Report on the Doctoral Dissertation

1–Motives and Albanese Maps in Arithmetic Geometry

submitted to the Hungarian Academy of Sciences

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This dissertation concerns three important topics in arithmetic geometry related to 1–motives. The first one gives a generalization of a famous theorem of Roitman, the second one establishes new arithmetic duality theorems for 1–motives, and the third is concerned with local–global principles. The results obtained are very substantial and innovative.

1.) On the Albanese map and Suslin homology

Let k be an algebraically closed field, and let X be a variety defined over k. We denote by $CH_0(X)^0$ the degree zero part of the Chow group of zero–cycles defined over X, and by Alb_X the Albanese variety of X. The following result is due to Roitman :

Theorem 1.1 *For X smooth and projective, the Albanese map*

$$\alpha_X : CH_0(X)^0 \to \mathrm{Alb}_X$$

induces an isomorphism on torsion elements of order prime to the characteristic of k.

This implies that the n–torsion subgroup of $CH_0(X)^0$ is finite for all $n > 0$.

In [1], Spieß and Szamuely generalize this result to open subvarieties of projective varieties. Assume that X is smooth and projective, and let U be an open subvariety of X. Then one can consider the generalized Albanese map $U \to \widetilde{\mathrm{Alb}}_U$ of Serre. Moreover, the group $CH_0(X)^0$ has to be replaced by a group $h_0(U)$ obtained via Suslin homology. They prove the following :

Theorem 1.2 *For U an open subvariety of a smooth projective variety, the generalized Albanese map*

$$h_0(U) \to \widetilde{\mathrm{Alb}}_U$$

induces an isomorphism on torsion elements of order prime to the characteristic of k.

The proof reduces the statement to a cohomological comparison isomorphism. This provides a more conceptual proof of Roitman's theorem as well – Roitman's original proof, as well as Bloch's subsequent approach, use several ad hoc arguments. Therefore Theorem 1.2 provides not only an important new result, but it also gives a new and more enlightening approach to Roitman's original theorem.

Moreover, Spieß and Szamuely also prove the following

Theorem 1.3 *Suppose that k is the algebraic closure of a finite field, and let U be an open subvariety of a smooth projective variety defined over k. Then the generalized Albanese map*

$$h_0(U) \to \widetilde{\mathrm{Alb}}_U$$

induces an isomorphism of torsion groups.

Note that the prime-to-the-characteristic part follows from Theorem 1.2. However, if $\mathrm{char}(k) = p$, then the statement concerning the p–part of the torsion is not covered by this theorem. The proof is completely different, and of a more arithmetic nature.

Bibliography

[1] M. Spieß , T. Szamuely, On Albanese maps for smooth quasi–projective varieties, *Math. Ann.* **325**, 1–17.

2.) Arithmetic duality theorems for 1–motives

Duality theorems for Galois cohomology are among the most fundamental results in arithmetic. One of the earliest ones is the following : let K be a p–adic field, and let T be an algebraic torus defined over K. Let us denote by T^* the character group of T. Consider the Galois cohomology groups $H^i(K, T)$ and $H^{2-i}(K, T^*)$ of these group schemes for $i = 0, 1, 2$. Then we get canonical pairings

$$H^i(K, T) \times H^{2-i}(K, T^*) \to \mathbf{Q}/\mathbf{Z}$$

for $i = 0, 1, 2$. The Tate–Nakayama duality theorem asserts that these pairings become perfect if in the cases $i \neq 1$ we replace the groups H^0 by their profinite completions.

Later, Tate observed that given an abelian variety A over K, one obtains similar pairings

$$H^i(K, A) \times H^{1-i}(K, A^*) \to \mathbf{Q}/\mathbf{Z}$$

for $i = 0, 1$, and he proved that these pairings are also perfect. Moreover, he constructed a similar duality pairing for Tate–Shafarevich groups over number fields.

In his joint work with D. Harari [2], Tamás Szamuely obtained a common generalization to the above duality theorems for 1–motives in the sense of Deligne. Recall that a 1–motive over a field F is a two–term complex M of F–group schemes $[Y \to G]$ (placed in degrees -1 and 0), where Y is the F–group scheme associated to a finitely generated free abelian group equipped with a continuous $\mathrm{Gal}(F)$–action and G is a semi–abelian variety over F. They prove the following

Theorem 2.1 *Let K be a local field, and let $M = [Y \to G]$ be a 1–motive over K. For $i = -1, 0, 1, 2$ there are canonical pairings*

$$H^i(K, M) \times H^{1-i}(K, M^*) \to \mathbf{Q}/\mathbf{Z}$$

inducing perfect pairings between

1. *the profinite group $H^{-1}_{\Lambda}(K, M)$ and the discrete group $H^2(K, M^*)$;*

2. *the profinite group $H^0(K, M)^{\Lambda}$ and the discrete group $H^1(K, M^*)$.*

They also obtain important duality results for the Tate–Shafarevich groups of 1–motives defined over algebraic number fields.

These results present natural and very significant generalizations of classical duality results, and they have inspired a fair amount of later research. One of these, concerning 1–motives with torsion, gave rise to the PhD thesis of P. Jossen, supervised by T. Szamuely.

Bibliography

[2]. D. Harari, T. Szamuely, Arithmetic duality theorems for 1–motives, *J. reine angew. Math.* **632** (2009), 233-236.

3.) Local–global principles for 1–motifs

Let k be an algebraic number field, and let X be a variety defined over k. We say that the local–global principle (or Hasse principle) does not hold if X has points locally everywhere, but no global points. There are several classical examples where the local–global principle does hold, for instance quadrics (Hasse–Minkowski theorem), but there are also many counter–examples. In 1970, Manin proposed a method that explains some of the cases where the local–global principle does not hold. Let us denote by $X(A_k)$ the adelic points of the variety X, and by $\mathrm{Br}(X)$ its Brauer group. For a smooth variety X Manin defines a pairing

$$X(A_k) \times \mathrm{Br}(X) \to \mathbf{Q}/\mathbf{Z}.$$

Let us denote by $X(A_k)^{\mathrm{Br}}$ the left kernel of this pairing. This defines an obstruction to the local–global principle, called the Manin obstruction (or Brauer–Manin obstruction). This explains many (though not all) of the counter–examples to the Hasse principle. It is an interesting open problem to decide whether the Manin obstruction is the only one in the case of rational varieties.

In [3], D. Harari and T. Szamuely prove the following

Theorem 3.1. *Given a torsor X under a semi–abelian variety G over a number field whose abelian quotient has finite Tate–Shafarevich group, the Manin obstruction is the only obstruction to the local–global principle.*

This result was known when G is an abelian variety (Manin) or a torus (Sansuc), but the general case is much harder. It was a long–standing open question, raised by Skorobogatov. The proof uses the duality theorems of the previous section.

Theorem 3.1 gave rise to several applications by other mathematicians. In particular, Borovoi, Colliot–Thélène and Skorobogatov generalized it to homogeneous spaces under an arbitrary connected algebraic group.

Bibliography

[3] D. Harari, T. Szamuely, Local–global principles for 1–motives, *Duke Math. J.* **143** (2008), 531–557.

The results of this doctoral dissertation are beautiful and important. They have a considerable impact in the scientific community. For all these reasons, I strongly recommend this doctoral dissertation to be accepted.

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