mathcal{K}$.

The notations in Theorem 3.1 are used throughout, without further explanation.

Lemma 3.2. *Consider a 2-category $\mathcal{K}$ which admits EM constructions for monads. For any 1-cell $(t,\mu,\eta)\xrightarrow{(V,\psi)}(t',\mu',\eta')$ in $\mathsf{EM}^w(\mathcal{K})$, the 2-cell $Vv\epsilon * \psi v * \eta' Vv : Vv\Rightarrow Vv$ in $\mathcal{K}$ is idempotent, and obeys the following identities:*

$$V\mu * \psi t * \eta' Vt * \psi = \psi;\tag{3.2}$$

$$Vv\epsilon * \psi v * t'Vv\epsilon * t'\psi v * t'\eta' Vv = Vv\epsilon * \psi v.\tag{3.3}$$

Proof. All statements follow easily by applying the interchange law, (1.1) and unitality of the monad t'. □

The idempotent 2-cell in Lemma 3.2, corresponding to a 1-cell $(t,\mu,\eta)\xrightarrow{(V,\psi)}(t',\mu',\eta')$ in $\mathsf{EM}^w(\mathcal{K})$, is an identity 2-cell if and only if $\psi * \eta' V = V\eta$, i.e. (V,ψ) is a 1-cell in $\mathsf{EM}(\mathcal{K})$.

Our next aim is to extend the 2-functor J in Theorem 3.1 to $\mathsf{EM}^w(\mathcal{K})$. Our method is reminiscent to the way J is obtained from the right adjoint of the inclusion 2-functor $\mathcal{K}\to\mathsf{Mnd}(\mathcal{K})$.

Lemma 3.3. *Consider a 2-category $\mathcal{K}$ which admits EM constructions for monads and in which idempotent 2-cells split. For a 1-cell $(t,\mu,\eta)\xrightarrow{(V,\psi)}(t',\mu',\eta')$ in $\mathsf{EM}^w(\mathcal{K})$, denote a chosen splitting of the idempotent 2-cell in Lemma 3.2 by $Vv\overset{\pi}{\Rightarrow}\tilde V\overset{\iota}{\Rightarrow}Vv$. Then $(\tilde V,\tilde\psi := \pi * Vv\epsilon * \psi v * t'\iota)$ is a 1-cell $IJ(t)\to t'$ in $\mathsf{EM}(\mathcal{K})$.*

Proof. By the interchange law, $\tilde\psi * \eta'\tilde V = \pi * \iota * \pi * \iota = \tilde V$. Furthermore, by (3.3) and (1.1),

$$\begin{aligned}\tilde\psi * t'\tilde\psi &= \pi * Vv\epsilon * \psi v * t'Vv\epsilon * t'\psi v * t't'\iota = \pi * Vv\epsilon * V\mu v * \psi tv * t'\psi v * t't'\iota\\ &= \pi * Vv\epsilon * \psi v * \mu' Vv * t't'\iota = \tilde\psi * \mu'\tilde V. \qquad □\end{aligned}$$

Lemma 3.4. *Consider a 2-category $\mathcal{K}$ which admits EM constructions for monads and in which idempotent 2-cells split. For any 2-cell $(V,\psi)\overset{\varrho}{\Rightarrow}(W,\phi)$ in $\mathsf{EM}^w(\mathcal{K})$, $\tilde\varrho := \pi * Wv\epsilon * \varrho v * \iota$ is a 2-cell in $\mathsf{EM}(\mathcal{K})$, between the 1-cells $(\tilde V,\tilde\psi)$ and $(\tilde W,\tilde\phi)$ in Lemma 3.3 (where π and ι denote chosen splittings of both idempotent 2-cells in Lemma 3.2, corresponding to the 1-cells (V,ψ) and (W,ϕ)).*

Proof. Apply (3.2) (in the first equality), (1.2) (in the third equality) and (1.3) (in the penultimate equality) to conclude that

$$\begin{aligned}\tilde{\varrho} * \tilde{\psi} &= \pi * Wv\epsilon * \varrho v * Vv\epsilon * \psi v * t'\iota = \pi * Wv\epsilon * W\mu v * \varrho t v * \psi v * t'\iota \\ &= \pi * Wv\epsilon * W\mu v * \phi t v * t'\varrho v * t'\iota = \pi * Wv\epsilon * \phi v * t'\iota * t'\pi * t'Wv\epsilon * t'\varrho v * t'\iota \\ &= \tilde{\phi} * t'\tilde{\varrho}. \qquad \square\end{aligned}$$

Theorem 3.5. *Consider a* 2*-category* $\mathcal{K}$ *which admits EM constructions for monads and in which idempotent* 2*-cells split. The following maps determine a pseudo-functor* $J^w : \mathsf{EM}^w(\mathcal{K}) \to \mathcal{K}$.

For a 0-cell t, $J^w(t) := J(t)$.
For a 1-cell $t \xrightarrow{(V,\psi)} t'$, $J^w(V,\psi) := J(\tilde{V},\tilde{\psi})$, *where the* 1*-cell* $(\tilde{V},\tilde{\psi})$ *in* $\mathsf{EM}(\mathcal{K})$ *is described in Lemma* 3.3*. That is, denoting by* ι *and* π *a chosen splitting of the idempotent* 2*-cell in Lemma* 3.2*,* $J^w(V,\psi)$ *is the unique* 1*-cell* $J^w(t) \to J^w(t')$ *in* $\mathcal{K}$ *for which* $v'\epsilon' J^w(V,\psi) = \pi * Vv\epsilon * \psi v * t'\iota$.
For a 2-cell $(V,\psi) \overset{\varrho}{\Rightarrow} (W,\phi)$, $J^w(\varrho) := J(\tilde{\varrho})$, *where the* 2*-cell* $\tilde{\varrho}$ *in* $\mathsf{EM}(\mathcal{K})$ *is described in Lemma* 3.4*. That is,* $J^w(\varrho)$ *is the unique* 2*-cell* $J^w(V,\psi) \Rightarrow J^w(W,\phi)$ *in* $\mathcal{K}$ *for which* $v'J^w(\varrho) = \pi * Wv\epsilon * \varrho v * \iota$.

The pseudo-natural isomorphism class of J^w *is independent of the choice of the* 2*-cells* ι *and* π *in its construction.*

Proof. Let us fix splittings (π,ι) of the idempotent 2-cells in Lemma 3.2, for all 1-cells (V,ψ) in $\mathsf{EM}^w(\mathcal{K})$.

By construction, $v'J^w(\psi * \eta' V) = \pi * \iota * \pi * \iota = \tilde{V}$, for any 1-cell $t \xrightarrow{(V,\psi)} t'$ in $\mathsf{EM}^w(\mathcal{K})$. Hence J^w preserves identity 2-cells $(V,\psi) \xRightarrow{\psi * \eta' V} (V,\psi)$. For 2-cells $(V,\psi) \overset{\varrho}{\Rightarrow} (W,\phi) \overset{\tau}{\Rightarrow} (U,\theta)$ in $\mathsf{EM}^w(\mathcal{K})$, it follows by (1.3) (applied to ϱ) that

$$\begin{aligned}v'J^w(\tau) * v'J^w(\varrho) &= \pi * Uv\epsilon * \tau v * Wv\epsilon * \varrho v * \iota \\ &= \pi * Uv\epsilon * U\mu v * \tau t v * \varrho v * \iota = v'J^w(\tau \bullet \varrho).\end{aligned}$$

We conclude by the isomorphism (3.1) that J^w preserves the vertical composition.

For an identity 1-cell $t \xrightarrow{(k,t)} t$, the idempotent 2-cell in Lemma 3.2 is the identity 2-cell v by the adjunction relation $v\epsilon * \eta v = v$. Hence any splitting of it yields mutually inverse isomorphisms $v \xRightarrow{\pi_k} \tilde{k}$ and $\tilde{k} \xRightarrow{\iota_k} v$. They give rise to an isomorphism $J^w(t) = J(v, v\epsilon) \xRightarrow{J(\pi_k)} J^w(k,t) = J(\tilde{k}, \pi_k * v\epsilon * t\iota_k)$ with the inverse $J(\iota_k)$. Thus J^w preserves identity 1-cells up to isomorphism. (In particular, we can choose for the definition of J^w a trivial splitting $v \overset{v}{\Rightarrow} v \overset{v}{\Rightarrow} v$, in which case the 1-cell $(\tilde{k},\tilde{t})$ in Lemma 3.3 is equal to $(v, v\epsilon)$. Applying the isomorphism (3.1), we conclude that with this choice, J^w strictly preserves identity 1-cells, i.e. $J^w(k,t) = J(v,v\epsilon) = J^w(t)$.)

In order to investigate the preservation of the horizontal composition, consider different splittings (π,ι) and (π',ι') of the idempotent 2-cell in Lemma 3.2, for some 1-cell (V,ψ), and denote the corresponding 1-cells in Lemma 3.3 by $(\tilde{V},\tilde{\psi})$ and $(\tilde{V}',\tilde{\psi}')$, respectively. Applying (3.3) (for (π',ι')) and (3.2) (for (π,ι)),

$$\pi' * Vv\epsilon * \psi v * t'\iota' * t'\pi' * t'\iota = \pi' * Vv\epsilon * \psi v * t'\iota = \pi' * \iota * \pi * Vv\epsilon * \psi v * t'\iota.$$

Hence $(\tilde{V},\tilde{\psi}) \overset{\pi' * \iota}{\Longrightarrow} (\tilde{V}',\tilde{\psi}')$ is an iso 2-cell in $\mathsf{EM}(\mathcal{K})$, so $J(\tilde{V},\tilde{\psi}) \overset{J(\pi' * \iota)}{\Longrightarrow} J(\tilde{V}',\tilde{\psi}')$ is an iso 2-cell in $\mathcal{K}$.

For 1-cells $(V,\psi),(W,\phi): t \to t'$ and $(V',\psi'),(W',\phi'): t' \to t''$ and 2-cells $(V,\psi) \overset{\varrho}{\Rightarrow} (W,\phi)$ and $(V',\psi') \overset{\varrho'}{\Rightarrow} (W',\phi')$ in $\mathsf{EM}^w(\mathcal{K})$, the idempotent 2-cell in Lemma 3.2 corresponding to the 1-cell $(V',\psi') \circ (V,\psi)$ comes out as $V'Vv\epsilon * V'\psi v * \psi' V v * \eta'' V'Vv$. We claim that it has a splitting given by the mono 2-cell $V'\iota * \iota' J^w(V,\psi)$ and the epi 2-cell $\pi' J^w(V,\psi) * V'\pi$, where (π,ι) and (π',ι') are the chosen splittings of the idempotent 2-cells in Lemma 3.2, corresponding to the 1-cells (V,ψ) and (V',ψ') in $\mathsf{EM}^w(\mathcal{K})$, respectively. Indeed, by construction of J^w (its action on a 1-cell (V,ψ)), (3.2) and (3.3),

$$\begin{aligned} &V'\iota * \iota' J^w(V,\psi) * \pi' J^w(V,\psi) * V'\pi \\ &\quad = V'\iota * V'\pi * V'Vv\epsilon * V'\psi v * V't'\iota * V't'\pi * \psi' V v * \eta'' V'Vv \\ &\quad = V'Vv\epsilon * V'\psi v * \psi' V v * \eta'' V'Vv. \end{aligned} \tag{3.4}$$

Denote by $V'Vv \overset{\pi_2}{\Longrightarrow} \widetilde{V'V} \overset{\iota_2}{\Longrightarrow} V'Vv$ the canonical splitting of this idempotent which was chosen to define J^w on the 1-cell $(V',\psi') \circ (V,\psi) = (V'V, V'\psi * \psi' V)$. By considerations in the previous paragraph, there are mutually inverse iso 2-cells $J(\pi' J^w(V,\psi) * V'\pi * \iota_2): J^w(V'V, V'\psi * \psi' V) \Rightarrow J^w(V',\psi')J^w(V,\psi)$ and $J(\pi_2 * V'\iota * \iota' J^w(V,\psi)): J^w(V',\psi')J^w(V,\psi) \Rightarrow J^w(V'V, V'\psi * \psi' V)$. In order to see their naturality, observe that

$$v'' J^w(\varrho' \circ \varrho) = \pi_2 * W'Wv\epsilon * W'W\mu v * W'\varrho t v * W'\psi v * \varrho' V v * \iota_2.$$

On the other hand,

$$\begin{aligned} &v'' J^w(\varrho') J^w(\varrho) \\ &\quad = \pi' J^w(W,\phi) * W'\pi * W'Wv\epsilon * W'\phi v * \varrho' W v * V'\iota * V'\pi * V'Wv\epsilon * V'\varrho v \\ &\qquad * V'\iota * \iota' J^w(V,\psi) \\ &\quad = \pi' J^w(W,\phi) * W'\pi * W'Wv\epsilon * W'W\mu v * W'\phi t v * W't'\varrho v * \varrho' V v * V'\iota * \iota' J^w(V,\psi) \\ &\quad = \pi' J^w(W,\phi) * W'\pi * W'Wv\epsilon * W'W\mu v * W'\varrho t v * W'\psi v * \varrho' V v * V'\iota * \iota' J^w(V,\psi). \end{aligned}$$

The second equality follows by applying (1.3), and the third one follows by applying (1.2), for ϱ. With this information in mind, we conclude that

$$\begin{aligned} &v'' J^w(\varrho' \circ \varrho) * \pi_2 * V'\iota * \iota' J^w(V,\psi) \\ &\quad = \pi_2 * W'Wv\epsilon * W'W\mu v * W'\varrho t v * W'\psi v * \varrho' V v * V'\iota * \iota' J^w(V,\psi) \\ &\quad = \pi_2 * W'\iota * \iota' J^w(W,\phi) * v'' J^w(\varrho') J^w(\varrho). \end{aligned}$$

Thus naturality of $J(\pi_2 * V'\iota * \iota' J^w(V,\psi))$ follows by the isomorphism (3.1). It remains to check its associativity and unitality. For a further 1-cell $(V'',\psi''): t'' \to t'''$ in $\mathsf{EM}^w(\mathcal{K})$, use the notation $V''V'Vv \overset{\pi_3}{\Longrightarrow} \widetilde{V''V'V} \overset{\iota_3}{\Longrightarrow} V''V'Vv$ for the canonically split idempotent in the construction of J^w on $(V'', \psi'') \circ (V', \psi') \circ (V, \psi) = (V''V'V, V''V'\psi * V''\psi'V * \psi''V'V)$. By (3.4), the associativity condition

$$\begin{aligned}
&\pi_3 * V''\iota_2 * \iota'' J^w\big(V'V, V'\psi * \psi'V\big) * v''' J^w\big(V'', \psi''\big) J\big(\pi_2 * V'\iota * \iota' J^w(V, \psi)\big)\\
&\quad = \pi_3 * V''\iota_2 * V''\pi_2 * V''V'\iota * V''\iota' J^w(V, \psi) * \iota'' J^w\big(V', \psi'\big) J^w(V, \psi)\\
&\quad = \pi_3 * V''V'\iota * V''\iota' J^w(V, \psi) * \iota'' J^w\big(V', \psi'\big) J^w(V, \psi)\\
&\quad = \pi_3 * V''V'\iota * \iota'_2 J^w(V, \psi) * \pi'_2 J^w(V, \psi) * V''\iota' J^w(V, \psi) * \iota'' J^w\big(V', \psi'\big) J^w(V, \psi)
\end{aligned}$$

holds. The 2-cells ι_k and π_k, splitting the idempotent (identity) 2-cell in Lemma 3.2 corresponding to a unit 1-cell (k, t), are mutual inverses. Hence also the unitality conditions

$$\begin{aligned}
&v' J\big(\pi * V\iota_k * \iota J^w(k, t)\big) * v' J^w(V, \psi) J(\pi_k) = \pi * V\iota_k * V\pi_k * \iota = v' J^w(V, \psi),\\
&v' J\big(\iota_{k'} J^w(V, \psi)\big) * v' J(\pi_{k'}) J^w(V, \psi) = \iota_{k'} J^w(V, \psi) * \pi_{k'} J^w(V, \psi) = v' J^w(V, \psi)
\end{aligned}$$

hold. Thus we conclude by the isomorphism (3.1) that J^w preserves also the horizontal composition up to a coherent family of iso 2-cells, i.e. that it is a pseudo-functor.

Finally, we investigate the ambiguity of the pseudo-functor J^w, caused by a free choice of the splittings of the idempotent 2-cells in Lemma 3.2. Take two collections $\{(\pi, \iota)\}$ and $\{(\pi', \iota')\}$ of splittings (indexed by the 1-cells in $\mathsf{EM}^w(\mathcal{K})$). The pseudo-functors J^w and J'^w, associated to both families of splittings, are pseudo-naturally isomorphic via $J^w(t) = J'^w(t)$ and $J^w(V, \psi) \xRightarrow{J(\pi' * \iota)} J'^w(V, \psi)$, for any 0-cell t and 1-cell (V, ψ) in $\mathsf{EM}^w(\mathcal{K})$. □

Consider a 2-category $\mathcal{K}$ which admits EM constructions for monads and in which idempotent 2-cells split. We can regard a 1-cell $t \xrightarrow{(V,\psi)} t'$ in $\mathsf{EM}(\mathcal{K})$ as a 1-cell in $\mathsf{EM}^w(\mathcal{K})$. Choosing a trivial splitting $Vv \xRightarrow{Vv} Vv \xRightarrow{Vv} Vv$ of the identity 2-cell, the corresponding 1-cell $IJ(t) \xrightarrow{(\tilde{V},\tilde{\psi})} t'$ in Lemma 3.3 comes out as the 1-cell $(Vv, Vv\epsilon * \psi v)$ in $\mathsf{EM}(\mathcal{K})$. By 1-naturality of the counit of the 2-adjunction (I, J), we have $(v' J(V, \psi), v'\epsilon' J(V, \psi)) = (Vv, Vv\epsilon * \psi v)$. From this and from the isomorphism (3.1) it follows that

$$J^w(V, \psi) = J(Vv, Vv\epsilon * \psi v) = J(V, \psi).$$

Similarly, we can regard a 2-cell $(V, \psi) \overset{\varrho}{\Rrightarrow} (W, \phi)$ in $\mathsf{EM}(\mathcal{K})$ as a 2-cell in $\mathsf{EM}^w(\mathcal{K})$. The corresponding 2-cell $(\tilde{V}, \tilde{\psi}) \overset{\tilde{\varrho}}{\Rrightarrow} (\tilde{W}, \tilde{\phi})$ in Lemma 3.4 is equal to the 2-cell $Wv\epsilon * \varrho v : (Vv, Vv\epsilon * \psi v) \Rightarrow (Wv, Wv\epsilon * \phi v)$ in $\mathsf{EM}(\mathcal{K})$. By the 2-naturality condition $v' J(\varrho) = Wv\epsilon * \varrho v$ and the isomorphism (3.1) we obtain that

$$J^w(\varrho) = J(Wv\epsilon * \varrho v) = J(\varrho).$$

Summarizing, we proved that the pseudo-functor $J^w : \mathsf{EM}^w(\mathcal{K}) \to \mathcal{K}$ in Theorem 3.5 can be chosen such that the 2-functor $J : \mathsf{EM}(\mathcal{K}) \to \mathcal{K}$ in Theorem 3.1 factorizes through the obvious inclusion $\mathsf{EM}(\mathcal{K}) \hookrightarrow \mathsf{EM}^w(\mathcal{K})$ and J^w.

The pseudo-functor J^w in Theorem 3.5 takes a monad $((s, \psi), \nu, \vartheta)$ in $\mathsf{EM}^w(\mathcal{K})$ to a monad $J^w(s, \psi)$ in $\mathcal{K}$, with multiplication $J^w(s, \psi) J^w(s, \psi) \overset{\cong}{\Rrightarrow} J^w((s, \psi) \circ (s, \psi)) \xRightarrow{J^w(\nu)} J^w(s, \psi)$

and unit $J^w(t) \stackrel{\cong}{\Rightarrow} J^w(k,t) \xrightarrow{J^w(\vartheta)} J^w(s,\psi)$. Applying to this monad in $\mathcal{K}$ a hom 2-functor $\mathcal{K}(l,-):\mathcal{K}\to \mathsf{CAT}$ (for any 0-cell l in $\mathcal{K}$), we obtain a monad in CAT. Our next aim is to describe its Eilenberg–Moore category.

Lemma 3.6. *Consider a 2-category $\mathcal{K}$ which admits EM constructions for monads and in which idempotent 2-cells split. Let l be a 0-cell and $(k \xrightarrow{t} k, \mu, \eta)$ be a monad in $\mathcal{K}$ and let $t \xrightarrow{(s,\psi)} t$ be a 1-cell in $\mathsf{EM}^w(\mathcal{K})$. There is a bijective correspondence between the following structures:*

(i) *Pairs $(l \xrightarrow{V} J^w(t), J^w(s,\psi)V \stackrel{\zeta}{\Rightarrow} V)$, consisting of a 1-cell V and a 2-cell ζ in $\mathcal{K}$;*
(ii) *Pairs $((l \xrightarrow{W} k, tW \stackrel{\varrho}{\Rightarrow} W), sW \stackrel{\lambda}{\Rightarrow} W)$, consisting of a 1-cell $I(l) \xrightarrow{(W,\varrho)} t$ in $\mathsf{EM}(\mathcal{K})$ and (regarding (W,ϱ) as a 1-cell in $\mathsf{EM}^w(\mathcal{K})$), a 2-cell $(s,\psi)\circ(W,\varrho) \stackrel{\lambda}{\Rightarrow} (W,\varrho)$ in $\mathsf{EM}^w(\mathcal{K})$.*

Proof. Denote by ι and π the splitting of the idempotent 2-cell in Lemma 3.2, corresponding to the 1-cell (s,ψ) in $\mathsf{EM}^w(\mathcal{K})$, that was chosen to construct $J^w(s,\psi)$. For the 1-cell (W,ϱ) in $\mathsf{EM}^w(\mathcal{K})$, choose the trivial splitting of the identity 2-cell $W \stackrel{W}{\Rightarrow} W$, so that $J^w(W,\varrho) = J(W,\varrho)$.

By (3.1), there is a bijection between the 1-cells $I(l) \xrightarrow{(W,\varrho)} t$ in $\mathsf{EM}(\mathcal{K})$ as in part (ii), and the 1-cells $V := J(W,\varrho) = J^w(W,\varrho): l \to J(t) = J^w(t)$ in $\mathcal{K}$ as in part (i). In order to see that it extends to the stated bijection, take first a 2-cell λ in $\mathsf{EM}^w(\mathcal{K})$ as in part (ii). Then there is a 2-cell $\zeta := (J^w(s,\psi)V \stackrel{\cong}{\Rightarrow} J^w((s,\psi)\circ\ (W,\varrho)) \xrightarrow{J^w(\lambda)} V)$ in $\mathcal{K}$ as in part (i). Conversely, for a 2-cell ζ in $\mathcal{K}$ as in part (i), put $\lambda := v\zeta * \pi V$. It satisfies

$$\begin{aligned} \varrho * t\lambda &= v\epsilon V * tv\zeta * t\pi V = v\zeta * v\epsilon J^w(s,\psi)V * t\pi V \\ &= v\zeta * \pi V * sv\epsilon V * \psi v V * t\iota V * t\pi V = \lambda * s\varrho * \psi W. \end{aligned} \tag{3.5}$$

The second equality follows by the interchange law and the third one follows by construction of the pseudo-functor J^w, cf. Theorem 3.5. The last equality follows by (3.3). Together with the unitality of ϱ, this proves that λ is a 2-cell in $\mathsf{EM}^w(\mathcal{K})$, as needed.

The above two constructions can be seen to be mutual inverses. Take first a pair (V,ζ) as in part (i) and iterate both constructions. The result is $(V, J^w(s,\psi)V \stackrel{\cong}{\Rightarrow} J^w((s,\psi)\circ (W,\varrho)) \xrightarrow{J^w(v\zeta*\pi v)} V) = (V,\zeta)$. In the opposite order, a datum $((W,\varrho),\lambda)$ is taken to $((W,\varrho), sW \stackrel{\pi V}{\Longrightarrow} vJ^w(s,\psi)V \stackrel{\cong}{\Rightarrow} vJ^w((s,\psi)\circ(W,\varrho)) \xrightarrow{vJ^w(\lambda)} W) = ((W,\varrho), \lambda * \iota V * \pi V)$. The resulting 2-cell $\lambda * \iota V * \pi V$ in $\mathcal{K}$ is equal to λ since by (3.5) and unitality of ϱ,

$$\lambda * \iota V * \pi V = \lambda * s\varrho * \psi W * \eta s W = \varrho * t\lambda * \eta s W = \lambda. \qquad \square \tag{3.6}$$

The following extends [16, Proposition 3.1] and also [9, Theorem 3.4].

Proposition 3.7. *Consider a 2-category $\mathcal{K}$ which admits EM constructions for monads and in which idempotent 2-cells split. Let l be a 0-cell and $(k \xrightarrow{t} k, \mu, \eta)$ be a monad in $\mathcal{K}$ and let $(t \xrightarrow{(s,\psi)} t, v, \vartheta)$ be a monad in $\mathsf{EM}^w(\mathcal{K})$. The following categories are isomorphic:*

(i) *The Eilenberg–Moore category $\mathsf{EM}(\mathcal{K})(I(l), J^w(s,\psi))$ of the monad $\mathcal{K}(l, J^w(s,\psi))$: $\mathcal{K}(l, J^w(t)) \to \mathcal{K}(l, J^w(t))$;*

(ii) *The Eilenberg–Moore category* $\mathsf{EM}(\mathcal{K})(I(l),\widehat{st})$ *of the monad* $\mathcal{K}(l,\widehat{st}):\mathcal{K}(l,k)\to\mathcal{K}(l,k)$, *where the monad* $\widehat{st}$ *is obtained from the pre-monad* st *in Theorem* 2.3 *in the way described in Lemma* 2.2*;*

(iii) *The category* $\mathcal{C}$, *with*

objects that are pairs $((W,\varrho),\lambda)$, *consisting of a* 1*-cell* $I(l)\xrightarrow{(W,\varrho)} t$ *in* $\mathsf{EM}(\mathcal{K})$ *and a* 2*-cell* $(s,\psi)\circ(W,\varrho)\overset{\lambda}{\Rightarrow}(W,\varrho)$ *in* $\mathsf{EM}^w(\mathcal{K})$, *satisfying*

$$\lambda\bullet\big((s,\psi)\circ\lambda\big)=\lambda\bullet\big(\nu\circ(W,\varrho)\big); \tag{3.7}$$

$$(W,\varrho)=\lambda\bullet\big(\vartheta\circ(W,\varrho)\big); \tag{3.8}$$

morphisms $((W,\varrho),\lambda)\to((W',\varrho'),\lambda')$ *that are* 2*-cells* $(W,\varrho)\overset{\alpha}{\Rightarrow}(W',\varrho')$ *in* $\mathsf{EM}(\mathcal{K})$ *such that*

$$\lambda'\bullet\big((s,\psi)\circ\alpha\big)=\alpha\bullet\lambda. \tag{3.9}$$

Proof. Denote by ι and π the splitting of the idempotent 2-cell in Lemma 3.2, corresponding to the 1-cell (s,ψ) in $\mathsf{EM}^w(\mathcal{K})$, that was chosen to construct $J^w(s,\psi)$. For the 1-cell (W,ϱ) in $\mathsf{EM}^w(\mathcal{K})$, choose the trivial splitting of the identity 2-cell $W\overset{W}{\Rightarrow}W$, so that $J^w(W,\varrho)=J(W,\varrho)$. Introduce shorthand notations $\bar{s}:=J^w(s,\psi)$ and $V:=J^w(W,\varrho)$.

Isomorphism of $\mathsf{EM}(\mathcal{K})(I(l),J^w(s,\psi))$ and $\mathcal{C}$. In light of Lemma 3.6, any object in $\mathsf{EM}(\mathcal{K})(I(l),J^w(s,\psi))$ is of the form $(J^w(W,\varrho),\bar{s}V\overset{\cong}{\Rightarrow}J^w((s,\psi)\circ(W,\varrho))\overset{J^w(\lambda)}{\Longrightarrow}V)$, for a unique 1-cell $I(l)\xrightarrow{(W,\varrho)} t$ in $\mathsf{EM}(\mathcal{K})$ and a unique 2-cell $(s,\psi)\circ(W,\varrho)\overset{\lambda}{\Rightarrow}(W,\varrho)$ in $\mathsf{EM}^w(\mathcal{K})$. So we only need to show that λ satisfies (3.7), i.e. the equality

$$\lambda*s\lambda=\lambda*s\varrho*\nu W \tag{3.10}$$

of 2-cells in $\mathcal{K}$, if and only if $\bar{s}V\overset{\cong}{\Rightarrow}J^w((s,\psi)\circ(W,\varrho))\overset{J^w(\lambda)}{\Longrightarrow}V$ is an associative action, and λ satisfies (3.8), i.e.

$$W=\lambda*s\varrho*\vartheta W \tag{3.11}$$

if and only if this $\bar{s}$-action on V is unital. Compose the associativity condition $J^w(\lambda)*J^w((s,\psi)\circ\lambda)=J^w(\lambda)*J^w(\nu\circ(W,\varrho))$ with v horizontally on the left, and compose it with the chosen split epimorphism $ssW\to vJ^w(ssW,ss\varrho*s\psi W*\psi sW)$ on the right (i.e. on the 'top'). It yields the equality

$$\lambda*s\lambda*(ss\varrho*s\psi W*\psi sW*\eta ssW)=\lambda*s\varrho*\nu W*(ss\varrho*s\psi W*\psi sW*\eta ssW).$$

Making use of (3.5), the left-hand side is easily shown to be equal to $\lambda*s\lambda$. As far as the right-hand side is concerned, use associativity of ϱ (in the first equality), (2.2) and (2.3) (in the second and third equalities, respectively) to see that

$$\lambda * s\varrho * \nu W * ss\varrho * s\psi W * \psi s W * \eta ss W = \lambda * s\varrho * s\mu W * \nu t W * s\psi W * \psi s W * \eta ss W$$
$$= \lambda * s\varrho * s\mu W * \psi t W * t\nu W * \eta ss W = \lambda * s\varrho * \nu W.$$

This proves that the $\bar{s}$-action on V is associative if and only if (3.10) holds. Similarly, the unitality condition $J^w(\lambda) * J^w(\vartheta \circ (W, \varrho)) = V$ is equivalent to $\lambda * \iota V * \pi V * s\varrho * \vartheta W = W$, hence by (3.6) it is equivalent to (3.11). Thus the bijection in Lemma 3.6 restricts to a bijection between the objects in $\mathsf{EM}(\mathcal{K})(I(l), J^w(s,\psi))$ and the objects in $\mathcal{C}$.

For a 2-cell $(W,\varrho) \overset{\alpha}{\Rightarrow} (W', \varrho')$ in $\mathsf{EM}(\mathcal{K})$, the condition $J^w(\lambda') * J^w((s,\psi)\circ\alpha) = J^w(\alpha) * J^w(\lambda)$ (expressing that $J^w(\alpha) = J(\alpha)$ is a morphism in $\mathsf{EM}(\mathcal{K})(I(l), J^w(s,\psi))$) is equivalent to $\lambda' * s\alpha * \iota V * \pi V = \alpha * \lambda * \iota V * \pi V$. The right-hand side is equal to $\alpha * \lambda$ by (3.6) and the left-hand side is equal to

$$\lambda' * s\alpha * s\varrho * \psi W * \eta s W = \lambda' * s\varrho' * \psi W' * \eta s W' * s\alpha = \varrho' * t\lambda' * \eta s W' * s\alpha = \lambda' * s\alpha,$$

using that α is a 2-cell in $\mathsf{EM}(\mathcal{K})$, (3.5) and unitality of ϱ'. Hence $J^w(\alpha) = J(\alpha)$ is a morphism in $\mathsf{EM}(\mathcal{K})(I(l), J^w(s,\psi))$ if and only if (3.9) holds. Thus we conclude by the isomorphism (3.1) that the 2-functor J induces an (obviously functorial) bijection between the morphisms in $\mathsf{EM}(\mathcal{K})(I(l), J^w(s,\psi))$ and the morphisms in $\mathcal{C}$.

Isomorphism of $\underline{\mathsf{EM}(\mathcal{K})(I(l), \widehat{st})}$ and $\underline{\mathcal{C}}$. In view of (2.14), we can choose $\widehat{st} = v\bar{s}f$ as 1-cells in $\mathcal{K}$. Moreover, taking axioms (2.8), (2.9) and (2.11) of a pre-monad into account,

$$\iota f * \pi f * \Theta = \Theta * \iota f st * \pi f st = \Theta * st\iota f * st\pi f = \Theta. \tag{3.12}$$

For an object $(l \xrightarrow{W} k, v\bar{s}fW \overset{\gamma}{\Rightarrow} W)$ in $\mathsf{EM}(\mathcal{K})(I(l), \widehat{st})$, put

$$\varrho := \gamma * \pi f W * s\mu W * \vartheta t W \quad \text{and} \quad \lambda := \gamma * \pi f W * s\eta W. \tag{3.13}$$

We show that $((W,\varrho),\lambda)$ is an object in $\mathcal{C}$. Recall that associativity and unitality of γ read as

$$\gamma * v\bar{s}f\gamma = \gamma * \pi f W * \Theta W * \iota f \iota f W \quad \text{and} \quad W = \gamma * \pi f W * \vartheta W,$$

respectively. Hence using associativity of γ and (3.12) (in the second equality) and applying the first identity in (2.16) (in the last equality),

$$\varrho * t\gamma * t\pi f W = \gamma * v\bar{s}f\gamma * \pi f \pi f W * s\mu st W * \vartheta tst W$$
$$= \gamma * \pi f W * \Theta W * s\mu st W * \vartheta tst W = \gamma * \pi f W * s\mu W * \psi t W. \tag{3.14}$$

Moreover, apply associativity of γ and (3.12) (in the second equality) and use (2.12) (in the third equality) to obtain

$$\lambda * s\varrho = \gamma * v\bar{s}f\gamma * \pi f \pi f W * s\eta st W * ss\mu W * s\vartheta t W$$
$$= \gamma * \pi f W * \Theta W * sts\mu W * st\vartheta t W * s\eta t W$$
$$= \gamma * \pi f W * s\mu W * \Theta t W * st\vartheta t W * s\eta t W = \gamma * \pi f W. \tag{3.15}$$

In the last equality we used (2.14) and that (since $\mu = v\epsilon f$) the interchange law yields $s\mu * \iota f t * \pi f t = \iota f * \pi f * s\mu$. With these identities at hand, associativity of ϱ is checked as

$$\begin{aligned}\varrho * t\varrho &= \varrho * t\gamma * t\pi f W * ts\mu W * t\vartheta t W = \gamma * \pi f W * s\mu W * s\mu t W * \psi tt W * t\vartheta t W\\ &= \gamma * \pi f W * s\mu W * s\mu t W * \vartheta tt W = \varrho * \mu W.\end{aligned}$$

The second equality follows by (3.14) and by associativity of μ. In the third equality we applied (2.4). The last equality follows by associativity of μ and the form of ϱ in (3.13). The unitality condition $\varrho * \eta W = W$ follows by unitality of μ and unitality of γ. Conditions (3.5), (3.10) and (3.11) are proven by

$$\begin{aligned}&\varrho * t\lambda = \varrho * t\gamma * t\pi f W * ts\eta W = \gamma * \pi f W * s\mu W * \psi t W * ts\eta W = \lambda * s\varrho * \psi W;\\ &\lambda * s\lambda = \gamma * v\bar{s} f\gamma * \pi f \pi f W * s\eta s\eta W = \gamma * \pi f W * \nu W = \lambda * s\varrho * \nu W;\\ &\quad W = \gamma * \pi f W * \vartheta W = \lambda * s\varrho * \vartheta W.\end{aligned}$$

In each case, the last equality follows by (3.15). In the first computation, the second equality follows by (3.14). In the second equality of the second computation we used associativity of γ together with (3.12) and we applied the expression of ν in (2.15). In the first equality of the last computation we used unitality of γ. This proves that $((W,\varrho),\lambda)$ is an object in $\mathcal{C}$.

Conversely, for an object $((W,\varrho),\lambda)$ in $\mathcal{C}$, put

$$\gamma := \lambda * s\varrho * \iota f W. \tag{3.16}$$

It is associative as

$$\begin{aligned}\gamma * \pi f W * \Theta W * \iota f \iota f W &= \lambda * s\varrho * \Theta W * \iota f \iota f W\\ &= \lambda * s\varrho * \nu W * ss\varrho * s\psi W * sts\varrho * \iota f \iota f W\\ &= \lambda * s\lambda * ss\varrho * s\psi W * sts\varrho * \iota f \iota f W\\ &= \lambda * s\varrho * st\lambda * sts\varrho * \iota f \iota f W = \gamma * v\bar{s} f\gamma.\end{aligned}$$

The first equality follows by (3.16) and (3.12). In the second equality we substituted Θ by its expression in (2.13) and we used associativity of ϱ twice. In the third equality we applied (3.10) and in the fourth one we used (3.5). By (2.4) and unitality of μ, $\iota f * \pi f * \vartheta = s\mu * \psi t * \eta st * \vartheta = \vartheta$. Hence the unitality condition $\gamma * \pi f W * \vartheta W = W$ follows by (3.11). This proves that (W,γ) is an object in $\mathsf{EM}(\mathcal{K})(I(l), \widehat{st})$.

Let us see that the above constructions are mutual inverses. Starting with an object (W,γ) of $\mathsf{EM}(\mathcal{K})(I(l), \widehat{st})$ and iterating the above constructions, we re-obtain (W,γ) by (3.15). In the opposite order, applying both constructions to an object $((W,\varrho),\lambda)$ of $\mathcal{C}$, we obtain $((W, \lambda * s\varrho * \iota f W * \pi f W * s\mu W * \vartheta t W), \lambda * s\varrho * \iota f W * \pi f W * s\eta W)$. Since $\iota f * \pi f * s\mu = s\mu * \iota f t * \pi f t = s\mu * \Theta t * \vartheta stt$, axiom (2.10) of a pre-monad, associativity of ϱ and (3.11) imply that

$$\lambda * s\varrho * \iota f W * \pi f W * s\mu W * \vartheta t W = \lambda * s\varrho * s\mu W * \vartheta t W = \lambda * s\varrho * \vartheta W * \varrho = \varrho.$$

Also, by (2.14), unitality of μ, (3.5) and unitality of ϱ,

$$\lambda * s\varrho * \iota f W * \pi f W * s\eta W = \lambda * s\varrho * s\mu W * \psi t W * \eta s\eta W = \lambda * s\varrho * \psi W * \eta s W = \lambda.$$

Hence we constructed a bijection between the objects of $\mathsf{EM}(\mathcal{K})(I(l), \widehat{st})$ and $\mathcal{C}$. It is immediate by the form of the bijection between the objects that a 2-cell $W \overset{\alpha}{\Rightarrow} W'$ in $\mathcal{K}$ is a morphism in $\mathsf{EM}(\mathcal{K})(I(l), \widehat{st})$ if and only if it is a morphism in $\mathcal{C}$. □

4. Weak liftings

If $\mathcal{K}$ is a 2-category which admits EM constructions for monads, then 'liftings' of 1- and 2-cells for monads in $\mathcal{K}$ arise as images under the right 2-adjoint J of the inclusion 2-functor $\mathcal{K} \to \mathsf{EM}(\mathcal{K})$, see [17]. In this section we discuss 'weak' liftings and the role what the pseudo-functor J^w plays in their description.

Definition 4.1. Consider a 2-category $\mathcal{K}$ which admits EM constructions for monads. We say that a 1-cell $k \overset{V}{\to} k'$ in $\mathcal{K}$ possesses a *weak lifting* for some monads $(k \overset{t}{\to} k, \mu, \eta)$ and $(k' \overset{t'}{\to} k', \mu', \eta')$ in $\mathcal{K}$ if there exist a 1-cell $J(t) \overset{\overline{V}}{\longrightarrow} J(t')$ and a split mono 2-cell $v'\overline{V} \overset{\iota}{\Rightarrow} Vv$ (with a retraction denoted by $Vv \overset{\pi}{\Rightarrow} v'\overline{V}$).

If in a 2-category $\mathcal{K}$ which admits EM constructions for monads also idempotent 2-cells split, then we know by Theorem 3.5 that, for every 1-cell $t \xrightarrow{(V,\psi)} t'$ in $\mathsf{EM}^w(\mathcal{K})$, the underlying 1-cell $k \overset{V}{\to} k'$ in $\mathcal{K}$ possesses a weak lifting $J^w(V,\psi)$ for t and t'. As we will see later in this section, in fact in such a 2-category $\mathcal{K}$, up-to an isomorphism, every weak lifting arises in this way. This extends assertions about 1-cells in [17, Lemma 3.9 and Theorem 3.10].

Definition 4.2. Consider a 2-category $\mathcal{K}$ which admits EM constructions for monads. Let $(k \overset{t}{\to} k, \mu, \eta)$ and $(k' \overset{t'}{\to} k', \mu', \eta')$ be monads, and $k \overset{V}{\to} k'$ and $k \overset{W}{\to} k'$ be 1-cells in $\mathcal{K}$, such that there exist their weak liftings $(J(t) \overset{\overline{V}}{\longrightarrow} J(t'), \iota_V, \pi_V)$ and $(J(t) \overset{\overline{W}}{\longrightarrow} J(t'), \iota_W, \pi_W)$ for t and t'. For a 2-cell $V \overset{\omega}{\Rightarrow} W$ in $\mathcal{K}$, we say that

a 2-cell $\overline{V} \overset{\overrightarrow{\omega}}{\Rightarrow} \overline{W}$ is a *weak ι-lifting* of ω if $\iota_W * v'\overrightarrow{\omega} = \omega v * \iota_V$;
a 2-cell $\overline{V} \overset{\overleftarrow{\omega}}{\Rightarrow} \overline{W}$ is a *weak π-lifting* of ω if $v'\overleftarrow{\omega} * \pi_V = \pi_W * \omega v$.

Throughout, indices of ι and π are omitted, as they can be reconstructed without ambiguity from the context.

By the isomorphism (3.1), the weak ι-lifting or weak π-lifting of a 2-cell is unique, provided that it exists. Moreover, if a 2-cell ω in Definition 4.2 possesses both a weak ι-lifting $\overrightarrow{\omega}$ and a weak π-lifting $\overleftarrow{\omega}$, then

$$v'\overrightarrow{\omega} = \pi * \iota * v'\overrightarrow{\omega} = \pi * \omega v * \iota = v'\overleftarrow{\omega} * \pi * \iota = v'\overleftarrow{\omega}.$$

Hence in view of the isomorphism (3.1), $\overrightarrow{\omega} = \overleftarrow{\omega}$.

Proposition 4.3 below, about weak liftings of 2-cells in a (nice enough) 2-category, extends statements about 2-cells in [17, Lemma 3.9 and Theorem 10]. Therein, notions and notations introduced in Corollary 1.4 are used.

Proposition 4.3. *Consider a 2-category $\mathcal{K}$ which admits EM constructions for monads and in which idempotent 2-cells split. Let $t \xrightarrow{(V,\psi)} t'$ and $t \xrightarrow{(W,\phi)} t'$ be 1-cells in $\mathsf{EM}^w(\mathcal{K})$ and $V \overset{\omega}{\Rightarrow} W$ be a 2-cell in $\mathcal{K}$. Denote by ι and π the splittings of both idempotent 2-cells in Lemma 3.2, corresponding to the 1-cells (V,ψ) and (W,ϕ), that were chosen to construct $J^w(V,\psi)$ and $J^w(W,\phi)$, respectively.*

(1) *The following assertions are equivalent:*
 (i) *ω is a 2-cell $(V,\psi) \Rightarrow (W,\phi)$ in $\mathsf{Mnd}^\iota(\mathcal{K})$;*
 (ii) *$\omega t * \psi * \eta' V : (V,\psi) \Rightarrow (W,\phi)$ is a 2-cell in $\mathsf{EM}^w(\mathcal{K})$;*
 (iii) *$\pi * \omega v * \iota : (v'J^w(V,\psi), v'\epsilon' J^w(V,\psi)) \Rightarrow (v'J^w(W,\phi), v'\epsilon' J^w(W,\phi))$ is a 2-cell in $\mathsf{EM}(\mathcal{K})$ such that $\iota * \pi * \omega v * \iota = \omega v * \iota$;*
 (iv) *ω possesses a weak ι-lifting $\overrightarrow{\omega} : (J^w(V,\psi),\iota,\pi) \to (J^w(W,\phi),\iota,\pi)$.*
 If these equivalent statements hold, then $v'J^w G^\iota(\omega) = \pi * \omega v * \iota$, that is, $\overrightarrow{\omega} = J^w G^\iota(\omega)$.

(2) *The following assertions are equivalent:*
 (i) *ω is a 2-cell $(V,\psi) \Rightarrow (W,\phi)$ in $\mathsf{Mnd}^\pi(\mathcal{K})$;*
 (ii) *$\phi * \eta' W * \omega : (V,\psi) \Rightarrow (W,\phi)$ is a 2-cell in $\mathsf{EM}^w(\mathcal{K})$;*
 (iii) *$\pi * \omega v * \iota : (v'J^w(V,\psi), v'\epsilon' J^w(V,\psi)) \Rightarrow (v'J^w(W,\phi), v'\epsilon' J^w(W,\phi))$ is a 2-cell in $\mathsf{EM}(\mathcal{K})$ such that $\pi * \omega v * \iota * \pi = \pi * \omega v$;*
 (iv) *ω possesses a weak π-lifting $\overleftarrow{\omega} : (J^w(V,\psi),\iota,\pi) \to (J^w(W,\phi),\iota,\pi)$.*
 If these equivalent statements hold, then $v'J^w G^\pi(\omega) = \pi * \omega v * \iota$, that is, $\overleftarrow{\omega} = J^w G^\pi(\omega)$.

(3) *The following assertions are equivalent:*
 (i) *$\phi * t'\omega = \omega t * \psi$;*
 (ii) *$\phi * \eta' W * \omega$ and $\omega t * \psi * \eta' V$ are (necessarily equal) 2-cells $(V,\psi) \Rightarrow (W,\phi)$ in $\mathsf{EM}^w(\mathcal{K})$;*
 (iii) *$\pi * \omega v * \iota : (v'J^w(V,\psi), v'\epsilon' J^w(V,\psi)) \Rightarrow (v'J^w(W,\phi), v'\epsilon' J^w(W,\phi))$ is a 2-cell in $\mathsf{EM}(\mathcal{K})$ such that $\iota * \pi * \omega v = \omega v * \iota * \pi$;*
 (iv) *ω possesses both a weak ι-lifting and a weak π-lifting 2-cell $(J^w(V,\psi),\iota,\pi) \to (J^w(W,\phi),\iota,\pi)$ (which are necessarily equal).*

Proof. (1) (i) ⇔ (ii) This equivalence follows by Lemma 1.2(1).

(ii) ⇔ (iii) The 2-cell $\pi * \omega v * \iota$ in $\mathcal{K}$ is a 2-cell in $\mathsf{EM}(\mathcal{K})$ if and only if $v'\epsilon' J^w(W,\phi) * t'\pi * t'\omega v * t'\iota = \pi * \omega v * \iota * v'\epsilon' J^w(V,\psi)$. Compose this equality by f horizontally on the right, and compose it vertically by ιf on the left and by $t'\pi f * t'V\eta$ on the right. By virtue of (3.2), (3.3) and the adjunction relation $\epsilon f * f\eta = f$, the resulting equivalent condition is identical to (1.7). Property $\iota * \pi * \omega v * \iota = \omega v * \iota$ is equivalent to $\iota * \pi * \omega v * \iota * \pi = \omega v * \iota * \pi$, what is easily seen to be equivalent to (1.6). Thus we conclude by Lemma 1.2(1)(i) ⇔ (iii).

(iii) ⇔ (iv) By the isomorphism (3.1), $\pi * \omega v * \iota$ is a 2-cell in $\mathsf{EM}(\mathcal{K})$ if and only if there is a 2-cell $J^w(V,\psi) \overset{\overrightarrow{\omega}}{\Rightarrow} J^w(W,\phi)$ in $\mathcal{K}$ such that $v'\overrightarrow{\omega} = \pi * \omega v * \iota$. Clearly, $\iota * \pi * \omega v * \iota = \omega v * \iota$ if and only if $\overrightarrow{\omega}$ is a weak ι-lifting of ω.

If the equivalent statements (i)–(iv) hold, then

$$v'J^w\big(\omega t * \psi * \eta' V\big) = \pi * W v\epsilon * \omega t v * \psi v * \eta' V v * \iota = \pi * \omega v * \iota * \pi * \iota = \pi * \omega v * \iota.$$

(2) (i) ⇔ (ii) This equivalence follows by Lemma 1.2(2).

(ii) ⇔ (iii) As we have seen in the proof of part (1), $\pi * \omega v * \iota$ is a 2-cell in $\mathsf{EM}(\mathcal{K})$ if and only if (1.7) holds. Property $\pi * \omega v * \iota * \pi = \pi * \omega v$ is equivalent to $\iota * \pi * \omega v * \iota * \pi = \iota * \pi * \omega v$ hence to the first condition in Lemma 1.2(2)(iii). Thus we conclude by Lemma 1.2(2)(i) ⇔ (iii).

(iii) ⇔ (iv) is proven by the same reasoning as in part (1).

If the equivalent statements (i)–(iv) hold, then

$$v' J^w\big(\phi * \eta' W * \omega\big) = \pi * W v\epsilon * \phi v * \eta' W v * \omega v * \iota = \pi * \iota * \pi * \omega v * \iota = \pi * \omega v * \iota.$$

(3) These equivalences follow immediately by Lemma 1.2(3) and parts (1) and (2) in the current theorem. □

For suggesting the following theorem, the author is grateful to the referee.

Consider a 2-category $\mathcal{K}$ which admits EM constructions for monads. To any monads $(k \xrightarrow{t} k, \mu, \eta)$ and $(k' \xrightarrow{t'} k', \mu', \eta')$ in $\mathcal{K}$, we can associate categories $\mathsf{Lift}^\iota(t,t')$ and $\mathsf{Lift}^\pi(t,t')$, as follows. In both categories objects are quadruples $(V, \overline{V}, \iota, \pi)$ such that the 1-cell $J(t) \xrightarrow{\overline{V}} J(t')$ in $\mathcal{K}$ is a weak lifting of the 1-cell $k \xrightarrow{V} k'$, corresponding to the split monic 2-cell $v'\overline{V} \overset{\iota}{\Rightarrow} Vv$, with a retraction $Vv \overset{\pi}{\Rightarrow} v'\overline{V}$. Morphisms $(V, \overline{V}, \iota, \pi) \to (W, \overline{W}, \iota, \pi)$ are pairs of 2-cells $(\omega, \overline{\omega})$ in $\mathcal{K}$ such that $\overline{V} \overset{\overline{\omega}}{\Rightarrow} \overline{W}$ is a weak ι-lifting, respectively, a weak π-lifting, of $V \overset{\omega}{\Rightarrow} W$. Composition of morphisms is defined via component-wise composition of 2-cells in $\mathcal{K}$.

Theorem 4.4. *For any 2-category $\mathcal{K}$ which admits EM constructions for monads and in which idempotent 2-cells split, and for any monads $(k \xrightarrow{t} k, \mu, \eta)$ and $(k' \xrightarrow{t'} k', \mu', \eta')$ in $\mathcal{K}$, the following assertions hold.*

(1) $\mathsf{Lift}^\iota(t,t')$ *is equivalent to the category* $\mathsf{Mnd}^\iota(\mathcal{K})(t,t')$.
(2) $\mathsf{Lift}^\pi(t,t')$ *is equivalent to the category* $\mathsf{Mnd}^\pi(\mathcal{K})(t,t')$.

For $t = t'$, these equivalences are also strong monoidal, with respect to the monoidal structure of $\mathsf{Lift}^{\iota/\pi}(t,t)$ *induced by the horizontal composition in $\mathcal{K}$.*

Proof. (1) For any 1-cell $t \xrightarrow{(V,\psi)} t'$ in $\mathsf{EM}^w(\mathcal{K})$, denote by $Vv \overset{\pi_c}{\Longrightarrow} v'J^w(V,\psi) \overset{\iota_c}{\Longrightarrow} Vv$ the chosen splitting of the idempotent 2-cell in Lemma 3.2, used to construct $J^w(V,\psi)$.

By Corollary 1.4 and Theorem 3.5, there is a pseudo-functor $J^w G^\iota : \mathsf{Mnd}^\iota(\mathcal{K}) \to \mathcal{K}$. By Proposition 4.3(1)(i) ⇒ (iv), it induces a functor $G : \mathsf{Mnd}^\iota(\mathcal{K})(t,t') \to \mathsf{Lift}^\iota(t,t')$, with object map $(V,\psi) \mapsto (V, J^w(V,\psi), \iota_c, \pi_c)$ and morphism map $\omega \mapsto (\omega, J^w G^\iota(\omega))$.

In the opposite direction, consider a functor $F : \mathsf{Lift}^\iota(t,t') \to \mathsf{Mnd}^\iota(\mathcal{K})(t,t')$ with the object map

$$(V, \overline{V}, \iota, \pi) \mapsto \big(V, \psi := \iota f * v'\epsilon'\overline{V} f * t'\pi f * t'V\eta : t'V \Rightarrow Vt\big) \tag{4.1}$$

and morphism map $(\omega, \overline{\omega}) \mapsto \omega$. By the form of ψ in (4.1) and the adjunction relation $v\epsilon * \eta v = v$, it follows that

$$Vv\epsilon * \psi v = \iota * v'\epsilon'\overline{V} * t'\pi. \tag{4.2}$$

Using (4.2) together with the form of ψ in (4.1) and naturality, it is easily checked that ψ satisfies (1.1), i.e. (V,ψ) is an object in $\mathsf{Mnd}^\iota(\mathcal{K})(t,t')$. By (4.2), the associated idempotent in Lemma 3.2 obeys $\iota_c * \pi_c = Vv\epsilon * \psi v * \eta' Vv = \iota * \pi$. Applying (4.2) together with (3.2) and (3.3), respectively, we conclude that

$$v'\epsilon' J^w(V,\psi) * t'\pi_c * t'\iota = \pi_c * \iota * v'\epsilon'\overline{V} \quad \text{and} \quad \pi * \iota_c * v'\epsilon' J^w(V,\psi) = v'\epsilon'\overline{V} * t'\pi * t'\iota_c.$$

That is, there are 2-cells $(v'J^w(V,\psi),\ v'\epsilon' J^w(V,\psi)) \overset{\pi*\iota_c}{\Longrightarrow} (v'\overline{V}, v'\epsilon'\overline{V})$ and $(v'\overline{V}, v'\epsilon'\overline{V}) \overset{\pi_c*\iota}{\Longrightarrow} (v'J^w(V,\psi), v'\epsilon' J^w(V,\psi))$ in $\mathsf{EM}(\mathcal{K})$. By (3.1) they induce mutually inverse isomorphisms $J^w(V,\psi) \overset{J(\pi*\iota_c)}{\Longrightarrow} \overline{V}$ and $\overline{V} \overset{J(\pi_c*\iota)}{\Longrightarrow} J^w(V,\psi)$ in $\mathcal{K}$. Both of these 2-cells are weak ι-liftings of the identity 2-cell $V \overset{V}{\Rightarrow} V$. Hence, for any morphism $(V,\overline{V},\iota,\pi) \xrightarrow{(\omega,\overline{\omega})} (W,\overline{W},\iota,\pi)$ in $\mathsf{Lift}^\iota(t,t')$, the composite $J(\pi_c * \iota) * \overline{\omega} * J(\pi * \iota_c) : J^w(V,\psi) \Rightarrow J^w(W,\phi)$ is a weak ι_c-lifting of ω (where both ψ and ϕ are defined via (4.1)). Thus it follows by Proposition 4.3(1)(iv) $\Rightarrow$ (i) that ω is a morphism $(V,\psi) \to (W,\phi)$ in $\mathsf{Mnd}^\iota(\mathcal{K})(t,t')$. This proves that F is a well-defined functor.

For any object (V,ψ) in $\mathsf{Mnd}^\iota(\mathcal{K})(t,t')$, we obtain $FG(V,\psi) = (V,\psi)$ by (3.2), (3.3) and unitality of μ. Evidently, also $FG(\omega) = \omega$. For any object $(V,\overline{V},\iota,\pi)$ of $\mathsf{Lift}^\iota(t,t')$, we obtain $GF(V,\overline{V},\iota,\pi) = (V, J^w(V,\psi),\iota_c,\pi_c)$. The mutually inverse isomorphisms $(V,\overline{V},\iota,\pi) \xrightarrow{(V,J(\pi_c*\iota))} (V,J^w(V,\psi),\iota_c,\pi_c)$ and $(V,J^w(V,\psi),\iota_c,\pi_c) \xrightarrow{(V,J(\pi*\iota_c))} (V,\overline{V},\iota,\pi)$ in $\mathsf{Lift}^\iota(t,t')$ define, in turn, mutually inverse natural isomorphisms between the identity functor and GF. Indeed, for any morphism $(V,\overline{V},\iota,\pi) \xrightarrow{(\omega,\overline{\omega})} (W,\overline{W},\iota,\pi)$ in $\mathsf{Lift}^\iota(t,t')$, we conclude by Corollary 1.4 and Theorem 3.5 that

$$v'J^w G^\iota(\omega) * \pi_c * \iota = \pi_c * Wv\epsilon * \omega t v * \psi v * \eta' Vv * \iota = \pi_c * \omega v * \iota = \pi_c * \iota * v'\overline{\omega}.$$

Hence naturality follows by the isomorphism (3.1).

It remains to prove strong monoidality of G in the $t = t'$ case. Recall from the proof of Theorem 3.5 that the coherent natural isomorphisms $J^w(V',\psi')J^w(V,\psi) \overset{j_{V',V}}{\Longrightarrow} J^w((V',\psi') \circ (V,\psi))$ and $J^w(t) \overset{j_0}{\Longrightarrow} J^w(k,t)$, rendering J^w (hence $J^w G^\iota$) a pseudo-functor, arise as weak ι-liftings of identity 2-cells, for any 1-cells $(V',\psi'),(V,\psi) : t \to t$ in $\mathsf{EM}^w(\mathcal{K})$. Hence they induce a strong monoidal structure $G(V',\psi')G(V,\psi) \overset{(V'V,j_{V',V})}{\Longrightarrow} G((V',\psi') \circ (V,\psi))$ and $(k, J^w(t)) \overset{(k,j_0)}{\Longrightarrow} G(k,t)$ of G.

Part (2) is proven symmetrically. □

5. Applications

In this section we collect from the literature several situations where weak liftings occur. The following corollary is a consequence of Proposition 4.3 and Theorem 4.4.

Corollary 5.1. *Consider a 2-category $\mathcal{K}$ which admits EM constructions for monads and in which idempotent 2-cells split. Let $(k \xrightarrow{t} k,\mu,\eta)$ be a monad and $(k \xrightarrow{c} k,\delta,\varepsilon)$ be a comonad in $\mathcal{K}$.*

(1) *The following assertions are equivalent:*

(i) *There exists a comonad* $((c,\psi),\delta,\varepsilon)$ *in* $\mathsf{Mnd}^\iota(\mathcal{K})$. *That is, there exists a* 1*-cell* $t \xrightarrow{(c,\psi)} t$ *in* $\mathsf{EM}^w(\mathcal{K})$, *satisfying*

$$\delta t * \psi = cc\mu * c\psi t * \psi ct * t\delta t * t\psi * t\eta c; \tag{5.1}$$

$$\varepsilon t * \psi = \mu * t\varepsilon t * t\psi * t\eta c. \tag{5.2}$$

(ii) *There is a comonad* $(t \xrightarrow{(c,\psi)} t, \delta t * \psi * \eta c, \varepsilon t * \psi * \eta c)$ *in* $\mathsf{EM}^w(\mathcal{K})$.

(iii) *There are a comonad* $(J^w(t) \xrightarrow{\bar{c}} J^w(t), \overrightarrow{\delta}, \overrightarrow{\varepsilon})$ *and a split monic* 2*-cell* $v\bar{c} \stackrel{\iota}{\Rightarrow} cv$ *in* $\mathcal{K}$ *such that* $\overrightarrow{\delta}$ *is a weak* ι*-lifting of* δ *and* $\overrightarrow{\varepsilon}$ *is a weak* ι*-lifting of* ε.

If these equivalent statements hold, then we say shortly that the comonad $(\bar{c}, \overrightarrow{\delta}, \overrightarrow{\varepsilon})$ *is a weak* ι*-lifting of the comonad* (c,δ,ε) *for the monad* (t,μ,η).

(2) *The following assertions are equivalent:*

(i) *There exists a comonad* $((c,\psi),\delta,\varepsilon)$ *in* $\mathsf{Mnd}^\pi(\mathcal{K})$. *That is, there exists a* 1*-cell* $t \xrightarrow{(c,\psi)} t$ *in* $\mathsf{EM}^w(\mathcal{K})$, *satisfying*

$$c\psi * \psi c * t\delta = cc\mu * c\psi t * \psi ct * \eta cct * \delta t * \psi; \tag{5.3}$$

$$t\varepsilon = \varepsilon t * \psi. \tag{5.4}$$

(ii) *There is a comonad* $(t \xrightarrow{(c,\psi)} t, c\psi * \psi c * \eta cc * \delta, \eta * \varepsilon)$ *in* $\mathsf{EM}^w(\mathcal{K})$.

(iii) *There are a comonad* $(J^w(t) \xrightarrow{\bar{c}} J^w(t), \overleftarrow{\delta}, \overleftarrow{\varepsilon})$ *and a split epi* 2*-cell* $cv \stackrel{\pi}{\Rightarrow} v\bar{c}$ *in* $\mathcal{K}$ *such that* $\overleftarrow{\delta}$ *is a weak* π*-lifting of* δ *and* $\overleftarrow{\varepsilon}$ *is a weak* π*-lifting of* ε.

If these equivalent statements hold, then we say shortly that the comonad $(\bar{c}, \overleftarrow{\delta}, \overleftarrow{\varepsilon})$ *is a weak* π*-lifting of the comonad* (c,δ,ε) *for the monad* (t,μ,η).

(3) *The following assertions are equivalent:*

(i) *There exists a* 1*-cell* $t \xrightarrow{(c,\psi)} t$ *in* $\mathsf{EM}^w(\mathcal{K})$, *satisfying*

$$c\psi * \psi c * t\delta = \delta t * \psi; \tag{5.5}$$

$$t\varepsilon = \varepsilon t * \psi. \tag{5.6}$$

(ii) *There are a comonad* $(J^w(t) \xrightarrow{\bar{c}} J^w(t), \bar{\delta}, \bar{\varepsilon})$ *and a split epi-mono pair of* 2*-cells* $cv \stackrel{\pi}{\Rightarrow} v\bar{c} \stackrel{\iota}{\Rightarrow} cv$ *in* $\mathcal{K}$ *such that* $\bar{\delta}$ *is both a weak* ι*-lifting and a weak* π*-lifting of* δ *and* $\bar{\varepsilon}$ *is both a weak* ι*-lifting and a weak* π*-lifting of* ε.

Note that by Lemmas 1.2 and 1.3, in parts (1) and (2) of Corollary 5.1, assertions (i) and (ii) are equivalent in case of an arbitrary 2-category $\mathcal{K}$.

Let us stress the (tiny) difference between a 2-cell $tc \stackrel{\psi}{\Rightarrow} ct$ in $\mathcal{K}$ occurring in Corollary 5.1(3)(i), and a mixed distributive law. A 2-cell ψ in Corollary 5.1(3)(i) satisfies three of the identities defining a mixed distributive law: compatibility with the multiplication of the monad (as (c,ψ) is a 1-cell in $\mathsf{EM}^w(\mathcal{K})$), compatibility with the comultiplication of the comonad (by (5.5)) and compatibility with the counit of the comonad (by (5.6)). However, the fourth condition on a mixed distributive law, compatibility $\psi * \eta c = c\eta$ with the unit of the monad, does not appear in Corollary 5.1(3)(i) – it plays no role in a weak lifting.

Example 5.2. Generalizing a mixed distributive law of a monad and a comonad (in particular in the bicategory BIM), weak entwining structures were introduced by Caenepeel and De Groot in [9]. The axioms are obtained by weakening the compatibility conditions of a mixed distributive law with the unit of the monad and the counit of the comonad. Precisely, a *weak entwining structure* in an arbitrary 2-category $\mathcal{K}$ consists of a monad $(k \xrightarrow{t} k, \mu, \eta)$, a comonad $(k \xrightarrow{c} k, \delta, \varepsilon)$ and a 2-cell $tc \overset{\psi}{\Rightarrow} ct$ subject to the following conditions:

$$\psi * \mu c = c\mu * \psi t * t\psi; \tag{5.7}$$

$$\delta t * \psi = c\psi * \psi c * t\delta; \tag{5.8}$$

$$\psi * \eta c = c\varepsilon t * c\psi * c\eta c * \delta; \tag{5.9}$$

$$\varepsilon t * \psi = \mu * t\varepsilon t * t\psi * t\eta c. \tag{5.10}$$

We claim that under these assumptions $((c, \psi), \delta t * \psi * \eta c, \varepsilon t * \psi * \eta c)$ is a comonad in $\mathsf{EM}^w(\mathcal{K})$. For that, we need to show that axioms (5.7)–(5.10) imply (5.1). Indeed,

$$cc\mu * c\psi t * \psi ct * t\delta t * t\psi * t\eta c = cc\mu * \delta tt * \psi t * t\psi * t\eta c = \delta t * \psi.$$

The first equality follows by (5.8) and the second one follows by (5.7) and unitality of the monad t.

Hence if moreover $\mathcal{K}$ admits EM constructions for monads and idempotent 2-cells in $\mathcal{K}$ split (hence there exists the pseudo-functor J^w) then, by Corollary 5.1(1), the comonad c has a weak ι-lifting for the monad t.

For a commutative, associative and unital ring k, consider a k-algebra A and a k-coalgebra C. Let $\Psi : C \otimes_k A \to A \otimes_k C$ be a k-module map such that the triple $((-) \otimes_k A, (-) \otimes_k C, (-) \otimes_k \Psi)$ is a weak entwining structure in CAT. (If we are ready to cope with the more involved situation of a bicategory, we can say simply that (A, C, Ψ) is a weak entwining structure in BIM.) The corresponding weak ι-lifting of the comonad $(-) \otimes_k C$ for the monad $(-) \otimes_k A$ is studied in [9, Section 2]. Brzeziński showed in [8, Proposition 2.3] that it can be described as a comonad $(-) \otimes_A \overline{C}$ on the category of right A-modules, where the A-coring (i.e. comonad $A \to A$ in BIM) $\overline{C}$ is constructed as a k-module retract of $A \otimes_k C$.

Examples of weak entwining structures, thus examples of weak ι-liftings of comonads for monads, are provided by weak Doi–Koppinen data in [3] (see [9]), i.e. by comodule algebras and module coalgebras of weak bialgebras. Further examples are weak comodule algebras of bialgebras in [11, Proposition 2.3].

Example 5.3. Another generalization of a mixed distributive law, motivated by partial coactions of Hopf algebras, is due to Caenepeel and Janssen. Following [10, Proposition 2.6], a *partial entwining structure* in a 2-category $\mathcal{K}$ consists of a monad $(k \xrightarrow{t} k, \mu, \eta)$, a comonad $(k \xrightarrow{c} k, \delta, \varepsilon)$ and a 2-cell $tc \overset{\psi}{\Rightarrow} ct$ in $\mathcal{K}$, such that identities (5.4) and (5.7) hold, together with

$$cc\mu * c\psi t * c\eta ct * \delta t * \psi = c\psi * \psi c * t\delta. \tag{5.11}$$

Observe that axiom (5.11) implies (5.3):

$$cc\mu * c\psi t * \psi ct * \eta cct * \delta t * \psi = cc\mu * cc\mu t * c\psi tt * c\eta ctt * \delta tt * \psi t * \eta ct * \psi$$
$$= cc\mu * c\psi t * c\eta ct * \delta t * \psi = c\psi * \psi c * t\delta.$$

The first and last equalities follow by (5.11) and the second equality is obtained using associativity of μ and (3.2). This implies that $((c,\psi), c\psi * \psi c * \eta cc * \delta, \eta * \varepsilon)$ is a comonad in $\mathsf{EM}^w(\mathcal{K})$. Thus if moreover $\mathcal{K}$ is a 2-category which admits EM constructions for monads and in which idempotent 2-cells split, then we conclude by Corollary 5.1(2) that a partial entwining structure (t, c, ψ) in $\mathcal{K}$ induces a weak π-lifting of the comonad c for the monad t.

Consider the particular case when a monad $t := (-) \otimes_k A$ in CAT is induced by an algebra A over a commutative, associative and unital ring k, a comonad $c := (-) \otimes_k C$ is induced by a k-coalgebra C and a natural transformation $tc \overset{\psi}{\Rightarrow} ct$ is induced by a k-module map $C \otimes_k A \to A \otimes_k C$. Then the weak π-lifting of the comonad c for the monad t, induced by a partial entwining ψ, is a comonad $(-) \otimes_A \overline{C}$ on the category of right A-modules. The A-coring $\overline{C}$ was constructed in [10, Proposition 2.6] as a k-module retract of $A \otimes_k C$.

Examples of partial entwining structures (hence of weak π-liftings of a comonad for a monad) are provided by partial comodule algebras of bialgebras in [11, Proposition 2.6].

Example 5.4. Yet another way to generalize a mixed distributive law was proposed in [10]. Following [10, Proposition 2.5], a *lax entwining structure* in a 2-category $\mathcal{K}$ consists of a monad $(k \overset{t}{\to} k, \mu, \eta)$, a comonad $(k \overset{c}{\to} k, \delta, \varepsilon)$ and a 2-cell $tc \overset{\psi}{\Rightarrow} ct$ in $\mathcal{K}$, such that identities (5.7), (5.10) and (5.11) hold, together with

$$c\mu * ct\varepsilon t * ct\psi * ct\eta c * \psi c * t\delta * \eta c = \psi * \eta c.$$

As we observed in Example 5.3, (5.11) implies (5.3), and (5.10) is identical to (5.2). However, none of (5.1) and (5.4) seems to hold for an arbitrary lax entwining structure. Still, the axioms of a lax entwining structure allow us to prove that there is a comonad $((c,\psi), c\psi * \psi c * \eta cc * \delta, \varepsilon t * \psi * \eta c)$ in $\mathsf{EM}^w(\mathcal{K})$. Therefore, if $\mathcal{K}$ admits EM constructions for monads and idempotent 2-cells in $\mathcal{K}$ split, then J^w takes it to a comonad $(J^w(t) \overset{\bar{c}}{\to} J^w(t), \overleftarrow{\delta}, \overrightarrow{\varepsilon})$ in $\mathcal{K}$. However, it is neither a weak ι-lifting nor a weak π-lifting of the comonad c, it is of a mixed nature.

In the particular case when a lax entwining structure in CAT is induced by modules over a commutative associative and unital ring, the comonad $(\bar{c}, \overleftarrow{\delta}, \overrightarrow{\varepsilon})$ is induced by a coring, which was computed in [10, Proposition 2.5]. Examples of lax entwining structures are provided by lax comodule algebras of bialgebras in [11, Proposition 2.5].

A fourth logical possibility, to obtain a comonad structure on a weak lifting for a monad t of a 1-cell c underlying a comonad (c, δ, ε), is to allow the comultiplication to be a weak ι-lifting of δ and the counit to be a weak π-lifting of ε. That is, to require a 1-cell $t \xrightarrow{(c,\psi)} t$ in $\mathsf{EM}^w(\mathcal{K})$ to satisfy (5.1) and (5.4). By (the proof of) Lemma 1.3, it yields a coassociative and counital comonad $(\bar{c}, \overrightarrow{\delta}, \overleftarrow{\varepsilon})$ in $\mathcal{K}$.

For any 2-category $\mathcal{K}$, one may consider the vertically-opposite 2-category $\mathcal{K}_*$. The 2-category $\mathcal{K}_*$ has the same 0-, 1-, and 2-cells as $\mathcal{K}$, the same horizontal composition and the opposite vertical composition. Obviously, 2-cells in $\mathcal{K}$ split if and only if 2-cells in $\mathcal{K}_*$ split. Since monads in $\mathcal{K}_*$ are the same as the comonads in $\mathcal{K}$, the 2-category $\mathcal{K}_*$ admits EM constructions for monads if and only if $\mathcal{K}$ admits EM constructions for comonads, cf. [17]. In this case we denote by $J^w_* : \mathsf{EM}^w(\mathcal{K}_*)_* \to \mathcal{K}$ the pseudo-functor in Theorem 3.5.

Definition 5.5. Consider a 2-category $\mathcal{K}$ which admits EM constructions for comonads. We say that a 1-cell V in $\mathcal{K}$ possesses a *weak lifting* $\overline{V}$ *for some comonads* c and c', provided that, regarded as 1-cells in $\mathcal{K}_*$, $\overline{V}$ is a weak lifting of V for the monads c and c' in $\mathcal{K}_*$.

For a 2-cell ω in $\mathcal{K}$, a *weak* ι*-lifting* (resp. *weak* π*-lifting*) for some comonads c and c' in $\mathcal{K}$ is defined as a weak π-lifting (resp. weak ι-lifting) of ω, regarded as a 2-cell in $\mathcal{K}_*$, for the monads c and c' in $\mathcal{K}_*$.

The following corollary is obtained by applying Corollary 5.1 to the vertically-opposite of a 2-category. Therein, the symbol $*$ denotes the vertical composition in $\mathcal{K}$ (not its opposite).

Corollary 5.6. *Consider a* 2*-category* $\mathcal{K}$ *which admits EM constructions for comonads and in which idempotent* 2*-cells split. Let* $(k \xrightarrow{t} k, \mu, \eta)$ *be a monad and* $(k \xrightarrow{c} k, \delta, \varepsilon)$ *be a comonad in* $\mathcal{K}$.

(1) *The following assertions are equivalent:*
   (i) *There is a monad* $((t, \psi), \mu, \eta)$ *in* $\mathsf{Mnd}^{\iota}(\mathcal{K}_*)_*$. *That is, there exists a* 1*-cell* $c \xrightarrow{(t,\psi)} c$ *in* $\mathsf{EM}^{w}(\mathcal{K}_*)_*$ *(i.e. a* 2*-cell* $tc \overset{\psi}{\Rightarrow} ct$ *in* $\mathcal{K}$ *such that* $\delta t * \psi = c\psi * \psi c * t\delta$*), satisfying*

$$\psi * \mu c = c\varepsilon t * c\psi * c\mu c * \psi tc * t\psi c * tt\delta; \tag{5.12}$$

$$\psi * \eta c = c\varepsilon t * c\psi * c\eta c * \delta. \tag{5.13}$$

   (ii) *There is a monad* $(c \xrightarrow{(t,\psi)} c, \varepsilon t * \psi * \mu c, \varepsilon t * \psi * \eta c)$ *in* $\mathsf{EM}^{w}(\mathcal{K}_*)_*$.
   (iii) *There are a monad* $(J_*^{w}(c) \xrightarrow{\bar{t}} J_*^{w}(c), \overleftarrow{\mu}, \overleftarrow{\eta})$ *and a split epi* 2*-cell* π *in* $\mathcal{K}$ *such that* $\overleftarrow{\mu}$ *is a weak* π*-lifting of* μ *and* $\overleftarrow{\eta}$ *is a weak* π*-lifting of* η.

   If these equivalent statements hold, then we say shortly that the monad $(\bar{t}, \overleftarrow{\mu}, \overleftarrow{\eta})$ *is a weak* π*-lifting of the monad* (t, μ, η) *for the comonad* (c, δ, ε).

(2) *The following assertions are equivalent:*
   (i) *There is a monad* $((t, \psi), \mu, \eta)$ *in* $\mathsf{Mnd}^{\pi}(\mathcal{K}_*)_*$. *That is, there exists a* 1*-cell* $c \xrightarrow{(t,\psi)} c$ *in* $\mathsf{EM}^{w}(\mathcal{K}_*)_*$*, satisfying*

$$c\mu * \psi t * t\psi = \psi * \mu c * \varepsilon ttc * \psi tc * t\psi c * tt\delta; \tag{5.14}$$

$$c\eta = \psi * \eta c. \tag{5.15}$$

   (ii) *There is a monad* $(c \xrightarrow{(t,\psi)} c, \mu * \varepsilon tt * \psi t * t\psi, \eta * \varepsilon)$ *in* $\mathsf{EM}^{w}(\mathcal{K}_*)_*$.
   (iii) *There are a monad* $(J_*^{w}(c) \xrightarrow{\bar{t}} J_*^{w}(c), \overrightarrow{\mu}, \overrightarrow{\eta})$ *and a split monic* 2*-cell* ι *in* $\mathcal{K}$ *such that* $\overrightarrow{\mu}$ *is a weak* ι*-lifting of* μ *and* $\overrightarrow{\eta}$ *is a weak* ι*-lifting of* η.

   If these equivalent statements hold, then we say shortly that the monad $(\bar{t}, \overrightarrow{\mu}, \overrightarrow{\eta})$ *is a weak* ι*-lifting of the monad* (t, μ, η) *for the comonad* (c, δ, ε).

(3) *The following assertions are equivalent:*
   (i) *There exists a* 1*-cell* $t \xrightarrow{(c,\psi)} t$ *in* $\mathsf{EM}^{w}(\mathcal{K}_*)_*$*, satisfying*

$$c\mu * \psi t * t\psi = \psi * \mu c; \tag{5.16}$$

$$c\eta = \psi * \eta c. \tag{5.17}$$

(ii) *There are a monad* $(J_*^w(c) \xrightarrow{\bar{t}} J_*^w(c), \overline{\mu}, \overline{\eta})$ *and a split epi-mono pair* (π, ι) *of* 2*-cells in* $\mathcal{K}$ *such that* $\overline{\mu}$ *is both a weak* ι*-lifting and a weak* π*-lifting of* μ *and* $\overline{\eta}$ *is both a weak* ι*-lifting and a weak* π*-lifting of* η.

A 2-cell ψ in Corollary 5.6(3)(i) differs from a mixed distributive law by the compatibility condition with the counit of the comonad.

In a 2-category $\mathcal{K}$ which admits EM constructions for both monads and comonads and in which idempotent 2-cells split, one can say more about weak entwining structures than it was said in Example 5.2.

Proposition 5.7. *Consider a* 2*-category* $\mathcal{K}$ *which admits EM constructions for both monads and comonads and in which idempotent* 2*-cells split. For a monad* $(k \xrightarrow{t} k, \mu, \eta)$*, a comonad* $(k \xrightarrow{c} k, \delta, \varepsilon)$*, and a* 2*-cell* $tc \overset{\psi}{\Rightarrow} ct$ *in* $\mathcal{K}$ *(with a chosen splitting* (π, ι) *of the associated idempotent in Lemma* 3.2*), the following assertions are equivalent:*

(i) *The triple* (t, c, ψ) *is a weak entwining structure, that is, axioms* (5.7)–(5.10) *are satisfied.*
(ii) *The* 2*-cell* ψ *induces both a weak* ι*-lifting of the comonad* c *for the monad* t *and a weak* π*-lifting of the monad* t *for the comonad* c*. That is to say, the assertions in Corollary* 5.1(1) *and Corollary* 5.6(1) *hold.*

Proof. We have seen in Example 5.2 that axioms (5.7)–(5.10) imply (5.1). Similarly, they can be seen to imply (5.12) as well, applying first (5.7) and next (5.8). □

Proposition 5.7 is the basis of a construction in [4] of a 2-category of weak entwining structures in any 2-category. In that paper, for a weak entwining structure in a 2-category $\mathcal{K}$ which admits EM constructions for both monads and comonads and in which idempotent 2-cells split, it is proven that the weakly lifted monad, and the weakly lifted comonad, occurring in part (ii) of Proposition 5.7, possess equivalent Eilenberg–Moore objects.

The characterization of weak entwining structures in Proposition 5.7 can be used, in particular, to describe weak bialgebras [6] in terms of weak liftings. Recall that a *weak bialgebra* H over a commutative, associative and unital ring k, is a k-module which possesses both a k-algebra and a k-coalgebra structure, subject to the following compatibility conditions. Denote the multiplication $H \otimes_k H \to H$ in H by juxtaposition of elements. Write 1 for the unit element of the algebra H and write $\varepsilon : H \to k$ for the counit. For the comultiplication $H \to H \otimes_k H$, use a Sweedler type index notation $h \mapsto \sum h_1 \otimes_k h_2$. With these notations, the axioms

$$\sum (hh')_1 \otimes_k (hh')_2 = \sum h_1 h'_1 \otimes_k h_2 h'_2; \tag{5.18}$$

$$\sum 1_1 \otimes 1_2 1_{1'} \otimes 1_{2'} = \sum 1_1 \otimes_k 1_2 \otimes_k 1_3 = \sum 1_1 \otimes_k 1_{1'} 1_2 \otimes_k 1_{2'}; \tag{5.19}$$

$$\sum \varepsilon(h1_1)\varepsilon(1_2 h') = \varepsilon(hh') = \sum \varepsilon(h1_2)\varepsilon(1_1 h') \tag{5.20}$$

are required to hold, for all elements h and h' of H. However, the comultiplication is not required to preserve the unit, i.e. $\sum 1_1 \otimes_k 1_2$ is not required to be equal to $1 \otimes 1$ and the counit is not required to be multiplicative, i.e. $\varepsilon(hh')$ is not required to be equal to $\varepsilon(h)\varepsilon(h')$, for elements $h, h' \in H$.

In the following proposition we deal with the (co)monads $H \otimes_k (-)$ and $(-) \otimes_k H$, induced by a k-(co)algebra H on the category of modules over a commutative, associative and unital ring k.

Proposition 5.8. *For a commutative, associative and unital ring k, consider a k-module H which possesses both a k-algebra and a k-coalgebra structure. Using the notations introduced above the proposition, the following assertions are equivalent:*

(i) *The algebra and coalgebra structures of H constitute a weak bialgebra;*
(ii) *The k-module map*

$$\Psi_R : H \otimes_k H \to H \otimes_k H, \qquad h \otimes_k h' \mapsto \sum h'_1 \otimes_k hh'_2 \tag{5.21}$$

induces a weak ι-lifting of the comonad $(-) \otimes_k H$ for the monad $(-) \otimes_k H$ and a weak π-lifting of the monad $(-) \otimes_k H$ for the comonad $(-) \otimes_k H$, and the k-module map

$$\Psi_L : H \otimes_k H \to H \otimes_k H, \qquad h \otimes_k h' \mapsto \sum h_1 h' \otimes_k h_2 \tag{5.22}$$

induces a weak ι-lifting of the comonad $H \otimes_k (-)$ for the monad $H \otimes_k (-)$ and a weak π-lifting of the monad $H \otimes_k (-)$ for the comonad $H \otimes_k (-)$. That is to say,

$$\big((-) \otimes_k H, (-) \otimes_k \Psi_R\big) \quad \text{and} \quad \big(H \otimes_k (-), \Psi_L \otimes_k (-)\big)$$

are comonads in $\mathsf{Mnd}^\iota(\mathsf{CAT})$, via the comultiplication and counit induced by the coalgebra H, and they are monads in $\mathsf{Mnd}^\iota(\mathsf{CAT}_*)_*$, via the multiplication and unit induced by the algebra H.

Proof. Note first that assertion (ii) implies (5.18). Indeed, (5.21) determines a 1-cell $((-) \otimes_k H, (-) \otimes_k \Psi_R)$ in $\mathsf{EM}^w(\mathsf{CAT})$ if and only if

$$\sum \big(h'h''\big)_1 \otimes_k h\big(h'h''\big)_2 = \sum h'_1 h''_1 \otimes_k hh'_2 h''_2,$$

for any elements h, h' and h'' of H. Putting $h = 1$ we obtain (5.18).

By Proposition 5.7, assertion (ii) is equivalent to saying that $((-) \otimes_k H, (-) \otimes_k H, (-) \otimes_k \Psi_R)$ and $(H \otimes_k (-), H \otimes_k (-), \Psi_L \otimes_k (-))$ are weak entwining structures in CAT (or, in the terminology of [9], (H, H, Ψ_R) is a right–right weak entwining structure and (H, H, Ψ_L) is a left–left weak entwining structure in BIM). This statement was proven to be equivalent to (i) in [9, Theorem 4.7]. □

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7. fejezet

Weak bimonads and weak Hopf monads

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Weak bimonads and weak Hopf monads

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ABSTRACT

We define a *weak bimonad* as a monad T on a monoidal category $\mathscr{M}$ with the property that the Eilenberg–Moore category $\mathscr{M}^T$ is monoidal and the forgetful functor $\mathscr{M}^T \to \mathscr{M}$ is separable Frobenius. Whenever $\mathscr{M}$ is also Cauchy complete, a simple set of axioms is provided, that characterizes the monoidal structure of $\mathscr{M}^T$ as a weak lifting of the monoidal structure of $\mathscr{M}$. The relation to bimonads, and the relation to weak bimonoids in a braided monoidal category are revealed. We also discuss antipodes, obtaining the notion of weak Hopf monad.

Introduction

Bialgebras (say, over a field) have several equivalent characterizations. One of the most elegant is due to Pareigis, who proved that an algebra A over a field K is a bialgebra if and only if the category of (left or right) A-modules is monoidal and the forgetful functor from the category of A-modules to the category of K-vector spaces is strict monoidal. This fact extends to bialgebras in any braided monoidal category [12].

Pareigis' characterization of a bialgebra was the starting point of Moerdijk's generalization in [14] of bialgebras to monoidal categories possibly without a braiding. He defined a *bimonad* (originally

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called a Hopf monad) as a monad T on a monoidal category $\mathcal{M}$, such that the Eilenberg–Moore category $\mathcal{M}^T$ of T-algebras is monoidal and the forgetful functor $\mathcal{M}^T \to \mathcal{M}$ is strict monoidal. That is, the monoidal structure of $\mathcal{M}$ lifts to $\mathcal{M}^T$. Because liftings of functors (respectively, of natural transformations) are described by 1-cells (respectively, by 2-cells) in the 2-category Mnd(Cat) of monads (in the notation of [18]), Moerdijk's definition says that a monad is a bimonad if and only if the functor induced by the monoidal unit of $\mathcal{M}$, from the terminal category to $\mathcal{M}$, and the functor provided by the monoidal product of $\mathcal{M}$, from $\mathcal{M} \times \mathcal{M}$ to $\mathcal{M}$, both admit the structure of a 1-cell in Mnd(Cat), and the coherence natural isomorphisms in $\mathcal{M}$ are 2-cells in Mnd(Cat). In [11], McCrudden showed that a bimonad is the same as an opmonoidal monad, that is, a monad in the 2-category of monoidal categories, opmonoidal functors and opmonoidal natural transformations. Equivalently, bimonads are the same as monoids in a multicategory of monads on a monoidal category.

Pareigis' characterization of a bialgebra was generalized to a *weak bialgebra* [15,4] by Szlachányi in [21]. He proved that an algebra A over a field K is a weak bialgebra if and only if the category of (left or right) A-modules is monoidal and the forgetful functor from the category of A-modules to the category of K-vector spaces obeys the so-called *separable Frobenius* condition. The latter means that the forgetful functor admits both a monoidal and an opmonoidal structure that satisfy some compatibility relations: see Definition 1.1. These (op)monoidal structures are no longer strict. In particular, the monoidal unit of the category of A-modules is not K as a vector space but a non-trivial retract of A. Also, the monoidal product of two A-modules is not their K-module tensor product but a linear retract of it.

Weak bialgebras can be defined in any braided monoidal category, see [1] and [16], as objects possessing both a monoid and a comonoid structure, subject to compatibility axioms that generalize those in [15] and [4] in the case of a symmetric monoidal category of vector spaces. The resulting category of modules was investigated in [16].

The aim of this paper is to generalize weak bialgebras to monoidal categories possibly without a braiding. Inspired by Szlachányi's characterization of a weak bialgebra, we define a *weak bimonad* as a monad T on a monoidal category $\mathcal{M}$, with extra structure making $\mathcal{M}^T$ monoidal and the forgetful functor $\mathcal{M}^T \to \mathcal{M}$ separable Frobenius.

For a weak bimonad T, the forgetful functor $\mathcal{M}^T \to \mathcal{M}$ is no longer strict monoidal, hence the monoidal structure of the domain category $\mathcal{M}$ does not lift to $\mathcal{M}^T$, and so the monoidal unit and the monoidal product of $\mathcal{M}$ are no longer 1-cells in Mnd(Cat). However, the notion of lifting $\overline{F}: \mathcal{M}^T \to \mathcal{M}'^{T'}$ of a functor $F: \mathcal{M} \to \mathcal{M}'$ was weakened in [3] by replacing commutativity of the diagram of functors

$$\begin{array}{ccc} \mathcal{M}^T & \xrightarrow{\overline{F}} & \mathcal{M}'^{T'} \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U'} \\ \mathcal{M} & \xrightarrow{F} & \mathcal{M}' \end{array}$$

by the existence of a split natural monomorphism $i: U'\overline{F} \to FU$. A weak lifting of a natural transformation is defined as a natural transformation between the lifted functors that commutes with the natural monomorphisms i in the evident sense. Weak liftings of functors and of natural transformations in a locally Cauchy complete 2-subcategory of Cat, are related to 1-cells and 2-cells in a 2-category Mnd^i(Cat) in [3], extending Mnd(Cat).

In Section 1 we give an interpretation of the axioms of a weak bimonad (on a Cauchy complete monoidal category), similar to the interpretation of a bimonad in [14]. While for a bimonad T the monoidal structure of $\mathcal{M}^T$ is given by lifting of the monoidal structure in the domain category $\mathcal{M}$, for a weak bimonad T the monoidal product in $\mathcal{M}^T$ is a weak lifting of the monoidal product in $\mathcal{M}$, the monoidal unit is a weak lifting of the functor $1 \to \mathcal{M} \xrightarrow{T} \mathcal{M}$, the associativity constraint is a weak lifting of the associativity constraint in $\mathcal{M}$ and the unit constraints are weak liftings of certain morphisms in $\mathcal{M}$ constructed from the other data.

By results in [16], a weak bimonoid in a braided monoidal category can be described as a quantum category over a separable Frobenius base monoid. Extending this result in Section 2, we establish an

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equivalence between the category of weak bimonads on a Cauchy complete monoidal category $\mathscr{M}$ and the category of bimonads on bimodule categories over separable Frobenius monoids in $\mathscr{M}$.

In Section 3 we show that weak bimonoids in a braided monoidal category (cf. [16,1]) induce weak bimonads. In certain braided monoidal categories the converse can also be proved: if a monoid induces a weak bimonad then it admits the structure of a weak bimonoid.

In Section 4, using the result in Section 2 that any weak bimonad (on a Cauchy complete monoidal category) can be regarded as a bimonad (on another monoidal category), we define a *weak right Hopf monad* to be a weak bimonad such that the associated bimonad is a right Hopf monad in the sense of [5] and [7]; there is a companion result involving weak left Hopf monads and left Hopf monads. A weak bimonoid in a Cauchy complete braided monoidal category is shown to induce a weak right Hopf monad by tensoring with it on the right if and only if it is a weak Hopf monoid in the sense of [1] and [16]; once again there is a companion result with left in place of right.

Notation and conventions. The monoidal categories in this paper are not necessarily strict but in order to simplify our expressions, we omit explicit mention of their coherence isomorphisms wherever possible.

Recall that, for an opmonoidal functor $F:(\mathscr{N},\boxtimes,R)\to(\mathscr{M},\otimes,K)$ with opmonoidal structure $i_{X,Y}:F(X\boxtimes Y)\to FX\otimes FY$ and $i_0:FR\to K$, the diagram

$$\begin{array}{ccc}
F(X\boxtimes Y\boxtimes Z) & \xrightarrow{i_{X\boxtimes Y,Z}} & F(X\boxtimes Y)\boxtimes FZ \\
{\scriptstyle i_{X,Y\boxtimes Z}}\downarrow & & \downarrow{\scriptstyle i_{X,Y}\boxtimes FZ} \\
FX\boxtimes F(Y\boxtimes Z) & \xrightarrow[FX\boxtimes i_{Y,Z}]{} & FX\boxtimes FY\boxtimes FZ
\end{array}$$

commutes. We sometimes write $i^{(3)}_{X,Y,Z}$ for the common composite, and we use an analogous notation for monoidal functors.

We say that a category is *Cauchy complete* provided that idempotent morphisms in it split.

1. Weak bimonads and their Eilenberg–Moore category of algebras

The definition of weak bimonad is based on the notion of separable Frobenius functor introduced in [21]:

Definition 1.1. A functor F from a monoidal category $(\mathscr{N},\boxtimes,R)$ to a monoidal category $(\mathscr{M},\otimes,K)$ is said to be *separable Frobenius* when it is equipped with a monoidal structure $p_{X,Y}:FX\otimes FY\to F(X\boxtimes Y)$, $p_0:K\to FR$ and an opmonoidal structure $i_{X,Y}:F(X\boxtimes Y)\to FX\otimes FY$, $i_0:FR\to K$ such that, for all objects X, Y, Z in $\mathscr{N}$, the following diagrams commute:

$$\begin{array}{ccc}
FX\otimes F(Y\boxtimes Z) & \xrightarrow{FX\otimes i_{Y,Z}} & FX\otimes FY\otimes FZ \\
{\scriptstyle p_{X,Y\boxtimes Z}}\downarrow & & \downarrow{\scriptstyle p_{X,Y}\otimes FZ} \\
F(X\boxtimes Y\boxtimes Z) & \xrightarrow[i_{X\boxtimes Y,Z}]{} & F(X\boxtimes Y)\otimes FZ
\end{array}
\qquad
\begin{array}{ccc}
F(X\boxtimes Y)\otimes FZ & \xrightarrow{i_{X,Y}\otimes FZ} & FX\otimes FY\otimes FZ \\
{\scriptstyle p_{X\boxtimes Y,Z}}\downarrow & & \downarrow{\scriptstyle FX\otimes p_{Y,Z}} \\
F(X\boxtimes Y\boxtimes Z) & \xrightarrow[i_{X,Y\boxtimes Z}]{} & FX\otimes F(Y\boxtimes Z)
\end{array}$$

$$\begin{array}{ccccc}
 & & FX\otimes FY & & \\
 & {\scriptstyle i_{X,Y}}\nearrow & & \searrow{\scriptstyle p_{X,Y}} & \\
F(X\boxtimes Y) & & = & & F(X\boxtimes Y)
\end{array}$$

Example 1.2. (1) Strong monoidal functors are clearly separable Frobenius.

(2) The composite of separable Frobenius functors is separable Frobenius, cf. [8].

(3) In a monoidal category $(\mathcal{M}, \otimes, K)$ possessing (appropriate) coequalizers preserved by $\otimes$, one may consider the monoidal category ${}_R\mathcal{M}_R$ of bimodules over a monoid R in $\mathcal{M}$. The monoidal product is provided by the R-module tensor product and the monoidal unit is R. Justifying the terminology, the forgetful functor ${}_R\mathcal{M}_R \to \mathcal{M}$ is separable Frobenius if and only if R is a separable Frobenius monoid; that is, a Frobenius monoid in the sense of [19] such that, in addition, composing its comultiplication $R \to R \otimes R$ with its multiplication $R \otimes R \to R$ yields the identity morphism R.

Definition 1.3. A *weak bimonad* on a monoidal category $(\mathcal{M}, \otimes, K)$ is a monad (T, m, u) on $\mathcal{M}$ equipped with a monoidal structure on the Eilenberg–Moore category $\mathcal{M}^T$ and a separable Frobenius structure on the forgetful functor $\mathcal{M}^T \to \mathcal{M}$.

The main aim of this section is to find an equivalent formulation of Definition 1.3 – in the spirit of the descriptions of bimonads in [14] and [11] (there called Hopf monads).

If a monad T possesses a monoidal Eilenberg–Moore category $(\mathcal{M}^T, \boxtimes, (R, r))$ then, for any T-algebras (A, a) and (B, b), there is a T-algebra $(A, a) \boxtimes (B, b)$ that we denote by $(A \Box B, a \Box b)$. (Note that by definition $a \Box b$ is a morphism $T(A \Box B) \to A \Box B$ in $\mathcal{M}$, while $a \boxtimes b$ is a morphism $(TA, m_A) \boxtimes (TB, m_B) \to (A, a) \boxtimes (B, b)$ in $\mathcal{M}^T$; that is, a morphism $TA \Box TB \to A \Box B$ in $\mathcal{M}$. Note also that $A \Box B$ depends not just on A and B but on the algebras (A, a) and (B, b).)

In order to get started, we need the following basic observation:

Proposition 1.4. *Consider a monad (T, m, u) on a monoidal category $(\mathcal{M}, \otimes, K)$ equipped with a monoidal Eilenberg–Moore category $(\mathcal{M}^T, \boxtimes, (R, r))$. If the forgetful functor $U : \mathcal{M}^T \to \mathcal{M}$ admits both a monoidal structure (p, p_0) and an opmonoidal structure (i, i_0) then T is opmonoidal, with τ_0 and $\tau_{X,Y}$ given, respectively, by the composite morphisms*

$$TK \xrightarrow{Tp_0} TR \xrightarrow{r} R \xrightarrow{i_0} K \quad \text{and} \tag{1.1}$$

$$T(X \otimes Y) \xrightarrow{T(u_X \otimes u_Y)} T(TX \otimes TY) \xrightarrow{Tp_{TX,TY}} T(TX \Box TY) \xrightarrow{m_X \Box m_Y} TX \Box TY \xrightarrow{i_{TX,TY}} TX \otimes TY. \tag{1.2}$$

Proof. Since U is monoidal, its left adjoint F is opmonoidal. Since U is also opmonoidal, so is $T = UF$. The explicit form of the structure morphisms (1.1) and (1.2) is immediate. □

At this point we can now state one characterization of weak bimonads:

Theorem 1.5. *Let $T = (T, m, u)$ be a monad on a monoidal category $(\mathcal{M}, \otimes, K)$ in which idempotents split. To give T the structure of a weak bimonad is equivalently to give the endofunctor T the structure of an opmonoidal functor (T, τ, τ_0) in such a way that the following conditions hold:*

$$\begin{array}{ccccccc}
T^2(X \otimes TK) & \xrightarrow{T\tau_{X,TK}} & T(TX \otimes T^2K) & \xrightarrow{T(TX \otimes m_K)} & T(TX \otimes TK) & \xrightarrow{T(TX \otimes \tau_0)} & T^2X \\
\uparrow {\scriptstyle Tu_{X \otimes TK}} & & & & & & \downarrow {\scriptstyle m_X} \\
T(X \otimes TK) & \xrightarrow[\tau_{X,TK}]{} & TX \otimes T^2K & \xrightarrow[TX \otimes m_K]{} & TX \otimes TK & \xrightarrow[TX \otimes \tau_0]{} & TX
\end{array} \tag{1.3}$$

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$$\begin{array}{ccccccc}
T^2(TK \otimes X) & \xrightarrow{T\tau_{TK,X}} & T(T^2K \otimes TX) & \xrightarrow{T(m_K \otimes TX)} & T(TK \otimes TX) & \xrightarrow{T(\tau_0 \otimes TX)} & T^2X \\
{\scriptstyle Tu_{TK\otimes X}}\uparrow & & & & & & \downarrow{\scriptstyle m_X} \\
T(TK \otimes X) & \xrightarrow[\tau_{TK,X}]{} & T^2K \otimes TX & \xrightarrow[m_K \otimes TX]{} & TK \otimes TX & \xrightarrow[\tau_0 \otimes TX]{} & TX
\end{array} \tag{1.4}$$

$$\begin{array}{ccccccc}
X \otimes T(Y \otimes Z) & \xrightarrow{X \otimes \tau_{Y,Z}} & X \otimes TY \otimes TZ & \xrightarrow{u_{X \otimes TY} \otimes TZ} & T(X \otimes TY) \otimes TZ & \xrightarrow{\tau_{X,TY} \otimes TZ} & TX \otimes T^2Y \otimes TZ \\
{\scriptstyle X \otimes u_{Y \otimes Z}}\uparrow & & & & & & \downarrow{\scriptstyle TX \otimes m_Y \otimes TZ} \\
X \otimes Y \otimes Z & \xrightarrow[u_{X \otimes Y \otimes Z}]{} & T(X \otimes Y \otimes Z) & \xrightarrow[\tau_{X \otimes Y,Z}]{} & T(X \otimes Y) \otimes TZ & \xrightarrow[\tau_{X,Y} \otimes TZ]{} & TX \otimes TY \otimes TZ
\end{array} \tag{1.5}$$

$$\begin{array}{ccccccc}
T(X \otimes Y) \otimes Z & \xrightarrow{\tau_{X,Y} \otimes Z} & TX \otimes TY \otimes Z & \xrightarrow{TX \otimes u_{TY \otimes Z}} & TX \otimes T(TY \otimes Z) & \xrightarrow{TX \otimes \tau_{TY,Z}} & TX \otimes T^2Y \otimes TZ \\
{\scriptstyle u_{X \otimes Y} \otimes Z}\uparrow & & & & & & \downarrow{\scriptstyle TX \otimes m_Y \otimes TZ} \\
X \otimes Y \otimes Z & \xrightarrow[u_{X \otimes Y \otimes Z}]{} & T(X \otimes Y \otimes Z) & \xrightarrow[\tau_{X \otimes Y,Z}]{} & T(X \otimes Y) \otimes TZ & \xrightarrow[\tau_{X,Y} \otimes TZ]{} & TX \otimes TY \otimes TZ
\end{array} \tag{1.6}$$

$$\begin{array}{ccccc}
T^2(X \otimes Y) & \xrightarrow{T\tau_{X,Y}} & T(TX \otimes TY) & \xrightarrow{\tau_{TX,TY}} & T^2X \otimes T^2Y \\
{\scriptstyle m_{X \otimes Y}}\downarrow & & & & \downarrow{\scriptstyle m_X \otimes m_Y} \\
T(X \otimes Y) & & \xrightarrow[\tau_{X,Y}]{} & & TX \otimes TY
\end{array} \tag{1.7}$$

We shall spend the rest of the section proving this theorem as well as formulating a further characterization in terms of weak lifting. One half of the theorem we prove immediately:

Proposition 1.6. *For any weak bimonad, Eqs.* (1.3)–(1.7) *hold when the endofunctor is given the opmonoidal structure of Proposition* 1.4*.*

Proof. (R, r) is the monoidal unit in $\mathscr{M}^T$ and the coherence natural isomorphisms in $\mathscr{M}^T$ are T-algebra morphisms. For any morphisms $f:(A,a) \to (A',a')$ and $g:(B,b) \to (B',b')$ of T-algebras, $f \boxtimes g$ is a morphism of T-algebras, so that

$$\begin{array}{ccc}
T(A \,\square\, B) & \xrightarrow{T(f \boxtimes g)} & T(A' \,\square\, B') \\
{\scriptstyle a \square b}\downarrow & & \downarrow{\scriptstyle a' \square b'} \\
A \,\square\, B & \xrightarrow[f \boxtimes g]{} & A' \,\square\, B'
\end{array} \tag{1.8}$$

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commutes. By (1.2) and unitality of the T-action $mX \,\square\, mY$, the diagram

$$\begin{array}{ccccc} X \otimes Y & \xrightarrow{u_X \otimes u_Y} & TX \otimes TY & \xrightarrow{p_{TX,TY}} & TX \,\square\, TY \\ \Big\downarrow{\scriptstyle u_{X\otimes Y}} & & & & \Big\downarrow{\scriptstyle i_{TX,TY}} \\ T(X \otimes Y) & & \xrightarrow[\tau_{X,Y}]{} & & TX \otimes TY \end{array} \tag{1.9}$$

commutes, for any objects X, Y of $\mathcal{M}$. Hence a straightforward computation, using these facts together with the unitality of m and with the opmonoidality of (U, i, i_0), shows that both routes around (1.3) are equal to

$$T(X \otimes TK) \xrightarrow{T(X \otimes Tp_0)} T(X \otimes TR) \xrightarrow{T(X \otimes r)} T(X \otimes R) \xrightarrow{T(u_X \otimes R)} T(TX \otimes R) \xrightarrow{Tp_{TX,R}} T^2X \xrightarrow{m_X} TX.$$

Equality (1.4) is proved symmetrically. In view of (1.9), the bottom path of (1.5) is equal to

$$X \otimes Y \otimes Z \xrightarrow{u_X \otimes u_Y \otimes u_Z} TX \otimes TY \otimes TZ \xrightarrow{p^{(3)}_{TX,TY,TZ}} TX \,\square\, TY \,\square\, TZ \xrightarrow{i^{(3)}_{TX,TY,TZ}} TX \otimes TY \otimes TZ.$$

This expression is checked to be equal also to the upper path of (1.5), by applying (1.9) repeatedly, and using monoidality of (U, p, p_0) and the first property in Definition 1.1 of the separable Frobenius functor U. Equality (1.6) is proved symmetrically, using the second property in Definition 1.1 of the separable Frobenius functor U instead of the first one. Finally, by (1.2), by naturality of i and p, by (1.8), and by unitality of m, we deduce that $(m_X \otimes m_Y) \circ \tau_{TX,TY} = i_{TX,TY} \circ (m_X \,\square\, m_Y) \circ Tp_{TX,TY}$. Hence (1.7) follows by the third property in Definition 1.1 of the separable Frobenius functor U and associativity of the action $m_X \,\square\, m_Y : T(TX \,\square\, TY) \to TX \,\square\, TY$. $\square$

Lemma 1.7. *Let (T, τ, τ_0) be an opmonoidal endofunctor of a monoidal category $(\mathcal{M}, \otimes, K)$, and (T, m, u) a monad on $\mathcal{M}$, and suppose that Eq.* (1.3) *holds. Then the morphism*

$$\sqcap := \big(TK \xrightarrow{u_{TK}} T^2K \xrightarrow{\tau_{K,TK}} TK \otimes T^2K \xrightarrow{TK \otimes m_K} TK \otimes TK \xrightarrow{TK \otimes \tau_0} TK\big)$$

is idempotent, and the diagram

$$\begin{array}{ccccc} T^2K & \xrightarrow{T\sqcap} & T^2K & \xrightarrow{m_K} & TK \\ \Big\downarrow{\scriptstyle m_K} & & & & \Big\downarrow{\scriptstyle \sqcap} \\ TK & & \xrightarrow[\sqcap]{} & & TK \end{array} \tag{1.10}$$

commutes.

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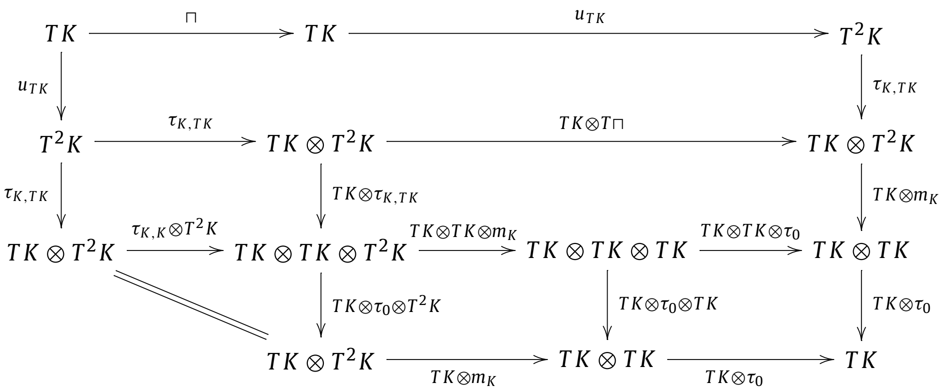
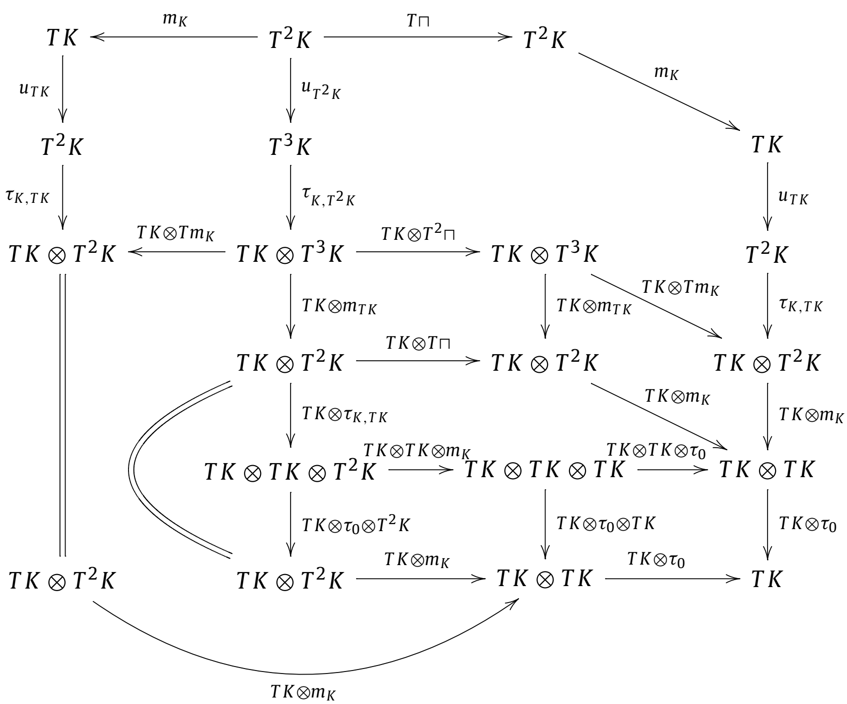
Proof. In the diagram



the squares at the bottom and the region at the top commute by naturality, the triangle and the square above it commute since (T, τ, τ_0) is opmonoidal, and the remaining region is seen to commute by taking $X = K$ in Eq. (1.3) and then tensoring on the left by TK. The composite of the top path is $\sqcap\sqcap$, and that of the bottom path is $\sqcap$.

As for commutativity of the displayed diagram, in the following diagram



the large regions at the top commute by naturality, the pentagonal region in the middle commutes by the case $X = K$ of Eq. (1.3), and remaining regions commute by naturality, associativity of m, and the opmonoidal functor axioms. □

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Lemma 1.8. *Consider a weak bimonad (T, m, u) on a monoidal category $(\mathcal{M}, \otimes, K)$, with opmonoidal structure τ_0 in (1.1) and τ in (1.2). Then the idempotent morphism*

$$\sqcap := \big(TK \xrightarrow{u_{TK}} T^2K \xrightarrow{\tau_{K,TK}} TK \otimes T^2K \xrightarrow{TK \otimes m_K} TK \otimes TK \xrightarrow{TK \otimes \tau_0} TK\big)$$

factorizes through an epimorphism $TK \to R$ and a section of it, where R denotes the object in $\mathcal{M}$ underlying the monoidal unit (R, r) of $\mathcal{M}^T$.

Proof. The desired epimorphism is constructed as

$$P := (TK \xrightarrow{Tp_0} TR \xrightarrow{r} R) \tag{1.11}$$

with a section

$$I := (R \xrightarrow{u_R} TR \xrightarrow{\tau_{K,R}} TK \otimes TR \xrightarrow{TK \otimes r} TK \otimes R \xrightarrow{TK \otimes i_0} TK), \tag{1.12}$$

where (p, p_0) denotes the monoidal structure and (i, i_0) denotes the opmonoidal structure of the forgetful functor $U : \mathcal{M}^T \to \mathcal{M}$. □

Lemma 1.9. *For a weak bimonad T and any T-algebras (A, a) and (B, b), the idempotent morphism*

$$E_{A,B} := \big(A \otimes B \xrightarrow{u_{A \otimes B}} T(A \otimes B) \xrightarrow{\tau_{A,B}} TA \otimes TB \xrightarrow{a \otimes b} A \otimes B\big)$$

is equal to $i_{A,B} \circ p_{A,B}$, where τ is the natural transformation (1.2), (p, p_0) denotes the monoidal structure and (i, i_0) denotes the opmonoidal structure of the forgetful functor $U : \mathcal{M}^T \to \mathcal{M}$.

Proof. This is immediate by (1.2) and unitality of the T-actions a, b and $m_A \,\square\, m_B$. □

The 2-category Mnd(Cat) of monads was extended in [3] to a 2-category Mnd^i(Cat), as follows. The objects of Mnd^i(Cat) are the monads in Cat. The 1-cells from a monad (T, m, u) on a category $\mathcal{C}$ to a monad (T', m', u') on $\mathcal{C}'$, are pairs consisting of a functor $V : \mathcal{C} \to \mathcal{C}'$ and a natural transformation $\psi : T'V \to VT$, such that the diagram below on the left commutes, while the 2-cells $(V, \psi) \to (W, \varphi)$ are natural transformations $\omega : V \to W$ such that the diagram on the right commutes.

$$\begin{array}{ccccc} T'T'V & \xrightarrow{T'\psi} & T'VT & \xrightarrow{\psi T} & VTT \\ \downarrow{\scriptstyle m'V} & & & & \downarrow{\scriptstyle Vm} \\ T'V & & \xrightarrow[\psi]{\qquad\qquad} & & VT \end{array} \qquad \begin{array}{ccccccc} T'T'V & \xrightarrow{T'\psi} & T'VT & \xrightarrow{T'\omega T} & T'WT & \xrightarrow{\varphi T} & WTT \\ \uparrow{\scriptstyle T'u'V} & & & & & & \downarrow{\scriptstyle Wm} \\ T'V & \xrightarrow[\psi]{} & VT & & \xrightarrow[\omega T]{\qquad\qquad} & & WT \end{array}$$

There is a variant, Mnd^p(Cat), of this 2-category which has the same objects and 1-cells but in which a 2-cell $(V, \psi) \to (W, \varphi)$ is a natural transformation $\omega : V \to W$ such that $\varphi \circ T'\omega = Wm \circ \varphi T \circ u'WT \circ \omega T \circ \psi$.

For a monad T on $\mathcal{C}$ and a monad T' on $\mathcal{C}'$, we say that a functor $\overline{V} : \mathcal{C}^T \to \mathcal{C}'^{T'}$ is a *weak lifting* of a functor $V : \mathcal{C} \to \mathcal{C}'$ if there exists a split natural monomorphism $i : U'\overline{V} \to VU$ (where $U : \mathcal{C}^T \to \mathcal{C}$ and $U' : \mathcal{C}'^{T'} \to \mathcal{C}'$ are the forgetful functors). Associated to a 1-cell (V, ψ) in Mnd^i(Cat), there is an idempotent natural transformation $VU \to VU$. Evaluated on a T-algebra (A, a), it is the morphism $Va \circ \psi A \circ u'VA : VA \to VA$. Whenever it splits (that is, it factorizes through some natural epimorphism $VU \to V_0$ and a section), the resulting functor $V_0 : \mathscr{C}^T \to \mathscr{C}'$ has a lifting to a functor $\overline{V} : \mathscr{C}^T \to \mathscr{C}'^{T'}$, which is clearly a weak lifting of V. Conversely, every weak lifting $\overline{V} : \mathscr{C}^T \to \mathscr{C}'^{T'}$ of a functor $V : \mathscr{C} \to \mathscr{C}'$ arises in this way from a unique 1-cell (V, ψ) in $\mathsf{Mnd}^i(\mathsf{Cat})$ such that the corresponding idempotent natural transformation splits: see [3, Theorem 4.4].

A natural transformation $\overline{\omega} : \overline{V} \to \overline{W}$ between weakly lifted functors is said to be a *weak i-lifting* of a natural transformation $\omega : V \to W$ provided that $\omega U \circ i = i \circ U'\overline{\omega}$. By [3, Proposition 4.3], a natural transformation has a weak i-lifting if and only if it is a 2-cell in $\mathsf{Mnd}^i(\mathsf{Cat})$. Symmetrically, $\overline{\omega}$ is said to be a *weak p-lifting* of ω provided that $p \circ \omega U = U'\overline{\omega} \circ p$, in terms of a natural retraction p of i. By [3, Proposition 4.3], a natural transformation has a weak p-lifting if and only if it is a 2-cell in $\mathsf{Mnd}^p(\mathsf{Cat})$.

Theorem 1.10. *Let (T, m, u) be a monad on a monoidal category $(\mathscr{M}, \otimes, K)$, where $(T, \tau, \tau_0) : (\mathscr{M}, \otimes, K) \to (\mathscr{M}, \otimes, K)$ is an opmonoidal functor. Then Eqs.* (1.3)–(1.7) *hold if and only if the following conditions are satisfied*:

(i) *The functor $1 \xrightarrow{K} \mathscr{M} \xrightarrow{T} \mathscr{M}$ and the natural transformation $T^2K \xrightarrow{m_K} TK \xrightarrow{\sqcap} TK$ constitute a 1-cell $1 \to T$ in* $\mathsf{Mnd}^i(\mathsf{Cat})$.

(ii) *The functor $\mathscr{M} \times \mathscr{M} \xrightarrow{\otimes} \mathscr{M}$ and the natural transformation $T(\bullet \otimes \bullet) \xrightarrow{\tau} T(\bullet) \otimes T(\bullet)$ constitute a 1-cell $T \times T \to T$ in* $\mathsf{Mnd}^i(\mathsf{Cat})$.

(iii) *The natural transformations*

$$\begin{array}{ccc} \mathscr{M} \times \mathscr{M} & \xrightarrow{\mathscr{M} \times T} & \mathscr{M} \times \mathscr{M} \\ {\scriptstyle \mathscr{M} \times K} \uparrow & \Downarrow \mathscr{M} \times \tau_0 & \downarrow {\scriptstyle \otimes} \\ \mathscr{M} & \xrightarrow[\mathscr{M}]{} & \mathscr{M} \end{array} \qquad \begin{array}{ccc} \mathscr{M} \times \mathscr{M} & \xrightarrow{T \times \mathscr{M}} & \mathscr{M} \times \mathscr{M} \\ {\scriptstyle K \times \mathscr{M}} \uparrow & \Downarrow \tau_0 \times \mathscr{M} & \downarrow {\scriptstyle \otimes} \\ \mathscr{M} & \xrightarrow[\mathscr{M}]{} & \mathscr{M} \end{array}$$

are 2-cells in $\mathsf{Mnd}^i(\mathsf{Cat})$.

(iv) *The idempotent natural transformations $E_{TX,TY}$ and $E^{(3)}_{TX,TY,TZ}$ which are respectively the composites*

$$TX \otimes TY \xrightarrow{u_{TX \otimes TY}} T(TX \otimes TY) \xrightarrow{\tau_{TX,TY}} T^2X \otimes T^2Y \xrightarrow{m_X \otimes m_Y} TX \otimes TY \quad \text{and}$$

$$TX \otimes TY \otimes TZ \xrightarrow{u_{TX \otimes TY \otimes TZ}} T(TX \otimes TY \otimes TZ) \xrightarrow{\tau^{(3)}_{TX,TY,TZ}} T^2X \otimes T^2Y \otimes T^2Z \xrightarrow{m_X \otimes m_Y \otimes m_Z} TX \otimes TY \otimes TZ,$$

make the following diagram commute:

$$\begin{array}{ccc} TX \otimes TY \otimes TZ & \xrightarrow{E_{TX,TY} \otimes TZ} & TX \otimes TY \otimes TZ \\ {\scriptstyle TX \otimes E_{TY,TZ}} \downarrow & \searrow {\scriptstyle E^{(3)}_{TX,TY,TZ}} & \downarrow {\scriptstyle TX \otimes E_{TY,TZ}} \\ TX \otimes TY \otimes TZ & \xrightarrow[E_{TX,TY} \otimes TZ]{} & TX \otimes TY \otimes TZ \end{array}$$

for any objects X, Y, Z in $\mathscr{M}$.

Proof. Using the definition of 1-cells in $\mathsf{Mnd}^i(\mathsf{Cat})$, assertion (i) is seen to be equivalent to the equation $\sqcap \circ m_K \circ T\sqcap = \sqcap \circ m_K$ which by Lemma 1.7 holds whenever Eq. (1.3) does. Assertion (ii) is clearly equivalent to (1.7). Condition (iii) depends on the 1-cells in $\mathsf{Mnd}^i(\mathsf{Cat})$ constructed in parts (i) and (ii). Now $\tau_0 \circ \sqcap = \tau_0$ by opmonoidality of T, and then the two conditions in (iii) are equivalent to (1.3) and (1.4). Similarly the two conditions in (iv) are equivalent to (1.5) and (1.6). □

Our next aim is to prove the other half of Theorem 1.5, which we state as:

Proposition 1.11. *Consider a monad (T, m, u) on a monoidal category $(\mathcal{M}, \otimes, K)$ and an opmonoidal structure (τ, τ_0) on the functor T. Assume that the identities* (1.3)–(1.7) *hold and that the following idempotent morphisms split*:

$$\sqcap := \big(TK \xrightarrow{u_{TK}} T^2K \xrightarrow{\tau_{K,TK}} TK \otimes T^2K \xrightarrow{TK\otimes m_K} TK \otimes TK \xrightarrow{TK\otimes\tau_0} TK\big) \tag{1.13}$$

and

$$E_{A,B} := \big(A \otimes B \xrightarrow{u_{A\otimes B}} T(A \otimes B) \xrightarrow{\tau_{A,B}} TA \otimes TB \xrightarrow{a\otimes b} A \otimes B\big), \tag{1.14}$$

for any T-algebras (A, a) and (B, b). Then T is a weak bimonad.

Proof. We prove this claim by constructing a monoidal structure on $\mathcal{M}^T$ weakly lifting that of $\mathcal{M}$, and by showing that with respect to this monoidal structure the forgetful functor $U : \mathcal{M}^T \to \mathcal{M}$ is separable Frobenius.

By Theorem 1.10 (ii), $(\otimes, \tau)$ is a 1-cell in $\mathsf{Mnd}^i(\mathsf{Cat})$ (equivalently, in $\mathsf{Mnd}^p(\mathsf{Cat})$) and so induces a weak lifting $\boxtimes : \mathcal{M}^T \times \mathcal{M}^T \to \mathcal{M}^T$; that is, a functor $\boxtimes$ equipped with natural transformations

$$\begin{array}{ccc} \mathcal{M}^T \times \mathcal{M}^T & \xrightarrow{\boxtimes} & \mathcal{M}^T \\ {\scriptstyle U\times U}\downarrow & p\Uparrow \quad \Downarrow i & \downarrow{\scriptstyle U} \\ \mathcal{M} \times \mathcal{M} & \xrightarrow[\otimes]{} & \mathcal{M} \end{array}$$

such that $p \circ i$ is the identity natural transformation. Explicitly, for T-algebras (A, a) and (B, b), the to-be-tensor product $(A, a) \boxtimes (B, b) = (A \,\square\, B, a \,\square\, b)$ is given by splitting the idempotent (1.14) to obtain $A \,\square\, B$, via maps $i_{A,B} : A \,\square\, B \to A \otimes B$ and $p_{A,B} : A \otimes B \to A \,\square\, B$, and then $a \,\square\, b$ is the composite

$$T(A \,\square\, B) \xrightarrow{Ti_{A,B}} T(A \otimes B) \xrightarrow{\tau_{A,B}} TA \otimes TB \xrightarrow{a\otimes b} A \otimes B \xrightarrow{p_{A,B}} A \,\square\, B.$$

By coassociativity of τ, the associativity isomorphism in $\mathcal{M}$ is an invertible 2-cell both in $\mathsf{Mnd}^i(\mathsf{Cat})$ and $\mathsf{Mnd}^p(\mathsf{Cat})$. So it weakly lifts to an associativity isomorphism for $\boxtimes$ such that the following diagrams, with the associativity isomorphisms on the vertical arrows, commute.

$$\begin{array}{ccccc} (A \,\square\, B) \,\square\, C & \xrightarrow{i_{A\square B,C}} & (A \,\square\, B) \otimes C & \xrightarrow{i_{A,B}\otimes C} & (A \otimes B) \otimes C \\ \downarrow & & & & \downarrow \\ A \,\square\, (B \,\square\, C) & \xrightarrow[i_{A,B\square C}]{} & A \otimes (B \,\square\, C) & \xrightarrow[A\otimes i_{B,C}]{} & A \otimes (B \otimes C) \end{array} \qquad \begin{array}{ccccc} (A \otimes B) \otimes C & \xrightarrow{p_{A,B}\otimes C} & (A \,\square\, B) \otimes C & \xrightarrow{p_{A\square B,C}} & (A \,\square\, B) \,\square\, C \\ \downarrow & & & & \downarrow \\ A \otimes (B \otimes C) & \xrightarrow[A\otimes p_{B,C}]{} & A \otimes (B \,\square\, C) & \xrightarrow[p_{A,B\square C}]{} & A \,\square\, (B \,\square\, C) \end{array}$$

That is, p and i satisfy the associativity conditions that will be needed to make U a monoidal and an opmonoidal functor. The pentagon identity for $\boxtimes$ follows from that for $\otimes$ and commutativity of either of the diagrams above.

Next we construct the unit object for the monoidal category $\mathscr{M}^T$. By Theorem 1.10 (i), $(TK, \sqcap \circ m_K)$ is a 1-cell in $\mathsf{Mnd}^i(\mathsf{Cat})$ and so gives an object (R, r) of $\mathscr{M}^T$. Explicitly, R is obtained by splitting the idempotent $\sqcap$ via maps $I : R \to TK$ and $P : TK \to R$, and r is the composite

$$TR \xrightarrow{TI} T^2K \xrightarrow{m_K} TK \xrightarrow{P} R.$$

The unit constraints for $\mathscr{M}^T$ are constructed by applying Theorem 1.10 (iii). The 2-cells of $\mathsf{Mnd}^i(\mathsf{Cat})$ therein induce morphisms $\varrho_A : A \,\Box\, R \to A$ and $\lambda_A : R \,\Box\, A \to A$ of T-algebras, natural in the T-algebra (A, a). Explicitly, ϱ_A and λ_A are given by the composites

$$A \,\Box\, R \xrightarrow{i_{A,R}} A \otimes R \xrightarrow{A \otimes I} A \otimes TK \xrightarrow{A \otimes \tau_0} A,$$

$$R \,\Box\, A \xrightarrow{i_{R,A}} R \otimes A \xrightarrow{I \otimes A} TK \otimes A \xrightarrow{\tau_0 \otimes A} A, \tag{1.15}$$

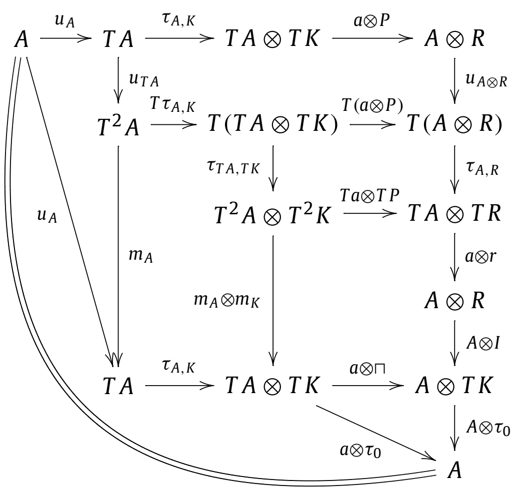
respectively. Since $A \,\Box\, R$ was constructed by splitting the idempotent $E_{A,R}$, to show that ϱ_A is invertible, it will suffice to show that

$$A \otimes R \xrightarrow{E_{A,R}} A \otimes R \xrightarrow{A \otimes I} A \otimes TK \xrightarrow{A \otimes \tau_0} A$$

is the epimorphism part of a splitting for $E_{A,R}$. We claim that the other half of the splitting can be taken to be

$$A \xrightarrow{u_A} TA \xrightarrow{\tau_{A,K}} TA \otimes TK \xrightarrow{a \otimes P} A \otimes R.$$

By commutativity of the following diagram, one composite yields the identity morphism on A.



The top two squares on the right-hand side commute by naturality. The third (pentagonal) region below them commutes by the associativity of a, definition of r and (1.10). The other pentagonal region

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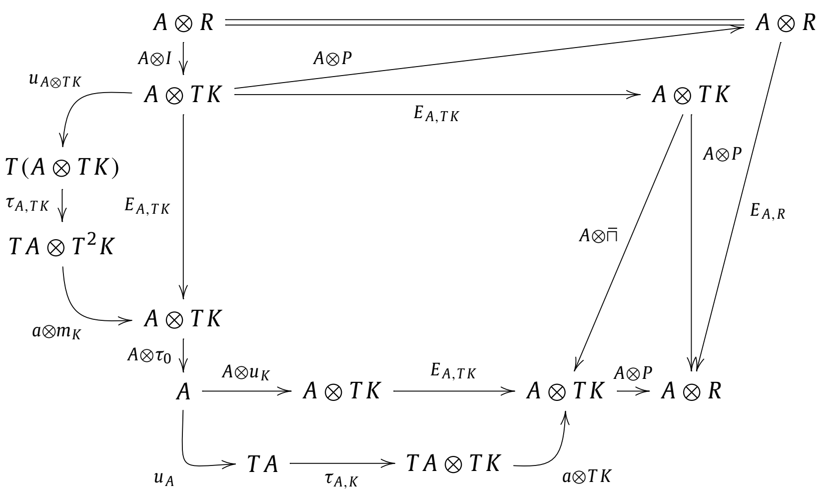
on its left commutes by (1.7) and the region on the left of that commutes by the unitality condition on a monad. The triangle at the bottom commutes by the (straightforward) fact that $\tau_0 \circ \sqcap = \tau_0$. The bottom-left region commutes by the opmonoidality of T and unitality of a.

Composition in the opposite order yields $E_{A,R}$ by commutativity of



The leftmost vertical path is equal to $(A \otimes \tau_0) \circ (A \otimes I) \circ E_{A,R}$ since by the definition of r, $\tau_0 \circ I \circ r = \tau_0 \circ m_K \circ TI$. The regions surrounded by the curved arrows commute by (1.14). The triangle at the top commutes by the definitions of I and P. The concave quadrangle below it commutes by naturality of E, since P is a morphism of T-algebras by (1.10). The large polygon at the bottom-left involves the morphism

$$\bar{\sqcap} := (\tau_0 \otimes TK) \circ E_{TK,TK} \circ (TK \otimes u_K) = (\tau_0 \otimes TK) \circ (m_K \otimes TK) \circ \tau_{TK,K} \circ u_{TK},$$

where the last equality follows by (1.14). In order to see that this polygon commutes, note that associativity of a together with (1.3) implies

$$E_{A,TK} \circ (A \otimes \tau_0 \otimes TK) \circ (E_{A,TK} \otimes TK) = (A \otimes \tau_0 \otimes TK) \circ E^{(3)}_{A,TK,TK}.$$

Using the first one of the equivalent forms of $\bar{\sqcap}$ above, this implies commutativity of the bottom-left polygon. In order to see that the triangle on its right commutes, use the second form of $\bar{\sqcap}$. By (1.4) and opmonoidality of T, it obeys $\tau_0 \circ m_K \circ T\bar{\sqcap} = \tau_0 \circ m_K$ which implies $\sqcap \circ \bar{\sqcap} = \sqcap$ hence commutativity of the triangle in question.

The case of λ is similar. We record here the explicit forms of ϱ_A^{-1} and λ_A^{-1} as

$$A \xrightarrow{u_A} TA \xrightarrow{\tau_{A,K}} TA \otimes TK \xrightarrow{a \otimes P} A \otimes R \xrightarrow{p_{A,R}} A \,\square\, R,$$

$$A \xrightarrow{u_A} TA \xrightarrow{\tau_{K,A}} TK \otimes TA \xrightarrow{P \otimes a} R \otimes A \xrightarrow{p_{R,A}} R \,\square\, A, \qquad (1.16)$$

respectively.

To conclude that $\mathscr{M}^T$ is a monoidal category, we only need to prove that the triangle condition holds. This follows by functoriality of weak lifting because both morphisms

$$(A \,\square\, R) \,\square\, B \longrightarrow A \,\square\, (R \,\square\, B) \xrightarrow{A \square \lambda_B} A \,\square\, B \quad \text{and} \quad (A \,\square\, R) \,\square\, B \xrightarrow{\varrho_A \square B} A \,\square\, B$$

are weak i-liftings of $(A \otimes \tau_0) \otimes B : (A \otimes TK) \otimes B \to A \otimes B$, for any T-algebras A and B.

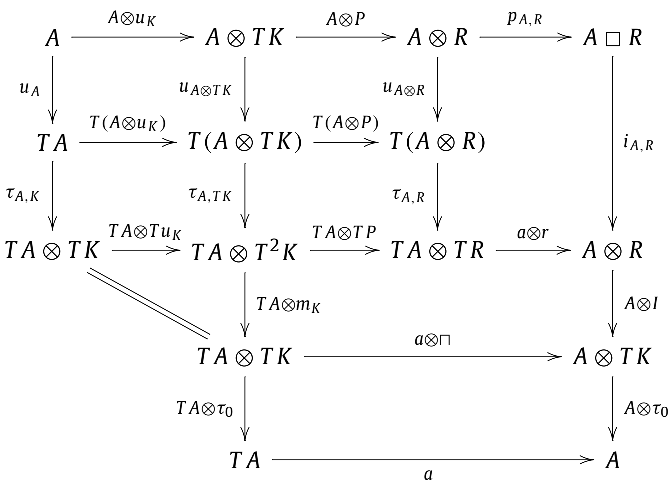
It remains to show that the forgetful functor $U : \mathscr{M}^T \to \mathscr{M}$ is separable Frobenius. We already have the binary parts of the monoidal and opmonoidal structures, in the form of morphisms $p_{A,B} : A \otimes B \to A \,\square\, B$ and $i_{A,B} : A \,\square\, B \to A \otimes B$. We already proved that they satisfy the associativity, respectively, coassociativity conditions. A counit i_0 for U, so that (U, i, i_0) becomes an opmonoidal functor, is constructed as the composite

$$R \xrightarrow{I} TK \xrightarrow{\tau_0} K$$

and the counit laws then reduce to Eqs. (1.15) defining λ and ϱ. The unit p_0 for the monoidal structure of U will be the composite

$$K \xrightarrow{u_K} TK \xrightarrow{P} R.$$

One of the unit laws follows by commutativity of the diagram below; the other is similar and left to the reader.



The four squares in the top left corner commute by naturality; the large region in the top right corner by definition of i and p; the triangle by one of the unit laws for a monad, the region to its right by definition of r and (1.10), and the bottom region by the equation $\tau_0 \circ \sqcap = \tau_0$ once again. The left/bottom path yields an identity morphism by opmonoidality of T and unitality of a.

The separability condition $p_{A,B} \circ i_{A,B} = A \,\square\, B$ holds by construction. As for the Frobenius conditions in Definition 1.1, by (1.14) we have $E_{A,B\square C} = (A \otimes p_{B,C}) \circ E^{(3)}_{A,B,C} \circ (A \otimes i_{B,C})$, and now the first Frobenius condition follows by Theorem 1.10 (iv); the other Frobenius condition is proved similarly. □

2. Weak bimonads vs. bimonads over a separable Frobenius base

The aim of this section is to study the category of weak bimonads on a given Cauchy complete monoidal category $\mathscr{M}$. As a main result, we prove that it is equivalent to an appropriate category of bimonads on bimodule categories over separable Frobenius monoids in $\mathscr{M}$.

Recall that if T is a bimonad then $\mathcal{M}^T$ can be given a monoidal structure so that the forgetful functor $U:\mathcal{M}^T\to\mathcal{M}$ is strict monoidal. Conversely, any monad T for which $U:\mathcal{M}^T\to\mathcal{M}$ is strong monoidal can be made into a bimonad; and these two processes are, in a suitable sense, mutually inverse. Similarly, if $g:T\to T'$ is a morphism of bimonads and $\mathcal{M}^T$ and $\mathcal{M}^{T'}$ are made monoidal as above, then the induced functor $g^*:\mathcal{M}^{T'}\to\mathcal{M}^T$ is strict monoidal; and conversely if $g:T\to T'$ is a morphism of monads for which the induced functor g^* is opmonoidal, compatibly with the forgetful functors, then g can be made into a morphism of bimonads.

For the entire section, we introduce the following notation. We work in a monoidal category $\mathcal{M}$, with monoidal product $\otimes$ and monoidal unit K. For a weak bimonad T, the monad structure is denoted by $m:T^2\to T$ and $u:\mathcal{M}\to T$. The opmonoidal structure of T is denoted by $\tau_{X,Y}:T(X\otimes Y)\to TX\otimes TY$ and $\tau_0:TK\to K$. The forgetful functor $\mathcal{M}^T\to\mathcal{M}$ is called U. The monoidal unit of $\mathcal{M}^T$ is denoted by (R,r). By the separable Frobenius property of U, R is a separable Frobenius monoid in $\mathcal{M}$. Its monoid structure is denoted by $(\mu:R\otimes R\to R,\eta:K\to R)$ and for the comonoid structure we write $(\delta:R\to R\otimes R,\varepsilon:R\to K)$. (See their explicit expressions in terms of (m,u) and (τ,τ_0) below.) For the monoidal category of R-bimodules, the forgetful functor is denoted by $V:{}_R\mathcal{M}_R\to\mathcal{M}$. We use the notation $\sqcap$ introduced in (1.13), E in (1.14) and $E^{(3)}$ in Theorem 1.10 (iv). For other (weak) bimonads T', $\widetilde{T}$, etc., we use the same symbols introduced for T, distinguished by prime, tilde, etc.

Our starting point is the following result due to Szlachányi.

Theorem 2.1. *(See [21, Theorem 2.2 and Lemma 6.2].) Any separable Frobenius functor U, from a monoidal category $\mathcal{N}$ with unit R, to a Cauchy complete monoidal category $\mathcal{M}$, factorizes through the forgetful functor ${}_{UR}\mathcal{M}_{UR}\to\mathcal{M}$ via a strong monoidal functor $\mathcal{N}\to{}_{UR}\mathcal{M}_{UR}$.*

In particular, for a weak bimonad T on a Cauchy complete monoidal category $\mathcal{M}$, the forgetful functor $U:\mathcal{M}^T\to\mathcal{M}$ factorizes through a strong monoidal functor $\widetilde{U}$ from $\mathcal{M}^T$ to the bimodule category ${}_R\mathcal{M}_R$ for the monoidal unit R of $\mathcal{M}^T$ and the forgetful functor $V:{}_R\mathcal{M}_R\to\mathcal{M}$. Explicitly, the monoid structure of R comes out as

$$\mu:=(R\otimes R\xrightarrow{E_{R,R}}R\otimes R\xrightarrow{R\otimes I}R\otimes TK\xrightarrow{R\otimes\tau_0}R),\qquad \eta:=(K\xrightarrow{u_K}TK\xrightarrow{P}R) \tag{2.1}$$

and its comonoid structure is given by

$$\delta:=(R\xrightarrow{R\otimes u_K}R\otimes TK\xrightarrow{R\otimes P}R\otimes R\xrightarrow{E_{R,R}}R\otimes R),\qquad \varepsilon:=(R\xrightarrow{I}TK\xrightarrow{\tau_0}K). \tag{2.2}$$

By (1.15), $\widetilde{U}$ takes a T-algebra (A,a) to the $R\otimes\bullet\otimes R$-algebra (A,ϱ_A) with the structure morphism

$$\varrho_A=(R\otimes A\otimes R\xrightarrow{E^{(3)}_{R,A,R}}R\otimes A\otimes R\xrightarrow{I\otimes A\otimes I}TK\otimes A\otimes TK\xrightarrow{\tau_0\otimes A\otimes\tau_0}A), \tag{2.3}$$

where $TK\overset{P}{\twoheadrightarrow}R\overset{I}{\rightarrowtail}TK$ denotes a chosen splitting of the idempotent morphism $\sqcap$ of (1.13). (Recall that $R\otimes\bullet\otimes R$-algebras (M,ϱ_M) are in bijection with R-bimodules $(M,\alpha_M:M\otimes R\to M,\beta_M:R\otimes M\to M)$ via the correspondence $\varrho_M=\alpha_M\circ(\beta_M\otimes R)=\beta_M\circ(R\otimes\alpha_M)$.)

Next we compare the monadicity properties of the functors in the factorization in Theorem 2.1.

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Lemma 2.2. *For a separable Frobenius monoid R in a Cauchy complete monoidal category $\mathcal{M}$ with forgetful functor $V : {}_R\mathcal{M}_R \to \mathcal{M}$, any V-contractible pair is a split coequalizer pair.* (*For the terminology we refer to* [2].)

Proof. Consider a Cauchy complete category $\mathcal{C}$ and an adjunction $L \dashv V : \mathcal{C} \to \mathcal{M}$ in which the counit $n : LV \to 1$ is split by a natural monomorphism $\bar{n}$. Under these assumptions, any V-contractible pair is a split coequalizer pair. Indeed, if for some morphisms $\mu, \nu : M \to N$ in $\mathcal{C}$, the first diagram in

$$VM \underset{V\nu}{\overset{V\mu}{\rightrightarrows}} VN \quad (\xi : VN \to VM), \qquad M \underset{\nu}{\overset{\mu}{\rightrightarrows}} N \quad (nM \circ L\xi \circ \bar{n}N : N \to M)$$

is a contractible pair, then so is the second one. Hence by Cauchy completeness of $\mathcal{C}$, the coequalizer of μ and ν exists.

We conclude by applying this observation to the adjunction $R \otimes \bullet \otimes R \dashv V : {}_R\mathcal{M}_R \to \mathcal{M}$, whose counit is given by the $R \otimes \bullet \otimes R$-action $\varrho_M : R \otimes M \otimes R \to M$, for any object (M, ϱ_M) of ${}_R\mathcal{M}_R$, hence by separable Frobenius property of R it is split by $(R \otimes \varrho_M \otimes R) \circ (\delta \circ \eta \otimes M \otimes \delta \circ \eta)$. □

Proposition 2.3. *Let R be a separable Frobenius monoid in a Cauchy complete monoidal category $\mathcal{M}$. Then for a Cauchy complete category $\mathcal{C}$, a functor $W : \mathcal{C} \to {}_R\mathcal{M}_R$ is monadic if and only if its composite with the forgetful functor $V : {}_R\mathcal{M}_R \to \mathcal{M}$ is monadic.*

Proof. This is proved by applying Beck's theorem [2, Theorem 3.14].

Assume first that W is monadic. Then it is immediate by monadicity of V that VW has a left adjoint and that it is conservative. It remains to show that the Beck condition holds. For a VW-split coequalizer pair (α, β) in $\mathcal{C}$, $(W\alpha, W\beta)$ is in particular V-contractible. Hence by Lemma 2.2 it is a split coequalizer pair (evidently preserved by V). Then by monadicity of W, there exists the coequalizer of α and β and it is preserved by W, so also by VW.

Conversely, assume that VW is monadic. Since VW is conservative by assumption, so is W. As for the Beck condition, a W-split coequalizer pair (α, β) is also a VW-split coequalizer pair. Hence by monadicity of VW, its coequalizer exists and it is preserved by VW. Since V is faithful, this implies that W preserves the coequalizer of α and β. Thus we need only to check that W has a left adjoint. This holds by a standard adjoint-lifting argument [2], made particularly simple here since the relevant coequalizers are split. In more detail, let L be the left adjoint of VW, with counit $n : LVW \to 1$ and unit $u : 1 \to VWL$. Consider the mate of $V\varrho W$ for $\varrho : R \otimes V(\bullet) \otimes R \to {}_R\mathcal{M}_R$ under the adjunction $L \dashv VW$; that is, the morphism

$$\lambda_X := \left(L(R \otimes X \otimes R) \xrightarrow{L(R \otimes u_X \otimes R)} L(R \otimes VWLX \otimes R) \xrightarrow{LV\rho_{WLX}} LVWLX \xrightarrow{n_{LX}} LX \right) \tag{2.4}$$

for any object X of $\mathcal{M}$. Whenever the coequalizer of $L\varrho_M, \lambda_M : L(R \otimes M \otimes R) \to LM$ exists for any object (M, ϱ_M) of ${}_R\mathcal{M}_R$, it defines a left adjoint for the lifting W of VW; see [2]. By the separable Frobenius property of R, the morphism λ_X is split by the natural monomorphism $\lambda_{R \otimes X \otimes R} \circ L(\delta \circ \eta \otimes X \otimes \delta \circ \eta)$. Since the diagram

$$\begin{array}{ccc} L(R \otimes M \otimes R) & \underset{\lambda_M}{\overset{L\varrho_M}{\rightrightarrows}} & LM \\ \uparrow {\scriptstyle \lambda_{R\otimes M\otimes R} \circ L(\delta\circ\eta\otimes M\otimes\delta\circ\eta)} & & \| \\ LM & \underset{LM}{\overset{\lambda_M \circ L((R\otimes\varrho_M\otimes R)\circ(\delta\circ\eta\otimes M\otimes\delta\circ\eta))}{\rightrightarrows}} & LM \end{array}$$

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is serially commutative, and the coequalizer of the bottom pair exists by Cauchy completeness of $\mathscr{C}$, it follows that it is also a coequalizer of the top pair, defining a left adjoint of W. □

Remark 2.4. Proposition 2.3 implies a relation between weak bimonads on a Cauchy complete monoidal category $\mathscr{M}$ and bimonads on ${}_R\mathscr{M}_R$, for some separable Frobenius monoid R in $\mathscr{M}$. Namely, for a weak bimonad T, the separable Frobenius forgetful functor $U : \mathscr{M}^T \to \mathscr{M}$ factorizes through a strong monoidal functor $\widetilde{U} : \mathscr{M}^T \to {}_R\mathscr{M}_R$ for a separable Frobenius monoid R, and the forgetful functor $V : {}_R\mathscr{M}_R \to \mathscr{M}$; see Theorem 2.1. By Proposition 2.3, $\widetilde{U}$ is also monadic hence together with its left adjoint $\widetilde{L}$, it induces a bimonad $\widetilde{T} := \widetilde{U}\widetilde{L}$ on ${}_R\mathscr{M}_R$, whose Eilenberg–Moore category is equivalent to $\mathscr{M}^T$; see [9]. Conversely, for a bimonad $\widetilde{T}$ on a bimodule category ${}_R\mathscr{M}_R$ over a separable Frobenius monoid R, the composite U of the forgetful functor $\widetilde{U} : ({}_R\mathscr{M}_R)^{\widetilde{T}} \to {}_R\mathscr{M}_R$ and the forgetful functor $V : {}_R\mathscr{M}_R \to \mathscr{M}$ is separable Frobenius; cf. Example 1.2. It is also monadic by Proposition 2.3, hence together with its left adjoint L, it induces a weak bimonad $T := UL$ on $\mathscr{M}$ such that $\mathscr{M}^T$ is equivalent to $({}_R\mathscr{M}_R)^{\widetilde{T}}$. What is more, by uniqueness of a left adjoint up to natural isomorphism, T and $V\widetilde{T}(R \otimes \bullet \otimes R)$ differ by an opmonoidal isomorphism of monads (or in fact they can be chosen equal).

Remark 2.5. Consider a weak bimonad T on a Cauchy complete monoidal category $\mathscr{M}$. By (the proof of) Proposition 2.3, the left adjoint $\widetilde{L}$ of the strong monoidal functor $\widetilde{U} : \mathscr{M}^T \to {}_R\mathscr{M}_R$ (occurring in the factorization of the forgetful functor $U : \mathscr{M}^T \to \mathscr{M}$) is constructed by choosing a splitting $LM \twoheadrightarrow \widetilde{L}(M, \varrho_M) \rightarrowtail LM$ of the idempotent natural transformation

$$\lambda_M \circ L\big((R \otimes \varrho_M \otimes R) \circ (\delta \circ \eta \otimes M \otimes \delta \circ \eta)\big) : LM \to LM, \tag{2.5}$$

for any object (M, ϱ_M) of ${}_R\mathscr{M}_R$, where λ_M is as in (2.4). Applying U, this yields a split idempotent natural transformation $ULV = TV \overset{q}{\twoheadrightarrow} U\widetilde{L} = V\widetilde{T} \overset{j}{\rightarrowtail} TV = ULV$. What is more, also as a monad, $\widetilde{T} = \widetilde{U}\widetilde{L}$ is a weak q-lifting of T hence by [3, Proposition 3.7], the Eilenberg–Moore categories $\mathscr{M}^T$ and $({}_R\mathscr{M}_R)^{\widetilde{T}}$ are in fact isomorphic. Explicitly, there is an isomorphism $\Xi : \mathscr{M}^T \to ({}_R\mathscr{M}_R)^{\widetilde{T}}$, taking a T-algebra (A, a) to the $R \otimes \bullet \otimes R$-algebra A described in (2.3), with a $\widetilde{T}$-algebra structure provided by the unique morphism $\widetilde{a} : \widetilde{T}A \to A$ for which $\widetilde{a} \circ q_A = a$. On the morphisms, Ξ acts as the identity map. The inverse of Ξ takes an object $((A, \varrho_A), \widetilde{a})$ of $({}_R\mathscr{M}_R)^{\widetilde{T}}$ to the T-algebra A, with structure morphism $TA \xrightarrow{q_A} V\widetilde{T}A \xrightarrow{\widetilde{a}} A$, and it also acts on the morphisms as an identity map. In particular, also the Eilenberg–Moore categories $({}_R\mathscr{M}_R)^{\widetilde{T}}$ and $\mathscr{M}^{V\widetilde{T}(R\otimes\bullet\otimes R)}$ are isomorphic, for any bimonad $\widetilde{T}$ on a bimodule category ${}_R\mathscr{M}_R$ over a separable Frobenius monoid R.

The final aim of this section is to extend the correspondence in Remark 2.4 between weak bimonads on one hand, and bimonads over separable Frobenius base monoids on the other hand, to an equivalence of categories.

Definition 2.6. A *morphism of weak bimonads* on a monoidal category $\mathscr{M}$ is defined as an opmonoidal morphism of monads; that is, as a natural transformation $g : T \to T'$ which is opmonoidal in the sense that, for any objects X and Y in $\mathscr{M}$,

$$\begin{array}{ccc} T(X \otimes Y) & \xrightarrow{g_{X\otimes Y}} & T'(X \otimes Y) \\ {\scriptstyle \tau_{X,Y}}\downarrow & & \downarrow{\scriptstyle \tau'_{X,Y}} \\ TX \otimes TY & \xrightarrow[g_X \otimes g_Y]{} & T'X \otimes T'Y \end{array} \qquad \begin{array}{ccc} TK & \xrightarrow{g_K} & T'K \\ {\scriptstyle \tau_0}\downarrow & & \downarrow{\scriptstyle \tau'_0} \\ K & = & K \end{array}$$

and which is a morphism of monads in the sense that, for any object X in $\mathcal{M}$,

$$\begin{array}{ccccc} T^2X & \xrightarrow{Tg_X} & TT'X & \xrightarrow{g_{T'X}} & T'^2X \\ {\scriptstyle m_X}\downarrow & & & & \downarrow{\scriptstyle m'_X} \\ TX & & \xrightarrow[g_X]{} & & T'X \end{array} \qquad \begin{array}{ccc} X & = & X \\ {\scriptstyle u_X}\downarrow & & \downarrow{\scriptstyle u'_X} \\ TX & \xrightarrow[g_X]{} & T'X\,. \end{array}$$

Weak bimonads on $\mathcal{M}$ (as objects) and their morphisms (as arrows) constitute a category $\mathsf{Wbm}(\mathcal{M})$, which contains the category of bimonads on $\mathcal{M}$ as a full subcategory.

Example 2.7. Any monoid R in a monoidal category $\mathcal{M}$, induces a monad $R \otimes \bullet \otimes R$ on $\mathcal{M}$. If R is a separable Frobenius monoid in a Cauchy complete monoidal category $\mathcal{M}$, then $R \otimes \bullet \otimes R$ is a weak bimonad; see Example 1.2 (3). Its opmonoidal structure is provided by the maps

$$R \otimes R \xrightarrow{\varepsilon \circ \mu} K \quad \text{and} \quad R \otimes X \otimes Y \otimes R \xrightarrow{R \otimes X \otimes \delta \circ \eta \otimes Y \otimes R} R \otimes X \otimes R \otimes R \otimes Y \otimes R,$$

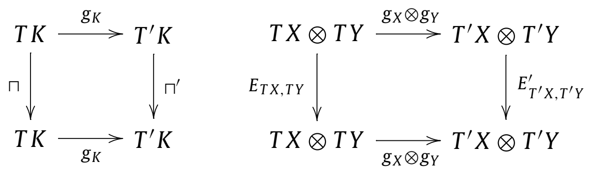
for any objects X, Y in $\mathcal{M}$, where (μ, η) and (δ, ε) denote the monoid and comonoid structures of R, respectively. A morphism $\gamma : R \to R'$ of separable Frobenius monoids in $\mathcal{M}$ induces a morphism of weak bimonads $\gamma \otimes \bullet \otimes \gamma : R \otimes \bullet \otimes R \to R' \otimes \bullet \otimes R'$.

Lemma 2.8. *Consider any morphism $g : T \to T'$ of weak bimonads on a Cauchy complete monoidal category $\mathcal{M}$, with monoidal units R, respectively R', of the Eilenberg–Moore categories $\mathcal{M}^T$ and $\mathcal{M}^{T'}$. There is a unique isomorphism $\gamma : R \to R'$ of separable Frobenius monoids such that the following diagram of functors commutes,*

$$\begin{array}{ccccc} ({}_{R'}\mathcal{M}_{R'})^{\widetilde{T}'} \cong \mathcal{M}^{T'} & & \xrightarrow{g^*} & & \mathcal{M}^T \cong ({}_R\mathcal{M}_R)^{\widetilde{T}} \\ {\scriptstyle \widetilde{U}'}\downarrow & & & & \downarrow{\scriptstyle \widetilde{U}} \\ {}_{R'}\mathcal{M}_{R'} & & \xrightarrow{\gamma^*} & & {}_R\mathcal{M}_R \\ & {\scriptstyle V'}\searrow & & \swarrow{\scriptstyle V} & \\ & & \mathcal{M} & & \end{array} \tag{2.6}$$

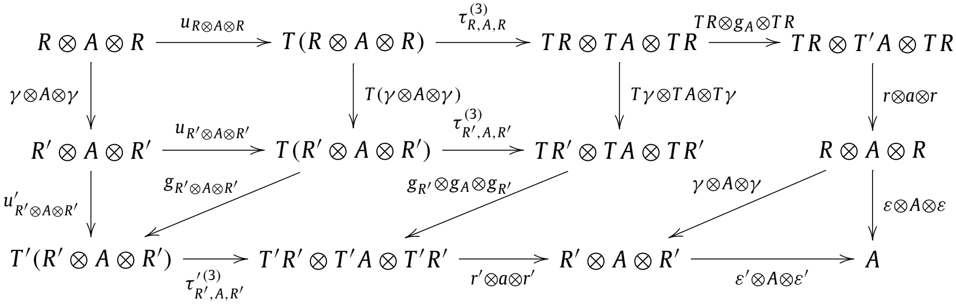
where the bimonads $\widetilde{T}$ and $\widetilde{T}'$ are associated to the weak bimonads T and T' as in Remark 2.4.

Proof. By Lemma 1.8, for weak bimonads T and T', the associated idempotent morphisms $\sqcap$ and $\sqcap'$ in (1.13) split through R and R', respectively; thus there are epi-mono pairs $TK \overset{P}{\twoheadrightarrow} R \overset{I}{\rightarrowtail} TK$ and $T'K \overset{P'}{\twoheadrightarrow} R' \overset{I'}{\rightarrowtail} T'K$. Using the fact that a morphism $g : T \to T'$ of weak bimonads is an opmonoidal natural transformation as well as a morphism of monads, one checks that for any objects X, Y in $\mathcal{M}$, the two diagrams on the left



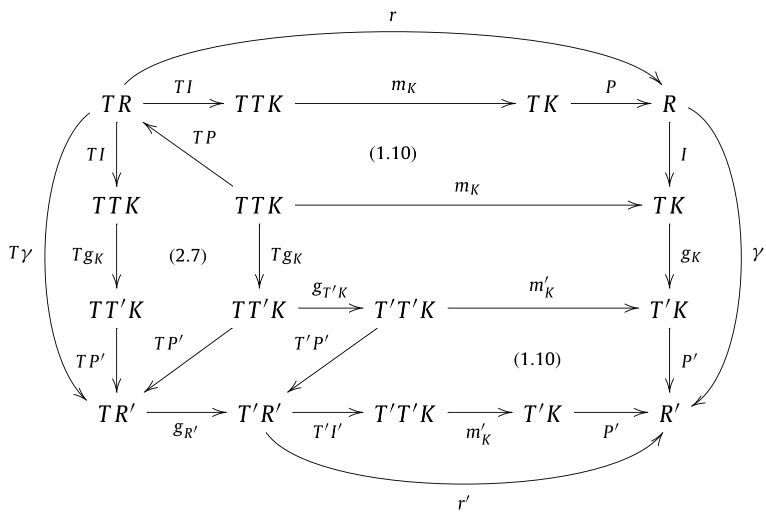
(2.7)

commute and so the morphism γ defined by the diagram on the right is compatible both with the monoid and with the comonoid structures of R and R', written out explicitly in (2.1) and (2.2). That is, γ is a morphism of separable Frobenius monoids. By [16, Proposition A.3], any morphism of Frobenius monoids is an isomorphism hence so is γ. It obviously renders commutative the lower triangle in (2.6). It renders commutative also the upper square by commutativity of the following diagram, for any T'-algebra (A, a).



(2.8)

The two squares in the upper left corner commute by naturality and the square below them commutes by the opmonoidality of g. The triangles in the bottom row commute since g is a monad morphism, and since γ is a comonoid morphism, respectively. The remaining region commutes by commutativity of the following diagram,



(2.9)

where the undecorated region in the middle commutes since g is a morphism of monads. It remains to show that γ is unique with the stated property. The upper part of the diagram in (2.6) commutes

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if and only if, for any T'-algebra (A,a) and the corresponding T-algebra $(A, a \circ g_A)$, the $R \otimes \bullet \otimes R$-action ϱ_A and the $R' \otimes \bullet \otimes R'$-action ϱ'_A in (2.3) obey $\varrho_A = \varrho'_A \circ (\gamma \otimes A \otimes \gamma)$; see (2.8). Applying it in the case where (A,a) is the monoidal unit (R', r') of $\mathscr{M}^{T'}$ and composing the resulting identity on the right with $\eta \otimes \eta' \otimes R$, we deduce that $\gamma = (R' \otimes \varepsilon) \circ E_{R',R} \circ (\eta' \otimes R)$. Thus γ renders commutative the last diagram in (2.7) by the forms of P' and I in (1.11) and (1.12), the form of $E_{R',R}$ in (1.14), and naturality. □

On the other hand, any isomorphism of separable Frobenius monoids clearly induces a strict monoidal isomorphism between the categories of bimodules, which can be seen as a particular case of the following:

Lemma 2.9. *If $g: T \to T'$ is a morphism of weak bimonads on a Cauchy complete monoidal category $\mathscr{M}$ then the induced functor $g^*: \mathscr{M}^{T'} \to \mathscr{M}^T$ is strong monoidal and* (2.6) *is a commutative diagram of separable Frobenius monoidal functors.*

Proof. If T and T' are bimonads then the monoidal structures on $\mathscr{M}^T$ and $\mathscr{M}^{T'}$ are lifted from that on $\mathscr{M}$, and so clearly $g^*: \mathscr{M}^{T'} \to \mathscr{M}^T$ is strong (in fact strict) monoidal. If T is only a weak bimonad then the monoidal structure on $\mathscr{M}^T$ is only weakly lifted from that on $\mathscr{M}$, but it is lifted, up to equivalence, from that on ${}_R\mathscr{M}_R$. Thus if $g: T \to T'$ is a morphism of weak bimonads, then the isomorphism $\gamma: R \to R'$ of the previous lemma induces a strict monoidal isomorphism ${}_{R'}\mathscr{M}_{R'} \to {}_R\mathscr{M}_R$, and now the strong monoidal structure on the composite $\mathscr{M}^{T'} \to {}_{R'}\mathscr{M}_{R'} \to {}_R\mathscr{M}_R$ lifts to a strong monoidal structure on $g^*: \mathscr{M}^{T'} \to \mathscr{M}^T$. Explicitly, this is given by

$$\gamma_{A,B} := \big(A \,\Box\, B \xrightarrow{i_{A,B}} A \otimes B \xrightarrow{p'_{A,B}} A \,\Box'\, B\big),$$

and $\gamma: R \to R'$. □

We now turn to our category of bimonads on categories of bimodules over separable Frobenius monoids in $\mathscr{M}$.

Definition 2.10. For a monoidal category $\mathscr{M}$, the category $\mathsf{Sfbm}(\mathscr{M})$ is defined to have *objects* which are pairs $(R, \widetilde{T})$, consisting of a separable Frobenius monoid R in $\mathscr{M}$ and a bimonad $\widetilde{T}$ on ${}_R\mathscr{M}_R$. *Morphisms* $(R, \widetilde{T}) \to (R', \widetilde{T}')$ are pairs (γ, Γ), consisting of an isomorphism $\gamma: R \to R'$ of separable Frobenius monoids (inducing a strong monoidal isomorphism $\gamma^*: {}_{R'}\mathscr{M}_{R'} \to {}_R\mathscr{M}_R$), and a morphism of bimonads $\Gamma: \widetilde{T} \to \gamma^* \widetilde{T}' (\gamma^*)^{-1}$ (that is, an opmonoidal morphism of monads, in the sense explicated in Definition 2.6).

There is an evident functor $\Psi: \mathsf{Sfbm}(\mathscr{M}) \to \mathsf{Wbm}(\mathscr{M})$ defined as follows. The object map is given by associating to a pair $(R, \widetilde{T})$ the weak bimonad induced by the composite of the forgetful functors $\widetilde{U}: ({}_R\mathscr{M}_R)^{\widetilde{T}} \to {}_R\mathscr{M}_R$ and $V: {}_R\mathscr{M}_R \to \mathscr{M}$, as described in Remark 2.4; explicitly, $\Psi(R, \widetilde{T})X = V\widetilde{T}(R \otimes X \otimes R)$. A morphism $(\gamma, \Gamma): (R, \widetilde{T}) \to (R', \widetilde{T}')$ in $\mathsf{Sfbm}(\mathscr{M})$ gives rise to a commutative diagram of functors

$$\begin{array}{ccc}
\mathscr{M}^{\Psi(R',\widetilde{T}')} \cong ({}_{R'}\mathscr{M}_{R'})^{\widetilde{T}'} & \xrightarrow{\Gamma^*} & ({}_R\mathscr{M}_R)^{\widetilde{T}} \cong \mathscr{M}^{\Psi(R,\widetilde{T})} \\
\Big\downarrow{\scriptstyle \widetilde{U}'} & & \Big\downarrow{\scriptstyle \widetilde{U}} \\
{}_{R'}\mathscr{M}_{R'} & \xrightarrow{\gamma^*} & {}_R\mathscr{M}_R \\
& {\scriptstyle V'} \searrow \quad \swarrow {\scriptstyle V} & \\
& \mathscr{M} &
\end{array}$$

and so in particular to a morphism of monads $g : \Psi(R, \widetilde{T}) \to \Psi(R', \widetilde{T}')$, explicitly, $g_X : V\widetilde{T}(R \otimes X \otimes R) \to V'\widetilde{T}'(R' \otimes X \otimes R')$ is given by

$$\begin{array}{ccc} V\widetilde{T}(R\otimes X\otimes R) \xrightarrow{V\Gamma_{R\otimes X\otimes R}} V\gamma^*\widetilde{T}'(\gamma^*)^{-1}(R\otimes X\otimes R) & & \\ \| & & \\ V'\widetilde{T}'(\gamma^*)^{-1}(R\otimes X\otimes R) & \xrightarrow{V'\widetilde{T}'(\gamma\otimes X\otimes\gamma)} & V'\widetilde{T}'(R'\otimes X\otimes R') \end{array}$$

and it is opmonoidal since V, V', and $\widetilde{T}'$ are opmonoidal functors, and Γ and $\gamma \otimes X \otimes \gamma : (\gamma^*)^{-1}(R \otimes X \otimes R) \to R' \otimes X \otimes R'$ are opmonoidal natural transformations.

Theorem 2.11. *If $\mathscr{M}$ is a Cauchy complete monoidal category, the functor $\Psi : \mathsf{Sfbm}(\mathscr{M}) \to \mathsf{Wbm}(\mathscr{M})$ is an equivalence of categories.*

Proof. First we show that Ψ is fully faithful. Suppose then that objects $(R, \widetilde{T})$ and $(R', \widetilde{T}')$ of $\mathsf{Sfbm}(\mathscr{M})$ are given. We must show that any morphism $g : \Psi(R, \widetilde{T}) \to \Psi(R', \widetilde{T}')$ of weak bimonads is induced by a unique morphism $(\gamma, \Gamma) : (R, \widetilde{T}) \to (R', \widetilde{T}')$ in $\mathsf{Sfbm}(\mathscr{M})$. The existence of a unique isomorphism $\gamma : R \to R'$ of monoids, inducing an isomorphism $\gamma^* : {}_{R'}\mathscr{M}_{R'} \to {}_R\mathscr{M}_R$ of categories rendering commutative (2.6), is given by Lemma 2.8. By Lemma 2.9, g induces a strong monoidal functor $g^* : \mathscr{M}^{T'} \to \mathscr{M}^T$. By commutativity of the upper square in (2.6) as a diagram of strong monoidal functors, it is necessarily of the form Γ^* for a unique opmonoidal monad morphism $\Gamma : \widetilde{T} \to \gamma^*\widetilde{T}'(\gamma^*)^{-1}$. This proves that Ψ is fully faithful. It is essentially surjective on objects by Remark 2.4. □

Bimonads are monads in the 2-category OpMon of monoidal categories, opmonoidal functors and opmonoidal natural transformations; cf. [11]. That is, they can be regarded as 0-cells in the 2-category $\mathsf{Mnd}(\mathsf{OpMon})$. Clearly, for a Cauchy complete monoidal category $\mathscr{M}$, the category $\mathsf{Sfbm}(\mathscr{M})$ is a subcategory in the opposite of the category underlying $\mathsf{Mnd}(\mathsf{OpMon})$. We may consider also the full subcategory of the underlying category of $\mathsf{Mnd}(\mathsf{OpMon})$, with objects the bimonads on bimodule categories over separable Frobenius monoids R in Cauchy complete monoidal categories. In this way (using the correspondence in Remark 2.4), we can define more general morphisms between weak bimonads than the arrows in $\mathsf{Wbm}(\mathscr{M})$ for a fixed $\mathscr{M}$. These more general morphisms do not need to preserve the underlying separable Frobenius monoid R.

3. An example: Weak bimonoids in braided monoidal categories

In this section we show that weak bimonoids in a Cauchy complete braided monoidal category $\mathscr{M}$ induce weak bimonads on $\mathscr{M}$.

Theorem 3.1. *For a monoid (B, μ, η) in a Cauchy complete braided monoidal category $(\mathscr{M}, \otimes, K, c)$, there is a bijection between*

(1) *weak bimonoids of the form $(B, \mu, \eta, \delta, \varepsilon)$ in $\mathscr{M}$;*
(2) *weak bimonads $(\bullet \otimes B, \bullet \otimes \mu, \bullet \otimes \eta, \tau, \tau_0)$ on $\mathscr{M}$ for which the diagram*

$$\begin{array}{ccc} X\otimes Y\otimes B & \xrightarrow{X\otimes Y\otimes \tau_{K,K}} & X\otimes Y\otimes B\otimes B \\ & \searrow^{\tau_{X,Y}} & \downarrow{\scriptstyle X\otimes c_{Y,B}\otimes B} \\ & & X\otimes B\otimes Y\otimes B \end{array} \tag{3.1}$$

commutes for all objects X, Y of $\mathscr{M}$.

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Remark 3.2. Consider a monoid B in a Cauchy complete braided monoidal category $(\mathscr{M}, \otimes K, c)$ such that $\bullet \otimes B$ is a weak bimonad on $\mathscr{M}$. By naturality, for all morphisms $f : K \to X$, $g : K \to Y$, $h : K \to B$, the natural transformation $\tau_{X,Y} : X \otimes Y \otimes B \to X \otimes B \otimes Y \otimes B$ makes

$$\begin{array}{ccccc}
K & \xrightarrow{f\otimes g\otimes h} & X\otimes Y\otimes B & \xrightarrow{X\otimes Y\otimes \tau_{K,K}} & X\otimes Y\otimes B\otimes B\\
{\scriptstyle h}\downarrow & \searrow^{f\otimes g\otimes h} & & & \\
B & & X\otimes Y\otimes B & & \downarrow{\scriptstyle X\otimes c_{Y,B}\otimes B}\\
{\scriptstyle \tau_{K,K}}\downarrow & & & \searrow^{\tau_{X,Y}} & \\
B\otimes B & & \xrightarrow[f\otimes B\otimes g\otimes B]{} & & X\otimes B\otimes Y\otimes B
\end{array}$$

commute. Hence (3.1) holds provided that the monoidal unit K is a '*cubic generator*' in the following sense: If, for some morphisms $p, q : X \otimes Y \otimes Z \to W$ in $\mathscr{M}$, the equality $p \circ (f \otimes g \otimes h) = q \circ (f \otimes g \otimes h)$ holds, for *all* morphisms $f : K \to X$, $g : K \to Y$, $h : K \to Z$, then $p = q$.

The monoidal unit is a 'cubic generator', for example, in the symmetric monoidal category $\mathsf{Mod}(k)$ of modules over a commutative ring k. With this observation in mind, Theorem 3.1 includes Szlachányi's description in [21, Corollary 6.5] of *weak bialgebras* over k as *weak bimonads* on $\mathsf{Mod}(k)$.

Proof of Theorem 3.1. Suppose that $(B, \mu, \eta, \delta, \varepsilon)$ is a weak bimonoid in $\mathscr{M}$. By [16, Proposition 3.8], the category of B-modules is monoidal and there is a strong monoidal functor from it to a certain bimodule category ${}_R\mathscr{M}_R$. Furthermore, the resulting monoid R is a separable Frobenius monoid by [16, Proposition 1.4]. In view of Example 1.2, this proves that $\bullet \otimes B$ is a weak bimonad. Its opmonoidal structure comes out with $\tau_{X,Y}$ equal to the composite

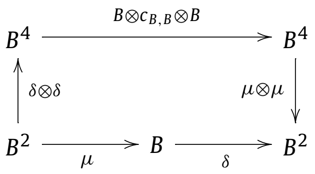
$$X \otimes Y \otimes B \xrightarrow{X\otimes Y\otimes\delta} X \otimes Y \otimes B \otimes B \xrightarrow{X\otimes c_{Y,B}\otimes B} X \otimes B \otimes Y \otimes B$$

and $\tau_0 = \varepsilon$. Hence (3.1) is satisfied.

Assume conversely that (2) holds. We claim that $(B, \mu, \eta, \delta := \tau_{K,K}, \varepsilon := \tau_0)$ is a weak bimonoid in $\mathscr{M}$. The functor $\bullet \otimes B$ is opmonoidal and so sends comonoids to comonoids; in particular, it sends the comonoid K in $\mathscr{M}$ to a comonoid, which turns out to be (B, δ, ε). Use (3.1) to write $\tau_{X,Y}$ as $(X \otimes c_{Y,B} \otimes B) \circ (X \otimes Y \otimes \delta)$, for any objects X, Y of $\mathscr{M}$. Substituting this expression in conditions (1.3)–(1.7), we obtain the following commutative diagrams.

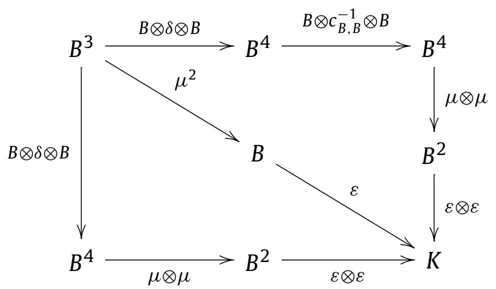
$$\begin{array}{ccccccc}
B^2 & \xrightarrow{B\otimes\eta\otimes B} & B^3 & \xrightarrow{B\otimes\delta\otimes B} & B^4 & \xrightarrow{B\otimes c^{-1}_{B,B}\otimes B} & B^4\\
\downarrow{\scriptstyle B\otimes\delta} & & & & & & \downarrow{\scriptstyle \mu\otimes\mu}\\
B^3 & & & & & & B^2\\
\downarrow{\scriptstyle B\otimes c^{-1}_{B,B}} & & & & & & \downarrow{\scriptstyle \varepsilon\otimes B}\\
B^3 & & \xrightarrow[\mu\otimes B]{} & B^2 & \xrightarrow[\varepsilon\otimes B]{} & & B
\end{array}
\qquad
\begin{array}{ccccc}
B^2 & \xrightarrow{B\otimes\eta\otimes B} & B^3 & \xrightarrow{B\otimes\delta\otimes B} & B^4\\
 & & & & \downarrow{\scriptstyle \mu\otimes\mu}\\
\downarrow{\scriptstyle B\otimes\delta} & & & & B^2\\
 & & & & \downarrow{\scriptstyle \varepsilon\otimes B}\\
B^3 & \xrightarrow[\mu\otimes B]{} & B^2 & \xrightarrow[\varepsilon\otimes B]{} & B
\end{array}
\tag{3.2}$$

$$\begin{array}{ccccccc}
K & & \xrightarrow{\eta} & B & \xrightarrow{\delta} & & B^2\\
\downarrow{\scriptstyle \eta\otimes\eta} & & & & & & \downarrow{\scriptstyle \delta\otimes B}\\
B^2 & \xrightarrow[\delta\otimes\delta]{} & B^4 & \xrightarrow[B\otimes c^{-1}_{B,B}\otimes B]{} & B^4 & \xrightarrow[B\otimes\mu\otimes B]{} & B^3
\end{array}
\qquad
\begin{array}{ccccc}
K & \xrightarrow{\eta} & B & \xrightarrow{\delta} & B^2\\
\downarrow{\scriptstyle \eta\otimes\eta} & & & & \downarrow{\scriptstyle \delta\otimes B}\\
B^2 & \xrightarrow[\delta\otimes\delta]{} & B^4 & \xrightarrow[B\otimes\mu\otimes B]{} & B^3
\end{array}
\tag{3.3}$$



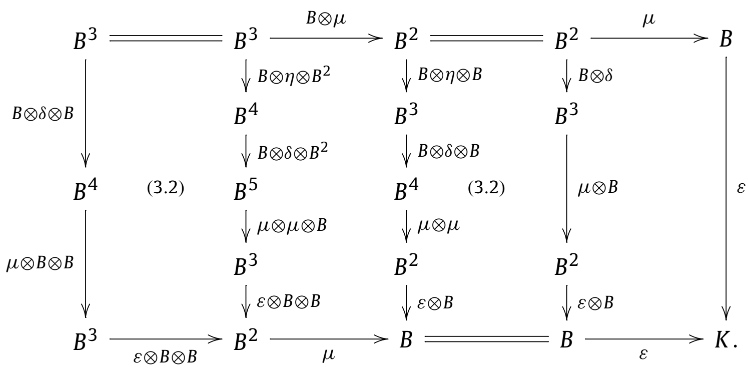
(3.4)

Condition (3.4) is identical to axiom (b), and the identities in (3.3) are identical to axiom (w) in the definition of a weak bimonoid in [16, Definition 2.1]. Thus we only need to show that, whenever the diagrams in (3.2), (3.3), and (3.4) commute, then axiom (v) in [16, Definition 2.1] holds; that is, the following diagram commutes.



(3.5)

Commutativity of the lower triangle in (3.5) follows by commutativity of



The undecorated regions commute by naturality, counitality of δ and by associativity of μ. Commutativity of the upper triangle in (3.5) is proved similarly, making use of the first identity in (3.2). □

4. The antipode

The first attempt to equip Moerdijk's bimonad with an antipode, i.e. to define a Hopf monad, was made by Bruguières and Virelizier in [6]. Here the authors studied bimonads on autonomous monoidal categories such that the (left/right) duals lift to the Eilenberg–Moore category. This generalizes finite dimensional Hopf algebras to the categorical setting.

A more general notion of Hopf monad was introduced in [5] (see also [7]). This is based on the observation of Lawvere [10] that a right adjoint preserves internal homs precisely when Frobenius reciprocity holds; this Frobenius reciprocity condition also appeared in [17, Theorem & Definition 3.5] in the context of Takeuchi bialgebroids. Based on this result, the following definition was proposed

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also for monoidal categories which are not necessarily closed. For any monad (T, m, u) on a monoidal category $(\mathscr{M}, \otimes, K)$ such that T admits an opmonoidal structure (τ, τ_0) (hence in particular for any (weak) bimonad T on $\mathscr{M}$), there is a canonical natural transformation, given for any objects X, Y of $\mathscr{M}$ by

$$\mathsf{can}_{X,Y} := \big(T(TX \otimes Y) \xrightarrow{\tau_{TX,Y}} T^2X \otimes TY \xrightarrow{m_X \otimes TY} TX \otimes TY\big). \tag{4.1}$$

By the terminology in [5], a bimonad is called a *right Hopf monad* whenever the associated natural transformation (4.1) is invertible. Similarly, a bimonad is a *left Hopf monad* when the analogous natural transformation $T(X \otimes TY) \to TX \otimes TY$ is invertible, and a *Hopf monad* when it is both left and right Hopf.

In this section we propose a definition of a *right weak Hopf monad* T on a monoidal category $\mathscr{M}$ – characterized by the property that, whenever $\mathscr{M}$ is also Cauchy complete, the associated bimonad $\widetilde{T}$ (on another monoidal category) in Remark 2.4, is a right Hopf monad, with analogous definitions for left weak Hopf monads and weak Hopf monads.

Suppose that T is a weak bimonad on a Cauchy complete monoidal category $\mathscr{M}$, and R the corresponding separable Frobenius monoid, with forgetful functor $V : {}_R\mathscr{M}_R \to \mathscr{M}$ and $G \dashv V$. To say that $\widetilde{T}$ is right Hopf is to say that for all $\widetilde{X}, \widetilde{Y} \in {}_R\mathscr{M}_R$, the canonical morphism

$$\widetilde{T}(\widetilde{T}\widetilde{X} \otimes_R \widetilde{Y}) \xrightarrow{\widetilde{\mathsf{can}}_{\widetilde{X},\widetilde{Y}}} \widetilde{T}\widetilde{X} \otimes_R \widetilde{T}\widetilde{Y}$$

is invertible. Since every $\widetilde{X} \in {}_R\mathscr{M}_R$ is (naturally) a retract of one of the form GX, this will be the case precisely when

$$\widetilde{T}(\widetilde{T}GX \otimes_R GY) \xrightarrow{\widetilde{\mathsf{can}}_{GX,GY}} \widetilde{T}GX \otimes_R \widetilde{T}GY$$

is invertible. Now

$$\widetilde{T}(\widetilde{T}GX \otimes_R GY) \cong \widetilde{T}\big(TX \otimes_R (R \otimes Y \otimes R)\big) \cong \widetilde{T}(TX \otimes Y \otimes R)$$

and $V\widetilde{T}(TX \otimes Y \otimes R)$ is a retract of $T(TX \otimes Y)$ by construction of $\widetilde{T}$, while

$$V(\widetilde{T}GX \otimes_R \widetilde{T}GY) \cong TX \,\square\, TY$$

which is a retract of $TX \otimes TY$. Thus we obtain a composite map

$$T(TX \otimes Y) \xrightarrow{q_{X,Y}} V\widetilde{T}(\widetilde{T}GX \otimes_R GY) \xrightarrow{\widetilde{\mathsf{can}}_{GX,GY}} V(\widetilde{T}GX \otimes_R \widetilde{T}GY) \xrightarrow{i_{TX,TY}} TX \otimes TY \tag{4.2}$$

which turns out to be the canonical map $\mathsf{can}_{X,Y}$ associated to T itself.

Now the inclusion $TX \,\square\, TY \to TX \otimes TY$ is the section for a splitting of the idempotent $E_{TX,TY}$ on $TX \otimes TY$ defined in (1.14).

On the other hand, the quotient $T(TX \otimes Y) \to V\widetilde{T}(\widetilde{T}GX \otimes_R GY)$ is the retraction of a splitting of an idempotent $F_{X,Y}$ on $T(TX \otimes Y)$ defined by

$$T(TX \otimes Y) \xrightarrow{T(\delta\circ\eta\otimes TX\otimes Y\otimes\eta)} T(R \otimes R \otimes TX \otimes Y \otimes R) \xrightarrow{T(R\otimes\beta_{TX}\otimes Y\otimes R)} T(R \otimes TX \otimes Y \otimes R)$$
$$\xrightarrow{\lambda_{TX\otimes Y}} T(TX \otimes Y), \tag{4.3}$$

where β_{TX} denotes the left R-action on TX and λ is the natural transformation in (2.4).

To say that $\widetilde{\mathsf{can}}$ is invertible, is to say that can induces an isomorphism between the splittings of the idempotents $F_{X,Y}$ and $E_{TX,TY}$. We then call T a weak right Hopf monad:

Definition 4.1. A weak bimonad T on a monoidal category $(\mathscr{M}, \otimes, K)$ is said to be a *weak right Hopf monad* provided that there is a natural transformation $\chi_{X,Y} : TX \otimes TY \to T(TX \otimes Y)$ such that, for the canonical natural transformation can of T in (4.1), for the idempotent morphisms $E_{TX,TY}$ and $F_{X,Y}$ (4.3), and for any objects X, Y of $\mathscr{M}$,

$$\chi_{X,Y} \circ E_{TX,TY} = \chi_{X,Y} = F_{X,Y} \circ \chi_{X,Y}, \qquad \chi_{X,Y} \circ \mathsf{can}_{X,Y} = F_{X,Y}, \qquad \mathsf{can}_{X,Y} \circ \chi_{X,Y} = E_{TX,TY}. \tag{4.4}$$

The definition just given makes sense for any monoidal category $\mathscr{M}$, but is motivated by the following theorem, which requires $\mathscr{M}$ to be Cauchy complete.

Theorem 4.2. *For any weak bimonad T on a Cauchy complete monoidal category $(\mathscr{M}, \otimes, K)$, and the associated bimonad $\widetilde{T}$ in Remark 2.4, the following assertions are equivalent.*

(1) *The canonical natural transformation $\widetilde{\mathsf{can}}$ of $\widetilde{T}$ as in (4.1), is an isomorphism; that is, $\widetilde{T}$ is a right Hopf monad.*
(2) *There is a natural transformation $\chi_{X,Y} : TX \otimes TY \to T(TX \otimes Y)$ obeying (4.4); that is, T is a weak right Hopf monad.*

Proof. The equations in (4.4) state exactly that the morphism induced by $\chi_{X,Y}$ between the splittings of $F_{X,Y}$ and $E_{TX,TY}$ is inverse to the morphism $\widetilde{\mathsf{can}}_{X,Y}$ induced by $\mathsf{can}_{X,Y}$ between the splittings of $E_{TX,TY}$ and $F_{X,Y}$. □

Remark 4.3. Consider a weak right Hopf monad T on a Cauchy complete monoidal category $\mathscr{M}$ with corresponding separable Frobenius monoid R. By Theorem 4.2 and [5, Theorem 3.6] we conclude that whenever the category of R-bimodules is right closed, this closed structure lifts to the Eilenberg–Moore category $\mathscr{M}^T$. The category of R-bimodules is right closed whenever $\mathscr{M}$ is right closed (in which case the internal homs are defined by splitting an appropriate idempotent natural transformation).

Next we show that, as expected, a weak bimonoid B in a Cauchy complete braided monoidal category $\mathscr{M}$, induces a right weak Hopf monad $\bullet \otimes B$ on $\mathscr{M}$ if and only if it is a weak Hopf monoid in the sense of [1,16].

Lemma 4.4. *For an arbitrary category $\mathscr{C}$, consider a functor $T : \mathscr{C} \to \mathscr{C}$ which admits both a monad structure $\underline{T} = (T, m, u)$ and a comonad structure $\overline{T} = (T, d, e)$. Denote by $\underline{U} : \mathscr{C}^{\underline{T}} \to \mathscr{C}$ and by $\overline{U} : \mathscr{C}^{\overline{T}} \to \mathscr{C}$ the corresponding forgetful functors with respective left adjoint $\underline{F} : \mathscr{C} \to \mathscr{C}^{\underline{T}}$ and right adjoint $\overline{F} : \mathscr{C} \to \mathscr{C}^{\overline{T}}$. The following monoids (in* Set*) are isomorphic.*

(1) *The monoid of natural transformations $\overline{F}\underline{U} \to \overline{F}\underline{U}$, with multiplication given by the composition of natural transformations.*
(2) *The monoid of those natural transformations $\gamma : \overline{F}T \to \overline{F}T$ for which $\overline{F}m \circ \gamma T = \gamma \circ \overline{F}m$, with multiplication given by the composition of natural transformations.*
(3) *The monoid of natural transformations $T \to T$, with multiplication given by the 'convolution product' $\varphi * \varphi' := m \circ T\varphi' \circ \varphi T \circ d$.*

Proof. (1) ≅ (2) The stated isomorphism is given by the maps $\mathsf{Nat}(\overline{F}\underline{U}, \overline{F}\underline{U}) \ni \beta \mapsto \beta\underline{F}$, with the inverse $\gamma \mapsto \overline{F}\underline{U}\underline{\kappa} \circ \gamma\underline{U} \circ \overline{F}u\underline{U}$, where $\underline{\kappa}$ is the counit of the adjunction $\underline{F} \dashv \underline{U}$.

(1) ≅ (3) This is the adjunction isomorphism $\mathsf{Nat}(\overline{F}\underline{U}, \overline{F}\underline{U}) \cong \mathsf{Nat}(\overline{U}\,\overline{F}, \underline{U}\underline{F})$. □

For a functor T as in Lemma 4.4, one may consider the so-called '*fusion operator*' in [20],

$$\gamma := \big(T^2 \xrightarrow{dT} T^3 \xrightarrow{Tm} T^2\big). \tag{4.5}$$

Clearly, it belongs to the monoid in Lemma 4.4 (2). The corresponding element of the isomorphic monoid in Lemma 4.4 (3) is the identity natural transformation $T \to T$. (Hence, incidentally, Lemma 4.4 provides an alternative proof of [13, Theorem 5.5].)

Lemma 4.5. *For a weak bimonoid $(B, \mu, \eta, \delta, \varepsilon)$ in a Cauchy complete braided monoidal category $(\mathscr{M}, \otimes, K, c)$, and its induced weak bimonad $T := \bullet \otimes B$, the following assertions hold, for any objects X, Y of $\mathscr{M}$.*

(i) *For the natural transformation* (4.1) *of $T = \bullet \otimes B$,*

$$\mathsf{can}_{X,Y} = (X \otimes c_{Y,B} \otimes B) \circ (X \otimes Y \otimes \gamma_K) \otimes \big(X \otimes c^{-1}_{Y,B} \otimes B\big),$$

where γ is the fusion operator (4.5) *for the monad and comonad $\bullet \otimes B$.*

(ii) *The idempotent natural transformation $E_{TX,TY}$ on $TX \otimes TY$* (1.14) *satisfies*

$$E_{TX,TY} = (X \otimes c_{Y,B} \otimes B) \circ (X \otimes Y \otimes E_{TK,TK}) \otimes \big(X \otimes c^{-1}_{Y,B} \otimes B\big). \tag{4.6}$$

Moreover, $\bullet \otimes E_{TK,TK}$ belongs to the monoid in Lemma 4.4 (2) *and the corresponding element of the isomorphic monoid in Lemma* 4.4 (3) *is $\bullet \otimes t$, where t is the composite*

$$B \xrightarrow{B\otimes\eta} B^2 \xrightarrow{B\otimes\delta} B^3 \xrightarrow{c_{B,B}\otimes B} B^3 \xrightarrow{B\otimes\mu} B^2 \xrightarrow{B\otimes\varepsilon} B.$$

(iii) *The idempotent natural transformation $F_{X,Y}$ on $T(TX \otimes Y)$* (4.3) *satisfies*

$$F_{X,Y} = (X \otimes c_{Y,B} \otimes B) \circ (X \otimes Y \otimes F_{K,K}) \circ \big(X \otimes c^{-1}_{Y,B} \otimes B\big). \tag{4.7}$$

Moreover, $\bullet \otimes F_{K,K}$ belongs to the monoid in Lemma 4.4 (2) *and the corresponding element of the isomorphic monoid in Lemma* 4.4 (3) *is $\bullet \otimes r$, where r is the composite*

$$B \xrightarrow{\eta\otimes B} B^2 \xrightarrow{\delta\otimes B} B^3 \xrightarrow{B\otimes c_{B,B}} B^3 \xrightarrow{\mu\otimes B} B^2 \xrightarrow{\varepsilon\otimes B} B.$$

(iv) *If in addition $T := \bullet \otimes B$ is a weak right Hopf monad; that is, there exists a natural transformation χ obeying* (4.4)*, then*

$$\chi_{X,Y} = (X \otimes c_{Y,B} \otimes B) \circ (X \otimes Y \otimes \chi_{K,K}) \otimes \big(X \otimes c^{-1}_{Y,B} \otimes B\big) \tag{4.8}$$

and $\bullet \otimes \chi_{K,K}$ belongs to the monoid in Lemma 4.4 (2).

Proof. Assertion (i) is immediate by relation (3.1) between the opmonoidal structure $\tau_{X,Y}$ of $T = \bullet \otimes B$ and the comultiplication $\delta = \tau_{K,K}$ in $B = TK$.

(ii) Eq. (4.6) follows from the formula (1.14) for $E_{TX,TY}$ and $E_{TK,TK}$. Then the morphism

$$E_{TK,TK} = \big(B^2 \xrightarrow{B^2\otimes\eta} B^3 \xrightarrow{B^2\otimes\delta} B^4 \xrightarrow{B\otimes c_{B,B}\otimes B} B^4 \xrightarrow{\mu\otimes\mu} B^2\big)$$

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renders commutative the first diagram in

$$\begin{array}{ccc} B^3 & \xrightarrow{B\otimes E_{TK,TK}} & B^3 \\ {\scriptstyle \mu\otimes B}\downarrow & & \downarrow{\scriptstyle \mu\otimes B} \\ B^2 & \xrightarrow[E_{TK,TK}]{} & B^2 \end{array} \qquad \begin{array}{ccc} B^2 & \xrightarrow{E_{TK,TK}} & B^2 \\ {\scriptstyle B\otimes\delta}\downarrow & & \downarrow{\scriptstyle B\otimes\delta} \\ B^3 & \xrightarrow[E_{TK,TK}\otimes B]{} & B^3 \end{array} \tag{4.9}$$

by associativity of μ. By self-duality of the axioms of a weak bimonoid, the dual of (3.2) holds; that is, the first diagram in

$$\begin{array}{ccccccc} B & \xrightarrow{\eta\otimes B} & B^2 & \xrightarrow{\delta\otimes B} & B^3 & \xrightarrow{B\otimes c^{-1}_{B,B}} & B^3 \\ {\scriptstyle \eta\otimes B}\downarrow & & & & & & \\ B^2 & & & & & & \downarrow{\scriptstyle B\otimes\mu} \\ {\scriptstyle \delta\otimes\delta}\downarrow & & & & & & \\ B^4 & \xrightarrow[B\otimes c^{-1}_{B,B}\otimes B]{} & B^4 & \xrightarrow[B\otimes\mu\otimes B]{} & B^3 & \xrightarrow[B\otimes\varepsilon\otimes B]{} & B^2 \end{array} \qquad \begin{array}{ccc} B^3 & \xrightarrow{E_{TK,TK}\otimes B} & B^3 \\ {\scriptstyle B\otimes\delta}\uparrow & & \downarrow{\scriptstyle B\otimes\varepsilon\otimes B} \\ B^2 & \xrightarrow[E_{TK,TK}]{} & B^2 \end{array}$$

commutes. Tensoring on the left with B and then composing with $\mu \otimes B$ gives commutativity of the diagram on the right. It follows by coassociativity of δ that also the second diagram in (4.9) commutes. This proves that $\bullet \otimes E_{TK,TK}$ belongs to the monoid in Lemma 4.4 (2). The corresponding element of the isomorphic monoid in Lemma 4.4 (3) is $(B \otimes \varepsilon) \circ E_{TK,TK} \circ (\eta \otimes B) = t$ as stated, by unitality of μ.

(iii) Similarly to part (ii), one easily checks that

$$F_{X,Y} = (X \otimes c_{Y,B} \otimes B) \circ \big(X \otimes Y \otimes (\mu \otimes \varepsilon \circ \mu \otimes B) \circ \big(B \otimes c^{-1}_{B,B} \circ \delta \circ \eta \otimes \delta\big)\big) \circ \big(X \otimes c^{-1}_{Y,B} \otimes B\big),$$

which proves (4.8). By associativity of μ and by coassociativity of δ,

$$F_{K,K} = \big(B^2 \xrightarrow{B\otimes\eta\otimes B} B^3 \xrightarrow{B\otimes\delta\otimes\delta} B^5 \xrightarrow{B\otimes c^{-1}_{B,B}\otimes B^2} B^5 \xrightarrow{\mu\otimes\mu\otimes B} B^3 \xrightarrow{B\otimes\varepsilon\otimes B} B^2\big)$$

makes commute both diagrams in

$$\begin{array}{ccc} B^3 & \xrightarrow{B\otimes F_{K,K}} & B^3 \\ {\scriptstyle \mu\otimes B}\downarrow & & \downarrow{\scriptstyle \mu\otimes B} \\ B^2 & \xrightarrow[F_{K,K}]{} & B^2 \end{array} \qquad \begin{array}{ccc} B^2 & \xrightarrow{F_{K,K}} & B^2 \\ {\scriptstyle B\otimes\delta}\downarrow & & \downarrow{\scriptstyle B\otimes\delta} \\ B^3 & \xrightarrow[F_{K,K}\otimes B]{} & B^3. \end{array} \tag{4.10}$$

Thus $\bullet \otimes F_{K,K}$ is an element of the monoid in Lemma 4.4 (2). By unitality of μ and counitality of δ, the corresponding element $(B \otimes \varepsilon) \circ F_{K,K} \circ (\eta \otimes B)$ of the isomorphic monoid in Lemma 4.4 (3) is the stated morphism r.

(iv) In (4.2) we have seen a relationship between the canonical morphism can of T and the canonical morphism $\widetilde{\mathsf{can}}$ of the weakly lifted bimonad $\widetilde{T}$. Using this along with part (i), we deduce that $\widetilde{\mathsf{can}}_{R\otimes X\otimes R, R\otimes Y\otimes R}$ is equal to

$$p_{TX,TY} \circ (X \otimes c_{Y,B} \otimes B) \circ (X \otimes Y \otimes i_{TK,TK} \circ \widetilde{\mathsf{can}}_{R\otimes R,R\otimes R} \circ q_{K,K}) \circ \big(X \otimes c^{-1}_{Y,B} \otimes B\big) \circ j_{X,Y}$$

where $p_{TX,TY}$ is the epi part of the splitting of $E_{TX,TY}$, and $j_{X,Y}$ is the mono part of the splitting of $F_{X,Y}$. Hence in view of (4.6) and (4.7), $\widetilde{\mathsf{can}}^{-1}_{R\otimes X\otimes R,R\otimes Y\otimes R}$ is equal to

$$q_{X,Y}\circ(X\otimes c_{Y,B}\otimes B)\circ\big(X\otimes Y\otimes j_{K,K}\circ\widetilde{\mathsf{can}}^{-1}_{R\otimes R,R\otimes R}\circ p_{TK,TK}\big)\circ\big(X\otimes c^{-1}_{Y,B}\otimes B\big)\circ i_{TX,TY}.$$

Thus for $\chi_{X,Y}=j_{X,Y}\circ\widetilde{\mathsf{can}}^{-1}_{R\otimes X\otimes R,R\otimes Y\otimes R}\circ p_{TX,TY}$, the required condition (4.8) holds.

We need to show that $\chi_{K,K}$ induces a natural transformation as in Lemma 4.4 (2). By part (i), $\mathsf{can}_{K,K}=\gamma_K$ induces such a natural transformation. Hence in view of (4.2), since $i_{TX,TY}$ is a morphism of left B-modules and of right B-comodules, and by (4.10),

$$(\mu\otimes B)\circ(B\otimes j_{K,K})\circ\big(B\otimes\widetilde{\mathsf{can}}^{-1}_{R\otimes R,R\otimes R}\big)=j_{K,K}\circ\widetilde{\mathsf{can}}^{-1}_{R\otimes R,R\otimes R}\circ(\mu\otimes_R B)\quad\text{and}$$
$$(B\otimes\delta)\circ j_{K,K}\circ\widetilde{\mathsf{can}}^{-1}_{R\otimes R,R\otimes R}=(j_{K,K}\otimes B)\circ\big(\widetilde{\mathsf{can}}^{-1}_{R\otimes R,R\otimes R}\otimes B\big)\circ(B\otimes_R\delta).$$

Since $p_{TK,TK}$ is a morphism of left B-modules and of right B-comodules, this implies that $\chi_{K,K}=j_{K,K}\circ\widetilde{\mathsf{can}}^{-1}_{R\otimes R,R\otimes R}\circ p_{TK,TK}$ belongs to the monoid in Lemma 4.4 (2). □

Theorem 4.6. *For a weak bimonoid B in a Cauchy complete braided monoidal category $(\mathcal{M},\otimes,K,c)$, the induced functor $\bullet\otimes B$ is a weak right Hopf monad if and only if B is a weak Hopf monoid.*

Proof. By Lemma 4.5, $\bullet\otimes B$ is a weak right Hopf monad if and only if (using the same notation in the lemma) there is an element $X\otimes Y\otimes\chi_{K,K}:X\otimes Y\otimes B\otimes B\to X\otimes Y\otimes B\otimes B$ of the monoid in Lemma 4.4 (2), such that

$$\chi_{K,K}\circ E_{TK,TK}=\chi_{K,K}=F_{K,K}\circ\chi_{K,K},\qquad\chi_{K,K}\circ\mathsf{can}_{K,K}=F_{K,K},\qquad\mathsf{can}_{K,K}\circ\chi_{K,K}=E_{TK,TK}.$$

By Lemma 4.4, this is equivalent to the existence of a morphism $\nu:B\to B$, such that

$$\nu*r=\nu=t*\nu,\qquad\nu*B=t,\qquad B*\nu=r,$$

where the morphisms $t,r:B\to B$ are introduced in Lemma 4.5 and $*$ denotes the convolution product $f*g=\mu\circ(f\otimes g)\circ\delta$, for any morphisms $f,g:B\to B$ in $\mathcal{M}$. □

Finally we turn to connections between right weak Hopf monads and left weak Hopf monads. Conditions (1.3)–(1.7) are invariant under replacing the monoidal product $\otimes$ with the opposite product $\overline{\otimes}$. That is, if (T,m,u,τ_0,τ) is a weak bimonad on a monoidal category $(\mathcal{M},\otimes,K)$, then $(T,m,u,\overline{\tau}_0,\overline{\tau})$ is a weak bimonad on $(\mathcal{M},\overline{\otimes},K)$, where $\overline{\tau}_0=\tau_0:TK\to K$ and $\overline{\tau}_{X,Y}=\tau_{Y,X}:T(X\,\overline{\otimes}\,Y)=T(Y\otimes X)\to TY\otimes TX=TX\,\overline{\otimes}\,TY$. We say that a weak bimonad (T,m,u,τ_0,τ) is a *left weak Hopf monad* on a monoidal category $(\mathcal{M},\otimes,K)$ provided that $(T,m,u,\overline{\tau}_0,\overline{\tau})$ is a right weak Hopf monad on $(\mathcal{M},\overline{\otimes},K)$. Clearly, this means that the *left canonical map*

$$T(X\otimes TY)\xrightarrow{\tau_{X,TY}}TX\otimes T^2Y\xrightarrow{TX\otimes m_Y}TX\otimes TY$$

induces an isomorphism between the retracts of $T(X\otimes TY)$ and $TX\otimes TY$ defined as above.

Some known facts about right weak Hopf monads immediately translate to left weak Hopf monads: obviously, for a weak Hopf monoid $(B,\mu,\eta,\delta,\varepsilon,\nu)$ in a braided monoidal category $(\mathcal{M},\otimes,K,c)$, the same data $(B,\mu,\eta,\delta,\varepsilon,\nu)$ describe a weak Hopf monoid in $(\mathcal{M},\overline{\otimes},K,\bar{c})$, where the braiding is given by $\bar{c}_{X,Y}=c_{Y,X}:X\,\overline{\otimes}\,Y=Y\otimes X\to X\otimes Y=Y\,\overline{\otimes}\,X$. From Theorem 4.6 we deduce

Proposition 4.7. *For a weak bimonoid B in a Cauchy complete braided monoidal category $(\mathcal{M},\otimes,K,c)$, the following assertions are equivalent*:

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(1) *the weak bimonad* $\bullet \otimes B$ *on* $(\mathscr{M}, \otimes, K)$ *is a right weak Hopf monad*;
(2) *the weak bimonad* $\bullet \otimes B = B \,\overline{\otimes}\, \bullet$ *on* $(\mathscr{M}, \overline{\otimes}, K)$ *is a left weak Hopf monad*;
(3) B *is a weak Hopf monoid in* $(\mathscr{M}, \otimes, K, c)$;
(4) B *is a weak Hopf monoid in* $(\mathscr{M}, \overline{\otimes}, K, \bar{c})$.

In particular, the equivalence of the second and fourth assertions says that a weak bimonoid in a Cauchy complete braided monoidal category is a weak Hopf monoid if and only if it induces a left weak Hopf monad by tensoring on the left.

Our next aim is to describe those weak bimonoids $(B, \mu, \eta, \delta, \varepsilon)$ in a Cauchy complete braided monoidal category $(\mathscr{M}, \otimes, K, c)$ for which $\bullet \otimes B$ is both a right and a left weak Hopf monad.

Consider the weak bimonoid B^{op} in $(\mathscr{M}, \otimes, K, c^{-1})$ with the same comonoid structure (δ, ε) of B, multiplication $\mu^{op} := \mu \circ c^{-1}_{B,B}$ and unit η. Observe that via $c_{\bullet,B} : \bullet \otimes B \to B \otimes \bullet$, the weak bimonads $\bullet \otimes B$ and $B^{op} \otimes \bullet$ are isomorphic. Hence the following assertions on B are equivalent:

(1) the weak bimonad $\bullet \otimes B$ on $(\mathscr{M}, \otimes, K)$ is a left weak Hopf monad;
(2) the weak bimonad $B^{op} \otimes \bullet$ on $(\mathscr{M}, \otimes, K)$ is a left weak Hopf monad;
(3) B^{op} is a weak Hopf monoid in $(\mathscr{M}, \otimes, K, c^{-1})$;
(4) there is a morphism $\nu^{op} : B \to B$ (the *antipode* for B^{op}) such that the following diagrams commute.

$$
\begin{array}{ccccc}
B & \xrightarrow{\delta} & B^2 & \xrightarrow{\nu^{op}\otimes B} & B^2 \\
{\scriptstyle \eta\otimes B}\downarrow & & & & \downarrow{\scriptstyle c^{-1}_{B,B}} \\
B^2 & & & & B^2 \\
{\scriptstyle \delta\otimes B}\downarrow & & & & \downarrow{\scriptstyle \mu} \\
B^3 & \xrightarrow[B\otimes\mu]{} & B^2 & \xrightarrow[B\otimes\varepsilon]{} & B
\end{array}
\qquad
\begin{array}{ccccc}
B & \xrightarrow{\delta} & B^2 & \xrightarrow{B\otimes\nu^{op}} & B^2 \\
{\scriptstyle B\otimes\eta}\downarrow & & & & \downarrow{\scriptstyle c^{-1}_{B,B}} \\
B^2 & & & & B^2 \\
{\scriptstyle B\otimes\delta}\downarrow & & & & \downarrow{\scriptstyle \mu} \\
B^3 & \xrightarrow[\mu\otimes B]{} & B^2 & \xrightarrow[\varepsilon\otimes B]{} & B
\end{array}
\qquad
\begin{array}{ccc}
B & \xrightarrow{\delta^2} & B^3 \\
 & & \downarrow{\scriptstyle \nu^{op}\otimes B\otimes\nu^{op}} \\
{\scriptstyle \nu^{op}}\downarrow & & B^3 \\
 & & \downarrow{\scriptstyle (\mu\circ c^{-1}_{B,B})^2} \\
B & = & B
\end{array}
$$

We shall use the notations s, r, t in weak Hopf monoids, as in [16]; the forms of t and r are recalled in Lemma 4.5 above. The left-bottom path in the first diagram in (4) above (playing the role of t^{op}) is equal to s. The left-bottom path in the second diagram is conveniently denoted by r^{op}. The four morphisms s, t, r, r^{op} obey the following four equations

$$\nu \circ s = r, \qquad \nu \circ r^{op} = t, \qquad s \circ \nu = t, \qquad r^{op} \circ \nu = r. \tag{4.11}$$

The first one is (15) in Appendix B of [16], and the others are proved by similar steps.

Finally we are ready to provide the desired characterization:

Theorem 4.8. *For a weak bimonoid* B *in a Cauchy complete braided monoidal category* $(\mathscr{M}, \otimes, K, c)$, *the following conditions are equivalent*:

(1) *the weak bimonad* $\bullet \otimes B$ *on* $(\mathscr{M}, \otimes, K)$ *is both a right and a left weak Hopf monad*;
(2) B *is a weak Hopf monoid in* $(\mathscr{M}, \otimes, K, c)$ *and* B^{op} *is a weak Hopf monoid in* $(\mathscr{M}, \otimes, K, c^{-1})$;
(3) B *is a weak Hopf monoid in* $(\mathscr{M}, \otimes, K, c)$ *with an invertible antipode* ν;
(4) B^{op} *is a weak Hopf monoid in* $(\mathscr{M}, \otimes, K, c^{-1})$ *with an invertible antipode* ν^{op}.

In case (3), ν^{-1} *will be an antipode for* B^{op}; *in case* (4), $(\nu^{op})^{-1}$ *will be an antipode for* B.

Proof. We have already seen that (1) and (2) are equivalent. We show that (3) is equivalent to (2); the equivalence of (4) and (2) is similar.

Assume first that the (3) holds: B is a weak Hopf monoid in $(\mathscr{M}, \otimes, K, c)$ with an invertible antipode ν. In order to see that ν^{-1} provides an antipode for the weak Hopf monoid B^{op}, compose with ν^{-1} on the left the antipode axioms for B. The first two antipode axioms for B^{op} follow from

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the respective axiom for B, using the anti-multiplicativity of ν [16, (17)] and the identities $\nu \circ s = r$ and $\nu \circ r^{op} = t$, respectively. The third antipode axiom for B^{op} follows from the corresponding axiom for B by [16, (17), (6)]. Thus (2) holds.

Conversely, assume that (2) holds: B admits an antipode ν and B^{op} admits an antipode ν^{op}. In order to see that ν^{op} is a left inverse of ν, use associativity of the multiplication, and coassociativity of the comultiplication in B to compute the convolution product $(\nu^{op} \circ \nu) * \nu * B = \mu^2 \circ ((\nu^{op} \circ \nu) \otimes \nu \otimes B) \circ \delta^2$ in two different ways. On one hand,

$$\big(\big(\nu^{op} \circ \nu\big) * \nu\big) * B = \big(\mu \circ c^{-1}_{B,B} \circ \big(B \otimes \nu^{op}\big) \circ \delta \circ \nu\big) * B = \big(r^{op} \circ \nu\big) * B = r * B = B.$$

The first equality follows by anti-comultiplicativity of ν, cf. [16, (16)]. The second equality is a consequence of one of the antipode axioms for B^{op}. The third equality follows by the identity $r^{op} \circ \nu = r$ (4.11) and the last equality is easily derived from the form of r and axiom (b) in [16]. On the other hand,

$$\big(\nu^{op} \circ \nu\big) * (\nu * B) = \big(\nu^{op} \circ \nu\big) * t = \big(\nu^{op} \circ \nu\big) * (s \circ \nu) = \mu \circ c^{-1}_{B,B} \circ \big(s \otimes \nu^{op}\big) \circ \delta \circ \nu = \nu^{op} \circ \nu.$$

The first equality follows by one of the antipode axioms for B. The second equality follows by the identity $s \circ \nu = t$ (4.11) and the third one follows by anti-comultiplicativity of ν, cf. [16, (16)]. The last equality follows by the weak Hopf monoid identity $t * \nu = \nu$ applied to B^{op}. A symmetrical reasoning shows that ν^{op} is also a right inverse of ν: By [16, (17)], one of the antipode axioms for B^{op}, the identities $\nu \circ s = r$ (4.11) and $r * B = B$,

$$\big(\big(\nu \circ \nu^{op}\big) * \nu\big) * B = B.$$

On the other hand, by one of the antipode axioms for B, the identity $\nu \circ r^{op} = t$ (4.11), by [16, (17)] and the weak Hopf monoid identity $\nu * r = \nu$ applied to B^{op},

$$\big(\nu \circ \nu^{op}\big) * (\nu * B) = \nu \circ \nu^{op}.$$

Thus (3) holds. □

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